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Intuition and conceptual construction in Weyl's analysis of the problem of space

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ABSTRACT. Hermann Weyl adopted the Kantian definition of space as a form of intuition and referred to Edmund Husserl's phenomenological approach for the philosophical characterization of space in the introduction to *Raum-Zeit-Materie* (1918) and other writings from the same period (1918–1923). At the same time, Weyl emphasized that subjective factors are completely excluded from the mathematical construction of physical reality in Albert Einstein's general theory of relativity, with the sole exception of the setting of a coordinate system, which for Weyl is what remains of the original perspective of the self in becoming aware of one's own intuitions. This paper reconsiders Weyl's philosophical position as a possible response to the earlier debate on the relation between intuition and conceptual construction in the foundation of geometry, key figures of which, besides Husserl, included Hermann von Helmholtz, Felix Klein, and Moritz Schlick.

Keywords: Classical and relativistic problems of space; form of intuition; Hermann von Helmholtz; Hermann Weyl.

1. Introduction

Weyl adopted the Kantian definition of space as a form of intuition and referred to Husserl's phenomenological approach for the philosophical characterization of space in the introduction to *Raum-Zeit-Materie* (1918) and in other writings from the same period (1918–1923). At the same time, Weyl emphasized that subjective factors are completely excluded from the mathematical construction of physical reality in Einstein's general theory of relativity, with the sole exception of the setting of a coordinate system, which for Weyl is what remains of the original perspective of the self in becoming aware of one's own spatial intuition. With regard to physical space or space-time, the problem arose to establish the conditions for determining the underlying geometrical structure, which became known as the problem of space, in its different formulations.[1]

Weyl's approach to the problem of space in *Mathematische Analyse des Raumproblems* (1923) reflects his philosophical considerations, insofar as he indicates increasingly higher levels of abstraction in the mathematical analysis of the structure of space. According to Weyl, a decisive step in this direction was taken by Helmholtz and Lie, with the group-theoretical deduction of the conditions for such a structure to satisfy the law of homogeneity. Given the heterogeneity of space-time in general relativity, Weyl's own solution of the problem of space required him to reformulate the invariants of the space-time continuum in differential geometry. In this respect, Weyl relied on Riemann. I believe that, nevertheless, the debate about the geometrical structure of space initiated by Helmholtz (1868, 1870) played an important role in Weyl's understanding of the relation between intuition and conceptual construction in the foundation of geometry.

In order to better appreciate this aspect of Weyl's work, this paper contrasts his reconstruction of the classical problem of space with Helmholtz's empiricist arguments against the Kantian theory of pure intuitions.[2] Helmholtz's arguments were twofold: as a physiologist of vision, he argued that the formation of spatial intuition deserved an empirical explanation; as a physicist and an epistemologist, he urged a generalization of the notion of the form of intuition to all possible contents that can enter the relevant form of perception. In Helmholtz's view, such a generalization ought to include the different hypotheses about metrical geometry formulated by Riemann. The corresponding (classical) problem of space was to establish necessary and sufficient conditions for a Riemannian metric of constant curvature, which included non-Euclidean geometries as special cases. Therefore, in 1878, Helmholtz reformulated his second argument by distinguishing between the general properties of the form of space (which he identified as a threefold extended manifold of constant curvature) and the specific axiomatic systems of metrical geometry, which correspond to the assumption of a flat or positively or negatively curved space. As pointed out by Ryckman, Helmholtz argued against the Kantian philosophy of geometry while retaining an inherently Kantian theory of space, according to which the former properties provide us with conditions of the

[1] I refer to Scholz (forthcoming) for a survey of different problems of space from the classical problem posed by Helmholtz to Weyl's and Cartan's problem of reformulating the older criteria for determining the structure of space in the light of differential geometry and the general theory of relativity (i.e., the modern or relativistic problem of space).

[2] On Helmholtz's empiricist philosophy of mathematics, see esp. DiSalle (1993).

possibility of geometrical measurement.[3] In other words, Helmholtz distanced himself from the received view of geometrical axioms as evident and necessary truths and reformulated the notion of the form of intuition in terms of a conceptual and hypothetical construction that lies at the foundation of measurement.

It is well known that the required mathematical tools for the solution of Helmholtz's problem of space were developed only later, in Lie's theory of continuous transformation groups. But this mathematical development showed another aspect of the epistemological problem as well: Is it still possible to draw a distinction between general and specific properties of space, given the fact that both kinds of properties can find an exact expression in axiomatic terms? An additional problem, after general relativity, is that the space-time structure under consideration would deserve a further generalization in order to include the hypotheses of variably curved spaces disregarded by Helmholtz.

Therefore, commenting on the centenary edition of Helmholtz's epistemological writings of 1921, Moritz Schlick maintained that, in order to uphold Helmholtz's original distinction, a distinction ought to be made between intuitive and indescribable factors of spatial perception and all kinds of axioms. Schlick ruled out the synthetic aprioricity of geometrical knowledge in Kant's sense by sharply distinguishing between acquaintance with sense qualities, which for Schlick pertain to the contents rather than the form of perception, on the one hand, and geometrical knowledge, which for Schlick is analytic, on the other. However, it has been question whether Schlick's distinction does justice to Helmholtz's account of localization as a construction of the form of space, which presupposes the interdependence of intuition and understanding.[4]

With reference to this debate, I address the question whether the forms of intuition in Kant's sense admit of a mathematical characterization, which is crucial to Weyl's connection between the philosophical and the mathematical aspects of the problem of space. One of the problems of this account is that it is not always possible to determine which of the earlier philosophical positions were known to Weyl. Ryckman has offered a detailed reconstruction of Weyl's relation to Husserl.[5] I will give evidence of the fact that even the wider debate finds an echo in Weyl's discussion, although sometimes more indirectly. Furthermore, I believe that some of the motivations for Weyl's defense of the phenomenological approach go back to the same debate. My

[3] Ryckman (2005, pp.73–74).

[4] See Friedman (1997).

