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"Emergent Gravity in two dimensions"

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1 Introduction

One of the biggest remaining challenges for theoretical physics is the unification of general relativity and quantum field theory. Till now there is no complete satisfying theory that incorporates these two pillars of theoretical physics. There exist many problems in unifying those two theories. One of the main problems considers the dynamics of spacetime. In general relativity spacetime is not just a stage on which dynamics take place, but rather a participant, which can be understood through the famous Einstein field equations: $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}$. The field equations link the geometry of spacetime with the energy momentum tensor. Quantum field theories on the other hand as QED (quantum electrodynamics) or QCD (quantum chromodynamics) work with spacetime just as a stage where dynamics takes place. Another technical problem for example is that the right hand side of the Einstein field equations contain the energy momentum tensor which is a quantum quantity, while the left handside describes gravitation as a classical field. If $T_{\mu\nu}$ is represented as quantized matter the field equations as they stand, are inconsistent.

One big problem considers the picture of spacetime at small scales. John A. Wheeler argued that the classical picture of spacetime should break down at very short distances of the order of the Planck-length $l_p = \sqrt{G\hbar/c^3}$. The Planck-length is also the distance where it is generally believed, that quantum fluctuations of spacetime, due to the interplay between gravity and quantum mechanics become important.

There are many approaches for example string theory or Ashtekars approach to quantum gravity, to mention just a few. Another beautiful approach would be to quantize spacetime itself, by imposing uncertainty relations between time and space coordinates, and to formulate a (hopefully) renormalisable QFT which contains gravity. At first it seemed to be a very pleasant approach, though it provides a natural UV-cutoff. But as the authors in [6] showed, not only is the ϕ^4 model not finite in the UV regime, but the model also exhibits a new type of divergences, the so called UV/IR mixing, that make the model nonrenormalisable. The situation remained so till H. Grosse and R. Wulkenhaar [4,5] discovered a way to define a renormalisable non commutative model, by adding a harmonic term to the Lagrangian. This was a great breakthrough, though it provides the way towards other non-commutative field theories.

After remarkable progress was done in understanding field theory on a fixed NC space, the next step in progress would be to try to formulate a dynamical structure of NC space time, to enable the incorporation of GR with QFT.

A recent realization of this idea was proposed by H.Steinacker with a matrix model and has been published under the name "emergent gravity from noncommutative gauge theory". The basic observation is that matrix models that define noncommutative (NC) gauge theory contain a specific version of gravity. This provides a dynamical theory for noncommutative spaces. The connection between gravity and NC gauge theory was first observed in [2]. In particular, it was pointed out in [1] that the Einstein-Hilbert action will be induced upon quantization, and that it should amount to the UV/IR mixing in noncommutative gauge theory. This observation is quite astonishing and the prediction is supported by the fact that both gravity and UV/IR mixing occur only in the U(1) sector of NC gauge theory. Emergent gravity explains the strange IR behavior of the would-be photons: they are not photons but gravitons defining a nontrivial geometric background. The would-be U(1) gauge fields are re-interpreted in terms of geometry and absorbed in the effective metric. The effective metric couples to all other fields.

In this work, we elaborate and verify this explanation of UV/IR mixing in terms of gravity in 2 dimensions. We will perform a one-loop quantization of a scalar field coupled to the matrix model of NC gauge theory resp. gravity in two different ways. First in the geometrical point of view, we interpret the action as scalar field coupled to gravity, which leads using standard arguments to an induced Einstein-Hilbert action. Second, we use the more conventional interpretation of the same matrix model in terms of

NC gauge theory, where integrating out the scalar field leads to an effective action for the NC gauge fields involving the well-known UV/IR mixing terms. These two computations should agree at least in the IR regime, where the geometrical picture is expected to make sense. The same procedure in this framework is also done for the matrix model coupled to spinors.

The 2 dimensional case is different from the 4 dimensional in few ways. First of all the calculations are easier, but we must deal in the action with an extra dilaton term which does not appear in 4 dimensions. This problem is rather nontrivial. Many authors [15-21] have tried to solve this problem, but till now there is no solution that fits in our framework. The problem of quantization with an extra dilaton term appears often in the dimensional reduction of a problem from 4 to 2 dimensions. Our first approach was to redefine the fields and take the Jacobian of the measure. This approach suffered from a divergence, which could not be treated with the heat kernel expansion just as in flat space. We were able to proceed with the quantization procedure by calculating the heat kernel perturbatively.

2 Matrix models, quantization and effective geometry

2.1 Yang Mills action

The starting action is the Yang Mills action that is defined as

$$S_{YM}[X] = -\frac{1}{2}Tr[X^a, X^b][X^c, X^d]g_{ac}g_{bd}.$$

$$(1)$$

The metric here is euclidean or minkowski, although during this framework we will use the euclidean metric.

$$g_{ac} = \delta_{ac}$$

$$(2)$$

or

$$g_{ac} = \eta_{ac}$$

$$(3)$$

X^a are hermitian matrices or operators acting on some Hilbert space $\mathcal{H}$.

The symmetries of this action are the following:

1. Gauge symmetry

$$X^a \to \mathcal{U}X^a\mathcal{U}^{-1}$$

$$(4)$$

$$\mathcal{U} \in \mathcal{U}(\mathcal{H})$$

$$(5)$$

Where the matrices $\mathcal{U}$ are unitary operators on some Hilbert space.

2. Translation invariance

$$X^a \to X^a + c^a$$

$$(6)$$

$$c^a \in \mathbb{R}^2$$

$$(7)$$

3. SO(4) resp. SO(3,1) invariance

2.1.1 Equations of motion

The equations of motion for this action are:

$$[X^a, [X^{a'}, X^{b'}]]g_{aa'} = 0 \tag{8}$$

One special solution is the case of a flat space

$$X^a = \bar{Y}^a \tag{9}$$

$$[\bar{Y}^a, \bar{Y}^b] = i\bar{\theta}^{ab}. \tag{10}$$

$\bar{Y}^a$ generates the algebra $\mathcal{A} \cong \mathbb{R}^2_{\bar{\theta}}$ of functions on the Moyal Weyl space(quantum plane). Where $\bar{\theta}^{ab}$ is constant and non degenerate.

Another solution is given by

$$X^a = \bar{Y}^a \otimes \mathbb{1}_n \tag{11}$$

which will lead to u(n) gauge theory. Now consider small fluctations of the form

$$X^a = \bar{Y}^a \otimes \mathbb{1}_n + \mathcal{A}^a(\bar{Y}). \tag{12}$$

In the emergent gravity model we focus on configurations which are close to the vacuum plus small fluctuations $A_0^a(\bar{Y})$, which is the new vacuum $X^a = \bar{Y}^a \otimes \mathbb{1}_n$.

$$\mathcal{A}^a(\bar{Y}) = A_0^a(\bar{Y}) \otimes \mathbb{1}_n + A_\alpha^a(\bar{Y}) \otimes \tau_\alpha \tag{13}$$

where $\mathcal{A}^a(\bar{Y}) \in \mathcal{A} \otimes M_n(\mathbb{C})$ and τ_α is the basis of su(n).

A different approach is to seperate the trace of the U(1) part and the remaining non abelian part as follows

$$X^a = Y^a \otimes \mathbb{1}_n + \mathcal{A}^a(Y) \tag{14}$$

$$= Y^a \otimes \mathbb{1}_n + A_\alpha^a(Y) \otimes \tau_\alpha. \tag{15}$$

$$Y^a = \bar{Y}^a + A_o^a(\bar{Y}) \tag{16}$$

That means Y^a are the generators of a non commutative space $\mathcal{M}_{\bar{\theta}}$ with general non commutativity:

$$[Y^a, Y^b] \equiv i\theta^{ab}(Y) \approx i\theta^{ab}(y). \tag{17}$$

Emergent gravity questions the usual interpretation of A_0^a as an abelian gauge field. In emergent gravity A_0^a is understood as a fluctuation in the NC space, which determines a Poisson structure $\theta^{ab}(y)$ and leads to an effective metric $G^{ab}(y)$.

2.2 Deformation quantization of Poisson manifolds

The emerging picture is that the u(1) sector of the matrix model describes a dynamical theory of Poisson manifolds. Now let us consider generators Y^a of $\mathcal{A}$ satisfying (17). This defines a Poisson manifold $(\mathcal{M}, \theta^{ab}(y))$, whose quantization is given by Y^a. Maxim Kontsevich proved in [3] that any finite dimensional Poisson manifold can be canonically quantized in the sense of deformation quantization, so we can quantize any poisson structure with the generators Y^a.

Before proceeding any further let us describe deformation quantization.

Let $(\mathcal{M}, \{.,.\})$ be a poisson manifold, $\mathcal{C}^\infty(\mathcal{M})$ the algebra of complex valued arbitrary often differentiable functions on $\mathcal{M}$ with the pointwise product as commutative algebra structure on the space $\mathcal{C}^\infty(\mathcal{M})$ fullfilling the Leibnitzrule

$$\{f \cdot g, h\} = f \cdot \{g, h\} + \{f, h\} \cdot g \tag{18}$$

Denote by $\mathcal{A} = \mathcal{C}^\infty(\mathcal{M})[[\theta]]$ the vector space of formal power series in the variable θ with differentiable function on $\mathcal{M}$ as coefficients. A formal deformation quantization is a multiplication $*$ defined on $\mathcal{A}$ which is associative, $\mathcal{C}[[\theta]]$ the vector space bilinear, and θ-adically continous, such that for f, g $\in \mathcal{C}^\infty(M)$

$$f * g \mid_{\theta=0} = f \cdot g \tag{19}$$

$$\frac{1}{\theta}(f * g - g * f) \mid_{\theta=0} = \mathrm{i}\{f, g\} \tag{20}$$

The multiplication $*$ is called a star multiplication, and the term star product is used as is used as synonym for a formal deformation quantization. The multiplication $*$ can also be described with the following formula

$$f * g = \sum_{m=0}^{\infty} C_k(f, g)\theta^k \tag{21}$$

with bilinear maps $C_k : \mathcal{C}^\infty(\mathcal{M}) \times \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})$ for k$\in \mathbb{N}_0$. By the θ-adical continuity this extends to

$$\left(\sum_{j=0}^{\infty} f_j \theta^j\right) * \left(\sum_{l=0}^{\infty} g_l \theta^l\right) = \sum_{j=0}^{\infty} \left(\sum_{k+j+l=m} C_k(f_j, g_l)\right)\theta^m \tag{22}$$

2.2.1 The Moyal product

The simplest example of a deformation quantization is the Moyal product for the Poisson sturcture on $\mathbb{R}^d$ with constant coefficients:

$$\alpha = \sum_{i,j} \alpha^{ij} \partial_i \wedge \partial_j \tag{23}$$

$$\alpha^{ij} = -\alpha^{ij} \tag{24}$$

Where $\partial_i = \frac{\partial}{\partial x^i}$ is the partial derivative with respect to x and $i = 1, \ldots., d$.
The formula for the Moyal product is

$$f * g = f \cdot g + \theta \sum_{i,j} \alpha^{ij} \partial_i(f) \partial_j(g) + \frac{\theta^2}{2} \sum_{i,j,k,l} \alpha^{ij} \alpha^{kl} \partial_i \partial_k(f) \partial_j \partial_l(g) + \ldots. = \tag{25}$$

$$= \sum_{0}^{\infty} \frac{\theta^n}{n!} \sum_{i_1,\ldots,i_n;j_1,\ldots,j_n} \prod_{k=1}^{n} \alpha^{i_k} \alpha^{j_k} \left(\prod_{k=1}^{n} \partial_{i_k}(f) \right) \cdot \left(\prod_{k=1}^{n} \partial_{j_k}(g) \right). \tag{26}$$

The associativity means that for any 3 functions f, g, h one has:

$$(f * g) * h = f * (g * h). \tag{27}$$

Let $\alpha = \sum_{i,j} \alpha^{ij} \partial_i \wedge \partial_j$ be a Poisson bracket with variable coefficients, then the following formula gives an associative product modulo $\mathrm{O}(\theta^2)$:

$$f * g = f \cdot g + \theta \sum_{i,j} \alpha^{ij} \partial_i(f) \partial_j(g) + \mathrm{O}(\theta^2). \tag{28}$$

The last equation implies that if Y^a operates on a function one obtains the following

$$[Y^a, f(y)] = i\theta^{ab}(y)\partial_b f(y) + \mathrm{O}(\theta^2). \tag{29}$$

2.3 The noncommutative metrik G^{ij}

In order to derive the metric we couple a scalar field ϕ to the matrix model. The most obvious way to couple ϕ is the following.

$$S[\phi] = -\frac{1}{2}\mathrm{Tr}[Y^b, \phi][Y^b, \phi]g_{ab} \tag{30}$$

$$S[\phi] = \frac{1}{2}\mathrm{Tr}\left(\theta^{ac}(y)\theta^{bd}(y)\partial_c \phi \partial_d \phi g_{ab} \right) + \mathrm{O}(\theta^3) \tag{31}$$

$$S[\phi] = \frac{1}{2}\mathrm{Tr}\left(G^{cd}\partial_c \phi \partial_d \phi \right) + \mathrm{O}(\theta^3) \tag{32}$$

The metric that one obtains from matrix models is given by

$$G^{ij} = \theta^{ik}(y)\theta^{jl}(y)g_{kl}. \tag{33}$$

From this point of view matrix models are very elegant, though they provide a simple prescription to quantize space time. And in addition they provide a simple prescription of coupling fields to this model.

2.4 Curvature in 2 dimensions after a conformal transformation

In this section we calculate the curvature of a metric after a conformal transformation. Every 2 dimensional Riemannian manifold is conformally flat. And the metric can allways be brought to the following form

$$G^{kl} = e^{\phi} g^{kl}. \tag{34}$$

$$g^{kl} = \delta^{kl} \tag{35}$$

The inverse metric is given by

$$G_{kl} = e^{-\phi} g_{kl}. \tag{36}$$

Where

$$\phi_{,b} = \partial_b \phi = \frac{\partial \phi}{\partial x^b} \tag{37}$$

$$A_{(ab)} = \frac{1}{2}(A_{ab} + A_{ba}) \tag{38}$$

$$A_{[ab]} = \frac{1}{2}(A_{ab} - A_{ba}) \tag{39}$$

The torsion free connection is given by

$$\Gamma^c_{ab} = \frac{1}{2}G^{cd}(\partial_b G_{ad} + \partial_a G_{bd} - \partial_d G_{ab}). \tag{40}$$

$$= -\frac{1}{2}G^{cd}(G_{ad}\partial_b\phi + G_{bd}\partial_a\phi - G_{ab}\partial_d\phi) \tag{41}$$

$$= -\frac{1}{2}(\delta^c_a \phi_{,b} + \delta^c_b \phi_{,a} - g_{ab}\phi^c) \tag{42}$$

The Riemann Tensor can be obtained by the following formula

$$R^c_{abd} = \Gamma^c_{ad,b} - \Gamma^c_{ab,d} + \Gamma^c_{eb}\Gamma^e_{ad} - \Gamma^c_{ed}\Gamma^e_{ab}. \tag{43}$$

$$R^c_{abd} = R^{(1)c}_{abd} + R^{(2)c}_{abd}. \tag{44}$$

$$R^{(1)c}_{abd} = \partial_b \Gamma^c_{ad} - \partial_d \Gamma^c_{ab} = \delta^c_{[b}\partial_{d]}\partial_a\phi + g_{a[d}\partial_{b]}\partial^c\phi \tag{45}$$

$$R^{(2)c}_{abd} = \Gamma^c_{eb}\Gamma^e_{ad} - \Gamma^c_{ed}\Gamma^e_{ab} = \frac{1}{2}(\delta^c_{[b}\partial_{d]}\phi\partial_a\phi - g_{a[b}\partial_{d]}\phi\partial^c\phi - g_{a[d}\delta^c_{b]}\partial_e\phi\partial^e\phi) \tag{46}$$

