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Valuing Interest Rate Derivatives using Monte Carlo Simulation

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(Martin Grasslober Bakk.)

Acknowledgments

As we express our gratitude, we must never forget that the highest appreciation is not to utter words, but to live by them. (J. F. Kennedy)

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$c(t, T, s)$	Price of a call option at time t, that matures at time T on a derivative that matures at time s
dz	Wiener process in continuous time
ε	Standard normally distributed increments of a Wiener process in discrete time
$f(t, s)$	Interest rate for an infinitesimal small period at time s observed at time t
$F(0, T)$	Current forward price for an asset that matures at time T
K	strike price of an option
λ	Market price of risk
$N()$	is the cumulative standard normal distribution
$p(t, T, s)$	Price of a put option at time t, that matures at time T on a derivative that matures at time s
$P(t, s)$	Price of a zero coupon bond at time t with maturity s
$\phi(\mu, \sigma)$	Value of the normal distribution with an expected value of μ and a standard deviation of σ
ψ	Relative price of an asset
$r(t)$	Interest rate for an infinitesimal small period at time t
r_{rf}	Risk free rate for a given period
$\overline{r}$	arithmetic mean of the short rate for a given period
$R(t, s)$	Continously compounded spot rate for the period between time t and time s
R_{Cap}	Cap rate for a given period
φ	Payoff of a derivative

1 Introduction

Monte Carlo (MC) methods are widely-used in finance, in particular for the valuation of a multiplicity of exotic financial products. In contrast to derivatives on stocks and currencies, the valuation of interest rate derivatives includes some peculiarities. These special characteristics are discussed and implemented in this thesis.

The goal of this thesis is to study and implement MC methods for the valuation of interest rate derivatives. Due to the wide range of different interest rate models, the scope of this thesis is restricted to Gaussian short rate models. Hence, I will price bonds, bond options and cap agreements according to the Vasicek model, the Ho-Lee model and the Hull-White model, by applying MC methods. As all three models are calibrated to market data, the resulting prices will be compared with market data. Furthermore, I will examine path-dependent interest rate derivatives briefly. Exemplary, I will present the valuation of a periodic cap instrument based on the Hull-White model.

This thesis is structured as follows; In Section 2, I will present the main characteristics of interest rates, as they form the underlying variables of interest rate derivatives. Furthermore, I will discuss the most important interest rate sensitive products and survey the markets in which these derivatives are traded. At this point, it is especially accounted for the Austrian market. In order to apply MC methods, stochastic processes describing the underlying variables have to be defined. Hence, I will present basics on stochastic processes in Section 3. Section 4 discusses the most important concept of derivative pricing, namely risk neutral valuation. This concept is crucial, as it provides a general framework to work with. In Section 5, models for the short rate are presented and discussed. Thus, I will go beyond Gaussian models in order to study the different approaches. In addition, I will generally examine how to determine the corresponding model parameters. After discussing the underlying short rate models, pricing methods are described in Section 6. Besides the Black model, which is based on the well known Black-Scholes-Merton model, numerical methods are presented in this section. Hence, I will discuss MC methods, lattice methods and finite difference methods. In Section 7, I will consolidate all acquired acknowledgements and value cap agreements based on to the Vasicek model, the Ho-Lee model and the Hull-White model using MC methods. As these short rate models have been calibrated to market prices, I will compare the simulated prices with observed ones. Moreover, I will price a periodic cap agreement after reviewing its main characteristics. In addition to this path-dependent interest rate derivative, some other important ones are examined in Section 8.

2 Interest Rates and Interest Rate Derivatives

According to Hull [1], derivatives can be described as financial contracts, whose values depend on the price of the object of purpose, the underlying. For example, the price for an option to buy a ton of corn in one year depends on the price of corn. The value of this derivative -the option- therefore depends on the price of corn in one year. The option will only be executed, if the corn spot price in the future is above the predefined price, the strike price. In the case of interest rate derivatives, the underlying variables are interest rates. Hence the prices for these contracts depend on the underlying interest rates[1]. The option on buying one ton of corn enables the buyer of this contract to hedge against facing a too high spot price in one year. The main purpose of interest rate derivatives is to hedge oneself against interest rates that are too high, too low or too volatile etc. In accordance to this, Obst and Hinter [2] note that the possibility of hedging against extreme price and interest rate fluctuations are the main goals of future and option markets. Since the 1960ies, interest rates in general are much more volatile and the levels that are reached are quite often above the ones before the 1960ies. Reißner [3] notes that the increased volatility was due to the rising inflation in the 1960ies, to the relaxation of interest rate regulations and also to the breakdown of the Bretton-Woods-System. Thus, interest rate derivatives became more and more popular, whereas they boomed in the 1980s and 1990s.

Besides hedgers, whose aim is to insure themselves against movements of interest rates, stock prices, exchange rates, etc., that influence their financial situation negatively, there are two other types of actors in derivative markets, namely speculators and arbitrageurs. Speculators have no position they want to hedge. It is their goal to profit from expected changes in prices. In fact, speculators play an important role in derivative markets as they are willing to except risk in order to make profit. As a hedger might want to buy the right to sell one of his assets in the future for a predefined price, there must be another trader who is willing to enter this contract. In most cases, the counterparts in such situations are speculators. Whereas the

[1] As it will be concretized in Section 2.2, this also covers bonds as possible underlying variables.

hedger expects that the price of the underlying asset will decrease, the speculator bets on increasing prices for the underlying asset. The latter tries to profit from the lower price assumed by the hedger. He buys the asset according to the contract specifications and tries to sell the asset for the expected higher market price in the future. As a result, speculators are needed to make hedging possible as they are willing to take risk hedgers do not want to carry[2]. Arbitrageurs are participants in the futures and option markets that are only trying to capitalize differences in prices on different markets. Therefore, an arbitrageur might buy an asset in one market, in order to sell it for a higher price in another market. Arbitrage is mostly possible because of differing information levels of the respective market participants.

2.1 Interest Rates

Before discussing derivatives on interest rates, it is necessary to be more specific about the underlying. In general, money can be invested over different periods, at different interest rates [3]. Plotting the interest rates against time gives the so called term structure or yield curve. The ECB (European Central Bank) provides daily spot rates based on European central government bonds[4]. Figure 1 displays the term structure according to bond prices for the 26^{th} of March 2009. The underlying data are taken from the ECB web site [5]. As the spot rates are based on the market prices of zero coupon bonds, as indicated in Equation (1), spot rates can not be computed for an arbitrary time to maturity. As a result, the missing spot rates have to be estimated, whereas proper estimation techniques are still widely discussed.

Zero coupon or pure discount bonds return a single unit of cash flow at the maturity date without paying anything in-between. Thus, observing the price of a zero coupon bond at time t that matures at time s can be expressed as follows:

$$P(t,s) = e^{-R(t,s)(s-t)} \tag{1}$$

[2]It would also be possible that such a contract is concluded by two hedgers, due to differing expectations. Such situations are rather rare, as Hull [4] notes.

[3]On a small scale, this means that it is possible to establish several saving accounts for different time horizons, at different interest rates.

[4]All of these bonds have to have an AAA rating and have to fulfill several criteria published on the ECB web site [5].

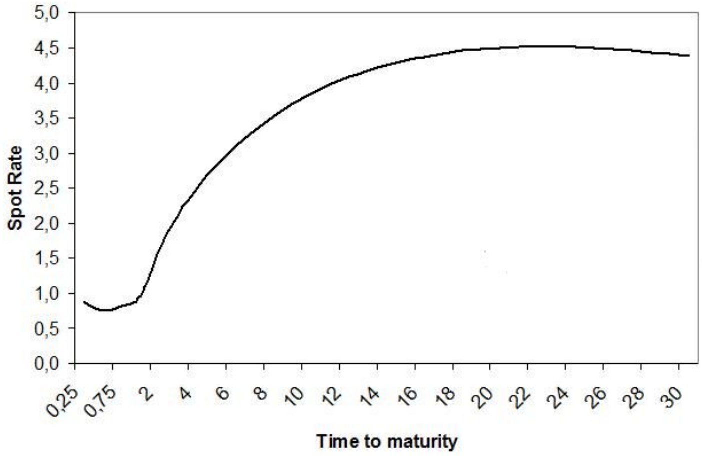

Figure 1: Yield curve for the 26^{th} of March, based on data provided by the ECB

After rearranging Equation (1), the corresponding spot rate $R(t,s)$ can be written as:

$$R(t,s) = -\frac{\ln P(t,s)}{(s-t)} \tag{2}$$

For these representations a continuously compounded spot rate $R(t,s)$ is assumed. The time to maturity $(s-t)$ is of special interest as it also incorporates the day count convention. This convention clarifies for how many days an invested amount of money is compounded per year. Due to different usances in markets there are several different day count convention types. Hull [1] presents the following three exemplary conventions:

$$\frac{Actual}{Actual \quad in \quad period} \qquad \frac{Actual}{360} \qquad \frac{30}{360}$$

Actual indicates the number of days that actually go by. Thus the actual number of days per year is assumed. In some markets it is common to assume trading days only. Hence, the number of days per year when assets are traded on exchanges are assumed. In Austria the average number of trading days is about 252 per year. For simplicity all upcoming simulations and calculations will be performed by applying the $\frac{30}{360}$ convention. Consequently, it is assumed that every month comprises of 30 days and one year of 360 days. Hence, challenges due to differing numbers of trading days are avoided by this selection of day count convention type.

A crucial concept, which is on the bottom of valuing interest rate derivatives using MC simulation, is the theoretical concept of the short rate $r(t)$. This is the yield on an instantly maturing bond, therefore:

$$r(t) = \lim_{t \to s} R(t, s) \tag{3}$$

As a result, the short rate represents the interest rate at time t for an infinitesimal small period. It has to be emphasized that the short rate is only a theoretical concept. Hence, it can not be observed in the market. In accordance to the short rate, the instantaneous forward rate $f(t, s)$ can be introduced. This rate is denoted in terms of the forward curve as it represents the interest rate for an infinitesimal small time period at time s observed at time t. The instantaneous forward rate therefore represents the interest rate for an infinitesimal small period in the future (beginning at time s), whereas this interest rate is determined ahead at time t. Thus, $f(t, t)$ and $r(t)$ are equivalent. Combining the definition of the instantaneous forward rate and Equation (2) gives the following representation of the spot rate:

$$R(t, s) = \frac{1}{(s - t)} \left(\int_t^s f(t, \tau) d\tau \right) \tag{4}$$

As a result, the instantaneous forward rates can be deduced from observed discount bond prices as in Equation (1) by setting

$$f(t, s) = -\frac{\partial}{\partial s} \ln P(t, s) \tag{5}$$

It follows that the term structure can be determined by the instantaneous forward curve as well as by future short rates. As already mentioned, these interest rates are not observable in the market, but they form crucial theoretic elements in the context of modelling the term structure. In accordance to Equation (4) and by assuming a market without any arbitrage possibilities, the interest payment for a given period has to be the same irrespective whether an overall interest rate or several interest rates for an arbitrary number of subperiods are applied. Figure 2 depicts a period of length $T_1 + T_2 + T_3 = T$. The T_i, for $i = 1, 2, 3$, represent the length of equal sized subperiods. The r_i, for $i = 1, 2, 3$, indicate the continuously compounded interest rates for the three subperiods. Thus, the value of a zero coupon bond that pays

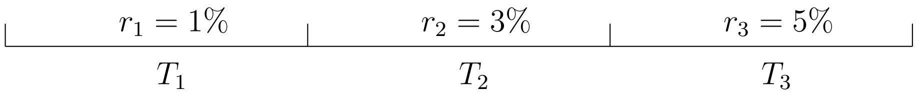

Figure 2: Representation of continuously compounding

one Euro after T has to equal:

$$P(0,T) = 1 \cdot e^{-0.01 \cdot T_1} e^{-0.03 \cdot T_2} e^{-0.05 \cdot T_3}$$

As the subperiods in the example of Figure 2 are all equally sized, the zero coupon bond price at time zero simplifies to:

$$P(0,T) = 1 \cdot e^{-0.09 \cdot T_i} = 1 \cdot e^{-0.09 \cdot T_i \frac{T}{T}} = 1 \cdot e^{-\bar{r}T} = 1 \cdot e^{-0.03 \cdot 3} = 0.9139$$

As a result, one would have to pay 91 Cents at time zero in order to receive one Euro at time T. $\bar{r}$ represents the arithmetically averaged interest rate for the whole period. Thus, observing the interest rates for an arbitrary number of sub periods, facilitates calculating the interest rate for the overall period as the arithmetic average of the subperiods interest rates. As this also has to hold for infinitesimal small periods, the continuously compounded one year rate for example can be deduced from all the short rates within the upcoming year. Hence, by simulating the short rate the whole term structure can be determined. This already shows the importance of the short rate. In order to achieve realistic yield curves via simulation, so called short rate models will be introduced in Section 5. As mentioned in the introduction, in this thesis short rate models are of main interest in order to narrow the wide range of modelling approaches.

After having discussed some basics on interest rates, financial instruments that depend on interest rates or bonds are introduced in the next section.

2.2 Interest Rate Sensitive Financial Instruments

As mentioned in Section 2, the prices of interest rate derivatives depend somehow on interest rates. This rather unspecific definition will be now concretized by discussing the main interest rate sensitive instruments. This discussion is geared to the representations of Branger and Schlag [6].

2.2.1 Unconditional Contracts

Investing in shares includes the participation in the profit and the loss of the company. As this investment might be too risky for some investors or as it might be unfavorable for the issuer, governments and companies provide *bonds*. From a theoretical point of view a bank deposit can also be interpreted as a bond. The bank costumer buys a theoretical bond issued by the bank when putting his money on an account and is payed out when closing the account. Disregarding the default risk, the main advantage of a basic bond is the certainty of all payments, as all coupon payments[5] and the nominal value are determined when the bond is issued. This is done by defining the number of coupon payments and by determining their amount, which is usually represented in percentages of the nominal value. *Zero Coupon bonds* repay the nominal value and the compensation for lending the money to the company or the government at maturity at once. As a result, zero coupon bonds do not imply any interest rate depending risk. If fixed coupon payments are settled, it seems rather controversial that interest rate depending risk is present. The interest rate depending risk follows from the coupon payments that can be reinvested until the date of maturity. As these payments can be reinvested at variable interest rates the price for the bond varies. Another possibility when issuing bonds are coupon payments that depend on variable interest rates, as for example the LIBOR[6] (London Interbank Offered Rate). Such bonds are called *floaters*. According to the conception, the coupon payments directly depend on a reference rate. The valuation of a bond also has to incorporate the default risk of the issuer. It might be possible that companies or governments can not afford the coupon payments and/or the nominal value at the date of maturity.

[5]Coupon payments are payments that compensate for lending the money to the government (Treasury bonds) or companies (Corporate bonds).

[6]The LIBOR is a floating reference rate, which is determined by the trading of deposits between banks on the Eurocurrency market. LIBOR reference rates are quoted for maturities ranging from over night and one year.

Forward Rate Agreements (FRA's) are similar to zero coupon bonds. The significant difference is that Forward Rate Agreements define the assessment of a certain amount of money for a period that starts at a certain point in the future and not immediately, as when investing in bonds. As the payments at maturity are determined beforehand, the prices for Forward Rate Agreements do not depend directly on varying interest rates. The interest rate sensitivity relies on the possibility of speculation by betting on a certain evolution of the interest rate.

Swaps are agreements on exchanging cash flows in the future. In the fundamental case, an investor agrees on paying a predetermined cash flow sequence in order to receive variable cash flows that depend on a reference rate such as the LIBOR. Reitz, Schwarz and Martin [7] note that Plain Vanilla interest rate swaps, which are equivalent to the ones just described, are the classical instrument to hedge against the risk of changing interest rates. The authors argue that this was the reason for the increasing popularity of these contracts. Nowadays, the number of different swaps is therefore unmanageable, they cite.

Forward contracts determine the exchange of an asset at a certain time in the future for a predefined price. These contracts can be individually established by the exchanging parties. As a result such contracts cannot be easily traded on an exchange. Forwards are mainly traded OTC[7].

Future contracts are very similar to forward contracts. Differently to forwards, future contracts are traded mainly on exchanges. This difference in the initiation of the contract brings along a crucial restriction for future contracts. The exchanging parties are no longer able to specify certain articles of agreement, as all future contracts have to be standardized in order to ensure smooth trading. Although this might seem to be a huge drawback, futures are popular as it is much easier to find an exchange partner. Moreover, Beike and Schlütz [8] note that it is nearly impossible to do not find a counterpart for a future contract at the EUREX (Future exchange established in a cooperation by the 'Deutschen Terminbörse' and the 'Swiss Options and Financial Futures Exchange)[8]. Beside the advantage of easy matching,

[7]OTC stands for Over-The-Counter, which means that the buying and the selling parties are in direct contact with each other to specify the details of the contract.

[8]Beike and Schlütz [8] note that this is mainly due to the Market Maker System at the EUREX,

future contracts imply no default risk, as both parties need to have a so called margin account. These accounts guarantee every counterpart the completion of the contract, as changes in the claims have to be booked daily to these margin accounts. Concluding the differences between futures and forwards, it can be said that futures imply lower transaction costs by passing on the possibility of individual arrangements of the contracts.

2.2.2 Conditional Contracts

All the instruments presented up to now are based on an unconditional execution of the contract. The parties in these contracts have not got the possibility to decide whether the exchange is conducted or not. Differently to these contracts, the instruments discussed in this section include the possibility for one party to decide upon the execution. Pricing such instruments is therefore more complicating and goes beyond discounting future cash flows. Although stochastic modelling of the interest rate dynamics is already needed when pricing swaps and futures. In this work, only contracts that include the possibility of deciding upon the execution will be discussed, as there is no need to use rather complicated models to price rather simple derivatives. Especially, when pricing forwards there is no need to assign a specific term structure in advance. Hence, I will focus on the valuation of conditional contracts. The most popular OTC interest rate options and therefore conditional contracts according to Hull [4] are: bond options, interest rate caps/floors and swap options[9].

In general, *options* constitute the right, but not the obligation to sell or buy an asset at a predefined point in time for a predefined price. The object of purchase is called the *underlying* and the predefined price the *strike price*. To complete the introductory terminology for options, the right to buy the underlying is constituted by a *call* option, where a *put* option[10] entitles to sell the underlying. Options, irrespective of the underlying, can be classified by their date of execution. *European*

where some participants have committed themselves to always take the counterpart, for reasonable contracts.

[9]For swap options, the abbreviated form *swaption* is also common.

[10]It has become common to use the abbreviations call and put, thus to omit 'option'.

options represent the simplest form of all options, as the holder has got the right to sell or buy the underlying at only one determined date in the future, the *maturity* or *expiration* date. *American* options incorporate the right for the holder to sell or buy the underlying at any point in time until the maturity date. As a result, the holder of an American option can exercise his option within a whole period instead of at a single point in time. American style options therefore provide more freedom in reacting on price changes of the underlying. *Bermudan* options are in the middle of European and American options, just like the Bermudan islands are. Bermudan options incorporate the right to sell or buy the underlying at several predefined points in time.

Besides the classification on the basis of execution dates, options can also be grouped according to their payoff function. Plain Vanilla options describe 'basic' options. They are traded mainly on exchanges, therefore the prices for these products are quoted regularly. The payoff φ of a Plain Vanilla option only depends on the price S of the underlying at maturity T and the strike price K. Therefore the payoff function for a call option can be written as:

$$\varphi(S_T) = \max\left(S_T - K, 0\right) = \begin{pmatrix} 0 & , & \text{if } S_T < K \\ S_T - K & , & \text{if } S_T \geq K \end{pmatrix} \tag{6}$$

For a put option the payoff function looks like:

$$\varphi(S_T) = \max\left(K - S_T, 0\right) = \begin{pmatrix} K - S_T & , & \text{if } S_T \leq K \\ 0 & , & \text{if } S_T > K \end{pmatrix} \tag{7}$$

In the case of a call option a rational investor would not execute the option, if the market price of the underlying is below the predefined strike price. A put option would not be executed, if the market price of the underlying is above the strike price[11]. For pricing Plain Vanilla options there exist comprehensive closed formulas that were presented by Fischer Black [9]. For Exotic options on the other hand, more complicating models are necessary.

[11]These considerations neither take into account the payed price of the option beforehand, nor dues that arise because of the transaction.

Exotic options are options that were modified with respect to their payoff function or their conditions of execution. The largest group of exotic options are the path-dependent ones. Differently to European, American or Bermudan options, the payoff function of e.g. *Asian* options depend on the average price of the underlying within a predefined period. As a result, the price of the underlying has to be investigated for the whole specified period. The payoff of *lookback* options also depends on the price of the underlying before maturity. The payoff of a lookback call when exercised, is the final price of the underlying, minus the minimum price of the underlying till maturity. For a lookback put the payoff is the maximum price of the underlying till maturity, minus the final price of the underlying at maturity. Another modification to standard options are incorporated by *knock-in* and *knock-out* options. The payoff of these derivatives depends on the fact, whether the price of the underlying has exceeded or undershot a specific level. These options are also called barrier options. As there are many possible conditions (combining barriers for example) the underlying price might have to fulfill the variety of these derivatives is immense.

Bond options incorporate the right to sell or buy a bond at a predefined point in time for a certain price. As a result, the payoff function for zero coupon bond options is equivalent to Equation (6) and Equation (7) by replacing $S(T)$ with $P(T, s)$. As Jamshidian [10] showed, options on coupon paying options can be interpreted as a portfolio of options on pure discount bonds as the ones in Equation (1). Concerning the issuer, Beike and Schlütz [8] note that government bonds are favored by investors. Corporate bonds on the other hand are irrelevant. They provide the following reason for Germany: The market for bonds issued by banks is not that liquid and there are nearly no corporate bonds. The interest for such options is therefore quite small. As government bonds, especially the ones issued by the USA, Germany, Japan and Great Britain are quite popular, options on these bonds are quite common. Furthermore, Beike and Schlütz state that bond options are favored for bonds where the maturity date is far away in the future. The latter fact might be explained with the argument that prices for bonds that last longer vary much more than those which mature in the near future. If the bond for example matures in 30 years the prices might vary much more, as people have more different expectations about the future. These expectations also include a potential default risk which will be incorporated in the prices of bonds, and which will then influence the option price. In theory, the default risk for government bonds is said to be zero, as it can be assumed that all governmental financial commitments are secured by tax income.

Some bonds include the right for the issuer to pay back the issued bond for a fixed price before the bond expires. In this case such *callable bonds* incorporate a call option on the bond. Bonds are issued to increase debenture capital. Callable bonds are issued as it might be possible that the debt can be payed back earlier. As the discounted value of the payoffs of the bond decreases with time, the prices the issuer would have to afford decrease as well. Bonds with call features generally offer higher yields than bonds without a call feature, as investors have to be compensated for a possible early payback. Of course the right to sell the bond earlier can also be embedded. In this case the bond is called a *puttable bond.* The holder of a puttable bond has bought the bond as such, as well as a put option on this bond that allows her to sell the bond at a given date for a given price. As puttable bonds incorporate the possibility of selling the bond at a predefined price before it matures, prices for such bonds are usually lower. This can be justified with the lower risk such bonds carry, as the bond can be disbursed before its maturity date. Hull [4] states, that a five-year fixed-rate deposit with the possibility of an immediate account closing can be seen as an embedded put option on a bond[12], as it contains an American put option on the bond. On the other hand, Hull [4] states that mortgages or loans that include the right to pay back the loan before it is due can be interpreted as they include a call option on the loan. This argument arises by assuming that a loan disbursed by the bank is the same as a bond issued by the debtor sold to the bank. As a result, if the debtor pays back his bond earlier by paying back his loan he executes a call option on the bond.

