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INNOVATIVE DYNAMICS FOR BIMATRIX GAMES

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Verfasser:	MAHDI RAHIMI
Matrikel-Nummer:	0025002
Studienrichtung:	Mathematik
Betreuer:	Univ.-Prof. Dr. Josef Hofbauer

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Abstract

Innovative dynamics are an interesting class of evolutionary dynamics, which include the Best-response dynamics, the Brown-von Neumann-Nash dynamics (in short BNN) and the Pairwise Difference dynamics (in short PD). Their models are more similar to the behavior of rational players and while the BR dynamics requires the theory of differential inclusions, the latter two dynamics fulfill also the conditions for the existence and uniqueness theorem of ordinary differential equations. We classify the behavior of these two dynamics for all 2×2 bimatrix games. We will see that under each dynamics every game has at least one reachable equilibrium and analyze stability with Liapunov functions. Further we will prove the non-existence of periodic orbits through Bendixson-Dulac.

Deutsche Zusammenfassung

Innovative Dynamiken, wie die Best response Dynamik, die Brown-von Neumann-Nash Dynamik und die Pairwise-Difference Dynamik sind eine interessante Klasse der evolutionären Dynamiken. Ihre Modelle richten sich mehr nach dem Verhalten von rationalen Spielern und während die Best response Dynamik eine Differential Inklusion ist, erfüllen die letzten beiden den Eindeutigkeitssatz für Gewöhnliche Differentialgleichungen. Wir untersuchen das Verhalten dieser Dynamiken für alle 2×2 Bimatrix-Spiele. Wir werden sehen, dass jedes Spiel zumindest ein erreichbares Gleichgewicht hat und untersuchen die Stabilität der Gleichgewichte mittels Liapunovfunktionen. Weiters werden wir mittels Bendixson-Dulac zeigen, dass es keine periodischen Orbits gibt.

Acknowledgments

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I dedicate this work to the memory of my late grandfathers Mr.Mashallah Rahimi and Mr.Nosratollah Hajmomenian, who both paid a lot of attention to the education of their kids and paved the way for us grandchildren to have real role models to look up to.

Contents

Introduction

The purpose of this work is to tackle two issues in Evolutionary Game Theory that are not thoroughly researched but offer some quite interesting results. The first issue are *bimatrix games*, among them the two most famous examples for *degenerate games*, the *Centipede Game* and the *Chain-Store Game*. The second issue are two cases of innovative dynamics, namely the *Brown-von Neumann-Nash Dynamics*, throughout shortened as *BNN* and the *Pairwise Difference Dynamics*, also known as *Pairwise Comparison Dynamics* or *Smith dynamics*[16].
The 13 possible cases of generic and degenerate 2×2 bimatrix games have been studied already by Cressman[9] and Hofbauer[12] for replicator dynamics and fictitious play. Binmore[5] studies the problems of the Chain-Store Game and Centipede Game in real life issues, such as entering a market competing with a rival or a drug deal, which raises new questions on the change of dynamics by mutation and there's also the famous Chain Store Paradox by Selten[18]. Our work continues the tradition of Zeeman[24] and Bomze[6] who studied all cases of the replicator dynamics for symmetrical games and Stejskal[21] for 2×2 *Best-response dynamics*. Little is known however on the behavior of degenerate games under the BNN- and Pairwise Difference dynamics, or, as a matter of fact, any innovative dynamics[23] for higher dimensional games at all.

Historically, BNN has been used by both Brown and von Neumann[8] and Nash[15] in the early days of Game Theory. Brown and von Neumann considered the special case of symmetric two person zero sum games to show that for such games solutions converge to the set of equilibria[3] and Nash used the dynamics in discrete times to prove the existence of equilibria. Later on, Arrow and Debreu[1] used this Nash map for an existence proof in General Equilibrium Theory.
The above mentioned examples from economic theory show the great benefit of the BNN; while it is not a smooth dynamics, unlike the replicator equation, the right hand side of it is Lipschitz and it fulfills the conditions for the existence and uniqueness theorem for differential equations[3], while the Best-Response dynamics is discontinuous and requires the theoretical framework of differential inclusions.

The Pairwise Difference dynamics is another innovative dynamics with similarities to the BNN dynamics. It was introduced by M J Smith[20] for a traffic congestion problem and later studied by Sandholm[16], [17] but otherwise, little attention has been paid to it in mathematics or economics.

We will give a definition of innovative dynamics, then look at the 13 different classes of generic and degenerate 2×2 games under the Pairwise Difference dynamics and the BNN dynamics, focus on the differences and similarities between the two in terms of dynamic behavior and Nash Equilibria and offer an outlook into future questions and problems for further research.

1. Innovative Dynamics

It might be a good start to give an introduction and theoretical framework on innovative dynamics and what they are all about. We take an $n \times m$-game with two players and mixed strategies $x \in \Delta_n$ and $y \in \Delta_m$ with $x = (x_1, x_2, \cdots, x_n)$, where x_i denotes the frequency of pure strategy i for player 1, and $y = (y_1, y_2, \cdots, y_m)$ with y_j respectively denoting the frequency of pure strategy j for player 2, and the respective payoff matrices given by

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} \tag{1.1}$$

The payoff for the pure strategy i for player 1 and j of player 2 are given by

$$a_i = (Ay)_i = \sum_{k=1}^{m} a_{ik} y_k \qquad b_j = (Bx)_j = \sum_{l=1}^{n} b_{jl} x_l$$

The average payoffs $\bar{a}$ and $\bar{b}$ are given by

$$\bar{a} = \sum_{i=1}^{n} x_i a_i = \sum_{i=1}^{n} \sum_{k=1}^{m} x_i a_{ik} y_k \qquad \bar{b} = \sum_{j=1}^{m} y_j b_j = \sum_{j=1}^{m} \sum_{l=1}^{n} y_j b_{jl} x_l$$

Dynamics for games can be divided into two groups, dynamics based on *imitative adaptation* and dynamics based on *innovative adaptation*. To give a heuristic explanation for the names, we might say that for *imitative dynamics* model agents imitate existing strategies and hence only pure strategies "known", "in use" or non-extinct are played while *innovative dynamics* are just that, *innovative*, and model agents play all pure strategies. Among *imitative dynamics* are the well known and thoroughly researched dynamics, like the *Replicator dynamics* while for the *innovative dynamics* we have the *Best-response dynamics*, the *Brown-von Neumann-Nash dynamics* and *Pairwise Difference dynamics*.

The division was first suggested by Weibull [23] and he also gives the following definition of innovative dynamics, which we will also use here.

Definition 1.1. For the population dynamics

$$\dot{x}_i = f_i(x, y) \qquad \dot{y}_j = g_j(x, y) \tag{1.2}$$

let $\Gamma_1(x, y)$ be the subset of *Better than Average* pure strategies of player 1, $\Gamma_1(x, y) = \{i : a_i > \bar{a}\}$ and respectively $\Gamma_2(x, y)$ for player 2. We call the dynamics *innovative* if its vector field satisfies the following axiom:
(**IN**): If $\Gamma_1(x, y) \neq \emptyset$ then $f_i(x, y) > 0$ for some $i \in \Gamma_1(x, y)$ and if $\Gamma_2(x, y) \neq 0$ then $g_j(x, y) > 0$ for some $j \in \Gamma_2(x, y)$.

A corollary of this definition is the following property of innovative dynamics, which is called *Nash Stationarity* by Sandholm[16].

Corollary 1.2. *All rest points of innovative dynamics are equivalent to Nash Equilibria*

Proof. Let us assume that we have a pair $(x, y) \in \Delta_n \times \Delta_m$ that is a rest point but not a Nash Equilibrium. Then $\dot{x}_i = f_i(x, y) = 0$, $\dot{y}_j = g_j(x, y) = 0$ by definition of a rest point. For (x, y) not being a Nash Equilibrium, we have that either for player 1 $a_i > \bar{a}$ for some i or for player 2 $b_j > \bar{b}$ for some j. Therefore $i \in \Gamma_1(x, y)$(or $j \in \Gamma_2(x, y)$) and $f_i(x, y) > 0$ ($g_j(x, y) > 0$), which contradicts (x, y) being a rest point. □

The Pairwise Difference dynamics, in short PD, is given by

$$\begin{aligned}
\dot{x}_i &= \sum_{k=1}^{n} x_k(a_i - a_k)_+ - x_i \sum_{k=1}^{n}(a_k - a_i)_+ \\
\dot{y}_j &= \sum_{l=1}^{m} y_l(b_j - b_l)_+ - y_j \sum_{l=1}^{m}(b_l - b_j)_+
\end{aligned} \tag{1.3}$$

As we can see, the name is self explanatory. Since it's not obvious from the equation that the PD dynamics fulfills the requirements of the definition for innovative dynamics, we will prove this in the following Lemma.

Lemma 1.3. *The PD dynamics satisfies axiom (IN).*

Proof. Without loss of generality, let $a_1 \leq a_2 \leq \cdots \leq a_n$ and let $\Gamma_1(x, y) \neq \emptyset$, hence at least $a_n > \bar{a}$.
$\dot{x}_i$ expanded is given by

$$\dot{x}_i = x_1(a_i - a_1)_+ + x_2(a_i - a_2)_+ + \cdots + x_n(a_i - a_n)_+ - x_i\left[(a_1 - a_i)_+ + \cdots + (a_n - a_i)_+\right]$$

In particular

$$\dot{x}_n = x_1(a_n - a_1)_+ + x_2(a_n - a_2)_+ + \cdots + x_n(a_n - a_n)_+ - x_n\left[(a_1 - a_n)_+ + \cdots + (a_n - a_n)_+\right]$$

Since $a_n \geq a_i \forall i$ with $i \in \{1, \cdots, n\}$, this leads us to

$$\begin{aligned}\dot{x}_n &= x_1(a_n - a_1) + x_2(a_n - a_2) + \cdots + x_n(a_n - a_n) \\ &= x_1 a_n + x_2 a_n + \cdots + x_n a_n - \sum_{j=1}^{n} x_j a_j\end{aligned}$$

Since $\sum_{i=1}^{n} x_i = 1$,we are left with

$$\dot{x}_n = a_n - \sum_{j=1}^{n} x_j a_j = a_n - \bar{a} > 0$$

Analogously for player 2. □

The BNN dynamics is given by

$$\begin{aligned}\dot{x}_i &= \alpha_i - x_i \sum_{k=1}^{n} \alpha_k \\ \dot{y}_j &= \beta_j - y_j \sum_{l=1}^{m} \beta_l\end{aligned} \tag{1.4}$$

where $\alpha_i = (a_i - \bar{a})_+$ and $\beta_j = (b_j - \bar{b})_+$.
Since, just like for the PD dynamics, the right hand side of the dynamics is Lipschitz, it also fulfills the conditions for the existence and uniqueness theorem for differential equations. As we will see, the dynamics satisfies **(IN)**. However, the best strategy does not always increase in BNN, therefore the proof is different.

Lemma 1.4. *The BNN dynamics satisfies axiom (IN).*

Proof. Suppose that $\Gamma_1(x, y) \neq \emptyset$, but $f_i(x, y) \leq 0 \; \forall i \in \Gamma_1(x, y)$. Since for $i \notin \Gamma_1(x, y)$, $a_i \leq \bar{a}$ and therefore $f_i(x, y) \leq 0$ and with $\sum_{i=1}^{n} f_i(x, y) = 0$, we have that $f_i(x, y) = 0 \forall i$.
This leads us to

$$\dot{x}_i = (a_i - \bar{a})_+ - x_i \sum_{k=1}^{n} (a_k - \bar{a})_+ = 0$$

For $\dot{x}_i$ to be 0, we require

$$(a_i - \bar{a})_+ = x_i \sum_{k=1}^{n} (a_k - \bar{a})_+ \tag{1.5}$$

If $a_i \leq \bar{a} \quad \forall i$, then $\Gamma_1(x, y) = \emptyset$ and we have a contradiction. Suppose now that $(a_i - \bar{a}) > 0$ for some i, then $\sum_{k=1}^{n} (a_k - \bar{a})_+ > 0$. We have that

$$0 = \sum_{i=1}^{n} x_i(a_i - \bar{a})$$

which leads us by using (1.5) to

$$\sum_{i:a_i>\bar{a}} x_i(a_i-\bar{a})_+$$

and since $x_i \neq 0$ for $i \in \Gamma_1(x,y)$ and $(a_i-\bar{a}) > 0$ for all $i \in \Gamma_1(x,y)$ it is $\neq 0$ and hence we have a contradiction. □

Before we start, we will introduce a result which might look trivial at first but is of great use later on.

Lemma 1.5. *Adding a constant c to a column of a payoff matrix does not change the BNN and PD dynamics.*

Proof. We have the matrix

$$A' = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j}+c & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2j}+c & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nj}+c & \cdots & a_{nm} \end{pmatrix}$$

which is matrix A from (1.1)with a constant c added to the j-th column.
We have $a'_i = \sum_{k=1}^m a'_{ik}y_k = \sum_{k=1}^m a_{ik}y_k + cy_j = a_i + cy_j$ and $\bar{a}' = \sum_{i=1}^n \sum_{k=1}^m x_i a'_{ik} y_k = \bar{a} - cy_j$.
We can see that

$$\begin{aligned} a'_i - a'_k &= a_i - a_k + cy_j - cy_j \\ a'_i - \bar{a}' &= a_i - \bar{a} + cy_j - cy_j \end{aligned}$$

Therefore, for the PD dynamics for the game (A', B'),

$$\begin{aligned} \dot{x_i} &= \sum_{k=1}^m x_k(a'_i - a'_k)_+ - x_i \sum_{k=1}^m (a'_k - a'_i)_+ \\ \Rightarrow \dot{x_i} &= \sum_{k=1}^m x_k(a_i - a_k)_+ - x_i \sum_{k=1}^m (a_k - a_i)_+ \end{aligned}$$

Likewise for the BNN dynamics. □

2. The Pairwise Difference Dynamics

We now look at bimatrix games with two strategies. Without loss of generality, we take for both PD- and BNN-dynamics the respective pay-off matrices

$$A = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & b_{12} \\ b_{21} & 0 \end{pmatrix} \tag{2.1}$$

We may use these matrices since because of *Lemma 1.5* we may add a constant to each column without changing the dynamics and therefore we may transform every 2×2 matrix into a matrix of the above form.

