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APPROXIMATION OF GENERALIZED STOCHASTIC PROCESSES

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Preface

The goal of the following work will be to develop some approximation properties of generalized stochastic processes, a very flexible concept which can be defined in various ways. Therefore we introduce a generalization of ordinary stochastic processes which will be analogous to the generalization of ordinary functions to distributions. In particular we regard generalized stochastic processes as Hilbert space valued bounded linear operators on spaces of test functions.
Historically the first works using this concept are due to K. Itô, who studied stationary random distributions (cf. [16]) and I. M. Gelfand, who developed generalized random processes as a part of his work on generalized functions (cf. [11, 12]). Both of them used the space of infinitely often differentiable functions with compact support over $\mathbb{R}$ and $\mathbb{R}^d$ as their space of test functions. A problem appears, as this space is not invariant under Fourier transform and thus the spectral process to an arbitrary given generalized stochastic process does not exist in general.
According to the works of W. Hörmann [8, 14, 15] we choose the Segal algebra S_0, a function space discovered by H. G. Feichtinger, as our space of test function. Therefore we may profit from the fact that S_0, nowadays often called Feichtinger algebra, is a Fourier invariant Banach space. Furthermore S_0 can be easily defined for locally compact Abelian groups and thus we are not restricted to $\mathbb{R}^d$.
After having fixed some notations we introduce generalized stochastic processes in chapter 1. Therefore we report the procedure to get from ordinary stochastic processes to the generalized ones. Furthermore we briefly note the ideas contained in the works of H. Niemi [18, 19, 20] which pointed the way to the papers of W. Hörmann. At the end of this chapter we give some important Definitions concerning our further theory.
In the second chapter we develop the preliminary theory which will be necessary sin the following chapters. Therefore this part of the work is strongly influenced by the thesis of W. Hörmann [15]. We start the chapter by studying the properties of the covariance distribution of a generalized stochastic process. This concept appears as one of the most important during the whole work. In the second section we introduce the spectral process, which exists generally, due to the properties of the Feichtinger algebra. Finally we study stationary processes and report some relations of our theory to ordinary stochastic processes and to vector measures. Therefore well known results concerning these theories will follow directly from our calculations and thus complicated proofs reduce to simple ones.

The first part of chapter 3 deals with some calculations concerning filtered generalized stochastic processes. This concept can be introduced in a natural way by using adjoint operators. Furthermore filtered processes will be very important for the first section of chapter 4. The second section of this chapter contains generalizations of stationary generalized stochastic processes, that will be the V-bounded and the harmonizable ones. Furthermore we note some interesting facts concerning the dilation theory.
The final chapter 4 represents the main part of this work. During the first section we report a remarkable result which states that V-bounded generalized stochastic process can be approximated by harmonizable ones, though there are V-bounded processes which are not harmonizable. In the second part of this chapter we will involve some new convergence results for the Schoenberg operator and quasi interpolation operators, as noted in [9], and apply them to generalized stochastic processes. This means in particular, that we will derive new results, which state that some natural conditions on a generator function will imply (pointwise) convergence of the covariance distribution of an arbitrary process. The third section justifies, that these new results also hold for spectral processes. Furthermore we give an alternative Definition of S_0, prove the invariance under Fourier transform and finally we point out an idea, which was found in [10], to a further approximation result using the short time Fourier transform. The final section of this chapter represents an additional report concerning convergence properties of generalized stochastic processes on the Zemanian space. This section shall only denote a further information for interested readers.
An abstract of this work in German language can be found in the Appendix.

The author wants to thank his advisor H. G. Feichtinger for important hints and for all the time he invested during this work.

Chapter 1

Introduction

Before we start with the theory of generalized stochastic processes we have to fix some notations, and thus the first section deals with different concepts, especially with basic facts about the Feichtinger algebra $S_0(G)$, which we need for our further calculations. We do not report the different proofs, but we always mention the relevant references.

1.1 Notations

First we recall some useful facts about Fourier-analysis on groups, refering to [23] for more details:
We always assume that G is a **locally compact Abelian group** (LCA group) with a **Haar measure** dx and addition as group operation. (e.g. If $G{=}\mathbb{R}^d$ it is the usual addition of vectors).
The **dual group** to G is denoted as $\hat{G}$ and χ_x is a **character** on $\hat{G}$.

We will use the following spaces of complex-valued continuous functions:

$C_0(G) := \{f : G \to \mathbb{C} \mid f$ is continuous and $\lim_{|x|\to\infty} = 0\}$ respectively $C_b(G) := \{f : G \to \mathbb{C} \mid f$ is continuous and bounded$\}$, both endowed with the infinity norm $\|f\|_\infty := \sup_{x\in G}|f(x)|$.

$C_c(G) := \{f : G \to \mathbb{C} \mid f$ is continuous and $\mathrm{supp}(f)$ is compact$\}$, endowed with the inductive limit topology. (The support of a continuous function $f : G \to \mathbb{C}$ is defined as: $\text{supp}(f) := \{x \in G \mid f(x) \neq 0\}^{-}$.)

Definition 1.1.1. Let $x, y \in G$ and $t \in \hat{G}$. The **Translation operator** $T_y f(x)$ is defined by:

$$T_y f(x) := f(x - y)$$

and the **Multiplication operator** $M_t f(x)$ is given by:

$$M_t f(x) := t(x) f(x).$$

Remark 1.1.2. These operators are isometric mappings on

$$L^1(G) := \{f : G \to \mathbb{C} \mid \ \|f\|_1 := \int_G |f(x)|\, dx < \infty\}, \tag{1.1}$$

which is a Banach-algebra with respect to convolution. $C_c(G)$ is a dense subspace of $L^1(G)$.

Definition 1.1.3. For $f, g \in C_c(G)$ we define the **convolution** by:

$$f * g(x) := \int_G T_x f(y) g(y)\, dy \tag{1.2}$$

and for $f \in L^1(G)$ we define the **Fourier transform** of f by:

$$\mathcal{F}f(t) := \hat{f}(t) := \int_G t(y) f(x)\, dx; \quad t \in \hat{G}. \tag{1.3}$$

Theorem 1.1.4. (Convolution theorem)
One can show, that: $(f * g)\hat{} = \hat{f} \cdot \hat{g}$.

Definition 1.1.5. The **Fourier-algebra** $A(G)$ is defined as:

$$A(G) := \mathcal{F}(L^1(\hat{G})) = \{f \mid f = \hat{g} \text{ with } g \in L^1(\hat{G})\}.$$

Next we note the important facts about the Segal algebra $S_0(G)$. For more details the reader is referred to [4, 6, 7].

Definition 1.1.6. Let $k \in A(G) \cap C_c(G)$ be arbitrary but fixed. The **Feichtinger algebra** $S_0(G)$ is defined by

$$S_0(G) := \{f \in A(G) \mid \ \|f\|_{S_0} := \int_G \|T_y k \cdot f\|_A \, dy < \infty\}.$$

Remark 1.1.7. $S_0(G)$ is the minimal Banach-space among all Banach-spaces which are isometrically invariant under translation and character multiplication and it contains all $f \in L^1(G)$ with compactly supported Fourier-transform. An other property of $S_0(G)$ - the invariance under Fourier-transform - is very important for this work. The details of this fact will be worked out in section 4.3.

Definition 1.1.8. We will often use the elements of $S_0'(G)$, i.e the bounded linear functionals on $S_0(G)$, which we call **distributions**.

Notation 1.1.9. For convenience we introduce for $\sigma \in S_0'(G)$ and $f \in S_0(G)$ the following notation: $\langle \sigma, f \rangle := \sigma(f)$. To avoid confusions, we denote the sesquilinear inner product in Hilbert-spaces, which we always symbolize with $\mathcal{H}$, by $(. \mid .)$.

Definition 1.1.10. For distributions $\sigma \in S_0'(G)$ we define the following operators (cf. [7] paragraph 3):

$$\langle T_x\sigma, f\rangle := \langle \sigma, T_{-x} f\rangle \quad \text{for } x \in G$$

$$\langle g * \sigma, f\rangle := \langle \sigma, \check{g} * f\rangle \quad \text{for } g \in L^1(G)$$

$$\langle h\sigma, f\rangle := \langle \sigma, hf\rangle \quad \text{for } h \in A(G)$$

$$\langle \hat{\sigma}, f\rangle := \langle \sigma, \hat{f}\rangle$$

$$\langle \check{\sigma}, f\rangle := \langle \sigma, \check{f}\rangle$$

Definition 1.1.11. A distribution $\sigma \in S_0'(G)$ is called **positive**, if:

$$f \geq 0 \Longrightarrow \langle \sigma, f\rangle \geq 0.$$

Definition 1.1.12. Let $f \in S_0(G_1)$ and $g \in S_0(G_2)$. The **tensor product** of f and g is the function $f \otimes g \in S_0(G_1 \times G_2)$ given by

$$f \otimes g(x, y) := f(x) \cdot g(y) \quad x \in G_1, y \in G_2. \tag{1.4}$$

Furthermore let B_1 and B_2 be two Banach-spaces, which are continuously embedded into $C_b(G_1)$ respectively $C_b(G_2)$. The **projective tensor product** of B_1 and B_2 is defined as

$$B_1 \hat{\otimes} B_2 := \{f \mid f = \sum_{n=1}^{\infty} f_n \otimes g_n \text{ such that } \sum_{n=1}^{\infty} \|f_n\|_{B_1} \|g_n\|_{B_2} < \infty\}. \quad (1.5)$$

Remark 1.1.13. $B_1 \hat{\otimes} B_2$ is a Banach-space which is continuously embedded in $C_b(G_1 \times G_2)$, endowed with the norm

$$\|f\|_{\hat{\otimes}} := \inf \{\sum_{n=1}^{\infty} \|f_n\|_{B_1} \|g_n\|_{B_2} \text{ with } f = \sum_{n=1}^{\infty} f_n \otimes g_n\}.$$

Furthermore we denote $M(G) := C_0'(G)$ which is the **space of bounded measures**. As $S_0(G)$ lies dense in $C_0(G)$ we get, that $M(G) \subseteq S_0'(G)$. The elements of $C_c'(G)$ are called **Radon-measures**.

Definition 1.1.14. A Radon-measure μ is called **translation bounded** if for any

$$\sup_{x \in G} |\mu(T_x(k))| < \infty \quad \forall\, k \in C_c(G).$$

Definition 1.1.15. A bounded set $S \subseteq M(G)$ is called **tight** if $\forall\, \varepsilon > 0$ there $\exists\, k \in C_c(G)$, such that:

$$\|k \cdot \mu - \mu\|_M \leq \varepsilon \quad \forall\, \mu \in S.$$

A bounded net $(e_\eta)_{\eta \in E} \in L^1(G)$ is called a **bounded approximate unit** for $L^1(G)$, if

$$\lim_{\eta \in E} \|e_\eta * f - f\|_1 = 0 \quad \forall\, f \in L^1(G).$$

Furthermore we define $\delta_0 := \lim_{\eta \in E} e_\eta$, which is called (Diracs) **Delta distribution**. (For more details cf. section 4.2)

Definition 1.1.16. A net $(\mu_\eta)_{\eta \in E} \in M(G)$ is called **vaguely convergent** with limit μ_0, if:

$$\lim_{\eta \in E} \mu_\eta(k) = \mu_0(k) \quad \forall\, k \in C_c(G).$$

Next we note some facts concerning bimeasures, taken from [13]. This concept will be very important in section 2.1.

Definition 1.1.17. We define the **test function space** $V_0(G_1 \times G_2)$ **of bimeasues** by:

$$V_0(G_1 \times G_2) := C_0(G_1)\hat{\otimes}C_0(G_2) =$$

$$= \{f \in C_b(G_1 \times G_2) \mid f = \sum_{n=1}^{\infty} f_n \otimes g_n \text{ such that } \sum_{n=1}^{\infty} \|f_n\|_\infty \|g_n\|_\infty < \infty\}$$

endowed with the norm :

$$\|f\|_{V_0} := \inf \{\sum_{n=1}^{\infty} \|f_n\|_\infty \|g_n\|_\infty \text{ with } f = \sum_{n=1}^{\infty} f_n \otimes g_n\}.$$

Lemma 1.1.18. (Properties of V_0)
(1) $S_0(G_1 \times G_2)$ lies dense in $V_0(G_1 \times G_2)$.
(2) $V_0(G_1 \times G_2)$ lies dense in $C_0(G_1 \times G_2)$.

Definition 1.1.19. The dual of $V_0(G_1 \times G_2)$ is called the **space of bimeasures**, symbolized with $BM(G_1 \times G_2)$.

Definition 1.1.20. Let $\mu \in BM(G_1 \times G_2)$. We define the **Fourier-transform of** μ by:

$$\hat{\mu}(t_1, t_2) := \mu(\bar{t}_1 \otimes \bar{t}_2) \quad t_1 \in G_1, t_2 \in G_2.$$

Lemma 1.1.21. The Fourier-transform maps $BM(G_1 \times G_2)$ into $C_b(\hat{G}_1 \times \hat{G}_2)$.

For our further calculations, especially in section 2.1, it will be necessary to restrict a distribution $\sigma \in S_0'(G)$ with $\text{supp}(\sigma) \subseteq E \subseteq G$ to the set E. Therefore we choose an arbitrary extension of $f_E \in S_0(E)$ and define

$$\sigma_E(f_E) := \sigma(f) \ \text{ for } \ f \in S_0(G)$$

which is an extension of f_E, satisfying $\|f\|_{S_0} \leq c_E \|f_E\|_{S_0}$. Now we have to show, that the definition above is indeed well defined. Therefore we have to show, that $\text{Restr}_E f = 0 \Longrightarrow \sigma(f) = 0$. First assume that $\text{supp}(f) \subseteq G \backslash \bar{E}$. Now we get:

Lemma 1.1.22. Let $\sigma \in S_0'(G)$ and $f \in S_0(G)$. Then $\sigma(f) = 0$, if

$$\text{supp}(\sigma) \cap \text{supp}(f) = \emptyset.$$

As $\mathrm{Restr}_E(f) = 0$ means $f(x) = 0\ \forall x \in E$, but not $\mathrm{supp}(f) \cap E = \emptyset$ we have to introduce a further concept, the so called spectral synthesis. Therefore we regard the pointwise Ideals I with $\mathrm{cosp}(I) = \{x \mid f(x) = 0, \forall x \in I\} =: E$, with the maximal ideal $k(E) := \{f \mid f \in S_0(G),\ f(x) = 0\ \forall x \in E\}$ and the minimal one $j(E) := \{f \mid f \in S_0(G),\ \ \mathrm{supp}(f) \cap E = \emptyset\}$.

Definition 1.1.23. Let $E \subseteq G$. We call E a **spectral synthesis** if $k(E)$ is equal to the closure of $j(E)$ with respect to the topology of $S_0(G)$.

Now we have left to note the following needed results:

Theorem 1.1.24. Let $\sigma \in S_0'(G)$ with $\mathrm{supp}(\sigma) \subseteq E$. Then σ can be restricted to E if and only if E denotes a spectral synthesis.

Theorem 1.1.25. Let G_1, G_2 denote lcA groups. Then $\{1\} \times G_2$ is a spectral synthesis.

Corollary 1.1.26. Let G denote a lcA group. Then $\Delta_G := \{(x, x) \mid x \in G\}$ is a spectral synthesis.

For further informations concerning the spectral synthesis, we refer to the books of H. Reiter [21, 22].

1.2 Generalizing Stochastic Processes

The aim of this section is to show, that generalized stochastic processes can be introduced in various ways. To illustrate this we regard a few different estimates. At the end of the chapter we define concepts like stationarity or boundedness of generalized stochastic processes, which will be very important in this work.

We start with a few notations about classical stochastic processes, which are taken from [2]. Definition 1.2.1. can be found on page 46.

Let (Ω, Σ, P) be an arbitrary Probability space, i.e. Ω is a set with a $\sigma-$Algebra Σ and P is a Probability measure with $P(\Omega) = 1$.
A measurable function $X : \Omega \to \mathbb{C}$ is called a **random variable**.

The vector space of all complex-valued random variables X on (Ω, Σ, P) is denoted as $\mathcal{L}(\Omega, \Sigma)$.
For $X \in \mathcal{L}(\Omega, \Sigma)$ the **expectation** (or **mean value**) of X is given by

$$E(X) := \int_\Omega X \, dP,$$

if this integral exists.

Definition 1.2.1. We define a **stochastic process** as any family of random variables $(X_t)_{t \in T}$, where X_t is the observation at time t and T is the time range.
Is T an interval (resp. an infinite sequence) the process is called **continuous** (resp. **discrete**) **parameter process**.

In other words a stochastic is a mapping[1] from $\mathbb{R}^d$ (resp. $\mathbb{Z}^d$) into a space of random variables over an arbitrary Probability space, which is commonly $L^2(\Omega, \Sigma, P)$.

Remark 1.2.2. To get more generality, we regard a stochastic process as a mapping from an arbitrary locally compact Abelian group G into the space

$$L^2(\Omega, \Sigma, P) := \{X \in \mathcal{L}(\Omega, \Sigma) \mid E(|X^2|) < \infty\}. \tag{1.6}$$

This is a (function-)space of random variables over an arbitrary Probability space which is even a Hilbert space, with the inner product

$$(X_1|X_2)_{L^2} := E(X_1 \bar{X_2}) \tag{1.7}$$

where $\bar{x_2}$ is the complex conjugate of x_2, and with the norm

$$\|X\|_{L^2} := (X|X)^{1/2}. \tag{1.8}$$

In the following we will only deal with the Hilbert space properties of L^2. Therefore we get the following alternative Definition of 1.2.1.

Definition 1.2.3. Let $t \in G$ and $\mathcal{H}$ be an arbitrary Hilbert space. A mapping $X_t : G \longrightarrow \mathcal{H}$ is called a **stochastic process.**

[1] Sometimes called a stochastic mapping (cf. [18] p. 7)

Yet we are in a position to define a generalized stochastic process. Therefore we regard the idea of generalizing a function. Reducing the knowledge about an "ordinary" function to that of certain averages leads us to the concept of generalized functions as continuous linear functionals on spaces of test functions. Now according to 1.2.3. the obvious generalizations of stochastic processes will be Hilbert space valued bounded linear operators on spaces of test functions. An illustrating diagram of this procedure one can find in [14] p.8 respectively in [15] p.9. Now we define:

Definition 1.2.4. A bounded linear mapping ρ from an arbitrary space of test-functions $S(G)$ into a Hilbert-space $\mathcal{H}$ is called a **generalized stochastic process** (GSP).

In view of Definition 1.2.4. it is obvious, that the theory of GSPs always depends on the choice of the space of test functions. Historically there exist a lot of references concerning generalized stochastic processes using different test functions. In the following we briefly note two examples of them. The first, one of the oldest works, is due to I.M. Gelfand. The second example, which is due to H. Niemi, actually deals not with GSPs. But the works of this author concerning stochastic processes as Fourier transforms of stochastic measures pointed the way to the works of W. Hörmann, our main references.

