Paolo Mancosu, *Abstraction and Infinity,* Oxford University Press, 2016, viii + 222pp.

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Paolo Mancosu‘s recent book investigates two central concepts in the foundations of mathematics, namely abstraction and infinity. The main focus of the book concerns so-called abstraction principles or definitions by abstraction. Roughly put, an abstraction principle is a biconditional statement of the form: For all a, b in D: F(a) = F(b) if and only if a $\sim$ b, where F presents a function or an abstraction operator and sim an equivalence relation between objects in a given set D. As is well known, such definitions became prominent in Frege’s work, in particular in his *Grundlagen der Arithmetik* §64 of 1884, and play a crucial role in current debates on neologicism, in particular on the logic and epistemological status of Hume’s Principle (and related principles). The central claim developed in the book is that the discussion of such abstraction principles did not originate with Frege’s work, but that there is rich and multifaceted mathematical prehistory of uses of “definitions by abstraction” in different mathematical fields. The first part of Mancosu’s book gives the first systematic study of abstraction principles in mathematical practice before Frege.

The book has four chapters and an introduction. The first two chapters are mainly historical in character and survey different uses of abstraction principles in mathematical practice until the early twentieth century. In particular, chapter 1 presents a rich study of the use of definitions by abstraction in number theory, geometry, the study of number systems, and set theory, including work by Euclid, Desargues, Gauss, Kronecker, Grassmann, Cantor, Hausdorff, and Weyl (among many others). Based on a number of examples---from Euclid’s definitions of "same ratio" to the definition of cardinal numbers---Mancosu convincingly shows that Frege's use of abstraction principles in *Grundlagen* is in fact directly rooted in a "rich mathematical practice" of using definitions by abstraction. This is an astonishing historical insight that has been so far ignored in Frege scholarship and also in the vast literature on abstraction principles in neo-logicism.

Based on his survey of different definitions of abstraction in mathematics and also in physics, Mancosu makes several highly insightful observations concerning the nature of abstraction principles, for instance on the relation between abstraction and invariants (based on a discussion of Kronecker's work); on the distinction between abstraction in a ``thick'' sense and a ``thin'' sense; as well as on the difference between a "substantial" and a "relational" understanding of abstraction principles. Arguably the most important conceptual point stressed by Mancosu is a distinction between "three theoretical options" in which the abstraction operator or its values can be (and have been) understood in mathematics: (i) as yielding (canonical) representatives for equivalent objects of the original domain (typical in number theory in the nineteenth century); (ii) as yielding equivalence classes; and (iii) as giving new, *sui generis* objects or *abstracta* not contained in original domain. Examples for the third understanding of the process of abstraction are found in geometry, e.g. in the introduction of points at infinity as intersection points of parallel lines in projective geometry, as well as in set theory in the early twentieth century. Given this survey of the widespread practice of definitions by abstraction in different mathematical disciplines, the second chapter focuses on the immediate mathematical background of Frege's famous discussion of abstraction principles in §64 of his *Grundlagen*, as well as on the "foundational reflection" on such definitions in work by members of the Peano school, by Russell, Couturat and others. Concerning the first thematic part, Mancosu argues convincingly that Frege general discussion of such principles are directly influenced by Grassmann's account developed in his book *Geometrische Analyse* of 1847.

A second important line of influence retraced by Mancosu concerns Frege's specific geometrical example of the definition by abstraction of the direction of lines in terms of the equivalence relation of parallelity. Mancosu shows that the conceptual priority of parallelity over the sameness of directions assumed by Frege was subject to intensive debate in textbook treatments of geometry in the nineteenth century. Mancosu identifies the discussion of this issue in Oscar Xaver Schlömilch's monograph *Grundzüge einer wissenschaftlichen Darstellung der Geometrie des Maasses* of 1849 as well as its critical reception in reviews by Reuschle and Gugler as the likely background of Frege's own treatment.

The second part of the chapter then turns to a discussion of the extensive ``logical discussion'' of definitions by abstraction in work by Frege's contemporaries, in particular, by Russell, Couturat and by members of the Peano school such as Peano himself, Burali -Forti, Vailati, and Padoa from the late 1880s until the 1900s. This discussion given here again focuses on several logical and semantical aspects related to such definitions, in particular, the issue of the proper understanding of the values of abstraction functions prior to and after the discovery of Russell's paradox and related set-theoretic paradoxes.

The second thematic part of book then turns to the concept of mathematical infinity and to the idea of comparing different sizes of infinite sets.

Chapter 3 presents different ways of measuring the size of infinite sets of natural numbers. In particular, a history of the mathematical reflection on paradoxes of infinity up to recent work on non-Archimedean mathematics and work on theory of counting infinite sets by Benci, Forti, and Di Nasso. As an alternative to Cantoiran set theory.

