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Weakly differential Curves

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Zusammenfassung

Die vorliegende Arbeit widmet sich den verschiedenen möglichen Arten, im Mehrdimensionalen *reell analytisch* zu definieren und deren Zusammenhang mit der Eigenschaft (DN). Das Ziel ist es zu ermitteln, unter welchen Bedingungen die verschiedenen Begriffe reell analytischer Abbildungen zusammenfallen. Unglücklicherweise reicht dazu die Eigenschaft (DN) alleine nicht aus.

Der erste Teil des Werkes dient der Einführung in die verwendete Notation und der benötigten Grundlagen. Kapitel 1 widmet sich grundlegenden Eigenschaften lokalkonvexer Räume und (beschränkter) stetiger linearer Abbildungen zwischen diesen. Kapitel 2 führt induktive und projektive Limiten lokalkonvexer Räume ein. Das letzte einleitende Kapitel befasst sich mit der Charakterisierung des Raumes holomorpher Funktionen.

Die nächsten beiden Kapitel liefern wichtige nicht elementare Ergebnisse für die primären Aussagen der Arbeit. Kapitel 4 befasst sich mit der Grothendieck-Köthe-Silva Dualität, einer Charakterisierung des Dualraums des Fréchet Raumes $H(U)$. Kapitel 5 führt den Raum der schnell fallenden Folgen ein und geht auf den Zusammenhang mit der Nuklearität ein.

Im folgenden Teil wurden die unabdingbaren Ideen und Methoden zur Lösung der grundlegenden Fragestellung zusammengefasst. Während Kapitel 6 reell analytische Funktionen (von $\mathbb{R}$ nach $\mathbb{R}$) behandelt, erweitert Kapitel 7 die Betrachtung auf reell analytische Abbildungen von $\mathbb{R}$ in einen lokalkonvexen Raum E. Schließlich geht Kapitel 8 auf Folgenräume und exakte Sequenzen von Folgenräumen ein.

Kapitel 9 zeigt die Korrelation zwischen Folgenräumen und der Eigenschaft (Ω). Insbesonders ist ein Fréchet Raum mit der Eigenschaft (Ω) isomorph zu einem Quotientenraum von $\ell^1(I)\hat{\otimes}s$ (für geeignetes I).

Die Kapitel 10 bis 12 widmen sich den Bedingungen an Fréchet Räume, unter denen jede stetige lineare Abbildung beschränkt ist. Für die Resultate muss entweder der Definitions- oder der Zielraum ein Folgenraum sein. Diese Einschränkung wird allerdings im nächsten Kapitel behoben. Kapitel 13 liefert diese wichtigen Zwischenresultate, die für sich alleine betrachtet werden können. So etwa die Verallgemeinerung des Hauptresultats von Kapitel 9, $C^\omega(\mathbb{R}, F) = C^\omega_t(\mathbb{R}, F)$ and $H_\omega(B, F) = H(B, F)$ (unter dort spezifizierten Voraussetzungen).

Das letzte Kapitel zeigt schließlich die folgenden beide Sätze: Einen Fréchet Raum F hat die Eigenschaft (DN) genau dann, wenn $C^\omega(E, F) = C^\omega_t(E, F)$ für alle nuklearen Fréchet Räume E beziehungsweise Fréchet-Schwarz Räume mit Eigenschaft $(\tilde{\Omega})$ gilt. Und zweitens, dass für Fréchet Räume F mit Eigenschaft (LB_∞) $C^\omega(E, F) = C^\omega_t(E, F)$ für alle reellen Fréchet Räume E gilt.

Im Appendix finden sich jene technischen Details, die den Aufbau der Arbeit unnötig beschwert hätten. Appendix A liefert alle in der Arbeit verwendeten

äquivalenten Beschreibungen der Eigenschaften (DN) und $(\bar{\Omega})$. Außerdem werden die Implikationen zwischen den Eigenschaften bewiesen. Appendix B ist ein Exkurs. Er liefert eine Verallgemeinerung eines Satzes aus Kapitel 11 zusammen mit der Beweisskizze. Leider konnte der Beweis nicht in allen Details gegeben werden und wurde deshalb aus dem Hauptteil genommen. Appendix C listet die grundlegenden Sätze der Funktionalanalysis auf, die hier verwendet wurden.

Abstract

This work is about real analytic curves, their different definitions, and the property (DN). My aim is to give a complete record under which conditions the different notions of real analytic mappings coincide. Unfortunately, the property (DN) is not sufficient to accomplish this.

It is traditional to start books with an introductory chapter. I dedicated three chapters to introduce the notation and the basic principles needed thereafter. While Chapter 1 lays down the basic qualities of locally convex spaces and the (bounded) continuous linear mappings between them, Chapter 2 focuses on projective and inductive limits of locally convex spaces. Chapter 3 introduces holomorphic functions and characterises the space of holomorphic functions.

The next two chapters reproduce non-trivial findings which are essential later on. Chapter 4 proves the Grothendieck-Köthe-Silva duality, a characterisation of the dual space of the Fréchet space $H(U)$. In Chapter 5 the space of rapidly decreasing sequences is introduced and the connection to nuclear spaces is laid down.

In the following three chapters I have collected the indispensable ideas and tools necessary for the main findings. While in Chapter 6 real analytic functions (from $\mathbb{R}$ to $\mathbb{R}$) are set forth, Chapter 7 expands the range to real analytic curves (from $\mathbb{R}$ to a locally convex space E). Chapter 8 takes a step back and revisits sequence spaces and exact sequences of sequence spaces are considered for later use.

In Chapter 9 the correlation between quotients of sequence spaces and the property (Ω) is laid down. Notably, a Fréchet space with property (Ω) is isomorphic to a quotient space of $\ell^1(I)\hat{\otimes}s$ (for suited I).

The Chapters 10 to 12 present conditions for Fréchet spaces under which every continuous linear map is bounded. The results require a sequence space as either the domain or the co-domain, but this restriction will be relieved in the next chapter. As the name suggests, Chapter 13 lists those results that are noteworthy on their own, such as a generalisation of the main statement from chapter 9, $C^\omega(\mathbb{R}, F) = C^\omega_t(\mathbb{R}, F)$ and $H_\omega(B, F) = H(B, F)$. (Note that I have not mentioned here the prerequisite conditions.)

Chapter 14 finally states the two main findings: That for a Fréchet space F with property (DN) we have $C^\omega(E, F) = C^\omega_t(E, F)$ for all either nuclear Fréchet spaces or (FS)-spaces E with property $(\tilde{\Omega})$. And alternatively, that for a Fréchet space F with property (LB_∞) we get $C^\omega(E, F) = C^\omega_t(E, F)$ for all real Fréchet spaces E.

In order to avoid polluting the main work with lengthy technical details, I have transferred those chunks of knowledge into the Appendices. In Appendix A equivalent descriptions of the properties (DN) and $(\bar{\Omega})$ are presented in any case for the mentioned instances. Additionally the dependencies/implications among the properties are offered. Appendix B contains an excursion; a more general form of

a theorem from chapter 11 is presented together with an outline of its proof. Unfortunately, I was not able to give a complete proof and hence didn't include it in the main part of the work. Appendix C collects the well-known theorems which are referred to in the prior chapters.

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Contents

Chapter 1

Locally Convex Spaces

Definition 1.1 A vector space E is said to be a *locally convex topological vector space*, or simply *locally convex space*, if it is a topological space and has a family of semi-norms such that a net $(x_\nu)_{\nu\in\mathcal{I}}$ in E with index set I converges to x if and only if $\|x_\nu - x\|_p \underset{\nu\in I}{\longrightarrow} 0$ for all semi-norms $\|.\|_p$ in this family.

An arbitrary family of semi-norms on a vector space E determines a unique topology on E, given by the condition of the preceding paragraph. This topology is called the *locally convex topology.*

Definition 1.2 A locally convex topology defined by the family of semi-norms is called *Hausdorff* if it satisfies the norm condition

$$\|x\|_p = 0\ \forall \|.\|_p \ \Rightarrow\ x = 0.$$

All locally convex spaces are considered Hausdorff unless explicitly otherwise stated.

Definition 1.3 Let E be a locally convex space. A subset K is called *precompact* if for every neighbourhood U of 0 there exists a finite set F with $K \subseteq U + F$.

Lemma 1.4 ([Kri02], 6.14 Lemma) *Let E be a Fréchet space and A a subset of E. A is precompact if and only if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ converging to 0, such that A is contained in the closed convex hull of the sequence.*

Proof. Let $(U_n)_{n\in\mathbb{N}}$ be a neighbourhood basis of 0 consisting of absolutely convex closed subsets of a Fréchet space E such that $2U_{n+1} \subseteq U_n$ and $U_0 = E$. We now contrive a sequence $(A_n)_{n\in\mathbb{N}}$ of precompact subsets and finite sets $F_n \subseteq A_n$ for all $n \in \mathbb{N}$. Let $A_0 := A$ and A_n already be conceived. Then there exists a finite set $F_n \subseteq A_n$ with $A_n \subseteq F_n + \frac{1}{2^n}U_n$. We now put

$$A_{n+1} := (A_n - F_n) \cap \frac{1}{2^n}U_n.$$

This set is precompact. Let $x_{k_n+1}, \dots, x_{k_{n+1}}$ be the Elements of 2^nF_n. Since we have

$$F_n \subseteq A_n \subseteq \frac{1}{2^n}U_n \subseteq \frac{1}{2^{n-1}}U_{n-1}$$

and $(x_n)_{n\in\mathbb{N}}$ converges to 0. Now choose $a \in A = A_0 \subseteq F_0 + \frac{1}{2^0}U_0$. Then there exist $a_0 \in F_0$ and $u_0 \in U_0$ such that $a = a_0 + u_0$. Since $a - a_0 = u_0 \in (A_0 - F_0) \cap \frac{1}{2^0}U_0 =$

$A_1 \subseteq F_1 + \frac{1}{2^1}U_1$, there exist $a_1 \in F_1$ and $u_1 \in U_1$ such that $a = \sum_{i<n} a_i + \frac{1}{2^n}u_n$. Since $(u_n)_{n\in\mathbb{N}}$ converges to 0, $\sum_i a_i$ converges to a. Now there exist $k_i < k(i) \leq k_{i+1}$ such that $a_i = \frac{1}{2^i}x_{k(i)}$. If we put $\lambda_{k(i)} := \frac{1}{2^i}$ and otherwise 0, then $a = \sum_i \lambda_i x_i$ is contained in the closed convex hull of the x_i.

Conversely, we show that the closed absolutely convex hull $\overline{\langle B\rangle}$ of a precompact subset B is itself precompact. Therefore let B be precompact and put $V := \frac{1}{3}U$, where U is a closed and absolutely convex subset in E. Then, by definition, there exists a finite set M such that $B \subseteq M + V$ and hence

$$\langle B\rangle \subseteq \langle M\rangle + V,$$

since $\langle M\rangle$ is precompact as the continuous image of the unit ball in $\ell^1(M)$. Again by definition, we have a finite set M_1 with $\langle M\rangle \subseteq M_1 + V$. Finally we get

$$\overline{\langle B\rangle} \subseteq \overline{M_1 + 2V} \subseteq M_1 + 2V + V \subseteq M_1 + U,$$

hence $\overline{\langle B\rangle}$ being precompact. □

Definition 1.5 Let E be a locally convex space, A an absolutely convex subset of E. We define the *Minkowski functional* $\|.\|_A : \mathrm{span}(A) \to \mathbb{R}$ as

$$\|x\|_A := \inf\{\lambda > 0 : x \in \lambda A\}.$$

If additionally $\mathrm{span}(A) = E$, then $\|.\|_A$ is a semi-norm on E.
If A is bounded in E, then $\|.\|_A$ is a norm on $\mathrm{span}(A)$.

Definition 1.6 Let E be a locally convex space and B a closed absolutely convex bounded subset. The linear span of B in E equipped with the Minkowski functional of B as its norm is denoted by E_B.

In general, E_B is a normed space but if E is sequentially complete, E_B is a Banach space.

Remark 1.7 If not otherwise stated, sequences of scalars will be in $\mathbb{F}$, being one of the scalar fields $\mathbb{R}$ or $\mathbb{C}$.

Lemma 1.8 *For each $m \in \mathbb{N}$ let $\lim_{n\in\mathbb{N}} \mu_{n,m} = 0$. Then there exists a monotonous sequence $(\lambda_n)_{n\in\mathbb{N}}$ with $\lim_{n\in\mathbb{N}} \lambda_n = 0$ such that*

$$\left\{\frac{\mu_{n,m}}{\lambda_n} : n \in \mathbb{N}\right\}$$

is bounded for each $m \in \mathbb{N}$.

Proof. For $k \in \mathbb{N}$ put

$$n_k := \max\left\{k, \sup\left\{n : \exists\, m \leq k : |\mu_{n,m}| > \frac{1}{k}\right\}\right\},$$

i.e. $k|\mu_{n,m}| \leq 1$ for $m \leq k$ and $n \geq n_k$. Then $n_k \underset{k\to\infty}{\longrightarrow} \infty$ monotonously. Now define $\lambda_n := \frac{1}{k}$ for $n_k \leq n \leq n_{k+1}$. Clearly $\lambda_n \underset{n\to\infty}{\longrightarrow} 0$. Furthermore, for each m we have

$$\frac{|\mu_{n,m}|}{\lambda_n} = k|\mu_{n,m}| \leq 1$$

provided n is so large that $n_m < n$ and hence k with $n_k \leq n \leq n_{k+1}$ satisfies $m \leq k$. □

Lemma 1.9 *Let E be a Fréchet space. For each absolutely convex compact subset B of E we can find an absolutely convex compact subset B_1 of E such that B is compact in E_{B_1}.*

Proof. By 1.4 there exists a sequence $(x_n)_{n\in\mathbb{N}}$ converging to $0 \in E$ with B contained in the closed absolutely convex hull of the sequence. By 1.8, there exists a sequence $(\mu_n)_{n\in\mathbb{N}} \to \infty$ such that $y_n := \mu_n x_n$ still converges to 0. We put $B_1 := \overline{\langle (y_n)_{n\in\mathbb{N}} \rangle}$. Therefore B_1 is an absolutely convex bounded subset of E and $(x_n)_{n\in\mathbb{N}} \to 0 \in E_{B_1}$, since $\|x_n\|_{B_1} = \frac{1}{\mu_n}$. Ergo B is compact in E_{B_1}. □

Definition 1.10 Let E and F be two locally convex spaces, then we put

$$L(E,F) := \{A : E \to F : A \text{ is linear and continuous}\}$$

and $L(E) := L(E,E)$.

Furthermore we define

$$LB(E,F) := \{A \in L(E,F) : A \text{ is bounded on some neighbourhood of } 0\}$$

and $LB(E) := LB(E,E)$.

Remark 1.11 For E and F being two normed spaces with closed unit balls U and V, the topology of $L(E,F)$ is obtained from the norm

$$\|T\| := \inf\{\varrho > 0 : T(U) \subseteq \varrho V\}$$

for $T \in L(E,F)$. $\|.\|$ is called *operator norm* on $L(E,F)$.

Remark 1.12 Let E and F be two locally convex spaces then for every element $T \in LB(E,F)$ the following assertions hold:

(i) There exists a neighbourhood U of 0 in E such that $T(U)$ is bounded in F.

(ii) There exists a neighbourhood U of 0 in E and a bounded absolutely convex subset $B \subseteq F$ such that $T(U) \subseteq B$.

(iii) There exists a bounded absolutely convex subset $B \subseteq F$ such that the map $T_B : E \to F_B$ exists and is continuous.

(iv) For every linear map $L : E \to F$ there exists a bounded subset $B \subseteq F$ and a neighbourhood U of 0 in E such that $T(U) \subseteq B$.

Thus we have shown that

$$LB(E,F) = \bigcup_B L(E, F_B)$$

where B runs through all bounded absolutely convex subsets of F.

Definition 1.13 Let X and Y be locally convex spaces. A linear mapping $T : X \to Y$ is said to be *compact* if T maps a neighbourhood V of 0 in X to a relatively compact set in Y.

In this case, an arbitrary neighbourhood W of 0 in Y absorbs the bounded set $T(V)$, i.e. $[0,\rho]T(V) \subseteq W$ for some $\rho > 0$. And so $T^{-1}(W)$ absorbs V. Hence, T is continuous.

Let $A : X_1 \to X$ and $B : Y \to Y_1$ be continuous linear mappings. Then $A^{-1}(V)$ is a neighbourhood of 0 in X_1, and $B(T(V))$ is a relatively compact set in Y_1, and so the product BTA is also compact.

Lemma 1.14 ([Kom99], Lemma 1.2) *Let X and Y be locally convex spaces. A linear mapping $T : X \to Y$ is compact if and only if there is a Banach space N such that T can be decomposed into the product of two continuous linear mappings*

$$X \xrightarrow{T_1} N \xrightarrow{T_2} Y,$$

where the image $T_2(B)$ of the unit ball B in N is compact. Moreover, if T is injective, then T_1 and T_2 can be chosen to be injective.

Proof. Sufficiency is clear.

If T is compact, then T maps an absolutely convex neighbourhood V of 0 in X into a compact set $B = \overline{T(V)}$, where the right hand side denotes the closure in Y. Let N be the normed space generated by B with B as its unit ball. Then a Cauchy sequence in N converges under the weak topology in Y, and hence converges also in N, i.e. N is a Banach space. Letting $T_1 : X \to N$ be the mapping included by T, and $T_2 : N \to Y$ be the embedding mapping, we obtain the desired decomposition.□

Lemma 1.15 ([Kom99], Lemma 1.3) *Let Y be a linear subspace of a locally convex space X. If V is an absolutely convex neighbourhood of 0 in Y, then there is an absolutely convex neighbourhood U of 0 in X such that $V = U \cap Y$. If Y is closed, then for any $x_0 \in X \backslash Y$, the neighbourhood U may be chosen such that $x_0 \notin U$.*

Proof. Since V is a neighbourhood of 0 in the relative topology, there is a neighbourhood W of 0 in X such that $W \cap Y \subseteq V$. Without loss of generality, we may assume that V and W are absolutely convex. Let U be the convex hull of the union of V and W. Since V and W are absolutely convex, U consists of all elements u which can be expressed as $u = \lambda v + (1 - \lambda)w$, with $v \in V$, $w \in W$ and $0 < \lambda < 1$. Consequently, $U \cap Y$ is the convex hull of the union of V and $W \cap Y$, and hence coincides with V. On the other hand, U is a neighbourhood of 0 in X since it contains W, and it is also clear that U is absolutely convex.

If Y is closed, we can choose W so small such that the canonical image of W in the Hausdorff space $X \backslash Y$ does not contain the canonical image of x. Then the set U constructed above does not contain x_0. □

Chapter 2

Projective and Inductive Limits of Locally Convex Spaces

Note that in exception to the rest of this work, locally convex spaces are not necessarily Hausdorff in this chapter.

The statements in this chapter can be found in any profound book on functional analysis, e.g. [MV92], [Köt69a] or [Jar81].

Definition 2.1 Let $\{u_\alpha : X \to X_\alpha\}_{\alpha \in A}$ be a family of linear mappings of a vector space X into locally convex spaces X_α. Then there exists the weakest locally convex topology on X under which all u_α are continuous. This topology is called the *projective locally convex topology* relative to the system $(X_\alpha, u_\alpha)_{\alpha \in A}$. If $\wp_\alpha$ is a family of semi-norms defining the topology of an X_α, then

$$\wp = \{\|u_\alpha(.)\|_{p_\alpha} : \alpha \in A,\ \|.\|_{p_\alpha} \in \wp_\alpha\}$$

is a family of semi-norms defining the projective locally convex topology. This topology is not necessarily Hausdorff.

If $X = \prod X_\alpha$ is a vector space represented as the product of locally convex spaces X_α, and if $u_\alpha : X \to X_\alpha$ are the canonical projections, then the projective locally convex topology is called the *product locally convex topology*, and the product space X endowed with this topology is called the *product* of locally convex spaces X_α.

Furthermore, if X is a linear subspace of a locally convex space Y, and if $u : X \to Y$ is the inclusion, then the weakest locally convex topology under which u is continuous is no other than the relative topology on X as a subspace of Y.

If the projective locally convex topology is Hausdorff, then it is a combination of the above two special cases in the following sense: Define

$$u : X \to \prod_{\alpha \in A} X_\alpha : u(x) = (u_\alpha(x))_{\alpha \in A};$$

then u is an injection because of the assumption that X is Hausdorff. The projective locally convex topology on X is then the same as the relative topology on X identified with a linear subspace of $\prod_{\alpha \in A} X_\alpha$ under u. Conversely, if the mapping u defined above is injective, then the projective locally convex topology on X is Hausdorff.

On the other hand, projective limits of locally convex spaces are defined as follows. Let A be a directed set and suppose that for each $\alpha \in A$ a locally convex space X_α is specified together with continuous linear mappings $u_{\alpha,\beta} : X_\alpha \to X_\beta$ defined for all pairs (α, β) with $\alpha > \beta$, and satisfying $u_{\alpha,\gamma} = u_{\beta,\gamma} \circ u_{\alpha,\beta}$ whenever $\alpha, \beta, \gamma \in A$ and $\alpha > \beta > \gamma$. Such a system $(X_\alpha, u_{\alpha,\beta})$ is called a *projective system of locally convex spaces*. Then, we define the *projective limit* of locally convex spaces X_α to be the projective limit

$$\varprojlim_{\alpha \in A} X_\alpha := \left\{ (x_\alpha) \in \prod_{\alpha \in A} X_\alpha : u_{\alpha,\beta}(x_\alpha) = x_\beta \right\}$$

of vector spaces X_α equipped with the weakest locally convex topology under which the canonical mappings $u_\alpha : \varprojlim_{\alpha \in A} X_\alpha \to X_\alpha$ defined by $(x_\alpha) \mapsto x_\alpha$ are continuous.

Suppose that $(X_\alpha, u_{\alpha,\beta})$ and $(Y_\alpha, v_{\alpha,\beta})$ are projective systems of locally convex spaces with the same directed set A as their index set. If a continuous linear mapping $T_\alpha : X_\alpha \to Y_\alpha$ is given for each α, and satisfies $v_{\alpha,\beta} \circ T_\alpha = T_\beta \circ u_{\alpha,\beta}$ for all $\alpha > \beta$, then a continuous linear mapping $T : \varprojlim_{\alpha \in A} X_\alpha \to \varprojlim_{\alpha \in A} Y_\alpha$ is defined by $T(x_\alpha) = (T_\alpha x_\alpha)$, which we call the *projective limit* of the mappings T_α.

Let $(X_\alpha, \alpha \in A)$ be a projective system of locally convex spaces. If $\alpha : \Lambda \to A$ is an order-preserving mapping of a directed set Λ with a cofinal image $\alpha(\Lambda)$ in A, then the projective system $(Y_\lambda, \lambda \in \Lambda)$ defined by $Y_\lambda = X_{\alpha(\lambda)}$ is called a *subsystem* of the original projective system. In this case, it is easy to verify that

$$\varprojlim_{\lambda \in \Lambda} Y_\lambda = \varprojlim_{\alpha \in A} X_\alpha$$

in the sense of a canonical isomorphism.

It follows from the continuity of $u_{\alpha,\beta}$ that $\varprojlim_{\alpha \in A} X_\alpha$ is a closed linear subspace of $\prod_{\alpha \in A} X_\alpha$.

Remark 2.2 If X_α are complete (respectively quasi-complete or sequentially complete), then the projective limit $\varprojlim_{\alpha \in A} X_\alpha$ is also complete (respectively quasi-complete or sequentially complete).

If the directed set A is the set $\mathbb{N}$ of natural numbers, we only have to specify continuous linear mappings $u_{j+1,j} : X_{j+1} \to X_j$ for all $j \in \mathbb{N}$. The other mappings are determined as compositions of those mappings. In this case, we often denote the projective system by the diagram:

$$X_1 \overset{u_{2,1}}{\longleftarrow} X_2 \overset{u_{3,2}}{\longleftarrow} X_3 \longleftarrow \ldots \longleftarrow X_j \overset{u_{j+1,j}}{\longleftarrow} \ldots . \quad (1)$$

If all X_j are Banach spaces, then the projective limit $\varprojlim_{j \in \mathbb{N}} X_j$ is complete and metrizable as a locally convex space whose topology is defined by a countable family of semi-norms.

The projective limit of a sequence of Banach spaces X_j is a Fréchet space. Conversely, every Fréchet space can be expressed as the projective limit of a sequence of Banach spaces.

Definition 2.3 An arbitrary locally convex space X is isomorphic to a dense linear subspace of a projective limit $\varprojlim_{\alpha \in A} X_\alpha$ of Banach spaces. Moreover, if we can choose a projective system such that for any $\alpha > \beta$, $u_{\alpha,\beta} : X_\alpha \to X_\beta$ are not only continuous but also compact, then X is said to be a *Schwartz space*.

A Fréchet space, which is also a Schwartz space is called an (FS)*-space* for short.

Remark 2.4 A locally convex space X is an (FS)-space if and only if it can be expressed as the projective limit of a sequence of Banach spaces X_j such that every $u_{j+1,j} : X_{j+1} \to X_j$ in (1) is compact.

Closed linear subspaces and quotient spaces of (FS)-spaces are (FS)-spaces. Products $\prod_{j\in\mathbb{N}} X_j$ and projective limits $\varprojlim_{j\in\mathbb{N}} X_j$ of a countable number of (FS)-spaces are also (FS)-spaces.

Definition 2.5 Let X be a vector space and $\{u_\alpha : X_\alpha \to X\}_{\alpha\in A}$ be a family of linear mappings from locally convex spaces X_α. Then the strongest locally convex topology on X under which all u_α are continuous is called the (generalised) *inductive locally convex topology* of the system $(X_\alpha, u_\alpha)_{\alpha\in A}$. A semi-norm $\|.\|_p$ on X is continuous under this locally convex topology if and only if $\|u_\alpha\|_p$ is a continuous semi-norm on X_α for every $\alpha \in A$. However, this locally convex topology is not necessarily Hausdorff even if $\{u_\alpha(X_\alpha) : \alpha \in A\}$ generates X. We remark also that it is, in general, different from the inductive limit topology as a topological space, that is, the strongest topology under which all u_α are continuous.

Definition 2.6 Let $X = \bigoplus_{\alpha\in A} X_\alpha$ be a vector space which is expressed as a direct sum of locally convex spaces X_α and let $u_\alpha : X_\alpha \to X$ be the canonical injections. Then the inductive locally convex topology on X is called the *locally convex direct sum topology* and the direct sum X equipped with this topology is called the *direct sum* of the locally convex spaces X_α. If $\wp_\alpha$ is the family of all continuous semi-norms on X_α, then the locally convex direct sum topology is the locally convex topology defined by all semi-norms of the form

$$\|x_\alpha\|_p = \sum_{\alpha\in A} \|x_\alpha\|_{p_\alpha},$$

where $p_\alpha \in \wp_\alpha$. In particular, a locally convex direct sum topology is Hausdorff.

The quotient topology on a quotient space X/Y of a locally convex space X is also the inductive locally convex topology relative to the canonical projection $X \to X/Y$.

If $\{u_\alpha(X_\alpha) : \alpha \in A\}$ generates X then the general inductive locally convex topology relative to a system $(X_\alpha, u_\alpha)_{\alpha\in A}$ is a combination of the above two classes. Namely, in this case, the mapping

$$u : \bigoplus_{\alpha\in A} X_\alpha \to X \text{ with } u(\oplus x_\alpha) := \sum_{\alpha\in A} u_\alpha(x_\alpha)$$

is surjective, and X may be regarded as a quotient space of the direct sum $\bigoplus_{\alpha\in A} X_\alpha$. The inductive locally convex topology on X is then identified with the quotient topology of the locally convex direct sum topology.

Definition 2.7 Let $\{X_\alpha\}_{\alpha\in A}$ be a family of locally convex spaces with a directed set A as its index set and $\{u_{\alpha,\beta} : X_\alpha \to X_\beta\}_{\alpha<\beta}$ be a family of continuous linear mappings satisfying $u_{\alpha,\gamma} = u_{\beta,\gamma} \circ u_{\alpha,\beta}$ whenever $\alpha, \beta, \gamma \in A$ and $\alpha < \beta < \gamma$. Then the inductive limit of the vector spaces X_α is the quotient space

$$\varinjlim_{\alpha \in A} X_\alpha = \bigoplus_{\alpha \in A} X_\alpha / \sim, \tag{2}$$

where $\sim$ is the linear subspace generated by all elements whose entry at the index α is $x_\alpha \in X_\alpha$, the entry at the index β is $-u_{\alpha,\beta}(x_\alpha)$ for a pair $\alpha < \beta$, and all other entries are 0. We define the canonical mapping $u_\alpha : X_\alpha \to \varinjlim_{\alpha \in A} X_\alpha$ to be the mapping which sends $x_\alpha \in X_\alpha$ to the equivalence class containing the element whose entry at the index α is x_α, and 0 elsewhere. The inductive limit $\varinjlim_{\alpha \in A} X_\alpha$ equipped with the strongest locally convex topology under which all canonical mappings are continuous is called the *inductive limit* of locally convex spaces X_α. As can be seen from (2), this is the same as a quotient space of the direct sum of locally convex spaces. However, in this case, the continuity of $u_{\alpha,\beta}$ does not, in general, imply that the linear subspace $\sim$ is closed, and so the inductive locally convex topology is not necessarily Hausdorff.

Definition 2.8 Let X be a locally convex space and V a subspace of X. If for every $x \in X$ there exists an $\rho_x > 0$ such that $[0, \rho_x]x \subseteq V$, then V is called an *absorbent* subset of X. If V is an absolutely convex, closed and absorbent subset of X, then V is called a *barrel* in X.

A locally convex space X is called *barrelled*, if every barrel in X is a neighbourhood of 0. Furthermore, a locally convex space is called *quasi-barrelled* if every barrel which absorbs every bounded subset is a neighbourhood of 0. Trivially, every barrelled space is quasi-barrelled and every sequentially complete quasi-barrelled space is barrelled.

Strengthening the condition on quasi-barrelled spaces, a locally convex space is called *bornological* if every semi-norm is continuous provided that it is bounded on each bounded set. It is called *ultrabornological* if every semi-norm is continuous provided that it is bounded on each Banach disc.

Remark 2.9 If the locally convex spaces X_α are barrelled (respectively quasi-barrelled or bornological) for all $\alpha \in A$, then the inductive limit $\varinjlim_{\alpha \in A} X_\alpha$ is also barrelled (respective quasi-barrelled or bornological).

Definition 2.10 An inductive system of a sequence of locally convex spaces can be expressed by the diagram

$$X_1 \xrightarrow{u_{1,2}} X_2 \xrightarrow{u_{2,3}} X_3 \longrightarrow \ldots \longrightarrow X_j \xrightarrow{u_{j,j+1}} \ldots. \tag{3}$$

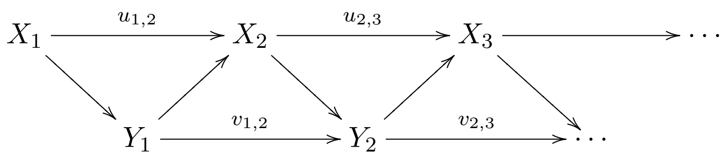
If $u_{j,j+1} : X_j \to X_{j+1}$ is an isomorphism onto its image for all j, then the sequence is called a *strict inductive sequence*. In this case, each X_j may be identified with a linear subspace of X_{j+1} under the isomorphism $u_{j,j+1}$, and the inductive limit X can be regarded as the union $X = \bigcup_{j \in \mathbb{N}} X_j$.

Remark 2.11 If (3) is a strict inductive sequence of locally convex spaces, then the canonical mapping $u_j : X_j \to \varinjlim_{j \in \mathbb{N}} X_j$ is an isomorphism onto its image for every j. In particular, the inductive limit $\varinjlim_{j \in \mathbb{N}} X_j$ is Hausdorff.

If, in addition, the image $u_{j,j+1}(X_j)$ is a closed linear subspace of X_{j+1} for every j, then the canonical image $u_j(X_j)$ in $\varinjlim_{j\in\mathbb{N}} X_j$ is a closed linear subspace for every j. Moreover, every bounded set B in $\varinjlim_{j\in\mathbb{N}} X_j$ is the image $u_j(B_j)$ of a bounded set B_j in X_j for some j.

Definition 2.12 A locally convex space X is said to be an (LF)-*space* if it can be expressed as the strict inductive limit of a sequence of Fréchet spaces X_j. In particular, if all X_j are (FS)-spaces, then X is said to be an (LFS)-*space*.

Remark 2.13 We now consider the case where the mappings $u_{j,j+1}$ in the inductive sequence (3) are compact linear injections. Then, by 1.14, we can find Banach spaces Y_j between X_j and X_{j+1} and injections $v_{j,j+1} : Y_j \to Y_{j+1}$ such that the diagram



commutes and that the image $v_{j,j+1}(B_j)$ of the unit ball B_j in Y_j is compact in Y_{j+1} for every j. Clearly we have the canonical isomorphism $\varinjlim_{j\in\mathbb{N}} X_j = \varinjlim_{j\in\mathbb{N}} Y_j$ including the topology.

Definition 2.14 If a locally convex space E has a sequence of B_j of bounded sets with the property that every bounded set in E is contained in some B_j, then the strong topology of E' is defined by a countable number of semi-norms

$$p_j(x') = p_{B_j}(x')$$

and so the strong dual E' is metrizable. A countable quasi-barrelled space satisfying this property is called a (DF)-*space*.

The strong dual of a (DF)-space is a Fréchet space. On the other hand, the strong dual of a metrizable locally convex space is a (DF)-space.

Definition 2.15 If a locally convex space X is expressed as above as an inductive limit of a sequence of locally convex spaces with compact injections $u_{j,j+1}$, we say that X is a (DFS)-*space*.

Without further explanation, please accept that these spaces are reflexive (DF)-spaces.

Definition 2.16 Let $Y = \varprojlim_{\alpha\in A} X_\alpha$. Then Y can be expressed as a limit of a projective system $(X_\alpha)_{\alpha\in A}$ in which the image $u_\alpha : \varprojlim_{\alpha\in A} X_\alpha \to X_\alpha$ is dense for all $\alpha \in A$. Such a projective system is said to be *reduced*.

Remark 2.17 If $(X_\alpha)_{\alpha\in A}$ is a reduced projective system of locally convex spaces, then the dual of its projective limit is, as a vector space, canonically isomorphic to the inductive limit of the duals X'_α of X_α

$$\left(\varprojlim_{\alpha\in A} X_\alpha\right)' = \varinjlim_{\alpha\in A} X'_\alpha.$$

Moreover, the weak topology on $\varprojlim_{\alpha\in A} X_\alpha$ coincides with the projective limit topology of the weak topologies of the X_α.

Remark 2.18 Let $(X_\alpha)_{\alpha\in A}$ be a reduced projective system of locally convex spaces X_α. If for each α there exists a $\beta \geq \alpha$ such that $u_{\beta,\alpha}$ maps every bounded set into a relatively weakly compact set, then $\varprojlim_{\alpha\in A} X_\alpha$ is semi-reflexive, and there is a natural isomorphism

$$\left(\varprojlim_{\alpha\in A} X_\alpha\right)'_\beta = \varinjlim_{\alpha\in A} (X_\alpha)'_\beta .$$

Remark 2.19 As for the dual of an inductive limit of locally convex spaces, there is a canonical isomorphism as vector spaces

$$\left(\varinjlim_{\alpha\in A} X_\alpha\right)' = \varprojlim_{\alpha\in A} X'_\alpha.$$

If every bounded set in $\varinjlim_{\alpha\in A} X_\alpha$ is the image of a bounded set in some X_α, then we have the natural isomorphism where the spaces considered are locally convex spaces

$$\left(\varinjlim_{\alpha\in A} X_\alpha\right)'_\beta = \varprojlim_{\alpha\in A} (X_\alpha)'_\beta .$$

Remark 2.20 The strong dual of an (FS)-space is a (DFS)-space. The strong dual of a (DFS)-space is an (FS)-space.

Since these spaces are reflexive and bornological, it follows also that they are complete as the strong dual of bornological spaces.

Remark 2.21 Closed linear subspaces and quotient spaces of (DFS)-spaces are also (DFS)-spaces. Countable direct sums $\bigoplus_{j\in\mathbb{N}} X_j$ and inductive limits $\varinjlim_{j\in\mathbb{N}} X_j$ of sequences of (DFS)-spaces are (DFS)-spaces.

Chapter 3

Holomorphic Functions

Definition 3.1 Let V be an open set in the complex plane $\mathbb{C}$. We denote by $H(V)$ the vector space of all holomorphic functions on V.