In contrast to loans where the interest rate is determined for the whole life-span of the loan, there are loans with flexible interest rates. In the latter case the interest rates and the cost for the debtor are adjusted to a predefined market spot rate, such

[12] It was already stated in Section that a bank deposit can be interpreted as a bond.

as the LIBOR or the EURIBOR[13]. Interest rates for loan and deposit contracts are usually not adjusted daily. Moreover, they are matched to spot rates in predefined time intervals such as three months. An *interest rate cap* (cap in short), which is a top-selling contract according to Reißner [3], can be used in such cases to impose a maximum of interest that has to be paid at these adjustment dates. As a result, a cap rate is defined. This rate is the highest interest rate that has to be afforded by the debtor. Thus, the debtor can hedge herself against the increase of the interest rate of his loan above the cap rate. Hull [11] stresses the practical issue that if a loan and a cap on that loan are provided by the same company, the value of the cap is already included in the charged interest rate. If that is not the case, the debtor has to afford the value of the cap agreement separately. In such a case, a cap does not reduce the interest rate payments for the debtor, but it compensates the debtor for the higher liabilities. If a bond with variable interest payments (which is in general the same as a loan, as mentioned earlier) that depend on the LIBOR pays 3.5% in three months is assumed, the debtor has to afford $0.25 \cdot 0.035 \cdot 100\ 000 = 875$ Euros, given a nominal value of 100 000 Euros. If a cap rate of 3% is assumed, the debtor would have to afford 750 Euros only, as the cap agreement compensates her for the 0.5 percentage points above the cap rate. An important feature of caps and floors is that the compensation for the differing interest rates is not payed at the reset days (the days when the interest rates are adjusted), but when the period for which the interest rate is adjusted expires (after three months, in the example above). Assuming a cap agreement with a volume of one unit of currency, with a cape rate of R_{Cap}, between the times t and s on the interest rate $R(t, s)$ being the realized interest rate for the very same period, whose length is denoted as $\Delta\tau = s - t$ gives a payoff at s of:

$$\Delta\tau \ max(R(t, s) - R_{Cap}, 0)$$

This formulation is geared to the representations of Clewlow and Strickland [12]. In order to receive the value of this cap at time t, this payoff has to be discounted. To discount the payoff, the spot rate $R(t, s)$ has to be assumed and the payoff of the

[13] The EURIBOR (Euro Interbank Offered Rate) is as the LIBOR an interest rate on debt between banks. The EURIBOR is calculated on the basis of interest rates offered by representative European banks for maturities from one month to one year.

cap at time t is then given by:

$$\frac{\Delta\tau}{1+R(t,s)\Delta\tau}\ \max\left(R(t,s)-R_{Cap},0\right)$$

which is equivalent to

$$(1+R_{Cap}\Delta\tau)\ \max\left(\frac{1}{1+R_{Cap}\Delta\tau}-\frac{1}{1+R(t,s)\Delta\tau},0\right)$$

Thus, an option that caps the interest rate at R_{Cap} between t and s is equivalent to $1+R_{Cap}\Delta\tau$ European put options with an exercise price of $\frac{1}{1+R_{Cap}\Delta\tau}$ on a discount bond with a face value of one unit of currency. In reality a cap agreement does not comprise of a single period of compensation, there are several of such periods. As a result, the summed up values of all *caplets*, as the agreements for the single periods are called, yields to the price of the whole cap agreement. Thus, the value of a cap agreement is a portfolio of European put options on a series of discount bonds. This result will not be important for the simulations, but it will be important when calibrating the investigated models to market data, as presented in Section 5.3. Figure 3 from Hull [11] shows the interest rate that a debtor would have to account for when repaying a Floating-rate loan combined with a long position in a cap rate agreement. Differently to cap agreements, *floor* contracts assure a compensation for

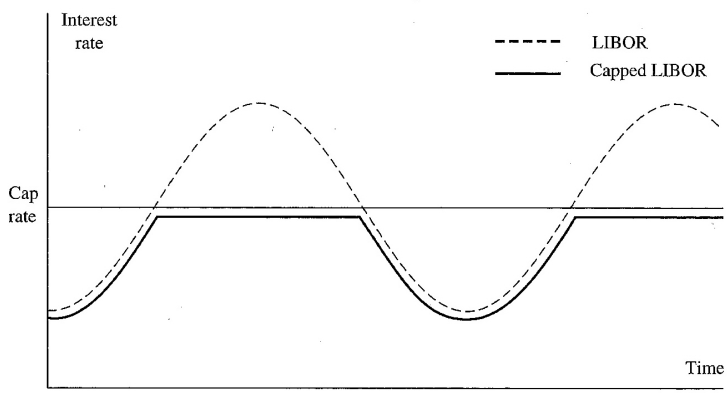

Figure 3: Borrower's effective interest rate with a floating- rate loan and an interest-rate cap (Presented by Hull [11])

interest payments that are below a certain rate, the floor rate. Hence, by holding a floor one can hedge oneself against too low interest payments that will be received.

The payoff function in t can be analogously written to the one of a cap as:

$$\frac{\Delta\tau}{1 + R(t,s)\Delta\tau} \ \max\left(R_{Floor} - R(t,s), 0\right)$$

which can be reformulated as

$$(1 + R_{Floor}\Delta\tau) \ \max\left(\frac{1}{1 + R(t,s)\Delta\tau} - \frac{1}{1 + R_{Floor}\Delta\tau}, 0\right)$$

Thus, a *floorlet* can be interpreted as $(1 + R_{Floor}\Delta\tau)$ European call options with an exercise price of $\frac{1}{1+R_{Floor}\Delta\tau}$ on a one unit paying bond expiring at time s.

Mixtures of caps and floors are called *collars*. These contracts ensure that the considered interest payments are always within a band, bounded by the cap and the floor rate. A collar is therefore a combination of a long position in a cap and a short position in a floor agreement. Hull [1] cites that collars are usually established such that the price of the cap equals the price of the collar. The cost for entering a collar is equal to zero in this case. The efforts for the collar therefore result from compensations that are due to interest rates below the floor rate.

As already stated in Section 2.2.1, swaps are contracts that convert variable into fixed interest payments. Differently to a forward swap, where a company has to fulfill the swap contract, *swap options* or *swaptions* in short, enable the company to decide, whether or not to execute the swap. As a result, if a company has to pay back a loan while the variable interest payments are high, the company can enter the swap contract; but it does not have to. Thus, if the variable interest payments are low the company can profit from the low market interest rates. Hull [11] gives the following example as an application of such swaptions: A company that will enter a five year loan in half a year is assumed. The assumed loan incorporates variable interest payments. In order to hedge against too high variable interest payments the company can buy a swaption. In this case the company would receive the variable interest payments from the swaption counterpart. These variable payments can then be used to pay back the loan. On the other hand, the company has to afford fixed interest payments to compensate the swaption counterpart. As swaptions provide the right, but not the obligation of execution, the company will only execute the swaption, if the variable interest rate is above the swap rate. In this example the swaption is called a *put-* or *payer-swaption*, as its holder has to afford the predefined fixed interest payments. If the holder of a swaption receives the predefined fixed interest payments, the swaption is called a *call-* or *receiver swaption*. An example for the latter agreement is presented by Beike and Schlütz [8]. If a fund manager is confronted with high fluctuations in interest rates he can buy a receiver swaption. This receiver swaption ensures her to receive at least a certain fixed interest. As a result, the fund manager will only execute the swaption, if the variable interest payments are below the fixed ones. A swap can be interpreted as the exchange of a bond with variable interest payments and a bond with a predetermined fixed interest payment. As a result, a swaption can be seen as an option on the exchange of a fixed interest paying bond and the nominal value of the swap. Thus, if a swap allows its holder to pay a fixed amount of interest and to receive a variable one, this contract can be seen as a put option on a fixed interest paying bond with a strike price equal to the nominal value of the swap. A call option on a fixed interest paying bond, with a strike price of the nominal value of the swap can be assumed, if the holder of this call option has the right to pay variable interest and to receive fixed interest payments.

As the features of swaptions and caps and floors look fairly the same, the difference shall be discussed briefly. In a cap and floor agreement the underlying interest rate is compared with the predefined cap or floor rates at several predefined times and whether the contract is a floor or a cap the option is executed. In the case of swaptions the interest payments are settled for a whole period. Therefore the underlying interest rate and the strike are compared only once[14]. As a result, it is possible to hold a cap that comprises of caplets with different strike prices and therefore provides differing cap rates. A payer swaption on the other hand guarantees only one swap rate and one payoff.

[14]It can be assumed that the holders of swaptions already include their expectations of the evolution of the underlying interest rate in the future.

2.3 Markets for Interest Rate Derivatives

According to Branger and Schlag [6], the most important derivative exchanges are the CBOT (Chicago Board of Table), the LIFFE (London International Financial Futures Exchange) and the EUREX in Frankfurt. Trading a derivative on an exchange is only possible, if the derivatives are standardized. Otherwise, there would be a large variety of different products that are not traded, as the date to maturity or the underlying does not fit the needs of the investors. As Hull [11] notes, the most popular interest rate options traded on exchanges are those on Treasury bonds futures, Treasury note futures and Eurodollar futures. All these contracts are highly standardized, very often demanded and can therefore be easily traded on exchanges.

On the other hand, there are completely customized products, such as swaptions and caps and floors. These contracts cannot be standardized as the needs of the specific investor have to be met. If the company from Section 2.2.2 wants to hedge itself against too high interest payments because of a floating rate loan, the counterpart for such an agreement would be a bank. In order to fit the needs of the company a swaption will be provided that ensures fixed interest payments of a specific volume for a specific period. As a result, such contracts can only be dealt Over The Counter (OTC). If the company would wish to resell the contract instead of using it, it would be rather burdensome to find a counterpart who is interested in this very specific agreement. The secondary market for such products will therefore also take place OTC.

Now the interest derivative market in Austria will be briefly highlighted. In 1991 the ÖTOB (Österreichische Termin- und Optionen Börse) was founded. The ÖTOB was then integrated in the Wienerbörse AG and is now part of the latter as the segment derivatives market.at. Nowadays, investors can buy futures and options on Austrian and central eastern European stocks and indices [13]. As a result, there is no Austrian exchange that supports trading of interest rate derivatives. In fact, in 1993 the trade of futures on Austrian government bonds (AGB) was started at the former ÖTOB. Up to September 1996, when the trading of AGB-options started, the AGB-Futures were the only interest rate derivatives traded at

	Interest Rate Swaps	Interest Rate Options	Forward Rate Agreements
1995	196	2	1927
1998	2080	113	1133
2001	2205	70	1962
2004	9338	288	3912

Table 1: Average daily turnover in millions of US Dollar at the Austrian derivative market (based on data from the Austrian Federal Bank [16])

the ÖTOB. Then AGB-options were introduced, which incorporated the right to sell or buy the underlying government bond at any point in time, as the option was American style. Trading on AGB-options was enabled by Sal Oppenheim Jr. & Cie. KGaA, as this independent private bank entered the ÖTOB as a General Clearing member to handle the exchange of AGB-options. For the ÖTOB the AGB-options were introduced in order to supplement its offer [14]. In fact, the ÖTOB ended its ambitions for the AGB-options already after 13 months. AGB-Futures on the other hand were traded from 1993 until 1999. The termination of trading the AGB-Futures also determines the termination of interest rate derivatives at an Austrian exchange [15]. Although there is no Austrian Exchange for interest rate derivatives anymore, Austrian banks are very active in trading them. Transactions for such contracts are carried out OTC or on foreign exchanges. To underpin the activities of Austrian banks some results of the Triennial Central Bank Survey 2004 [16] are presented. The whole survey, which is also known as the BIS-Survey, is carried out in 50 different countries. Its aim is it to gather information about the turnovers of foreign exchange contracts and contracts for derivatives. Thus, 13 private banks selected by the Österreichischen Nationalbank were surveyed. These 13 banks were investigated as they were responsible for 98% of the turnovers for derivative contracts in Austria. Table 1 presents the average daily turnovers for interest rate swaps, interest rate options and forward rate agreements (FRAs). As Table 1 shows, the daily turnovers for the three contracts are not negligible. Although the average daily turnover is rather small, compared with interest rate swaps and FRAs, Austrian banks seem to be fond of interest rate options. It is remarkable that the turnover for interest rate options increased by 400% from 2001 to 2004. These numbers as well as the fact that for example on the 13^{th} of November 2008 1.514.799 future contracts and 251.158 options on fixed income derivatives were traded at the EUREX [17] in Frankfurt should make clear that interest rate derivatives are widely used and that it is worth thinking about proper pricing models.

3 Stochastic Processes

In order to simulate changes of variables using MC techniques, stochastic processes have to be introduced. The presentations in this section are based on the descriptions in Hull [4].

In general, stochastic processes can be defined in discrete or in continuous time. For discrete processes, changes of the variable are only possible at certain points in time. On the other hand, when defining the process in continuous time, changes are possible at any point in time. The basic process employed in this thesis is the so called *Wiener process*[15] or *Brownian Motion*. A Wiener process is a special *Markov process*, thus the following feature is also valid for Wiener processes. In a Markov process the best predictor for the future value of a variable is it's current value. Hence, all previous values of the variable are irrelevant for determining future values. This feature is captured by Markov processes, as consecutive changes are independent of each other. As a result, Markov processes incorporate the weak form of capital market efficiency, as the prices of traded assets already reflect all the information about previous prices. Otherwise it would be possible to predict future prices by analyzing past ones. Hull [4] also notes that the weak form of capital market efficiency should be valid due to the trades in a market. If a certain chart of an asset price would indicate a specific movement of the future price, this movement would be anticipated and the possibility of making a profit out if it would diminish. For the upcoming simulations variables that follow Markov processes are employed. These variables change randomly by $\phi(\mu, \sigma)$ in a given period, where ϕ is the normal distribution, with an expected value of μ and a standard deviation of σ.

3.1 Wiener Processes

In the case of a Wiener process a variable z is assumed whose random change in discrete time can be expressed by $\phi(0, 1)$, in every period. As the consecutive changes in the variable of a Wiener process are independent, the probability distributions are also independent. Thus, for calculating the expected value of the process one

[15] This process is named after the American mathematician Norbert Wiener.

can sum up all expected values, which gives a value of zero. The standard deviation of this process for two periods would be $\sqrt{1+1}$, as the additivity is given for the variance, but not for the standard deviation. Hence, calculating the standard deviation for half a year gives $\sqrt{0.5}$. In order to do this, the changes of the variable z for a given change in time Δt can be written as:

$$\Delta z = \varepsilon\sqrt{\Delta t} \quad \text{with } \varepsilon \sim \phi(0,1) \tag{8}$$

In accordance to this, it can be assumed that the whole period under consideration T can be split up into N equally sized time periods of length Δt. In order to do this, $N = \frac{T}{\Delta t}$ changes are observed. The calculation of the change in z between $t = 0$ and $t = T$ is determined as:

$$z(T) - z(0) = \sum_{i=1}^{N} \varepsilon\sqrt{\Delta t}$$

The expected value of the change over the whole period is again zero and the variance is $N\Delta t = T$. The standard deviation of the change is therefore $\sqrt{T}$.

Figure 4 shows the evolution of two variables. Variable z changes 100 times a year, whereas variable k changes 1000 times a year. As a result the changes in the two variables can be written as:

$$\Delta z = \varepsilon\sqrt{\frac{1}{100}} \quad \text{with } \varepsilon \sim \phi(0,1)$$

$$\Delta k = \varepsilon\sqrt{\frac{1}{1000}} \quad \text{with } \varepsilon \sim \phi(0,1)$$

Thus, the time steps for the evolution of variable z are ten times longer, than the ones for k. Therefore, the possible changes in z are much larger than the ones of k. For the variables z and k two exchange traded goods can be assumed, where the first one is only traded a few times a year and the other one is traded every minute. The price for the first one would only change a few times a year, but the magnitude of the changes might be considerably high as some important factors have changed. As the second good is traded every minute the actual prices already incorporate all available information one minute ago. As a result, the increase of the magnitude of the changes, as the length of the time steps increases, is quite reasonable. The

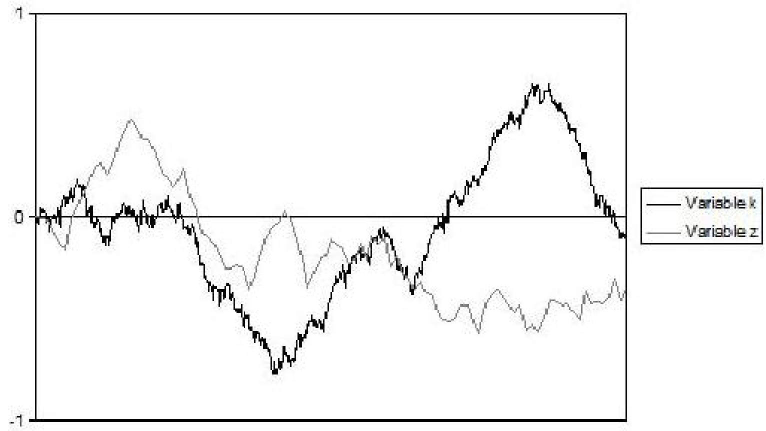

Figure 4: Two stochastic processes with different step length and variance

variables z and k were generated using Microsoft Visual Basic (VBA).

3.2 Generating Normal Random Variables

In order to simulate a variable that changes randomly within $\phi(0,1)$, standard normal pseudo random numbers have to be generated. Clewlow and Strickland [12] state that during MC simulations 30% of the execution time is needed for generating random numbers. Hence, it is worth having a closer look at this part. As C++ only provides a generator for standard uniform random numbers, a transformation has to be applied. Clewlow and Strickland [12] propose three alternatives for generating standard normal pseudo random numbers, when a generator for standard uniform pseudo random numbers is available. The first one is only an approximation, where twelve uniform numbers are generated, summed up and then six is subtracted from the total. In this case twelve standard uniform random numbers have to be generated, in order to receive one standard normal random number. As a result, this procedure is rather inefficient. A more efficient alternative to generate standard normal pseudo random numbers is the Box-Muller transformation. This algorithm was presented by Box and Muller [18] in 1958 and is based on sampling independent standard uniform numbers and projecting them on a circle, whose radius is based on one of these random numbers. Then a random angle between zero and 2π is set using a second random number. This procedure defines two random points at the boundary of the assumed circle using the sine and cosine function. Upon the three methods presented by Clewlow and Strickland, the polar rejection is the most efficient one. The polar rejection is also known as the Marsaglia-Bray algorithm, developed by G. Marsaglia and T. Bray [19]. Their algorithm is a modification of the Box-Muller algorithm. For this method two independent uniform random numbers are necessary in order to generate two standard normal pseudo random numbers. According to Clewlow and Strickland [12], the algorithm for the polar rejection can be written in the following manner:

repeat

$$x_1 = standard\ uniform\ random\ number$$

$$x_2 = standard\ uniform\ random\ number$$

$$w = x_1^2 + x_2^2$$

until $w < 1$

$$c = \sqrt{-2\frac{\ln(w)}{w}}$$

$$z_1 = cx_1$$

$$z_2 = cx_2$$

Comparing the three methods, Clewlow and Strickland [12] conclude that the polar rejection is nearly three times faster than the first alternative and still slightly faster than the Box-Muller transformation. Glasserman [20] notes that avoiding to incorporate an evaluation of the sine and cosine function reduces the computing time noticeable. As the polar rejection is faster and more accurate, this transformation technique will be applied in the upcoming simulations. As the polar rejection has to be implemented separately, the transformation from standard uniform to standard normal random numbers will be named random() in the algorithms in appendix. The algorithm itself is presented in Appendix B.12.

Figure 5 shows the distribution of 10000 values that have been generated according to the polar rejection method. For these random numbers a mean of -0.03 and a standard deviation of 1.07 were found. The shape of the distribution function and the distribution parameters are in favor of the hypothesis that the generated values are standard normally distributed. Furthermore, it was tested, wether the 10.000 values are standard normally distributed or not, using a χ^2 test on distributions.

According to the test statistic, the Null-Hypothesis of standard normally distributed values could not be rejected, even at a significance level of 0.5%. Thus, it can be assumed that the algorithm works well.

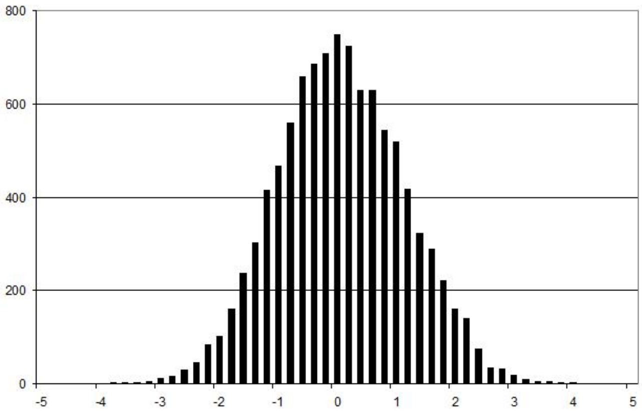

Figure 5: Histogram for 10.000 pseudo random numbers, generated via Visual Basic

3.3 Generalized Wiener Process

Up to now, processes were assumed that do not change deterministically. Accordingly, they did not incorporate a drift a for example, which describes a constant change of the variable. Furthermore, one can modify the standard deviation of the process by multiplying the increments of the Wiener process, $\Delta z = \epsilon\sqrt{\Delta t}$, by a constant b. Such a process with dz[16] defining a Generalized Wiener process in continuous time can be written as:

$$dx = a\,dt + b\,dz \tag{9}$$

For small changes in time the change in the variable x can also be written as:

$$\Delta x = a\Delta t + b\varepsilon\sqrt{\Delta t} \tag{10}$$

Figure 6 shows four different Wiener processes. $dx = dz$ describes a standard Wiener process without a drift parameter and with a standard deviation equal to

[16] In order to define a Wiener process in continuous time, the step length Δt (as in Equation (8)) has to become infinitesimal small.

one for the observed period. $dx = 2\,dz$ on the other hand, describes the increasing magnitude of the random changes by two. The variance of the process is therefore increasing by four. Moreover, one can claim a deterministic drift in the process, by increasing the variable x by 0.1 per time increment. This is described by the process $dx = 0.1\,dt + 2\,dz$. $dx = 0.1\,dt$ describes the situation where x does not rely on any random changes. In both cases the expected value of the change in the variable x is no more equal to zero. In the case of an investigated period of length one, the expected value of the changes in x would be equal to 0.1. As it will be shown later

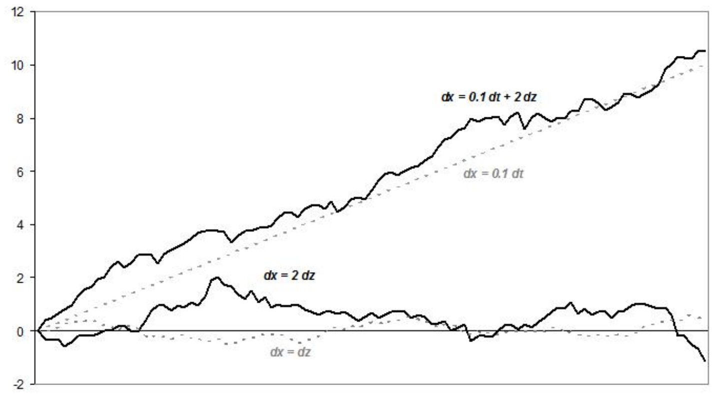

Figure 6: General Wiener processes with a=0.1 and b=2

on, a and b in Equation (9) and (10) might depend on the variable x and on time, thus, such a process looks like:

$$dx = a(x,t)\,dt + b(x,t)\,dz$$

In this case a and b change with t and x. These processes are called *Itô-Processes* and are of special interest when pricing derivatives applying MC simulations, as it will be shown in the next section.

4 Pricing Interest Rate Derivatives

An elementary question that has to be answered is: Why do we need pricing formulas for interest rate options, as there are market prices for them? Reitz, Schwarz and Martin [7] present the following arguments: Although prices for interest rate options result from the demand and supply on exchanges and between banks, it is necessary to calculate theoretical prices for these products. These theoretical prices are needed, as it is not possible to observe market prices for all products. Furthermore, these theoretical prices are necessary to analyze the determinants of these derivatives plainly, in order to calculate potential changes in market prices and to hedge oneself against these changes.

Before presenting valuation methods for interest rate derivatives (see Section 6), two fundamental concepts have to be introduced. The concept of risk neutral valuation constitutes a conceptual framework for pricing derivatives, irrespective of investors' risk preferences. Martingales on the other hand, are necessary when defining stochastic processes of variables in such a risk neutral framework.

4.1 Risk Neutral Valuation

The most important concept for pricing derivatives is the one of Risk Neutral Valuation. For the time being the Risk Neutral Valuation will be discussed in the light of stock prices, as the concept is more intuitive in this case. The presented approach is based on the findings of Cox, Ross and Rubinstein [21].