Example 1.2.5. As we already mentioned before, I.M. Gelfand was one of the first mathematicians, who worked with GSPs, cf. [11, 12]. In [12], Chapter 3 p. 227 he introduced them as continuous linear functionals

$$\Phi \in C_0^\infty(\mathbb{R}^d) := \{f \in C_0(\mathbb{R}^d) \mid f \text{ is smooth}\}.$$

His motivation using this concept was a mathematical and a physical as well. Therefore we imagine, that if we work with classical stochastic processes $(X_t)_{t\in T}$, we know the exact values X_t at each $t \in T$, independent of the values at other times. But in practical each measuring instrument will only show us an average value

$$\Phi(\varphi) := \int \varphi(t) X_t \, dt$$

instead of the exact value X_t. We regard, that $\Phi(\varphi)$ depends linearly on a function $\varphi(t)$, which characterizes the instrument. Using different instruments, we get a linear mapping from the characteristic functions of the instruments into a set of random variables. In other words we get a GSP. A disadvantage of Gelfands estimate appears, as $C_0^\infty(\mathbb{R}^d)$ is not invariant under Fourier transform and thus the spectral process to a given GSP does not exist in general.

Example 1.2.6. Now we regard the works [18, 19, 20] of H. Niemi concerning stochastic processes. He used the space of all continuous functions with compact support over a locally compact Hausdorff-space T (that is $C_C(T)$) to define vector measures (c.f. [18] p. 15):
Let F be a locally convex topological vector space. A **vector measure** on T is a continuous linear mapping $\mu : C_C(T) \to F$.
Now he worked with stochastic processes as Fourier-transforms of vector measures. As we already mentioned, that this work is partly influenced of Niemis papers, many results of [18, 19, 20] will also appear in our following calculations, but then in the case of GSPs. In section 2.5 we will show, that under some assumptions the concepts of GSPs and vector measure are equivalent. Therefore we are able to proof many of Niemis properties in a new way. Again the disadvantage of this estimate is, that $C_C(T)$ is not invariant under Fourier transform and so you need a lot of integration theory.

In this work (except section 4.4) we will use $S_0(G)$ as our space of test-functions. Therefore we may profit from the properties of this Segal algebra, as already explained in the references [8, 14, 15], to get very simple proofs without using a lot of integration theory. Thus the following Definition will be used from now on:

Definition 1.2.7. Let $\mathcal{H}$ be an arbitrary Hilbert space. A bounded linear mapping $\rho : S_0(G) \to \mathcal{H}$ is called a **generalized stochastic process** (GSP).

Remark 1.2.8. According to the classical theory, we introduce for $\mathcal{H} = L^2(\Omega, \Sigma, P)$ an analogous concept to the expectation of a stochastic process. A bounded linear operator $E_\rho(f) : S_0(G) \longrightarrow \mathbb{C}$, defined by

$$E_\rho(f) := \int_\Omega \rho(f)\, dP \quad \forall\, f \in S_0(G) \tag{1.9}$$

is called **expectation distribution** of ρ.
In this work we will consider, as usual, only random variables in

$$L_0^2(\Omega, \Sigma, P) := \{X \in L^2(\Omega, \Sigma, P) \mid E(X) = 0\}.$$

L_0^2 is a closed subspace of L^2 and thus also a Hilbert space so we can profit from the fact, that uncorrelated random variables are always orthogonal in L^2_0. In general this property is not true for elements of L^2.

In the following we define the basic properties of GSPs which are of course analogous to the same concepts for classical stochastic processes.

Definition 1.2.9. Let ρ be a GSP.
ρ is called **(wide sense time-) stationary**, if $(\rho(f)|\rho(g)) = (\rho(T_x f)|\rho(T_x g))$ $\forall\, x \in G$ and $\forall\, f, g \in S_0(G)$.
ρ is called **frequency stationary**, if $(\rho(f)|\rho(g)) = (\rho(M_t f)|\rho(M_t g))$ $\forall\, t \in \hat{G}$ and $\forall\, f, g \in S_0(G)$.
A time- and frequency-stationary GSP ρ is called **white noise**.

Definition 1.2.10. Let ρ be a GSP.
ρ is called **bounded**, if there $\exists\, c > 0$ such that $\|\rho(f)\|_{\mathcal{H}} \leq c\|f\|_\infty \forall\, f \in S_0(G)$.
ρ is called **variation bounded (V-bounded)**, if there $\exists\, c > 0$ such that $\|\rho(f)\|_{\mathcal{H}} \leq c\|\hat{f}\|_\infty\ \forall\, f \in S_0(G)$.

Definition 1.2.11. Let ρ be a GSP. ρ is called **orthogonally scattered** if $\text{supp}(f) \cap \text{supp}(g) = \emptyset \Longrightarrow \rho(f) \perp \rho(g)$ for $f, g \in S_0(G)$.

Chapter 2

Basic facts about GSPs

In this second and also in the following third chapter we develop the preliminary theory about GSPs using the Feichtinger algebra S_0 as our space of test functions. Our main reference for these preliminaries will be the thesis of W. Hörmann and therefore the next two chapters will follow essentially the chapters 2 - 5 of [15]. As we will not report every single proof, we sometimes refer the reader to this work, to get all the details.

2.1 Covariance

In this section we introduce and characterize the important concept of the covariance distribution to an arbitrary given GSP. In Remark 2.1.2. we will see, that through the following definition we get a uniquely determined element of $S_0'(G \times G)$, i.e. a distribution.

Definition 2.1.1. Let ρ be a GSP. The **covariance** (or **auto-correlation**) **distribution**[1] σ_ρ is defined as:

$$\langle \sigma_\rho, f \otimes g \rangle := (\rho(f)|\rho(\bar{g})) \quad \forall f, g \in S_0(G). \tag{2.1}$$

[1] Because of practical reasons and for convenience of the reader we will call σ_ρ from now on short covariance instead of covariance-distribution.

Remark 2.1.2. One can show, that the covariance σ_ρ is a well defined bounded linear functional on $S_0(G) \otimes S_0(G)$. Furthermore, as we want σ_ρ to be an element of $S_0'(G \times G)$, i.e. a bounded linear functional on $S_0(G \times G)$, we can extend (2.1) to a (uniquely determined) bounded linear functional $h := \sum_{n=1}^{\infty} f_n \otimes g_n$ on $S_0(G) \hat{\otimes} S_0(G)$. This equals to $S_0(G \times G)$ because of the tensor product property[2] of S_0 and hence $\sigma_\rho \in S_0'(G \times G)$. The proof can be found in [15] p. 16 & 17.

Theorem 2.1.3. (Relations between a GSP an its covariance)
Let ρ be a GSP. Then:
(1) ρ stationary $\Longleftrightarrow$ σ_ρ diagonally invariant, i.e. $T_{(x,x)}\sigma_\rho = \sigma_\rho \; \forall\, x \in G$.
(2) ρ bounded $\Longleftrightarrow$ σ_ρ extends in a unique way to a bimeasure on $G \times G$.
(3) ρ orthogonally scattered
$\Longleftrightarrow$ σ_ρ is supported by the diagonal, i.e. $\mathrm{supp}(\sigma_\rho) \subseteq \Delta_G := \{(x,x) \mid x \in G\}$
$\Longleftrightarrow$ there exists a positive and translation bounded measure τ_ρ with:

$$\langle \sigma_\rho, f \otimes g \rangle = \langle \tau_\rho, fg \rangle \quad \forall\, f, g \in S_0(G). \tag{2.2}$$

Proof.
(1) If ρ is stationary, then the following estimate holds:

$$\langle \sigma_\rho, f \otimes g \rangle = (\rho(f) | \rho(\bar{g})) = (\rho(T_x f) | \rho(T_x \bar{g})) =$$

$$= \langle \sigma_\rho, T_{(x,x)} f \otimes g \rangle = \langle T_{(-x,-x)} \sigma_\rho, f \otimes g \rangle.$$

Now the last term equals to $\langle \sigma_\rho, f \otimes g \rangle$ if ρ is diagonally invariant.
(2) ($\Rightarrow$) Using the continuity of σ_ρ and the definition of a bounded GSP we see that the following holds:

$$|\langle \sigma_\rho, \sum_{n=1}^{\infty} f_n \otimes g_n \rangle| \leq \sum_{n=1}^{\infty} |\langle \sigma_\rho, f_n \otimes g_n \rangle| =$$

$$= \sum_{n=1}^{\infty} |(\rho(f_n) | \rho(\bar{g}_n))| \leq \sum_{n=1}^{\infty} \|\rho(f_n)\|_{\mathcal{H}} \|\rho(\bar{g}_n)\|_{\mathcal{H}} \leq c^2 \sum_{n=1}^{\infty} \|f_n\|_\infty \|g_n\|_\infty \tag{2.3}$$

for all admissible representations $\sum_{n=1}^{\infty} f_n \otimes g_n$ of $h \in S_0(G \times G)$.
Hence $|\langle \sigma_\rho, h \rangle| \leq c^2 \|h\|_{V_0} \quad \forall\, h \in S_0(G \times G)$.
The density of S_0 in V_0 (cf. section 1.1) now implies that σ_ρ extends to a uniquely determined bimeasure on $G \times G$.

[2] The details can be found in [6] Theorem 7D

($\Leftarrow$) It is $\|\rho(f)\|_{\mathcal{H}}^2 = (\rho(f)|\rho(f)) = \langle\sigma_\rho, f\otimes\bar{f}\rangle \le c\|f\otimes\bar{f}\|_{V_0} \le c\|f\|_\infty^2$, i.e. ρ is bounded.
(3) (first equivalence $\Leftarrow$)
It is clear, that $\mathrm{supp}(f\otimes g)\cap\Delta_G = \emptyset \Longleftrightarrow \mathrm{supp}(f)\cap\mathrm{supp}(g) = \emptyset$. If σ_ρ is supported by Δ_G and $\mathrm{supp}(f)\cap\mathrm{supp}(g) = \emptyset \Longrightarrow (\rho(f)|\rho(g)) = \langle\sigma_\rho, f\otimes\bar{g}\rangle = 0$, i.e. ρ is orthogonally scattered.
(first equivalence $\Rightarrow$)
Assume that $\langle\sigma_\rho, f\otimes g\rangle = 0$ whenever $\big(\mathrm{supp}(f)\times\mathrm{supp}(g)\big)\cap\Delta_G = \emptyset$. Using the tensor product property of S_0, i.e. $S_0(G)\hat{\otimes}S_0(G) = S_0(G\times G)$ and suitably refined partitions of unity (in both factors) we get $\langle\sigma_\rho, h\rangle = 0$ for any $h\in S_0(G\times G)$ having compact support disjoint to $\Delta_G \Longrightarrow \mathrm{supp}(\sigma_\rho)\subseteq\Delta_G$.
(second equivalence $\Rightarrow$)
Because Δ_G is a set of spectral synthesis (cf. section 1.1 or [21] Chapter 7 Theorem 4.1 and Chapter 6 Remark 1.5), a distribution σ_ρ with support on Δ_G satisfies $\sigma_\rho(F) = \sigma_\rho(H) \Longleftrightarrow \mathrm{Restr}_{\Delta_G}(F) = \mathrm{Restr}_{\Delta_G}(H)$ ($\mathrm{Restr}_{\Delta_G}$ maps $S_0(G\times G)$ onto $S_0(\Delta_G)$ by [6] Theorem 7 C). Furthermore, to get (2.2), we use the canonical identification j_G of G and Δ_G and the following estimate:

$$\langle\sigma_\rho, f\otimes g\rangle = \langle\tau_\rho, (\mathrm{Restr}_{\Delta_G}(f\otimes g))\circ j_G\rangle = \langle\tau_\rho, fg\rangle. \tag{2.4}$$

Yet we have left to show, that τ_ρ is positive. Therefore we take a net $(f_\alpha)_{\alpha\in A}\in S_0(G)$ with $|f_\alpha|^2 \longrightarrow \delta_0$ and define:

$$\langle(\tau_\rho)_\alpha, g\rangle := \langle\tau_\rho * |f_\alpha|^2, g\rangle = \langle\tau_\rho, |f_\alpha|^{2\check{}} * g\rangle. \tag{2.5}$$

Then $\langle(\tau_\rho)_\alpha, g\rangle \longrightarrow \langle\tau_\rho, g\rangle\ \forall\, g\in S_0(G)$. In addition $(\tau_\rho)_{\alpha\in A}$ can be identified with the bounded function $h_\alpha(x) := \langle\tau_\rho, T_x|f_\alpha|^{2\check{}}\rangle = \langle\tau_\rho, |T_x\check{f}_\alpha|^2\rangle$. As $\langle\tau_\rho, f\bar{f}\rangle = (\rho(f)|\rho(f)) \ge 0\ \forall\, f\in S_0$ it is obvious that $h_\alpha(x)\ge 0\ \forall\, x\in G \Longrightarrow \langle(\tau_\rho)_\alpha, g\rangle \ge 0\ \forall\, g\in S_0$ with $g\ge 0 \Longrightarrow \tau_\rho$ is positive, and the positive elements of $S_0'(G)$ are translation bounded measures (cf. section 1.1). The opposite direction is obvious. □

If we combine the claims (2) and (3) of the previous theorem 2.1.3. we get the following consequence:

Corollary 2.1.4. Let ρ be a GSP. Then:
ρ is bounded and orthogonally scattered $\Longleftrightarrow$ there $\exists$ a bounded measure μ_ρ on G such that:

$$\langle\sigma_\rho, f\otimes g\rangle = \langle\mu_\rho, fg\rangle = \int_G f(x)g(x)\,d\mu_\rho(x) \quad \forall\, f,g\in S_0(G). \tag{2.6}$$

Proof. ($\Leftarrow$) Follows by Theorem 2.1.3.(3) and (2).
($\Rightarrow$) Theorem 2.1.3.(3) implies the first part of (2.6) for a $\tau_\rho \in S_0(G)$, which is a diagonally supported bimeasure, as ρ is bounded. Yet we have left to show that τ_ρ is bounded with respect to $\|.\|_\infty$:
It is possible to write $f \in C^0(G)$ as $f = f_1 f_2$ with $f_1, f_2 \in C^0(G)$ and $\|f\|_\infty = \|f_1\|_\infty \|f_2\|_\infty$ (e.g. $f_1(x) := \arg(f(x))\sqrt{|f(x)|}$, $f_2(x) := \sqrt{|f(x)|}$). Then: $|\langle \tau_\rho, f\rangle| = |\langle \tau_\rho, \mathrm{Restr}_{\Delta_G}(f_1 \otimes f_2)\rangle| = |\langle \sigma_\rho, f_1 \otimes f_2\rangle| \leq \|\sigma_\rho\|_{BM} \|f_1\|_\infty \|f_2\|_\infty = c\|f\|_\infty$. □

2.2 Spectral Process

An important part of the classical theory is the "spectral representation" of stochastic processes. Now we introduce the analogous concept of the spectral process $\hat{\rho}$ to a given GSP ρ which can be defined in the same way as the Fourier-transform of distributions.

Definition 2.2.1. Let ρ be a GSP and $f \in S_0(G)$. The **spectral process** $\hat{\rho}$ to ρ is defined as $\hat{\rho}(f) := \rho(\hat{f})$.

As $f : S_0(G) \to S_0(\hat{G})$ defined by $f \mapsto \hat{f}$ denotes an isomorphism it is obvious that $\hat{\rho}$ is actual a GSP.

Definition 2.2.2. Let $f \in S_0(G)$ and ρ be a GSP. We define the mapping $\check{f} : S_0(G) \to S_0(G)$ by $\check{f}(x) := f(-x)$, and furthermore we denote $\check{\rho}(f) := \rho(\check{f})$.

As $\check{f}$ is an automorphism of $S_0(G)$ it follows, that $\check{\rho}$ is also a GSP.

Proposition 2.2.3. (Characterizing the spectral process)
Let ρ be a GSP. Then:
(1) ρ resp. $\hat{\rho}$ is bounded $\Longleftrightarrow$ $\hat{\rho}$ resp. ρ is V-bounded.
(2) ρ resp. $\hat{\rho}$ is stationary $\Longleftrightarrow$ $\hat{\rho}$ resp. ρ is frequency-stationary.
(3) $\rho = \hat{\tau} \Longleftrightarrow \check{\tau} = \hat{\rho}$.

Proof. These properties follow directly from the definitions. For illustration

we prove the first relation of **(2)**:
$(\Rightarrow)\,(\rho(f)\mid\rho(g)) = (\rho(T_xf)\mid\rho(T_xg)) \Longrightarrow (\hat\rho(f)\mid\hat\rho(g)) = (\rho(\hat f)\mid\rho(\hat g)) = (\rho(T_x\hat f)\mid\rho(T_x\hat g)) = (\rho(M_tf)\hat{}\mid\rho(M_tg)\hat{}) = (\hat\rho(M_tf)\mid\hat\rho(M_tg))$, i.e. $\hat\rho$ is frequency-stationary.
$(\Leftarrow)\,(\hat\rho(M_tf)\mid\hat\rho(M_tg)) = (\hat\rho(f)\mid\hat\rho(g)) \Longrightarrow (\rho(f)\mid\rho(g)) = (\hat\rho(\hat f\check{})\mid\hat\rho(\hat g\check{})) = (\hat\rho(M_t\hat f\check{})\mid\hat\rho(M_t\hat g\check{})) = (\rho(M_t\hat f\check{})\hat{}\mid\rho(M_t\hat g\check{})\hat{}) = (\rho(T_xf)\mid\rho(T_xg))$, i.e. ρ is stationary. □

Remark 2.2.4. Part (3) of Proposition 2.2.3. shows, that $\rho\mapsto(\check\rho)\hat{}$ is the inverse mapping of $\rho\mapsto\hat\rho$. This implies that the Fourier transform is a bijective mapping between the GSPs over G and the GSPs over $\hat G$.

Before we characterize the covariance of spectral processes we have to note a few facts about the support of a distribution.

Definition 2.2.5. Let $\sigma\in S_0'(G)$. The **support** $\mathrm{supp}(\sigma)$ of σ is defined as the complement of the open set of all points $x\in G$ such that the "action of σ near x is trivial", i.e. let N_x be a neighborhood of x, then $x\notin\mathrm{supp}(\sigma) :\Longleftrightarrow \exists\, N_x$ with $\langle\sigma,f\rangle=0\quad\forall\, f\in S_0(G)$ with $\mathrm{supp}(f)\subseteq N_x$.

Definition 2.2.6. Let H be a subgroup of G, then we call σ **H-invariant** if $T_x\sigma=\sigma\quad\forall\, x\in H$. The **orthogonal group of** H is defined as

$$H^\perp := \{t\in\hat G \mid \langle h,t\rangle = 1\quad\forall\, h\in H\}. \tag{2.7}$$

Lemma 2.2.7. Let H be a subgroup of G and $\sigma\in S_0(G)$. Then:
σ H-invariant $\Longleftrightarrow \mathrm{supp}(\hat\sigma)\subseteq H^\perp$.

Proof. We refer to [5] Theorem 3.4 A. □

Yet we may return back to our theory and study the covariance of the spectral process.

