

DISSERTATION

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Functions of Variable Bandwidth - a Time-Frequency Approach

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Mag. Roza Aceska

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Abstract

Although many methods are tailored towards the use of band-limited signals, for applications in signal processing it would be more appropriate to work with functions that have variable bandwidth. From a practical point of view, the concept of variable bandwidth sounds like a very natural one and has an intuitive meaning. Moreover, from a strictly mathematical point of view there is a bit of an inconsistency: functions of limited bandwidth are entire functions of time, that is, the represented signals go on forever and cannot have compact support (in a strict sense). Real-world signals are, by necessity, time limited; this means using functions which are not band limited. Various natural approaches found in the literature, trying to describe the idea of functions of variable bandwidth in one or the other way have serious shortcomings. This might also be the reason why one finds little theoretical work on this topic.

We suggest a new approach to this concept, using the theory of modulation spaces, suitably adapted by working with customized weights which help to express the idea in a mathematical sound way. The general theory grants that variable bandwidth (VB) spaces are well-defined independent from the used Gabor system, invariant under time-frequency shifts and possess suitable atomic decompositions. Starting from a moderate function describing the behavior of the local bandwidth one obtains a Banach space, which behaves locally more or less like a Sobolev space (of some fixed, high order).

More precisely, we define the variable bandwidth via a special weight on the time-frequency plane. We consider functions with essential time-frequency in a strip which varies through time and control the outside area with a very strong weight. We give the conditions that ensure that a function belongs to

a space of variable bandwidth as well as the criteria for two of these spaces to coincide, with equivalent norms. We show that a particular space is not too sensitive to bandwidth changes and we use atomic decompositions to achieve good approximations. For the L^2 -case, we take a look at the reproducing kernel property.

Zusammenfassung

In Anwendungen wie Signalverarbeitung ist es wirkungsvoller mit Funktionen mit variabler Bandbreite zu arbeiten. Das Konzept der variablen Bandbreite scheint, von einem praktischen Gesichtspunkt aus, natürlich zu sein. Die Verwendung von bandbeschränkten Funktionen birgt einige Inkonsistenzen: Diese Funktionen sind holomorphe Funktionen der Zeit, weswegen die repräsentierten Signale niemals verschwinden können. Klarerweise aber sind physikalische Signale stets zeitlich begrenzt, welches die Verwendung von nicht-bandbeschränkten Funktionen motiviert. Viele Ansätze, die die Idee der Funktionen mit variabler Bandbreite zu beschreiben versuchen, haben ernste Defizite. Das könnte ein Grund sein, warum es erst wenig theoretische Arbeit auf diesem Gebiet gibt.

Wir schlagen vor, gewichtete Modulationsräume zu verwenden. Die Theorie garantiert, dass Räume von Funktionen mit variabler Bandbreite unabhängig vom verwendeten Gabor-System wohldefiniert sind, invariant unter Zeit-Frequenz-Verschiebungen sind, eine geeignete atomare Zerlegung besitzen, und, am wichtigsten, Banachräume sind.

Variable Bandbreite wird durch Gewichte auf der Zeit-Frequenz-Ebene definiert. Wir verwenden Funktionen mit Zeit-Frequenz-Träger in einem Streifen, welcher mit der Zeit variieren kann. Das Gebiet ausserhalb wird mit einem starken Gewicht kontrolliert. Wir geben Bedingungen an, die versichern, dass Funktionen zu einem dieser Banachräume von Funktionen mit variabler Bandbreite gehören, sowie Kriterien, wann zwei solche Räume übereinstimmen, und unter welchen Umständen die Normen äquivalent sind. Wir zeigen, dass ein bestimmter solcher Banachraum nicht zu sensibel bezüglich Bandbreitenänderungen ist, und wir werden die atomare Zerlegung verwenden.

den, um gute Approximationen zu gewinnen. Im L^2 -Fall betrachten wir die reproducing-kernel Eigenschaft.

Chapter 1

Motivation: Why Variable Bandwidth?

The well-developed theory of band limited functions was induced by the real-life fact that signal's frequencies higher than some finite cut-off in practice are considered not important. However, from a mathematical point of view, functions of limited bandwidth possess derivatives of all orders, which implies that they are entire functions of time, i.e. the represented signals go on forever. Real-world signals are, by necessity, time limited, so it is more appropriate to describe them as such. By the uncertainty principle [83], this means using functions which are not band limited. Then again, functions with unbounded spectrum are not a good fit - as explained before - for a signal happening in reality.

This thesis is motivated by a suggestion of Slepian in [88] for resolving 'this seeming paradox': we study functions of *variable bandwidth* (as named in [19], [31]). Some authors refer to the phenomena as *warped frequency* [81], *time-varying spectrum* [29], [11], [59], or, *approximately band limited* functions.

As it is clear that there is no such thing as an *instantaneous bandwidth* ([10]: p.I, ch.4) but one can only have an average frequency spread at a fixed point in time, we need to seek other ways to describe a bandwidth that varies through time. The phrase "variable bandwidth" has a very obvious practical meaning, but the approaches studied so far are either not well defined (in

a strict mathematical sense) or have the flaw to result in mathematically nonequivalent although similar systems. The main difficulty is that it is not possible to give a sharp, point-wise definition of the local bandwidth of a function, as well as its variation in time cannot be precisely captured in mathematical terms.

The concept presented here goes beyond the naive description of time-varying signals as only smooth deformations of band-limited functions (as suggested by Xian and Lin in [95]). We suggest to pursue functions with *essential* time-frequency in a strip, bordered by a function which *mildly* varies through time. We are not excluding the possibility that the considered function has non-zero values outside this strip, but we minimize such occurrences with a very strong weight. In that way, the spectrum would involve all possible frequencies, but with graded importance. Of course, the 'out of band' values - if occurring - would lead to a reasonably large norm, due to the large weight. We shall use the STFT as a basic tool on the time-frequency plane and naturally employ weighted modulation spaces, a topic on which rich literature exists (some of earliest papers are by Junek and Vuong[74] and Feichtinger [38]).

The goal of this thesis is to show that (customized) modulation spaces are fitting well for describing functions of variable bandwidth. Each of these variable bandwidth (VB) spaces would certainly involve band limited functions as well, if a suitable weight is applied. Some of the questions this thesis considers are: What are the conditions that ensure that a function belongs to a space of variable bandwidth? Given a L^2 function, what is the smallest VB space that would contain this function? What is the criteria for two of these spaces to coincide, with equivalent norms? How sensitive is a particular space to mild changes of the bandwidth? Would these spaces be translation invariant and do they have atomic decompositions? Among the tasks that need to be realized is to study the reconstruction from samples, which might not be a perfect reconstruction as in the band limited case. The reconstruction from samples is possible and almost perfect - up to a fairly small error. As for the L^2 modulation spaces, the reproducing kernel Hilbert space structure is an important result as it should allow to look for a minimal norm solution [48].

Outline

This thesis is organized as follows: in the remain of this chapter we overview several approaches to the concept of variable bandwidth. In chapter 2 we give a summary on bases and frames, with a particular accent on Gabor frames and reduced multi-Gabor frames. In chapter 3 we explore the properties of variable bandwidth weights on the time-frequency plane. Chapter 4 is an introduction to modulation spaces and the standard time-frequency tools used: the STFT, Gabor frames and Wilson bases. In chapter 5 we develop the concept of variable bandwidth spaces and explore their properties. The next chapter studies the reproducing kernel property, followed by a chapter on further research prospects and introducing the idea of warping on the time-frequency side. The appendix has several Matlab codes, used to produce the images in this thesis.

Alternative approaches

From a practical point of view, the concept of variable bandwidth sounds like a very natural one. For example, consider any melody played on a piano: the lowest C on a standard 88-key piano is 32.70Hz; highest is 4186Hz, with different parts of the melody played in different frequency ranges. However, there is not a satisfactory strict mathematical definition of the concept of variable bandwidth, with well-developed mathematical theory.

Time-variant filters

One approach to the concept of variable bandwidth is via time-variant filters: Take a window h_t whose bandwidth is changing in time t . While moving the filter over a signal f , the local bandwidth is changing. More precisely, convolve the signal with a time-varying window, then the outcome signal $F_h(f) := f * h_t$ will have variable bandwidth.

The frequency filter needs to have two dimensions $h_t(x) = H(t, x)$ as its support width on the frequency side should vary, that is, $\hat{h}_t(\omega)$ has time-

varying support Ω_t . Such slowly varying filters have already been used in [49], while a similar concept was developed in [71]. By Young's theorem, applying a L^1 -time-variant filter h_t on L^p functions would produce L^p functions with variable bandwidth. The class of functions defined via a time-variant filter h_t would be

$$Filt_h(L^p) := \{F_h(f) : f \in L^p(\mathbb{R}^d)\}$$

and is obviously linear and translations invariant.

It is practical to expect that minor differences between two distinct filters produce equivalent spaces. Let g_t and h_t be two similar filters, in the sense that

$$|g_t(x) - h_t(x)| < \epsilon$$

for a small ϵ . Then

$$|F_g(f)(0) - F_h(f)(0)| \leq \int |f(x)| |g_t(x) - h_t(x)| dx < \epsilon \|f\|_1,$$

that is, the point-wise difference between two filtered versions of a signal f would be proportionally small. Still, even if using different filters g_t and h_t is defining the same space $Filt_h(L^p) \equiv Filt_g(L^p)$, it would be difficult to calculate the sum of $F_g(f_1)$ and $F_h(f_2)$. That is, by definition one would have $f_3 = f_1 + f_2$ in the L^p space, while in the filtered space one needs to have a resulting filter l so that point-wise it holds

$$f_3 * l_t = f_1 * g_t + f_2 * h_t.$$

This only complicates the whole concept and is not practical to work with. In addition, this production does not cover L^p functions with almost vanishing areas of 'unlimited' bandwidths. Also, the convolution is a smoothing process so discontinuities would be impossible to describe.

Local deformations of band-limited functions

Another natural approach is to take a local deformation of the graph of a band-limited function f . This can be achieved via a smooth function ρ that causes time deformation and substitute the band-limited function f with a warped version f_ρ , that is, point-wise it means $f(t) \mapsto f(\rho(t))$. The outcome

would be a function with variable bandwidth. The pool of functions may then be

$$\{f_\rho : f \in L^p(\mathbb{R}^d)\}$$

and would have strong dependence on the choice of the time-deformation function ρ . The motivation behind this idea is to use the inverse deformation ρ^{-1} to go back to a band-limited function and then apply the known band-limited functions tools like the sampling theorem.

This idea has already been used by Costello in [18] (also see [96], or [95], who refer to the setting as *time warping*), but it has some drawbacks. For one, the sum of two functions having "almost the same" deformations ρ_1 and ρ_2 , is expected to be a function that belongs in the same space, with a deformation ρ_3 , but it is imprecise what ρ_3 would be. Thus not even addition is a clearly defined operation. Only asymptotically equal deformations would, in theory, give rise to quite different function spaces. On the computational level, it is highly unpractical to expect a good estimate of the time warping function ρ .

Subsets of Gabor lattices

Another approach is to introduce variable bandwidth spaces through atomic decompositions (in particular, Gabor expansions) in the sense of considering only those expansions that use coefficients from a certain band-strip in the TF-plane. These spaces would be closed, but its construction depends on the used (Gabor) atom and the TF-lattice chosen to construct the Gabor family, so it has lots of restrictions. For instance, an element generated on a slightly different lattice, or by a slightly different atom may not be in the constructed space. Also, translation invariance would not be possible. However, this idea may be good for approximation of functions within the spaces we define in the chapters to follow.

Weighted time-frequency plane

We suggest to use *modulation spaces* (originally developed by Feichtinger and Gröchenig in [43], [44] and many others) with customized weights. The

general theory grants that these spaces are well-defined independent from the used Gabor system, invariant under time-frequency shifts and posses suitable atomic decompositions.

The working domain is to be the time-frequency generalized plane $(\mathbb{R}^d \times \hat{\mathbb{R}}^d)$ with the *short-time Fourier transform* (STFT) as the basic tool. Put in simple words, the short-time Fourier transform is describing the behavior of a function f in time and frequency simultaneously. As an instantaneous time-frequency description is impossible (due to the uncertainty principle), one can think of the STFT as the best possible tool for time-frequency description.

Another name for the STFT is sliding-window Fourier transform, which is a very clear description. Namely, it is a Fourier transform of a local cut-off, produced via a window function g . Thus the STFT has excellent localization properties. When fixed a non-zero Schwartz function g (called the window function), the STFT of a function or distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ (with respect to g) is

$$V_g f(x, \omega) := \langle f, M_\omega T_x g \rangle .$$

To avoid artificial discontinuities that occur with sharp cut-offs, one usually uses a smooth g . If g is compactly supported with support centered around 0, then $V_g f(x, \cdot)$ is the Fourier transform of the local version of f around x . Changing x means sliding the window g along the x -axis into different positions (thus comes the name 'sliding window Fourier transform'). Clearly, the narrower the support of g , the more localized in time results: In the extreme case, for $g = \delta$, we have pure temporal behavior, that is

$$V_\delta f(x, \omega) = M_{-\omega} f(x).$$

Opposed to this, pure frequency behavior is the outcome when considering the other extreme window

$$g = 1_{\mathbb{R}^d} : V_{1_{\mathbb{R}^d}} f(x, \omega) = \hat{f}(\omega).$$

We suggest to pursue functions with essential time-frequency support of their STFT within a strip, bordered out by a mildly time-varying function. The whole time-frequency domain is greater than this strip, but the out-of-strip values are 'small enough'. We control the outer values by a very strong weight whose value rises in proportion with the distance from the strip. In that way, the spectrum would involve all possible frequencies, but with graded

importance. Of course, the out-of-band values would lead to a reasonably large norm, due to the weight.

What does a strip with variable bandwidth mean? What would *essential* time-frequency support concentrated within a such a strip mean? Observe band-limited functions on \mathbb{R} , as they provide the simplest example of a strip support. Take a band-limited function f and a band-limited window g , then $V_g f$ has non-zero values in

$$ST_{r+R} := \mathbb{R} \times [-r - R, r + R]$$

for $[-R, R]$ and $[-r, r]$ being the spectra of f and g respectively. It is handy to treat approximately band-limited functions as members of the same pool as band-limited functions. In this example, an approximately band limited function h in the same pool with f would have most of its STFT energy within the strip ST_{r+R} and for instance, only 1% outside of ST_{r+R} . The 1% energy over-leak should decay strong enough, say faster than a polynomial of degree s . This decay can be controlled by a polynomial weight, applied outside of ST_{r+R} . Then we say that h has the same essential time-frequency support as f .

In a more general setting, let the function $b \geq 0$ describe the bandwidth of a time-varying strip ST_b in the time-frequency plane:

$$ST_b := \{(x, \omega) \in \mathbb{R} \times \hat{\mathbb{R}} : |\omega| \leq b(x)\}.$$

We call ST_b a strip with variable bandwidth (with time-varying width b). Notice that this strip is symmetric with respect to the time axis. We will limit ourselves in working with only symmetric strips. If we only think of L^2 -functions with TF domain ST_b , we seemingly get a net of nested subspaces by moving the bandwidth b . As we shall see later, small changes of b would not define a new class of functions, which is very practical. In other words, moving the bandwidth mildly is not going to change the work setting; that is, estimates of the bandwidth need **not** to be too precise, which gives an extra dimension of freedom in applications.

Thus signals with variable bandwidth are signals with essential TF support concentrated within the strip ST_b , i.e. such signals may be non-zero beyond the strip ST_b , but those values are assigned with a strong weight. For practical purposes, one can shift the time-frequency content of the function within

ST_b , or one can shift the time-frequency plane so that $V_g f$ is contained in ST_b , as suggested in [56].

Our setting on the time-frequency plane are the mixed-norm weighted spaces $L_m^{p,q}(\mathbb{R}^{2d})$: The idea is to measure L^p -norm in one parameter (time) and L^q -norm in the other parameter (frequency). That is, given $1 \leq p, q \leq \infty$ and a v -moderate weight m (with v being sub-multiplicative), the mixed-norm weighted space is the collection of all measurable functions F for which the norm

$$\|F\|_{L_m^{p,q}(\mathbb{R}^{2d})} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |F(x, \omega)|^p m(x, \omega)^p dx \right)^{q/p} d\omega \right)^{1/q}$$

is finite.

From here we may introduce generalized modulation spaces: We take a function or a distribution, calculate its STFT and measure the mixed norm of it on the $2d$ -domain. Or, more precisely, for a fixed non-zero Schwartz window g , the modulation space $M_m^{p,q}(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ for which $V_g f \in L_m^{p,q}(\mathbb{R}^{2d})$. Note that the norm is measured on the time-frequency side. $M_m^{p,q}(\mathbb{R}^d)$ is a Banach space whose definition is independent of the choice of the window g . A generalization of the classical modulation spaces theory is the co-orbit theory ([37],[42], [44], [43]). It is shown that different test functions define the same spaces and equivalent norms, and that this family of Banach spaces is essentially closed with respect to duality and complex interpolation.

The modulation spaces $M_m^{p,q}(\mathbb{R}^d)$, $1 \leq p, q \leq \infty$ are Banach spaces of tempered distributions σ on \mathbb{R}^d , characterized by the behavior (for $t \rightarrow \infty$) of the convolution product $M_t g * \sigma$ in L^p , where the window g is a Schwartz function. The behavior of this family is similar to the one of Besov spaces, when duality, interpolation, embedding and trace theorems are concerned. Furthermore, the classical Sobolev spaces and the remarkable Segal Algebra S_o are examples of modulation spaces. As a result, our approach gives Banach space structure to variable bandwidth functions, something that was impossible to have in the other approaches (mentioned in the previous sections).

The approach to modulation spaces here is through uniform decompositions of the Fourier transforms of their elements [34, 38, 39]. Such decompositions

correspond to uniform coverings (dyadic decompositions in Besov spaces), as they are obtained by translations (translation of \hat{g} corresponds to multiplication of g with a character). Of great use here are the techniques of Wiener type spaces, as they are Banach spaces of distributions, characterized by uniform decompositions. In fact, modulation spaces can be defined for a class of solid BF-spaces (L^p are such) on a locally compact Abelian group (\mathbb{R}^d is such). So the general modulation spaces $M(B, L_v^q)(G)$ consist of those distributions σ on g for which $t \rightarrow \|M_t g * \sigma\|_B, t \in G$, satisfies a weighted q -integrability condition. The relevant facts concerning modulation spaces can thus be drawn from corresponding properties of Wiener-type spaces $W(\mathcal{F}_G B, L_v^q)(\hat{G})$.

Chapter 2

Discrete representations

2.1 Bases

Bases are essential tools in the study of Banach and Hilbert spaces. We review here what (unconditional [67]) bases are.

Definition 1. A countable set $\{e_n\}_{n=1}^\infty$ in a Banach space $(B, \|\cdot\|_B)$ is called an unconditional basis for B if

(1) the finite linear combinations of $\{e_n\}_{n=1}^\infty$ span a dense subspace of B

and

(2) there exists $C \geq 0$ such that for all multipliers $\mu = (\mu_n)_{n \in \mathbb{N}} \in l^\infty(\mathbb{N})$ and all finite sequences $(c_n)_{n \in \mathbb{N}}$ it is true that

$$\left\| \sum_{n=1}^{\infty} \mu_n c_n e_n \right\|_B \leq C \|\mu\|_\infty \cdot \left\| \sum_{n=1}^{\infty} c_n e_n \right\|_B \quad (2.1)$$

In a Hilbert space, $\{e_n\}_{n=1}^\infty$ is an unconditional basis if and only if it is a Riesz basis. Condition (2) implies the unconditional convergence of the corresponding series.

Lemma 1. If $\{e_n\}$ is an unconditional basis of B then every $f \in B$ has a series expansion of the form

$$f = \sum_{n=1}^{\infty} c_n e_n .$$

The coefficients $c_n \in C$ are uniquely determined and the series converges unconditionally in B .

Proof. Let B_o denote the dense subspace of finite linear combinations of the e_n . Given a finite subset $F \subseteq \mathbb{N}$ we define an operator P_F on B_o by means of the multiplier $\chi_F(n)$

$$P_F(\sum_{n=1}^{\infty} c_n e_n) = \sum_{n \in F} c_n e_n .$$

Then $P_F P_F = P_F$ and by the definition $\|P_F f\|_B \leq C \|f\|_B$ for $f \in B_o$. therefore P_F extends to a bounded projection from B onto the finite span of $\{e_n\}_{n \in F}$ with a bound $\|P_F\|_B \leq C$ independent of F .

Each $P_{\{n\}} f = c_n e_n$ defines a bounded linear functional $c_n = \lambda_n(f)$ on B and thus there exists $e_n^* \in B^*$ such that $P_{\{n\}} f = \langle f, e_n^* \rangle e_n$. More explicitly, $P_F(f) = \sum_{n \in F} \langle f, e_n^* \rangle e_n$ and $f = \sum_{n \in \mathbb{N}} \langle f, e_n^* \rangle e_n$. The last claim is true as for any positive ε there exists a finite $F_o \subseteq \mathbb{N}$ and an element $p = \sum_{n \in F_o} c_n e_n$ such that $\|f - p\|_B < \varepsilon$. As $P_F p = p$ whenever $F \supseteq F_o$ we have

$$\begin{aligned} \left\| f - \sum_{n \in F} \langle f, e_n^* \rangle e_n \right\|_B &= \|f - P_F f\|_B \\ &\leq \|f - p\|_B + \|P_F p - P_F f\|_B \leq (1 + C) \|f - p\|_B < (1 + C)\varepsilon. \end{aligned}$$

This gives unconditional convergence. □

2.2 Frames and Riesz bases

The primary task of signal analysis is to extract information from a signal (with small storage/transport requirements) and reconstruct/approximate, using this information, via a set of known, simple functions, that is, via a frame. Excellent guiding though frames provide Christensen's books [13, 15]; also, [50, 51] are solid collections of important Gabor frames results. An alternative approach to Gabor frames brings [55].

Let $\{g_k\}_{k \in \mathbb{Z}}$ be a family in an infinite dimensional (separable) Hilbert space \mathcal{H} with inner product $\langle \cdot, \cdot \rangle$ and inner product-induced norm $\|\cdot\|$ (the classical example is the Hilbert space L^2 of functions of finite energy). We want to represent a signal f in \mathcal{H} as a linear combination of the form

$$f = \sum_k c_k g_k.$$

This representation should satisfy several conditions; for one, the sum should *converge in the Hilbert-norm*, i.e.,

$$\lim_{K \rightarrow \infty} \left\| f - \sum_{k=1}^K c_k g_k \right\| \rightarrow 0.$$

Also, the sum should *converge unconditionally*, that is, to the same limit f , regardless of the summation order we choose. Another desired feature is *numerical stability*: we want a continuous linear dependency between f and the summation coefficients c_k , as then small alterations in the signal would result in controllable changes in the corresponding coefficient sequence and vice-versa. For an infinite family $\{g_k\}$ these assumptions (unconditional convergence, numerical stability) have to be ensured before dealing with decomposition and reconstruction.

A family $\{g_k\}$ of a Hilbert space \mathcal{H} is *complete* in \mathcal{H} , if $\text{span}(g_k)$ is dense in \mathcal{H} , that is, every f in \mathcal{H} can be arbitrarily well approximated by elements in $\text{span}(g_k)$ with respect to the \mathcal{H} -norm. However, complete families do not necessarily allow a series expansion of arbitrary elements from the given Hilbert space.

Definition 2. *The family $\{g_k\}$ of a Hilbert space \mathcal{H} is a basis for \mathcal{H} if for all $f \in \mathcal{H}$ there exist unique scalars $c_k(f)$ such that*

$$f = \sum_k c_k(f) g_k.$$

Unlike complete families, a basis always induces a series expansion.

Definition 3. *The sequence $\{g_k\}$ in a Hilbert space \mathcal{H} is called a Bessel sequence if*

$$\sum_k |\langle f, g_k \rangle|^2 < \infty, \quad f \in \mathcal{H}.$$

A basis is always a Bessel sequence.

Definition 4. A family $\{g_k\}$ of a Hilbert space \mathcal{H} is a Riesz sequence if there exist bounds $A, B > 0$ such that for all finite sequences $c \in \ell^2$,

$$A\|c\|_{\ell^2}^2 \leq \left\| \sum_k c_k g_k \right\|^2 \leq B\|c\|_{\ell^2}^2.$$

A Riesz sequence which generates the whole space \mathcal{H} is called a *Riesz basis* for \mathcal{H} . Riesz bases are only modified orthonormal bases, as seen in the following lemma [16].

Lemma 2. Let $\{g_k\}$ be a sequence in a Hilbert space \mathcal{H} . The following are equivalent.

- (1) $\{g_k\}$ is a Riesz basis for \mathcal{H} .
- (2) $\{g_k\}$ is an unconditional basis for \mathcal{H} and g_k are uniformly bounded.
- (3) $\{g_k\}$ is a basis for \mathcal{H} , and $\sum_k c_k g_k$ converges if and only if $\sum_k |c_k|^2$ converges.
- (4) There is an equivalent inner product on \mathcal{H} for which $\{g_k\}$ is an orthonormal basis for \mathcal{H} .
- (5) $\{g_k\}$ is a complete Bessel sequence and possesses a bi-orthogonal system $\{h_k\}$ that is also a complete Bessel sequence.