Through Contraction one gets the Ricci Tensor

$$R^c_{abc} = R_{ab} = R^{(1)}_{ab} + R^{(2)}_{ab}. \tag{47}$$

$$R^{(1)}_{ab} = -\frac{1}{2}g_{ab}\partial_c\partial^c\phi \tag{48}$$

$$R^{(2)}_{ab} = 0 \tag{49}$$

$$R_{ab} = -\frac{1}{2}g_{ab}\partial_c\partial^c\phi \tag{50}$$

The Ricci scalar is given through the contraction with the metric

$$R = R_{ad}G^{ad}. \tag{51}$$

$$R = -G^{ab}\partial_a\partial_b\phi \tag{52}$$

3 The bosonic case

Let us now consider a massive scalar field coupled to the matrix model.

$$S[\phi] = -(2\pi)\text{Tr}\frac{1}{2}\left(g_{aa'}[Y^a,\phi][Y^{a'},\phi] - m^2\phi^2\right) \tag{53}$$

We want to write the trace as an integral. As pointed out in [1], the relation is the following:

$$(2\pi)\text{Tr}f(y) \sim \int \omega f(y) = \int d^2y\rho(y)f(y) \tag{54}$$

where $\omega = i\theta_{ab}^{-1}(y)dy^a dy^b$ is the symplectic form, and $\omega = \rho(y)d^2y$ the symplectic volume element. The density factor is given by

$$\rho(y) = \text{Pfaff}(i\theta_{ab}^{-1}(y)) = \sqrt{\det\theta_{ab}^{-1}(y)} = \left(\det G^{ab}(y)\right)^{-1/4} \tag{55}$$

So now we can write

$$S[\phi] = -(2\pi)\text{Tr}\frac{1}{2}\left(g_{aa'}[Y^a,\phi][Y^{a'},\phi] - m^2\phi^2\right) \tag{56}$$

as

$$\sim \int d^2y \frac{1}{2}\left(\rho(y)G^{ab}(y)\partial_a\phi(y)\partial_b\phi(y) + \rho(y)m^2\phi^2\right). \tag{57}$$

The metric G^{ab} plays the role of a gravitational metric. The metric enters the kinetic term for any matter coupled to the matrix model, though the only way to couple matter to this model is done by $\text{Tr}[Y^a,.][Y^{a'},.]g_{aa'}$.

$$G^{ab} = \theta^{ac}(y)\theta^{bd}(y)g_{cd} \tag{58}$$

$\sim$ indicates the leading contribution in semi-classical expansion in powers of θ^{ab}.
The metric g_{ab} is the following one

$$g_{ab} = \delta_{ab} \tag{59}$$

or

$$g_{ab} = \eta_{ab}. \tag{60}$$

So g_{ab} is the flat Euclidean or Minkowski metric, allthough during this framework we remain euclidean. The partial derivatives denote the derivation with respect to y.

$$\partial_b = \frac{\partial}{\partial y^b} \tag{61}$$

3.1 The non trivial geometric background in 2 dimensions

The metric G^{ab} which plays the role of a non trivial geometric background is given as

$$G^{ab} = \theta^{ac}(y)\theta^{bd}(y)g_{cd}. \tag{62}$$

In 2 dimensions there is a more sufficient way to write this metric, due to the fact that $\theta^{ab}(y)$ is a Poisson tensor. So it is an antisymmetric tensor which has quite a nice representation in 2 dimensions.

$$\theta^{ab}(y) = \varepsilon^{ab}\theta(y) \tag{63}$$

Where ε^{ab} is the epsilon tensor. So the metric can be rewritten as

$$G^{ab} = \varepsilon^{ac}\theta(y)\varepsilon^{bd}\theta(y)g_{cd} := \theta^2(y)\bar{g}^{ab}. \tag{64}$$

Where $\bar{g}^{ab}$ is the flat noncommutative metric and is defined as follows

$$\bar{g}^{ab} := \varepsilon^{ac}\varepsilon^{bd}g_{cd}. \tag{65}$$

So we can rewrite the metric in the following way

$$G^{ab} = e^{2\sigma(y)}\bar{g}^{cd}. \tag{66}$$

Where σ is given as

$$\sigma(y) = \log(\theta(y)). \tag{67}$$

3.2   Quantization and induced gravity

Again the action that we consider after coupling a massive scalar field is the following one

$$S[\phi] = \frac{1}{2}\int d^2y\left(\rho(y)G^{ab}(y)\partial_a\phi(y)\partial_b\phi(y) + \rho(y)\tilde{m}^2\phi^2\right) \tag{68}$$

The density factor is in the notation we introduced above given as

$$\rho(y) = (\det G_{ab}(y))^{1/4} = e^{-\sigma}. \tag{69}$$

So the action becomes the following one

$$S[\phi] = \frac{1}{2}\int d^2y\sqrt{\tilde{G}}\left(\tilde{G}^{\mu\nu}e^{\sigma(y)}\partial_\mu\phi(y)\partial_\nu\phi(y) + \tilde{m}^2\phi^2(y)\right). \tag{70}$$

In the last step we introduced the following metric.

$$\tilde{G}^{ab} := \bar{g}^{ab}e^{\sigma}. \tag{71}$$

We cannot proceed with the heat kernel expansion due to the dilaton field e^{σ}. So we introduce the following fields:

$$\phi = \tilde{\phi}e^{-\sigma/2}. \tag{72}$$

The action is the following in terms of the new fields

$$S[\phi] = \frac{1}{2}\int d^2y\sqrt{\tilde{G}}\left(\tilde{G}^{\mu\nu}\partial_\mu\tilde{\phi}\partial_\nu\tilde{\phi} - \tilde{\phi}\partial_\mu\sigma\partial_\nu\tilde{\phi} + \frac{1}{4}\tilde{\phi}^2\partial_\mu\sigma\partial_\nu\sigma + e^{-\sigma}\tilde{m}^2\tilde{\phi}^2\right). \tag{73}$$

After integrating by parts the action reads

$$S[\tilde{\phi}] = -\frac{1}{2}\int d^2y\sqrt{\tilde{G}}\left(\tilde{G}^{\mu\nu}\tilde{\phi}\partial_\mu\partial_\nu\tilde{\phi} + \tilde{\phi}\partial_\mu\sigma\partial_\nu\tilde{\phi} - \frac{1}{4}\tilde{\phi}^2\partial_\mu\sigma\partial_\nu\sigma - e^{-\sigma}\tilde{m}^2\tilde{\phi}^2\right). \tag{74}$$

The Laplace type operator is the following one

$$\Delta_{\tilde{G}} = -\left(\tilde{G}^{\mu\nu}\partial_\mu\partial_\nu + \tilde{G}^{\mu\nu}\partial_\mu\sigma\partial_\nu - \tilde{G}^{\mu\nu}\frac{1}{4}\partial_\mu\sigma\partial_\nu\sigma\right) \tag{75}$$

Due to the fact that we work with the redefined fields $\tilde{\phi}$ we first have to calculate the functional determinant.

$$\mathrm{e}^{-\Gamma_{\tilde{\phi}}} = \mathrm{J}\int D\tilde{\phi}\mathrm{e}^{-S[\tilde{\phi}]} \tag{76}$$

In order to define the path integral more precise we decompose ϕ and $\tilde{\phi}$ into eigenfunctions of the Laplace operator $\Delta_{\tilde{G}}$.

$$\phi(y) = \sum_n a_n\phi_n(y) = \sum_n a_n <x|n> \tag{77}$$

$$\tilde{\phi}(y) = \sum_n \tilde{a}_n\phi_n(y) = \sum_n \tilde{a}_n <x|n> \tag{78}$$

where a_n and $\tilde{a}_n$ are coefficients. The Laplace operator $\Delta_{\tilde{G}}$ has real eigenvalues λ_n

$$\Delta_{\tilde{G}}\phi_n(y) = \lambda_n\phi_n(y) \tag{79}$$

and the set of eigenfunctions $\{\phi_n(y)\}$ is orthonormal and complete

$$\int d^2y\sqrt{\tilde{G}}\phi_m(y)\phi_n(y) = \delta_{mn} \tag{80}$$

$$\sum_n \phi_m(x)\phi_m(y) = \delta(y-x). \tag{81}$$

The path integral measure is defined as

$$d\mu(\phi) = \prod_n da_n = \prod_x D\phi(x). \tag{82}$$

Let us consider the redefined field in terms of the expansion

$$\phi(y) = e^{\sigma/2}\tilde{\phi}(y) = \sum_n a_n\phi_n(y) = \sum_n \tilde{a}_n\phi_n(y))e^{\sigma/2}. \tag{83}$$

Due to the orthonormality of the eigenfunctions we obtain

$$a_n = \sum_m \left(\int d^2y\sqrt{\tilde{G}}\phi_m(y)e^{\sigma/2}\phi_n(y)\right)\tilde{a}_m = \sum_m C_{nm}\tilde{a}_m \tag{84}$$

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$$C_{nm} = \int d^2y \sqrt{\tilde{G}} \phi_m(y) e^{\sigma/2} \phi_n(y) \tag{85}$$

after expanding $e^{\sigma/2}$, the transformation matrix reads

$$C_{nm} = \delta_{mn} - \frac{1}{2} \int d^2y \sqrt{\tilde{G}} \phi_m(y) \sigma(y) \phi_n(y) \tag{86}$$

So we find for the change in the path integral measure

$$\prod_n da_n = \det(C_{nm}) \prod_m d\tilde{a}_m \tag{87}$$

After taking the jacobian into account the effective action reads

$$\Gamma_{\tilde{\phi}} = -\text{logdet}(C_{nm}) + \frac{1}{2}\text{logdet}\left(\Delta_{\tilde{G}} + e^{-\sigma}\tilde{m}^2\right). \tag{88}$$

We rewrite the last equation and obtain

$$\Gamma_{\tilde{\phi}} = -\text{trlog}(C_{nm}) + \frac{1}{2}\text{Trlog}\frac{1}{2}(\Delta_{\tilde{G}} + e^{-\sigma}\tilde{m}^2). \tag{89}$$

Induced gravity:
We now focus on the geometry point of view. For this we rewrite the trace of the operator in a different way, to calculate it later on with the heat kernel expansion. The Jacobian will be treated in the next section.

$$\text{Tr}\left(\log\frac{1}{2}(\Delta_{\tilde{G}} + e^{-\sigma}\tilde{m}^2)\right) \sim -\text{Tr}\int_0^\infty \frac{d\alpha}{\alpha}\left(e^{-\alpha\frac{1}{2}\Delta_{\tilde{G}}}\right)e^{-\frac{1}{2}e^{-\sigma}\tilde{m}^2\alpha} \tag{90}$$

$$\equiv -\text{Tr}\int_0^\infty \frac{d\alpha}{\alpha}\left(e^{-\alpha\frac{1}{2}\Delta_{\tilde{G}}}\right)e^{-\frac{1}{\alpha\tilde{\Lambda}^2} - \frac{1}{2}e^{-\sigma}\tilde{m}^2\alpha} \tag{91}$$

where the small α divergence is regularized using an UV cutoff $\tilde{\Lambda}$ and the big α divergence is regularized using a cutoff $\tilde{m}^2$. Now we use the heat kernel expansion

$$\text{Tr}e^{-\frac{1}{2}\alpha\Delta_{\tilde{G}}} \sim \sum_{n\geq 0}(\frac{\alpha}{2})^{\frac{n-2}{2}}\int_{\mathcal{M}} d^2y\sqrt{\tilde{G}}a_n(y, \Delta_{\tilde{G}}) \tag{92}$$

where $a_n(y, \Delta_{\tilde{G}})$ are known as the Seeley de Witt (or Duhamel) coefficients. So the effective action in this case takes the following form

$$\Gamma_\phi = -\int d^2y\sqrt{\tilde{G}}\left(a_0(\tilde{\Lambda}^2 - e^{-\sigma}\frac{\tilde{m}^2}{2}\log\frac{2\tilde{\Lambda}^2}{\tilde{m}^2}) + a_2\log(\frac{\sqrt{2}\tilde{\Lambda}}{\tilde{m}})\right) \tag{93}$$

One can drop finte terms due to the fact that we are just interested in the divergent part.

$$\Gamma_\phi = -\int d^2y\sqrt{\tilde{G}}\left(a_0(\tilde{\Lambda}^2 - e^{-\sigma}\frac{\tilde{m}^2}{2}\log\tilde{\Lambda}^2) + a_2\log(\frac{\tilde{\Lambda}}{\tilde{m}})\right) \tag{94}$$

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3.2.1 Heat kernel expansion and the Seeley de Witt coefficients

For a given action of the following form

$$S[\phi] = \int d^n x \sqrt{G} \phi(x) D \phi(x) \tag{95}$$

$$G = \det(G_{ab}) \tag{96}$$

it is possible to calculate the effective action with the heat kernel expansion.

Let $\mathcal{M}$ be a smooth compact Riemannian manifold of dimension n and let $\mathcal{V}$ be a vetor bundle over $\mathcal{M}$. This means that there is a vector space attached to each point of the manifold. We study differential operators on $\mathcal{V}$. Restricting ourselves only to second order differential operators of Laplace type, such operators can be represented as

$$D = -(G^{ab}\partial_a \partial_b + a^d \partial_d + b). \tag{97}$$

Where a^d and b are matrix valued functions on $\mathcal{M}$ and G^{ab} is the metric on $\mathcal{M}$. Due to the fundamental theorem of Riemannian geometry there is a unique connection on $\mathcal{V}$ and a unique endomorphism E of $\mathcal{V}$ so that

$$D = -(G^{ab}\nabla_a \nabla_b + E). \tag{98}$$

∇ is the covariante derivative which contains Riemann derivatives and the gauge ω parts.
One can now express

$$\omega_d = \frac{1}{2} G_{ed}(a^e + G^{ab}\Gamma^e_{ab}) \tag{99}$$

$$E = b - G^{cd}(\partial_c \omega_d + \omega_c \omega_d - \omega_b \Gamma^b_{cd}) \tag{100}$$

Where the Seeley de Witt coefficients read

$$a_0(y) = \frac{1}{4\pi} \tag{101}$$

$$a_2(y) = \frac{1}{24\pi}(R[G] + 6E). \tag{102}$$

Again there are few restrictions that the operator D has to fulfill. First D must be a partial second order operator. Second the coefficient a^d is contracted with the metric. The third restriction is the selfadjointness of the operator with respect to the scalar product.

3.2.2 Heat kernel expansion for the induced action

In the 2 dimensional case we have further simplifications for example

$$\tilde{G}^{ab}\Gamma^e_{ab} = -\frac{1}{2}\tilde{G}^{ab}\tilde{G}^{ed}(\tilde{G}_{ad}\phi_{,b} + \tilde{G}_{bd}\phi_{,a} - \tilde{G}_{ab}\phi_{,d}) \tag{103}$$

$$= -\frac{1}{2}\tilde{G}^{ab}(\delta^e_a \phi_{,b} + \delta^e_b \phi_{,a} - \tilde{G}_{ab}\phi^e) \tag{104}$$

$$= -\frac{1}{2}(\tilde{G}^{eb}\phi_{,b} + \tilde{G}^{ea}\phi_{,a} - 2\phi^e) = 0. \tag{105}$$

The curvature in 2 dimensions is

$$R = -\tilde{G}^{ab}\partial_a\partial_b\sigma. \tag{106}$$

The gauge part ω is given by

$$\omega_a = \frac{1}{2}\tilde{G}_{ab}a^b. \tag{107}$$

$$a^b = \tilde{G}^{bc}\partial_c\sigma \tag{108}$$

$$\omega_a = \frac{1}{2}\partial_a\sigma. \tag{109}$$

And E reads

$$E = -\frac{1}{2}\tilde{G}^{ab}\Big(\partial_a\partial_b\sigma + \partial_a\sigma\partial_b\sigma\Big). \tag{110}$$

So the second seeley de witt coefficient is the following one

$$a_2 = R[\tilde{G}] + 6E = R[\tilde{G}] + 3\Delta_{\tilde{G}}\sigma - 3\tilde{G}^{ab}\partial_a\sigma\partial_b\sigma \tag{111}$$

Where

$$\Delta_{\tilde{G}}\sigma = -\tilde{G}^{ab}\partial_a\partial_b\sigma + \Gamma^c\partial_c\sigma = -\frac{1}{\sqrt{\tilde{G}_{ab}}}\partial_a\Big(\sqrt{\tilde{G}_{ab}}\tilde{G}^{ab}\partial_b\sigma\Big). \tag{112}$$

$$\Gamma^c = \tilde{G}^{ab}\Gamma^c_{ab} \tag{113}$$

So the effective action reads

$$\Gamma_\phi = -\frac{1}{4\pi}\int d^2y\sqrt{\tilde{G}}(\tilde{\Lambda}^2 - e^{-\sigma(y)}\frac{\tilde{m}^2}{2}\log\tilde{\Lambda}^2) - \frac{1}{(24\pi)}\int d^2y\sqrt{\tilde{G}}\Big(R[\tilde{G}] + 3\Delta_{\tilde{G}}\sigma - 3\tilde{G}^{ab}\partial_a\sigma\partial_b\sigma\Big)\log(\frac{\tilde{\Lambda}}{\tilde{m}}). \tag{114}$$

3.2.3   The Jacobian

There is a nontrivial problem concerning the Jacobian. By taking the trace of the Jacobian we found a new divergence $\delta(x-x)$. The problem of treating this divergence will be the subject of the following section.