Since the strategy $x_2 = (1 - x_1)$ and $y_2 = (1 - y_1)$, it is enough to consider the strategies x_1 and y_1 only. For $x_1 = x$, $x_2 = (1 - x)$ and for $y_1 = y, y_2 = (1 - y)$ likewise, the equations (1.3) transform to

$$\begin{aligned} \dot{x} &= (1 - x)(a_1 - a_2)_+ - x(a_2 - a_1)_+ \\ \dot{y} &= (1 - y)(b_1 - b_2)_+ - y(b_2 - b_1)_+ \end{aligned} \tag{2.2}$$

where

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix} \begin{pmatrix} y \\ 1 - y \end{pmatrix}$$

and

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 & b_{12} \\ b_{21} & 0 \end{pmatrix} \begin{pmatrix} x \\ 1 - x \end{pmatrix}$$

We use the terminology as in [12], which contains a similar discussion of the *replicator dynamics* and the *BR-dynamics*.

2.1. The Four Cases of Generic Games

First we define a generic game for the 2×2 case.

Definition 2.2. A bimatrix game (2.1) is called generic if no column of a payoff matrix is all zero. (Alternative definition: In a generic game, the payoff of two different strategies for one player is never equal.)

2.2.1. 0a and 0b

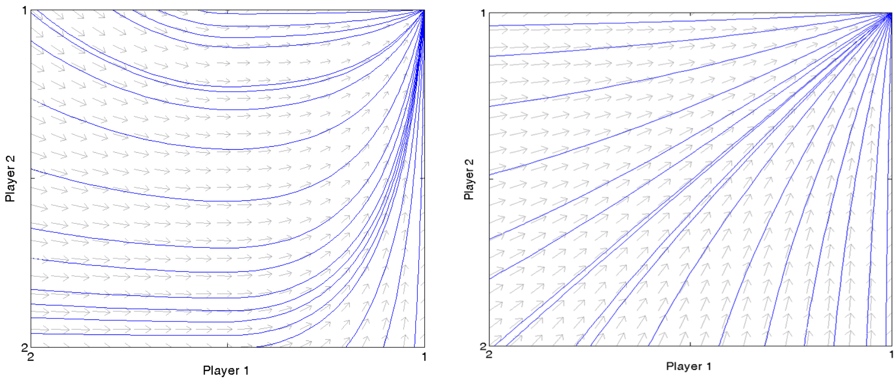

Figure 2.1.: 0a and 0b

The games (2.2) in case 0a are given by $a_{12}a_{21} < 0$ and $b_{12}b_{21} > 0$. For case 0b, which includes the Prisoner's Dilemma game, we have $a_{12}a_{21} < 0$ and $b_{12}b_{21} < 0$.
In short, the equations are for $0a$

$$\dot{x} = (1-x)(a_{12} - (a_{12}+a_{21})y)$$
$$\dot{y} = (1-y)(b_{12} - (b_{12}+b_{21})x)_+ - y(-b_{12} + (b_{12}+b_{21})x)_+$$

and for $0b$

$$\dot{x} = (1-x)(a_{12} - (a_{12}+a_{21})y)$$
$$\dot{y} = (1-y)(b_{12} - (b_{12}+b_{21})x)$$

In both cases strategy 1 for player 1 dominates strategy 2 and we have a unique strict Nash Equilibrium and asymptotic stability for $(1, 1)$.

2.2.2. 0c

The cyclic game 2×2 game 0c is also known as the "Buyer-Seller game" or "Matching Pennies game". We have $a_{12}a_{21} > 0$, $b_{12}b_{21} > 0$ and $a_{12}b_{12} < 0$. We get the unique equilibrium point $E = (\frac{b_{12}}{b_{12}+b_{21}}, \frac{a_{12}}{a_{12}+a_{21}})$. For the equations (2.2) we study stability.

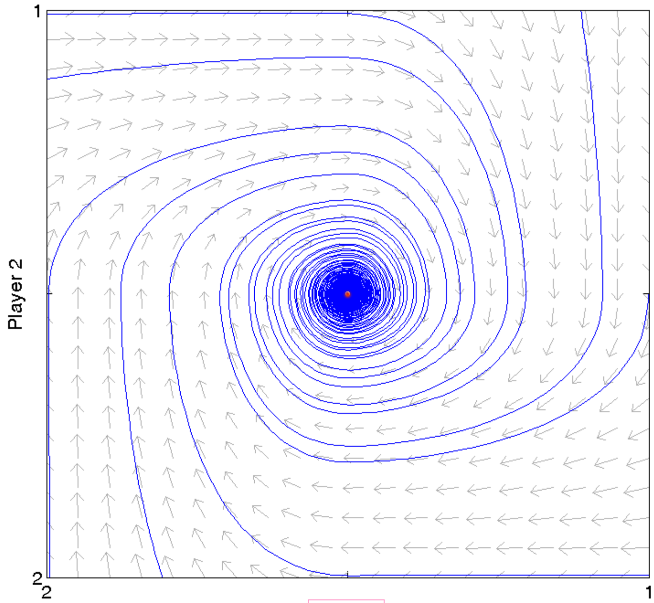

Figure 2.2.: 0c or Buyer-Seller

In order to prove stability, we will use a similar Liapunov function as the one from [13], [16], [20] which is given for single population games by

$$V(x) = \frac{1}{2}\sum_{i\in S}\sum_{j\in S} x_i \left[F_j(x) - F_i(x)\right]_+^2 \tag{2.3}$$

where $F_i(x)$ denotes the payoff of strategy i.

Proof of stability through Liapunov

The PD dynamics for the bimatrix game 0c is given by

$$\begin{aligned} \dot{x} &= (1-x)(a_1-a_2)_+ - x(a_2-a_1)_+ \\ \dot{y} &= (1-y)(b_1-b_2)_+ - y(b_2-b_1)_+ \end{aligned} \tag{2.4}$$

The Liapunov function for the PD dynamics as written in [13], [16] and [20] for stable games is given for a bimatrix game by the following equation (note that we need different weights for different players):

$$\begin{aligned} V(x,y) = &(b_{12}+b_{21})(x(a_2-a_1)_+^2 + (1-x)(a_1-a_2)_+^2) \\ &-(a_{12}+a_{21})(y(b_2-b_1)_+^2 + (1-y)(b_1-b_2)_+^2) \end{aligned} \tag{2.5}$$

We look at the Liapunov function in the first quadrant, that is where

$$x < \bar{x}, y < \bar{y}$$

and $\dot{x} < 0$ and $\dot{y} > 0$. In that region, V is given by

$$V = (b_{12}+b_{21})(x(a_2-a_1)^2) - (a_{12}+a_{21})((1-y)(b_1-b_2)^2)$$

Therefore

$$\dot{V} = (b_{12} + b_{21})(a_2 - a_1)^2\dot{x} + (a_{12} + a_{21})(b_1 - b_2)^2\dot{y}$$
$$+ 2(b_{12} + b_{21})(x(a_2 - a_1))(a_2 - a_1)^{\bullet} - 2(a_{12} + a_{21})((1 - y)(b_1 - b_2)(b_1 - b_2)^{\bullet})$$

and we have that

$$\begin{aligned} &(b_{12} + b_{21})(x(a_2 - a_1))(a_2 - a_1)^{\bullet} - (a_{12} + a_{21})((1 - y)(b_1 - b_2)(b_1 - b_2)^{\bullet}) \\ =&(b_{12} + b_{21})(a_{12} + a_{21})x(a_2 - a_1)(1 - y)(b_1 - b_2) \\ -&(b_{12} + b_{21})(a_{12} + a_{21})((1 - y)(b_1 - b_2))(x(a_2 - a_1)) \\ =& \quad 0 \end{aligned}$$

and hence

$$\dot{V} = (b_{12} + b_{21})(a_2 - a_1)^2\dot{x} + (a_{12} + a_{21})(b_1 - b_2)^2\dot{y} < 0$$

for $(x, y) \neq E$. The same computation can be done for the other 3 quadrants with the same result.
For the Liapunov function, $\omega(x, y) \subseteq \{\bar{x}, \bar{y}\}$ and therefore, by A.8 we have a globally asymptotically stable Nash Equilibrium in $E = (\bar{x}, \bar{y})$. For different proofs regarding the stability of the Nash Equilibrium, we refer to *Chapter 4* and *Chapter 5*.

2.2.3. 0d

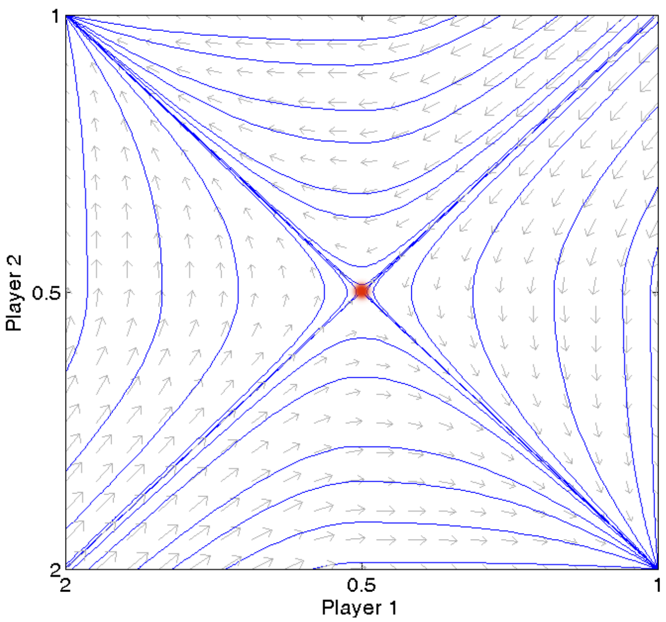

Figure 2.3.: 0d or Battle of Sexes

Case 0d contains the "Battle of the Sexes", the "Hawk and Dove" and the "Owner-Intruder game". We have $a_{12}a_{21} > 0$ and $b_{12}b_{21} > 0$, however with $a_{12}b_{12} > 0$ as well. We get three rest points.

The two corners $(1,0)$ and $(0,1)$ are both strict Nash Equilibria and we also have the interior equilibrium $E = (\frac{b_{12}}{b_{12}+b_{21}}, \frac{a_{12}}{a_{12}+a_{21}})$. We use linearization around the equilibrium E and get for the four quadrants four matrix of the form

$$\begin{pmatrix} 0 & - \\ - & 0 \end{pmatrix}$$

which leads to a positive and a negative eigenvalue in each quadrant. This suggests that E is a saddle point. It will be interesting to know whether there is any orbit leading to the Nash Equilibrium E. We will take the equations of the dynamics (2.2) and compute the orbits. We will look at the left-bottom quadrant where $x < \bar{x}, \dot{x} > 0$ and $y < \bar{y}, \dot{y} > 0$

$$\frac{dx}{dy} = \frac{(1-x)(a_{12} - (a_{12}+a_{21})y)}{(1-y)(b_{21} - (b_{12}+b_{21})x)}$$

which leads us to

$$\frac{b_{21} - (b_{12}+b_{21})x}{(1-x)} dx = \frac{a_{12} - (a_{12}+a_{21})y}{1-y} dy$$

and after integration to

$$b_{21} \log(1-x) - (b_{12}+b_{21})(1-x) + C = a_{21} \log(1-y) - (a_{12}+a_{21})(1-y).$$

By inserting $\bar{x}$ and $\bar{y}$ for (x,y), we get

$$b_{21} \log(1 - \frac{b_{12}}{b_{12}+b_{21}}) + b_{21} + C = a_{21} \log(1 - \frac{a_{12}}{a_{12}+a_{21}}) - a_{21}$$

which leads to

$$\begin{aligned} C &= a_{21} \log(\frac{a_{12}}{a_{12}+a_{21}}) - a_{21} - b_{21} \log(1 - \frac{b_{12}}{b_{12}+b_{21}}) - b_{21} \\ &= a_{21} \log(\frac{a_{12}}{a_{12}+a_{21}}) - a_{21} - b_{21} \log(\frac{b_{21}}{b_{12}+b_{21}}) - b_{21} \end{aligned}$$

hence

$$H(x,y) = b_{21} \log(1-x) - (b_{12}+b_{21})(1-x) - a_{21} \log(1-y) + (a_{12}+a_{21})(1-y) + C$$

is the unique orbit leading to E in the first quadrant. The same way, we will find out that exactly one orbit from the third quadrant also leads to the equilibrium, while for the second and fourth quadrant, we have exactly one orbit leading away from it in each quadrant.

2.3. One Degenerate Strategy

There are 4 cases with 1 degenerate strategy, where 2 cases are pretty famous and known as the Centipede Game and the Chain-Store Game. In each phase plot, the degenerate edge is always colored, with red indicating Nash Equilibria and yellow the non-equilibria.