From the lemma we conclude that for a Riesz basis $\{g_k\}$, there exists a unique bi-orthogonal sequence $\{h_k\}$ and any $f \in \mathcal{H}$ has a representation of form

$$f = \sum_k \langle f, h_k \rangle g_k = \sum_k \langle f, g_k \rangle h_k, \quad f \in \mathcal{H},$$

and the coefficients sequences $\{\langle f, g_k \rangle\}$ and $\{\langle f, h_k \rangle\}$ are square summable. Thus, Riesz bases are good candidates for to represent signals with, except that they allow only unique expansions with respect to the coefficients. In real-life applications it is useful to weaken this property, which is provided in the concept of frames (Duffin and Schaeffer, 1952 [30]).

Definition 5. A family $\{g_k\}$ of a Hilbert space \mathcal{H} is a frame of \mathcal{H} if there exist bounds $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{k \in \mathbb{Z}} |\langle f, g_k \rangle|^2 \leq B\|f\|^2, \quad f \in \mathcal{H}. \quad (2.2)$$

If we can take $A = B$, then $\{g_k\}_{k \in \mathbb{Z}}$ is called a *tight* frame. The *synthesis* map $D : \ell^2 \rightarrow \mathcal{H}$ of a frame $\{g_k\}$ is defined by

$$D : (c_k) \rightarrow \sum_k c_k g_k.$$

The *analysis* operator $Cf = (\langle f, g_k \rangle)$ is simply the adjoint of the synthesis operator $C = D^*$. The *frame operator* S is defined by

$$Sf = DD^*f = C^*Cf = \sum_{k \in \mathbb{Z}} \langle f, g_k \rangle g_k, \quad f \in \mathcal{H}.$$

Due to (2.2), the frame operator satisfies

$$A\langle f, f \rangle \leq \langle Sf, f \rangle \leq B\langle f, f \rangle, \quad f \in \mathcal{H}.$$

It is a positive, bounded, and invertible operator. Its inverse, the operator S^{-1} is also positive and thus has a square root $S^{-1/2}$, which is self-adjoint [86]. An important consequence is that the sequence $\{S^{-1/2}g_k\}$ is a tight frame.

An orthonormal basis of \mathcal{H} is also a Riesz basis of \mathcal{H} ; a Riesz basis of \mathcal{H} is also a frame. The important difference between a Riesz basis and a frame is that the range of the analysis map D^* is a (closed) proper subspace of ℓ^2 . This is why the null space $\mathcal{N}(D)$ of the synthesis map D of a frame $\{g_k\}$ is in general non-trivial.

The sequence $\{S^{-1}g_k\}$ is also a frame, it is called a *canonical dual* frame for $\{g_k\}$ in the sense that

$$f = \sum_k \langle f, S^{-1}g_k \rangle g_k = \sum_k \langle f, g_k \rangle S^{-1}g_k, \quad f \in \mathcal{H}.$$

It produces minimal ℓ^2 coefficients (see [30]) and its frame bounds are $1/B$ and $1/A$.

We call inequality(2.2) the frame inequality for L^2 .

Unlike Riesz bases, frames have no bi-orthogonal relation in general; so, the dual frame is not unique. For alternative dual frames there exist constructive approaches that rely on the canonical dual [14, 75]. In addition, a frame is still a frame when discarding single frame elements [1, 2], but not a Riesz basis. Interested readers in the relation between HS operators and frames should refer to [4].

2.3 Gabor analysis on L^2

Gabor frames are produced from a single function g , called an *atom*, via *time-frequency shifts* $\pi(\lambda) = M_\omega T_x$, $\lambda(x, \omega) \in \mathbb{R}^{2d}$. The time-frequency shifts have to satisfy

$$\pi(\lambda_2)\pi(\lambda_1) = e^{2\pi i(x_1\omega_2 - x_2\omega_1)}\pi(\lambda_1)\pi(\lambda_2)$$

for $\lambda_l = (x_l, \omega_l) \in \mathbb{R}^{2d}$, $l = 1, 2$.

Here Λ is a time-frequency *lattice*, more precisely, $\Lambda = A\mathbf{Z}^{2d}$ is a discrete subgroup of \mathbb{R}^{2d} for A being an invertible, real $d \times d$ -matrix. Its dual, that is, its *adjoint lattice* is $\Lambda^\circ = (A^T)^{-1}\mathbf{Z}^{2d}$, which by definition is the set of all $\lambda^\circ \in \mathbb{R}^{2d}$ so that for all $\lambda \in \Lambda$ it holds

$$\pi(\lambda^\circ)\pi(\lambda) = \pi(\lambda)\pi(\lambda^\circ),$$

that is,

$$e^{-2\pi i\lambda\lambda^\circ} = 1.$$

A special case of a lattice is of form $a\mathbf{Z}^d \times b\mathbf{Z}^d$, it is called *separable*. The parameters a and b are called *lattice parameters*. The adjoint lattice is then $\Lambda^\circ = \frac{1}{b}\mathbf{Z}^d \times \frac{1}{a}\mathbf{Z}^d$.

Given a lattice Λ in \mathbb{R}^{2d} and a Gabor atom $g \in L^2$, the associated Gabor family is defined by

$$\mathcal{G}(g, \Lambda) = \{\pi(\lambda)g\}_{\lambda \in \Lambda}.$$

If $\mathcal{G}(g, \Lambda)$ is a frame for L^2 , we call it a *Gabor frame*. The frame operator

$$Sf = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)g$$

commutes with all time-frequency shifts $\pi(\lambda)$ for $\lambda \in \Lambda$. A very important result is that the canonical dual frame of $\mathcal{G}(g, \Lambda)$ is $\mathcal{G}(h, \Lambda)$ with $h = S^{-1}g$; so, it has Gabor structure, too. This fact reduces computational issues to solving the linear system $Sh = g$.

A typical Gabor atom is the Gaussian

$$\psi(x) = e^{-\pi x^2 q^2}. \quad (2.3)$$

for some real $q \neq 0$. The Gaussian generates a Gabor frame on a separable lattice with lattice parameters a and b if and only if $ab < 1$ [78, 87]. Whenever $ab = 1$ the generated Gabor family is complete but coefficient sequences must not be bounded, that is, the system is unstable. A central result is the so-called *density theorem*, see [62] or [73].

Theorem 1. *Let $\mathcal{G}(g, a, b)$ be a frame for $L^2(\mathbb{R}^d)$. Then, $ab \leq 1$. Moreover, $\mathcal{G}(g, a, b)$ is a Riesz basis for $L^2(\mathbb{R}^d)$ if and only if $ab = 1$; $\mathcal{G}(g, a, b)$ is an orthonormal basis for $L^2(\mathbb{R}^d)$ if and only if $ab = 1$, $\|g\|_2 = 1$ and the frame coefficients are $A = B = 1$.*

D. Gabor used $a = b = 1$ and the Gaussian in [57], with the goal to achieve a Gabor system with maximal time-frequency localization. As previously mentioned, this system is complete, but is no longer stable. The Balian-Low Theorem [6, 77] states that it is impossible to have both good time-frequency localization and Gabor Riesz bases.

Theorem 2. (Balian-Low) *If $\mathcal{G}(g, 1, 1)$ is a Riesz basis for $L^2(\mathbb{R}^d)$, then either*

$$\int_{\mathbb{R}^d} |g(t)|^2 t^2 dt = \infty \text{ or } \int_{\mathbb{R}^d} |\hat{g}(\omega)|^2 \omega^2 d\omega = \infty.$$

As mentioned before, the dual of a Gabor frame is also a Gabor frame and is defined over the adjoint lattice; therefore, the frame operator $S = S_{g, \gamma, \Lambda}$ associated to the pair (g, γ) , where γ performs the analysis and g performs the synthesis is:

$$Sf = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\gamma \rangle \pi(\lambda)g, \quad f \in L^2.$$

Unconditional convergence is satisfied whenever both atoms g, γ are elements of Feichtinger's algebra $S_0 = M^{1,1}$. Changing the roles of an analysis and

synthesis window is then possible. We summarize several crucial results ([22, 47, 72, 93]) in one theorem, similarly as given in [53]:

Theorem 3. *Given atoms $g, h \in S_0$ and a lattice $\Lambda \subseteq \mathbb{R}^{2d}$ with adjoint lattice Λ° , the following hold.*

1. *(Fundamental Identity of Gabor Analysis)*

$$\sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\gamma \rangle \langle \pi(\lambda)g, h \rangle = |\Lambda|^{-1} \sum_{\lambda^\circ \in \Lambda^\circ} \langle g, \pi(\lambda^\circ)\gamma \rangle \langle \pi(\lambda^\circ)f, h \rangle \quad (2.4)$$

for all $f, h \in L^2$, where both sides converge absolutely.

2. *(Wexler-Raz Identity)*

$$S_{g,\gamma,\Lambda}f = |\Lambda|^{-1} \cdot S_{f,\gamma,\Lambda^\circ}g \quad (2.5)$$

for all $f \in L^2$.

3. *(Janssen Representation)*

$$S_{g,\gamma,\Lambda} = |\Lambda|^{-1} \sum_{\lambda^\circ \in \Lambda^\circ} \langle \gamma, \pi(\lambda^\circ)g \rangle \pi(\lambda^\circ) \quad (2.6)$$

where the series converges unconditionally in the strong operator sense.

Assuming that both synthesis operators D_g and D_γ are bounded, it holds:

$$S_{g,\gamma,\Lambda} = Id \text{ if and only if } \langle \gamma, \pi(\lambda^\circ)g \rangle = |\Lambda|^{-1} \delta_{0,\lambda^\circ}.$$

This means that g and γ are dual Gabor windows if and only if the corresponding Gabor systems are bi-orthogonal

$$\langle \pi(\lambda^{\circ'})\gamma, \pi(\lambda^\circ)g \rangle = |\Lambda|^{-1} \delta_{\lambda^{\circ'}, \lambda^\circ}.$$

Another important result is the Ron-Shen Duality Principle ([84][85]).

Theorem 4. *Ron-Shen Duality Principle Let $g \in L^2$ and Λ be a lattice in \mathbb{R}^{2d} with adjoint Λ° . Then the Gabor system $\mathcal{G}(g, \Lambda)$ is a frame for L^2 if and only if $\mathcal{G}(g, \Lambda^\circ)$ is a Riesz basis for its closed linear span.*

The following result gives explicit and simple way to construct Gabor frames [20]:

Theorem 5. *Let $g \in L^\infty$ with support in Q_L . If $\alpha \leq L$ and $\beta \leq \frac{1}{L}$ then the frame operator $S_{g,g}$ is the multiplication operator*

$$Sf(x) = \left(\beta \sum_{k \in \mathbf{Z}} |g(x - \alpha k)|^2\right) f(x). \quad (2.7)$$

As a consequence, $\mathcal{G}(g, \alpha, \beta)$ is a frame with frame bounds a/β and b/β if and only if

$$a \leq \sum_{k \in \mathbf{Z}} |g(x - \alpha k)|^2 \leq a.e.$$

In addition, $\mathcal{G}(g, \alpha, \beta)$ is a tight frame if and only if $\sum_{k \in \mathbf{Z}} |g(x - \alpha k)|^2 = \text{const}$ a.e.

2.3.1 Reduced Multi-Gabor frames

Here we review a simplified idea of *quilted Gabor frames* [28], [27]. In its core, this concept allows using different (multi-) Gabor systems in different regions of the time-frequency plane. Applicable for our purposes is a special case of such a construction, the so-called *reduced multi-Gabor frames*.

Reduced multi-Gabor systems are obtained by a partition of the time - frequency plane which is adapted to audio signals of practical interest [25],[28]. In fact, the original motivation to construct such frames comes from the processing of music signals, where the trade-off between time and frequency resolution has a strong influence in the results of analysis and synthesis, see [24]. The union of “localized parts” from different Gabor frames forms a frame. The frame reconstruction operator employs the partition of unity as well as the (abstract) dual frame. A more general idea is the concept of *quilted frames*, which allow arbitrary tilings of the time-frequency plane, using a partition of unity in the time-frequency plane [27]. A related approach to this problem is the concept of fusion frames [12], which in its core is splitting a large frame system into a set of overlapping, small frame systems that are used for local processing.

The construction of reduced multi-Gabor frames is with the help of amalgam spaces techniques: *Wiener amalgam spaces* were introduced by H. Feichtinger in 1980 (see [35, 46]). The purpose of amalgam spaces is decoupling of local and global properties of \mathbf{L}^p -spaces.

Let $Q = [0, 1]^d$ and let χ_{k+Q} be the identity function on its translates. The Wiener amalgam space $W(\mathbf{L}^p, \ell^q)$ is defined as follows:

$$W(\mathbf{L}^p, \ell^q)(\mathbb{R}^d) = \left\{ f \in L^p_{loc}(\mathbb{R}^d) : \|f\|_{W(\mathbf{L}^p, \ell^q)} = \left(\sum_{k \in \mathbb{Z}^d} \|f \cdot \chi_{k+Q}\|_p^q \right)^{\frac{1}{q}} < \infty \right\}.$$

Here \mathbf{L}^p is the local component and ℓ^q is the global component, in the sense that the function f is chopped up in pieces, each piece is measured (locally) in the \mathbf{L}^p -norm and this sequence of local norms is in ℓ^q . Spaces $W(\mathbf{L}^\infty, \ell^q)$ and $W(\mathbf{L}^p, \ell^\infty)$ are defined with the usual adjustment to the sup norm. A comprehensive review of (weighted) Wiener amalgam spaces in their most general form ($W(B, C)$, with local component B and global component C) can be found in [68, 69].

Here we list some properties which will be needed later on:

- If $B_1 * B_2 \subseteq B_3$, $C_1 * C_2 \subseteq C_3$, then $W(B_1, C_1) * W(B_2, C_2) \subseteq W(B_3, C_3)$.
- If $B_1 \subseteq B_2$, $C_1 \subseteq C_2$, then $W(B_1, C_1) \subseteq W(B_2, C_2)$.
- In particular, if $1 \leq p \leq q \leq \infty$, then $W(L^q, \ell^p) \subseteq L^p + L^q \subseteq W(L^p, \ell^q)$.
- and, $W(\mathbf{L}^\infty, \ell^2) \subseteq L^2 + \mathbf{L}^\infty \subseteq W(L^2, \ell^\infty)$.

A more general partition of unity than the simple $\cup_k \chi_{k+Q} \equiv 1$ is the following: The family $(\Omega_r = B_{R_r}(x_r))_{r \in \mathcal{I}}$ is an *admissible covering* if

$$\cup_{r \in \mathcal{I}} \Omega_r = \mathbb{R}^d$$

and the number of overlapping Ω_r is finite (*admissibility condition*). That is, there exists $n_0 \in \mathbb{N}$ such that $|r^*| \leq n_0$ for all $r \in \mathcal{I}$, where

$$r^* := \{s : s \in \mathcal{I}, \Omega_r \cap \Omega_s \neq \emptyset\}.$$

A family $(\psi_r)_{r \in \mathcal{I}}$ of non - negative functions with

$$\sum_r \psi_r(x) \equiv 1$$

is called *bounded admissible partition of unity* (for short, we refer to it as a BAPU) subordinate to $(\Omega_r)_{r \in \mathcal{I}}$, if

- the support of (ψ_r) is Ω_r for $r \in \mathcal{I}$, and
- $(\Omega_r)_{r \in \mathcal{I}}$ is an admissible covering.

An equivalent definition of Wiener amalgam spaces is obtained via BAPUs. A formal theory of decomposition spaces based on such BAPUs has been developed in [41, 36].

In short, this is the core idea: Assume that a BAPU $(\psi_r)_{r \in \mathcal{I}}$ of \mathbb{R}^d (time-domain) or $(\varphi_r)_{r \in \mathcal{I}}$ of $\hat{\mathbb{R}}$ (frequency-domain) is given, so that

$$f = \sum_{r \in \mathcal{I}} \psi_r \cdot f \quad \text{for all } f \in L^2(\mathbb{R}^d),$$

and $\text{supp}(\psi_r) \subseteq \Omega_r$ or alternatively

$$\hat{f} = \sum_{r \in \mathcal{I}} \varphi_r \cdot \hat{f} \quad \text{for all } f \in L^2(\mathbb{R}^d),$$

and $\text{supp}(\varphi_r) \subseteq \Omega'_r$, respectively.

Assuming, now, that different given Gabor frames have been assigned to the localized regions determined by the support of ψ_r for all $r \in \mathcal{I}$, the new system will constitute a frame again. The resulting frame is called a *reduced multi-Gabor frame*.

Definition 6. Let the index sets \mathcal{X}^r be chosen by cutting off Λ^r using BAPU elements ψ_r for all $r \in \mathcal{I}$, i.e.

$$\mathcal{X}^r = \Lambda^r \cap (\Omega_r \times 1).$$

Then the system

$$\mathcal{G}_{g^r, \mathcal{I}^r} = \cup_{r \in \mathcal{I}} \{(g_\lambda^r) : \lambda \in \mathcal{X}^r\} \quad (2.8)$$

is called a *reduced multi-Gabor family*.

For practical purposes we shall only deal with BAPUs with compact support on the time or frequency side. More general results are given in [27]; we extract the following theorem, adapted to compactly supported ψ_r s.

Theorem 6. (*Reduced multi-Gabor frames*)

Say that a family of Gabor frames $(g^r, \Lambda^r)_{r \in \mathcal{I}}$ and suitable admissible coverings $(\Omega_r)_{r \in \mathcal{I}}$ exist, satisfying the following properties:

(i) $g^r \in H_{s,C}$ for all r , where

$$H_{s,C} = \{g \in L^2 : |\mathcal{V}_{g_0}g|(z) \leq C(1 + |z|^2)^{-\frac{s}{2}}\}, \quad s > 2d, \quad C > 0.$$

(ii) The lattice constants α^r, β^r are chosen from a compact set in $\mathbb{R}^+ \times \mathbb{R}^+$, i.e. $\alpha^r \subseteq [\alpha_0, \alpha_1] \subset (0, \infty)$ and $\beta^r \subseteq [\beta_0, \beta_1] \subset (0, \infty)$.

(iii) The regions assigned to the different Gabor systems correspond to a BAPU $(\psi_r)_{r \in \mathcal{I}}$, for which $\text{supp}(\psi_r) \subseteq \Omega_r$ for each $r \in \mathcal{I}$ is compact.

Then one can determine for each $r \in \mathcal{I}$ a sampling set $\mathcal{X}^r \subset \Lambda^r$ such that the overall family, i.e.

$$\mathcal{G}_{g^r, \mathcal{I}^r} = \cup_{r \in \mathcal{I}} \{(g_\lambda^r) : \lambda \in \mathcal{X}^r\} \quad (2.9)$$

is a frame for $\mathbf{L}^2(\mathbb{R}^d)$.

The following theorem concerns with building multi-Gabor frames out of tight Gabor frames. In practice, we can use BAPUs on the time or frequency side, depending on the situation. For our purposes of describing variable bandwidth, using band-limited Gabor atoms on different time sections is more practical. Here we adapt the argument from [28] for band-limited Gabor atoms:

Theorem 7. Given tight Gabor frames $\mathcal{G}_j = \mathcal{G}(g_j, \Lambda_j)$, $j \in \mathcal{I}$ with band-limited atoms g_j for $L^2(\mathbb{R}^d)$ and a family of functions $\Psi = \{\psi_j\}_{j \in \mathcal{I}}$ such that

(i) $\{\hat{\psi}_j\}_{j \in \mathcal{I}}$ form a BAPU (with admissibility condition number n_0) and

$$(ii) (\forall j \in \mathcal{I}) (\exists \chi_j \subseteq \Lambda_j) (\forall \lambda \notin \chi_j) \text{ supp}(\hat{\psi}_j) \cap \text{supp}(\pi(\lambda')\hat{g}_j) = \emptyset,$$

then

$$\cup_{j \in \mathcal{I}} (\mathcal{G}_j, \chi_j) \quad (2.10)$$

is a frame for $L^2(\mathbb{R}^d)$.

Proof. Given a tight Gabor frame $\mathcal{G}_j = \mathcal{G}(g_j, \Lambda_j)$, any function $f \in L^2(\mathbb{R}^d)$ has a representation

$$f = \sum_{\Lambda_j} \langle f, \pi(\lambda)g_j \rangle \pi(\lambda)g_j. \quad (2.11)$$

Notice that for $\lambda = (x, \omega) \in \Lambda_j$ and $\lambda' = (\omega, -x)$, the Fourier transform of the Gabor atoms gives

$$\mathcal{F}(\pi(\lambda)g_j) = e^{2\pi i x \omega} \pi(\lambda')\hat{g}_j.$$

Thus

$$\hat{f} = \sum_{\lambda \in \Lambda_j} \langle f, \pi(\lambda)g_j \rangle e^{2\pi i x \omega} \pi(\lambda')\hat{g}_j.$$

As Ψ forms a BAPU on the frequency side, we have $\sum_{j \in \mathcal{I}} \hat{\psi}_j \hat{f} = \hat{f}$, and

$$\hat{\psi}_j \hat{f} = \sum_{\chi_j} \langle f, \pi(\lambda)g_j \rangle e^{2\pi i x \omega} \hat{\psi}_j \pi(\lambda')\hat{g}_j,$$

where $\chi_j \subset \Lambda_j$ corresponds to the non-zero contributions. Then

$$\begin{aligned} \|f\|_2^2 &= \|\hat{f}\|_2^2 \leq n_0 \sum_{j \in \mathcal{I}} \|\hat{\psi}_j \hat{f}\|_2^2 = n_0 \sum_{j \in \mathcal{I}} \left\| \sum_{\chi_j} \langle f, \pi(\lambda)g_j \rangle \hat{\psi}_j \mathcal{F}(\pi(\lambda)g_j) \right\|_2^2 \\ &\leq n_0 \sum_{j \in \mathcal{I}} \left\| \sum_{\chi_j} \langle f, \pi(\lambda)g_j \rangle \mathcal{F}(\pi(\lambda)g_j) \right\|_2^2 \\ &\leq n_0 \sum_{j \in \mathcal{I}} \|\mathcal{F}(\pi(\lambda)g_j)\|_{s_o} \left| \sum_{\chi_j} \langle f, \pi(\lambda)g_j \rangle \mathcal{F}(\pi(\lambda)g_j) \right|^2 \\ &\leq n_0 C_{\max_j \{\|g_j\|_{s_o}\}} \sum_{j \in \mathcal{I}} \sum_{\chi_j} |\langle f, \pi(\lambda)g_j \rangle|^2 \end{aligned}$$

due to the boundedness of the coefficients operators $C_j : l^2 \rightarrow L^2$, $j \in \mathcal{I}$.

The upper frame bound is simply a multiple of the maximum of all the upper frame bounds B_j :

$$\begin{aligned} \sum_j \sum_{\lambda \in \chi_j} |\langle f, \pi(\lambda)g_j \rangle|^2 &\leq \sum_j \sum_{\lambda \in \Lambda_j} |\langle f\psi_{j^*}, \pi(\lambda)g_j \rangle|^2 \\ &\leq B \sum_j \|f\psi_{j^*}\|_2^2 \leq n_o B \|f\|_2^2. \end{aligned}$$

The last inequality is due to the admissibility condition

(as then $\sum_j \|f\psi_{j^*}\|_2^2 \leq n_o \|f\|_2^2$). □

Further in this thesis, we shall look into the possibility of developing the multi-Gabor frames concept for weighted modulation spaces.

Chapter 3

Weights on the time-frequency plane

Weight functions are a commonly used tool in time-frequency analysis and occur in several contexts: in the definition of modulation spaces where the weight controls the time-frequency concentration of a function [65], in the theory of Gabor frames and time-frequency expansions where the weight measures the quality of time-frequency concentration [63, 5] and in the definition of symbol classes for pseudo-differential operators to describe the smoothness in the Sjöstrand class [54, 91, 90].

In general, weights are used to quantify growth and decay conditions. One standard example is the polynomial growth weight function

$$v(t) = v_s(t) = (1 + |t|)^s. \quad (3.1)$$

The weighted norm of a function tells more about the behavior of that function. For instance, let $\|f\|_{L_v^\infty} = \sup_{t \in \mathbb{R}^d} |f(t)|v(t) < \infty$. Then obviously

$$|f(t)| < c(1 + |t|)^{-s}.$$

For $s < 0$ the last inequality says that f grows at most like a polynomial of degree s . On the other hand, if $s > 0$, then f decays polynomially.

A weight function v on \mathbb{R}^n is called *sub-multiplicative*, if

$$v(z_1 + z_2) \leq v(z_1)v(z_2), \quad (3.2)$$

for all $z_1, z_2 \in \mathbb{R}^n$.

Given a sub-multiplicative weight v , a weight function m on \mathbb{R}^n is *v-moderate* if

$$m(z_1 + z_2) \leq cv(z_1)m(z_2) \quad (3.3)$$

for all $z_1, z_2 \in \mathbb{R}^n$ and some positive constant c . Surely v_r , as defined in (3.1), is v_s -moderate for $|r| \leq s$.

A weight v is called *subconvolutive*, if $v^{-1} \in L^1(\mathbb{R}^n)$ and it holds point-wise

$$v^{-1} * v^{-1} \leq cv^{-1} \quad (3.4)$$

for some positive constant c .

Two weights v_1 and v_2 are *equivalent*, written $v_1 \asymp v_2$, if

$$C^{-1}v_1(z) \leq v_2(z) \leq Cv_1(z) \quad (3.5)$$

for all $z = (x, \xi) \in \mathbb{R}^{2d}$ and some positive constant C .

The weight v_s (see (3.1)) is a typical example of a sub-multiplicative weight for $s \geq 0$. Any weight that is moderate with respect to v_s , we shall call v_s -moderate or s -moderate.

