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Generic Core-Emptiness in Spatial Models of Voting

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Michael Greinecker

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Contents

1 Introduction

In 1967, Charles Plott, showed in [27] that when an odd number of people with smooth utility functions vote over arbitrary alternatives in Euclidean space, each alternative is dominated by another alternative under majority rule unless one voter gets her preferred alternative and the other voters can be paired of so as to having utility gradients pointing in opposite directions (See Figure 1).

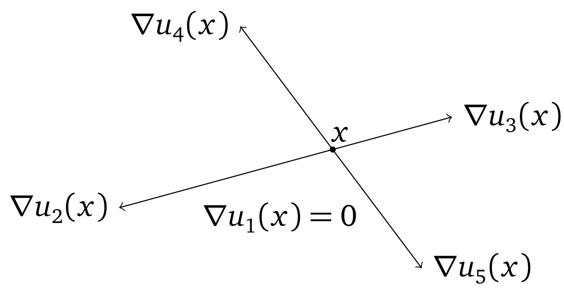

Figure 1: Point x satisfies Plotts condition.

This situation seems highly unstable and such alternatives may seldom exist. Almost never. That one may give a precise notion of "almost never"' became clear to economists after Gerard Debreu introduced economists in 1970 in [11] to a generic way of reasoning. Debreus analysis was based on the notion of full Lebesgue measure. Something holding for generic economies could be interpreted as holding for a randomly chosen economy with probability one. But these techniques were useless for studying Plotts problem. There is no such thing as Lebesgue measure for spaces of utility functions, so researchers had to use a topological notion of genericity. The first results obtained on the core of some voting games being generically empty, that used this topological notion of genericity, turned out to be wrong, and it was in 1995 when Jeffrey Banks gave the first rigorous theorem on generic core-emptiness, something we will explain in more detail later. There were still some problems. The topology on the space of utility functions being used was quite unintuitive, corresponding little to our intuitive notion of space. In addition, topological notions of genericity don't correspond to any probabilistic notion of genericity. Help comes from the theory of prevalence, developed by Brian Hunt, Tim Sauer and James Yorke. Their notion of genericity has a nice probabilistic interpretation in terms of random distortions and corresponds to full

Lebesgue measure in the finite dimensional case. This thesis shows that one can give prevalent versions of the topological results, and discusses the ways one can formulate and interpret notions of genericity. Furthermore, a very simple to prove new generic core-emptiness result for Euclidean preferences is given. The method of proof is instructive for the case of more general utility functions. It is not the most powerful result one can give though. It makes use of a well known result (Corollary 6.1, usually proven like Theorem 7.1) for which an easy new proof is given, that draws on intuition from high school geometry. Readers not interested in mathematical details can read the section without following first a long discussion on topologies on function spaces for the general case. They simply have to take it on face value that "almost all" $n \times n$-matrices have full rank. The prevalent core-nonemptiness results are new, but straightforward modifications of existing results.

In the few cases were a gender pronoun is called for in a sentence, I use female pronouns, which have as much claim to being generic as male ones.

2 Mathematical Preliminaries

This section is supposed to make the text reasonably self-contained. What is presumed to be known is basic set theoretic notation and vocabulary, some linear algebra and multivariable calculus. Remarks in footnotes may relate to mathematical ideas not explained in the preliminaries, but the main text should be readible after working through the mathematical preliminaries. They are however meant to be more of a refresher than a solid introduction. In particular little motivation and no proofs are given. Readers looking for additional sources will find the material on topology and measure theory in solid graduate textbooks on real analysis such as [31] or [7]. A readable introduction to topics of the geometry chapter is [41], more on the projection theorem can be found in [21], which covers the issues in a more general setting. The literature on differential topology is much less accessible. One can find lots of motivation in [28], but for the meat one has to go elsewhere. A more solid book is [14], but there's little about the Thom transversality theorem and nothing about jets. For this, [16] and the very demanding [13] are good sources.

2.1 Topology

A *topology* on a set X is a family $\tau \subset 2^X$ such that:

(i) $X \in \tau$ and $\emptyset \in \tau$.

(ii) $O_1 \in \tau$ and $O_2 \in \tau$ implies $O_1 \cap O_2 \in \tau$.

(iii) $\mathscr{O} \subset \tau$ implies $\bigcup \mathscr{O} \in \tau$.

If τ is a topology on X, we call the pair (X, τ) a *topological space*[1]. The elements of τ are the *open* sets. The complement (in X) of an open set is a *closed* set. For any subset S of a topological space, there is a largest open set contained in S, its *interior*, $\mathrm{int}(S)$, and a smallest closed superset of S, its *closure*, $\mathrm{cl}(S)$. A set S is a *neighborhood* of a point x if there exists an open set O such that $x \in O \subset S$. If τ_1 and τ_2 are topologies on X then τ_1 is *weaker* than τ_2 or τ_2 *stronger* than τ_1 if $\tau_1 \subset \tau_2$.

Let $\mathscr{F}$ be a family of subsets of X such that $A \in \mathscr{F}$ and $B \in \mathscr{F}$ implies $A \cap B \in \mathscr{F}$. Then the family of unions of elements of $\mathscr{F}$ is a topology on X and $\mathscr{F}$ is a *basis* for that topology. The collection of all open intervals forms a basis for a topology on $\mathbb{R}$, the *natural topology*. Every family of subsets S of a set X generates a topology on X if one takes the set of all finite intersections of elements of $\mathscr{S}$ as a basis. We call S a *subbasis* for the topology thus created. If (X, τ_1) and (Y, τ_2) are topological spaces, $\tau_1 \times \tau_2$ is the basis for a topology, the *product topology*, on $X \times Y$. We will always endow the product of finitely many topological spaces with the product topology. We use the product topology to define the *natural topology* on $\mathbb{R}^n$ for $n > 1$. A topology has the *Hausdorff property* if for any two points $x \neq y$ there exists open sets O_1 and O_2 such that $x \in O_1$, $y \in O_2$ and $O_1 \cap O_2 = \emptyset$. The natural topology on $\mathbb{R}^n$ has the Hausdorff property. If S is a subset of a topological space (X, τ), $(S, \{S \cap O : O \in \tau\})$ is a *topological subspace*. We always endow subsets with their subset topology.

A function $f : X \to Y$ from a topological space (X, τ_1) to a topological space (Y, τ_2) is *continuous* if we have $f^{-1}(O) \in \tau_1$ for every $O \in \tau_2$. A bijective continuous function with a continuous inverse is a *homeomorphism*. If there is a homeomorphism between two topological spaces, we call them *homeomorphic* and view them as topologically indistinguishable. A subset K of a topological space is *compact* if we have for every family of open sets $\mathscr{C}$ with $K \subset \bigcup \mathscr{C}$ a finite family $\mathscr{F} \subset \mathscr{C}$ with $K \subset \bigcup \mathscr{F}$. A topological space (X, τ) is separable if there exists a countable set $D \subset X$ such that $X = \mathrm{cl}(D)$.

[1] We will often be sloppy and call X itself a topological space when it is clear or not important what topology is being used. We will do the same thing with other spaces, like metric spaces or measure spaces.

One way to define a topology on a set is by using a metric. A *metric* (or distance) d on a set X is a function $d : X \times X \to \mathbb{R}$ such that:

(i) $d(x, y) = 0$ if and only if $x = 0$.

(ii) $d(x, y) = d(y, x)$.

(iii) $d(x,z) \leq d(x, y) + d(y,z)$

for all $x, y,z \in X$. One can easily show that a metric takes on only nonnegative values. If d is a metric on X, the pair (X, d) is a metric space. For every $x \in X$ and $\epsilon > 0$ we define the *open ball* with center x and radius ϵ to be $B(x,\epsilon) = \{y \in X : d(x, y) < \epsilon\}$. We call a subset of a metric space containing an open ball around each of its points open. The collection of all open sets of a metric space form a Hausdorff topology. If a topology can be obtained this way from some metric, we call the topological space *metrizable*.

A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space *converges* to a point x if we have for every $\epsilon > 0$ a natural number N such that $d(x_n, x) < \epsilon$ for all $n > N$. A sequence in a metric space can converge to at most one point. A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space is a *Cauchy sequence* if for every $\epsilon > 0$ there is a natural number N such that $(x_n, x_m) < \epsilon$ for all $n,m > N$. A metric space is *complete* is every Cauchy sequence converges to some point.

A vector space[2] V endowed with a topology that makes both addition $+ : V \times V \to V$ and scalar multiplication $\mathbb{R} \times V \to V$ continuous, is a *topological vector space*. There is only one topology with the Hausdorff property on a finite dimensional vector space that makes it a topological vector space, and that is the natural topology. This means that an n-dimensional topological vector space has linear homeomorphism to $\mathbb{R}^n$. An important class of topological vector spaces is the class of normed spaces. A *norm* N on a vector space V is a function $N : V \to \mathbb{R}$ such that:

(i) $N(x) = 0$ implies $x = 0$.

(ii) $N(\alpha x) = |\alpha| N(x)$.

(iii) $N(x + y) \leq N(x) + N(y)$.

It is easily shown that a norm takes on only nonnegative values. If N is a norm on V, one can define a metric $d : V \times V \to \mathbb{R}$ by setting $d(x, y) = N(x - y)$.

[2] We will assume all vector spaces to be real vector spaces.