Theorem 2.2.8. Let ρ be a GSP. Then:
(1) $\langle\hat\sigma_\rho, f\otimes g\rangle = \langle\sigma_{\hat\rho}, f\otimes\check g\rangle$
(2) ρ orthogonally scattered $\Longleftrightarrow T_{(t,t)}\sigma_{\hat\rho}=\sigma_{\hat\rho}\ \forall\, t\in\hat G$

Proof. **(1)** $\langle\hat\sigma_\rho, f\otimes g\rangle = \langle\sigma_\rho,\hat f\otimes\hat g\rangle = (\rho(\hat f)\mid\rho(\bar{\hat g})) = (\hat\rho(f)\mid\hat\rho(\check{\bar{\check g}})) =$

$(\hat{\rho}(f) \mid \hat{\rho}(\bar{g}^{\smile})) = \langle \sigma_{\hat{\rho}}, f \otimes \bar{g}^{\smile\smile} \rangle = \langle \sigma_{\hat{\rho}}, f \otimes \check{g} \rangle$.
(2) Using the previous Lemma 2.2.7. and Theorem 2.1.3.(3), we get the following fact: ρ orthogonally scattered $\Longleftrightarrow T_{(t,-t)}\hat{\sigma}_\rho = \hat{\sigma}_\rho \quad \forall\, t \in \hat{G}$.
We calculate: $\langle \sigma_{\hat{\rho}}, f \otimes g \rangle = \langle \hat{\sigma}_\rho, f \otimes \check{g} \rangle = \langle T_{(t,-t)}\hat{\sigma}_\rho, f \otimes \check{g} \rangle = \langle \hat{\sigma}_\rho, T_{(-t,t)} f \otimes \check{g} \rangle = \langle \sigma_{\hat{\rho}}, T_{(-t,-t)} f \otimes g \rangle = \langle T_{(t,t)}\sigma_{\hat{\rho}}, f \otimes g \rangle \quad \forall\, t \in \hat{G}$ and $\forall\, f, g \in S_0(G)$. □

Corollary 2.2.9. Let ρ be a GSP. Then:
ρ is V-bounded $\Longleftrightarrow \hat{\sigma}_\rho$ extends to a bimeasure.

Proof. It is ρ V-bounded $\Longleftrightarrow \hat{\rho}$ is bounded (by Proposition 2.2.3.(1)) $\Longleftrightarrow \sigma_{\hat{\rho}}$ extends to a bimeasure (by Theorem 2.1.3.(2)). Applying Theorem 2.2.8.(1) we get the claim. □

2.3 Stationary Processes

As in the classical case, the concept of stationarity plays also a very important role in the theory of GSPs. Part (1) of the following proposition is the "spectral representation theorem of stationary processes" (c.f. [2] p. 527) for GSPs. The easiness of the proof shows the advantages of using GSPs, especially in this way.

Proposition 2.3.1. (Characterizing stationary processes)
Let ρ be a GSP. Then:
(1) ρ frequency-stationary $\Longleftrightarrow \rho$ orthogonally scattered.
(2) ρ stationary $\Longleftrightarrow \exists$ a positive translation bounded measure $\tau_{\hat{\rho}}$ on G with

$$\langle \sigma_{\hat{\rho}}, f \otimes g \rangle = \langle \tau_{\hat{\rho}}, f \cdot g \rangle \quad \forall\, f, g \in S_0(G).$$

Proof.
(1) ρ frequency-stationary $\Longleftrightarrow \hat{\rho}$ stationary (by Proposition 2.2.3.(2)) $\Longleftrightarrow \sigma_{\hat{\rho}}$ is diagonally invariant (by Theorem 2.1.3.(1)) $\Longleftrightarrow \rho$ orthogonally scattered (by Theorem 2.2.8.(2)).
(2) ρ stationary $\Longleftrightarrow \hat{\rho}$ frequency-stationary (by Proposition 2.2.3.(2)) $\Longleftrightarrow \hat{\rho}$ orthogonally scattered (by part (1) of this Proposition) $\Longleftrightarrow \exists$ a positive translation bounded measure $\tau_{\hat{\rho}}$ on G , which satisfies the equality (by Theorem 2.1.3.(3)). □

Definition 2.3.2. The measure $\tau_{\hat\rho}$, defined by part (2) of the previous Proposition, is called the **spectral measure** of ρ.

Corollary 2.3.3. Let ρ be a GSP. Then:
ρ is a white noise $\Longleftrightarrow$ ρ is stationary and orthogonally scattered.

Proof. By Definition ρ is white noise if ρ is stationary and frequency stationary. Now we apply Proposition 2.3.1.(1) to get the claim. □

The next result is analogous to classical calculations, (cf. [2] p. 519 Theorem 3.1.) but it has to be formulated in a slightly different way, as the covariance of a stationary stochastic process is written as a function of only one variable. But if we add diagonal invariance of the covariance, we get the following criterion, which also shows us the existence of stationary GSPs.

Theorem 2.3.4. (Characterizing the covariance of stationary GSPs)
Let $\sigma \in S_0'(G \times G)$. Then σ denotes the covariance of a stationary GSP if and only if σ is diagonally invariant and positive definite.

Proof. ($\Rightarrow$) By Theorem 2.1.3.(1) we get that σ is diagonally invariant. Applying Proposition 2.3.1.(2) it is clear, that the covariance distribution $\sigma_{\hat\rho}$ of $\hat\rho$ is positive. Theorem 2.2.8.(1) implies, that $\hat\sigma$ is also positive and this is equivalent to σ is positive definite.
($\Leftarrow$) Let σ be diagonally invariant and positive definite. By Theorem 3.4 A of [5] it follows, that $\hat\sigma$ is supported by

$$\nabla\hat G = \Big\{(t| -t) \mid t \in \hat G\Big\}$$

Furthermore, as $f \otimes \bar f\check{}$ is non-negative on $\nabla\hat G$ we may conclude, that:

$$\langle\hat\sigma, f \otimes \bar f\check{}\,\rangle \geq 0 \qquad \forall\, f \in S_0(\hat G).$$

This implies $\langle\sigma, \hat f \otimes \bar f\check{}\,\hat{}\,\rangle \geq 0 \quad \forall\, f \in S_0(\hat G)$ and on the other hand this is equavalent to $\langle\sigma, f \otimes \bar f\rangle \geq 0 \quad \forall\, f \in S_0(G)$, as $\bar f\check{}\,\hat{} = \hat f^-$. We have proved that an express of the form $Q(f,g) := \langle\sigma, f \otimes \bar g\rangle$ is positive semi-definite and sesquilinear on $S_0(G) \times S_0(G)$. Since $N := \big\{f \mid \langle\sigma, f \otimes \bar f\rangle = 0\big\}$ is a linear subspace of $S_0(G)$, Q defines a canonical inner product on $H_1 := S_0(G)/N$. Let $\mathcal{H}$ denote the Hilbert space which we obtain by completion. Then the canonical projection, followed by the embedding of H_1 into $\mathcal{H}$ defines a bounded, linear mapping $\rho : S_0(G) \longrightarrow \mathcal{H}$, i.e. ρ is a GSP and σ coincides with σ_ρ. Stationarity follows by the diagonal invariance of σ. □

Corollary 2.3.5. Let $\sigma \in S_0'(G\times G)$. Then σ is covariance of an orthogonally scattered GSP ρ if and only if there $\exists$ a positive and translation bounded measure τ with:

$$\langle \sigma, f \otimes g \rangle = \int_G fg \, d\tau \qquad \forall \, f, g \in S_0(G).$$

Proof. ($\Rightarrow$) cf. Theorem 2.1.3.(3).
($\Leftarrow$) We define a distribution $\omega \in S_0'(\hat{G} \times \hat{G})$ by:

$$\langle \omega, f \otimes g \rangle := \langle \hat{\sigma}, f \otimes \check{g} \rangle = \langle \sigma, \hat{f} \otimes \check{g}\hat{\ } \rangle = \int_G \hat{f}\check{g}\hat{\ } d\tau \quad \forall \, f, g \in S_0(\hat{G}).$$

The following calculation shows us the diagonal invariance of ω:

$$\langle \omega, T_x f \otimes T_x g \rangle = \int_G (T_x f)\hat{\ } T_x(g)\check{\ }\hat{\ } d\tau = \int_G M_x \hat{f} \, M_{-x}(\check{g}\hat{\ }) d\tau = \langle \omega, f \otimes g \rangle.$$

As σ is positive it follows, that ω is positive definite. Yet, by the previous Theorem 2.3.4., there $\exists$ a stationary GSP $\hat{\rho}$ over $\hat{G}$ with covariance ω. Thus, by part (1) of Theorem 2.2.8., ρ is a GSP over G with covariance σ, which is orthogonally scattered because of Proposition 2.2.3.(2) and Proposition 2.3.1.(1). □

Remark 2.3.6. Yet we have left to study some properties of white noise. Therefore we briefly note 3 criteria (cf. [15] Theorem 10). We assume, that ρ is an arbitrary GSP. Then ρ is white noise if and only if one of the following conditions is satisfied:
(1) There $\exists \, c \geq 0$ such that: $(\rho(f)|\rho(\bar{g})) = c \int_G f(t)g(t) \, dt \quad \forall \, f, g \in S_0(G)$.
This follows from Proposition 2.3.1.(1) together with the fact, that ρ is stationary and orthogonally scattered if and only if there $\exists \, c > 0$, such that:

$$\langle \sigma_\rho, f \otimes g \rangle = c \int_G f(x)g(x) \, dx,$$

where σ_ρ is the covariance and dx denotes the Haar measure (cf [15] Corollary 2 p. 19).
(2)$\|\rho(f)\|_{\mathcal{H}} = c\|f\|_2$ for some $c \geq 0$ and $\forall \, f \in S_0(G)$, i.e. ρ is a scalar multiple of an isometry between $L^2(G)$ and $\mathcal{H}$.
(3) For some $\gamma > 0$, the GSP ρ extends to an (Hilbert space) isomorphism between $(L^2(G), \gamma\|.\|_2)$and $\mathcal{H}$.
The proof of (2) and (3) makes use of the following:

$$(a|b) = \frac{1}{4}\big(\|a+b\|_{\mathcal{H}}^2 - \|a-b\|_{\mathcal{H}}^2 + i\|a+ib\|_{\mathcal{H}}^2 - i\|a-ib\|_{\mathcal{H}}^2\big) \quad \forall \, a, b \in \mathcal{H}.$$

According to this identity $(\rho(f)|\rho(\bar{g}))$ can be expressed with help of $\|\rho(f)\|_{\mathcal{H}}$ and $\|\rho(\bar{g})\|_{\mathcal{H}}$ and consequently with $c\|f\|_{\mathcal{H}}$ and $c\|g\|_2$. Therefore we may refer to criterion (1) to get the claims.

2.4 GSPs and stochastic processes

In this section we want to study some relations of GSPs to classical stochastic processes.
As there are GSPs with a covariance distribution which can not be represented by an ordinary function, it is obvious, that we can not associate GSPs with stochastic processes in general. But in the following we will prove, that any GSP with a covariance distribution induced by a function in $C^b(G \times G)$ can be identified with a uniquely determined stochastic process in the classical sense. On the other hand any mean square continuous stochastic process can be identified with a uniquely determined GSP. The exact formulation of this fact is contained in Theorem 2.4.3., but before we have to regard two preliminary results.

Lemma 2.4.1. Let ρ be a GSP with covariance $\sigma_\rho \in S_0'(G \times G)$. If σ_ρ is represented by some $h \in C^b(G \times G)$ then $\rho(f_\alpha)$ is a Cauchy net in $\mathcal{H}$, whenever $(f_\alpha)_{\alpha \in A}$ is a vaguely convergent, L^1 - bounded and tight net in $S_0(G)$.

Proof. Let $(f_\alpha)_{\alpha \in A}$ be a vaguely convergent, L^1 - bounded and tight net. Then there exists a $k \in C_c(G)$ such that

$$\|(1-k)(f_\alpha - f_\beta)\|_1 < \varepsilon \quad \forall\, \alpha \in A. \tag{2.8}$$

To show that $\rho(f_\alpha)$ is a Cauchy net in $\mathcal{H}$, we may now calculate:

$$\|\rho(f_\alpha) - \rho(f_\beta)\|_{\mathcal{H}}^2 = (\rho(f_\alpha - f_\beta)|\rho(f_\alpha - f_\beta)) = \langle \sigma_\rho, (f_\alpha - f_\beta) \otimes (\bar{f}_\alpha - \bar{f}_\beta)\rangle =$$

$$= \langle \sigma_\rho \cdot k \otimes k, (f_\alpha - f_\beta) \otimes (\bar{f}_\alpha - \bar{f}_\beta)\rangle + \langle \sigma_\rho, k(f_\alpha - f_\beta) \otimes (1-k)(\bar{f}_\alpha - \bar{f}_\beta)\rangle +$$
$$+\langle \sigma_\rho, (1-k)(f_\alpha - f_\beta) \otimes k(\bar{f}_\alpha - \bar{f}_\beta)\rangle. \tag{2.9}$$

The second term of (2.9) can be handled in the following way:

$$|\langle \sigma_\rho, k(f_\alpha - f_\beta) \otimes (1-k)(\bar{f}_\alpha - \bar{f}_\beta)\rangle| =$$

$$= \left| \int_G \int_G h_\rho(x,y) k(x)(f_\alpha(x) - f_\beta(x))dx(1-k(y))(\bar{f}_\alpha(y) - \bar{f}_\beta(y))dy \right| \leq$$

$$\leq \int_G \int_G \|h_\rho\|_\infty |k(x)(f_\alpha(x) - f_\beta(x))|dx|(1-k(y))(\bar{f}_\alpha(y) - \bar{f}_\beta(y))|dy \leq$$

$$\leq \int_G \|h_\rho\|_\infty C|(1-k(y))(\bar{f}_\alpha(y) - \bar{f}_\beta(y))|dy \tag{2.10}$$

The last integral of (2.10) can be made arbitrarily small by suitable choice of k in (2.8), independent of α and β. The third term of (2.9) can be treated in the same way as the second. Finally, by the vague convergency of f_α, the first term of (2.9) converges to 0 and thus it follows

$$\|\rho(f_\alpha) - \rho(f_\beta)\|_{\mathcal{H}} \longrightarrow 0,$$

i.e. $(\rho(f_\alpha))_{\alpha \in A}$ is a Cauchy-net in the norm topology. □

Lemma 2.4.2. In the situation of Lemma 2.4.1, ρ extends to a bounded linear operator $\tilde{\rho} : M(G) \longrightarrow \mathcal{H}$, which is σ - norm continuous on tight subsets of $M(G)$, especially: $\lim_{y \to x} \tilde{\rho}(\delta_y) = \tilde{\rho}(\delta_x)$. $\tilde{\rho}$ is uniquely determined, if $\{\rho(f) \mid f \in S_0(G)\}$ is dense in $\mathcal{H}$.

Proof. We assume, that $(f_\alpha)_{\alpha \in A}$ is a vaguely convergent, L^1 - bounded and tight net with $\lim_{\alpha \in A} f_\alpha := \mu$. Due to the density of $\{\rho(f) \mid f \in S_0(G)\}$ in $\mathcal{H}$ we may conclude, that there exists an uniquely determined element $\tilde{\rho}(\mu) \in \mathcal{H}$, such that:

$$(\tilde{\rho}(\mu)|\rho(g)) := \lim_{\alpha \in A} (\rho(f_\alpha)|\rho(g)) \quad \forall\, g \in S_0(G). \tag{2.11}$$

If we apply now the proof of Lemma 2.4.1. to two different L^1 - bounded and tight nets with the same vague limit, we get independence from the choice of the net $(f_\alpha)_{\alpha \in A}$ and the Definition of $\tilde{\rho}(\mu)$ by (2.11) is justified.
To show the continuity of $\tilde{\rho}$, we assume that $(\mu_\beta)_{\beta \in B}$ is a L^1 - bounded, tight and w^*-convergent net in $M(G)$ with limit μ. Now for any $U = U(k_1, k_2, \ldots, k_n, \varepsilon) \in \mathcal{U}$ with $(k_i)_{i=1}^n \in S_0(G)$ there exists a net $(f_U)_{U \in \mathcal{U}} \in S_0(G)$, such that the following holds:

$$|\langle \mu_\beta, k_i \rangle - \langle f_U, k_i \rangle| < \frac{\varepsilon}{2} \quad \forall\, i = 1, \ldots, n \text{ and } \forall\, \beta \in B$$

$$\text{and } |\langle \mu_\beta, k_i \rangle - \langle \mu, k_i \rangle| < \frac{\varepsilon}{2} \quad \forall\, \beta > \beta_0.$$

Now it follows, that:

$$|\langle \mu, k_i \rangle - \langle f_U, k_i \rangle| < \varepsilon \quad \forall\, i = 1, \dots, n \text{ and } \forall\, \beta > \beta_0, \tag{2.12}$$

which finally shows us that $(f_U)_{U \in \mathcal{U}}$ is a w^*-convergent, L^1 - bounded and tight net in $S_0(G)$ with limit μ. Lemma 2.4.1. implies that $\rho(f_U)$ converges in norm with limit $\tilde{\rho}(\mu)$ according to the above Definition (2.11). Due to the construction of f_U it is easy to see that $\tilde{\rho}(\mu_\beta)$ converges to $\tilde{\rho}(\mu)$ as well and this shows the σ - norm continuity of $\tilde{\rho}$ on bounded and tight subsets. □

Theorem 2.4.3. (Relations to stochastic processes)
(1) The mapping $\rho_G(x) := \tilde{\rho}(\delta_x)$ ($\tilde{\rho}$ as in Lemma 2.4.2.) denotes a bounded, continuous stochastic process on G with covariance

$$h(x, y) = (\tilde{\rho}(\delta_x)|\tilde{\rho}(\delta_y)). \tag{2.13}$$

(2) If $\rho_1 : G \longrightarrow \mathcal{H}$ denotes a continuous and bounded stochastic process, then the covariance

$$h(x, y) := (\rho_1(x)|\rho_1(y))$$

of ρ_1 is also bounded and continuous on $G \times G$. By vector-valued integration ρ_1 may be lifted to a bounded linear mapping $\tilde{\rho}_1 : M(G) \longrightarrow \mathcal{H}$, which is σ-norm continuous on bounded tight subsets, and the restriction $\tilde{\rho}_1|_{S_0(G)}$ may be interpreted as a GSP with covariance distribution h.