A stock, whose current price is S_0 and an option on this stock that matures in T, which is worth h today is assumed. The price of this stock might increase or decrease until T. In order to this, the stock price at T, might be S_0u, if the stock price increases and S_0d, if it decreases. Where u and d represent the percentage change of the stock price plus 100%. In T the option has a payoff which is set equal to h_u, if the stock price increased and h_d, if it decreased. Now a portfolio is considered that consists of Δ shares and a short position in an European call option. As a result, the right to buy a certain amount of shares at T for a given price is sold. Thus, the portfolio will be worth $S_0 u\Delta - h_u$ in T, if the stock price increases and $S_0 d\Delta - h_d$, if the stock price decreases. Setting the values of this portfolio equal gives:

$$S_0 u\Delta - h_u = S_0 d\Delta - h_d$$

or equivalently

$$\Delta = \frac{h_u - h_d}{S_0 u - S_0 d} \tag{11}$$

As stated above, Δ represents the amount of shares in the portfolio. As the payoffs for an upward and downward movement are equal in the case of setting Δ equal to (11), the payoff of the portfolio is risk free. At this point, it has to be assumed that there are no arbitrage opportunities. Therefore, it is not possible to invest zero today and receive a positive amount tomorrow with a positive probability. In accordance to that, portfolios with the same payoff at a future date have to have the same price today. In the case of the portfolio that consists of Δ shares and the short position in an European call option on the same shares, there is no risk at all. According to the assumption that there are no arbitrage opportunities, this portfolio can only return the risk free rate r_{rf}. Δ therefore ensures that the value of the portfolio is the same, no matter if the stock prices rise or decline. As a result it can be stated that the value of the risk free portfolio is equal to $S_0 u\Delta - h_u$, irrespective of the evolution of the stock price. Assuming that there are no arbitrage opportunities, the discounted value (using the risk free rate) of the portfolio in T, must be equal to the cost for setting up the portfolio today.

$$S_0\Delta - h = (S_0 u\Delta - h_u)e^{-r_{rf}T}$$

Inserting Δ from Equation (11) and rearranging this equation gives:

$$h = e^{-r_{rf}T}[ph_u + (1-p)h_d]$$

where

$$p = \frac{e^{r_{rf}T} - d}{u - d}$$

The term p can be interpreted as a probability in this case. In accordance to this, p is known as the risk neutral probability. The risk neutral probabilities are computed without making any assumptions about the real world probabilities of up- or downward moves of the stock price. This is due to the fact that the current prices of the shares should already include all future considerations about the evolution of the price. The only assumptions that have to be made concern the possible changes in the stock price and the strike price. The risk free rate can be observed in the market. Thus, the value of the option can be deduced from the changed stock price, the strike price and the time to maturity T.

For the upcoming deviations $\widehat{\mathbb{E}}$ will denote the expected value in a risk neutral world, as described. In such a world all investors are indifferent towards risk. Hence, there is no need to compensate them for the risk they are taking. As already mentioned before, the return of every portfolio can only be the risk free rate r_{rf}. The expected return of the stock in a risk neutral world is simply the risk free rate[17]. Therefore one can write $\widehat{\mathbb{E}}(S_T) = S_0 e^{r_{rf}T}$. Applying p and (1-p) as the risk neutral probabilities for the up and downward movement of the price, it follows that the investment will only pay the risk free rate. Concluding these findings Hull states:

> 'In a risk-neutral world all individuals are indifferent to risk. In such a world investors require no compensation for risk, and the expected return on all securities is the risk free interest rate. [...] This result is an example of an important general principle in option pricing known as risk-neutral valuation. The principle states that we can assume the world is risk neutral when pricing an option. The price we obtain is correct not just in a risk-neutral world, but in the real world as well.' [4]

The concept of Risk Neutral Valuation is incorporated in the famous Black-Scholes-Merton model through the stochastic differential equation, which can be deduced from the evolution of the stock price (presented in Equation (38)) they assumed. By deriving the price for a derivative on the share, the expected value of the share price is cancelled out.

[17] This can be also shown by computing the expected return of the stock using the risk neutral probabilities p.

4.2 Martingale Measures

For the discussion of the martingale measure it will be followed the representation in Hull [1]. In Section 4.1 it was assumed that the risk free interest rate is constant over time. In fact, interest rates change over time (The yield curve that is presented in Figure 1 for example, only represents the spot rates at the 26^{th} of March 2009). In Section 4.1 it was shown, that the expected returns of all securities in a risk neutral world have to equal the risk free rate and that future payoffs can be discounted using the risk free rate. Thus, how can these changes in interest rates be described?

The unifying characteristics of derivatives is the dependence on an underlying variable, the price of a share or an interest rate for example. To model these variables (indicated as θ) one can assume that they follow a stochastic process such as:

$$\frac{d\theta}{\theta} = m\,dt + s\,dz \tag{12}$$

dz again indicates a Wiener process. m indicates the mean of the changes or the drift. As it was already discussed in Section 3, multiplying the increments of the Wiener process by a certain value, changes the volatility of the process in the same proportion, as the increments of the Wiener process are standard normally distributed. Furthermore, it is assumed that m (the expected value of θ) and s (the volatility of θ) only depend on time and θ.

Now two derivatives g_1 and g_2 on θ are assumed, which follow the processes:

$$\frac{dg_1}{g_1} = \mu_1\,dt + \sigma_1\,dz \quad \text{and} \quad \frac{dg_2}{g_2} = \mu_2\,dt + \sigma_2\,dz \tag{13}$$

μ_1, μ_2, σ_1 and σ_2 are functions of θ and t. The two functions in Equation (13) represent the percentage change of the derivative prices in a continuous setting. In a discrete setting the absolute changes of the derivatives' prices would look like:

$$\Delta g_1 = \mu_1 g_1 \Delta t + \sigma_1 g_1 \Delta z \quad \text{and} \quad \Delta g_2 = \mu_2 g_2 \Delta t + \sigma_2 g_2 \Delta z$$

In all the presented formulas the only source of uncertainty lies in the increments of the Wiener process dz. Eliminating these increments by building a portfolio Π of the two derivatives leads to an invested value of $\sigma_2 g_2$ in derivative g_1 and $-\sigma_1 g_1$ in derivative g_2. The value of this portfolio Π can therefore be written as:

$$\Pi = (\sigma_2 g_2) g_1 - (\sigma_1 g_1) g_2$$

or

$$\Delta\Pi = (\sigma_2 g_2)\Delta g_1 - (\sigma_1 g_1)\Delta g_2$$

As this portfolio is risk free (the increments of the Wiener process cancels out) and due to the assumption that there are no arbitrage possibilities, the return of this portfolio has got to equal the risk free rate r_{rf}. Therefore the following has got to hold:

$$\Delta\Pi = (\sigma_2 g_2)\Delta g_1 - (\sigma_1 g_1)\Delta g_2 = r_{rf}\Pi\Delta t$$

or equivalently

$$\Delta\Pi = (\sigma_2 g_2)(\mu_1 g_1 \Delta t + \sigma_1 g_1 \Delta z) - (\sigma_1 g_1)(\mu_2 g_2 \Delta t + \sigma_2 g_2 \Delta z) = r_{rf}\Pi\Delta t$$

which simplifies to

$$\frac{\mu_1 - r_{rf}}{\sigma_1} = \frac{\mu_2 - r_{rf}}{\sigma_2}$$

The left and the right hand side of the last equation represent the market price of risk for θ. This price represents how much risk one has to take in order to increase the return of the asset that depends on θ. Moreover, this shows that for two derivatives with the very same underlying, the market price of risk for this underlying has to be same irrespective of the two derivatives. Defining the market price of risk of θ as λ gives:

$$\frac{\mu - r_{rf}}{\sigma} = \lambda \quad \text{or equivalently} \quad \mu = r_{rf} + \lambda\sigma \tag{14}$$

Thus, the mean return of a derivative is defined by the risk free rate plus one part that depends on the market price of risk and the volatility of the derivative.

A martingale is a stochastic process that has a drift equal to zero. A variable θ

follows a martingale, if the evolution of θ can be written as

$$d\theta = \sigma\, dz$$

where dz is again a Wiener process. As there is no drift parameter, the expected value of the variable θ_T for all $T \geq 0$ has got to equal the initial value θ_0, by the law of large numbers. This is due to the fact that the increments of a Wiener process are standard normally distributed.

Assuming two assets that only depend on one source of uncertainty with the prices g and k, one can define the relative price ψ of g in terms of k as $\psi = \frac{g}{k}$. Thus, the asset price k is used as a numeraire, the value of g is represented in terms of k. Now it is assumed that the market price of risk is equivalent to the standard deviation of the second asset k, $\lambda = \sigma_k$. According to Equation (14), μ, the expected value of a derivative, which depends on θ and t, therefore equals $\mu_k = r_{rf} + \sigma_k^2$, for a derivative k and $\mu_g = r_{rf} + \sigma_g \sigma_k$, for a derivative g. Combining these assumptions with Equation (13) defines the changes of the derivative prices for g and k as:

$$dg = (r_{rf} + \sigma_k \sigma_g) g\, dt + \sigma_g g\, dz$$

$$dk = (r_{rf} + \sigma_k^2) k\, dt + \sigma_k k\, dz$$

These two formulas imply that the changes in the asset price for g and k also depend on the current absolute values of the prices. In this case it is useful to apply the natural logarithm. Applying Itô's Lemma (A derivation of Itô's Lemma using some results from differential calculus is presented in Appendix A.), where $a(x,t)$ and $b(x,t)$ from the Appendix are equal to $(r_{rf} + \sigma_k \sigma_g) g$ and $\sigma_g g$ for the variable g gives:

$$d \ln g = \left(\frac{1}{g} (r_{rf} + \sigma_k \sigma_g) g - \frac{1}{2g^2} \sigma_g^2 g^2 \right) dt + \frac{1}{g} \sigma_g g\, dz$$

which simplifies to

$$d \ln g = \left(r_{rf} + \sigma_k \sigma_g - \frac{\sigma_g^2}{2} \right) dt + \sigma_g\, dz \tag{15}$$

Similarly, applying Itô's Lemma to the process of the variable k gives:

$$d \ln k = \left(r_{rf} + \frac{\sigma_k^2}{2} \right) dt + \sigma_k \, dz \tag{16}$$

By subtracting Equation (16) from Equation (15) gives:

$$d(\ln g - \ln k) = \left(\sigma_k \sigma_g - \frac{\sigma_g^2}{2} - \frac{\sigma_k^2}{2} \right) dt + (\sigma_g - \sigma_k) \, dz$$

which is equivalent to

$$d \left(\ln \frac{g}{k} \right) = -\frac{(\sigma_g - \sigma_k)^2}{2} \, dt + (\sigma_g - \sigma_k) \, dz$$

Applying once again Itô's Lemma, as indicated in Appendix A, with the exponential function as the function G, $-\frac{(\sigma_g - \sigma_k)^2}{2}$ as a and $(\sigma_g - \sigma_k)$ as b gives:

$$d \left(\frac{g}{k} \right) = \underbrace{-\frac{g}{k} \frac{(\sigma_g - \sigma_k)^2}{2} + \frac{g}{k} \frac{(\sigma_g - \sigma_k)^2}{2}}_{0} \, dt + \frac{g}{k} (\sigma_g - \sigma_k) \, dz$$

As a result the process of $\frac{g}{k}$ has no drift and follows a process like $d\theta = \sigma \, dz$. Thus, $\frac{g}{k}$ is a Martingale as defined above. As the best predictor for a Martingale is the initial value of the process, the expected value of $\frac{g}{k}$ has to equal:

$$\frac{g_0}{k_0} = \mathbb{E}_k \left(\frac{g_T}{k_T} \right) \quad \text{or equivalently} \quad g_0 = k_0 \mathbb{E}_k \left(\frac{g_T}{k_T} \right) \tag{17}$$

Where the indices indicate the initial values and the values at the maturity date T. The expectation function $\mathbb{E}_k$ stands for the expected value in the case of a risk neutral world in terms of k. Hence, for calculating the expected value not the 'real world' probabilities are used, but the risk neutral ones. In the case of assuming the standard deviation σ_k, as the market price of risk, one calls this measure a forward risk neutral measure with respect to k. From Equation (17) it follows that the initial price of the asset g can be calculated by describing the evolution of g and k in a risk neutral world without making assumptions about the real probabilities of an up- or downward movement of the prices for the two assets. As a bank account is also a tradable asset, it can also be applied as a numeraire. In this case the initial value k_0 can be set equal to one. The changes in the value of a bank account are described by the interest rate $R(t, s)$, the value of the bank account therefore increases with every time step by $r(t)$. As it is assumed that a bank account does not bear any risk, σ_k equals zero. By writing the value of the bank account at the time to maturity T as $k_T = exp\left(\int_0^T r(t)\,dt\right)$, the initial value of a derivative g can be defined as:

$$g_0 = \widehat{\mathbb{E}}(e^{-\overline{r}T} g_T) \tag{18}$$

$\overline{r}$ indicates the mean value of the short rate $r(t)$ over the whole period and $\widehat{\mathbb{E}}$ indicates the expected value in the classic risk neutral world. An interest rate derivative can therefore be priced by simulating the evolution of the short rate $r(t)$ in a risk neutral setting. In this case the price of the derivative at the time to maturity g_T is calculated[18] and discounted using the mean of the short rate for the random path, as it was proposed in Figure 2 and the accompanying calculations in Section 2.1. Pricing for example a bond in this classic risk neutral world can be carried out by

$$P(t, s) = \widehat{\mathbb{E}}\left[exp\left(\int_t^s r(t)\,dt\right)\right]$$

Thus, one can simulate the short rate $r(t)$ in the classic risk neutral world and is then able to calculate several different payoffs according to the simulated short rate paths and use the mean value of these payoffs, as the price for this derivative. In order to apply MC methods, it is necessary to describe the evolution of the variables of interest as stochastic processes. Thus, for the simulation of interest rates it is necessary to define a stochastic process that describes the changes in the short rate such that the resulting term structure at least resembles the actual one. In the next section such models will be presented, the short rate models.

[18] The price of this derivative might depend on the value of the short rate at a certain point in time or at the whole path that is described by the risk neutral evolution of it.

5 Short Rate Models

The short rate (or instantaneous interest rate) $r(t)$, introduced in Section 2.1, can be interpreted as the interest rate for an infinitesimal small time period. In Section 2.1 it was also shown that in the case of continuous compounding the interest rate for a period that comprises of several subperiods is determined as the average interest rate of all subperiods. As this is also valid for the short rate, stochastic processes in terms of the short rate can be defined to simulate term structures. There are several proposals for processes the short rate might follow. This thesis concentrates on Equilibrium and No-Arbitrage models, as Hull [1] calls them. Figure 7, which was deduced from Clewlow, Strickland [12], summarizes all common approaches for valuing interest rate derivatives. The distinction Hull made for the models is equivalent to the categories 'Traditional term structure models' for Equilibrium models and 'Equilibrium term structure volatility models' for No-Arbitrage models. The 'Fit term structure volatility models' are beyond the scope of this thesis in order to concentrate on a narrow clipping of a vast topic. The 'Model bond prices' models on the other hand will be discussed briefly in Section 6.1.

5.1 Equilibrium Models

Hull [4] describes equilibrium models as models that are based on assumptions about economic variables and that derive a process for the short rate, $r(t)$. These models then explore what the assumed process for $r(t)$ implies for bond and option prices.

The theory of interest rate modelling is based on the findings of Vasicek [22]. In general short rate models assume that changes in the short rate depend on the short rate's mean a (the drift) and its variance b^2 (or diffusion function), which both depend on the level of $r(t)$ and time t. This assumption is equivalent to the presentation in Section 3. Thus, changes in the short rate can be represented as:

$$dr = a(r,t)\,dt + b(r,t)\,dz \tag{19}$$

This is equivalent to say that the short rate $r(t)$ follows a continuous Markov process.

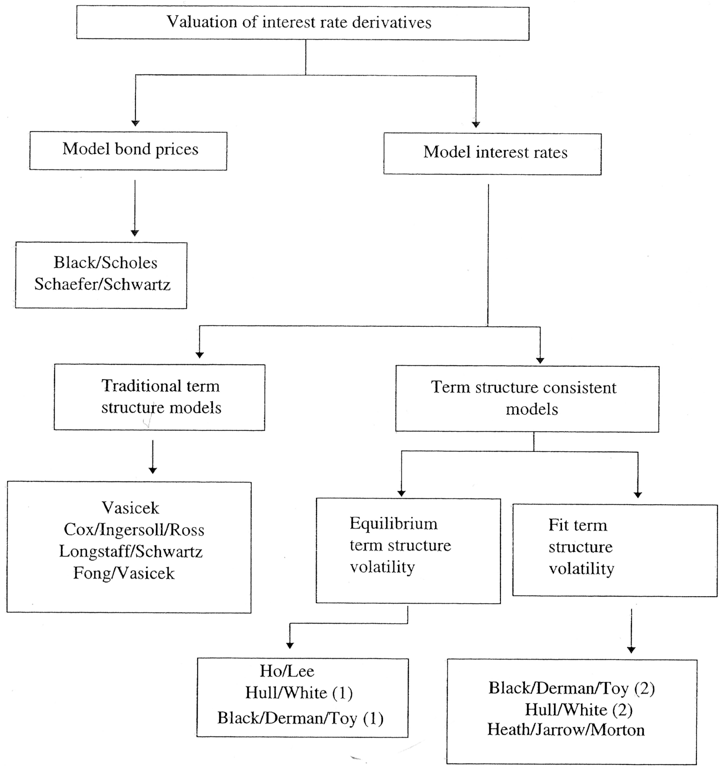

Figure 7: Representation of the possibilities to value interest rate derivatives (deduced from Clewlow, Strickland [12] p. 189)

The short rate process is therefore characterized by a single state variable, namely its current value. It has to be noted, that this is one out of three assumptions made by Vasicek. The second assumptions states that the price of a discount bond is determined by the process of the short rate over the bond's time to maturity. This ensures that the interest paid for a certain investment has got to be the same no matter whether the whole period is divided in infinitesimal small periods or the period is examined as one (see Figure 2 in Section 2.1). The last assumption made by Vasicek concerns the market. Vasicek assumed that the market is efficient. Thus, there are no transaction cost, information is available for all investors simultaneously and every investor acts rationally. This assumption ensures that investors have homogenous expectations and that no profitable risk-free arbitrage is possible.

As mentioned before, the price of a pure discount bond is assumed to depend on the short rate $r(t)$. For the derivation of the risk neutral interdependence between the term structure and the short rate, Vasicek did not assume any specific function. In a classical risk neutral world (as discussed in Section 4.1), where no investor has to be compensated for taking risk, the dependence between the short rate and bond prices can be described as follows:

$$P(t,s) = \widehat{\mathbb{E}}[e^{-\overline{r}(s-t)}]$$

where $\widehat{\mathbb{E}}$ is the risk neutral expected value. Thus, the discount bond price can be interpreted as the expected value of a function of $r(t)$. Applying now Itô's lemma as presented in Appendix A, while the process of $r(t)$ is similar to the one for x in Equation (46) and the bond price $P(t,s,r(t))$ will represent the continuously differentiable function G, shows that the bond price satisfies a stochastic differential equation like:

$$dP = P(\mu(t,s,r(t))\,dt - P\sigma(t,s,r(t))\,dz \tag{20}$$

$\mu(t,s,r(t))$, $\sigma(t,s,r(t))$ are the mean and the variance of the instantaneous rate of return at time t, on a bond with maturity time s. Considering now an investor who issues an amount of W_1 of a bond and simultaneously buys an amount W_2 of another bond, gives a portfolio of these two bonds that is now worth $W = W_2 - W_1$. Applying now Equation (20) to the value of this portfolio and assuming that the values W_1 and W_2 are set proportional to $\sigma(t,s_2)$ and $\sigma(t,s_1)$[19] shows that the value of the constructed portfolio changes over time according to:

$$dW = W\frac{\mu(t,s_2)\sigma(t,s_1) - \mu(t,s_1)\sigma(t,s_2)}{(\sigma(t,s_1) - \sigma(t,s_2))}\,dt$$

As changes in the value of the portfolio do not depend on the stochastic element dz anymore, the return of the portfolio W has got to equal the risk free rate $r(t)$, as profitable risk-free arbitrage is not possible (Third assumption of Vasicek):

$$\frac{\mu(t,s_1) - r(t)}{\sigma(t,s_1)} = \frac{\mu(t,s_2) - r(t)}{\sigma(t,s_2)}$$

As a result the market prices of risk for the two bonds have to be equivalent. As the times to maturity s_1 and s_2 are set arbitrary, the market price of risk $q(t,r(t))$

[19]It has to be noted that this convention for the values W_1 and W_2 is similar to the selection of numeraire as it was presented in section 4.2.

can also be written as:

$$q(t, r(t)) = \frac{\mu(t, s, r(t)) - r(t)}{\sigma(t, s)}$$

Substituting here the formulas for the mean $\mu(t, s, r(t))$ and $\sigma(t, s, r(t))$ that resulted from applying Itô's lemma, to the bond price formula which lead to Equation (20), gives:

$$\frac{\partial P}{\partial t} + (f + \rho q)\frac{\partial P}{\partial r(t)} + \frac{1}{2}\rho^2 \frac{\partial^2 P}{\partial r(t)^2} - r(t), P = 0 \quad \text{for} \quad t \leq s \tag{21}$$

Vasicek calls Equation (21) the term structure equation, as one obtains bond prices after defining the process of the short rate $r(t)$ and the market price of risk $q(t, r(t))$.

As it was shown in Equation (18), the price of an interest rate derivative, that pays φ_s at time s, is in t equivalent to: $\widehat{\mathbb{E}}[e^{-\bar{r}(s-t)}\varphi_s]$. Where $\widehat{\mathbb{E}}$ is the risk neutral expected value. As above a zero coupon bond is introduced that has a price of $P(t, s)$ in t and pays 1\$ in s, therefore $\varphi_s = 1$. For the case that the market price of risk is equal to zero ($q = 0$), the classical risk neutral world is assumed. Consequently, bond prices can be calculated as

$$P(t, s) = \widehat{\mathbb{E}}[e^{-\bar{r}(s-t)}] \tag{22}$$

The term structure can therefore be deduced from Equation (1) by substituting $P(t, s)$ by Equation (22), thus

$$R(t, s) = -\frac{1}{(s-t)} \ln \widehat{\mathbb{E}}[e^{-\bar{r}(s-t)}] \tag{23}$$

As a result, if the risk neutral process of the short rate is determined the whole structure of $R(t, s)$ is determined as well.

5.1.1 Vasicek Model

Vasicek [22], who presented the derivation of an arbitrage-free price of a derivative as shown in Section 5.1, started his analysis by assuming that the short or instantaneous spot rate follows a stochastic process, which is determined by the increments of a

Wiener process, a drift and a diffusion parameter, as shown in Equation (19). The derivation of an arbitrage-free price followed the ideas of Black and Scholes [23]. Therefore, by constructing a suitable risk-free portfolio Vasicek showed how to range between a measure in the 'real' world and a risk-neutral one. These changes between the measures describe a Girsanov change of measure. As Vasicek showed that this is true for a general setting of Equation (19), the implementation of a special model is based on defining a risk neutral process for the short rate.

Vasicek proposed to assume a constant market price of risk λ. For pricing derivatives, the value of the market price of risk is actually irrelevant, as shown in Section 4.2. If the risk neutral process is defined, derivatives can be priced without having any idea of the value of λ. The value of the market price of risk is only necessary for moving from the real to the risk-neutral measure and vice versa. The risk neutral process Vasicek assumed is equivalent to:

$$dr = \alpha(\gamma - r)\,dt + \rho\,dz \tag{24}$$

The short rate therefore follows an Ornstein-Uhlenbeck process [24]. As a result a drift of $\alpha(\gamma - r)$ and a time independent variance ρ^2 are assumed. γ represents the long-term mean of the short rate and α the speed, at which the short rate r returns to its long-term mean. This feature is know as mean reversion. Thus, the interest rate will return to a long term average level. If r is above this mean reverting level, r will decrease. The opposite will happen, if r is below this level. As already mentioned, α represents the speed at which the short rate returns to the average long-term level. The speed of return is given as a proportional factor per time interval. There are also economic arguments that are in favor of mean reverting interest rates. If the interest rates are too high, it is more expensive to borrow money. Hence, investments will decrease and therefore economic growth will decrease as well. Thus, the demand for money will decrease, which will lead to decreasing interest rates. If interest rates are too low, the demand for money will increase which will then lead to increasing interest rates.