[5] Ryckman (2005, Ch.5).

suggestion is that the analysis of essence in Husserl's sense provided a plausible solution to the problem concerning the possibility of a mathematical characterization of space. The fact that essences in Husserl's sense are not given in isolation but only in their entanglement with external factors enabled Weyl to account for the fact that the new insights into the structure of space-time resulted from a long and complex historical development of geometry from Euclid to Riemann. At the same time, the clarification of the notions of space and time in general relativity provided a principled distinction between essential and accidental aspects in the geometrical representation of space, time, and matter. Whereas as a matter of fact the notion of the a priori form of intuition emerged from a conceptual construction of different hypotheses about the structure of space, as Helmholtz suggested, the form thus obtained retained the role of synthetic a priori knowledge in Kant's sense, namely, as knowledge that is independent of experience, because it lies at the foundation of the possibility of experience itself.

The first part of the paper provides a general account of the shift from the classical to the modern analysis of the problem of space, with a special focus on the Kantian framework of Weyl's analysis. The second part offers a discussion of the epistemological problems concerning the distinction between intuitions and conceptual constructions by reconsidering different arguments for and against the possibility of a mathematical characterization of what Helmholtz called the form of spatial intuition in group-theoretical terms. I argue in the concluding section that the philosophical background can help us to better appreciate the originality of Weyl's position by providing a plausible reconstruction of his argument for the analysis of essence when it comes to discerning the form of spatial intuition from the external factors of perception.

2. From the classical to the modern analysis of the problem of space

It is well known that Weyl was one of the first to account for the fact that the problem of determining the geometrical structure of space, as addressed in different ways from Kant to Helmholtz–Lie, deserved a new formulation after general relativity. In other words, Weyl recognized that, mathematically speaking, there are different problems of space, and worked on a rigorous solution of the relativistic problem of space from 1918 to 1923 and again from 1928 to 1929, in the context of the discussion about the foundations of quantum physics.[6] At the same time, Weyl emphasized some continuity in the shift from the classical to the relativistic analysis of the problem of space with regard to the philosophical dimension of the problem. As Weyl put it, the philosophical

[6] See Scholz (2004)

problem of space consists in establishing the relations between the form and the matter of appearances, which he conceives of in Kant's sense as the extensive medium that makes it possible to individuate different qualitative contents.[7] The philosophical problem guides the mathematical analysis, insofar as this deals with the structure of space and its generalization from the classical to the relativistic view.

This section offers a brief reconstruction of the Kantian framework of Weyl's analysis of the problem of space in his lectures of 1923 and investigates the question: In what sense and to what extent does the 1923 formulation of the problem integrate Weyl's earlier relativistic account of metrical geometry in *Raum-Zeit-Materie*? In order to highlight the philosophical aspect of the problem, the second part of the section draws a comparison with the earlier generalization of the Kantian form of spatial intuition by Helmholtz.

2.1. The Kantian framework of Weyl's analysis

After analyzing the mathematical form of Euclidean space and its role in classical physics in *Raum-Zeit-Materie*, Weyl emphasized that the shift from classical to relativistic physics entails a corresponding transformation of geometry, which was foreshadowed only by Riemann. Weyl compared the transition from Euclidean to Riemannian geometry to the shift from action at distance to infinitely near action in physics. The fundamental fact of Euclidean geometry is Pythagoras's theorem, according to which the square of the distance between two points is a quadratic form of the relative coordinates of the two points. Intuitively, the fundamental idea of Riemann's geometry was to interpret the same law as strictly valid only when the points under consideration are infinitely near. The mathematical expression of this fact gives us a Riemannian metric of constant curvature, that is, the condition that is satisfied by both Euclidean geometry (when the curvature equals 0) and by the other two classical cases of such manifolds (when the curvature is greater or less than 0). The latter case was proved by Beltrami (1869) to correspond to the non-Euclidean geometry of Bolyai-Lobachevsky and played a central role in Helmholtz's imagination of the observations on a convex mirror that would be compatible with the homogeneity of space.[8] Therefore, Helmholtz argued that

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[7] Weyl 1923, p.1.

[8] Helmholtz (1870). We have already mentioned that Helmholtz had posed the classical problem of space as the problem of establishing the necessary and sufficient conditions for obtaining a Riemannian metric of constant curvature in Helmholtz (1868). However, it was only after his correspondence with Beltrami, in 1869, that he became aware of the fact that his previous

the particular value of curvature is a matter for empirical science, which in the late nineteenth century became known as the “Riemann-Helmholtz” theory of space.[9]

However, in the context of Weyl’s interpretation of general relativity, Weyl’s emphasis is on the novelty of Riemann’s approach, even when compared to his contemporaries:

> Space is a form of phenomena, and, by being so, is necessarily homogeneous. It would appear from this that out of the rich abundance of possible geometries included in Riemann’s conception only the three special cases mentioned [i.e., spherical and Bolyai-Lobachevsky] come into consideration from the outset, and that all the others must be rejected without further examination as being of no account: *parturiunt montes, nascetur ridiculus mus!* Riemann held a different opinion, as is evidenced by the concluding remarks of his essay. Their full purport was not grasped by his contemporaries, and his words died away almost unheard (with the exception of a solitary echo in the writings of W. K. Clifford). Only now that Einstein has removed the scales from our eyes by the magic light of his theory of gravitation do we see what these words actually mean.[10]

Not only was Riemann the first to consider the hypothesis of variably curved spaces in Riemann (1854/1867), but it followed from his distinction between discrete and continuous manifolds that only a manifold of the former kind entails the principle of its metrical relations in itself or, as Weyl put it, “a priori.”[11] According to Riemann, under the assumption that space is continuous, we would have to seek the ground of its metric relations outside it, in the binding forces that act upon it. The importance of the latter claim, in Weyl’s view, lies in the fact that Riemann, contrary to his contemporaries, acknowledged the possibility that the form of space depends on its material contents, insofar as this determines metrical relations. In doing so, Riemann distanced himself from the received view that the metrical structure of space is fixed and inherently independent of the physical phenomena for which it serves as a background. More precisely, the equivalence between

characterization of space included non-Euclidean manifolds of constant curvature. The relevant correspondence between Beltrami and Helmholtz is now available in Boi et al. (1998, pp.204–205).

[9] See, for example, Benno Erdmann’s (then popular) exposition in Erdmann (1877).

[10] Weyl (1921/1952, pp.96–98).