The space $H(V)$ is usually endowed with the topology of uniform convergence on compact sets; i.e. the locally convex topology determined by the family of semi-norms

$$\|\phi\|_K := \sup \{|\phi(z)| : z \in K\},$$

as K runs through the family of all compact sets in V. In practice, choose a sequence $K_1 \subseteq K_2 \subseteq \ldots \subseteq V$ of compact sets K_j with K_j compact in K_{j+1} such that $\bigcup_{j\in\mathbb{N}} K_j = V$. Then, the topology of $H(V)$ is determined by the sequence of semi-norms $\|.\|_{K_j}$ since every compact set in V is contained in some K_j.

Definition 3.2 For a compact set K in $\mathbb{C}$, we denote by $H_C(K)$ the vector space of all continuous functions on K which are holomorphic in the interior of K. This is a Banach space with the norm

$$\|\phi\|_{H_C(K)} := \sup_{z\in K} |\phi(z)|.$$

Remark 3.3 By the definition of the topology of $H(V)$, we have the isomorphisms

$$H(V) = \varprojlim_{K\subset V} H_C(K) = \varprojlim_{j\in\mathbb{N}} H_C(K_j)$$

as locally convex spaces (note that here K runs through all compact subsets of V and $(K_j)_{j\in\mathbb{N}}$ be a sequence of compact sets like in the definition). The restriction mappings $H_C(K_{j+1}) \to H_C(K_j)$ are compact by Montel's theorem (the classical result of complex analysis will work in this settings). Thus $H(V)$ can be expressed as the projective limit of a sequence of Banach spaces with compact linear mappings.

Hence the vector space $H(V)$ of all holomorphic functions on an open set $V \subseteq \mathbb{C}$ is an (FS)-space under the topology of uniform convergence on compact sets.

Definition 3.4 Let M be a topological, N a locally convex space, and let $A \subseteq M$ be an arbitrary subset. We consider all continuous mappings $f : U_f \to N$, where U_f is some open neighbourhood of A in M. Then we put $f \sim_A g$ if there is some open neighbourhood V of A with $f\,|_V = g\,|_V$. This defines an equivalence relation on the set of functions considered. The equivalence class of a function f is called the *germ* of f (along A) and we denote it by $[f]$.

Analogously, we can consider germs of smooth, holomorphic or real analytic mappings, sufficiently nice M provided.

Definition 3.5 For a compact subset $K \subset \mathbb{C}$, we denote by $H(K)$ the vector space of all germs of holomorphic functions on a neighbourhood V of K. In other words, we have

$$H(K) = \varinjlim_{V \supset K} H(V) \tag{1}$$

in the sense of the inductive limit relative to the direct set of all open neighbourhoods V of K. Here, we may take only relative compact open neighbourhoods V, and then $H(V)$ may be replaced by $H_C(\overline{V})$.

Furthermore, if we choose a sequence of compact sets $K_1 \supseteq K_2 \supseteq \ldots \supseteq K$ such that $\bigcap_{j \in \mathbb{N}} K_j = K$, where K_j are closed subsets with non-empty interior, K_j compact in K_{j-1}, and each connected component of the interior of K_j intersects K, then we have the representation

$$H(K) = \varinjlim_{j \in \mathbb{N}} H_C(K_j), \tag{2}$$

where the restriction mappings $H_C(K_j) \to H_C(K_{j+1})$ are compact linear injections. Such a sequence of compact neighbourhoods can be constructed as follows. Assuming that K_j has already been constructed, choose, for each $x \in K$, a closed disc D_x of radius less than $1/j$, with centre at x, and contained in the interior of K_j. Since K is compact, there is a finite number of D_x whose interiors cover K. Then let K_{j+1} be the union of these D_x.

The space $H(K)$ is endowed with the locally convex topology as the inductive limit (1) of locally convex spaces $H(V)$. This topology coincides with the inductive limit locally convex topology defined by (2) because (1) and (2) are equivalent inductive limits.

Remark 3.6 Let $K \subseteq \mathbb{C}$ be a compact set. Then, the space $H(K)$ equipped with the above inductive limit locally convex topology is a (DFS)-space.

A sequence $(\phi_j)_{j \in \mathbb{N}}$ in $H(K)$ converges if and only if the ϕ_j are represented by holomorphic functions defined on a common open set $V \supset K$ and the representatives converge uniformly on V.

A set $B \subseteq H(K)$ is bounded if and only if there is a common neighbourhood $V \supset K$ such that each $\phi \in B$ is represented by a holomorphic function on V which is uniformly bounded on V.

A (linear) mapping f from $H(K)$ into a locally convex space X is continuous if and only if for each convergent sequence $(\phi_j)_{j \in \mathbb{N}} \to \phi$, $f(\phi_j)$ converges to $f(\phi)$.

If $H(K)$ is represented as in (2), the mapping $H_C(K_j) \to H_C(K)$ is injective, so that $\phi \in H(K)$ and its representative holomorphic function in $H_C(K_j)$ are in one to one correspondence.

Definition 3.7 We introduce some standard notations.

$$\begin{aligned}
C(\mathbb{R}^m, \mathbb{F}) &:= \{f : \mathbb{R}^m \to \mathbb{F} : f \text{ is continuous}\} \\
C^1(\mathbb{R}^m, \mathbb{F}) &:= \left\{f : \mathbb{R}^m \to \mathbb{F} : f' \text{ exists and is continuous}\right\} \\
&\ \ \vdots \\
C^\infty(\mathbb{R}^m, \mathbb{F}) &:= \{f : \mathbb{R}^m \to \mathbb{F} : f \text{ is smooth}\}
\end{aligned}$$

Alternatively,

$$C^\infty(\mathbb{R}^m, \mathbb{F}) := \bigcap_{i\in\mathbb{N}} C^i(\mathbb{R}^m, \mathbb{F}).$$

Let $K \subset \mathbb{R}^m$ be a compact subset of $\mathbb{R}^m$.

$$C_K^n(\mathbb{R}^m, \mathbb{F}) := \{f \in C^n(\mathbb{R}^m, \mathbb{F}) : \operatorname{supp}(f) \subseteq K\}$$

Furthermore, we define

$$\mathcal{S} := \left\{ f \in C^\infty(\mathbb{R}^m, \mathbb{F}) : \max_{s\leq k} \max_{x\in\mathbb{R}^m} \left((1 + \|x\|^2)^n |f^{(k)}(x)| \right) < \infty \ \forall n \in \mathbb{N} \right\}.$$

And finally

$$\mathcal{D}(\mathbb{R}^m, \mathbb{F}) := \bigcup_{K\subseteq\mathbb{R}^m} C_K^\infty(\mathbb{R}^m, \mathbb{F})$$

as K runs through all compact subsets of $\mathbb{R}^m$ and

$$\mathcal{D}_K(\mathbb{R}^m, \mathbb{F}) := \{f \in \mathcal{D}(\mathbb{R}^m, \mathbb{F}) : \operatorname{supp}(f) \subseteq K\} = C_K^\infty(\mathbb{R}^{m,\mathbb{F}}).$$

Definition 3.8 Let $U \subseteq \mathbb{C}$ be an open subset. By $H^\infty(U)$ we denote the space of bounded holomorphic functions.

Equipped with the supremum norm, $H^\infty(U)$ is a Banach space.

Definition 3.9 Let $U \subseteq \mathbb{C}$ be open, E a locally convex space and let E' denote its dual. A function $f : U \to E$ is called *weakly holomorphic* if for every $x' \in E'$

$$x' \circ f : U \to \mathbb{C}$$

is holomorphic in the usual sense.

By $H_\omega(U, E)$ we denote the space of weakly holomorphic functions from U into E.

Definition 3.10 A subset U of a vector space E over $\mathbb{C}$ is said to be *finitely open* if $U \cap F$ is open in the Euclidean topology of F for each finite dimensional subspace F of E.

Definition 3.11 Let U be a finitely open subset of a vector space E over $\mathbb{C}$ and F a locally convex space. A function $f : U \to F$ is *Gâteaux holomorphic* if for each $\xi \in U, \nu \in E$ and $\varphi \in F'$ the $\mathbb{C}$ valued function of one complex variable

$$\lambda \to (\varphi \circ f)(\xi + \lambda\nu)$$

is holomorphic on some neighbourhood of 0 in $\mathbb{C}$. We let $H_G(U, F)$ denote the set of all Gâteaux holomorphic mappings from U into F and write $H_G(U)$ in place of $H_G(U, \mathbb{C})$.

Definition 3.12 Let U be an open subset of a Fréchet space E and F a locally convex space. A function $f : U \to F$ is called *holomorphic* if f is continuous and Gâteaux holomorphic on U.

We let $H(U, F)$ denote the set of all holomorphic mappings from U into F and write $H(U)$ in place of $H(U, \mathbb{C})$.

Now let B be a compact subset in a Fréchet space E and F a locally convex space. By the standard notation $H(B, F)$ denotes the space of germs of holomorphic functions on B with values in F equipped with the inductive limit topology.

Recall that $f \in H(B, F)$ if there exists a neighbourhood V of B in E and a holomorphic function $\hat{f} : V \to F$ whose germ on B is f.

Definition 3.13 Let U and V denote open subsets of Banach spaces E and F, respectively, and suppose $f : U \times V \to \mathbb{C}$. The function f is called *separately holomorphic* if for each $x \in U$ the function $f_x : y \in V \to f(x, y)$ is holomorphic and for each $y \in V$ the function $f^y : x \in U \to f(x, y)$ is holomorphic.

Chapter 4

The Grothendieck Köthe Silva Duality

Lemma 4.1 ([Kri02], first Lemma in 4.6) *Let E be a one dimensional locally convex space and $a \in E$, $a \neq 0$, then the mapping $f : \mathbb{F} \to E : t \to at$ is an isomorphism of locally convex spaces.*

Proof. Since $\{a\}$ is a basis of the vector space E, f is bijective and every linear isomorphism $f : \mathbb{F} \to E$ looks like this with $a := f(1)$. Because the scalar multiplication is continuous, so is f. Using that E is separated there exists a semi-norm $\|.\|_q$ with $\|a\|_q \geq 1$. Then

$$|f^{-1}(at)| = |t| = \frac{\|at\|_q}{\|a\|_q} \leq \|at\|_q,$$

hence $|f^{-1}| \leq \|.\|_q$. Therefore f^{-1} is continuous. □

Lemma 4.2 (Continuous linear functionals) ([Jar81], Proposition 2.3.4 or cf. [Kri02], second Lemma in 4.6) *Let E be a locally convex space and $f : E \to \mathbb{F}$ a non-trivial linear functional on E. The following statements are equivalent:*

(i) f *is continuous.*

(ii) $|f|$ *is a continuous semi-norm.*

(iii) $\ker(f)$ *is closed.*

Proof. (i) ⇒ (ii) is clear, since $|.|$ is a continuous norm on $\mathbb{F}$.

(ii) ⇒ (iii) follows from the fact that $\ker(f) = \ker(|f|)$.

(iii) ⇒ (i) Since f is non-trivial, it maps E onto $\mathbb{F}$. $\ker(f)$ is closed by hypothesis, hence $E/\ker(f)$ is a locally convex space, too. Since $f|_{\ker(f)}= 0$, f factorizes over the according quotient map $\pi : E \to E/\ker(f)$ to a linear map $\tilde{f} : E/\ker(f) \to \mathbb{F}$. Because f is onto, so is $\tilde{f}$. Moreover, $\tilde{f}$ is one to one since $0 = \tilde{f}(\pi(x)) = f(x)$, hence $x \in \ker(f)$ and therefore $\pi(x) = 0$. By 4.1 $\tilde{f}$ is an isomorphism of locally convex spaces. Finally $f = \tilde{f} \circ \pi$ is continuous as composition of continuous mappings.

Now let f be non-continuous, i.e. $\ker(f)$ not closed and $a \in \overline{\ker(f)} \setminus \ker(f)$. Without loss of generality, let $f(a) = 1$. The map $\hat{f} : \ker(f) \times \mathbb{F} \to E : (x, t) \to x + at$ is continuous, linear and bijective, since $E \to \ker(f) \times \mathbb{F} : y \to (y - af(y), f(y))$ is clearly the right-inverse to $\hat{f}$. The image of $\hat{f}$ is in $\overline{\ker(f)}$, hence it is all of E. □

Remark 4.3 In the following, we will denote by $\mathbb{C}_\infty$ the complex manifold $\mathbb{C}\cup\{\infty\}$ also known as the Riemannian sphere. By $\overline{\mathbb{R}} := \mathbb{R}\cup\{\infty\} \cong S^1$ we denote the closure of $\mathbb{R}$ in $\mathbb{C}_\infty$. And by $\mathbb{D}$ we denote the unit disk in $\mathbb{F}$.

Lemma 4.4 ([Con90], III.8.2 or [Kri02], 7.18 Sublemma) *If μ is a complex-valued regular Borel measure on a compact subset K of $\mathbb{C}$, then*

$$\widehat{\mu}(w) = \int \frac{d\mu(z)}{z-w}$$

is holomorphic on $\mathbb{C}_\infty\backslash K$, and $\widehat{\mu}(\infty) = 0$.

Proof. To show that $\widehat{\mu}$ is holomorphic on $\mathbb{C}_\infty\backslash K$, let $w, w_0 \in \mathbb{C}\backslash K$ and note that

$$\frac{\widehat{\mu}(w) - \widehat{\mu}(w_0)}{w - w_0} = \int_K \frac{d\mu(z)}{(z-w)(z-w_0)}.$$

As $w \to w_0$, $\frac{1}{(z-w)(z-w_0)} \to \frac{1}{(z-w_0)^2}$ converges uniformly for z in K, so that $\widehat{\mu}$ has a derivative at w_0 and

$$\frac{d\widehat{\mu}}{dw}(w_0) = \int_K \frac{d\widehat{\mu}(z)}{(z-w_0)^2}.$$

So $\widehat{\mu}$ is holomorphic on $\mathbb{C}\backslash K$. To show that it is holomorphic at infinity, note that $\widehat{\mu} \to 0$ as $z \to \infty$, so infinity is a removable singularity. □

Theorem 4.5 ([Con90], IV.4.1 or [Kri02], 7.18) *Let X be completely regular. If $L : C(X) \to \mathbb{C}$ is a continuous linear functional, then there is a compact set K and a regular Borel measure μ on K such that $L(f) = \int_K f d\mu$ for every f in $C(X)$. Conversely, each such measure defines an element of $C(X)'$.*

Proof. By C.6, each measure μ supported on a compact set K defines an element of $C(X)'$. In fact, if $L(f) = \int_K f d\mu$, then

$$|L(f)| \leq \|\mu\| \|f\,|_K\,\|_\infty,$$

and so L is continuous.

Now assume $L \in C(X)'$. There are compact sets $K_1, \ldots, K_n$ and positive numbers $\alpha_1, \ldots, \alpha_n$ such that

$$|L(f)| \leq \sum_{j=1}^{n} \alpha_j \|f\,|_{K_j}\,\|_\infty$$

(see 4.2). Let $K = \bigcup_{j=1}^n K_j$ and $\alpha = \max\{\alpha_j : 1 \leq j \leq n\}$. Then

$$|L(f)| \leq \sum_{j=1}^{n} \alpha \|f\,|_K\,\|_\infty.$$

Hence if $f \in C(X)$ and $f\,|_K = 0$, then $L(f) = 0$.

Define $F : C(K) \to \mathbb{F}$ as follows. If $g \in C(K)$, let $\tilde{g}$ be any continuous extension of g to X and put $F(g) = L(\tilde{g})$. To check that F is well defined, suppose that $\tilde{g}_1$ and

$\tilde{g}_2$ are both extensions of g to X. Then $\tilde{g}_1 - \tilde{g}_2 = 0$ on K, and hence $L(\tilde{g}_1) = L(\tilde{g}_2)$. Thus F is well defined.

Since L is a continuous linear functional and the restriction to K is surjective, $F : C(K) \to \mathbb{F}$ is linear. If $g \in C(K)$ and $\tilde{g}$ is an extension on $C(X)$, then

$$|F(g)| = |L(\tilde{g})| \leq \alpha \|\tilde{g}\,|_K\,\|_\infty = \alpha\|g\|,$$

where the norm is the norm of $C(X)$. By C.6 there is a measure μ on K such that $F(g) = \int_K g d\mu$. If $f \in C(X)$, then $g = f\,|_K \in C(K)$ and so

$$L(f) = F(g) = \int_K f d\mu.$$

Theorem 4.6 (GKS-duality) ([Con90], IV.4.2) $L \in H(\mathbb{D})'$ *if and only if there is an* $r < 1$ *and a unique function* g *holomorphic on* $\mathbb{C}_\infty \backslash \overline{r\mathbb{D}}$ *with* $g(\infty) = 0$ *such that*

$$L(f) = \frac{1}{2\pi\,\mathrm{i}} \int_\gamma fg \tag{1}$$

for every f *in* $H(\mathbb{D})$*, where* $\gamma(t) = \rho e^{\mathrm{i}\,t}, 0 \leq t \leq 2\pi$*, and* $r < \rho < 1$.

Proof. Let g be given and define L as in (1). If $K = \{z \in \mathbb{C} : |z| = \rho\}$, then

$$|L(f)| = \frac{1}{2\pi\,\mathrm{i}} \left| \int_0^{2\pi} f(\rho e^{\mathrm{i}\,t}) g(\rho e^{\mathrm{i}\,t})\,\mathrm{i}\,\rho e^{\mathrm{i}\,t} dt \right| \leq \frac{1}{2\pi\,\mathrm{i}} \|f\,|_K\,\|_\infty \|g\,|_K\,\|_\infty 2\pi\rho.$$

So if $c = \rho\|g\,|_K\,\|_\infty$ then $|L(f)| \leq c\|f\,|_K\,\|_\infty$ and $L \in H(\mathbb{D})'$.

Now assume that $L \in H(\mathbb{D})'$. The theorem of Hahn-Banach C.2 implies there is an F in $C(\mathbb{D})'$ such that $F\,|_{H(\mathbb{D})} = L$. By 4.5 there is a compact set K contained in $\mathbb{D}$ and a measure μ on K such that $L(f) = \int_K f d\mu$ for every f in $H(\mathbb{D})$. Define $g : \mathbb{C}_\infty \backslash K \to \mathbb{C}$ by $g(\infty) := 0$ and

$$g(z) := -\int_K \frac{1}{w - z} d\mu(w)$$

for $z \in \mathbb{C}\backslash K$. By 4.4, g is holomorphic on $\mathbb{C}_\infty \backslash K$. Let $\rho < 1$ such that $K \subseteq \rho\mathbb{D}$. If $\gamma(t) = \rho e^{\mathrm{i}\,t}, 0 \leq t \leq 2\pi$, then Cauchy's integral formula implies

$$f(w) = \frac{1}{2\pi\,\mathrm{i}} \int_\gamma \frac{f(z)}{z - w} dz$$

for $|w| < \rho$; in particular, this is true for $w \in K$. Thus,

$$\begin{aligned} L(f) &= \int_K f(w) d\mu(w) \\ &= \int_K \left(\frac{\rho}{2\pi} \int_0^{2\pi} \frac{f(\rho e^{\mathrm{i}\,t})}{\rho e^{\mathrm{i}\,t} - w} e^{\mathrm{i}\,t} dt \right) d\mu(w) \\ &= \frac{\rho}{2\pi} \int_0^{2\pi} f(\rho e^{\mathrm{i}\,t}) e^{\mathrm{i}\,t} \left(\int_K \frac{1}{\rho e^{\mathrm{i}\,t} - w} d\mu(w) \right) dt \\ &= \frac{1}{2\pi\,\mathrm{i}} \int_\gamma f(z) g(z) dz. \end{aligned}$$

To prove the uniqueness of g, we show that from $\int_\gamma fg = 0$ for all $f \in H(\mathbb{D})$ it follows that $g = 0$. We set $f(z) := \frac{1}{z-a}$. For $a \notin \mathbb{D}$ it is $\frac{1}{z-a} \in \mathbb{D}$. Hence

$$0 = \int_\gamma \frac{g(x)}{z-a} dz = 2\pi \mathrm{i}\, g(a),$$

the latter from Cauchy's integral theorem (see 4.9). □

Definition 4.7 Let $U \subseteq \mathbb{C}$ be open. A 1-*chain* is a formal linear combination $c := \sum_j k_j c_j$ of curves $c_j : [0,1] \to U$ with coefficients $k_j \in \mathbb{Z}$. The set of all 1-chains forms an Abelian group regarding the component-wise addition. The boundary ∂c of a 1-chain is a 0-*chain*, i.e. a formal linear combination of points defined by $\partial c := \sum_j k_j(c_j(1) - c_j(0))$. A 1-chain is called a *cycle* if $\partial c = 0$. That is, if all c_j are closed curves. The set of all cycles is a subset of the 1-chains.

The integral for 1-forms ω on 1-cycles c is defined as

$$\int_c \omega := \sum_j k_j \int_{c_j} \omega,$$

the index is defined respectively as

$$n(c,z) := \sum_j k_j \frac{1}{2\pi \mathrm{i}} \int_{c_j} \frac{d\omega}{\omega - z}$$

for all $z \notin c([0,1]) := \bigcup_j c_j([0,1])$.

Definition 4.8 Let $U \subseteq \mathbb{C}$ be open. A 1-cycle c is called 0-*homologous* in U, if $n(c,z) = 0$ holds for all $z \notin U$.

Two cycles c_1 and c_2 are called *homologous* in U, if $n(c_1,z) = n(c_2,z)$ holds for all $z \notin U$. The 0-homologous cycles form a subgroup of the 0-cycles.

The quotient group $H_1(U,\mathbb{Z})$ is called 1st *homology group* of U with quotients in $\mathbb{Z}$.

Theorem 4.9 (Integral theorem and integral formula of Cauchy) *Let $U \subseteq \mathbb{C}$ be open, $f : U \to F$ holomorphic. For arbitrary cycles c_1 and c_2 homologous in U*

$$\int_{c_1} f(z)dz = \int_{c_2} f(z)dz$$

holds. For a 0-homologous cycle c in U

$$f(z)n(c,z) = \frac{1}{2\pi \mathrm{i}} \int_c \frac{f(\omega)}{\omega - z} d\omega$$

holds for all $z \in U \backslash \mathrm{im}(c)$.

Reference. A proof can be found in [Kri03], Theorem 9.21.

Lemma 4.10 (Jordan-System) *Let $U \subseteq \mathbb{C}$ be open and $K \subseteq U$ compact. Then there exists a 1-cycle $c = \sum_j c_j$ of smooth closed curves c_j in $U \backslash K$ with pairwise* *disjoint images such that* $n(c,z) \in \{0,1\}$ *for all* $z \notin \operatorname{im}(c)$. *We define the inside and the outside of* c *by*

$$\begin{aligned}\operatorname{inn}(c) &:= \{z \notin \operatorname{im}(c) : n(c,z) = 1\} \\ \operatorname{out}(c) &:= \{z \notin \operatorname{im}(c) : n(c,z) = 0\}\end{aligned}$$

respectively.

Then $K \subseteq \operatorname{inn}(c) \subseteq U$ *or equivalently* $\mathbb{C}\backslash U \subseteq \operatorname{out}(c) \subseteq \mathbb{C}\backslash K$. *This is named* Jordan-System*.*

Reference. A proof can be found in [Kri03], Lemma 9.22.

Theorem 4.11 (GKS-duality) ([Kri02], 7.19) *Let* $U \subseteq \mathbb{C}$ *be open. The dual space of the Fréchet space* $H(U)$ *can be identified with* $H_0(\mathbb{C}_\infty\backslash U)$, *the space of the germs of holomorphic functions* f *on* $\mathbb{C}_\infty\backslash U$ *with* $f(\infty) = 0$.

Proof. Let $[g] \in H_0(\mathbb{C}_\infty\backslash U)$, i.e. g holomorphic on a neighbourhood W of $\mathbb{C}_\infty\backslash U$. Without loss of generality, let the boundary of W be parameterisable by finitely many C^1-curves c_k. Resulting,

$$\mu_g(f) := \sum_k \frac{1}{2\pi\,\mathrm{i}} \int_{c_k} \frac{f(z)}{z-w} dz$$

defines a linear functional on $C(U) \supseteq H(U)$. This definition depends only on the germ $[g]$ of g following from Cauchy's integral theorem 4.9.

Conversely let $\mu \in H(U)'$ and according to C.2 $\mu \in C(U,\mathbb{C})'$. From $C(K,\mathbb{C}) \subseteq C(U,\mathbb{C})$ for a compact subset $K \subseteq U$ we get $C(U,\mathbb{C})' \subseteq C(K,\mathbb{C})'$. Thus the support of μ is a compact subset $K \subseteq U$, i.e. $\mu \in C(K,\mathbb{C})'$. Due to 4.4 the map $\widehat{\mu} : \mathbb{C}_\infty\backslash K \to \mathbb{C}$ is holomorphic and according to Cauchy's integral formula 4.9

$$\begin{aligned}\mu(f) &= -\sum_{k=1}^{n} \frac{1}{2\pi\,\mathrm{i}} \mu\left(z \to \int_{c_k} \frac{f(\omega)}{\omega - z} d\omega\right) \\ &= -\sum_{k=1}^{n} \frac{1}{2\pi\,\mathrm{i}} \mu\left(\sum_i \frac{f(c(t_i))}{c(t_i) - z} c'(t_i)|I_i|\right) \\ &= -\sum_{k=1}^{n} \frac{1}{2\pi\,\mathrm{i}} \sum_i f(c(t_i))\mu\left(\frac{1}{c(t_i) - z}\right) c'(t_i)|I_i| \\ &= -\sum_{k=1}^{n} \frac{1}{2\pi\,\mathrm{i}} \int_{c_k} f(\omega)\mu\left(z \to \frac{1}{\omega - z}\right) d\omega \\ &= -\sum_{k} \frac{1}{2\pi\,\mathrm{i}} \int_{c_k} f(\omega)\widehat{\mu}(\omega) d\omega\end{aligned}$$

with $f \in H(U)$ holds. Therefore μ is given by an "inner product" with $\widehat{\mu} \in H_0(C_\infty\backslash K)$. □

Chapter 5

Nuclear Spaces

Definition 5.1 For two arbitrary normed spaces E and F, $\mathcal{A}_n(E,F)$ for $n \in \mathbb{N}$ denotes the collection of all mappings $A \in L(E,F)$ whose range is at most n-dimensional.

For an arbitrary mapping $T \in L(E,F)$ we designate

$$\alpha_n(T) := \inf \{\|T - A\| : A \in \mathcal{A}_n(E,F)\}$$

as the *n-th approximation number* of T. Clearly, we always have

$$\|T\| = \alpha_0(T) \geq \alpha_1(T) \geq \ldots \geq 0.$$

Theorem 5.2 ([Pie72], 8.1.2 Proposition 5) *Let E, F and G be normed spaces. For two mappings $T \in L(E,F)$ and $S \in L(F,G)$ we have*

$$\alpha_{n+m}(ST) \leq \alpha_n(S)\alpha_m(T).$$

Proof. For an arbitrary positive number σ we determine mappings $B \in \mathcal{A}_m(E,F)$ and $A \in \mathcal{A}_n(F,G)$ with

$$\|T - B\| \leq \alpha_m(T) + \sigma$$

and

$$\|S - A\| \leq \alpha_n(S) + \sigma.$$

Then since $A(T - B) + SB \in \mathcal{A}_{n+m}(E,G)$, we have the estimate

$$\begin{aligned}
\alpha_{n+m}(ST) &\leq \|ST - A(T-B) - SB\| \\
&= \|(S-A)(T-B)\| \\
&\leq \|S-A\|\|T-B\| \\
&\leq (\alpha_n(S) + \sigma)(\alpha_m(T) + \sigma),
\end{aligned}$$

from which we get the required inequality by taking the limit as $\sigma \to 0$. □

Definition 5.3 Let E and F be normed spaces and p a positive number. We consider the collection $l^p(E,F)$ of all mappings $T \in L(E,F)$ for which

$$\sum_{n \in \mathbb{N}} \alpha_n(T)^p < \infty.$$

Clearly, $l^p(E,F)$ is a linear space (cf. [Pie72], 8.2.2).

Definition 5.4 By

$$\varrho_p(T) := \left(\sum_n \alpha_n(T)^p\right)^{1/p}$$

we define on $l^p(E,F)$ a real valued function with the following properties

(i) $\varrho_p(T) \geq 0$,

(ii) from $\varrho_p(T) = 0$ it follows that $T = 0$,

(iii) for all numbers λ we have $\varrho_p(\lambda T) = |\lambda|\varrho_p(T)$, and

(iv) for some number $\sigma_p \geq 1$ we have the inequality

$$\varrho_p(S+T) \leq \sigma_p\left(\varrho_p(S) + \varrho_p(T)\right)$$

for $S, T \in l^p(E,F)$.

On the basis of the properties stated above, $\varrho_p(T)$ will be designated as a quasi-norm. We obtain a metric topology on $l^p(E,F)$ by using the sets

$$U_\varepsilon(T) := \{S \in l^p(E,F) : \varrho_p(S-T) \leq \varepsilon\}$$

with $\varepsilon > 0$ as a fundamental system of neighbourhoods of the mapping T.

Theorem 5.5 ([Pie72], 8.2.7) *For E, F and G three normed spaces, $T \in l^p(E,F)$, and $S \in l^q(F,G)$ it follows that $ST \in l^s(E,G)$ with*

$$\frac{1}{s} = \frac{1}{p} + \frac{1}{q}.$$

Proof. By applying the generalised Hölder inequality (cf. e.g. [Kri02], 2.3)

$$\left(\sum_{n\in\mathbb{N}} |\xi_n\eta_n|^s\right)^{1/s} \leq \left(\sum_{n\in\mathbb{N}} |\xi_n|^p\right)^{1/p} \left(\sum_{n\in\mathbb{N}} |\eta_n|^q\right)^{1/q}$$

and 5.2 we get the estimate

$$\begin{aligned}
\varrho_s(ST) &= \left(\sum_{n\in\mathbb{N}} \alpha_n(ST)^s\right)^{1/s} \\
&\leq \left(2\sum_{n\in\mathbb{N}} \alpha_{2n}(ST)^s\right)^{1/s} \\
&\leq \left(2\sum_{n\in\mathbb{N}} (\alpha_n(S)\alpha_n(T))^s\right)^{1/s} \\
&\leq 2^{1/s}\left(\sum_{n\in\mathbb{N}} \alpha_n(S)^q\right)^{1/q} \left(\sum_{n\in\mathbb{N}} \alpha_n(T)^p\right)^{1/p} \\
&= 2^{1/s}\varrho_q(S)\varrho_p(T).
\end{aligned}$$

Therefore the product ST belongs to $l^s(E,G)$. □

Definition 5.6 Let E be a locally convex space and E' its topological dual. Further let $A \subseteq E$ and $B \subseteq E'$ be two subsets. Then by

$$A^\circ := \{x' \in E' : |x'(x)| \le 1, \forall x \in A\}$$

respectively

$$B^\circ := \{x \in E : |x'(x)| \le 1, \forall x' \in B\}$$

we define the *polar* of A respectively B.

By $A^{\circ\circ} := (A^\circ)^\circ$ we denote the *bi-polar* of A. Obviously, we have $A \subseteq A^{\circ\circ}$.

Lemma 5.7 ([Pie72], 8.4.1 Lemma 1) *Let E be a normed space and U its closed unit ball. For each n-dimensional linear subspace F of E there are elements $x_1, \dots, x_n \in F$ and linear forms $a_1, \dots, a_n \in E'$ with $\|x_i\|_U = 1$, $\|a_k\|_{U^\circ} = 1$ and $a_k(x_i) = \delta_{i,k}$, where $\delta_{i,k}$ denotes the Kronecker delta. Then we have*

$$x = \sum_{i=1}^{n} a_i(x) x_i$$

for all $x \in F$.

Proof. We consider an arbitrary system of linearly independent elements $y_1, \dots, y_n$ in F and set

$$\delta(b_1, \dots, b_n) := |\det(b_k(y_i))|$$

for $b_1, \dots, b_n \in U^\circ$. Then δ is a continuous function on the compact n-fold topological product of the weakly compact unit ball U° of E'. Consequently, there must exist elements $a_1, \dots, a_n \in U^\circ$ for which $\delta(a_1, \dots, a_n)$ takes the maximum δ_0, which certainly must be greater than 0, since expanding $y_1, \dots, y_n$ to a basis and getting the dual basis $b_1, \dots, b_n$ yields $\delta(b_1, \dots, b_n) = 1$.

If the elements $x_1, \dots, x_n \in F$ are the uniquely determined elements of the solution set of the system of equations

$$\sum_{j=1}^{n} a_j(y_i) x_j = y_i$$

for $i = 1, \dots, n$, we then have

$$a_k(x_j) = \delta_{j,k}.$$

Since

$$\sum_{j=1}^{n} a_j(y_i) b_k(x_j) = b_k(y_i)$$

for $b_1, \dots, b_n \in U^\circ$, we get

$$\delta(a_1, \dots, a_n) \, |\det(b_k(x_j))| = \delta(b_1, \dots, b_n)$$

from the multiplication of determinants. Therefore, the inequality

$$|\det(b_k(x_j))| \le 1$$

holds for $b_1, \ldots, b_n \in U^\circ$. If for fixed i we set $b_k = a_k$ with $k \neq i$ and $b_i = b \in U^\circ$, we then obtain

$$|b(x_i)| \leq 1$$

for $b \in U^\circ$ or $\|x_i\|_U \leq 1$. Since $\|a_i\|_{U^\circ} \leq 1$ holds by hypothesis, it follows from

$$1 = a_i(x_i) \leq \|x_i\|_U \|a_i\|_{U^\circ},$$

that $\|x_i\|_U = 1$ and $\|a_i\|_{U^\circ} = 1$.

Since the elements $x_1, \ldots, x_n$ form a linearly independent basis in F we can write each element $x \in F$ uniquely as a linear combination

$$x = \sum_{i=1}^{n} \xi_i x_i.$$

Here we have

$$a_k(x) = \sum_{i=1}^{n} \xi_i a_k(x_i) = \xi_k$$

and hence the desired result. □

Lemma 5.8 ([Pie72], 8.4.1 Lemma 2) *Let E and F be two normed spaces and U, V the closed unit balls in E respectively F. Each mapping $T \in \mathcal{A}_n(E, F)$ can be represented in the form*

$$Tx = \sum_{i=1}^{n} \lambda_i a_i(x) y_i$$

with linear forms $a_i \in U^\circ$ and elements $y_i \in V$ so that the inequality

$$|\lambda_i| \leq \|T\|$$

holds for the numbers λ_i, $1 \leq i \leq n$.

Proof. For the n-dimensional range of T we determine elements $y_i \in \text{range}(T)$ and linear forms $b_i \in F'$ with the properties presented in 5.7. The identity

$$Tx = \sum_{i=1}^{n} T' b_i(x) y_i$$

then holds for $x \in E$ and we have

$$\lambda_i = \|T' b_i\|_{U^\circ} > 0.$$

Finally, if we set

$$a_i = \frac{T' b_i}{\lambda_i},$$

we get the representation

$$Tx = \sum_{i=1}^{n} \lambda_i a_i(x) y_i$$

for the mapping T. Moreover, we have $a_i \in U^\circ$, $y_i \in V$ and $|\lambda_i| \leq \|T\|$. □

Theorem 5.9 ([Pie72], 8.4.2) *Let E, F be normed spaces with closed unit balls U and V. Each mapping $T \in l^p(E,F)$ with $0 < p \leq 1$ can be represented as*

$$Tx = \sum_{n\in\mathbb{N}} \lambda_n a_n(x) y_n$$

with linear forms $a_n \in U^\circ$ and elements $y_n \in V$, such that the inequality

$$\left(\sum_{n\in\mathbb{N}} |\lambda_n|^p\right)^{1/p} \leq 2^{2+3/p} \varrho_p(T)$$

holds for the numbers λ_n, $n \in \mathbb{N}$.

Proof. For $n \in \mathbb{N}$ we determine the mappings $A_n \in \mathcal{A}_{2^n-2}(E,F)$ with

$$\|T - A_n\| \leq 2\alpha_{2^n-2}(T)$$

and set

$$B_n := A_{n+1} - A_n.$$

Then the statements

$$d_n = \dim(\text{range}(B_n)) \leq 2^{n+2}$$

and

$$\|B_n\| \leq \|T - A_n\| + \|T - A_{n+1}\| \leq 4\alpha_{2^n-2}(T)$$

are valid. Consequently, we have

$$d_n\|B_n\|^p \leq 2^{2p+n+2}\alpha_{2^n-2}(T)^p.$$

Since the sequence $(\alpha_m(T))_m$ decreases monotonically, the inequality

$$\sum_{n\in\mathbb{N}} 2^{n-1}\alpha_{2^n-2}(T)^p \leq \sum_{n\in\mathbb{N}} \sum_{m=2^{n-1}-1}^{2^n-2} \alpha_m(T)^p = \sum_{m\in\mathbb{N}} \alpha_m(T)^p = \varrho_p(T)^p$$

holds. Therefore, the estimate

$$\sum_{n\in\mathbb{N}} d_n\|B_n\|^p \leq 2^{2p+3}\varrho_p(T)^p$$

is valid.