In the Vasicek model the term structure of interest rates and the associated volatility structure are both determined by the model after fixing the constant parameters α, γ, σ and the initial value of the short rate r. Bond prices and the corresponding yields can be therefore verified as:

$$P(t,s) = A(t,s)e^{-rB(t,s)} \tag{25}$$

$$R(t,s) = -\frac{\ln A(t,s)}{s-t} + \frac{B(t,s)}{s-t}r$$

where

$$B(t,s) = \frac{1}{\alpha}(1 - e^{-\alpha(s-t)})$$

$$\ln A(t,s) = \frac{R_\infty}{\alpha}(1 - e^{-\alpha(s-t)}) - (s-t)R_\infty - \frac{\rho^2}{4\alpha^3}(1 - e^{-\alpha(s-t)})^2$$

where

$$R_\infty = lim_{\tau\to\infty}R(t,\tau) = \gamma - \frac{\rho^2}{2\alpha^2}$$

The volatility of the spot rates for maturity s at time t is given by:

$$\rho_R(t,s) = \frac{\rho}{\alpha(s-t)}(1 - e^{-\alpha(s-t)})$$

Thus, by varying the parameters α, γ, σ and r, the yield curve can have several different shapes, as shown in Figure 8. In order to simulate the short rate r according to Equation (24), it is necessary to formulate the process in a discrete setting beginning at $t = 0$. According to the Euler scheme Equation (24) can be written as:

$$r_{t+1} = r_t + \alpha(\gamma - r_t)\Delta t + \rho\varepsilon_{t+1}\sqrt{\Delta t} \tag{26}$$

In the setting presented by Vasicek [22] the coefficients α, γ and ρ are all assumed to be constant over time. Glasserman [20] shows that for this case Equation (24) can be written as:

$$r_{t+1} = e^{-\alpha(\Delta t)}r_t + \gamma(1 - e^{-\alpha(\Delta t)}) + \rho\sqrt{\frac{1}{2\alpha}(1 - e^{-2\alpha\Delta t})}\varepsilon_{t+1} \tag{27}$$

The advantage of using Equation (27) lies in the of the simulation. While Equation (26) is only an approximation, Equation (27) enables one to simulate the the

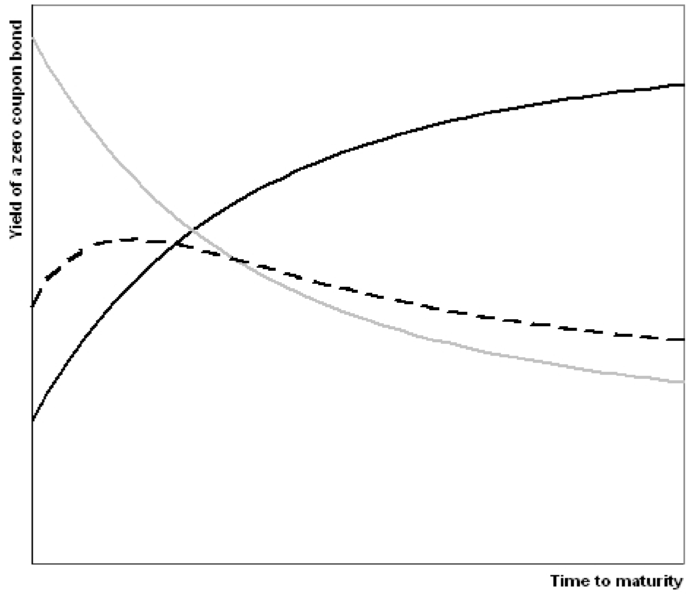

Figure 8: Three different yield curves of a zero coupon bond, due to different combinations of the parameters proposed by Vasicek.

process assumed by Vasicek exactly, without discretization error. For the term structure that will be presented in Figure 14, the average deviation of the spot rates deduced from Equation (26), from the spot rates simulated via Equation (27) is only −0.004 percentage points. Thus, the approximation error is negligible. For all upcoming simulations I will apply exact formulas, if available.

5.1.2 CIR Model

One severe drawback of the Vasicek model is that the short rate might become negative. As a result the interest payed for such an infinitesimal small time period would be negative. Thus, one would have to pay for lending money to someone else. Cox, Ingersoll and Ross [25] therefore presented an adaption of the Vasicek model. By including the square root of the short rate into the diffusion process, they overcame this problem. As a result, the formulation of the short rate process under the CIR model in a risk neutral world looks like:

$$dr = \alpha(\gamma - r)\,dt + \rho\sqrt{r}\,dz \tag{28}$$

The constants α, γ and ρ are the same as in Equation (24). In order to ensure that the short rate cannot become negative, $2\alpha\gamma > \rho^2$ has to hold. An other feature concerns the volatility of the process. The higher the short rate becomes, the more volatile the process will be. Figure 9 contrasts the difference in the processes described by the Vasicek and the CIR model. For these sample paths Euler approximations were used in order to define the processes (24) and (28) in discrete time. The Euler approximation for the CIR model is equivalent to the one for the Vasicek model as in Equation (26), but the diffusion process is multiplied by the square root of the short rate one period ahead. This also led to the sobriquet 'square-root' process of the CIR model. For the two paths the increments of the Wiener process are equivalent for every time step, thus the different shapes are only due to the different diffusion processes. As the short rate is in this case always smaller than 100% the square root term decreases the effect of the diffusion term, thus the path for the CIR model is less volatile. As the long term average short rate is positive and above the initial value of 0.979% the square root term also ensures that the short rate does not become negative. As the short rate approaches zero the effect of the increments of the Wiener process decreases and might even get equal to zero. In this case the long-term positive effect dominates. The parameters of the CIR can

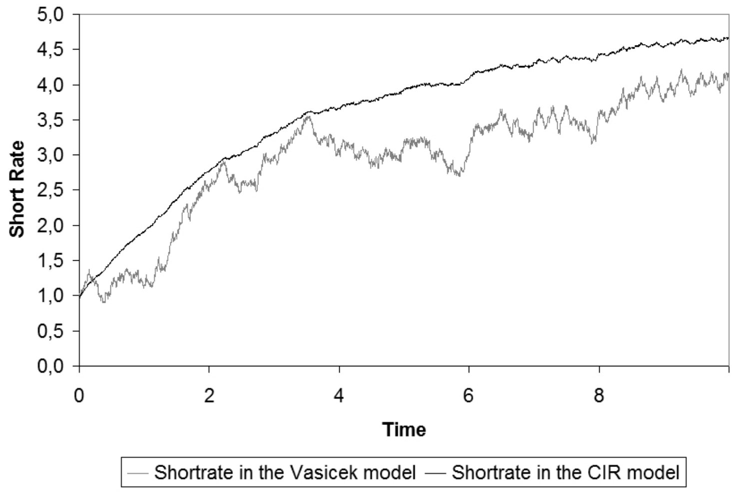

Figure 9: Sample short rate paths of the Vasicek and the CIR model with equivalent increments of the Wiener process with $\gamma = 0.05$, $\alpha = 0.3$ and $r_0 = 0.00979$

again be deduced from calibrating the model to real market data (see Section 5.3). As the CIR and the Vasicek model are both relatively strict concerning the possible paths of the short rate, the lineup of possible shapes of the zero coupon curve can be widened by using for example time depending long-term average short rate values. Similar to the Vasicek model there are analytical models for bonds and bond option prices in the CIR model. Thus, no simulations for such contracts are necessary.

5.1.3 Two Factor Short Rate Models

The CIR as well as the Vasicek model are both one-factor short rate models, as the whole zero coupon interest-rate curve is characterized by the single factor $r(t)$, the short rate. Thus, if a poor model is selected to simulate the term structure, the resulting estimates for prices of derivatives will also be poor. Hull [1] mentions, that a model that leads to a one percent deviating price of a bond from the real price, might lead to a 25% deviation in the option price. As a result a proper model is essential for pricing interest rate derivatives. Jamshidian and Zhu [26], who considered data on the Japanese Yen, the U.S. Dollar and the German Mark, showed that when using only one explaining component 68%-76% of the total variation in the term structure can be explained. Using two components already increases the explained part to 85%-90%, whereas three components already explain 93%-94% in the total variation. As Brigo and Mercurio note:

> 'The choice of the number of factors then involves a compromise between numerically-efficient implementation and capability of the model to represent realistic correlation patterns (and covariance structures in general) and to fit satisfactory enough market data in most concrete situations.' [27]

Thus, besides the fact that a multi factor model increases the explained total variation of a given term structure, they also incorporate more realistic assumptions concerning the correlations between interest rates for different times to maturity. Assuming a thirty-year (which equals the longest time to maturity assumed in the Vasicek model in Section 5.1.1) and a one-year interest rate, according to the Vasicek model a shock in the interest rate curve at time t is transmitted through all maturities equally, as the correlation coefficient is equal to one for two different times to maturity[20]. As a result, one-factor short rate models can be applied for products that only depend on a single rate of the term structure and also for products that depend on interest rates that are close to each other, according to their maturity dates. The latter assumption is due to the fact that for example a one-year and a six-month interest rate will surely be highly correlated.

Brennan and Schwartz [28] for example presented a two factor model where a short term interest rate tends to the level of a long-term one, where the latter follows a stochastic process as well. As a long term interest rate, the return of a bond with infinite time to maturity was used in their model. In fact the maturity date for such bonds is not determined beforehand. The invested amount would be payed back when the corporation is liquidated. In reality the interest that is payed for these bonds is either determined beforehand for the whole time to maturity or for predefined periods such as ten years. As such bonds are traded assets, the return of such a bond has to equal the risk free rate in a risk neutral world. The process for the payed interest can be computed according to the process for the bond prices. Brennan and Schwartz therefore proposed to model the short rate as well as a long-term interest rate explicitly.

5.1.4 Drawbacks of Equilibrium Models

A huge drawback becomes apparent when determining the process of the short rate, described by the Vasicek model. The variance, the long term mean and the speed of mean reversion have to be set such that an observed term structure is replicated. As a result the observed term structure is not an input, but rather an output of equilibrium models. As the long term mean and the speed of mean reversion are both time invariant, zero coupon bond prices cannot be reproduced exactly. By relaxing the assumption of time invariant parameters term structures can be modelled more accurately. This is the main difference of equilibrium and no-arbitrage models that will be discussed in the next section.

[20] see Brigo and Mercurio [27] p. 138

5.2 No-Arbitrage Models

As discussed in the previous section equilibrium models usually cannot replicate a given term structure perfectly as their determining parameters do not depend on time. As a result, huge discrepancies between the modelled term structure and the observed one might occur. By implementing a time depending long-term level, a good fit to market data can be achieved. Besides the fact that zero coupon bond prices are replicated correctly, models for the short rate also have to take into account the observed volatilities for interest rate derivatives. Instead of presenting and discussing several no-arbitrage models, the generalized Hull-White model presented by Hull and White [29] will be introduced. The advantage of this approach is the possibility of deducing the most common no-arbitrage models as special cases of the generalized Hull-White model. In Hull and White's model some function of the short-rate follows a Gaussian diffusion process of the following form:

$$df(r) = [\theta(t) - a(t)f(r)]\,dt + \sigma(t)\,dz \tag{29}$$

As mentioned earlier, equilibrium models cannot replicate an observed term structure satisfactorily due to their inflexible conception. For no-arbitrage models the variable $\theta(t)$ incorporates a more flexible process, in order to fit an observed term structure. The parameters $a(t)$ and $\sigma(t)$ are usually called volatility parameters, as they have to be set such that market prices of a set of actively traded interest-rate derivatives can be properly replicated. For the case that these volatility parameters as well as the parameter $\theta(t)$ are all time independent and that $f(r) = r$ the generalized Hull-White model is equivalent to the Vasicek model, as in Equation (24).

5.2.1 The Ho-Lee Model

The Ho-Lee model, which was presented by Ho and Lee in 1986 [30], was the first no-arbitrage model for the term structure. Initially Ho and Lee presented their model by applying a binomial tree, as it will be discussed briefly in section 6, but they also showed that the model converges to the generalized Hull-White model with

$f(r) = r$, $a(t) = 0$ and σ being constant in continuous time.

$$dr = \theta(t)\, dt + \sigma\, dz \tag{30}$$

Thus, by defining this process in a discrete manner, paths for the short rate can be simulated as in Section 5.1. Differently to the Vasicek model, the Ho-Lee model incorporates a time depending term $\theta(t)$. In the case of the Ho-Lee model this parameter $\theta(t)$ can be computed analytically as:

$$\theta(t) = \frac{\partial f(0,t)}{\partial t} + \sigma^2 t \tag{31}$$

$f(0,t)$ is the instantaneous forward rate at time t. Hence, the slope of the forward curve determines the direction where the short rate is heading to. A derivation of this relation was presented by Glasserman[21]. Thus, the exact discrete process of the short rate in the Ho-Lee model can be written as:

$$r(t_{i+1}) = r(t_i) + [f(0,t_{i+1}) - f(0,t_i)] + \frac{\sigma^2}{2}[t_{i+1}^2 - t_i^2] + \sigma\sqrt{t_{i+1} - t_i}\varepsilon_{i+1} \tag{32}$$

ε_{i+1} is again an independent standard normally distributed random variable. The forward rates $f(0,t)$ can be deduced from the spot rate curve by stressing the fact that the interest payed for a certain period has to be the same no matter, if it is computed by a zero coupon bond that matures at the end of the period or by assuming the interest rate for a shorter period and a forward rate, that covers the remaining time of the period. As a result, by applying the forward rates observed in the market the Ho-Lee model will perfectly fit the observed term structure. The only still unknown parameter in Equation (32) is σ, the volatility of the short rate. The Ho-Lee model does not incorporate a mean reversion variable, as a result, this model has one fewer variable as the Hull-White model, that will be discussed in the next section. Although the Ho-Lee model fits an observed term structure, it is not necessarily the fact that it also replicates observed market prices. Discrepancies between model implied prices and market data might be due to a poor replication

[21]see Glasserman [20] pp. 112

of the volatility observed in the market.

As the short rate is normally distributed in the Ho-Lee model, European call and put options on a s-maturity pure discount bond can be calculated as:

$$c(t,T,s) = P(t,s)N(d_1) - KP(t,T)N(d_2)$$

$$p(t,T,s) = KP(t,T)N(-d_2) - P(t,s)N(-d_1)$$

with

$$d_1 = \frac{\ln\left(\frac{P(t,s)}{KP(t,s)}\right)}{\sigma_p} + \frac{\sigma_p}{2}$$

$$d_2 = d_1 - \sigma_p$$

with the standard deviation of the bond price in T:

$$\sigma_p = \sigma(s-T)\sqrt{T-t}$$

This explicit result will be important when calibrating the model to market data in Section 5.3.

5.2.2 The Hull-White Model

The process Hull and White [31] proposed for the short rate can be written as:

$$dr = [\theta(t) - \alpha r]\, dt + \sigma\, d \tag{33}$$

Differently to the Vasicek model, the mean reversion level is now scaled by the speed of mean reversion and moreover it is time dependent in order to fit an observed term structure. Thus, the processes look similar, but in the Hull-White model the level of the short rate is adjusted to an observed term structure as $\theta(t)$ is determined as:

$$\theta(t) = \frac{\partial f(0,t)}{\partial t} + \alpha f(0,t) + \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}) \tag{34}$$

As a result the Hull-White model can be seen as a Vasicek model fitted to the term structure or as a Ho-Lee model with mean reversion. As in the Ho-Lee model the short rate follows the gradient of the forward curve. Thus, if the forward rate curve has a positive slope the short rate will increase. If the short rate deviates from the one implied by the initial forward rate curve, the short rate will be pulled back at the rate of α. Differently to the Vasicek and the Ho-Lee model there is no exact time-discrete version of the short rate process. As a result the Euler scheme (see Equation (40)) will be applied. The Hull-White model can therefore be implemented as:

$$\begin{aligned} r(t_{i+1}) &= r(t_i) \\ &+ \left([f(0, t_{i+1}) - f(0, t_i)] + \alpha f(0, t_i) + \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t_i}) - \alpha r(t_i) \right) [t_{i+1} - t_i] \\ &+ \sigma \sqrt{t_{i+1} - t_i} \varepsilon_{i+1} \end{aligned} \tag{35}$$

As for the Ho-Lee model there exist explicit formulas for European call and put options on pure discount bonds, thus:

$$c(t, T, s) = P(t, s)N(d_1) - KP(t, T)N(d_2$$

$$P(t, T, s) = KP(t, T)N(-d_2 - P(t, s)N(-d_1)$$

with

$$d_1 = \frac{\ln\left(\frac{P(t,s)}{KP(t,s)}\right)}{\sigma_p} + \frac{\sigma_p}{2}$$

$$d_2 = d_1 - \sigma_p$$

with the standard deviation of the bond price in T:

$$\sigma_p^2 = \frac{\sigma^2}{2\alpha^3}(1 - e^{-2\alpha(T-t)})(1 - e^{-\alpha(s-T)})^2$$

Equivalently to the Ho-Lee model this formula will be necessary in section 5.3.

5.2.3 The Black-Karasinski Model

The Black-Karasinski model [32] was proposed by Black and Karasinski in 1991. Differently to the Ho-Lee and the Hull-White model, this model does not allow for negative short rates, as the function $f(r)$ in the generalized Hull-White model is assumed to be the natural logarithm. Thus, the short rate is lognormally distributed, instead of normal as in the Ho-Lee and in the Hull-White model. In the Black-Karasinski model it is not possible to deduce prices of bonds from the short rate, moreover Hull [1] notes that it is more difficult to manage the Black-Karasinski model analytically. As a result, it will be concentrated on Gaussian models such as the Vasicek, the Ho-Lee and the Hull-White model in this thesis.

5.3 Model Calibration

As a next step in the employment of the short rate models presented in Section 5, the process defining parameters have to be determined. For the valuation of the bond option in Section 6.2.2, I have evaluated the parameters experimentally such that the simulated yield curve resembles the observed one sufficiently accurate. This approach for determining the parameters will be refined in this section, by introducing the method of calibration.

The method of calibration is based on the existence of closed formulas for interest rate derivatives in the respective models. By pricing derivatives according to these formulas and varying the process determining parameters, the combination of parameters can be found that minimizes the difference between model generated prices and market prices. In order to verify the difference between market and model implied prices, a proper measure has to be found. Hull [1] for example proposes to apply the sum of squared residuals as a measure for the differences. In this case the sum of squared absolute deviations has to be minimized. Clewlow and Strickland [12] on the other hand, propose to apply a measure that is based on the minimization of proportional differences. Correspondingly to this, the deviations are evaluated according to relative prices. As this ensures that differences in prices are minimized independent of the prices' levels the latter approach will be applied in this thesis.

In accordance to this, market data for interest rate derivatives have to be found, whereas the selected derivatives should resemble the derivatives that shall be prices as much as possible, as Hull [1] notes. Finally, it shall be noted that whenever the number of instruments that has to be calibrated is larger than one, the calibration can only provide an overall approximation for the short rate processes.

In Section 7, the Vasicek, the Ho-Lee and the Hull-White model will be calibrated in order to replicate observed market prices. For the Vasicek model the parameter selection approach presented in Section 5.1.1 will be refined.

6 Methods for Valuing Interest Rate Derivatives

In this section valuation methods for interest rate derivatives are presented. In general, there are two approaches for pricing interest rate derivatives (see Figure 7). The first one is based one modelling bond prices. This approach was presented by Fischer Black [9] and is an extension to the well known Black-Scholes-Merton model [23]. After discussing the Black model and pointing out the disadvantages of applying it, valuation methods based on modelling interest rates are presented. Hence, tree building methods and finite difference methods are reviewed briefly. Finally the method of MC simulations is introduced.

6.1 Black Model

The Black model is based on the well known Black-Scholes-Merton model developed in the 1970ies, by Fischer Black, Myron Scholes [23] and Robert Merton [33]. The adapted version for interest rate derivatives was then presented by Fischer Black in 1976 [9]. According to the findings of Gjukez [34] the Black model is the favored model by European companies that offer interest rate derivatives. One possible reason for this might be the easy computation of prices, without modelling the whole term structure. As a result, this rather simple pricing method is not applicable to path-dependent derivatives.

The Black-Scholes-Merton model was developed to price stock options. Extensions to the original model allow for the pricing of options on foreign exchange, options on indices and options on future contracts. The Black model [9], was developed to price futures on commodities. Thus, an option can be priced, which gives its holder the right to buy or sell a future on a commodity at a certain date in the future. Moreover, if the future contract and the option have the same maturity date, the Black model prices an option on the future contract as well as an option on the underlying of the future contract. This is due to the fact that the future price of a commodity will converge to the spot price of the commodity as the maturity date approaches.[22] The assumptions of the Black model can be summarized as follows:

[22] Assuming that the future price is above the spot price at the time to maturity, arbitrageurs

1. The market is arbitrage free. Thus, it is not possible to invest zero today and receive a positive amount tomorrow with a positive probability.

2. Transaction costs and similar costs are not accounted for.

3. The assumed instruments can be traded in unrestricted volumes.

4. There exists a risk free rate[23], which is constant over time.

5. The underlying variable of the option is lognormally distributed at maturity of the option.

(ad 1) This assumption is essential, as it ensures that two instruments with the same cash flows have the same price. This result is important for Risk Neutral Valuation (see Section 4.1). As real markets are surely not arbitrage free, due to differing information levels, this assumption does not meet reality. At this point it is pointed out that model prices are only consistent with real prices, if all assumptions made are fulfilled in real markets.

(ad 2) As transaction and similar costs are either fixed or variable according to the contract value, this assumption should not cause too much harm.

(ad 3) The assumption of unrestricted trading possibilities is also surely not fulfilled in reality, but as derivative markets are in general quite liquid, this difference should be negligible.

(ad 4) This is a very problematic assumption in the sense that the underlying variable might be a forward rate as in the case of pricing a cap which is assumed to be stochastic, but for calculating the discounted value of the payoff a constant and therefore non-stochastic interest rate is applied.

(ad 5) When pricing a single instrument this assumption is not that problematic. But, if for example a bond option, a cap agreement and a swaption, all with the

can sell future contracts, buy the underlying at the lower spot price at the time to maturity and deliver the underlying to the buyer of the future contract. As a result the future prices will decrease as long as this arbitrage opportunity is persistent. If the future price is below the spot price at the time to maturity it will be profitable to enter in a future contract, the prices of these contracts will rise.

[23]Which is the same as assuming that there is a bond with an appropriate time to maturity that has no default risk.

same maturity date, are priced, the Black model leads to an inconsistent valuation, as all underlying variables are assumed to be lognormally distributed at once. In the case of the bond option, it has to be assumed that the bond prices at maturity are lognormally distributed, for the cap agreement on the other hand, it has to be assumed that the interest rate is lognormally distributed, in the case of the swaption, it has to be assumed that the swap rate is lognormally distributed. As these three assumptions cannot be met at the very same point in time the Black model can only be regarded as a 'one model-one product' approach as Clewlow and Strickland [12] put it.

According to Reißner [3] the Black model is not appropriate to value interest rate options, as it does not take into account the term structure of the underlying interest rate, although this is a large source of risk for the option. This critic arises as Black sets the behavior of the interest rate as constant. Another point that is mentioned by Reißner is that bonds and future prices differ fundamentally from stocks which were the basis for the remarks of Black. Beside all these inconsistencies the Black model is widely used by practitioners as shown by Gjukez [34]. As some calculations according to the Black model will be necessary when dealing with market data, as in Section 7, the closed formula for pricing an option on a variable V will be now presented.

Similar to the payoff function for a call option on stocks, the payoff function of a call option on V at maturity, can be written as $\max(V_T - K, 0)$. Assuming that V is lognormally distributed with a standard deviation s equal to $\sigma\sqrt{T}$ and an expected value equal to $F(0, T)$ the expected value of an option with strike K on V can be written as:

$$\mathbb{E}[\max(V_T - K, 0)] = \mathbb{E}(V_T)N(d_1) - KN(d_2)$$

where $\mathbb{E}$ is the usual expected value and N() is the cumulative standard normal distribution. The functions $N(d_1)$ and $N(d_2)$ can be interpreted as the risk neutral probabilities (see Section 4.1) and are defined as follows:

$$d_1 = \frac{\ln[\mathbb{E}(V_T)/K] + s^2/2}{s}$$

$$d_2 = \frac{\ln[\mathbb{E}(V_T)/K] - s^2/2}{s}$$

As discussed earlier, at time to maturity the option price on a future contract has to equal the option price on the underlying of the future contract, as a result V_T can be replaced by its forward price. Here it shall be noted again that future contracts do not require any investment beforehand. In a risk neutral world (see 4.1) a future contract therefore has to have a return equal to zero. As a result one can state that $\mathbb{E}[F(0,T)] = F(0,T)$. The payoff of the option in T can therefore be written as $F(0,T)N(d_1) - KN(d_2)$. Furthermore, the value of the option has to be discounted in order to evaluate it today. Therefore the price of a zero coupon bond is needed. This zero coupon bond has to have the same maturity date as the option. Here a zero coupon bond is introduced that pays 1 unit of currency at T and whose value is known in $t = 0$ which is donated as $P(0,T)$. Thus, one can just multiply the value of the option in T with $P(0,T)$ and receive the discounted call option price in $t = 0$:

$$c = P(0,T)[F(0,T)N(d_1) - KN(d_2] \tag{36}$$

$$d_1 = \frac{\ln[F(0,T)/K] + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln[F(0,T)/K] - \sigma^2 T}{\sigma\sqrt{T}}$$

In the case of a put option the value of the option can be expressed as $\max(K - V, 0)$. In this case Equation (36) changes and the value of the put option in $t = 0$ can be expressed as:

$$p = P(0,T)[KN(-d_2) - F(0,T)N(-d_1] \tag{37}$$

Thus, in order to price a bond option in the Black setting the variable V has to be replaced by the forward bond price. As it was mentioned above, the Black model assumes that the underlying is lognormally distributed. Thus, in order to price a European bond option, it has to be assumed that the bond price is lognormally distributed at maturity. For caps and floors it has to be assumed that the underlying forward rate is lognormally distributed at maturity. And for swaptions it has to be assumed that the swap rate is lognormally distributed at maturity.