[11] Weyl (1921/1952, p.97).

gravitational and inertial phenomena established by Einstein ruled out the view that space is necessarily homogeneous, and therefore the conception of space as a form of appearance in Kant's sense or even its generalization to the class of the manifolds of constant curvature in the works of Helmholtz, Klein, Lie, and Poincaré.

Nevertheless, there is evidence that one of Weyl's motivation in dealing with the relativistic problem of space was to vindicate the Kantian view of space in some respects. The fourth edition of *Raum-Zeit-Materie* contains a new section on "Space Metric from the Group-Theoretical View," which is characteristic of the tradition inaugurated by Helmholtz.[12] But the most comprehensive treatment of the problem of space from this point of view is found in Weyl's 1922 lectures at the Institut d'Estudios Catalans in Barcelona and at the Universidad Central in Madrid. Weyl held these lectures in French and Spanish. A German version of the lectures appeared in 1923 under the title "Mathematische Analyse des Raumproblems."[13] In the Preface, Weyl presented this work as an "integration" of *Raum-Zeit-Materie*: "The deeper, group-theoretical conception of the problem of space was sketched only briefly in the latter book, because the main focus there was on physics and relativity theory and their immediate presuppositions."[14] Weyl's goal in 1923 was to catch up with the problem of space.

Weyl used his infinitesimal geometry to generalize the problem of space so as to encompass general relativity. Firstly, he allowed for indefinite metrics ("postulate of freedom"). Secondly, he postulated a metric connection according to which to each congruent transfer there exists exactly one equivalent affine connection ("postulate of coherence"). Weyl's goal was to show that these principles are necessary and sufficient to individuate the subgroups of the general linear group that correspond to rotations in relativistic physics. He identified the geometry that satisfies the principle of freedom with the type of Finsler metrics also called by him Pythagorean.[15]

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[12] Weyl (1921/1952, pp.138–148). The question whether the group-theoretical view provides a suitable interpretation of Helmholtz's thought experiments, as Weyl assumes, is discussed in the next section.

[13] The French text has been made available only recently in the critical edition by Audureau and Bernard (Weyl 2015).

[14] Weyl 1923, Vorrede.

[15] For a detailed discussion of Weyl's approach to the problem of space and its development from 1921 to 1923, see Coleman and Korté (2001).

Without entering into the details of Weyl's analysis of the problem of space, the relevant point for my present purpose is that Weyl's new formulation of the problem enabled him to emphasize some continuity in the historical development from the classical to the relativistic view. Weyl described such a development as a progressive generalization of the a priori component of the representation of space (or space-time) that provides us with preconditions for the possibility of measurement. In the context of Newtonian physics, Euclidean geometry satisfies this purpose by postulating the homogeneity of space via Pythagoras's theorem. Kant's Transcendental Aesthetic provides a philosophical account of the role of Euclidean geometry in physics, according to which homogeneity pertains to the nature of space as a form of appearance. Therefore, the spatial properties of a physical object do not depend on its location and the metrical structure of space can be established a priori, independently of its material contents.

The classical problem of space that occupied scientists and philosophers after the development of non-Euclidean geometry was to individuate the class of metrical spaces that are homogeneous in the same sense. Therefore, Helmholtz postulated that a rigid body in space possesses the degree of free mobility that corresponds to a Euclidean metric. We have already mentioned that Helmholtz in 1868 restricted his consideration to Euclidean geometry. After his correspondence with Beltrami, in Helmholtz (1870), he acknowledged that the Riemannian metric of constant curvature thus characterized included different possible metrical geometries as special cases and maintained that the measure of curvature is a matter for empirical investigation. Weyl argued that, mathematically speaking, Helmholtz's requirement of homogeneity received a precise interpretation only in Sophus Lie's theory of continuous transformation groups.[16] Lie identified the spaces that satisfy this requirement as the group of congruent transformations. It followed that Helmholtz's conjecture was confirmed, at least insofar as the specific value of curvature was left undetermined by the mathematical analysis. Regarding the philosophical problem of space, Weyl concluded his discussion of the classical view as follows:

> Under these circumstances, one is led almost necessarily to ask herself whether in the end actual space could be not Euclidean at all, but rather a spherical space with non-vanishing curvature λ. Owing to its metrical homogeneity, such a space would be

[16] Lie (1893, pp.437–471).

> equally suitable as the Euclidean to serve as a form of the appearances. With this question, we turn back from the mathematical analysis to reality.[17]

Weyl went on to present the relativistic view as a further generalization of the homogeneity requirement. In terms of Weyl's infinitesimal geometry, the same requirement corresponds to the fact that the metric at a point *P* and the metrical relation of *P* to its neighboring points is everywhere the same, essentially single and absolutely determined. The mutual orientation of the metrics in different points, on the other hand, is contingent and capable of continuous variations depending on its material content. Weyl used the Kantian framework of his analysis to identify the former requirement as the "a priori essence of the space-time structure" and the variability of metrics as a posteriori or "impossible to fully grasp in a rational manner, but only approximately and with the help of immediate and intuitive reference to reality."[18] Weyl identified the postulate of freedom in particular (i.e., the possibility that the metric field is subject to arbitrary changes while the nature of the metric remains fixed) as a sort of relativistic equivalent of Helmholtz's homogeneity requirement.[19] Insofar as both of Weyl's postulates (i.e., freedom and coherence) provided an interpretation of the conceptual analysis of the problem of space, he called this the "the synthetic part of the investigation in Kant's sense."[20]

To sum up, Weyl relied on Lie for the mathematical analysis of the classical problem while looking at Helmholtz's approach to the philosophical problem. The Kantian theory of space need not be rejected in the light of the later scientific developments, insofar as the requirement of homogeneity can be generalized accordingly. The most explicit indication of Weyl's reliance on the Kantian conception is his attempt to redraw the line between the a priori and the a posteriori components of the structure of space-time as follows:

> Even Einstein upholds the view that the metrical structure of the world is everywhere of the same kind, as our general metrical infinitesimal geometry presupposes it to be. It is not simply denied that something in the structure of the extensive medium of the

[17] Weyl (1923, p.43).

[18] Weyl (1923, p.45).

[19] Weyl (1923, p.46).

[20] Weyl (1923, p.49).