Proof. **(1)** As we have already shown the σ-norm continuity of $\tilde{\rho}$ on tight subsets, the continuity of ρ_G follows from Lemma 2.4.2. To get boundedness we use the fact, that $\tilde{\rho}$ is bounded with respect to $\|.\|_M$, i.e.

$$\|\tilde{\rho}(f)\|_{\mathcal{H}} \le c\|f\|_M \quad \forall\, f \in M(G),$$

and then apply

$$\|\rho_G(x)\|_{\mathcal{H}} = \|\tilde{\rho}(\delta_x)\|_{\mathcal{H}} \le c \cdot 1 = c \quad \forall\, x \in G.$$

To show (2.13), we assume that $(f_\alpha)_{\alpha \in A}$ is L^1 - bounded and vaguely convergent net with limit δ_0, i.e. $(f_\alpha)_{\alpha \in A}$ is a kind of a generalized "Dirac-sequence". Then the following holds:

$$h(x, y) = \int_{G \times G} h(t)(\delta_x \otimes \delta_y) dt = \lim_{\alpha \in A} \langle h, T_x f_\alpha \otimes T_y f_\alpha \rangle =$$

$$= \lim_{\alpha \in A} \langle \sigma_\rho, T_x f_\alpha \otimes T_y f_\alpha \rangle = (\tilde{\rho}(\delta_x)|\tilde{\rho}(\bar{\delta}_y)) = (\tilde{\rho}(\delta_x)|\tilde{\rho}(\delta_y)),$$

which proves our claim.
(2) For the proof of the first part, we first use the boundedness of ρ_1 to justify the following estimate:

$$h(x,y) = (\rho_1(x)|\rho_1(y)) = \|\rho_1(x)\|_{\mathcal{H}}\|\rho_1(y)\|_{\mathcal{H}} \leq c^2,$$

which proves that h is bounded. The continuity of h results from the continuity of ρ_1 and of the inner product.
Now we assume, that $l \in \mathcal{H}$ and $\mu \in M(G)$. With the help of vector - valued integration we may define:

$$\big(\tilde{\rho}_1(\mu)|l\big) := \int_G (\rho_1(x)|l)\, d\mu. \tag{2.14}$$

By the Riesz representation theorem (cf. [24] p. 62) $\tilde{\rho}_1(\mu)$ is a well defined element of $\mathcal{H}$, as

$$|(\tilde{\rho}_1(\mu)|l)| \ \leq \ c\|\mu\|_M\|l\|_{\mathcal{H}}. \tag{2.15}$$

Now (2.15) implies $\|\tilde{\rho}_1(\mu)\|_{\mathcal{H}} \leq c\|\mu\|_M$ and thus the boundedness of $\tilde{\rho}_1$ with respect to $\|.\|_M$ follows.
Now let $(\mu_\alpha)_{\alpha\in A}$ be a bounded, tight w^* - convergent net in $M(G)$ with limit μ. As $x \longmapsto (\rho_1(x)|l)$ is continuous and bounded for any $l \in \mathcal{H}$ and $(\mu_\alpha)_{\alpha\in A}$ is tight we get:

$$\lim_{\alpha\in A} (\tilde{\rho}_1(\mu_\alpha)|l) = \lim_{\alpha\in A} \int_G (\rho_1(x)|l)\ d\mu_\alpha = \int_G (\rho_1(x)|l)\ d\mu = (\tilde{\rho}_1(\mu)|l)$$

which shows the σ - norm continuity of $\tilde{\rho}_1$.
Now as $(\rho_1(x)|\rho_1(y))$ is continuous and bounded and $(\mu_\alpha)_{\alpha\in A}$ is w^* - convergent, bounded and tight, the following equality is true:

$$\lim_{\alpha\in A} (\tilde{\rho}_1(\mu_\alpha)|\tilde{\rho}_1(\mu_\alpha)) = \lim_{\alpha\in A} \int_G \int_G (\rho_1(x)|\rho_1(y))\, d\mu_\alpha d\mu_\alpha =$$

$$= \int_G \int_G (\rho_1(x)|\rho_1(y))\, d\mu d\mu = \|\tilde{\rho}_1(\mu)\|^2_{\mathcal{H}}$$

and thus the convergence of $\|\tilde{\rho}_1(\mu_\alpha)\|_{\mathcal{H}}$ is shown. We get the required GSP $\rho := \tilde{\rho}_1|_{S_0}$ by restriction to $S_0(G)$.
Yet we have left to show, that $h(x,y) = (\rho_1(x)|\rho_1(y))$ represents the covariance distribution σ_ρ of ρ. According to (2.14) we get for an arbitrary $f \in S_0(G)$:

$$(\rho(f)|l) = \int_G (\rho_1(x)|l)\, f(x)\, dx \quad \forall\, l \in \mathcal{H}.$$

This implies the following identity:

$$\langle \sigma_\rho, f \otimes g \rangle = (\rho(f)|\rho(\bar{g})) = \int_G (\rho_1(x)|\rho(\bar{g}))\, f(x)\, dx =$$

$$= \int_G \int_G g(y)\, (\rho_1(x)|\rho_1(y))\, dy\, f(x)\, dx =$$

$$= \int_G \int_G h(x,y)\, f(x)\, g(y)\, dx\, dy = \langle h, f \otimes g \rangle$$

for $f, g \in S_0(G)$ and the proof is complete. □

Remark 2.4.4. The above Theorem also describes a bijective identification between continuous bounded stochastic processes and GSPs with continuous bounded covariance. This can be easily seen as any measure in $M(G)$ can be represented as the w^* - limit of a bounded tight net of functions in S_0 or of discrete measures. Now the σ - norm continuity of the mappings $\tilde{\rho}$ and $\tilde{\rho_1}$ on tight, bounded subsets implies the uniqueness of these extensions from $S_0(G)$ or G to $M(G)$.

Corollary 2.4.5. Let ρ be a V-bounded GSP. Then:
(1) ρ can be identified with an uniquely determined stochastic process.
(2) ρ extends to $M(G)$ and $\hat{\rho}$ to $\mathcal{F}(M(G))$. Thus:

$$\rho(\mu) = \hat{\rho}(h) \quad \text{ if } \hat{\mu} = \check{h},$$

and in particular: $\rho(\delta_x) = \hat{\rho}(\chi_x)$.

Proof. It is ρ V-bounded $\Longleftrightarrow$ $\hat{\sigma}_\rho$ extends to a bimeasure (by Corollary 2.2.9.). Now the Fourier transform of a bimeasure is a bounded, continuous function (cf. section 1.1 respectively [13] Theorem 2.4i and Definition 2.1) and so the claims of the Corollary follow from the results stated above. □

Yet we have shown, that there are indeed very strong relations between GSPs and classical stochastic processes. Together with the conditions stated in our results so far these concepts are even equivalent.
This means in particular: If we assume, that the covariance distribution σ_ρ is represented by a bounded, continuous function, all the results we have proved during this chapter also hold for "ordinary" stochastic processes and thus we get in many cases short and clear proofs of classical theorems, as

the Definitions of certain properties for GSPs and stochastic processes are the same. We also will use this considerations in the sections 3.2 and 4.1 to prove some results on V-bounded and harmonizable GSPs also in the case of stochastic processes.

2.5 GSPs and vector measures

We close this chapter by comparing GSPs with vector measures, as defined by Niemi (cf. Example 1.2.6.), i.e. continuous and linear mappings

$$\mu : C_c(G) \longrightarrow \mathcal{H}.$$

But there exists no relation by inclusion of $C_c(G)$ and $S_0(G)$ and so a general comparison is impossible. The following results, first noted in [14] p.19, show that under various assumptions this two concepts are in fact strongly related to each other, though the concept of GSPs seems to be more general than vector measures.

Remark 2.5.1. For the proofs it will be necessary to deal with the **Wiener Algebra**, which is given by:

$$W(G) := \{f \in C_b(G) \mid \|f\|_W := \int_G \|T_y k \cdot f\|_\infty dy < \infty\}$$

where $k \in C_c(G)$. $W(G)$ has the local properties of $C_0(G)$ and the global properties of $L^1(G)$. The definition is independent of the choice of k. Furthermore $S_0(G)$ lies dense in $W(G)$.

Theorem 2.5.2. (Relations to vector measures)
(1) Under the assumption of boundedness or V-boundedness the concepts of vector measures and GSPs are equivalent. This means in particular, that these mappings are uniquely determined by their restriction to $A(G) \cap C_c(G)$, which lies dense in $C_c(G)$ and in $S_0(G)$.
(2) Any stationary vector measure determines a unique stationary GSP.

Proof.
(1) If we recall the Definition 1.2.10. of boundedness and V-boundedness it is obvious, that bounded GSPs extend to a bounded linear mappings on

$C_0(G)$ and V-bounded ones extend to bounded linear mappings on $\mathcal{F}(C_0(\hat{G}))$ respectively. The equivalence of both concepts follows, since $C_c(G)$ and $S_0(G)$ are dense in these spaces.
(2) As $S_0(G)$ lies dense in the Wiener Algebra $W(G)$ we shall show, that any stationary vector measure $\mu : C_c(G) \longrightarrow \mathcal{H}$ extends to a bounded linear mapping on $W(G)$. Because of the inductive limit topology of $C_c(G)$ there exists for any compact set $Q \subseteq G$ a constant $c > 0$ such that

$$\|\mu(f)\|_{\mathcal{H}} \leq c\|f\|_\infty \quad \forall\, f \in C_c(G) \tag{2.16}$$

with $\operatorname{supp}(f) \subseteq Q$. The stationarity of μ, i.e. $\|\mu(T_x f)\|_{\mathcal{H}} = \|\mu(f)\|_{\mathcal{H}}\, \forall\, x \in G$ and $\forall f \in C_c(G)$, implies that for any $k \in C_c(G)$ and any representation $k := \sum_{i=1}^N T_{x_i} f_i$ of k with $\operatorname{supp}(f_i) \subseteq Q$ with $1 \leq i \leq N$ the following holds:

$$\begin{aligned} \|\mu(k)\|_{\mathcal{H}} = \|\mu(\sum_{i=1}^N T_{x_i} f_i)\|_{\mathcal{H}} \leq \sum_{i=1}^N \|\mu(T_{x_i} f_i)\|_{\mathcal{H}} = \\ = \sum_{i=1}^N \|\mu(f_i)\|_{\mathcal{H}} \leq \sum_{i=1}^N \|f_i\|_\infty. \end{aligned} \tag{2.17}$$

Now we choose an appropriate partition of unity $(\psi_i)_{i\in I}$ to get:

$$\|\mu(k)\|_{\mathcal{H}} \leq 2c \sum_{i=1}^N \|f_i \psi_i\|_\infty = 2c\|k\|_W \quad \forall\, K \in C_c(G). \tag{2.18}$$

As $C_c(G)$ lies dense in $W(G)$, we get the claim. □

The following result justifies that the concept of stationary GSPs is in fact more general then a stationary vector measure. This can be easily seen as

$$\|\rho(f)\|_{\mathcal{H}} \leq c\|f\|_{S_0} \tag{2.19}$$

is true for any GSP. Now as $S_0(G) \subseteq W(G)$ and $\|f\|_W \leq c\|f\|_{S_0} \forall\, f \in S_0(G)$, and this is the reason, why the assumption (2.20) of the following Corollary is "stronger" than (2.19).

Corollary 2.5.3. Let ρ be a stationary GSP. ρ determines a vector measure if and only if:

$$\|\rho(f)\|_{\mathcal{H}} \leq c\|f\|_W \quad \forall\, f \in S_0(G). \tag{2.20}$$

Proof. ($\Rightarrow$) already shown in part (2) of the previous proof.
($\Leftarrow$) As $S_0(G)$ lies dense in $W(G)$ it remains to show, that $\tilde{\rho}|_{C_c(G)}$ is a bounded

linear mapping on $C_c(G)$, where $\tilde{\rho}$ determines the extension of ρ to $W(G)$. Therefore we choose an arbitrary $h \in C_c(G)$ and assume, that

$$\|\rho(h)\|_{\mathcal{H}} \leq c\|h\|_W \leq c\sum_{i=1}^{l} \|T_{x_i}k\|_\infty \tag{2.21}$$

where $\text{supp}(k) \subseteq Q$ and Q is a compact subset of G. The second inequality of (2.21) is due to the definition of the norm in $W(G)$ (cf. [15] p. 74). Now as ρ is stationary, (2.21) equals to $lc_\rho\|k\|_\infty = c_Q\|k\|_\infty$ where c_Q depends on a fixed compact subset Q of G. We finally may conclude, that $\tilde{\rho}|_{C_c(G)}$ is bounded with respect to the inductive limit topology of $C_c(G)$. □

Remark 2.5.4. Using the results concerning the relations of GSPs to vector measures, the formula $\rho(\delta_x) = \hat{\rho}(\chi_x)$ in Corollary 2.4.5. now can be seen as an alternative formulation of Niemis representation theorem of V-bounded stochastic processes, which one can find in [18], Theorem 3.2.1. on p. 35.

Chapter 3

Filtered and harmonizable GSPs

Before we get to the main part of this work, we have to regard some facts about filtered respectively harmonizable GSPs. After a short report concerning filtered GSPs, we introduce the concept of harmonizability, which appears as a generalization of stationarity. Finally we conclude the chapter with some interesting remarks about dilations of GSPs.

3.1 Filtered GSPs

In this section we deal with linear smoothing operators. In the following we will see, that there is a natural way to define this concept for GSPs.

Now let $f \in S_0(G)$ be an arbitrary function. We introduce linear smoothing operators for f by the following:

$$f \longmapsto k * f, \qquad k \in L^1(G)$$

respectively, if we use elements of the Fourier algebra $A(G)$ (cf. section 1.1):

$$f \longmapsto hf, \qquad h \in A(G).$$

As these operators map the elements of $S_0(G)$ into $S_0(G)$, we may extend these concepts, with the help of adjoint operators, to linear smoothing operators on $S_0'(G)$.

Definition 3.1.1. Let $\sigma \in S_0'(G)$ and $f \in S_0(G)$. We define the following smoothing operators on $S_0'(G)$ which we call **filters** in accordance to signal analysis:

$$\langle k * \sigma, f\rangle := \langle \sigma, \check{k} * f\rangle, \qquad k \in L^1(G)$$

respectively

$$\langle h\sigma, f\rangle := \langle \sigma, hf\rangle, \qquad h \in A(G).$$

As every distribution in $S_0'(G)$ denotes a GSP with $\mathcal{H} = \mathbb{C}$ we may introduce:

Definition 3.1.2. Let ρ be a GSP and $f \in S_0(G)$. In accordance to Definition 3.1.1. we introduce **filtered GSPs** by:

$$k * \rho(f) := \rho(k * f) \qquad k \in L^1(G)$$

$$h\rho(f) := \rho(hf) \qquad h \in A(G).$$

These operators denote indeed bounded linear mappings from $S_0(G)$ into $\mathcal{H}$. (cf. [15] Remark on p. 46)

Before we prove some properties, we note different facts about the spectral process and the covariance of filtered GSPs. The following Lemma contains very important facts for section 4.1.

Lemma 3.1.3. Let ρ be a GSP, $k \in L^1(G)$ and $h \in A(G)$. Then:
(1) $(k * \rho)\hat{} = \hat{k}\hat{\rho}$
(2) $\sigma_{h\rho} = (h \otimes \bar{h})\sigma_\rho$
(3) $\sigma_{k*\rho} = (k \otimes \bar{k}) * \sigma_\rho$

Proof. **(1)** $(k * \rho)\hat{}(f) = k * \rho(\hat{f}) = \rho(\check{k} * \hat{f}) = \hat{\rho}(\check{k} * \hat{f})\hat{}\check{} = \hat{\rho}(\check{k}\hat{}\hat{f}\hat{})\check{} = \hat{\rho}(\hat{k}f) = \hat{k}\hat{\rho}(f) \quad \forall f \in S_0(G)$. We used the fact that $\rho(f) = \hat{\rho}(\hat{f}\check{})$ and the convolution theorem. (cf. section 1.1)
(2) $\langle \sigma_{h\rho}, f \otimes g\rangle = (h\rho(f) \mid h\rho(\bar{g})) = (\rho(hf) \mid \rho(h\bar{g})) = \langle \sigma_\rho, hf \otimes \bar{h}g\rangle = \langle (h \otimes \bar{h})\sigma_\rho, f \otimes g\rangle$.
(3) We show, that $\hat{\sigma}_{k*\rho} = ((k \otimes \bar{k}) * \sigma_\rho)\hat{}$ which implies (3) as the Fourier transform is bijective:

$\langle\hat{\sigma}_{k*\rho}, f \otimes g\rangle = \langle\sigma_{(k*\rho)\hat{}}, f \otimes \check{g}\rangle = \langle\sigma_{\hat{k}\hat{\rho}}, f \otimes \check{g}\rangle =_{(2)} \langle(\hat{k} \otimes \hat{k}^{-})\sigma_{\hat{\rho}}, f \otimes \check{g}\rangle =$ $\langle\sigma_{\hat{\rho}}, \hat{k}f \otimes \hat{k}^{-}\check{g}\rangle = \langle\hat{\sigma}_{\rho}, \hat{k}f \otimes \hat{k}^{-\check{}}g\rangle = \langle(\hat{k} \otimes \bar{k}\hat{})\hat{\sigma}_{\rho}, f \otimes g\rangle = \langle((k \otimes \bar{k}) * \sigma_{\rho})\hat{}, f \otimes g\rangle$.
For the last equation we used the convolution theorem for distributions. □

Theorem 3.1.4. (Properties of filtered GSPs)
Let ρ be a GSP, $h \in A(G)$ and $k \in L^1(G)$. Then:
(1) ρ is bounded $\Longleftrightarrow k * \rho$ is bounded $\forall\, k \in L^1(G)$.
(2) ρ is V-bounded $\Longleftrightarrow k * \rho$ is V-bounded $\forall\, k \in L^1(G)$.
(3) ρ is bounded $\Longleftrightarrow h\rho$ is bounded $\forall\, h \in A(G)$.
(4) ρ is V-bounded $\Longleftrightarrow h\rho$ is V-bounded $\forall\, h \in A(G)$.
(5) ρ is orthogonally scattered $\Longleftrightarrow h\rho$ is orthogonally scattered $\forall\, h \in A(G)$.
(6) ρ is stationary $\Longleftrightarrow k * \rho$ is stationary $\forall\, k \in L^1(G)$.

Proof. **(1)** ($\Rightarrow$) Let ρ be bounded, then we may calculate:

$$\|k * \rho(f)\|_{\mathcal{H}} = \|\rho(\check{k} * f)\|_{\mathcal{H}} \leq c\|\check{k} * f\|_\infty \leq c\|k\|_1\|f\|_\infty = c'\|f\|_\infty$$

i.e. $k * \rho$ is bounded.
($\Leftarrow$) Let $k * \rho$ be bounded, i.e. $\forall\, k \in L^1(G)$, there $\exists$ a constant $c_k > 0$, such that: $\|k * \rho(f)\|_{\mathcal{H}} \leq c_k\|f\|_\infty \quad \forall\, f \in S_0(G)$. First we have to show, that:

$$\|k * \rho(f)\|_{\mathcal{H}} \leq c\|k\|_1\|f\|_\infty \quad \forall\, k \in L^1(G),\, \forall\, f \in S_0(G),$$

i.e. the mapping $\phi_\rho : L^1(G) \longrightarrow \mathcal{L}(S_0(G), \mathcal{H})$, defined by $k \longmapsto k * \rho$ has to be a bounded linear operator. The details of this fact can be found in [15] p.38. Then we choose $g_n \in L^1(G)$ with $\|\check{g}_n * f - f\|_{S_0} \longrightarrow 0$ and $\|g_n\|_1 = 1 \quad \forall\, n \in \mathbb{N}$ and a fixed but arbitrary $f \in S_0(G)$. Then there $\exists\, c > 0$ such that $\|g_n * \rho(f)\|_{\mathcal{H}} \leq c\|f\|_\infty \quad \forall n \in \mathbb{N}$, i.e. $(g_n * \rho(f))$ is uniformly bounded. It follows, that $\|\rho(f)\|_{\mathcal{H}} \leq c\|f\|_\infty$ (because $g_n * \rho$ converges to ρ in the pointwise operator topology), i.e. ρ is bounded.
(3) analogous to (1) - replace the convolution by pointwise multiplication.
(2) follows from (3).
(4) follows from (1).
(5) ($\Rightarrow$) The second part of the previous Lemma 3.1.3. implies:

$$\mathrm{supp}(\sigma_{h\rho}) = \mathrm{supp}((h \otimes \bar{h})\sigma_\rho) = \mathrm{supp}((h \otimes h)\sigma_\rho) \subseteq \mathrm{supp}(\sigma_\rho).$$

Because of ρ being orthogonally scattered and Theorem 2.1.3.(3) it follows, that $\mathrm{supp}(\sigma_\rho) \subseteq \Delta_G \Longrightarrow \mathrm{supp}(h\rho) \subseteq \Delta_G \Longrightarrow h\rho$ is orthogonally scattered (again by Theorem 2.1.3.(3)).
($\Leftarrow$) If $h\rho$ is orthogonally scattered $\forall h \in A(G)$, it follows:

$$\mathrm{supp}(\sigma_{h\rho}) = \mathrm{supp}((h \otimes h)\sigma_\rho) \subseteq \Delta_G \quad \forall\, h \in A(G).$$

Now this implies $\mathrm{supp}(\sigma_\rho) \subseteq \Delta_G$, i.e. ρ is orthogonally scattered.
(6) Follows from (5) and Proposition 2.3.1.(1). □

3.2 Harmonizable GSPs

For many, especially practical, problems the concept of stationary processes, which was introduced by Definition 1.2.9. and characterized in section 2.3, is a too strong restriction. Therefore we start this final section of chapter 3 with generalizations of stationary GSPs. We will work with two different estimates. The first generalization are the V-bounded GSPs, which we have already introduced by Definition 1.2.10. We recall: A GSP ρ is called **V-bounded**, if there $\exists\, c > 0$ such that:

$$\|\rho(f)\|_{\mathcal{H}} \leq c\|\hat{f}\|_\infty \quad \forall\, f \in S_0(G).$$

In addition to V-bounded GSPs we now introduce, according to the classical theory, a second generalization of stationarity, the so called harmonizable GSPs. Our definition is due to H. Niemi (cf. [18] p. 35). In the case of GSPs this concept was used in [15] p. 41-45.