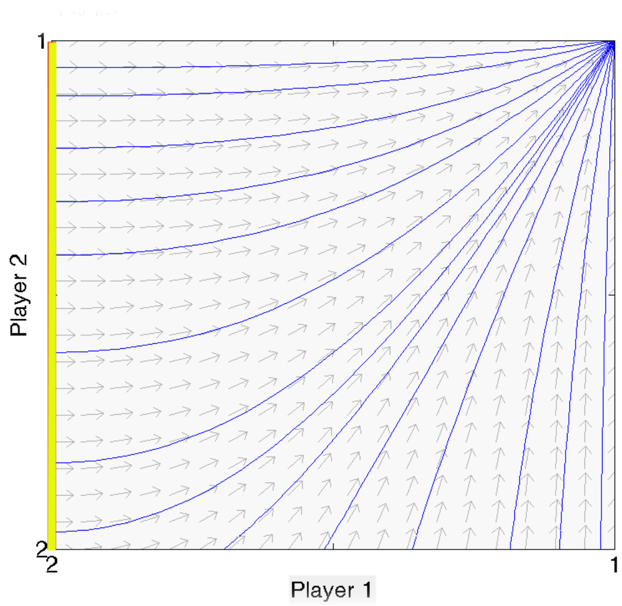

Figure 2.4.: 1a for PD

2.3.1. 1a

We have the case that $a_{12} > 0, a_{21} < 0, b_{21} < 0, b_{12} = 0$ which implies $a_2 > a_1, b_1 > b_2$. This turns (2.2) into

$$\dot{x} = (1-x)(a_{12} - (a_{12} + a_{21})y)$$
$$\dot{y} = (1-y)(-b_{21}x)$$

since $-a_{12}+(a_{12}+a_{21})y \leq 0$ and equally $-b_{12}+(b_{12}+b_{21})x \leq 0$. Hence, we have for the Nash Equilibrium $(1,1)$ the Jacobi-Matrix $\begin{pmatrix} a_{21} & 0 \\ 0 & b_{21} \end{pmatrix}$ and therefore an asymptotically stable equilibrium point. The degenerate strategy, strategy 2, is dominated for player 2, just as player 1's strategy 2 is dominated. Further, if we take the Liapunov function

$$V(x,y) = x$$

defined on $G = [0,1] \times [0,1]$, we see that $\dot{V}(x,y) > 0 \qquad \forall x \neq 1$.
Therefore $\omega(x,y) \subseteq \{x = 1\}$ according to A.9. Since the only invariant set on $x = 1$ is the point $(1,1)$, according to A.8 $(1,1)$ is even globally asymptotically stable.

2.3.2. 1b

We have the case that $a_{12} < 0, a_{21} > 0, b_{21} < 0, b_{12} = 0$. Thus (2.2) is

$$\dot{x} = -x(-a_{12} + (a_{12} + a_{21})y)$$
$$\dot{y} = (1-y)(-b_{21}x)$$

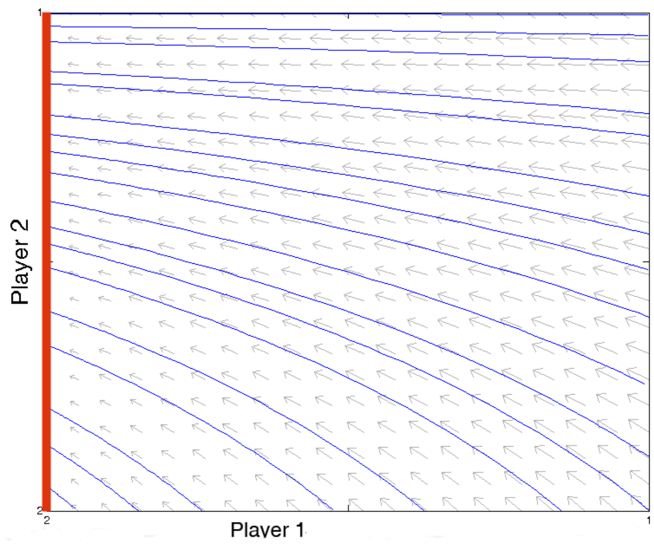

Figure 2.5.: PD for 1b

The Nash Equilibria of this game are $(0, \hat{y})$ with $\hat{y} \in [0, 1]$. This leads to the Jacobian

$$\begin{pmatrix} -(-a_{12} + (a_{12} + a_{21})y) & 0 \\ * & 0 \end{pmatrix}$$

equaling a negative eigenvalue and one eigenvalue 0 at each equilibrium point $(0, \hat{y})$. By using

$$\frac{\dot{x}}{\dot{y}} = \frac{-a_{12} + (a_{12} + a_{21}y)}{(1-y)(-b_{21})}$$

we get by separation of variables the level curves

$$H = -b_{21}x + a_{21}\ln(1-y) - (a_{12} + a_{21})(1-y)$$

or

$$-b_{21}x = -a_{21}\ln(1-y) + (a_{12} + a_{21})(1-y) + C.$$

Hence, for $x_0 = 0$ and $y_0 \in [0, 1]$, we have that

$$a_{21}\ln(1-y_0) = (a_{12} + a_{21})(1-y_0) + C$$

which is solvable with regard to C, i.e.

$$C = a_{21}\ln(1-y_0) - (a_{12} + a_{21})(1-y_0)$$

which solves the equation for $x = 0$. Therefore, we have an orbit leading to every point on the degenerate edge, except for $(0, 0)$, which is a minimum for $H(x, y)$ and we have a continuum of stable Nash equilibria given by $(0, \hat{y})$ with $\hat{y} \in [0, 1]$.

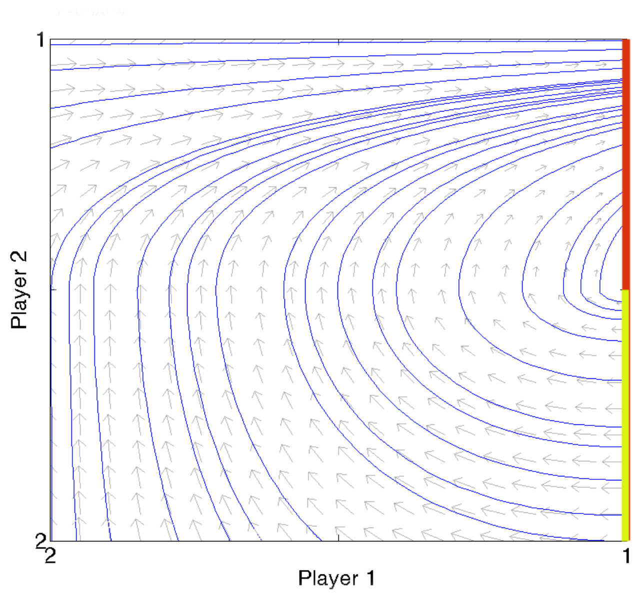

Figure 2.6.: Centipede Game 1c

2.3.3. 1c or Centipede Game

For the Centipede Game, we have that $a_{12} < 0, a_{21} < 0, b_{12} > 0, b_{21} = 0$. This turns (2.2) to the form

$$\begin{aligned}\dot{x} &= (1-x)(a_{12} - (a_{12}+a_{21})y)_+ - x(-a_{12} + (a_{12}+a_{21}y))_+ \\ \dot{y} &= (1-y)(1-x)b_{12}\end{aligned}$$

We have for $(1, \hat{y})$ with $\hat{y} \in [0,1]$ a degenerate edge. By linearizing around $x = 1$ and $y > \frac{a_{12}}{a_{12}+a_{21}}$ we get the Jacobi matrix

$$\begin{pmatrix} -(a_{12} - (a_{12}+a_{21})y) & 0 \\ * & 0 \end{pmatrix}$$

which leads to a matrix of the form

$$\begin{pmatrix} - & 0 \\ * & 0 \end{pmatrix} \tag{2.6}$$

This leads to a negative eigenvalue and one eigenvalue 0 for $(1, y > \frac{a_{12}}{a_{12}+a_{21}})$. By separation of variables, we get for $y > \frac{a_{12}}{a_{12}+a_{21}}$ the level curves

$$\begin{aligned}H(x,y) &= b_{12}x - a_{12}\log(1-y) - (a_{12}+a_{21})\left[(1-y) - \log(1-y)\right] \\ &= b_{12}x - (2a_{12}+a_{21})\log(1-y) - (a_{12}+a_{21})(1-y)\end{aligned}$$

As we can see, there is a maximum for $x = 1$, while for y it is increasing for $y \to 1$. For $x_0 = 1$ and $y_0 \in \left[\frac{a_{12}}{a_{12}+a_{21}}, 1\right]$ we have

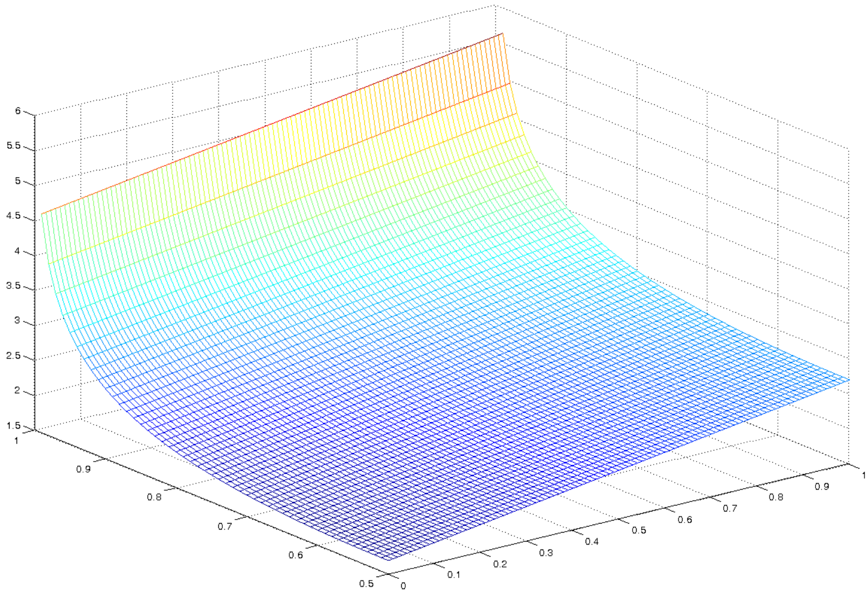

Figure 2.7.: Plot of H(x,y) for 1c

$$C = b_{12} - a_{12}\log(1-y_0) + (a_{12}+a_{21})\left[(1-y_0) - \log(1-y_0)\right]$$

and hence we have at least one orbit converging to each $(1,\hat{y}), \hat{y} > \frac{a_{12}}{a_{12}+a_{21}}$. Further we have a continuum of stable equilibria for $(1,\hat{y}), \hat{y} \geq \frac{a_{12}}{a_{12}+a_{21}}$.

Looking at the phase plane and the equations for the orbits, we can see that the phase portrait is the same as the phase portrait of the simple *SIR model from epidemiology* (see [7]). For $y \geq \bar{y}$ even the differential equations and hence the level curves are equivalent, modulo some constants. Furthermore, the *SIR model* is a special case of the classical *Lotka-Volterra predator prey equation*, where one species, the prey, have no growth and hence this leads to the population of the prey minimizing but the predators dying out. Likewise, for the *SIR-model*, the number of susceptibles does not grow, which leads after an initial rise of the infected population to a fall and a population of susceptible but non-infected individuals remaining.

1d or Chain-Store Game

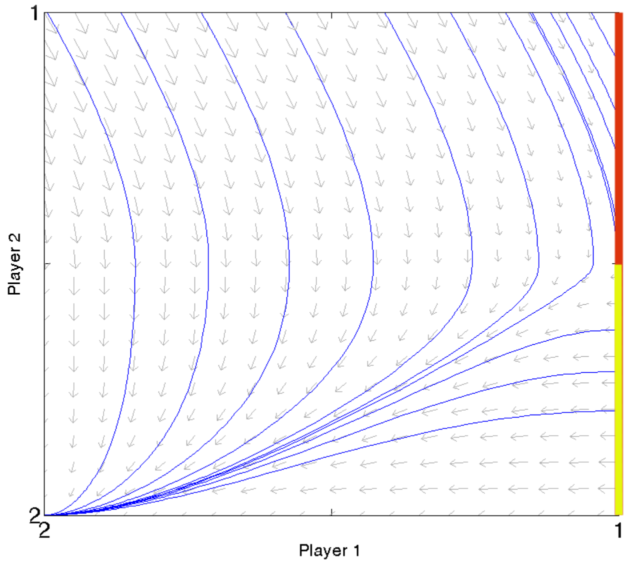

Figure 2.8.: 1d or Chain-Store Game

The only difference in the equation for the Chain-Store Game compared to the Centipede Game lies with $b_{12} < 0$. Hence we get

$$\dot{x} = (1-x)(a_{12} - (a_{12}+a_{21})y)_+ - x(-a_{12} + (a_{12} + a_{21}y))_+$$
$$\dot{y} = -y(1-x)b_{12}$$

We have Nash Equilibria at $(1, \hat{y})$ with $\frac{a_{12}}{a_{12}+a_{21}} \leq \hat{y} \leq 1$ and at $(0,0)$ a strict Nash Equilibrium.

Therefore we need to look at two parts for stability. For $(1,\hat{y}), \hat{y} > \frac{a_{12}}{a_{12}+a_{21}}$ we have the same Jacobian matrix as above in 1c. Therefore we have a set of stable equilibria for $(1,\hat{y})$. By separation of variables, we get the constant of motion for $y > \frac{a_{12}}{a_{12}+a_{21}}$

$$H(x,y) = -b_{12}x + (a_{12}\log y - (a_{12}+a_{21})y) = const$$

For $x_0 = 1$ and $y_0 \in \left[\frac{a_{12}}{a_{12}+a_{21}}, 1\right]$ we have

$$C = -b_{12} + (a_{12}\log y_0 - (a_{12}+a_{21})y_0)$$

and hence we have interior orbits converging to each $(1,\hat{y}), \hat{y} \geq \frac{a_{12}}{a_{12}+a_{21}}$. Further, we have a continuum of stable equilibria for $(1,\hat{y}), \hat{y} > \frac{a_{12}}{a_{12}+a_{21}}$.

This leads to an interesting case by comparison with the centipede game, as for the centipede game, we had that the Nash Equilibrium $(1, \frac{a_{12}}{a_{12}+a_{21}})$ was stable but not reachable, while for the chain-store game, the equilibrium is reachable but not stable.

Therefore we see that a reachable equilibrium does not need to be stable, while a stable equilibrium does not need to be reachable.

For $y < \frac{a_{12}}{a_{12}+a_{21}}$, we have the Jacobian matrix

$$\begin{pmatrix} a_{12} & 0 \\ * & b_{12} \end{pmatrix}$$

which has two negative eigenvalues and hence asymptotic stability for $E = (0, 0)$.