> external worlds is a priori; however, the borderline between what is a priori and a posteriori is set somewhere else.[21]

This passage suggests that the appropriate strategy for dealing with the relativistic problem of space lies in nuce in Helmholtz's interpretation of the form of spatial intuition as dependent in its content for the determination of the specific metrical structure of space.

In order to better understand Weyl's position, the second part of this section contrasts it with Helmholtz's own considerations on the form of intuition.[22] The following section turns back to Weyl's further claim that the group-theoretical view provides a conceptual analysis of spatial intuition, and therefore the appropriate means to implement the strategy above.

2.2. Helmholtz's and Weyl's accounts of the form of intuition

As suggested by Scholz (forthcoming), Weyl's connection with Helmholtz becomes apparent in the light of a manuscript from the 1840s "On the General Natural Concepts," which was made available in the second volume of Königsberger's intellectual biography of Helmholtz.[23] (1903, 126–138). Helmholtz's aim was to determine the concept of space "in such a way that it can comprehend all possible changes of matter, which are obviously to be considered here only as changes in spatial relations, i.e., as motions."[24] The same requirement received a precise formulation in Helmholtz's geometrical papers as the free mobility of rigid bodies.[25] However, as Scholz admits, there is no information about the possibility that Weyl had read this text, nor is it possible to determine precisely which of Helmholtz's works Weyl might have known. It might be added that the debate on non-Euclidean geometry and the Kantian theory of space at that time focused on Helmholtz's geometrical papers. To my knowledge, Helmholtz's earlier manuscript did not receive attention until only recently.

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[21] Weyl (1923, pp.44–45).

[22] See Biagioli (2014b) for a more thorough discussion of Helmholtz's stance towards the Kantian theory of space.

[23] Königsberger (1903, pp.126–138).

[24] Königsberger (1903, p.34). The English translation of this and other passages, along with a detailed discussion of Helmholtz's manuscript is found in Hyder (2009, pp.140–146).

[25] Scholz (forthcoming). Cf. Hyder (2009, Ch.5) for a more detailed account of the different phases in Helmholtz's thought.

Since there is no evidence that Weyl had read this text, my suggestion is to restrict the consideration to the works cited by Weyl. At that time, there were several editions of Helmholtz's geometrical papers available, including the centenary edition of Helmholtz's *Schriften zur Erkenntnistheorie* (1921) by Paul Hertz and Moritz Schlick. However, Weyl quotes Helmholtz's "Ueber die Thatsachen, die der Geometrie zugrunde liegen" (which first appeared in 1868) from the earlier edition of Helmholtz's *Wissenschaftliche Abhandlungen* (vol. 2, 1883). Amongst Helmholtz's epistemological works, the same volume included "Ueber die thatsächlischen Grundlagen der Geometrie"[26] and "Ueber den Ursprung und Sinn der geometrischen Sätze" (1878), the follow-up paper to the English edition of Helmholtz (1870), which had appeared in *Mind* in 1876. In 1878, Helmholtz published a second paper in the same journal in response to Land (1877). The 1883 version is the German translation of Helmholtz (1878), which was incorporated in "Die Tatsachen in der Wahrnehmung" (1878). All this to say that, arguably, Weyl was acquainted with the latter paper. This might shed some light on Weyl's reading of Helmholtz's stance on the classical problem of space, as this paper contains Helmholtz's most comprehensive discussion of the Kantian themes already addressed in the 1840s manuscript. In particular, the concluding paragraph of the German version of the paper describes the notion of the form of intuition as follows:

> Kant's doctrine of the a priori given forms of intuition is a very fortunate and clear expression of the state of affairs; but these forms must be devoid of content and free to an extent sufficient for absorbing any content whatsoever that can enter the relevant form of perception. But the axioms of geometry limit the form of intuition of space in such a way that it can no longer absorb every thinkable content, if geometry is at all supposed to be applicable to the actual world. If we drop them, the doctrine of the transcendentality of the form of intuition of space is without any taint. Here Kant was not critical enough in his critique; but this is admittedly a matter of theses coming from mathematics, and this bit of critical work had to be dealt with by the mathematicians.[27]

[26] Here dated 1866. However, this paper also appeared in 1868 (see Hyder 2009, p.165).

[27] In Helmholtz (1883, p.660). I quoted from the English translation of the same passage in Helmholtz (1921/1977, pp.162–163).

Not only does this passage contain the same claim that Weyl compared to the postulate of freedom, but it clearly suggests the above strategy for dealing with the philosophical problem of space. Helmholtz's starting point was the consideration that a generalization of Kant's form of intuition was made necessary by the mathematical discovery of non-Euclidean geometry as a possible application of the free mobility of rigid bodies. In order to show this possibility, in Helmholtz (1870), Helmholtz described an imaginary world in a convex mirror in line with Beltrami's interpretation of non-Euclidean geometry. Helmholtz constructed a series of thought experiment by assuming that for every measurement in our world, a hypothetical inhabitant of the mirror would be able to carry out a corresponding measurement. However, this would appear to us to be subject to some contraction according to a hyperbolic geometry in the vicinity of the border of the mirror. This fiction shows that, in principle, it would be possible to interpret the same operations in both ways. Helmholtz concluded that it is impossible to make a choice on purely mathematical grounds, regardless of some further knowledge of the laws of mechanics. The axioms of geometry as propositions concerning physical quantities – he concluded – must be empirical propositions and can be subject to revision, contrary to Kant's view of (Euclidean) axioms as evident propositions that lie at the foundation of the theory of motion.

In 1878, Helmholtz expressed the same idea by calling the type of knowledge that results from analytic geometry, on the one hand, and from observations on rigid bodies, on the other, "physical geometry."[28] In response to Land and other orthodox Kantians, who defended the aprioricity of Euclidean geometry as grounded in the form of spatial intuition, Helmholtz emphasized that the metrical structure of space is a posteriori in the sense of physical geometry. Nevertheless, he drew on his physiological work to explore the possibility of deriving a more general form of spatial intuition from the laws governing the localization of particular objects in space. It is in this sense that Helmholtz deemed the mathematical formulation of the free mobility of rigid bodies a fundamental "fact" induced by repeated observations on solid bodies. The same fact generalized provides us with a precondition for the possibility of measurement that is compatible with different metrical geometries, as in Helmholtz's thought experiment.