This metric makes the normed space into a Hausdorff topological vector space. If the corresponding metric space is complete, we call the normed space a *Banach space*. The Euclidean Norm of an $x \in \mathbb{R}^n$ is $\|x\| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$. It gives a complete metric. A useful result for normed spaces is *Riesz' lemma*. It says that if S is a closed proper subspace of a normed space and $\theta \in (0,1)$, then there exists a vector x of norm 1 such that $N(x-y) > \theta$ for all $y \in S$. Together with the fact that finite dimensional subspaces of normed spaces are closed, this implies that compact subsets of infinite dimensional normed spaces contain no open ball. A subset of $\mathbb{R}^n$ on the other hand is, according to the *Heine-Borel theorem*, compact if and only if it is a closed set contained in a sufficiently large open ball.

2.2 Geometry

Let V be a vector space and $v_1, v_2, \dots, v_n$ be a finite number of elements of V. A vector $v \in V$ is an *affine combination* of $v_1, v_2, \dots, v_n$ if there are real numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ with $\sum_{i=1}^n \alpha_i = 1$ such that $v = \sum_{i=1}^n \alpha_i v_i$. Le S be a subset of V. The *affine span* of S, denoted by $\mathrm{aff}(S)$, is the set of all affine combinations of elements of S. A set that equals its affine hull is called a *flat*. A nonempty set is a flat if and only if it is the translate of a vector subspace. A set of vectors is *affinely independent* if no vector in it is an affine combination of other vectors in the set. The *dimension* of a flat F is the (unique) number of elements[3] in a minimal set of vectors having affine span F.

Let V be a vector space and $v_1, v_2, \dots, v_n$ be a finite number of elements of V. A vector $v \in V$ is a *convex combination* of $v_1, v_2, \dots, v_n$ if there are real numbers $\alpha_1 \geq 0, \alpha_2 \geq 0, \dots, \alpha_n \geq 0$ with $\sum_{i=1}^n \alpha_i = 1$ such that $v = \sum_{i=1}^n \alpha_i v_i$. The *convex hull* of S, denoted by $\mathrm{con}(S)$, is the set of all convex combinations of elements of S. Clearly $\mathrm{con}(S) \subset \mathrm{aff}(S)$. A *convex* set is a set that equals its convex hull.

A *hyperplane* H is a maximal flat smaller than V. That is H is a flat other than V and any flat containing H equals either H or V. In an n-dimensional vector space, hyperplanes have dimension $(n-1)$. A real valued linear function p on V is a *linear functional*. A linear functional is *nondegenerate* if it takes on

[3] At least in the finite dimensional case. But even for infinite dimensional flats, the cardinality of a minimal set having the flat as affine span is uniquely determined. This follows from the corresponding theorem for the dimension of a vector space and the fact that each flat is the translate of a vector subspace.

values other than zero. If p is a nondegenerate linear functional on V, the set $\{x \in V : p(x) = c\}$ for some $c \in \mathbb{R}$ is a hyperplane. Any hyperplane can be written this way as the level set of a linear functional. A *closed halfspace* is a set of the form $\{x \in V : p(x) \leq c\}$ with p being a nondegenerate linear functional and c a real number.

For the rest of this section let $V = \mathbb{R}^n$, endowed with the natural topology. The convex hull of a finite set is compact. Given any point x and a closed convex set C, there is a unique point $c \in C$ that is closest to x. Given a closed, convex set C and a convex, compact set K such that $C \cap K = \emptyset$, there is a linear functional p and a constant c such that $p(x) < c < p(y)$ for all $x \in C$ and $y \in K$. This result is known as the *separating hyperplane theorem*. An important consequence is that a convex, closed set C equals the intersection of all halfspaces containing C.

Two vectors $x, y \in \mathbb{R}^n$ are *normal* to each other if their inner product $\sum_{i=1}^n x_i y_i$ is 0. The *Pythagorean theorem* says that if x and y are normal to each other, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$. Let S be a proper vector subspace and $x \notin S$. A necessary and sufficient condition for a point y in S being closest to x is that $(x - y)$ is normal to each element of S. This is known as the *projection theorem*.

A *cone* is a subset V of a vector space such that $V + V \subset V$, $V \cap -V = \{0\}$ and $\alpha V \subset V$ for all $\alpha \geq 0$.[4] An *open cone* is the interior of a cone. A *semi-positive combination* of vectors $v_1, \dots, v_n$ is a linear combination with nonnegative weights of which at lest one is larger than zero. The *semi-positive span* of a subset S of a vector space is the set of all semi-positive combinations of elements of S.

2.3 Measure Theory

A *σ-algebra* $\mathscr{B}$ on a set X is a nonempty family of subsets of X closed under complementation and countable unions. That is

(i) $\mathscr{B} \neq \emptyset$.

(ii) $B \in \mathscr{B}$ implies $X - B \in \mathscr{B}$.

(iii) If $B_i \in \mathscr{B}$ for all $i \in \mathbb{N}$ then $\bigcup_{i=1}^{\infty} B_i \in \mathscr{B}$.

[4] Many authors call that a proper convex cone pointed at zero.

For any family $\mathscr{F}$ of subsets of X, there is a smallest σ-algebra containing all elements of $\mathscr{F}$. We call it the σ-algebra *generated* by $\mathscr{F}$. If (X, τ) is a topological space, then the σ-algebra generated by τ is called the *Borel σ-algebra* on (X, τ) and its elements are the *Borel sets*. A pair $(X, \mathscr{B})$ with X being a set and $\mathscr{B}$ being a σ-algebra on X is a *measurable space* and the elements of $\mathscr{B}$ are *measurable sets*.

The *extended real line* $\overline{\mathbb{R}}$ is the set $\mathbb{R} \cup \{-\infty, \infty\}$ endowed with an extension of the usual order on $\mathbb{R}$ so that $-\infty < r < \infty$ for all $r \in \mathbb{R}$.

A *measure* μ on a measurable space $(X, \mathscr{B})$ is a function $\mu : \mathscr{B} \to \overline{\mathbb{R}}$ such that:

(i) $\mu(B) \geq 0$ for all $B \in \mathscr{B}$.

(ii) $\mu(\emptyset) = 0$.

(iii) If $B_i \in \mathscr{B}$ for all $i \in \mathbb{N}$ and $B_i \cap B_j = \emptyset$ for $i \neq j$ then $\mu(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} \mu(B_i)$.

If μ is a measure on $(X, \mathscr{B})$, we call the triple $(X, \mathscr{B}, \mu)$ a *measure space*. We call μ a *probability measure* if $\mu(X) = 1$. A measure defined on the Borel σ-algebra of a topological space is a *Borel measure*. A measure space $(X, \mathscr{B}, \mu)$ is *complete* if any subset of a set with measure zero is measurable.

An *outer measure* μ^* on a set X is a function $\mu^* : 2^X \to \overline{\mathbb{R}}$ such that:

(i) $\mu^*(\emptyset) = 0$.

(ii) $A \subset B$ implies $\mu^*(A) \leq \mu^*(B)$.

(ii) $B \subset \bigcup_{i=1}^{\infty} B_i)$ implies $\mu^*(B) \leq \sum_{i=1}^{\infty} \mu^*(B_i)$.

According to *Caratheodory's extension theorem*, the family

$$\mathscr{B} = \{A \subset X : \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap X - E) \text{ for all } E \subset X\}$$

is a σ-algebra on X and $(X, \mathscr{B}, \mu^*)$ is a complete measure space if μ^* is an outer measure on X.

2.4 Differential Topology

A function from an open set of $\mathbb{R}^n$ to $\mathbb{R}^p$ is said to be C^r if all partial derivatives up to order r exist at all points and are continuous. A function is C^∞ or *smooth* if it is C^r for all $r \in \mathbb{N}$. A C^r *diffeomorphism* is a C^r homeomorphism with a C^r inverse. A subset M of $\mathbb{R}^n$ is a C^r *manifold* of dimension d if every point in M has an open neighborhood U that has a diffeomorphism ϕ with an open subset V of $\mathbb{R}^n$ so that $\phi(M \cap U) = (\mathbb{R}^d \times \{0\}) \cap V$, where $0 \in \mathbb{R}^{n-d}$.[5] So a manifold has a local diffeomorphism around each point. If $M \subset \mathbb{R}^n$ is a C^r manifold of dimension d, its *codimension* is $n - d$. Manifolds can locally be approximated by some $\mathbb{R}^d$. The surface of the earth looks pretty flat, so a sphere is a two-dimensional manifold. To say what space a manifold looks like locally, we use the notion of a tangent space. The *tangent space* of a C^r manifold M at a point $x \in M$, denoted by T_xM, is the inverse image of the differential of a local diffeomorphism. One can show that the choice of the diffeomorphism doesn't matter which makes the tangent space well defined. We write the *differential* of a C^1 map from $\mathbb{R}^n$ to a manifold $M \subset \mathbb{R}^m$ as a linear map with $Df_x = \mathbb{R}^m \to T_{f(x)}M$.