2.3.4. A few remarks on the Centipede and Chain-Store Game

The Centipede Game and the Chain-Store Game are the two most famous cases of degenerate 2×2 games which arise from extensive-form games. Contrary to normal-form games, the extensive-form allows explicit modeling of interactions and therefore, each player has perfect information.

The Chain-Store Game was introduced by Reinhard Selten[18] and we will also use his description of the game. We consider a fictitious market situation, where a chain store called player **A** has branches in n towns, numbered from 1 to n. In each town, there is a potential competitor who might raise money at a local bank and establish a second shop of the same kind. The competitor at town k is called player k. Thus the game has $n + 1$ players, the chain-store A and the players k, with $k = 1, \ldots, n$. As time goes on, each player will have saved enough to increase his owned capital to a required amount to establish a second shop. As soon as the time comes up for player k, he must decide whether he wants to establish a second shop, or whether he wants to use the owned capital in a different way. If he chooses latter possibility, he stops being a competitor of player A.

If a second shop is established in town k, then player A has to choose between two price policies for town k. He may respond by being 'cooperative' or 'aggressive'. Since the game is played in a non-cooperative way, no binding contracts, cartels, commitments and other issues are allowed.

player k's decision	player A's decision period k	player k's payoff	player A's partial payoff for period k
IN	COOPERATIVE	2	2
IN	AGGRESSIVE	0	0
OUT	-	1	5

Table 2.1.: Player A's partial payoffs and player k's payoff

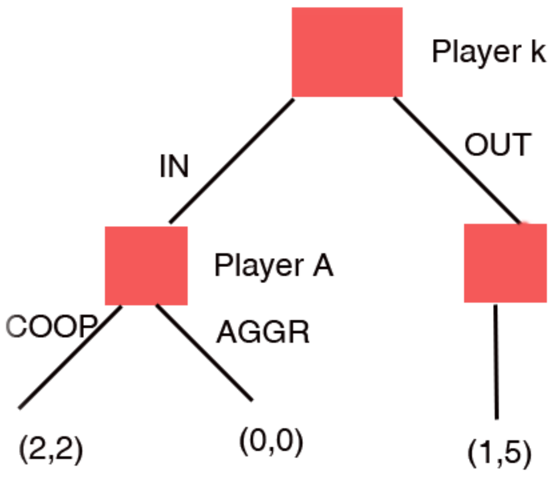

Figure 2.9.: Extensive Form of Chain-Store Game

Looking at the table, we can give the payoff matrix of the game the following way

$$P = \begin{pmatrix} (5,1) & (5,1) \\ (0,0) & (2,2) \end{pmatrix}$$

where player A's payoff is indicated by the row and player k's by the column. which for the bimatrix game gives us the respective pay-offs

$$A = \begin{pmatrix} 5 & 5 \\ 0 & 2 \end{pmatrix} \qquad B^T = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

As we can see, this game belongs to case 1d.

This game is played with the n other players too over n rounds. After each round, player A's decision is immediately made known to all other players and for $k = 1, \ldots, n-1$ period $k+1$ begins and is played accordingly. Player A's payoff is the sum of n partial payoffs for the periods $1, \ldots, n$. If we look at the game inductively, then if in period n player n selects **IN**, then the best choice for player A is to cooperate, since it yields a higher payoff. The strategic situation obviously does not depend on the players' decisions in previous periods.
Now, if we consider period $n-1$, then similarly, if player $n-1$ selects **IN**, then it is again the best strategy for A to cooperate. This way we can conclude by induction that each player k should choose **IN** and each time player A should cooperate. This way, each player k receives a payoff of 2 and A a total payoff of $2n$.
However, this does not really work that way in real life. In real life, player A can receive a higher pay off by being deterrent. As an example, if player A in the first few m rounds, with $1 < m < n$ behaves aggressively against competitors, this will scare off his competitors in other towns from stepping into the market as for them no gain or a

loss of money weighs much more than for player A, since for him this is just a partial payoff while for player k in town k it is the whole payoff. As an example, if we take $n = 30$ and $m = 6$ and from the remaining 24 towns, 5 competitors are not impressed by player A's bullying, then player A can still get a minimum payoff of $19 \times 5 = 95$, which is much higher than the 60 he would get for always cooperating. So while from an inductive standpoint it is rational for player A to cooperate in every round, it will not be a strategy played by too many businesses in reality. This phenomenon where a logical inescability is beaten by a plausible argument is called 'Chain Store Paradox' by Selten.

A similar case of a, as one might call it *"Equilibrium Fallacy"* comes with the Centipede-Game. We will use the example from Binmore[5] where he takes the example of a drug trade. We have *Bubbles* the customer and *Avon* the drug dealer. Avon has 100 grains of Heroin, Bubbles has say 100 Dollars. Bubbles does not trust Avon and Avon does not trust Bubbles (as a rule of thumb, you should neither trust a criminal nor a junkie, both have very bad reputations), but they need to pull the deal through since Avon has no use for heroin, except for selling it and Bubbles as a junkie has no use for money, except for buying heroin. Therefore, we may give the respective payoffs in the following way:

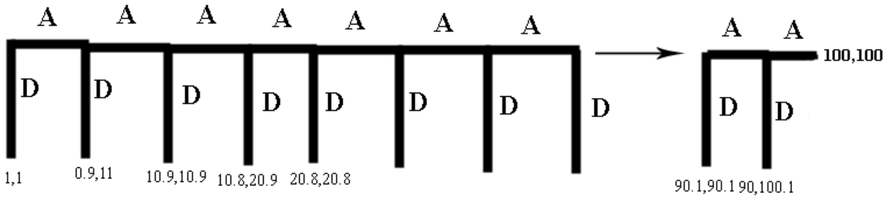

Figure 2.10.: A sketch of the extensive form of the Centipede-Game

Bubbles' payoff is given by $\pi_1 = 0.01m + h$ and Avon's is given by $\pi_2 = 0.01h + m$ where m stands for *money* and h for *heroin*. Bubbles starts first and can either draw out or play across, which would be passing over 10 Dollars for buying 10 grains of heroin. If the game is played through, each player receives a payoff of 100. Once a player draws out, the game is over and everyone goes with the current payoff at hand. However, on the last move of Avon, he has a payoff of 100.1, while Bubbles has a payoff of 90. Obviously he would have a better payoff by withdrawing and not playing the last 10 grains across. Therefore, Bubbles would be better off by not paying the last 10 dollars and staying at a payoff of 90.1. Iterating backwards, Avon has before his first move a total payoff of 11 while Bubbles has 0.9. If he plays across however, his payoff will be 10.9 and Bubbles will be at 10.9 too. Obviously 11 is larger than 10.9 and he would have a higher payoff if he withdraws. Likewise, given this, Bubbles

would be better off not playing at all and withdrawing in the beginning, meaning that his best strategy is to play D in his first move, but this would be unsatisfactory for both.
Summarized with payoff matrices, it would look the following:

Bubbles decision	Avon's decision	Bubbles payoff	Avon's payoff
DRAW	-	1	1
Always ACROSS	Always-1 ACROSS	90	100.1
Always ACROSS	Always ACROSS	100	100

Table 2.2.: Avon and Bubbles payoff after 10 deals

Bubbles decision	Avon's decision	Bubbles payoff	Avon's payoff
DRAW	-	1	1
ACROSS	DRAW	0.9	11
ACROSS	ACROSS	10.9	10.9

Table 2.3.: Avon and Bubbles payoff after 1 deal

or as a payoff-matrix:

$$P = \begin{pmatrix} (1,1) & (1,1) \\ (90, 100.1) & (100, 100) \end{pmatrix}$$

As we can see, the payoff of this matrix would be the equivalent of case *1c*.

2.4. Two Degenerate Strategies

2.4.1. 2a

We have that $b_{12}, b_{21} = 0$ and $a_{12} < 0, a_{21} > 0$. Hence $\dot{y} = 0$, and we have a continuum of Nash Equilibria for $(0, \hat{y})$ with $\hat{y} \in [0, 1]$.

$$\begin{aligned} \dot{x} &= -x(-a_{12} + (a_{12} + a_{21})y) \\ \dot{y} &= 0 \end{aligned}$$

Solving the equation, we get

$$x(t) = Ce^{(a_{12} - (a_{12} + a_{21})y)t}$$

Given that $a_{12} - (a_{12} + a_{21})y < 0$ and $y \in [0, 1]$ we have $\lim_{t \to \infty} x(t) = 0$ Hence, for $(0, \hat{y})$ we have a continuum of stable and reachable Nash equilibria.

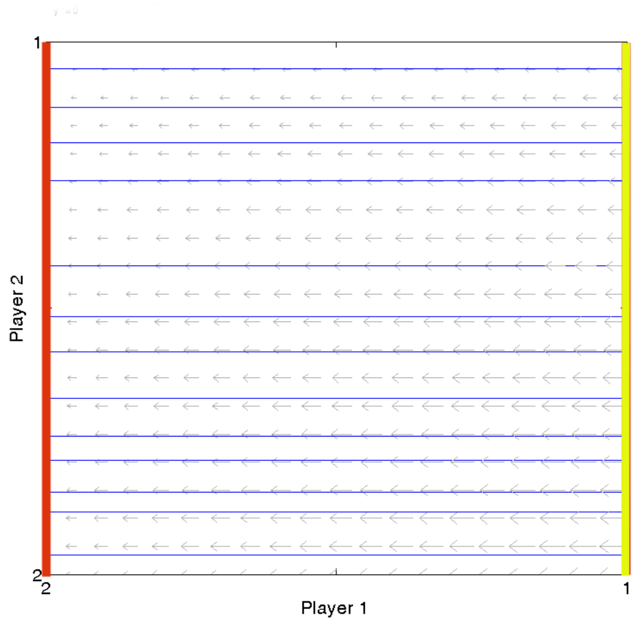

Figure 2.11.: 2a

2.4.2. 2b

Again, we have the zero payoff matrix for Player 2 as in 2a but for Player 1, the payoffs are $a_{12} > 0, a_{21} > 0$. This changes the equation to

$$\dot{x} = (1-x)(a_{12} - (a_{12} + a_{21})y)_+ - x(-a_{12} + (a_{12} + a_{21})y)_+$$
$$\dot{y} = 0$$

With the same computations as above we have stable and reachable Nash equilibria for $(1, \bar{y})$ with $\bar{y} > \frac{a_{12}}{a_{12}+a_{21}}$ and for $(0, \hat{y})$ with $\hat{y} < \frac{a_{12}}{a_{12}+a_{21}}$, while for $(\hat{x}, \frac{a_{12}}{a_{12}+a_{21}})$ $\hat{x} \in [0, 1]$ we have a continuum of Nash Equilibria that are not stable and not reachable either.

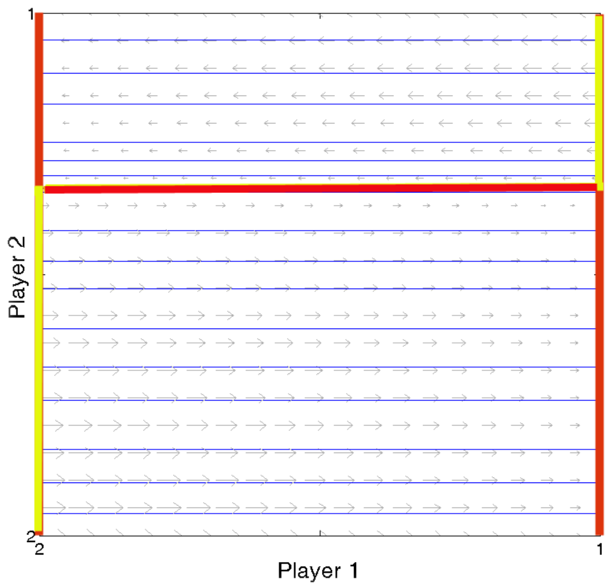

Figure 2.12.: 2b

2.4.3. 2c

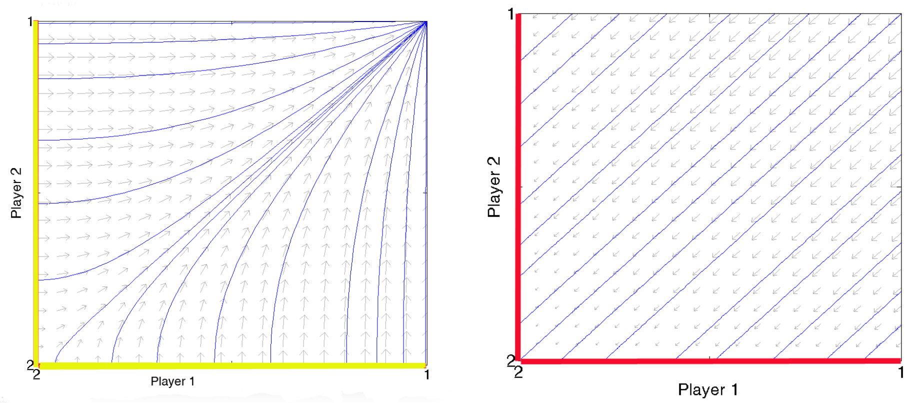

Figure 2.13.: 2c and 2d

We have for each player one degenerate strategy that is dominated, e.g. $a_{12} = 0, a_{21} < 0$ and $b_{12} = 0, b_{21} < 0$. In this case for both players strategy 2 is degenerate and strategy 1 dominates it. Hence, the equation reads

$$\dot{x} = -a_{21}(1-x)y$$
$$\dot{y} = -b_{21}(1-y)x$$

The Nash Equilibria are $(0,0)$ and $(1,1)$. We have for the equilibrium point $(1,1)$ the Jacobian matrix

$$\begin{pmatrix} a_{21} & 0 \\ 0 & b_{21} \end{pmatrix}$$

which leads to 2 eigenvalues with negative real part and therefore asymptotic stability at the equilibrium $(1,1)$. Since $\dot{x}, \dot{y} \geq 0$ we have one Nash Equilibrium $(0,0)$ that is not reachable and not stable and one asymptotically stable Nash equilibrium $(1,1)$. Given that $\omega(x,y) \subseteq \{(1,1)\}$ for all interior orbits, by A.8 $(1,1)$ is also globally asymptotically stable.