Using 5.8 we write the mappings B_n in the form

$$B_n x = \sum_{i=1}^{d_n} \lambda_i^n a_i^n(x) y_i^n$$

where $a_i^n \in U^\circ$, $y_i^n \in V$ and $|\lambda_i^n| \leq \|B_n\|$ for any given n. Consequently, we have

$$\sum_{n\in\mathbb{N}} \sum_{i=1}^{d_n} |\lambda_i^n|^p \leq \sum_{n\in\mathbb{N}} d_n\|B_n\|^p \leq 2^{2p+3}\varrho_p(T)^p.$$

Our assertion is thus proved because for all $x \in E$ the identity

$$Tx = \lim_{m\to\infty} A_{m+1}x = \sum_{n\in\mathbb{N}} B_n x = \sum_{n\in\mathbb{N}} \sum_{i=1}^{d_n} \lambda_i^n a_i^n(x) y_i^n$$

is true. □

Definition 5.10 Let E and F be two arbitrary normed spaces with closed unit balls U and V. A mapping $T \in L(E,F)$ is called *nuclear* if there are continuous linear forms $a_n \in E'$ and elements $y_n \in F$ with

$$\sum_{n\in\mathbb{N}} \|a_n\|_{U^\circ}\|y_n\|_V < \infty$$

such that T has the form

$$Tx = \sum_{n\in\mathbb{N}} a_n(x)y_n$$

for $x \in E$.

For each nuclear mapping T we set

$$\nu(T) := \inf\left\{\sum_{n\in\mathbb{N}} \|a_n\|_{U^\circ}\|y_n\|_V\right\},$$

where the infimum is taken over all possible representations of T.

Lemma 5.11 ([Pie72], 8.4.3) *Let E and F be two normed spaces. Every mapping $T \in \ell^1(E,F)$ is nuclear and we have $\nu(T) \leq 2^5\varrho_1(T)$.*

Proof. This is an immediate consequence of 5.9 and definition 5.10. □

Definition 5.12 A locally convex space E is called *nuclear* if for any convex balanced neighbourhood V of 0 there exists another convex balanced neighbourhood $U \subseteq V$ of 0 such that the canonical mapping from E_U onto E_V is nuclear.

We designate as *dual nuclear* all locally convex spaces whose strong topological dual is nuclear.

Definition 5.13 A sequence of numbers $(\lambda_i)_{i\in\mathbb{N}}$ is called *rapidly decreasing* if the sequences of numbers $\big((i+1)^k\lambda_i\big)_{i\in\mathbb{N}}$ is bounded for all $k \in \mathbb{N}$.

By a small computation, we can conclude that if $(\lambda_i)_{i\in\mathbb{N}}$ is a rapidly decreasing sequence of numbers the inequality

$$\sum_{i\in\mathbb{N}} (i+1)^k|\lambda_i|^p < \infty$$

holds for all $k \in \mathbb{N}$ and $p > 0$.

The space of the rapidly decreasing sequences is denoted by s (see 8.4).

Definition 5.14 For two normed spaces E and F we consider the collection $s(E,F)$ of all mappings $T \in L(E,F)$ for which the inequality

$$\sum_{n\in\mathbb{N}} \alpha_n(T)^p < \infty$$

holds for every positive number p, and we say that these mappings are of *type s*.

We have

$$s(E,F) = \bigcap_{p>0} l^p(E,F)$$

and therefore $s(E,F)$ is a linear space.

Remark 5.15 Let E and F be two normed spaces. A mapping $T \in L(E,F)$ is of type s if and only if the sequence of its approximation numbers is rapidly decreasing.

Theorem 5.16 ([Pie72], 8.5.6) *Let E and F be two normed spaces with closed unit balls U and V. A mapping $T \in L(E,F)$ is of type s if and only if it can be represented in the form*

$$Tx = \sum_{n\in\mathbb{N}} \lambda_n a_n(x) y_n$$

with linear forms $a_n \in U^\circ$, elements $y_n \in V$ and a rapidly decreasing sequence of numbers $(\lambda_n)_{n\in\mathbb{N}}$.

Proof. First we prove necessity. If T is a mapping in $s(E,F)$, we determine the mappings $A_n \in \mathcal{A}_n(E,F)$ with

$$\|T - A_n\| \leq 2\alpha_n(T)$$

for $n \in \mathbb{N}$. Then the statements

$$d_n := \dim(\text{range}(B_n)) \leq 2n + 1$$

and

$$\|B_n\| \leq 4\alpha_n(T)$$

are valid for the mappings

$$B_n := A_{n+1} - A_n,$$

and we have

$$\sum_{n\in\mathbb{N}} d_n \|B_n\| \leq 2\ 4^p \sum_{n\in\mathbb{N}} (n+1)\alpha_n(T)^p < \infty$$

for all $p > 0$. Using 5.8, we put the mapping B_n in the form

$$B_n x = \sum_{i=1}^{d_n} \lambda_i^n a_i^n(x) y_i^n$$

where $a_i^n \in U^\circ$, $y_i^n \in V$ and $|\lambda_i^n| \leq \|B_n\|$. Consequently, we have

$$\sum_{n\in\mathbb{N}} \sum_{i=1}^{d_n} |\lambda_i^n|^p \leq \sum_{n\in\mathbb{N}} d_n \|B_n\|^p < \infty$$

for all $p > 0$, and for all $x \in E$ the identity

$$Tx = \lim_{m\to\infty} A_{m+1}x = \sum_{n\in\mathbb{N}} B_n x = \sum_{n\in\mathbb{N}} \sum_{i=1}^{d_n} \lambda_i^n a_i^n(x) y_i^n$$

holds. We have thus shown that the mapping T can be represented as

$$Tx = \sum_{m\in\mathbb{N}} \lambda_m a_m(x) y_m$$

with linear forms $a_m \in U^\circ$ and elements $y_m \in V$ such that for each integer m the inequality

$$\sum_{m\in\mathbb{N}} |\lambda_m|^p < \infty$$

holds. Since we can always reorder $\mathbb{N}$ such that $|\lambda_0| \geq |\lambda_1| \geq \ldots \geq 0$, the sequence $(\lambda_m)_{m\in\mathbb{N}}$ is rapidly decreasing.

Now to sufficiency. We consider a mapping $T \in L(E, F)$ which can be represented in the given way. Since the mapping A with

$$Ax = \sum_{m=0}^{n-1} \lambda_m a_m(x) y_m$$

for $x \in E$ belongs to $\mathcal{A}_n(E, F)$, we have

$$\alpha_n(T) \leq \|T - A\| \leq \sum_{m=n}^{\infty} |\lambda_m|.$$

Consequently, for all p between 0 and 1, the inequality

$$\alpha_n(T)^p \leq \left(\sum_{m=n}^{\infty} |\lambda_m| \right)^p \leq \sum_{m=n}^{\infty} |\lambda_m|^p$$

is valid, and we get the estimate

$$\sum_{n\in\mathbb{N}} \alpha_n(T)^p \leq \sum_{n\in\mathbb{N}} \sum_{m=n}^{\infty} |\lambda_m|^p = \sum_{m\in\mathbb{N}} (m+1)|\lambda_m|^p < \infty.$$

Therefore T is of type s. □

Theorem 5.17 ([Pie72], 8.6.1) *A locally convex space E is nuclear if and only if for some, hence each, positive number p the following statement is valid.*

For each neighbourhood U of 0 in E there is a neighbourhood V of 0 in E with $V \subseteq U$ such that the canonical mapping from E_V onto E_U is of type l^p.

Proof. To prove necessity, we determine for an arbitrary neighbourhood $U = U_0$ of 0 neighbourhoods $U_1, \ldots, U_{4n}$ of 0 with $U_{4n} \subseteq \ldots \subseteq U_1 \subseteq U_0$ such that the canonical mappings $E_{U_k} \to E_{U_{k-1}}$ are absolutely summing, and set $V = U_{4n}$. Then the mapping $E_V \to E_U$ equals

$$E_{U_{4n}} \to E_{U_{4n-1}} \to \ldots \to E_{U_1} \to E_{U_0}$$

and is of type $l^{1/n}$ by 5.5, because we can combine each pair of consecutive mappings in the sequence to obtain a mapping of type l^2.

If p is an arbitrary positive number, we choose n greater than $1/p$. Then the canonical mapping $E_V \to E_U$ is of type l^p.

We now prove sufficiency. If the stated assertion is satisfied for some positive number p, we determine for a neighbourhood $U = U_0$ of 0 neighbourhoods $U_1, \ldots, U_{4n}$ of 0 with $U_{4n} \subseteq \ldots \subseteq U_1 \subseteq U_0$ such that the canonical mappings $E_{U_k} \to E_{U_{k-1}}$ are of type l^p. Here, the natural number n is assumed to be greater than p. We now set $V = U_n$. Then the mapping $E_V \to E_U$ equals

$$E_{U_n} \to E_{U_{n-1}} \to \ldots \to E_{U_1} \to E_{U_0}$$

and is of type $l^{p/n}$ by 5.5. But since $p/n < 1$, $E_V \to E_U$ must be nuclear by 5.11. □

Theorem 5.18 ([Pie72], 8.6.6) *For a nuclear or dual nuclear locally convex space E, all canonical mappings $E_A \to E_U$ with A bounded in E and U a neighbourhood of* 0 *in E are of type s.*

Proof. If V is an arbitrary neighbourhood of 0 in E with $V \subseteq U$ we have

$$E_A \to E_V \to E_U.$$

On the basis of 5.17 we can, in the case of a nuclear space, choose V in such a way that $E_V \to E_U$ is of type l^p for an arbitrary positive number p. Consequently (see 5.14), the canonical mapping $E_A \to E_U$ must be of type s.

The proof of our assertion for dual nuclear locally convex spaces proceeds in the same way if we write the canonical mapping $E_A \to E_U$ in the form

$$E_A \to E_B \to E_U$$

with a bounded set B in E. □

Chapter 6

Real Analytic Functions

Definition 6.1 A function f, with domain $U \subseteq \mathbb{R}$ open and range $\mathbb{F}$, is called *real analytic* at $\alpha \in U$ if the function f may be represented by a convergent power series

$$f(x) = \sum_{j=0}^{\infty} a_j (x - \alpha)^j$$

on some interval of positive radius centred at α. The function is said to be *real analytic* on $V \subseteq U$ if it is real analytic at each $\alpha \in V$.

The linear space of all real analytic functions $f : \mathbb{R} \to \mathbb{C}$ will be denoted by $C^{\omega}(\mathbb{R})$.

Now we want to provide a topology on $C^{\omega}(\mathbb{R})$. But there is no elementary topology at hand which results in a complete linear Fréchet space. The next obvious thing to do is to consider inductive and projective limit topologies.

Definition 6.2 Let A be an arbitrary subset of a complex analytic manifold V.

By $H_{P,A}(V)$ we denote $\varprojlim_{K \subseteq A} \varinjlim_{U \supseteq K} H(U)$ as U runs through all open neighbourhoods of K and K runs through all compact subsets of A in V.

And by $H_{I,A}(V)$ we denote $\varinjlim_{U \supseteq A} H(U)$, as U runs through all open neighbourhoods of A in V.

Lemma 6.3 ([Mar63], Proposition 1.1) *Let V be a complex analytic manifold. Every element of $H(V)'$ can be represented by a measure with compact support, i.e. there exists a measure μ with compact support such that for all $\phi \in H(V)$ we have*

$$\langle T, \phi \rangle = \int_V \phi \, d\mu.$$

Proof. This is an application of the theorem of Hahn-Banach C.2. More precisely, the map $\mu \to T(\mu)$ is a topological vector homomorphism from $C(V)'$, equipped with its strong dual topology, to $H(V)'$. One easily can verify that $C(V)'$ is an (LF)-space. As we know, $H(V)'$ is itself an (LF)-space and hence the application $\mu \to T(\mu)$ which is surjective by C.3, is a homomorphism. □

Lemma 6.4 ([Mar63], Proposition 1.2) *If A is a compact or open subset of a complex analytic manifold V, then we have $H_{I,A}(V)' = H_{P,A}(V)'$.*

Proof. The injection from $H_{I,A}(V)$ to $H_{P,A}(V)$ being continuous, it follows immediately that $H_{P,A}(V)' \subseteq H_{I,A}(V)'$.

If A is compact we have by definition $H_{P,A}(V) = H_{I,A}(V)$. If A is an open subset of V, by definition we have $H_{I,A}(V) = H(A)$. If $T \in H(A)'$, by 6.3 there exists a measure with compact support K included in A and μ_T such that for all $\phi \in H(A)$ we have

$$\langle T, \phi \rangle = \int_A \phi d\mu_T.$$

Hence T is continuous on $H(A)$ through the inductive topology on $H_K(V)$, in consequence belonging to $H_{P,A}(V)'$. □

Since we now know that the two possible approaches to equip the space of real analytic functions with a topology coincide, we are free to choose.

Remark 6.5 The topology we will use is given by the following equality

$$C^\omega(\mathbb{R}) := \varprojlim_{K \subseteq \mathbb{R}} \varinjlim_{U \supseteq K} H(U),$$

as K runs through all compact subsets of $\mathbb{R}$ and U runs over all open neighbourhoods of K in $\mathbb{C}_\infty$. We can consider only sequences of K's and U's and we usually put

$$C^\omega(\mathbb{R}) := \varprojlim_{N \in \mathbb{N}} H([-N, N]) = \varprojlim_{N \in \mathbb{N}} \varinjlim_{n \in \mathbb{N}} H(U_{N,n}),$$

where $U_{N,n} := [-N, N] + \frac{1}{n}\mathbb{D}$.

The inductive topology on $C^\omega(\mathbb{R})$ will be denoted by

$$C_i^\omega(\mathbb{R}) := \varinjlim_{U \supseteq \mathbb{R}} H(U),$$

as U runs through all open neighbourhoods of $\mathbb{R}$ in $\mathbb{C}$.

Theorem 6.6 ([BD98], Proposition 4 (1)) *A subset $B \subseteq C^\omega(\mathbb{R})$ is bounded if and only if for every compact interval I in $\mathbb{R}$ there is an $n \in \mathbb{N}$ such that*

$$C := \sup \left\{ \left| \frac{\phi^{(i)}(x)}{i! \; n^i} \right| : \phi \in B, x \in I, i \in \mathbb{N} \right\} < \infty.$$

Proof. If B is bounded, for every compact interval $I \subseteq \mathbb{R}$ the set B is bounded in $H(I) = \varinjlim_{n \in \mathbb{N}} H^\infty\left(I + \frac{1}{n}\mathbb{D}\right)$. Since this inductive limit is regular, there are $n \in \mathbb{N}$ and $M > 0$ such that each $\phi \in B$ can be extended to a

$$\tilde{\phi} \in H^\infty(I + \frac{1}{n}\mathbb{D})$$

with

$$\sup_{z \in I + \frac{1}{n}\mathbb{D}} |\tilde{\phi}(z)| \leq M.$$

By the Cauchy inequalities, for $x \in I$ and $i \in \mathbb{N}$, we have

$$\left\| \frac{\phi^{(i)}(x)}{i! \; (2n)^i} \right\| \leq \max_{|z| = \frac{1}{2n}} |\tilde{\phi}(x + z)| \leq M.$$

Conversely, we fix a compact interval I in $\mathbb{R}$ and select n satisfying the prerequisite. Define

$$\tilde{\phi}(z) := \sum_{i=0}^{\infty} \frac{1}{i!}\phi^{(i)}(x)(z-x)^i$$

for $x \in I$, $z \in I + \frac{1}{2n} \subseteq \mathbb{C}$. Since ϕ is real analytic,

$$\tilde{\phi} \in H^{\infty}(I + \frac{1}{2n}\mathbb{D})$$

is well defined and extends ϕ with norm less or equal to C. Thus B is bounded in $C^{\omega}(\mathbb{R})$. □

Theorem 6.7 ([Mar66], Proposition 1.2 and Proposition 1.7) *The strong dual space $C^{\omega}(\mathbb{R})'_{\beta}$ coincides topologically with $\varinjlim_{N\in\mathbb{N}} H([-N,N])'_{\beta}$ and it is a complete nuclear (LF)-space. Both $C^{\omega}(\mathbb{R})$ and its strong dual $C^{\omega}(\mathbb{R})'_{\beta}$ are reflexive.*

Proof. Since $C^{\omega}(\mathbb{R})$ is a complete Schwartz space, $C^{\omega}(\mathbb{R})'_{\beta}$ is ultrabornological by C.8.

The topology of the dual space $C^{\omega}(\mathbb{R})'_{\beta}$ is weaker than the inductive limit topology of $\varinjlim_{N\in\mathbb{N}} H([-N,N])'_{\beta}$; i.e. the map from $\varinjlim_{N\in\mathbb{N}} H([-N,N])'_{\beta}$ to $C^{\omega}(\mathbb{R})'_{\beta}$ is continuous and hence has a closed graph. By Runge's theorem (cf. [Kri02], 7.18) the projective limit $\varprojlim_{N\in\mathbb{N}} H([-N,N])$ is a reduced projective system and we can apply 2.17 to obtain that $\varinjlim_{N\in\mathbb{N}} H([-N,N])'_{\beta} \cong C^{\omega}(\mathbb{R})'_{\beta}$ is a bijection. Therefore the inverse exists and has a closed graph. Since $C^{\omega}(\mathbb{R})'_{\beta}$ is ultrabornological, by Grothendieck's closed graph theorem the inverse is continuous too. Hence the spaces must coincide.

Since $C^{\omega}(\mathbb{R})$ is ultrabornological, $C^{\omega}(\mathbb{R})'_{\beta} = L(C^{\omega}(\mathbb{R}),\mathbb{R})$ and hence the strong dual is complete.

The nuclearity of $H(\mathbb{C}_{\infty}\backslash[-N,N])$ implies the reflexivity and nuclearity of both $C^{\omega}(\mathbb{R})$ and $C^{\omega}(\mathbb{R})'_{\beta}$ via 4.11. □

Theorem 6.8 ([BD98], Proposition 4 (2)) *Let δ_x be the* Dirac functional *associated to x which is given by $\delta_x(\varphi) = \varphi(x)$. The linear span H of the set $\{\delta_x : x \in \mathbb{R}\}$ is a sequentially dense subset of $C^{\omega}(\mathbb{R})'_{\beta}$.*

Proof. Fix $u \in C^{\omega}(\mathbb{R})'_{\beta}$. By 6.7 there is $N \in \mathbb{N}$ with $u \in H([-N,N])'_{\beta}$. The linear span H_N of $\{\delta_x : x \in [-N,N]\}$ is dense in $H([-N,N])'_{\beta}$, because it is clearly weak-star dense and $H([-N,N])$ is a reflexive nuclear (DF)-space. Since $H([-N,N])'_{\beta}$ is a Fréchet space, we can find a sequence $(u_j)_{j\in\mathbb{N}}$ in H_N, hence in H, such that the sequence converges to $u \in H([-N,N])'_{\beta}$. Therefore $(u_j)_{j\in\mathbb{N}}$ converges to $u \in C^{\omega}(\mathbb{R})'_{\beta}$. □

Theorem 6.9 ([BD98], Proposition 3) *The space $C^{\omega}(\mathbb{R})$ is isomorphic to a projective limit of a sequence of spaces isomorphic to $H(\overline{\mathbb{D}})$ and $C^{\omega}(\mathbb{R})'_{\beta}$ is isomorphic to an inductive limit of a sequence of spaces isomorphic to $H(\mathbb{D})$.*

Proof. First consider the composition operator with the map

$$\varphi : \mathbb{D} \to \mathbb{C}_{\infty}\backslash[-N,N] : \varphi(z) := \frac{N}{2}\left(z + \frac{1}{z}\right), \tag{1}$$

which yields the following isomorphisms

$$H(\mathbb{C}_\infty \backslash [-N,N]) \simeq H(\mathbb{D}) \simeq H(\mathbb{C}_\infty \backslash \overline{\mathbb{D}}).$$

From the GKS-duality 4.11 we have $H([-N,N])'_\beta \simeq H(\mathbb{C}_\infty \backslash [-N,N])$ which, by (1), is isomorphic to $H(\mathbb{C}_\infty \backslash \overline{\mathbb{D}})$. Applying the GKS-duality again gives

$$H(\mathbb{C}_\infty \backslash \overline{\mathbb{D}}) \cong H(\overline{\mathbb{D}})'_\beta.$$

Utilising the reflexivity of $H(\overline{\mathbb{D}})'_\beta$ to use the bi-polar theorem, we get

$$H([-N,N]) \cong H(\overline{\mathbb{D}}).$$

The conclusion for $C^\omega(\mathbb{R})'_\beta$ follows from the GKS-duality 4.11 to get

$$H([-N,N])'_\beta \simeq H(\mathbb{C}_\infty \backslash [-N,N])$$

and 6.7. □

Lemma 6.10 *Let $I \subseteq \mathbb{R}$ be an open interval and $\mu \in C^\omega(\mathbb{R})'_\beta$. If $I \neq \mathbb{R}$ suppose* $\operatorname{supp}(\mu) = \{0\}$. *Assume that for some $r : \mathbb{C} \to \mathbb{R}^+$ with $\frac{r(x)}{x} \to 0$ for $|x| \to \infty$ μ satisfies*

(*)

$$|\Im(z)| \leq r(|z|)\ \forall z \in \mathbb{C} \textit{ with } \hat{\mu}(z) = 0,$$

where $\hat{\mu}$ denotes the Fourier transform of μ.

(E) For all $x \in \mathbb{R}$ exists $t \in \mathbb{C}$ such that

$$|x - t| \leq r(x) \textit{ and } |\hat{\mu}(t)| \geq e^{-r(t)}$$

(3.10) For every $x \in \mathrm{i}\,\mathbb{R}_\pm$ exists $t \in \mathbb{C}$ such that

$$|x - t| \leq r(x) \textit{ and } |\hat{\mu}(t)| \geq e^{H_G(t) - r(t)}$$

where $G := \langle \operatorname{supp}(\mu) \rangle$ denotes the convex hull of $\operatorname{supp}(\mu)$ *and* $H_G(z) := \sup\{\Im(\xi z) : \xi \in G\}$.

Then $T_\mu : C^\omega(I - G) \to C^\omega(I)$ has a continuous right inverse, where T_μ is the convolution operator given by $T_\mu(f)(x) := \langle y, f(x-y) \rangle$.

Reference. The proof can be found in [Lan94], 3.2 Lemma.

Theorem 6.11 ([BD98], Proposition 5) *The space of the periodic real analytic functions $C^\omega_{2\pi}(\mathbb{R})$ is complemented in $C^\omega(\mathbb{R})$.*

Proof. Let $\mu := \delta_\pi - \delta_{-\pi}$. Then $T_\mu(f)(x) = (\mu \star f)(x) = f(x-\pi)\ f(x+\pi)$ and hence $C^\omega_{2\pi}(\mathbb{R}) = \ker(T_\mu)$. Now we apply 6.10 to $I = \mathbb{R}$ and constant $r = 1$. We have $\hat{\mu} = T_\pi(\delta) - T_{-\pi}(\delta) : z \mapsto e^{\mathrm{i}\,\pi z} - e^{-\mathrm{i}\,\pi z} = 2\,\mathrm{i}\sin(\pi z)$ and $\hat{\mu}^{-1}(0) = \mathbb{Z}$. Thus the first condition is satisfied.

We have $\mathrm{supp}(\mu) = \{\pi, -\pi\}$, hence $G = [-\pi, \pi]$ and $I - G = \mathbb{R}$. Furthermore $H_G(z) = \pi|\Im(z)|$. Note that

$$\begin{aligned} |\hat{\mu}(x + \mathrm{i}\, y) &= |e^{\pi(\mathrm{i}\, x - y)} - e^{-\pi(\mathrm{i}\, x - y)}| \\ &= |e^{-\pi y}(\cos(\pi x) + \mathrm{i}\sin(\pi x)) - e^{\pi y}(\cos(\pi x) - \mathrm{i}\sin(\pi x))| \\ &\geq e^{-\pi y}|\cos(\pi x) + \mathrm{i}\sin(\pi x)| - e^{\pi y}|\cos(\pi x) - \mathrm{i}\sin(\pi x)| \\ &= e^{-\pi y} - e^{\pi y} \\ &= -\sinh(\pi y). \end{aligned}$$

Hence, the two other conditions are satisfied.

Now let $\sigma : C^\omega(\mathbb{R}) \to C^\omega(\mathbb{R})$ be a continuous linear right inverse for T_μ given by 6.10. Then $q := 1 - \sigma \circ T_\mu$ has image in $\ker(T_\mu)$, since $T_\mu \circ q = T_\mu - T_\mu \circ \sigma \circ T_\mu = 0$, and is a left-inverse for the inclusion of $\ker(T_\mu)$ in $C^\infty(\mathbb{R})$, since $q\,|_{\ker(T_\mu)} = 1 - 0 = 1$.□

Theorem 6.12 *The space $C^\omega_{2\pi}(\mathbb{R})$ of 2π periodic real analytic functions is isomorphic to $H(\overline{\mathbb{D}})$.*

Proof. $C^\omega_{2\pi}(\mathbb{R})$ is isomorphic to the space of real analytic functions on the unit circle. By using the Laurent series representation, this is isomorphic to $H(\overline{\mathbb{D}}) \times H(\mathbb{C}_\infty \backslash \mathbb{D}) \cong H(\overline{\mathbb{D}}) \times H(\overline{\mathbb{D}})$.

Now we identify $H(\mathbb{D})$ with $\Lambda_1(\alpha)$ by [MV92] 29.4.(3), following from 27.27, 27.25 and 27.16.(1) for which the isomorphism $\Lambda_1(\alpha) \times \Lambda_1(\alpha) \cong \Lambda_1(\alpha)$ (for shift-stable α) holds by [MV92] §29, example 3(b). From the GKS-duality 4.11 we obtain $H(\mathbb{D}) \times H(\mathbb{D}) \cong H(\mathbb{D})$. Finally, take the duals. □

Theorem 6.13 *$C^\omega(\mathbb{R})'_\beta$ has a complemented subspace isomorphic to $H(\mathbb{D})$.*

Proof. By 6.12 $C^\omega(\mathbb{R})$ has a complemented subspace isomorphic to $H(\overline{\mathbb{D}})$. Taking the duals and applying the GKS-duality 4.11 gives the desired result. □

Chapter 7

Real Analytic Curves

Definition 7.1 Let E be a locally convex space. A curve $c : \mathbb{R} \to E$ is called *real analytic* if $l \circ c : \mathbb{R} \to \mathbb{R}$ is real analytic for all $l \in E'$.

By $C^\omega(\mathbb{R}, E)$ we denote the space of all real analytic curves from $\mathbb{R}$ to E.

Definition 7.2 Let E be a locally convex space. A curve $c : \mathbb{R} \to E$ is called *topologically real analytic* if c is locally given by a power series converging with respect to the locally convex topology.

By $C_t^\omega(\mathbb{R}, E)$ we denote the space of all topologically real analytic curves from $\mathbb{R}$ to E.

Definition 7.3 Let E be a locally convex space. A curve $c : \mathbb{R} \to E$ is called *bornologically real analytic* if c factors locally over a topologically real analytic curve into E_B for some bounded absolutely convex set $B \subseteq E$.

By $C_b^\omega(\mathbb{R}, E)$ we denote the space of all bornologically real analytic curves from $\mathbb{R}$ to E.

Theorem 7.4 ([BD98], Proposition 10.(2)) *Let E be a complete locally convex space, $f : \mathbb{R} \to E$ be given. The following assertions are equivalent.*

(i) $f \in C_t^\omega(\mathbb{R}, E)$.

(ii) $f \in C^\omega(\mathbb{R}, E)$ *and for every compact interval $I \subseteq \mathbb{R}$ there is an $n \in \mathbb{N}$ such that for all continuous semi-norms $\|.\|_p$ on E*

$$\sup_{x \in I} \sup_{i \in \mathbb{N}} \left| \frac{\|f^{(i)}(x))\|_p}{i! n^i} \right| < \infty.$$

Proof. (i) $\Rightarrow$ (ii) By hypothesis, for all $x \in I$ there exists an $r > 0$ depending on x such that for all $h > 0$ with $|h| \leq 2r$ the power series $\sum_{i \in \mathbb{N}} \frac{f^{(i)}(x)}{i!} h^i$ converges. Put

$$U_r(x) := \{y \in \mathbb{R} : |y - x| < r\}.$$

Then $\{U_r(x) : x \in I\}$ is an open covering of I. Hence there exists a $\rho > 0$, called Lebesgue-number, such that for every subset A with a diameter less than ρ there exists an $x \in I$ such that $A \subseteq U_r(x)$. Therefore for all $x \in I$ the Taylor series converges for $|h| < \frac{\varrho}{2}$. Using Cauchy's integral formula, we get

$$\frac{f^{(n)}(\omega)}{n!}h^n = \int_\gamma 2\pi \frac{f(z)}{(z-\omega)^{n+1}} h^n dz = 2\pi \|u \circ f\|_\infty$$

where γ is a circle with radius h around ω. Applying $\|.\|_p$ proves the statement.

(ii) $\Rightarrow$ (i) Let n be as in the hypothesis and $|h| \leq r < \frac{1}{n}$. Then $\sum_{i\in\mathbb{N}} \frac{f^{(i)}(x)}{i!}h^i$ converges uniformly and absolutely for $|h| < r$ and $x \in I$. In fact, consider

$$\sum_{i\in\mathbb{N}} \left\| \frac{f^{(i)}(x)}{i!} h^i \right\|_p \leq \sum_{i\in\mathbb{N}} \left\| f^{(i)}(x) \right\|_p \frac{1}{i!}(rn)^i \frac{1}{n^i}.$$

The right side converges absolutely, since

$$\frac{\|f^{(i)}(x)\|_p}{i!\; n^i}$$

is bounded. □

Lemma 7.5 ([Köt69a], 29.1.(5)) *Let E be a locally convex and metrizable space and $(B_n)_{n\in\mathbb{N}}$ a sequence of bounded subsets of E. Then there exist positive scalars $\rho_n, n \in \mathbb{N}$ such that $\bigcup_{n\in\mathbb{N}} \rho_n B_n$ is also bounded.*

Proof. If $V_1 \supseteq V_2 \supseteq \cdots$ is a base of absolutely convex neighbourhoods of 0 in E, and if $\rho_n B_n \subseteq V_n$, then $\bigcup_{n=m}^\infty \rho_n B_n \subseteq V_m$ for each $m \in \mathbb{N}$, and hence $\bigcup_{n\in\mathbb{N}} \rho_n B_n$ is bounded. □

Theorem 7.6 ([BD98], Proposition 12) *If F is a Fréchet space, then the spaces $C_t^\omega(\mathbb{R}, F)$ and $C_b^\omega(\mathbb{R}, F)$ coincide.*

Proof. First we observe that every $f \in C_t^\omega(\mathbb{R}, F)$ is locally a bornologically real analytic function for an arbitrary locally convex space F. Indeed, from 7.4, for an arbitrary compact interval $I \subseteq \mathbb{R}$, we get $n \in \mathbb{N}$ such that

$$C := \left\{ \frac{f^{(i)}(x)}{i!\; n^i} : x \in I, i \in \mathbb{N} \right\}$$

is bounded in F. We denote by B the closed absolutely convex hull of C. Then it is easy to see that for all $x \in I$ and a suitable ε

$$f(t) = \sum_{i=0}^\infty f^{(i)}(x)(t-x)^i \tag{1}$$

for all $t \in \,]x - \varepsilon, x + \varepsilon[$, and the series converges in F_B.

To conclude, we fix $f \in C_t^\omega(\mathbb{R}, F)$ and we assume that F is a Fréchet space. For each $n \in \mathbb{N}$ there are a closed absolutely convex bounded subset B_n of F and an $\varepsilon_n > 0$ such that for all $x \in [-n, n]$ equation (1) holds for $\varepsilon = \varepsilon_n$ and the series converges in F_{B_n}. By 7.5 in metrizable spaces there is for each sequence of bounded sets another bounded set absorbing all the sets in the sequence. Hence there is a closed absolutely convex bounded set $B \subseteq F$ and $\lambda_n > 0$ such that $B_n \subseteq \lambda_n B$ for all $n \in N$. Thus, for all $x \in \mathbb{R}$, the series in (1) converges in F_B with a positive radius of convergence. □

Theorem 7.7 ([BD98], Lemma 14) *If $f \in C^\omega(\mathbb{R}, E)$, then for every compact interval $I \subseteq \mathbb{R}$ and every continuous semi-norm $\|.\|_p$ on E there is an $n \in \mathbb{N}$ such that*

$$\sup\left\{\left|\frac{(u\circ f)^{(i)}(x)}{i!\ n^i}\right| : |u(x)| \le \|x\|_p, u \in E', i \in \mathbb{N}, t \in I\right\} < \infty.$$

In particular,

$$B := \left\{u \circ f : |u(x)| \le \|x\|_p, u \in E'\right\}$$

is bounded in $C^\omega(\mathbb{R})$ for all continuous semi-norms $\|.\|_p$ on E.

Proof. Since f is smooth, $u \circ f$ is smooth for all bounded $u : E \to \mathbb{R}$ and satisfies $(u \circ f)^{(i)}(t) = u(f^{(i)}(t))$. Furthermore, by hypothesis we have $|u(x)| \le \|x\|_p$. Hence it suffices to show

$$\sup\left\{\left\|\frac{f^{(i)}(x)}{i!n^i}\right\|_q : |u(x)| \le \|x\|_p, u \in E', i \in \mathbb{N}, t \in I\right\} < \infty$$

for $f : \mathbb{R} \to E/\widehat{\ker(\|.\|_q)}$. The range being a Banach-space, we can apply [KM97] 9.6. to ensure that $f \in C^\omega_t(\mathbb{R}, E)$.

From 7.4, for an arbitrary compact interval $I \subseteq \mathbb{R}$, we get $n \in \mathbb{N}$ such that

$$\left\{\frac{f^{(i)}(t)}{i!\ n^i} : t \in I, i \in \mathbb{N}\right\}$$

is bounded in E. Hence we have

$$u\left(\left\{\frac{f^{(i)}(t)}{i!\ n^i} : t \in I, i \in \mathbb{N}\right\}\right) = \left\{\frac{(u\circ f)^{(i)}(t)}{i!\ n^i} : t \in I, i \in \mathbb{N}\right\}.$$

This proves the hypothesis, since the right hand side is bounded as the image of a bounded set under a bounded map.

Inserting $(u \circ f)$ into 6.6 assures that the set B is bounded in $C^\omega(\mathbb{R})$. □

Theorem 7.8 ([BD98], Theorem 16) *Let E be a sequentially complete locally convex space. The spaces $C^\omega(\mathbb{R}, E)$ and $C^\omega(\mathbb{R})\hat{\otimes}E = L(C^\omega(\mathbb{R})'_\beta, E)$ are algebraically isomorphic in a canonical way.*

Moreover, this isomorphism maps $C^\omega_b(\mathbb{R}, E)$ onto $LB(C^\omega(\mathbb{R})'_\beta, E)$.

Proof. We define

$$\Delta : \mathbb{R} \to C^\omega(\mathbb{R})'_\beta : \Delta(x) := \delta_x.$$

Since $g \circ \Delta = g$ for all $g \in C^\omega(\mathbb{R}) = \left(C^\omega(\mathbb{R})'_\beta\right)'$, we have $\Delta \in C^\omega(\mathbb{R}, C^\omega(\mathbb{R})'_\beta)$. This clearly implies that the map

$$\phi : L(C^\omega(\mathbb{R})'_\beta, E) \to C^\omega(\mathbb{R}, E) : \phi(W) := W \circ \Delta$$

is well-defined and linear (see [KM90], 1.9).

We put

$$H := \operatorname{span}\left(\{\delta_x : x \in \mathbb{R}\}\right) \subseteq C^\omega(\mathbb{R})'_\beta,$$

and endow it with the topology induced by $C^\omega(\mathbb{R})'_\beta$. For $f \in C^\omega(\mathbb{R}, E)$ we denote by

$$\psi(f) : H \to E : \psi(f)(\delta_x) := f(x),$$

for all $x \in \mathbb{R}$. We check that $\psi(f)$ is continuous. Fix a continuous semi-norm $\|.\|_p$ on E. By 7.7 and 6.6,

$$B := \left\{u \circ f : |u(x)| \le \|x\|_p, u \in E'\right\}$$

is bounded in $C^\omega(\mathbb{R})$. If $y \in H$ belongs to the polar B° taken in $C^\omega(\mathbb{R})'_\beta$ we have

$$\sup_{|u(x)| \le \|x\|_p} |u(\psi(f)(y))| = \sup_{|u(x)| \le \|x\|_p} |\langle y, u \circ f\rangle| \le 1.$$

By 6.8, H is sequentially dense in $C^\omega(\mathbb{R})'_\beta$. Since E is sequentially complete, there is the unique continuous extension $\psi(f) : C^\omega(\mathbb{R})'_\beta \to E$. Clearly, $\psi(f) \in C^\omega(\mathbb{R})\hat{\otimes}E$. Furthermore $\psi : C^\omega(\mathbb{R}, E) \to C^\omega(\mathbb{R})\hat{\otimes}E$ is well-defined and linear. Now, both $\phi \circ \psi$ and $\psi \circ \phi$ coincide with the identity in the corresponding space. Therefore both are linear isomorphisms. Observe that $(\psi \circ \phi)(W) = W$ has to be checked only on $\{\delta_x : x \in \mathbb{R}\}$ for each $W \in L(C^\omega(\mathbb{R})'_\beta, E)$.