As mentioned before, a huge drawback of the Black model is that it does not take into account the evolution of interest rates. As a result the evolution of the forward rate for example is assumed to be stochastic, but the risk-free rate is assumed to be deterministic. Moreover, all interest rate derivatives that depend on the path of the underlying can not be priced using the Black-Model. Therefore models for the behavior of interest rates, as presented in Section 5, were developed. As already mentioned there, for some of the presented short rate models there exist explicit formulas. In these cases European bond options can be priced using explicit formulas, which is in consequence (see Section 2.2.2) also true for swaptions and cap agreements. As explicit formulas are not available in many cases, numerical methods for valuing interest rate derivatives are presented In the upcoming two subsections.

6.2 Numerical Methods for the Valuation of Interest Rate Derivatives

Besides the explicit prices the Black model returns, interest rate derivatives can also be valued by applying numerical methods. Moreover for a wide range of interest rate sensitive instruments numerical methods are inevitable.

6.2.1 Lattice and Finite Difference Methods

Lattice methods are based on a time discrete representation of stochastic processes. In a so called binomial tree it is assumed that the underlying interest rate is heading towards two directions only during one subperiod. It can be assumed, for example that the interest rate for the observed period is either increasing or decreasing. The probabilities for both events can be deduced from risk neutral valuation, as discussed in Section 4.1. Hull [1] notes that it is sometimes favorable to apply trinomial trees for interest rate derivatives, as they provide one additional degree of freedom. In the case of a trinomial tree it is possible to assume an additional; 'middle' movement of the interest rate. Hull and White [35] presented an approach, how to implement trinomial trees for one-factor short rate models. After deciding how the interest rate can change within one subperiod, the constructed lattice is adjusted for all subperiods in order to fit an observed term structure. The underlying parameters for the short rate models are found via calibration.

Finite difference methods on the other hand are based on solving a system of partial differential equations, which are met by the derivative. The explicit finite difference method (based on a mathematical relation between the current option prices and three option prices one time step ahead[24]) is equivalent to the construction of a trinomial tree. Differently to the method of lattices and finite-differences, MC methods simulate the evolution of stochastic variables.

6.2.2 Monte Carlo Simulation

The technique of MC simulation is a numerical tool to simulate uncertain events. By simulating these uncertain events sufficiently often, it is possible to obtain information about the distribution of the investigated variable. Winston [36] presents some examples for the practical applicability of MC simulation. Well known corporations such as General Motors, Pfizer or Procter and Gamble use MC methods to simulate the average return and the risk factor for new products. Thus, MC simulation methods are helpful tools for launching new products. On the other hand MC methods can be used to predict the net income or potential costs, as it is done by General Motors. It is also possible to simulate the optimal plant capacity for a certain product, as it is done by Lilly, a pharmaceutic corporation. These examples should give an insight in the wide range of applications for MC methods in economics. The application of interest in this thesis is the financial one. The technique of MC simulation can be used to value options on different kinds of assets and contracts. Glasserman [20] states, that the technique of MC simulation has become an essential tool for pricing of derivative securities and risk management. Boyle [37] already presented the approach of valuing options via MC simulations in 1977. As the MC simulation as such is a rather costly tool for option valuation at first sight, techniques for improving the efficiency are necessary. Boyle already suggested to use antithetic variates in order to reduce the variance and therefore improve the simulations. This variance reducing technique as well as others will be

[24]The implicit finite difference method applies the other way around

discussed and applied later in this section.

> 'The name Monte Carlo simulation comes from the fact that during the 1930s and 1940s, many computer simulations were performed to estimate the probability that the chain reaction needed for the atom bomb would work successfully. The physicists involved in this work were big fans of gambling, so they gave the simulations the code name Monte Carlo.' [36]

Basics on Monte Carlo Methods For the upcoming technical discussions of MC methods it will be followed the comprehensive presentation in Glasserman [20]. MC methods are based on the common statistical idea of deducing the probability of events from their frequency. For example, by asking a sample of students about their grade point average, one can hypothesize about the probability that a randomly picked student has a grade point average lower than two. MC methods use this relationship in reverse. By sampling randomly from a universe of possible outcomes and taking the fraction of random draws that fall in a given set, one can infer on the sets volume. Thus, by sampling the grade point average of the students given a realistic model, it would be possible to estimate the frequency of a certain grade average point. Due to the law of large numbers these estimates converge to their correct values as the number of draws increases.

As the volume can also be seen as the integral, Glasserman [20] presents the problem of estimating the integral of a function h as follows: The integral over the unit interval of a function $h(x)$ can be written as:

$$\alpha = \int_0^1 h(x)\, dx$$

To estimate this integral one can sample uniformly distributed values U in-between the unit interval, apply the function h to them and divide the sum of all these values by the number of randomly determined numbers. Thus, the estimate would look like:

$$\widehat{\alpha}_n = \frac{1}{n} \sum_{i=1}^{n} h(U_i)$$

If $h(x)$ is integrable over [0,1] then by the law of large numbers the estimated value $\widehat{\alpha}_n$ has to converge to the real value α. Assuming for example $h(x) = x$ as the underlying function, one can easily estimate the integral over the unit interval for this function. Generating 500 hundred independently and uniformly distributed values in the interval [0,1] and calculating successively the mean for these values, shows that the mean is actually converging to the true value of 0.5. Figure 10 represents the mean of the value in dependence of the number of generated values. Another important question that arises concerns the deviation of the simulated value

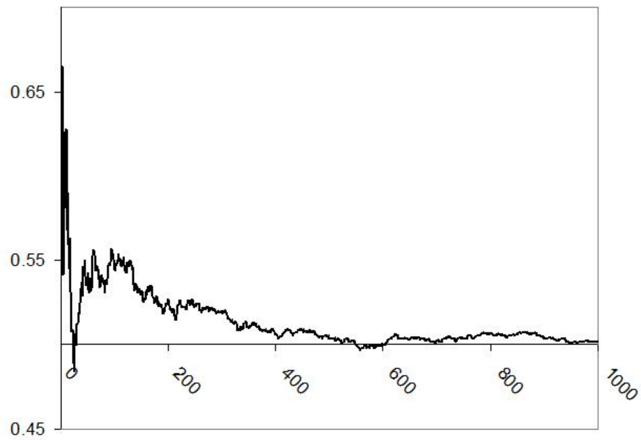

Figure 10: Mean of independently and uniformly distributed values in the unit interval, in dependence of the number of generated values.

from the real value. For the variance of the values of $h(x)$ one can therefore write:

$$\sigma_h^2 = \int_0^1 (h(x) - \alpha)^2 \, dx$$

The estimation error $\widehat{\alpha} - \alpha$ of the MC simulation is approximately normally distributed with a mean equal to zero and a standard deviation (standard error) of:

$$\frac{\sigma_h}{\sqrt{n}}$$

Thus, the standard deviation of the estimation error is decreasing as n increases. As a result, the more random numbers (later on it will be whole paths) are simulated the smaller the standard deviation of the estimation error will become. According to this formula, it is necessary to sample 100 times more random numbers in order to increase the precision by one decimal place. In Figure 11 the reduction of the sample standard deviation is shown as the size of the sample increases. Besides the cases as above, where the integral can be solved analytically, α will not be known usually. Thus, σ_h can not be calculated. Using the estimate for α one can calculate the sample standard deviation or standard error as an estimate for σ_h as:

$$s_h = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(h(U_i) - \widehat{\alpha}_n)^2}$$

In Figure 11 this estimation error is plotted against the volume of sampled numbers by applying the simple function $h(x) = x$. Glasserman [20] notes that MC

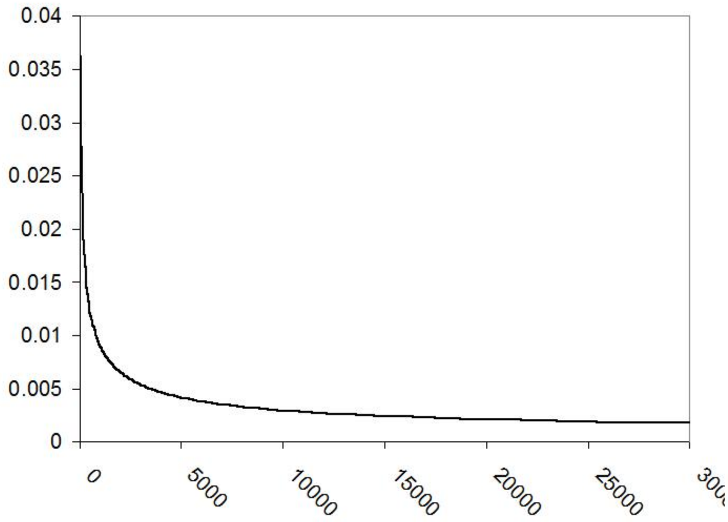

Figure 11: Estimation error based on estimating the integral over the unit interval, for the function $h(x) = x$, in dependence of the number of samples.

methods are generally not competitive when calculating one-dimensional integrals. The advantages of MC methods arises when calculating integrals for higher dimensions, because the dimensionality does not influence the estimation error. Moreover, MC methods are not restricted to the unit interval, that is why the method of MC simulation is a practical tool for valuing derivatives.

In order to value derivatives it is necessary to sample the evolution of their underlying and to calculate the price of the derivative based on the values of the underlying at maturity. Finally, one can calculate the discounted expectation for these prices. In accordance to this, it is necessary to sample the whole evolution of a variable instead of a single random number. Among others, potential underlying assets are interest rates and stock prices. As the latter was the underlying presented by Boyle [37] and as the valuation of derivatives on stocks was the starting point, an example will be presented. As it was discussed in Section 2.2, a put option on a stock would be either worth $S(T) - K$, if the stock price is above the strike price K, or equal to zero, if it is not. The Black-Scholes-Merton model, as presented in Section 6.1, describes the evolution of stock prices through the stochastic differential equation:

$$\frac{dS(t)}{S(t)} = r\,dt + \sigma dz(t) \tag{38}$$

The percentage change in the asset price can be interpreted as the mean rate of return per time step plus the increment of a Wiener process $dz(t)$. The mean rate of return is equal to the risk free rate r as the Black-Scholes formula is based on the assumption of risk neutrality (see Section 4.1). The solution of Equation (38) can be written as follows:

$$S_T = S(0)e^{(r\frac{1}{2}\sigma^2 T+\sigma W(T))} \tag{39}$$

Thus, when the initial stock price and the risk free rate r is known, one can simulate the stock price in T quite easily. As Equation (39) indicates, it is possible to simulate the change of the asset price for the whole period at once. Thus, the evolution of the asset price is not taken into account. If the changes in the asset price can be expressed as in Equation (38), the resulting European call option prices will be equivalent to the ones calculated according to the famous Black-Scholes formula. Figure 12 shows one hundred different paths that describe the possible evolution of a stock price for one year by splitting the whole year in one thousand time steps. Calculating the payoff for a European call and discounting it gives in this case one hundred different values for the option. By the law of large numbers the mean of all discounted values is an unbiased estimator for the real value. Clewlow and Strickland [12] note that in order to get an acceptably accurate estimate one has typically got to simulate more than one million paths. According to the Black-Scholes formula, a European call option on a stock that has a current price of one hundred units of currency and a volatility of 20%, with a strike price of ninety and one year to maturity has to have a current price of 15.42 units of currency. Modelling one million paths, that are split in ten equally sized time periods returns a price that deviates by about 0.16 from the real price. To be more precise, the standard error for this simulation is already only different from zero at the fifth decimal place. Therefore accuracy can be increased by increasing the number of paths. Whereas, this is equivalent to the results presented in Figure 11.

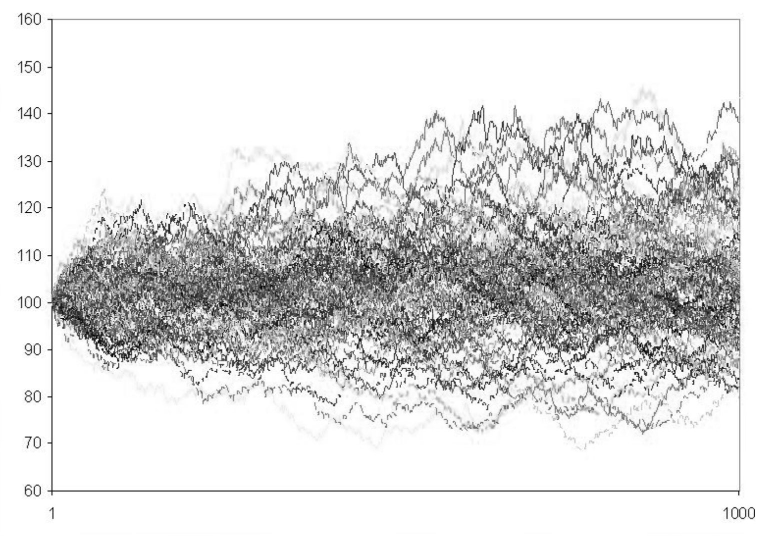

Figure 12: One hundred sample paths of a stock price

In examples different from the one above, it is not always possible to formulate the continuous evolution of a variable in an exact discrete form. Glasserman [20] notes that such models are exceptional and that most derivative models can only be simulated approximately[25]. Thus, the joint distribution of the simulated values does not coincide with the values of the continuous-time model. For all models where there is no exact process the discrete Euler approximation will be applied. The reason for selecting this approach was the easy implementation and the universal applicability of this method. Considering a stochastic differential Equation for a process X of the form:

$$dX(t) = a(X(t))\,dt + b(X(t))dz(t)$$

where $X(0)$ is a fixed value, z is a Wiener process and a and b take real values, the

[25] If the underlying SDE is not integrable and/or the derivative is path dependent

Euler scheme can be written as:

$$\widehat{X}(t_{i+1}) = \widehat{X}(t_i) + a(\widehat{X}(t_i))[t_{i+1} - t_i] + b(\widehat{X}(t_i))\sqrt{t_{i+1} - t_i}\varepsilon_{i+1} \tag{40}$$

where $\widehat{X}$ indicates a time-discretized approximation to X and $Z_1, Z_2, ...$ are independent standard normal numbers. As a result, this discretized form can easily be implemented. Glasserman notes that the Euler scheme is not sufficiently accurate in certain cases and that the Euler scheme has to be improved (For a discussion of possible refinements it is referred to Glasserman [20]). In order to decrease the possible discretization error due to using the Euler scheme, it is necessary to include as many time steps as possible in order to achieve a good approximation for the continuous formulation. In the last example one million paths were simulated, so that the standard error was negligible. By decreasing the length of the time steps, the simulation time extends enormously.

Differently to derivatives on stocks and currencies, interest rate derivatives are more difficult to evaluate. Hull [1] gives the following four reasons:

1. The behavior over time of a single interest rate is more complex than the one of a stock price or an exchange rate.
2. In order to price some specific derivatives it is necessary to model the behavior of the whole term structure. Thus, the behavior of a collection of spot rates for different times to maturity has to be simulated.
3. The volatilities of the interest rates might be different at different points in time.
4. In order to price interest rate derivatives, models for the evolution of interest rates have to be found.

In the upcoming sections all these hurdles are taken, which will result in pricing derivatives on the basis of real market data.

Simulating Short Rate Paths According to the Vasicek Model As Haug [38] points out, implementing MC simulations in C++ or any other lower level computer language decreases the computation time dramatically, all further simulations will be carried out using C++. Although VBA is widely used in practice, I will stick to C++, as simulating 1 million paths of a short rate process, as presented in Section 5, took about four hours in VBA, but only about half an hour in C++.

The first example for simulating short rate paths is based on the Vasicek model as presented in Section 5.1.1. As Equation (26) and (27) show, it is necessary to evaluate the speed of mean reversion, the long-term mean and the standard deviation of the process of the short rate process. Brigo and Mercurio [27] note, that it is possible to deduce the model parameters that define the process of the short rate from a series of daily quoted interest rates. Thus, by applying an appropriate proxy variable for the short rate, like a daily series of the interest rate for one month, the model parameters can be estimated. As the applied data set is collected in the real world and not in a risk neutral one, the market price of risk has also got to be estimated. On the other hand derivative prices such as bond prices are equivalent in the real and in a risk-neutral world. As a result observed prices of bonds for example, can be used to calibrate the model. Brigo and Mercurio [27], then propose to combine the two approaches in order to fit the short rate process to market data. The two authors cite that the diffusion process is the same in the real as in a risk-neutral world. As a result ρ, the standard deviation of the generalized Wiener process, can be estimated from historical data by a maximum-likelihood estimator. The coefficients for the speed of mean reversion and for the long-term mean, on the other hand, can be found by calibrating (see Section 5.3) the process to observed derivative prices.

In order to replicate the Euro yield curve from Figure 1 a proxy for the short rate had to be found. Differently to the proposal of Brigo and Mercurio [27] and differently to the approach of Treepongkaruna and Gray [39], I applied the EONIA (Euro Over Night Index Average) to estimate the standard deviation for the process of the short rate (instead of a three month interest rate). The EONIA- Interest rate is a weighted average of interest rates for interbank lending in the Euro money market for one night. I selected this interest rate as it is the one with the shortest time to maturity available, namely one night. In accordance to the recommendations of Brigo and Mercurio [27], I estimated the standard deviation using maximum likelihood estimators[26]. The estimation lead to a annual standard deviation[27] of 0.5 percentage points. The values of the speed of mean reversion and the long-term average interest rate on the other hand were determined experimentally for the time being. Hence, I set the parameters such that the simulated yield curve at least resembles the observed one[28]. As a result I determined a long-term mean of 5%, a mean reverting speed of 30% and an initial short rate value of 0.979%[29]. Figure 14 represents the resulting yield curve. In order to incorporate the $\frac{30}{360}$ day count convention, I assumed 360 time steps per year, which facilitates computing interest rates for periods less than one year. At this point it shall be pointed out again, that the number of time steps can be set arbitrarily, as the number of time steps only influences the constant parameter that is multiplied with the diffusion coefficient, the standard deviation of the process for one year still equals the square root of one times ρ. Nevertheless, when presenting some basics on Monte Carlo methods, it was briefly discussed that using as many time steps as possible is preferable, when simulating approximations for short rate processes.

Figure 13 represents 100 different paths of the proposed short rate process. As it can be seen, the short rate tends to increase until the long-term level of 5% is reached. This increase of the short rate ensures that the spot rate will also increase as time to maturity increases. In order to price zero coupon bonds, with these short rate paths, I computed today's bond prices according to Equation (22). By simulating 1 million paths with 360 time steps per year, as for the yield curve presented in Figure 14, I had to determine the values of 10800 bond prices (for every time to maturity). In accordance to the remarks while presenting some basics on MC methods, I simulated 1 million different bond prices for these 10800 times to maturity. By calculating the average value of the bond prices at every maturity date, I deduced the simulated bond prices. Transforming these bond prices, as indicated in Equation (23), leads

[26]see Brigo and Mercurio [27] p. 62

[27]The data set comprised of EONIA interest rates from the 26^{th} of March 2008 till the 26^{th} of March 2009

[28]A more accurate and scientific approach will be presented in Section 5.3

[29]The initial short rate value was set equal to the last daily EONIA interest rate, namely the one for the 26^{th} of March 2009.

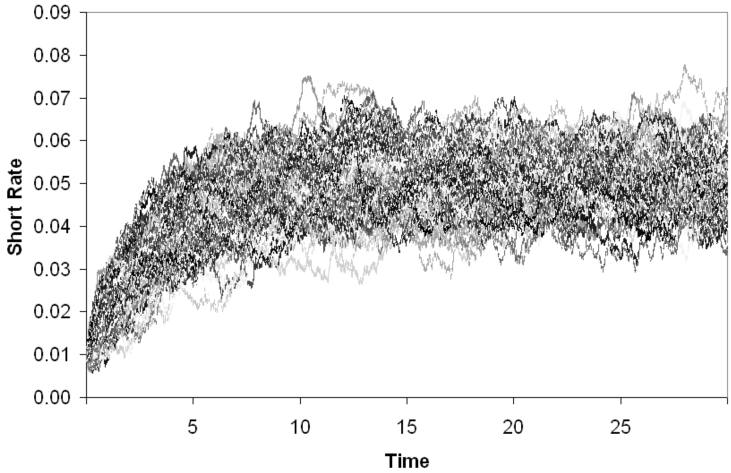

Figure 13: Representation of 100 short rate paths according to the Vasicek model with $\gamma = 0.05$, $\alpha = 0.3$ and $r_0 = 0.00979$

to the presentation of the spot rates for times to maturity from one to thirty years as presented in Figure 14.

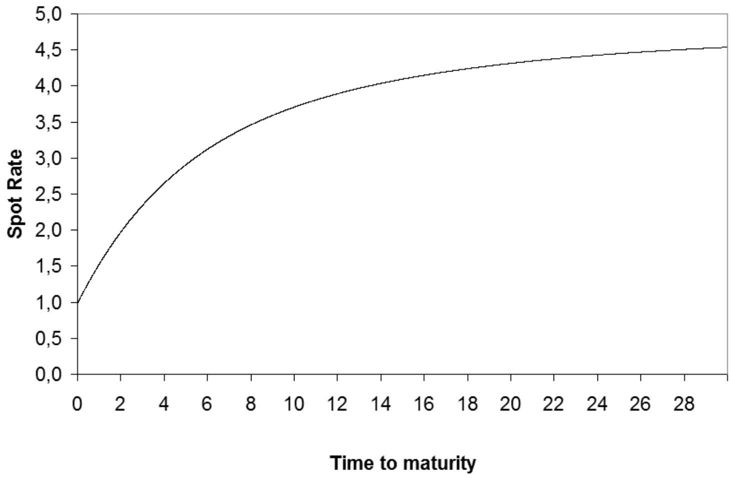

Figure 14: Spot rates for maturities up to 30 years, simulated using 1 million paths of the short rate with 360 time steps for each year, based on the Vasicek model with $\gamma = 0.05$, $\alpha = 0.3$ and $r_0 = 0.00979$

As a first example for interest derivative valuation using MC simulation, a rather simple derivative will be priced, namely a European call option on a bond. For the underlying short rate process according to the Vasicek model I assumed the same parameters as above. In this example an European call option with a strike price of $K = 0.5$ and a time to maturity of one year, on a 10 year zero coupon bond was priced. Thus, this contract gives its owner the right to buy a zero coupon bond in one year that matures in ten years. In order to calculate the discounted payoff of this bond option, I had to simulate the values of two zero coupon bonds. The first one concerns the period starting in one year and ending in ten years. The second starts today and matures next year. The first one determines the payoff of the option, whereas the second ensures proper discounting. As a matter of fact, the initial value for the simulations one year ahead are different in every path, thus the value of the option has to be discounted applying a zero coupon bond according to the former evolution of the short rate. Consequently, I set the initial value equal to the last simulated increment of the short rate process for pricing the one year zero coupon bond. For every single initial value one year ahead, I simulated a short rate path until maturity. Thus, one million paths for the evolution of the ten year bond were simulated. I calculated the one-year-option price for every single path by applying $max[P(1,10)-0.5,0]$. Afterwards I discounted these payoffs using the one year zero coupon bond calculated before. As there are 1 million different discounted values, the mean value was calculated as an estimate for today's option value, as it discussed earlier. The simulation of today's option price resulted in a value of 0.196966. Thus, in order to have the right to buy a 10 year zero coupon bond, with a face value of one Euro in one year for a price of 50 Cents one has to pay 19.6966 Cents today. The algorithm for pricing this bond option is presented in Appendix B.1.

Jamshidian [10] showed that under the short rate process described by the Vasicek model, European discount bond call and put options can be priced using a closed formula as:

$$c(t,T,s) = P(t,s)N(d_1) - KP(t,T)N(d_2$$

$$p(t,T,s) = KP(t,T)N(-d_2) - P(t,s)N(-d_1)$$

with

$$d_1 = \frac{\ln\left(\frac{P(t,s)}{KP(t,s)}\right)}{\sigma_p} + \frac{\sigma_p}{2}$$

$$d_2 = d_1 - \sigma_p$$

with the standard deviation of the bond price in T:

$$\sigma_p = \frac{\nu(t,T)(1 - e^{-\alpha(s-T)})}{\alpha}$$

$$\nu(t,T) = \sqrt{\frac{\sigma^2(1 - e^{-2\alpha(T-t)})}{2\alpha}}$$

As a result, the price of the just simulated bond option can be calculated analytically and leads to a value of 19.6994. In Figure 15 the convergence of the simulated price towards the exact one is plotted for up to one hundred thousand paths. In order to decrease the number of paths while achieving a certain accuracy of the simulation, variance reducing techniques will be introduced next. For such simple contracts and

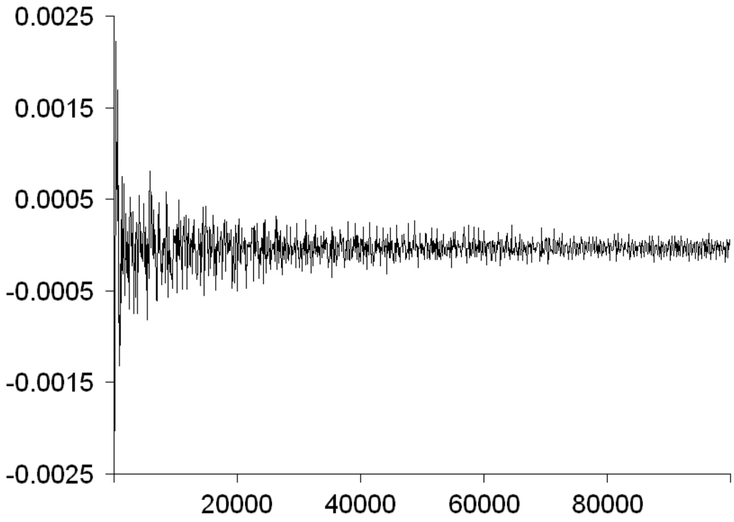

Figure 15: Convergence of the simulated price of an European call option

under the assumption that the Vasicek model replicates the real world perfectly, no simulations would be necessary as a closed formula is available. As it was discussed in Section 2.2.2, cap agreements and swaptions can also be represented as European bond options. Thus, they can also be priced according to the formula presented by Jamshidian. Hence, as it will be shown in Section 8, MC methods are inevitable in many cases.