Definition 3.2.1. Let ρ be a GSP with covariance σ_ρ. We call ρ (**strongly**) **harmonizable** $:\Longleftrightarrow$ $\hat{\sigma}_\rho$ can be identified with a bounded measure.

Proposition 3.2.2. (Characterizing harmonizable GSPs)
Let ρ be a GSP with covariance σ_ρ. Then:
(1) ρ is harmonizable $\Longleftrightarrow$ σ_ρ lies in the Fourier-Stieltjes algebra $B(G\times G) := \mathcal{F}(M(\hat{G}\times\hat{G}))$. Thus we may note σ_ρ as:

$$\sigma_\rho = h_\rho(x,y) = \langle \nu, \bar{\chi}_x \otimes \bar{\chi}_y \rangle \quad \forall\, x,y \in G \text{ and } \nu \in M(\hat{G}\times\hat{G}).$$

(2) ρ is harmonizable $\Longrightarrow$ ρ is V-bounded.
(3) Let ρ be stationary. Then: ρ is harmonizable $\Longleftrightarrow$ ρ is V-bounded.

Proof.
(1) $h_\rho \in B(G\times G) \Longleftrightarrow \hat{\sigma}_\rho$ denotes bounded measure on $\hat{G}\times\hat{G} \Longleftrightarrow \rho$ is

harmonizable.
(2) ρ harmonizable $\Longrightarrow \hat{\sigma}_\rho$ extends to a bounded measure on $C_0(G\times G)$. The fact, that $S_0(G\times G)$ lies dense in $V_0(G\times G)$ which lies dense in $C_0(G\times G)$ (cf. section 1.1), implies that $\hat{\sigma}_\rho$ extends to a bimeasure. Applying now Corollary 2.2.9. we get the claim.
(3) ($\Rightarrow$) Already shown in part (2).
($\Leftarrow$) Let ρ be stationary and V-bounded $\Longrightarrow \hat{\rho}$ is bounded and orthogonally scattered. Corollary 2.1.4. implies, that there $\exists$ a bounded measure $\mu_{\hat{\rho}}$ on $\hat{G}$ with:

$$\langle \sigma_{\hat{\rho}}, f \otimes g \rangle = \langle \mu_{\hat{\rho}}, fg \rangle.$$

Therefore we can identify $\mu_{\hat{\rho}}$ with $\sigma_{\hat{\rho}}$ which is a bounded measure, i.e. ρ is harmonizable. □

Remark 3.2.3. As $B(G\times G) \subseteq C_b(G\times G)$, the previous Proposition together with our calculations in section 2.4 shows, that any harmonizable GSP can be identified with a (strongly) harmonizable stochastic process. The converse is obvious.

Now we could ask ourselves, if every stationary GSP has to be harmonizable or V-bounded. This fact is non trivial, as a GSP has a covariance distribution which cannot be identified with a continuous function. But if we assume, that $\sigma_\rho \in C_b(G \times G)$ we get:

Proposition 3.2.4. Every stationary GSP ρ with covariance $\sigma_\rho \in C_b(G\times G)$ is harmonizable.

Proof. Let ρ be stationary and $\sigma_\rho \in C_b(G \times G)$. It follows by Theorem 2.3.4, that σ_ρ can be identified with a continuous positive definite function. Bochner's theorem (cf. [23] p.19) implies $\sigma_\rho \in B(G \times G)$. □

Corollary 3.2.5. Let X_t be a continuous stochastic process. Then the following chain of implications is true:
X_t stationary $\Longrightarrow X_t$ harmonizable $\Longrightarrow X_t$ V-bounded.

Proof. As any stationary continuous stochastic process can be identified with a stationary GSP with $\sigma_\rho \in C^b(G \times G)$ (cf. section 2.4) and any harmonizable (resp. V-bounded) stochastic process can be identified with a harmonizable (resp. V-bounded) GSP (cf. Remark 3.2.3.), the claim follows from Proposition 3.2.2.(2) and Proposition 3.2.4. □

We close this chapter with some very interesting remarks concerning the dilation theory for GSPs, starting with the following Definition:

Definition 3.2.6. Let ρ be a GSP into $\mathcal{H} := \{\rho(f) \mid f \in S_0(G)\}^-$ and $\tilde{\rho}$ a GSP into $\tilde{\mathcal{H}}$, with $\mathcal{H} \subset \tilde{\mathcal{H}}$. The GSP $\tilde{\rho}$ is called a **dilation** of ρ if:

$$\rho(f) = \mathcal{P}\big(\tilde{\rho}(f)\big) \qquad \forall\, f \in S_0(G). \tag{3.1}$$

$\mathcal{P}$ denotes the orthogonal projection from $\tilde{\mathcal{H}}$ into $\mathcal{H}$.

Our aim is to show, that any V-bounded GSP ρ appears as the projection of a stationary GSP $\tilde{\rho}$. This means, we prove the existence of a stationary dilation $\tilde{\rho}$ of ρ. Therefore our result, first proved by W. Hörmann in [15] p. 49, is a generalization of the result for stochastic processes which was shown by H. Niemi in [19] using the main result of [20]. As a first step we want to prove the existance of orthogonally scattered dilation (cf. Corollary 3.2.8.) but therefore we need the following:

Theorem 3.2.7. Let ρ be an arbitrary bounded GSP into $\mathcal{H}$ ($\mathcal{H}$ as in Definition 3.2.6.) with covariance σ_ρ. Then $\exists$ a dilation $\tilde{\rho}$ into $\tilde{\mathcal{H}} \supset \mathcal{H}$ which is orthogonally scattered and bounded if and only if there $\exists$ a positive, bounded measure μ such that:

$$\|\rho(f)\|^2_{\mathcal{H}} \le \langle \mu, |f|^2 \rangle \quad \forall\, f \in S_0(G). \tag{3.2}$$

Proof. ($\Rightarrow$) By Corollary 2.1.4. there $\exists$ a bounded measure $\mu \in M(G)$, such that:

$$(\|\tilde{\rho}(f)\|_{\tilde{\mathcal{H}}})^2 = (\tilde{\rho}(f)|\tilde{\rho}(f))_{\tilde{\mathcal{H}}} = \langle \mu, f\bar{f} \rangle = \langle \mu, |f|^2 \rangle$$

Now if $\tilde{\rho}$ denotes a dilation of ρ, we get the following estimate:

$$\|\rho(f)\|^2_{\mathcal{H}} = \|\mathcal{P}(\tilde{\rho}(f))\|^2_{\mathcal{H}} \le (\|\tilde{\rho}(f)\|_{\tilde{\mathcal{H}}})$$

and this implies $\|\rho(f)\|^2_{\mathcal{H}} \le \langle \mu, |f|^2 \rangle$.
($\Leftarrow$) If we assume, that $\|\rho(f)\|^2_{\mathcal{H}} = \langle \mu, |f|^2 \rangle \quad \forall\, f \in S_0(G)$, we may apply the so called polarization identity to compute:

$$(\rho(f)|\rho(g)) = \frac{1}{4}\big(\langle \mu, |f+g|^2 \rangle - \langle \mu, |f-g|^2 \rangle +$$

$$+ i\langle \mu, |f+ig|^2 \rangle - i\langle \mu, |f-ig|^2 \rangle\big) = \langle \mu, f\bar{g} \rangle. \tag{3.3}$$

The estimate (3.3), together with Corollary 2.1.4., shows that ρ is orthogonally scattered and moreover it is its own orthogonally scattered dilation. Now let $f_0 \in S_0(G)$ denote a function with: $\|\rho(f_0)\|_{\mathcal{H}}^2 \leq \langle \mu, |f_0|^2 \rangle$. We define an element $\sigma \in S_0'(G \times G)$, such that:

$$\langle \sigma, f \otimes g \rangle := \langle \mu, fg \rangle - \langle \sigma_\rho, f \otimes g \rangle \tag{3.4}$$

As $\langle \sigma, f \otimes \bar{f} \rangle = \langle \mu, |f|^2 \rangle - \langle \sigma_\rho, f \otimes \bar{f} \rangle \geq 0 \quad \forall f \in S_0(G)$ we conclude, that $Q(f,g) := \langle \sigma, f \otimes \bar{g} \rangle$ denotes a positive semi definite sesquilinear form on $S_0(G)$. Furthermore Q defines a canonical inner product on $\mathcal{H}_1 := S_0(G)/\mathcal{N}$, where $\mathcal{N} := \{f \in S_0(G) \mid \langle \sigma, f \otimes \bar{f} \rangle = 0\}$, which is a linear subspace of $S_0(G)$. The Hilbert space, we obtain by completion of $\mathcal{H}_1$ shall be denoted as $\mathcal{H}'$. Now the canonical projection $\rho' : S_0 \longrightarrow \mathcal{H}_1$, followed by the embedding of $\mathcal{H}_1$ into $\mathcal{H}'$ is a bounded linear mapping from S_0 into $\mathcal{H}'$. Thus ρ' denotes a GSP with covariance σ, by construction in (3.4).
Yet we are in a position to introduce a new Hilbert space

$$\tilde{\mathcal{H}} := \mathcal{H} \oplus \mathcal{H}' = \{(x,y) \mid x \in \mathcal{H}, y \in \mathcal{H}'\} \tag{3.5}$$

with the inner product

$$\big((x_1,y_1)|(x_2,y_2)\big)_{\tilde{\mathcal{H}}} := (x_1,x_2)_{\mathcal{H}} + (y_1,y_2)_{\mathcal{H}'}. \tag{3.6}$$

According to (3.5) and (3.6) we define the following GSP

$$\tilde{\rho}(f) := (\rho(f), 0) + (0, \rho'(f)),$$

which is obviously a dilation of ρ. Furthermore $\tilde{\rho}$ is orthogonally scattered by Corollary 2.1.4., as:

$$(\tilde{\rho}(f)|\tilde{\rho}(g))_{\tilde{\mathcal{H}}} = (\rho(f)|\rho(g))_{\mathcal{H}} + (\rho'(f)|\rho'(g))_{\mathcal{H}'} =$$

$$= \langle \sigma_\rho, f \otimes \bar{g} \rangle + \langle \sigma, f \otimes \bar{g} \rangle = \langle \mu, f\bar{g} \rangle,$$

and thus the proof is complete. □

Corollary 3.2.8. Let ρ be a GSP. ρ is bounded if and only if there $\exists$ a dilation $\tilde{\rho}$ which is orthogonally scattered and bounded.

Proof. ($\Rightarrow$) This follows from the previous Theorem together with the fact, that if ρ is bounded, then there $\exists$ a bounded and positive measure μ with $\|\rho(f)\|_{\mathcal{H}}^2 \leq \langle \mu, |f|^2 \rangle \quad \forall f \in S_0(G)$ (cf. [15] Lemma 21.).

($\Leftarrow$) Because of $\|\rho(f)\|_{\mathcal{H}} = \|\mathcal{P}(\tilde{\rho}(f))\|_{\mathcal{H}} \leq \|\tilde{\rho}(f)\|_{\tilde{\mathcal{H}}} \quad \forall f \in S_0(G)$ it is clear, that any GSP with a bounded dilation is bounded itself. $\square$

Now we are almost in a position to prove the main result. Before we note the following simple fact:

Lemma 3.2.9. If ρ is a GSP with dilation $\tilde{\rho}$ into $\tilde{\mathcal{H}}$ then $(\tilde{\rho})\hat{}$ is a dilation of $\hat{\rho}$ into $\tilde{\mathcal{H}}$.

Proof. For $\hat{\rho} : S_0(G) \longrightarrow \mathcal{H}$ and $(\tilde{\rho})\hat{}: S_0(\hat{G}) \longrightarrow \tilde{\mathcal{H}}$, we calculate:

$$\hat{\rho}(f) = \rho(\hat{f}) = \mathcal{P}(\tilde{\rho}(\hat{f})) = \mathcal{P}((\tilde{\rho})\hat{}(f)) \quad \forall f \in S_0(\hat{G}),$$

i.e. $(\tilde{\rho})\hat{}$ denotes a dilation of $\hat{\rho}$. $\square$

If we recall our calculations from section 2.2 the following result is a direct consequence:

Theorem 3.2.10. Let ρ be a GSP. ρ is V-bounded if and only if there $\exists$ a dilation $\tilde{\rho}$ which is V-bounded and stationary .

Proof. ($\Rightarrow$) ρ is V-bounded $\Longrightarrow$ $\hat{\rho}$ is bounded (by Proposition 2.2.3.(1)) $\Longrightarrow$ there $\exists$ a dilation $(\hat{\rho})\tilde{}$ which is bounded and orthogonally scattered (by Corollary 3.2.8. together with Lemma 3.2.9.). Yet it follows, that $\tilde{\rho} := ((\hat{\rho})\tilde{})\tilde{}\tilde{}$ is a V-bounded and stationary dilation of ρ.
($\Leftarrow$) Because of $\|\rho(f)\|_{\mathcal{H}} \leq \|\tilde{\rho}(f)\|_{\tilde{\mathcal{H}}} \quad \forall f \in S_0(G)$ it is obvious that a GSP with a V-bounded dilation is V-bounded itself. $\square$

Together with our calculations in section 2.4 we finally get Theorem 3.2.10., in the case of classical stochastic processes, from Corollary 3.2.5.

Corollary 3.2.11. Let X_t be a continuous stochastic process. Then X_t is V-bounded if and only if there $\exists$ a stationary dilation of X_t.

Chapter 4

Approximation of GSPs

In this final chapter we regard different Approximation properties of GSPs. We start with a well known result concerning the V-bounded ones. By our calculations of section 2.4 the result will also hold for classical stochastic processes. During the second section we derive new approximation results, by using the results of [9]. In the third section we justify, that the results, developed in section 4.2 also hold for spectral processes. The last section denotes an additional report concerning some convergence concepts of GSPs on the Zemanian space.

4.1 Approximation of V-bounded GSPs

As a first result we get the following Theorem which states, that any V-bounded GSP can be approximated by harmonizable ones. For stochastic processes it was first proved by H. Niemi (c.f. [18] p. 44). The proof for GSPs is due to W. Hörmann (c.f. [15] p. 44).
This result is remarkable as there are GSPs which are V-bounded but not harmonizable.

Theorem 4.1.1. (Approximating V-bounded GSPs)
Let ρ be a V-bounded GSP. There $\exists$ a net $(\rho_\eta)_{\eta \in E}$ of harmonizable GSPs such that $\sigma_{\rho_\eta}(x, y) \longrightarrow \sigma_\rho(x, y)$ for $\eta \longrightarrow \infty$ uniformly on compact sets.

Proof. Let $(e_\alpha)_{\alpha\in A}$ be a net in $S_0(G)$ constituting a tight, L^1-bounded approximate unit for $L^1(G)$, and let $(u_\beta)_{\beta\in B}$ be a bounded approximate unit for the Fourier-algebra $A(G)$ in $S_0(G)$. Then we set $\rho_\eta := u_\beta(e_\alpha * \rho)$, with $\eta := (\alpha,\beta) \in E := A \times B$. Because of Lemma 3.1.3.(2) and (3) the following is justified:

$$\sigma_{\rho_\eta} := (u_\beta \otimes \bar{u}_\beta)[(e_\alpha \otimes \bar{e}_\alpha) * \sigma_\rho]. \tag{4.1}$$

Furthermore $d_\alpha := e_\alpha \otimes \bar{e}_\alpha$ is a tight, L^1-bounded approximate unit for $L^1(G\times G)$ and $v_\beta := u_\beta \otimes \bar{u}_\beta$ is a bounded approximate unit in $A(G\times G)$. Let K be an arbitrary but fixed compact subset of $G\times G$. Our aim is to show that σ_{ρ_η} converges uniformly on K, i.e. for $\|f\|_{K,\infty} := \sup_{x\in K}|f(x)|$ we have to verify that for any $\varepsilon > 0$ there $\exists\, \eta_0 \in E$ such that:

$$\|\sigma_{\rho_\eta} - \sigma_\rho\|_{K,\infty} \le \varepsilon \quad \forall\, \eta \ge \eta_0.$$

Therefore we may use the following estimate:

$$\|\sigma_{\rho_\eta} - \sigma_\rho\|_{K,\infty} = \|v_\beta(d_\alpha * \sigma_\rho) - \sigma_\rho\|_{K,\infty} \le$$

$$\le \|v_\beta(d_\alpha * \sigma_\rho) - d_\alpha * \sigma_\rho\|_{K,\infty} + \|d_\alpha * \sigma_\rho - \sigma_\rho\|_{K,\infty}. \tag{4.2}$$

As σ_ρ is continuous and bounded (cf. Proof of Corollary 2.4.5.) we can find a $p \in C_0(G)$ with $p(x) \equiv 1 \quad \forall\, x \in O \supset K$, such that:

$$d_\alpha * p\,\sigma_\rho(x) = d_\alpha * \sigma_\rho\,(x) \quad \forall\, x \in K.$$

It follows that there $\exists\, \alpha_0 \in A$ with $\|d_\alpha * p\,\sigma_\rho - p\,\sigma_\rho\|_\infty \le \frac{\varepsilon}{2} \quad \forall\ \alpha \ge \alpha_0$ ($p\,\sigma_\rho \in C^0(G)$ respectively $d_\alpha * p\,\sigma_\rho$ converges with respect to $\|.\|_\infty$) $\Longrightarrow$ $\|d_\alpha * \sigma_\rho - \sigma_\rho\|_{K,\infty} \le \frac{\varepsilon}{2} \quad \forall\, \alpha \ge \alpha_0$, i.e. the second term of (4.2) tends to zero for $\alpha \longrightarrow \infty$.

