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Proof of a basic hypergeometric supercongruence modulo the fifth power of a cyclotomic polynomial

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ABSTRACT

By means of the q -Zeilberger algorithm, we prove a basic hypergeometric supercongruence modulo the fifth power of the cyclotomic polynomial $\Phi_n(q)$. This result appears to be quite unique, as in the existing literature so far no basic hypergeometric supercongruences modulo a power greater than the fourth of a cyclotomic polynomial have been proved. We also establish a couple of related results, including a parametric supercongruence.

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1. Introduction



In 1997, Van Hamme [27] conjectured that 13 Ramanujan-type series including

$$\sum_{k=0}^{\infty} (-1)^k (4k+1) \frac{(\frac{1}{2})_k^3}{k!^3} = \frac{2}{\pi}$$

admit nice p -adic analogues, such as

$$\sum_{k=0}^{\frac{p-1}{2}} (-1)^k (4k+1) \frac{(\frac{1}{2})_k^3}{k!^3} \equiv p (-1)^{\frac{p-1}{2}} \pmod{p^3},$$

where $(a)_n = a(a+1) \dots (a+n-1)$ denotes the Pochhammer symbol and p is an odd prime. Up to present, all of the 13 supercongruences have been confirmed. See [21,24] for historic remarks on these supercongruences. Recently, q -analogues of congruences and supercongruences have caught the interests of many authors [1–7,8–20,23,25,26,29]. In particular, the first author and Zudilin [16] devised a method, called ‘creative microscoping’, to prove quite a few q -supercongruences by introducing an additional parameter a . In [13], the authors of this paper proved many additional q -supercongruences by the creative microscoping method. Supercongruences modulo a higher integer power of a prime, or, in

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the q -case, of a cyclotomic polynomial, are very special and usually difficult to prove. As far as we know, until now the result

$$\sum_{k=0}^{\frac{n-1}{2}} [4k+1] \frac{(q; q^2)_k^4}{(q^2; q^2)_k^4} \equiv q^{\frac{1-n}{2}} [n] + \frac{(n^2-1)(1-q)^2}{24} q^{\frac{1-n}{2}} [n]^3 \pmod{[n]\Phi_n(q)^3}, \quad (1)$$

for an odd positive integer n , due to the first author and Wang [15], is the unique q -supercongruence modulo $[n]\Phi_n(q)^3$ in the literature that was completely proved. (Several similar conjectural q -supercongruences are stated in [13] and in [16].) The purpose of this paper is to establish an even higher q -congruence, namely modulo a fifth power of a cyclotomic polynomial. Specifically, we prove the following three theorems. (The first two together confirm a conjecture by the authors [13, Conjecture 5.4].)

Theorem 1.1: *Let $n > 1$ be a positive odd integer. Then*

$$\sum_{k=0}^{\frac{n+1}{2}} [4k-1] \frac{(q^{-1}; q^2)_k^4}{(q^2; q^2)_k^4} q^{4k} \equiv -(1+3q+q^2)[n]^4 \pmod{[n]^4\Phi_n(q)} \quad (2a)$$

and

$$\sum_{k=0}^{n-1} [4k-1] \frac{(q^{-1}; q^2)_k^4}{(q^2; q^2)_k^4} q^{4k} \equiv -(1+3q+q^2)[n]^4 \pmod{[n]^4\Phi_n(q)}. \quad (2b)$$

Theorem 1.2: *Let $n > 1$ be a positive odd integer. Then*

$$\sum_{k=0}^{\frac{n+1}{2}} [4k-1] \frac{(aq^{-1}; q^2)_k (q^{-1}/a; q^2)_k (q^{-1}; q^2)_k^2}{(aq^2; q^2)_k (q^2/a; q^2)_k (q^2; q^2)_k^2} q^{4k} \equiv 0 \pmod{[n]^2(1-aq^n)(a-q^n)}$$

and

$$\sum_{k=0}^{n-1} [4k-1] \frac{(aq^{-1}; q^2)_k (q^{-1}/a; q^2)_k (q^{-1}; q^2)_k^2}{(aq^2; q^2)_k (q^2/a; q^2)_k (q^2; q^2)_k^2} q^{4k} \equiv 0 \pmod{[n]^2(1-aq^n)(a-q^n)}.$$

The $a = -1$ case of Theorem 1.2 admits an even stronger q -congruence.