Recall the Jacobian:

$$C_{nm} = \delta_{mn} - \frac{1}{2}\int d^2y\sqrt{\tilde{G}}\phi_m(y)\sigma(y)\phi_n(y) \tag{115}$$

During the calculation of the effective action the term concerning the Jacobian is the following one

$$\Gamma_\phi^J = -\text{trlog}(C_{nm}) \tag{116}$$

Where the upper index J denotes the part of the effective action that only considers the Jacobian. We now use the following Taylor expansion

$$\log(1 + \beta(y)) = \beta(y) + \text{O}(\beta^2). \tag{117}$$

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The term that emerge after the expansion is the following

$$\Gamma_\phi^J = \frac{1}{2}\mathrm{tr}(\int d^2y\sqrt{\tilde{G}}\phi_m(y)\sigma(y)\phi_n(y)) = \frac{1}{2}\int d^2y\sqrt{\tilde{G}}\sigma(y)\sum_n \phi_n(y)\phi_n(y) \tag{118}$$

The sum that enters the integral is not well defined

$$\sum_n \phi_n(y)\phi_n(y) = \delta(y-y) = \delta(0) \tag{119}$$

There exists many rigorous methods to regularize this expression. We the choose the heat kernel to regularize this sum, and even in this regularization inconsistencies appear.

The standard method is to introduce a heat kernel and it is done as follows

$$\Gamma_{\tilde{\phi}}^J = \lim_{t\to 0}\frac{1}{2}\int d^2y\sqrt{\tilde{G}}\sigma(y)\sum_n \phi_n(y)\exp(-t\Delta_{\tilde{G}})\phi_n(y) \tag{120}$$

The sum is now simply given through the Seeley de Witt coefficients:

$$\lim_{t\to 0}\sum_n \phi_n(y)\exp(-t\Delta_{\tilde{G}})\phi_n(y) = \lim_{t\to 0}\left(\frac{a_0(y,y)}{4\pi t} + \frac{1}{4\pi}a_2(y,y)\right) \tag{121}$$

As one can easily see we would have a divergence in the first term by taking $\lim_{t\to 0}$. In quantum field theory on flat spaces this is usually renormalized by subtracting the flat and free Laplacian Δ_0 from the initial expression. This can not be done in our case, because the scalar products are differently defined for the operators $\Delta_{\tilde{G}}$ and Δ_0 and the first term does not disappear. One could renormalize the divergence in the first term bei subtracting the Laplacian $\Delta_{\tilde{G}_0} = -\tilde{G}^{\mu\nu}\partial_\mu\partial_n u$ from the initial expression. To avoid any ambiguity we derive the heat kernel for our operator perturbatively.

3.3 Evaluation of the heat kernel

3.3.1 Emergent gravity operator

Our starting action is the following

$$S[\phi] = \frac{1}{2}\int d^2y\sqrt{\tilde{G}}\left(\tilde{G}^{\mu\nu}e^{\sigma(y)}\partial_\mu\phi(y)\partial_\nu\phi(y)\right). \tag{122}$$

Note: We took out the mass for simplicity.

After integrating by parts we obtain the following operator

$$\Delta_{\tilde{G}} = -\frac{1}{\sqrt{\tilde{G}_{ab}}}\partial_a\left(\sqrt{\tilde{G}_{ab}}\tilde{G}^{ab}e^\sigma\partial_b\right) = -e^{2\sigma}\partial^2 - e^{2\sigma}\partial^a\sigma\partial_a. \tag{123}$$

Where ∂^2 is defined as

$$\partial^2 = \partial_a\partial^a. \tag{124}$$

As we showed above the effective action is given by

$$\Gamma_\phi = \frac{1}{2}\mathrm{Tr}\log\frac{1}{2}\Delta_{\tilde{G}} \sim -\mathrm{Tr}\int_0^\infty \frac{d\alpha}{\alpha}\left(e^{-\alpha\frac{1}{2}\Delta_{\tilde{G}}}\right)e^{-\frac{1}{2}\tilde{m}^2\alpha} \tag{125}$$

$$\equiv -\text{Tr} \int_0^\infty \frac{d\alpha}{\alpha} \left(e^{-\alpha \frac{1}{2}\Delta_{\tilde{G}}} \right) e^{-\frac{1}{\alpha \tilde{\Lambda}^2} - \frac{1}{2}\tilde{m}^2 \alpha} \tag{126}$$

where the small α divergence is regularized using an UV cutoff $\tilde{\Lambda}$ and the big α divergence is regularized using a cutoff $\tilde{m}^2$. Note that we allready had the mass so we can use it as an IR regulator.

The calculations simplfy if we calculate the heat kernel for $\Delta_{\tilde{G}}$ instead for $\frac{1}{2}\Delta_{\tilde{G}}$. To get rid of the factor $1/2$ we simply shift the integration variable $\alpha \to 2t$. After the shift the effective action is given by

$$\Gamma_\phi \equiv -\frac{1}{2}\text{Tr} \int_0^\infty \frac{dt}{t} \left(e^{-t\Delta_{\tilde{G}}} \right) e^{-\frac{1}{2t\tilde{\Lambda}^2} - \tilde{m}^2 t}. \tag{127}$$

The main problem during this work was the calculation of the trace of our operator. Due to the extra dilaton field we were not able to proceed with the standard heat kernel procedure, where the trace is simply given by the Seeley de Witt coefficients.

3.3.2 The heat kernel

To calculate the heat kernel expansion perturbatively we expand the operator $\Delta_{\tilde{G}}$ around the flat Laplacian,

$$\Delta_{\tilde{G}} = \Delta_0 + V(y) \tag{128}$$

When $V(y) = 0$ the heat kernel in 2 dimensions is

$$K_0(y, y', t) = \frac{1}{4\pi t} e^{-\frac{|y-y'|^2}{4t}} \tag{129}$$

and satisfies the initial condition

$$K_0(y, y') = \lim_{t \to 0} \delta^2(y - y'). \tag{130}$$

Let K(y,y',t) denote the heat kernel of $\Delta_{\tilde{G}}$, that means that K(y,y',t) satisfies the heat equation for the operator $\Delta_{\tilde{G}}$. K(y,y',t) is also the solution of the integral equation

$$K(y, y', t) = K_0(y, y', t) - \int_0^t dt_1 \int d^2 y_1 K_0(y, y_1, t - t_1) V(y_1) . K(y_1, y', t_1) \tag{131}$$

The integral equation can be solved perturbatively in a Neumann series by successive approximation if we write

$$K(y, y', t) = K_0(y, y', t) + K_1(y, y', t) + K_2(y, y', t) + \dots \tag{132}$$

then at coinciding points the heat kernel reads

$$K(y, y, t) = \frac{1}{4\pi t} + \int_0^t dt' \int d^2 y' K(y, y', t - t') V(y') K(y, y', t') + \dots \tag{133}$$

The trace of the operator is given by the heat kernel at coinciding points

$$\text{Tr}\left(e^{-t\Delta_{\tilde{G}}} \right) = \int d^2 y \sqrt{\tilde{G}} K(y, y, t) \tag{134}$$

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We now use the pertubative heat kernel expansion to evaluate K(y,y,t).

$$K(y, y, t) = \frac{1}{4\pi t} + \int_0^t dt' \int d^2y' K(y, y', t - t')V(y')K(y, y', t') + \dots \tag{135}$$

The perturbation V(y) of our operator is the following

$$V(y) = -(e^{2\sigma} - 1)\partial^2 - e^{2\sigma}\partial^a\sigma\partial_a \tag{136}$$

To the second order in $\sigma(y)$, this is just

$$V(y) = -(2\sigma(y) + 2\sigma(y)^2)\partial^2 - (1 + 2\sigma(y))\partial^a\sigma(y)\partial_a \tag{137}$$

Now we perform a Taylor expansion of $\sigma(y)$ about $y = y_0$

$$\sigma(y) = \sigma(y_0) + \partial_a\sigma(y_0)(y - y_0)^a + \frac{1}{2}\partial_a\partial_b\sigma(y_0)(y - y_0)^a(y - y_0)^b + \dots \tag{138}$$

Note that we pull indices with the metric $G^{ab} = e^{2\sigma}\delta^{ab}$. If we choose Riemannian normal coordinates the condition $\partial_a\tilde{G}^{ab} = 0$ implies $\sigma(y_0) = 0$ and $\partial_a\sigma(y_0) = 0$.

The perturbation V(y) reads after the Taylor expansion and the choice of the Riemannian normal coordinates

$$V(y) = -\partial_a\partial_b\sigma(y_0)(y - y_0)^a(y - y_0)^b\partial^2 - 2\partial^a\partial_b\sigma(y_0)(y - y_0)^b\partial_a \tag{139}$$

The heat kernel reads

$$K(y, y, t) = \frac{1}{4\pi t} + K_1(y, y, t) + K_2(y, y, t) \tag{140}$$

$$K_1(y, y, t) = -\frac{1}{16\pi^2}\partial_a\partial_b\sigma(y_0) \int_0^t dt' \frac{1}{(t - t')t'} \int d^2y'(y - y_0)^a(y - y_0)^b e^{-\frac{|y-y'|^2}{4(t-t')}}\partial^2 e^{-\frac{|y'-y|^2}{4(t')}} \tag{141}$$

$$K_2(y, y, t) = -\frac{1}{8\pi^2}\partial^a\partial_b\sigma(y_0) \int_0^t dt' \frac{1}{(t - t')t'} \int d^2y'(y - y_0)^b e^{-\frac{|y-y'|^2}{4(t-t')}}\partial_a e^{-\frac{|y'-y|^2}{4(t')}} \tag{142}$$

Besides basic integration one only needs the following integrals to to solve the heat kernel:

$$\int d^2k e^{-tk^2} = \frac{\pi}{t} \tag{143}$$

$$\int d^2k e^{-tk^2} k^a k^b = \frac{\pi}{2t^2} g^{ab} \tag{144}$$

$$\int d^2k e^{-tk^2} k^a k^b k^c k^d = \frac{\pi}{4t^3}(g^{ab}g^{cd} + g^{ac}g^{bd} + g^{ad}g^{bc}) \tag{145}$$

The solutions for K_1 and K_2 are the following

$$K_1(y, y, t) = -\frac{1}{12\pi}\partial_a\partial^a\sigma(y_0) \tag{146}$$

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$$K_2(y, y, t) = \frac{1}{4\pi}\partial_a\partial^a\sigma(y_0). \tag{147}$$

After adding all contributions one obtains the following heat kernel

$$K(y, y, t) = \frac{1}{4\pi t} + \frac{1}{6\pi}\partial_a\partial^a\sigma(y_0) \tag{148}$$

$$\Gamma_\phi = -\frac{1}{4\pi}\int d^2y\sqrt{\tilde{G}}(\tilde{\Lambda}^2 - \tilde{m}^2\log\tilde{\Lambda}^2) - \frac{1}{6\pi}\int d^2y\sqrt{\tilde{G}}G^{ab}\partial_a\partial_b\sigma\log(\frac{\tilde{\Lambda}}{\tilde{m}}). \tag{149}$$

Where G^{ab} is defined as

$$G^{ab} = e^{2\sigma}\bar{g}^{ab}. \tag{150}$$

3.4 Geometry from u(1) gauge fields

3.4.1 Moyal Weyl point of view

Now we rewrite the bosonic one loop effective action in terms of the u(1) gauge fields on the flat Moyal Weyl background $\mathbb{R}^2_{\bar\theta}$ with generators X^a. This means that we consider small fluctuations

$$Y^a = X^a + \mathcal{A}^a. \tag{151}$$

around the Moyal Weyl Generators X^a, which are solutions of the equations of motion and satisfy:

$$[X^a, X^b] = i\bar{\theta}^{ab} \tag{152}$$

Where $\bar{\theta}^{ab}$ is a constant antisymmetric tensor.
So lets take a look at the matrix model coupled to a scalar field

$$S[\phi] = -\frac{1}{2}Tr[Y^a, f(y)][Y^b, f(y)]g_{ab} \tag{153}$$

$$= \frac{1}{2}Tr[X^a + \mathcal{A}^a, f(y)][X^b + \mathcal{A}^b, f(y)]g_{ab} = \frac{1}{2}Tr[X^a + \bar{\theta}^{ac}A_c, f(y)][X^b + \bar{\theta}^{bd}A_d, f(y)]g_{ab} \tag{154}$$

$$\tag{155}$$

$$= \frac{1}{2}Tr\bar{g}^{cd}D_c\phi D_d\phi = \int d^2x\bar{g}^{cd}D_c\phi D_d\phi = \int d^2x\phi\Delta_A\phi \tag{156}$$

Where D_a and $\bar{g}^{cd}$ are given as follows

$$D_a = \frac{\partial}{\partial x^a} + i[A_a, .] \tag{157}$$

$$\bar{g}^{cd} = \bar{\theta}^{ac}\bar{\theta}^{bd}g_{ab}. \tag{158}$$

These formulas are exact if interpreted as non commutative gauge theory on $\mathbb{R}^2_{\bar\theta}$ where D_a is interpreted as covariant derivative withe u(1) gauge field $A_a(x)$.

3.4.2 Coordinate Transformation

To compare the results from the non commutative gauge theory point of view with the results of emergent gravity one has first to transform the coordinates from y to x.

$$y^a = x^a - \bar{\theta}^{ab} A_b \tag{159}$$

So the Jacobian is given by:

$$\left| \frac{\partial y^a}{\partial x^b} \right| = |\delta^a_b - V^a_b| = 1 - \bar{\theta}^{ac} \frac{\partial A_c}{\partial x^a} + O(\bar{\theta}^3) = 1 - \frac{1}{2} \bar{\theta}^{ac} \bar{F}_{ac} \tag{160}$$

We will use the following notation

$$\bar{\partial}_a = \frac{\partial}{\partial x^a} \tag{161}$$

$$\partial_a = \frac{\partial}{\partial y^a} \tag{162}$$

$$\partial_a = \frac{\partial x^c}{\partial y^a} \frac{\partial}{\partial x^c} = \bar{\partial}_a + V^c_a \bar{\partial}_c \tag{163}$$

$$V^c_a = \bar{\theta}^{cf} \frac{\partial A_f}{\partial x^a} \tag{164}$$

One wants to transform the following action from y to x coordinates.

$$\Gamma_\phi = -\frac{1}{4\pi} \int d^2 y \sqrt{\tilde{G}} (\tilde{\Lambda}^2 - \tilde{m}^2 \log \tilde{\Lambda}^2) - \frac{1}{6\pi} \int d^2 y \sqrt{\tilde{G}} G^{ab} \partial_a \partial_b \sigma \log(\frac{\tilde{\Lambda}}{\tilde{m}}). \tag{165}$$

The metric is given by

$$G^{ab}(y) = \theta^{ac}(y)\theta^{bd}(y)g_{cd} = (\bar{\theta}^{ac} - \bar{\theta}^{ae}\bar{\theta}^{ch}\bar{F}_{eh})(\bar{\theta}^{bd} - \bar{\theta}^{bf}\bar{\theta}^{dg}\bar{F}_{fg})g_{cd} \tag{166}$$

This metric can be splitted in 2 parts the flat part and the perturbation part, as allways done in the linearized version of general relativity.

$$G^{ab} = \bar{g}^{ab} - \tilde{h}^{ab} \tag{167}$$

$$\tilde{h}^{ab} = -\bar{g}^{ad}\bar{\theta}^{fb}\bar{F}_{df} - \bar{g}^{bd}\bar{\theta}^{fa}\bar{F}_{df} \tag{168}$$

One should notice that F^{ac} and $\bar{\theta}^{ch}$ are tensors in x coordinates G^{ab} is a tensor in y coordinates. So one has to be careful to the change of variables.