For the first term of (4.2) there $\exists\, \beta_0 \in B$, such that:

$$\|v_\beta(d_\alpha * \sigma_\rho) - d_\alpha * \sigma_\rho\|_{K,\infty} \le \|v_\beta(d_\alpha * p\,\sigma_\rho) - d_\alpha * p\,\sigma_\rho\|_{K,\infty} \le$$

$$\le \|v_\beta - 1\|_{K,\infty} \cdot \|d_\alpha * p\,\sigma_\rho\|_{K,\infty} \le$$

$$\le \|v_\beta - 1\|_{K,\infty} \cdot \|d_\alpha\|_1 \cdot \|p\,\sigma_\rho\|_{K,\infty} \le \frac{\varepsilon}{2} \quad \forall\, \beta \ge \beta_0. \tag{4.3}$$

As the first term of (4.3) does not depend on α, we conclude that $\forall\, \varepsilon > 0$ we can find a $\eta_0 := (\alpha_0, \beta_0)$, such that $\|\sigma_{\rho_\eta} - \sigma_\rho\|_{K,\infty} \le \varepsilon \quad \forall\, \eta \ge \eta_0$. This proves the uniform convergence of σ_{ρ_η} over an arbitrary compact set K. Furthermore, as $\sigma_{\rho_\eta} \in S_0 * S_0(G\times G) \subseteq S_0(G\times G) \Longrightarrow \hat{\sigma}_{\rho_\eta} \in S_0(\hat{G}\times\hat{G}) \subseteq M(\hat{G}\times\hat{G})$, i.e. ρ_η is harmonizable $\forall\, \eta \in E$. □

Using the previous Theorem and the theory developed in section 2.4, we get Niemis result (c.f. [18] p. 44) as a Corollary.

Corollary 4.1.2. (Approximating V-bounded stochastic processes)
Any continuous V-bounded stochastic process can be approximated by harmonizable processes uniformly over compact sets.

Proof. Due to section 2.4, we may use the fact that any continuous V-bounded stochastic process can be identified with a uniquely determined V-bounded GSP and then apply Theorem 4.1.1. The approximating harmonizable GSPs can be identified with harmonizable stochastic processes. By the proof of Theorem 2.4.3. the following holds:

$$\tilde{\rho}_\eta(\delta_x) = \tilde{\rho}(u_\beta(e_\alpha * \delta_x)) = \tilde{\rho}(u_\beta(T_x e_\alpha)) \longrightarrow \tilde{\rho}(\delta_x)$$

uniformly on compact sets by the vague continuity of $\tilde{\rho}$. □

4.2 Approximation via Discretization I

In the previous section we dealt with V-bounded GSPs by approximating them with harmonizable GSPs. In this section we take a totally different path by involving new convergence results for the Schoenberg operator and the more general Quasi interpolation operators as noted in reference [9], which was motivated by the results of [17]. In particular, we will prove, that some natural conditions on a generator function will imply pointwise, i.e. weak star, convergence of the covariance of an arbitrary GSP. Let us mention here, that we will assume $G = \mathbb{R}^d$ from now on.

The Schoenberg operator is an important tool for approximation of a continuous function from uniform samples.
This means in particular: For a given continuous function $f \in S_0(\mathbb{R}^d)$ and $h > 0$ the Schoenberg approximant, symbolized with $Q_h^\varphi f$, denotes a superposition of dilated and shifted versions of a given generator function $\varphi \in S_0(\mathbb{R}^d)$ using the samples of f on the lattice $h\mathbb{Z}^d$ as coefficients.
Thus we get the following exact Definition:

Definition 4.2.1. Let $\mathbb{R} \ni h > 0$ characterizing the lattice $h\mathbb{Z}^d$ and $f \in S_0(\mathbb{R}^d)$. For any generator function $\varphi \in S_0(\mathbb{R}^d)$, the **Schoenberg operator**

is given by:

$$Q_h^{\varphi} f(x) := \sum_{k\in\mathbb{Z}^d} f(hk)\varphi(\frac{x}{h} - k) \qquad x \in \mathbb{R}^d. \tag{4.4}$$

Remark 4.2.2. If we set $d = 1$ in (4.4) we get for example piecewise linear interpolation or spline interpolation respectively. This is obvious, because of the well known fact from numerical mathematics, that the B-Splines from uniform samples on $\mathbb{R}$ can be written as

$$\mathcal{B}(x) = \varphi(\frac{x}{h} - k),$$

where $h > 0$ characterizes the lattice $h\mathbb{Z}$ and $k \in \mathbb{Z}$.

If we replace the sampling of f by the application of a linear functional $\mu \in M(\mathbb{R}^d) := C_0'(\mathbb{R}^d)$ to $f \in C_0(\mathbb{R}^d)$, which we usually write as $\int_{\mathbb{R}^d} f(t)\, d\mu(t)$, we get the more general type of Quasi interpolation.
Before we introduce this concept exactly we fix the following Notation.

Notation 4.2.3. Let $j > 0$ and $x \in \mathbb{R}^d$. We denote the dilation $f^{[j]}$ of $f \in S_0(\mathbb{R}^d)$ by $f^{[j]}(x) := j^{-d} f(\frac{x}{j})$. This notion can be extended to measures or distributions by $\int_{\mathbb{R}^d} f(t)\, d\mu^{[j]}(t) := \int_{\mathbb{R}^d} f(jt)\, d\mu(t)$.

Definition 4.2.4. Let $\mu \in M(\mathbb{R}^d)$ and $h, j > 0$. The **Quasi interpolation operator** $Q_{h,j}^{\varphi} f(x)$ is defined by

$$Q_{h,j}^{\varphi} f(x) := \sum_{k\in\mathbb{Z}^d} \big(\int_{\mathbb{R}^d} f(t + hk)\, d\mu^{[j]}(t)\big)\, \varphi(\frac{x}{h} - k), \tag{4.5}$$

where $x \in \mathbb{R}^d$ and $\varphi \in S_0(\mathbb{R}^d)$ denotes a (suitable) generator function.

Remark 4.2.5. If we set $\mu = \delta_x \in M(\mathbb{R}^d)$, i.e. the Delta distribution, then (4.5) is independent of j and reduces to the Schoenberg operator Q_h^{φ}.

Next we shall prove, that the operators (4.4) and (4.5) are indeed well defined and belong to S_0 for $f \in S_0$.

Lemma 4.2.6. Let $\varphi, f \in S_0(\mathbb{R}^d)$ and $h > 0$. Then:
(1) $Q_{h,j}^{\varphi} f \in S_0(\mathbb{R}^d)$ and the series (4.5) converges absolutely in $S_0(\mathbb{R}^d)$.

(2) Furthermore the Fourier transform of $Q_h^\varphi f$, given by:

$$\hat{Q}_h^\varphi f(t) := \sum_{k\in\mathbb{Z}^d} \hat{\varphi}(ht)\hat{f}(t-\frac{k}{h}), \qquad t\in\mathbb{R}^d$$

converges absolutely in $S_0(\mathbb{R}^d)$.

Proof. (1) Since all spaces involved are dilation invariant, we may assume $h = 1$. The claim follows, as the estimate

$$\sum_{k\in\mathbb{Z}^d} \|f(k)T_k\varphi\|_{S_0} \le \sum_{k\in\mathbb{Z}^d} |f(k)|\, \|T_k\varphi\|_{S_0} \le C\|f\|_{S_0}\|\varphi\|_{S_0}$$

holds, which implies absolute convergence of $\sum_{k\in\mathbb{Z}^d} f(k)T_k\varphi$. Because of Remark 4.2.5. the Lemma is also true for the Schoenberg operator Q_h^φ.
(2) The discrete norm of $S_0(\mathbb{R}^d)$ (cf. [9], equation (7)) implies absolute convergence of $\sum_{k\in\mathbb{Z}^d}(\hat{\varphi}\cdot T_k\hat{f})$. Therefore we may calculate:

$$\sum_{k\in\mathbb{Z}^d} \|(\hat{\varphi}\cdot T_k\hat{f})\|_{S_0} \le C\|\hat{\varphi}\|_{S_0}\|\hat{f}\|_{S_0} = C\|\varphi\|_{S_0}\|f\|_{S_0}.$$

□

The following Remark concerning Strang-Fix conditions, a tool to characterize the approximating properties of a shift invariant localized operator by its ability to reconstruct polynomials, is from importance for our further calculations.
Remark 4.2.7. One can show (cf. [9], section 5), that the following two forms of Strang-Fix conditions:

$$\hat{\varphi}(k) = \delta_{k,0} \quad k\in\mathbb{Z}^d \quad \text{and} \quad \sum_{k\in\mathbb{Z}^d}\varphi(x-k) = 1 \quad x\in\mathbb{R}^d \qquad (4.6)$$

are equivalent for some generator $\varphi \in S_0(\mathbb{R}^d)$. A procedure to construct functions that satisfy (4.6) is proved in [9] Lemma 5.1.:
For a given $\psi \in S_0(\mathbb{R}^d)$ the series $\Psi(x) := \sum_{k\in\mathbb{Z}^d}\psi(x-k)$ converges absolutely in $x \in \mathbb{R}^d$ and uniformly on compact sets. Now if $\Psi(x) \neq 0 \quad \forall\, x \in [0,1]^d$, we define:

$$\varphi(x) := \frac{\psi(x)}{\Psi(x)} \qquad x\in\mathbb{R}^d.$$

Then $\varphi \in S_0(\mathbb{R}^d)$ and φ satisfies the conditions (4.6).
As an example for functions satisfying (4.6) we note tensor products of the

symmetric B-splines, defined by $\hat{B_n}(t) := (\frac{\sin \pi t}{\pi t})^{n+1} \quad t \in \mathbb{R}$. As these functions belong to $S_0(\mathbb{R})$ for $n \geq 1$ we may conclude, that the following results also include the case of spline approximation and also the piecewise linear approximation on $\mathbb{R}$.

Before we note a first main result of [9] we need some preliminary facts. The first part of the following Lemma deals with special classes of generators. In the second part we replace the approximation $Q_h^{\varphi} f \longrightarrow f$ for a given $\varphi \in \Phi^{\circ}$ by a sampling Theorem for bandlimited functions, i.e. $Q_h^{\varphi} f = f$ for sufficiently small $h > 0$. Finally the third part shows the uniform boundedness of the Schoenberg operator on S_0, for generators $\varphi \in S_0$.

Lemma 4.2.8.
(1) We define:

$$\Phi := \{\varphi \in S_0(\mathbb{R}^d) \mid \hat{\varphi}(k) = \delta_{k,0}, \quad k \in \mathbb{Z}^d\} \qquad \text{and}$$

$$\Phi^{\circ} := \{\varphi \in S_0(\mathbb{R}^d) \mid \hat{\varphi}|_{k+[-\varepsilon,\varepsilon]^d} = \delta_{k,0}, \quad k \in \mathbb{Z}^d \text{ for some } \varepsilon > 0\}.$$

Then Φ° lies dense in Φ with respect to $\|.\|_{S_0}$. Φ is even the closure of Φ° in S_0.
(2) Let $\varphi \in \Phi^{\circ}$ and $f \in L^1$ such that $\text{supp}(\hat{f})$ is compact. Now there $\exists\, h_0 > 0$ such that $Q_h^{\varphi} f = f \quad \forall\, h \leq h_0$.
(3) Let $\varphi, f \in S_0(\mathbb{R}^d)$. Then for $h \leq 1$ there $\exists\, C > 0$ such that:

$$\|Q_h^{\varphi} f\|_{S_0} \leq C \|\varphi\|_{S_0} \|f\|_{S_0}.$$

Proof.
(1) First we introduce the following subsets of $S_0(\mathbb{R}^d)$:

$$I := \{f \in S_0(\mathbb{R}^d) \mid f(k) = 0,\, k \in \mathbb{Z}^d\} \tag{4.7}$$

$$I^{\circ} := \{f \in S_0(\mathbb{R}^d) \mid f|_{k+[-\varepsilon,\varepsilon]^d} = 0,\, k \in \mathbb{Z}^d,\, \varepsilon > 0\}. \tag{4.8}$$

Now we use the fact, that $\mathbb{Z}^d$ is a set of spectral synthesis for the Fourier algebra (cf. [22] Theorem 2.4.16. or Corollary 6.1.8.) in combination with the ideal theorem for Segal algebras (cf. [22], Theorem 6.2.9.) to conclude, that I° lies dense in I with respect to the S_0 - norm. I is even the closure of I° in $S_0(\mathbb{R}^d)$. Furthermore we define

$$I^{\circ\circ} := \{f \in I^{\circ} \mid \text{supp}(f) \text{ is compact }\}, \tag{4.9}$$

and the closure of $I^{\circ\circ}$ in $S_0(\mathbb{R}^d)$ is denoted as I_1. Now I and I_1 are closed Ideals in S_0. They coincide if and only if their closures are equal in the Fourier algebra $A(\mathbb{R}^d)$. This follows again by the ideal theorem for Segal algebras. As $\mathbb{Z}^d$ is a closed subgroup of $\mathbb{R}^d$ and therefore a set of spectral synthesis, there exists only one closed ideal with cospectrum $\mathrm{cosp}(I) = \mathbb{Z}^d$. Let $f \in A(\mathbb{R}^d)$ with $\mathrm{supp}(f) \subseteq [-\frac{1}{2}, \frac{1}{2}]^d$ and $f(x) = 1$ for all $x \in [-\frac{1}{4}, \frac{1}{4}]^d$. Then $f \in S_0(\mathbb{R}^d)$ since it is compactly supported in $A(\mathbb{R}^d)$. Therefore we may define $\Phi := \mathcal{F}(f+I)$ and $\Phi^\circ := \mathcal{F}(f+I^\circ)$, for I and I° given in (4.7) and (4.8) respectively. As the Fourier transform denotes an isometric mapping on $S_0(\mathbb{R}^d)$ (this fact will be exactly worked out in the next section 4.3) we conclude that Φ is the closure of Φ° in $S_0(\mathbb{R}^d)$, since I is the closure of I° in $S_0(\mathbb{R}^d)$.

(2) By part (1) we get for some $a > 0$, that $\hat{\varphi}|_{k+[-a,a]^d} = \delta_{k,0}$. Furthermore we assume $\mathrm{supp}(\hat{f}) \subseteq [-r, r]$ for some $r > 0$. Now we apply Lemma 4.2.6.(2) for $h \le h_0 := \frac{a}{r}$ to compute:

$$\hat{Q}_h^{\varphi} f(t) = \hat{\varphi}(ht) \sum_{k\in\mathbb{Z}^d} \hat{f}(t - \frac{k}{h}) =$$

$$= \hat{\varphi}(ht)\hat{f}(t) + \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \hat{\varphi}(ht)\hat{f}(t - \frac{k}{h}) = \hat{f}(t) + 0 = \hat{f}(t), \quad t \in \mathbb{R}^d.$$

(3) Let $g \in S_0(\mathbb{R}^d)$ be a function with $\mathrm{supp}(g) \subseteq [-1, 1]^d$ and $\sum_{k\in\mathbb{Z}^d} T_k g = 1$. For $k \in \mathbb{Z}^d$, we define $f_k := f \cdot T_k g$ and $\varphi_k := \varphi \cdot T_k g$. Therefore we get:

$$f = \sum_{k\in\mathbb{Z}^d} f_k \quad \text{and} \quad \varphi = \sum_{k\in\mathbb{Z}^d} \varphi_k,$$

where $\mathrm{supp}(f_k) \subseteq k + [-1, 1]^d$ and $\mathrm{supp}(\varphi_k) \subseteq k + [-1, 1]^d$. Now there $\exists\, C_1 > 0$, such that

$$\sum_{k\in\mathbb{Z}^d} \|f_k\|_A \le C_1\|f\|_{S_0} \quad \text{and} \quad \sum_{k\in\mathbb{Z}^d} \|\varphi_k\|_A \le C_1\|\varphi\|_{S_0}. \tag{4.10}$$

This can be seen since the discrete norm (cf. [9], equation (7)) for g, as defined above, is an equivalent norm. Now we use the fact, that f_k and φ_l have compact support to conclude that $Q_h^{\varphi_l} f_k$ is also compactly supported for $k, l \in \mathbb{Z}^d$. This means in particular:

$$\mathrm{supp}(Q_h^{\varphi_l} f_k) \subseteq \mathrm{supp}(f_k) + h\,\mathrm{supp}(\varphi_l). \tag{4.11}$$

For $h \le 1$ we get $\mathrm{supp}(Q_h^{\varphi_l} f_k) \subseteq k+l+[-2, 2]^d$. Furthermore there $\exists\, C_2 > 0$, such that

$$\|Q_h^{\varphi_l} f_k\|_{S_0} \le C_2\|Q_h^{\varphi_l} f_k\|_A. \tag{4.12}$$

This estimate makes use of the fact, that the S_0 - norm is equivalent to the A - norm (for the Definitions of these norms cf. section 1.1) for those functions in S_0, whose supports do not exceed a fixed diameter. Next we combine (4.10), (4.12) and the uniform boundedness of $Q_h^\varphi f$ on $A(\mathbb{R}^d)$ (cf. [9], Lemma 3.6. with $p = 1$), using the fact that $Q_h^\varphi f$ is bilinear in f and φ, to compute:

$$\|Q_h^{\varphi_l} f_k\|_{S_0} = \|\sum_{k,l\in\mathbb{Z}^d} Q_h^{\varphi_l} f_k\|_{S_0} \le \sum_{k,l\in\mathbb{Z}^d} \|Q_h^{\varphi_l} f_k\|_{S_0} \le C_2 \sum_{k,l\in\mathbb{Z}^d} \|Q_h^{\varphi_l} f_k\|_A \le$$

$$\le C_2 C' \sum_{k,l\in\mathbb{Z}^d} \|f_k\|_A \|\varphi_l\|_A \le C_2 C' C_1^2 \|f\|_{S_0} \|\varphi\|_{S_0}.$$

□

Theorem 4.2.9. Let $k \in \mathbb{Z}^d$ and $\varphi \in S_0(\mathbb{R}^d)$, such that $\hat{\varphi}(k) = \delta_{k,0}$. Then for any $f \in S_0(\mathbb{R}^d)$ the following holds:

$$\|f - Q_h^\varphi f\|_{S_0} \longrightarrow 0 \tag{4.13}$$

as $h \longrightarrow 0$.

Proof. If we view $Q_h^\varphi f$ as a bilinear operator, i.e.

$$Q_h^\varphi : S_0 \times S_0 \longrightarrow S_0,\ (\varphi, f) \longmapsto Q_h^\varphi f$$

we may conclude by Lemma 4.2.8.(3), that $Q_h^\varphi f$ is uniformly bounded for $h \le 1$. By part (1) of 4.2.8. Φ° is dense in Φ. Furthermore the set of all $f \in L^1(\mathbb{R}^d)$ with compactly supported Fourier transform lies dense in $S_0(\mathbb{R}^d)$ (cf. [6]). Thus $\forall\, h \le 1$ there $\exists\, \varphi_1 \in \Phi^\circ,\ f_1 \in L^1(\mathbb{R}^d)$, such that

$$\|f - Q_h^\varphi f\|_{S_0} \le \|f_1 - Q_h^{\varphi_1} f_1\|_{S_0} + \varepsilon, \tag{4.14}$$

where $\mathrm{supp}(\hat{f}_1)$ is compact. Applying 4.2.8.(2) to φ_1 and f_1 we find some $h_0 > 0$ such that the term $\|f_1 - Q_h^{\varphi_1} f_1\|_{S_0}$ of (4.14) vanishes for $h \le h_0$. □

The next result is a generalization of the previous Theorem, i.e. we study the same problem, but now in the case of Quasi interpolation operators.