Theorem 1.3: *Let $n > 1$ be a positive odd integer. Then*

$$\sum_{k=0}^{\frac{n+1}{2}} [4k-1] \frac{(q^{-2}; q^4)_k^2}{(q^4; q^4)_k^2} q^{4k} \equiv -q^n(1-q+q^2)[n]_{q^2}^2 \pmod{[n]_{q^2}^2\Phi_n(q^2)} \quad (3a)$$

and

$$\sum_{k=0}^{n-1} [4k-1] \frac{(q^{-2}; q^4)_k^2}{(q^4; q^4)_k^2} q^{4k} \equiv -(1-q+q^2)[n]_{q^2}^2 \pmod{[n]_{q^2}^2\Phi_n(q^2)}. \quad (3b)$$

In the above q -supercongruences and in what follows:

$$(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$$

is the q -shifted factorial,

$$[n] = [n]_q = 1 + q + \cdots + q^{n-1}$$

is the q -number,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}$$

is the q -binomial coefficient and $\Phi_n(q)$ is the n th cyclotomic polynomial of q . Note that the congruences in Theorem 1.1 modulo $[n]\Phi_n(q)^2$ and the congruences in Theorem 1.2 modulo $[n](1 - aq^n)(a - q^n)$ have already been proved by the authors in [13, Equations (5.5) and (5.10)].

2. Proof of Theorem 1.1 by the Zeilberger algorithm

The Zeilberger algorithm [cf. 22] can be used to find that the functions

$$f(n, k) = (-1)^k \frac{(4n-1)(-\frac{1}{2})_n^3 (-\frac{1}{2})_{n+k}}{(1)_n^3 (1)_{n-k} (-\frac{1}{2})_k^2},$$

$$g(n, k) = (-1)^{k-1} \frac{4(-\frac{1}{2})_n^3 (-\frac{1}{2})_{n+k-1}}{(1)_{n-1}^3 (1)_{n-k} (-\frac{1}{2})_k^2}$$

satisfy the relation

$$(2k-3)f(n, k-1) - (2k-4)f(n, k) = g(n+1, k) - g(n, k).$$

Of course, given this relation, it is not difficult to verify by hand that it is satisfied by the above pair of doubly indexed sequences $f(n, k)$ and $g(n, k)$.

Here we use the convention $1/(1)_m = 0$ for all negative integers m . We now define the q -analogues of $f(n, k)$ and $g(n, k)$ as follows:

$$F(n, k) = (-1)^k q^{(k-2)(k-2n+1)} \frac{[4n-1](q^{-1}; q^2)_n^3 (q^{-1}; q^2)_{n+k}}{(q^2; q^2)_n^3 (q^2; q^2)_{n-k} (q^{-1}; q^2)_k^2},$$

$$G(n, k) = \frac{(-1)^{k-1} q^{(k-2)(k-2n+3)} (q^{-1}; q^2)_n^3 (q^{-1}; q^2)_{n+k-1}}{(1-q)^2 (q^2; q^2)_{n-1}^3 (q^2; q^2)_{n-k} (q^{-1}; q^2)_k^2},$$

where we have used the convention that $1/(q^2; q^2)_m = 0$ for $m = -1, -2, \dots$. Then the functions $F(n, k)$ and $G(n, k)$ satisfy the relation

$$[2k-3]F(n, k-1) - [2k-4]F(n, k) = G(n+1, k) - G(n, k). \quad (4)$$

Indeed, it is straightforward to obtain the following expressions:

$$\begin{aligned}\frac{F(n, k-1)}{G(n, k)} &= \frac{q^{2n-4k+6}(1-q)(1-q^{4n-1})(1-q^{2k-3})^2}{(1-q^{2n-2k+2})(1-q^{2n})^3}, \\ \frac{F(n, k)}{G(n, k)} &= -\frac{q^{4-2k}(1-q)(1-q^{4n-1})(1-q^{2n+2k-3})}{(1-q^{2n})^3}, \\ \frac{G(n+1, k)}{G(n, k)} &= \frac{q^{4-2k}(1-q^{2n-1})^3(1-q^{2n+2k-3})}{(1-q^{2n})^3(1-q^{2n-2k+2})}.\end{aligned}$$