Weight conditions, often explored, are:

i) Gelfand-Reikov-Shilov condition (GRS)

$$\lim_{n \rightarrow \infty} v(nz)^{1/n} = 1 \text{ for all } z \in \mathbb{R}^n.$$

ii) Beurling-Domar condition (BD)

$$\sum_{n=0}^{\infty} \frac{\log v(nz)}{n^2} < \infty \text{ for all } z \in \mathbb{R}^n.$$

It is almost trivial to see that a weight satisfying the BD condition, also satisfies the GRS condition. A more general example of a weight function is

$$m(x)_{a,b,s,t} = e^{a|x|^b} (1 + |x|)^s (\log(e + |x|))^t, \quad (3.6)$$

which is a combination of a polynomial, exponential and logarithmic weight. This class contains the polynomial, the exponential and the sub-exponential weights. We summarize here the results from [32] and [65] for a weight of type (3.6).

Lemma 3. (*Properties of $m = m_{a,b,s,t}$*)

1. If $a, s, t \geq 0$ and $0 \leq b \leq 1$, then m is sub-multiplicative
2. If $a, s, t \in \mathbb{R}$ and $|b| \leq 1$ then m is moderate.
3. If either $0 < b < 1, a > 0, s, t \in \mathbb{R}$ or $b \in \{0, 1\}$ and $s > d$, then m is subconvolutive.
4. If $a, s, t \geq 0$ and $0 \leq b < 1$, then m satisfies the GRS and BG - conditions.

3.1 Variable bandwidth weights

Accepting that - due to the uncertainty principle - the concept of variable bandwidth cannot be too strict, a reasonable way to capture the idea is the following: we expect that any "good" time-frequency representation will be 'essentially concentrated' in a strip of variable bandwidth. It will become clear that this width needs not to change drastically from point to point. We shall describe the local STFT support of signals f as concentrated in a strip ST_b in the time-frequency plane (where b is the local width). We shall use weights to describe the notion of essential concentration in a certain area.

Definition 7. Let $b(x) \geq 0, x \in \mathbb{R}^d$. We call

$$ST_b := \{(x, \omega) \in \mathbb{R}^d \times \hat{\mathbb{R}}^d : |\xi| \leq b(x)\} \quad (3.7)$$

a strip with variable bandwidth, given by the bandwidth function b .

Note. Other names for ST_b are a time-varying strip, or a variable band with width b .

We denote the graph of b by

$$\Gamma_b := \{(x, b(x)) : x \in \mathbb{R}^d\}.$$

Notice that ST_b is symmetric with respect to the x -axis; we will work with only symmetric strips.

We shall use weights to measure the quality of time-frequency concentration, localized within the strip ST_b .

Definition 8. *Given a variable strip ST_b , a variable bandwidth weight is*

$$m_{b,s}(z) = (1 + d(z, ST_b))^s, \quad (3.8)$$

where d is a function, which quantifies the distance between $z = (x, \omega) \in \mathbb{R}^{2d}$ and ST_b .

On ST_b the weight is trivial and has value 1; otherwise (when $z \notin ST_b$), $m_{b,s}(z) = (1 + d(z, ST_b))^s > 1$ in case $s > 0$. In this case, the weight is assigning large values to the area outside the strip ST_b , that is, a $2d$ -function F , when multiplied by this weight, would have greater values outside the strip than its original. In case $s < 0$, the weight multiplication $F \cdot m_{b,s}$ results with decreased value on $\mathbb{R}^{2d} \setminus ST_{m_{b,s}}$.

When the power s is known, we write m_b instead $m_{b,s}$. If the bandwidth b is fixed as well, we write m instead of m_b .

As for the distance function d , we may choose a vertically measured distance or a minimal distance in the Euclidean sense for example.

Property 1. *$m_{b,s}$ is a sub-multiplicative weight, if the bandwidth b is such that for all $z_1, z_2 \in \mathbb{R}^{2d}$ it holds*

$$d(z_1 + z_2, ST_b) \leq d(z_1, ST_b) + d(z_2, ST_b) + d(z_1, ST_b)d(z_2, ST_b). \quad (3.9)$$

Proof.

$$\begin{aligned} m_{b,s}(z_1 + z_2) &= (1 + d(z_1 + z_2, ST_b))^s \\ &\leq (1 + d(z_1, ST_b) + d(z_2, ST_b) + d(z_1, ST_b)d(z_2, ST_b))^s \\ &= (1 + d(z_1, ST_b))^s (1 + d(z_2, ST_b))^s \\ &= m_{b,s}(z_1) m_{b,s}(z_2). \end{aligned}$$

□

If we choose $b \equiv 0$, then $m_{b,s} = v_s$, which is a classical example of a sub-multiplicative weight. Obviously, the required condition on the bandwidth in

the last property is not satisfied for a general choice of b . To verify this, we give an easy example of a constant bandwidth $b \equiv 2$ in \mathbb{R}^2 . In this setting, we choose the distance function

$$d(z, ST_b) = |\omega| - b(x)$$

to be the vertical distance for any time-frequency location. Consider the locations $z_1(2, 2)$ and $z_2(4, 2)$ on the graph Γ_b . We have $d(z_1, ST_b) = d(z_2, ST_b) = 0$, but $d(z_1 + z_2, ST_b) = d((6, 4), ST_b) = 2$ and the condition (3.9) is not satisfied. More generally, we can always choose a time-varying bandwidth b that would not satisfy this condition. At most we can hope that $m_{b,s}$ is moderate under less restrictive conditions.

Comment 1. *Band-limited functions can be classified as functions of variable bandwidth!*

From a practical point of view, when working with band-limited functions, we can choose a strip ST_b wide enough so that the weight would not affect their STFT in the time-frequency plane. If we choose to work with a band-limited window g in the STFT, all we need to do in this case is to take the strip boundary $b \geq \omega_o + r$, where $\text{spec}(f) \subseteq B_{\omega_o}(0)$ and $\text{spec}(g) \supseteq B_r(0)$. Then $V_g f \equiv 0$ outside the strip and the weight has no influence.

3.1.1 Vertical distance

As a first example of the specially designed weight class, we shall use vertical distance in the weight definition:

Proposition 1. *Let $s > 0$, $b(x) \geq 0$ and choose*

$$m_{b,s}(x, \omega) = \begin{cases} 1, & |\omega| \leq b(x) \\ (1 + |\omega| - b(x))^s, & |\omega| > b(x). \end{cases} \quad (3.10)$$

The weight m_b is moderate with respect of $v_s(x, \omega) = (1 + |x| + |\omega|)^s$, if the boundary function b satisfies the following condition:

$$(\forall x, y \in \mathbb{R}^d) \quad |b(x) - b(y)| \leq k|x - y| \quad (3.11)$$

for some $k \geq 1$.

Proof. Let $s > 0$. Whenever $z_1 + z_2 \in ST_b$, $z_l = (x_l, \omega_l)$, $l = 1, 2$, the weight at that point is 1 and the inequality

$$m_b(x_1 + x_2, \omega_1 + \omega_2) \leq C v_s(x_1, \omega_1) m_b(x_2, \omega_2)$$

is trivially satisfied for any $C \geq 1$, as both $v_s(x_1, \omega_1)$, $m_b(x_2, \omega_2) \geq 1$. Therefore, we only need to explore the case when $(x_1 + x_2, \omega_1 + \omega_2) \notin ST_b$. In this case, let us first take $(x_2, \omega_2) \in ST_b$. Then

$$m_b(x_1 + x_2, \omega_1 + \omega_2) = (1 + |\omega_1 + \omega_2| - b(x_1 + x_2))^s$$

and $m_b(x_2, \omega_2) = 1$; respectively, $|\omega_2| \leq b(x_2)$. We estimate

$$\begin{aligned} m_b(x_1 + x_2, \omega_1 + \omega_2) &\leq (1 + |\omega_1| + |\omega_2| - b(x_1 + x_2))^s \\ &\leq (1 + |\omega_1| + b(x_2) - b(x_1 + x_2))^s \\ &\leq (1 + |\omega_1| + |b(x_2) - b(x_1 + x_2)|)^s \\ &\leq (1 + |\omega_1| + k|x_1|)^s \\ &\leq k^s \cdot (1 + |x_1| + |\omega_1|)^s \cdot 1 \\ &\leq k^s \cdot (1 + |x_1| + |\omega_1|)^s m_b(x_2, \omega_2). \end{aligned}$$

The other possibility is that $(x_2, \omega_2) \notin ST_b$, that is, $|\omega_2| > b(x_2)$. We have

$$\begin{aligned} m_b(x_1 + x_2, \omega_1 + \omega_2) &\leq (1 + |\omega_1| + |\omega_2| - b(x_1 + x_2) \pm b(x_2))^s \\ &\leq (|\omega_1| + (1 + |\omega_2| - b(x_2)) + |b(x_2) - b(x_1 + x_2)|)^s \\ &\leq (|\omega_1| + 1 + |b(x_2) - b(x_1 + x_2)|)^s \cdot (1 + |\omega_2| - b(x_2))^s \\ &\leq (|\omega_1| + 1 + k \cdot |x_1|)^s \cdot m_b(x_2, \omega_2) \\ &\leq k^s (1 + |x_1| + |\omega_1|)^s \cdot m_b(x_2, \omega_2). \end{aligned}$$

□

Comment 2. *The condition on the bandwidth function in the previous proposition is the Lipschitz condition [94].*

Comment 3. *Alternatively, in the conditions of the last proposition, we can choose b to be differentiable and b' to be bounded. Then the obvious choice for k is $k = \max_{\xi \in \mathbb{R}^d} \{1, |b'(\xi)|\}$, as by the mean value theorem, it would hold*

$$|b(x_2) - b(x_1 + x_2)| \leq k |x_1|$$

A large class of functions that describe a bandwidth that satisfies (3.11) and therefore defines a moderate weight $m = m_b$, satisfy $0 \leq b(x) \leq k$, as then $|b(x) - b(y)| \leq k$. In fact,

Corollary 1. *If there exists a constant $k \geq 1$ such that $|b(x) - b(y)| \leq k$ for all $x, y \in \mathbb{R}^d$, then m_b is moderate with respect to $v_s(x, \omega) = (1 + |\omega|)^s$.*

Proof. In case $\omega + \xi \in ST_b$, it is trivial to verify

$$m(x + y, \omega + \xi) \leq (1 + |\omega|)^s m(x, \omega).$$

Take $\omega + \xi \notin ST_b$ and $\xi \in ST_b$. Then

$$\begin{aligned} 1 &\leq 1 + |\omega + \xi| - b(x + y) \\ &\leq 1 + |\omega| + |\xi| - b(x + y) \\ &\leq 1 + |\omega| + b(y) - b(x + y) \leq 1 + |\omega| + k \\ &\leq (k + 1)(1 + |\omega|), \end{aligned}$$

thus,

$$(1 + |\omega + \xi| - b(x + y))^s \leq (k + 1)^s (1 + |\omega|)^s.$$

If both $\omega + \xi$ and ξ are not in ST_b ,

$$\begin{aligned} 1 &\leq 1 + |\omega + \xi| - b(x + y) \mp b(y) \\ &\leq |\omega| + \underbrace{(1 + |\xi| - b(y))}_{\geq 0} + b(y) - b(x + y) \\ &\leq |\omega| + (1 + |\xi| - b(y)) + |b(y) - b(x + y)| \\ &\leq |\omega| + (1 + |\xi| - b(y)) + k \\ &\leq (k + 1)(1 + |\omega|)(1 + |\xi| - b(y)), \end{aligned}$$

which comes to

$$m(x + y, \omega + \xi) \leq (1 + |\omega|)^s m(x, \omega).$$

□

We shall see in chapter 4 that the special case of bounded bandwidth is characterized with equivalent norms, that is, both weights describe the same

Sobolev space. We emphasize here that we shall be interested in a more general choice of the bandwidth. For instance, we may choose the bandwidth b to follow the behavior of $y = a_1|x| + a_2$. Then $b' \approx a_1$ so we have bounded derivative and thus, m_b is a moderate weight. The weight $m = m_b$ is similar to v_s so we can expect that it satisfies the GRS condition. To simplify things, we may choose that the bandwidth grows with order one: for large values of x we take $\omega = x$ to describe the bandwidth. Then $(1 + |n\omega| - b(nx))^{1/n} \rightarrow 1$. Then m_b satisfies GRS condition for all s .

The next proposition is used in a result on dilation invariance later on in this thesis.

Proposition 2. *If $1 \leq b(x) \leq B$ for all $x \in \mathbb{R}^d$ and the weight (3.10) is moderate, then*

$$m_b(\mu^{-1}x, \mu\omega) \leq (|\mu|(B+1))^s m_b(x, \omega). \quad (3.12)$$

Proof. Whenever $m(\mu^{-1}x, \mu\omega) = 1$, the inequality (3.12) is trivially satisfied. Consider the case when $|\omega| \leq b(x)$ and $|\mu\omega| > b(\mu^{-1}x)$, that is,

$$m_b(x, \omega) = 1, \quad m_b(\mu^{-1}x, \mu\omega) = (1 + |\mu\omega| - b(\mu^{-1}x))^s.$$

We use $|\mu\omega| = |\mu||\omega| \leq |\mu|b(x)$ and obtain

$$m_b(\mu^{-1}x, \mu\omega) \leq (|\mu|(b(x) + 1))^s \leq (|\mu|(B + 1))^s = (|\mu|(B + 1))^s \cdot m_b(x, \omega)$$

If the weight at both positions has non-trivial values, that is, $|\omega| > b(x)$ and $|\mu\omega| > b(\mu^{-1}x)$, we have

$$\begin{aligned} m_b(\mu^{-1}x, \mu\omega) &= (1 + |\mu\omega| - b(\mu^{-1}x))^s \\ &\leq (1 + |\mu|(1 + |\omega| - b(x) + b(x)) - b(\mu^{-1}x))^s \end{aligned}$$

(use $1 - b(\mu^{-1}x) \leq 0$)

$$\begin{aligned} &\leq (|\mu|(1 + |\omega| - b(x)) + |\mu|b(x))^s \\ &= (|\mu| \cdot m_b^{1/s}(x, \omega) + |\mu|b(x))^s \\ &\leq (|\mu| \cdot m_b^{1/s}(x, \omega) + |\mu| \cdot B)^s \\ &\leq (|\mu| \cdot (1 + B))^s \cdot m_b(x, \omega). \end{aligned}$$

□

We will show that small modifications of the bandwidth $b(x)$ produce equivalent weights. This is most desirable in the practical setting as it allows a certain level of in-precise measurement of the bandwidth.

Consider moving the bandwidth function b for a small, non-negative step h , $0 \leq h(x) < 1$. We shall check the relation between the corresponding weights m_b and m_{b+h} . There should be a significant similarity in point-wise values, as the bandwidths b and $b + h$ are near by.

It holds trivially that the related strips are contained one into another, that is, $ST_b \subseteq ST_{b+h}$, as $b(x) \leq b(x) + h(x)$ for all x . The weight, corresponding to the band ST_{b+h} is

$$m_{b+h}(x, \omega) = \begin{cases} 1, & |\omega| \leq b(x) + h(x) \\ (1 + |\omega| - b(x) - h(x))^s, & |\omega| > b(x) + h(x). \end{cases}$$

Proposition 3. *Let $0 \leq h(x) < 1$ for all $x \in \mathbb{R}^d$, and assume that $b(x)$ generates a moderate weight $m_b = m_{b,s}$.*

Then $b(x) + h(x)$ generates an equivalent weight m_{b+h} and

$$2^{-s}m_{b+h}(x, \omega) \leq m_b(x, \omega) \leq 2^s m_{b+h}(x, \omega), \quad (3.13)$$

for all $(x, \omega) \in \mathbb{R}^{2d}$.

Proof. On ST_{b+h} , we have $m_b = m_{b+h} = 1$, as $|\omega| < b(x) < b(x) + h(x)$, and (3.13) is trivially satisfied.

Let $b(x) < |\omega| \leq b(x) + h(x)$. Then $|\omega| - b(x) \leq h(x)$ and it holds

$$2^{-s}m_{b+h}(x, \omega) = 2^{-s} < 1 < m_b(x, \omega) \leq (1 + h(x))^s \leq 2^s = 2^s m_{b+h}(x, \omega).$$

If $|\omega| - b(x) - h(x) > 0$, then both weights have non-trivial values,

$$m_b(x, \omega) = (1 + |\omega| - b(x))^s \text{ and } m_{b+h}(x, \omega) = (1 + |\omega| - b(x) - h(x))^s.$$

We use the estimation

$$\begin{aligned} \left(\frac{1 + |\omega| - b(x)}{1 + |\omega| - b(x) - h(x)} \right)^s &= \left(1 + \frac{h(x)}{1 + |\omega| - b(x) - h(x)} \right)^s \\ &\leq \left(1 + \frac{1}{\underbrace{1 + |\omega| - b(x) - h(x)}_{>0}} \right)^s \\ &\leq 2^s. \end{aligned}$$

So

$$(1 + |\omega| - b(x))^s \leq 2^s (1 + |\omega| - b(x) - h(x))^s,$$

which is the right inequality in (3.13).

On the other hand, $2^{-s}(1 + |\omega| - b(x) - h(x))^s \leq (1 + |\omega| - b(x))^s$, which is the left inequality in (3.13). \square

Only a finite number of shifting the bandwidth is allowed to preserve equivalence of weights; in fact,

Corollary 2. *If there is a positive constant C so that the bandwidth shift is $0 \leq h \leq C$, then the weight equivalence constant is $(C + 1)^s$.*

When moving the bandwidth inwards ($b(x) \rightarrow 0$) we end up with a polynomial weight $m_b(x, \omega) \rightarrow (1 + |\omega|)^s$. The finite bandwidth shift property may give the impression that there is only one such weight up-to equivalence, but this is not the case. Consider a bandwidth function of form $b(x) = (1 + |x|)^a$ and take $a \in (0, 1]$. Such a bandwidth gives a moderate weight as b' is bounded for each a , but the generated weights for different values of a are **not** equivalent. The reason for it is that b tends to ∞ when $x \rightarrow \infty$.

Comment 1. *Due to the 'finite steps' argument, one can expect different bandwidths with same asymptotic behavior at ∞ to define equivalent weights. This will later on generate norm-equivalent spaces.*

3.1.2 Equivalent variable bandwidth weights

Another example of a sub-multiplicative, polynomial weight is

$$v_2(x, \omega) = (1 + |\omega|^2)^{s/2}.$$

It is of significant importance in calculating the reproducing kernel of Sobolev spaces and is in fact, equivalent to $v_1(x, \omega) = (1 + |\omega|)^s$, that is, it holds

$$c^{-1}v_1 \leq v_2 \leq cv_1.$$

The variable bandwidth weight designed from this weight by inserting an unweighted region in the time-frequency plane, is moderate. In fact, the corresponding variable bandwidths are equivalent to one another:

$$3m_2(x, \omega) \geq m_1(x, \omega) \geq m_2(x, \omega).$$

When $z = (x, \omega) \in ST_b$, both variable bandwidth weights have value 1 and the equivalence inequalities hold trivially. Otherwise, when $|\omega| - b(x) > 0$, we have

$$m_1(x, \omega) = ((1 + |\omega| - b(x))^s) \quad (3.14)$$

and

$$m_2(x, \omega) = ((1 + (|\omega| - b(x))^2)^{s/2}). \quad (3.15)$$

As

$$(1 + (|\omega| - b(x))^2) = (1 + |\omega|^2 - 2|\omega|b(x) + b(x)^2) \quad (3.16)$$

$$\leq 1 + |\omega|^2 - 2|\omega|b(x) + b(x)^2 + |\omega| - b(x) \quad (3.17)$$

$$= ((1 + |\omega| - b(x))^2), \quad (3.18)$$

it holds that $m_1(x, \omega) \geq m_2(x, \omega)$.

The inequality $3m_2(x, \omega) \geq m_1(x, \omega)$ is equivalent to

$$2(1 + (|\omega| - b(x))^2) \geq 2(|\omega| - b(x)),$$

which is always true. Then

$$m_2(z_1 + z_2) \leq m_1(z_1 + z_2) \leq v_1(z_1)m_1(z_2) \leq 2cv_2(z_1)m_2(z_2),$$

for c being the constant occurring in the inequality that quantifies the equivalence of weights v_1 and v_2 . We conclude

Corollary 3. (*Properties of m_2*)

a) *If the bandwidth b is bounded, then*

$$m_2(x, \omega) = \begin{cases} 1, & |\omega| \leq b(x) \\ ((1 + (|\omega| - b(x))^2)^{s/2}), & \text{otw.} \end{cases} \quad (3.19)$$

is a moderate weight with respect to $v_2(x, \omega) = (1 + |\omega|^2)^{s/2}$.

b) *Similarly, m_2 is moderate with respect to $v_2(x, \omega) = (1 + |x|^2 + |\omega|^2)^{s/2}$ in the general case.*

c) *Moving the bandwidth for a finite step generates an equivalent weight.*

The version m_2 of a variable bandwidth weight shall be important further on when we construct a reproducing kernel for the corresponding space.

3.1.3 Minimal distance

When the weight is defined with respect to the minimal distance to the strip, we can make use of the *triangle inequality*

$$d(z_1 + z_2, ST_b) \leq d(z_1, ST_b) + d(z_2, ST_b). \quad (3.20)$$

If the distance function d satisfies (3.20), then

$$\begin{aligned} (1 + d(z_1 + z_2, ST_b))^s &\leq (1 + d(z_1, ST_b) + d(z_2, ST_b))^s \\ &\leq (1 + |z_1| + d(z_2, ST_b))^s \\ &\leq (1 + |x_1| + |\omega_1| + d(z_2, ST_b))^s \\ &\leq (1 + |x_1| + |\omega_1|)^s (1 + d(z_2, ST_b))^s. \end{aligned} \quad (3.21)$$

Hence the weight (3.8) is moderate with respect to $(1 + |x| + |\omega|)^s$ without any additional condition on the bandwidth b , apart of the implicit convention that it satisfies the triangle inequality (3.20) and $b(x) \geq 0$. Although weight (3.8) is more elegant and at first look, more appropriate for applications, the vertical distance weight (3.10) has an advantage of easier computation: in the Matlab experiments we shall only make use of the vertical distance.

The dilation property in this case is not clear: when the weight at both positions has non-trivial values, satisfying the inequality

$$m_b(\mu^{-1}x, \mu\omega) \leq c|\mu|m_b(x, \omega),$$

for some constant c means requiring

$$1 + d((\mu^{-1}x, \mu\omega), ST_b) \leq c|\mu|(1 + d((x, \omega), ST_b)),$$

which is not a simple test.

As for the bandwidth move by $0 \leq h < 1$, by the triangular inequality it is true that $d(z, ST_{b+h}) \leq d(z, ST_b) \leq 1 + d(z, ST_{b+h})$ so we conclude

Proposition 4. *Let $0 \leq h(x) < 1$ for all $x \in \mathbb{R}^d$, and assume that $b(x)$ generates a moderate weight $m_b = m_{b,s}$.*

Then $b(x) + h(x)$ generates an equivalent weight m_{b+h} and

$$m_{b+h}(x, \omega) \leq m_b(x, \omega) \leq 2^s m_{b+h}(x, \omega), \quad (3.22)$$

for all $(x, \omega) \in \mathbb{R}^{2d}$.

Similar to in the previous case, for a positive constant C so that the bandwidth shift is $0 \leq h \leq C$, the weight equivalence constant is $(C + 1)^s$.

3.1.4 Piecewise constant bandwidth

We study a piecewise constant bandwidth that gradually grows toward infinity and is symmetric with respect to the time-frequency axes. We take the bandwidth to be $b(x) = a_n$, for each $x \in [n, n + 1)^d$. For $n \geq 0$, we choose $a_{n+1} \geq a_n$, $a_n \rightarrow \infty$ and $a_{-n} = a_n$. Equivalently, we may write

$$b_a(x) = \sum_n a_n \cdot 1_{[n, n+1)^d}(x). \quad (3.23)$$

The purpose of designing a weight with respect to this particular bandwidth is to locally be able to work with a weight with constant bandwidth, which is equivalent to a Sobolev space weight.

Property 2. *The weight defined with respect to the bandwidth (3.23)*

$$m(x, \omega) = \begin{cases} 1 & \text{if } |\omega| \leq b_a(|x|), \\ (1 + |\omega| - b_a(|x|))^s & \text{if } |\omega| > b_a(|x|). \end{cases} \quad (3.24)$$

is moderate with respect to $v(x, \omega) = (1 + |\omega|)^s$, if all $a_n, n \in \mathbf{Z}^d$ are bounded.

Notice that the weight (3.24) is certainly moderate with respect to $(1 + |x| + |\omega|)^s$ as $b'_a = 0$. We give here an alternative proof, that employs sectioned vertical strips in the time-frequency plane.

Proof. On each section $x \in [n, n+1)^d$, we have $m(x, \omega) = m_{a_n}(x, \omega)$, that is

$$m(x, \omega) = \sum_n m_{a_n}(x, \omega) \cdot 1_{[n, n+1)}(x)$$

The weight should satisfy

$$m(x + y, \omega + \xi) \leq C \cdot v(x, \omega) \cdot m(y, \xi).$$

In the trivial case when the left side of this inequality is 1, there is nothing to check as $C \geq 1$. So, let

$$m(x + y, \omega + \xi) = (1 + |\omega + \xi| - a_p)^s,$$

that is, $x + y \in [p, p+1)^d$ and $|\omega + \xi| > a_p$. If we choose $x, y \geq 0$, clearly we have $a_n \leq a_p$, as $n \leq p$. If $m(y, \xi) = 1$, where $y \in [n, n+1)^d$, $|\xi| \leq a_n$; as it is clear that

$$1 + |\omega + \xi| - a_p \leq 1 + |\omega| + \underbrace{|\xi|}_{\leq a_n} - a_p \leq 1 + |\omega| + \underbrace{a_n - a_p}_{\leq 0} \leq 1 + |\omega|, \quad (3.25)$$

we have $(1 + |\omega + \xi| - a_p)^s \leq (1 + |\omega|)^s \cdot 1$.