Let $O \subset \mathbb{R}^n$ be open and $f : O \to \mathbb{R}^m$ a C^r function. For $k \leq r$ define the *k-jet* of f at x, denoted by $j^k f(x)$ to be $(x, f(x), Df(x), \dots, D^k f(x))$, which is x together with its k^{th} order Taylor expansion. We can view $j^k f$ as a C^{r-k} function from O to the *jet space* (of order k) $J^k(\mathbb{R}^n, \mathbb{R}^m) = \mathbb{R}^n \times P^k(\mathbb{R}^n, \mathbb{R}^m)$ with $P^k(\mathbb{R}^n, \mathbb{R}^m)$ being the space of polynomials from $\mathbb{R}^n$ to $\mathbb{R}^m$ with degree at most k. The jet space $J^k(\mathbb{R}^n, \mathbb{R}^m)$ is a finite dimensional vector space and will therefore be endowed with the natural topology. It is possible to define a jet space of infinite order. The construction is a little bit involved (see [24]). For our purposes it suffices to know that it admits a completely metrizable topology.

Two lines in a plane intersect usually in exactly one point, and two lines in three dimensional space not at all. Two planes in three dimensional space usually intersect on a whole line and a plane and a line in three dimensional space usually intersect in a point. These intersections have the property of being transversal.

Definition 2.1 *Let $M, N \subset \mathbb{R}^n$ be C^1 manifolds. They intersect* transversally *if their tangent spaces jointly span $\mathbb{R}^n$ at each point of intersection. So for all*

[5] There is a more involved definition of a manifold that does not assume the manifold to be embedded in some $\mathbb{R}^n$, but in our applications this will never be necessary.

$x \in M \cap N$ one has:

$$T_x M + T_x N = \mathbb{R}^n.$$

Observe that two manifolds intersect transversally if they do not intersect at all. If the sum of their dimensions is smaller than n, this is the only possible type of tranversal intersection. The condition that the dimensions of the manifolds is smaller than n is equivalent to one manifold having codimension larger than the dimension of the other manifold. One can also define what it means for a manifold and a function to intersect transversally.

Definition 2.2 *Let $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ be C^1 manifolds and $f : M \to \mathbb{R}^n$ be a C^1 function. We say that f intersects N* transversally *if for all $x \in f^{-1}(N) \cap M$ one has*

$$T_{f(x)} N + Df_x T_x M.$$

This definitions are basically equivalent. If $M, N \subset \mathbb{R}^n$ intersect transversally, than the inclusion map $\iota_M : M \to \mathbb{R}^n$ given by $\iota(x) = x$ intersects N transversally. Conversely, one can interpret a transversal intersection of a function and a manifold as the transversal intersection of its graph and some manifold. We will later give theorems that will give mathematical content to the idea that intersections are usually transversal.

3 Notions of Genericity

3.1 Lebesgue Measure

It's natural to define the length of an interval $[a, b]$ in $\mathbb{R}$ as $b-a$, the area of an rectangle $[a_1, b_1] \times [a_2, b_2]$ in $\mathbb{R}^2$ as $(b_1-a_1)(b_2-a_2)$ and the volume of a block $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ in $\mathbb{R}^3$ as the product $(b_1 - a_1)(b_2 - a_2)(b_3 - a_3)$. Generalizing this, the measure λ of a n-fold Cartesian product of intervals in $\mathbb{R}^n$ is the product of the lengths of these intervals, so that

$$\lambda\left(\prod_{i=1}^{n} [a_i, b_i] \right) = \prod_{i=1}^{n} (b_i - a_i).$$

Let $\mathscr{I}$ be the collection of n-fold products of intervals in $\mathbb{R}^n$. So far, λ is a function defined on $\mathscr{I}$. We can extend λ to all of $2^{\mathbb{R}^n}$ by defining

$$\lambda(S) = \inf\left\{ \sum_{I \in \mathscr{C}} \lambda(I) : \mathscr{C} \subset \mathscr{I}, S \subset \bigcup \mathscr{C}, \text{ and } \mathscr{C} \text{ countable} \right\} \text{ for every } S \subset \mathbb{R}^n.$$

One can verify that λ is an outer measure, it is called Lebesgue outer measure. We can use Caratheodory's extension theorem to define a complete measure from it, *Lebesgue measure*. It is straightforward to show that all sets in $\mathscr{I}$ are measurable. Since countable unions of elements of $\mathscr{I}$ form a basis for the natural topology on $\mathbb{R}^n$, every Borel set is measurable too.

Moving objects around in space shouldn't change their volume. Lebesgue measure[6] λ respects this requirement by being translation invariant; that is for every measurable set S and every $x \in \mathbb{R}^n$ we have $\lambda(S) = \lambda(S + x)$. A (tedious) way to see this is by noting that Lebesgue outer measure is obviously translation invariant, and showing that the construction used in the proof of Caratheodory's extension theorem is invariant with respect to translations. We will not bother with the details of the verification here.

If a property P of elements of $\mathbb{R}^n$ holds only on a set of Lebesgue measure zero, that is $\lambda(\{x \in \mathbb{R}^n : P(x)\}) = 0$, we say that P holds *almost nowhere* and non-P holds *almost everywhere* and that $\{x \in \mathbb{R}^n : \neg P(x)\}$ has *full Lebesgue measure*. One can easily show that every set having Lebesgue measure zero is contained in a Borel set having Lebesgue measure zero.

Full Lebesgue measure is the natural candidate for genericity for finite dimensional parameter spaces. We run, however, into trouble when trying to define an infinite dimensional version of Lebesgue measure[7]:

Theorem 3.1 *If μ is a translation-invariant measure on an infinite-dimensional, separable, normed space, then μ is identically zero or infinite on every open set.*

Proof:

Suppose O is an open set with finite measure. O contains an open ball with radius ϵ. By Riesz' Lemma, there is a sequence of disjoint open balls with radius $\epsilon/4$ contained in this ball. Since O has finite measure, these balls all have the same measure, zero, and so does every ball with radius $\epsilon/4$ since μ is translation invariant. Now the whole space is covered by countably many balls with radius $\epsilon/4$, since it is separable, and has therefore measure zero.

■

If we drop separability, we still have nonempty open sets with zero measure, something very strange.

[6] From now on we will reserve the letter λ for Lebesgue measure.

[7] The result is taken from [17].

3.2 Residual Sets

The lack of a translation invariant measure for infinite dimensional spaces has led mathematicians to define topological notions of genericity. A *residual* set is the countable intersection of open dense sets. The Baire category theorem asserts that residual subsets of complete metric spaces are dense. Any space which has this property is called a *Baire space*.

Residual sets have often been used in economics as the notion of genericity. Donald Saari for example wrote (in [33]) that something true for a residual set of preferences should be interpreted as

> everything except improbable, carefully concocted examples which are not indicative of what can happen because the conclusion can fail with even a slight change in the preferences.

If we accept full Lebesgue measure as the natural notion of genericity in the finite dimensional case, residual sets should have full Lebesgue measure there. This can fail completely[8]:

Theorem 3.2 *There exists a residual Lebesgue measure zero subset S of $\mathbb{R}$.*

Proof:

Let $D = \{d_1, d_2, \ldots\}$ be a countable dense subset of $\mathbb{R}$. Define $I_{i,j}$ to be the open interval with center d_i and length $1/2^{i+j}$, and let $S = \bigcap_{j=1}^{\infty}(\bigcup_{i=1}^{\infty} I_{i,j})$.

Let $\epsilon > 0$ and choose j so that $1/2^j < \epsilon$. Then $\lambda(S) \leq \lambda(\bigcup_{i=1}^{\infty} I_{i,j}) \leq \sum_{i=1}^{\infty} \lambda(I_{i,j}) = \sum_{i=1}^{\infty} 1/2^{j+i} = 1/2^j < \epsilon$. So S has Lebesgue measure zero.

The set $\bigcup_{i=1}^{\infty} I_{i,j}$ is open and dense for every j, so S is residual as the intersection of countably many of these sets.

■

A somewhat stronger topological notion of genericity is given by identifying open dense sets with generic sets. Still, it is clear from the proof of Theorem 3.2 that open dense subsets of $\mathbb{R}$ can have arbitrarily small positive Lebesgue measure.

3.3 Prevalence and Shyness

While it is not possible to generalize Lebesgue measure to infinite dimensional spaces, one can generalize the notion of Lebesgue measure zero using a char-

[8] The exaple is taken from [26].

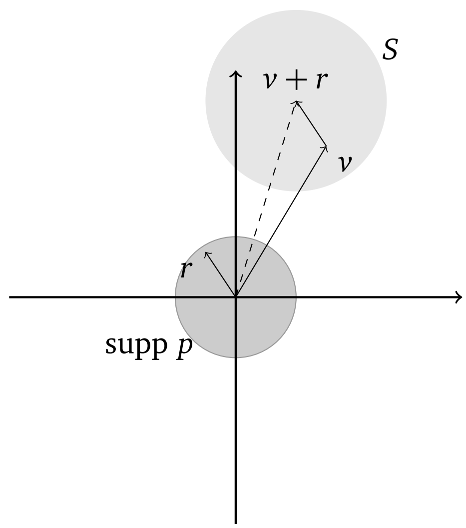

Figure 2: Any vector v plus a distortion lies almost surely outside S.

acterization that is applicable in both settings. This is what Brian Hunt, Tim Sauer and James Yorke did (in [17]).