The chapter consists of three thematic parts. The first part is again mainly historical in nature. It surveys different contributions to the paradoxes related to the different sizes of infinites sets, i.e. different sizes of infinity. This includes the discussion of contributions of thinkers form different periods, such as Thabit ibn Qurra; Galileo, Leibniz and Maignan, Bolzano, and Cantor. Mancosu shows that in the discussion of such paradoxes, four different approaches to the question whether there are eunqual infinites can be identified, namely (i) to deny existence of such infinites as a results of paradoxes, (ii) accepts infinites, but deny that they can be compared or ordered, (iii) accepts them and hold the have different part whole relations than finite sets, (iv) "develop an arithmetic of the infinite in analogy to the arithmetic of the finite". The latter option is the focus of Mancosu, this inclused the idea of comparing different sizes of infinite sets. A theory of different sizes of infinity. As is well known, this idea developed in Cantor and Bolzano.

Second part of chapter then turns to modern systematic work on arithmetic, measuring the size of of infinite sets of natural numbers in mathematics. Starting point or conceptual motivation here is relation between two intuitions or principles how to characterize the sizes of sets. The first Cantorian principle is that two infinite sets have the same size if there exists a one to one correspondence between them. The second idea is based on the part-whole relation: this is the idea that the sizes of a set is strictly less than that of a set B is A is properly included, strictly contained in B. These principles are perfelctly consists with each other in the case of finite sets, but inconsistent with each other for infinite sets if beoth are understood as definitions of the same concept of size. This has led to paradoxes.

Mancosu presents two modern mathematical theories of sets of natural numbers are presented as alternatives to the standard Cantorian account that also preserve the second intuition, namely a theory Fred Katz presented in his 1981 dissertation titled “Set and their Sizes” and second, a theory of “numerosities” first developed in work by Benco, Di Nasso, and Forti in the 1990s and 2000s. This account differs from Cantor in the sense that counable sets come in difference sizes. Given these modern theories and their mathematical prehistory, the third part then turns to a philosophical discussion based on an argument by Gödel on the inevitability of the Cantorian conception of cardinal number and Kitcher’s theory of generalization in mathematica.

The fourth and final chapter of the book focuses on the relation between the concept of numerosities (as an alternative or refinement of Cantorian infinite cardinalites) and abstraction principles. Specifically, recent work on numerosites and infinity outlined in chapter 3 applied to neologicist debates on status of HP. Chapter has two parts, a historical and a logical-mathematical part. Starting with a brief overview over debates in neo-logicism and status of Himes principles, the historical part focused on question whether there where alternative to Cantorian account of assigning numbers to finite sets vis 1-1 correspondence. In particular, section contains a discussion of different treatments of numerosity functions in work by SChröder, by Peano and by Bolzano. What unites these contributions is that the all differ (in different respects) from the Canorian account of treatement of assigning cardinal numbers to infinites sets via one-one correlation.

These non-Cantorian accounts (angelegt in these historical positions) are then allplied to the contemporary neo-logicist debate. Mancosu first (in section 4.4.) presents a number (in fact countably many) of “good” abstraction principles as alternatives to HP, including several principles that are inconsistent with HP in that they satisfy the part-whole principles underlying the theory of numerosites (described in third chapter).These principles meet the forml conrtainst usually discussed in treatments of HP (e.g. consisteny, etc.) and also allow one to derive the axioms of PA2 and thus form adequate basis for Frege these, and alternative to HP.

The central result of the chapter is a new argument or objection (as a generalization of a argument by Richard heck in a recent article) against the neologicist approach and in particular, of taking HP as a conceptual or analytic truth. This is what Mancosu terms the new company objection (building on the rich literature of the bad company of HP). The argument runs as follows: HP has countably many good companions, i.e. logical abstraction principles of the form of HP but based on the theory of numerosities (NUM) that are consistent and allow to derive PA2. As such they function as possible substitutes (as analytic statements) for HP in neo-logicist project. However, they are inconsistent, or better put, conceptually in conflict with HP if both principles are taken as “explications” of the “intuitive” notion of infinite cardinalities, given that one (HP) assigns same number or size to countably infinite sets) whereas in other case, countably infinite sets come in different sizes. Consequently, given this conceptual conflict between abstraction principles with HP, it is conceptually possible that either HP or cases of Num are not true. Hence, neither principle has the status of a conceptual or analytic truth.

Mancosu end the chapter with a more general discussion of possible consequence of this objection for the neologicist project as well as with a more technical discussion o the problem of “cross-sortal identity” and the relation to the GC objection. Different versions (a liber, moderate and conservative) of neo-logicism will be able t counter the argument or respond differently.

Systematic part focuses on what Mancosu calls "Good company" objection. A significat part of neologicist debate deals with bad company objection.

In conclusion, Mancosu's masterly book presents an invaluable contribution both to the historical and logical-mathematical study of abstraction principles. It should be read by any scholar seriously interested in the history and philosophy of modern mathematics as well as those interested in the related foundational and methodological debates at the turn of the twentieth century. Moreover, the second part of book (in particular the forth chapter) will likely also turn out to be an invaluable read for systematic philosophers working on neologicism and abstraction principles. It should also be added that Mancosu's writing style is clear and well-structured and makes his book a pleasure to read.