It is easy to verify the identity

$$\begin{aligned}&\frac{q^{2n-4k+6}(1-q^{4n-1})(1-q^{2k-3})^3}{(1-q^{2n-2k+2})(1-q^{2n})^3} + \frac{q^{4-2k}(1-q^{2k-4})(1-q^{4n-1})(1-q^{2n+2k-3})}{(1-q^{2n})^3} \\ &= \frac{q^{4-2k}(1-q^{2n-1})^3(1-q^{2n+2k-3})}{(1-q^{2n})^3(1-q^{2n-2k+2})} - 1,\end{aligned}$$

which is equivalent to (4). (Alternatively, we could have established (4) by only guessing $F(n, k)$ and invoking the q -Zeilberger algorithm [28].)

Let $m > 1$ be an odd integer. Summing (4) over n from 0 to $(m+1)/2$, we get

$$\begin{aligned}[2k-3] \sum_{n=0}^{\frac{m+1}{2}} F(n, k-1) - [2k-4] \sum_{n=0}^{\frac{m+1}{2}} F(n, k) &= G\left(\frac{m+3}{2}, k\right) - G(0, k) \\ &= G\left(\frac{m+3}{2}, k\right).\end{aligned}\tag{5}$$

We readily compute

$$\begin{aligned}G\left(\frac{m+3}{2}, 1\right) &= \frac{q^{m-1}(q^{-1}; q^2)_{(m+3)/2}^4}{(1-q)^2(q^2; q^2)_{(m+1)/2}^4(1-q^{-1})^2} \\ &= \frac{q^{m-3}[m]^4}{[m+1]^4(-q; q)_{(m-1)/2}^8} \left[\frac{m-1}{(m-1)/2} \right]^4\end{aligned}\tag{6a}$$

and

$$\begin{aligned}G\left(\frac{m+3}{2}, 2\right) &= -\frac{(q^{-1}; q^2)_{(m+3)/2}^3(q^{-1}; q^2)_{(m+5)/2}}{(1-q)^2(q^2; q^2)_{(m+1)/2}^3(q^2; q^2)_{(m-1)/2}(q^{-1}; q^2)_2^2} \\ &= -\frac{q^{-2}[m]^4[m+2]}{[m+1]^3(-q; q)_{(m-1)/2}^8} \left[\frac{m-1}{(m-1)/2} \right]^4.\end{aligned}\tag{6b}$$

Combining (5) and (6), we have

$$\begin{aligned}\sum_{n=0}^{\frac{m+1}{2}} F(n, 0) &= \frac{[-2]}{[-1]} \sum_{n=0}^{\frac{m+1}{2}} F(n, 1) + \frac{1}{[-1]} G\left(\frac{m+3}{2}, 1\right) \\ &= \frac{1+q}{q} G\left(\frac{m+3}{2}, 2\right) - qG\left(\frac{m+3}{2}, 1\right) \\ &= -\frac{(1+q)[m]^4[m+1][m+2] + q^{m+1}[m]^4}{q^3[m+1]^4(-q; q)_{(m-1)/2}^8} \left[\frac{m-1}{(m-1)/2} \right]^4,\end{aligned}$$

i.e.

$$\sum_{n=0}^{\frac{m+1}{2}} [4n-1] \frac{(q^{-1}; q^2)_n^4}{(q^2; q^2)_n^4} q^{4n} = -\frac{(1+q)[m]^4[m+1][m+2] + q^{m+1}[m]^4}{q[m+1]^4(-q; q)_{(m-1)/2}^8} \left[\frac{m-1}{(m-1)/2} \right]^4. \quad (7)$$

By [4, Lemma 2.1] (or [3, Lemma 2.1]), we have $(-q; q)_{(m-1)/2}^2 \equiv q^{(m^2-1)/8} \pmod{\Phi_m(q)}$. Moreover, it is easy to see that

$$\begin{aligned}\left[\frac{m-1}{(m-1)/2} \right] &= \prod_{k=1}^{(m-1)/2} \frac{1-q^{m-k}}{1-q^k} \\ &\equiv \prod_{k=1}^{(m-1)/2} \frac{1-q^{-k}}{1-q^k} = (-1)^{(m-1)/2} q^{(1-m^2)/8} \pmod{\Phi_m(q)},\end{aligned}$$

and $[m]$ is relatively prime to $(-q; q)_{(m-1)/2}$. It follows from (7) that

$$\sum_{n=0}^{\frac{m+1}{2}} [4n-1] \frac{(q^{-1}; q^2)_n^4}{(q^2; q^2)_n^4} q^{4n} \equiv -((1+q)^2 + q)[m]^4 \pmod{[m]^4 \Phi_m(q)}.$$

Concluding, the congruence (2a) holds.