To compute the determinant one uses the following formula:

$$\det(1 + X) = 1 + \mathrm{tr}X + \frac{1}{2}\left((\mathrm{tr}X)^2 - \mathrm{tr}(X^2)\right) + O(X^3) \tag{169}$$

Therefore we rewrite the metric in a slightly other way.

$$G^{ab}(y) = \bar{g}^{ar}(\delta^b_r + X^b_r) \tag{170}$$

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$$X_r^b = \bar{\theta}^{fb}\bar{F}_{rf} + \bar{g}_{rm}\bar{\theta}^{mf}\bar{F}_{fd}\bar{g}^{db} + \bar{g}_{rm}\bar{\theta}^{me}\bar{F}_{eh}\bar{g}^{hg}\bar{\theta}^{fb}\bar{F}_{gf} \tag{171}$$

The relation between σ and the metric:

$$(\det G^{ab}) = (\det \bar{g}^{ab})(1 - 2\bar{F}_{rf}\bar{\theta}^{rf} + \frac{3}{2}(\bar{F}_{rf}\bar{\theta}^{rf})^2) \tag{172}$$

$$e^\sigma = (\det G^{ab})^{1/4} = (\det \bar{g}^{ab})^{1/4}(1 - \frac{1}{2}\bar{F}_{rf}\bar{\theta}^{rf} + \mathrm{O}(\bar{\theta}^3)) \tag{173}$$

So σ is given as

$$\sigma = \frac{1}{4}\mathrm{logdet}(\bar{g}^{ab}) - \frac{1}{2}\bar{\theta}^{ac}\bar{F}_{ac} - \frac{1}{8}(\bar{\theta}^{ac}\bar{F}_{ac})^2 \tag{174}$$

As pointed out before G^{ab} is not the metric one proceeds with. The metric we use has an extra term, which comes from the density factor. $\tilde{G}^{ab}$ is the metric that will be used.

$$\tilde{G}^{ab} = e^{-\sigma}G^{ab} = (1 + \frac{1}{2}\bar{\theta}^{kl}\bar{F}_{kl})(\bar{g}^{ab} - \tilde{h}^{ab}) = \bar{g}^{ab} + \frac{1}{2}\bar{g}^{ab}\bar{\theta}^{kl}\bar{F}_{kl} - \tilde{h}^{ab} \tag{175}$$

$$\tilde{G}^{ab} = \bar{g}^{ab} - h^{ab} \tag{176}$$

Where we redefined the perturbation as follows

$$h^{ab} := \tilde{h}^{ab} - \frac{1}{2}\bar{g}^{ab}\bar{\theta}^{kl}\bar{F}_{kl} \tag{177}$$

$$h^{ab} := -\bar{g}^{ad}\bar{\theta}^{fb}\bar{F}_{df} - \bar{g}^{bd}\bar{\theta}^{fa}\bar{F}_{df} - \frac{1}{2}\bar{g}^{ab}\bar{\theta}^{kl}\bar{F}_{kl} \tag{178}$$

In the 2 dimensional case calculations simplify further.

$$\bar{\theta}^{ca} = \varepsilon^{ca}\bar{\theta} \tag{179}$$

$$\bar{F}_{am} = \varepsilon_{am}F(x) \tag{180}$$

$$\bar{\theta}^{ca}\bar{F}_{am} = \varepsilon^{ca}\varepsilon_{am}\bar{\theta}F(x) = -\delta_m^c\bar{\theta}F(x) \tag{181}$$

Using this simple properties the metric perturbation reads

$$h^{ab} = -\bar{g}^{ad}\varepsilon^{bf}\varepsilon_{fd}\bar{\theta}F(x) - \bar{g}^{bd}\varepsilon^{af}\varepsilon_{fd}\bar{\theta}F(x) - \frac{1}{2}\bar{g}^{ab}\varepsilon^{kl}\varepsilon_{kl}\bar{\theta}F(x) \tag{182}$$

$$= \bar{g}^{ab}\bar{\theta}F(x). \tag{183}$$

So the metric is the following one

$$\tilde{G}^{ab} = \bar{g}^{ab}(1 - \bar{\theta}F(x)). \tag{184}$$

The inverse metric is given as

$$\tilde{G}_{ab} = \bar{g}_{ab}(1 + \bar{\theta}F(x)) + \mathrm{O}(A^2). \tag{185}$$

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The partial derivative is after the coordinate transformation

$$\partial_a \bar{\theta} F(x) = (\bar{\partial}_a + V_a^c \bar{\partial}_c) \bar{\theta} F(x) = (\bar{\partial}_a + \bar{\theta}^{cn} \bar{\partial}_a A_n \bar{\partial}_c) \bar{\theta} F(x). \tag{186}$$

The dilaton field is given as

$$\sigma = -\frac{1}{2} \bar{\theta}^{mn} \bar{F}_{mn} = -\bar{\theta} F(x). \tag{187}$$

One has first to transform the partial derivatives from y to x coordinates.

$$\partial_c \partial_d (\bar{\theta} F) = (\bar{\partial}_c + V_c^a \bar{\partial}_a)(\bar{\partial}_d + V_d^b \bar{\partial}_b)(\bar{\theta} F) \tag{188}$$

$$= \bar{\partial}_c \bar{\partial}_d (\bar{\theta} F) + \bar{\theta}^{bg} \bar{\partial}_c \bar{\partial}_d A_g \bar{\partial}_b (\bar{\theta} F) + \bar{\theta}^{bg} \bar{\partial}_d A_g \bar{\partial}_b \bar{\partial}_c (\bar{\theta} F) + \bar{\theta}^{af} \bar{\partial}_c A_f \bar{\partial}_a \bar{\partial}_d (\bar{\theta} F) \tag{189}$$

Ommiting terms $O((\bar{\theta} F)^3)$ the second and third eliminate each other through partial integration and the remaining terms are

$$\int d^2 x \sqrt{\tilde{G}} |J| \tilde{G}^{cd} \partial_c \partial_d (\bar{\theta} F) = \int d^2 x \sqrt{\bar{g}} \left(\bar{\partial}_c \bar{\partial}^c (\bar{\theta} F) + \bar{\theta}^{af} \bar{\partial}_c A_f \bar{\partial}_a \bar{\partial}^c (\bar{\theta} F) \right) \tag{190}$$

$$= \int d^2 x \sqrt{\bar{g}} \left(\bar{\partial}_c \bar{\partial}^c (\bar{\theta} F) + \bar{\theta}^{af} \bar{\partial}_a A_f \bar{\partial}_c \bar{\partial}^c (\bar{\theta} F) \right) = \int d^2 x \sqrt{\bar{g}} \left(\bar{\partial}_c \bar{\partial}^c (\bar{\theta} F) + \bar{\theta} F \bar{\partial}_c \bar{\partial}^c (\bar{\theta} F) \right) + O((\bar{\theta} F)^3) \tag{191}$$

$$\sqrt{\tilde{G}} |J| = \sqrt{\tilde{G}} e^{\sigma} = \sqrt{\bar{g}} e^{-\sigma} e^{\sigma} = \sqrt{\bar{g}} \tag{192}$$

We can drop the difference between ∂_c and $\bar{\partial}_c$ in terms which involve $O((\bar{\theta} F)^2)$, due to the fact that we negelect terms of $O((\bar{\theta} F)^3)$. Transforming the bosonic effective action and taking the Jacobian in to account one obtains the following:

$$\Gamma_\phi = -\frac{1}{4\pi} \int d^2 x \sqrt{\bar{g}} (\tilde{\Lambda}^2 - \tilde{m}^2 \log \tilde{\Lambda}^2) + \frac{1}{6\pi} \int d^2 x \left(\bar{g}^{ab} (1 - 2\bar{\theta} F) \partial_a \partial_b (\bar{\theta} F) \right) \log(\frac{\tilde{\Lambda}}{\tilde{m}}) \tag{193}$$

$$= -\frac{1}{4\pi} \int d^2 x \sqrt{\bar{g}} \tilde{\Lambda}^2 - \tilde{m}^2 \log \tilde{\Lambda}^2) + \frac{1}{6\pi} \int d^2 x \bar{g}^{ab} \left(\partial_a \partial_b (\bar{\theta} F) - 2\bar{\theta} F \bar{\partial}_a \bar{\partial}_b (\bar{\theta} F) \right) \log(\frac{\tilde{\Lambda}}{\tilde{m}}) \tag{194}$$

Now we transform the partial derivatives and obtain

$$= -\frac{1}{4\pi} \int d^2 x \sqrt{\bar{g}} (\tilde{\Lambda}^2 - \tilde{m}^2 \log \tilde{\Lambda}^2) + \frac{1}{6\pi} \int d^2 x \bar{g}^{ab} \left(\bar{\partial}_a \bar{\partial}_b (\bar{\theta} F) + \bar{\theta} F \bar{\partial}_a \bar{\partial}_b (\bar{\theta} F) - 2\bar{\theta} F \bar{\partial}_a \bar{\partial}_b (\bar{\theta} F) \right) \log(\frac{\tilde{\Lambda}}{\tilde{m}}) \tag{195}$$

The term $\int d^2 x \bar{\partial}_c \bar{\partial}^c (\bar{\theta} F)$ vanishes due to Stokes theorem and finally the effective action reads

$$= -\frac{1}{4\pi} \int d^2 x \sqrt{\bar{g}} (\tilde{\Lambda}^2 - \tilde{m}^2 \log \tilde{\Lambda}^2) - \frac{1}{6\pi} \int d^2 x \left(\bar{\theta} F \bar{\partial}_a \bar{\partial}^a (\bar{\theta} F) \right) \log(\frac{\tilde{\Lambda}}{\tilde{m}}) \tag{196}$$

There is detail concerning the cutoffs: $\tilde{\Lambda}$ is the effective cutoff for $\Delta_{\tilde{G}}$, which acts on a Hilbert space

of functions with inner product $(f,g) = \int d^2y f^*(y)g(y)$. From the gauge theory point of view, we have an effective cutoff Λ for $\Delta_A = [Y^a, [Y_a, .]]$ which acts on a Hilbert space of functions with inner product $(f,g) = \mathrm{Tr} f^*(y)g(y) = \int d^2y \rho(y) f^*(y)g(y)$. The relation between $\Delta_{\tilde{G}}$ and Δ_A can be easily understood if we write the action in 2 equivalent ways.

$$S[\phi] = -\frac{1}{2}\int d^2y \left(\phi\partial_a(\tilde{G}^{ab}(y)\partial_b\phi) + \rho(y)m^2\phi^2\right) = \frac{1}{2}\int d^2y \phi \left(\Delta_{\tilde{G}} + \rho(y)\tilde{m}^2\right)\phi \tag{197}$$

$$= \frac{1}{2}\mathrm{Tr}\left(-\phi[Y^a, [Y_a, \phi]] + \tilde{m}^2\phi^2\right) = \frac{1}{2}\int d^2y \phi \left(\Delta_A + \tilde{m}^2\right)\phi \tag{198}$$

This means the effective cutoffs are related as

$$\Delta_{\tilde{G}} = \rho(y)\Delta_A. \tag{199}$$

Which means that the effective cutoffs we implented to regularize the small α divergence are related as

$$\tilde{\Lambda}^2 = \rho(y)\Lambda^2. \tag{200}$$

And in order to be consistent with the cutoff for $\frac{1}{2}\Delta^2$ for scalar fields (from the gauge theory point of view) we replace as in [1] Λ^2 with $2\Lambda^2$. The same is also done for $\tilde{m}^2$.
So the effective action reads

$$\Gamma_\phi = -\frac{1}{2\pi}\int d^2x \sqrt{\bar{g}}(\Lambda^2 - m^2\mathrm{log}\Lambda^2)(1 + (\bar{\theta}F)^2) - \frac{1}{6\pi}\int d^2x \left(\bar{\theta}F\bar{\partial}_c\bar{\partial}^c(\bar{\theta}F)\right)\mathrm{log}(\frac{\Lambda}{m}). \tag{201}$$

The effective action we obtained from the geometrical point of view is exactly the effective action we obtained from the gauge theory point of view.

3.5 One-loop computation for the bosonic case

Consider now the action for a scalar coupled to the u(1) gauge field, written in Moyal-Weyl space. The Action has the following form.

$$S[\phi] = \int d^2x \frac{1}{2}\tilde{g}^{ab}(\partial_a + ig[A_a, .])\phi(\partial_b + ig[A_b, .])\phi + \frac{1}{2}m^2\phi^2 = S_0[\phi] + S_{int}[\phi] \tag{202}$$

$$S_0[\phi] = \int d^2x \frac{1}{2}\tilde{g}^{ab}\partial_a\phi\partial_b\phi + \frac{1}{2}m^2\phi^2 \tag{203}$$
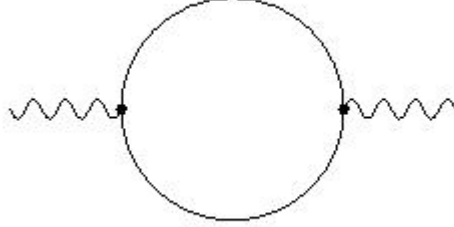
$$det(\tilde{g}_{ab}) = 1 \tag{204}$$

$$k.k = k_i k_j \tilde{g}^{ij} \tag{205}$$
$$k^2 = k_i k_j g^{ij} \tag{206}$$

We therefore compute

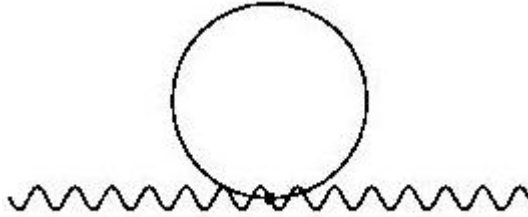$$\Gamma_\phi = \frac{1}{2}\mathrm{Tr}\log\frac{1}{2}\Delta_0 + \Gamma_\phi^{(1)} + \Gamma_\phi^{(2)}. \tag{207}$$

The first contribution is given by the following diagramm.

$$\Gamma_\phi^{(1)} = -\frac{g^2}{2} \int \frac{d^2p}{(2\pi)^2} A_{a'}(p) A_{b'}(-p) \tilde{g}^{a'a} \tilde{g}^{b'b} \int \frac{d^2k}{(2\pi)^2} \frac{4k_a k_b + 2k_a p_b + 2p_a k_b + p_a p_b}{(k.k + m^2)((k+p).(k+p) + m^2)} (1 - e^{ik_i \theta^{ij} p_j})$$

(208)

$$= \Gamma_\phi^{(1),P} + \Gamma_\phi^{(1),NP}$$

(209)

The second contribution is given by the following diagramm.