To see the second part, observe that $f \in C^\omega_b(\mathbb{R}, E)$ if and only if $f \in C^\omega(\mathbb{R}, E_B)$ for some closed absolutely convex bounded subset B of E. By the proof given above, for the Banach space E_B,

$$\psi(f) \in L(C^\omega(\mathbb{R})'_\beta, E_B) \subseteq LB(C^\omega(\mathbb{R})'_\beta, E).$$

Conversely, if $W \in LB(C^\omega(\mathbb{R})'_\beta, E)$ there is a B with

$$W \in L(C^\omega(\mathbb{R})'_\beta, E_B).$$

Thus $\phi(W) \in C^\omega(\mathbb{R}, E_B) \subseteq C^\omega_b(\mathbb{R}, E)$. □

Chapter 8

Sequence Spaces

Definition 8.1 Let M be a set, $a : M \to \mathbb{F}$ with $a(t) \geq 1$ for all $t \in M$. We set

$$\Lambda^\infty(M, a) := \left\{ x : M \to \mathbb{F} : \|x\|_k := \sup_{t \in M} |x(t)| a(t)^k < \infty \ \forall k \right\}$$

and

$$\Lambda^1(M, a) := \left\{ x : M \to \mathbb{F} : \|x\|_k := \sum_{t \in M} |x(t)| a(t)^k < \infty \ \forall k \right\}.$$

Remark 8.2 Equipped with the respective semi-norms $\|.\|_k, k \in \mathbb{N}$, $\Lambda^\infty(M, a)$ and $\Lambda^1(M, a)$ are Fréchet spaces, since for each semi-norm k the mapping $E_k \to E_{k+1}$ is compact and hence the spaces are projective limits of Fréchet spaces.

Remark 8.3 In definition 8.1, put $M = \mathbb{N}$ (here $\mathbb{N}$ is meant explicitly without 0) and $a(n) = e^{\alpha_n}$, where $0 < \alpha_n \leq \alpha_{n+1}$, for all $n \in \mathbb{N}$ and $\sup_{n \in \mathbb{N}} \frac{\log(n)}{\alpha_n} =: q < \infty$. Then

$$\Lambda^\infty(\mathbb{N}, (e^{\alpha_n})_{n \in \mathbb{N}}) = \Lambda^1(\mathbb{N}, (e^{\alpha_n})_{n \in \mathbb{N}}) =: \Lambda_\infty(\alpha)$$

is called *power series space* of infinite type. $\Lambda_\infty(\alpha)$ is nuclear (see [MV92], 29.6.(1)).

Remark 8.4 If we put $M = \mathbb{N}$ and $a(n) = n$ in the definition 8.1, then

$$\Lambda^\infty(\mathbb{N}, (n)_{n \in \mathbb{N}}) = \Lambda^1(\mathbb{N}, (n)_{n \in \mathbb{N}}) =: s$$

is the space of *rapidly decreasing sequences*. It is the special case $\alpha_n = \log(n+1)$ of 8.3.

Remark 8.5 For $M = I \times \mathbb{N}$, where I is an arbitrary non-empty set and $a(i, n) = e^{\alpha_n}$ with α as in 8.3 we get the following isomorphisms in a canonical way (see [Vog85], 1. Example (3)).

$$\begin{aligned} \Lambda^\infty(I \times \mathbb{N}, (e^{\alpha_n})_{i \in I, n \in \mathbb{N}}) &\cong \ell^\infty(I) \hat{\otimes} \Lambda_\infty(\alpha) \\ \Lambda^1(I \times \mathbb{N}, (e^{\alpha_n})_{i \in I, n \in \mathbb{N}}) &\cong \ell^1(I) \hat{\otimes} \Lambda_\infty(\alpha) \end{aligned}$$

Because of the nuclearity of $\Lambda_\infty(\alpha)$ the tensor products $\hat{\otimes}_\varepsilon$ and $\hat{\otimes}_\pi$ coincide. Therefore we can write in this case $\hat{\otimes}$ for both.

Theorem 8.6 ([Vog85], Proposition 1.1) *Let I be an infinite index set and $(\alpha_n)_{n\in\mathbb{N}}$ an increasing sequence with $\sup_{n\in\mathbb{N}} \frac{\alpha_{n+1}}{\alpha_n} =: p < \infty$. Then the following isomorphisms hold:*

$$\begin{aligned} \ell^\infty(I)\hat{\otimes}\Lambda_\infty(\alpha) &\cong \ell^\infty(I)\hat{\otimes}s, \\ \ell^1(I)\hat{\otimes}\Lambda_\infty(\alpha) &\cong \ell^1(I)\hat{\otimes}s. \end{aligned}$$

Proof. By 8.5 it suffices to identify the spaces $\Lambda^\infty(M,a)$ and $\Lambda^\infty(M,b)$ (respectively $\Lambda^1(M,a)$ and $\Lambda^1(M,b)$) where $M = I \times \mathbb{N}$ and $a(i,k) := e^{\alpha_k}$, $b(i,k) := k$. We give the isomorphism by means of a bijection of M onto itself.

We put $n_0 = 0$, $n_k := [ke^{\alpha_k}]$ and $m_k := n_k - n_{k-1}$ for all $k \in \mathbb{N}$. For every $k \in \mathbb{N}$ the set I can be written as a disjoint union of subsets $I_{k,j}$, $j \in I$ with $|I_{k,j}| = m_k$. We put $I_{k,j} = \{i_{k,j,\mu} : \mu = 1, \ldots, m_k\}$ and define

$$\phi(i_{k,j,\mu}, k) := (j, n_k + \mu)$$

for $i_{k,j,\mu} \in I_{k,j}$. Furthermore put $q := \sup_{n\in\mathbb{N}} \frac{\log(n)}{\alpha_n}$. Then ϕ is a bijection of M onto itself with the following properties

$$\begin{aligned} b(\phi(i,k)) &= n_k + \mu \le n_{k+1} \le (k+1)e^{\alpha_{k+1}} \le e^{(q+1)\alpha_{k+1}} \le e^{(q+1)p\alpha_k} \\ &\le (a(i,k))^{(q+1)p}, \\ b(\phi(i,k)) &= n_k + \mu \ge n_k \ge \frac{1}{2}(n_k+1) \ge \frac{k}{2}e^{\alpha_k} \ge \frac{1}{2}e^{\alpha_k} \\ &\ge \frac{1}{2}a(i,k). \end{aligned}$$

Hence the map $x \to x \circ \phi$ defines an isomorphism of $\Lambda^\infty(M,a)$ onto $\Lambda^\infty(M,b)$ (respectively $\Lambda^1(M,a)$ onto $\Lambda^1(M,b)$). □

Definition 8.7 A sequence $(e_n)_{n\in\mathbb{N}}$ of elements in a locally convex space E is called *basis* if for each element $x \in E$ there is a uniquely determined sequence of numbers $(\xi_n)_{n\in\mathbb{N}}$ such that

$$x = \lim_{m\to\infty} \sum_{n=0}^{m} \xi_n e_n.$$

For each basis the correspondence $x \mapsto \xi_m$ defines linear forms f_m on E with $\xi_m = \langle x, f_m \rangle$. Here we have

$$\langle e_n, f_m \rangle = \delta_{n,m}$$

for $n, m \in \mathbb{N}$.

Definition 8.8 We say that a basis $(e_n)_{n\in\mathbb{N}}$ of a locally convex space E is *equicontinuous* if for each zero neighbourhood U there exists a zero neighbourhood V for which the inequalities

$$|\langle x, f_n \rangle| \|e_n\|_U \le \|x\|_V$$

are valid for all $x \in E$ and all $n \in \mathbb{N}$. In particular, all linear forms f_n are continuous.

Remark 8.9 Each basis in a Fréchet space is equicontinuous (c.f. [Pie72], 10.1.2. Theorem).

Definition 8.10 We call an equicontinuous basis $(e_n)_{n\in\mathbb{N}}$ *absolute* if for each zero neighbourhood U there is a zero neighbourhood V for which the inequalities

$$\sum_{n\in\mathbb{N}} |\langle x, f_n\rangle| \|e_n\|_U \leq \|x\|_V$$

hold for all $x \in E$. Then for all elements $x \in E$ we have the identity

$$x = \sum_{n\in\mathbb{N}} \langle x, f_n\rangle\, e_n,$$

where the series on the right hand side is absolutely summable.

Theorem 8.11 ([Pie72], 10.1.4. Theorem) *Each complete locally convex space E with an absolute basis $(e_n)_{n\in\mathbb{N}}$ can be identified with a sequence space Λ.*

Proof. We set

$$P := \{(\|e_n\|_U)_{n\in\mathbb{N}} : U \in \mathcal{U}(E)\}$$

where $\mathcal{U}(E)$ denotes the set of zero neighbourhoods in E. We now construct the associated sequence space Λ, whose locally convex topology is obtained from the semi-norms

$$\|(\xi_n)_{n\in\mathbb{N}}\|_U := \sum_{n\in\mathbb{N}} |\xi_n|\, \|e_n\|_U$$

with $U \in \mathcal{U}(E)$. Since, by hypothesis, there is for each zero neighbourhood $U \in \mathcal{U}(E)$ a zero neighbourhood $V \in \mathcal{U}(E)$ with

$$\left\|(\langle x, f_n\rangle)_{n\in\mathbb{N}}\right\|_U = \sum_{n\in\mathbb{N}} |\langle x, f_n\rangle|\, \|x\|_U \leq \|x\|_V$$

for all $x \in E$, the expression

$$Ax := (\langle x, f_n\rangle)_{n\in\mathbb{N}}$$

defines a one to one continuous linear mapping from E into Λ. Since all families $(\xi_n e_n)_{n\in\mathbb{N}}$ with $(\xi_n)_{n\in\mathbb{N}} \in \Lambda$ in E are absolutely summable, we can set

$$x = \sum_{n\in\mathbb{N}} \xi_n e_n.$$

But then the relation $Ax = (\xi_n)_{n\in\mathbb{N}}$ is valid and we have shown that A is also a mapping onto Λ. Finally, the continuity of the inverse mapping A^{-1} follows from the inequality

$$\|\sum_{n\in\mathbb{N}} \xi_n e_n\|_U \leq \sum_{n\in\mathbb{N}} |\xi_n|\, \|e_n\|_U = \|(\xi_n)_{n\in\mathbb{N}}\|_U$$

which is valid for $(\xi_n)_{n\in\mathbb{N}} \in \Lambda$ and $U \in \mathcal{U}(E)$. $\square$

Remark 8.12 Since the unit vectors $e_n := (\delta_{n,m})_{m\in\mathbb{N}}$ form an absolute basis in every sequence space Λ, we get from 8.11 that the collection of all complete locally convex spaces in which there is an absolute basis coincides with the collection of all sequence spaces.

Remark 8.13 ([Pie72], 10.3.4. - 10.3.9.) In the nuclear locally convex space $\mathcal{S}$ the Hermitian functions

$$h_r(t) = e^{t^2/2} \frac{d^r}{dt^r}(e^{-t^2})$$

form a basis. See [Jar81], 14.8.5.(d)

Remark 8.14 In the nuclear locally convex space $\mathcal{D}_{[-1,1]}$ the transformed Hermitian functions

$$g_r(t) = h_r\left(\tan\left(\frac{\pi}{2}t\right)\right)$$

form a basis since the correspondence

$$[f(t)] \to \left[f\left(\tan\left(\frac{\pi}{2}t\right)\right)\right]$$

is an isomorphism between the spaces $\mathcal{S}$ and $\mathcal{D}_{[-1,1]}$.

Lemma 8.15 *The nuclear locally convex spaces $C^\infty_{[-1,1]}, \mathcal{D}_{[-1,1]}$ and $\mathcal{S}$ can all be identified with the nuclear sequence space s. In other words we have*

$$s \cong C^\infty_{[-1,1]} \cong \mathcal{D}_{[-1,1]} \cong \mathcal{S}$$

Outline of the proof. Using the transformation in 8.11 we immediately get the desired result.

Reference. A complete proof is given in [MV92], 29.5.(2)-(4).

Theorem 8.16 (Borel's theorem) ([KM97]) *If $(a_k)_{k\in\mathbb{N}}$ is an arbitrary sequence of real numbers, then there exists a smooth function F such that $F^{(k)}(0) = a_k$ for all $k \in \mathbb{N}$.*

Proof. Let $\phi \in C^\infty$ with

$$\phi(x) = \begin{cases} 0 & : \quad |x| > 1 \\ 1 & : \quad |x| < \frac{1}{2} \end{cases}$$

and let

$$b_k := k + \sum_{m=0}^{k} |a_k|.$$

The function

$$F(x) := \sum_{m=0}^{\infty} \frac{a_k}{k!} x^k \phi(b_k x)$$

has the required properties. Only finitely many terms of the series are nonzero on any closed interval $[c, d]$ not containing the origin, since $\phi(b_k x)$ vanishes for $|x| > \frac{1}{b_k}$, a quantity which converges to 0. Thus F is smooth in a neighbourhood of any nonzero x, and we have to show that it is equally regular at the origin. We form the derivatives. If x is not 0 these are given by the convergent series

$$F^{(n)}(x) = \sum_{k=0}^{\infty} \sum_{j=0}^{n} \frac{n! a_k}{(n-j)!\ j!\ (k-j)!} \phi^{(n-j)}(b_k x) b_k^{n-j}$$

and we shall show that this series of continuous functions converges uniformly on the real axis using the Weierstraß M-test.

Let $M_n = \max_{j\le n} \|\phi^{(j)}\|_\infty$ and suppose $k \ge n+1$. Since only terms for which $|x|b_k < 1$ will contribute to the sum we have

$$|a_k||x|^{k-j} b_k^{n-j} \le |a_k| b_k^{n-k} \le b_k^{n+1-k} \le 1.$$

Accordingly

$$\left| \sum_{k=0}^{\infty} \sum_{j=0}^{n} \frac{n! a_k}{(n-j)!\ j!\ (k-j)!} x^{k-j} \phi^{(n-j)}(b_k x) b_k^{n-j} \right| \le \sum_{k=0}^{\infty} 2^n M_n \frac{1}{(k-n)!}.$$

Hence the sum of the terms is given for which $k \ge n+1$ is bounded by

$$2^n M_n \sum_{k=n}^{\infty} \frac{1}{(k-n)!} = e 2^n M_n.$$

Since the series giving $F^{(n)}$ converges uniformly on the axis, that function may be extended to $x = 0$ in such a way that it becomes continuous and $F^{(n-1)}$, if so extended, is differentiable at the origin and its derivative is $F^{(n)}(0)$. But this number is just a_n. $\square$

Definition 8.17 By ω we denote the space of all sequences (sometimes referred to as $\mathbb{F}^{\mathbb{N}}$).

Lemma 8.18 ([Vog85], Lemma 1.4 or [Vog77a], Lemma 1.6 or [Vog77b], Lemma 3.1) *There exists an exact sequence*

$$0 \to s \to s \to \omega \to 0.$$

Proof. Let

$$\Delta : \mathcal{D}_{[-1,1]} \to \omega : \phi \to (\phi(0), \phi'(0), \ldots).$$

Then according to 8.16 Δ is surjective. The kernel of Δ is isomorphic to $\mathcal{D}_{[-1,0]} \times \mathcal{D}_{[0,1]}$. Since $\mathcal{D}_{[a,b]}$ can be isomorphically mapped to $\mathcal{D}_{[-1,1]}$ for all $a < b$ then by 8.15 we have $\mathcal{D}_{[a,b]} \cong s$ for all $a < b$. Furthermore we have $s \times s \cong s$ and hence the exact sequence

$$0 \to \mathcal{D}_{[-1,0]} \times \mathcal{D}_{[0,1]} \to \mathcal{D}_{[-1,1]} \to \omega \to 0$$

leads to an exact sequence

$$0 \to s \to s \to \omega \to 0.$$

Lemma 8.19 *Let E, F and G be Fréchet spaces, $A \in L(E,F)$ and $B \in L(F,G)$. The sequence*

$$0 \longrightarrow E \xrightarrow{A} F \xrightarrow{B} G \longrightarrow 0$$

is exact if and only if the dual sequence

$$0 \longrightarrow G' \xrightarrow{B'} F' \xrightarrow{A'} E' \longrightarrow 0$$

is exact.

Reference. The proof can be found in [MV92] 26.4.

Lemma 8.20 (Canonical Resolution) ([Vog85], Lemma 1.5, [MV92], 26.14 Definition and following) *Let E be a Fréchet space. There is an exact sequence, called* canonical resolution

$$0 \to E \xrightarrow{\pi} \prod_{k\in\mathbb{N}} \widehat{E_k} \xrightarrow{\sigma} \prod_{k\in\mathbb{N}} \widehat{E_k} \to 0$$

defined by $\pi(x) := (\pi_k(x))_k$ where $\pi_k : E \to \widehat{E_k}$ is the canonical dense mapping and by $\sigma((x_k)_k) := (x_k - \pi_{k+1,k}(x_{k+1}))_k$ where $\pi_{k+1,k} \circ \pi_{k+1} = \pi_k$ with $\|\pi_{k+1,k}\| \le 1$.

Proof. Clearly, $\prod_{k\in\mathbb{N}} \widehat{E_k}$ is a Fréchet space. From the continuity of π_k and $\pi_{k+1,k}$ and the properties of the product topology we get as a consequence the continuity of π and σ. By 8.19, it suffices to prove the hypothesis for the dual exact sequence

$$0 \to \left(\prod_{k\in\mathbb{N}} \widehat{E_k}\right)' \xrightarrow{\sigma'} \left(\prod_{k\in\mathbb{N}} \widehat{E_k}\right)' \xrightarrow{\pi'} E' \to 0.$$

We recall that

$$\left(\prod_{k\in\mathbb{N}} \widehat{E_k}\right)' = \bigoplus_{k\in\mathbb{N}} \widehat{E_k}'$$

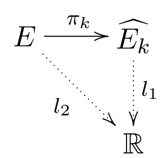
with

$$y(x) = \sum_{k\in\mathbb{N}} y_k(x_k)$$

for $y \in \bigoplus_{k\in\mathbb{N}} \widehat{E_k}'$ and $x \in \prod_{k\in\mathbb{N}} \widehat{E_k}$. If we put

$$U_k := \{x \in E : \|x\|_k \le 1\},$$

then $\pi_k' : \widehat{E_k}' \to E'$ is an isometric bijection between $\widehat{E_k}'$ and $E'_{U_k^\circ}$ as can be seen from the following diagram



since $l_1 \circ \pi_k \in E'_{U_k^\circ}$ if and only if $l_1 \circ \pi_k |_{U_k}$ is bounded, which it is, as the conjunction of bounded functions. On the other hand, if l_2 is bounded on U_k we have $l_2 |_{\ker(\pi_k)} = 0$ hence there exists an $l_1 \in \widehat{E_k}'$. We therefore identify $\widehat{E_k}'$ with $E'_{U_k^\circ}$. Hence, for $y \in \bigoplus_{k\in\mathbb{N}} \widehat{E_k}'$ we have

$$\pi'(y) = \sum_{k\in\mathbb{N}} y_k$$

and

$$\sigma'(y) = (y_k - y_{k-1})_{k\in\mathbb{N}}$$

with $y_0 := 0$. From this, we immediately get that π' is onto and σ' is one to one. Clearly, $\pi' \circ \sigma' = 0$ and therefore $\sigma'\left(\bigoplus_{k\in\mathbb{N}} \widehat{E_k}'\right) \subseteq \ker(\pi')$. To prove $\ker\pi' \subseteq \sigma'\left(\bigoplus_{k\in\mathbb{N}} \widehat{E_k}'\right)$, choose a $y \in \ker(\pi')$. Then

$$\eta := \left(\eta_k := \sum_{j=1}^{k} y_j\right)_{k\in\mathbb{N}}$$

is in $\prod_{k\in\mathbb{N}} \widehat{E_k}'$. If $y_k = 0$ for all $k > m$, we have

$$\eta_k = \sum_{j=1}^{k} y_j = \sum_{j=1}^{m} y_j = \pi'(y) = 0.$$

Hence, $\eta \in \bigoplus_{k\in\mathbb{N}} \widehat{E_k}'$ and obviously we have $\sigma'(\eta) = (y_k)_{k\in\mathbb{N}} = y$. $\square$

Chapter 9

The Property (Ω)

Definition 9.1 Let F be a Fréchet space with a topology defined by an increasing system of semi-norms $(\|.\|_k)_{k\in\mathbb{N}}$. We say that F has *property* (Ω) if for every p there exists a q such that for all k there exists a $C > 0$ and we have

$$\|u\|_{U_q}^2 \leq C\|u\|_{U_k}\|u\|_{U_p}$$

for $u \in F'$.

Recall that $\|u\|_{U_k} := \sup\{|u(x)| : x \in U_k\} \in [0, \infty]$ where U_k is the unit ball in F, i.e. $U_k := \{x \in F : \|x\|_k \leq 1\}$. These $\|.\|_{U_k}$ are not to be confused with the continuous semi-norms on F'.

Lemma 9.2 ([Vog85], Lemma 1.3 and [Vog77b], Theorem 2.3) *If*

$$0 \to E \to F \xrightarrow{q} \Lambda^1(M, a) \to 0$$

is an exact sequence of Fréchet spaces, E having property (Ω) and $\Lambda^1(M, a)$ being nuclear, then the sequence splits, i.e. the mapping $q : F \to \Lambda^1(M, a)$ has a continuous right inverse.

Proof. For sake of simplicity, we only consider the case $\Lambda^1(M, a) = s$. We assume that E is a subspace of F and let $W_k \supseteq W_{k+1}$ be a basis of absolutely convex neighbourhoods of 0 in F. Then we define inductively V_k as a neighbourhood basis of 0 in E by

$$V_k := W_k \cap E,$$

such that

$$V_k \subseteq r^{\nu_k} V_{k+1} + \frac{1}{r} V_{k-1} \tag{1}$$

for all $k \in \mathbb{N}$ and $r > 2$ by using property (Ω) in an equivalent form derived from A.10 (ii"') and with appropriate $\nu_k \in \mathbb{N}$. Note that in the iteration step $k-1$ we can choose V_k freely (by property (Ω)) but we must take care that for V_{k+1} the conditions are met in respect to the next iteration step.

If e_j is the j-th unit vector in s, then, using the canonical norms,

$$\|x\|_k := \sum_j j^k |x_j|$$

and we have $\|e_j\|_k = j^k$. By the open mapping theorem C.15, $q(W_k) \subseteq s$ is open. Hence, for every k there exists an $n_k \in \mathbb{N}$ and a $C_k \geq 1$ with

$$\{x : \|x\|_{n_k} \leq 1\} \subseteq C_k q(W_k).$$

Using

$$\frac{e_j}{j^{n_k}} \in C_k q(W_k)$$

we find a sequence $(d_j^k)_{j\in\mathbb{N}}$ in F such that

$$d_j^k \in C_k j^{n_k} W_k,$$

$$q(d_j^k) = e_j$$

for all $j \in \mathbb{N}$. We can assume that $n_k \leq n_{k+1}$, $C_k \leq C_{k+1}$ and that

$$(\nu_k + 1)n_{k+1} \leq n_{k+2},$$

$$C_k^{\nu_k+1} 2^{(k+1)\nu_k+1} \leq C_{k+1}$$

for all $k \in \mathbb{N}$.

Multiplying equation (1) with $2C_k j^{n_{k+1}}$ and choosing $r = 2^{k+1}C_k j^{n_{k+1}}$ we obtain

$$\begin{aligned} 2C_k j^{n_{k+1}} V_k &\subseteq 2C_k^{\nu_k+1} j^{n_{k+1}+\nu_k n_{k+1}} 2^{(k+1)\nu_k} V_{k+1} + 2^{-k} V_{k-1} \\ &\subseteq C_{k+1} j^{n_{k+2}} V_{k+1} + 2^{-k} V_{k-1} \end{aligned} \tag{2}$$

Since

$$d_j^{k+1} - d_j^k \in 2C_{k+1} j^{n_{k+1}} W_k \cap E = 2C_k j^{n_{k+1}} V_k,$$

we can choose inductively a sequence $(a_j^k)_{k\in\mathbb{N}}$ in E in the following way

$$a_j^0 = 0,$$

$$a_j^k \in C_k j^{n_{k+1}} V_k,$$

then $d_j^{k+1} - d_j^k + a_j^k \in 2C_k j^{n_{k+1}} V_k$ and according to (2) we can find $a_j^{k+1} \in C_{k+1} j^{n_{k+2}} V_{k+1}$ such that $d_j^{k+1} - d_j^k + a_j^k - a_j^{k+1} \in 2^{-k} V_{k-1}$.

We define

$$R_j^k := d_j^k - a_j^k \in 2C_k j^{n_{k+1}} W_k,$$

then we have

$$R_j^{k+1} - R_j^k = d_j^{k+1} - d_j^k + a_j^k - a_j^{k+1} \in 2^{-k} V_{k-1}$$

for all $j, k \in \mathbb{N}$. It follows that

$$\lim_{k\to\infty} R_j^k =: R_j$$

exists and $R_j \in 2C_k j^{n_{k+1}} W_k + W_{k-1} \subseteq 3C_k j^{n_{k+1}} W_{k-1}$. So we can define

$$R(x) := \sum_j x_j R_j$$

for $x := (x_j)_{j\in\mathbb{N}} \in s$ and with $\|R(x)\|_k$ denoting the semi-norm on F belonging to W_k, respectively $\|x\|_k$ the canonical norm on s, we obtain

$$\|R(x)\|_{k-1} \leq 3C_k\|x\|_{n_{k+1}}.$$

Therefore $R \in L(s, F)$ and, because

$$q(R_j) = \lim_{k\to\infty} q(R_j^k) = \lim_{k\to\infty} q(d_j^k) = e_j,$$

we get $q \circ R = \mathrm{id}$. □

Theorem 9.3 ([Vog85], Lemma 3.1) *Let E be a Fréchet space equipped with an increasing fundamental system of semi-norms. If E has property (Ω) and $\{x_i : i \in I\}$ is dense in E, then E is isomorphic to a quotient space of $\ell^1(I)\hat{\otimes}s$.*

Proof. The case of a finite index set I is trivial. So let I be infinite. From 8.20 we use the canonical resolution

$$0 \to E \to \prod_{k\in\mathbb{N}} \widehat{E_k} \to \prod_{k\in\mathbb{N}} \widehat{E_k} \to 0.$$

We choose a Banach space F such that every $\widehat{E_k}$ is isomorphic to a complemented subspace of F and such that F has a dense subset of a cardinality less than that of I, e.g.

$$F := \left\{ x := (x_k)_k \in \prod_{k\in\mathbb{N}} \widehat{E_k} : \|x\| := \sum_{k\in\mathbb{N}} \|x_k\|_k < \infty \right\}.$$

For any k let F_k be a topological complement of $\widehat{E_k}$ in F, i.e.

$$F = \widehat{E_k} \oplus F_k.$$

The direct sum of the canonical resolution above with the exact sequence

$$0 \to 0 \to \prod_{k\in\mathbb{N}} F_k \xrightarrow{\mathrm{id}} \prod_{k\in\mathbb{N}} F_k \to 0$$

can, by utilising

$$\prod_{k\in\mathbb{N}} E_k \oplus \prod_{k\in\mathbb{N}} F_k = \prod_{k\in\mathbb{N}} E_k \oplus F_k = F^{\mathbb{N}},$$

be considered as an exact sequence

$$0 \to E \to F^{\mathbb{N}} \xrightarrow{q_1} F^{\mathbb{N}} \to 0.$$

We now consider the continuous linear mapping

$$\ell^1(I) \to F : \sum_{i\in I} \lambda_i e_i \to \sum_{i\in I} \lambda_i f_i$$

for $(\lambda_i)_{i\in I} \in \ell^1(I)$, i.e. $\sum_{i\in I} |\lambda_i| < \infty$, $\{e_i : i \in I\}$ the canonical basis in $\ell^1(I)$ and $\{f_i \in F : i \in I, \|f_i\| \leq 1\}$ dense in the unit ball of F. This map is clearly onto, since for each $y \in F$ there exists an f_{i_0} such that $\|y - f_{i_0}\| \leq \frac{1}{2}$ and an f_{i_1} such that $\|y - f_{i_0} - \frac{1}{2} f_{i_1}\| \leq \frac{1}{4}$; iteration leads to

$$\left\| y - \sum_{k=0}^{n} \frac{1}{2^k} f_{i_k} \right\| \leq \frac{1}{2^{n+1}}.$$

Therefore we have an exact sequence

$$0 \to K \to \ell^1(I) \to F \to 0$$

where the kernel K is a Banach space which has a dense subset of cardinality $|J| \leq |I|$ and hence has a map $\ell^1(J) \to K$ which is onto.

According to 8.18

$$0 \to s \to s \to \omega \to 0$$

is an exact sequence. We tensor it with the previous one considered as a column and obtain the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F\hat{\otimes}s & \longrightarrow & F\hat{\otimes}s & \longrightarrow & F^{\mathbb{N}} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \ell^1(I)\hat{\otimes}s & \xrightarrow{\imath_1} & \ell^1(I)\hat{\otimes}s & \longrightarrow & \ell^1(I)^{\mathbb{N}} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle \imath_2} & & \uparrow & & \\
0 & \longrightarrow & K\hat{\otimes}s & \xrightarrow[\jmath_1]{} & K\hat{\otimes}s & \longrightarrow & K^{\mathbb{N}} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Consider the quotient space

$$\left(\ell^1(I)\hat{\otimes}s\right) / \left(\imath_1\left(\ell^1(I)\hat{\otimes}s\right) \oplus \imath_2\left(K\hat{\otimes}s\right)\right)$$

which, by the second isomorphism theorem, equals

$$\begin{aligned}
\left(\ell^1(I)\hat{\otimes}s/\imath_1\left(\ell^1(I)\hat{\otimes}s\right)\right) / \left(K\hat{\otimes}s/\jmath_1\left(K\hat{\otimes}s\right)\right) &= \left(\ell^1(I)\right)^{\mathbb{N}} / K^{\mathbb{N}} \\
&= \left(\ell^1(I)/K\right)^{\mathbb{N}} \\
&= F^{\mathbb{N}}.
\end{aligned}$$

This results in an exact sequence

$$\left(\ell^1(I)\hat{\otimes}s\right) \oplus \left(K\hat{\otimes}s\right) \xrightarrow{\imath_1 \oplus \imath_2} \ell^1(I)\hat{\otimes}s \xrightarrow{q_2} F^{\mathbb{N}} \to 0.$$

We denote by N the kernel of q_2. Thus N is a quotient of $\left(\ell^1(I)\hat{\otimes}s\right) \oplus \left(K\hat{\otimes}s\right)$.

We now consider the following diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & E & \longrightarrow & F^{\mathbb{N}} & \xrightarrow{q_1} & F^{\mathbb{N}} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle p_1} & & \uparrow{\scriptstyle q_2} & & \\
0 & \longrightarrow & E & \longrightarrow & H & \xrightarrow[p_2]{} & \ell^1(I)\hat{\otimes}s & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & N & \xrightarrow[\mathrm{id}]{} & N & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

where

$$H = \Big\{(x,y) \in F^{\mathbb{N}} \times \big(\ell^1(I)\hat{\otimes}s\big) : q_1(x) = q_2(y)\Big\}$$

and

$$p_1(x,y) = x, \ p_2(x,y) = y.$$

Using 9.2 with $\Lambda^1(I \times \mathbb{N}, (n)_{(i,n)\in I\times\mathbb{N}}) = \ell^1(I)\hat{\otimes}s$ (see 8.5 and 8.6) we know that the second row splits and we obtain from the first column the first row of the following diagram. The right column is the same as before. The rest of the diagram is constructed as in the previous one.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & N & \longrightarrow & H & \longrightarrow & F^{\mathbb{N}} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & \ell^1(I)\hat{\otimes}s & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & N & \xrightarrow[\mathrm{id}]{} & N & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Since N is a quotient of $\big(\ell^1(I)\hat{\otimes}s\big) \oplus \big(K\hat{\otimes}s\big) \cong \big(\ell^1(I) \oplus K\big)\,\hat{\otimes}s$ and hence of $\big(\ell^1(I) \oplus \ell^1(J)\big)\,\hat{\otimes}s$ which is isomorphic to $\Lambda^1(M,a)$ for suited M and a (see 8.5 and 8.6), N has property (Ω). Therefore the second row splits and we obtain from the first column

$$0 \to N \to G \to H \to 0$$

which can be written as

$$0 \to N \to N \oplus \big(\ell^1(I)\hat{\otimes}s\big) \to E \oplus \big(\ell^1(I)\hat{\otimes}s\big) \to 0.$$

Hence E is a quotient of $N \oplus \big(\ell^1(I)\hat{\otimes}s\big)$ and therefore of

$$\big(\ell^1(I) \oplus K \oplus \ell^1(I)\big)\,\hat{\otimes}s.$$

We have chosen K such that it contains a dense subset of cardinality $|J|$. Hence it is a quotient of $\ell^1(I)$. I is infinite, so we have $\ell^1(I) \oplus \ell^1(I) \oplus \ell^1(I) \cong \ell^1(I)$ and therefore E is a quotient of $\ell^1(I)\hat{\otimes}s$. $\square$

Theorem 9.4 *Let E be a Fréchet space. E has property (Ω) if and only if $H(K)'$ has property (Ω) for some non-empty (respectively all) compact sets K in E.*

Reference. The proof can be found in [KD97], Theorem 1.

Chapter 10

The Property (DN)

Definition 10.1 Let E be a metrizable locally convex space with the topology defined by an increasing system of semi-norms $\|.\|_1 \leq \|.\|_2 \leq \ldots$. We say that E has *property* (DN) if there exists a semi-norm $\|.\|$ on E such that for all $k \in \mathbb{N}$ there exist a $p \in \mathbb{N}$ and a $C > 0$ with

$$\|\cdot\|_k \leq r\|.\| + \frac{C}{r}\|\cdot\|_{k+p}$$

for all $r > 0$.

Clearly, the property (DN) does solely depend on the topology and not on the system of semi-norms. From the postulated inequality it follows that $\|.\|$ has to be a norm.

Remark 10.2 Let E be a metrizable locally convex space with property (DN). Then every subspace of E has the property (DN). This follows directly from the definition.

Lemma 10.3 ([Vog77a], 1.2. Bemerkung) *The space of rapidly decreasing sequences s has property (DN).*

Proof. We check, whether for an element x of s the conditions of the property (DN) are met. Let $\|x\|_k = \sum_{j=1}^{\infty} j^k|x_j|$ and $\|x\| = \sum_{j=1}^{\infty} |x_j|$. We then get for $j_0^k \leq r \leq (j_0+1)^k$

$$\begin{aligned} \|x\|_k &= \sum_{j=1}^{j_0} j^k|x_j| + \sum_{j=j_0+1}^{\infty} j^k|x_j| \\ &\leq j_0^k\|x\| + (j_0+1)^{-k}\|x\|_{2k} \\ &\leq r\|x\| + \frac{1}{r}\|x\|_{2k}. \end{aligned}$$

Thus we have proved the inequality for $r \geq 1$ and $p = k$. The case $0 < r < 1$ is obvious. □

Definition 10.4 Let E and F be arbitrary Fréchet spaces, each equipped with an ascending fundamental system of continuous semi-norms. For each $A \in L(E, F)$ we define

$$\|A\|_{n,k} := \sup\{\|Ax\|_n : \|x\|_k \leq 1\} \in \mathbb{R} \cup \{\infty\}.$$

Where applicable, we use this as a semi-norm on $L(E, F)$.

Lemma 10.5 ([Vog83], 1.1) *Let E and F be Fréchet spaces equipped with ascending fundamental systems of continuous semi-norms. The following assertions are equivalent.*

(i) $L(E,F) = LB(E,F)$.

(ii) *For each sequence $(k_m)_{m\in\mathbb{N}}$ in $\mathbb{N}$ there exists a $k_0 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ exist $n_0 \in \mathbb{N}$ and $C > 0$ with*

$$\|A\|_{n,k_0} \leq C \max_{m=1,\ldots,n_0} \|A\|_{m,k_m}$$

for all $A \in L(E,F)$.