Theorem 4.2.10. Let $k \in \mathbb{Z}^d$ and $\varphi \in S_0(\mathbb{R}^d)$, such that $\hat{\varphi}(k) = \delta_{k,0}$. Furthermore we assume, that $\mu \in M(\mathbb{R}^d)$ satisfies $\hat{\mu}(0) = 1$. Then for any $f \in S_0(\mathbb{R}^d)$ the following holds:

$$\|f - Q_{h,j}^\varphi f\|_{S_0} \longrightarrow 0 \tag{4.15}$$

as $h, j \longrightarrow 0$.

Remark 4.2.11. The exact proof of this Theorem one can find in [9], section 4.2. In this work we only want to point out its basic ideas. Therefore we need at first a result about the action of approximate units with respect to convolution on elements of $S_0(\mathbb{R}^d)$ for the extension to general $\mu \in M(\mathbb{R}^d)$ as noted in [9], Lemma 4.7.:
Let $f \in S_0(\mathbb{R}^d)$ and $\mu \in M(\mathbb{R}^d)$ such that $\hat{\mu}(0) = 1$. Then

$$\|f * \mu^{[j]} - f\|_{S_0} \longrightarrow 0 \tag{4.16}$$

as $j \longrightarrow 0$. This result uses the fact that $\|f * \mu\|_{S_0} \leq \|f\|_{S_0}\|\mu\|_M$.
By Lemma 4.2.6. we have the fact, that $Q^{\varphi}_{h,j}f$ belongs to $S_0(\mathbb{R}^d)$ for fixed $h, j > 0$. Yet we need the following alternative view of the quasi-interpolation operator (cf. [9], section 3.2.):

$$Q^{\varphi}_{h,j}f(x) = Q^{\varphi}_h(f * \mu^{[j]})(x), \qquad x \in \mathbb{R}^d \tag{4.17}$$

Here Q^{φ}_h denotes the Schoenberg operator and the operation "$*$" describes the convolution of a function with a measure, i.e.

$$(f * \mu^{[j]})(x) = \int_{\mathbb{R}^d} f(x-t)d\mu^{[j]}(t), \qquad x \in \mathbb{R}^d.$$

Now we use (4.17) and combine Lemma 4.2.8.(3), Theorem 4.2.9 and (4.16) to get finally:

$$\|f - Q^{\varphi}_{h,j}f\|_{S_0} = \|f - Q^{\varphi}_h(f * \mu^{[j]})\|_{S_0} \leq \|f - Q^{\varphi}_h f\|_{S_0} + \|Q^{\varphi}_h(f - f * \mu^{[j]})\|_{S_0} \leq$$

$$\leq \|f - Q^{\varphi}_h f\|_{S_0} + C\|f - f * \mu^{[j]}\|_{S_0} \longrightarrow 0$$

as $h, j \longrightarrow 0$ and this was exactly the claim of Theorem 4.2.10.

Before we study the dual space of $S_0(\mathbb{R}^d)$ we note a useful consequence of our results so far. In particular we regard the behavior of piecwise linear interpolation on $\mathbb{R}$ in the situation of Theorem 4.2.9 and 4.2.10 respectively. Therefore we denote the piecewise linear interpolant to the sequence $\{hk, f(hk)\}_{k\in\mathbb{Z}}$ with f_h. Now we get:

Corollary 4.2.12. Let $h > 0$. Then for any $f \in S_0(\mathbb{R})$ the following holds:

$$\|f - f_h\|_{S_0} \longrightarrow 0, \tag{4.18}$$

as $h \longrightarrow 0$.

Next we shall note several facts concerning the concept of convergence in $S_0'(\mathbb{R}^d)$. We mention the following references [3], [4] and [7]. The dual-space to $S_0(\mathbb{R}^d)$ is endowed with two natural convergence properties:

Lemma 4.2.13. $S_0'(\mathbb{R}^d)$ is a Banach-space with respect to its natural norm

$$\|\sigma\|_{S_0'} := \sup_{\|f\|_{S_0}=1}|\sigma(f)|.$$

Lemma 4.2.14. $S_0'(\mathbb{R}^d)$ is a topological vector space with respect to the weak star (symb.: w^*) - topology (i.e. the topology of pointwise convergence).

As we will most deal with the w^*-convergence we have to be more precisely concerning the description of this concept:

Remark 4.2.15. Let $\varepsilon > 0$ and F be an arbitrary finite subset of $S_0(\mathbb{R}^d)$. Then the following subset-system of $S_0'(\mathbb{R}^d)$ describes a basis for the neighborhood of $\sigma_0 \in S_0'(\mathbb{R}^d)$:

$$U(\sigma_0, F, \varepsilon) := \{\sigma \in S_0'(\mathbb{R}^d) \mid |\sigma(f) - \sigma_0(f)| < \varepsilon \quad \forall f \in F\}. \tag{4.19}$$

This implies the following:

Lemma 4.2.16. (Characterizing the w^*-convergence)
Let $(\sigma_\alpha)_{\alpha\in A}$ be a net of functionals in $S_0'(\mathbb{R}^d)$ and $\sigma_0 \in S_0'(\mathbb{R}^d)$. Then we have $w^* - \lim_{\alpha\in A} \sigma_\alpha = \sigma_0$ if and only if $\forall\, \varepsilon > 0$ and for any finite family $(f_k)_{1\le k\le K} \subset S_0(\mathbb{R}^d)$ there $\exists\, \alpha_0 \in A$ such that for any $\alpha \succeq \alpha_0$ we get:

$$|\sigma_n(f_k) - \sigma_0(f_k)| < \varepsilon \tag{4.20}$$

for $1 \le k \le K$.

Now we want to return back to the theory of GSPs. By our calculations so far, we want to modify Theorem 4.2.9. and 4.2.10. (and consequently also Corollary 4.2.12.) to be able to compute the covariances of GSPs. As we will work with elements of $S_0'(\mathbb{R}^d)$ we have to introduce "new" operators, in particular we regard the adjoint operators of (4.4) and (4.5). Therefore we note the following (very general) Definition, taken from [24]:

Definition 4.2.17. Let X and Y denote some arbitrary normed spaces with the Dual-spaces X' respectively Y' and let $T \in \mathcal{L}(X, Y)$. The **adjoint operator** $T' : Y' \longrightarrow X'$ is given by

$$(T'y')(x) := y'(Tx) \quad \forall\, y' \in Y'. \tag{4.21}$$

Notation 4.2.18. In accordance with modern literature we denote the adjoint operators from now on with T^*.

Now with the help of (4.21) we get the following "adjoint versions" of (4.4) and (4.5):

$$Q_h^{\varphi *}\sigma(f) := \sigma(Q_h^{\varphi} f) \tag{4.22}$$

$$Q_{h,j}^{\varphi}{}^{*}\sigma(f) := \sigma(Q_{h,j}^{\varphi} f) \tag{4.23}$$

In view of these operators we make use of the following, simple consequence of Definition 4.2.17. and the convergence properties of $S_0'(\mathbb{R}^d)$, stated before:

Lemma 4.2.19. Let $f \in S_0(\mathbb{R}^d)$ and $\sigma \in S_0'(\mathbb{R}^d)$. Let $(T_n)_{n\in\mathbb{N}}$ denote a sequence of operators on $S_0(\mathbb{R}^d)$, which converges to the operator $T_0(f)$. Then $T_n^*(\sigma(f)) \longrightarrow T_0^*(\sigma(f))$ in the w^* - sense.

Proof. As the elements of $S_0'(\mathbb{R}^d)$ are continuous it follows directly from Definition 4.2.17, that:

$$T_n^*(\sigma(f)) = \sigma(T_n(f)) \longrightarrow \sigma(T_0(f)) = T_0^*(\sigma(f)),$$

which gives us the claim. □

Remark 4.2.20. For further information of the reader we mention the obvious fact, that the previous Lemma also holds for arbitrary Banach spaces.

Finally we are in a position to report the following two approximation identities, which are already shown by our calculations so far. We note, that these results also contain the special case of (adjoint) piecewise linear interpolation operators (cf. Remark 4.2.2. and Corollary 4.2.12.).

Theorem 4.2.21. (Approximation via Schoenberg operator)
Let ρ be a GSP with covariance $\sigma_\rho \in S_0'(\mathbb{R}^{2d})$. Furthermore we assume

$k \in \mathbb{Z}^d$ and $\varphi \in S_0(\mathbb{R}^d)$, such that $\hat{\varphi}(k) = \delta_{k,0}$. Then for any $f, g \in S_0(\mathbb{R}^d)$ the following holds:

$$w^* - \lim_{h \to 0} Q_h^{\varphi *} \langle \sigma_\rho, f \otimes g \rangle = \langle \sigma_\rho, f \otimes g \rangle. \tag{4.24}$$

Theorem 4.2.22. (Approximation via Quasi interpolation operator)
Let ρ be a GSP with covariance $\sigma_\rho \in S_0'(\mathbb{R}^{2d})$. Furthermore we assume $k \in \mathbb{Z}^d$, $\varphi \in S_0(\mathbb{R}^d)$, such that $\hat{\varphi}(k) = \delta_{k,0}$ and $\mu \in M(\mathbb{R}^d)$ satisfying $\hat{\mu}(0) = 1$. Then for any $f, g \in S_0(\mathbb{R}^d)$ the following holds:

$$w^* - \lim_{h \to 0} Q_{h,j}^{\varphi}{}^* \langle \sigma_\rho, f \otimes g \rangle = \langle \sigma_\rho, f \otimes g \rangle. \tag{4.25}$$

Now we could ask ourselves, if these results are also suitable for spectral processes. This will be the content of the next section.

4.3 Approximation via Discretization II

In this section we want to regard the same problems, which appeared in the previous section, but now in the case of the spectral process. Therefore we will show, that the adjoint Fourier-transform $\mathcal{F}^* : S_0'(\mathbb{R}^d) \longrightarrow S_0'(\hat{\mathbb{R}}^d)$ is a linear isometric mapping, i.e. $\|\mathcal{F}\|_{S_0} = \|\mathcal{F}^*\|_{S_0'}$. By the way we will also show the invariance under Fourier transform of $S_0(G)$. At the end of the section we note a further idea, in particular we will approximate GSPs via the short time Fourier transform. We start with two Definitions, the second denotes an alternative introduction of the Feichtinger space.

Definition 4.3.1. Let $f, g \in L^2(\mathbb{R}^d)$. The **Short Time Fourier Transform** (STFT) $\mathcal{V}_g(f) : \mathbb{R}^d \times \hat{\mathbb{R}}^d \longrightarrow \mathbb{C}$ of f with window g is defined by

$$(\mathcal{V}_g f)(x, \xi) := \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-2\pi i \xi t} dt = \langle f, M_\xi T_x g \rangle \tag{4.26}$$

where $(M_\xi f)(x) := e^{2\pi i \xi t} f(x)$.

Definition 4.3.2. Let g_0 be the Gauss function, i.e. $g_0(x) := e^{-\pi |x|^2}$. The **Feichtinger space** $S_0(\mathbb{R}^d)$ is given by

$$S_0(\mathbb{R}^d) := \{ f \in L^2(\mathbb{R}^d) \mid \|f\|_{S_0} = \|\mathcal{V}_{g_0} f\|_{L^1(\mathbb{R}^{2d})} < \infty \}. \tag{4.27}$$

With this alternative view it is now very easy to prove the Fourier invariance of S_0, a fundamental property of the Feichtinger space. We start with the following:

Lemma 4.3.3. Let $f, g \in L^1(\mathbb{R}^d)$. Then the following fundamental equation of the Fourier-transform is true:

$$\int_{\mathbb{R}^d} \hat{f}(t) g(t)\, dt = \int_{\mathbb{R}^d} f(t) \hat{g}(t)\, dt \tag{4.28}$$

Proof. This result is a consequence of Fubinis Theorem (cf. [24] p. 490). □

Theorem 4.3.4. (Plancherel)
The Fourier-transform $\mathcal{F} : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$ is an isometric isomorphism and the so called Plancherel equality holds:

$$(\mathcal{F}f|\mathcal{F}g)_{L^2} = (f|g)_{L^2} \quad \forall\, f, g \in L^2(\mathbb{R}^d). \tag{4.29}$$

Remark 4.3.5. The Plancherel Theorem can be proved in various ways. We give a brief outline of the proof by following the calculations in [24] p. 207-212. The Schwartz-space is given by:

$$\mathcal{S}(\mathbb{R}^d) := \{f \in C^\infty(\mathbb{R}^d) \mid \lim_{|x|\to\infty} D^\alpha f(x) = 0\}$$

for every multi index $\alpha \in \mathbb{N}_0^n$ (for the multi index notation cf. [24] p. 7). One can show, that $\mathcal{S}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ for all $p \geq 1$ and that $f \in \mathcal{S}(\mathbb{R}^d) \Longrightarrow \mathcal{F}f \in \mathcal{S}(\mathbb{R}^d)$ respectively. Now we may use Lemma 4.3.3. to prove the following fact (cf. [24] Theorem V.2.8.): $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathcal{S}(\mathbb{R}^d)$ is bijective and

$$(\mathcal{F}f|\mathcal{F}g)_{L^2} = (f|g)_{L^2} \quad \forall\, f, g \in \mathcal{S}(\mathbb{R}^d).$$

It follows, that

$$\|\mathcal{F}f\|_{L^2} = \|f\|_{L^2} \quad \forall\, f \in \mathcal{S}(\mathbb{R}^d).$$

Now the operator $\mathcal{F}$ is well defined, bijective and isometric with respect to $\|.\|_{L^2}$ on $\mathcal{S}(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d)$. As $\mathcal{S}(\mathbb{R}^d)$ lies dense in $L^2(\mathbb{R}^d)$ (cf. [24] Lemma V.1.10.), we can extend $\mathcal{F}$ to an operator on $L^2(\mathbb{R}^d)$ which is an isometric isomorphism by the above calculations and finally get:

$$(\mathcal{F}f|\mathcal{F}g)_{L^2} = (f|g)_{L^2} \quad \forall\, f, g \in L^2(\mathbb{R}^d),$$

which was our second claim.

As $S_0(G) \subseteq L^2(G)$ by Definition 4.3.2. we get the invariance under Fourier transform with the help of 4.3.4. Now we return back to main part of this section. Therefore we denote the **adjoint Fourier transform** for elements $\sigma \in S_0'(\mathbb{R}^d)$ by:

$$\mathcal{F}^*\sigma(f) := \sigma(\mathcal{F}f), \qquad f \in S_0(G). \tag{4.30}$$

The following Theorem (cf. [24] p. 110) is a consequence of the so called Hahn-Banach Theorem, one of the "big" results within the theory of functionals. In general this result is true for arbitrary adjoint operators, we only regard the case of the adjoint Fourier transform for elements of $S_0(\mathbb{R}^d)$.

Theorem 4.3.6. $\mathcal{F} \longmapsto \mathcal{F}^* : \mathcal{L}(S_0(\mathbb{R}^d)) \longrightarrow \mathcal{L}(S_0'(\mathbb{R}^d))$ is a linear isometric mapping, i.e. $\|\mathcal{F}\|_{S_0} = \|\mathcal{F}^*\|_{S_0'}$.

Proof. The linearity is obvious. Because of (4.30) we get:

$$\|\mathcal{F}^*\sigma\|_{S_0'} = \|\sigma \circ \mathcal{F}\|_{S_0'} \leq \|\sigma\|_{S_0'}\|\mathcal{F}\|_{S_0},$$

i.e. $\|\mathcal{F}^*\|_{S_0'} \leq \|\mathcal{F}\|_{S_0}$. With help of the Hahn-Banach Theorem and its consequences (cf. [24] chapter 3) we get equality:

$$\|\mathcal{F}\|_{S_0} = \sup_{\|f\|\leq 1} \|\mathcal{F}f\|_{S_0} = \sup_{\|f\|\leq 1} \sup_{\|\sigma\|\leq 1} |\sigma(\mathcal{F}f)| =$$

$$= \sup_{\|\sigma\|\leq 1} \sup_{\|f\|\leq 1} |\sigma(\mathcal{F}f)| = \sup_{\|\sigma\|\leq 1} \|\mathcal{F}^*\sigma\|_{S_0'} = \|\mathcal{F}^*\|_{S_0'},$$

where $f \in S_0(\mathbb{R}^d)$. □

Yet we are in a position to note the results we aimed. By our calculations so far, the following results are a direct consequences of 4.2.21 and 4.2.22.

Theorem 4.3.7. (Approximating spectral GSPs via Schoenberg operator) Let $\hat{\rho}$ be the spectral process of a GSP ρ with covariance $\hat{\sigma}_\rho = \sigma_{\hat{\rho}} \in S_0'(\mathbb{R}^{2d})$. Furthermore we assume $k \in \mathbb{Z}^d$ and $\varphi \in S_0(\mathbb{R}^d)$, such that $\hat{\varphi}(k) = \delta_{k,0}$. Then for any $f, g \in S_0(\mathbb{R}^d)$ the following holds:

$$w^* - \lim_{h\to 0} Q_h^{\varphi *}\langle \sigma_{\hat{\rho}}, f \otimes g\rangle = \langle \sigma_{\hat{\rho}}, f \otimes g\rangle. \tag{4.31}$$

Theorem 4.3.8. (Approximating spectral GSPs via quasi interpolation) Let $\hat{\rho}$ be the spectral process of a GSP ρ with covariance $\hat{\sigma}_\rho = \sigma_{\hat{\rho}} \in S_0'(\mathbb{R}^{2d})$.

Furthermore we assume $k \in \mathbb{Z}^d$, $\varphi \in S_0(\mathbb{R}^d)$, such that $\hat{\varphi}(k) = \delta_{k,0}$ and $\mu \in M(\mathbb{R}^d)$ satisfying $\hat{\mu}(0) = 1$. Then for any $f, g \in S_0(\mathbb{R}^d)$ the following holds:

$$w^* - \lim_{h \to 0} {Q^{\varphi}_{h,j}}^* \langle \sigma_{\hat{\rho}}, f \otimes g \rangle = \langle \sigma_{\hat{\rho}}, f \otimes g \rangle. \tag{4.32}$$

Now we want to briefly point out a further idea, found in [10] Proposition 6.12., a started book project about S_0. The following result is based on the density of time frequency shifts in the Feichtinger algebra (cf. section 3.1. in this reference). Furthermore we note that, by our calculations of this section, the following result also holds for spectral processes.