$$\Gamma_\phi^{(2)} = g^2 \int \frac{d^2p}{(2\pi)^2} A_a(p) A_b(-p) \tilde{g}^{ab} \int \frac{d^2k}{(2\pi)^2} \frac{1}{(k.k + m^2)} (1 - e^{ik_i \theta^{ij} p_j})$$

(210)

$$= \Gamma_\phi^{(2),P} + \Gamma_\phi^{(2),NP}$$

(211)

3.5.1 IR regulator and UV cutoff

We use the Schwinger representation for the propagators.

$$\frac{1}{k.k + m^2} = \int_0^\infty d\alpha e^{-\alpha(k.k + m^2)},$$

(212)

$$\frac{1}{(k.k + m^2)^2} = \int_0^\infty d\alpha \alpha e^{-\alpha(k.k + m^2)},$$

(213)

One puts a small mass as an IR regulator and the UV cutoff is implemented as follows:

$$\frac{1}{k.k} \to \int_0^\infty d\alpha e^{-\alpha k.k - \frac{1}{\alpha \Lambda^2}}$$

(214)

which removes the UV singularity at $\alpha = 0$. For this regularization one needs the following integrals:

$$\int_0^\infty d\alpha \frac{1}{\alpha} e^{-\alpha m^2 - \frac{1}{\Lambda^2 \alpha}} = 2K_0(2\sqrt{\frac{m^2}{\Lambda^2}}) = -2\left(\gamma + \log(\sqrt{\frac{m^2}{\Lambda^2}})\right) + \mathrm{O}(\frac{m^2}{\Lambda^2}\log(\frac{\Lambda}{\mathrm{m}})) \tag{215}$$

$$\int_0^\infty d\alpha \frac{1}{\alpha^2} e^{-\alpha m^2 - \frac{1}{\Lambda^2 \alpha}} = 2\sqrt{m^2 \Lambda^2} K_1(2\sqrt{\frac{m^2}{\Lambda^2}}) = \Lambda^2 - 2m^2 \log(\sqrt{\frac{\Lambda^2}{m^2}}) + m^2(2\gamma + 1)$$
$$+ \mathrm{O}(\frac{m^4}{\Lambda^2}\log(\frac{\Lambda}{\mathrm{m}})) \tag{216}$$

3.5.2 $\Gamma_\phi^{(1)}$

It is convenient to write (116) using a Feynman parameter.

$$\frac{1}{(k.k + m^2)((k+p).(k+p) + m^2)} = \int_0^1 dz \frac{1}{(l.l + z(1-z)p.p + m^2)^2} \tag{217}$$

$$= \int_0^1 dz \int_0^\infty d\alpha\, \alpha\, e^{-\alpha(l.l + z(1-z)p.p + m^2)} \tag{218}$$

where

$$l = k + zp \tag{219}$$

We need

$$\int \frac{d^2 k}{(2\pi)^2} \frac{P(k)}{(k.k + m^2)((k+p).(k+p) + m^2)}(1 - e^{ik_i \theta^{ij} p_j}) \tag{220}$$

$$= \int \frac{d^2 k}{(2\pi)^2} P(k) \int_0^1 dz \int_0^\infty d\alpha\, \alpha\, e^{-\alpha(l.l + z(1-z)p.p + m^2) - \frac{1}{\Lambda^2 \alpha}}(1 - e^{ik_i \theta^{ij} p_j}) \tag{221}$$

$$= \int_0^1 dz \int_0^\infty d\alpha\, \alpha\, e^{-\alpha(z(1-z)p.p + m^2) - \frac{1}{\Lambda^2 \alpha}} \int \frac{d^2 l}{(2\pi)^2} P(l - zp)(e^{-\alpha l.l} - e^{-\alpha(l_i.l_j + il_i \frac{\tilde{p}_j}{\alpha})\tilde{g}^{ij}}) \tag{222}$$

$$= \int_0^1 dz \int_0^\infty d\alpha\, \alpha\, e^{-\alpha(z(1-z)p.p + m^2) - \frac{1}{\Lambda^2 \alpha}} \int \frac{d^2 l}{(2\pi)^2} (P(l - zp)e^{-\alpha l.l} - P(l - zp + i\frac{\tilde{p}_j}{\alpha})e^{-\alpha l.l - \frac{\tilde{p}.\tilde{p}}{4\alpha}}) \tag{223}$$

We completed the square and shifted the integration in the last expression, where

$$\tilde{p}_i = \tilde{g}_{ij}\tilde{p}^j = \tilde{g}_{ij}\bar{\theta}^{jk} p_k. \tag{224}$$

For our purpose, P(k) is a polynomial which is at most quadratic. So one has

$$\int d^2 l\, e^{-\alpha l.l} = \frac{\pi}{\alpha} \tag{225}$$

$$\int d^2 l\, l_i e^{-\alpha l.l} = 0 \tag{226}$$

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$$\int d^2 l \, l_i l_j e^{-\alpha l.l} = \frac{\pi}{2} \frac{1}{\alpha^2} \tilde{g}_{ij} \tag{227}$$

Therefore

$$
\begin{aligned}
= \quad & \frac{1}{4\pi} \int_0^1 dz \int_0^\infty d\alpha e^{-\alpha(z(1-z)p.p+m^2) - \frac{1}{\Lambda^2 \alpha}} \left(p_a p_b (1 - 4z + 4z^2) + \frac{2}{\alpha} \tilde{g}_{ab} - \right. \\
& \left. - \left(p_a p_b (1 - 4z + 4z^2) + 2i(\tilde{p}_a p_b + p_a \tilde{p}_b) \frac{1 - 2z}{\alpha} - \frac{4\tilde{p}_a \tilde{p}_b}{\alpha^2} + \frac{2}{\alpha} \tilde{g}_{ab} \right) e^{-\frac{\tilde{p}.\tilde{p}}{4\alpha}} \right)
\end{aligned} \tag{228}
$$

$$m(z)^2 := z(1-z)p.p + m^2 \tag{229}$$

$$
\begin{aligned}
= \quad & \frac{1}{4\pi} \int_0^1 dz \left(p_a p_b (1 - 4z + 4z^2) \left(\frac{1}{m(z)^2 \Lambda^2} (2\sqrt{m(z)^2 \Lambda^2} K_1(2\sqrt{\frac{m(z)^2}{\Lambda^2}})) - \right. \right. \\
& \left. - \frac{1}{m(z)^2 \Lambda_{eff}^2} (2\sqrt{m(z)^2 \Lambda_{eff}^2} K_1(2\sqrt{\frac{m(z)^2}{\Lambda_{eff}^2}})) \right) + 2\tilde{g}_{ab} \left(2K_0(2\sqrt{\frac{m(z)^2}{\Lambda^2}}) - 2K_0(2\sqrt{\frac{m(z)^2}{\Lambda_{eff}^2}}) \right) + \\
& \left. + 4\tilde{p}_a \tilde{p}_b \left(2\sqrt{m(z)^2 \Lambda_{eff}^2} K_1(2\sqrt{\frac{m(z)^2}{\Lambda_{eff}^2}}) \right) \right)
\end{aligned} \tag{230}
$$

Here

$$\Lambda_{eff}^2 = \frac{1}{\frac{1}{\Lambda^2} + \frac{1}{4}\tilde{p}.\tilde{p}} = \frac{1}{\frac{1}{\Lambda^2} + \frac{1}{4}\frac{p^2}{\Lambda_{NC}^4}} \tag{231}$$

is the effective cutoff for non-planar graphs.
Note that

$$\tilde{p}.\tilde{p} = \tilde{p}_a \tilde{p}_b \tilde{g}^{ab} = \bar{\theta}^{ac} \bar{\theta}^{bd} p_c p_d \tilde{g}_{ab} = g^{cd} p_c p_d = \frac{p^2}{\Lambda_{NC}^4} \tag{232}$$

$$
\begin{aligned}
= \quad & \frac{1}{4\pi} \left(\frac{p_a p_b}{3} \left(\frac{1}{\Lambda_{eff}^2} \log(\Lambda_{eff}^2) - \frac{1}{\Lambda^2} log(\Lambda^2) + \right. \right. \\
& \left. + \frac{1}{3} (\frac{1}{\Lambda^2} - \frac{1}{\Lambda_{eff}^2})(3\log(p.p) - 47 + 6\gamma) \right) - 2\tilde{g}_{ab} (\log \frac{\Lambda_{eff}^2}{\Lambda^2}) + \\
& + 4\tilde{p}_a \tilde{p}_b \left(\Lambda_{eff}^2 - m^2 log(\Lambda_{eff}^2) - \frac{1}{6} p.p log(\Lambda_{eff}^2) + \right. \\
& \left. \left. + \int_0^1 dz (z(1-z)p.p + m^2) log(z(1-z)p.p + m^2) \right) - 12\gamma + 2 \right) \right).
\end{aligned} \tag{233}
$$

$$
\begin{aligned}
= \quad & \frac{1}{4\pi} \left(-2\tilde{g}_{ab} (\log \frac{\Lambda_{eff}^2}{\Lambda^2}) + 4\tilde{p}_a \tilde{p}_b \left(\Lambda_{eff}^2 - m^2 log(\Lambda_{eff}^2) - \frac{1}{6} p.p log(\Lambda_{eff}^2) + \right. \right. \\
& \left. \left. + \int_0^1 dz (z(1-z)p.p + m^2) log(z(1-z)p.p + m^2) \right) \right).
\end{aligned} \tag{234}
$$

Taking the limit m to 0 is not possible in this case and the reason is the following integral.

$$\int_0^1 dz (z(1-z)p.p + m^2)\log(z(1-z)p.p + m^2) = \tag{235}$$

$$= \frac{1}{3p}(4m^2 + p.p)^{\frac{3}{2}}\operatorname{arctanh}\left(\frac{p}{\sqrt{4m^2 + p.p}}\right) - \frac{1}{18}\left(24m^2 + 5p.p - (18m^2 + 3p.p)\log(m^2)\right) \tag{236}$$

$$= \frac{1}{6}p.p\log(m^2) + \text{finite} \tag{237}$$

By taking the limit m to 0 we would have a logarithmic divergence.

3.5.3 $\Gamma_\phi^{(2)}$

The second contribution is given by the following integral.

$$= \tilde{g}_{ab}\int \frac{d^2k}{(2\pi)^2}\frac{1}{(k.k+m^2)}(e^{ik_i\theta^{ij}p_j} - 1) = \tilde{g}_{ab}\int \frac{d^2k}{(2\pi)^2}\int_0^\infty d\alpha e^{-\alpha(k.k+m^2) - \frac{1}{\Lambda^2\alpha}}(e^{ik_i\theta^{ij}p_j} - 1) \tag{238}$$

$$= \tilde{g}_{ab}\int_0^\infty d\alpha e^{-\alpha m^2 - \frac{1}{\Lambda^2\alpha}}\int \frac{d^2k}{(2\pi)^2}\left(e^{-\alpha(k-i\frac{\tilde{p}}{2\alpha}).(k-i\frac{\tilde{p}}{2\alpha}) - \frac{\tilde{p}.\tilde{p}}{4\alpha}} - e^{-\alpha k.k}\right) \tag{239}$$

$$= \tilde{g}_{ab}\frac{1}{4\pi}\int_0^\infty d\alpha \frac{1}{\alpha}e^{-\alpha m^2 - \frac{1}{\Lambda^2\alpha}}\left(e^{-\frac{\tilde{p}.\tilde{p}}{4\alpha}} - 1\right) \tag{240}$$

$$= \frac{1}{4\pi}\tilde{g}_{ab}\left(2K_0(2\sqrt{\frac{m^2}{\Lambda_{eff}^2}}) - 2K_0(2\sqrt{\frac{m^2}{\Lambda^2}})\right) \tag{241}$$

$$= \frac{1}{4\pi}\tilde{g}_{ab}\log\frac{\Lambda_{eff}^2}{\Lambda^2} \tag{242}$$

The $\tilde{g}_{ab}$ term in Γ_ϕ^1 eliminates Γ_ϕ^2.

3.5.4 Effective Action

Combining the results and using

$$\bar{\theta}\bar{F}(p)\bar{\theta}\bar{F}(-p) = 4(\tilde{p}^a A_a(p))(\tilde{p}^b A_b(-p)). \tag{243}$$

One obtains the induced Action

$$\Gamma_\phi = \Gamma_\phi^{(1)} + \Gamma_\phi^{(2)}. \tag{244}$$

$$= -\frac{g^2}{2}\frac{1}{4\pi}\int \frac{d^2p}{(2\pi)^2}\bar{\theta}\bar{F}(p)\bar{\theta}\bar{F}(-p)\left(\Lambda_{eff}^2 + \frac{1}{6}p.p\log(m^2) + \frac{p.p}{6}\log(\frac{p.p}{\Lambda_{eff}^2})\right) \tag{245}$$

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To make contact with the geometrical action one uses the following expansion:

$$\Lambda_{eff}^2 = \Lambda^2 - p^2 \frac{\Lambda^4}{4\Lambda_{NC}^4} + \dots \tag{246}$$

which is valid in the IR regime. Assume the first case

$$\underline{\Lambda << \Lambda_{NC}}. \tag{247}$$

In this case we can assume

$$\Lambda_{eff}^2 \sim \Lambda^2 \tag{248}$$

Therefore one obtains the following.

$$\Gamma_\phi \quad \sim \quad -\frac{g^2}{2}\frac{1}{4\pi}\int \frac{d^2p}{(2\pi)^2}\bar\theta\bar F(p)\bar\theta\bar F(-p)\Big(\Lambda^2 - m^2\mathrm{log}(\Lambda^2) + \frac{1}{6}p.p\mathrm{log}(m^2) - \frac{1}{6}p.p\mathrm{log}(\Lambda^2) + \mathrm{finite}\Big) \tag{249}$$

$$= \quad -\frac{g^2}{2}\frac{1}{4\pi}\int \frac{d^2p}{(2\pi)^2}\bar\theta\bar F(p)\bar\theta\bar F(-p)\Big(\Lambda^2 - m^2\mathrm{log}(\Lambda^2) - \frac{1}{6}p.p\mathrm{log}(\frac{\Lambda^2}{m^2}) + \mathrm{finite}\Big) \tag{250}$$

$$= \quad -\frac{g^2}{8\pi}\int d^2x\Big(\bar\theta\bar F\bar\theta\bar F\Lambda^2 - \bar\theta\bar F\bar\theta\bar Fm^2\mathrm{log}(\Lambda^2) + \frac{1}{3}\bar\theta\bar F\bar\partial^a\bar\partial_a(\bar\theta\bar F)\mathrm{log}\frac{\Lambda}{m} + \mathrm{finite}\Big) \tag{251}$$

Using the notation from the last chapter the effective action reads

$$\Gamma_\phi = -\frac{g^2}{2\pi}\int d^2x\Big(\bar\theta F\bar\theta F\Lambda^2 - \bar\theta F\bar\theta Fm^2\mathrm{log}(\Lambda^2) + \frac{1}{3}\bar\theta F\bar\partial^a\bar\partial_a(\bar\theta F)\mathrm{log}\frac{\Lambda}{m} + \mathrm{finite}\Big). \tag{252}$$

The constant term in (201) represents the phase space volume of states with $\Delta_0 < \Lambda$. It can be obtained using the same regularization as above with the heat kernel expansion.

$$\frac{1}{2}\mathrm{Trlog}(\frac{1}{2}\Delta_0) \sim -\mathrm{Tr}\int_0^\infty \frac{d\alpha}{\alpha}\Big(e^{-\alpha\frac{1}{2}\Delta_0}\Big)e^{-\frac{1}{2\alpha\Lambda^2}-m^2\alpha} \tag{253}$$

We use the following expansion

$$\mathrm{Tr}e^{-\frac{1}{2}\alpha\Delta_0} \sim \sum_{n\geq 0}(\frac{\alpha}{2})^{\frac{n-2}{2}}\int_\mathcal{M} d^2x\sqrt{\bar g}a_n(x,\Delta_0). \tag{254}$$

where $a_n(x,\Delta_0)$ are known as the Seeley de Witt (or Duhamel) coefficients. The only non zero coefficient for this flat and free operator is the first Seeley de Witt (or Duhamel) coefficient.

$$a_0 = \frac{1}{4\pi} \tag{255}$$

So the constant term reads

$$\frac{1}{2}\mathrm{Trlog}(\frac{1}{2}\Delta_0) = -\frac{1}{2\pi}\int d^2x\sqrt{\bar g}\Big(\Lambda^2 - m^2\mathrm{log}\Lambda^2\Big). \tag{256}$$

4 The Fermionic case

Let us now consider a Dirac spinor coupled to the matrix model.

$$S = (2\pi)Tr\left(\bar{\psi}\gamma_a[Y^a, \psi] \right) \tag{257}$$

$$\sim \int d^2 y \rho(y)\left(\bar{\psi}i\gamma_a\theta^{ab}\partial_b\psi \right) \tag{258}$$

This is written for the Minkowski case the Euclidean Version is obtained through the replacement $\bar{\psi} \to \psi^\dagger$. The Dirac operator is then defined through:

$$\slashed{D}\psi = \gamma_a[Y^a, \psi] \sim i\gamma_a\theta^{ab}(y)\partial_b\psi \tag{259}$$

4.1 Quantization and induced gravity

As in the bosonic case, we couple fermions to the Matrix Model and quantize with the path integral formalism.
The one loop effective action is given by:

$$e^{-\Gamma_\psi} = \int d\psi d\psi^\dagger e^{-S[\psi]} \tag{260}$$

$$\Gamma_\psi = -\frac{1}{2}\text{Trlog}\left(\slashed{D}^2 \right) \tag{261}$$

Also as in the bosonic case we compute the one loop effective action in 2 different ways, we first compute Γ_ψ as induced gravity action, using the geometric heat kernel method. This is then compared with the one loop effective action of NC u(1) gauge theory in the IR Regime.