If $m(y, \xi) = (1 + |\xi| - a_n)^s$ for $y \in [n, n+1)^d$, $|\xi| > a_n$, we use the simple fact that $-a_p \leq -a_n$ and get

$$\begin{aligned} 1 + |\omega + \xi| - a_p &\leq 1 + |\omega| + |\xi| - a_p \\ &= (1 + |\omega|) \left(1 + \frac{|\xi| - a_p}{1 + |\omega|} \right) \\ &\leq (1 + |\omega|) (1 + |\xi| - a_p) \\ &\leq (1 + |\omega|) (1 + |\xi| - a_n). \end{aligned}$$

Therefore

$$(1 + |\omega + \xi| - a_p)^s \leq (1 + |\omega|)^s \cdot (1 + |\xi| - a_n)^s,$$

which concludes the proof. \square

The weight, generated from a piecewise constant bandwidth is therefore describing a Sobolev function on a local level, see chapter 4. We shall also work with these kind of weights in chapter 6. However, the vertical distance causes weight discontinuities and is not too convenient when there are huge discontinuities in the strip border function b .

Figure 3.1: Frequency-only weight vs. constant band weight vs. variable bandwidth weight. $wgt = tfbwgt(n, xs, ys)$

Chapter 4

Modulation spaces

4.1 STFT

The basic flaw of the Fourier transform when extracting the frequency content of a signal f is that it contains no time information, i.e. how long each frequency appears in the signal f . This information is often needed in practice; H. Feichtinger's favorite example that describes this need is "a musician playing a piece of music": when playing, a musician has to know which tune (frequency) to play at what moment (time). Therefore, it is useful to describe the time-frequency content of a signal f , as its frequency content is to change in time.

The short-time Fourier transform (in short notation, STFT) is a standard tool for time-frequency analysis. It measures the time-frequency content of a signal f (energy distribution) and provides information about local (smoothness) properties of the signal f . This is achieved by localization of f near t through multiplication with some *window/test function* g and subsequently applying the Fourier transform, thus providing information about the frequency content of f in this segment.

Definition 9. Fix a function $g \neq 0$. The short-time Fourier transform of a function f with respect to g is defined as

$$V_g f(x, \omega) = \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-2\pi i t \omega} dt, \quad \text{for } (x, \omega) \in \mathbb{R}^{2d}. \quad (4.1)$$

Whenever $f, g \in L^2$, $V_g f$ is obviously well-defined. Typically, g is chosen so that it is concentrated around the origin; then its translates $T_x g$ are concentrated around x . To avoid artificial discontinuities that occur with sharp cut-offs, it is better to use a continuous g .

In general, g can be any non-zero Schwartz function such as the Gaussian $g(t) = e^{-\pi t^2}$; the Gaussian in fact is a pretty good choice as it provides for good time and frequency localization since then it is rapidly decreasing, Fourier invariant function. If g is compactly supported only a segment of f in some neighbourhood around t is relevant; in fact, shrinking the window support provides good time resolution while widening gives good frequency resolution.

Given a compactly supported window g with support centered around 0, $V_g f(x, \cdot)$ is the Fourier transform of the localized f around x . Changing x means translating/sliding the window g along the x -axis into different positions (thus comes the name 'sliding window Fourier transform'). Clearly, the narrower the support of g , the more localized in time results: in the extreme case, for $g = \delta$, we have pure temporal behavior, that is

$$V_\delta f(x, \omega) = M_{-\omega} f(x).$$

Hence $|V_\delta f(x, \omega)| = |f(x)|$. Opposed to this, pure frequency behavior is the outcome when considering the other extreme window $g = 1_{\mathbb{R}^d}$:

$$V_{1_{\mathbb{R}^d}} f(x, \omega) = \hat{f}(\omega).$$

Hence $|V_{1_{\mathbb{R}^d}} f(x, \omega)| = |\hat{f}(\omega)|$.

Whenever the scalar product is defined (in the sense of duality of Banach spaces), we have

$$V_g f(x, \omega) = \langle f, \pi(x, \omega)g \rangle.$$

Here $\pi(x, \omega) = M_\omega T_x$ denotes a *time-frequency shift* in the *time-frequency plane*; due to Weyl's' commutation relation it holds

$$\pi(x, \omega) = e^{2\pi i x \omega} T_x M_\omega.$$

The time-frequency shifts $M_\omega T_x$ for $(x, \omega) \in \mathbb{R}^{2d}$ satisfy the following *composition law*:

$$\pi(x, \omega)\pi(y, \eta) = e^{-2\pi i x \cdot \eta} \pi(x + y, \omega + \eta), \quad (4.2)$$

for $(x, \omega), (y, \eta)$ in the time-frequency plane $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. This implies another important property of the STFT, the *covariance property*:

$$V_g(T_u M_\eta f)(x, \omega) = e^{-2\pi i u \omega} V_g f(x - u, \omega - \eta). \quad (4.3)$$

To put it in words, when viewed by absolute value, a time-frequency shift of the function f is simply a translation on the time-frequency plane.

$V_g f$ is linear in f and conjugate linear in g ; for $f, g \in L^2(\mathbb{R}^d)$ the STFT is uniformly continuous on \mathbb{R}^{2d} , i.e., we can sample the $V_g f$ without a problem. The choice of the window function g influences the properties of the STFT remarkably. One example of a good window class is the Schwartz space of rapidly decreasing functions. A broader function space, which is perfectly suited as a good class of windows is the *Feichtinger's algebra* \mathcal{S}_o [33]. It is most remarkable as, by equations (4.4) and (4.3), it is isometric Fourier invariant and time-frequency-shift invariant.

By Parseval's theorem (the Fourier transform is unitary) and using the commutation relations, we derive the following relation

$$V_g f(x, \omega) = e^{-2\pi i x \omega} V_{\hat{g}} \hat{f}(\omega, -x). \quad (4.4)$$

The last equation is often referred to as the *fundamental identity* of time-frequency analysis [62]. It states that the STFT is a joint time-frequency representation and that the Fourier transform amounts to a rotation of the time-frequency plane $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ by an angle of $\frac{\pi}{2}$ whenever the window g is Fourier invariant.

As for the Fourier transform there is also a Parseval's equation for the STFT which is referred to as *Moyal's formula*.

Lemma 4. (*Moyal's Formula*) *Let $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d)$. Then $V_{g_1} f_1$ and $V_{g_2} f_2$ are in $L^2(\mathbb{R}^{2d})$ and the following identity holds:*

$$\langle V_{g_1} f_1, V_{g_2} f_2 \rangle_{L^2(\mathbb{R}^{2d})} = \langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle}. \quad (4.5)$$

Moyal's formula implies that orthogonality of windows g_1, g_2 resp. of signals f_1, f_2 implies orthogonality of their STFT's. For normalized $g \in L^2(\mathbb{R}^d)$ (i.e., with $\|g\|_2 = 1$) one has:

$$\|V_g f\|_{L^2(\mathbb{R}^{2d})} = \|f\|_{L^2(\mathbb{R}^d)},$$

for all $f \in L^2(\mathbb{R}^d)$, i.e., the STFT is an isometry from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^{2d})$.

Using the tensor product $f \otimes g(x, t) := f(x)g(t)$, the asymmetric coordinate transform $\mathcal{T}_a F(x, t) := F(T, t - x)$ and the partial Fourier transform $\mathcal{F}_2 F(x, \omega) := \int F(x, t) e^{-2\pi i t \cdot \omega} dt$, we can rewrite the STFT as

$$(f, g) \mapsto V_g f = \mathcal{F}_2 \mathcal{T}_a (f \otimes g).$$

Both \mathcal{F}_2 and \mathcal{T}_a are isomorphisms on $\mathcal{S}'(\mathbb{R}^{2d})$ and, provided $f, g \in \mathcal{S}'(\mathbb{R}^d)$, we have $f \otimes g \in \mathcal{S}'(\mathbb{R}^{2d})$. Thus, the domain of the STFT as a sesquilinear form is $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d)$.

Using these unitary operators is quite handy: for instance, the classical proof of Moyal's' formula involves Parseval's formula and Fubini's theorem. Instead, we give an almost one-line proof:

$$\begin{aligned} \langle V_{g_1} f_1, V_{g_2} f_2 \rangle &= \langle \mathcal{F}_2 \mathcal{T}_a (f_1 \otimes g_1), \mathcal{F}_2 \mathcal{T}_a (f_2 \otimes g_2) \rangle \\ &= \langle f_1 \otimes g_1, f_2 \otimes g_2 \rangle = \langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle}. \end{aligned}$$

By the Heisenberg uncertainty principle, a function f and its Fourier transform \hat{f} cannot be both localized simultaneously. As the STFT is a localized Fourier transform, that is,

$$V_g f(x, \omega) = \widehat{(f \cdot T_x \bar{g})}(\omega),$$

one would expect a similar result on the time-frequency plane. Lieb's result gives the answer.

Lemma 5. *Lieb [76]*

If $f, g \in L^2(\mathbb{R}^d)$ and $1 \leq p < 2$, then

$$\left(\iint_{\mathbb{R}^{2d}} |V_g f(x, \omega)|^p dx d\omega \right)^{1/p} \geq \left(\frac{2}{p} \right)^{d/p} \|f\|_2 \|g\|_2. \quad (4.6)$$

The inequality is reversed for $2 \leq p < \infty$.

The last result may be a motivation to pursue function spaces, characterized by decay properties of the STFT of their elements. In 1983 Feichtinger introduced such a family of Banach spaces, the so-called *modulation spaces*, the

right setup for time-frequency analysis. Typical examples are Feichtinger's algebra S_0 and its dual space S'_0 . In fact, Lieb's inequality expresses just embeddings of certain modulation spaces into L^2 . Gröchenig and some of his collaborators have extensively studied uncertainty principles as embeddings of certain weighted L^p -spaces into modulation spaces, [61, 64].

We gather all the faces of the STFT in a lemma:

Lemma 6. *If $f, g \in L^2(\mathbb{R}^d)$, then $V_g f$ is uniformly continuous on \mathbb{R}^{2d} , and*

$$\begin{aligned} V_g f(x, \omega) &= \mathcal{F}(f \cdot T_x \bar{g})(\omega) = \langle f, M_\omega T_x g \rangle \\ &= e^{-2\pi i x \cdot \omega} \langle \hat{f}, M_{-x} T_\omega \hat{g} \rangle = e^{-2\pi i x \cdot \omega} \mathcal{F}(\hat{f} \cdot T_\omega \bar{\hat{g}})(-x) \\ &= e^{-2\pi i x \cdot \omega} V_{\hat{g}} \hat{f}(\omega, -x) = (\hat{f} * M_{-x} \hat{g}^*)(\omega). \end{aligned} \quad (4.7)$$

Another consequence of Moyal's formula is an *inversion formula* for the STFT. Given an analysis window $g \in L^2(\mathbb{R}^d)$ and a synthesis window $\gamma \in L^2(\mathbb{R}^d)$ such that $\langle g, \gamma \rangle \neq 0$, for $f \in L^2(\mathbb{R}^d)$ it holds

$$f = \frac{1}{\langle g, \gamma \rangle} \iint_{\mathbb{R}^{2d}} \langle f, \pi(x, \omega) g \rangle \pi(x, \omega) \gamma \, dx d\omega. \quad (4.8)$$

In contrast to the Fourier inversion, the building blocks of the STFT inversion formula are simply time-frequency shifts of a square-integrable function. Thus, the Riemannian sums corresponding to this inversion integral are functions in L^2 and are even norm convergent in L^2 for nice windows [92].

Comment 4. *Formula (4.8) reveals what kind of TF-analysis is possible despite the uncertainty principle. If we choose γ to be mostly concentrated in $K \subseteq \mathbb{R}^d$ and its spectrum to be around $\Omega \subseteq \mathbb{R}^d$, then $M_\omega T_x \gamma$ 'occupies' the cell*

$$(x + K) \times (\omega + \Omega)$$

in the TF-plane, and its contribution is then measured by $V_g f(x, \omega)$. Good time resolution demands a window with small support $|K|$, which by the uncertainty principle implies poor frequency resolution as $|\Omega|$ becomes large. And vice versa, a band-limited window shall provide poor time resolution from the same reason. Windows with rapid decay in both time and frequency (Schwartz class) give optimal results.

In practice, given a signal f , its STFT is computed and considered as a joint TF information for f . Then, $V_g f$ is processed into an altered function $F(x, \omega)$ (most common is truncating $V_g f$ to a region of interest [3]). Then, reconstruction is performed using the modified inversion formula

$$\tilde{f} = \iint F(x, \omega) M_\omega T_x \gamma dx d\omega. \quad (4.9)$$

This procedure has mathematical justification in the following result [62]

Theorem 8. *Let $K_n \subseteq \mathbb{R}^{2d}$ be a nested exhausting sequence of compact sets. given $g, \gamma \in L^2(\mathbb{R}^d)$, define f_n to be*

$$f_n = \iint_{K_n} V_g f(x, \omega) M_\omega T_x \gamma dx d\omega.$$

Then $\|f - f_n\|_2 \rightarrow 0, n \rightarrow \infty$.

Another related result is using approximate units (Benedetto, Heil, Walnut [70], [7]):

Theorem 9. *Let (u_n) be an approximate unit in $L^1 \cap \mathcal{FL}^1$, that is, let $\int u_n(t) dt = 1$, $\sup_n \|u_n\|_1 < \infty$ and for any $r > 0$,*

$$\lim_{n \rightarrow \infty} \int_{|t| > r} |u_n(t)| dt = 0.$$

Let $g, \gamma \in L^1 \cap L^\infty$. Given $f \in L^p, 1 \leq p < \infty$ we define

$$f_n = \iint V_g f(x, \omega) M_\omega T_x \gamma \cdot \hat{u}_n(\omega) d\omega dx.$$

Then

$$\|f - f_n\|_p \rightarrow 0, n \rightarrow \infty.$$

Under additional assumptions, the inverse STFT can be extended to other function spaces, namely to modulation spaces.

4.2 Modulation spaces

The STFT describes the global TF-distribution of a function/signal. Assuming that $V_g f \in L^2(\mathbb{R}^d)$ we still cannot estimate well the TF-localization of f . When we employ weights in the time-frequency plane though, the STFT decay is more accurately measured and the signal is better described. We shall use the following norm

$$\|f\|_{M_m^{p,q}} := \left(\int \left(\int |V_g f(x, \omega)|^p m(x, \omega)^p dx \right)^{q/p} d\omega \right)^{1/q} \quad (4.10)$$

to describe different behavior in time (L^p) and in frequency (L^q). In 1983 H.G. Feichtinger used this weighted norm to define Banach spaces of functions with a given/wanted TF-behavior [35]. These spaces are basically mixed-norm weighted spaces on the TF-side. Later on came the realization that modulation spaces are part of an even bigger theory, namely the general co-orbit theory.

Definition 10. *Given a fixed non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$, a moderate weight function m on \mathbb{R}^{2d} and $1 \leq p, q \leq \infty$, the modulation space $M_m^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $V_g f \in L_m^{p,q}(\mathbb{R}^{2d})$. The norm is then*

$$\|f\|_{M_m^{p,q}} := \|V_g f\|_{L_m^{p,q}}$$

The definition as it is, depends on the particular choice of a window g . We shall later see that choosing a different window brings us to equivalent norms.

To achieve better understanding of modulation spaces, let us recall what the STFT measures. In particular, if the window g is well-centered around the origin and has small support, then $V_g f(x, \omega)$ measures the magnitude of f near x . As $V_g f(x, \omega) = \langle \hat{f}, T_\omega M_{-x} \hat{g} \rangle$, we have the measure of \hat{f} near ω . Thus, decay of f or \hat{f} imply decay of $V_g f$. Even more, for particular choices of the weight, the modulation spaces coincide with some known spaces.

Proposition 5. *Let $g \in \mathcal{S}(R^d)$.*

1. *If $s > d$ and $|f(x)| \leq c(1 + |x|)^{-s}$ then $|V_g f(x, \omega)| \leq c'(1 + |x|)^{-s}$.*
2. *If $m(x, \omega) = m(x)$, then $M_m^2 = L_m^2$.*

3. If $m(x, \omega) = m(\omega)$, then $M_m^2 = \mathcal{FL}_m^2$. If $m(\omega) = (1 + |\omega|)^s$ for some $s \in \mathbb{R}$ then we have exactly the Bessel potential space $H^s(\mathbb{R}^d)$.
4. $\mathcal{S}(\mathbb{R}^d) = \cap_{s \geq 0} M_{v_s}^\infty$ and $\mathcal{S}'(\mathbb{R}^d) = \cup_{s \geq 0} M_{1/v_s}^\infty$ when

$$v_s(x, \omega) = (1 + |x| + |\omega|)^s.$$

Other useful, already known function spaces that are covered with the theory of modulation spaces are the Feichtinger's' algebra $M^1 = \mathcal{S}_0$ [33] and the spaces $M_{v_s}^2$, used in the theory of pseudodifferential operators

We shall use the extended definition of the inverse STFT to explore the basic properties of modulation spaces.

Definition 11. Fix a non-zero window γ and a function F on \mathbb{R}^{2d} . Then

$$V_\gamma^* F := \iint F(x, \omega) M_\omega T_x \gamma dx d\omega.$$

This integral is to be interpreted weakly, that is

$$\begin{aligned} \langle V_\gamma^* F, f \rangle &= \iint F(x, \omega) \langle M_\omega T_x \gamma, f \rangle dx d\omega \\ &= \iint F(x, \omega) \overline{V_\gamma f(x, \omega)} dx d\omega \\ &= \langle F, V_\gamma f \rangle. \end{aligned}$$

V_γ^* is a well-defined operator under the conditions given in the following proposition:

Proposition 6. Let m be moderate and $\gamma \in \mathcal{S}(R^d)$. Then

1. V_γ^* maps $L_m^{p,q}$ into $M_m^{p,q}(\mathbb{R}^d)$ and satisfies

$$\|V_\gamma^* F\|_{M_m^{p,q}} \leq c \|V_{g_0} \gamma\|_{L_v^1} \|F\|_{L_m^{p,q}}.$$

2. If $F = V_g f$ then the inversion formula on $M_m^{p,q}(\mathbb{R}^d)$ is

$$f = \frac{1}{\langle \gamma, g \rangle} \iint V_g f(x, \omega) M_\omega T_x \gamma dx d\omega.$$

3. The definition of $M_m^{p,q}(\mathbb{R}^d)$ is independent of the choice of g , as different windows yield equivalent norms.
4. For $g_o, g, \gamma \in \mathcal{S}(\mathbb{R}^d)$ such that $\langle g, \gamma \rangle \neq 0$ and $f \in \mathcal{S}'(\mathbb{R}^d)$ it holds

$$|V_{g_o}f(x, \omega)| \leq \frac{1}{|\langle \gamma, g \rangle|} (|V_gf| * |V_{g_o}\gamma|)(x, \omega).$$

For our purposes we shall only use polynomial-like weights. The extra convenience we have with this choice is that, for weights with polynomial decay, $\mathcal{S}(\mathbb{R}^d)$ is a dense subspace of each $M_m^{p,q}(\mathbb{R}^d)$ when $1 \leq p, q < \infty$, and $M_m^{p,q}(\mathbb{R}^d)$ are dense in $\mathcal{S}'(\mathbb{R}^d)$. The inclusion and density are easy to prove with the formalism of the inversion formula; however, the proofs were non-trivial at the beginning of the development of the theory [33].

Theorem 10. [*The Banach space property and TF-shift invariance.*]

1. $M_m^{p,q}(\mathbb{R}^d)$ is a Banach space for $1 \leq p, q \leq \infty$.
2. $M_m^{p,q}(\mathbb{R}^d)$ is invariant under time-frequency shifts and

$$\|M_\omega T_x f\|_{M_m^{p,q}} \leq cv(x, \omega) \|f\|_{M_m^{p,q}}.$$

Just like the mixed-norm spaces $L_m^{p,q}$ and $L_{1/m}^{p',q'}$ being dual to one another, a similar statement holds for modulation spaces.

Theorem 11. *If $1 \leq p, q < \infty$, then $(M_m^{p,q})' = M_{1/m}^{p',q'}$ and*

$$\langle f, h \rangle = \int V_{g_o}f(z) \overline{V_{g_o}h(z)} dz$$

for $f \in M_m^{p,q}$ and $h \in (M_m^{p,q})'$.

The core of the proof is using a bounded linear functional

$$l_h(f) = \int V_{g_o}f(z) \overline{V_{g_o}h(z)} dz$$

on $M_m^{p,q}$. Hölder's inequality implies that

$$|l_h(f)| \leq \|V_{g_o}f\|_{L_m^{p,q}} \|V_{g_o}h\|_{L_{1/m}^{p',q'}} = \|f\|_{M_m^{p,q}} \|h\|_{M_{1/m}^{p',q'}}.$$

As $M_m^{p,q}$ is isometrically isomorphic to the closed subspace

$$V = \{F \in L_m^{p,q} : F = V_{g_o}f\}$$

of $L_m^{p,q}$, any $l \in (M_m^{p,q})'$ induces a linear functional \tilde{l} on V by $l(f) := \tilde{l}(V_{g_o}f)$. By Hahn-Banach, \tilde{l} extends to a continuous functional on $L_m^{p,q}$. By the properties of mixed norm spaces, there exists a function $H \in L_{1/m}^{p',q'}$ such that

$$\tilde{l}(V_{g_o}f) = \int V_{g_o}f(z)\overline{H(z)}dz$$

Take $h = V_{g_o}^*H$ Then $h \in M_{1/m}^{p',q'}$ and

$$\langle f, h \rangle = l(f).$$

The space of admissible windows can be expanded from $\mathcal{S}(\mathbb{R}^d)$ to $M_v^1(\mathbb{R}^d)$; the inversion formula shall still hold and any window $\gamma \in M_v^1$ defines an equivalent norm on $M_m^{p,q}$. In this case however, the pool of functions/distributions of which the STFT is well-defined, is $(M_v^1)'$.

Theorem 12. *Let m be a v -moderate weight and let $g, \gamma \in M_v^1 \setminus \{0\}$. Then*

1. V_γ^* is bounded from $L_m^{p,q}$ into $M_m^{p,q}$ and

$$\|V_\gamma^*F\|_{M_m^{p,q}} \leq c\|V_{g_o}\gamma\|_{L_v^1}\|F\|_{L_m^{p,q}}.$$

2. The inversion formula holds on $M_m^{p,q}$.
3. $\|V_g f\|_{L_m^{p,q}}$ is an equivalent norm on $M_m^{p,q}$.

4.3 Gabor analysis on Modulation spaces

As both the coefficient and the reconstruction operators are bounded for an analysis/synthesis window in M_v^1 , we have

Theorem 13. *If $g, \gamma \in M_v^1$, then the Gabor frame operator $S_{g,\gamma} = D_\gamma C_g$ is bounded on $M_m^{p,q}$ for all $1 \leq p, q \leq \infty$, all v -moderate weights m and all lattice constants $\alpha, \beta > 0$. It holds*

$$\|S_{g,\gamma}\|_{op} \leq c_{v,\alpha,\beta} \|V_g g\|_{W(L_v^1)} \|V_\gamma \gamma\|_{W(L_v^1)}.$$

Further on, assuming $S_{g,\gamma} = I$ on L^2 , it is true that

$$f = \sum_{\Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)\gamma \quad (4.11)$$

$$= \sum_{\Lambda} \langle f, \pi(\lambda)\gamma \rangle \pi(\lambda)g, \quad (4.12)$$

with unconditional convergence in $M_m^{p,q}$ for all $1 \leq p, q < \infty$ and weak* convergence otherwise. In other words, there are constants $A, B > 0$ such that for all $f \in M_m^{p,q}$

$$A \|f\|_{M_m^{p,q}} \leq \left(\sum_n \left(\sum_k |\langle f, \pi(\lambda)g \rangle|^p m(\lambda)^p \right)^{q/p} \right)^{1/q} \leq B \|f\|_{M_m^{p,q}}, \quad (4.13)$$

for $\lambda = (\alpha k, \beta n) \in \Lambda$. Also, the norm equivalence

$$A' \|f\|_{M_m^{p,q}} \leq \|\langle f, \pi(\lambda)g \rangle_{\Lambda}\|_{\ell_m^{p,q}} \leq B' \|f\|_{M_m^{p,q}}, \quad (4.14)$$

holds on $M_m^{p,q}$.

We call inequality (4.13) a Gabor frame inequality on $M_m^{p,q}$.

The last result says that a function or distribution belongs to $M_m^{p,q}$ if and only if the sequence of Gabor coefficients $C_g f$ belongs to $\ell_m^{p,q}$. That is, the decay and summability of the Gabor coefficients characterize the time-frequency concentration as it is measured with modulation space norm. Therefore, modulation spaces are the right choice of spaces for quantitative Time-Frequency Analysis.