Summing up, Weyl's relativistic approach to the problem of space can be compared to Helmholtz's in several respects. Firstly, both Helmholtz and Weyl rely upon a conceptual analysis of the mathematical problem to show that not all the possible hypotheses had been considered in the previous definition of the a priori form of space. Consequently, they argue for a generalization of

[28] Helmholtz (1921/1977, p.153).

the previous view. Secondly, given the variability of metrics established a priori, they maintain that the specific metrical structure of space is a matter for empirical science. Thirdly, they deal with a philosophical problem of space that goes back to Kant and consists in accounting for the possibility of physico-geometrical knowledge. More specifically, the form of intuition figures in both accounts as the a priori component of the investigation that lies at the foundation of synthetic principles such as the free mobility of rigid bodies or its relativistic equivalent. To conclude this section, however, it is important to notice that there is an important difference between Helmholtz's and Weyl's accounts. Whereas Helmholtz proposed an explanation of how the form of intuition is acquired, Weyl defended a Kantian view of a priori intuition also with regard to its evidence. The difference is apparent in the following passage from Helmholtz (1878):

> I have […] frequently emphasized in my previous studies the agreement between recent physiology of the senses and Kant's doctrine, although this admittedly does not mean that I had to swear by the master's words in all subordinate matters too. I believe the resolution of the concept of intuition into the elementary processes of thought as the most essential advancement in the recent period. This resolution is still absent in Kant, which is something that then also conditions his conception of the axioms of geometry as transcendental propositions.[29]

Such an interpretation of intuition in terms of more fundamental conceptual processes was considered to be an advancement over Kant by most of Helmholtz's contemporaries.[30] So the question arises whether Weyl's sort of return to intuition as a (distinct) basis for conceptual construction is a regressive view. I will argue in the second part of the paper that this is not the case by showing that Weyl's literal interpretation of the phenomenological aspects of intuition offered a plausible solution to the problem of giving a more precise mathematical characterization of Helmholtz's form of intuition. My suggestion is that the background for Weyl's group-theoretical

[29] Helmholtz (1921/1977, p.143). This passage was added in the full paper "Die Tatsachen in der Wahrnehmung," but not in the shortened version of the paper that appeared in Helmholtz (1883).

[30] Arguably, this was one of the main reasons for Helmholtz's influence on the renewal of Kant's transcendental philosophy proposed by different directions of neo-Kantianism (Biagioli 2014a, 2016, Ch.1–2).

approach is provided by an earlier debate on the same subject. In particular, I will consider two opposed readings of Helmholtz by Klein and Schlick.

3. Group theory in the reception of Helmholtz's work

As it emerges from Weyl's historical account in Weyl (1923), by the end of the nineteenth century it was a commonplace that group theory provided a rigorous analysis of Helmholtz's problem of space. However, Weyl does not discuss the fact that Helmholtz himself made no use of the concept of group. The mathematical analysis considered by Weyl emerged only later, in the context of Lie's theory of continuous transformation groups.

My suggestion is that, nevertheless, the debate on the origin of the group-theoretical representation of space played an important role in Weyl's defense of a phenomenological approach to Kant's spatial intuition. A more comprehensive account of the relevant positions would require us to take into consideration a much wider debate from Helmholtz to Poincaré. For reasons of space, this section is restricted to the more specific question whether group theory provides a plausible interpretation of Helmholtz's view. Klein (1897) was one of the first mathematicians to answer this question affirmatively by relying on Lie's theory of transformation groups. Twenty-four years later, commenting on the centenary edition of Helmholtz's epistemological writing, Schlick called into question this reading, which had become commonplace, by pointing out that it remained unclear how the conceptual constructions of mathematics should represent something intuitive. Weyl avoided this difficulty by adopting the phenomenological approach.

3.1. Klein

It is well known that Felix Klein was one of the first to envision a unified approach to geometry based on group theory in his "Vergleichende Betrachtungen über neuere geometrische Forschungen" (1872). This pamphlet was distributed during Klein's inaugural address as newly appointed Professor at the University of Erlangen, and therefore is best known today as the "Erlangen Program." Although the ideas of the Erlangen Program have been considered to be very influential in retrospect, there is evidence that Klein himself did not draw much attention to it until only after the development of essential requirement for the implementation of such a project by other mathematicians.[31] One of this contributions is certainly Lie's theory of transformation group, which appeared in three volumes between 1888 and 1893. In the same years, Klein resumed his work on non-Euclidean geometry and promoted his early ideas by publishing a revised version of the Erlangen Program in *Mathematische Annalen* (1893) and other writings and lectures on related subjects. His most detailed discussion of Helmholtz is found in Klein's review of the third volume of Lie's *Theorie der Transformationsgruppen*, where Lie dealt with the classical problem of space.[32] Klein delivered his review at the University of Kazan in 1897 when Lie was offered the first Lobachevsky prize.[33]

On this occasion, Klein emphasized the broader significance of the group-theoretical approach for the theory of measurement. Klein began with a general consideration about the numerical representation of spatial relations. Such a representation presupposes the use of axioms, which Klein here and elsewhere defined as "the postulates by which we read exact assertions into inexact intuition."[34] The need to introduce postulates corresponds to the fact that there is a lower limit to empirical measurement. Klein's approach to the problem of space emerges from the following consideration about the upper limit:

> Correspondingly, when it comes to take into consideration the topologically different forms of space for the determination of the geometry of actual space, we are faced not so much with an arbitrary but with an inner consequence. Our empirical measurement has also an upper limit, which is given by the dimensions of the objects accessible to us or to our observation. What do we know about spatial relations in the infinitely large? To begin with, nothing. Therefore, we have to formulate postulates.[35]

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[31] The latter aspect has been emphasized by several historical studies (see esp. Hawkins 1984, Rowe 1992). Cf. Birkhoff and Bennett (1988) for the opposing view that the Erlangen Program was very influential.

[32] Lie (1893, pp.437–471).

[33] In the following, I refer to the reprinted version of Klein's review in *Mathematische Annalen* (1898).

[34] Klein (1890, p.572).

[35] Klein (1898, p.595).