Similarly, summing (4) over n from 0 to $m-1$, we get

$$[2k-3] \sum_{n=0}^{m-1} F(n, k-1) - [2k-4] \sum_{n=0}^{m-1} F(n, k) = G(m, k),$$

and so

$$\begin{aligned}\sum_{n=0}^{m-1} [4n-1] \frac{(q^{-1}; q^2)_n^4}{(q^2; q^2)_n^4} q^{4n} &= \frac{1+q}{q} G(m, 2) - qG(m, 1) \\ &= -\frac{(1+q)[2m-2][2m-1] + q^{2m-2}}{q(-q; q)_{m-1}^8} \left[\frac{2m-2}{m-1} \right]^4. \quad (8)\end{aligned}$$

It is easy to see that

$$\frac{1}{[m]} \begin{bmatrix} 2m-2 \\ m-1 \end{bmatrix} = \frac{1}{[m-1]} \begin{bmatrix} 2m-2 \\ m-2 \end{bmatrix} \equiv (-1)^{m-2} q^{2-\binom{m-1}{2}} \pmod{\Phi_m(q)},$$

and $(-q; q)_{m-1} \equiv 1 \pmod{\Phi_m(q)}$ [4]. The proof of (2b) then follows easily from (8).

3. Proof of Theorems 1.2 and 1.3

Proof of Theorem 1.2: It is easy to see by induction on N that

$$\begin{aligned} & \sum_{k=0}^N [4k-1] \frac{(aq^{-1}; q^2)_k (q^{-1}/a; q^2)_k (q^{-1}; q^2)_k^2}{(aq^2; q^2)_k (q^2/a; q^2)_k (q^2; q^2)_k^2} q^{4k} \\ &= \frac{(aq; q^2)_N (q/a; q^2)_N ((a+1)^2 q^{2N+1} - a(1+q)(1+q^{4N+1}))}{q(a-q)(1-aq)(aq^2; q^2)_N (q^2/a; q^2)_N (-q; q)_N^4} \begin{bmatrix} 2N \\ N \end{bmatrix}^2. \end{aligned} \quad (9)$$

For $N = (n+1)/2$ or $N = n-1$, we see that $(aq; q^2)_N (q/a; q^2)_N$ contains the factor $(1-aq^n)(1-q^n/a)$. Moreover,

$$\frac{[(n+1)/2]}{[n]} \begin{bmatrix} n \\ (n-1)/2 \end{bmatrix} = \begin{bmatrix} n-1 \\ (n-1)/2 \end{bmatrix}$$

is a polynomial in q . Since $[(n+1)/2]$ and $[n]$ are relatively prime, we conclude that $\begin{bmatrix} n \\ (n-1)/2 \end{bmatrix}$ is divisible by $[n]$. Therefore, $\begin{bmatrix} n+1 \\ (n+1)/2 \end{bmatrix} = (1+q^{(n+1)/2}) \begin{bmatrix} n \\ (n-1)/2 \end{bmatrix}$ is also divisible by $[n]$. It is also well known that $\begin{bmatrix} 2n-2 \\ n-1 \end{bmatrix}$ is divisible by $[n]$. Moreover, it is easy to see that $[n]$ is relatively prime to $1+q^m$ for any non-negative integer m . The proof then follows from (9) by taking $N = (n+1)/2$ and $N = n-1$. ■

Proof of Theorem 1.3: For $a = -1$, the identity (9) reduces to

$$\begin{aligned} & \sum_{k=0}^N [4k-1] \frac{(q^{-2}; q^4)_k^2}{(q^4; q^4)_k^2} q^{4k} = - \frac{(-q; q^2)_N^2 (1+q^{4N+1})}{q(1+q)(-q^2; q^2)_N^2 (-q; q)_N^4} \begin{bmatrix} 2N \\ N \end{bmatrix}^2 \\ &= - \frac{(1+q^{4N+1})}{q(1+q)(-q^2; q^2)_N^4} \begin{bmatrix} 2N \\ N \end{bmatrix}_{q^2}^2. \end{aligned} \quad (10)$$