2.4.4. 2d

Again, each player has a degenerate strategy but this time the strategies are dominating, $a_{12} = 0, a_{21} > 0$ and $b_{12} = 0, b_{21} > 0$ which turns the equation to

$$\dot{x} = -a_{21}xy$$
$$\dot{y} = -b_{21}yx$$

The equilibria are $(0,\hat{y})$ and $(\hat{x},0)$ with $\hat{x},\hat{y} \in [0,1]$. By separation of variables, we get the constant of motion

$$b_{21}x = a_{21}y + C$$

and therefore for either $(0,\hat{y})$ or $(\hat{x},0)$, we have a continuum of stable Nash equilibria. However, no interior orbit leads to $(0,1)$ and $(1,0)$, as it is also visible from figure 2.13, and therefore, these 2 points are not reachable, while all other stable equilibria are.

2.4.5. 2e

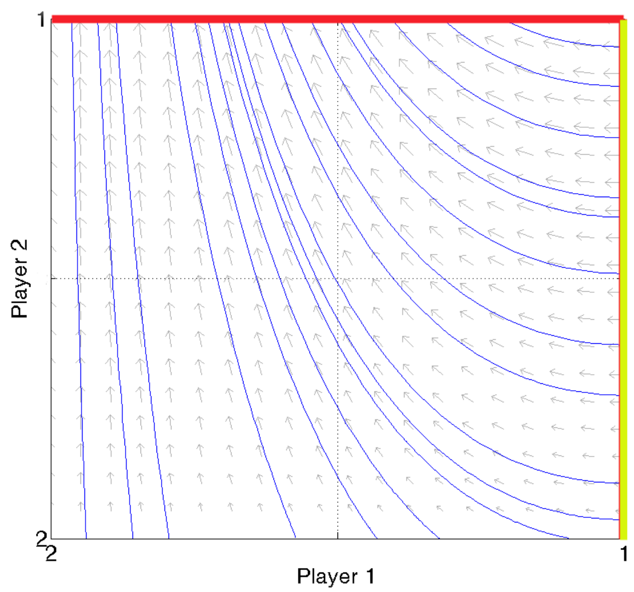

Figure 2.14.: 2e

In this case, one degenerate strategy is dominated and the other one dominant, as an example if strategy 2 of player 1 is dominant, strategy 1 of player 2 is dominant, e.g. $a_{12} < 0, a_{21} = 0$ and $b_{12} > 0, b_{21} = 0$. The equation reads

$$\dot{x} = a_{12}x(1-y)$$
$$\dot{y} = b_{12}(1-y)(1-x)$$

which leads us to Nash Equilibria for $(\hat{x}, 1)$. By finding the Jacobian for all $(\hat{x}, 1)$, we get

$$J_{(\hat{x},1)} = \begin{pmatrix} a_{21} & 0 \\ 0 & b_{21} \end{pmatrix}$$

which leads us to an eigenvalue smaller 0 and one equal 0 and hence stability.
By separation of variables, we get the level curves

$$y = \frac{b_{12}}{a_{12}}(\log(x) - x) + C \tag{2.7}$$

and by using the Liapunov-function

$$V(x, y) = x + y$$

we have a continuum of stable Nash equilibria for $(\hat{x}, 1)$ for $0 \leq \hat{x} \leq 1$. Further, all Nash equilibria, except for $(0, 1)$ and $(1, 1)$ are also reachable. $(0, 1)$ is not reachable, because the only orbit leading to it is on the edge with $x = 0$, while $(1, 1)$ is a minimum for (2.7) and therefore no interior orbit leads to it.

2.5. Three Degenerate Strategies

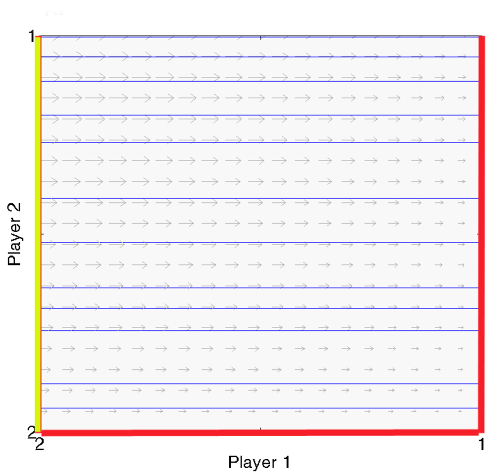

Figure 2.15.: 3

We only have one case here to look at. We take $a_{21} < 0$ and everything else 0, which leads us to the equation $\dot{x} = -a_{21}y(1-x)$ and $\dot{y} = 0$ and Nash Equilibria for $(\hat{x}, 0)$ and $(1, \hat{y})$. We get

$$x(t) = 1 - Ce^{a_{21}yt}$$

and hence a continuum of stable and reachable Nash equilibria at $(1, \hat{y})$. Likewise for $y = 0$, we also have a set of Nash Equilibria with $(\hat{x}, 0)$, however no orbit leads to any of them We can finish this chapter with a final conclusion observed from the results.

Remark 2.6. *Every* 2×2 *Bimatrix game has at least one reachable Nash Equilibrium for the PD dynamics and every quasi-strict Nash Equilibrium is reachable for the PD dynamics.*

3. BNN Dynamics

As with (2.2), the BNN dynamics turns in the 2×2 case to

$$\begin{aligned} \dot{x} &= (1-x)^2(a_1-a_2)_+ - x^2(a_2-a_1)_+ \\ \dot{y} &= (1-y)^2(b_1-b_2)_+ - y^2(b_2-b_1)_+ \end{aligned} \tag{3.1}$$

3.1. The Generic Case

As we can see, the main difference between the two equations (2.2) and (3.1) is with the quadratic terms. This also makes the case of *0a, 0b* rather uninteresting as nothing really changes. However, we will take a look at cases *0c* and *0d* as these were also the most interesting cases for the PD dynamics.

3.1.1. 0c

We use the same payoff matrices (2.1) as we used for the PD dynamics. Hence we have

$$\begin{aligned} \dot{x} &= (1-x)^2(a_{12}-(a_{12}+a_{21})y)_+ - x^2(-a_{12}+(a_{12}+a_{21})y)_+ \\ \dot{y} &= (1-y)^2(b_{12}-(b_{12}+b_{21})x)_+ - y^2(-b_{12}+(b_{12}+b_{21})x)_+ \end{aligned}$$

As we can see, the Nash Equilibrium is also the rest point $E = (\frac{b_{12}}{b_{12}+b_{21}}, \frac{a_{12}}{a_{12}+a_{21}})$ as it was for the PD dynamics.
We use again a Liapunov function inspired by Brown, von Neumann [8] and Sandholm [16]:

$$\begin{aligned} V(x,y) = &(b_{12}+b_{21})((a_1-\bar{a})_+^2 + (a_2-\bar{a})_+^2) \\ &- (a_{12}+a_{21})((b_1-\bar{b})_+^2 + (b_2-\bar{b})_+^2) \end{aligned} \tag{3.2}$$

In the first quadrant with $x < \bar{x}, \dot{x} < 0$, $y < \bar{y}, \dot{y} > 0$ with $a_2 > a_1, b_1 > b_2$ this leads to

$$\begin{aligned} V(x,y) &= (b_{12}+b_{21})(a_2-\bar{a})^2 - (a_{12}+a_{21})(b_1-\bar{b})^2 \\ &= (b_{12}+b_{21})^2(x(a_2-a_1)^2) - (a_{12}+a_{21})(1-y)^2(b_1-b_2)^2 \end{aligned}$$

for the Liapunov function. Therefore we get

$$\begin{aligned} \dot{V} =& 2(b_{12}+b_{21})(a_2-a_1)^2 x\dot{x} + 2(-(a_{12}+a_{21})(b_1-b_2)^2(1-y)(-\dot{y}) \\ &+ \underbrace{2(b_{12}+b_{21})x^2(a_2-a_1)(a_2-a_1)^{\bullet} - 2(a_{12}+a_{21})(1-y)^2(b_1-b_2)(b_1-b_2)^{\bullet}}_{=0} < 0 \end{aligned}$$

and thus global asymptotic stability for the equilibrium E.

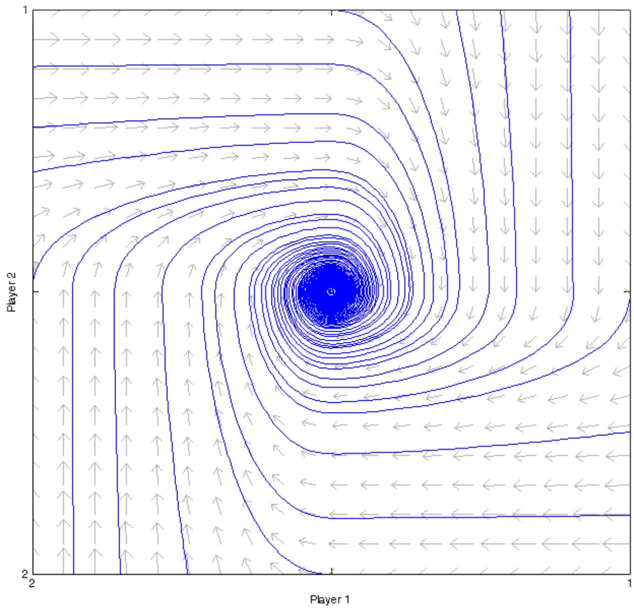

Figure 3.1.: 0c for BNN Dynamics

3.1.2. 0d

By using the same parameters as in case *0d* for the PD dynamics, we get

$$\dot{x} = (1-x)^2(a_{12} - (a_{12}+a_{21})y)_+ - x^2(-a_{12} + (a_{12}+a_{21})y)_+$$
$$\dot{y} = (1-y)^2(b_{12} - (b_{12}+b_{21})x)_+ - y^2(-b_{12} + (b_{12}+b_{21})x)_+$$

We have the same 2 strict Nash Equilibria $(0,1)$ and $(1,0)$ which are asymptotically stable and a mixed Nash Equilibrium in E. Once again, what would be interesting to know for us would be whether there is any orbit leading to the point E.

We use the same procedure as before and compute in the first quadrant

$$\frac{dx}{dy} = \frac{(1-x)^2(a_{12} - (a_{12}+a_{21})y)}{(1-y)^2(b_{12} - (b_{12}+b_{21})x)}$$

This leads us to

$$(b_{12} - b_{21})\log(1-x) - \frac{b_{21}}{(1-x)} = -(a_{12} - a_{21})\log(1-y) + \frac{a_{21}}{(1-y)} + C$$

and since E lies on a level curve, we have exactly one orbit leading to it in the first quadrant. Similarly we can find exactly one orbit leading to E from the third quadrant.

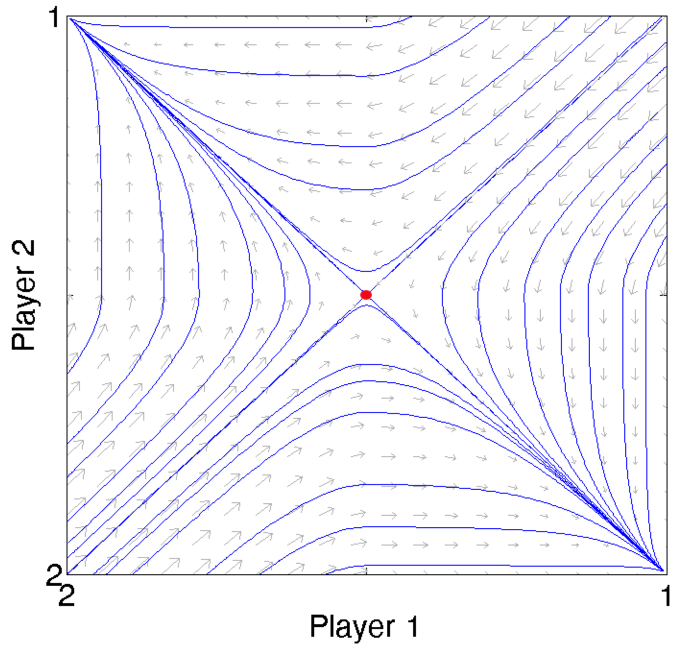

Figure 3.2.: Battle of the Sexes or 0d for BNN

3.2. One Degenerate Strategy

3.2.1. 1a

We have the same parameters as for the PD dynamics, and therefore (3.1) turns to

$$\dot{x} = (1-x)^2(a_{12} - (a_{12} + a_{21})y)$$
$$\dot{y} = -b_{21}(1-y)^2 x$$

In contrast to the PD dynamics linearization around the equilibrium point $E = (1, 1)$ does not really work well, since it leads to the 0-matrix and therefore is not hyperbolic and *Hartman-Grobman* cannot be applied. We may however use the function

$$V(x, y) = 2 - x - y$$

as a Liapunov function and we get

$$\dot{V} = -\dot{x} - \dot{y} < 0$$

for $(x, y) \neq (\bar{x}, \bar{y})$ and $\dot{V}(\bar{x}, \bar{y}) = 0$ and hence E is globally asymptotically stable.