Proof. We define to a given sequence $(k_m)_{m\in\mathbb{N}}$

$$G := \{A \in L(E,F) : \|A\|_{m,k_m} < \infty \ \forall m \in \mathbb{N}\}.$$

This is in a natural sense a Fréchet space. For each m we set

$$H_{k_m} := \{A \in L(E,F) : \|A\|_{n,k_m} < \infty \ \forall n \in \mathbb{N}\}.$$

H_{k_m} is also in a natural sense a Fréchet space. We have $H_{k_m} \subseteq H_{k_{m+1}}$ for all $m \in \mathbb{N}$ with continuous injection. By hypothesis we get

$$G \subseteq \bigcup_{m\in\mathbb{N}} H_{k_m}.$$

All occurring spaces are continuously embedded in $LB(E,F)$. From C.5 we get the existence of a k_0 such that $E \subseteq H_{k_0}$ and such that the embedding $E \subseteq H_{k_0}$ is continuous. For all n therefore exist n_0 and C such that

$$\|A\|_{n,k_0} \leq C \max_{m=1,\ldots,n_0} \|A\|_{m,k_m}.$$

Conversely, let $A \in L(E,F)$. From the continuity of A we get that for all $m \in \mathbb{N}$ exists a $k(m)$ such that

$$\|A\|_{m,k(m)} < \infty.$$

Therefore we can choose a sequence $(k_m)_{m\in\mathbb{N}}$ in $\mathbb{N}$ such that $\|A\|_{m,k_m} < \infty$. By hypothesis

$$\|A\|_{n,k_0} \leq C \max_{m=1,\ldots,n_0} \|A\|_{m,k_m}.$$

Hence there exists a $k \in \mathbb{N}$ such that for all $n \in \mathbb{N}$

$$\|A\|_{n,k} < \infty$$

holds. This is equivalent to

$$\|Ax\|_n < \infty$$

for all $\|x\|_k \leq 1$ and $n \in \mathbb{N}$ or, in other words, $A(\{x : \|x\|_k \leq 1\})$ is bounded. Since $\{x : \|x\|_k \leq 1\}$ is a neighbourhood of 0 and its image under A is bounded, A itself is bounded. □

Definition 10.6 Let $A = (a_{j,k})_{(j,k)\in\mathbb{N}\times\mathbb{N}}$ be a matrix with the following properties

(i) $a_{j,k} \geq 0$ for all $j, k \in \mathbb{N}$,

(ii) for all $j \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ with $a_{j,k} > 0$ and

(iii) $a_{j,k} \leq a_{j,k+1}$ for all $j, k \in \mathbb{N}$.

Then A is called *Köthe matrix*. We define

$$\begin{aligned} \lambda^1(A) &:= \left\{\xi = (\xi_n)_{n\in\mathbb{N}} : \|\xi\|_k := \textstyle\sum_{j=1}^{\infty} |\xi_j| a_{j,k} < \infty \ \forall k\right\}, \\ \lambda^\infty(A) &:= \left\{\xi = (\xi_n)_{n\in\mathbb{N}} : \|\xi\|_k := \sup_j |\xi_j| a_{j,k} < \infty \ \forall k\right\}. \end{aligned}$$

Remark 10.7 Equipped with their canonical semi-norms $\|.\|_k$, $\lambda^1(A)$ and $\lambda^\infty(A)$ are Fréchet spaces. First, put $\mathbb{N}_k := \{j \in \mathbb{N} : a_{j,k} \neq 0\}$. Then for every $k \in \mathbb{N}$ we get a linear function $f_k : \lambda^1(A) \to \ell^1(\mathbb{N}_k)$ and since for every $\xi \in \lambda^1(A)$ we have $\|\xi\|_k \leq \|\xi\|_{k+1}$, the map from $\ell^1(\mathbb{N}_{k+1})$ to $\ell^1(\mathbb{N}_k)$ is continuous and linear. The $\ell^1(\mathbb{N}_k)$ being Banach spaces and $\lambda^1(A)$ being embedded into $\prod_{k\in\mathbb{N}} \ell^1(\mathbb{N}_k)$, we get $\lambda^1(A) = \varprojlim_{k\in\mathbb{N}} \ell^1(\mathbb{N}_k)$. Finally, as the projective limit of Banach spaces, $\lambda^1(A)$ is a Fréchet space.

In the nuclear case we have $\lambda^1(A) = \lambda^\infty(A)$. For details see [MV92], 28.16.

Remark 10.8 Interesting special cases are $\Lambda^1_r(\alpha) := \lambda^1(A)$, respectively $\Lambda^\infty_r(\alpha) := \lambda^\infty(A)$ with $a_{j,k} := \rho_k^{\alpha_j}$, and $\alpha = (\alpha_j)_{j\in\mathbb{N}}$, $\alpha_j \underset{j\in\mathbb{N}}{\longrightarrow} \infty$ monotonously and $\lim_{k\in\mathbb{N}} \rho_k = r$ monotonously, $\rho_k > 0$ for $k \in \mathbb{N}$ with $0 < r \leq \infty$.

The spaces solely depend on α and r, not on the choice of $(\rho_k)_{k\in\mathbb{N}}$. For a fixed α all spaces $\Lambda^1_r(\alpha)$ (respectively $\Lambda^\infty_r(\alpha)$) with $r < \infty$ are isomorphic. Therefore only the cases $r = 1, \infty$ are of further interest.

Remark 10.9 The equivalence of the definitions of $\Lambda^1_\infty(\alpha)$ respectively $\Lambda^\infty_\infty(\alpha)$ in 8.1 and 10.6 is obvious via $\lambda^1((e^{2k\alpha_j})_{j,k\in\mathbb{N}}) = \Lambda^1(\mathbb{N}, (e^{2\alpha_n})_{n\in\mathbb{N}})$ respectively $\lambda^\infty((e^{2k\alpha_j})_{j,k\in\mathbb{N}}) = \Lambda^\infty(\mathbb{N}, (e^{2\alpha_n})_{n\in\mathbb{N}})$.

Definition 10.10 Let $(\alpha_j)_{j\in\mathbb{N}}$ be a sequence with $\alpha_j \underset{j\in\mathbb{N}}{\longrightarrow} \infty$ monotonously. The condition $\sup_n \frac{a_{n+1}}{a_n} < \infty$, called *shift-stability*, is equivalent to $\Lambda^1_r(\alpha) \cong \mathbb{F} \oplus \Lambda^1_r(\alpha)$ respectively $\Lambda^\infty_r(\alpha) \cong \mathbb{F} \oplus \Lambda^\infty_r(\alpha)$.

Lemma 10.11 ([Vog83], 1.3) *Let B be a Köthe matrix, F a Fréchet space. The following assertions are equivalent.*

(i) $L(\lambda^1(B), F) = LB(\lambda^1(B), F)$.

(ii) *For each sequence $(k_m)_{m\in\mathbb{N}}$ in $\mathbb{N}$ there exists a $k_0 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ exist $n_0 \in \mathbb{N}$ and $C > 0$ with*

$$\frac{\|x\|_n}{b_{j,k_0}} \leq C \max_{m=1,\dots,n_0} \frac{\|x\|_m}{b_{j,k_m}} \tag{1}$$

for all $j \in \mathbb{N}$ and $x \in F$.

Proof. The first implication follows from the corresponding implication in 10.5 as we apply (ii) to $A : \xi \to f_j \cdot x$ with $x \in F$ and $f_j(\xi) := \xi_j$ for $\xi = (\xi_n)_{n\in\mathbb{N}} \in \lambda^1(B)$.

Conversely let $A \in L(E, F)$. Then A is of the form

$$A\xi = \sum_j x_j \xi_j$$

with $x_j = Ae_j$, where $e_j = (\delta_{j,v})_{v\in\mathbb{N}}$ is the j^{th} unit vector in $\lambda^1(B)$. There exists a sequence $(k_m)_{m\in\mathbb{N}}$ and a family of constants $\{C_m\}_{m\in\mathbb{N}}$ with

$$\|x_j\|_m \le C_m \|e_j\|_{k_m} = C_m b_{j,k_m} \tag{2}$$

for all $m \in \mathbb{N}$. We get for arbitrary n an n_0 and a C such that

$$\begin{aligned} \|A\xi\|_n &\le \sup_{j\in\mathbb{N}} \left(\frac{\|x_j\|_n}{b_{j,k_0}}\right) \sum_{j\in\mathbb{N}} b_{j,k_0}|\xi_j| = \sup_{j\in\mathbb{N}} \left(\frac{\|x_j\|_n}{b_{j,k_0}}\right) \|\xi\|_{k_0} \\ &\le C \sup_{j\in\mathbb{N}} \left(\max_{m=1,\dots,n_0} \frac{\|x_j\|_m}{b_{j,k_m}}\right) \|\xi\|_{k_0} \\ &\le \left(C \max_{m=1,\dots,n_0} C_m\right) \|\xi\|_{k_0} \end{aligned}$$

where the second inequality follows from (1) and the third from (2). Hence A is bounded. □

Proposition 10.12 ([Vog83], 1.4) *The following assertions are equivalent.*

(i) $L(E, \lambda^\infty(A)) = LB(E, \lambda^\infty(A))$.

(ii) *For each sequence $(k_m)_{m\in\mathbb{N}}$ of integers exists a $k_0 \in \mathbb{N}$, such that for each $n \in \mathbb{N}$ exist $n_0 \in \mathbb{N}$ and $C > 0$ such that*

$$a_{j,n}\|y\|_{U_{k_0}} \le C \max_{m=1,\dots,n_0} a_{j,m}\|y\|_{U_{k_m}}$$

for all $j \in \mathbb{N}$ and $y \in E'$.

Proof. This proof is similar to the proof of 10.11. Again, let $e_j = (\delta_{j,v})_{v\in\mathbb{N}}$ be the j^{th} unit vector in $\lambda^1(B)$. One direction can be obtained by inserting $A = e_j \otimes y$ into 10.5 for all $j \in \mathbb{N}$ and all $y \in E'$.

The other direction follows also from 10.5 by

$$\begin{aligned} \|Ax\|_n &\le \sup_{j\in\mathbb{N}} a_{j,n}\|y_j\|_{U_{k_0}} \|x\|_{k_0} \\ &\le C \max_{m=1,\dots,n_0} \left(\sup_{j\in\mathbb{N}} a_{j,n}\|y_j\|_{U_{k_0}}\right) \|x\|_{k_0}, \end{aligned}$$

where $y_j = f_j \circ A$. This implies

$$\|A\|_{n,k_0} \le C \max_{m=1,\dots,n_0} \|A\|_{m,k_m}$$

and thus proves the assertion. □

Remark 10.13 That the sequence of integers in 10.12.(ii) can be chosen to be the identity is equivalent to the condition that 10.12.(ii) holds for each system of semi-norms on E.

Theorem 10.14 ([Vog83], 2.1) *Let $\beta := (\beta_j)_{j\in\mathbb{N}}$ be a shift-stable sequence. The following assertions are equivalent.*

(i) $L(\Lambda_1^1(\beta), F) = LB(\Lambda_1^1(\beta), F)$.

(ii) *F has the property (DN).*

Here follows (i) from (ii) without assumptions on β.

Proof. We apply 10.11(ii) on the sequence $(k_m := m)_{m\in\mathbb{N}}$ and receive a $k_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ there exists an $n_0 \in \mathbb{N}$ and a $C > 0$ with

$$\|x\|_n e^{-\rho_{k_0}\beta_j} \leq C \max_{m=1,\ldots,n_0} \|x\|_m e^{-\rho_m\beta_j} \tag{3}$$

for all $x \in F$ and for all $j \in \mathbb{N}$. Thereby $\lim_{m\in\mathbb{N}} \rho_m = 1$ monotonously, $\rho_m > 0$, $m \in \mathbb{N}$ denotes an arbitrary but fixed sequence (of radii). Therefore we have replaced $\Lambda_1^1(\beta)$ by the isomorphic space $\Lambda_e^1(\beta)$.

We can suppose that $n_0 \geq n, k_0$ and from (3) we get

$$\begin{aligned} \|x\|_n e^{-\rho_{k_0}\beta_j} &\leq C \max_{m=1,\ldots,n_0} \|x\|_m e^{-\rho_m\beta_j} \\ &\leq C \max_{m=k_0,\ldots,n_0} \|x\|_m e^{-\rho_m\beta_j} \\ &\leq C \max\left\{\|x\|_{k_0}, \|x\|_{n_0} e^{-\rho_{k_0+1}\beta_j}\right\}. \end{aligned}$$

For a fixed x we choose a $j \in \mathbb{N}$ with

$$\|x\|_n e^{-\rho_{k_0}\beta_{j+1}} \leq C\|x\|_{k_0} < \|x\|_n e^{-\rho_{k_0}\beta_j}$$

if there is such a j. Following, we get

$$\begin{aligned} \|x\|_n &\leq C\|x\|_{n_0} e^{(\rho_{k_0}-\rho_{k_0+1})\beta_j} \\ &\leq C\|x\|_{n_0} e^{\frac{\rho_{k_0+1}-\rho_{k_0}}{p\rho_{k_0}}(-\rho_{k_0}\beta_{j+1})} \\ &\leq C\|x\|_{n_0} \left(C\frac{\|x\|_{k_0}}{\|x\|_n}\right)^d, \end{aligned}$$

where $d = \frac{\rho_{k_0+1}-\rho_{k_0}}{p\rho_{k_0}}$. With $D = C^{1+d}$ we obtain

$$\|x\|_n^{1+d} \leq D\|x\|_{k_0}^d \|x\|_{n_0}. \tag{4}$$

If no such j exists, we get

$$\|x\|_n \leq Ce^{\rho_{k_0}\beta_1}\|x\|_{k_0}$$

and by increasing D (such that $D \geq C^d e^{d\rho_{k_0}\beta_1}$) (4) also holds. Hence

$$\|x\|_n \leq D\|x\|_n\|x\|_{k_0}^d \leq D\|x\|_{n_0}\|x\|_{k_0}^d$$

which is equivalent to property (DN) by A.5 (iv).

Conversely, to the given sequence $(k_m)_{m\in\mathbb{N}}$ we put $\sigma_m := \rho_{k_m}$ and by the property (DN) (here used in the form A.5 (iv)) we can choose $m_0 \in \mathbb{N}$ and $d \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ there exists an $n_0 \in \mathbb{N}$ with $m_0 \leq n_0$ and a $D > 0$ with

$$\|x\|_n^{1+d} \leq D\|x\|_{m_0}^d \|x\|_{n_0}.$$

Then we choose a $k_0 > k_{m_0}$ such that

$$\frac{1-\rho_{k_0}}{\rho_{k_0}-\sigma_{m_0}} < d.$$

For $x \in F$ and $j \in \mathbb{N}$ we get either

$$\|x\|_n e^{-\rho_{k_0}\beta_j} \leq \|x\|_{m_0} e^{-\sigma_{m_0}\beta_j}$$

or

$$\|x\|_{m_0} < e^{(\sigma_{m_0}-\rho_{k_0})\beta_j}\|x\|_n.$$

In the second case

$$\|x\|_n^{1+d} \leq De^{d(\sigma_{m_0}-\rho_{k_0})\beta_j}\|x\|_n^d\|x\|_{n_0}.$$

By hypothesis $d(\sigma_{m_0}-\rho_{k_0}) \leq \rho_{k_0}-1 \leq \rho_{k_0}-\rho_{k_{n_0}}$ and we get

$$\|x\|_n \leq D\|x\|_{n_0} e^{(\rho_{k_0}-\rho_{k_{n_0}})\beta_j}.$$

In any case we get

$$\|x\|_n e^{-\rho_{k_0}\beta_j} \leq D \max_{m=1,\ldots,n_0} \|x\|_m e^{-\rho_{k_m}\beta_j}$$

and by 10.11(ii) finish the proof. □

Chapter 11

The Property (LB^∞)

Definition 11.1 Let F be a Fréchet space equipped with a topology defined by an increasing system of semi-norms. We say that F has the *property* (LB^∞) if for every strict monotonous sequence $(\rho_n)_{n\in\mathbb{N}}$ with $\rho_n \underset{n\in\mathbb{N}}{\longrightarrow} \infty$ and for every p there exists a $q \geq p$ such that for all $n_0 \in \mathbb{N}$ there exists an $N_0 \geq n_0$ and a $C > 0$ such that for all $u \in F'$ there exists an m with $n_0 \leq m \leq N_0$ such that

$$\|u\|_{U_q}^{1+\rho_m} \leq C\|u\|_{U_m}\|u\|_{U_p}^{\rho_m}.$$

Theorem 11.2 ([Vog83], Satz 5.2) *Let E be a Fréchet space and $\alpha = (\alpha_j)_{j\in\mathbb{N}}$ a shift-stable sequence. The following assertions are equivalent.*

(i) $L(E, \Lambda_\infty^\infty(\alpha)) = LB(E, \Lambda_\infty^\infty(\alpha))$.

(ii) *E has the property (LB^∞).*

Here follows (i) from (ii) without prerequisite on α.

Proof. Let $\rho_m \underset{m\in\mathbb{N}}{\longrightarrow} \infty$ be a given monotonous sequence and without loss of generality let $p = 1$ in the property (LB^∞). We apply 10.12(ii) to the spaces E and Λ_∞^∞, the latter equipped with the norms $\|\xi\|_m = \sup_{j\in\mathbb{N}} |\xi_j| e^{\rho_m \alpha_j}$. By 10.12 we get a $k_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ exists an $n_0 \in \mathbb{N}$ and a $C > 0$ with

$$e^{\rho_n \alpha_j}\|y\|_{U_{k_0}} \leq C \max_{m=1,\dots,n_0} e^{\rho_m \alpha_j}\|y\|_{U_m} \tag{1}$$

for $y \in F'$. Contemplating on $n > k_0$, we pick a j_0 such that for $j \geq j_0$

$$Ce^{\rho_{n-1}\alpha_j} < e^{\rho_n \alpha_j} \tag{2}$$

holds. Hence we have

$$e^{\rho_n \alpha_j}\|y\|_{U_{k_0}} > Ce^{\rho_m \alpha_j}\|y\|_{U_m},$$

for $k_0 \leq m \leq n-1$ and therefore (1) transforms into

$$e^{\rho_n \alpha_j}\|y\|_{U_{k_0}} \leq C \max_{m=1,\dots,k_0-1,n,\dots,n_0} e^{\rho_m \alpha_j}\|y\|_{U_m}. \tag{3}$$

Either we can choose a $j \geq j_0$ (in order to exploit (2)) with

$$e^{(\rho_n - \rho_{k_0-1})\alpha_{j-1}}\|y\|_{U_{k_0}} \leq C\|y\|_{U_1} \tag{4}$$

and

$$C\|y\|_{U_1} \le e^{(\rho_n-\rho_{k_0-1})\alpha_j}\|y\|_{U_{k_0}} \tag{5}$$

which holds since $e^{\rho_{k_0-1}-1}\alpha_j \ge e^{\rho_m\alpha_j}$ for $m < k_0$ and $\|y\|_{U_1} \ge \|y\|_{U_m}$ for $m \ge 1$. We suppose that the maximum is assumed on the right hand side of (3) at m. Then by (5) $m \ge n$ and we get

$$\begin{aligned}\|y\|_{U_{k_0}} &\le Ce^{(\rho_m-\rho_n)\alpha_j}\|y\|_{U_m}\\ &\le Ce^{s\frac{\rho_m-\rho_n}{\rho_n-\rho_{k_0-1}}(\rho_n-\rho_{k_0-1})\alpha_{j-1}}\|y\|_{U_m}\\ &\le C\|y\|_{U_m}\left(C\frac{\|y\|_{U_1}}{\|y\|_{U_{k_0}}}\right)^d, \end{aligned}\tag{6}$$

where $d := s\frac{\rho_m-\rho_n}{\rho_n-\rho_{k_0-1}}$ with $s := \sup_{j\to\infty}\frac{\alpha_{j+1}}{\alpha_j}$ and (6) follows from (4). This can be written as

$$\|y\|_{U_{k_0}}^{1+d} \le D\|y\|_{U_m}\|y\|_{U_1}^d,$$

where $D := C^{d+1}$ and $d \le \left(\frac{s}{\rho_n-\rho_{k_0-1}}\right)\rho_m$. We have $m \in \{n,\dots,N_0\}$, since from $m < k_0$ follows

$$Ce^{(\rho_m-\rho_n)\alpha_j}\|y\|_{U_m} \le Ce^{-(\rho_n-\rho_{k_0-1})\alpha_j}\|y\|_{U_1} < \|y\|_{U_1}^d$$

which is a contradiction.

Otherwise we have

$$e^{(\rho_n-\rho_{k_0-1})\alpha_{j_0}}\|y\|_{U_{k_0}} \le C\|y\|_{U_1}$$

which leads to

$$\begin{aligned}\|y\|_{U_{k_0}}^{1+d} &\le \|y\|_{U_{k_0}}\|y\|_{U_{k_0}}^d\\ &\le \|y\|_{U_{k_0}}\left(Ce^{(\rho_{k_0-1}-\rho_n)\alpha_{j_0}}\right)^d\|y\|_{U_1}^d\\ &= D\|y\|_{U_{k_0}}\|y\|_{U_1}^d.\end{aligned}$$

In both cases we have $\exists k_0\ \forall n \in \mathbb{N}\ \exists N_0, C > 0\ \exists m$ with $n \le m \le N_0$ such that

$$\|.\|_{U_{k_0}}^{1+\rho_m} \le C\|.\|_{U_m}\|.\|_{U_1}^{\rho_m},$$

i.e. property (LB^∞).

Conversely, note that the property (LB^∞) does not depend on a special system of semi-norms. We therefore can, by 10.13, assume without loss of generality $(k_m := m)_{m\in\mathbb{N}}$ in 10.12(ii). Let $\Lambda_\infty^\infty(\alpha)$ be endowed with the norms

$$\|\xi\|_n := \sup_{j\in\mathbb{N}}|\xi_j|e^{n\alpha_j}.$$

We choose a sequence $(\rho_m)_{m\in\mathbb{N}}$ with $\lim_{m\to\infty}\frac{\rho_m}{m} = 0$ and insert it and $p = 1$ into (LB^∞). We receive a q and for each given n_c an n_{c_0} and aC with the desired property.

If, for the verification of 10.12(ii), there is a given n_c, we choose n_{c_0} such that $\rho_m(n-1) \leq m-n$ for $m \geq n_{c_0}$. For $y \in E'$ there exists, by property (LB^∞), an $m \in \mathbb{N}$ with $n_{c_0} \leq m \leq n_0$ such that

$$\|y\|_{U_{k_0}}^{1+\rho_m} \leq C\|y\|_{U_m}\|y\|_{U_1}^{\rho_m}.$$

Then either

$$e^{n\alpha_j}\|y\|_{U_{k_0}} \leq e^{\alpha_j}\|y\|_{U_1}$$

or

$$\begin{aligned}\|y\|_{U_{k_0}}^{1+\rho_m} &\leq C\|y\|_{U_m}\|y\|_{U_1}^{\rho_m}\\ &\leq C\|y\|_{U_m}\|y\|_{U_{k_0}}^{\rho_m}e^{\rho_m(n-1)\alpha_j}\\ &\leq C\|y\|_{U_m}\|y\|_{U_{k_0}}^{\rho_m}e^{(m-n)\alpha_j}\end{aligned}$$

holds. In both cases

$$e^{n\alpha_j}\|y\|_{U_{k_0}} \leq C \max_{m=1,\ldots,n_0} e^{m\alpha_j}\|y\|_{U_m}.$$

This finishes the proof. □

Definition 11.3 A smooth function h on an open set $U \subseteq \mathbb{C}$ is called *harmonic* if $\Delta h = \frac{4\partial^2 h}{\partial z \partial \bar{z}} = 0$ on U.

A function u defined on an open set $U \subseteq \mathbb{C}$ and with values in $\mathbb{R} \cup \{-\infty\}$ is called *upper-semi-continuous* if the set $\{z \in \mathbb{C} : u(z) < s\}$ is open for every $s \in \mathbb{R}$.

An upper-semi-continuous function $u : U \to \mathbb{R} \cup \{-\infty\}$ is called *sub-harmonic* if for every compact $K \subset U$ and every continuous function h on K, which is harmonic in the interior of K and with $h \geq u$ on the boundary ∂K of K, we have $u \leq h$ in K.

Definition 11.4 An upper-semi-continuous function $\varphi : E \to \mathbb{R}$ is called *pluri-sub-harmonic* if φ is sub-harmonic on every complex line in E.

Definition 11.5 A subset $B \subseteq E$ is said to be *pluripolar* if there exists a pluri-sub-harmonic function φ on E such that $\varphi \neq -\infty$ and $\varphi|_B = -\infty$.

Theorem 11.6 *Let D be a domain in $\mathbb{C}^n$, E a compact non-pluripolar subset of D and F a compact non-pluripolar subset in $\mathbb{C}^m$. Then every separately analytic function f defined on $(D \times F) \cup (E \times \mathbb{C}^m)$ has an analytic extension on $D \times \mathbb{C}^m$.*

Reference. The proof can be found in [VZ83], Théorème 4.1.

Definition 11.7 Let F be a Fréchet space with a topology defined by an increasing system of semi-norms. We say that F has *property* $(\tilde{\Omega})$ if for every p there exists a q such that for all k exists a $C > 0$ and we have

$$\|\cdot\|_{U_q}^2 \leq C\|\cdot\|_{U_k}\|\cdot\|_{U_p}.$$

We say that F has *property* $(\tilde{\Omega}_B)$ if for every p exist q and $C > 0$ such that

$$\|\cdot\|_{U_q}^2 \leq C\|\cdot\|_B\|\cdot\|_{U_p}$$

where $\|u\|_B := \sup\{|u(x)| : x \in B\}$ for $u \in F'$.

Note that in the definition of $(\tilde{\Omega}_B)$, by choosing q sufficiently large, we may assume that $C = 1$.

Definition 11.8 Let K be a compact subset in a complex Fréchet space E. We say that K is a *set of uniqueness* if for all $f \in H(K)$ with $f\mid_K = 0$ follows that $f = 0$.

Lemma 11.9 ([Lan00], Lemma 2.2) *Let E be a nuclear Fréchet space and B a balanced convex compact subset in E. Suppose that E has property $(\tilde{\Omega}_B)$. Then B is a set of uniqueness.*

Proof. Let E have property $(\tilde{\Omega}_B)$. Hence, for every $x' \in E'$ with $x'\mid_{\text{span}(B)} = 0$ we have $\forall p\ \exists q, C > 0$ such that

$$0 \le \|x'\|_{U_q}^2 \le C\|x'\|_B\|x'\|_{U_p} = 0.$$

Which means $\|x'\|_{U_q} = 0$ and since x' is linear and $\{x : q(x) < 1\}$ is absorbent, we get $x' = 0$. From the theorem of Hahn-Banach C.2 we conclude that span(B) is dense in E.

Now given $f \in H(B)$ with $f\mid_B = 0$, consider the Taylor expansion of f at $0 \in B$ in a balanced convex neighbourhood W of B in E

$$f(x) = \sum_{n\ge 0} P_n f(x)$$

for $x \in W$, where

$$P_n f(x) = \frac{1}{2\pi\,\mathrm{i}} \int_{|\lambda|=\delta_x>0} \frac{f(\lambda x)}{\lambda^{n+1}} d\lambda$$

for $x \in E$.

Since $P_n f$ are homogeneous polynomials of degree n and $P_n f\mid_B = 0$, it follows that $P_n f\mid_{\text{span}(B)} = 0$. By the continuity of $P_n f$ and by $\overline{\text{span}(B)} = E$, we have $P_n f = 0$ for $n \ge 0$. Thus $f = 0$ in W and hence B is a set of uniqueness. □

Theorem 11.10 ([Lan00], Theorem 2.1) *Let E be a nuclear Fréchet space and B a balanced convex compact subset in E. Assume that E has $(\tilde{\Omega}_B)$. Then $H(B)'_\beta$ has property (LB^∞).*

Proof. By 11.2 it suffices to show that every continuous linear map $T : H(B)' \to H(\mathbb{C})$ is compact.

Consider the function $f : B \to H(\mathbb{C})$ defined by

$$f(x)(\lambda) = T(\delta_x)(\lambda)$$

for $x \in B, \lambda \in \mathbb{C}$, where $\delta_x \in H(B)'_\beta$ is the *Dirac functional* associated to x which is given by

$$\delta_x(\varphi) = \varphi(x),$$

with $\varphi \in H(B)$. It follows that f is weakly holomorphic, because $T'(\mu) \in H(B)'' \cong H(B)$. By Grothendieck's factorisation theorem C.5, this yields that $f : B \to H^\infty(2\mathbb{D})$, where $\mathbb{D}$ is the open unit disc in $\mathbb{C}$, is extended to a holomorphic function $\hat{f}$ on a neighbourhood W of B in E.

Let $g : (B \times \mathbb{C}) \cup (W \times \overline{\mathbb{D}}) \to \mathbb{C}$ given by

$$g(x,\lambda) = \begin{cases} f(x)(\lambda) & : \quad x \in B,\ \lambda \in \mathbb{C} \\ \hat{f}(x)(\lambda) & : \quad x \in W,\ \lambda \in \overline{\mathbb{D}} \end{cases}.$$

Obviously, g is separately holomorphic. We denote by $\mathcal{F}$ the family of all non-empty finite dimensional subspaces P of E_B. For each $P \in \mathcal{F}$ consider

$$g_P := g\,|_{((B\cap P)\times\mathbb{C})\cup((W\cap P)\times\overline{\mathbb{D}})}\,.$$

Since $B\cap P$ is the unit ball in P and $\overline{\mathbb{D}}$ is not polar, by 11.6, g_P is uniquely extended to a holomorphic $\tilde{g}_P$ on $(W\cap P)\times\mathbb{C}$. The uniqueness implies that the family $\{\tilde{g}_P\}_{P\in\mathcal{F}}$ defines a Gâteaux holomorphic function $\tilde{g}$ on $(W\cap E_B)\times\mathbb{C}$. On the other hand, since $\tilde{g}$ is holomorphic on $(W\cap E_B)\times\mathbb{D}$, 11.6 implies that $\tilde{g}$ is holomorphic on $(W\cap E_B)\times\mathbb{C}$. Consider the holomorphic function $\hat{g} : (W\cap E_B)\to H(\mathbb{C})$ associated to $\tilde{g}$. We prove that $\hat{g}$ can be extended to a bounded holomorphic function on a neighbourhood of B with values in $H(\mathbb{C})$.

Let $\{\|.\|_\gamma\}_{\gamma\in\mathbb{N}}$ and $\{\|.\|_k\}_{k\in\mathbb{N}}$ be two fundamental systems of semi-norms of E and $H(\mathbb{C})$ respectively. Since $H(\mathbb{C})$ has (DN) we have $\exists p\ \forall q, d>0\ \exists k, C>0$ such that $\|.\|_q^{1+d}\le\|.\|_k\|.\|_p^d$.

Note that by replacing k by some $k'>k$, we always may assume that $C=1$. Choose α such that $U_\alpha\subseteq W$ and

$$M(\alpha,p) := \sup\{\|\hat{g}(x)\|_p : x\in U_\alpha\cap E_B\}<\infty.$$

Let ω_α be the canonical map from E into E_α, the Banach space associated to $\|.\|_\alpha$ and

$$A := \omega_\alpha\,|_{E_B} : E_B\to E_\alpha.$$

Since E is nuclear, without loss of generality we may assume that E_B and E_α are Hilbert spaces. Then, by 5.18 and 5.16, A can be written in the form

$$A(x) = \sum_{j\ge 1}\lambda_j y_j(x) z_j$$

where $\lambda := (\lambda_j)_{j\in\mathbb{N}}$ is a rapidly decreasing sequence with $\lambda_j>0$ for $j\ge 1$, $(y_j)_{j\in\mathbb{N}}$ is a complete orthonormal system in $(E_B)^*$, and $(z_j)_{j\in\mathbb{N}}$ an orthonormal system in E_α.

Since

$$A\left(\frac{y_j}{\lambda_j}\right) = z_j\in\omega_\alpha(U_\alpha)$$

for all $j\ge 1$, we have

$$\frac{y_j}{\lambda_j}\in U_\alpha$$

for all $j\ge 1$.

It follows that

$$\sum_{j=1}^{m}\frac{\mu_j}{\lambda_j}y_j\in U_\alpha$$

for all $m\ge 1$, where $\mu_j=\frac{\delta}{j^k}$ and $\delta>0$ are chosen such that

$$\left\{u\in E_\alpha : u=\sum_{j=1}^{\infty}\xi_j z_j,\ |\xi_j|<\mu_j\ \forall j\ge 1\right\}\subseteq\omega_\alpha(U_\alpha)$$

and

$$\delta \sum_{j\geq 1}^{\infty} \frac{1}{j^k} \leq 1.$$

We put $\langle z_k, z\rangle_\alpha$ as the scalar product in E_α. Then $\sqrt{\langle z_k, z\rangle_\alpha} = 1$ for all $k \geq 1$ and

$$\begin{aligned} \|A^* \langle z_k, z\rangle_\alpha \|_B &= \sup_{\|x\|\leq 1} | \langle z_k, A(x)\rangle_\alpha | \\ &= \sup_{\|x\|\leq 1} \left| \left\langle z_k, \sum_j \lambda_j y_j(x) z_j \right\rangle_\alpha \right| \\ &= \sup_{\|x\|\leq 1} | \langle z_k, \lambda_k y_k(x) z_k\rangle_\alpha | \\ &= \lambda_k \end{aligned} \tag{7}$$

for all $k \geq 1$. Recall that by the Bessel inequality $|y_k(x)| \leq \|x\|$. Now put

$$\varphi_k = \omega_\alpha^* \langle z_k, z\rangle_\alpha, \tag{8}$$

and choose β such that

$$\exists C > 0 \ \|.\|_{U_\beta}^2 \leq C\|.\|_B\|.\|_{U_\alpha}. \tag{9}$$

For β sufficiently large, we can choose $C = 1$.

From (7), (8) and (9) we have

$$\|\varphi_k\|_{U_\beta}^2 = \|\omega_\alpha^* \langle z_k, z\rangle_\alpha \|_{U_\beta}^2 \leq \|A^* \langle z_k, z\rangle_\alpha \|_B \| \langle z_k, z\rangle_\alpha \|_{U_\alpha} \leq \lambda_k$$

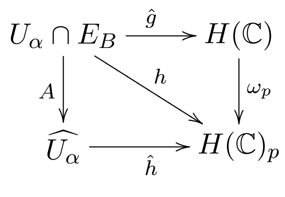
for all $k \geq 1$. Hence for all $k \geq 1$ we get

$$\|\varphi_k\|_{U_\beta} \leq \lambda_k^{\frac{1}{2}}.$$

Let

$$h := \omega_p \hat{g}.$$

Since $M(\alpha, p) < \infty$ and $A(U_\alpha \cap E_B)$ is dense in $\omega_\alpha(U_\alpha)$, h is holomorphically factorized through $A : U_\alpha \cap E_B \to \widehat{U_\alpha}$ by $\hat{h} : \widehat{U_\alpha} \to H(\mathbb{C})_p$, where $\widehat{U_\alpha}$ denotes the unit ball in E_α. This may be illustrated by the following diagram.