Note that, in the proof of Theorem 1.2, we have proved that $\begin{bmatrix} 2N \\ N \end{bmatrix}_{q^2}$ is divisible by $[n]_{q^2}$ for both $N = (n+1)/2$ and $N = n-1$. Moreover, $[n]_{q^2}$ is relatively prime to $(-q^2; q^2)_m$ for $m \geq 0$. Hence the right-hand side of (10) is congruent to 0 modulo $[n]_{q^2}^2$ for $N = (n+1)/2$ or $N = n-1$. To further determine the right-hand side of (10) modulo $[n]_{q^2}^2 \Phi_n(q^2)$, we need only to use the same congruences (with $q \mapsto q^2$) used in the proof of Theorem 1.1. ■

4. Immediate consequences

Notice that for $n = p^r$ being an odd prime power, $\Phi_{p^r}(q) = [p]_{q^{p^{r-1}}}$ holds. This observation was used in [15] to extend (1) to a supercongruence modulo $[p^r][p]_{q^{p^{r-1}}}^3$. In the same vein, we immediately deduce from Theorem 1.1 the following result:

Corollary 4.1: Let p be an odd prime and r a positive integer. Then

$$\sum_{k=0}^{\frac{p^r+1}{2}} [4k-1] \frac{(q^{-1}; q^2)_k^4}{(q^2; q^2)_k^4} q^{4k} \equiv -(1+3q+q^2)[p^r]^4 \pmod{[p^r]^4 [p]_{q^{p^r-1}}} \quad (11a)$$

and

$$\sum_{k=0}^{p^r-1} [4k-1] \frac{(q^{-1}; q^2)_k^4}{(q^2; q^2)_k^4} q^{4k} \equiv -(1+3q+q^2)[p^r]^4 \pmod{[p^r]^4 [p]_{q^{p^r-1}}}. \quad (11b)$$

The $q \rightarrow 1$ limiting cases of these two identities yield the following supercongruences:

Corollary 4.2: Let p be an odd prime and r a positive integer. Then

$$\sum_{k=0}^{\frac{p^r-1}{2}} \frac{4k+3}{16(k+1)^4 256^k} \binom{2k}{k}^4 \equiv 1 - 5p^{4r} \pmod{p^{4r+1}} \quad (12a)$$

and

$$\sum_{k=0}^{p^r-2} \frac{4k+3}{16(k+1)^4 256^k} \binom{2k}{k}^4 \equiv 1 - 5p^{4r} \pmod{p^{4r+1}}. \quad (12b)$$

Similarly, we deduce from Theorem 1.3 the following result:

Corollary 4.3: Let p be an odd prime and r a positive integer. Then

$$\sum_{k=0}^{\frac{p^r+1}{2}} [4k-1] \frac{(q^{-2}; q^4)_k^2}{(q^4; q^4)_k^2} q^{4k} \equiv -q^{p^r} (1-q+q^2)[p^r]_{q^2}^2 \pmod{[p^r]_{q^2}^2 [p]_{q^{2p^r-1}}} \quad (13a)$$

and

$$\sum_{k=0}^{p^r-1} [4k-1] \frac{(q^{-2}; q^4)_k^2}{(q^4; q^4)_k^2} q^{4k} \equiv -(1-q+q^2)[p^r]_{q^2}^2 \pmod{[p^r]_{q^2}^2 [p]_{q^{2p^r-1}}}. \quad (13b)$$

The $q \rightarrow 1$ limiting cases of these two identities yield the following supercongruences:

Corollary 4.4: Let p be an odd prime and r a positive integer. Then

$$\sum_{k=0}^{\frac{p^r-1}{2}} \frac{4k+3}{4(k+1)^2 16^k} \binom{2k}{k}^2 \equiv 1 - p^{2r} \pmod{p^{2r+1}} \quad (14a)$$

and

$$\sum_{k=0}^{p^r-2} \frac{4k+3}{4(k+1)^2 16^k} \binom{2k}{k}^2 \equiv 1 - p^{2r} \pmod{p^{2r+1}}. \quad (14b)$$

The supercongruences in Corollaries 4.2 and 4.4 are remarkable since they are valid for arbitrarily high prime powers. Swisher [24] had empirically observed several similar but different hypergeometric supercongruences and stated them without proof.

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