The assumptions $g, \gamma \in M_v^1$ and $S_{g,\gamma} = I$ on L^2 are satisfied for an invertible Gabor frame operator $S_{g,g}$ on M_v^1 and for the canonical dual window $\gamma^\circ = S_{g,g}^{-1}g$. If $S_{g,g}$ is invertible on M_v^1 , then $\gamma^\circ \in M_v^1$ as well; by construction, $S_{g,\gamma^\circ} = I$ on L^2 and $S_{g,g}^{-1} = S_{\gamma^\circ,\gamma^\circ}$. Therefore $S_{g,g}^{-1}$ is bounded on all $M_m^{p,q}$.

The invertibility problem is irrelevant for tight Gabor frames [60], simply because in that case $S_{g,g}^{-1} = AI$ and $\gamma^o = A^{-1}g$ for A being the frame bound. A full answer to the invertibility of the Gabor frame operator on modulation spaces is given in chapter 13 of [62]:

Theorem 14. *Assume $\mathcal{G}(g, \alpha, \beta)$ is a frame for $L^2(\mathbb{R}^d)$ and that either*

- $\alpha\beta \in Q$ and $g \in M_v^1$, or
- $\alpha\beta$ is arbitrary and $g \in M_{v_s+2d+\epsilon}^\infty$ for some $\epsilon > 0$.

Then there exists a constant $C \geq 1$ such that for all $f \in M_m^{p,q}$, $1 \leq p, q \leq \infty$, it holds

$$C^{-1}\|f\|_{M_m^{p,q}} \leq \left(\sum_n \left(\sum_k |\langle f, T_{\alpha k} M_{\beta n} g \rangle|^p m(\alpha k, \beta n)^p \right)^{q/p} \right)^{1/q} \leq C\|f\|_{M_m^{p,q}} \quad (4.15)$$

where m is a v -moderate weight (in the second case, $v = v_s$).

There exists a dual window $\gamma \in M_v^1$ such that $f \in M_m^{p,q}$ can be recovered from the frame coefficients $(\langle f, T_{\alpha k} M_{\beta n} g \rangle)$ via a Gabor expansion.

4.4 Wilson Basis

Although there are very simple and natural windows which produce a complete orthogonal basis for $L^2(\mathbb{R})$ (for instance, the indicator function on $[0, 1]$), they cannot have good joint time-frequency localization, due to Balian-Low. In the case of $1_{[0,1]}$, its Fourier transform is the *sinc* function, which is badly concentrated. The same holds for Gabor frames that are orthogonal bases for $L^2(\mathbb{R})$.

This is good reason why to explore signal representations, other than Gabor. The time-frequency localization difficulties may be overcome by re-arranging the Gabor atoms into a *Wilson basis*. This is analogue to the fact that wavelet bases are also unconditional bases for Besov spaces [79].

The new basis is symmetric in frequency, so instead of using $T_x M_\omega g$ as atoms, one combines them into new atoms of form

$$T_x(M_\omega g \pm M_{-\omega} g).$$

We shall only give an overview of the one-dimensional case and keep in mind that Wilson bases are possible in dimension $d > 1$ via tensor products.

Definition 12. *If $\mathcal{G}(g, \frac{1}{2}, 1)$ is a Gabor system (of redundancy 2), then the associated Wilson system $\mathcal{W}(g)$ consists of functions of form*

$$\begin{aligned} \psi_{k,0} &= T_k g, \\ \psi_{k,n} &= \frac{1}{\sqrt{2}} T_{\frac{k}{2}} (M_n + (-1)^{k+n} M_{-n}) g, \quad (k, n) \in \Lambda_{\mathcal{W}} = \mathbf{Z} \times \mathbf{N}. \end{aligned}$$

Imposing smoothness and decay to the window g , this re-arranging delivers a orthonormal basis with desired TF localization, therefore Wilson bases overcome the problem arising with standard Gabor systems (the Balian-Low theorem). It was shown in [21] that the transition from a Gabor frame $\mathcal{G}(g, \frac{1}{2}, 1)$ of density $(\alpha\beta)^{-1} = 2$ to a Wilson system $\mathcal{W}(g)$ gives an orthonormal basis of $L^2(\mathbb{R})$ (removes all redundancy). Another proof (less computational) of this statement is given in [62] and is based on the Gabor frame structure. For discrete time Wilson expansions we refer the reader to [9, 8]

4.4.1 Schwartz generator window

A simple way to construct a Schwartz window that gives a Wilson basis is the following:

Let $g_o = g_o^* \in \mathcal{S}(\mathbb{R})$ and define

$$G(x, \omega) := |Zg_o(x, \omega)|^2 + |Zg_o(x - \frac{1}{2}, \omega)|^2 > 0,$$

where $Zf(x, \omega) = \sum_{k \in \mathbf{Z}} M_{k\omega} T_k f(x)$ is the *Zak transform*. Choose such an atom g so that

$$Zg(x, \omega) := 2^{-1/2} G(x, \omega)^{-1/2} Zg_o(x, \omega)$$

Then

$$|Zg(x, \omega)|^2 + |Zg(x - \frac{1}{2}, \omega)|^2 = \frac{1}{2},$$

which suffices for $\mathcal{W}(g)$ to be an orthonormal basis for $L^2(\mathbb{R})$. Due to the properties of the Zak transform and its inverse (smoothness is preserved), the atom g is also in $\mathcal{S}(\mathbb{R})$.

4.4.2 Compactly supported generator window

We explain here another easy method to produce a window $g \in C_c^\infty$ (or $\hat{g} \in C_c^\infty$) such that $\mathcal{W}(g)$ is orthonormal basis for $L^2(\mathbb{R})$.

Take $g_o = g_o^* \in C_c^\infty$ with support in $[-1, 1]$ such that

$$G_o(x) = |g_o(x)|^2 + |g_o(x - \frac{1}{2})|^2 > 0$$

on $[-\frac{1}{2}, \frac{1}{2}]$. Set $g = G_o^{-1/2} g_o$. Then $g = g^*$, $g \in C_c^\infty$ with support in $[-1, 1]$ hence the Gabor system, translated by 1, is a tight frame by Theorem 6.4.1 from time-frequency analysis (as the translated atom $T_{-1}g$ has support in $[0, 2]$). As translation makes no difference, the original system is also a tight frame. Then $\mathcal{W}(g)$ is orthonormal basis for $L^2(\mathbb{R})$.

4.4.3 Wilson basis for modulation spaces

Here we explain the conditions under which a Wilson basis for L^2 is also a basis for $M_m^{p,q}$, $1 \leq p, q \leq \infty$. We shall use a shorter notation for the Wilson system $\mathcal{W}(g)$:

$$\psi_{k,n} = c_n T_{k/2} (M_n + (-1)^{k+n} M_{-n}) g, \quad (4.16)$$

$(k, n) \in \Lambda_{\mathcal{W}}$, with $c_0 = 1$ and $c_n = 2^{-1/2}$ for $n \neq 0$. Notice that with this formula we get $\psi_{2s+1,0} = 0$.

The associated *analysis/synthesis* operators

$$C_\psi f = (\langle f, \psi_{k,n} \rangle)_{\Lambda_{\mathcal{W}}}$$

and

$$D_\psi a = \sum_{k \in \mathbf{Z}} \sum_{n \in \mathbf{Z}^+} a_{k,n} \psi_{k,n}.$$

are continuous and it holds

$$\|C_\psi f\|_{\ell_m^{p,q}} \leq 2C_1 \|f\|_{M_m^{p,q}}, \quad (4.17)$$

$$\|D_\psi a\|_{M_m^{p,q}} \leq \sqrt{2}C_2 \|a\|_{\ell_m^{p,q}}, \quad (4.18)$$

where $C_{1,2}$ are the constants occurring in the corresponding inequality of the Gabor analysis/synthesis operators (see [62], pg. 265). The frame operator is a simple multiplication operator, see equation (4.19).

Provided that $\mathcal{W}(g)$ is a orthonormal basis for $L^2(\mathbb{R})$ (in particular, $g = g^*$ and the corresponding Gabor system is a tight frame for $L^2(\mathbb{R})$), we may consider Wilson expansions

$$f = \sum_{k \in \mathbf{Z}} \sum_{n \in \mathbb{N}} \langle f, \psi_{k,n} \rangle \psi_{k,n} \quad (4.19)$$

in modulation spaces.

The next result follows almost directly from the continuity of the analysis and synthesis operators. As a consequence, the isomorphism between $M_m^{p,q}$ and $\ell_m^{p,q}$ holds ([45]). The weighted case was first considered in [45] (only for polynomial weights $(1 + |\omega|)^s$) and generalized for any moderate weight w in [62]. Namely, if $p, q < \infty$, then the series

$$f = \sum_{k \in \mathbf{Z}} \sum_{n \in \mathbb{N}} \langle f, \psi_{k,n} \rangle \psi_{k,n}$$

converges unconditionally in $M_m^{p,q}$ (and weak* in otherwise). Thus, given $\epsilon > 0$, there exists a finite set $F \subseteq \Lambda_{\mathcal{W}}$ so that

$$\|f - \sum_F \langle f, \psi_{k,n} \rangle \psi_{k,n}\|_{M_m^{p,q}} < \epsilon,$$

that is, finite linear combinations of $\mathcal{W}(g)$ are dense in $M_m^{p,q}$.

Set

$$f = \sum_F a_{k,n} \psi_{k,n}$$

and

$$f_\mu = \sum_F \mu_{k,n} a_{k,n} \psi_{k,n}.$$

for $\mu = (\mu_{k,n}) \in \ell^\infty$. As D_ψ is bounded, it holds

$$\|f_\mu\|_{M_m^{p,q}} \leq C \|(\mu_{k,n} a_{k,n})_{\Lambda_{\mathcal{W}}}\|_{\ell_m^{p,q}} \leq C \|\mu\|_\infty \|a\|_{\ell_m^{p,q}}. \quad (4.20)$$

But $\|a\|_{\ell_m^{p,q}} = \|D_\psi f\|_{\ell_m^{p,q}} \leq C \|f\|_{M_m^{p,q}}$, so

$$\|f_\mu\|_{M_m^{p,q}} \leq C^2 \|\mu\|_\infty \|f\|_{M_m^{p,q}}$$

which means that $\mathcal{W}(g)$ is an unconditional basis for $M_m^{p,q}$.

Similar to the frame characterization result for modulation spaces in [60], it holds

Theorem 15. *Assume that $\mathcal{W}(g)$ is a orthonormal basis for $L^2(\mathbb{R})$, let $g \in M_v^1$ and let m be a v -moderate weight. Then*

1. *the Banach spaces $M_m^{p,q}(\mathbb{R})$ and $\ell_m^{p,q}(\Lambda_{\mathcal{W}})$ are isomorphic, the isomorphism being provided by the coefficient operator C_ψ .*
2. *there exists $c > 0$ so that*

$$\frac{1}{c} \|f\|_{M_m^{p,q}} \leq \left(\sum_{n \in \mathbb{N}} \left(\sum_{k \in \mathbf{Z}} |\langle f, \psi_{k,n} \rangle|^p \cdot m(k/2, n)^p \right)^{q/p} \right)^{1/q} \leq c \|f\|_{M_m^{p,q}}. \quad (4.21)$$

3. *the orthogonal expansion*

$$f = \sum_{k \in \mathbf{Z}} \sum_{n \in \mathbb{N}} \langle f, \psi_{k,n} \rangle \psi_{k,n}$$

converges unconditionally in the $M_m^{p,q}$ -norm if $p, q < \infty$ and weak in $M_{1/v}^\infty$ otherwise.*

Moreover, (this is a generalized statement of the one in [45]), $f \in M_m^{p,q}(\mathbb{R})$ if and only if

$$\left(\sum_{n=0}^{\infty} \left(\sum_{k \in \mathbf{Z}} |c_{k,n}|^p \right)^{q/p} m(k/2, n)^q \right)^{1/q} < \infty \quad (4.22)$$

and the sequence space norm is an equivalent norm on $M_{p,q}^w(\mathbb{R})$. As a consequence, $M_{p,q}^w$ and $l_{p,q}^w$ are isomorphic spaces.

Note. Modulation spaces and Wilson bases are only an example of a wider concept, namely, of *Banach spaces with unconditional bases* [67]. A Banach space B with an unconditional basis can be identified with a ‘solid’ sequence space B_d ; namely, the uniqueness of the coefficients implies that B is isomorphic to the sequence space

$$B_d = \left\{ c = (c_\lambda)_{\lambda \in \mathcal{I}} : \sum_{\lambda \in \mathcal{I}} c_\lambda e_\lambda \in B \right\}.$$

The norm on B_d depends only on the absolute value of the coefficients, that is,

$$\|c\|_{B_d} = \left\| \sum_{\lambda \in \mathcal{I}} |c_\lambda| e_\lambda \right\|_B,$$

which defines an equivalent norm on B . To verify the last claim, consider a sequence $c \in B_d$, let $f = \sum_{\lambda \in \mathcal{I}} c_\lambda e_\lambda \in B$ and define μ_λ by $c_\lambda \mu_\lambda = |c_\lambda|$ if $c_\lambda \neq 0$ and $\mu_\lambda = 1$ otherwise. then both $\|\mu\|_\infty = 1$ and $\|(\mu)^{-1}\|_\infty = 1$ and from the definition of unconditional basis it follows that

$$\left\| \sum_{\lambda \in \mathcal{I}} |c_\lambda| e_\lambda \right\|_B \leq C \left\| \sum_{\lambda \in \mathcal{I}} c_\lambda e_\lambda \right\|_B \leq C^2 \left\| \sum_{\lambda \in \mathcal{I}} |c_\lambda| e_\lambda \right\|_B$$

that is

$$C^{-1} \|c\|_{B_d} \leq \|f\|_B \leq C \|c\|_{B_d}.$$

Chapter 5

Variable bandwidth spaces

Now that we have studied weights (chapter 3), defined with respect to a variable bandwidth strip ST_b , we can define functions of variable bandwidth, using the tools of weighted modulation spaces theory. The only requirement in the definition is that b is such that the related weight $m_{b,s}$ is moderate.

We have seen that the variable bandwidth weights provide for a certain flexibility, that is, the precise knowledge of the bandwidth is not necessary as finite changes of the bandwidth give equivalent weights. This shall provide for equivalent norms on the function spaces level. We shall only require that the bandwidth function is $b \geq 0$, defined on \mathbb{R}^d and take ST_b

$$ST_b = \{z = (x, \omega) \in \mathbb{R}^{2d} : |\omega| \leq b(x)\}.$$

to describe an interest area in the time-frequency plane. The corresponding weight function $m_{b,s}$ is defined via a distance function $d_b(z)$, no matter if it denotes the minimal/vertical distance function from a point z to the graph of b :

$$m_{b,s}(z) = (1 + d_b(z))^s \text{ with } d_b(z) = 0 \text{ if } z \in ST_b.$$

We have seen that this weight is moderate with respect to the related polynomial weight $v_s(z)$.

5.0.4 Function of variable bandwidth

Definition 13. We call f a function of variable bandwidth if it belongs to a weighted modulation space $M_{m_b,s}^{p,q}$ for some $1 \leq p, q \leq \infty$ and a moderate weight $m_{b,s}$, related to a bandwidth function b and $s > 0$.

We shall call the space $M_{m_b,s}^{p,q}$ a *variable bandwidth space* and denote it by $VB_{m_b,s}^{p,q}$, or simply $VB_{m_b,s}$ when parameters p, q are known. If the weight $m_{b,s}$ is also fixed, we only write VB and its norm is then marked with

$$\|\cdot\|_{VB} = \|\cdot\|_{M_{m_b,s}^{p,q}}.$$

In this particular setting, a function $f \in VB_{m_b,s}^{p,q}$ is such that its STFT $V_g f$ (with respect to a window g) is weighted mixed-norm Lebesgue - integrable. Due to the special design of the weight, $V_g f$ is decreasing faster than a polynomial of degree s on $\mathbb{R}^{2d} \setminus ST_b$ in the frequency direction.

From a practical point of view, it makes no sense to work with $s < 0$ in $m_{b,s}$, as that would allow polynomial growth of the STFT outside of ST_b .

5.1 Basic properties of VB

5.1.1 Inclusions

We shall first list out some properties that are basically consequences of the general modulation spaces theory.

Proposition 7. a) If $0 \leq s_2 \leq s_1$, then

$$VB_{m_b,s_1}^{p,q}(\mathbb{R}^d) \subseteq VB_{m_b,s_2}^{p,q}(\mathbb{R}^d).$$

b) If $b_1 \leq b_2$, then

$$VB_{m_{b_1},s}^{p,q}(\mathbb{R}^d) \subseteq VB_{m_{b_2},s}^{p,q}(\mathbb{R}^d).$$

c) If $b_1 \leq b_2$ and $0 \leq s_2 \leq s_1$, then

$$VB_{m_{b_1, s_1}}^{p, q}(\mathbb{R}^d) \subseteq VB_{m_{b_2, s_2}}^{p, q}(\mathbb{R}^d).$$

d) All L^2 -band limited functions are functions of variable bandwidth.

e) If $p_1 \leq p_2$, $q_1 \leq q_2$, $b_1 \leq b_2$ and $0 \leq s_2 \leq s_1$, then

$$VB_{m_{b_1, s_1}}^{p_1, q_1}(\mathbb{R}^d) \subseteq VB_{m_{b_2, s_2}}^{p_2, q_2}(\mathbb{R}^d).$$

Proof. The first property follows directly from

$$(1 + d(z, ST_b))^{s_2} \leq (1 + d(z, ST_b))^{s_1}$$

as then

$$\|f\|_{VB_{s_2}^b} \leq \|f\|_{VB_{s_1}^b}.$$

Similarly, property b) follows from

$$(1 + d(z, ST_{b_2}))^s \leq (1 + d(z, ST_{b_1}))^s,$$

that is, the wider the chosen strip, the weight values at point z are smaller.

Property c) is a combination of a) and b).

d) Let $f \in L^2(\mathbb{R}^d)$ and $\text{supp}(\hat{f}) \subseteq [-r, r]^d$. We choose a band limited window g so that $\text{supp}(\hat{g}) \subseteq [-a, a]^d$ for some $a > 0$. Then

$$|V_g f(x, \omega)| = |\langle \hat{f}, M_{-x} T_\omega \hat{g} \rangle| = 0$$

for all $(x, \omega) \notin \mathbb{R}^d \times [-a - r, a + r]^d$. Then the weight m_b has no influence on the time-frequency content and

$$\|V_g f\|_{L_{m_b}^2} = \|f\|_2 \|g\|_2 < \infty.$$

We conclude that $f \in VB_s^b(\mathbb{R}^d)$ for a bandwidth $b \geq r + a$ and $s > 0$.

e) follows directly from a) – c), combined with Theorem 12.2.2 from [62]. \square

5.1.2 Move the bandwidth

As we have seen in chapter 3, working with respect to a variable bandwidth b or a slightly changed $b+h$, produces equivalent weights on the time-frequency plane. This gives norm equivalence, that is:

Theorem 16. *Moving the bandwidth b for a step function h such that*

$$|h(x)| < 1, x \in \mathbb{R}^d,$$

results with equivalent variable bandwidth norms:

$$2^{-s} \|V_g f \cdot m_{b+h}\|_{L^{p,q}} \leq \|V_g f \cdot m_b\|_{L^{p,q}} \leq 2^s \|V_g f \cdot m_{b+h}\|_{L^{p,q}}. \quad (5.1)$$

That is, $VB_{m_b,s}^{p,q}$ is norm-equivalent to $VB_{m_{b+h},s}^{p,q}$.

Proof. As the two weights m_b and m_{b+h} satisfy the inequality

$$2^{-s} m_{b+h} \leq m_b(x, \omega) \leq 2^s m_{b+h}(x, \omega),$$

the result follows. □

A finite number of bandwidth shift is allowed and does not harm the structure of the function space.

Corollary 4. *a) If b is piecewise constant, then $VB_{m_b,s}^{2,2} = \mathcal{FL}_{m_b,s}^2$.*

b) If b -bounded, then $VB_{m_{b+h},s}^{2,2} \simeq VB_{m_b,s}^{2,2} \simeq \mathcal{H}^s$, where \mathcal{H}^s is the Sobolev space with weight v_s .

Proof. Recall proposition 5: whenever the weight depends on ω only, the corresponding modulation space is $\mathcal{FL}_{m_b,s}^2$. If $v_s(x, \omega) = (1 + |\omega|^2)^{s/2}$ (which is equivalent to m_b for b -bounded), then it follows that $M_{v_s}^2$ coincides with $\mathcal{H}^s(\mathbb{R}^d)$. □

The last result cannot be generalized on p or $q \neq 2$.

5.1.3 Signal processing techniques

We shall employ signal processing techniques to produce variably bandwidth functions out of L^2 functions. Given a L^2 function, we can produce functions of variable bandwidth by moderating the STFT content via an inverse weight.

Proposition 8. *The variable bandwidth space $VB_{m,(b,s)}^{2,2}(\mathbb{R}^d)$ is a processed $L^2(\mathbb{R}^{2d})$.*

Proof. Take any $f \in L^2(\mathbb{R}^d)$ and normalized Schwartz windows $g, \gamma \in S(\mathbb{R}^d)$. It surely holds that

$$F = V_g f \in L^2(\mathbb{R}^{2d}).$$

We make use of a decaying weight function on the time-frequency -plane

$$P(x, \omega) = P_b^s(x, \omega) := m_{b,s}(x, \omega)^{-1}.$$

Surely $FP \equiv F$ on the strip ST_b and decays polynomially in the frequency direction. In fact⁽¹⁾, $FP \in L_{m_{b,s}}^2(\mathbb{R}^{2d})$ because

$$\iint |FP(x, \omega)|^2 m_{b,s}(x, \omega)^2 dx d\omega = \iint |F(x, \omega)|^2 dx d\omega < \infty.$$

We claim that

$$f_1 = \iint F(x, \omega) P(x, \omega) M_\omega T_x \gamma dx d\omega \in VB_{m,(b,s)}^{2,2}(\mathbb{R}^d), \quad (5.2)$$

as a vector-valued integral. That is, $V_g f_1$ would be essentially supported within the same strip ST_b .

$$\begin{aligned} V_g f_1(y, \eta) &= \int f_1(t) e^{-2\pi i \eta t} \overline{g(t-y)} dt \\ &= \iiint F(x, \omega) P(x, \omega) M_\omega T_x \gamma dx d\omega e^{-2\pi i \eta t} \overline{g(t-y)} dt \\ &= \iiint F(x, \omega) P(x, \omega) \int \gamma(t-x) \overline{g(t-y)} e^{-2\pi i(\eta-\omega)t} dt dx d\omega \\ &= \iint F(x, \omega) P(x, \omega) V_g \gamma(y-x, \eta-\omega) e^{-2\pi i(\eta-\omega)x} dx d\omega. \end{aligned}$$

¹We can also choose any $F \in L^2(\mathbb{R}^{2d})$, not only $F = V_g f$.

Then for $m = m_{b,s}$ we have

$$\begin{aligned}
\|f_1\|_{VB}^2 &= \\
&= \iint \left| \iint F(x, \omega) P(x, \omega) V_g \gamma(y - x, \eta - \omega) e^{-2\pi i(\eta - \omega)x} dx d\omega \right|^2 \\
&\quad \cdot m^2(y, \eta) dy d\eta \\
&= \iint |(FP) \natural V_g \gamma(y, \eta)|^2 m^2(y, \eta) dy d\eta \\
&\leq \iint (|FP| * |V_g \gamma|(y, \eta))^2 m^2(y, \eta) dy d\eta < \infty.
\end{aligned}$$

(here \natural denotes the twisted convolution)

The last integral is finite because of the weighted Young's inequality: we have $|FP| * |V_g \gamma|$ is in $L^2_{m_{b,s}}(\mathbb{R}^{2d})$, as $|FP| \in L^2_{m_{b,s}}(\mathbb{R}^{2d})$ and $V_g \gamma \in S(\mathbb{R}^{2d})$ (see Prop.11.1.3. in [62]). \square

In general, using

$$\|H * F\|_{L^p_m} \leq c \|H\|_{L^1_b} \|F\|_{L^p_m},$$

we have

Corollary 5. *The variable bandwidth space $VB^{p,q}_{m,(b,s)}(\mathbb{R}^d)$ is a processed $L^{p,q}(\mathbb{R}^{2d})$.*

5.1.4 Patchwork of various band-limited functions

Another way to construct functions of variable bandwidth is by cutting parts of band limited functions via a BUPU: We shall make use of a countable family of smooth functions

$$\{\psi_n = T_{x_n} \psi\}_{n \in I},$$

produced by translation from a single, compactly supported function ψ , with frequency decay sufficient to fight the weight on the frequency side. Let $0 \leq \psi \leq 1$, $\text{supp}(\psi) \subseteq U$, $\text{supp}(\psi_n) \subseteq x_n + U$ and

$$(x_n + U) \cap (x_{n+p} + U) = \emptyset$$

for some finite p . Here

$$X = (\dots < x_{n-1} < x_n < x_{n+1} < \dots)_{n \in \mathcal{I}}$$

is a well-spread countable family so that $\mathbb{R}^d = \cup_{n \in \mathcal{I}} (x_n + U)$ and U is a neighborhood of 0. To simplify things, we shall choose $U = [-1, 1]^d$. We take $\sum_{i \in \mathcal{I}} \psi_i \equiv 1$ so that any function f of interest can be written as a sum

$$f = \sum_{i \in \mathcal{I}} \psi_i f.$$

Let B_Ω denote the set of band-limited functions h in L^2 so that, for a band-limited window γ with $\|\gamma\|_2 = 1$, it holds

$$\text{supp}(V_\gamma h) \subseteq \mathbb{R}^d \times \Omega \quad (5.3)$$

and $\Omega = [-b, b]^d$ is compact for a constant $b > 0$. In particular, if

$$\text{supp}(\hat{h}) \subseteq [-b+1, b-1]^d \text{ and } \text{supp}(\hat{\gamma}) \subseteq [-1, 1]^d,$$

then condition (5.3) is satisfied. Let $m_b = m_{b,s}$ be a weight created with respect to a constant bandwidth b . Since there is no overlap between the support of $V_\gamma h$ and the region of non-trivial values of the weight, it holds

$$\|V_\gamma h \cdot m_{b,s}\|_2 = \|V_\gamma h\|_2 = \|h\|_2.$$

As we saw previously, it holds

$$\|V_\gamma h \cdot m_b\|_2 \simeq \|V_\gamma h \cdot m_{b_1,s}\|_2$$

for any band b_1 close to b , so we can adapt the bandwidth if necessary. Surely then $h \in VB_{m_b}^2$ and $\|h\|_{VB_{m_b}^2}$ is equivalent to $\|V_\gamma h \cdot m_b\|_2$ upto a constant.