4.1.1 Square of the Dirac operator and induced Action

Starting with the following Action

$$S[\psi] = (2\pi)Tr\left(\psi^\dagger\gamma_a[Y^a, \psi] \right) \tag{262}$$

we want to study:

$$e^{-\Gamma_\psi} = \int d\psi d\psi^\dagger e^{(2\pi)Tr\left(\bar{\psi}^\dagger\gamma_a[Y^a, \psi] \right)} \tag{263}$$

$$= \exp(\ln(\det \slashed{D})) = \exp(\frac{1}{2}\text{logdet}(\slashed{D}^2)) \tag{264}$$

$$= \exp(\frac{1}{2}\text{Trlog}(\slashed{D}^2)) \tag{265}$$

at one loop, and consider the euclidean case. The square of the Dirac operator has the following form:

$$\slashed{D}^2\psi = \gamma_a\gamma_b[Y^a, [Y^b, \psi]] \tag{266}$$

$$= -\gamma_a \gamma_b \theta^{ac} \partial_c (\theta^{bd} \partial_d \psi) \tag{267}$$

$$= -G^{cd} \partial_c \partial_d \psi - a^d \partial_d \psi \tag{268}$$

Where a^d is:

$$a^d = \gamma_a \gamma_b \theta^{ma} \partial_m \theta^{db} \tag{269}$$

After taking $\rho(y) \, (= \sqrt{\tilde{G}})$ in to account the action reads

$$S[\psi] = \int d^2 y \sqrt{\tilde{G}} \bar{\psi} \left(\slashed{D}^2 \right) \psi. \tag{270}$$

Where the operator $\slashed{D}^2$ is the following one

$$\slashed{D}^2 = -G^{cd} \partial_c \partial_d - a^d \partial_d \tag{271}$$

$$= -G^{cd} \partial_c \partial_d - \gamma_a \gamma_b \bar{\theta}^{ma} \bar{\theta}^{db} e^{2\sigma(y)} \partial_m \sigma \partial_d := -G^{cd} \partial_c \partial_d - \Gamma^{md} e^{2\sigma(y)} \partial_m \sigma \partial_d \tag{272}$$

We define Γ^{md} as

$$\Gamma^{md} =: \gamma_a \gamma_b \bar{\theta}^{ma} \bar{\theta}^{db}. \tag{273}$$

We now use the explicit heat kernel procedure to calculate the trace of the following operator

$$\slashed{D}^2 = -e^{2\sigma} \partial_c \partial^c - \Gamma^{md} e^{2\sigma(y)} \partial_m \sigma \partial_d. \tag{274}$$

4.1.2 Quantization

We are now interested in the quantization of our matrix model coupled to the fermions. For the quantization procedure we use the path integral formalism, where we take the path integral over all matrices Y^a, while the spinors can be integrated out in terms of a determinant.

As in the bosonic case we have to deal in the fermionic case with an extra dilaton field. Due to this dilaton field we are not able to proceed with the standard heat kernel expansion, where the effective action is simply given by Seeley de Witt coefficients. The one loop action will be calculated in the same way we proceeded with the Bosons.

The effective action for the Fermions in the path integral formalism is given as

$$e^{-\Gamma_\psi} = \int d\psi d\bar{\psi} e^{-S[\psi]}. \tag{275}$$

By integrating out the spinors one simply obtains

$$\Gamma_\psi = -\frac{1}{2} \mathrm{Trlog} \left(\slashed{D}^2 \right) \tag{276}$$

$$\sim \frac{1}{2} \mathrm{Tr} \int_0^\infty \frac{d\alpha}{\alpha} e^{-\slashed{D}^2 \alpha} \equiv \frac{1}{2} \mathrm{Tr} \int_0^\infty \frac{d\alpha}{\alpha} \left(e^{-\slashed{D}^2 \alpha} \right) e^{-\tilde{m}^2 \alpha - \frac{1}{\tilde{\Lambda}^2 \alpha}} \tag{277}$$

We introduced $\tilde{\Lambda}^2$ as cutoff for the UV regime and we use the mass as an IR regulator.

To be consistent with the change of variables that we used in the geometrical interpretation of the bosonic case we shift $\tilde{\Lambda}^2 \to 2\tilde{\Lambda}^2$. The same is done for the mass $\tilde{m}^2$.

$$\Gamma_\psi \sim \frac{1}{2}\mathrm{Tr}\int_0^\infty \frac{dt}{t}\left(e^{-\slashed{D}^2 t}\right)e^{-2\tilde{m}^2 t - \frac{1}{2\tilde{\Lambda}^2 t}} \tag{278}$$

4.2 Evaluation of the heat kernel

4.2.1 The heat kernel

In this section we calculate the pertubvative heat kernel at coinciding points as in the bosonic case.

The trace of the operator is given by the heat kernel

$$\mathrm{Tr}\left(\mathrm{e}^{-\slashed{D}^2 t}\right) = \int d^2 y\sqrt{\tilde{G}}K(y,y,t) \tag{279}$$

We now use the pertubativ heat kernel expansion to evaluate K(y,y,t).

$$K(y,y,t) = \frac{1}{4\pi t} + \int_0^t dt'\int d^2 y'\, K(y,y',t-t')V(y')K(y,y',t') + \dots \tag{280}$$

To calculate the heat kernel expansion explicitly we expand the operator $\slashed{D}^2$ around the flat Laplacian,

$$\slashed{D}^2 = \slashed{D}_0^2 + V(y) \tag{281}$$

The perturbation V(y) of our operator is the following

$$V(y) = -(e^{2\sigma} - 1)\partial^2 - e^{2\sigma}\Gamma^{md}\partial_m\sigma\partial_d \tag{282}$$

To the second order in $\sigma(y)$, this is just

$$V(y) = -(2\sigma(y) + 2\sigma(y)^2)\partial^2 - (1 + 2\sigma(y))\Gamma^{md}\partial_m\sigma\partial_d \tag{283}$$

Now we perform a Taylor expansion of $\sigma(y)$ about $y = y_0$

$$\sigma(y) = \sigma(y_0) + \partial_a\sigma(y_0)(y - y_0)^a + \frac{1}{2}\partial_a\partial_b\sigma(y_0)(y - y_0)^a(y - y_0)^b + \dots \tag{284}$$

Note that we pull indices with the metric $G^{ab} = e^{2\sigma}\delta^{ab}$. If we choose Riemannian normal coordinates the condition $\partial_a\tilde{G}^{ab} = 0$ implies $\sigma(y_0) = 0$ and $\partial_a\sigma(y_0) = 0$.

The perturbation V(y) reads after the Taylor expansion and the choice of the Riemannian normal coordinates

$$V(y) = -\partial_a\partial_b\sigma(y_0)(y - y_0)^a(y - y_0)^b\partial^2 - 2\Gamma^{md}\partial_m\partial_a\sigma(y_0)(y - y_0)^a\partial_d \tag{285}$$

The heat kernel reads

$$K(y,y,t) = \frac{1}{4\pi t} + K_1(y,y,t) + K_2(y,y,t) \tag{286}$$

$$K_1(y,y,t) = -\frac{1}{16\pi^2}\partial_a\partial_b\sigma(y_0)\int_0^t dt'\frac{1}{(t-t')t'}\int d^2y'(y'-y_0)^a(y'-y_0)^b e^{-\frac{|y-y'|^2}{4(t-t')}}\partial^2 e^{-\frac{|y'-y|^2}{4(t')}} \quad (287)$$

$$K_2(y,y,t) = -\frac{1}{8\pi^2}\Gamma^{ma}\partial_m\partial_b\sigma(y_0)\int_0^t dt'\frac{1}{(t-t')t'}\int d^2y'(y'-y_0)^b e^{-\frac{|y-y'|^2}{4(t-t')}}\partial_a e^{-\frac{|y'-y|^2}{4(t')}} \quad (288)$$

After adding all contributions one obtains the following heat kernel

$$K(y,y,t) = \frac{1}{4\pi t} - \frac{1}{12\pi}\partial_a\partial^a\sigma(y_0) + \frac{1}{4\pi}\Gamma^{ab}\partial_a\partial_b\sigma(y_0) \quad (289)$$

We obtain the following for the effective action

$$\Gamma_\psi = \frac{1}{4\pi}\int d^2y\sqrt{\tilde{G}}(\tilde{\Lambda}^2 - \tilde{m}^2\log\tilde{\Lambda}^2)\text{tr}(\mathbb{1}) + \int d^2y\sqrt{\tilde{G}}\left(-\frac{1}{12\pi}G^{ab}\partial_a\partial_b\sigma\text{tr}(\mathbb{1}) + \frac{1}{4\pi}e^{2\sigma}\text{tr}(\Gamma^{ab})\partial_a\partial_b\sigma\right)\log(\frac{\tilde{\Lambda}}{\tilde{m}}). \quad (290)$$

We first have to calculate the trace of Γ^{ab} and it is done as follows

$$\text{tr}(\Gamma^{ab}) = \bar{\theta}^{ac}\bar{\theta}^{bd}\text{tr}(\gamma_c\gamma_d). \quad (291)$$

The trace over the 2 dimensional gamma matrices in Euclidean space is given by

$$\text{tr}(\gamma_c\gamma_d) = 2\delta^{cd}. \quad (292)$$

The trace of Γ^{ab} is

$$\text{tr}(\Gamma^{ab}) = 2\bar{\theta}^{ac}\bar{\theta}^{bd}\delta^{cd} = 2\bar{g}^{cd}. \quad (293)$$

The trace for a 2 dimensional Dirac spinor is

$$\text{tr}(\mathbb{1}) = 2. \quad (294)$$

We finally obtain the following for the effective action:

$$\Gamma_\psi = \frac{1}{2\pi}\int d^2y\sqrt{\tilde{G}}(\tilde{\Lambda}^2 - \tilde{m}^2\log\tilde{\Lambda}^2) + \int d^2y\sqrt{\tilde{G}}G^{ab}\left(-\frac{1}{6\pi}\partial_a\partial_b\sigma + \frac{1}{2\pi}\partial_a\partial_b\sigma\right)\log(\frac{\tilde{\Lambda}}{\tilde{m}}). \quad (295)$$

$$= \frac{1}{2\pi}\int d^2y\sqrt{\tilde{G}}(\tilde{\Lambda}^2 - \tilde{m}^2\log\tilde{\Lambda}^2) + \frac{1}{3\pi}\int d^2y\sqrt{\tilde{G}}G^{ab}\partial_a\partial_b\sigma\log(\frac{\tilde{\Lambda}}{\tilde{m}}). \quad (296)$$

Where G^{ab} is defined as

$$G^{ab} = e^{2\sigma}\bar{g}^{ab}. \quad (297)$$

4.3 Geometry from u(1) gauge fields

4.3.1 Moyal Weyl point of view

Now we rewrite the action for a Dirac Fermion on the Moyal-Weyl quantum Plane $\mathbb{R}_\theta^2$ coupled to a u(1) gauge field in the adjoint.

As in the bosonic case we write the general covariant coordinate Y^a as

$$Y^a = X^a + \mathcal{A}^a \tag{298}$$

around the Moyal Weyl Generators X^a, which are solutions of the equations of motion and satisfy:

$$[X^a, X^b] = i\bar{\theta}^{ab} \tag{299}$$

Where $\bar{\theta}^{ab}$ is a constant antisymmetric Tensor.

So lets take a look at the Matrix model coupled to a spinor field

$$[Y^a, f(y)] \sim [X^a + \mathcal{A}^a, f(y)] = [X^a - \bar{\theta}^{ab} A_b(x), f(y)] \equiv i\bar{\theta}^{ab} D_b f(y) \tag{300}$$

giving for the quadratic form

$$S[\psi]_{square} = (2\pi) Tr \frac{1}{2} \psi^\dagger \gamma_a \gamma_b [Y^a, [Y^b, \psi]] \tag{301}$$

$$= -\int d^2x \, \psi^\dagger \gamma_a \gamma_b \bar{\theta}^{am} \bar{\theta}^{bn} D_m D_n \psi \tag{302}$$

$$= -\int d^2x \, \psi^\dagger \tilde{\slashed{D}}_A^2 \psi \tag{303}$$

Where

$$\tilde{\slashed{D}}_A^2 = -\gamma_a \gamma_b \bar{\theta}^{am} \bar{\theta}^{bn} D_m D_n \tag{304}$$

4.3.2 Coordinate Transformation

To compare the results from the non commutative gauge theory point of view with the results of emergent gravity one has first to transform the coordinates from y to x.

$$y^a = x^a - \bar{\theta}^{ab} A_b \tag{305}$$

So the Jacobian is given by:

$$|\frac{\partial y^a}{\partial x^b}| = |\delta_b^a - V_b^a| = 1 - \bar{\theta}^{ac} \frac{\partial A_c}{\partial x^a} + O(\bar{\theta}^3) = 1 - \frac{1}{2} \bar{\theta}^{ac} \bar{F}_{ac} \tag{306}$$

I will use the following notation:

$$\bar{\partial}_a = \frac{\partial}{\partial x^a} \tag{307}$$

$$\partial_a = \frac{\partial}{\partial y^a} \tag{308}$$

$$\partial_a = \frac{\partial x^c}{\partial y^a}\frac{\partial}{\partial x^c} = \bar{\partial}_a + V_a^c \bar{\partial}_c \tag{309}$$

$$V_a^c = \bar{\theta}^{cf}\frac{\partial A_f}{\partial x^a} \tag{310}$$

One wants to transform the following action from y to x coordinates:

$$\Gamma_\psi = \frac{1}{2\pi}\int d^2y\sqrt{\tilde{G}}(\tilde{\Lambda}^2 - \tilde{m}^2\log\tilde{\Lambda}^2) + \frac{1}{3\pi}\int d^2y\sqrt{\tilde{G}}G^{ab}\partial_a\partial_b\sigma\log(\frac{\tilde{\Lambda}}{\tilde{m}}). \tag{311}$$