Theorem 3.3 *A set $S \subset \mathbb{R}^n$ has Lebesgue measure zero if and only if there exists a Borel probability measure p with compact support such that every translate of S has p-measure zero.*

■

Theorem 3.3 provides a nice intuition on Lebesgue measure zero: Interpret the support of p as a set of distortions, and take any vector $v \in \mathbb{R}^n$ (see Figure 2). Then by distorting the vector v by a random distortion, we land outside S with probability one. More exactly, for all $v \in \mathbb{R}^n$, one has

$$p\Big(\{r \in \operatorname{supp} p : v + r \notin S\}\Big) = 1.$$

Following Hunt, Sauer and Yorke we can use this characterization as a definition of genericity based on probabilistic intuition:[9]

[9]Jens Peter Reusen Christensen has already defined an equivalent notion in the separable case in [9]. A *Haar zero* subset of a separable, completely metrizable topological vector space (or, more generally, topological abelian group) V is a universally measurable set S such that a measure μ on V exists with $\mu(S + x) = 0$ for all $x \in V$. The equivalence of shyness and Haar zero in the separable case is explained in [39].

Definition 3.1 *Let V be a completely metrizable topological vector space. A Borel set $E \subset V$ is said to be **shy** if there exists a probability measure μ with compact support such that $E + x$ has μ-measure zero for every $x \in V$. A subset of a shy set is also said to be shy. The complement of a shy set is said to be **prevalent**.*

Hunt, Sauer and Yorke prove that shyness has properties similar to the ones of Lebesgue measure zero:

Theorem 3.4 *Shyness satisfies the following conditions:*

(i) A shy set has empty interior.

(ii) Every translate of a shy set and every subset of a shy set is shy.

(iii) A countable union of shy sets is shy.

■

We omit the proof. But (ii) is trivial and (i) relatively easy. The proof of (iii) is rather complicated.

Even though prevalence and shyness have a probabilistic intuition behind them, prevalence is unrelated to probabilistic beliefs. Maxwell Stinchcombe pointed out (in [39]) that every probability measure on a separable, completely metrizable topological vector space assigns measure zero to some prevalent set.

It should be noted that we need the measure μ in the definition of shyness only to be positive and finite on some compact set. If the definition is satisfied with such a measure we can restrict it to a compact set and normalize it to be a probabilty measure on the compact set. A very useful measure for this purpose is Lebesgue measure on a finite dimensional subspace. We call such a subspace a *probe*. The nice thing with probes is that when working with probes, we can forget about the details of the vector space topology being used since the topology of a finite dimensional subspace is always Euclidean.

4 Spaces of Preferences and Utility Functions

Here we study ways of topologizing spaces of preferences and spaces of utility functions. We are mainly concerned with preferences and utility functions on

$\mathbb{R}^n$. We will see that spaces of preferences usually don't allow for the metrizable topological vector space structure needed to apply the theory of prevalence. For this reason we will directly work with spaces of utility functions[10]. These may seem to be less "fundamental" than preferences, but what space preferences are "really" drawn from is moot metaphysics, so we will not spend time on discussing this.

At the end of the section, we present transversality theorems, which are useful for proving results on the genericity of some property in spaces of utility functions.

4.1 Preferences

The most general class of preferences we will look at in this section is the class of continuous preferences on $\mathbb{R}^n$. A *continuous preference relation* is a complete and transitive relation on $\mathbb{R}^n$ that has a graph closed in $\mathbb{R}^{n\times n}$. We call the class of continuous preferences $\mathscr{C}$.

A topology on $\mathscr{C}$ that has been used for stuying core emptyness for majority rule is the *Kannai topology* introduced by Yakar Kannai in [18] for studying markets. The Kannai topology is the weakes topology that makes the set $\{(x,y,\preceq) : x \succ_{\preceq} y\}$ open in $\mathbb{R}^{n\times n}\times\mathscr{C}$ for all $P\in\mathscr{C}$. Equivalenty it is the weakest topology satisfying that if $x_n\to x$, $y_n\to y$, $\preceq_n\to\preceq$ and $x\succ y$ then $x_n\succ_n y_n$ for n large enough. The Kannai topology was used by Ariel Rubinstein in [32] who showed that if the choice space is a convex, compact subset of $\mathbb{R}^n$ with nonempty interior and there are at least three voters, then the majority rule core fails to exist on an open dense set of preference profiles in the Kannai topology. The Kannai topology has however a strong defect: $\mathscr{C}$ with the Kannai topology is not a Hausdorff space. The following example is inspired by [29]: Take $\preceq$ as the preference relation that ranks all alternatives as indifferent and let $(\preceq_n)$ be any (!) sequence in $\mathscr{C}$. Then $(\preceq_n)$ converges to $\preceq$. The problem with the Kannai topology is its reliance on the existence of enough strict rankings locally[11].

Another approach one can take is working with so called *hyperspace topologies*, these are topologies defined on subsets of some topological space. Since

[10]Another way of getting generic results in a probabilitsic sense would be to work with random variables taking values in a space of preferences as in [15]. It is not clear to the author of this thesis wether such an approach would be workable four our purposes.

[11]This is of course no problem for the theory of markets, where local non-satiation is usually assumed.

the graph of a continuous preference relation on a set X is a closed subset of $X \times X$, we can directly apply topologies on the set of closed subsets of $X \times X$. The most popular such topology in economics is the *Fell topology* developed by James M. G. Fell (in[12]), who called it the *H-topology*. We define this topology by giving a basis for it. Let X be a topological space and let $CL(X)$ be the set of closed subsets of X. For every compact subset C of X and every finite set $\mathscr{F}$ of nonempty open subsets of X let $U(C,\mathscr{F})$ be the set of all $Y \in CL(X)$ such that $Y \cap C = \emptyset$ and $Y \cap A \neq \emptyset$ for all $A \in \mathscr{F}$. It can be shown that the collection of all such sets $U(C,\mathscr{F})$ is a basis for a topology, namely the Fell topology. Provided every neighborhood of every point in X contains a compact neighborhood, we call X *locally compact*. The Fell topology makes $CL(X)$, for a locally compact space X, into a compact Hausdorff space. If X is completely metrizable and locally compact, then $CL(X)$ is compact, separable and completely metrizable (for a proof, see [2]). In this case, the Fell topology is also known as the *topology of closed convergence*. This is also the name usually used in the economics literature. When working with satiated preferences, the Fell topology is arguably too weak. There's an example in [29] where a sequence of prefences that is arguable not supposed to be convergent still converges.

Since we are eventually going to apply the powerful tools of differential topology, we are going to work with "smooth preferences". Preferences can basically be described by a partition giving all the indifference curves and a total order[12] on the partition cells. It is then possible to endow the indifference surfaces with some differential structure. One can for example demand that they are manifolds of a certain class. This approach is developed for example by Graciela Chichilnisky in [8], such topologies don't admit a linear structure though.

4.2 Utility Functions

If X and Y are topological spaces, denote the set of continuous functions from X to Y by $C(X,Y)$.

Definition 4.1 *Let X, Y be topological spaces. The* compact-open topology *on $C(X,Y)$ is generated by the subbasis*

$$\{\{f \in C(X,Y) : f(K) \subset O\} : K \subset X, O \subset Y \text{ with } K \text{ compact and } O \text{ open}\}.$$

[12]A *total order* is a preference ordering satifying anti-symmetry. That is, a preference ordering in which all indifference curves are singletons.

The compact-open topology was introduced by Richard Arens (in [3]), who called it the k-topology. Arens gave the following useful result:

Theorem 4.1 *If Y is metrizable by a metric d, then $C(X,Y)$ with the compact-open topology is metrizable if and only if there exists a countable family $\mathscr{K} = \{K_1, K_2, \ldots\}$ of compact subsets of X such that $X = \bigcup_{i=1}^{\infty} K_i$ and every compact subset of X is covered by finitely many elements of $\mathscr{K}$. It can be metrized by the metric*

$$m(f,g) = \sum_{i=1}^{\infty} \min\{1/2^n, \max\{d(f(x), g(x)) : x \in K_i\}\}.$$

The metric m makes $C(X,Y)$ into a complete metric space, if and only if d makes Y into a complete metric space.

■

From this we get that $C(\mathbb{R}^n, \mathbb{R})$ with the compact-open topology is a completely metrizable topological vector space. We simply take $\mathscr{K}$ to be the set $\{[0,1]^n + z : z \in \mathbb{Z}^n\}$. What we are really interested in are, however, not just continuous functions, but differentiable functions. There exists a natural mapping indentifiying $C^r(\mathbb{R}^n, \mathbb{R})$ with a subset of $C(\mathbb{R}^n, J^r(\mathbb{R}^n, \mathbb{R}))$ (see [16] for the details). When we endow the latter set with the compact-open topology we can endow $C^r(\mathbb{R}^n, \mathbb{R})$ with the subspace topology[13]. We call the resulting topology the C^k *compact-open topology* on $C^r(\mathbb{R}^n, \mathbb{R})$. Now we have the following useful result due to Peter W. Michor (from [24]):

Theorem 4.2 *The C^k* compact-open topology *on $C^r(\mathbb{R}^n, \mathbb{R})$ is completely metrizable if $k \leq r$. It makes $C^r(\mathbb{R}^n, \mathbb{R})$ into a topologial vector space.*

■

A sequence of functions (f_n) in $C^r(\mathbb{R}^n, \mathbb{R})$ converges to f in this topology if and only if all derivatives up to the r^{th} and converge in the sup-metric[14] on every compact set to the corresponding derivative of f and f_n on every compact set to f. We will use this topology when looking at prevalent and shy subsets of $C^r(\mathbb{R}^n, \mathbb{R})$.