For each $m = (m_1, \ldots, m_n, 0, \ldots) \in M$, with

$$M := \{m = (m_j)_{j\in\mathbb{N}} : m_j \neq 0 \text{ only for finitely many } j \in \mathbb{N}\},$$

we put

$$a_m := \left(\frac{1}{2\pi\,\mathrm{i}}\right)^n \int_{|\rho_1|=\mu_1} \int_{|\rho_2|=\mu_2} \cdots \int_{|\rho_n|=\mu_n} \frac{\hat{h}(\rho_1 z_1 + \cdots + \rho_n z_n)}{\rho^{m+1}} d\rho$$

where

$$\begin{array}{rcl} \rho^{m+1} & := & \rho_1^{m_1+1}\rho_2^{m_2+1}\dots\rho_n^{m_n+1}, \\ d\rho & := & d\rho_n d\rho_{n-1}\dots d\rho_1, \end{array}$$

then

$$\|a_m\| \leq \frac{M(\alpha,p)}{\mu^m} \ \forall m \in M.$$

From the relation

$$\sum_{j=1}^{k} \frac{\rho_j}{\lambda_j} y_j \in U_\alpha \cap E_B \ \forall k \geq 1,$$

we deduce that

$$\hat{h}\left(\sum_{j\geq 1} \rho_j z_j\right) = \hat{h}A\left(\sum_{j=1}^{k} \frac{\rho_j}{\lambda_j} y_j\right) = \omega_p \hat{g}\left(\frac{\rho_j}{\lambda_j} y_j\right).$$

On the other hand, by Cauchy's theorem 4.9, we get

$$a_m = \left(\frac{1}{2\pi\,\mathrm{i}}\right)^n \int_{|\rho_1|=\lambda_1\mu_1} \int_{|\rho_2|=\lambda_2\mu_2} \cdots \int_{|\rho_n|=\lambda_n\mu_n} \frac{\hat{h}(\rho_1 z_1 + \cdots + \rho_n z_n)}{\rho^{m+1}} d\rho.$$

It follows that

$$\begin{array}{rcl} a_m & = & \left(\frac{1}{2\pi\,\mathrm{i}}\right)^n \int_{|\rho_1|=\lambda_1\mu_1} \int_{|\rho_2|=\lambda_2\mu_2} \cdots \int_{|\rho_n|=\lambda_n\mu_n} \frac{\omega_p \hat{g}\left(\frac{\rho_j}{\lambda_j} y_j\right)}{\lambda^{m+1}\left(\frac{\rho}{\lambda}\right)^{m+1}} d\rho \\ & = & \omega_p \left(\frac{1}{\lambda^m}\left(\frac{1}{2\pi\,\mathrm{i}}\right)^n \int_{|\theta_1|=\mu_1} \cdots \int_{|\theta_n|=\mu_n} \frac{\hat{g}(\theta_1 y_1 + \cdots + \theta_n y_n)}{\theta m + 1} d\theta\right) \\ & =: & \omega_p(b_m) \end{array}$$

where

$$\theta_j := \frac{\rho_j}{\lambda_j} \ \forall j \geq 1.$$

We have

$$\|b_m\|_q \leq \frac{N(q)}{\lambda^m \mu^m} \ \forall m \in M, \ \forall q \geq p,$$

where

$$N(q) := \sup\left\{ \|\hat{h}(x)\|_q : x = \sum_{j=1}^{\infty} \xi_j y_j, \ |\xi_j| \leq \mu_j \ \forall j \geq 1 \right\} < \infty,$$

because the set

$$\left\{ x = \sum_{j=1}^{\infty} \xi_j y_j : |\xi_j| \leq \mu_j \ \forall j \geq 1 \right\}$$

is compact in E_B.

Since $H(\mathbb{C})$ has the property (DN), for every $q \geq p$ and $\bar{d} := \frac{d}{\delta}$ there exists a $k \geq q$ and a $C > 0$ such that

$$\|.\|_q^{1+\bar{d}} \leq C\|.\|_k\|.\|_p^{\bar{d}},$$

where $0 < \delta < 1$ is chosen such that

$$\varepsilon := \frac{d(1-2\delta)}{2(1+d)(\delta+d)} > 0.$$

We may assume $C = 1$ again. Then

$$\begin{aligned}
S &:= \sum_{m\in M} r^m \|b_m\|_q \prod_{j=1}^{\infty} \|\varphi_j\|_{U_\beta}^{m_j} \\
&\leq \sum_{m\in M} r^m \|b_m\|_q \prod_{j=1}^{\infty} (\lambda_j)^{\frac{m_j}{1+d}} \\
&= \sum_{m\in M} r^m \|b_m\|_q \lambda^{2tm} \\
&= \sum_{m\in M} r^m \left(\lambda^m \|b_m\|_q\right)^t \lambda^{tm} \|b_m\|_q^{1-t} \\
&\leq N(q)^t N(k)^{\frac{1-t}{1+d}} M(\alpha,p)^{\frac{(1-t)\bar{d}}{1+d}} \sum_{m\in M} r^m \frac{\lambda^{m\left(t-\frac{1-t}{1+d}\right)}}{\mu^{m\left(t+\frac{1-t}{1+d}+\frac{(1-t)\bar{d}}{1+d}\right)}} \\
&\leq N(q)^t N(k)^{\frac{1-t}{1+d}} M(\alpha,p)^{\frac{(1-t)\bar{d}}{1+d}} \sum_{m\in M} r^m \frac{\lambda^{m\left(t-\frac{1-t}{1+d}\right)}}{\mu^m}.
\end{aligned}$$

Since $\lambda \in s$, the sequence $\left(\frac{\lambda_j^\varepsilon}{\mu_j}\right)$ is in ℓ^1 and hence for $R := \sum_{j\geq 1}\left(\frac{\lambda_j^\varepsilon}{\mu_j}\right)$ we have

$$2R > R > \frac{\lambda_j^\varepsilon}{\mu_j}$$

for $j \geq 1$. This implies

$$0 < \sup\left\{\frac{\lambda_j^\varepsilon}{2R\mu_j}\right\} < \frac{1}{2}.$$

We have

$$\begin{aligned}
S &= \sum_{m\in M} r^m \|b_m\|_q \prod_{j=1}^{\infty} \|\varphi_j\|_{U_\beta}^{m_j} \\
&\leq N(q)^t N(k)^{\frac{1-t}{1+d}} M(\alpha,p)^{\frac{(1-t)\bar{d}}{1+d}} \sum_{m\in M} \left(\frac{r\lambda^\varepsilon}{\mu}\right) \\
&= N(q)^t N(k)^{\frac{1-t}{1+d}} M(\alpha,p)^{\frac{(1-t)\bar{d}}{1+d}} \prod_{j=1}^{\infty} \frac{1}{1-\frac{r\lambda_j^\varepsilon}{\mu_j}} \\
&< \infty.
\end{aligned}$$

Hence the form

$$x \longmapsto \sum_{m\in M} b_m \prod_{j\geq 1} (\varphi_j(x))^{m_j}$$

defines a bounded holomorphic function $\widehat{h_1}$ on δU_β with $\delta = \frac{1}{4R}$ such that $\widehat{h_1} \mid_{\delta U_\beta \cap B} = \hat{g} \mid_{\delta U_\beta \cap B}$, i.e. $\widehat{h_1}(z)(\lambda) = g(z, \lambda)$ for $z \in \delta U_\beta \cap B$ and $\lambda \in \overline{\mathbb{D}}$. Since $\overline{\text{span}(B)} = E$, by considering the Taylor expansion of $\widehat{h_1}(.)(\lambda) - g(., \lambda)$ in $z \in \text{span}(B)$ at $0 \in B$, we get $\widehat{h_1}(z)(\lambda) = g(z, \lambda)$ for $z \in \delta U_\beta \cap B$ and $\lambda \in \overline{\mathbb{D}}$.

Consider the separately holomorphic function h_1 on $(\delta U_\beta \times \mathbb{C}) \cup (W \times \overline{\mathbb{D}})$, induced by $\widehat{h_1}$ and g. As we have seen at the beginning of the proof, h_1 is holomorphically extended to a function $\overline{h_1}$ on $W \times \mathbb{C}$. Let $\widehat{\overline{h_1}} : W \to H(\mathbb{C})$ denote the holomorphic function associated to $\overline{h_1}$. Since B is convex, balanced and the equality $(\widehat{h_1} - \hat{g}) \mid_{\delta U_\beta \cap B} = 0$ holds, from the Taylor expansion of $(\widehat{h_1} - \hat{g}) \mid_B$ at $0 \in B$ it follows that $\widehat{\overline{h_1}} \mid_B = \hat{g} \mid_B$.

$\widehat{\overline{h_1}}$ is locally bounded. Thus, by shrinking W, without loss of generality, we may assume that $\widehat{\overline{h_1}}(W)$ is bounded. Define the continuous linear map $S : H^\infty(W)' \to H(\mathbb{C})$ as

$$S(\mu)(\lambda) = \mu(\widehat{\overline{h_1}}(.)(\lambda))$$

for $\mu \in H^\infty(W)'$ and $\lambda \in \mathbb{C}$. We have

$$\begin{aligned} T\left(\sum_{j=1}^m \alpha_j \delta_{x_j}\right)(\lambda) &= \sum_{j=1}^m \alpha_j T(\delta_{x_j})(\lambda) = \sum_{j=1}^m \alpha_j f(x_j)(\lambda) \\ &= \sum_{j=1}^m \alpha_j \hat{g}(x_j)(\lambda) = \sum_{j=1}^m \alpha_j \widehat{\overline{h_1}}(x_j)(\lambda) \\ &= \sum_{j=1}^m \alpha_j S(\delta_{x_j})(\lambda) = S\left(\sum_{j=1}^m \alpha_j \delta_{x_j}\right) \end{aligned}$$

for $x_1, x_2, \ldots, x_m \in B$ and $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{C}$.

On the other hand, since B is a set of uniqueness and $H(B)$ is reflexive, it follows that $S = T$. Hence T is compact. $\square$

Chapter 12

The Property (LB_∞)

Definition 12.1 Let F be a Fréchet space with the topology defined by an increasing system of semi-norms.

We say that F has *property* (LB_∞) if for every sequence $(\rho_n)_{n\in\mathbb{N}}$ with $\rho_n > 0$ for all $n \in \mathbb{N}$ and $\rho_n \underset{n\in\mathbb{N}}{\longrightarrow} \infty$ there exists a $p \in \mathbb{N}$ such that for all $q \in \mathbb{N}$ there exist $n_0 \in \mathbb{N}$ and $C > 0$ such that for all $x \in F$ there exists an m with $q \leq m \leq n_0$ and that

$$\|x\|_q^{1+\rho_m} \leq C\|x\|_m\|x\|_p^{\rho_m}$$

holds.

Lemma 12.2 *A Fréchet space F has the property (LB_∞) if and only if for every sequence $(\rho_n)_{n\in\mathbb{N}}$ with $\rho_n > 0$ for $n \in \mathbb{N}$ and $\rho_n \underset{n\in\mathbb{N}}{\longrightarrow} \infty$ there exists a $p \in \mathbb{N}$ such that for all $q \in \mathbb{N}$ there exists an $n_0 \in \mathbb{N}$ with $n_0 \geq q$ and a $C > 0$ such that we have*

$$\|x\|_q \leq C \max_{q\leq m\leq n_0} \left(r_m^{\rho_m}\|x\|_m + \frac{1}{r_m}\|x\|_p \right)$$

for $x \in F$, $r_q > 0, \cdots, r_{n_0} > 0$.

Proof. We give an indirect proof. Therefore assume that F has property (LB_∞) but for all $q \in \mathbb{N}$ there exists an $n_0 \in \mathbb{N}$ with $n_0 \geq q$ and a $C > 0$ such that we have

$$\|x\|_q > C \max_{q\leq m\leq n_0} \left(r_m^{\rho_m}\|x\|_m + \frac{1}{r_m}\|x\|_p \right)$$

for $x \in F$, $r_q > 0, \cdots, r_{n_0} > 0$. It follows that for all m with $q \leq m \leq n_0$ we have

$$\begin{aligned} \|x\|_q &> C\left(r_m^{\rho_m}\|x\|_m + \frac{1}{r_m}\|x\|_p \right) \\ &\geq C\frac{1+\rho_m}{\rho_m}(\rho_m\|x\|_m)^{\frac{1}{1+\rho_m}}\|x\|_p^{\frac{\rho_m}{1+\rho_m}} \end{aligned}$$

which is the minimum of the right hand side with respect to r_m. This implies

$$\|x\|_q^{1+\rho_m} > D\|x\|_m\|x\|_p^{\rho_m}$$

which leads to the desired contradiction.

Conversely, assume that for all m with $q \le m \le n_0$ we have

$$\|x\|_q^{1+\rho_m} > C\|x\|_m\|x\|_p^{\rho_m}.$$

Or in other words

$$\|x\|_q > C^{\frac{1}{1+\rho_m}}\|x\|_m^{\frac{1}{1+\rho_m}}\|x\|_p^{\frac{\rho_m}{1+\rho_m}}.$$

Then the right hand side from above can be seen as the minimum of

$$r_m^{\rho_m}\|x\|_m + \frac{1}{r_m}\|x\|_p$$

with respect to r_m. Finally, we get

$$\|x\|_q \ge C \max_{q\le m\le n_0}\left(r_m^{\rho_m}\|x\|_m + \frac{1}{r_m}\|x\|_p\right)$$

which concludes in the contradiction. □

Proposition 12.3 *Let F be a Fréchet space with property (LB_∞). Then $(F'_{\text{bor}})'_\beta$ also has property (LB_∞).*

Proof. By 12.2 we have that a Fréchet space F has the property (LB_∞) if and only if for every sequence $(\rho_n)_{n\in\mathbb{N}}$ with $\rho_n > 0$ for $n \in \mathbb{N}$ and $\rho_n \underset{n\in\mathbb{N}}{\longrightarrow} \infty$ there exists a $p \in \mathbb{N}$ such that for all $q \in \mathbb{N}$ there exists an $n_0 \in \mathbb{N}$ with $n_0 \ge q$ and a $C > 0$ such that we have

$$U_q \supseteq C \bigcap_{q\le m\le n_0}\left(r_m^{\rho_m}U_m \cap \frac{1}{r_m}U_p\right)$$

for $r_q > 0, \cdots, r_{n_0} > 0$. Applying the bi-polar theorem we get

$$U_q^\circ \subseteq D\left\langle \bigcup_{q\le m\le n_0}\left(r_m^{\rho_m}U_m^\circ + \frac{1}{r_m}U_p^\circ\right)\right\rangle$$

for appropriate $D > 0$. Here the angle brackets denote the absolutely convex hull of the inner expression.

Let $u \in (F'_{\text{bor}})'_\beta$. Then

$$\begin{aligned}
\|u\|_{U_q^\circ} &= \sup\left\{|u(y)| : y \in U_q^\circ\right\} \\
&\le \sup\left\{|u(y)| : y \in D\left\langle \bigcup_{q\le m\le n_0}\left(r_m^{\rho_m}U_m^\circ + \frac{1}{r_m}U_p^\circ\right)\right\rangle\right\} \\
&\le D\max_{q\le m\le n_0}\left\{r_m^{\rho_m}\sup_{y\in U_m^\circ}|u(y)| + \frac{1}{r_m}\sup_{y\in U_p^\circ}|u(y)|\right\} \\
&= D\max_{q\le m\le n_0}\left\{r_m^{\rho_m}\|u\|_{U_m^\circ} + \frac{1}{r_m}\|u\|_{U_p^\circ}\right\}.
\end{aligned}$$

Hence $(F'_{\text{bor}})'_\beta$ has property (LB_∞). □

Theorem 12.4 ([Vog83], 3.2 Satz) *Let $(\beta_j)_{j\in\mathbb{N}}$ be a shift-stable sequence and F a Fréchet space. The following assertions are equivalent.*

(i) $L(\Lambda^1_\infty(\beta), F) = LB(\Lambda^1_\infty(\beta), F)$.

(ii) F *has property* (LB_∞).

Here follows (i) from (ii) without prerequisite on β.

Proof. Let $(\rho_m)_{m\in\mathbb{N}}$ be a given monotonous sequence. Then we can choose a strictly monotonous sequence $(\sigma_m)_{m\in\mathbb{N}}$, $\sigma_m \underset{m\in\mathbb{N}}{\longrightarrow} \infty$ with

$$\lim_{m\to\infty} \frac{\rho_m}{\sigma_m} = 0.$$

Let further $\Lambda^1_\infty(\beta)$ be equipped with the semi-norms

$$\|\xi\|_k = \sum_{j\in\mathbb{N}} |\xi_j| e^{\sigma_k \beta_j}.$$

We apply 10.12(ii) on the sequence $(k_m := m)_{m\in\mathbb{N}}$ and get a $k_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ exists an $n_0 \in \mathbb{N}$ and a $C > 0$ with

$$\|x\|_n e^{-\sigma_{k_0}\beta_j} \leq C \max_{m=1,\dots,n_0} \|x\|_m e^{-\sigma_m \beta_j} \tag{1}$$

for all $x \in F$ and $j \in \mathbb{N}$. Henceforward we contemplate only on $n > k_0$. We pick a j_0, such that for $j \geq j_0$ we have

$$Ce^{(\sigma_{k_0} - \sigma_{k_0+1})\beta_j} < 1.$$

Resulting, (1) transforms into

$$\|x\|_n e^{-\sigma_{k_0}\beta_j} \leq C \max_{m=1,\dots,k_0,n+1,\dots,n_0} \|x\|_m e^{-\sigma_m \beta_j}.$$

With fixed x we choose a $j \geq j_0$ with

$$\|x\|_n e^{-\sigma_{k_0}\beta_{j+1}} \leq C\|x\|_{k_0} < \|x\|_n e^{-\sigma_{k_0}\beta_j}$$

if there exists such a j. In this case the maximum is taken at $n < m \leq n_0$ and we get

$$\begin{aligned} \|x\|_n &\leq C\|x\|_m e^{(\sigma_{k_0} - \sigma_m)\beta_j} \\ &\leq C\|x\|_m e^{\frac{\sigma_m - \sigma_{k_0}}{p\sigma_{k_0}}(-\sigma_{k_0}\beta_{j+1})} \\ &\leq C\|x\|_m \left(C\frac{\|x\|_{k_0}}{\|x\|_n} \right)^d, \end{aligned}$$

i.e.

$$\|x\|_n^{1+d} \leq D\|x\|_{k_0}^d \|x\|_m, \tag{2}$$

where $d := \frac{\sigma_m - \sigma_{k_0}}{p\sigma_{k_0}}$ and $D := C^{1+d}$.

If there is no such j, we have

$$\|x\|_n \leq Ce^{\sigma_{k_0}\beta_{j_0}} \|x\|_{k_0}$$

and hence (2) with an eventually bigger D.

For a fixed n', we apply the above to $n \geq n'$ such that for $m \geq n$ we have

$$d = \frac{\sigma_m - \sigma_{k_0}}{p\sigma_{k_0}} \geq \rho_m$$

and we get an m with $n' \leq n \leq m \leq n_0$ such that

$$\|x\|_{n'}^{1+\rho_m} \leq \|x\|_n^{1+\rho_m} \leq D\|x\|_{k_0}^{\rho_m}\|x\|_m.$$

Conversely, note that the property (LB_∞) does not depend on a special system of semi-norms. We therefore can assume $(k_m := m)_{m\in\mathbb{N}}$ in 10.12(ii) without loss of generality. Let $\Lambda_\infty^\infty(\alpha)$ be endowed with the norms $\|\xi\|_k := \sum_{j\in\mathbb{N}} |\xi_j| e^{k\beta_j}$.

We choose a sequence $(\rho_m := m)_{m\in\mathbb{N}}$ and insert this into the hypothesis of 10.12(ii). Hence, we receive a k with the desired properties. If we put $k_0 = k + 1$ and find to any fixed n a desired $n_0 \geq k$ and a D. Let $x \in F$. Then either

$$\|x\|_n e^{-k_0\beta_j} \leq \|x\|_k e^{-k\beta_j}$$

or, for suitable $n \leq m \leq n_0$,

$$\|x\|_n^{1+m} \leq D\|x\|_k^m \|x\|_m \leq De^{-m\beta_j}\|x\|_n^m \|x\|_m$$

holds. In the second case we have

$$\|x\|_n e^{-k_0\beta_j} \leq \|x\|_n \leq D\|x\|_m e^{-m\beta_j}.$$

In any case we have

$$\|x\|_n e^{-k_0\beta_j} \leq D \max_{m=1,\ldots,n_0} \|x\|_m e^{-m\beta_j}.$$

This proves the conversion. □

Corollary 12.5 *Let I be a fixed index set and F a Fréchet space. Then the following assertions are equivalent.*

(i) $L(\ell^1(I)\hat{\otimes}s, F) = LB(\ell^1(I)\hat{\otimes}s, F)$.

(ii) *F has property (LB_∞).*

Outline of the proof. It is easily seen that this is a straight forward generalisation of 12.4, since the necessary double indexing doesn't interfere with that proof nor the proof of 10.12.

Chapter 13

Intermediate Results

Lemma 13.1 ([Vog85], Lemma 1.3 and [Vog77a], Satz 1.5) *Let F and E be Fréchet spaces and assume that E has property (DN) and $\Lambda^\infty(M,a)$ is nuclear. Then the exact sequence*

$$0 \to \Lambda^\infty(M,a) \to F \xrightarrow{\varphi} E \to 0$$

splits.

Proof. For the sake of simplicity, we only consider the case $\Lambda^\infty(M,a) = s$. Let further be s a subspace of F. We now prove that s is continuously projected into F, i.e. there exists a subspace H of F such that $F = s \oplus H$. Thus $\varphi\,|_H$ is a bijection and hence has an inverse.

Let $f_j \in s'$ with $f_j(x) = x_j$ for $x = (x_1, x_2, \ldots)$. For each $k \in \mathbb{N}$ $\left\{j^k f_j : j \in \mathbb{N}\right\}$ is equicontinuous. By Hahn-Banach C.2 we can extend f_j to $f_j^k \in E'$ for each $k \in \mathbb{N}$ such that $\left\{j^k f_j^k : j \in \mathbb{N}\right\}$ is equicontinuous, thus contained in U_k° for a suitable neighbourhood U_k of $0 \in E$. We can assume that $U_{k+1} \subseteq U_k$ for all $k \in \mathbb{N}$.

If we put

$$g_j^k := f_j^{k+1} - f_j^k$$

then $g_j^k \in s^\circ \subseteq F'$ and we get

$$\left\{j^k g_j^k : j \in \mathbb{N}\right\} \subseteq 2U_{k+1}^\circ \cap s^\circ =: B_k.$$

Since $s^\circ \cong E'$ and E has property (DN), there exists a bounded set $B \subseteq s^\circ$ which satisfies the conditions of A.4 for a fixed fundamental system of bounded sets in s°. Without loss of generality we then can assume that

$$B_k \subseteq rB + \frac{2^{-k-2}}{r} B_{k+1}$$

for all $r > 0$ and $k \in \mathbb{N}$. In particular we have for $r = j2^{-k-1}$ and by multiplication with $2j^{-k}$

$$2j^{-k} B_k \subseteq j^{-k+1} 2^{-k} B + j^{-k-1} B_{k+1}. \tag{1}$$

We now choose for fixed j gradually a sequence a_j^k with $a_j^k \in j^{-k} B_k \subseteq s^\circ$. Hereby put $a_j^0 = 0$. If $a_j^{k+1} \in j^{-k} B_{k+1}$ is chosen, we have

$$g_j^k + a_j^k \in 2j^{-k} B_k.$$

Hence by (1) there exists an $a_j^{k+1} \in j^{-k-1}B_{k+1}$ such that

$$g_j^k + a_j^k \in 2^k j^{-k} B.$$

If we put

$$\phi_j^k := f_j^k - a_j^k,$$

we get for $k \geq 1$

$$\phi_j^{k+1} - \phi_j^k = g_j^k - a_j^{k+1} + a_j^k \in 2^{-k} j^{-k+1} B \subseteq 2^{-k} B.$$

Hence $(\phi_j^k)_{k\in\mathbb{N}}$ converges in F'. We put

$$\phi_j := \lim_{k\to\infty} \phi_j^k.$$

For $k > n$ by $\phi_j^{n+1} = f_j^{n+1} - a_j^{n+1} \in 3j^{-n}U_{n+2}^\circ$ we have

$$j^n \phi_j^k = j^n \phi_j^{n+1} + \sum_{\nu=n+1}^{k-1} j^n(\phi_j^{\nu+1} - \phi_j^\nu) \in 3U_{n+2}^\circ + 2^{-n}B.$$

Therewith $j^n\phi_j \in 3U_{n+1}^\circ + 2^{-n}B$, i.e. $\{j^n\phi_j : j \in \mathbb{N}\}$ is equicontinuous in F'.

By $x \to (\phi_j x)_{j\in\mathbb{N}}$ we define a continuous linear mapping $\phi : F \to s$. For $x \in s$ we have

$$(\phi x)_j = \phi_j(x) = \lim_{k\to\infty} f_j^k(x) - a_j^k(x) = f_j(x) = x_j,$$

where from $a_j^k \in s$ follows $a_j^k(x) = 0$ for $x \in s$ and $f_j^k \mid_s = f_j$. Therefore ϕ is a continuous projection from F to s. □

Proposition 13.2 ([Vog83], 6.1 Satz) *Let M be a set and let a be a function on M with $a(t) \geq 1$ for all $t \in M$. Recall that*

$$\Lambda^\infty(M, a) = \left\{ f \in \mathbb{F}^M : \|f\|_k = \sup_{t\in M} |f(t)| a(t)^k < \infty \ \forall k \in \mathbb{N} \right\}.$$

A Fréchet space F has property (DN) if and only if there exists a space $\Lambda^\infty(M, a)$ such that F is isomorphic to a subspace of $\Lambda^\infty(M, a)$.

Proof. Let $B_1 \subset B_2 \subset \ldots$ a fundamental system of equicontinuous sets in F', I a set such that $B_k \subset I$ for all $k \in \mathbb{N}$. Then F can in a natural way be embedded into $(l^\infty(I))^{\mathbb{N}}$.

By tensoring the exact sequence $0 \to s \to s \to \omega \to 0$ with $l^\infty(I)$ we get the exact sequence

$$0 \to s\hat{\otimes} l^\infty(I) \to s\hat{\otimes} l^\infty(I) \to (l^\infty(I))^{\mathbb{N}} \to 0$$

where $(l^\infty(I))^{\mathbb{N}} = \left(\mathbb{F}\hat{\otimes} l^\infty(I)\right)^{\mathbb{N}} = \mathbb{F}^{\mathbb{N}}\hat{\otimes} l^\infty(I) = \omega\hat{\otimes} l^\infty(I)$ or, since $s\hat{\otimes} l^\infty(I) \cong \Lambda^\infty(M, a)$ with $M = \mathbb{N} \times I$, $a(n, i) := n$ we have

$$0 \to \Lambda^\infty(M, a) \to \Lambda^\infty(M, a) \xrightarrow{q} (l^\infty(I))^{\mathbb{N}} \to 0.$$

If F is embedded in $(l^\infty(I))^{\mathbb{N}}$ and $\tilde{F}$ is the preimage of F under q, we get

$$0 \to \Lambda^\infty(M, a) \to \tilde{F} \to F \to 0.$$

By 13.1 the sequence splits if and only if F has property (DN). We therefore get the embedding $F \subseteq \tilde{F} \subseteq \Lambda^\infty(M, a)$.

Since $\Lambda^\infty(M, a)$ obviously has property (DN), the converse follows from 10.2. □

Theorem 13.3 ([Vog83], 6.2 Satz) *Let E and F be Fréchet spaces. If E has property (LB^∞) and F property (DN), then*

$$L(E,F) = LB(E,F).$$

Proof. It is obvious that 10.12 still holds if we replace $\lambda^\infty(A)$ by $\Lambda^\infty(M,a)$. And 11.2 holds if we replace $\Lambda^\infty_\infty(\alpha)$ by $\Lambda^\infty(M,a)$. Hence we have

$$L(E,\Lambda^\infty(M,a)) = LB(E,\Lambda^\infty(M,a)).$$

By 13.2 F is a closed subspace of a $\Lambda^\infty(M,a)$. □

Theorem 13.4 ([BD98], Theorem 18) *A Fréchet space F has the property (DN) if and only if $C^\omega(\mathbb{R},F) = C^\omega_t(\mathbb{R},F)$.*

Proof. The following assertion holds by 7.6 and 7.8: $C^\omega(\mathbb{R},F) = C^\omega_t(\mathbb{R},F)$ if and only if

$$L(C^\omega(\mathbb{R})'_\beta, F) = LB(C^\omega(\mathbb{R})'_\beta, F).$$

First, let us assume F to have property (DN). By 6.9, $C^\omega(\mathbb{R})'_\beta = \varinjlim_{n\in\mathbb{N}} G_n$, where G_n is isomorphic to $H(\mathbb{D})$ for each $n \in \mathbb{N}$. Given $h \in L(C^\omega(\mathbb{R})'_\beta, F)$, we can apply 10.14 to get, for each $n \in \mathbb{N}$, a neighbourhood $U_n \subseteq G_n$ of 0 such that $h(U_n)$ is bounded in F. Since F is metrizable, we can find a sequence of positive constants $(\lambda_n)_{n\in\mathbb{N}}$ such that $\bigcup_{n\in\mathbb{N}} \lambda_n h(U_n)$ is bounded. Then the absolutely convex hull of $\bigcup_{n\in\mathbb{N}} \lambda_n U_n$, which is a neighbourhood of 0 in $C^\omega(\mathbb{R})'_\beta$, is mapped by h into a bounded set in F.

Conversely, assume that this identity holds. By 6.12, there is a quotient map $q : C^\omega(\mathbb{R})'_\beta \to H(\mathbb{D})$ and hence we have

$$L(H(\mathbb{D},F)) = LB(H(\mathbb{D}),F).$$

It is easily seen that $H(\mathbb{D})$ is linearly homeomorphic to $\Lambda^1_1(\beta)$ (cf. [Jar81], 2.10.10), hence we can directly apply 10.14. As a result, F has the property (DN). □

Lemma 13.5 ([HK00], Lemma 2.2) *Every compact set B in a Fréchet space E for which $H(B)'_\beta$ has property (LB^∞) is a set of uniqueness.*

Proof. Let $\{V_n\}_{n\in\mathbb{N}}$ be a decreasing neighbourhood basis of B in E. Given $f \in H(B)$ with $f\,|_B = 0$, choose $p \geq 1$ such that $f \in H^\infty(V_p)$. For each $n \geq p$ put

$$\varepsilon_n := \|f\|_n = \sup\{|f(z)| : z \in V_n\}.$$

Then the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ converges to 0. By hypothesis $H(B)'_\beta$ has property (LB^∞). Employing this with $\left(\rho_n := \sqrt{\log \frac{1}{\varepsilon_n}}\right)_{n\in\mathbb{N}}$, $\rho_n \underset{n\in\mathbb{N}}{\longrightarrow} \infty$ we have $\exists q\, \forall n_0\, \exists N_0 \geq n_0, C_{n_0} > 0\ \forall m > 0\ \exists k_m : n_0 \leq k_m \leq N_0 :$

$$\|f^m\|_q^{1+\rho_{k_m}} \leq C_{n_0} \|f^m\|_{k_m} \|f^m\|_p^{\rho_{k_m}}$$

which yields

$$\|f\|_q^{1+\rho_{k_m}} \leq C_{n_0}^{\frac{1}{m}} \|f\|_{k_m} \|f\|_p^{\rho_{k_m}}.$$

Now choose a k with $n_0 \leq k \leq N_0$ such that

$$\operatorname{card}\{m : k_m = k\} = \infty.$$

Then putting $C_{n_0} = 1$ without loss of generality as $m \to \infty$ we get

$$\|f\|_q \leq \|f\|_k^{\frac{1}{1+\rho_k}} \|f\|_p^{\frac{\rho_k}{1+\rho_k}}.$$

This leads to

$$(\varepsilon_k)^{\frac{1}{1+\rho_k}} (\varepsilon_p)^{\frac{\rho_k}{1+\rho_k}} \to 0$$

as $k \to \infty$, particularly, the limit exists.

Hence $f\,|_{V_q} = 0$. □

Lemma 13.6 ([HK00], Lemma 2.3) *Let F be a Fréchet space having property (DN). Then $(F'_{\text{bor}})'_\beta$ has property (DN).*

Proof. Let $(U_n)_{n\in\mathbb{N}}$ be a decreasing neighbourhood basis of $0 \in F$. Since F has property (DN) by A.4, there exists a bounded absolutely convex set $B \subset F'$ such that for every $k \in \mathbb{N}$ there exist a $p \in \mathbb{N}$ and a $C > 0$ with

$$U_k^\circ \subseteq rB + \frac{C}{r} U_{k+p}^\circ$$

for all $r > 0$.

For $u \in (F'_{\text{bor}})'_\beta$ and $r > 0$ we have

$$\begin{aligned}
\|u\|_{U_k^\circ} &= \sup\left\{|u(x')| : x' \in U_k^\circ\right\} \\
&\leq \sup\left\{|u(x')| : x' \in rB + \frac{C}{r} U_{k+p}^\circ\right\} \\
&\leq r \sup\left\{|u(x')| : x' \in B\right\} + \frac{C}{r} \sup\left\{|u(x')| : x' \in U_{k+p}^\circ\right\} \\
&= r\|u\|_B + \frac{C}{r}\|u\|_{U_{k+p}^\circ}.
\end{aligned}$$

Hence $(F'_{\text{bor}})'_\beta$ has property (DN). □

Theorem 13.7 ([HK00], Theorem 2.1) *Let F be a Fréchet space. Then*

$$H_\omega(B, F) = H(B, F)$$

holds for every compact set B in a Fréchet space E for which $H(B)'_\beta$ has property (LB^∞) if and only if F has property (DN).

Proof. We first prove the sufficiency. It suffices to show that $H_\omega(B, F) \subseteq H(B, F)$. Let $f \in H_\omega(B, F)$. By hypothesis F has property (DN) and $H(B)'_\beta$ property (LB^∞). Since B is a compact subset in the Fréchet space E, by 13.5 it is a set of uniqueness. Hence, we can consider the linear map

$$\hat{f} : F'_{\text{bor}} \to H(B)$$

given by

$$\hat{f}(x') = \widehat{x' \circ f}$$

for $x' \in F'_{\text{bor}}$, where $\widehat{x' \circ f}$ is a holomorphic extension of $x' \circ f$ to some neighbourhood of B in E. Still by the uniqueness of B using C.4 it follows that $\hat{f}$ has closed graph. On the other hand, F'_{bor} is an inductive limit of Banach spaces, $H(B)$ is an (LF)-space, so by Grothendieck's closed graph theorem $\hat{f}$ is continuous. Since $\hat{f}$ maps bounded subsets of F'_{bor} to bounded subsets of $H(B)$, the dual map

$$\hat{f}^* : H(B)'_\beta \to (F'_{\text{bor}})'_\beta$$

is also continuous. By hypothesis $H(B)'_\beta$ has property (LB^∞) and by 13.6 $(F'_{\text{bor}})'_\beta$ has property (DN). From 13.3 it follows that there exists a bounded subset $L \subseteq H(B)$ such that $\hat{f}^*(L^\circ)$ is a bounded subset of $(F'_{\text{bor}})'_\beta$, where L° denotes the polar of L in $H(B)'_\beta$. Hence, $\left(\hat{f}^*\left(L^\circ\right)\right)^\circ$ is a neighbourhood of $0 \in \left((F'_{\text{bor}})'_\beta\right)'_\beta$. Put

$$W := \left(\hat{f}^*\left(L^\circ\right)\right)^\circ \cap F'_{\text{bor}}.$$

Then W is a neighbourhood of $0 \in F'_{\text{bor}}$. We have

$$\hat{f}(W) \subseteq L^{\circ\circ} \cap H(B)$$

where $L^{\circ\circ}$ is the bi-polar of L. However, $L^{\circ\circ} \cap H(B)$ is the closure of the absolutely convex envelope of L and hence it is a bounded subset of $H(B)$. This shows that $\hat{f}(W)$ is bounded in $H(B)$. By Grothendieck's factorisation theorem C.5 and since B is a set of uniqueness, there exists a neighbourhood U of B in E such that $\hat{f}(W)$ is contained and bounded in $H(U)$. From the absorption of W it follows that

$$\hat{f}(F'_{\text{bor}}) \subseteq H(U).$$

Now we can define a holomorphic function

$$g : U \to (F'_{\text{bor}})'_\beta$$

given by

$$g(z)(x') = \hat{f}(x')(z)$$

for $z \in U,\ x' \in F'_{\text{bor}}$.

We see that $g(z)(x') = \hat{f}(x')(z) = f(z)(x')$ for every $z \in B,\ x' \in F'$. This yields $g\,|_B = f$ and since B is a set of uniqueness, $g(U) \subseteq F$.

To prove the necessity, by 13.3 it suffices to show that every continuous linear map T from $H(\mathbb{D})$ to F is bounded on a neighbourhood of $0 \in H(\mathbb{D})$. Consider $T' : F'_\beta \to H(\mathbb{D})'_\beta \cong H(\overline{\mathbb{D}})$. Since $T'(x') \in H(\overline{\mathbb{D}})$ for all $x' \in F'_\beta$, we can define a map $f : \overline{\mathbb{D}} \to (F'_{\text{bor}})'_\beta$ given by

$$f(z)(x') = \delta_z(T'(x'))$$

for $x' \in F'_\beta,\ z \in \overline{\mathbb{D}}$, where δ_z is the Dirac functional defined by z.

From the weak continuity of T' and δ_z we infer that $f(z)$ is $\sigma(F', F)$-continuous and, hence, $f(z) \in F$. Moreover, $f \in H_\omega(\overline{\mathbb{D}}, F)$. Since $H(\overline{\mathbb{D}})'_\beta$ has property (LB^∞) it follows that $f \in H(\overline{\mathbb{D}}, F)$. Thus there exists a neighbourhood V of $\overline{\mathbb{D}}$ and we can consider $f \in H(V, F)$. Now we can choose an open subset V_1 which is compact in V such that $\overline{\mathbb{D}}\bar{V_1} \subset V$. Hence $f\,|_{\bar{V_1}}$ is bounded in F and also $B := f(V_1)$. It is easy to see that T' is bounded on B°, since $T'(B^\circ) \subseteq V^\circ$. Put $C = T'(B^\circ) \subseteq H(\overline{\mathbb{D}})'_\beta$ and $U := C^\circ$ with $\overline{\mathbb{D}} \subseteq U \subseteq \bar{V_1}$. Then U is a neighbourhood of $0 \in H(\mathbb{D})$ and $T(U) \subseteq B^{\circ\circ}$ is bounded in F. □

Chapter 14

The Main Theorems

Definition 14.1 Let E and F be locally convex spaces. A mapping $f : E \to F$ is called *real analytic*, denoted by $C^\omega(E, F)$ if f is real analytic along real analytic curves and smooth along smooth curves, i.e. $f \circ c \in C^\omega(\mathbb{R}, F)$ for all $c \in C^\omega(\mathbb{R}, E)$ and $f \circ c \in C^\infty(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, E)$.