Variance Reduction Techniques In this section two methods are introduced that can be applied in order to reduce the standard deviation of the simulated prices and therefore improve the convergence of the prices towards their exact values by applying the same number of simulated paths. The sample standard deviation for the simulated values can be written as:

$$= S_D = \frac{\sqrt{\sum_{j=1}^{Paths}(Caplet_j)^2 - \frac{1}{Paths}\left(\sum_{j=1}^{Paths} Caplet_j\right)^2}}{Paths - 1}$$

where $Caplet_j$ is the j^{th} caplet for one maturity date. Thus, by calculating the standard deviation for every simulated price, one can compare the efficiency of different simulation methods.

Antithetic Variates

> 'The method of antithetic variates attempts to reduce variance by introducing negative dependence between pairs of replications.' [20]

A very simple technique to reduce the variance and therefore the standard deviation of the estimators for derivative prices is the method of antithetic variates. For the presentation of this method it is followed the comprehensive presentation of Glasserman [20]. As it was remarked in Section 6.2.2, the aim of applying MC methods for pricing derivatives is to estimate the expected value of a certain random variable. In the case of a derivative this variable is the derivatives price. Differently to standard MC methods, where only single observations (per path) were made, the method of antithetic variates assumes that pairs $(Y_i, \widetilde{Y}_i)$ of observations (per path) are made. For these pairs it is assumed that they are independently, identically distributed and that the observations of one pair have the same distribution although they are ordinarily not independent. The antithetic variates estimator $\widehat{Y}_{AV}$ can therefore be written as:

$$\widehat{Y}_{AV} = \frac{1}{2n}\left(\sum_{i=1}^{n} Y_i + \sum_{i=1}^{n} \widetilde{Y}_i\right) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{Y_i + \widetilde{Y}_i}{2}\right)$$

The estimator for the expected value of a derivative is then calculated as the average of the average of a pair of observation. As Glasserman shows, by assuming that the computational effort of generating a pair $(Y_i, \widetilde{Y}_i)$ is twice as high than for a single observation, an antithetic variate will only reduce the standard deviation, if the covariance between Y_i and $\widetilde{Y}_i$ is negative.

As the standard deviation can only be decreased in the case of a negative covariance, a negative relationship between the observations is necessary. In accordance to Clewlow and Strickland [12] an option on an asset S_1 can be assumed. By the way, another option on a second asset S_2 is assumed that is perfectly negatively correlated with S_1 and which is currently worth exactly S_1. As the current prices and the volatilities for these two assets are the same the values of the options on these assets also have to equal (see Section 4.1). Assuming a portfolio of the two options on the assets leads to a much lower variability in the option price. This is due to the fact that whenever one option pays-off, the other one does not and vice versa. In order to generate perfectly negative correlated pairs of observations, the increments of the Wiener process are once applied with a negative and once with a positive sign for one time step. From the resulting values the average is computed. For the short rate processes according to the Vasicek model this would mean, that two process are defined at once like:

$$dr_1 = \alpha(\gamma - r_1)\,dt + \rho\,dz$$

$$dr_2 = \alpha(\gamma - r_2)\,dt - \rho\,dz$$

Control Variates As Glasserman [20] notes, the method of control variates is among the most effective and broadly applicable technique for improving MC simulation. The remarks in this section are based on the presentation of Glasserman [20]. The method of control variates is based on the knowledge of future values of a control variate.

The deviations of simulated future values from observed future values of the control variate can be incorporated in the simulation of the variable of interest, in order to improve the efficiency of the simulation. Assuming n outputs of the simulation of n paths that can be written as $Y_1, Y_2, Y_3, ..., Y_n$. These values might be the discounted payoffs of an option. For every path that was simulated, the variables $X_1, X_2, X_3, ..., X_n$ are computed as well. It is assumed that the pairs (Y_i, X_i) are independently, identically distributed and that the expected value $\mathbb{E}(X)$ is known. As a result:

$$Y_i(b) = Y_i - b(X_i - \mathbb{E}[X])$$

can be computed for every simulated path. Calculating the mean for this series, gives the control variate estimator:

$$\overline{Y}(b) = \overline{Y} - b(\overline{X} - \mathbb{E}[X]) = \frac{1}{n}\sum_{i=1}^{n}(Y_i - b(X_i - \mathbb{E}[X])) \tag{41}$$

As the expected value of X is known, this estimator is unbiased and consistent. As it can be shown, the value of b that minimizes the variance of the control variate estimator is equivalent to:

$$b^* = \frac{\sigma_Y}{\sigma_X}\rho_{XY} = \frac{COV[X,Y]}{Var[X]} \tag{42}$$

As $\mathbb{E}[Y]$ is usually not known, the optimal b^* will not be observed either. Nevertheless, Glasserman [20] notes, that the benefit of using an estimate of b^* as a control variate is still present. An obvious estimate for the optimal parameter b^* is to replace the population parameters in Equation (42) by their sample counterparts. In this case the estimator equals the slope coefficient in a linear regression analysis.

7 Valuing a Cap Agreement Using Monte Carlo Simulation

In this section a cap agreement will be priced, applying the Vasicek, the Ho-Lee and the Hull-White model. These models were selected as there exist explicit formulas for pricing caplets. Thus, the simulated prices can be verified by comparing them with the market data and the prices deduced from the explicit formulas. Moreover, it will be shown, how the simulations can be improved, in order to achieve faster convergence of the simulated prices towards the explicit ones.

7.1 Market Data

The underlying market data, were gathered at the 2^{nd} of June, from Reuters 3000 Xtra a real time prices providing platform of the Thomson Reuters corporation.[30] As this platform provides real time prices it is important to note that the prices were downloaded at nine o'clock in the morning. The cap agreement I picked out, lasts for five years, caps the EURIBOR-12M[31] at 2% and starts at the 4^{th} of June. As it is common practice that no caplet is assumed for the first period, there are four caplets overall that have to be priced in this agreement. As mentioned in Section 2.2.2, the maturity date of a caplet is the date when the actual spot rate is compared with the cap rate. The compensation, on the other hand, is paid at the next reset date, in this example this will be in one year. Figure 16 illustrates this contract.

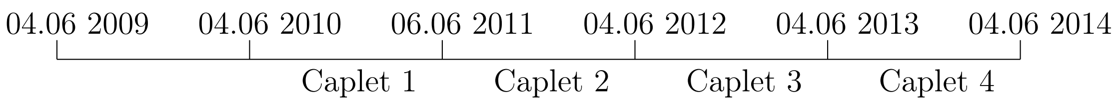

Figure 16: Representation of a cap agreement starting on the 4 June 2009 on the 12 month EURIBOR, which matures on the 4 June 2014.

In the year 2011 the reset date for the cap agreement will be the 6^{th} instead of the 4^{th} as the 4^{th} June 2011 is a Saturday. Hence, there is no trading of bonds which could verify the market rate for the next year. In accordance to all previous

[30] At this point I want to especially thank Arne Westerkamp for providing me with the access to this platform as well as for introducing it to me.

[31] This implies that the cap agreement is reset annually.

Caplet Maturity	Maturity in years	$P(0,t)$	$R(0,t)$	$f(0,t,t+1)$	Black(76) Volatilities	$caplet(0,t,t+1)$
04 June 2009	0	1				
04 June 2010	1	0.9828	0.0173	0.0348	0.4863	0.0153
06 June 2011	2	0.9491	0.026	0.0498	0.3862	0.0286
04 June 2012	3	0.903	0.034	0.0549	0.336	0.0318
04 June 2013	4	0.8547	0.0392	0.0569	0.3133	0.0319
04 June 2014	5	0.8074				

Table 2: Black volatilities and the resulting caplet prices, prices of pure discount bonds as well as the corresponding spot and forward rates for the 4^{th} June 2009

calculations and simulations I will again apply the $\frac{30}{360}$ day count convention.

Market data for caps and floors are usually not quoted in cash prices, instead in Black volatilities. This shows how widely used and accepted the Black model [9] is. Thus, the price of a cap or floor is not quoted in a specific currency, but is an input parameter of a model. In order to obtain cash prices, these volatilities with the corresponding bond and strike prices have to be plugged into the formula proposed by Black. The prices of a pure discount bond as well as the spot and forward rates, the Black volatilities and the resulting cash prices for the single caplets are presented in Table 2. In accordance to the Black model, the price of a pure discount bond at the expiry date of the cap is also needed which is stated in the line for the 4^{th} June 2014. In accordance to this, the first caplet is worth 1.53 Cents at the 2^{nd} of June. Thus, a contract that ensures that one has to pay 2% interest only, for a one-year investment starting in one year, at a nominal value of one Euro is worth 1.53 Cents. The prices for the other Caplets are given in the last column in Table 2.

7.2 Valuing a Cap Agreement

In this section the results from Section 5 and section 5.3 are consolidated in order to price a cap agreement. The procedure, of how to price a cap agreement, shall be briefly reviewed.

At first a model for the short rate has to be selected. As the main focus of this thesis lies on Gaussian short rate models, one of the following most popular short rate models was applied: the Vasicek, the Ho-Lee or the Hull-White model.

In the next stage, all necessary parameters that define the short rate process in each model have to be determined. In the case of the Vasicek model the standard deviation was estimated from a proxy variable for the short rate, the EONIA, the mean reversion level and the rate of mean reversion on the other hand were chosen in order to minimize the difference between the observed term structure and the one implied by the Vasicek model. Due to the rather simple process described by the Vasicek model, the current term structure is not used as an input parameter like in the Ho-Lee and the Hull-White model.

Subsequently, for the Ho-Lee and the Hull-White model the current term structure is incorporated by calculating forward rates from observed bond prices as indicated in Equation (5). As only a restricted number of bond prices (namely for every month) were available, I interpolated the 29 values (based on the assumption that every month has got 30 days and every year 360 days) in-between geometrically (every interpolated value s_n is equal to $s_n = \sum_{k=0}^{n} a_0 q^k$ where a_0 is the start value of the interpolated series and q is the increment that ensures that the last available value is reached.). Afterwards short rate paths have to be simulated. While simulating these paths, the average value of the short rate has to be calculated for every caplet till maturity date, in order to discount the value of the caplet. Moreover, the average short rate has to be computed for the time during the maturity dates of the caplets, as forward rates for this period are needed to calculate the payoff of the caplets. The discounted values of the four caplets were all computed at once when all short rate paths were simulated. Thus, from every started short rate path, I deduced prices for all four caplets. Consequently, the discounted values for every caplet for every path had to be summed up and stored. After simulating several paths, the caplet prices are calculated as the average prices for all paths.

The number of simulated paths in a MC simulation is usually above one million. As the average calculation time for the three models is about half an hour in the case of one million paths, I will only present results of simulations with one hundred thousand paths. This approach was selected in order to achieve faster results as this is essential in real time trading. As the difference in the simulated prices and the explicit ones might be big in this case, variance reducing techniques will also be introduced and implemented. Additionally, the computation time for every model will be presented, in order to compare the procedures according to the computational effort.

7.2.1 Valuing a Cap Agreement in the Vasicek Model

The continuous process for the short rate in the Vasicek model is given in Equation (24). In Section 6.2.2 the value of ρ was estimated by assuming the EONIA as a proxy for the short rate. The values α and γ on the other hand, were set experimentally. Now the approach proposed by Brigo and Mercurio [27] will be completely followed for fitting the model to the term structure observed for the 4^{th} of June. As the Vasicek model does not fit the initial term structure automatically, the aim of this section will be to replicate market bond prices by the model as good as possible. For the volatility ρ of the short rate process, the estimated value 0.5 percentage points, from Section 6.2.2 will be applied. The initial value of the short rate process was set equal the actual value of the EONIA from the 2^{nd} of June, 0.75%. This interest rate has the same maturity as the simulated short rates, namely one day. As a result, the only model parameters that have to be found are α and γ. Thus, by combining different values for these parameters the Vasicek model can be fitted to a given term structure. On way to verify the value of α and γ that fit the observed bond prices best, is to minimize the following function, which I will refer to as $SSPR(\alpha, \gamma)$ (Sum of Squared Percentaged Residuals):

$$SSPR(\alpha,\gamma) = \min_{\alpha,\gamma} \quad \sqrt{\sum_{i=1}^{M} \left(\frac{model_i(\alpha,\gamma) - market_i}{market_i} \right)^2} \tag{43}$$

$model_i(\alpha, \gamma)$ denotes the i^{th} model implied price and $market_i$ denotes the i^{th} corresponding market price. This approach was proposed by Clewlow and Strickland [12] and was selected as it measures the deviations irrespective of the size of the prices. The parameters that fit the overall term structure best were found numerically via Microsoft Excel Solver. Thus, the parameters that fit the given term structure best are $\alpha = 0.8553$ and $\gamma = 0.0577$. In Figure 17 the yield curves according to the market data and the data implied by the Vasicek model calibrated as described are plotted. As it can be seen, although the curves look similar there are still deviations. It has to be noted that these yield curves are only based on bonds with full year maturity. Thus, it is not accounted for deviations in-between. After all coefficients were found, I calculated the caplet prices, as discussed in Section 5.1.1. The resulting prices are presented in Table 3. The coefficient of determination R^2 for the explicitly calculated prices and the observed market data is about 94%. Thus, 94% of the variation in the market data is explained by the prices deduced from the Vasicek model with the determined parameters. In the case of the Vasicek model the

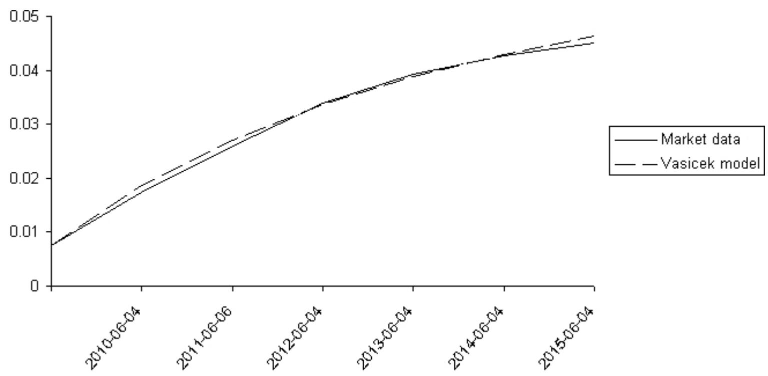

Figure 17: Yield curves implied by market data and the calibrated Vasicek model (based on bonds maturing at the maturity dates of the caplets)

short rate process was simulated one hundred thousand times according to Equation (27). The simulation results based on the calibrated Vasicek model and one hundred thousand short rate paths are presented in Table 3. The corresponding program code can be found in Appendix B.2.

In accordance to the remarks of Section 6.2.2, I also applied variance reduction techniques. The antithetic variate method was applied as discussed before. Thus, one step forward in a short rate paths comprises of the average change of the process based on the usual increment Wiener process and the very same increment multiplied by minus one. The program code for the simulation with antithetic variates in the Vasicek model is presented in Appendix B.3. For the control variate technique, I applied the observed bond prices as control variates. In order this, short rate paths were sampled beforehand to calculate the linear regression coefficient in-between the simulated bond and caplet prices. Afterwards, I simulated hundred thousand short rate paths, by applying the estimated optimal values for b in Equation (41). The program code for the simulation with control variates is presented in Appendix B.4.

	Caplet 1	Caplet 2	Caplet 3	Caplet 4	SSPR	Calculation time in seconds
Market prices	0.01541	0.02845	0.03162	0.0318		
Explicit prices	0.01518	0.02513	0.03057	0.03307	0.13487	
Simulated prices	0.0152	0.02514	0.03058	0.03308	0.13468	70
std. error	0.00405	0.0043	0.0042	0.00397		
Simulated prices (antithetic)	0.0152	0.02514	0.03057	0.03307	0.13442	74
std. error	0.00002	0.00004	0.00006	0.00007		
Simulated prices (control variate)	0.01517	0.02512	0.03056	0.03307	0.13525	72
std. error	0.00305	0.00333	0.00345	0.00344		

Table 3: Caplet prices simulated according to the Vasicek model

For the Vasicek model the simulated values do not fit the market data very well. The coefficient of determination for the simulated prices was about 94%, just as the one for the explicit prices. As the coefficient of determination is very similar for all three simulation results, the goodness of fit is determined according to the values of the $SSPR$ function. Although the explicit calculation of the caplet prices is supposed to return the most accurate prices, Table 3 shows that the simulation approaches (except for the one with control variates) returned smaller values of the $SSPR$. Thus, the simulated prices fit the market data better than the explicitly calculated prices. As the simulated prices are supposed to tend towards the explicit prices, it can be assumed that this improvement is coincidental. The benefits of using variance reducing techniques can be metered from the standard deviations of the simulated prices. As it can be seen, the standard deviations for the prices simulated with antithetic variates dropped sharply. Thus, this variance reducing technique is very powerful in this case. This is even more interesting as the change in computation time is negligible. For the caplet prices simulated with control variates, the results also indicate an improvement of the simulation as the standard deviations again decreased. Comparing this decrease with the one that followed from applying antithetic variates, it can be said that the latter was more intensive. Thus, the control variate technique does not seem to be that powerful in this case. This might be due to a poor control variate. Overall it can be said that the two variance reduction techniques improve the simulations, but at different magnitudes. The improvement is also visible, as the simulated prices show a tendency towards the explicit prices.

As only computation time hinders one to simulate more short rate paths, improvements of the program code are desirable. When simulating every single path exactly according to Equation (27), where the exponential function and the square root function have to be called at every single time step, the computation time is relatively high. Thus, by computing all constant values beforehand, the simulation time can be reduced by about two thirds.

7.2.2 Valuing a Cap Agreement in the Ho-Lee Model

The continuous formulation of the Ho-Lee model was given in Equation (30) and (31). As the forward rates can be deduced from the initial term structure, the only unknown parameter in this formulation is σ, the standard deviation of the short rate process. Differently to the Vasicek model, the Ho-Lee model already fits the initial term structure as the gradient of the forward curve is incorporated in discrete form. Thus, differently to the Vasicek model, the prices of the caplets and not bond prices are fitted to market prices. In order to calibrate the Ho-Lee model to the market data, the following function $SSPR(\sigma)$ has to be minimized:

$$SSPR(\sigma) = \min_{\sigma} \quad \sqrt{\sum_{i=1}^{M} \left(\frac{model_i(\sigma) - market_i}{market_i} \right)^2} \tag{44}$$

As for the Vasicek model, the solution for this problem was found by applying the Excel solver. As a result σ that minimizes this function is equal to 0.0126 percentage points. The coefficient of determination R^2 for the explicitly calculated prices and the observed market data is about 99.991%. Thus, 99.991% of the variation in the market data is explained by the prices deduced from the Ho-Lee model. As a result the Ho-Lee model fits the market data much better than the Vasicek model.

The simulations of the short rate paths, according to the Ho-Lee model were run by applying the exact discrete formulation of the continuous short rate process as given in Equation (32). The results of the simulation of one hundred thousand paths, with $\sigma = 0.01265$, is presented in Table 4. The corresponding program code is shown in Appendix B.5.

	Caplet 1	Caplet 2	Caplet 3	Caplet 4	SSPR	Calculation time in seconds
Market prices	0.01541	0.02845	0.03162	0.0318		
Explicit prices	0.01536	0.02844	0.03162	0.03183	0.01066	
Simulated prices	0.0156	0.0284	0.03154	0.03175	0.02144	123
std. error	0.0122	0.01639	0.01796	0.01858		
Simulated prices (antithetic)	0.01561	0.02842	0.03159	0.0318	0.02115	178
std. error	0.00225	0.0013	0.00134	0.0015		
Simulated prices (control variate)	0.01562	0.02842	0.0316	0.03181	0.02125	122
std. error	0.00838	0.01018	0.01098	0.01135		

Table 4: Caplet prices simulated according to the Ho-Lee model

In accordance to the implications of the coefficient of determination, $SSPR$ also indicates that the Ho-Lee model returns more accurate prices than the Vasicek model. Moreover, Table 4 shows that the explicit calculations lead to the best results, as $SSPR$ is the lowest in this case. The variance reduction techniques again improved the simulations as $SSPR$ decreased when applying them. In accordance to the results from the simulations in the Vasicek model, applying the bond prices as a control variate did not improve the simulations as much as the antithetic variates did. The very same story is told by the standard deviations. The standard deviations decreased for all prices whenever a variance reducing technique was applied, but the improvements achieved by the antithetic variates were much higher. As control variates, once again the bond prices were applied. The program code for the simulation with control variates can be found in Appendix B.7, whereas the one for the simulation with antithetic variates can be found in Appendix B.6.

Differently to the Vasicek model, there is no chance to avoid the call of a power function for every time step. Furthermore, the forward rate curve has to be included in the simulations. This amongst others is reflected in the increase of the computation time by about 70%. The computational burden due to functions, that have to be called in every time step, becomes even more apparent when applying antithetic variates. In this case the power function has to be called twice, hence the computation time increases by about 45%, compared with the standard simulation.

7.2.3 Valuing a Cap Agreement in the Hull-White Model

Differently to the Ho-Lee model, the Hull-White model has a parameter for mean reversion. The continuous Hull-White process for the short rate is presented in Equation (33) and (34). This process is again consistent with the initial term structure, due to the incorporation of the forward curve and its changes over time. σ and the speed of mean reversion have to be determined by calibrating the Hull-White model to a set of market prices. The function that has to be minimized now equals:

$$SSPR(\alpha, \sigma) = \min_{\alpha,\sigma} \sqrt{\sum_{i=1}^{M} \left(\frac{model_i(\alpha, \sigma) - market_i}{market_i} \right)^2} \tag{45}$$

Solving this equation, again via Excel Solver, lead to the values: $\alpha = 0.0213$ and $\sigma = 0.01317$. The coefficient of determination R^2 for the explicitly calculated prices and the observed market data is about 99.994%. Thus, 99.994% of the variation in the market data is explained by the prices deduced from the Hull-White model. As a result, the Hull-White model fits the market data much better than the Vasicek and slightly better than the Ho-Lee model. The value of the $SSPR$ function on the other hand indicates that the Ho-Lee model fits the market data better. This shows the incompatibility of these measures.

For the Hull-White model the discrete formulation of the short rate as in Equation (35) was applied. This formulation represents the discrete short rate process deduced from the continuous model, as in Equation (33), by applying the Euler scheme. As a result, the simulated values might incorporate a bias due to the discretization. Once again, I simulated one hundred thousand paths. The program code for the standard simulation can be found in Appendix B.8. The simulated caplet prices are presented in Table 5.

The values of the $SSPR$ function once again indicate that the explicitly computed prices are the most accurate ones. The simulated values once again return a worse fit. The variance reduction techniques return improvements as the values of $SSPR$ decrease. Moreover, the standard deviations also decreased notably. Differently to the results for the Vasicek and the Ho-Lee the Hull-White simulation

	Caplet 1	Caplet 2	Caplet 3	Caplet 4	SSPR	Calculation time in seconds
Market prices	0.01541	0.02845	0.03162	0.0318		
Explicit prices	0.01541	0.02845	0.03162	0.0318	0.01138	
Simulated prices	0.01555	0.02797	0.03081	0.03062	0.05867	70
std. error	0.01236	0.01649	0.01796	0.01842		
Simulated prices (antithetic)	0.01554	0.028	0.03081	0.03062	0.05825	79
std. error	0.00239	0.00139	0.00142	0.00159		
Simulated prices (control variate)	0.01554	0.02811	0.03107	0.03108	0.04232	78
std. error	0.00002	0.00003	0.00003	0.00003		

Table 5: Caplet prices simulated according to the Hull-White model

results imply that the control variate technique is more powerful than the antithetic variate technique. This can be deduced from the tremendously decreasing standard deviations, when applying observed bond prices as control variates. For the simulations in the Vasicek and the Ho-Lee model exact formulations in discrete time were applied. In the Hull-White model on the other hand, I applied an Euler approximation to simulate the process. This approximation might have cased an error that demonstrates in prices differing from the explicit ones, beyond the range of the standard deviations. According to the computational time it can be said, that the simulations according to the Hull-White model are nearly as fast as the ones in the Vasicek model. This might be caused by the rather simple process of the short rate due to the Euler approximation. Overall, it can be said that a tradeoff between accuracy and computational afford can be determined. Thus, in order to achieve more accurate results more computation time has to be accepted.