It is plausible to expect that ψh belongs to the same space as h : for instance, the L^2 -norm of ψh is smaller than of h . We have

$$\begin{aligned} |V_\gamma(\psi h)| &= |V_{\hat{\gamma}}(\widehat{\psi h})| \\ &\leq |\hat{\gamma}| * |\widehat{\psi h}| = |\hat{\gamma}| * |\hat{\psi} * \hat{h}| \leq |\hat{\gamma}| * |\hat{\psi}| * |\hat{h}| = (|\hat{\gamma}| * |\hat{h}|) * |\hat{\psi}| \end{aligned}$$

which means that the decay of $|V_\gamma(\psi h)|$ is controlled by the decay of

$$h_1 = (|\hat{\gamma}| * |\hat{h}|) * |\hat{\psi}|.$$

Here $(|\hat{\gamma}| * |\hat{h}|)$ is still band-limited and ψ is smooth and compactly supported, thus is a Schwartz function.

We observe a sequence of band-limited functions f_n with spectra in $\Omega_n = [-b_n + 1, b_n - 1]$ and cut out sections $f_n \psi_n$. The question here is: when is the sum $f = \sum f_n \psi_n$ a function of variable bandwidth? We aim at choosing the bandwidth function b such that its samples satisfy $b(n) \geq b_n$.

As explained before, $f_n \psi_n \in VB_{m_{b_n, s}}$, even though $f_n \psi_n$ is time-limited and therefore, cannot be band-limited.

We choose a normalized window g centered at 0, $\text{supp}(g) \subseteq [-1, 1]^d$. Then $V_g(f_n \psi_n)$ has its time support in $[-2, 2]^d$ and, for sufficiently large $k = n_o$, it has no overlap with $V_g(f_{n+k} \psi_{n+k})$. We shall use this property to separately measure the norm of f .

Also, observe that measuring the weighted norm of $V_g(f_n \psi_n)$ with respect to the general weight m_b is equivalent to measuring the norm of $V_g(f_n \psi_n) \cdot m_{b_n}$.

It holds

$$\|V_g(f) \cdot m_b\|_2 = \|V_g(\sum_{n \in \mathcal{I}} f_n \psi_n) m_b\|_2 \quad (5.4)$$

$$= \|V_g(\sum_{k=1}^{n_o} \sum_{n \in \mathcal{I}/n_o} f_{n+k} \psi_{n+k}) m_b\|_2 \quad (5.5)$$

$$= \|\sum_{k=1}^{n_o} V_g(\sum_{n \in \mathcal{I}/n_o} f_{n+k} \psi_{n+k}) m_b\|_2 \quad (5.6)$$

What we have done so far is simply re-arrange the summation and interchange the order of STFT and the finite sum. Now, by the basic properties of norms, we have

$$\|V_g f \cdot m_b\|_2 \leq \sum_{k=1}^{n_o} \|V_g(\sum_{n \in \mathcal{I}/n_o} f_{n+k} \psi_{n+k}) m_b\|_2 \quad (5.7)$$

$$= \sum_{k=1}^{n_o} \sum_{n \in \mathcal{I}/n_o} \|V_g(f_{n+k} \psi_{n+k}) m_b\|_2. \quad (5.8)$$

$$= \sum_{n \in \mathcal{I}} \|V_g(f_{n+k} \psi_{n+k}) m_b\|_2. \quad (5.9)$$

Note that locally m_b is close to $m_{b(n)}$ and it holds up to a maximal constant c_o ; to make sure we can control c_o we additionally choose $|b_n - b_{n+1}| < 1$ as then moving the bandwidth for a step less than 1 is giving a norm equivalence

constant at most 2, as seen in chapter 3:

$$\|V_g f \cdot m_b\|_2 \leq c_o \sum_{n \in \mathcal{I}} \|V_g(f_{n+k} \psi_{n+k}) m_{b(n)}\|_2. \quad (5.10)$$

What we have used previously are some basic properties of integrations and norms:

- $V_g(\sum_{n \in \mathcal{I}} h_n) = \sum_{n \in \mathcal{I}} V_g(h_n)$,
- $\|\sum_{n \in \mathcal{I}} H_n\|_2 \leq \sum_{n \in \mathcal{I}} \|H_n\|_2$,
- $\|\sum_{n \in \mathcal{I}} H_n\|_2 = \sum_{n \in \mathcal{I}} \|H_n\|_2$, for disjoint supports.

As the choice of window has no effect on the space up to norm equivalence, we conclude

Theorem 17. *For a countable sequence of band limited L^2 -functions $f_n \in B^{\Omega_n}$, $n \in \mathcal{I}$, with spectra $\text{supp } f_n \subseteq [-b_n + 1, b_n - 1]^d$ and $|b_n - b_{n+1}| < 1$, and a BUPU with finite overlap of compactly supported ψ_n , it holds:*

If $\sum_{n \in \mathcal{I}} \|V_g(f_n \psi_n) m_{b_n}\|_2 < \infty$, then

$$f = \sum_{n \in \mathcal{I}} \psi_n f_n \in VB_{m_b} = M_{m_b}^2$$

for b being a smooth bandwidth with samples $b(n) = b_n$ and

$$\|f\|_{VB} \leq c \sum_{n \in \mathcal{I}} \|V_g(f_n \psi_n) m_{b_n}\|_2.$$

Note. For band-limited L^p functions it holds

$$\|f\|_{M^{p,q}} \leq \|f\|_p + \|\hat{f}\|_q$$

but the question whether we can adapt the previous argument remains open.

5.2 Frames and Bases revisited

5.2.1 Reduced Multi-Gabor frames revisited

If we have some a priori knowledge about the structure of the analyzed signal, it is handy to use sections of several Gabor frames that would make a better local fit. This is in particular useful for variable bandwidth functions, since the VB weight is already emphasizing the varying importance of sections in the time-frequency area. The L^2 -concept of quilted Gabor frames is constructed in [27], while a more simple version (with compactly supported, or with band-limited windows) was described in chapter 4.

In general, we start with a family of Gabor frames $\mathcal{G}_j = \mathcal{G}(g_j, \Lambda_j)$, $j \in \mathbb{N}$, for a (weighted) modulation space, built out of different atoms g_j over different lattices Λ_j . We shall cut out the parts of each frame with the use of a BUPU, with a finite number of overlaps. Ideally, each lattice Λ_j and atom g_j are chosen to fit a local area of the signal.

We shall use parts of these frames in a new collection, that satisfy the frame inequality only if $p = q$. Namely, in [46] it was shown that $M^{p,p}$ coincide with $W(\mathcal{FL}^p, \ell^p)$, that is, with $W(M^{p,p}, \ell^p)$. For $p = 2$, in [89], this property of $L^2 = M^{2,2}$ was referred to as to a ℓ^2 -puzzle; in that terminology, $M^{p,p}$ are ℓ^p -puzzles. Fixing a BUPU $\Psi = \{\psi_j\}$, the weighted ℓ^p -puzzle property is:

$$c_{\Psi}^{-1} \|f\|_{M_m^{p,p}}^p \leq \sum_{j \in \mathbb{N}} \|\psi_j f\|_{M_m^{p,p}}^p \leq c_{\Psi} \|f\|_{M_m^{p,p}}^p. \quad (5.11)$$

Make notice of the fact that, for $p \neq q$, even though it is true that locally, $M^{p,q}$ coincides with \mathcal{FL}^q (see [80]), the Wiener amalgam $W(\mathcal{FL}_m^p, \ell^q)$ does not coincide with $M_m^{p,q}(\mathbb{R}^d)$. It is only true that $\mathcal{F}^{-1}(W(\mathcal{FL}_m^p, \ell^q)) = M_m^{p,q}$. In the mixed norm case then, quilted Gabor frames do not work as they do for L^2 .

We can only talk about a satisfied frame inequality on the Wiener amalgam spaces. For instance, in $W = W(M_m^{p,q}, \ell^1)$, almost directly from the definition

of W , would follow

$$A\|f\|_W \leq \sum_{j \in \mathbb{N}} \left(\sum_{\omega \in \mathcal{X}_{j_2}} \left(\sum_{x \in \mathcal{X}_{j_1}} |\langle f, \pi(\lambda)g_j \rangle|^p m(\lambda)^p \right)^{q/p} \right)^{1/q} \leq B\|f\|_W \quad (5.12)$$

for appropriately chosen A and B and subsets $\mathcal{X}_j = \mathcal{X}_{j_1} \times \mathcal{X}_{j_2}$ of the corresponding lattices Λ_j . To put it in words, this means that in W , it is possible to locally use vertical/horizontal strups of Gabor frames for W , and to summarize these parts. A similar inequality would hold on $W(M_m^{p,q}, \ell^q)$, for $q > 1$.

We make use of inequality (5.11) to prove the following result:

Theorem 18. *Let m be a v -moderate weight on \mathbb{R}^2 with respect to a submultiplicative weight v . Let $\mathcal{G}_j = \mathcal{G}(g_j, \Lambda_j)$, $j \in \mathbb{N}$, be a family of tight Gabor frames for $L^2(\mathbb{R})$ with compactly supported atoms $g_j \in M_v^1(\mathbb{R})$ and frame constants A_j .*

We choose a BUPU $\{\psi_j = T_j\psi\}_{j \in \mathbb{N}}$. In addition, we require that for all $j \in \mathbb{N}$ there exists a $\mathcal{X}_j \subseteq \Lambda_j$ so that

$$(\forall \lambda \notin \mathcal{X}_j) \text{ supp}(\psi_j) \cap \text{supp}(\pi(\lambda)g_j) = \emptyset. \quad (5.13)$$

Then there exist positive constants A', B' such that for all $f \in M_m^{p,p}$ it holds

$$A'\|f\|_{M_m^{p,p}} \leq \left(\sum_{j \in \mathbb{N}} \sum_{\lambda \in \mathcal{X}_j} |\langle f, \pi(\lambda)g_j \rangle|^p m(\lambda)^p \right)^{1/p} \leq B'\|f\|_{M_m^{p,p}}. \quad (5.14)$$

In words, the collection

$$\mathcal{G} = \{\pi(\lambda)g_j : \lambda \in \mathcal{X}_j, j \in \mathbb{N}\} \quad (5.15)$$

satisfies the frame inequality on $M_m^{p,p}(\mathbb{R})$.

Proof. Let $\mathcal{G}_j = \mathcal{G}(g_j, \Lambda_j)$, $j \in \mathbb{N}$, be tight Gabor frames for L^2 , with frame constant A_j respectively and let $A = \min\{A_j : j \in \mathbb{N}\}$ and $B = \max\{A_j : j \in \mathbb{N}\}$.

Any $f \in M_m^{p,p}(\mathbb{R})$ can be represented as

$$f = \sum_{\Lambda_j} \langle f, \pi(\lambda)g_j \rangle \pi(\lambda)g_j,$$

with $(\langle f, \pi(\lambda)g_j \rangle) \in l_m^{p,p}(\mathbf{Z}^2)$, $\lambda = (x, \omega) \in \Lambda_j$ and it holds

$$A^p \cdot \|f\|_{M_m^{p,p}}^p \leq \sum_{\lambda \in \Lambda_j} |\langle f, \pi(\lambda)g_j \rangle|^p \cdot m(\lambda)^p \leq B^p \cdot \|f\|_{M_m^{p,p}}^p, \quad (5.16)$$

for all $j \in \mathbb{N}$. Using $f = \sum_{j \in \mathbb{N}} \psi_j \cdot f$, and taking in account only the non-zero coefficients occurring, due to equation (5.13), we have

$$f = \sum_{j \in \mathbb{N}} \sum_{\Lambda_j} \langle f, \pi(\lambda)g_j \rangle \psi_j \cdot \pi(\lambda)g_j = \sum_{j \in \mathbb{N}} \sum_{\mathcal{X}_j} \langle f, \pi(\lambda)g_j \rangle \psi_j \cdot \pi(\lambda)g_j.$$

Then

$$L := \sum_{j \in \mathbb{N}} \sum_{\lambda \in \mathcal{X}_j} |\langle f, \pi(\lambda)g_j \rangle|^p m(\lambda)^p \quad (5.17)$$

$$= \sum_{j \in \mathbb{N}} \sum_{\lambda \in \mathcal{X}_j} \left| \sum_{k \in \mathbb{N}} \psi_k \cdot f, \pi(\lambda)g_j \right|^p m(\lambda)^p \quad (5.18)$$

$$= \sum_{j \in \mathbb{N}} \sum_{\lambda \in \Lambda_j} |\langle \psi_{j_*} f, \pi(\lambda)g_j \rangle|^p m(\lambda)^p. \quad (5.19)$$

In the last expression, ψ_{j_*} is denoting the finite sum of those ψ_j that overlap with the actual $\pi(\lambda)g_j$.

We now use inequality (5.16) to obtain

$$L \leq B^p \sum_{j \in \mathbb{N}} \|\psi_{j_*} f\|_{M_m^{p,p}}^p.$$

But, $M_m^{p,p}$ locally behave as Wiener amalgams and the property (5.11) holds, so it follows

$$L \leq c_* B^p \|f\|_{M_m^{p,p}}^p, \quad (5.20)$$

where c_* corresponds to the new BUPU, formed by ψ_{j_*} . Thus, it holds

$$\left(\sum_{j \in \mathbb{N}} \sum_{\mathcal{X}_j} |\langle f, \pi(\lambda)g_j \rangle|^p m(\lambda)^p \right)^{1/p} \leq B c_*^{1/p} \|f\|_{M_m^{p,p}}.$$

As for the lower bound, using again ψ_{j^*} , $j \in \mathbb{N}$, inequality (5.16) and the puzzle property (5.11), we have

$$\begin{aligned}
L &= \sum_{j \in \mathbb{N}} \sum_{\lambda \in \mathcal{X}_j} |\langle f, \pi(\lambda)g_j \rangle|^p m(\lambda)^p \\
&= \sum_{j \in \mathbb{N}} \sum_{\lambda \in \Lambda_j} |\langle \psi_{j^*} f, \pi(\lambda)g_j \rangle|^p m(\lambda)^p \\
&\geq A^p \sum_{j \in \mathbb{N}} \|\psi_{j^*} f\|_{M_m^{p,p}} \geq A^p c_*^{-1} \|f\|_{M_m^{p,p}}. \tag{5.21}
\end{aligned}$$

We conclude that

$$A c_*^{-1/p} \|f\|_{M_m^{p,p}} \leq \left(\sum_{j \in \mathbb{N}} \sum_{\lambda \in \mathcal{X}_j} |\langle f, \pi(\lambda)g_j \rangle|^p m(\lambda)^p \right)^{1/p} \leq B c_*^{1/p} \|f\|_{M_m^{p,p}}. \tag{5.22}$$

□

Similar to the proof given in the section on reduced multi-Gabor frames on L^2 , it is possible to adapt the argument for band-limited Gabor atoms.

If we define the analysis operator by From the last inequality it is clear that

$$C_G f = (\langle f, \pi(\lambda)g_j \rangle : \lambda \in \mathcal{X}_j, j \in \mathbb{N}), \tag{5.23}$$

from the last inequality it is clear that it is bounded. In fact,

$$\|C_G f\|_{\ell_m^{p,p}} = \left(\sum_{j \in \mathbb{N}} \sum_{\lambda \in \mathcal{X}_j} |\langle f, \pi(\lambda)g_j \rangle|^p m(\lambda)^p \right)^{1/p} \leq B c_*^{1/p} \|f\|_{M_m^{p,p}}.$$

Thus,

$$\|C_G\|_{op} \leq B c_*^{1/p}. \tag{5.24}$$

However, note that we are not proving here that the collection is a frame on $M_m^{p,p}$ nor we are suggesting a way to construct a dual frame for \mathcal{G} .

5.2.2 Wilson bases revisited

Wilson bases preserve the time-frequency localization, with a special property of frequency symmetry. This makes them most suitable for the variable bandwidth setting with symmetric stripes ST_b on the TF domain. As we have seen in chapter 4, Wilson bases are unconditional bases for mixed-norm weighted modulation spaces and provide an isomorphism map to $\ell_m^{p,q}$, $1 \leq p, q \leq \infty$. Therefore the same holds for variable bandwidth spaces $VB = M_{m_{b_o,s}}^{p,q}$, $1 \leq p, q \leq \infty$.

We shall pay special interest to function approximation by partial sums of Wilson atoms, using the Wilson coefficients from an expanded bandwidth strip. Such a setting surely provides for closed subspaces, as the corresponding discrete entity is a closed subspace of $\ell_{m_b}^{p,q}$.

In particular, we start with a Gabor frame $\mathcal{G}(g, \frac{1}{2}, 1)$ of redundancy 2 for L^2 , generated with a band-limited window g (we have demonstrated in chapter 4 that this is possible). This provides a Wilson basis \mathcal{W} such that we can use subsets of the Wilson basis to form closed subspaces of variable bandwidth spaces, by using a subset of the basis, corresponding to the TF area within an expanded strip.

The reasoning behind taking a band limited window g with spectrum $[-r_o, r_o]$ is that the Wilson atoms

$$\psi_{k,n} = 2^{-1/2} T_{k/2} (M_n + (-1)^{k+n} M_{-n}) g$$

have corresponding spectra $[-n - r_o, n + r_o]$. Then the restricted sum

$$\sum_{k \in \mathbf{Z}, |n| \leq b_o(k)} c_{k,n} \psi_{k,n}$$

would have TF support within a variable bandwidth strip in the time-frequency plane, that is,

$$V_g \left(\sum_{|n| \leq b_o(k)} c_{k,n} \psi_{k,n} \right) (x, \omega) = 0 \quad (5.25)$$

for such (x, ω) so that $|\omega| > r_o + b_o(x)$ holds. If we mark $b \geq r_o + b_o$, then (5.25) is true on the exterior of ST_b .

We then make use of a tailored multiplier $\mu_b \in \ell^\infty$,

$$\mu_b(x, \omega) = \begin{cases} 1 & \text{if } (x, \omega) \in ST_b, \\ 0 & \text{otherwise,} \end{cases} \quad (5.26)$$

to cut out an expanded strip with bandwidth $b \geq b_o + r$ from the time-frequency domain. This is done in order to construct an approximation, belonging to a closed subspace

$$VB(\mathcal{W}, ST_b) := \left\{ \sum c_{k,n} \cdot \mu_b(k, n) \cdot \psi_{k,n} : (c_{k,n}) \in \ell_{m_{b_o,s}}^{p,q} \right\}. \quad (5.27)$$

Notice that this multiplier is not a Gabor multiplier as we have left the Gabor frame setting. That is, it is true that μ_b is a projection operator, unlike with the case of 0/1 multiplier in the Gabor setting.

The sequence space $\mu_b \cdot \ell_{m_{b_o,s}}^{p,q}$ is a closed subspace of $\ell_{m_{b_o,s}}^{p,q}$, and it is isomorphic to $VB(\mathcal{W}, ST_b)$. Thus,

Lemma 7. *$VB(\mathcal{W}, ST_b)$ is a closed subspace of the starting variable bandwidth space $VB = M_{m_{b_o,s}}^{p,q}$.*

In addition, we get a nested sequence of closed subspaces:

$$VB(\mathcal{W}, ST_{b_1}) \subseteq VB(\mathcal{W}, ST_{b_2}),$$

if $b_1 \leq b_2$.

The approximation error of a function $f \in VB$ with

$$f_b = \sum_{(k,n) \in ST_b} c_{k,n} \psi_{k,n} \quad (5.28)$$

is easily estimated in the unweighted space $M^{p,q}$, knowing the weight estimate. In other words, the global estimate of the approximation error is easily constructed and depends exclusively on the bandwidth b_o , not on the particular choice of f .

Corollary 6. *For every $f \in VB = M_{m_{b_o,s}}^{p,q}$ it holds*

$$\|f - f_b\|_{M^{p,q}} \leq (1+r)^{-s} \|f\|_{VB}, \quad (5.29)$$

where $f_b = \sum_{(k,n) \in ST_b} c_{k,n} \psi_{k,n}$ and $b(x) = r + b_o(x)$.

Proof. The value of the weight $m = m|_{b_o, s}$ within the expanded strip ST_b , $b = b_o + r$, is at most $(1 + r)^s$. We make use of the inequality

$$|F(x, \omega)| = |F(x, \omega)m(x, \omega)m^{-1}(x, \omega)| \leq (1 + r)^{-s}|F(x, \omega)m(x, \omega)|$$

and (5.29) follows. \square

Due to (4.18) and (4.17), it holds

Corollary 7. *Let $a_{k,n} = \langle f, \psi_{k,n} \rangle$ be the Wilson coefficients of a function $f \in VB$, then the approximation error within the same space is*

$$\left\| \sum a_{k,n} \mu_b(k, n) \psi_{k,n} - f \right\|_{VB} \leq \sqrt{2} C_2 \|a|_{\Lambda_b^c}\|_{VB}, \quad (5.30)$$

where μ_b is defined by (5.26) and $a|_{\Lambda_b^c}$ denotes the Wilson coefficients on $\Lambda_{\mathcal{W}} \cap ST_b^c$.

Obviously, the wider the band b , the smaller the value of $\|a|_{\Lambda_b^c}\|_{VB}$ is.

5.3 Essential time-frequency support

In the frequency domain Giardina has explored the concept of effective bandwidth [58]. Donoho and Stark [23] speak of essential support in signal recovery. Here we shall look into developing a related concept at the time-frequency plane.

5.3.1 Band-limited functions in the TF plane

If $f \in L^2(\mathbb{R})$, $g \in \mathcal{S}(\mathbb{R})$ and

$$\text{supp}(\hat{f}) \subseteq [-a, a], \quad \text{supp}(\hat{g}) \subseteq [-S, S],$$

then $V_g f(x, \omega) = 0$ for $\omega \geq a + S$, that is,

$$\text{supp } V_g f \in \mathbb{R} \times [-a - S, a + S],$$

and surely $V_g f \in L^2(\mathbb{R}^2)$. We can say that f has *time-frequency representation* in the strip $ST_{a+S} = \mathbb{R} \times [-a - S, a + S]$. Given a weight defined with respect to a bandwidth $b \geq a + S$, it holds

$$\|V_g f \cdot m_b\|_{L^2(\mathbb{R}^2)} = \|V_g f \cdot 1\|_{L^2(\mathbb{R}^2)} = \|f\|_2 < \infty.$$

As choice of window is irrelevant, it is clear that $f \in VB_{m_b}^{2,2}$.

If the used window g is not band-limited, we cannot make such a precise estimate of the time-frequency representation of f . For a different choice of the window, the strip would vary, so this description requires to fix the window g . Still, most of the STFT's energy should be in the strip. We may say that $V_g f$ would be concentrated around $ST_b = \mathbb{R} \times [-b, b]$, in other words, $V_g f$ would be *essentially supported* in ST_b . In general, for a function f in $VB_{m_b}^2$, the decay of its STFT beyond the chosen bandwidth is polynomial, therefore is reasonably small.

5.3.2 Essential support in the TF plane

Let us explore a bit more in what way the term essential support is best defined. The uncertainty principle by Donoho and Stark speaks of essential support and ϵ -concentration of the Fourier transform; we would like to discuss an analogous situation on the STFT side.

Donoho and Stark define the term essential support for a function $f \in L^2(\mathbb{R})$ via ϵ -concentration: for some positive $\epsilon < \frac{1}{4}$, \hat{f} is ϵ -concentrated on $|\omega| \leq a$, if

$$\int_{|\xi| \geq a} |\hat{f}(\xi)|^2 d\xi \leq \epsilon^2 \|\hat{f}\|_2^2 = \epsilon^2 \|f\|_2^2.$$

Then

$$\text{ess.sup}(\hat{f}) \subseteq [-a, a].$$

Note that, if both f and g are band-limited like in the previous subsection, then in Donoho/Stark's sense, $\text{ess.sup}(\hat{f}) \subseteq [-a, a]$, $\text{ess.sup}(\hat{g}) \subseteq [-S, S]$ and $\text{supp } V_g f \subseteq \mathbb{R} \times [-a - S, a + S]$, for any ϵ . The question that arises is: What is $\text{ess.sup } V_g f$? It is probably contained in a smaller strip than $\mathbb{R} \times [-a - S, a + S]$, however it may be hard/impossible to estimate the exact strip in relation to a ϵ -environment for a whole class of functions.

Say \hat{f} is ϵ -concentrated in $[-a, a]$ and g is a window function such that \hat{g} is δ -concentrated on $[-S, S]$ for $\delta \leq \frac{1}{4}$; intuitively, we expect that $V_g f$ is ϵ -concentrated on the strip

$$ST_{a+S} = \mathbb{R} \times \{|\omega| \leq S + a\}.$$

What is the value of ϵ in relation of ϵ and δ ?