3.2.2. 1b

Now we get to the more interesting parts of the BNN dynamics, especially if we compare it with the PD dynamics. For case 1b, we have the equations given as

$$\dot{x} = -x^2(-a_{12} + (a_{12} + a_{21})y)$$
$$\dot{y} = (1-y)^2(-b_{21}x) \quad (3.3)$$

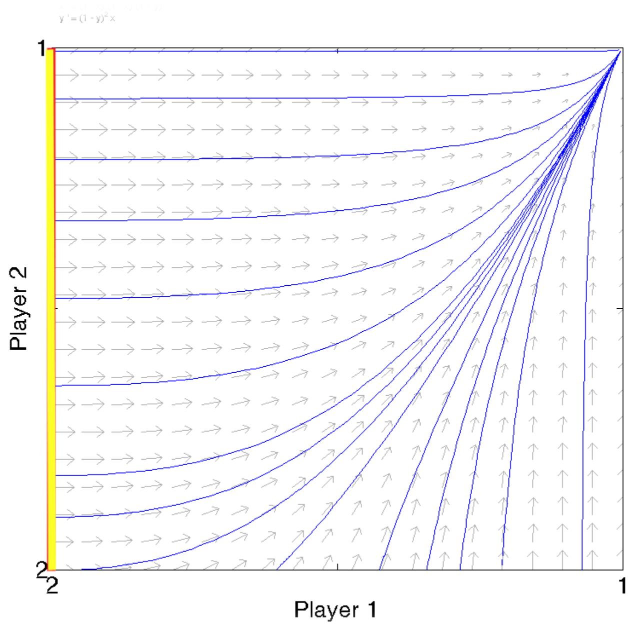

Figure 3.3.: 1a BNN

As we can see, the equilibrium points are all $(0, \hat{y})$ for $\hat{y} \in [0, 1]$. From the first look, we will have again stable equilibria on the degenerate edge. However, given that linearization does not work well with this system, since again the fixed points are not hyperbolic, we will use a proposition to simplify our system a bit.

Proposition 1. *If the function $B(\mathbf{x}, t)$ is strictly positive, then the solutions of the two differential equations $\dot{\mathbf{x}} = f(\mathbf{x}, t)$ and $\dot{\mathbf{x}} = B(\mathbf{x}, t)f(\mathbf{x}, t)$ can be transformed into each other by a strictly monotonic change in the time scale $\tau = \phi(t)$*

Proof. Let

$$\begin{aligned} \dot{x} &= f(x(t)) \\ \dot{y} &= f(y)B(y) \qquad y(t) = x(\phi(t)) \end{aligned}$$

Then $\dot{y} = x(\phi(t))\dot{\phi}(t)$ and for $\dot{\phi}(t) = B(y(t)) = B(x(\underbrace{\phi(t)}_{s})) = B(x(s))$ we get

$$\dot{y} = f(y(t))B(y(t))$$

□

By multiplying the right hand side of (3.3) with $\frac{1}{x}$ we get the new system

$$\begin{aligned} \dot{x} &= -x(-a_{12} + (a_{12} + a_{21})y) \\ \dot{y} &= (1 - y)^2(-b_{21}) \end{aligned} \tag{3.4}$$

and the same orbits for $x > 0$. This leads to the unique equilibrium point $E = (0, 1)$ with the following Jacobian matrix

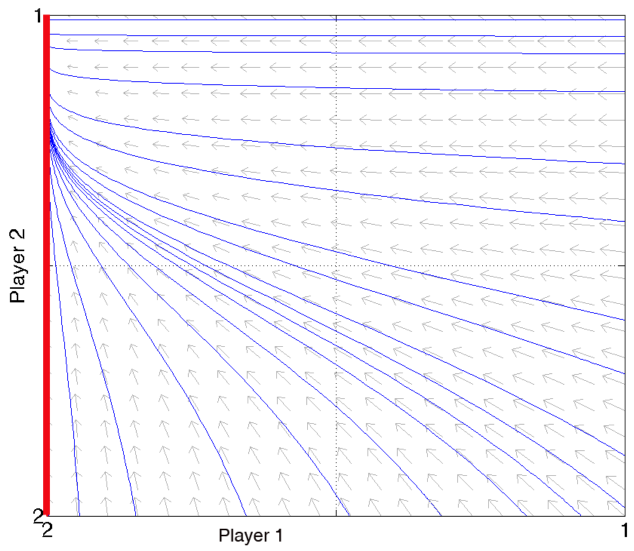

Figure 3.4.: 1b BNN

$$J_{(0,1)} = \begin{pmatrix} -a_{21} & 0 \\ * & 0 \end{pmatrix}$$

and hence one eigenvalue $-a_{21} < 0$ and one eigenvalue 0. By using the Liapunov function

$$V(x, y) = 1 - y + x$$

we have

$$\dot{V} = -\dot{y} + \dot{x} < 0 \quad \forall (x, y) \neq E$$

and we see that the ω-limit set of each interior orbit is E and we have a stable equilibrium for $E = (0, 1)$. Therefore and by the Theorem of Liapunov for ω-limits, we can say that E is the only reachable Nash Equilibrium and all interior orbits converge to E, while all $(0, \hat{y})$ are Nash Equilibria and hence restpoints.

3.2.3. 1c

The system reads

$$\begin{aligned} \dot{x} &= (1-x)^2(a_{12} - (a_{12} + a_{21})y)_+ - x^2(-a_{12} + (a_{12} + a_{21})y)_+ \\ \dot{y} &= (1-y)^2(1-x)b_{12} \end{aligned}$$

Our equilibria are all $(1, \hat{y})$ with $\hat{y} \in \left(\frac{a_{12}}{a_{12}+a_{21}}, 1\right)$. Since we do not have any equilibria for $y < \frac{a_{12}}{a_{12}+a_{21}}, \dot{x} < 0$, we may only focus on the $y \geq \frac{a_{12}}{a_{12}+a_{21}}, \dot{x} > 0$. By the same procedure as for (1b) and Proposition 1, we multiply the right hand side with $\frac{1}{1-x}$ and we get

$$\begin{aligned} \dot{x} &= (1-x)(a_{12} - (a_{12} + a_{21})y) \\ \dot{y} &= (1-y)^2 b_{12} \end{aligned}$$

This modified system has an equilibrium at $E = (1,1)$ with the Jacobian matrix

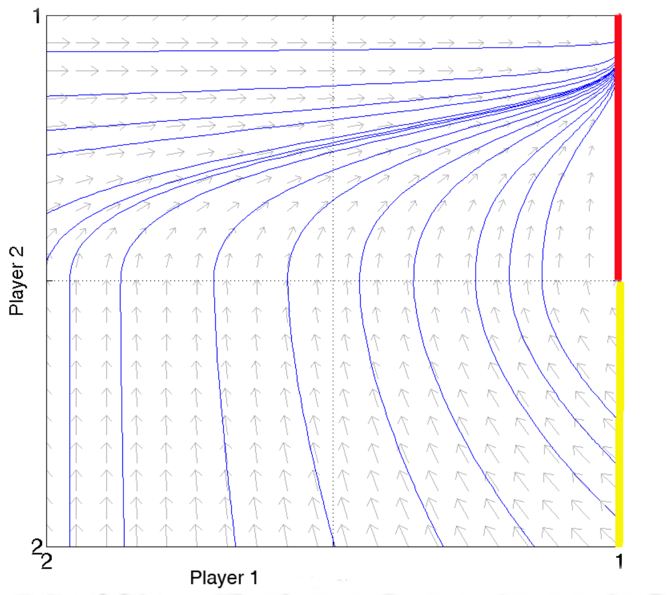

Figure 3.5.: Centipede Game for BNN

$$J_{(1,1)} = \begin{pmatrix} -a_{21} & 0 \\ * & 0 \end{pmatrix}$$

Using the Liapunovfunction

$$V(x,y) = 2 - x - y$$

we get stability for $E = (1,1)$. As in the previous case, this means that we have a set of Nash Equilibria for $(1, \hat{y})$ with $\frac{a_{12}}{a_{12}+a_{21}} \leq \hat{y} \leq 1$, while E is the only reachable Nash Equilibrium. The following example will also show that the equilibrium is also *perfect*.

Example 1. *We use the following payoff matrix*

$$P = \begin{pmatrix} (0,0) & (-1,0) \\ (-1,2) & (0,0) \end{pmatrix}$$

The payoffs are given by

$$a_1 = -(1-y) \qquad a_2 = -y$$

and

$$b_1 = 2(1-x) \qquad b_2 = 0$$

respectively. As we can see, player 2 has always a preference for strategy 1 as the payoff is always higher or equal to 0 *while player 1 has given player 2's preference a higher payoff for strategy 1. However, given that for player 2 strategy 1 is always a best reply against the other strategy, by definition of (A.5)* $(1,1)$ *is the perfect equilibrium.*

3.2.4. 1d

For the BNN dynamics of the Chain-Store Game, we have

$$\dot{x} = (1-x)^2(a_{12} - (a_{12} + a_{21})y)_+ - x^2(-a_{12} + (a_{12} + a_{21}y))_+$$
$$\dot{y} = -y(1-x)b_{12}$$

Again, we look at two parts, first for $y \geq \frac{a_{12}}{a_{12}+a_{21}}$ and then for $y \leq \frac{a_{12}}{a_{12}+a_{21}}$.
For $y \geq \frac{a_{12}}{a_{12}+a_{21}}$ we have

$$\dot{x} = (1-x)^2(a_{12} - (a_{12} + a_{21})y)$$
$$\dot{y} = -y^2(1-x)b_{12}$$

and Nash Equilibria on $(1, \hat{y})$ with $\hat{y} \in \left(\frac{a_{12}}{a_{12}+a_{21}}, 1\right)$.

By our usual procedure Proposition 1, we get the equivalent system

$$\dot{x} = (1-x)(a_{12} - (a_{12} + a_{21})y)$$
$$\dot{y} = -y^2 b_{12}$$

As we can see, we don't have any rest points with $y \geq \frac{a_{12}}{a_{12}+a_{21}}$ for this system and $x = 1$ is invariant. Hence no orbit converges to any of the Nash Equilibria on the degenerate edge $(1, \hat{y} \geq \frac{a_{12}}{a_{12}+a_{21}})$.

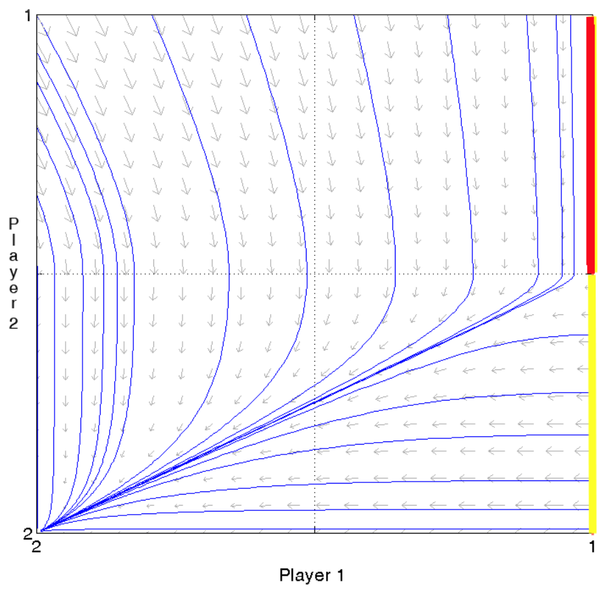

Figure 3.6.: Chain-Store Game for BNN dynamics

For $y \leq \frac{a_{12}}{a_{12}+a_{21}}$ we have

$$\dot{x} = -x^2(-a_{12} + (a_{12} + a_{21}y)) < 0$$
$$\dot{y} = -y^2(1-x)b_{12} < 0$$

and a non-hyperbolic equilibrium at $E = (0,0)$.
By using the Liapunov function

$$V(x,y) = x + y$$

we see that E is asymptotically stable. Summarizing, all interior orbits converge to E while we have a set of non-reachable and not stable Nash Equilibria for $(1,\hat{y})$ with $\hat{y} \geq \frac{a_{12}}{a_{12}+a_{21}}$.

3.3. Two and Three Degenerate Strategies

We leave out the cases *2a,2b and 3* since in each case, the BNN dynamics differs from the PD dynamics only by a quadratic term for one player and gives us the same qualitative results as for the PN dynamics. The other 3 cases however turn out quite interesting.

3.3.1. 2c

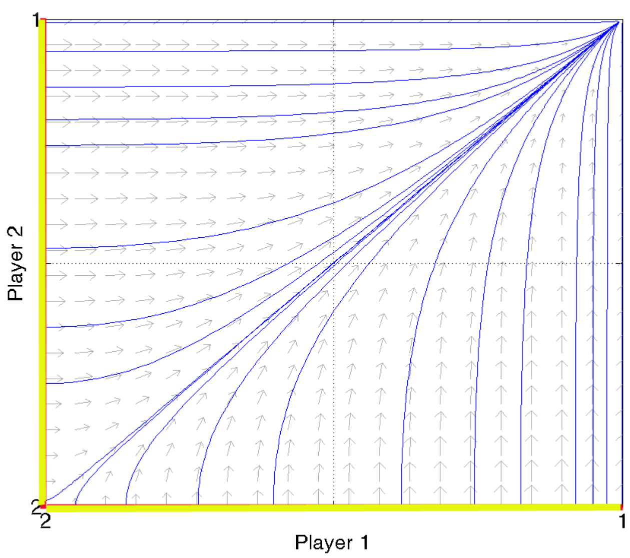

Figure 3.7.: 2c BNN

Again we have that for each player one strategy is degenerate and the degenerate strategy is dominated, e.g. $a_{12} = 0, a_{21} < 0$ and $b_{12} = 0, b_{21} < 0$. In this case for both players strategy 2 is degenerate and Strategy 1 dominates it. Hence, the equation reads

$$\dot{x} = -a_{21}y(1-x)^2$$
$$\dot{y} = -b_{21}x(1-y)^2$$

where we get two Nash equilibria, one at $(0,0)$ and one at $(1,1)$.

We use the Liapunov function $L(x,y) = x$, which leads us to

$$\dot{L}(x,y) = \dot{x} > 0 \qquad \forall (x,y) \neq (1,1) \text{ or } (0,0)$$

and hence we have asymptotic stability for $E = (1,1)$ and all interior orbits lead to E, while $(0,0)$ is not reachable and not stable.