Proposition 4.3.9. Let $(\rho_n)_{n \in \mathbb{N}}$ denote a sequence of GSPs, such that $(\sigma_n)_{n \in \mathbb{N}}$, i.e. the sequence of covariances, is bounded. Furthermore we assume $g \in S_0(\mathbb{R}^d) \backslash \{0\}$. Then there $\exists$ a GSP ρ_0 with covariance σ_0 such that $w^* - \lim_{n \to \infty} \sigma_n = \sigma_0$ if and only if $w^* - \lim_{n \to \infty} \mathcal{V}_g \sigma_n = \mathcal{V}_g \sigma_0$.

4.4 An additional report

We close this work with an interesting additional report concerning convergence properties of sequences of GSPs on the Zemanian space $\mathcal{A}$ as noted in reference [1]. This means in particular, that we will introduce different convergence concepts and study its properties. As this report can be viewed as a independent theory within this work, we will use the notion of [1] to avoid confusion. In the following we briefly introduce the Zemanian space:

Let $I \subseteq \mathbb{R}$ denote an open interval. We regard the spaces $L^2(I)$ and $C^\infty(I)$. Furthermore we define $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We use linear differential self adjoint operators $\mathcal{R}$ of the form

$$\mathcal{R} = \theta_0 \, D^{n_1} \theta_1 \, \dots \, D^{n_\nu} \theta_\nu$$

such that

$$\mathcal{R} = \bar{\theta}_\nu (-D)^{n_\nu} \dots (-D)^{n_2} \bar{\theta}_1 (-D)^{n_1} \bar{\theta}_0$$

where $D := \frac{d}{dx}$, $n_k \in \mathbb{N}_0$ for $k = 1, 2, \dots, \nu$ and $\theta_i \in C^\infty(I)$ such that $\theta_i(x) \neq 0 \quad \forall x \in I$ and $\forall i = 0, 1, \dots, \nu$. Furthermore the $\bar{\theta}_i$ denote the complex conjugates of θ_i.

Assume, that there $\exists$ a sequence $(\lambda_n)_{n\in\mathbb{N}_0} \in \mathbb{R}$ and a sequence $(\psi_n)_{n\in\mathbb{N}_0} \in C^\infty(I)$, such that $\mathcal{R}\psi_n = \lambda_n\psi_n$. If we assume additional, that (λ_n) monotonically tends to infinity and (ψ_n) denotes a complete orthonormal system in $L^2(I)$, we may write $|\lambda_0| \leq |\lambda_1| \leq |\lambda_2| \leq \ldots$. Now we introduce a new sequence $(\tilde{\lambda}_n)_{n\in\mathbb{N}_0}$ by setting $\tilde{\lambda}_n = \lambda_n$ if $\lambda_n \neq 0$ and $\tilde{\lambda}_n = 1$ if $\lambda_n = 0$. This is a nondecreasing sequence with $|\tilde{\lambda}_n| \longrightarrow \infty$. Furthermore assume, that $\mathcal{R}^0 := Id$, then we define the sequence $\mathcal{R}^k$ recursively by $\mathcal{R}^{k+1} := \mathcal{R}(\mathcal{R}^k)$. Yet we introduce the following spaces $\mathcal{A}_k$, where $k \in \mathbb{N}_0$:

$$\mathcal{A}_k := \{\phi \in L^2(I) \mid \phi = \sum_{m=0}^{\infty} a_m\psi_m,\ \|\phi\|_k := \sum_{m=0}^{\infty} \|a_m\|^2\|\tilde{\lambda}_m\|^{2k} < \infty\} \quad (4.33)$$

With the help of (4.33) we are now in a position to introduce the Zemanian space:

$$\mathcal{A} := \bigcap_{k=0}^{\infty} \mathcal{A}_k. \quad (4.34)$$

Furthermore the dual space of $\mathcal{A}$, symbolized as $\mathcal{A}'$ is given by:

$$\mathcal{A}' := \bigcup_{k=0}^{\infty} \mathcal{A}_k. \quad (4.35)$$

For further details cf. [1] and the references there.

We assume, that (Ω, Σ, P) is an arbitrary but fixed probability space. The following definition of GSPs, given in [1] p.221 is essentially different from the definition we gave in section 1.2.

Definition 4.4.1. A **GSP on $\mathcal{A}$** is a mapping $\xi : \Omega \times \mathcal{A} \longrightarrow \mathbb{C}$, such that:
(i) $\xi(\cdot, \phi)$ is a random variable on Ω for all $\phi \in \mathcal{A}$.
(ii) $\xi(\omega, \cdot) \in \mathcal{A}'$ for all $\omega \in \Omega$.

Now we give three definitions of different types of convergences of a sequence of GSPs on $\mathcal{A}$.

Definition 4.4.2. Let $(\xi_n)_{n\in\mathbb{N}_0}$ be a sequence of GSPs on $\mathcal{A}$. Then (ξ_n) is said to **converge in probability** to the GSP ξ, if $\forall\,\varepsilon > 0$ there $\exists\, k \in \mathbb{N}_0$ such that:

$$\lim_{n\to\infty} P\big(\{\omega \in \Omega \mid \sup_{\|\phi\|_k \leq 1} |\xi_n(\omega, \phi) - \xi(\omega, \phi)| \geq \varepsilon\}\big) = 0. \quad (4.36)$$

We symbolize this property with: $\xi_n \longrightarrow_P \xi$.

Definition 4.4.3. Let $(\xi_n)_{n\in\mathbb{N}_0}$ be a sequence of GSPs on $\mathcal{A}$. Then (ξ_n) is said to **converge in mean** to the GSP ξ, if there $\exists\, k \in \mathbb{N}_0$ such that:

$$\lim_{n\to\infty} \int_\Omega \sup_{\|\phi\|_k \le 1} |\xi_n(\omega,\phi) - \xi(\omega,\phi)|\, dP(\omega) = 0 \tag{4.37}$$

We symbolize this property with: $\xi_n \longrightarrow_1 \xi$.

Definition 4.4.4. Let $(\xi_n)_{n\in\mathbb{N}_0}$ be a sequence of GSPs on $\mathcal{A}$. Then (ξ_n) is said to **converge almost surely** to the GSP ξ, if there $\exists$ a set $Z \in \Sigma$ with $P(Z) = 0$ and for $\omega \in \Omega\backslash Z$ the sequence $\xi_n(\omega,\cdot)$ converges to $\xi(\omega,\cdot)$ in the weak sense.

Furthermore we note the following bounded versions of Definition 4.4.2. and 4.4.3.

Definition 4.4.5. Let $(\xi_n)_{n\in\mathbb{N}_0}$ be a sequence of GSPs on $\mathcal{A}$. Then (ξ_n) is said to **converge boundedly** in probability respectively mean to the GSP ξ, if:
(i) $\xi_n \longrightarrow_P \xi$ respectively $\xi_n \longrightarrow_1 \xi$.
(ii) There $\exists$ a set $Z \in \Sigma$ with $P(Z) = 0$ and for $\omega \in \Omega\backslash Z$ the sequence $\xi_n(\omega,\cdot)$ is bounded.
We symbolize these properties with: $\xi_n \longrightarrow_P^b \xi$ respectively $\xi_n \longrightarrow_1^b \xi$.

Remark 4.4.6. One can show, that condition (ii) of the previous Definition is equivalent to:
(ii') $\forall\, \varepsilon > 0$ there $\exists$ a set $B \in \Sigma$ with $P(B) \ge 1-\varepsilon$ and a $k \in \mathbb{N}_0$, independent of n, such that $|\xi_n(\omega,\phi)| \le k\|\phi\|_k \quad \forall \omega \in B,\ \phi \in \mathcal{A}$.
In the following we will always use this condition.

Finally we note several facts using the concepts we have introduced. For the different proofs we refer to [1].

Theorem 4.4.7. Let (ξ_n) denote a sequence of GSPs on $\mathcal{A}$. If $\xi_n \longrightarrow_P^b \xi$ then $\forall\, n \in \mathbb{N}_0$ there $\exists$ (in view of (ii')) a sequence $(c_m^n)_{m\in\mathbb{N}_0}$ of random variables on Ω, such that:

$$\xi_n(\omega,\phi) = \sum_{m=0}^{\infty} c_m^n(\omega)(\psi_m,\phi) \qquad \omega \in B,\ \phi \in \mathcal{A}$$

and $\left(\sum_{m=0}^{\infty} |c_m^n(\omega)|^2 |\tilde{\lambda}_m|^{-2k_0}\right)^{\frac{1}{2}} < k_0, \quad \omega \in B$. Then $\forall\, \delta > 0$ the following holds:

$$\lim_{n\to\infty} P\{\omega \in B \mid \big(\sum_{m=0}^{\infty} |c_m^n(\omega)|^2 |\tilde{\lambda}_m|^{-2k_0}\big)^{\frac{1}{2}} > \delta\} = 0.$$

Remark 4.4.8. If we assume, that there $\exists$ a $k \in \mathbb{N}_0$ such that $\forall\, p \in \mathbb{N}$ there $\exists$ a $B_p \in \Sigma$ with $P(B) \geq 1 - \frac{1}{p}$, so that we have

$$\xi_n(\omega, \phi) = \sum_{m=0}^{\infty} c_m^n(\omega)(\psi_m, \phi) \qquad \omega \in B_p,\ \phi \in \mathcal{A}$$

and $\left(\sum_{m=0}^{\infty} |c_m^n(\omega)|^2 |\tilde{\lambda}_m|^{-2k}\right)^{\frac{1}{2}} < k, \quad \omega \in B_p$ and cosequently

$$\lim_{n\to\infty} P\{\omega \in B_p \mid \big(\sum_{m=0}^{\infty} |c_m^n(\omega)|^2 |\tilde{\lambda}_m|^{-2k}\big)^{\frac{1}{2}} > \delta\} = 0,$$

then the converse of Theorem 4.4.7. is also true.

Theorem 4.4.9. Let (ξ_n) denote a sequence of GSPs on $\mathcal{A}$. If $\xi_n \longrightarrow_P^b \xi$ then $\forall\, n \in \mathbb{N}_0$ there $\exists$ (in view of (ii')) for each $m \in \Lambda := \{n \in \mathbb{N}_0 \mid \lambda_n = 0\}$ a sequence $(c_m^n)_{m\in\mathbb{N}_0}$ of random variables on Ω and $\forall\, k \geq k_0$ a sequence of functions X_n^k on $\Omega \times 1$, such that:

$$\xi_n(\omega, \phi) = \int_I X_n^k(\omega, t)\mathcal{R}^k \phi(t)\, dt + \sum_{m\in\Lambda} c_m^n(\omega)(\psi_m, \phi) \qquad \omega \in B,\ \phi \in \mathcal{A}$$

with $\|X_n^k(\omega, \cdot)\|_{L^2} < k, \quad \omega \in \Omega$ and $\|X_n^k(\omega, \cdot)\|_{L^2} \longrightarrow_P 0$. Then $\forall\, \delta > 0$ the following holds:

$$\lim_{n\to\infty} P\{\omega \in B \mid \big(\sum_{m\in\Lambda} |c_m^n(\omega)|\big) > \delta\} = 0.$$

Remark 4.4.10. Using some further assumptions, the converse of 4.4.9. is also true (cf. [1] Theorem 4.4.). Note, that all these results also hold for convergence in mean.

Bibliography

[1] Z. Crvenkovic. Convergence of a sequence of Generalized Random Processes on the Zemanian Space. *Zb. Rad. Prirod. - Mat. Fak. Ser. Mat*, 19,2:219–232, 1989.

[2] J. L. Doob. *Stochastic Processes.* Wiley Classics Library; A Wiley-Interscience Publication. John Wiley & Sons Ltd., New York etc., Paperback edition edition, 1990.

[3] H. G. Feichtinger. The minimal strongly character invariant Segal algebra, II.

[4] H. G. Feichtinger. Un espace de Banach de distributions tempérées sur les groupes localement compacts abéliens. *Compt. Rend. Acad. Sci. Paris, Ser.A*, 290(17):791–794, 1980.

[5] H. G. Feichtinger. The Banach space of translation bounded quasimeasures, 1980.

[6] H. G. Feichtinger. On a new Segal algebra. *Monatsh. Math.*, 92:269–289, 1981

[7] H. G. Feichtinger. An elementary approach to the generalized Fourier transform. In T. Rassias, editor, *Topics in Mathematical Analysis*, pages 246–272. World Sci.Pub., 1989.

[8] H. G. Feichtinger and W. Hörmann. Harmonic analysis of generalized stochastic processes on locally compact abelian groups, 1990

[9] H. G. Feichtinger and N. Kaiblinger. Quasi-interpolation in the Fourier algebra. *J. Approx. Theory*, 144(1):103–118, 2007

[10] H. G. Feichtinger and G. Zimmermann. *S_0 - A Banach Space of test functions.* Preprint.

[11] I. M. Gelfand. Generalized random processes. *Dokl. Akad. Nauk SSSR*, (100):853–856, 1955

[12] I. M. Gelfand and J. Wilenkin. *Verallgemeinerte Funktionen Distributionen IV.* Number 50 in Hochschulbücher f. Math. VEB Deutscher Verlag der Wissenschaften, Berlin, 1964.

[13] C. C. Graham and B. M. Schreiber. Bimeasure algebras on LCA groups. *Pacific J. Math.*, 115:91–127, 1984.

[14] W. Hörmann. Stochastic Processes and Vector Quasi-Measures. Master's thesis, University of Vienna, July 1987.

[15] W. Hörmann. *Generalized Stochastic Processes and Wigner Distribution.* PhD thesis, University of Vienna, (AUSTRIA), 1989.

[16] K. Itô. Stationary random distributions. *Mem. Coll. Sci. Univ. Kyoto, Ser. A*, 28:209–223, 1954.

[17] N. Kaiblinger. Approximation of the Fourier transform and the dual Gabor window. *J. Fourier Anal. Appl.*, 11(1):25–42, 2005.

[18] H. Niemi. Stochastic processes as Fourier transforms of stochastic measures. *Ann. Acad. Sci. Fenn., Ser.AI*, 591:1–47, 1975.

[19] H. Niemi. On stationary dilations and the linear prediction of certain stochastic processes. *Comment. Phys.-Math.*, 45(4):111–130, 1975.

[20] H. Niemi. On orthogonally scattered dilations of bounded vector measures. *Ann. Acad. Sci. Fenn., Ser.AI*, 3:44–51, 1977

[21] H. Reiter. *Classical Harmonic Analysis and Locally Compact Groups.* Clarendon Press, Oxford, 1968.

[22] H. Reiter and J.D. Stegeman. *Classical Harmonic Analysis and Locally Compact Groups. 2nd ed.* Clarendon Press, 2000.

[23] W. Rudin. *Fourier Analysis on Groups.* Interscience Publishers, New York, London, 1962.

[24] D. Werner. *Functional Analysis. (Funktionalanalysis) 5., Erweiterte Aufl.* Springer-Verlag, Berlin, 2002.

Appendix A

Abstract (in German)

Verallgemeinerte stochastische Prozesse (VSP) können auf verschiedenste Art und Weise eingeführt werden. In dieser Arbeit verwenden wir den Ansatz von W. Hörmann, der in seinen Arbeiten [8, 14, 15] diesen Begriff als beschränkte, lineare Operatoren vom Testfunktionenraum S_0 in einen beliebigen Hilbert Raum definiert hat. Es zeigt sich, dass die Nutzung dieser Segal Algebra, die oft als Feichtinger Algebra bezeichnet wird, sehr viele Vorteile hat. Da S_0 invariant unter der Fourier transformation ist, existiert zu jedem verallgemeinerten stochastischen Prozess automatisch auch dessen Spektral Prozess. Dies stellte etwa in den Arbeiten [11, 16] einen grossen Nachteil dar. Weiters lässt sich die Feichtinger Algebra, welche auch ein Banach Raum ist, einfach für lokalkompakte abelsche Gruppen definieren, sodass man sich nicht nur auf $\mathbb{R}^d$ beschränken muss. Das Ziel der vorliegenden Arbeit wird sein, dass wir einige Approximations Eigenschaften von VSP studieren. Das wird im abschliessenden vierten Kaptitel geschehen.
In Kapitel 1 werden wir zunächst einige Notationen festlegen, die im folgenden Text Verwendung finden. Im zweiten Abschnitt dieses Kapitels werden wir den Weg von gewöhnlichen Stochastischen Prozessen zu verallgemeinerten genau skizzieren. Weiters werden wir kurz auf die Arbeiten von H. Niemi [18, 19, 20] eingehen, die richtungsweisend für die Arbeiten von W. Hörmann waren. Am Ende des Kapitels definieren wir einige, für die weiteren Überlegungen, sehr wichtige Begriffe.
Das zweite Kapitel beinhaltet die vorbereitende Theorie für die weitere Arbeit. Schon im 1. Abschnitt wird die Kovarianz Distribution eingeführt und charakterisiert. Dieser Begriff wird einer der wichtigsten während der ganzen

Arbeit sein. Im nächsten Teil folgt ein Bericht über Spektral Prozesse, welche wie bereits erwähnt aufgrund der Eigenschaften von S_0 immer existieren. Wir beschliessen das Kapitel mit Abschnitten über stationäre Prozesse und weiters werden wir die Zusammenhänge von VSP mit stochastischen Prozessen beziehungsweise mit Vektor Maßen notieren. Dies ermöglicht uns bereits bekannte Resultate dieser Theorien auf neue und sehr einfache Weise zu beweisen.
Der erste Teil des dritten Kapitels betrifft gefilterte VSP. Dieser Begriff aus der Signal Analyse kann auf einfache Weise mittels adjugierter Operatoren auf VSP angewendet werden und wird eine wichtige Rolle im ersten Teil von Kapitel 4 spielen. Im 2. Abschnitt beschäftigen wir uns mit Verallgemeinerungen von stationären VSP, indem wir die Begriffe V - Beschränktheit und harmonisierbare VSP studieren. Das Kapitel wird durch einen Bericht über die "Ausdehnungstheorie" von VSP abgerundet.
Im vierten, dem Hauptkapitel dieser Arbeit, werden wir zunächst ein bereits bekanntes (cf. [15] p. 44) Resultat beweisen, das besagt, dass jeder V - beschränkte VSP durch harmonisierbare approximiert werden kann. Dieses Resultat ist deswegen bemerkenswert, da es V - beschränkte VSP gibt die aber nicht harmonisierbar sind. Im 2. Abschnitt werden wir neue Konvergenzresultate, bewiesen in [9], über Schoenberg Operatoren und die allgemeineren Quasi Interpolations Operatoren miteinbeziehen. Wir werden mittels diesen neue Resultate über VSP entwickeln, die besagen, dass bestimmte, natürliche Bedingungen an eine erzeugende Funktion punktweise Konvergenz der Kovarianz eines beliebigen VSP ergeben. Im dritten Teilabschnitt rechtfertigen wir, dass diese Resultate auch für Spektral Prozesse gelten. Weiters werden wir eine alternative Definition von S_0 angeben, die Fourier Invarianz beweisen und am Ende kurz eine Idee, die in [10] gefunden wurde skizzieren. Der letzte Teil besteht aus einem zusätzlichen Bericht über Konvergenz Eigenschaften von VSP auf dem Zemanian Raum. Dieser Bericht stellt eine eigene Theorie innerhalb dieser Arbeit dar und soll lediglich Zusatzinformation für interessierte Leser sein.

Appendix B

Curriculum vitae

Persönliche Daten:

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1999 - 2002 kaufm. Angestellter bei Firma Adler Optik in Krems/Donau
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