The problem under consideration consists in determining the class of all surfaces in elliptic, hyperbolic, and parabolic space that are locally isometric to the Euclidean plane.[36] Similar to Helmholtz before him, Klein restricted the consideration to the three classical cases of manifolds of constant curvature to then single out Euclidean geometry for reasons of convenience. The relevant aspect for Klein's comparison with Helmholtz's problem, however, lies in his methodological consideration about the need of introducing postulates to cope with the limits of intuitions.

Similarly, Klein introduced the distinction between metrical and projective geometry "not as arbitrary or indicated by the nature of the mathematical methods, but as corresponding to the actual formation of our space intuition, in which mechanical experiences (concerning the movement of rigid bodies) combine with experiences of visual space (concerning the different kind of projection of intuited objects)."[37] The introduction of numbers in projective space (e.g., the construction of a numerical scale on a projective line) presupposes that the indefinite divisibility of empirical intuition is replaced by the axiom of continuity, which was implicit in Christian von Staudt's foundation of projective geometry as an autonomous branch of geometry. As showed by Klein in Klein (1871), the numerical representation of space opens the door to the study of a projective metric, which contains elliptic, hyperbolic, and parabolic metrical geometries as specific cases. In other words, projective metrical geometry provides a model of non-Euclidean geometry or "Klein's model." The classification thus obtained is generalized further to the discussion of all the possible topological forms of space in the infinitely large.[38]

To conclude, Klein maintained that his model of non-Euclidean geometry provides an adequate interpretation of Helmholtz's thought experiments about free mobility in a hyperbolical space. The fundamental ideas of Helmholtz's argument find a precise expression in projective and group-theoretical terms by saying that the measurements in our world and the corresponding measurements in the mirror belong to the larger group of collineations.[39]

Klein based this interpretation on the following consideration about the mathematical aspects of Helmholtz's physiology of vision:

[36] Klein's solution to this problem is found in Klein (1890). This is now known as a distinct problem of space, which is called "Clifford-Klein" or "the problem of the form of space" (see Torretti 1978, p.151).

[37] Klein (1898, p.593).

[38] Klein (1898, p.597).

[39] Klein (1898, p.599).

> Helmholtz was presumably far from the typical projective way of thinking (in the sense of von Staudt). It must be added that, in the years of Helmholtz's mathematical work, projective geometry was usually considered to be a specialty; the insight into its foundational meaning for every geometrical speculation was not widespread at all. Or maybe Helmholtz, as a natural scientist, was fundamentally reluctant towards the abstraction that lies at the foundation of projective geometry. In the introduction to his Göttingen notice from 1868 he distances himself from a foundation of geometry that would put forward the properties of visual space, because even the blind can acquire correct representations of space. Interestingly, this is in contrast though with the fact that Helmholtz himself is continuously led to deal with projective questions by his extensive optical investigations. He deals with this question by auxiliary means of his own invention, but also sometimes by means of general reasoning.[40]

Not only does Helmholtz foreshadow the projective and group-theoretical view of geometry according to Klein, but the passage above suggests that there is some continuity between Helmholtz's psychological considerations on spatial intuitions and the generalization to the mathematical reasoning. This example shed further light on Klein's view of geometric knowledge as based on axioms, where axioms impose conceptual constraints on fundamentally imprecise intuitions.

The plausibility of this reading notwithstanding,[41] it remained unclear about the status of what Helmholtz, referring to Kant, called form of spatial intuition. If identified as imprecise intuitions, the form of intuition would lose its general character. If identified with the full-blown mathematical analysis of a projective metric, on the other hand, such a form would lose its immediacy and would coincide fundamentally with conceptual thinking. The latter option is suggested by Helmholtz's remark about the resolution of intuition into intellectual processes and inspired several strategies for an intellectualization of intuition in neo-Kantianism. The next section deals with a fundamental objection to this tradition by Moritz Schlick.

[40] Klein (1898, p.598).

[41] Helmholtz himself drew attention to the relevance of his psychological investigations to his mathematical considerations in Helmholtz (1870/1977, p.15).

3.2. Schlick

Schlick distanced himself from the received interpretation of Helmholtz in his comments on the centenary edition of Helmholtz's *Schriften zur Erkenntnistheorie* (1921). In particular, in what follows, I will refer to Schlick's comments on "Die Tatsachen in der Wahrnehmung," namely, the central work for the articulation of Helmholtz's view of the form of intuition.[42] On the one hand, Schlick and Paul Hertz, the coeditor of Helmholtz (1921), referred to Lie for a rigorous solution of the mathematical problem of space posed by Helmholtz. On the other hand, Schlick distanced himself from the philosophical interpretation of this result as providing a generalization of Kant's form of intuition. Schlick reconsidered an interpretative question concerning Helmholtz's distinction between the general properties of the form of spatial intuition and its narrower specifications by means of geometrical axioms. Helmholtz's examples suggest that the latter correspond to Euclidean axioms.[43] However, it remains unclear what properties exactly count as general and if they would admit an axiomatic formulation in the light of later mathematical developments. Schlick admitted that modern geometers (in particular Lie and Klein) tended to answer this question affirmatively, although not everyone agreed on the particular axiomatization.[44] However, he advocated a different interpretation, for the following reasons. Firstly, in order to uphold Helmholtz's distinction, "the

[42] Arguably, Schlick commented on Helmholtz's most philosophical papers (i.e., Helmholtz 1870, 1878) while leaving to Paul Hertz the comments on Helmholtz's mathematical papers (Helmholtz 1868, 1887). It might be objected that such a division obscures the connection between the philosophical and the mathematical considerations in Helmholtz's work. In the following, I suggest that this partly depends on Schlick's own attempt to clarify the different aspects of Helmholtz's notion of space.

[43] Helmholtz examples include such propositions as: Between two points only one straight line is possible; through any three points a plane can be placed; through any point only one line parallel to a given line is possible (Helmholtz 1878/1977, p.128).

[44] Schlick contrasts the received interpretation via a projective metric with Poincaré's identification of a purely qualitative geometry as his development of "analysis situs," which became known as "topology" (Schick in Helmholtz 1921/1977, pp.172–173). It might be added that even more recent axiomatic interpretations of Helmholtz's distinction differ slightly, although most interpreters identify Helmholtz's general characterization of space as a differentiable, three-dimensional manifold of constant curvature (Cf. Torretti 1978, Lenoir 2006).