Analogously, a mapping $f : E \to F$ is called *topologically real analytic* if $f \circ c \in C^\omega_t(\mathbb{R}, F)$ for all $c \in C^\omega_t(\mathbb{R}, E)$.

At last, $f : E \to F$ is called *bornologically real analytic* if $f \circ c \in C^\omega_b(\mathbb{R}, F)$ for all $c \in C^\omega_b(\mathbb{R}, E)$.

Lemma 14.2 ([Vog82], Theorem 1.4) *Let E be an (FS)-space with property $(\bar{\Omega})$. Then there exists a bounded (hence a relatively compact) subset B such that E has property $(\bar{\Omega}_B)$.*

Proof. By A.10 (iii"'), the property $(\bar{\Omega})$ of E can be written in the following way. For all p and μ with $0 < \mu < 1$ there exists a q such that for all k exists a $C > 0$ and we have

$$U_q \subseteq r^\mu U_k + \frac{C}{r^{1-\mu}} U_p$$

for $r > 0$. Or equivalently, for all $x \in U_q$ exists a $y \in U_k$ and a $z \in U_p$ such that

$$x = r^\mu y + \frac{C}{r^{1-\mu}} z.$$

This means, for every $x \in U_q$ there exists a $y \in U_k$ and a $z \in U_p$ such that

$$\|x - r^\mu y\|_p = \|\frac{C}{r^{1-\mu}} z\|_p = \frac{C}{r^{1-\mu}} \|z\|_p \leq \frac{C}{r^{1-\mu}}$$

which is equivalent to

$$\min\{\|x - r^\mu y\|_p : y \in U_k\} \leq \frac{C}{r^{1-\mu}}$$

for all $x \in U_q$. This can be rewritten as

$$r^{1-\mu} \sup\{\min\{\|x - r^\mu y\|_p : y \in U_k\} : x \in U_q\} \leq C.$$

Now we replace μ by $\frac{\mu}{2}$ and r by r^2, multiply with $\frac{1}{r}$, and obtain (strictly speaking we also get new C and U_q but this has no effect to our calculation)

$$\begin{aligned}\frac{1}{r}C &\geq \frac{1}{r}(r^2)^{1-\frac{\mu}{2}} \sup\left\{\min\left\{\|x - r^{2\frac{\mu}{2}} y\|_p : y \in U_k\right\} : x \in U_q\right\} \\ &= r^{1-\mu} \sup\left\{\min\left\{\|x - r^{\mu} y\|_p : y \in U_k\right\} : x \in U_q\right\}.\end{aligned}$$

Hence for each k we have

$$\varepsilon(r,k) := r^{1-\mu} \sup\left\{\min\left\{\|x - r^{\mu} y\|_p : y \in U_k\right\} : x \in U_q\right\} \to 0$$

for $r \to \infty$. Thus for all $k \in \mathbb{N}$ exists an r_k such that for all $r \geq r_k$ we have $\varepsilon(r,k) < \frac{1}{k}$. Without loss of generality we may assume $k \leq r_k < r_{k+1}$. Now let $k(n) := \max\{k : r_k \leq n\} < \infty$. Then $\varepsilon(n)$, defined by

$$\varepsilon(n) := \varepsilon(n, k(n)) < \frac{1}{k(n)}$$

converges to 0 since $r_{k(n)} \leq n$. Furthermore $k(n) \underset{n\to\infty}{\longrightarrow} \infty$, since otherwise there would exist a bound K and thus $r_{K+1} > n$ for infinitely many n, ergo $\varepsilon \to 0$, i.e.

$$U_q \subseteq n^{\mu} U_{k(n)} + \frac{\varepsilon(n)}{n^{1-\mu}} U_p.$$

Since E is an (FS)-space, we find finite sets $Z_n \subseteq U_{k(n)}$ with $U_{k(n)} \subseteq Z_n + \frac{\varepsilon(n)}{n} U_p$ and hence

$$\begin{aligned}U_q &\subseteq n^{\mu} U_{k(n)} + \frac{\varepsilon(n)}{n^{1-\mu}} U_p \\ &\subseteq \left(n^{\mu} Z_n + \frac{\varepsilon(n)}{n^{1-\mu}} U_p\right) + \frac{\varepsilon(n)}{n^{1-\mu}} U_p \\ &= n^{\mu} Z_n + \frac{2\varepsilon(n)}{n^{1-\mu}} U_p.\end{aligned}$$

Let B be the absolutely convex hull of $\bigcup_{n\in\mathbb{N}} Z_j$. Then B is bounded since it is contained in the absolutely convex hull of $U_{k(n)} \cup \bigcup_{j<n} Z_j$ and $k(n) \to \infty$. Resulting we have

$$\begin{aligned}U_q &\subseteq n^{\mu} Z_n + \frac{2\varepsilon(n)}{n^{1-\mu}} U_p \\ &\subseteq n^{\mu} B + \frac{2\|\varepsilon\|_\infty}{n^{1-\mu}} U_p.\end{aligned}$$

Note that without loss of generality we may assume $q \geq p$, i.e. $U_q \subseteq U_p$, and hence for $0 < r \leq 1$

$$U_q \subseteq U_p \subseteq r^{\mu} B + U_p \subseteq r^{\mu} B + \frac{1}{r^{1-\mu}} U_p.$$

Finally, if $n \leq r < n+1$ then we have

$$\begin{aligned}U_q &\subseteq n^{\mu} B + \frac{C}{n^{1-\mu}} U_p \\ &\subseteq r^{\mu} B + \left(\frac{n+1}{n}\right)^{1-\mu} \frac{C}{r^{1-\mu}} U_p,\end{aligned}$$

thus for $C \geq \max\{4\|\varepsilon\|_\infty, 1\}$ we have

$$U_q \subseteq r^\mu B + \frac{C}{r^{1-\mu}} U_p$$

for all $r > 0$. □

Corollary 14.3 *Let E be an (FS)-space with property $(\tilde{\Omega})$. Then there exists a bounded (hence a relatively compact) subset B such that E has property $(\tilde{\Omega}_B)$.*

Reference. The proof goes along the same lines as 14.2, as cited in [DMV84], Proposition 3b. In this case we put $\frac{\mu}{2}$ as the μ in the hypothesis and get $\mu = 1$ in the result.

Theorem 14.4 ([HK02], Theorem A) *Let F be a Fréchet space. Then the following assertions are equivalent.*

(i) *F has property (DN).*

(ii) *$C^\omega(E, F) = C^\omega_t(E, F)$ for every real nuclear Fréchet space E having property $(\tilde{\Omega})$.*

(iii) *$C^\omega(E, F) = C^\omega_t(E, F)$ for every real (FS)-space E having property $(\tilde{\Omega})$ and an absolute basis $\{e_j\}_{j\geq 1}$.*

Proof. The implications (ii) ⇒ (i) and (iii) ⇒ (i) are proven in 13.4 by choosing $E = \mathbb{R}$.

First we prove (i) ⇒ (ii). Let E, F be as in the statement of the theorem. By 14.3 there exists an absolutely convex compact subset $L \subseteq E$ such that E has property $(\tilde{\Omega}_L)$. It follows that E has the property $(\tilde{\Omega}_B)$ for all absolutely convex compact subsets B with $L \subseteq B$. Hence, so does $E \otimes \mathbb{C}$ and by 11.9 B is a set of uniqueness in $E \otimes \mathbb{C}$.

Thus we can define the linear map $S_B : F'_{\text{bor}} \to H(B)$ by

$$S_B(u) = \widehat{u \circ f}$$

for $f \in C^\omega(E, F)$ and $u \in F'_{\text{bor}}$, where $\widehat{u \circ f}$ is the unique holomorphic extension of $u \circ f$ to a neighbourhood of B in $E \otimes \mathbb{C}$. Again using the uniqueness of B, by C.4, we deduce that S_B has a closed graph and, hence, it is continuous by Grothendieck's closed graph theorem C.3. Now by 11.10 $H(B)'_\beta$ has property (LB^∞) and by 13.6 $(F'_{\text{bor}})'_\beta$ has property (DN). Then 13.3 implies that S_B is bounded on a neighbourhood of $0 \in F'_{\text{bor}}$. We deduce from 13.7 that there exists a convex neighbourhood W_B of B in $E \otimes \mathbb{C}$ and a holomorphic function

$$\tilde{f}_B : W_B \to F$$

such that

$$\tilde{f}_B |_B = f |_B \,.$$

Put W as the union of all W_B where B is an absolutely convex compact subset of E with $L \subset B$ and define the function

$$\tilde{f} : W \to F$$

given by

$$\tilde{f}\mid_{W_B}=\tilde{f}_B.$$

Now we show that the function $\tilde{f}$ is correctly defined in this way and, hence, is holomorphic on the interior of W in $E\otimes\mathbb{C}$. Indeed, let B and C be absolutely convex compact subsets of E with $L\subseteq B,C$. Then $\tilde{f}_B$, $\tilde{f}_C$ are holomorphic on $W_B\cap W_C$. On the other hand, $L\subseteq B\cap C$ then E has property $(\tilde{\Omega}_{B\cap C})$ and, at the same time,

$$\tilde{f}_B\mid_{B\cap C}=f\mid_{B\cap C}.$$

Now using the uniqueness of $B\cap C$, noticing that $W_B\cap W_C$ is connected and $B\cap C\subseteq W_B\cap W_C$, we deduce that

$$\tilde{f}_B\mid_{W_B\cap W_C}=\tilde{f}_C\mid_{W_B\cap W_C}.$$

It remains to check that the interior of W is contained in E and hence, $f\in C^\omega(E,F)$. Assume, for the sake of obtaining a contradiction, that there exists $x_0\in E$ and a sequence $\{(x_n+\mathrm{i}\,y_n)_{n\in\mathbb{N}}\}\subseteq E\otimes\mathbb{C}$ converging to x_0 but $x_n+\mathrm{i}\,y_n\notin W$ for $n\geq 1$. Put

$$B:=\overline{\langle\{(x_n)_{n\in\mathbb{N}},x_0,(y_n)_{n\in\mathbb{N}}\}\cup L\rangle}$$

as the closed convex hull of the mentioned elements. Since E is a Fréchet space, by 1.9 there exists an absolutely convex compact set $B_1\subseteq E$ containing B such that B is compact in E_{B_1}. Then B_1 is an absolutely convex compact subset of E with $L\subseteq B_1$ and W_{B_1} is a neighbourhood of $\{x_0\}\times\{0\}$ in $E\otimes\mathbb{C}$. Hence $\{x_n+\mathrm{i}\,y_n\}\subseteq W_{B_1}\subseteq W$ for sufficiently large n. This is a contradiction.

(i) ⇒ (iii) Let E be as in the hypothesis an (FS)-space with property $(\tilde{\Omega})$. Then by 14.3 there exists an absolutely convex compact subset $L\subseteq E$ such that E has property $(\tilde{\Omega}_L)$. Next, let B an absolutely convex compact subset and $\{e_j\}_{j\geq1}$ the absolute basis of E. Putting

$$\tilde{B}:=\overline{\left\langle B\cup L\cup\bigcup_{j\geq1}\|e'_j\|_L e_j\right\rangle}$$

we get that E has also the property $(\tilde{\Omega}_{\tilde{B}})$. Then by B.1 $H(B)'_\beta$ has property (LB^∞). From here on, the proof of the implication (i) ⇒ (iii) is analogous to the proof of (i) ⇒ (ii). □

Remark 14.5 Let E be a real Fréchet space and B an absolutely convex compact set in E. Then B is a set of uniqueness for $H(B\otimes\mathbb{C})$ in $E_B\otimes\mathbb{C}$.

Theorem 14.6 ([HK02], Theorem B) *Let F be a Fréchet space having property (LB_∞) then*

$$C^\omega(E,F)=C^\omega_t(E,F)$$

holds for all real Fréchet spaces E.

Proof. (i) By $\mathcal{B}(E)$ we denote the family of all absolutely convex compact subsets of E. Then by 1.9, for each $B\in\mathcal{B}(E)$ we can find a $B_1\in\mathcal{B}(E)$ such that B is compact in E_{B_1}. Let $\mathcal{F}(B_1)$ be the family of separable closed subspaces of E_{B_1} and let $M \in \mathcal{F}(B_1)$ be given. Choose a sequence $(x_n^M)_{n\in\mathbb{N}}$ in M converging to $0 \in M$ such that its span is dense in M. Put

$$B_M = \overline{\left\langle (B \cap M) \cup \bigcup_{n\geq 1} \{x_n^M\} \right\rangle}.$$

The convex hull of $\bigcup_{n\geq 1} \{x_n^M\}$ is compact, since $(x_n^M)_{n\in\mathbb{N}}$ is a sequence converging to 0. Then B_M is absolutely convex compact in M.

Furthermore, B_M is the set of uniqueness for holomorphic functions on neighbourhoods of B_M in $M \otimes \mathbb{C}$. Indeed, let f be a holomorphic function on a neighbourhood W_B of B_M in $M \otimes \mathbb{C}$ such that $f\mid_{B_M} = 0$. Then for each $x \in B_M$ and $\xi \in \operatorname{span}\left\{(x_n^M)_{n\in\mathbb{N}}\right\}$ we have

$$f'(x)(\xi) = \lim_{t\to 0} \frac{f(x+t\xi) - f(x)}{t} = 0$$

because $t\xi \in B_M$ for sufficiently small $t \in \mathbb{R}$. Hence $f'(x)(u) = 0$ for all $u \in M$ and it implies that $f'(x)(z) = 0$ for all $z \in M \otimes \mathbb{C}$. Thus $f' = 0$ on B_M. Replacing f by f' we get $f'' = 0$ on B_M and continuing this process we deduce that $f^{(n)} = 0$ on B_M for all $n \in \mathbb{N}$. Using the Taylor expansion of f we derive that $f = 0$ on a neighbourhood of B_M in $M \otimes \mathbb{C}$.

(ii) The uniqueness of B_M in $M \otimes \mathbb{C}$ allows us to define the linear map

$$\hat{f} : F'_{\text{bor}} \to H(B_M) : \hat{f}(u) = \widehat{u \circ f}$$

where $\widehat{u \circ f}$ denotes the holomorphic extension of $u \circ f$ to a neighbourhood of B_M in $M \otimes \mathbb{C}$ and $H(B_M)$ is the space of germs of holomorphic functions on neighbourhoods of B_M in $M \otimes \mathbb{C}$. Again using the uniqueness of B_M in $M \otimes \mathbb{C}$ we deduce that $\hat{f}$ has a closed graph and, by Grothendieck's closed graph theorem, it is continuous. Since M is a Banach space, it clearly has property (Ω) and hence by 9.4 $H(B_M)'_\beta$ has property (Ω). From 9.3 there exists an index set I such that $H(B_M)'_\beta$ is isomorphic to a quotient of $\ell^1(I)\hat{\otimes} s$. On the other hand, if F has property (LB_∞) then by 12.3 $(F'_{\text{bor}})'_\beta$ also has property (LB_∞). By 12.5 we derive that every continuous linear map from $\ell^1(I)\hat{\otimes} s$ and, hence, from $H(B_M)'_\beta$ to $(F'_{\text{bor}})'_\beta$ is bounded on a neighbourhood of $0 \in H(B_M)'_\beta$. Now as in the argument of 14.4 there exists a convex neighbourhood W_M of B_M in $M \otimes \mathbb{C}$ and a holomorphic function $f_M : W_M \to F$ such that

$$f_M \mid_{B_M} = f \mid_{B_M} .$$

Recall that $E_{B_M} \otimes \mathbb{C} = M \otimes \mathbb{C}$ where $E_{B_M} \otimes \mathbb{C}$ denotes the complexification of E_{B_M} and E_{B_M} denotes the Banach space induced by B_M in E_{B_1}. Put

$$W_B := \bigcup \{W_M : M \in \mathcal{F}(B_1)\}$$

and define the function $\tilde{f}_B : W_B \to F$ which is given by

$$\tilde{f}_B \mid_{W_M} = f_M.$$

First we check that $\tilde{f}_B$ is correctly defined in this way. Let M_1, M_2 be in $\mathcal{F}(B_1)$. Without loss of generality we may consider that $M_1 \subseteq M_2$. In the same notations

$B_{M_1} \subseteq B_{M_2}$ and, hence, $B_{M_1} \otimes \mathbb{C} \hookrightarrow B_{M_2} \otimes \mathbb{C}$. Then $W_{M_1} \cap W_{M_2} \supseteq B_{M_1} \cap B_{M_2} = B_{M_1}$ and is an open subset in $E_{B_{M_1}} \otimes \mathbb{C}$. At the same time

$$f_{M_1} |_{B_{M_1}} = f |_{B_{M_1}} = f_{M_2} |_{B_{M_1}} .$$

By 14.5 this implies that

$$f_{M_1} |_{W_{M_1} \cap W_{M_2}} = f_{M_2} |_{W_{M_1} \cap W_{M_2}} .$$

Next we show that B is contained in the interior of W_B in $E_{B_1} \otimes \mathbb{C}$.

Assume, for the sake of seeking a contradiction, that there would exist an $x_0 \in B$ and a sequence $(z_n := x_n + \mathrm{i}\, y_n)_{n\in\mathbb{N}}$ in $E_{B_1} \otimes \mathbb{C}$ converging to x_0 but $z_n \notin W_B$ for $n \geq 1$. Let

$$M := \overline{\operatorname{span}\{(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}, x_0\}}.$$

Then $M \in \mathcal{F}(B_1)$ and $\{(z_n)_{n\in\mathbb{N}}\} \subseteq M \otimes \mathbb{C}$ with $z_n \to z_0$ in $M \otimes \mathbb{C}$. Notice that $x_0 \in W_M$ implies $\{(z_n)_{n\geq N\in\mathbb{N}}\} \subseteq W_M \subseteq W_B$ for n sufficiently large. This is a contradiction.

Since $\tilde{f}_B$ is holomorphic on one $W_M \in W_B$, it is Gâteaux holomorphic on the interior of W_B. Let $\{(z_n)_{n\in\mathbb{N}}, z_0\} \subseteq W_B$ and $z_n \to z_0$. Put

$$M := \overline{\operatorname{span}\{(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}, x_0, y_0\}}.$$

Then $M \in \mathcal{F}(B_1)$ and let B_M, defined by

$$B_M := \overline{\langle (B \cap M) \cup \{(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}, x_0, y_0\}\rangle},$$

be the convex hull of the mentioned sets. Notice that the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ as well as the elements x_0, y_0 are contained in W_M and $x_n \to x_0$, $y_n \to y_0$. Hence $f_M(x_n, y_n) \to f_M(x_0, y_0)$. Thus $f_B(z_n) \to f_B(z_0)$ and consequently, f_B is holomorphic on the interior of W_B.

(iii) By (ii) for each $B \in \mathcal{B}(E)$ we can extend $f |_B$ to a holomorphic function $\tilde{f}_B$ on a convex neighbourhood W_B in $E_{B_1} \otimes \mathbb{C}$ where $B_1 \in \mathcal{B}(E)$, $B \subseteq B_1$ and B is compact in E_{B_1}. Again put

$$W = \bigcup_{B \in \mathcal{B}(E)} W_B.$$

Then $W \subseteq E \otimes \mathbb{C}$ and we can find the function $\tilde{f} : W \to E$ given by

$$\tilde{f} |_W = \tilde{f}_B.$$

As in (ii), the function $\tilde{f}$ defined in this way is a holomorphic extension of f to the interior of W, a neighbourhood of E in $E \otimes \mathbb{C}$. Hence, $f \in C_t^\omega(E, F)$. □

Appendix A

On equivalent descriptions of the Properties

Lemma A.1 *The following implication holds, the reverse implication does not.*

$$(LB_\infty) \Rightarrow (DN)$$

Proof. The implication is obvious.

The reverse implication does not hold, since if it would, by 12.4, the identity mapping would be compact in $\Lambda^1_\infty(\beta)$. □

Theorem A.2 (Characterisation of the property (DN)) *Let E be a Fréchet space with an increasing fundamental system of semi-norms. The following characterisations of the property (DN) are equivalent.*

(i) There exists a continuous semi-norm $\|.\|$ on E such that for all $k \in \mathbb{N}$ there exists a $p \in \mathbb{N}$ and a $C > 0$ with

$$\|.\|_k \leq r\|.\| + \frac{C}{r}\|.\|_{k+p}$$

for all $r > 0$.

(ii) There exists a continuous semi-norm $\|.\|$ on E such that for all $k \in \mathbb{N}$ there exists a $p \in \mathbb{N}$ and a $C > 0$ with

$$\|.\|_k^2 \leq C\|.\|\|.\|_{k+p}.$$

(iii) There exists a system of semi-norms such that for all $k \in \mathbb{N}$

$$\|.\|_k^2 \leq \|.\|_{k-1}\|.\|_{k+1}.$$

(iv) There exists a bounded, absolutely convex set B in E' such that for every $k \in \mathbb{N}$ there exists a $p \in \mathbb{N}$ and a $C > 0$ with

$$B_k \subseteq rB + \frac{C}{r}B_{k+p}$$

for all $r > 0$.

(v) There exists a continuous semi-norm $\|.\|$ on E such that for all $k \in \mathbb{N}$ there exists a $p \in \mathbb{N}$ and a $C > 0$ such that

$$\|.\|_k^2 \leq C\|.\|\|.\|_p.$$

(vi) There exists a continuous semi-norm $\|.\|$ on E such that for all $k \in \mathbb{N}$ and all μ with $0 < \mu < 1$ there exists a $p \in \mathbb{N}$ and a $C > 0$ with

$$\|.\|_k \leq C\|.\|^{1-\mu}\|.\|_p^{\mu}.$$

(vii) There exists a continuous semi-norm $\|.\|$ and a $d > 0$ such that for all $k \in \mathbb{N}$ exists a $p \in \mathbb{N}$ and a $C > 0$ such that

$$\|.\|_k^{1+d} \leq C\|.\|^d\|.\|_p.$$

Outline of the proof. In A.3, the equivalence of the items (i) to (iii) is proven. The proof of the equivalence of (i) and (iv) is given in A.4. Finally, A.5 shows the equivalence of (i) and (v) to (vii).

Lemma A.3 ([Vog77a], 2.1. Satz) *Let E be a Fréchet space. Then the following assertions are equivalent.*

(i) There exists a continuous semi-norm $\|.\|$ on E such that for all $k \in \mathbb{N}$ there exists a $p \in \mathbb{N}$ and a $C > 0$ with

$$\|.\|_k \leq r\|.\| + \frac{C}{r}\|.\|_{k+p}$$

for all $r > 0$.

(ii) There exists a continuous semi-norm $\|.\|$ on E such that for all $k \in \mathbb{N}$ there exists a $p \in \mathbb{N}$ and a $C > 0$ with

$$\|.\|_k^2 \leq C\|.\|\|.\|_{k+p}.$$

(iii) There exists a system of semi-norms such that for all $k \in \mathbb{N}$

$$\|.\|_k^2 \leq \|.\|_{k-1}\|.\|_{k+1}.$$

Proof. (i) $\Leftrightarrow$ (ii) Calculating the minimum of $r\|.\| + \frac{C}{r}\|.\|_{k+p}$ with respect to $r > 0$, we get

$$\min_{r>0}\left(r\|.\| + \frac{C}{r}\|.\|_{k+p}\right) = 2\sqrt{C\|.\|\|.\|_{k+p}}.$$

Hence, $\|.\|_k^2 \leq 4C\|.\|_k\|.\|_{k+p}$.

(ii) $\Rightarrow$ (iii) If $\|.\|_k^2 \leq C\|.\|\|.\|_{k+p}$ holds, we can assume without loss of generality that $\|.\| \leq \|.\|_k$ for all $k \in \mathbb{N}$ (as a continuous semi-norm, $\|.\| \leq \|.\|_{k_0}$ for some $k_0 \in \mathbb{N}$ and it suffices to show (iii) for all $k > k_0$). By hypothesis, we have

$$\|.\|_k^2 \leq C_k\|.\|\|.\|_{k+p} \leq C_k\|.\|_k\|.\|_{k+p},$$

hence

$$\|.\|_k \leq C_k\|.\|_{k+p} =: \|.\|_{k+1}$$

which leads to $\|.\|_k^2 \leq \|.\|_{k-1}\ \|.\|_{k+1}$.

(iii) $\Rightarrow$ (ii) From $\|.\|_k^2 \leq \|.\|_{k-1}\ \|.\|_{k+1}$ we obtain that all $\|.\|_k$ are norms and for $x \neq 0$ from

$$\frac{\|x\|_k}{\|x\|_{k-1}} \leq \frac{\|x\|_{k+1}}{\|x\|_k}$$

we get for all k the inequality

$$\frac{\|x\|_k}{\|x\|_0} = \prod_{j=1}^{k} \frac{\|x\|_j}{\|x\|_{j-1}} \leq \prod_{j=k+1}^{2k} \frac{\|x\|_j}{\|x\|_{j-1}} = \frac{\|x\|_{2k}}{\|x\|_k}$$

and hence

$$\|x\|_k^2 \leq \|x\|_0 \|x\|_{2k}$$

for all $k \in \mathbb{N}$. □

Lemma A.4 ([Vog77a], Lemma 1.4) *Let E be a Fréchet space and $B_1 \subseteq B_2 \subseteq \ldots$ be a fundamental system of absolutely convex bounded sets in E'. Then the following assertions are equivalent.*

(i) There exists a continuous semi-norm $\|.\|$ on E such that for all $k \in \mathbb{N}$ there exists a $p \in \mathbb{N}$ and a $C > 0$ with

$$\|.\|_k \leq r\|.\| + \frac{C}{r}\|.\|_{k+p}$$

for all $r > 0$.

(iv) There exists a bounded, absolutely convex set B in E' such that for every $k \in \mathbb{N}$ there exists a $p \in \mathbb{N}$ and a $C > 0$ with

$$B_k \subseteq rB + \frac{C}{r}B_{k+p}$$

for all $r > 0$.

Proof. From $\|.\|_k \leq r\|.\| + \frac{C}{r}\|.\|_{k+p}$ we get

$$\frac{1}{2r}U \cap \frac{r}{2C}U_{k+p} \subseteq U_k,$$

with

$$U := \{x : \|x\| \leq 1\}.$$

By taking the polars we get

$$U_k^\circ \subseteq 2rU^\circ + \frac{2C}{r}U_{k+p}^\circ.$$

To show the other direction, we put

$$B_k := U_k^\circ,$$

where

$$U_k := \{x : \|x\|_k \leq 1\},$$

since the condition does not depend on the choice of the fundamental system. If the condition is satisfied, there exists a bounded set B and for each $k \in \mathbb{N}$ there exists a $p \in \mathbb{N}$ and a $C > 0$ such that

$$U_k^\circ \subseteq rB + \frac{C}{r} U_{k+p}^\circ$$

for all $r > 0$. Therefore a $y \in U_k^\circ$ can be written as $y = rb + \frac{C}{r}u$ with $b \in B$, $u \in U_{k+p}^\circ$, i.e.

$$|y(x)| \leq r|b(x)| + \frac{C}{r}|u(x)| \leq r\|x\| + \frac{C}{r}\|x\|_{k+p}$$

for all $x \in E$. From this follows (i) by $\|x\| = \sup_{b \in B} |b(x)|$. □

Lemma A.5 ([MV92], Lemma 29.10) *Let E be a Fréchet space with an increasing fundamental system of semi-norms. The following assertions are equivalent.*

(ii) There exists a continuous semi-norm $\|.\|$ on E such that for all $k \in \mathbb{N}$ there exists a $p \in \mathbb{N}$ and a $C > 0$ with

$$\|.\|_k^2 \leq C\|.\|\|.\|_{k+p}.$$

(v) There exists a continuous semi-norm $\|.\|$ on E such that for all $k \in \mathbb{N}$ there exists a $p \in \mathbb{N}$ and a $C > 0$ such that

$$\|.\|_k^2 \leq C\|.\|\|.\|_p.$$

(vi) There exists a continuous semi-norm $\|.\|$ on E such that for all $k \in \mathbb{N}$ and all μ with $0 < \mu < 1$ there exists a $p \in \mathbb{N}$ and a $C > 0$ with

$$\|.\|_k \leq C\|.\|^{1-\mu}\|.\|_p^\mu.$$

(vii) There exists a continuous semi-norm $\|.\|$ on E and a $d > 0$ such that for all $k \in \mathbb{N}$ exists a $p \in \mathbb{N}$ and a $C > 0$ such that

$$\|.\|_k^{1+d} \leq C\|.\|^d\|.\|_p.$$

Proof. (ii) ⇔ (v) This is obvious.

(v) ⇒ (vi) In (v), fix $k \in \mathbb{N}$ with $\|.\|_k < \|.\|$. Put $n_1 := k$ and apply (v) iteratively to get $n_{\nu+1} > n_\nu$ and a C_ν with

$$\|.\|_{n_\nu}^2 \leq C_\nu\|.\|\|.\|_{n_{\nu+1}}.$$

Since $\|.\|$ is a norm, we have for all $m \in \mathbb{N}$

$$\begin{aligned} \left(\frac{\|.\|_k}{\|.\|}\right)^m &\leq \prod_{\nu=0}^{m-1} \frac{\|.\|_{n_\nu}}{\|.\|} \\ &\leq \prod_{\nu=0}^{m-1} C_\nu \frac{\|.\|_{n_{\nu+1}}}{\|.\|_{n_\nu}} \\ &\leq \left(\prod_{\nu=0}^{m-1} C_\nu\right) \frac{\|.\|_{n_m}}{\|.\|}. \end{aligned}$$

If we put $D_m := \left(\prod_{\nu=0}^{m-1} C_\nu\right)^{\frac{1}{m}}$, we get

$$\|.\|_k \leq D_m \|.\|^{1-\frac{1}{m}} \|.\|_{n_m}^{\frac{1}{m}}.$$

Finally, if $0 < \mu < 1$ is given, we choose $m \in \mathbb{N}$ with $\frac{1}{m} < \mu$ and get the desired inequality.

(vi) $\Rightarrow$ (v) Take $\mu = \frac{1}{2}$ and the square of the equation in (vi). The equivalence is now obvious.

(vi) $\Rightarrow$ (vii) This follows directly by putting $\frac{1}{1+d} = \mu$.

(vii) $\Rightarrow$ (v) With fixed $d > 0$, by hypothesis we have that for all k there exists a p such that $\|.\|_k^{1+d} \leq C\|.\|^d\|.\|_p$. Reinserting k into the hypothesis yields $\exists k\ \forall k'\ \exists p'\ \exists C' > 0$ such that $\|.\|_{k'}^{1+d} \leq C'\|.\|_k^d\|.\|_{p'}$. Combining these statements leads to:

$$\begin{aligned}
\|.\|_{k'}^{1+d} &\leq C'\|.\|_k^d\|.\|_{p'} \\
&\leq C'\left(\|.\|_k^{1+d}\right)^{\frac{d}{1+d}}\|.\|_{p'} \\
&\leq C'\left(C\|.\|^d\|.\|_p\right)^{\frac{d}{1+d}}\|.\|_{p'} \\
&\leq C'C^{\frac{d}{1+d}}\|.\|^{\frac{d^2}{1+d}}\|.\|_p^{\frac{d}{1+d}}\|.\|_{p'} \\
&\leq C'C^{\frac{d}{1+d}}\|.\|^{\frac{d^2}{1+d}}\|.\|_p^{\frac{1+2d}{1+d}}
\end{aligned}$$

where in the last step $p' = p$ is assumed. Thus the inequality can be transformed into

$$\|.\|_{k'}^{1+\frac{d^2}{1+2d}} \leq \left(C'C^{\frac{d}{1+d}}\right)^{\frac{1+d}{1+2d}}\|.\|^{\frac{d^2}{1+2d}}\|.\|_p.$$

Now putting $D := \left(C'C^{\frac{d}{1+d}}\right)^{\frac{1+d}{1+2d}}$ and $d' := \frac{d^2}{1+2d} \leq \frac{d}{2}$ gives (iv) by induction also for some $d < 1$. Hence also for $d = 1$. □

Lemma A.6 *The following chain of implications holds.*

$$(\bar{\Omega}) \Rightarrow (\tilde{\Omega}) \Rightarrow (LB^\infty) \Rightarrow (\Omega)$$

Proof. The implication $(\bar{\Omega}) \Rightarrow (\tilde{\Omega})$ is obvious.

Now we prove $(\tilde{\Omega}) \Rightarrow (LB^\infty)$. In the definition of (LB^∞), namely for every monotonous sequence $(\rho_n)_{n\in\mathbb{N}}$ with $\rho_n \underset{n\in\mathbb{N}}{\longrightarrow} \infty$ and for all p there exists a $q \geq p$ such that for all n_0 exists an $N_0 \geq n_0$ and a $C > 0$ such that for all $u \in F'$ exists an m with $n_0 \leq m \leq N_0$ such that

$$\|u\|_q'^{1+\rho_m} \leq C\|u\|_m'\|u\|_p'^{\rho_m},$$

choose N_0 such that $\rho_{N_0} \geq d$ and put $N = N_0$ for all $u \in F'$.

Finally to prove $(LB^\infty) \Rightarrow (\Omega)$, we fix a sequence, e.g. $(\rho_n = n)_{n\in\mathbb{N}}$ and observe that we get

$$\|u\|_{U_q}^{1+N_0} \leq C\|u\|_{U_{n_0}}\|u\|_{U_p}^{N_0}$$

which is property (Ω) for $k = n_0$ and $d = N_0$. □

Remark A.7 None of the reverse implications hold, as can be seen in [Vog83], the remark after proposition 5.3.

Lemma A.8 *Let* $k, p \geq 0$ *and* $a, b > 0$*. Then*

$$\inf_{r>0}\left(r^a k + \frac{1}{r^b} p\right) = \frac{a+b}{a^{\frac{a}{a+b}} b^{\frac{b}{a+b}}} k^{\frac{b}{a+b}} p^{\frac{a}{a+b}}.$$

Proof. The minimum of $r^a k + r^{-b} p$ in respect to r is assumed at

$$r_{\min} := \min_{r>0}\left(r^a k + r^{-b} p\right) = \left(\frac{bp}{ak}\right)^{\frac{1}{a+b}}$$

as can be seen from $\left(r^a k + r^{-b} p\right)' = a r^{a-1} k - b r^{-b-1} p = r^{a-1}\left(ak - bpr^{-(a+b)}\right)$. Evaluation gives

$$\begin{aligned}
r^a k + r^{-b} p \geq r_{\min}^a k + r_{\min}^{-b} p &= \left(\frac{bp}{ak}\right)^{\frac{a}{a+b}} k + \left(\frac{bp}{ak}\right)^{\frac{-b}{a+b}} p \\
&= \left(\frac{b^a p^a k^b}{a^a}\right)^{\frac{1}{a+b}} + \left(\frac{a^b k^b p^a}{b^b}\right)^{\frac{1}{a+b}} \\
&= \left(\left(\frac{b^a}{a^a}\right)^{\frac{1}{a+b}} + \left(\frac{a^b}{b^b}\right)^{\frac{1}{a+b}}\right)\left(k^b p^a\right)^{\frac{1}{a+b}} \\
&= \left(a^{\frac{-a}{a+b}} b^{\frac{a}{a+b}} + a^{\frac{b}{a+b}} b^{\frac{-b}{a+b}}\right) k^{\frac{b}{a+b}} p^{\frac{a}{a+b}} \\
&= \left((a+b) a^{\frac{-a}{a+b}} b^{\frac{-b}{a+b}}\right) k^{\frac{b}{a+b}} p^{\frac{a}{a+b}}.
\end{aligned}$$

The infimum is attained if $p, k > 0$, since $r^a k + r^{-b} p \to \infty$ for $r \to 0$ if $k > 0$ and for $r \to \infty$ if $p > 0$. The statement is valid if $p, k = 0$ as well.