Example 2. *In order to show that* $(0,0)$ *is dominated, we can look at the following example with the payoff matrix*

$$P = \begin{pmatrix} (0,0) & (0,-1) \\ (-1,0) & (0,0) \end{pmatrix}$$

As we can see, the payoffs are given by

$$a_1 = 0 \qquad a_2 = -y$$

and

$$b_1 = 0 \qquad b_2 = -x$$

respectively. As we can see, strategy 2 for each player is never a best reply to strategy 1, while strategy 1 is always a best reply and hence a perfect equilibrium.

3.3.2. 2d

For $a_{12} = 0, a_{21} > 0$ and $b_{12} = 0, b_{21} > 0$ we have

$$\dot{x} = -a_{21}yx^2$$
$$\dot{y} = -b_{21}xy^2$$

Again, as in (1) we multiply the right hand side with $\frac{1}{xy}$ and get for the new system

$$\dot{x} = -a_{21}x$$
$$\dot{y} = -b_{21}y$$

which leads us to the equilibrium $E = (0,0)$.

By either using the Jacobian matrix

$$\begin{pmatrix} -a_{21} & * \\ 0 & -b_{21} \end{pmatrix}$$

which leads us to two negative eigenvalues or by using the Liapunov function $L(x,y) = x$ we see that E is stable. Here we see that the dynamics differs from the PD dynamics, as in the PD dynamics we had two edges of reachable equilibria, while now we have one single reachable Nash Equilibrium.

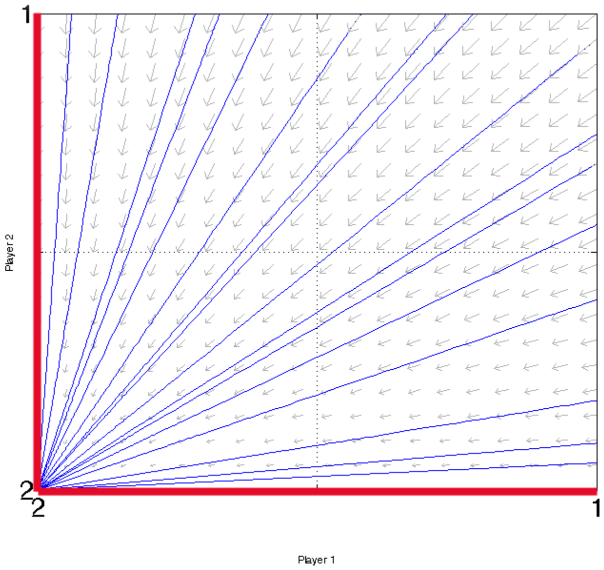

Figure 3.8.: 2d BNN

3.3.3. 2e

For one player the degenerate strategy is dominated and for the opposing player the degenerate strategy is dominant. As an example if strategy 1 of player 1 is dominant, strategy 2 of player 2 is dominant, e.g. $a_{12} = 0, a_{21} > 0$ and $b_{12} < 0, b_{21} = 0$. The equation reads

$$\dot{x} = -a_{21}(1-y)x^2$$
$$\dot{y} = b_{12}(1-x)(1-y)^2$$

We use again Proposition 1 and this time multiply with $\frac{1}{1-y}$, and we get

$$\dot{x} = -a_{21}x^2$$
$$\dot{y} = b_{12}(1-x)(1-y)$$

As we can see, the only equilibrium point of the new system is $E = (0, 1)$ and by using the Liapunov function $L(x, y) = 1 - y + x$, we have that

$$\dot{L}(x, y) = \dot{x} + \dot{y} < 0 \qquad (x, y) \neq (0, 1)$$

and therefore E is stable and all interior orbits converge to E. All other Nash Equilibria $(\hat{x}, 1)$ have no interior orbit converging to them.

Once again, the case for the BNN dynamics differs from the PD dynamics and we have only one reachable Nash Equilibrium.

As a conclusion we may remark the following three results.

Remark 3.4. *A pure Nash Equilibrium is asymptotically stable under the BNN dynamics if and only if it is strict.*

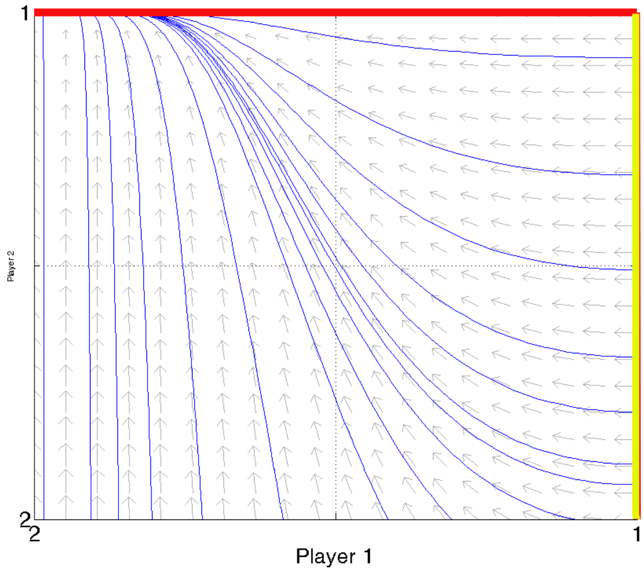

Figure 3.9.: 2e BNN

Remark 3.5. *All perfect equilibria are reachable for the BNN dynamics..*

Remark 3.6. *Every* 2×2 *bimatrix game has at least one reachable Nash Equilibrium for the BNN dynamics.*

4. Stability through Bendixson-Dulac

Generally, there is a much simpler way in 2 dimensions to prove the existence or non-existence of periodic orbits and stability of fixed points than by looking for Liapunov functions, which is through the *Poincaré-Bendixson Theorem* and the *Bendixson-Dulac Theorem*.

Theorem 4.1. (Poincaré-Bendixson)
Suppose that Ω is a ***nonempty, closed and bounded*** *limit set of an orbit of a planar system of differential equations that contains no equilibrium point. Then Ω is a closed orbit.*

A proof of this theorem can be found in [11]. In order to predict the absence of periodic orbits we need the *Bendixson-Dulac Theorem* which is also known as *Bendixson Theorem* or *Bendixson-Dulac criteria or method*[14].

Theorem 4.2. (Bendixson-Dulac)
Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be C^1 and consider the differential equation $\dot{x} = f(x)$. If $\operatorname{div} f < 0$ throughout the simply connected set Γ then Γ does not contain a closed orbit.

As we can see, we face a problem concerning C^1 for both *PD* and *BNN* since neither dynamics is C^1. However, both dynamics are Lipschitz continuous and their orbits are C^1 and by finding a way to prove the *Gauss-Green-Theorem* for Lipschitz-continuous functions, we may actually use the *Bendixson-Dulac Theorem* to prove stability. Fortunately, such a theorem exists, and was proved by Federer[10]. Federer has given a general proof for higher dimensions. We will use the Theorem for $n = 2$.

Theorem 4.3. (The Gauss-Green-Federer Theorem)
Let Γ be a compact region of $\mathbb{R}^2$ with the positively orientated C^1 boundary γ. Then for any Lipschitz vector field $f(x) = (f_1(x_1, x_2), f_2(x_1, x_2))$

$$\int_\gamma f_1 dx_1 + f_2 dx_2 = \int_\Gamma \left[\frac{\partial f_2}{\partial x_1}(x_1, x_2) - \frac{\partial f_1}{\partial x_2}(x_1, x_2) \right] dx_2 dx_1 \tag{4.1}$$

A proof of this theorem can be found in [10]. However, we will provide a simple proof of the theorem in a special case where Γ is a rectangle.

First, we need a couple of additional theorems to help us with the proof.

Definition 4.4. *(Absolute continuity)*
Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}(\text{or } \mathbb{R}^n)$. f is **absolutely continuous** on I if for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever a sequence (x_k, y_k) of pairwise disjoint subintervals of I satisfies $\sum_k |y_k - x_k| < \delta$ then $\sum_k |f(y_k) - f(x_k)| < \epsilon$.

Lemma 4.5. *Every Lipschitz continuous function f is absolutely continuous.*

Proof. For f defined over an interval $I = [a, b]$ and $(x, y) \in I$, we have that

$$|f(y) - f(x)| < L|y - x|$$

for a Lipschitz-constant $L \in \mathbb{R}_+$. For $\epsilon > 0$ let $\delta = \frac{\epsilon}{L}$ and

$$a \leq x_1 \leq y_1 \leq x_2 \leq y_2 \leq \ldots \leq x_n \leq y_n \leq b$$

Thus we have that

$$\sum_k |f(y_k) - f(x_k)| \leq \sum_k L|y_k - x_k| = L \sum_k |y_k - x_k| < L\delta = \epsilon$$

. □

Theorem 4.6. (Fundamental Theorem of Calculus) *For F absolutely continuous on I, we have that F is differentiable almost everywhere on I, $f(x) = F'(x)$ is defined almost everywhere and L^1 and for all $x \in I$ we have $F(x) = F(a) + \int_a^x f(t)dt$. Conversely, for $f \in L^1$, $F(x) = \int_a^x f(t)dt$ and therefore F is absolutely continuous and $F(x)' = f(x)$ almost everywhere.*

In order to have a stronger version of the theorem, we may also write down the case for *Lipschitz-continuous functions.*

Theorem 4.7. *For F Lipschitz on I, we have that F is differentiable almost everywhere on I, $f(x) = F'(x)$ is L^∞ and for all $x \in I$ we have $F(x) = F(a) + \int_a^x f(t)dt$. Conversely, for $f \in L^\infty$, F is Lipschitz continuous, $F(x) = \int_a^x f(t)dt$ and $F'(x) = f(x)$ almost everywhere.*

Now we prove Theorem 4.3 for a rectangle Γ.

Proof. Let Γ be a rectangle given by the 4 points $\alpha_1 = (a_1, b_1)$, $\alpha_2 = (a_2, b_1)$, $\alpha_3 = (a_2, b_2)$, $\alpha_4 = (a_1, b_2)$, with $a_1 \leq a_2$ $b_1 \leq b_2$. The boundary of Γ, B is given in 4 parts by the 4 sides of the rectangle, namely

$$B_1 = \overline{\alpha_1\alpha_2}, B_2 = \overline{\alpha_2\alpha_3}, B_3 = \overline{\alpha_3\alpha_4}, B_4 = \overline{\alpha_4\alpha_1}$$

For B_1 we have that $x_2 = b_1$ and $dx_2 = 0$ and we integrate over x_1. Likewise, we simplify for B_2, B_3, B_4. Therefore, we get

$$\begin{aligned} &\int_B f_1 dx_1 + f_2 dx_2 = \sum_{n=1}^{4} \int_{B_n} f_1 dx_1 + f_2 dx_2 = \\ &= \int_{a_1}^{a_2} f_1(x_1, b_1)dx_1 + \int_{b_1}^{b_2} f_2(a_2, x_2)dx_2 + \int_{a_2}^{a_1} f_1(x_1, b_2)dx_1 + \int_{b_2}^{b_1} f_2(a_1, x_2)dx_2 \end{aligned} \tag{4.2}$$

Figure 4.1.: Rectangle Γ

which leads to

$$\int_{a_1}^{a_2} \left[f_1(x_1, b_1) - f_1(x_1, b_2) \right] dx_1 + \int_{b_1}^{b_2} \left[f_2(x_2, a_2) - f_2(x_2, a_1) \right] dx_2 \tag{4.3}$$

By using the Fundamental Theorem of Calculus (4.7) for each integrand, we get

$$f_1(x_1, b_1) - f_1(x_1, b_2) = \int_{b_2}^{b_1} \frac{\partial f_1}{\partial x_2}(x_1, x_2) dx_2 = - \int_{b_1}^{b_2} \frac{\partial f_1}{\partial x_2}(x_1, x_2) dx_2$$

since f is differentiable almost everywhere. By doing the same as above for the other integrand, we finally get

$$\begin{gathered} \int_{a_1}^{a_2} \left[f_1(x_1, b_1) - f_1(x_1, b_2) \right] dx_1 + \int_{b_1}^{b_2} \left[f_2(x_2, a_2) - f_2(x_2, a_1) \right] dx_2 \\ \int_{a_1}^{a_2} - \int_{b_1}^{b_2} \frac{\partial f_1}{\partial x_2}(x_1, x_2) dx_2 dx_1 + \int_{b_1}^{b_2} \int_{a_1}^{a_2} \frac{\partial f_2}{\partial x_1}(x_1, x_2) dx_1 dx_2 \end{gathered}$$

which leads us to

$$\int_{a_1}^{a_2} \int_{b_1}^{b_2} \left[\frac{\partial f_2}{\partial x_1}(x_1, x_2) dx_1 dx_2 - \frac{\partial f_1}{\partial x_2}(x_1, x_2) \right] dx_2 dx_1$$

$\square$

With that proved, we can go on and prove the *Bendixson-Dulac Theorem* for Lipschitz-continuous functions.

Theorem 4.8. (Bendixson-Dulac for Lipschitz-continuous functions)
Suppose G is a simply connected domain with piecewise C^1 boundary in $\mathbb{R}^2$ and $f(x)$ is a Lipschitz-continuous vector field on G such that $\operatorname{div} f(x) < 0$ almost everywhere. Then G contains no closed trajectories of $\dot{x} = f(x)$.