The metric is given by

$$G^{ab}(y) = \theta^{ac}(y)\theta^{bd}(y)g_{cd} = (\bar{\theta}^{ac} - \bar{\theta}^{ae}\bar{\theta}^{ch}\bar{F}_{eh})(\bar{\theta}^{bd} - \bar{\theta}^{bf}\bar{\theta}^{dg}\bar{F}_{fg})g_{cd} \tag{312}$$

This metric can be splitted in 2 parts the flat part and the perturbation part, as allways done in the linearized version of general relativity.

$$G^{ab} = \bar{g}^{ab} - \tilde{h}^{ab} \tag{313}$$

$$\tilde{h}^{ab} = -\bar{g}^{ad}\bar{\theta}^{fb}\bar{F}_{df} - \bar{g}^{bd}\bar{\theta}^{fa}\bar{F}_{df} \tag{314}$$

One should notice that F^{ac} and $\bar{\theta}^{ch}$ are tensors in x coordinates G^{ab} is a tensor in y coordinates. So one has to be careful to the change of variables.
To compute the determinant one use the following formula:

$$\det(1 + X) = 1 + \text{tr}X + \frac{1}{2}\Big((\text{tr}X)^2 - \text{tr}(X^2)\Big) + \text{O}(X^3) \tag{315}$$

Therefore we rewrite the metric in a slightly other way.

$$G^{ab}(y) = \bar{g}^{ar}(\delta_r^b + X_r^b) \tag{316}$$

$$X_r^b = \bar{\theta}^{fb}\bar{F}_{rf} + \bar{g}_{rm}\bar{\theta}^{mf}\bar{F}_{fd}\bar{g}^{db} + \bar{g}_{rm}\bar{\theta}^{me}\bar{F}_{eh}\bar{g}^{hg}\bar{\theta}^{fb}\bar{F}_{gf} \tag{317}$$

The relation between σ and the metric:

$$(\det G^{ab}) = (\det\bar{g}^{ab})(1 - 2\bar{F}_{rf}\bar{\theta}^{rf} + \frac{3}{2}(\bar{F}_{rf}\bar{\theta}^{rf})^2) \tag{318}$$

$$e^\sigma = (\det G^{ab})^{1/4} = (\det\bar{g}^{ab})^{1/4}(1 - \frac{1}{2}\bar{F}_{rf}\bar{\theta}^{rf} + \text{O}(\bar{\theta}^3)) \tag{319}$$

So σ is given as

$$\sigma = \frac{1}{4}\log\det(\bar{g}^{ab}) - \frac{1}{2}\bar{\theta}^{ac}\bar{F}_{ac} - \frac{1}{8}(\bar{\theta}^{ac}\bar{F}_{ac})^2 \tag{320}$$

As pointed out before that is not the metric one proceeds with. The metric we use has an extra term, which comes from the density factor. This is the metric that will be used.

$$\tilde{G}^{ab} = \mathrm{e}^{-\sigma} G^{ab} = (1 + \frac{1}{2}\bar{\theta}^{kl}\bar{F}_{kl})(\bar{g}^{ab} - \tilde{h}^{ab}) = \bar{g}^{ab} + \frac{1}{2}\bar{g}^{ab}\bar{\theta}^{kl}\bar{F}_{kl} - \tilde{h}^{ab} \tag{321}$$

$$\tilde{G}^{ab} = \bar{g}^{ab} - h^{ab} \tag{322}$$

Where we redefined the perturbation as follows

$$h^{ab} := \tilde{h}^{ab} - \frac{1}{2}\bar{g}^{ab}\bar{\theta}^{kl}\bar{F}_{kl} \tag{323}$$

$$h^{ab} := -\bar{g}^{ad}\bar{\theta}^{fb}\bar{F}_{df} - \bar{g}^{bd}\bar{\theta}^{fa}\bar{F}_{df} - \frac{1}{2}\bar{g}^{ab}\bar{\theta}^{kl}\bar{F}_{kl} \tag{324}$$

In the 2 dimensional case calculations simplfy further.

$$\bar{\theta}^{ca} = \varepsilon^{ca}\bar{\theta} \tag{325}$$

$$\bar{F}_{am} = \varepsilon_{am}F(x) \tag{326}$$

$$\bar{\theta}^{ca}\bar{F}_{am} = \varepsilon^{ca}\varepsilon_{am}\bar{\theta}F(x) = -\delta^c_m\bar{\theta}F(x) \tag{327}$$

Using this simple properties the metric perturbation reads

$$h^{ab} = -\bar{g}^{ad}\varepsilon^{bf}\varepsilon_{fd}\bar{\theta}F(x) - \bar{g}^{bd}\varepsilon^{af}\varepsilon_{fd}\bar{\theta}F(x) - \frac{1}{2}\bar{g}^{ab}\varepsilon^{kl}\varepsilon_{kl}\bar{\theta}F(x) \tag{328}$$

$$= \bar{g}^{ab}\bar{\theta}F(x). \tag{329}$$

So the metric is the following one
$$\tilde{G}^{ab} = \bar{g}^{ab}(1 - \bar{\theta}F(x)). \tag{330}$$

The inverse metric is given as
$$\tilde{G}_{ab} = \bar{g}_{ab}(1 + \bar{\theta}F(x)) + \mathrm{O}(A^2). \tag{331}$$

The partial derivative is after the coordinate transformation

$$\partial_a\bar{\theta}F(x) = (\bar{\partial}_a + V^c_a\bar{\partial}_c)\bar{\theta}F(x) = (\bar{\partial}_a + \bar{\theta}^{cn}\bar{\partial}_a A_n\bar{\partial}_c)\bar{\theta}F(x). \tag{332}$$

The dilaton field is given as

$$\sigma = -\frac{1}{2}\bar{\theta}^{mn}\bar{F}_{mn} = -\bar{\theta}F(x). \tag{333}$$

Transforming the fermionic effective action and taking the Jacobian in to account one obtains the following

$$\Gamma_\psi = \frac{1}{2\pi}\int d^2x\sqrt{\tilde{G}}|J|(\tilde{\Lambda}^2 - \tilde{m}^2\mathrm{log}\tilde{\Lambda}^2) - \frac{1}{3\pi}\int d^2x\sqrt{\tilde{G}}|J|G^{ab}\partial_a\partial_b\bar{\theta}F(x)\mathrm{log}(\frac{\tilde{\Lambda}}{\tilde{m}}). \tag{334}$$

As pointed out in the last chapter $|J|\sqrt{\tilde{G}} = 1$. And one has first to transform the partial derivatives from

y to x coordinates. It is exactly done as in the previous chapter.
One obtains the following for the effective action

$$\Gamma_\psi = \frac{1}{2\pi} \int d^2x (\tilde{\Lambda}^2 - \tilde{m}^2 \log\tilde{\Lambda}^2) - \frac{1}{3\pi} \int d^2x \bar{g}^{ab}(1 - 2\bar{\theta}F(x))\partial_a\partial_b\bar{\theta}F(x)\log(\frac{\tilde{\Lambda}}{\tilde{m}}). \tag{335}$$

$$= \frac{1}{2\pi} \int d^2x (\tilde{\Lambda}^2 - \tilde{m}^2 \log\tilde{\Lambda}^2) - \frac{1}{3\pi} \int d^2x \bar{g}^{ab}\Big(\partial_a\partial_b\bar{\theta}F(x) - 2\bar{\theta}F(x)\bar{\partial}_a\bar{\partial}_b\bar{\theta}F(x)\Big)\log(\frac{\tilde{\Lambda}}{\tilde{m}}). \tag{336}$$

$$= \frac{1}{2\pi} \int d^2x (\tilde{\Lambda}^2 - \tilde{m}^2 \log\tilde{\Lambda}^2) - \frac{1}{3\pi} \int d^2x \bar{\partial}_a\bar{\partial}^a\bar{\theta}F(x) + \bar{\theta}F(x)\bar{\partial}_a\bar{\partial}^a\bar{\theta}F(x) - 2\bar{\theta}F(x)\bar{\partial}_a\bar{\partial}^a\bar{\theta}F(x)\log(\frac{\tilde{\Lambda}}{\tilde{m}}). \tag{337}$$

$$= \frac{1}{2\pi} \int d^2x (\tilde{\Lambda}^2 - \tilde{m}^2 \log\tilde{\Lambda}^2) + \frac{1}{3\pi} \int d^2x \bar{\theta}F(x)\bar{\partial}_a\bar{\partial}^a\bar{\theta}F(x)\log(\frac{\tilde{\Lambda}}{\tilde{m}}). \tag{338}$$

As pointed out in the last chapter there is a detail concerning the cutoffs if one compares it with the gauge theory point of view. Which means that the effective cutoffs we implented to regularize the small α divergence are related as

$$\tilde{\Lambda}^2 = \rho(y)\Lambda^2. \tag{339}$$

And in order to be consistent with the cutoff for $\frac{1}{2}\Delta^2$ for scalar fields (from the gauge theory point of view) we replace as in [1] Λ^2 with $2\Lambda^2$. The same is again done also for $\tilde{m}^2$.

So the effective action reads

$$\Gamma_\psi = \frac{1}{\pi} \int d^2x (\Lambda^2 - m^2 \log\Lambda^2)(1 + (\bar{\theta}F(x))^2) + \frac{1}{3\pi} \int d^2x \bar{\theta}F(x)\bar{\partial}_a\bar{\partial}^a\bar{\theta}F(x)\log(\frac{\Lambda}{m}). \tag{340}$$

The effective action we obtained from the geometrical point of view is exactly the effective action we obtained from the gauge theory point of view.

4.4   One-loop computation for the fermionic case

We now consider the Feynman diagram corresponding to the following Action

$$\Gamma_\psi = -\frac{1}{2}\text{Trlg}\Delta_0 - \frac{g^2}{2}\left\langle \int d^2x \bar{\rho}\bar{\psi}\tilde{\gamma}^a[A_a, \psi] \int d^2y \bar{\rho}\bar{\psi}\tilde{\gamma}^b[A_b, \psi] \right\rangle \tag{341}$$

$$= -\frac{1}{2}\text{Trlg}\Delta_0 + \Gamma_\psi(A) \tag{342}$$

This gives:

$$\Gamma_\psi = 2g^2 \int \frac{d^2p}{(2\pi)^2} A_{a'}(p)A_{b'}(-p)\tilde{g}^{a'a}\tilde{g}^{b'b} \int \frac{d^2k}{(2\pi)^2} \frac{2k_ak_b + k_ap_b + p_ak_b - \tilde{g}_{ab}k(k+p)}{(k.k)((k+p).(k+p))}(1 - e^{ik_i\theta^{ij}p_j}) \tag{343}$$

which is quite close to the bosonic case, using the notation

$$k.k = k_ik_j\tilde{g}^{ij} \tag{344}$$

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$$k^2 = k_i k_j g^{ij} \tag{345}$$

To evaluate this loop integral, we rewrite it in a different way as in [3]

$$
\begin{aligned}
= \quad & \int \frac{d^2k}{(2\pi)^2} \left(\frac{(2k_a + p_a)(2k_b + p_b)}{(k.k)((k+p).(k+p))} - \frac{2\tilde{g}_{ab}}{k.k} \right)(1 - e^{ik_i \theta^{ij} p_j}) + \\
& + (p_a p_b - \tilde{g}_{ab} p.p) \int \frac{d^2k}{(2\pi)^2} \frac{1}{(k.k)((k+p).(k+p))} (e^{ik_i \theta^{ij} p_j} - 1)
\end{aligned} \tag{346}
$$

Now the first term is precisely the induced action we obtained from the Bosonic case.

$$
\begin{aligned}
\Gamma_\psi = \quad & -2\Gamma_\phi + g^2 \int \frac{d^2p}{(2\pi)^2} A_{a'}(p) A_{b'}(-p) \tilde{g}^{a'a} \tilde{g}^{b'b} (p_a p_b - \tilde{g}_{ab} p.p) \\
& \int \frac{d^2k}{(2\pi)^2} \frac{1}{(k.k)((k+p).(k+p))} (e^{ik_i \theta^{ij} p_j} - 1)
\end{aligned} \tag{347}
$$

So lets take a look at the k part

$$\Gamma_\psi^{(2)} = -\int \frac{d^2k}{(2\pi)^2} \frac{1}{(k.k)((k+p).(k+p))} (1 - e^{ik_i \theta^{ij} p_j}) \tag{348}$$

$$= \int_0^1 dz \int_0^\infty d\alpha \, \alpha e^{-\alpha(z(1-z)p.p) - \frac{1}{\Lambda^2 \alpha}} \int \frac{d^2l}{(2\pi)^2} e^{-\alpha l.l} \left(1 - e^{-\frac{\tilde{p}\tilde{p}}{4\alpha}}\right) \tag{349}$$

$$m(z)^2 := z(1-z)p.p + m^2 \tag{350}$$

$$\Gamma_\psi^{(2)} = -\frac{1}{4\pi} \int_0^1 dz \left(\frac{1}{m(z)^2 \Lambda^2} (2\sqrt{m(z)^2 \Lambda^2} K_1(2\sqrt{\frac{m(z)^2}{\Lambda^2}})) - \frac{1}{m(z)^2 \Lambda_{eff}^2} (2\sqrt{m(z)^2 \Lambda_{eff}^2} K_1(2\sqrt{\frac{m(z)^2}{\Lambda_{eff}^2}})) \right) \tag{351}$$

$$= \frac{1}{4\pi} \left(\frac{1}{\Lambda^2} \log(\Lambda^2) - \frac{1}{\Lambda_{eff}^2} \log(\Lambda_{eff}^2) + (\frac{1}{\Lambda_{eff}^2} - \frac{1}{\Lambda^2})(2\gamma - 3 + \log p.p) \right) \tag{352}$$

Here

$$\Lambda_{eff}^2 = \frac{1}{\frac{1}{\Lambda^2} + \frac{1}{4}\tilde{p}.\tilde{p}} = \frac{1}{\frac{1}{\Lambda^2} + \frac{1}{4}\frac{p^2}{\Lambda_{NC}^4}} \tag{353}$$

is the effective cutoff for non-planar graphs.

4.4.1 Effective Action

Summing up all results we obtain the following for the fermionic effective Action

$$
\Gamma_\psi \;=\; -2\Gamma_\phi + \frac{1}{4\pi}g^2 \int \frac{d^2p}{(2\pi)^2} A_{a'}(p)A_{b'}(-p)\tilde{g}^{a'a}\tilde{g}^{b'b}(p_a p_b - \tilde{g}_{ab}p.p)
$$
$$
\left(\frac{1}{\Lambda^2}\log(\Lambda^2) - \frac{1}{\Lambda_{eff}^2}\log(\Lambda_{eff}^2) + (\frac{1}{\Lambda_{eff}^2} - \frac{1}{\Lambda^2})(2\gamma - 3 + \log p.p) \right) \tag{354}
$$

$$
\;=\; -2\Gamma_\phi - \frac{1}{2}\frac{1}{4\pi}g^2 \int \frac{d^2p}{(2\pi)^2} \bar{F}_{ab}\bar{F}_{a'b'}\tilde{g}^{a'a}\tilde{g}^{b'b}
$$
$$
\left(\frac{1}{\Lambda^2}\log(\Lambda^2) - \frac{1}{\Lambda_{eff}^2}\log(\Lambda_{eff}^2) + (\frac{1}{\Lambda_{eff}^2} - \frac{1}{\Lambda^2})(2\gamma - 3 + \log p.p) \right) \tag{355}
$$

To make contact with the geometrical action one uses the following expansion:

$$
\Lambda_{eff}^2 = \Lambda^2 - p^2 \frac{\Lambda^4}{4\Lambda_{NC}^4} + \dots \tag{356}
$$

which is valid in the IR regime. Assume the first case

$$
\underline{\Lambda << \Lambda_{NC}}. \tag{357}
$$

In this case we can assume

$$
\Lambda_{eff}^2 \sim \Lambda^2 \tag{358}
$$

Therefore one obtains the following:

$$
\Gamma_\psi^{(2)} = 0 \tag{359}
$$

Therefore

$$
\Gamma_\psi + 2\Gamma_\phi = 0 \tag{360}
$$

This result is quite surprising from a supersymmetric point of view. The supersymmetric non commutative case in 2 dimensions is the same as the supersymmetric commutative case in 2 dimensions. This result should also encourage the work on 2 dimensional NCQFT, due to the fact that in the supersymmetric case no UV/IR mixing appears.