'general form' will have to be understood as the indescribable psychological component of spatiality which is imbued in sense perception."[45]

Secondly, Schlick called into question Helmholtz's distinction between form and matter of spatial intuition. In Schlick's view, a more consistent naturalization of Kant's forms of intuition would reduce them to sensuous contents. The result of such a reduction is what Schlick identified as the subjective spatial intuition or acquaintance. Insofar as the physical concept of space can be characterized mathematically, on the other hand, Schlick identified it with a formal, conceptual construction.[46]

As pointed out by Friedman (1997), Schlick's reading of Helmholtz presupposes a very different scientific and philosophical context after the development of the axiomatic method in geometry, on the one hand, and general relativity, on the other. Furthermore, Schlick tends to ascribe to Helmholtz his own philosophical assumptions about causal realism, which are sometimes inconsistent with Helmholtz's remarks on the status of scientific knowledge as restricted to the phenomena in Kant's sense. Nevertheless, it is true that, by advocating the interpretation above, Schlick formulated a compelling objection against the received view about the form of spatial intuition. Whereas the mathematical tradition associated with Helmholtz underpinned the view that a mathematical description of what Kant called form of intuition would be made possible by the developments of nineteenth-century geometry, Schlick showed the impossibility of such a description, insofar as space had to retain both intuitive and a priori character in Kant's sense.

The following section considers Weyl's position in this debate as an original response to the same problem within a Kantian framework.

3.3. Weyl

As in the case of Weyl's reception of Helmholtz, it is not always possible to reconstruct what Weyl might have known directly or indirectly from the debate above. Nevertheless, it is safe to assume that Weyl was well acquainted with the Göttingen mathematical tradition, of which Klein was one of the leading figures.[47] Weyl studied in Göttingen from 1904 to 1908, when he completed his doctoral theses under Hilbert's supervision. Subsequently, Weyl habilitated in Göttingen in 1910

[45] Schlick in Helmholtz (1921/1977, p.172–173).

[46] Schlick in Helmholtz (1921/1977, p.167).

[47] On the development of this tradition from Riemann to Klein and Hilbert, see Rowe (1989). For further details about Weyl's relationship to the Göttingen community, see Sigurdsson (1994).

and was as a Privatdocent there until 1913, when he was offered a professorship at the Eidgenössische Technische Hochschule in Zürich.

Klein was Weyl's teacher in Göttingen and corresponded with him even later, while Weyl was working on his relativistic analysis of the problem of space.[48] Furthermore, Weyl's extensive references to Klein in Weyl (1923) are evidence that Weyl was well acquainted with Klein's works on non-Euclidean geometry, even though Weyl did not provide any detailed references. At other times, Weyl's considerations are clearly reminiscent of Klein's views, even when there is no explicit reference. For our present discussion, for example, it is worth noting that Weyl reproduced Klein's definition of geometrical axioms almost word for word in his own discussion about how numbers are introduced in measurement in *Das Kontinuum* (1918). The broader aim of this work was to provide an alternative foundation of analysis that would overcome the paradoxes of the standard, set-theoretical foundation. Therefore, Weyl embraced an intuitionist approach close to Brower's.[49] Although Klein was very far from such a view, Weyl clearly relied on Klein's concept of a projective metric when he maintained that the geometry of continuity proper can be developed only analytically, that is, by articulating analysis as a part of the pure theory of numbers in order to transfer its propositions to geometry. Weyl identified geometrical axioms as "the formulation of such transfer principles out of particular relations, which have to be regarded as immediately given."[50] As we saw in 3.1, a very similar definition of geometrical axiom is found in Klein (1890). In the same passage, Klein went on to say that "the theory of irrationals should be developed and delimited arithmetically, to be then transferred to geometry by means of axioms, and hereby enable that precision which is required for the mathematical consideration."[51] However, the imprecise spatial intuition according to Klein provides us at best with a trigger for the formulation of axioms. Therefore, he denied that intuition can suffice for establishing the existence of a mathematical

[48] A letter dated December 28, 1920, in which Weyl informs Klein about the group-theoretical treatment of metrical space in the fourth edition of *Raum-Zeit-Materie* is found in Klein's *Nachlass*. An extract of this letter is quoted by Scholz (2001, p.87).

[49]Weyl distanced himself from his earlier approach in the course of his mathematical analysis of the problem of space – which was largely based on the set-theoretic account of analysis – and later, more explicitly, in Weyl (1949, p.54), on account of the unjustified limits that Brower's intuitionism would impose on mathematical practice.

[50] Weyl (1918, p.73).

[51] Klein (1890, p.572).

object.[52] By contrast, Weyl's emphasis is on the inevitably subjective origin of what is immediately given for formulation of axioms.

Weyl developed his view in the following years by reconsidering the a priori function of spatial intuition. This does not imply that we are aware of what lies in our intuition before any geometric knowledge about space. Once the relevant axioms have been formulated, we can nonetheless ground such knowledge in what pertains to the form of intuition in opposition to its empirical contents. In other words, it does not follow from the subjectivity of spatial intuition that such a form would either coincide with its heterogeneous contents or be indescribable, as assumed by Schlick. Weyl did not discuss the aforementioned objection of Schlick against the possibility of generalizing the form of intuition in line with Helmholtz's homogeneity requirement. Nevertheless, Weyl was certainly acquainted with Schlick's argument against the possibility of a pure intuition in Kant's sense, which found one of its clearest formulations in Schlick's *Allgemeine Erkenntnistheorie*: "The validity of geometrical propositions cannot be grounded in a pure intuition, for the simple reason that the space of geometry is not intuitive at all."[53] As we have mentioned in 3.2, Schlick used the same distinction between intuition or acquaintance with the spatiality of sense qualities and the physico-geometrical construction of space to rule out a mathematical interpretation of the form of intuition.

Weyl wrote a review of Schlick's work in 1923, in which he especially criticized the lack of intuition as a mediating term between the semiotic character of cognition and the mere modality of sense experience. Referring to this passage, Ryckman argued that Weyl's motivation was to respond to Schlick's attack against one of the fundamental concepts of Husserl's phenomenology, that is, evidence as a source of insight.[54]

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[52] Ibid.