[13] What we really do is using an embedding of $C^r(\mathbb{R}^n, \mathbb{R})$ in $C(\mathbb{R}^n, J^r(\mathbb{R}^n, \mathbb{R}))$. We endow the image of the embedding with the subspace topology and give $C^r(\mathbb{R}^n, \mathbb{R})$ the topology that makes the embedding a homeomorphism with its image.

[14] The sup-metric for bounded functions on some set S is defined by the rule $d(f,g) = \sup\{|f(x) - g(x)| : x \in S\}$. Continuous, real-valued functions on compact sets are bounded, so we can apply the definition.

Another topology is widely used in the area this thesis is concerned with, the Whitney topology. We can construct it in a similar way. We start with another topology:

Definition 4.2 *Let X, Y be topological spaces. The* wholly open topology *on $C(X,Y)$ is generated by the basis*

$$\{\{f \in C(X,Y) : f(X) \subset O\} : O \subset Y \text{ with } O \text{ open}\}.$$

This topology isn't very nice, it's not even Hausdorff. But the Whitney topology is. The *Whitney C^k topology* on $C^r(\mathbb{R}^n, \mathbb{R})$ for $k < \infty$ is constructed in exactly the same way as we have constructed the C^k compact-open topology by replacing the compact-open topology by the wholly open topology (see [24]). We define the *Whitney C^∞ topology* or simply *Whitney topology* as the topology having the union of all Whitney C^k topologies with $k < \infty$ as a basis. The Whitney C^k topologies on $C^r(\mathbb{R}^n, \mathbb{R})$ are Hausdorff and have actually much more open sets than the compact-open C^k topologies. The Whitney C^k topologies are not metrizable, which makes it useless for studying prevalent and shy properties. A sequence of functions (f_n) in $C^r(\mathbb{R}^n, \mathbb{R})$ converges to f in the Whitney topology if and only if there exists a compact set K so that all $r+1$ first derivatives of (f_n) converge in the sup-metric to the corresponding derivative of f on K and all but finitely many elements of (f_n) equal f outside of K. This implies that $C^\infty(\mathbb{R}^n, \mathbb{R})$ is not a topological vector space. For example in $C^\infty(\mathbb{R}, \mathbb{R})$ the sequence of functions defined by $f_n(x) = x/n$ does not converge to 0 although we have $(1/n)f$ converging to 0 for every function f in a topological vector space. The reason that the Whitney topology is still very popular is that $C^\infty(\mathbb{R}^n, \mathbb{R})$ with the Whitney topology is a Baire space. So one has lots of open sets but it is still relatively easy to be dense. This is nice for topological genericity results.

Now we are ready to give theorems to the effect that a function usually intersects a manifold transversally. The classical transversality result is the Thom transversality theorem. Here's a slightly weaker version sufficient for our purposes:

Theorem 4.3 *Let $Z \subset \mathbb{R}^n$ be a smooth manifold, and $1 \leq r \leq \infty$. Then the set of functions in $C^r(\mathbb{R}^m, \mathbb{R}^n)$ that are transversal to Z is residual and hence dense in the space $C^r(\mathbb{R}^m, \mathbb{R}^n)$ endowed with the Whitney topology or the compact-open topology. Furthermore,if Z is closed, then the set of functions in $C^r(\mathbb{R}^m, \mathbb{R}^n)$ that* *are transversal to Z is open in the space $C^r(\mathbb{R}^m, \mathbb{R}^n)$ endowed with the Whitney topology.*

■

The proof can be found in [16]. We will also make use of the following prevalent version of the Thom transversality theorem due to Hunt, Sauer and Yorke:

Theorem 4.4 *Let $Z \subset \mathbb{R}^m$ be a C^r manifold of codimension c, with $r > \max\{n-c, 0\}$. For $\max\{n-c, 0\} < k \leq \infty$, a prevalent set of functions in $C^k(\mathbb{R}^n, \mathbb{R}^m)$ is transversal to Z.*

■

We don't give the proof here, but note that it relies completely on a probe. For this reason, the result is independent of the specific topology employed, as long as it makes the space into a completely metrizable topological vector space.

5 Basic Concepts from Social Choice Theory

This section introduces some basic notions used later on[15]. In the following, we will assume that there is a finite set I of agents, N in number, whose members have to make a collective choice from a set X of *alternatives*. Nonempty subsets of I are *coalitions*. Each agent i has a preference ordering $\preceq_i$ on X that is represented by an utility function $u_i : X \to \mathbb{R}$. When we specify a utility function for every agent, we have a *profile* $u : I \to \mathbb{R}^X$. There is a set of admissible profiles $\mathcal{U} \subset \mathbb{R}^{X^I}$. We will assume that all profiles we are talking about lie in such a set $\mathcal{U}$.

An *aggregation rule* $f : \mathcal{U} \to 2^{X \times X}$ associates an asymmetric relation $\prec$ on X, the *social ordering*, to every utility profile such that the ordering of two alternatives only depends on how each agent orders the two alternatives. An aggregation rule satisfies what is commonly called independence of irrelevant alternatives, the social ordering of two alternatives depends only on how the individuals in a profile rank the two alternatives.

The *core* of an aggregation rule f under a profile u is the set of elements in X that are maximal under the social ordering. An alternative is in the core if and only if no alternative is socially preferred.

[15]The material of this section can be found in [5] and [36].

Given an aggregation rule f and a profile u there is a set of *decisive coalitions* $\mathscr{D}(f,u) \subset 2^I$ such that $D \in \mathscr{D}(f,u)$ if and only if $a \prec b$ holds whenever $a \prec_i b$ holds for all $i \in D$ for any alternatives a, b. The intersection of all decisive coalitions for a utility profile u, $\bigcap_{D \in \mathscr{D}(f,u)} D$, is called the *collegium* of f under u. The collegium of f can be interpreted as the set of voters who can veto every policy change under f. An aggregation rule f is *non-collegial* if the collegium is empty for all $u \in \mathscr{U}$. Given a nonempty family $\mathscr{D} \subset 2^I$ such that $C \in \mathscr{D}$ implies $I - C \notin \mathscr{D}$ and such that every superset of a set in $\mathscr{D}$ is also in $\mathscr{D}$, we can define a aggregation rule $f_{\mathscr{D}}$ with decisive coalitions $\mathscr{D}$ for all $u \in \mathscr{U}$ such that for any alternatives a, b we have $a \prec b$ whenever $a \prec_i b$ for all i in an element of $\mathscr{D}$. An aggregation rule f is *simple* if $f = f_{\mathscr{D}(f,u)}$ for all u.

For any $r \in \mathbb{R}, r \geq 0$ let $[r]$ be the largest natural number not larger than r. A *q-rule* is defined by the set of decisive coalitions $\mathscr{D} = \{C \subset I : \mid C \mid \geq q\}$ for some $q \geq [N/2 + 1]$. It is clearly simple and is non-collegial if and only if $q \leq N - 1$. *Unanimity rule* is the q-rule with $q = N$ and *majority rule* is the q-rule with $q = [N/2 + 1]$.

An alternative a is (weakly) *Pareto optimal* for coalition D if there is no alternative b that is strictly preferred to a for all members of D. If we say a is Pareto optimal without mentioning for which coalition, it is understood that we mean the grand coalition I. If an alternative a is in the core, then a is Pareto optimal for all decisive coalitions. If a rule is simple, then this actually characterizes the core.

The following simple lemma will be useful later on. It is implicit in [6]:

Lemma 5.1 *Given an profile u, the core either exists for the* $(N-1)$*-rule or it exists for no non-collegial aggregation rule.*

Proof:

Let $D(f,u)$ be the set of decisive coalitions for a non-collegial aggregation rule f under profile u and suppose there is no core for the $(N-1)$-rule. Since f is non-collegial there exists a coalition not containing i for every $i \in I$. So $I - \{i\} \in D(f,u)$ for every $i \in I$. So every coalition with at least $N-1$ members is decisive. But no alternative is accepted by all such coalitions.

■

6 Instability in Voting: Euclidean Preferences

We first prove an instability theorem for the special parametric case of Euclidean preferences, in which each voter i has a favorite alternative, her *bliss point* $b_i \in X = \mathbb{R}^n$, and wants the chosen alternative to be as close as possible to b_i in Euclidean distance, so that her preferences can be represented by a utility function that gives utility $-\|x - b_i\|$ for alternative x. The approach will be instructive for what we are doing when working with general smooth utility functions. Since utility functions depend only on the bliss points, we can do with a finite dimensional parameter space and full Lebesgue measure as our notion of genericity. Euclidean preferences have been introduced by Otto Davis and Melvin Hinich in 1966 in [10] There are some problems with Euclidean preferences. Jeffrey Milyo for example has shown that standard economic preferenes over two public goods cannot be represented by Euclidean preferences in [25].

Theorem 6.1 *Let $C \subset \mathbb{R}^n$ be a nonempty, closed, covex set and $x \notin C$. Then there is a point $y \in C$ such that $\|y - z\| \leq \|x - z\|$ for all $z \in C$.*

Proof:

Let y be the unique point in C closest to x. Without loss of generality, we can assume C to be a closed halfspace bounded by a hyperplane H containing y, so that y is still closest to x in this halfspace. By the projection theorem, $(x - y)$ is normal to H. Now let z be any point in C and y_1 be the projection of z on H. The vectors $(y - y_1)$ and $(y_1 - z)$ are normal to each other, so by the Pythagorean theorem $\|y - z\| = \sqrt{\|y - y_1\|^2 + \|y_1 - z\|^2}$. Now let y_2 be the projection of z on $H + (x - y)$. Again, since $(x - y_2)$ and $(y_2 - z)$ are normal to each other, we have $\|x - z\| = \sqrt{\|x - y_2\|^2 + \|y_2 - z\|^2}$. Now since H and $H + (x - y)$ are parallel, we have $(x - y_2) = (y - y_1)$ and $\|y_2 - z\| = \|y_2 - y_1\| + \|y_1 - z\| > \|y_1 - z\|$.