Note that $1 \leq (a+b) a^{\frac{-a}{a+b}} b^{\frac{-b}{a+b}} \leq 2$ for all $a, b > 0$. □

Lemma A.9 (Interpolation Inequalities) *Let* E *be a Fréchet space with an increasing system of semi-norms,* $U_j = \{x : \|x\|_j \leq 1\}$*,* $a, b > 0$ *and* $\mu := \frac{b}{a+b}$*. Then the following statements are equivalent.*

(i) $\exists C_1$

$$\|.\|_{U_1} \leq C_1 \|.\|_{U_0}^{\mu} \|.\|_{U_2}^{1-\mu}$$

(ii) $\exists C_2\ \forall r$

$$\|.\|_{U_1} \leq C_2\left(r^a \|.\|_{U_0} + \frac{1}{r^b}\|.\|_{U_2}\right)$$

(ii') $\exists C_2'\ \forall r$

$$\|.\|_{U_1} \leq C_2' r^a \|.\|_{U_0} + \frac{1}{r^b}\|.\|_{U_2}$$

(ii") $\exists C_2''\ \forall r$

$$\|.\|_{U_1} \leq r^a \|.\|_{U_0} + C_2'' \frac{1}{r^b}\|.\|_{U_2}$$

(iii) $\exists C_3\ \forall r$

$$U_1 \subseteq C_3 \left(r^a U_0 + \frac{1}{r^b} U_2 \right)$$

(iii') $\exists C_3'\ \forall r$

$$U_1 \subseteq C_3' r^a U_0 + \frac{1}{r^b} U_2$$

(iii") $\exists C_3''\ \forall r$

$$U_1 \subseteq r^a U_0 + C_3'' \frac{1}{r^b} U_2$$

Proof. (i) $\Leftrightarrow$ (ii) This follows directly by A.8 with $C_1 = C_2(a+b)a^{\frac{-a}{a+b}} b^{\frac{-b}{a+b}}$.

(ii) $\Leftrightarrow$ (ii') Put $r = C_2^{-\frac{1}{a}} r'$. Then

$$\begin{aligned} \|.\|_{U_1} &\leq C_2 r^a \|.\|_{U_0} + C_2 r^{-b} \|.\|_{U_2} \\ &= C_2 \left(C_2^{-\frac{1}{a}} r' \right)^a \|.\|_{U_0} + C_2 \left(C_2^{-\frac{1}{a}} r' \right)^{-b} \|.\|_{U_2} \\ &= r'^a \|.\|_{U_0} + C_2^{1+\frac{b}{a}} r'^{-b} \|.\|_{U_2}. \end{aligned}$$

Finally, put $C_2' = C_2^{1+\frac{b}{a}}$.

(iii) $\Leftrightarrow$ (iii') Analogous to the equivalence above.

(ii) $\Leftrightarrow$ (ii") Put $r = C_2^{\frac{1}{b}} r'$. Then

$$\begin{aligned} \|.\|_{U_1} &\leq C_2 r^a \|.\|_{U_0} + C_2 r^{-b} \|.\|_{U_2} \\ &= C_2 \left(C_2^{\frac{1}{b}} r' \right)^a \|.\|_{U_0} + C_2 \left(C_2^{\frac{1}{b}} r' \right)^{-b} \|.\|_{U_2} \\ &= C_2^{1+\frac{a}{b}} r'^a \|.\|_{U_0} + r'^{-b} \|.\|_{U_2}. \end{aligned}$$

Finally, put $C_2'' = C_2^{1+\frac{a}{b}}$.

(iii) $\Leftrightarrow$ (iii") Analogous to the equivalence above.

(ii) $\Rightarrow$ (iii) From $\|.\|_{U_1} \leq C_2 \left(r^a \|.\|_{U_0} + \frac{1}{r^b} \|.\|_{U_2} \right)$ follows for $x' \in E'_{U_0^\circ}$

$$\begin{aligned} \sup_{x \in U_1} |x'(x)| &\leq C_2 \left(r^a \sup_{x \in U_0} |x'(x)| + r^{-b} \sup_{x \in U_2} |x'(x)| \right) \\ &= C_2 \sup_{r^a U_0 + r^{-b} U_2} |x'(x)|, \end{aligned}$$

and by the bi-polar theorem we get

$$U_1 \subseteq C_2 \overline{(r^a U_0 + r^{-b} U_2)},$$

where the closure is taken in respect to $\|.\|_0$. Finally, by choosing $C_3 > C_2$, we have $U_1 \subseteq C_2 \overline{(r^a U_0 + r^{-b} U_2)} \subseteq C_3 \left(r^a U_0 + r^{-b} U_2 \right)$.

(iii) $\Rightarrow$ (ii) Clearly, the U_j's are absolutely convex sets and by hypothesis we have $U_1 \subseteq U_0 + U_2$. Then for all $x' \in E'$ we have

$$\begin{aligned}
\|x'\|_{U_1} &= \sup\left\{|x'(u_1)| : u_1 \in U_1\right\} \\
&\leq \sup\left\{|x'(u_0+u_2)| : u_0 \in U_0, u_2 \in U_2\right\} \\
&= \|x'\|_{U_0+U_2} \\
&\leq \sup\left\{|x'(u_0)| + |x'(u_2)| : u_0 \in U_0, u_2 \in U_2\right\} \\
&= \sup\left\{|x'(u_0)| : u_0 \in U_0\right\} + \sup\left\{|x'(u_2)| : u_2 \in U_2\right\} \\
&= \|x'\|_{U_0} + \|x'\|_{U_2}
\end{aligned}$$

since $|x'(u_0+u_2)| \leq |x'(u_0)| + |x'(u_2)|$. □

Theorem A.10 (Characterisation of the property $(\bar{\Omega})$) *Let F be a Fréchet space. The following characterisations of the property $(\bar{\Omega})$ are equivalent.*

(i) $\forall p\ \exists q\ \forall k\ \exists C > 0$

$$\|.\|_{U_q}^2 \leq C\|.\|_{U_k}\|.\|_{U_p}.$$

(ii) $\exists d > 0\ \forall p\ \exists q\ \forall k\ \exists C > 0$

$$\|.\|_{U_q}^{1+d} \leq C\|.\|_{U_k}\|.\|_{U_p}^d.$$

(iii) $\forall d > 0\ \forall p\ \exists q\ \forall k\ \exists C > 0$

$$\|.\|_{U_q}^{1+d} \leq C\|.\|_{U_k}\|.\|_{U_p}^d.$$

(ii') $\exists\ \mu : 0 < \mu < 1\ \forall p\ \exists q\ \forall k\ \exists C > 0$

$$\|.\|_{U_q} \leq C\|.\|_{U_k}^{\mu}\|.\|_{U_p}^{1-\mu}$$

respectively

$$\|.\|_{U_q} \leq C\|.\|_{U_k}^{1-\mu}\|.\|_{U_p}^{\mu}.$$

(iii') $\forall\ \mu : 0 < \mu < 1\ \forall p\ \exists q\ \forall k\ \exists C > 0$

$$\|.\|_{U_q} \leq C\|.\|_{U_k}^{\mu}\|.\|_{U_p}^{1-\mu}$$

respectively

$$\|.\|_{U_q} \leq C\|.\|_{U_k}^{1-\mu}\|.\|_{U_p}^{\mu}.$$

(i") $\forall p\ \exists q\ \forall k\ \exists C > 0\ \forall r > 0$

$$\|.\|_{U_q} \leq C\left(r\|.\|_{U_k} + \frac{1}{r}\|.\|_{U_p}\right).$$

(ii") $\exists a, b > 0\ \forall p\ \exists q\ \forall k\ \exists C > 0\ \forall r > 0$

$$\|.\|_{U_q} \leq C\left(r^a\|.\|_{U_k} + \frac{1}{r^b}\|.\|_{U_p}\right).$$

(iii") $\forall a, b > 0 \ \forall p \ \exists q \ \forall k \ \exists C > 0 \ \forall r > 0$

$$\|.\|_{U_q} \leq C \left(r^a \|.\|_{U_k} + \frac{1}{r^b} \|.\|_{U_p} \right).$$

(i"') $\forall p \ \exists q \ \forall k \ \exists C > 0 \ \forall r > 0$

$$U_q \subseteq C \left(r U_k + \frac{1}{r} U_p \right).$$

(ii"') $\exists a, b > 0 \ \forall p \ \exists q \ \forall k \ \exists C > 0 \ \forall r > 0$

$$U_q \subseteq C \left(r^a U_k + \frac{1}{r^b} U_p \right).$$

(iii"') $\forall a, b > 0 \ \forall p \ \exists q \ \forall k \ \exists C > 0 \ \forall r > 0$

$$U_q \subseteq C \left(r^a U_k + \frac{1}{r^b} U_p \right).$$

(ii"") $\exists \mu : 0 < \mu < 1 \ \forall p \ \exists q \ \forall k \ \exists C > 0 \ \forall r > 0$

$$U_q \subseteq C \left(r^{\mu} U_k + \frac{1}{r^{1-\mu}} U_p \right)$$

respectively

$$U_q \subseteq C \left(r^{1-\mu} U_k + \frac{1}{r^{\mu}} U_p \right).$$

(iii"") $\forall \mu : 0 < \mu < 1 \ \forall p \ \exists q \ \forall k \ \exists C > 0 \ \forall r > 0$

$$U_q \subseteq C \left(r^{\mu} U_k + \frac{1}{r^{1-\mu}} U_p \right)$$

respectively

$$U_q \subseteq C \left(r^{1-\mu} U_k + \frac{1}{r^{\mu}} U_p \right).$$

Proof. (i) $\Rightarrow$ (ii) Put $d := 1$

(ii) $\Rightarrow$ (iii) For fixed d from (ii) we derive the assertion for every $d' \geq d$ since clearly $\|.\|_{U_q} \leq C \|.\|_{U_k} \left(\frac{\|.\|_{U_p}}{\|.\|_{U_q}} \right)^d \leq C \|.\|_{U_k} \left(\frac{\|.\|_{U_p}}{\|.\|_{U_q}} \right)^{d'}$.

By hypothesis, $\exists d > 0 \ \forall p \ \exists q \ \forall k \ \exists C > 0$ such that $\|.\|_{U_q}^{1+d} \leq C \|.\|_{U_k} \|.\|_{U_p}^d$ and $\forall q \ \exists q' \ \forall k' \ \exists C'$ such that $\|.\|_{U_{q'}}^{1+d} \leq C' \|.\|_{U_{k'}} \|.\|_{U_q}^d$. Combining these two statements, we get $\forall p \ \exists q, q' \ \forall k, k' \ \exists C, C'$

$$\begin{aligned} \|.\|_{U_{q'}}^{1+d} &\leq C' \|.\|_{U_{k'}} \|.\|_{U_q}^d \\ &= C' \|.\|_{U_{k'}} \left(\|.\|_{U_q}^{1+d} \right)^{\frac{d}{1+d}} \\ &\leq C' \|.\|_{U_{k'}} \left(C \|.\|_{U_k} \|.\|_{U_p}^d \right)^{\frac{d}{1+d}}. \end{aligned}$$

Putting $k = k'$ and evaluating the expressions, we get $\forall q\ \exists q'\ \forall k\ \exists C, C'$

$$\|.\|_{U_{q'}}^{1+d} \leq C' C^{\frac{d}{1+d}} \|.\|_{U_k}^{\frac{1+2d}{1+d}} \|.\|_{U_p}^{\frac{d^2}{1+d}}.$$

This leads to $\forall q\ \exists q'\ \forall k\ \exists C, C'$

$$\left(\|.\|_{U_{q'}}^{1+d}\right)^{\frac{1+d}{1+2d}} \leq \left(C' C^{\frac{d}{1+d}}\right)^{\frac{1+d}{1+2d}} \|.\|_{U_k} \left(\|.\|_{U_p}^{\frac{d^2}{1+d}}\right)^{\frac{1+d}{1+2d}},$$

simplified as

$$\|.\|_{U_{q'}}^{1+\frac{d^2}{1+2d}} \leq C'' \|.\|_{U_k} \|.\|_{U_p}^{\frac{d^2}{1+2d}}.$$

Thus we have proved (iii) also for $d' := \frac{d^2}{1+2d} \leq \frac{d}{2}$.

By induction, for any given $d' > 0$ we can find an $n \in \mathbb{N}$ such that $\frac{d}{2^n} \leq d'$ for the d from the hypothesis. In this way, every $d' > 0$ can be reached.

(iii) $\Rightarrow$ (ii) Trivial.

(ii) $\Leftrightarrow$ (ii') For $\mu = \frac{1}{1+d}$ we have

$$\|.\|_{U_q}^{1+d} \leq C \|.\|_{U_k} \|.\|_{U_p}^{d} \ \Leftrightarrow\ \|.\|_{U_q} \leq C \|.\|_{U_k}^{\mu} \|.\|_{U_p}^{1-\mu}$$

respectively for $\mu = \frac{d}{1+d}$ we get

$$\|.\|_{U_q}^{1+d} \leq C \|.\|_{U_k} \|.\|_{U_p}^{d} \ \Leftrightarrow\ \|.\|_{U_q} \leq C \|.\|_{U_k}^{1-\mu} \|.\|_{U_p}^{\mu}.$$

(iii) $\Leftrightarrow$ (iii') Analogous to the equivalence above.

(ii') $\Leftrightarrow$ (ii") and (iii') $\Leftrightarrow$ (iii") Follow directly from A.9 (i) $\Leftrightarrow$ (ii).

(ii") $\Leftrightarrow$ (ii"') and (iii") $\Leftrightarrow$ (iii"') Follow directly from A.9 (ii) $\Leftrightarrow$ (iii).

(iii") $\Rightarrow$ (i") $\Rightarrow$ (ii") and (iii"') $\Rightarrow$ (i"') $\Rightarrow$ (ii"') Trivial. □

Appendix B

An equivalent result to Proposition 11.10

Theorem B.1 ([HK00], Proposition 3.4) *Let E be an (FS)-space with an absolute basis. If E has the property $(\tilde{\Omega})$ then there exists a balanced convex compact subset B of E such that $H(B)'_\beta$ has property (LB^∞).*

Outline of the proof. Let $\{e_j : j \geq 1\}$ be an absolute basis for E. From the hypothesis, by 14.3 there exists a balanced convex compact set B_1 in E such that

$$\forall p \; \exists q, d > 0, C > 0 : \; \|.\|_{U_q}^{1+d} \leq C \|.\|_{B_1} \|.\|_{U_p}^{d} \tag{1}$$

On the other hand, since $\{e_j\}_{j \geq 1}$ is an absolute basis it follows that $\left(\|e'_j\|_{B_1} e_j \right)_{j \geq 1}$ converges to $0 \in E$. Put

$$B := \overline{\left\langle B_1 \cup \bigcup_{j \geq 1} \left\{ \|e'_j\|_{B_1} e_j \right\} \right\rangle}$$

as the closure of the convex hull of the union of B_1 and the sequence.

Now we prove that $H(B)'_\beta$ has property (LB^∞). In order to prove this, by 11.2 it suffices to show that every continuous linear map $T : H(B)'_\beta \to H(\mathbb{C})$ is bounded on a neighbourhood of $0 \in H(B)'_\beta$. Let

$$T : H(B)'_\beta \to H(\mathbb{C})$$

be given. Consider the function

$$f : B \to H(\mathbb{C})$$

defined by

$$f(x)(\lambda) = T(\delta_x)(\lambda)$$

for $x \in B, \lambda \in \mathbb{C}$, where $\delta_x \in H(B)'_\beta$ is the Dirac functional associated to x which is given by $\delta_x(\varphi) = \varphi(x)$, with $\varphi \in H(B)$. We claim that f is weakly holomorphic. Indeed, since E is an (FS)-space, so is $H(B)'_\beta$, by [BM77] 7(a), and hence it is reflexive. Now let $\mu \in H(\mathbb{C})'_\beta$ then $\mu \circ T \in (H(B)'_\beta)'_\beta \cong H(B)$ which gives a holomorphic extension of $\mu \circ f$. For each $s > 0$ let $R^s : H(\mathbb{C}) \to H^\infty(2s\mathbb{D})$ be the restriction map where $\mathbb{D}$ is the unit disc in $\mathbb{C}$. Recall that $H^\infty(2s\mathbb{D})$ is a Banach space and hence in order for a function to be holomorphic it suffices that it is weakly holomorphic and we can consider the function $h^s := R^s \circ f$ which hereby extends to a bounded holomorphic function $\widehat{h^s} : V^s \to H^\infty(2s\mathbb{D})$ on a neighbourhood V^s of B in E. Take $p \geq 1$ such that $B + U_p \subseteq V^1$ where $U_p := \{x \in E : \|x\|_p \leq 1\}$ and $(\tilde{\Omega}_{B_1})$ holds for E with this p. Let

$$V_1 := B + U_p$$

and define the function

$$\bar{g} : (B \times \mathbb{C}) \cup (V_1 \times \overline{\mathbb{D}}) \to \mathbb{C}$$

as follows

$$\bar{g}(x,\lambda) = \begin{cases} f(x)(\lambda) & : \quad x \in B,\ \lambda \in \mathbb{C} \\ \widehat{h^1}(x)(\lambda) & : \quad x \in V_1,\ \lambda \in \overline{\mathbb{D}} \end{cases} .$$

Obviously, $\bar{g}$ is separately holomorphic (see 3.13). Let $\mathcal{F}$ denote the family of all finite dimensional subspaces P of E_B, the Banach space induced by B. Put

$$\bar{g}_P = \bar{g}\,|_{((B\cap P)\times\mathbb{C})\cup((V_1\cap P)\times\overline{\mathbb{D}})} .$$

$B \cap P$ and $\overline{\mathbb{D}}$ have non-empty interiors in $V_1 \cap P$ and $\mathbb{C}$ respectively and hence are not pluripolar. By 11.6, $\bar{g}_P$ extends uniquely to a holomorphic function $\hat{g}_P$ on $(V_1 \cap P) \times \mathbb{C}$. Since $V_1 \cap E_B = \bigcup\{V_1 \cap P : P \in \mathcal{F}\}$ the family $\{\hat{g}_P\}_{P\in\mathcal{F}}$ defines a Gâteaux holomorphic function $\hat{g}$ on $(V_1 \cap E_B) \times \mathbb{C}$. On the other hand, $\bar{g}$ is holomorphic on $\{x \in B : \|x\|_B < 1\} \times \mathbb{D}$ and $\hat{g}$, by Zorn's Lemma, is holomorphic on $(V_1 \cap E_B) \times \mathbb{C}$, where $V_1 \cap E_B$ is equipped with the topology of E_B.

Now we prove that $\hat{g}$ can be extended holomorphically to $\hat{g}_1$ on $W \times \mathbb{C}$, a neighbourhood of $B \times \mathbb{C}$ in $E \times \mathbb{C}$ such that $\hat{g}_1(W \times r\mathbb{D})$ is bounded for $r > 0$. Let $q \geq p,\ d > 0,\ C > 0$ be chosen such that $(\tilde{\Omega}_{B_1})$ holds (see (1)).

Since $B = \overline{\left\langle B_1 \cup \bigcup_{j\geq 1} \left\{\|e'_j\|_{B_1} e_j\right\}\right\rangle}$ we have

$$\|e'_j\|_{B_1}\|e_j\|_B \leq 1$$

for $j \geq 1$. From the condition $(\tilde{\Omega}_{B_1})$ in (1) we have

$$\left(\frac{1}{\|e_j\|_q}\right)^{1+d} \leq \frac{C}{\|e_j\|_B\|e_j\|_p^d}. \tag{2}$$

Now let $\delta = \frac{1}{2}\left(C^{\frac{1}{1+d}}e\right)^{-1}$. Given $r > 0,\ d > 0$ we can find $s, D > 0$ such that

$$\|\sigma\|_r^{1+d} \leq D\|\sigma\|_s\|\sigma\|_1^d \tag{3}$$

for $\sigma \in H(\mathbb{C})$, where

$$\|\sigma\|_k = \sup\{|\sigma(z)| : |z| \leq k\}.$$

Write the Taylor expansion of $g : V_1 \cap E_B \to H(\mathbb{C})$, the function associated to $\hat{g} : (V_1 \cap E_B) \times \mathbb{C} \to \mathbb{C}$ at $0 \in E_B$

$$g(x) = \sum_{n=0}^{\infty} P_n g(x)$$

where

$$P_n g(x)(\lambda) = \frac{1}{2\pi\,\mathrm{i}} \int_{|t|=1} \frac{\hat{g}(tx,\lambda)}{t^{n+1}} dt$$

for $x \in V_1 \cap E_B,\ \lambda \in \mathbb{C}$.

Since $\widehat{h^s}$ is holomorphic at $0 \in E$ for every $s > 0$, we infer that $P_n g(.)(\lambda)$ is continuous on E for every λ. Let $\widehat{P_n g}$ be the symmetric n-linear form associated with $P_n g$. We have

$$\sum_{n\geq 0} |P_n g(x)(\lambda)| \leq \sum_{n\geq 0} \sum_{j_1,\ldots,j_n\geq 1} \frac{|e'_{j_1}(x)|\|e_{j_1}\|_q \cdots |e'_{j_n}(x)|\|e_{j_n}\|_q}{\|e_{j_1}\|_q \cdots \|e_{j_n}\|_q} \times |\widehat{P_n g}(e_{j_1},\ldots,e_{j_n})(\lambda)|. \quad (4)$$

Using (2), (3) and (4) we get

$$\begin{aligned}
\sum_{n\geq 0} |P_n g(x)(\lambda)| &\leq \sum_{n\geq 0} \sum_{j_1,\ldots,j_n\geq 1} \frac{D^{\frac{1}{1+d}} C^{\frac{n}{1+d}} |e'_{j_1}(x)|\|e_{j_1}\|_q \cdots |e'_{j_n}(x)|\|e_{j_n}\|_q}{\|e_{j_1}\|_B^{\frac{1}{1+d}} \cdots \|e_{j_n}\|_B^{\frac{1}{1+d}} \|e_{j_1}\|_p^{\frac{d}{1+d}} \cdots \|e_{j_n}\|_p^{\frac{d}{1+d}}} \\
&\quad \times \|\widehat{P_n g}(e_{j_1},\ldots,e_{j_n})\|_s^{\frac{1}{1+d}} \|\widehat{P_n g}(e_{j_1},\ldots,e_{j_n})\|_1^{\frac{1}{1+d}} \\
&\leq D^{\frac{1}{1+d}} \sum_{n\geq 0} C^{\frac{n}{1+d}} \frac{n^n}{n!} \|P_n g\|_{s,B}^{\frac{1}{1+d}} \|P_n g\|_{1,p}^{\frac{d}{1+d}} \|x\|_q^n \\
&\leq D^{\frac{1}{1+d}} \|g\|_{B\times s\mathbb{D}}^{\frac{1}{1+d}} \|g\|_{U_p\times\mathbb{D}}^{\frac{d}{1+d}} \sum_{n\geq 0} C^{\frac{n}{1+d}} \frac{n^n}{n!} \delta^n \\
&< \infty
\end{aligned}$$

for $x \in \delta U_p$ and $|\lambda| < r$.

Thus $\hat{g}$ is extended holomorphically to $(\delta U_q \times \mathbb{C}) \cup (V_1 \times \overline{\mathbb{D}})$. By the same argument as above g is extended holomorphically to g_1 on $V_1 \times \mathbb{C}$. Consider $\widehat{g_1} : V_1 \to H(\mathbb{C})$ associated with g_1. By the same argument as above it follows that $\widehat{g_1}$ is locally bounded. Hence there exists a neighbourhood W of B in V_1 such that $\widehat{g_1}(W)$ is bounded. Define a continuous linear map $S : H^\infty(W)' \to H(\mathbb{C})$ as

$$S(\mu)(\lambda) = \mu(\mathrm{ev}_\lambda \circ \widehat{g_1}).$$

Put $\delta : B \to H(B)'_\beta$ with $\delta(x)(\varphi) = \varphi(x),\ x \in B,\ \varphi \in H(B)$. Since (1) holds for B_1 it holds for B, i.e. E has property $(\tilde{\Omega}_B)$. This shows that B is a set of uniqueness and we infer that $\mathrm{span}(\delta(B))$ is weakly dense in $H(B)'_\beta$. Because $H(B)'_\beta$ is reflexive, $\mathrm{span}(\delta(B))$ is dense in $H(B)'_\beta$. Now we have

$$\begin{aligned}
T\left(\sum_{j=1}^m \lambda_j \delta_{z_j}\right)(\lambda) &= \sum_{j=1}^m \lambda_j T(\delta_{z_j})(\lambda) = \sum_{j=1}^m \lambda_j f(z_j,\lambda) \\
= \sum_{j=1}^m \lambda_j \widehat{g_1}(z_j,\lambda) &= \sum_{j=1}^m \lambda_j S(\delta_{z_j})(\lambda) = S\left(\sum_{j=1}^m \lambda_j \delta_{z_j}\right)(\lambda)
\end{aligned}$$

for $\lambda \in \mathbb{C}$.

Hence $S\mid_{H(B)'_\beta} = T$. Now reflect on the restriction map

$$\mathrm{res} : H^\infty(W) \to H(B).$$

Since res is continuous, it maps the unit ball U of $H^\infty(W)$ to a bounded set $\mathrm{res}(U)$. The polar $\mathrm{res}(U)^\circ$ is a neighbourhood of $0 \in H(B)'_\beta$. Since $\mathrm{res}(U)^\circ \subseteq \mathrm{res}^*(U^\circ)$ and $T \circ res^* = S$, $T\,(\mathrm{res}(U)^\circ)$ is bounded in $S(U^\circ)$. Thus $H(B)'_\beta$ has property (LB^∞).

Appendix C

Some Well-Known Theorems

Theorem C.1 *Suppose that E is a locally convex space with the topology defined by a family of semi-norms. A linear functional or a semi-norm $\|.\|_q$ on E is continuous if and only if there are a finite number of semi-norms $\|.\|_{p_1}, \ldots, \|.\|_{p_n}$ and real numbers $\lambda_1, \ldots, \lambda_n$ such that*

$$\|x\|_q \leq \lambda_1 \|x\|_{p_1}, \ldots, \lambda_n \|x\|_{p_n}$$

for all $x \in E$.

Reference. A proof can be found in [Kom99], Proposition 1.1.

Theorem C.2 (Hahn-Banach) *Suppose that $q(x)$ is a positive homogeneous sub-additive function on a real vector space E. If a linear functional $l(z)$, defined on a linear subspace F, satisfies*

$$l(z) \leq q(z) \ \forall z \in F,$$

then $l(z)$ can be extended to a linear functional u defined on the whole of E, which satisfies

$$u(x) \leq q(x) \ \forall x \in E.$$

If E is a locally convex space and $q(x)$ is continuous at 0, then u is also continuous.

Reference. A proof can be found in [Köt69a], 17.3.

Theorem C.3 (Grothendieck's closed graph theorem) *Let E and F be two separated locally convex spaces, E equipped with a topology less fine than that of an (LF)-space and F ultrabornological.*

(i) *Every continuous linear application from E to F is a homomorphism.*

(ii) *For a linear application from F to E to be continuous, it suffices that it has a closed graph.*

Reference. The original proof can be found in [Gro55].

Remark C.4 In order to show that the graph of an application $f : E \to F$ is closed, one often verifies that for a sequence $(x_i)_{n\in\mathbb{N}}$ in E with the limits $x = \lim_{n\in\mathbb{N}} x_n$ and $y = \lim_{n\in\mathbb{N}} f(x_n)$ one gets $y = f(x)$.

Theorem C.5 (Grothendieck's factorisation theorem) ([MV92], 24.33 or cf. [Gro55], I) *Let E be a separated locally convex space, F a Fréchet space, $(F_i)_{i\in\mathbb{N}}$ a sequence of Fréchet spaces, u a linear continuous map from F to E and for all $i \in \mathbb{N}$ be u_i a linear continuous map from F_i to E. Assume that $u(F) \subseteq \bigcup_i u_i(F_i)$. Then exists an index i such that $u(F) \subseteq u_i(F_i)$, and u_i is onto. There is a linear continuous map v from F to F_i such that $u = u_i \circ v$.*

Proof. Let $u : F \to E$, $u_i : F_i \to E$ as in the hypothesis above. For all i, define $H_i := \{(x, y) \in F \times F_i : u(x) = u_i(y)\}$. Since they are closed subspaces of (a product of) Fréchet space(s), they are also Fréchet spaces. For all i, let $p_i : F \times F_i \to F$ denote the projection to the first entry. We get $p_i(H_i) = \{x \in F : u(x) \in u_i(F_i)\}$ by the following computation

$$\begin{aligned} z \in p_i(H_i) &\Leftrightarrow \exists (x, y) \in H_i : z = p_i(x, y) = x \\ &\Leftrightarrow \exists (x, y) \in F \times F_i : u(x) = u_i(y) \text{ and } z = x \\ &\Leftrightarrow \exists y \in F_i : u(z) = u_i(y) \\ &\Leftrightarrow \exists y \in F_i : z \in u^{-1}(u_i(y)) \\ &\Leftrightarrow z \in u^{-1}(u_i(F_i)). \end{aligned}$$

By hypothesis we get $F = \bigcup_i p_i(H_i)$ and by C.15 also $F = p_i(H_i)$, i.e. $u(F) = u_i(F_i)$. Suppose that u_i is bijective, thereby for all $x \in F$ there exists a unique $y \in F_i$ such that $u_i(y) = u(x)$, i.e. $(x, y) \in H_i$. This y depends obviously linearly on x, even $y = v(x)$. The map $x \to v(x) : F \to F_i$ is continuous by Grothendieck's closed graph theorem since its graph H_i is closed. □

Theorem C.6 (Riesz' representation theorem) *Let u be a continuous linear functional on $C(E)$, where E is a compact space. Then there exists a unique regular Borel measure μ on E such that*

$$u(f) = \int_E f d\mu$$

for all $f \in C(E)$.

μ is positive if and only if u is positive. Moreover, the map $u \to \mu$ establishes an isometric isomorphism of $C(E)'$ onto the space of the regular Borel measures on E.

Reference. A proof can be found in [Jar81], 7.6.1 or in [Köt69a], 17.7.4 for E a real interval.

Definition C.7 Let E be a Hausdorff locally convex space. E is called a *Schwartz space* if for every balanced, closed convex neighbourhood U of 0 in E there exists a neighbourhood V of 0 in E such that for every $\alpha > 0$ the set V can be covered by finitely many translates of αU, i.e. there are $x_1, \dots, x_n \in V$ such that $V \subseteq \bigcup_{i=1}^n (x_i + U)$.

Theorem C.8 *Let E be a complete Schwartz space. Then E' is ultrabornological.*

Reference. The original proof can be found in [Sch57a]. A modern version in [MV92], 24.23.

Theorem C.9 *If U is an open subset of a locally convex space E and F is a normed linear space then $f \in H_G(U,F)$ is holomorphic if and only if it is locally bounded.*

Reference. A proof can be found in [Din99], Proposition 3.7.

Lemma C.10 *The quotient space of a metrizable locally convex space L over a closed subspace M is metrizable, and if L is complete then L/M is complete.*

Reference. A proof can be found in [Sch71], I.6.3.

Lemma C.11 ([Sch71], III.2) *Let L and M be metrizable locally convex spaces (e.g. Fréchet spaces with $|x| := \sum_{n\in\mathbb{N}} \frac{1}{2^n}\frac{\|x\|_n}{1+\|x\|_n}$). We denote by $U_r := \{x \in L : |x| \le r\}$ and $U_\rho := \{y \in M : |y| \le \rho\}$ the closed balls of centre 0 and radius r, ρ. Let L be complete and let u be a continuous linear map of L into M satisfying*

$$\forall\ r > 0\ \exists \rho = \rho(r) > 0 : \overline{u(U_r)} \supset U_\rho. \tag{1}$$

Then $u(U_t) \supset U_\rho$ for each $t > r$.

Proof. Let r and t be fixed, $t > r > 0$ and denote by $(r_n)_{n\in\mathbb{N}}$ a sequence of positive real numbers such that $r_1 = r$ and $\sum_{n=1}^\infty r_n = t$. Let $(\rho_n)_{n\in\mathbb{N}}$ be a null sequence of positive numbers such that $\rho_1 = \rho$ and for each $n \in \mathbb{N}, \rho_n$ satisfies $\overline{u(U_{r_n})} \supset U_{\rho_n}$. For each $y \in U_\rho$, we must establish the existence of $z \in U_t$ with $u(z) = y$.

We define inductively a sequence $(x_n)_{n\in\mathbb{N}}$ such that for all $n \ge 1$

$$|x_n - x_{n-1}| \le r_n, \tag{2}$$

$$|u(x_n) - y| \le \rho_{n+1}. \tag{3}$$

Set $x_0 = 0$ and assume that for $k \ge 1$ $x_1, x_2, \ldots, x_{k-1}$ have been selected to satisfy (2) and (3). By (1), the set $u(x_{k-1} + U_{r_k})$ is dense with respect to $u(x_{k-1}) + U_{\rho_k}$. From (3) we conclude that $y \in u(x_{k-1}) + U_{\rho_k}$; thus there exists x_k satisfying $|x_k - x_{k-1}| \le r_k$ and $|u(x_k) - y| \le \rho_{k+1}$.

Since $\sum_{n=1}^\infty r_n$ converges, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete space L and thus converges to some $z \in L$. Clearly, $|z| \le t$, and $u(z) = y$ follows from the continuity of u and (3), since $(\rho_n)_{n\in\mathbb{N}}$ was chosen to be a null sequence. □

Definition C.12 Let L and M be locally convex spaces. A continuous linear map $u : L \to M$ is called *homomorphism* if for each open subset $G \subset L$ the image $u(G)$ is an open subset of $u(L)$ for the topology induced by M.

Examples of homomorphisms are for any subspace H of L the canonical quotient map $u : L \to L/H$ and the canonical imbedding $u : H \to L$.

Lemma C.13 ([Sch71], III.1.2.a/b) *Let L and M be locally convex spaces and let $u : L \to M$ be a continuous linear map. With the aid of the canonical quotient map φ and the canonical embedding ψ we can decompose u into*

$$L \xrightarrow{\varphi} L/u^{-1}(0) \xrightarrow{u_0} u(L) \xrightarrow{\psi} M,$$

where u_0 is a bijective map and called the associated *map with u.*

Then the following assertions are equivalent.

(i) *u is a homomorphism.*

(ii) *For every neighbourhood base $\mathcal{U}$ of 0 in L, $u(\mathcal{U})$ is a neighbourhood base of 0 in $u(L)$.*

(iii) *The map u_0 associated with u is an isomorphism.*

Proof. Since u is open, every element of $u(\mathcal{U})$ is a neighbourhood of 0, and $u(\mathcal{U})$ is a base at 0 in $u(L)$ since u is continuous.

Since $\varphi(\mathcal{U})$ is a neighbourhood base of 0 in $L/u^{-1}(0)$ for any neighbourhood base of 0 in L, u_0 has the second property and is consequently an isomorphism.

Since φ and u_0 are continuous and open, so is $u = \psi \circ u_0 \circ \varphi$, and hence is a homomorphism. □

Remark C.14 A Baire space is, by definition, a topological space in which every non-empty open subset is not meager. This implies that every locally convex space L over $\mathbb{F}$ which is non-meager (of second category) in itself, is a Baire space. Otherwise, there would exist a meager, non-empty, open subset of L and hence a meager neighbourhood U of 0. Since L is a countable union of homothetic images of U (hence of meager subsets), we arrive at a contradiction.

Theorem C.15 (Open mapping theorem) ([Sch71], III.2.1) *Let L, M be complete metrizable locally convex spaces and let u be a continuous linear map of L with range dense in M. Then either $u(L)$ is meager (of first category) in M, or else $u(L) = M$ and u is a homomorphism.*

Proof. Suppose that $u(L)$ is not meager in M. We continue to use the notation of C.11. The family $\{S_r\}_{r>0}$ is a neighbourhood of 0 in L. For fixed r, let $U := S_r, V := S_{\frac{r}{2}}$; then $V + V \subseteq U$ and $u(L) = \bigcup_{n=1}^{\infty} nu(V)$, since V is absorbent, i.e. for every $x \in L$ there exists $\rho_x > 0$ such that $[0, \rho_x]x \subseteq V$. Since, by assumption, $u(L)$ is a Baire space, there exists an $n \in \mathbb{N}$ such that $\overline{nu(V)}$ has an interior point. Hence $\overline{u(V)}$ has an interior point. Now

$$\overline{u(V)} + \overline{u(V)} \subseteq \overline{u(V) + u(V)} = \overline{u(V+V)} \subseteq \overline{u(U)}$$

and thus $\overline{u(U)}$ is a neighbourhood of 0 in $u(L)$, since 0 is an interior point to $\overline{u(V)} + \overline{u(V)}$. Hence there exists a $\rho > 0$ such that $u(L) \cap S_\rho \subseteq u(S_{r+\varepsilon})$ for every $\varepsilon > 0$. Thus $\{u(S_t) : t > 0\}$ is a neighbourhood base of 0 in $u(L)$, hence by C.13, u is a homomorphism. The quotient space of a complete metrizable locally convex space over a closed subspace is itself complete and metrizable. Therefore u_0 is an isomorphism of the space $L/u^{-1}(0)$ onto $u(L)$. Again from C.13 it follows that $u(L) = M$. □

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Index