[53] Schlick (1918, p.301).

[54] Ryckman (2005, pp.113–114). Weyl's quotation, in the English translation provided by Ryckman, reads: "In Schlick's opinion, the essence [*Wesen*] of the process of cognition is exhausted by [the semiotic character of cognition]. To the reviewer, it is incomprehensible how anyone, who has ever striven for insight [*Einsicht*], can be satisfied with this. To be sure, Schlick also speaks of 'acquaintance' ['*Kennen*', in opposition to cognizing, *Erkennen*] as the mere intuitive grasping of the given; but he says nothing of its structure, also nothing of the grounding connections between the given and the meanings giving it expression. To the extent that he ignores intuition, in so far as it

Weyl's distinction between evidence and acquaintance enabled him to argue that the conceptual articulation of intuition is always possible in principle, although the historical study of the problem of space shows that in fact it took centuries and great efforts to correctly identifying the essential characteristics of space. Weyl believed to have proved such a characterization to be possible in the second chapter of Weyl (1921) on the metrical continuum. Therefore, he called this chapter a good example of analysis of essence in Husserl's sense and concluded the analysis thus provided with the following historical consideration:

> The historical development of the problem of space teaches how difficult it is for us human beings entangled in external reality to reach a definite conclusion. A prolonged phase of mathematical development, the great expansion of geometry dating from Euclid to Riemann, the discovery of the physical facts of nature and their underlying laws from the time of Galilei, together with the incessant impulses imparted by new empirical data, finally the genius of individual great minds – Newton, Gauss, Riemann, Einstein – all these factors were necessary to set us free from the external, accidental, non-essential characteristics which would otherwise have been held us captive. Certainly, once the true point of view has been adopted reason becomes flooded with light, and it recognises and appreciates what is of itself intelligible to it. Nevertheless, although reason was, so to speak, always conscious of this point of view in the whole development of the problem, it had not the power to penetrate into it with one flash. This reproach must be directed at the impatience of those philosophers who believe it possible to describe adequately the mode of existence on the basis of a single act of typical presentation: in principle they are right: yet from the point of view of human nature, how utterly they are wrong![55]

Weyl elaborated on this view in Weyl (1923) by proposing a mathematical description of the form of space in group-theoretical terms in line with Klein's reading of Helmholtz. At the same time, Weyl escaped the charge of relying on arbitrary assumptions in the distinction between the general and the special properties. The historical shift from the classical to the relativistic view reveals the true

ranges beyond the mere modalities of sense experience, he outrightly rejects self-evidence [*die Evidenz*] which is still the sole source of all insight."

[55] Weyl (1921/1952, pp.147–148).

a priori component of the form of intuition, insofar as the mathematical analysis of the problem of space presents the latter view as a generalization of the previous one. In this sense, the mathematical problem is an essential development of the philosophical problem of space and offers a principled solution against Schlick's empiricist objection.

4. Concluding remarks

The question remains whether Weyl offered a plausible account of Helmholtz's view. Our previous considerations suggest that Weyl captured well Helmholtz's emphasis on the wider scope of his homogeneity requirement when compared to Kant's. I argued in Section 2 that the first analogy between Helmholtz and Weyl is a similar strategy for dealing with the philosophical problem of space: instead of rejecting any a priori intuition, the borderline between the a priori and the a posteriori is set somewhere else. As argued in the present section, a second analogy derives from Weyl's reliance on the Göttingen mathematical tradition initiated by Riemann. Weyl was very clear about the wider scope of Riemann's geometry, despite the common way to refer to the "Riemann-Helmholtz" theory of space. It remains true that nevertheless, as Weyl suggested, both Riemann and Helmholtz belong to a tradition that looked at the conceptual and hypothetical constructions of mathematics as a necessary presupposition of physical geometry.[56] In line with the same tradition, which can be traced back to Gauss, Riemann and Dedekind, Klein maintained that conceptual postulates are necessary to overcome the limits of our spatial intuitions.[57]

Weyl departed from this tradition, insofar as he grounded the possibility of measurement in an inevitable rest of subjectivity. In this sense, he contrasted the intuitive and the mathematical continuum in 1918.[58] He called into question the full-blown objectivity of physical geometry in *Raum-Zeit-Materie* by saying that "the objectivity of things conferred by the exclusion of the ego and its data derived directly from intuition, is not entirely satisfactory; the co-ordinate system which can only be specified by an individual act (and then only approximately) remains as an inevitable

[56] In the literature, a closer connection between Riemann and Helmholtz as the proponents of an empiricist view has been emphasized by DiSalle (2008, p.91): "The empiricist view […] was that dynamical principles – principles involving time as well as space – could force revision of the spatial geometry that had been originally assumed in their development. We might say that this view acknowledges the possibility, at least, that space-time is more fundamental than space."

[57] See Section 3.2.

[58] Weyl (1918, pp.70–71).

residuum of this elimination of the percipient."[59] Weyl's work provides a subtle analysis of the conceptual conditions of measurement in the relativistic context, and therefore an important development of the classical theory of measurement. However, his fundamental assumptions clearly contradict Helmholtz's view that physical equivalence is "a completely determinate, unambiguous objective property of spatial magnitudes," insofar the results of measurement hold true in any frame of reference.[60] In other words, in Helmholtz's view, complete objectivity extends insofar as the laws of (Newtonian) physics apply. He acknowledged that the domain of validity of these laws cannot be fixed once and for all. Nevertheless, he argued for the possibility of a progressive extension of this domain to all known physical phenomena according to the demand of the comprehensibility of nature. By contrast, Weyl retained a foundational role of some subjective structures of cognition while restricting the domain of a priori knowledge to the highly generalized homogeneity requirement of his infinitesimal geometry.

I argued that Weyl's phenomenological standpoint enabled him to overcome Schlick's empiricist objection against the mathematical representation of spatial intuition in the period between 1918 and 1923. A further question is whether Weyl himself reconsidered his aprioristic arguments in favor of a more empiricist approach to the problem of space in his later work on the new wave functions of Dirac.[61] But that is a question for another paper.

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[59] Weyl (1921/1952, p.8).

[60] Helmholtz (1878/1977, p.158).

[61] See Scholz (2005).

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