7.3 Pricing a Periodic Cap Agreement in the Hull-White Model

In this section the applicability of MC methods are demonstrated for pricing periodic cap agreements, in the Hull-White model. The short rate process from the Hull-White model was selected, as it fits the observed market data from Section 5.3 best (according to the coefficient of determination). A periodic cap agreement will be priced as the Hull-White model is already calibrated to a cap agreement. It shall be mentioned here again, that the applied calibration instruments should resemble the derivative that has be priced as much as possible. Thus, in order to price knock in/out swaps (see Section 8) the models should be calibrated to observed swap prices and their volatility structure.

Periodic caps and periodic floors incorporate a relative cap rate, as the latter depends on the evolution of the observed interest rate. The cap rate in a periodic cap agreement is determined by the variable interest rate, with a maturity equal to the time in-between the reset dates, plus an arbitrary percentage amount. At the maturity date of the first caplet, the cap rate is determined by the initial interest rate for one year plus the arbitrary amount. For the next caplet the cap rate is determined by the one year interest rate observed at the maturity date of the first caplet, plus the arbitrary amount. Thus, the cap rates are changing relatively to the evolution of the interest rates (see Dash [40]). As a result the owner of such a periodic cap can hedge herself against to intensive jumps in the interest rates he has to afford.

As an example a period cap agreement was assumed for the next five years. Differently to the example in Section 7.2.3, the cap rate is now flexible, for all caplets with maturity dates above one year in the future. The cap rate for the first caplet on the other hand is already determined, as 1.73%, which is the one-year rate, plus the predefined amount, which is assumed to be 1% in this example. Thus, the cap rate for the first caplet is equal to 2.73%. For all other caplets the cap rate differs for every single path, as the cap rate will then be determined by the one year forward rate plus 1%. Once again I simulated one hundred thousand short rate paths. The results for this periodic cap agreement are presented in Table 6. The program code for this example can be found in Appendix B.11.

	Caplet 1	**Caplet 2**	**Caplet 3**	**Caplet 4**	**Calculation time in seconds**
Simulated prices	0.00874	0.00675	0.00196	0.00116	74
std. error	0.00818	0.00726	0.004	0.00296	

Table 6: Periodic caplet prices simulated according to the Hull-White model

The price for the one year 1% periodic caplet with a nominal value of one Euro for example, would be 0.874 Cents. Comparing the prices of the standard caplets and the periodic ones, it can be said that the prices of the latter ones are much lower. This is due to the flexible cap rate that is always above 2%, which was the cap rate in the standard cap agreement. Consequently, the compensation payments for the holder of the periodic cap agreement are always smaller. In order to this the values of the single caplets and therefore the whole cap agreement has to be lower. As periodic cap agreements are path-dependent interest rate derivatives there are no closed formulas for them. Furthermore, no market data for such contracts were available. Thus, a comparison according to $SSPR$ is not possible.

8 Advantages of Using Monte Carlo Methods for the Valuation of Interest Rate Derivatives

An important question that has not been posed jet concerns the necessity of MC methods for valuing interest rate derivatives. In Section 5.3 I determined explicit solutions for the three presented models. Besides MC methods, interest rate derivatives can also be valued using lattice methods or finite difference methods, as mentioned in Section 6.2.1. Hence, why is it necessary to apply MC methods?

8.1 Necessity for Monte Carlo Methods

From a theoretical point of view the Black model does not return satisfying prices, as the valuation of different derivatives with differing underlyings leads to an inconsistent overall approach (see Section 6.1). Finite difference methods are based on solving iteratively a system of stochastic differential equations by approximating partial differentials through discrete formulations. Thus, by decreasing the step length, the solution converges to the solution of the differential equations. Lattice methods work out similar to finite difference methods, moreover the trinomial lattice is equivalent to apply finite difference methods. As MC simulations are comparatively inefficient according to Greco [41], the necessity of MC methods is questioned. In fact, finite difference methods start at the maturity date of a derivative and work through to the present value of the derivative. As a result, finite difference and lattice methods can be easily applied to derivatives of European and American style. MC methods, on the other hand, simulate the underlying value starting with the initial value of the process. Thus, especially path-dependent derivatives can be easily priced by applying MC methods. Greco [41] presents the following path-dependent interest rate derivatives:

- In the case of *index amortizing swaps* the principal declines (amortizes), when interest rates decrease. One period ahead, it is always determined whether the underlying interest rate has decreased or not. If it has decreased, the principal value decreases at the next time to maturity by a predefined percentage amount. If the underlying has not changed or even increased, the principal value stays the same (see London [42]).

- As it was discussed in Section 2.2.2, *Asian* options are options where the final payoff depends on the average value of the underlying.
- The payoff of *structured notes* does not depend solely on one underlying, but on several different indices, which would also have to be simulated.
- *Range notes* incorporate the possibility to earn a higher interest rate than the one observed in the market, if the spot interest rate is within a predefined range. If the spot rate lies outside this range the derivative pays nothing.
- In the case of *Knock in/out* swaps, the payoff depends on whether the underlying interest rate has exceeded a predefined level or not. It has to be noted that the payoff might be conditional on several predefined levels that have to be crossed.
- And Periodic caps and floors, as presented earlier.

For all these and similar products, MC methods are inevitable. The advantages of MC methods rely on their facile and flexible applicability, their independence of the dimensionality and the smooth parallelization. As the variety of interest rate derivatives is broad and the possibilities for new products are immense, applications of Monte Carlo methods will surely be of interest in the future.

9 Conclusion

The aim of this thesis was to discuss and implement Monte Carlo methods for pricing interest rate derivatives. Due to the wide range of different interest rate models, I focused on Gaussian short rate models. In consequence to this, short rate processes according to the Vasicek, the Ho-Lee and the Hull-White model were discussed and implemented in C++. I adjusted these models such that they replicate observed market prices best. As there are closed formulas available for caplets in all the three investigated models, it is easy to compare the explicit solutions. I showed that the Ho-Lee and the Hull-White model fitted the observed market prices much better than the Vasicek model. This result is not surprising as the Vasicek model is a rather rigid model in contrast to the Ho-Lee and the Hull-White model.

The additional effort for the explicit calculations in the Ho-Lee and the Hull-White model, was confined as the explicit formulas only incorporate single bond prices, that can easily be observed in the market. For the simulation of short rate paths on the other hand, the whole observed term structure had to be included. Thus, the changes of the instantaneous forward rate had to be incorporated for every single time step. Taken together, the implementation of the Vasicek model was rather plain, compared to the Ho-Lee and Hull-White model. The complexity of the implementation of the latter two models paid off, when comparing the simulations results. As for the explicit calculations, the Ho-Lee and the Hull-White model provided results much closer to the market data. Comparing the simulation results of the Ho-Lee and the Hull-White model it can be said that the Ho-Lee model lead to better results than Hull-White model, although the latter incorporates one additional parameter for fitting observed market prices. This result corresponds to the application of an approximation for the simulation of the continuous process in the Hull-White model. Thus, for further analysis techniques for reducing the discretization error are recommended.

As trading on interest rate derivatives proceeds continuously, it is important to achieve fast simulation results. Thus, the computational effort was an important aspect in this thesis. The discussion of this aspect has occurred in nearly every section, as there are multiple starting points to achieve faster simulations. The discussion ranges from applying a suitable transformation algorithm for uniform random numbers, to the selection of an adequate computer language, to the optimization of the program codes and finally to the implementation of variance reducing techniques. For transforming uniform random number, I applied the polar rejection. I adopted this method, as it is supposed to be the fastest and most accurate upon the investigated ones. At the beginning of this thesis, I was convinced of implementing the investigated short rate models in Visual Basic. For the first few examples it seemed to be sufficient to apply Visual Basic. Nevertheless, when I increased the number of simulated paths or when I carried out a convergence analysis, the implementation in Visual Basic was rather burdensome. In consequence to this, I implemented all models in C++. This lead to an enormous reduction of simulation time. In order to receive a first simulation result, I always implemented the short rate processes as presented. Afterwards, I sought for possible improvements of the program code. Consequently, I computed constant terms beforehand. This lead to noteworthy reductions in simulation time, especially when power, exponential or square root functions were computed beforehand. Finally, I implemented variance reductions techniques in order to improve the simulations. Thus, by applying antithetic and control variates, the sample standard deviations were reduced for all models. Although, the control variate technique is said to be one of the most powerful variance reduction techniques, the implementation of antithetic variates lead to much lower sample standard deviations in the Vasicek and the Ho-Lee model for these instruments. Thus, for further investigations it might be interesting wether a better control variate than observed bond prices can be applied for interest rate derivatives. Summarizing the simulation results applying variance reduction techniques it can be said, that the implementation of antithetic variates lead to remarkable improvements of the simulations by increasing simulation time imperceptibly. The poor results from applying control variates might be due to the low correlation in-between the control variates, the observed bond prices, and the cap prices.

As there are explicit formulas for pricing interest rate cap agreements in all three discussed models, the question arose whether it is necessary to apply a method that is computationally highly intensive or not. In Section 8, I presented the most common path-dependent interest rate derivatives. For pricing all these derivatives MC methods are inevitable. Thus, there is no other possibility for pricing these derivatives. Hull [1] cites that the underlying short rate models should always be calibrated to market prices of derivatives that resemble the ones of interest as much as possible. In order to this, I concentrated on pricing a periodic cap agreement. Consequently, I showed that the implementation is rather straight forward when the underlying short rate process is already defined.

Concluding this thesis, it can be said that MC methods are inevitable for most path-dependent interest rate derivatives. In order to price these derivatives, the whole simulation approach has to be questioned. For proper improvements of the latter, the trade-off between computational accuracy and computational effort has to be considered persistently.

A Itô's Lemma

Instead of deriving Itô's Lemma it will be shown here that the lemma can be deduced from results of the differential calculus [1]. A continuously differentiable function G on the variables x and y is assumed, where Δx indicates a small change in x analogously for y and ΔG the corresponding change in G. Using a Taylor series expansion ΔG can be represented as:

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial y}\Delta y + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\Delta x^2 + \frac{1}{2}\frac{\partial^2 G}{\partial y^2}\Delta y^2 + \frac{\partial^2 G}{\partial x \partial y}\Delta x \Delta y + \ldots$$

$$\text{for} \quad \lim_{\Delta x, \Delta y \to 0} \quad dG = \frac{\partial G}{\partial x}\,dx + \frac{\partial G}{\partial y}dy$$

Now it is assumed that x follows an Itô process, therefore:

$$dx = a(x,t)\,dt + b(x,t)\,dz \tag{46}$$

The change in a function G of this variable x and time t can therefore be written as:

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\Delta x^2 + \frac{1}{2}\frac{\partial^2 G}{\partial t^2}\Delta t^2 + \frac{\partial^2 G}{\partial x \partial t}\Delta x \Delta t + \ldots \tag{47}$$

In a discrete setting Equation (46) would look like:

$$\Delta x = a(x,t)\Delta t + b(x,t)\varepsilon\sqrt{\Delta t}$$

For the Taylor series expansion of the changes in G there is a quadratic term of Δx. Among others, this latter term consists of Δt. Although one can assume that the changes in x will be infinitesimal small, the quadratic term ofΔx will not be negligible. Moreover this term includes the quadratic increments of the Wiener process. As these increments are standard normally distributed the following has to hold: $\mathbb{E}(\varepsilon^2) - [\mathbb{E}(\varepsilon)]^2 = 1$. As the expected vale of ε is equal to zero, $\mathbb{E}(\varepsilon^2) = 1$ has to hold. The variance of $\varepsilon^2 \Delta t$ consists of quadratic terms of Δt, which diminishes when the time steps are assumed to be infinitesimal small. In accordance to this, the quadratic term of Δx will not be stochastic as $\Delta t \to 0$ and will therefore be equal to $\Delta x^2 = b^2 \Delta t$. Applying these acknowledgements to Equation (47) it follows that:

$$dG = \frac{\partial G}{\partial x}\, dx + \frac{\partial G}{\partial t}\, dt + \frac{1}{2}\frac{\partial^2 G}{\partial x^2} b^2\, dt$$

Substituting here dx with Equation (46), it follows:

$$dG = \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b\, dz$$

B Program Codes

In this section the applied program codes are presented. It has to be noted that the program codes of the main program were shortened for the preprocessors. The subprogram random() is presented in Section B.12, whereas the subprogram max() is denoted in Section B.13.

B.1 Pricing an European call option on a zero coupon bond

```
int main()
{
ofstream fout( "optionprice.txt" );
time_t start,end;
time(&start);
srand(time(0));
int paths=100000;
long double doublepaths=paths;
long double sigma=0.005;
long double alpha=0.3;
long double longr=0.05;
long double sigma2=pow(sigma,2)/2;
long double steplength=0.00277777777777777780000000000000;
long double term1=exp(-alpha*steplength);
long double term2=longr*(1-exp(-alpha*steplength));
long double term3=sigma*sqrt((1/(2*alpha))*(1-exp(-2*alpha*steplength)));
long double sigmasqrdt=sigma*sqrt(steplength);
long double sumoption=0;
for(int h=0; h<paths; h++)
{
    long double r=0.00979;
    long double bondr=0;
    long double discount=0;
    for (int i=0; i < 360; ++i)
    {
        r=term1*r+term2+term3*random();
        discount=discount+r;
    }
    for (int i=360; i < 3600; ++i)
    {
        r=term1*r+term2+term3*random();
        bondr=bondr+r;
    }
    sumoption = sumoption +exp(-discount/360)*max(exp(-bondr/360)-0.5,0);
}
fout << sumoption/doublepaths;
fout << endl;
time(&end);
double dif=difftime(end,start);
cout << "Calculation Time  "<<dif<<"\n";
cin.get();
}
```

B.2 Vasicek Model: Simulation of a 5 year cap agreement

```
int main()
{
time_t start,end;
time(&start);
srand(time(0));
int paths=100000;
long double pathsdouble=paths;
long double sigma=0.005;
long double alpha=0.408391354946227000000000000000;
long double longr=0.069370233769936100000000000000;
long double steplength=0.002777777777777780000000000000;
long double sumrpayoff;
long double sigmasqrdt=sigma*sqrt(steplength);
long double payoff[7]={0};
long double payoffsquared[7]={0};
long double timesteps=1800;
long double term1= exp(-alpha*steplength);
long double term2=longr*(1-term1);
long double term3=sigma*sqrt((1/(2*alpha))*(1-exp(-2*alpha*steplength)));
long double capamount=exp(-0.02);
long double discountcap=1/capamount;
long double payoffamount;
for(int h=0; h<paths; h++)
{
    long double r=0.0075;
    long double bond=0;
    long double sumr=0;
    long double sumrpayoff=0;
    long double discount=0;
    for (int i=0; i < timesteps; ++i)
    {
        long double idouble=i;
        r=term1*r+term2+term3*random();//r=r+alpha*(longr-r)*steplength+sigmasqrdt*random();
        sumr=sumr+r;
        sumrpayoff=sumrpayoff+r;
        if ((i+1)%360==0)
        {
            int o=(i+1)/360;
            if ((i+1)/360>1)
            {
                payoffamount=exp(-sumrpayoff/360);
                payoff[o-2]=payoff[o-2]+(1/capamount)*max(capamount-payoffamount,0)*bond;
                payoffsquared[o-2]=payoffsquared[o-2]+pow(discountcap*max(capamount-payoffamount,0)*bond,2);
            }
            bond=exp(-sumr/360);
            sumrpayoff=0;
        }
    }
}
cout << "Caplet 1:   "<<payoff[0]/paths<<"\n";
cout << "Standard Error:"<<sqrt((payoffsquared[0]-(payoff[0]*payoff[0])/paths)/(paths-1))<<"\n";
cout << "Caplet 2:   "<<payoff[1]/paths<<"\n";
cout << "Standard Error:"<<sqrt((payoffsquared[1]-(payoff[1]*payoff[1])/paths)/(paths-1))<<"\n";
cout << "Caplet 3:   "<<payoff[2]/paths<<"\n";
cout << "Standard Error:"<<sqrt((payoffsquared[2]-(payoff[2]*payoff[2])/paths)/(paths-1))<<"\n";
cout << "Caplet 4:   "<<payoff[3]/paths<<"\n";
cout << "Standard Error:"<<sqrt((payoffsquared[3]-(payoff[3]*payoff[3])/paths)/(paths-1))<<"\n";
time(&end); double dif=difftime(end,start);
cout << "Calculation Time  "<<dif<<"\n";
cin.get();
}
```

B.3 Vasicek Model: Simulation of a 5 year cap agreement with antithetic variates

```
int main()
{
time_t start,end;time(&start);
srand(time(0));
int paths=100000;
long double pathsdouble=paths;
long double sigma=0.005;
long double alpha=0.40839135494622700000000000000;
long double longr=0.069370233769936100000000000000;
long double steplength=0.0027777777777777800000000000000;
int timesteps=1800;
long double sumrpayoff;
long double sigmasqrdt=sigma*sqrt(steplength);
long double payoff[7]={0};
long double payoffsquared[7]={0};
long double term1= exp(-alpha*steplength);
long double term2=longr*(1-term1);
long double term3=sigma*sqrt((1/(2*alpha))*(1-exp(-2*alpha*steplength)));
long double capamount=exp(-0.02);
long double discountcap=1/capamount;
long double payoffamount1;
long double payoffamount2;
for(int h=0;h<paths; h++)
{
    long double r1=0.0075;
    long double bond1=0;
    long double sumr1=0;
    long double r2=0.0075;
    long double bond2=0;
    long double sumr2=0;
    long double sumrpayoff1=0;
    long double sumrpayoff2=0;
    for (int i=0; i < timesteps; ++i)
    {
        long double idouble=i;
        long double epsilon=random();
        r1=term1*r1+term2+term3*epsilon;//r1=r1+alpha*(longr-r1)*steplength+sigmasqrdt*epsilon;
        r2=term1*r2+term2-term3*epsilon;//r2=r2+alpha*(longr-r2)*steplength-sigmasqrdt*epsilon;
        sumr1=sumr1+r1;
        sumr2=sumr2+r2;
        sumrpayoff1=sumrpayoff1+r1;
        sumrpayoff2=sumrpayoff2+r2;
        if ((i+1)%360==0)
        {
            int o=(i+1)/360;
            if ((i+1)/360>1)
            {
                payoffamount1=exp(-sumrpayoff1/360);
                payoffamount2=exp(-sumrpayoff2/360);
                payoff[o-2]=    payoff[o-2]+0.5*((1/capamount)*max(capamount-payoffamount1,0)*bond1
                                +(1/capamount)*max(capamount-payoffamount2,0)*bond2);
                payoffsquared[o-2]= payoffsquared[o-2]+pow(0.5*((1/capamount)*max(capamount-payoffamount1,0)*bond1+
                                    (1/capamount)*max(capamount-payoffamount2,0)*bond2),2);
            }
            bond1=exp(-sumr1/360);
            bond2=exp(-sumr2/360);
            sumrpayoff1=0;
            sumrpayoff2=0;
        }
    }
}
cout << "Caplet 1:"<<payoff[0]/pathsdouble<<"\n";
cout <<"Standard Error:"<<sqrt((payoffsquared[0]-(payoff[0]*payoff[0])/pathsdouble)/(pathsdouble-1))<<"\n";
cout << "Caplet 2:"<<payoff[1]/pathsdouble<<"\n";
cout <<"Standard Error:"<<sqrt((payoffsquared[1]-(payoff[1]*payoff[1])/pathsdouble)/(pathsdouble-1))<<"\n";
cout << "Caplet 3:"<<payoff[2]/pathsdouble<<"\n";
cout <<"Standard Error:"<<sqrt((payoffsquared[2]-(payoff[2]*payoff[2])/pathsdouble)/(pathsdouble-1))<<"\n";
cout << "Caplet 4:"<<payoff[3]/pathsdouble<<"\n";
cout <<"Standard Error:"<<sqrt((payoffsquared[3]-(payoff[3]*payoff[3])/pathsdouble)/(pathsdouble-1))<<"\n";
time(&end);
double dif=difftime(end,start);
cout << "Calculation Time"<<dif<<"\n";
cin.get();
}
```

B.4 Vasicek Model: Simulation of a 5 year cap agreement with a control variate

```
int main()
{
time_t start,end;
time(&start);
srand(time(0));
int paths=100000;
long double doublepaths=paths;
long double sigma=0.005;
long double alpha=0.408391354946227000000000000000;
long double longr=0.069370233769936100000000000000;
long double steplength=0.002777777777777780000000000000;
long double term1=exp(-alpha*steplength);
long double term2=longr*(1-term1);
long double term3=sigma*sqrt((1/(2*alpha))*(1-exp(-2*alpha*steplength)));
long double capamount=exp(-0.02);
long double discountcap=1/capamount;
long double payoffamount;
long double bond;
long double explicitbond[]={0.981598533, 0.947277262, 0.903879138,
0.856014443};
signed long double b[]={1.0807,0.466422,0.264835,0.167488};
long double sumrpayoff;
long double payoff [5]={0};
long double payoffsquared [5]={0};
long double timesteps=1800;
for(int h=0; h<paths; h++)
{
    long double r=0.0075;
    long double sumr=0;
    long double sumrpayoff=0;
    long double discount=0;
    for (int i=0; i < timesteps; ++i)
    {
        long double idouble=i;
        r=term1*r+term2+term3*random();//r=r+alpha*(longr-r)*steplength+sigmasqrdt*random();
        sumr=sumr+r;
        sumrpayoff=sumrpayoff+r;
        if ((i+1)%360==0)
        {
            int o=(i+1)/360;
            if ((i+1)/360>1)
            {
                payoffamount=exp(-sumrpayoff/360);
                payoff[o-2]=payoff[o-2]+(1/capamount)*max(capamount-payoffamount,0)*bond+b[o-2]*(bond-explicitbond[o-2]);
                payoffsquared[o-2]=payoffsquared[o-2]
                +pow((1/capamount)*max(capamount-payoffamount,0)*bond+b[o-2]*(bond-explicitbond[o-2]),2);
            }
            bond=exp(-sumr/360);
            sumrpayoff=0;
        }
    }
}
cout << "Caplet 1:"<<payoff[0]/paths<<"\n";
cout << "Standard Error:"<<sqrt((payoffsquared[0]-(payoff[0]*payoff[0])/paths)/(paths-1))<<"\n";
cout << "Caplet 2:"<<payoff[1]/paths<<"\n";
cout << "Standard Error:"<<sqrt((payoffsquared[1]-(payoff[1]*payoff[1])/paths)/(paths-1))<<"\n";
cout << "Caplet 3:"<<payoff[2]/paths<<"\n";
cout << "Standard Error:"<<sqrt((payoffsquared[2]-(payoff[2]*payoff[2])/paths)/(paths-1))<<"\n";
cout << "Caplet 4:"<<payoff[3]/paths<<"\n";
cout << "Standard Error:"<<sqrt((payoffsquared[3]-(payoff[3]*payoff[3])/paths)/(paths-1))<<"\n";
time(&end);
time(&end);
double dif=difftime(end,start);
cout <<"Calculation Time"<<dif<<"\n";
cin.get();
}
```