■

The following corollary is common knowledge in the literature:

Corollary 6.1 *If an alternative x is Pareto optimal for coalition L, then x is in $con\{b_i : i \in L\}$.*

Proof:

Suppose $X \notin \text{con}\{b_i : i \in L\}$. By Theorem 6.1 there is a point y in $\text{con}\{b_i : i \in L\}$ closer to every point in $\text{con}\{b_i : i \in L\}$ than x, especially closer to

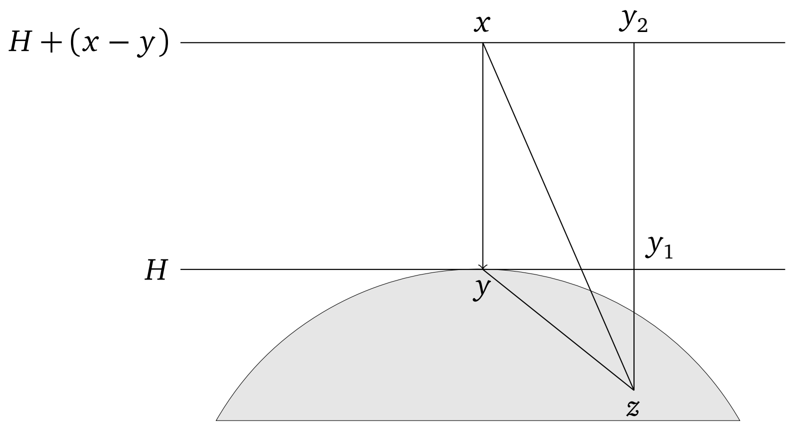

Figure 3: Theorem 6.1 illustrated.

$\{b_i : i \in L\}$. So all members of L prefer y to x and x cannot be Pareto optimal for L.
■

This implies that for simple rules with D being the set of decisive coalitions, the core is a subset[16] of

$$\bigcap_{C \in D} \text{con}\{b_i : i \in D\} \subset \bigcap_{C \in D} \text{aff}\{b_i : i \in D\}.$$

So if $X = \mathbb{R}^2$ and there are three voters, the majority rule core will be empty if the bliss points are not lying on a line. Generalizing this example leads to a generic core-emptyness theorem.

Lemma 6.1 *Take $N, n \in \mathbb{N}$ with $n \neq N$ and let $\pi_1, \ldots, \pi_N$ be projections, where π_j projects $\mathbb{R}^{nN}$ to the j^{th} $\mathbb{R}^n$. For almost every point $p \in \mathbb{R}^{nN}$ it is true that any n points in $U = \{\pi_i(p) : 1 \leq i \leq N\}$ are affinely independent.*

Proof:

It clearly suffices to show that they are linearly independent. Take S to be any subset of $\{1, \ldots, N\}$ with n elements. Now $\mathbb{R}^{nS}$ has rank n almost everywhere in $\mathbb{R}^{nS}$, a space of $n \times n$ matrices, because according to Proposition

[16]Actually, the core equals $\bigcap_{C \in D} \text{con}\{b_i : i \in D\}$ (see [33]), but we are only interested in core-emptiness results here and make no use of the full equality.

2 in [20], all subsets of lower rank $n \times n$ matrices are lower dimensional manifolds which have measure zero according to Lemma 1.5 in [13]. Let $\mathscr{S}$ be the set of n-element subsets of $\{1, \ldots, N\}$, and R_S be the set of non-full rank matrices in $\mathbb{R}^{nS}$ for all $S \in \mathscr{S}$. So for all $S \in \mathscr{S}$ we have $R_S \times \mathbb{R}^{Nn-n}$ has measure zero, by Fubinis theorem (see for example [31]). Now $\mathscr{S}$ has finitely many elements, so $\bigcup_{S \in \mathscr{S}} R_S \times \mathbb{R}^{Nn-n}$ has measure zero.
■

Theorem 6.2 *Suppose X is $\mathbb{R}^n$. For almost all profiles, the core of the q-rule with $q \leq n < N$ is empty.*

Proof:

There are at least N pivotal coalitions. The affine span of the blisspoints in these N coalitions has dimension at most $q - 1 < n$. Without loss of generality, assume that there are N such coalitions and that the affine span of the bliss points of its members, denoted by A_i has dimension $n - 1$. Now for any $m + 1$ such flats, the dimension of $\bigcap_{i=1}^{m} A_i \cap A_{m+1}$ is at most $\dim(\bigcap_{i=1}^{m} A_i) - 1$, by Lemma 6.1. So $\dim(\bigcap_{i=1}^{N} A_i) \leq (n-1) - (N-1) < 0$. It follows that this intersection is empty.
■

The intuition behin the theorem is straightforward. Intersecting more than n flats of dimension $n-1$ leads to an empty intersection, for every intersection makes one lose one dimension. The intersection of two plane is a line that intersected with a plane again is a point that has no intersection with the next plane. That this works follows from the affine independence of the points.

Corollary 6.2 *Suppose X is $\mathbb{R}^n$ and $n = N - 1$. Then the core of any non-collegial rule is empty for almost all profiles.*

Proof:

From Theorem 6.2 we know that in this case, the core is generically empty for the $N - 1$-rule. By Lemma 5.1, it is then empty for all non-collegial rules.
■

One can easily strengthen the theorem to allow any convex subset of R^n with nonempty interior as the alternative space X.[17] If one restricts intersections to lie in a smaller set, they are certainly not becoming more likely. Convexity is needed in order to get Lemma 6.1. In order to show that the

[17] This has been suggested by Egbert Dierker.

blisspoints are almost always affinely independent, it suffices to note that nonempty open subsets of R^n have positive Lebesgue measure and using the fact that the boundary of a convex set has Lebesgue measure zero[18].

The emptiness of the core has grave consequences in the model with Euclidean preferences. Richard McKelvey has shown in [22] that the emptiness of the core for simple majority voting implies that there exists a "path of improvement" in the social preference ordering between any two points of $\mathbb{R}^n$. So for any alternatives a and b there exists finitely many elements $a_1, \dots, a_n$ such that $a \prec a_1 \prec \dots \prec a_n \prec b$ for the social preference ordering $\prec$. Maria Tataru has generalized this result to hold for arbitrary q-rules in [40].

7 Instability in Voting: Smooth Preferences

In this section, we give a necessary condition for the core of a q-rule to be empty and use it to show that the core is almost always empty. The condition was initially derived by Richard McKelvey and Norman Schofield (in [23]), who tried to show that core emptiness holds on a residual set in the Whitney topology for certain dimensionality conditions. Their proof contained a mistake that was pointed out by Jeffrey Banks (in [6]) who also showed that the correct, implied, dimensionality conditions are less tight. We will follow Banks in this section. Giving a measure theoretic version of the result is a straightforward extension. Tighter conditions have been derived by Donald Saari (in [33]). We will not give the (extremely lengthy) proof for his conditions[19], but they allow for stronger results on the generic non-emptiness of the core.

Theorem 7.1 *If an alternative x is Pareto optimal for coalition L, then 0 is in the semi-positive span of all gradient vectors at x of members of L.*

Proof:

Suppose 0 is not in the semi-positive span of all gradient vectors at x of members of L. Then 0 is not an element of $Y = \text{con}\{\nabla u_i(x) : i \in L\}$. Since $\{0\}$ is convex and compact and Y convex and closed, we can apply the separating hyperplane theorem and get a vector $p \in \mathbb{R}^n$ such that $p \cdot y > 0$ for all $y \in Y$. In particular, we have $p \cdot \nabla u_i(x) > 0$ for all $i \in L$. Setting $tp \equiv h$ we get by the

[18] See [19] for a proof of this fact.

[19] An exposition of the central ideas can be found in [35].

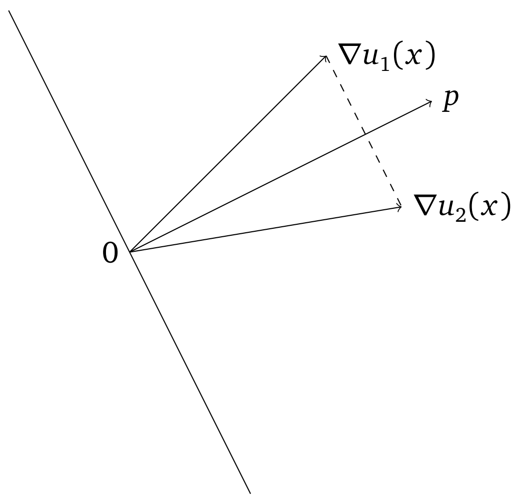

Figure 4: Theorem 7.1 illustrated.

definition of the derivative for all $i \in L$

$$0 < p \cdot \nabla u_i(x) = p \lim_{h \to 0} \frac{u_i(x+h) - u_i(x)}{h} = \lim_{t \to 0} \frac{u_i(x+tp) - u_i(x)}{t},$$

so a small move in the direction p is a Pareto improvement over x for members of L.