5   Conclusion

The basic intention of this paper is the verification of emergent gravity in 2 dimensions. Emergent gravity is interesting from many points of view, first the u(1) sector of NC gauge theory is interpreted as terms of gravity. Gravity is obtained from the Yang Mills matrix model which is similiar to actions that arise in the context of string theory, such as the IKKT Model. It differs from string theory, in claiming backgroung independency. We first derived the model with the heat kernel expansion where the coupling metric G^{ab} is interpreted as gravity. The next step was to compare the gravitational interpretation with the u(1) sector of NC gauge theory. The effective action is obtained in the gauge theory point of view in terms of Feynman diagrams. Even though the two effective actions are obtained from a different point of view, they agree in the IR regime.

Another exciting issue is that the metric couples to all matter, though a reasonable kinetic term is always of the from: $[X^a, .][X^b, .]g_{ab}$. Further in [1], it was pointed out that the Einstein Hilbert action will be induced upon quantization and will amount to the UV/IR mixing. The first fact can be easily understood by looking at the second Seeley de Witt coefficient: $a_2 = R[G] + 6E$. This means that every gravitational coupling to matter, will lead to an induced Einstein Hilbert action. But the second fact is surprising, though it gives an alternative explanation to the UV/IR mixing. The "would be photons" are not photons but rather gravitons defined on a nontrivial noncommutative background.

The bosonic case was a non trivial problem in this paper, since we had to deal with an extra dilaton term. This means we end up with a non minimal operator. Many authors [4,9] dealt with the same problem, but till now no solution was satisfying enough to fit in our framework. The procedure of the heat kernel expansion is not that "simple" as in the 4 dimensional case, though in 4 dimensions one has a minimal operator. The interesting part is that the dilaton factor often appears in the dimensional reduction from 4 to 2 dimensions, but we started with a matrix model that is in particular defined in 2 dimensions.

We simply solved the problem by calculating the heat kernel perturbatively instead of using the standard procedure of the heat kernel expansion, where the effective action is given through the Seeley de Witt coefficients. We finally compared the results with the NC gauge theory point of view and the two results coincide in the IR regime. The agreement of the 2 results was very pleasing, due to the fact that we had a non trivial check to the solution of the heat kernel expansion.

An important observation occured during this framework considering the limes m $\rightarrow$ 0, we explicitly showed that IR divergences would occur in this limes. One could care less and think of it as anomaly in 2 dimensions, this is simply not true it holds in any dimension. This is not just an effect that occurs in NCQFT, but as well in usual quantum field theories it is a well known fact that this particular limes is hard to handle. To be more precise the heat kernel expansion is not the best way to proceed for a massless scalar field. One can avoid IR divergences with the zeta function regularization. For a great insight to this problem view [18].

Another exciting observation was done in 2-dimensional NC gauge theory: $\Gamma_\psi + 2\Gamma_\phi = 0$. This in particular means that in the supersymmetric case all divergences would cancel each other, just as in the commutative case. Not only the divergences kill each other but the UV/IR mixing phenomena is totally absent. This opens the way towards a renormalizable model in 2 dimensions that incorporates gravity as an intristic part.

Danksagung

6 Appendix

Heat kernel expansion for the induced action

We described in the last chapter how to quantize a given action with the heat kernel expansion. Note that during the whole procedure the action was covariant, and the meausure that restored covariance was $\sqrt{G}$. In the action that we obtained from the matrix model we do not have such a factor.

The action that emerged from the matrix model is the following one

$$S[\phi] = -\frac{1}{2} \int d^2 y \sqrt{g} \phi(y)(\bar{g}^{ab} e^{\sigma(y)} \partial_a \partial_b + \bar{g}^{ab} e^{\sigma(y)} \partial_a \sigma \partial_b - e^{-\sigma(y)} m^2) \phi(y). \tag{361}$$

and the Laplace type operator is

$$A = -\bar{g}^{ab} e^{\sigma(y)}(\partial_a \partial_b + \partial_a \sigma \partial_b) = -(\tilde{G}^{ab} \partial_a \partial_b + \tilde{G}^{cd} \partial_c \sigma \partial_d). \tag{362}$$

$$\tilde{G}^{ab} = \bar{g}^{ab} e^{\sigma(y)} \tag{363}$$

Note: We do not include the mass to the operator, though we took it out during the heat kernel expansion to regularize the IR divergence.
In the 2 dimensional case many things simplfy for example

$$\tilde{G}^{ab} \Gamma^e_{ab} = -\frac{1}{2} \tilde{G}^{ab} \tilde{G}^{ed}(\tilde{G}_{ad} \phi_{,b} + \tilde{G}_{bd} \phi_{,a} - \tilde{G}_{ab} \phi_{,d}) \tag{364}$$

$$= -\frac{1}{2} \tilde{G}^{ab}(\delta^e_a \phi_{,b} + \delta^e_b \phi_{,a} - \tilde{G}_{ab} \phi^e) \tag{365}$$

$$= -\frac{1}{2}(\tilde{G}^{eb} \phi_{,b} + \tilde{G}^{ea} \phi_{,a} - 2\phi^e) = 0. \tag{366}$$

The curvature in 2 dimensions is

$$R = -\tilde{G}^{ab} \partial_a \partial_b \sigma. \tag{367}$$

a^d is given as

$$a^d = \tilde{G}^{cd} \partial_c \sigma. \tag{368}$$

In our case ω_d is

$$\omega_d = \frac{1}{2} \tilde{G}_{ed} a^e = \frac{1}{2} \tilde{G}_{ed} \tilde{G}^{ce} \partial_c \sigma = \frac{1}{2} \partial_d \sigma. \tag{369}$$

And E reads

$$E = -\tilde{G}^{cd}(\partial_c \omega_d + \omega_c \omega_d) \tag{370}$$

$$\partial_c \omega_d = \frac{1}{2} \partial_c \partial_d \sigma \tag{371}$$

$$\omega_c \omega_d = \frac{1}{4} \partial_c \sigma \partial_d \sigma \tag{372}$$

$$E = -\frac{1}{2} \tilde{G}^{cd} (\partial_c \partial_d \sigma + \frac{1}{2} \partial_c \sigma \partial_d \sigma). \tag{373}$$

So the second coefficient is the following one

$$a_2(y) = \frac{1}{(24\pi)} \left(R[\tilde{G}] + 3 \Delta_{\tilde{G}} \sigma - \frac{3}{2} \tilde{G}^{cd} \partial_c \sigma \partial_d \sigma \right) \tag{374}$$

After taking all coefficients in to account one obtains for the effective action the following.

$$\Gamma_\phi = -\frac{1}{4\pi} \int d^2 y \sqrt{\tilde{G}} (\tilde{\Lambda}^2 - e^{-\sigma(y)} \frac{\tilde{m}^2}{2} \log \tilde{\Lambda}^2) - \frac{1}{(24\pi)} \int d^2 y \sqrt{\tilde{G}} \left(R[\tilde{G}] + 3 \Delta_{\tilde{G}} \sigma - \frac{3}{2} \tilde{G}^{cd} \partial_c \sigma \partial_d \sigma \right) \log(\frac{\tilde{\Lambda}}{\tilde{m}}) \tag{375}$$

Where

$$\Delta_{\tilde{G}} \sigma = -\tilde{G}^{ab} \partial_a \partial_b \sigma + \Gamma^c \partial_c \sigma = -\frac{1}{\sqrt{\tilde{G}_{ab}}} \partial_a \left(\sqrt{\tilde{G}_{ab}} \tilde{G}^{ab} \partial_b \sigma \right). \tag{376}$$

$$\Gamma^c = \tilde{G}^{ab} \Gamma^c_{ab} \tag{377}$$

A bit of surprise is the second Seeley de Witt coefficient, because it is exactly the same as in the 4 dimensional case.

Now let us consider S_ϕ again. During the following calculations we will negelect the mass term for clarity.

$$S[\phi] = -\frac{1}{2} \int d^2 y \sqrt{\bar{g}} \phi(y) (\bar{g}^{ab} e^{\sigma(y)} \partial_a \partial_b + \bar{g}^{ab} e^{\sigma(y)} \partial_a \sigma \partial_b) \phi(y) = S^{(1)}[\phi] + S^{(2)}[\phi] \tag{378}$$

$$S^{(1)}[\phi] := -\frac{1}{2} \int d^2 y \sqrt{\bar{g}} \phi(y) (\bar{g}^{ab} e^{\sigma(y)} \partial_a \partial_b) \phi(y) \tag{379}$$

$$S^{(2)}[\phi] := -\frac{1}{2} \int d^2 y \sqrt{\bar{g}} \phi(y) (\bar{g}^{ab} e^{\sigma(y)} \partial_a \sigma \partial_b) \phi(y) \tag{380}$$

We now take a closer look at the second term of the action. After one partial integration one obtains

$$S^{(2)}[\phi] := \frac{1}{2} \int d^2 y \sqrt{\bar{g}} \bar{g}^{ab} e^{\sigma(y)} \phi(y) (\partial_a \sigma \partial_b + \partial_a \sigma \partial_b \sigma + \partial_a \partial_b \sigma) \phi(y). \tag{381}$$

Comparing the two actions gives us the following

$$S^{(2)}[\phi] := \frac{1}{4} \int d^2 y \sqrt{\bar{g}} \bar{g}^{ab} e^{\sigma(y)} \phi(y) (\partial_a \sigma \partial_b \sigma + \partial_a \partial_b \sigma) \phi(y). \tag{382}$$

So after taking this in to account

$$S[\phi] = -\frac{1}{2} \int d^2 y \sqrt{\bar{g}} \phi(y) (\bar{g}^{ab} e^{\sigma(y)} \partial_a \partial_b - \frac{1}{2} \bar{g}^{ab} e^{\sigma(y)} \partial_a \sigma \partial_b \sigma - \frac{1}{2} \bar{g}^{ab} e^{\sigma(y)} \partial_a \partial_b \sigma) \phi(y) \tag{383}$$

The Laplace type operator is the following

$$A = -(\tilde{G}^{ab}\partial_a\partial_b - \frac{1}{2}\tilde{G}^{ab}\partial_a\sigma\partial_b\sigma - \frac{1}{2}\tilde{G}^{ab}\partial_a\partial_b\sigma). \tag{384}$$

The Endomorphism E reads

$$E = b \tag{385}$$

due to the fact that in this case ω_d is 0.

$$E = -\frac{1}{2}\tilde{G}^{ab}\left(\partial_a\sigma\partial_b\sigma + \partial_a\partial_b\sigma\right) \tag{386}$$

So the second coefficient is the following one

$$a_2(y) = \frac{1}{(24\pi)}\left(R[\tilde{G}] + 3\Delta_{\tilde{G}}\sigma - 3\tilde{G}^{cd}\partial_c\sigma\partial_d\sigma\right) \tag{387}$$

The effective action reads

$$\Gamma_\phi = -\frac{1}{4\pi}\int d^2y\sqrt{\tilde{G}}(\tilde{\Lambda}^2 - e^{-\sigma(y)}\frac{\tilde{m}^2}{2}\log\tilde{\Lambda}^2) - \frac{1}{(24\pi)}\int d^2y\sqrt{\tilde{G}}\left(R[\tilde{G}] + 3\Delta_{\tilde{G}} - 3\tilde{G}^{cd}\partial_c\sigma\partial_d\sigma\sigma\right)\log(\frac{\tilde{\Lambda}}{\tilde{m}}). \tag{388}$$

Which is a clear contradiction to the previous calculation. The condradiction is in the term $\partial_c\sigma\partial_d\sigma$. Allthough we calculated the effective action from the same action, with the same procedure we obtain two different results.

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Zusammenfassung

Die grösste Herausforderung der heutigen modernen Physik ist die Vereinheitlichung der Allgemeinen Relativitätstheorie (ART) mit der Quantenfeldtheorie (QFT). Bis zum heutigen Stand gibt es keine vollständige Theorie die diese 2 Grundpfeiler der modernen Physik vereinigt. Es treten viele Probleme auf beim Versuch die Theorien zu vereinen. Einer der grössten Probleme betrifft die Dynamik der Raumzeit. In der ART ist die Raumzeit nicht bloss eine Bühne wo die Dynamik stattfindet, die Raumzeit trägt gundlegend zur Dynamik bei. Dieser Sachverhalt kann durch die berühmten Einsteinschen Feldgleichungen verstanden werden: $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}$. Die Einsteinschen Feldgleichungen verknüpfen die Geometrie der Raumzeit mit dem Energie-Impuls-Tensor. Quantenfeldtheorien wie QED (Quantenelekrodynamik) oder QCD (Quantenchromodynamik) hingegen arbeiten mit der Raumzeit bloss als eine Bühne wo die Dynamik stattfindet.

Die Vereinheitlichung scheitert nicht am Aufwand, es gibt zahlreiche wunderschöne Ansätze, z.B. Stringtheorie oder Ashtekars Ansatz zur Quantengravitation, um nur einige Wenige zu nennen. Der nichtkommutative Ansatz ist die Raumzeit selbst zu quantisieren, und eine (hoffentlich) renormalisierbare QFT auf solch eine Raumzeit zu definieren, die nach Möglichkeit auch Gravitation beinhaltet. Anfangs erschien der Ansatz erfreulich, da die nichtkomutative QFT (NCQFT) einen natürlichen UV-Cutoff enthält. Es stellte sich heraus [6], dass das nichtkommutative ϕ^4 Modell nicht endlich im UV Bereich ist und auch neue Divergenztypen beinhaltet, die sogenannten UV/IR mixing Divergenzen. Die Situation blieb unverändert bis H. Grosse und R. Wulkenhaar [4,5] ein renormalisierbares nichtkommutatives Modell definierten. Das war ein Durchbruch in der NCQFT.

Erst kürzlich realisierte H. Steinacker mit einem Matrix Modell NCQFT auf einen dynamischen nichtkommutativen Raum zu definieren. Die Idee wurde unter den Namen "emergent gravity from noncommutative gauge theory" publiziert. Steinacker erkannte, dass Matrix Modelle (MM) die nichtkommutative Eichtheorie (NKET) definieren eine spezifische Version der Gravitation beinhalten. Diese Erkenntnis legt den Grundstein für eine dynamische Theorie der nichtkommutativen Räume. Im Wesentlichen wurde in [1] behauptet, dass die Quantisierung zur Einstein Hilbert Wirkung führen wird. Es sollte auch das berühmte UV/IR mixing in NKET erklären. Diese Behauptung war überraschend und die Aussage wird durch die Tatsache gestützt, dass UV/IR mixing nur im U(1) Sektor der NCET auftritt. Emergent Gravity erklärt auch das seltsame Verhalten der Photonen: es sind nicht Photonen sondern Gravitonen, die einen nicht trivialen Hintergrund definieren. Die U(1) Eichfelder werden in der effektiven Metrik absorbiert. Alle Felder koppeln an die effektive Metrik.

In dieser Arbeit verifizieren wir die Erklärung des UV/IR mixings durch die Gravitation. Die Rechnungen werden in 2 Dimensionen durchgeführt. Wir koppeln das skalare Feld an das MM und führen eine 1 Loop Quantisierung durch. In der geometrischen Interpretation interpretieren wir die Wirkung als ein Skalarfeld gekoppelt an Gravitation. Diese Interpretation führt zur Einstein Hilbert Wirkung. In der 2. Interpretation interpretieren wir dasselbe MM als NCET. Die 1 Loop Quantisierung wird mit den Feynmanregeln berechnet. Diese 2 Interpretationen sollten zumindest im IR Bereich übereinstimmen. Dieselbe Prozedur wird für Fermionen gekoppelt am MM durchgeführt.

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