B.5 Ho-Lee Model: Simulation of a 5 year cap agreement

Note: The increments of $\theta(t)$, stored in difference[], were deleted from this code.

```
int main()
{
time_t
start,end;
time(&start);
srand(time(0));
int paths=100000;
long double pathsdouble=paths;
long double sigma=0.012653215911395000000000000000;
long double sigma2=pow(sigma,2)/2;
long double steplength=0.00277777777777777780000000000000;
long double sigmasqrdt=sigma*sqrt(steplength);
long double capamount=exp(-0.02);
long double discountcap=1/capamount;
long double payoffamount;
signed long double difference[]={};
long double sumrpayoff;
long double payoff[7]={0};
long double payoffsquared[7]={0};
long double timesteps=1800;
for(int h=0; h<paths; h++)
{
    long double r=0.0075;
    long double sumr=0;
    long double bond=0;
    long double sumrpayoff=0;
    long double discount=0;
    for (int i=0; i < timesteps; ++i)
    {
        long double idouble=i;
        r=r+difference[i]+sigma2*(pow((idouble+1)/360,2)-pow((idouble)/360,2))+sigmasqrdt*random();
        sumr=sumr+r;
        sumrpayoff=sumrpayoff+r;
        if ((i+1)%360==0)
        {
            int o=(i+1)/360;
            if ((i+1)/360>1)
            {
                payoffamount=exp(-sumrpayoff/360);
                payoff[o-2]=payoff[o-2]+(1/capamount)*max(capamount-payoffamount,0)*bond;
                payoffsquared[o-2]=payoffsquared[o-2]+pow(discountcap*max(capamount-payoffamount,0)*bond,2);
            }
            bond=exp(-sumr/360);
            sumrpayoff=0;
        }
    }
}
cout << "Caplet 1:"<<payoff[0]/pathsdouble<<"\n";
cout <<"Standard Error:"<<sqrt((payoffsquared[0]-payoff[0]*payoff[0]/pathsdouble)/(pathsdouble-1))<<"\n";
cout << "Caplet 2:"<<payoff[1]/pathsdouble<<"\n";
cout <<"Standard Error:"<<sqrt((payoffsquared[1]-payoff[1]*payoff[1]/pathsdouble)/(pathsdouble-1))<<"\n";
cout << "Caplet 3:"<<payoff[2]/pathsdouble<<"\n";
cout <<"Standard Error:"<<sqrt((payoffsquared[2]-payoff[2]*payoff[2]/pathsdouble)/(pathsdouble-1))<<"\n";
cout << "Caplet 4:"<<payoff[3]/pathsdouble<<"\n";
cout <<"Standard Error:"<<sqrt((payoffsquared[3]-payoff[3]*payoff[3]/pathsdouble)/(pathsdouble-1))<<"\n";
time(&end); double dif=difftime(end,start);
cout << "Calculation Time  "<<dif<<"\n";
cin.get();
}
```

B.6 Ho-Lee Model: Simulation of a 5 year cap agreement with antithetic variates

Note: The increments of $\theta(t)$, stored in difference[], were deleted from this code.

```
int main()
{
time_t start,end;
time(&start);
srand(time(0));
int paths=100000;
long double pathsdouble=paths;
long double sigma=0.0126532159113950000000000000;
long double sigma2=pow(sigma,2)/2;
long double steplength=0.0027777777777777780000000000000;
long double sigmasqrdt=sigma*sqrt(steplength);
long double capamount=exp(-0.02);
long double discountcap=1/capamount;
long double payoffamount1;
long double payoffamount2;
signed long double difference[]={};
long double sumrpayoff;
long double payoff[7]={0};
long double payoffsquared[7]={0};
long double timesteps=1800;
for(int h=0; h<paths; h++)
{
    long double bond1=0;
    long double bond2=0;
    long double r1=0.0075;
    long double r2=0.0075;
    long double sumr1=0;
    long double sumr2=0;
    long double sumrpayoff1=0;
    long double sumrpayoff2=0;
    for (int i=0; i < timesteps; ++i)
    {
        long double idouble=i;
        long double epsilon=random();
        r1=r1+difference[i]+sigma2*(pow((idouble+1)/360,2)-pow((idouble)/360,2))+sigmasqrdt*epsilon;
        r2=r2+difference[i]+sigma2*(pow((idouble+1)/360,2)-pow((idouble)/360,2))-sigmasqrdt*epsilon;
        sumr1=sumr1+r1;
        sumr2=sumr2+r2;
        sumrpayoff1=sumrpayoff1+r1;
        sumrpayoff2=sumrpayoff2+r2;
        if ((i+1)%360==0)
        {
            int o=(i+1)/360;
            if ((i+1)/360>1)
            {
                payoffamount1=exp(-sumrpayoff1/360);
                payoffamount2=exp(-sumrpayoff2/360);
                payoff[o-2]=payoff[o-2]+0.5*(discountcap*max(capamount-payoffamount1,0)*bond1
                            +discountcap*max(capamount-payoffamount2,0)*bond2);
                payoffsquared[o-2]= payoffsquared[o-2]+pow(0.5*(discountcap*max(capamount-payoffamount1,0)*bond1
                                    +discountcap*max(capamount-payoffamount2,0)*bond2),2);
            }
            bond1=exp(-sumr1/360);
            bond2=exp(-sumr2/360);
            sumrpayoff1=0;
            sumrpayoff2=0;
        }
    }
}
cout << "Caplet 1:"<<payoff[0]/pathsdouble<<"\n";
cout <<"Standard Error:"<<sqrt((payoffsquared[0]-payoff[0]*payoff[0]/pathsdouble)/(pathsdouble-1))<<"\n";
cout << "Caplet 2:"<<payoff[1]/pathsdouble<<"\n";
cout <<"Standard Error:"<<sqrt((payoffsquared[1]-payoff[1]*payoff[1]/pathsdouble)/(pathsdouble-1))<<"\n";
cout << "Caplet 3:"<<payoff[2]/pathsdouble<<"\n";
cout <<"Standard Error:"<<sqrt((payoffsquared[2]-payoff[2]*payoff[2]/pathsdouble)/(pathsdouble-1))<<"\n";
cout << "Caplet 4:"<<payoff[3]/pathsdouble<<"\n";
cout <<"Standard Error:"<<sqrt((payoffsquared[3]-payoff[3]*payoff[3]/pathsdouble)/(pathsdouble-1))<<"\n";
time(&end);
double dif=difftime(end,start);
cout << "Calculation Time"<<dif<<"\n";
cin.get();
}
```

B.7 Ho-Lee Model: Simulation of a 5 year cap agreement with a control variate

Note: The increments of $\theta(t)$, stored in difference[], were deleted from this code.

```
int main()
{
time_t start,end;
time(&start); srand(time(0));
int paths=100000;
long double doublepaths=paths;
long double sigma=0.0126532159113950000000000000;
long double sigma2=pow(sigma,2)/2;
long double steplength=0.0027777777777777800000000000000;
long double sigmasqrdt=sigma*sqrt(steplength);
long double capamount=exp(-0.02);
long double discountcap=1/capamount;
long double payoffamount;
long double bond;
long double explicitbond[]={0.982838484311106000000000000000,
0.949199967737070000000000000000,
0.903014155972148000000000000000,
0.854738281615469000000000000000,
0.807421108299888000000000000000};
signed long double b[]={1.22418,0.65277,0.415419, 0.293846};
signed long double difference[]={};
long double sumrpayoff;
long double payoff [5]={0};
long double payoffsquared [5]={0};
long double timesteps=1800;
for(int h=0; h<paths; h++)
    {
    long double r=0.0075;
    long double sumr=0;
    long double sumrpayoff=0;
    long double discount=0;
    for (int i=0; i < timesteps; ++i)
    {
        long double idouble=i;
        r=r+difference[i]+sigma2*(pow((idouble+1)/360,2)-pow((idouble)/360,2))+sigmasqrdt*random();
        sumr=sumr+r;
        sumrpayoff=sumrpayoff+r;
        if ((i+1)%360==0)
        {
            int o=(i+1)/360;
            if ((i+1)/360>1)
            {
                payoffamount=exp(-sumrpayoff/360);
                payoff[o-2]=payoff[o-2]+(1/capamount)*max(capamount-payoffamount,0)*bond+b[o-2]*(bond-explicitbond[o-2]);
                payoffsquared[o-2]=payoffsquared[o-2]
                +pow((1/capamount)*max(capamount-payoffamount,0)*bond+b[o-2]*(bond-explicitbond[o-2]),2);
            }
            bond=exp(-sumr/360);
            sumrpayoff=0;
        }
    }
}
cout << "Caplet 1:   "<<payoff[0]/paths<<"\n";
cout << "Standard Error: "<<(sqrt(payoffsquared[0]-(payoff[0]*payoff[0])/paths)/(paths-1))<<"\n";
cout << "Caplet 2:   "<<payoff[1]/paths<<"\n";
cout << "Standard Error: "<<(sqrt(payoffsquared[1]-(payoff[1]*payoff[1])/paths)/(paths-1))<<"\n";
cout << "Caplet 3:   "<<payoff[2]/paths<<"\n";
cout << "Standard Error: "<<(sqrt(payoffsquared[2]-(payoff[2]*payoff[2])/paths)/(paths-1))<<"\n";
cout << "Caplet 4:   "<<payoff[3]/paths<<"\n";
cout << "Standard Error: "<<(sqrt(payoffsquared[3]-(payoff[3]*payoff[3])/paths)/(paths-1))<<"\n";
time(&end);
double dif=difftime(end,start);
cout << "Calculation
Time  "<<dif<<"\n";
cin.get();
}
```

B.8 Hull-White Model: Simulation of a 5 year cap agreement

Note: The increments of $\theta(t)$ were deleted from this code.

```
int main()
{
time_t start,end;
time(&start);
srand(time(0));
long double paths=100000;
long double sigma=0.013170825943613200000000000000;
long double alpha=0.021352409494617600000000000000;
long double sigma2=pow(sigma,2)/2;
long double steplength=0.002777777777777778000000000000;
long double sigmasqrdt=sigma*sqrt(steplength);
long double capamount=exp(-0.02);
long double discountcap=1/capamount;
long double payoffamount;
signed long double theta[]={};
long double sumrpayoff;
long double payoff[7]={0};
long double payoffsquared[7]={0};
long double timesteps=1800;
long double bond=0;
for(int h=0;h<paths; h++)
{
    long double r=0.0075;
    long double sumr=0;
    long double sumrpayoff=0;
    for (int i=0; i < timesteps; ++i)
    {
        long double idouble=i;
        r=r+(theta[i]-alpha*r)*steplength+sigmasqrdt*random();
        sumr=sumr+r;
        sumrpayoff=sumrpayoff+r;
        if ((i+1)%360==0)
        {
            int o=(i+1)/360;
            if ((i+1)/360>1)
            {
                payoffamount=exp(-sumrpayoff/360);
                payoff[o-2]=payoff[o-2]+(1/capamount)*max(capamount-payoffamount,0)*bond;
                payoffsquared[o-2]=payoffsquared[o-2]+pow(discountcap*max(capamount-payoffamount,0)*bond,2);
            }
            bond=exp(-sumr/360);
            sumrpayoff=0;
        }
    }
}
cout << "Caplet 1:"<<payoff[0]/paths<<"\n";
cout << "Standard Error:"<<sqrt((payoffsquared[0]-(payoff[0]*payoff[0])/paths)/(paths-1))<<"\n";
cout << "Caplet 2:"<<payoff[1]/paths<<"\n";
cout << "Standard Error:"<<sqrt((payoffsquared[1]-(payoff[1]*payoff[1])/paths)/(paths-1))<<"\n";
cout << "Caplet 3:"<<payoff[2]/paths<<"\n";
cout << "Standard Error:"<<sqrt((payoffsquared[2]-(payoff[2]*payoff[2])/paths)/(paths-1))<<"\n";
cout << "Caplet 4:"<<payoff[3]/paths<<"\n";
cout << "Standard Error:"<<sqrt((payoffsquared[3]-(payoff[3]*payoff[3])/paths)/(paths-1))<<"\n";
time(&end);
time(&end);
double dif=difftime(end,start);
cout <<"Calculation Time"<<dif<<"\n";
cin.get();
}
```

B.9 Hull-White Model: Simulation of a 5 year cap agreement with antithetic variates

Note: The increments of $\theta(t)$ were deleted from this code.

```
int main()
{
time_t start,end;
time(&start);
srand(time(0));
long double paths=100000;
long double pathsdouble=paths;
long double
sigma=0.0131708259436132000000000000000;
long double alpha=0.0213524094946176000000000000000;
long double sigma2=pow(sigma,2)/2;
long double steplength=0.0027777777777777780000000000000;
long double sigmasqrdt=sigma*sqrt(steplength);
long double capamount=exp(-0.02);
long double discountcap=1/capamount;
long double payoffamount1;
long double payoffamount2;
long double bond1;
long double bond2;
signed long double theta[]={};
long double sumrpayoff;
long double payoff[7]={0};
long double payoffsquared[7]={0};
long double timesteps=1800;
for(int h=0; h<paths; h++)
{
    long double r1=0.0075;
    long double r2=0.0075;
    long double sumr1=0;
    long double sumr2=0;
    long double sumrpayoff1=0;
    long double sumrpayoff2=0;
    long double discount1=0;
    long double discount2=0;
    for (int i=0; i < timesteps; ++i)
    {
        long double idouble=i;
        long double epsilon=random();
        r1=r1+(theta[i]-alpha*r1)*steplength+sigmasqrdt*epsilon;
        r2=r2+(theta[i]-alpha*r2)*steplength-sigmasqrdt*epsilon;
        sumr1=sumr1+r1;
        sumr2=sumr2+r2;
        sumrpayoff1=sumrpayoff1+r1;
        sumrpayoff2=sumrpayoff2+r2;
        if ((i+1)%360==0)
        {
            int o=(i+1)/360;
            if ((i+1)/360>1)
            {
                payoffamount1=exp(-sumrpayoff1/360);
                payoffamount2=exp(-sumrpayoff2/360);
                payoff[o-2]=payoff[o-2]+0.5*((1/capamount)*max(capamount-payoffamount1,0)*bond1
                            +(1/capamount)*max(capamount-payoffamount2,0)*bond2);
                payoffsquared[o-2]= payoffsquared[o-2]+pow(0.5*((1/capamount)*max(capamount-payoffamount1,0)*bond1
                                    +(1/capamount)*max(capamount-payoffamount2,0)*bond2),2);
            }
            bond1=exp(-sumr1/360);
            bond2=exp(-sumr2/360);
            sumrpayoff1=0;
            sumrpayoff2=0;
        }
    }
}
cout << "Caplet 1:"<<payoff[0]/pathsdouble<<"\n";
cout << "Standard Error:"<<sqrt((payoffsquared[0]-(payoff[0]*payoff[0])/pathsdouble)/(pathsdouble-1))<<"\n";
cout << "Caplet 2:"<<payoff[1]/pathsdouble<<"\n";
cout <<"Standard Error:"<<sqrt((payoffsquared[1]-(payoff[1]*payoff[1])/pathsdouble)/(pathsdouble-1))<<"\n";
cout << "Caplet 3:"<<payoff[2]/pathsdouble<<"\n";
cout <<"Standard Error:"<<sqrt((payoffsquared[2]-(payoff[2]*payoff[2])/pathsdouble)/(pathsdouble-1))<<"\n";
cout << "Caplet 4:"<<payoff[3]/pathsdouble<<"\n";
cout <<"Standard Error:"<<sqrt((payoffsquared[3]-(payoff[3]*payoff[3])/pathsdouble)/(pathsdouble-1))<<"\n";
time(&end);
double dif=difftime(end,start);
cout << "Calculation Time"<<dif<<"\n";
cin.get();
}
```

B.10 Hull-White Model: Simulation of a 5 year cap agreement with a control variate

Note: The increments of $\theta(t)$ were deleted from this code.

```
int main()
{
time_t start,end;
time(&start);
srand(time(0));
int paths=100000;
long double doublepaths=paths;
long double sigma=0.01317082594361320000000000000;
long double alpha=0.02135240949461760000000000000;
long double sigma2=pow(sigma,2)/2;
long double steplength=0.00277777777777778000000000000;
long double sigmasqrdt=sigma*sqrt(steplength);
long double capamount=exp(-0.02);
long double discountcap=1/capamount;
long double payoffamount;
long double bond; long double
explicitbond[]={0.98283848431110600000000000000,
0.94919996773707000000000000000,
0.90301415597214800000000000000,
0.85473828161546900000000000000,
0.80742110829988800000000000000};
signed long double b[]={1.20546,0.635602,0.401353, 0.282529};
signed long double theta[]={};
long double sumrpayoff;
long double payoff [5]={0};
long double payoffsquared [5]={0};
long double timesteps=1800;
for(int h=0; h<paths; h++)
{
    long double r=0.0075;
    long double sumr=0;
    long double sumrpayoff=0;
    long double discount=0;
    for (int i=0; i < timesteps; ++i)
    {
        long double idouble=i;
        r=r+(theta[i]-alpha*r)*steplength+sigmasqrdt*random();
        sumr=sumr+r;
        sumrpayoff=sumrpayoff+r;
        if ((i+1)%360==0)
        {
            int o=(i+1)/360;
            if ((i+1)/360>1)
            {
                payoffamount=exp(-sumrpayoff/360);
                payoff[o-2]=payoff[o-2]+(1/capamount)*max(capamount-payoffamount,0)*bond+b[o-2]*(bond-explicitbond[o-2]);
                payoffsquared[o-2]=payoffsquared[o-2]
                +pow((1/capamount)*max(capamount-payoffamount,0)*bond+b[o-2]*(bond-explicitbond[o-2]),2);
            }
            bond=exp(-sumr/360);
            sumrpayoff=0;
        }
    }
}
cout << "Caplet 1:"<<payoff[0]/paths<<"\n";
cout << "Standard Error:"<<(sqrt(payoffsquared[0]-(payoff[0]*payoff[0])/paths)/(paths-1))<<"\n";
cout << "Caplet 2:"<<payoff[1]/paths<<"\n";
cout << "Standard Error:"<<(sqrt(payoffsquared[1]-(payoff[1]*payoff[1])/paths)/(paths-1))<<"\n";
cout << "Caplet 3:"<<payoff[2]/paths<<"\n";
cout << "Standard Error:"<<(sqrt(payoffsquared[2]-(payoff[2]*payoff[2])/paths)/(paths-1))<<"\n";
cout << "Caplet 4:"<<payoff[3]/paths<<"\n";
cout << "Standard Error:"<<(sqrt(payoffsquared[3]-(payoff[3]*payoff[3])/paths)/(paths-1))<<"\n";
time(&end);
double dif=difftime(end,start);
cout << "Calculation Time  "<<dif<<"\n";
cin.get();
}
```

B.11 Hull-White Model: Simulation of a periodic cap agreement

Note: The increments of $\theta(t)$ were deleted from this code.

```
int main()
{
time_t start,end;
time(&start);
srand(time(0));
long double paths=100000;
long double sigma=0.01317082594361320000000000000;
long double alpha=0.02135240949461760000000000000;
long double sigma2=pow(sigma,2)/2;
long double steplength=0.00277777777777778000000000000;
long double sigmasqrdt=sigma*sqrt(steplength);
long double capamount;
long double payoffamount;
signed long double theta[]={};
long double sumrpayoff;
long double payoff[7]={0};
long double payoffsquared[7]={0};
long double timesteps=1800;
long double bond=0;
for(int h=0; h<paths; h++)
{
    long double r=0.0075;
    long double caprate=0.0273;
    long double sumr=0;
    long double sumrpayoff=0;
    for (int i=0; i < timesteps; ++i)
    {
        long double idouble=i;
        r=r+(theta[i]-alpha*r)*steplength+sigmasqrdt*random();
        sumr=sumr+r;
        sumrpayoff=sumrpayoff+r;
        if ((i+1)%360==0)
        {
            int o=(i+1)/360;
            if ((i+1)/360>1)
            {
                payoffamount=exp(-sumrpayoff/360);
                payoff[o-2]=payoff[o-2]+(1/capamount)*max(capamount-payoffamount,0)*bond;
                payoffsquared[o-2]=payoffsquared[o-2]+pow((1/capamount)*max(capamount-payoffamount,0)*bond,2);
            }
            capamount=exp(-sumrpayoff/360-0.01);
            bond=exp(-sumr/360);
            sumrpayoff=0;
        }
    }
}
cout << "Caplet 1:   "<<payoff[0]/paths<<"\n";
cout << "Standard Error:"<<sqrt((payoffsquared[0]-(payoff[0]*payoff[0])/paths)/(paths-1))<<"\n";
cout << "Caplet 2:   "<<payoff[1]/paths<<"\n";
cout << "Standard Error:"<<sqrt((payoffsquared[1]-(payoff[1]*payoff[1])/paths)/(paths-1))<<"\n";
cout << "Caplet 3:   "<<payoff[2]/paths<<"\n";
cout << "Standard Error:"<<sqrt((payoffsquared[2]-(payoff[2]*payoff[2])/paths)/(paths-1))<<"\n";
cout << "Caplet 4:   "<<payoff[3]/paths<<"\n";
cout << "Standard Error:"<<sqrt((payoffsquared[3]-(payoff[3]*payoff[3])/paths)/(paths-1))<<"\n";
time(&end);
time(&end);
double dif=difftime(end,start);
cout <<"Calculation Time  "<<dif<<"\n";
cin.get();
}
```

B.12 The subprogram random()

```
static long double z2;
static bool  second=false;
long double random()
{
long double x1;
long double x2;
long double w;
long double c;
long double snd;
if (second==true)
{
    snd= z2;
    second=false;
}
else
{
    do
    {
        x1=2*rand()/static_cast<double>(RAND_MAX)-1;
        x2=2*rand()/static_cast<double>(RAND_MAX)-1;
        w=x1*x1+x2*x2;
    }
    while (w>=1);
    c =sqrt(-2*log(w)/w);
    snd=c*x2;
    z2=c*x1;
    second=true;
}
return snd;
}
```

B.13 The subprogram max()

```
long double  max(long double x, long double y)
{
    if(x<y)
    {
        return y;
    }
    else
    {
        return x;
    }
}
```

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Abstract

This thesis presents the applicability of Monte Carlo methods for the valuation of interest rate derivatives. As the spectrum of interest rate models is wide, I focus on the Vasicek model, the Ho-Lee model and the Hull-White model. These three Gaussian short rate models are implemented in C++. In the course of that, I present improvements of the corresponding simulations. These improvements range from selecting suitable algorithms for transforming uniform pseudo random numbers to standard normal pseudo random numbers, to the selection of a suitable programming language, to the optimization of the program codes and finally to the implementation of variance reducing techniques. In order to improve the standard deviation of the simulated interest rate derivative prices, I apply the variance reducing techniques of antithetic and control variates. For the latter I selected observed bond prices.

In order to implement the Vasicek model, the Ho-Lee model and the Hull-White model, I calibrate them to observed market prices of an interest rate cap agreement. The calibration suggests that the Ho-Lee and the Hull-White model fit observed market prices much better than the Vasicek model. Subsequently, I simulate prices for the very same cap agreement and compare them with the ones calculated via closed formulas. The results indicate that improvements in the simulated prices, due to variance reduction techniques, are always accompanied with an increase in the computational burden. Thus, it has always got to be accounted for the trade-off between computational accuracy and computational efficiency.

In conclusion, I implement a periodic cap agreement based on the Hull-White model as a case study. Exemplary, I show that after the parameters of the short rate processes are defined, path-dependent interest rate derivatives can be priced easily. Eventually, I account for the importance of Monte Carlo methods for pricing path-dependent interest rate derivatives.

Zusammenfassung

In dieser Diplomarbeit wird die Anwendung von Monte Carlo Verfahren für die Bewertung von Zinsderivaten diskutiert. Aufgrund der Vielzahl von unterschiedlichen Zinsmodellen, liegt das Hauptaugenmerk dieser Arbeit auf dem Vasicek, dem Ho-Lee und dem Hull-White Modell. Diese drei Short Rate Modelle werden in der Programmiersprache C++ implementiert. In weiterer Folge werden Verbesserungen der jeweiligen Simulationen diskutiert. Diese Verbesserungen reichen von der Auswahl eines Algorithmus zur Umwandlung von gleich verteilten Pseudo-Zufallszahlen in standardnormal verteilte Pseudo-Zufallszahlen, über die Wahl einer geeigneten Programmiersprache, hin zu möglichen Optimierungen des Quellcodes bis hin zur Implementierung von Varianz reduzierenden Verfahren. Um die Standardabweichung der simulierten Preise zu verbessern, werden antithetische Zufallsvariablen und Kontrollvariate angewendet. Für Letztere wurden beobachtete Preise von Anleihen gewählt.

Um das Vasicek Modell, das Ho-Lee Model und das Hull-White Modell zu implementieren, werden alle drei Modelle gemäß beobachteten Marktpreisen eines Interest Rate Cap Agreements kalibriert. Die Kalibrierung zeigt, dass das Ho-Lee sowie das Hull-White Modell die beobachteten Marktpreise besser repliziert als das Vasicek Modell. In weiterer Folge werden die Preise eines Interest Rate Cap Agreements simuliert. Die simulierten Preise werden anschließend mit explizit berechneten Preisen und beobachteten Marktpreisen verglichen. Die Ergebnisse zeigen, dass Verbesserungen der Simulationen aufgrund von Varianz reduzierenden Verfahren stets mit höherem rechnerischen Aufwand verbunden sind.

Schließlich wird ein Periodic Cap Agreement auf Basis des Hull-White Modells als Fallstudie implementiert. Exemplarisch wird gezeigt, dass pfadabhängige Zinsderivate einfach bewertet werden können, sofern die Parameter des zugrunde liegenden Short Rate Prozesses bereits definiert sind. Schlussendlich wird die Relevanz von Monte Carlo Methoden für die Bewertung von pfadabhängigen Zinsderivaten diskutiert.

Lebenslauf

Seit Oktober 2003	Diplomstudium der Volkswirtschaftslehre an der Universität Wien
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Februar 2008 - Juni 2009	Studienassistent am Institut für Statistik und Decision Support Systems (ISDS) der Universität Wien
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