■

This was first shown by Stephen Smale (in [38]). Geometrically, this means that the utility gradients are not allowed to lie in an open cone. Any vector in such a cone would be a direction of common improvement for all members of the coalition. Under appropriate convexity conditions, the converse holds too.

The core is the set of all alternatives that are Pareto optimal for all decisive coalitions. So if an alternative x is in the core, the set of utility gradients is semi-positively dependent for every decisive coalition and hence linearly dependent. Checking this for q-rules is relatively easy. So, given a q-rule, let $\mathscr{D}$ be the set of all decisive coalitions and for each $K \in \mathscr{D}$ define

$$\Lambda_K(u) \equiv \{x \in X : \{\nabla u_i(x) : i \in K\} \text{ is linearly dependent}\}.$$

The core is a subset of $\Lambda(u) \equiv \cap_{K \in \mathscr{D}} \Lambda_K(u)$. We want to show that $\Lambda(u)$, and hence the core, is empty for most utility profiles. For this we show that $\Lambda(u)$

is a set of certain singularities of the Jacobian matrix

$$J_u(x) = \begin{pmatrix} \nabla u_1(x) \\ \vdots \\ \nabla u_n(x) \end{pmatrix}$$

Define $S_p(u)$ to be the set of alternatives at which the Jacobian matrix has rank p given the profile u. We have the following result:

Lemma 7.1 *$\Lambda(u) = \cup_{p=0}^{q-1} S_p(u)$ holds for every $q - rule$.*

Proof:

All decisive coalitions have at least q members, so any set of at least q gradients in the profile is linearly dependent at an alternative $x \in \Lambda(u)$. This is the same as all sets of at least q rows of $J_u(x)$ being linearly dependent. This is the same as the rank of $J_u(x)$ being at most $q-1$ or x being in $S_p(u)$ for some $p \leq q-1$.

■

A Jacobian matrix $J_u(x)$ can be interpreted as the value of the *Jacobian map*[20] $J_u : \mathbb{R}^n \to \mathbb{R}^{Nn}$ at x. Define L_p to be the subset of $\mathbb{R}^{Nn}$ whose elements, interpreted as $N \times n$-matrices, have rank p. Then

$$S_p(u) = J_u^{-1}(L_p).$$

This means that x is in $S_p(u)$ if the function J_u intersects L_p at x. According to Proposition 2 in [20], L_p is a smooth manifold of codimension equal to $(n-p)(N-p)$[21].

Theorem 7.2 *Suppose $n > \frac{(N-q+1)(q-1)}{(N-q)}$ and J_u is transversal to L_p for $p = 0, \ldots, q-1$. Then the core is empty.*

Proof:

Since L_p has codimension $(n-p)(N-p)$, the transversal intersection $J_u^{-1}(L_p) = S_p$ is empty or has codimension $(n-p)(N-p)$ and hence dimension $n-(n-p)(N-p)$. If $n < (n-p)(N-p)$, the intersection has to be empty. Now we are only interested in $S_p(u)$ for $p \leq n, N$, for no larger dimension can

[20] $J_u(x)$ is simply the Fréchet differential of u at x.

[21] The proof is nontrivial. Levine shows that a certain point lies in L_p and that L_p is the orbit of a group preserving rank and having differentiable orbits

the rank be full. In that range $(n-p)(N-p)$ is nonincreasing in p. By 7.1, we have this to check for $p = q-1$ in order to have an empty core. This gives us the following equivalent inequalities

$$n < (n-q+1)(N-q+1)$$

$$n < n(N-q+1) - (q-1)(N-q+1)$$
$$n - n(N-q+1) < -(q-1)(N-q+1)$$
$$n(N-q+1) - n > (q-1)(N-q+1)$$
$$n(N-q+1-1) > (q-1)(N-q+1)$$
$$n > \frac{(N-q+1)(q-1)}{(N-q)}.$$

■

We are now ready for Banks main theorem:

Theorem 7.3 *Suppose $n > \frac{(N-q+1)(q-1)}{(N-q)}$ and $\mathscr{U}$ is the set of C^∞ functions on $\mathbb{R}^n$. Then the core is empty for an open dense subset of $\mathscr{U}^N$ in the Whitney topology.*

Proof:

By Theorem 7.2 and the Thom transversality theorem, the core is empty on a residual and hence dense subset as the intersection of residual sets in a Baire space. Now according to Proposition 1 in [20], we have $\mathrm{cl}(L_k) = \bigcup_{j=0}^{k} L_j$. Together, these results establish that the core fails to exist on a open dense subset of $\mathscr{U}^n$ in the Whitney topology.

■

We can easily give a prevalent version of Banks result:

Theorem 7.4 *Suppose $n > \frac{(N-q+1)(q-1)}{(N-q)}$ and $\mathscr{U}$ is the set of C^∞ functions on $\mathbb{R}^n$. Then the core is empty for a prevalent subset of $\mathscr{U}^N$ for any completely metrizable (such as the compact-open topology) topology.*

Proof:

Replace the Thom transversality theorem by its prevalent version.

■

Banks result is not the strongest result available. The sharpest results were obtained by Donald Saari in [33]. Here's the main result of Saari (slightly simplified):

Theorem 7.5 *The core of a $q-$rule rule fails to exist in the Whitney topology for residual set of utility profiles if $n > 2q - N$.*
■

Moreover Saari shows that these results are as strong as possible. If $n \leq 2q-N$, there exists examples of nonempty open sets of preferences on which the core exists. Since these examples are constructed with Euclidean preferencces, one cannot get stronger results with the compact-open topology on utility profiles. The genericity proof is based on the jet-transversality theorem, a slightly stronger version of the Thom transversality theorem we used. This result works with prevalence as our notion of genericity, as shown in [17]. Saaris proof is considerably more complicated, since his approach requires some case distinctions. He's working with normalized utility gradients, and analyses the resulting patterns on the unit sphere. Since one cannot normalize the 0-vector, he has to treat the case in which one voter may be infinitesimally satiated, so called *bliss core points* independently of the non-bliss core points. A readable overview of the central ideas of the proof can be found in [35].

We could strengthen our results by replacing the space of alternatives $\mathbb{R}^n$ by some n-dimensional manifold. This allows us to use open sets, which may be bounded, as spaces of alternatives, since unbounded sets of alternatives are hard to interpret. If we want to strengthen the results to compact sets with nonempty interior, we can can replace "residual" in Saaris theorem by "open". When we want to apply prevalence methods in the case of compact sets with nonempty interior, we have to work with so-called manifolds with boundary, and the existing prevalence transversality theorems are not adopted for this case. The complication is that for boundary points, utility gradients that point outside the choice space are no hindrance to core existence.

8 Interpretation

So far we have shown that, generically, no alternative is undominated under the ordering of a q-rule, given the preferences of the agents. In the first part of this section we discuss what this implies for the desirability of certain ways of social decision making and what boundaries this sets on normative criteria. In the second part we discuss what the implications are for a positive theory of political decision making.

8.1 Normative Issues

The largest class of aggregation rules we have shown to have empty cores generically are non-collegial aggregation rules. We discuss their normative desirabilty and discuss alternative means of social decision making. There is no apparent reason why the complicated structure of our choice space should matter here, so we can discuss the major issues with examples with finitely many alternatives and no particular algebraic or topological structure on them. The normative content of non-collegial aggregation rule is basically this.

(i) The outcome of aggregation is an ordering of alternatives.

(ii) The preference ordering depends only on the preferences of the agents.

(iii) The nature of the alternatives does not matter.

(iv) The ranking of a pair of alternatives does not depend on how other alternatives are ranked.

(v) Nobody can veto any change.

We will not spend much time discussing (ii) [22], (iii) and (v). There may be reasons in which they do not seem reasonable. They ignore issues such as rights or differences between altruistic or sadistic preferences. In many contexts, these issues shouldn't matter.

(i) is a little bit more controversial. An ordering is not really necessary, a more general approach would be possible. Let X be some nonempty set and define $\mathscr{X} = 2^X/\{\emptyset\}$. A *choice function* is a function $C : \mathscr{X} \to 2^X$ satisfying $C(X) \subset X$. We say that a choice function C is *rationalizable* by a relation $R \subset X \times X$ if $C(X)$ is the set of R-maximal elements of X for all $X \in \mathscr{X}$. One can show (see [1] for further information) that a choice function is rationalizable if and only if for all $X, X', X'' \in \mathscr{X}$:

(H) $X' \subset X$ implies $C(X') \supset C(X) \cap X'$.

(C) $X' \cup X'' = X$ implies $C(X') \cap C(X'') \subset C(X)$.

[22] An important class of ways of voting that violate this condition are systems of *range voting*, the most important example being *approval voting*. In approval voting, each voter can either approve or disapprove of each alternative. Alternatives are then ranked according to how many voters approve of them.

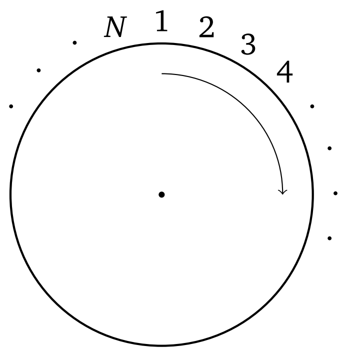

Figure 5: The ranking wheel.