We need to make sure the following holds

$$\iint_{\mathbb{R} \times \{|\omega| \geq S+a\}} |V_g f(x, \omega)|^2 dx d\omega \leq \epsilon^2 \iint_{\mathbb{R} \times \mathbb{R}} |V_g f(x, \omega)|^2 dx d\omega. \quad (5.31)$$

For a non-band limited function, by ϵ - TF essential support (with respect to a fixed Schwartz window) we shall mean that

$$\iint_{ST_b^c} |V_g f(x, \omega)|^2 dx d\omega \leq \epsilon^2 \iint_{\mathbb{R} \times \mathbb{R}} |V_g f(x, \omega)|^2 dx d\omega. \quad (5.32)$$

That is, $f \in L^2$ needs to satisfy

$$\|V_g f|_{ST_b^c}\|_{L^2} \leq \epsilon \|V_g f\|_{L^2}. \quad (5.33)$$

In that sense, it is reasonable to work with customized weight modulation spaces, for the weight would describe the decay of the signal's STFT beyond the variable strip area. In general,

Definition 14. *Let g be a fixed Schwartz window. A function $f \in M_m^{p,q}$ is time-frequency essentially supported within ST_b if for some positive $\epsilon < \frac{1}{4}$, it holds*

$$\|V_g f|_{ST_b^c}\|_{L_m^{p,q}} \leq \epsilon \|V_g f\|_{L_m^{p,q}}. \quad (5.34)$$

Note. The last definition ensures that $\|V_g f|_{ST_b}\|_{L_m^{p,q}}$ has a very small value and can be neglected in practice.

5.3.3 Essential TF-support via weighted Gabor frames

We shall work with the standard weight

$$v(\lambda) = v_s(\lambda) = (1 + |\omega|)^s$$

with $\lambda = (x, \omega) \in \mathbb{R} \times \hat{\mathbb{R}}$ and consider a function $f \in M_v^2$. Given a frame $\{\pi(\lambda)g\}_{\lambda \in \Lambda}$ for $M_v^2(\mathbb{R})$ and a dual atom γ , the reconstruction coefficients $c_\lambda = \langle f, \pi(\lambda)\gamma \rangle$ are in $\ell_v^2(\Lambda)$, due to the Gabor frame theory for weighted modulation spaces. As v grows polynomially with frequency growth, we expect that a reduced sub-lattice

$$\Lambda_M := \{\lambda \in \Lambda : |\omega| \leq M\}$$

would give a good reconstruction.

In fact, as a consequence of decaying properties of sequences in $\ell_v^2(\Lambda)$, for $\epsilon > 0$ there exists $M > 0$, called a *bar*, such that

$$\left(\sum_{\lambda \notin \Lambda_M} |c_\lambda|^2 v^2(\lambda) \right)^{1/2} < \epsilon \|f\|_{M_v^2}.$$

That is, the coefficients beyond a constant bandwidth strip ST_{2M} have neglectful values.

We denote by $C := \inf\{v(\lambda)^{-1} : \lambda \in (\Lambda_M)^c\} \leq (1 + M)^{-s}$. Let

$$f_M := \sum_{\lambda \in \Lambda_M} c_\lambda \pi(\lambda)g;$$

Now we can estimate $\|f - f_M\|_2$, for a properly chosen M . The discrete evaluation is

$$\begin{aligned} \left(\sum_{\lambda \notin \Lambda_M} |c_\lambda|^2 \right)^{1/2} &= \left(\sum_{\lambda \notin \Lambda_M} |c_\lambda|^2 v^2(\lambda) \frac{1}{v^2(\lambda)} \right)^{1/2} \\ &\leq C \left(\sum_{\lambda \notin \Lambda_M} |c_\lambda|^2 v^2(\lambda) \right)^{1/2} < C\epsilon \|f\|_{M_v^2}. \end{aligned}$$

Also, from the frame inequality, it surely holds that

$$\left(\sum_{\lambda \notin \Lambda_M} |c_\lambda|^2 v^2(\lambda) \right)^{1/2} \leq B \|f\|_{M_v^2}.$$

Then

$$\left(\sum_{\lambda \notin \Lambda_M} |c_\lambda|^2 \right)^{1/2} \leq \frac{B}{(1+M)^s} \|f\|_{M_v^2}.$$

The beauty of this estimate is that M does not depend on the quality of the particular function f , but only on the decay properties of the weight v .

Proposition 9. *Let $\{\pi(\lambda)g\}_{\lambda \in \Lambda}$ be a frame for M_v^2 with constants A, B , let $f \in M_v^2$ and let $f = \sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda)g$ be its Gabor expansion.*

Then, L^2 estimate for the essential support within a bar M is

$$\left(\sum_{\lambda \notin \Lambda_M} |c_\lambda|^2 \right)^{1/2} \leq \frac{B}{(1+M)^s} \|f\|_{M_v^2}.$$

Also, $f_M := \sum_{\lambda \in \Lambda_M} c_\lambda \pi(\lambda)g$ is a good approximation of f in L^2 , that is

$$\|f - f_M\|_2 \leq c \|f\|_{M_v^2} / (1+M)^s.$$

Obviously, the wider we take M , the closer the approximation f_M gets to f . We can also say that f has a *time-frequency essential support* in $\mathbb{R} \times [-M, M]$.

In general, for a function f in a variable bandwidth space, related to a strip $ST_{b_o, s}$, we would choose to work with an extended sub-lattice $\Lambda_b = \Lambda \cap ST_b$ and estimate

$$\left(\sum_{\lambda \notin \Lambda_b} |c_\lambda|^2 \right)^{1/2} \leq c \|f\|_{M_{m_{b_o, s}}^2}.$$

5.4 Approximate reconstruction

For band-limited functions, we can expect an estimate, similar to the result in [53]: the reconstruction of band-limited functions within $\mathcal{S}_o = M^{1,1}$ or L^2 does not require the whole lattice and can be performed from incomplete

data; one can perform a perfect reconstruction with coefficients from within a band in the time-frequency plane.

Theorem 19. [53] *Given a Λ -dual pair of windows (g, γ) in S_o , for every $\epsilon > 0$ there exists $r > 0$ such that for all $f \in \mathcal{S}_o$ with $\text{supp } \hat{f} \subseteq B_R(0)$, it holds*

$$\|f - \sum_{\substack{|\omega| \leq R+r \\ \lambda=(x,\omega) \in \Lambda}} \langle f, \pi(\lambda)\gamma \rangle \pi(\lambda)g\|_{S_o} \leq \epsilon \|f\|_{S_o}. \quad (5.35)$$

For band-limited L^2 -functions, the same estimate holds in the L^2 -norm.

In addition,

If we assume on a closed subspace $H \subseteq L^2$ that $J \subseteq \Lambda$ is an index set such that for some $\epsilon < 1$, for all $f \in H$

$$\|f - \sum_{\lambda \in J} \langle f, \pi(\lambda)\gamma \rangle \pi(\lambda)g\|_2 \leq \epsilon \|f\|_2, \quad (5.36)$$

then $f \in H$ can be completely reconstructed from $(\langle f, \pi(\lambda)\gamma \rangle)_{J}$.

In general, for non-band limited functions, there is an error in approximating from an incomplete set of data. The relative error margin depends only on the frame structure and not on the individual function that is to be approximated.

5.4.1 Approximation in $M_m^{p,q}$

When working with essential supports, we have the implication that the STFT values outside the strip ST_{b_o} are decaying polynomially. Therefore, we can expect that an approximation using coefficients from within an expanded strip ST_b , for b wide enough, is sufficiently accurate. As we are working with essential supports (our functions are not band-limited in general), we cannot expect a complete reconstruction as given in the result just cited from [53].

Let us see why the type of result as in theorem 19 holds in weighted modulation spaces:

Corollary 8. *Let (g, γ) be a Λ -dual pair of windows in M_v^1 for a submultiplicative weight v , let $1 \leq p, q < \infty$ and let m be a v moderate weight. Then for every $\epsilon > 0$ there exists $r > 0$ such that for all $f \in M_m^{p,q}$ with $\text{supp } \hat{f} \subseteq B_R(0)$, it holds*

$$\|f - \sum_{\substack{|\omega| \leq R+r \\ \lambda=(x,\omega) \in \Lambda}} \langle f, \pi(\lambda)\gamma \rangle \pi(\lambda)g\|_{M_m^{p,q}} \leq \epsilon \|f\|_{M_m^{p,q}}. \quad (5.37)$$

Proof. The analysis Gabor operator C_γ is bounded on modulation spaces as well, so (Theorem 12.2.3, [62]) for an analysis window $\gamma \in M_v^1$ it holds

$$\|C_\gamma f\|_{\ell_m^{p,q}} \leq c \|\gamma\|_{M_v^1} \|f\|_{M_m^{p,q}}.$$

Given a small, positive ϵ , we choose a band-limited window γ_c close to γ so that

$$\|C_{\gamma-\gamma_c} f\|_{\ell_m^{p,q}} \leq \epsilon \|f\|_{M_m^{p,q}} / \|g\|_{M_v^1}.$$

This is possible as band-limited, M_v^1 -functions are dense in M_v^1 . Let

$$\text{supp}(\hat{\gamma}_c) \subseteq B_r(0) \text{ and } \text{supp}(\hat{f}) \subseteq B_R(0).$$

We have $V_{\gamma_c} f(\lambda) = 0$, for $\lambda = (x, \omega)$, $|\omega| > R + r$.

The total Gabor expansion for f (with respect to the Λ -dual pair (g, γ)) is

$$f = \sum_{\Lambda} \langle f, \pi(\lambda)\gamma \rangle \pi(\lambda)g.$$

We want to estimate the "left-over" of this expansion beyond a wider band in the TF plane (with coefficients out of the relevant strip $\Lambda_b = \Lambda \cap ST_b$ for $b = r + R$) so we define

$$c_\lambda = \langle f, \pi(\lambda)\gamma \rangle, \text{ for all } |\omega| > R + r$$

and $c_\lambda = 0$ otherwise. Then $f - \sum_{\Lambda_b} \langle f, \pi(\lambda)\gamma \rangle \pi(\lambda)g = \sum_{\Lambda} c_\lambda \pi(\lambda)g$. Surely it holds $|c_\lambda| \leq |\langle f, \pi(\lambda)\gamma - \pi(\lambda)\gamma_c \rangle|$ as c_λ are the minimal coefficients.

Therefore

$$\|f - \sum_{\Lambda_b} \langle f, \pi(\lambda)\gamma \rangle \pi(\lambda)g\|_{M_m^{p,q}} \leq \|c\|_{\mathbf{e}_m^{p,q}} \|g\|_{M_v^1} \quad (5.38)$$

$$\leq \|C_{\gamma-\gamma_c} f\|_{\mathbf{e}_m^{p,q}} \|g\|_{M_v^1} \quad (5.39)$$

$$\leq \epsilon \|f\|_{M_m^{p,q}}. \quad (5.40)$$

□

The proof for a special case of a variable bandwidth function f is analogous: We choose a variable bandwidth $\omega = b(x)$ so that

$$V_{\gamma_c} f(x, \omega) = 0, \text{ for all } (x, \omega) \notin ST_b,$$

with respect to a band-limited window γ_c , close to γ as described previously.

This type of a function f certainly exists: we can construct it by using the same band-limited Gabor atom γ_c as a building block ($\text{supp } \gamma_c \subseteq [-r, r]$) and coefficients a_λ on locations only within $\Lambda \cap ST_{b-r}$.

Then we work with a subset $\Lambda_b = \Lambda \cap ST_b$. We have the result

Corollary 9. *Let (g, γ) in M_v^1 be a Λ -dual pair of windows. For every $\epsilon > 0$ there exists $r > 0$ such that for all $f \in M_m^{p,q}$ with*

$$\text{supp}(V_{g_c} f) \subseteq ST_{b_o},$$

with respect to a band-limited window g_c , it holds

$$\|f - \sum_{\Lambda \cap ST_{b_o+r}} \langle f, \pi(\lambda)\gamma \rangle \pi(\lambda)g\|_{M_m^{p,q}} \leq \epsilon \|f\|_{M_m^{p,q}} \quad (5.41)$$

For $f \in VB_{m_b}^{p,q}$, the analysis coefficients $C_\gamma f(\lambda)$ have polynomial decay on the exterior of ST_b , as they are controlled by the variable bandwidth weight m_b . It is worth pursuing the relation between the bandwidth b and the coefficients decay that would give a general result of type

$$\|f - \sum_{\Lambda \cap ST_b} \langle f, \pi(\lambda)\gamma \rangle \pi(\lambda)g\|_{VB} \leq \epsilon \|f\|_{VB}. \quad (5.42)$$

5.4.2 Approximation error with respect to a weaker weight

Reconstruction estimates are very easy to do in a space with weaker constraints (weight). We start with a simple estimate of the inverse weight.

Lemma 8. *Given $1 > \epsilon > 0$, there is $r > 0$ such that for all $\lambda = (x, \omega) \notin ST_{b+r}$, it holds $m_b^{-1}(\lambda) \leq \epsilon$.*

Proof. At any $\lambda = (x, \omega)$ so that $d(\lambda, ST_b) = r$ we have $m_b(\lambda) = (1+r)^s$. Given $1 > \epsilon > 0$, we choose any $r \geq \epsilon^{-1/s} - 1$ and obtain

$$\frac{1}{m_b(\lambda)} = \frac{1}{(1+r)^s} < \epsilon.$$

Then for any $\lambda \notin ST_{b+r}$ the same estimate holds. \square

We shall use the weight estimate to measure the L^2 -norm of the spectrogram's out-of-band energy.

Lemma 9. *For all $f \in VB = VB_{m_b}^{2,2}(\mathbb{R}^d)$ and any $1 > \epsilon > 0$ there exists $r > 0$ so that*

$$\|V_g f|_{ST_{b+r}^c}\|_{L^2} \leq \epsilon \|f\|_{VB}. \quad (5.43)$$

Proof. For a given ϵ , we choose r so that for all $(x, \omega) \in ST_{b+r}^c$ it holds

$$m_b^{-1}(x, \omega) \leq \epsilon,$$

which is possible as m_b^{-1} is decaying beyond ST_b . Then

$$\begin{aligned} \iint_{ST_{b+r}^c} |V_g f(x, \omega)|^2 dx d\omega &= \iint_{ST_{b+r}^c} |V_g f(x, \omega)|^2 m_b^2(x, \omega) m_b^{-2}(x, \omega) dx d\omega \\ &\leq \iint_{ST_{b+r}^c} |V_g f(x, \omega)|^2 m_b^2(x, \omega) \epsilon^2 dx d\omega \\ &\leq \epsilon^2 \iint_{\mathbb{R}^d \times \mathbb{R}^d} |V_g f(x, \omega)|^2 m_b^2(x, \omega) dx d\omega = \epsilon^2 \|f\|_{VB}^2. \end{aligned}$$

\square

Applying the last lemma to the approximation problem, we easily obtain:

Corollary 10. *Let (g, γ) be a Λ -dual pair of windows in $M_v^1(\mathbb{R}^d)$. Then for all $\epsilon > 0$ there exists $r > 0$ such that for all $f \in VB = VB_{m_b}^{2,2}(\mathbb{R}^d)$ it holds*

$$\|f - \sum_{\Lambda \cap ST_{b+r}} \langle f, \pi(\lambda)\gamma \rangle \pi(\lambda)g\|_2 \leq \epsilon \|f\|_{VB}. \quad (5.44)$$

Similarly, for a weaker weight, it holds:

Corollary 11. *Given a Λ -dual pair of windows (g, γ) in $M_v^1(\mathbb{R}^d)$ and $0 < a < s$, for all $\epsilon > 0$ there exists $r > 0$ so that for all $f \in VB = VB_{m_b,s}^{2,2}(\mathbb{R}^d)$, the approximation error with respect to the $VB_{m_b,a}^{2,2}$ -norm is*

$$\|f - \sum_{\Lambda \cap ST_{b+r}} \langle f, \pi(\lambda)\gamma \rangle \pi(\lambda)g\|_{VB_{m_b,a}^{2,2}} \leq \epsilon \|f\|_{VB_{m_b,s}^{2,2}}. \quad (5.45)$$

Figure 5.1: Spectrograms of six functions, with different bandwidths

Figure 5.2: Bounded uniform partition of unity, six sections

Figure 5.3: Spectrograms of six functions, with different bandwidths, cut out with a BUPU

Figure 5.4: Variable bandwidth function, a composition of the previous six cut-out functions

Chapter 6

Reproducing Kernel

Given a sub-multiplicative weight $v(x, \omega) = (1 + |\omega|^2)^{s/2}$, $s > d/2$, the associated Sobolev space $\mathcal{H}_v^2(\mathbb{R}^d)$ is a reproducing kernel Hilbert space [52]. Knowing that weighted modulation spaces $M_v^2(\mathbb{R}^d)$ (with the same weight) coincide with $\mathcal{H}_v^2(\mathbb{R}^d)$ (Prop.11.3.1, [62]) and are therefore reproducing kernel Hilbert spaces, it is reasonable to ask whether there would be a reproducing kernel (RK) for a weighted modulation space M_m^2 with another weight m .

In particular, it is reasonable to ask the following questions: What happens to the reproducing kernel if the weight mildly varies from the sub-multiplicative weight v ? Or, more precisely, are spaces of variable bandwidth reproducing kernel Hilbert spaces for some weights and, if so, under what conditions? It is obvious that we shall only deal with spaces of type $M_m^2(\mathbb{R}^d)$ for any moderate weight m (and having variable bandwidth weights in mind) as $M_m^{p,q}$ for $p \neq 2$ or $q \neq 2$ are only Banach spaces and do not possess inner product.

By Riesz representations theorem [86], we shall seek for a function $\Phi_y \in M_m^2(\mathbb{R}^d)$, $y \in \mathbb{R}^d$ that satisfies the reproducing property

$$f(y) = \langle f, \Phi_y \rangle_{VB}$$

for each $f \in VB = M_m^2(\mathbb{R}^d)$. Then it should hold

$$\Phi_t(y) = \langle \Phi_t, \Phi_y \rangle_{VB}.$$

The work is here organized as follows: in the first section we observe the reproducing kernel for $M_v^2(\mathbb{R}^d) = \mathcal{H}_v^2(\mathbb{R}^d)$ with respect to the two different

inner products, associated to each setting. We notice that the reproducing kernel with respect to the new inner product has an extra multiplier. Next, we work with an altered weight that originates from the sub-multiplicative one by inserting an unweighted strip; here the reproducing kernel still resembles the starting one. We then follow by working with a weight with respect to a piece-wise linear bandwidth. Eventually, we work with a setting in which the variable bandwidth b is generally with a polynomial growth. For simplicity, we choose the window g in the STFT and its inverse to have L^2 - norm 1.

6.1 RK for Sobolev space

a) Let $v(x, \omega) = (1 + |\omega|^2)^{s/2}$, where $s > d/2$. The reproducing kernel (RK) for the Sobolev space $\mathcal{H}_v^2(\mathbb{R}^d)$ with respect to the inner product of form

$$\langle f, h \rangle_{\mathcal{H}_v^2} = \langle \hat{f} \cdot v, \hat{h} \cdot v \rangle_{L^2(\mathbb{R}^d)},$$

is

$$\Phi_y(t) = \Phi(t, y) = T_y \mathcal{F}^{-1}(v^{-2})(t),$$

where T_y is the translation operator, see [52].

Here is only a sketch of the proof published in [52]: In terms of an inner product, the inverse Fourier transform can be written as

$$f(y) = \langle \hat{f}, e^{-2\pi i y \cdot} \rangle.$$

More precisely, if we want $f(y) = \langle f, \Phi_y \rangle_{\mathcal{H}_v^2}$, we would have

$$f(y) = \langle f, \Phi_y \rangle_{\mathcal{H}_v^2} = \langle \hat{f} \cdot v, \mathcal{F}(\Phi_y) \cdot v \rangle_{L^2} = \langle \hat{f}, \mathcal{F}(\Phi_y) \cdot v^2 \rangle_{L^2}.$$

Thus $\mathcal{F}(\Phi_y)v^2 = e^{-2\pi i y \cdot}$, so $\Phi_y = \mathcal{F}^{-1}(v^{-2}e^{-2\pi i y \cdot})$ and the result follows. Note that $\mathcal{F}^{-1}(v^{-2})$ does exist; it can be calculated via residues of the analytic continuation, see pg. 123 in [82] for more details.

b) For comparison, let's calculate the reproducing kernel for the equivalent modulation space $M_v^2(\mathbb{R}^d)$: for any $f, h \in M_v^2$ the corresponding inner product is

$$\langle f, h \rangle_{M_v^2(\mathbb{R}^d)} = \langle V_g f \cdot v, V_g h \cdot v \rangle_{L^2(\mathbb{R}^{2d})}. \quad (6.1)$$

Let the reproducing kernel function be denoted by $\Phi_y(t) = \Phi(t, y)$ again. We require $f(y) = \langle f, \Phi_y \rangle_{M_v^2(\mathbb{R}^d)}$, therefore

$$f(y) = \langle f, \Phi_y \rangle_{M_v^2(\mathbb{R}^d)} = \langle V_g f \cdot v, V_g \Phi_y \cdot v \rangle_{L^2(\mathbb{R}^{2d})} = \langle f, V_g^* \{v^2 \cdot V_g(\Phi_y)\} \rangle_{L^2(\mathbb{R}^d)}.$$

Knowing that $f(y) = \langle f, \delta_y \rangle_{L^2(\mathbb{R}^d)}$, we have

$$V_g^* \{v^2 \cdot V_g(\Phi_y)\} = \delta_y,$$

that is, $v^2 \cdot V_g(\Phi_y) = V_g(\delta_y)$. As $V_g(\delta_y)(x, \omega) = \langle \delta_y, M_\omega T_x g \rangle = \overline{M_\omega T_x g(y)}$, we conclude that

Proposition 10. *The reproducing kernel of the space M_v^2 is*

$$\Phi(t, y) = T_y \{ \mathcal{F}^{-1}(v^{-2})(t) \} \cdot G(t, y), \quad (6.2)$$

where

$$G(t, y) = \int g(t-x) \overline{g(y-x)} dx = \langle g, T_{t-y} g \rangle \quad (6.3)$$

and it depends on the analysis window g .

More precisely,

$$\begin{aligned} \Phi(t, y) &= \iint \frac{1}{(1+|\omega|^2)^s} e^{-2\pi i \omega y} \overline{g(y-x)} e^{2\pi i \omega t} g(t-x) dx d\omega \\ &= \int \frac{e^{2\pi i \omega(t-y)}}{(1+|\omega|^2)^s} d\omega \int \overline{g(y-x)} g(t-x) dx, \end{aligned}$$

which comes to the result we stated.

Proof. For y -fixed, Φ_y is in $M_v^2(\mathbb{R}^d)$ as

$$\begin{aligned} \|\Phi_y\|^2 &= \langle V_g(\Phi_y) \cdot v, V_g(\Phi_y) \cdot v \rangle \\ &= \langle \frac{1}{v^2} V_g(\delta_y) \cdot v, \frac{1}{v^2} V_g(\delta_y) \cdot v \rangle \\ &= \iint_{\mathbb{R}^{2d}} \frac{1}{v(x, \omega)^2} M_\omega T_x g(y) \overline{M_\omega T_x g(y)} dx d\omega \end{aligned} \quad (6.4)$$

$$\begin{aligned}
&= \iint_{\mathbb{R}^{2d}} \frac{1}{(1 + |\omega|^2)^s} |g(y - x)|^2 dx d\omega & (6.5) \\
&= \int_{\mathbb{R}^d} \frac{1}{(1 + |\omega|^2)^s} d\omega \cdot \int_{\mathbb{R}^d} |g(y - x)|^2 dx \\
&= \frac{2}{2s - 1} \|g\|_2^2 = \frac{2}{2s - 1}.
\end{aligned}$$

□

Note that G is decaying fast for a fast decaying window g . For instance, if $g(t) = e^{-\pi t^2}$ we have

$$G(t, y) = e^{-\frac{\pi}{2}(t-y)^2} \int e^{-2\pi(x - \frac{t+y}{2})^2} dx = \frac{\sqrt{2}}{2} e^{-\frac{\pi}{2}(t-y)^2},$$

which decays fast the further away are t and y . In the discrete setting, G is a diagonal-like matrix with fast off-diagonal decay.