Both conditions have a natural interpretation as some form of optimality. (H) means that any alternative that is optimal in the larger set X will be chosen in the smaller set X' when available. (C) says that a choice that is optimal for both X' and X'' has to be optimal when there are no elements available better than the ones in X' and X''. Both seem to be necessary conditions for rational decision making, so we gain little by using this more general framework. Ideally, we would actually like to have transitivity, but that's not possible with non-collegial aggregation rules. We can show this by giving a preference profile in which the $(N-1)$- rule has a cyclic outcome. Now a strict relation on a finite set allows for maximal elements on all non-empty subsets if and only if is acyclic (the proof is straightforward and standard). So Lemma 5.1 implies then the existence of a cycle for every non-collegial aggregation rule for this preference profile. We take $N > 2$ and $X = \{1, \dots, N\}$. We illustrate the construction of our preference profile with a "ranking wheel", an idea due to Donald Saari (for example [35]). The ranking wheel is shown in Figure 5. One may think of the problem as a group voting for a representative, which explains why voters and alternatives coincide. Voter 1 has preferences $1 \succ 2 \succ \dots \succ N$, voter 2 has preferences $2 \succ 3 \succ \dots \succ N \succ 1$ and so on. Each voter i has preferences with alternative i on top, followed by all other alternatives ordered clockwise on the ranking wheel. Now for every alternative, the alternative that is next counterclockwise is preferred by $N-1$ voters. So there's a cycle in the social preference. Donald Saari has actually shown that every intransitivity in every aggregation rule comes necessarily from a preference profile in which a subset of voters has preferences from a ranking wheel on some subset of alternatives (at least if all voters have linear preference orderings). The proof of this fact uses a heavy dose of linear algebra and can be found in [34].

Now we arrive naturally at point (iv). The reason why non-collegial aggregation rules necessarily lead to intransitivities under some profile of preferences is that the ranking of any pair of alternatives depends only on how all voters rank that pair, and these are exactly the rules that are influenced by "ranking wheel profiles". But let's take a look at why one may want to have "independence of irrelevant alternatives", as this property is usually called, in the first place. Here is what Kenneth Arrow, who popularized the condition, wrote in its defense when he introduced it (in [4]):

> [S]uppose an election is held, with a certain number of candidates in the field, each individual filing his list of preferences, and then one of the candidates dies. Surely, the social choice should be made by taking each of the individual's preference lists, blotting out completely the dead candidate's name, and considering only the orderings of the remaining names in going through the procedure of determining the winner. That is, the choice to be made among the set of surviving candidates should be independent of the preferences of individuals for the nonsurviving candidates. To assume otherwise would be to make the result of the election dependent on the obviously accidental circumstance of whether a candidate died before or after the date of polling. Therefore, we may require of our social welfare function that the choice made by society from a given set of alternatives depend only on the orderings of individuals among those alternatives.

So is there any reason we should rely on the ranking of dead candidates?[23] One reason is that ignoring dead candidates forces us not to ignore the rankings of a group of people whose preferences come from the ranking wheel. A large class of voting rules, so called positional rules, do ignore them. A positional rule for a problem with n alternatives consists of a list of weights $w_1, \ldots, w_n$ and the alternative ranked m^{th} by an individual gets w_m points by that individual[24]. Alternatives are socially ranked according to how many

[23]State laws in Missouri allow dead people to be elected, which happened in the case of Melvin Eugene Carnahan, who was elected posthumously to the Senate (Guardian, November 8. 2000).

[24]If there are ties, there are various ways of extending these rules, we'll ignore this issue here.

points they got in total then. Since an ordering of numbers is always transitive, the corresponding ranking is transitive. A profile from the ranking wheel gives the same number of points to each alternative, so positional rules are unaffected by the ranking wheel. Positional rules suffer usually from another defect. If two voters have "opposite" preferences (the preference ordering of one voter reverses the ordering of the others), their absense can still effect the outcome. A rule that avoids this problem is the *Borda count*. For the Borda count $w_i - w_{i-1}$ is the same for $i = 2, \ldots, n$. The Borda count is the rule least affected by subprofiles that ought to "cancel out" (see [34]). On the other hand, it is very easy to vote strategically with the Borda count, so it is seldom used in actual decision making.

8.2 Positive Issues

Apart from normative questions, it is worth asking what our results tell us about positive political theory. Insofar as the results have been negative and considering the absence of total political chaos, the results tell us mainly what models shouldn't be used.

The easiest model we can think of is a model of two-party competition. Two political parties position themselves in the "issue space" $\mathbb{R}^n$. If q voters prefer the position of one party to the positon of the other party, the first party wins. If no q voters prefer one party over the other, there is a draw. Each utility profile for voters defines such a game and a generic core-emptiness result can be interpreted as proving the non-existence of pure-strategy Nash equilibria for generic games of this type.

Cycles in voting are usually giving rise to agenda manipulation. If there are n alternatives to be voted on and the social preference has a cycle of full length n, someone can set an *agenda*, an order in which alternatives are pairwise eliminated as in a tennis competition, so that her preferred alternative comes out on top. Since one cannot vote on all, uncountably many, alternatives, this way of manipulating agendas will not work directly. But it would be possible to make "amendments". But in practice, proposing several competing amendments seems implausible. But if all voters can make amendments to some decision, we can expect an outcome with little structure. Donald Saari observed that amendments seem inevitable in the real world too:

> It always seems to be the case. No matter how hard you might work on a proposal, no matter how polished and complete the

> final product may be, when it is presented to a group for approval, there always seems to be a majority who wants to "improve" it. [35]

Since the concept of a core is originally from cooperative game theory, one might try to use weaker solutions concepts from cooperative game theory, such as the bargaining set. Interestingly, this route hasn't been followed much. Positive political science has come up with many special purpose solution concepts for spatial voting. Most of them are inspired by some notion of strategic voting. Some of them work only with Euclidean preferences or in low dimensions. See [37] for a recent survey. So far, there is no consensus on an appropriate solution concept. For applied researchers, a solution concept has not only to have nice characteristica, it should be computable from empirical data.

All solution concepts are based on alternatives and preferences. But one can also explain political stability by looking at additional factors. William Riker argues in [30] that the instability results simply force us to take a closer look at institutions again. I think he's right on that.

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Abstract

This work surveys work on the core of voting rules when the issue space can be identified with some finite dimensional Euclidean space. It turns out that the core is almost always empty. Making this precise has proven to be mathematically challenging. The most commonly used notion of "almost always" is the topological notion of being residual. This notion is hard to interpret and depends strongly on the topology one imposes on the space of profiles. We show that the major results obtained this way can be reformulated using a more natural notion of "almost always", the notion of prevalence. Prevalence was introduced by Hunt, Sauer and Yorke in 1992 and is easily interpreted in terms of random distortions to a system. These techniques matter only for large classes of preferences and utility function. In the case in which preferences are Euclidean and can be specified by a point in issue space, things are much easier. A new theorem on the generic emptiness of the core of majority and some supermajority rules is obtained in that case. The theorem, albeit weak, has an easy proof, drawing on intuitions from elementary highschool geometry, and is instructive in how one can prove related results in the case where profiles lie in an infinite-dimensional space.

Zusammenfassung

Diese Arbeit gibt eine Übersicht über verschiedene Arbeiten über den Kern von Wahlregeln wenn der Alternativenraum Euklidisch ist. Der Kern existiert fast nie und es gibt mathematisch genaue Formulierungen dieser Aussage. Es wird argumentiert dass die Standardformulierung konzeptionell unbefriedigend ist und gezeigt dass eine befriedigendere Formulierung möglich ist. Für den Spezialfall, in dem Präferenzen durch den Abstand zu einem Idealpunkt charakterisiert werden können, wird ein neuer Satz über die gewöhliche Leere des Kerns gegeben, dessen Beweis intuitiv verständlich ist und instruktiv für den Fall allgemeinerer Präferenzen.

Curriculum Vitae

CITIZENSHIP Austria

BIRTH Hallein, April 26, 1982

RESEARCH INTERESTS Microeconomic Theory, Mathematical Economics, Equilibrium Theory, Social Choice, Games with many Agents, Public Goods

EDUCATION *Rudolf Steiner Schule Salzburg* **until June 2001**

University of Vienna
October 2001 to June 2004 and October 2005 to present

Magister Program in Economics

- Thesis Topic: Generic Core-emptiness in Spatial Models of Voting
- Advisor: Assistant Professor Dr. Konrad Podczeck
- Area of Study: Social Choice, Mathematical Economics
- The one year break comes from my civil service, which lasted from October 2004 to September 2005. I worked for the ambulance service of the Red Cross Rauris.

TEACHING EXPERIENCE University of Vienna, Department of Economics

Teaching Assistant **March 2008 to June 2008**

- Tutor for the course *Advanced Microeconomics*.

Tutor **Octobert 2007 to January 2008**

- Tutor for the course *Mathematical Economics*.

Technical University Vienna, Department of Economics

Teaching Assistant **Oktober 2004 to January 2005**

- Instructing the exercise part of a principles of economics course for first year students of various technical disciplines.
- Tasks included lecturing, designing the syllabus and exams, grading essays and exams.

OTHER ACTIVITIES

Student Representative **October 2001 to present**

- Elected representative at the study level from summer terms 2003 to 2004. In that time I was also a vice head of the studenr representatives on the faculty level. Involved in decision making of student representatives during whole study.

- Activities included among others reading circles, moderation of a panel discussion, writing and layouting for a students magazine, negotiations with faculty, tutoring of beginning students in economics