Compared to the RK with respect to the Sobolev norm, the 'new' RK has an extra multiplier G , which is different for different windows g . A simple estimation calculation gives the expected answer that minor changes of the analysis window g provide for minor changes within G :

Proposition 11. *Let $g_1, g_2 \in L^1 \cap L^\infty$ and $G_{1,2}(t, y) = \langle g_{1,2}, T_{t-y} g_{1,2} \rangle$. For every $\epsilon > 0$ there exists $\delta > 0$ so that whenever $\|g_1 - g_2\|_\infty < \delta / \|g_1\|_1$, it holds*

$$\|G_1 - G_2\|_\infty \leq \|g_1\|_1 \delta = \epsilon.$$

The corresponding change in the reproducing kernel is also minor, namely

$$\|\Phi_1 - \Phi_2\|_{(L^\infty, \mathcal{FL}^\infty)} \leq (\|g_1\|_1 + \|g_2\|_1) \|g_1 - g_2\|_\infty.$$

Proof.

$$\begin{aligned}
|G_1(t, y) - G_2(t, y)| &\leq \int |\overline{g_1(y-x)} g_1(t-x) - \overline{g_2(y-x)} g_2(t-x)| dx \\
&\leq \int |\overline{g_1(y-x)} g_1(t-x) - \overline{g_1(y-x)} g_2(t-x)| dx \\
&\quad + \int |\overline{g_2(y-x)} g_2(t-x) - \overline{g_1(y-x)} g_2(t-x)| dx
\end{aligned}$$

$$\leq (\|g_1\|_1 + \|g_2\|_1)\|g_1 - g_2\|_\infty.$$

The estimate of the change in the reproducing kernel is as follows:

$$\begin{aligned} |\Phi_1(t, y) - \Phi_2(t, y)| &= |T_y\{\mathcal{F}^{-1}(v^{-2})(t)\}| \cdot |G_1(t, y) - G_2(t, y)| \\ &\leq |T_y\{\mathcal{F}^{-1}(v^{-2})(t)\}|(\|g_1\|_1 + \|g_2\|_1)\|g_1 - g_2\|_\infty \end{aligned}$$

Then

$$\|\Phi_1 - \Phi_2\|_{(L^\infty, \mathcal{FL}^\infty)} \leq (\|g_1\|_1 + \|g_2\|_1)\|g_1 - g_2\|_\infty.$$

□

In practice this implies that working with an approximate reproducing kernel would be possible, that is, one could use a reproducing kernel calculated with respect to one window g_1 with an inner product that depends on another window g_2 and still derive good results. Let us denote the RK with respect to a window g_l with Φ_y^l , $l = 1, 2$ and have

$$\tilde{f}(y) = \langle V_{g_2} f \cdot v, V_{g_2} \Phi_y^1 \cdot v \rangle \approx f(y).$$

This is true as

$$|f(y) - \tilde{f}(y)| = |\langle V_{g_2} f \cdot v, (V_{g_2} \Phi_y^2 - V_{g_2} \Phi_y^1) \cdot v \rangle| \leq c\|f\|_2 \cdot \|\Phi_y^1 - \Phi_y^2\|_2.$$

6.2 RK: constant bandwidth

A similar split of integration is possible for a weight derived from v by cutting it at the origin and translating it by a

$$m_a(x, \omega) = \begin{cases} 1 & \text{if } |\omega| \leq a, \\ (1 + (|\omega| - a)^2)^{s/2} & \text{if } |\omega| > a, s > 0. \end{cases} \quad (6.6)$$

This weight is moderate w.r.t. $v(x, \omega) = (1 + |\omega|^2)^{s/2}$; recall that the corresponding weighted modulation spaces have equivalent norms for finite changes of the bandwidth. The reproducing kernel is easy to compute following the previous proof, because the weight m_a depends on ω only.

Proposition 12. *The reproducing kernel for the space $M_{m_a}^2$ is*

$$\Phi(t, y) = T_y\{\mathcal{F}^{-1}(m_a^{-2})(t)\}G(t, y), \quad (6.7)$$

where

$$G(t, y) = \langle g, T_{t-y}g \rangle$$

and it varies with the analysis window g .

We check if it holds $\Phi(y, b) = \langle \Phi_b, \Phi_y \rangle$:

$$\begin{aligned} \langle \Phi_b, \Phi_y \rangle &= \langle V_g(\Phi_b) \cdot w, V_g(\Phi_y) \cdot w \rangle & (6.8) \\ &= \langle V_g(\Phi_b), V_g(\Phi_y) \cdot m^2 \rangle = \langle V_g(\delta_b) \cdot m^{-2}, V_g(\delta_y) \cdot m^{-2} \cdot m^2 \rangle \\ &= \langle V_g^*[V_g(\delta_b) \cdot m^{-2}], \delta_y \rangle = V_g^*[V_g(\delta_b) \cdot m^{-2}](y) \end{aligned}$$

$$= \iint V_g(\delta_b)(x, \omega) \cdot m^{-2}(x, \omega) M_\omega T_x g(y) dx d\omega \quad (6.9)$$

$$= \iint \overline{M_\omega T_x g(b)} \cdot m^{-2}(x, \omega) M_\omega T_x g(y) dx d\omega$$

$$= \iint e^{-2\pi i \omega b} \overline{g(b-x)} \cdot m^{-2}(x, \omega) e^{2\pi i \omega y} g(y-x) dx d\omega$$

$$= \int \int e^{2\pi i \omega (y-b)} \cdot m^{-2}(x, \omega) d\omega g(y-x) \overline{g(b-x)} dx$$

$$= \int T_b \mathcal{F}^{-1}(m^{-2})(y) g(y-x) \overline{g(b-x)} dx$$

$$= T_b \mathcal{F}^{-1}(m^{-2})(y) G(y, b) = \Phi(y, b). \quad (6.10)$$

Comment: Small changes of the window cause small changes in the kernel. Similarly, small changes of the bandwidth cause small changes in the reproducing kernel:

Proposition 13. *If $a \rightarrow 0$, then $m_{a+a_0}^{-2} \rightarrow m_{a_0}^{-2}$. Therefore, if Φ_1 and Φ_2 are the reproducing kernel functions defined for m_{a+a_0} and m_{a_0} , it holds*

$$|\Phi_1(t, y) - \Phi_2(t, y)| \rightarrow 0.$$

6.3 RK: piecewise constant bandwidth

Say the bandwidth b is piecewise constant, and symmetric with respect to the time-frequency axes: $b(x, \omega) = a_n > 1$ for each $x \in [n, n+1)$, $n \in \mathbf{Z}$ so that $a_{-n} = a_n$. Recall that the difference $|a_n - a_{n+1}|$ must be controlled from the above by a constant c , as this provides a moderateness (as then $b'(x) = 0$, or the left and right differentials would be at most $\pm c$; apply the generalized version of Rolle's theorem). We shall write in shorter notation

$$b(x) = \sum_n a_n \cdot 1_{[n, n+1)}(x). \quad (6.11)$$

The (moderate) weight defined with respect to this bandwidth is

$$m(x, \omega) = \begin{cases} 1 & \text{if } |\omega| \leq b(|x|), \\ (1 + (|\omega| - b(|x|))^2)^{s/2} & \text{if } |\omega| > b(|x|). \end{cases} \quad (6.12)$$

In practice, if we denote by d_v the vertical distance, we have $m(x, \omega) = m_{a_n}(x, \omega) = (1 + d_v(|\omega|, a_n)^2)^{s/2}$ for $x \in [n, n+1)$, $n \in \mathbf{Z}$, that is

$$m(x, \omega) = \sum_n m_{a_n}(x, \omega) \cdot 1_{[n, n+1)}(x).$$

What this weight does to the time-frequency plane is cutting it into vertical strips of length $[n-1, n)$ with graded importance (described by the weight on each strip). We have $1 \equiv \sum_n s_n$, where s_n denote the indicator function on $[n, n+1) \times \mathbb{R}$. If we denote by Φ_y the reproducing kernel, then

$$f(y) = \langle f, \Phi_y \rangle_{VB} = \langle V_g f \cdot w, V_g(\Phi_y) \cdot w \rangle = \langle f, V_g^*[V_g(\Phi_y) \cdot m^2] \rangle.$$

In the distributional sense,

$$\delta_y = V^*[(\sum_n s_n m_{a_n})^2 V_g(\Phi_y)].$$

Notice that $(\sum_n s_n m_{a_n})^{-2} = \sum_n s_n m_{a_n}^{-2}$, because only the term $s_n m_{a_n} \neq 0$ on $[n, n+1) \times \mathbb{R}$. Then

$$\overline{M_\omega T_x g(y)} \sum_n s_n(x, \omega) m_{a_n}(x, \omega)^{-2} = V_g(\Phi_y)(x, \omega).$$

Thus

$$\begin{aligned}
\Phi_y(t) &= \iint \overline{M_\omega T_x g(y)} \sum_n s_n(x, \omega) m_{a_n}(x, \omega)^{-2} M_\omega T_x g(t) dx d\omega \\
&= \sum_n \int_n^{n+1} \int_{\mathbb{R}} \underbrace{s_n(x, \omega) m_{a_n}(x, \omega)^{-2}}_{\equiv m_{a_n}(x, \omega)^{-2}} e^{2\pi i \omega(t-y)} d\omega g(t-x) \overline{g(y-x)} dx \\
&= \sum_n \int_n^{n+1} T_y \mathcal{F}^{-1}(m_{a_n}^{-2})(t) g(t-x) \overline{g(y-x)} dx \\
&= \sum_n \int_n^{n+1} \frac{\Phi_{a_n}(t, y)}{G(t, y)} g(t-x) \overline{g(y-x)} dx \\
&= \sum_n \frac{\Phi_{a_n}(t, y)}{G(t, y)} \int_n^{n+1} g(t-x) \overline{g(y-x)} dx \\
&= \sum_n \frac{\Phi_{a_n}(t, y)}{G(t, y)} \int_{[n, n+1)} g(t-x) \overline{g(y-x)} dx. \tag{6.13}
\end{aligned}$$

Denoting

$$G_n(t, y) = \int_{[n, n+1)} g(t-x) \overline{g(y-x)} dx,$$

G_n is localized as for far away values of t, y it is practically 0; for example, if g is compactly supported in $[-a, a]$ then $G_n(t, y) = 0$ out of the area

$$(n-a, n-a+1) \times (n-a, n-a+a) \cup (n+a, n+a+1) \times (n+a, n+a+1), t \leq y.$$

we have the following result:

Theorem 20. *The reproducing kernel for a space of variable bandwidth, with piecewise constant bandwidth marked out via a_n , is*

$$\Phi(t, y) = \sum_n \Phi_{a_n}(t, y) \frac{G_n(t, y)}{G(t, y)}. \tag{6.14}$$

Note. The structure of the RK has some relation to localization operators [26] and/or the finite section method [66].

As expected, if a minor change on the bandwidth occurs, ex. say in section $[n_o, n_{o+1}]$ the bandwidth changes from a_{n_o} to $b_{n_o} = a_{n_o} + \varepsilon$, then the reproducing kernel is only slightly changed from Φ to Φ_1 :

$$\Phi(t, y) - \Phi_1(t, y) = T_y(\mathcal{F}^{-1}(m_{a_{n_o}}^{-2} - m_{b_{n_o}}^{-2})(t))G_{n_o}(t, y)$$

6.4 RK: variable bandwidth

We take $\omega = b(x), x \in \mathbb{R}^d$ to describe the bandwidth in the time-frequency domain. For practical reasons, it is enough that b is continuous as discontinuities with minor jumps can be easily overcome (see chapter 3). We work here with the weight defined via a vertical distance with respect to this bandwidth and we denote it by m . We will use the notation $\pi^*g(y) := \overline{M_\omega T_x g(y)}$. The inverse STFT is well-defined for functions or distributions in \mathcal{S}' (see [62])

$$f(y) = \frac{1}{\langle g, g \rangle} \iint_{\mathbb{R}^{2d}} V_g f(x, \omega) M_\omega T_x g(y) dx d\omega = \langle V_g f, \pi^*g(y) \rangle_{L^2(\mathbb{R}^{2d})}.$$

We seek a $2d$ -function $\Phi_y(t) := \Phi(t, y)$ such that

$$f(y) = \langle f, \Phi_y \rangle_{M_m^2(\mathbb{R}^d)},$$

that is

$$f(y) = \langle V_g f \cdot m, V_g \Phi_y \cdot m \rangle_{L^2(\mathbb{R}^{2d})} = \langle V_g f, m^2 \cdot V_g \Phi_y \rangle_{L^2(\mathbb{R}^{2d})}.$$

We conclude $\pi^*g(y) = m^2 \cdot V_g \Phi_y$, i.e.

Theorem 21. *The reproducing kernel for the space of variable bandwidth $M_m^2(\mathbb{R}^d)$ defined with respect to the variable bandwidth weight $m = m_b$ is*

$$\Phi_y = V_g^* \left(\frac{1}{m^2} \cdot \pi^*g(y) \right).$$

For y -fixed,

$$\|\Phi_y\| = 2 \|g(y - \cdot) \sqrt{m(\cdot)}\|_2^2 + \frac{2}{2s-1} \|g\|_2^2.$$

Proof. In only remains to calculate the norm of Φ_y :

Let y be fixed, we have

$$\begin{aligned}
\|\Phi_y\| &= \langle V_g \Phi_y \cdot m, V_g \Phi_y \cdot m \rangle & (6.15) \\
&= \langle V_g V_g^* \left(\frac{1}{m^2} \cdot \pi^* g(y) \right) \cdot m, V_g V_g^* \left(\frac{1}{m^2} \cdot \pi^* g(y) \right) \cdot m \rangle \\
&= \langle \frac{1}{m^2} \cdot \pi^* g(y) \cdot m, \frac{1}{m^2} \cdot \pi^* g(y) \cdot m \rangle
\end{aligned}$$

$$\begin{aligned}
&= \langle \frac{1}{m^2} \cdot \pi^* g(y), \pi^* g(y) \rangle & (6.16) \\
&= \iint \frac{1}{m(x, \omega)^2} \cdot M_\omega T_x g(y) \overline{M_\omega T_x g(y)} dx d\omega \\
&= \iint \frac{1}{m(x, \omega)^2} |g(y - x)|^2 dx d\omega \\
&= \int |g(y - x)|^2 \left\{ \int_{-b(x)}^{b(x)} d\omega + 2 \int_{b(x)}^{+\infty} \frac{1}{(1 + \omega - b(x))^{2s}} d\omega \right\} dx \\
&= 2 \int |g(y - x)|^2 \left\{ b(x) + \int_1^{+\infty} \frac{1}{\xi^{2s}} d\xi \right\} dx \\
&= 2 \int |g(y - x)|^2 \left\{ b(x) + \frac{1}{2s - 1} \right\} dx \\
&= 2 \|g(y - \cdot) \sqrt{b(\cdot)}\|_2^2 + \frac{2}{2s - 1} \|g\|_2^2. & (6.17)
\end{aligned}$$

□

Note. The difference

$$\Phi_y(t) - \Phi_0(t) = \iint \frac{1}{m^2(x, \omega)} e^{2\pi i t \omega} g(t-x) \overline{(g(y-x)e^{2\pi i y \omega} - g(-x))} dx d\omega$$

is small for y close to 0.

$$\begin{aligned} \|\Phi_y - \Phi_0\|_{S_o} &= \|V_g(\Phi_y - \Phi_0)\|_1 \\ &= \iint \frac{|g(y-x) - g(-x)|}{m^2(x, \omega)} dx d\omega \\ &\leq \iint \frac{|g(y-x) - g(-x)|}{v^2(x, \omega)} dx d\omega \end{aligned} \quad (6.18)$$

$$= \int |g(y-x) - g(-x)| dx \int v^{-2}(x, \omega) d\omega. \quad (6.19)$$

As expected, for s big enough, all functions in our space are continuous:

$$|f(y) - f(y')| = |\langle f, \Phi_y - \Phi_{y'} \rangle| \leq \|f\| \|\Phi_y - \Phi_{y'}\|.$$

Chapter 7

Prospects: Time warping

The concept of time warping has been independently explored by many authors, mostly with the idea to develop an irregular sampling procedure, using the classical sampling theorem [17, 95]. A worthwhile pursue in the line of this project would be time-warping of band-limited functions on the STFT side and obtaining (sub)spaces of variable bandwidth.

7.1 Warping the Gabor window

In time-frequency analysis, reconstruction is performed using the inverse short-time Fourier transform. The discrete version gives us a reconstruction formula

$$h(\tau) = \sum_{m,n} V_g(h)(ma, nb)\pi(ma, nb)g(\tau). \quad (7.1)$$

Notice that the sampling is performed on an equidistant lattice, defined with shift parameters a and b .

Let $\gamma(t) = \tau$ be a warping function; we observe a warped function

$$f := \gamma^*(h),$$

that is $f(t) = h(\gamma(t)) = h(\tau)$ (in the line of Clark's setting in [17]). We have:

$$f(t) = \sum_{m,n} V_g(h)(ma, nb) \gamma^*(\pi(ma, nb)g(t)). \quad (7.2)$$

That is, we would construct f using the old STFT samples of h and time-warped atoms

$$\gamma^*(\pi(ma, nb)g(t)) = e^{2\pi i nb \gamma(t)} g(\gamma(t - ma));$$

in a general case, the building blocs $\pi(ma, nb)g$ would have different warping on different locations (ma, nb) , simply because the warping function γ is varying in time. Each of the warped atoms would have different time-frequency ratio, thus, the warped function f would have variable bandwidth.

Some questions worth answering are: What is a warped Gabor analysis? What would g_γ have to satisfy to be a (warped) STFT/Gabor window?

Another question is with the motif to simplify things: Can we claim (and under which conditions) that the equality

$$e^{2\pi i \omega t} g(\gamma(t - x)) = e^{2\pi i \omega \gamma(t)} g(\gamma(t) - x)$$

holds? If so, then the relation between the original function h and the warped f on the TF domain would be

$$V_g h(x, \omega) = V_{g_\gamma}(f\gamma')(x, \omega).$$

7.2 Warping the lattice

A possibly easier approach to the time warping on the TF plane is the following: Say that the sampling rate in time is regular, but is irregular in frequency (i.e. we sample $V_g f$ at each ma , $m \in \mathbb{Z}$ and at an irregular sequence ω_n). We would keep the original building blocks, but on a new (warped) lattice. Is it possible to use a frequency transformation that satisfies

$$\gamma(\omega_n) = nb$$

and

$$f(t) = \sum_{m,n} V_g(f)(ma, nb) \pi(ma, nb) g(t) \quad (7.3)$$

$$= \sum_{m,n} V_g(f)(ma, \gamma(\omega_n)) T_{ma} M_{\gamma(\omega_n)} g(t)? \quad (7.4)$$

Then, irregular sampling on the TF lattice is a warped version of regular sampling on the TF lattice. This will again have the effect of variable bandwidth. To understand this, let us consider a simple case of warped lattice: let us assume there is a good way of warping the lattice so that one area of it stays the same (with fixed parameters (a, b)), then there is an area of change (irregular lattice locally), and then an area of locally regular lattice, but with new parameters. If we keep the same lattice with parameters (a, b) on the area right from 0 on the TF plane, and modify the lattice on the area left from 0 with parameters $(a, b/2)$, then a band-limited synthesis atom g for a band-limited function f would give the effect of twice as narrow bandwidth on the right of 0. That is, the new function would have variable bandwidth.

7.3 Matlab demonstration

The following figures are produced, using the Nuhag Matlab toolbox, see subsection 8.3. We start with a band-limited function f , obtain its STFT with respect to two Gaussian windows g_1, g_2 with different TF ratios tfr_1 (time-stretched) and tfr_2 (frequency stretched), see first two images in Figure 7.1. The third image in Figure 7.1 is the warped STFT with respect to a warped Gaussian g_w , whose TF ratio varies from tfr_1 to tfr_2 . The warped function has a variable bandwidth in the TF domain.

In Figure 7.2 we can see the reconstructed function f_a from the warped stft samples, using a window g_1 with small TF ratio tfr_1 ; notice that locally, the reconstruction is perfect on the area where the the warped window g_w was matching the window g_1 . Observe in Figure 7.3 that a local perfect reconstruction happens in the case where we have reconstructed from the warped stft samples, using a window g_2 with big TF ratio tfr_2 .

Figure 7.1: Comparison of the stft of a band-limited function with respect to a window with small TF ratio $tfr1$, a big TF ratio $tfr2$ and a warped window

Figure 7.2: Reconstruction from warped stft samples, using a synthesis window with low TF ratio. The first row plots are of the original, band-limited function x ; the second row plots are of the difference between the original and the warped reconstruction.

Figure 7.3: Reconstruction from warped stft samples, using a synthesis window with big TF ratio. The first row plots are of the inverse stft of the warped function; the second row plots are of the difference between the original and the warped reconstruction.

Chapter 8

Appendix: Matlab codes

8.1 Variable bandwidth weight

This code was used to build Figure 3.1. It generates a indicator area of a TF band, from samples (xs, ms). First, it interpolates the partially linear band. Then the weight is calculated using vertical distance. The indicator area is created first in the 1st quadrant, then by symmetry in the other quadrants.

```
%Input parameters: xs,ys
%%%%%%%% - samples of half the band b=b(x) (from 1 to n/2)
%n - dimension of the weight (n x n)

function wgt = tfbwgt(n,xs, ys)
wgt=ones(n,n);
%generates a band B in the 1st quadrant
mx=1:(n/2); B=interp1(xs,ys, mx);
%generates ind.area in the 1st quadrant
for ii=1 : (n/2);
    for jj=1 : (n/2);
        if (B(ii) ≤ jj )
            wgt(ii, jj) =1 + 0.1*(jj -B(ii));
        else
            wgt(ii, jj) =1;
        end
    end
end
```

```

end
%symmetry in other quadrants
for ii=1:(n/2);
    for jj=1 : (n/2); wgt(n-ii+1,jj)=wgt(ii,jj);
    end
end
for jj=1:(n/2);
    for ii=1:n; wgt(ii, n-jj+1)= wgt(ii,jj);
    end
end
wgt=wgt';
imgc(wgt); colorbar; shg
end

```

8.2 Variable bandwidth via a BUPU: a patchwork

This code was used to make Figures 5.1, 5.3, 5.4 in subsection 5.1.4. The standard Nuhag Matlab toolbox was used.

```

n=960; g=gaussnk(n); figure; plotc(g); hold; title('Gaussian')
figure;
BUPU = bupuspln(n,160,69,4); plot(BUPU');
ax20; hold; plot(sum(BUPU)); title('BUPU')

figure;
for jj = 1:6; ZZ(jj,:) = lowsigr(n,30+30*min(jj,6-jj)); end;
zztest = sum(BUPU.*ZZ);
subplot(1,2,1); imgc(stft(zztest, g));
title('the spectrogramme');
subplot(1,2,2); plotc(zztest); title('the signal');

figure;
secmatc(ZZ); secmatc(BUPU.*ZZ);
figure;
secstfm(ZZ,g,4,4,3,2); hold;
title('6 band-limited functions')
figure;
secstfm(BUPU.*ZZ,g,4,4,3,2);

```

8.3 Warped STFT

This code was used to build Figures 7.1, 7.2, 7.3. It produces the STFT of x with warped Gaussian window w_1 . Warping is performed via a linear change of the TF ratio of w_1 . See also: `stft.m`, `pgauss.m`. The LTFAT toolbox was used.

```

ltfatstart
n = 960; x=lowsign(n,65); res = zeros(n,n);
%c is starting TF ratio of window w1
c=5;
disp('max value of TF ratio of warped Gaussian is ');
disp(c)
w1 = pgauss(n,c)';   ww1 = [w1,w1];   c0=c;
%c0 is the min TF ratio of window w1
for jj = 1 : n;
    y = x.* conj(ww1( (n+1-(jj-1)) : (2*n - (jj-1)))) ;
    v = fft(y);
    res(jj,:) = v;
    if (n/3-jj> 0 && jj-n/6> 0)           c=c*9790/10000;
    end
    if (-2*n/3+jj> 0 && 5*n/6-jj> 0)     c=c/(9790/10000);
    end
    if (c0 > c)           c0=c;
    end
    w1 = pgauss(n,c)';   ww1 = [w1,w1];
end;
disp('min value of TF ratio of warped Gaussian is ');
disp(c0); stf = res.';

figure; subplot(3,1,1); imgc(stft(x,pgauss(n,5)'));
title('stft w.r.t. time-stretched g');
subplot(3,1,2); imgc(stft(x,pgauss(n,c0)'));
title('stft w.r.t. freq.stretched g')
subplot(3,1,3); imgc(stf); title('stft w.r.t. a warped g')

fa=istft(stf,pgauss(n,c)');   fb=istft(stf,pgauss(n,c0)');
figure; f2sp(x,x-fa);         figure; f2sp(x,x-fb);

```


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Curriculum Vitae

Name: Roza Aceska
Date and Place of birth: 13.12.1974, Skopje
Nationality: Macedonian
Email: roza.aceska@gmail.com

Education

2009 Uni. Vienna: PhD in Mathematics.
2004 UKIM, R. Macedonia: Masters in Mathematics
1998 UKIM, R. Macedonia: Diploma in Mathematics
1993 High school Nikola Karev, Skopje, R. Macedonia

Work experience

2006-2009 Uni. Vienna, Faculty of Mathematics, Research Assistant
1999-2006 UKIM, Faculty of Mechanical Engineering Skopje, Teaching Assistant
2001-2004 UKIM, Faculty for Pedagogy Skopje, Teaching Assistant
1997 University of Oviedo, Spain, Research Assistant (IAESTE)

Selected publications and presentations

1. Immediate calculation of the integral Theodorescu; MSDE 1998, R. Macedonia
2. Orthogonal Analytic Wavelets; Wavelets And Multifractal Analysis Summer School, 2004, Corsica, France
3. Some properties of a class of starlike functions; III Congress of Mathematics, 2005, R. Macedonia
4. Functions of variable bandwidth; Trends in Harmonic Analysis Strobl07, 2007, Austria
5. Mathematical model of the incubation of penguin eggs; European Consortium for Mathematics in Industry - 21st Modelling Worekshop, Rouen, 2007, France
6. Variable bandwidth; ICMS08, Classical and Modern Harmonic Analysis: from Theory to Numerical Computation, 2008, Edinburgh, UK
7. Variable bandwidth: a time-frequency approach; 5th European Congress of Mathematics, 2008, The Netherlands
8. L^1 Approximation: Jackson Theorems revisited; Approx.Theory Seminar, Univie, 2009, Austria
9. Variable Bandwidth on the TF plane; EWM 2009 - 14th meeting, 2009, Serbia

Memberships: SIAM, ECMI, IMU