Proof. Suppose there exists a closed orbit $\gamma = \{x(t) : t \in \mathbb{R}\}$ and Γ is its interior. Following the *Gauss-Green-Federer Theorem*, we have that

$$\int_\Gamma \operatorname{div} f(x) d(x_1, x_2) = \pm \int_0^T (f_2(x(t))\dot{x_1}(t) - f_1(x(t))\dot{x_2}(t))dt$$

where T is the period of γ. Therefore, the left-hand side is negative, while the right-hand side is 0, since $f_1 = \dot{x_1}$ and $f_2 = \dot{x_2}$ and we have a contradiction. □

Example 3. *For the PD dynamics we have*

$$\dot{x} = (1-x)(a_{12} - (a_{12}+a_{21})y)_+ - x(-a_{12} + (a_{12}+a_{21})y)_+$$
$$\dot{y} = (1-y)(b_{12} - (b_{12}+b_{21})x)_+ - y(-b_{12} + (b_{12}+b_{21})x)_+$$

Applying the Bendixson-Dulac Theorem, we get

$$\frac{df_1}{dx_1} = \frac{\partial \dot{x}}{\partial x} = -(a_{12} - (a_{12}+a_{21})y)_+ - (-a_{12} + (a_{12}+a_{21})y)_+$$
$$\frac{df_2}{dx_2} = \frac{\partial \dot{y}}{\partial y} = -(b_{12} - (b_{12}+b_{21})x)_+ - (-b_{12} + (b_{12}+b_{21})x)_+$$

and therefore, since $\frac{df_1}{dx_1} < 0$ almost everywhere except for $y = \frac{a_{12}}{a_{12}+a_{21}}$ and $\frac{df_2}{dx_2} < 0$ almost everywhere except for $x = \frac{b_{12}}{b_{12}+b_{21}}$, no periodic orbit exists. Further, by 4.1, since the ω-limit set is not a closed orbit, we have an interior fixed point where all orbits converge to and therefore by A.8, we have a globally asymptotically stable equilibrium.

Example 4. *Similarly for the BNN dynamics we have*

$$\dot{x} = (1-x)^2(a_{12} - (a_{12}+a_{21})y)_+ - x^2(-a_{12} + (a_{12}+a_{21})y)_+$$
$$\dot{y} = (1-y)^2(b_{12} - (b_{12}+b_{21})x)_+ - y^2(-b_{12} + (b_{12}+b_{21})x)_+$$

Applying Bendixson-Dulac, we get

$$\frac{df_1}{dx_1} = \frac{\partial \dot{x}}{\partial x} = -2(1-x)(a_{12} - (a_{12}+a_{21})y)_+ - 2x(-a_{12} + (a_{12}+a_{21})y)_+ < 0$$
$$\frac{df_2}{dx_2} = \frac{\partial \dot{y}}{\partial y} = -2(1-y)(b_{12} - (b_{12}+b_{21})x)_+ - 2y(-b_{12} + (b_{12}+b_{21})x)_+ < 0$$

and we have no periodic orbit. Likewise for the BNN dynamics, the interior fixed point is globally asymptotically stable.

As a conclusion, we can see that neither the PD dynamics nor the BNN dynamics allow periodic orbits around the interior equilibrium.

5. Computing Quadratic forms as Liapunov functions

Given that we will not always be lucky enough to find a Liapunov function somewhere, there is a different way to build such a function $L(x, y)$ fulfilling the requirements for both PD and BNN dynamics, by using a transformation of variables. We will see the relationship of that function with the Jacobian matrix of the dynamics and also how it is related to the Liapunov function of Sandholm. We consider here only the case of 0c, with $a_{12}, a_{21} < 0$ and $b_{12}, b_{21} > 0$, first for the PD dynamics and later for the BNN dynamics.

5.1. Constructing a Liapunov function for the PD-dynamics

We will try to find a quadratic form as Liapunov function

$$L(u, v) = \alpha_1 \frac{u^2}{2} + \beta_1 \frac{v^2}{2} \tag{5.1}$$

with $\alpha_1, \beta_1 > 0$ fulfilling the criteria for Liapunov stability. In the first quadrant with $x < \bar{x}$ and $y < \bar{y}$ transforming x and y to u and v, we get

$$\begin{aligned}
\dot{u} &= (u + \bar{x})(a_{12} + a_{21})v \\
\dot{v} &= -(1 - \bar{y} - v)(b_{12} + b_{21})u
\end{aligned}$$

with $u = x - \bar{x}$ and $v = y - \bar{y}$. Hence after the transformation $u < 0, v < 0$.

We need to find the right parameters for α_1 and β_1. We have that

$$\begin{aligned}
\dot{L}(u, v) &= \alpha_1 u\dot{u} + \beta_1 v\dot{v} \\
&= -\alpha_1(u + \bar{x})(a_{12} + a_{21})uv - \beta_1(1 - \bar{y} - v)(b_{12} + b_{21})uv
\end{aligned}$$

In order for $\dot{L}$ to be < 0 and given that $uv > 0$, we need

$$-\alpha_1(u + \bar{x})(a_{12} + a_{21}) - \beta_1(1 - \bar{y} - v)(b_{12} + b_{21}) < 0$$

By using $\bar{x} = \frac{b_{12}}{b_{12}+b_{21}}$, $\bar{y} = \frac{a_{12}}{a_{12}+a_{21}}$ and $D = \frac{b_{12}+b_{21}}{a_{12}+a_{21}}$ we have that

$$-\alpha_1(u(a_{12} + a_{21}) + \frac{b_{12}}{D}) - \beta_1(a_{21}D - v(b_{12} + b_{21}) < 0$$

Since $-\alpha_1 u(a_{12}+a_{21}) + \beta_1 v(b_{12}+b_{21}) < 0$, we have to find $\alpha_1 > 0$ and $\beta_1 > 0$ in such a way, that

$$-\alpha_1 \frac{b_{12}}{D} = \beta_1 a_{21} D$$

and hence $\alpha_1 = a_{21}D$ and $\beta_1 = -\frac{b_{12}}{D}$.

If we look back at (5.1) we see that

$$\dot{u} = \frac{b_{12}}{D} v \qquad \dot{v} = -a_{21} D u \tag{5.2}$$

the constants can be found in the Jacobian. For the other 3 quadrants it works analogously. Further, one can easily calculate that the function is continuous in all four quadrants and the level sets are closed curves.
The question remaining is about the relationship between the two Liapunov functions. For (2.5) we have in the first quadrant that

$$V(u,v) = \frac{1}{2}((u+\bar{x})(a_{12}+a_{21})^2 v^2 + (1-\bar{y}-v)(b_{12}+b_{21})^2 u^2)$$

By leaving the mixed terms out, this leads to the following form

$$V^*(u,v) = \frac{1}{2}(a_{21}(b_{12}+b_{21})Du^2 + \frac{b_{12}}{D}(a_{12}+a_{21})v^2)$$

where we can see that

$$V^*(u,v) = \frac{1}{2}(\bar{y}(b_{12}+b_{21})^2 u^2 + \bar{x}(a_{12}+a_{21})^2 v^2)$$

and

$$L(u,v) = \frac{1}{2}(\bar{y}(b_{12}+b_{21})u^2 - \bar{x}(a_{12}+a_{21})v^2)$$

hence $V^* = -(a_{12}+a_{21})(b_{12}+b_{21})L$

5.2. Liapunov Function for the BNN-dynamics

We use a similar trick for the BNN-dynamics. Again, we look for a function

$$L(u,v) = \alpha_1 \frac{u^2}{2} + \beta_1 \frac{v^2}{2} \tag{5.3}$$

that fulfills the criteria for Liapunov stability. As we know, one cannot linearize the BNN dynamics, so trying to find a relationship with the Jacobian makes little sense. However, we will still use the same transformation of variables as in the previous case with

$$u = x - \bar{x}$$
$$v = y - \bar{y}$$

This turns our dynamics in the first quadrant to

$$\begin{aligned} \dot{u} &= -(u+\bar{x})^2(a_{12}+a_{21})v \\ \dot{v} &= -(1-\bar{y}-v)^2(b_{12}+b_{21})u \end{aligned} \tag{5.4}$$

Therefore, we get

$$\begin{aligned} \dot{L}(u,v) &= \alpha_1 u\dot{u} + \beta_1 v\dot{v} \\ &= -\alpha_1 u(u+\bar{x})^2(a_{12}+a_{21})v - \beta_1(1-\bar{y}-v)^2(b_{12}+b_{21})u \\ &= -uv(\alpha_1(u+\frac{b_{12}}{b_{12}+b_{21}})^2(a_{12}+a_{21}) + \beta_1(\frac{a_{21}}{a_{12}+a_{21}}-v)^2(b_{12}+b_{21})) \end{aligned}$$

Given that $-uv<0$ we have to show that the bracket is >0. Therefore we say

$$\alpha_1(a_{12}+a_{21})(u^2+2\frac{b_{12}}{b_{12}+b_{21}}u+\frac{b_{12}^2}{(b_{12}+b_{21})^2})+\beta_1(b_{12}+b_{21})(\frac{a_{21}^2}{(a_{12}+a_{21})^2}-2\frac{a_{21}}{a_{12}+a_{21}}v+v^2)>0$$

For $u,v\leq 1$ we have that

$$\alpha_1(a_{12}+a_{21})(u^2+2\frac{b_{12}}{b_{12}+b_{21}}u)+\beta_1(b_{12}+b_{21})(-2\frac{a_{21}}{a_{12}+a_{21}}v+v^2)>0$$

because since $\alpha_1(a_{12}+a_{21})<0$ we need to look at

$$u^2+2\frac{b_{12}}{b_{12}+b_{21}}u = \qquad u(u+2\frac{b_{12}}{b_{12}+b_{21}})$$

and with $u+2\frac{b_{12}}{b_{12}+b_{21}}$ never smaller than 0 (say $u+2\frac{b_{12}}{b_{12}+b_{21}}<0$ then $u<-2\frac{b_{12}}{b_{12}+b_{21}}$ and therefore $x<-\frac{b_{12}}{b_{12}+b_{21}}$ which contradicts $x>0$) we only need to look at

$$\alpha_1(a_{12}+a_{21})\frac{b_{12}^2}{(b_{12}+b_{21})^2}+\beta_1(b_{12}+b_{21})\frac{a_{21}^2}{(a_{12}+a_{21})^2}=0$$

which equals

$$\alpha_1\frac{b_{12}}{D}\bar{x}=-\beta_1 a_{21}\bar{y}D$$

which leads to

$$\alpha_1 = a_{21}\bar{y}D \qquad \beta_1 = -\frac{b_{12}}{D}\bar{x}$$

It works analogously for the other 3 quadrants. Overall, the Liapunov function is given by

$$L(u,v)=\lambda_1\bar{y}D\frac{u^2}{2}+\lambda_2\frac{\bar{x}}{D}\frac{v^2}{2} \tag{5.5}$$

with λ_1 and λ_2 depending on the quadrants. Again, by calculations, one can reach the conclusion that the function is continuous and the level sets are closed curves.

6. Conclusion and Further Outlook

Conclusion

We have studied all cases of 2×2 games for the PD and BNN dynamics. Overall we may summarize with these main results.

- For both dynamics, just like for all innovative dynamics, the rest points or stationary points are equal to the Nash Equilibria.
- For the PD dynamics all quasi-strict Nash Equilibria are reachable. This behavior bears similarities with the Replicator dynamics(see [9], [12]).
- For the BNN dynamics the perfect equilibria are reachable, similar as in the case for the Best-response dynamics [9], [12]. This is quite surprising as from an intuitive viewpoint by looking at the equations, one might suggest that the BNN dynamics would behave more like the Replicator dynamics, while the PD dynamics would behave more like the Best-response dynamics.
- For both dynamics a pure Nash Equilibrium is asymptotically stable iff it is strict.
- Both dynamics do not have periodic orbits around interior equilibria.

Outlook

Several questions arise from the work and are left open. The obvious question would be about the behavior of the dynamics for $2 \times n$ games. Obviously, the number of equivalence classes of $2 \times n$ games is much larger than the number of 2×2 games so it would lead to a more extensive analysis. Berger proves in [2] the convergence of BR dynamics to equilibria in $2 \times n$ games. It is tempting to conjecture that the same holds for BNN and PD dynamics. What would be of further interest in higher dimensions is the behavior of the perfect and quasi-strict Nash Equilibria and whether they would still be reachable or not for the BNN- and PD dynamics respectively.

A. Additional Definitions and Theorems

Game Theory

We start with some definitions and results from Game Theory.

Remark A.1. *The best reply or best response is the strategy(or strategies) which produces the most favorable outcome for a player, taking other players' strategies as given.*

Definition A.2. (Nash Equilibrium)
Let (A, B) be a Bimatrix game. $E = (\hat{x}, \hat{y})$ is a *Nash Equilibrium*(NE), if

$$\begin{aligned} \hat{x}A\hat{y} &\geq xA\hat{y} \qquad \forall x \in \Delta_n \\ \hat{y}B\hat{x} &\geq yB\hat{x} \qquad \forall y \in \Delta_m \end{aligned} \tag{A.1}$$

Definition A.3. (Strict Nash Equilibrium)
A *Nash Equilibrium* is considered *strict* if it is the unique best-reply to itself i.e. for $(x, y) \in \Delta_n \times \Delta_m$

$$\begin{aligned} \hat{x}A\hat{y} &> xA\hat{y} \qquad \forall x \neq \hat{x} \\ \hat{y}B\hat{x} &> yB\hat{x} \qquad \forall y \neq \hat{y} \end{aligned}$$

Definition A.4. (Quasi-strict Nash Equilibrium)
A Nash Equilibrium (x, y) is considered quasi-strict if not used strategies are inferior replies, i.e. for $x_i = 0$, i is not a *best reply* to y and $y_j = 0$ implies j is not a *best response* to x.

Definition A.5. (Perfect Equilibrium)
$\hat{z}$ is a *perfect equilibrium* if for a sequence $z^k \in \text{int}(\Delta_n \times \Delta_m)$ $z^k \to \hat{z}, k \to \infty$ and $\hat{z}$ is a best reply against z^k $\forall k = 1, 2, \ldots$.

Ordinary Differential Equations

We will look at results for Ordinary Differential Equations given by $\dot{x} = f(x)$ with the solution $x(t)$.

Definition A.6. (ω-limit)
The ω-limit of $\mathbf{x}$ is the set of all accumulation points of $\mathbf{x}(t)$, for $t \to +\infty$: $\omega(x) = \{y \in \mathbb{R}^n : x(t_k) \to y \text{ for some sequence } t_k \to +\infty\}$

Remark A.7. *If the orbit remains in some compact set K, then every sequence $x(t_k)$ must admit accumulation points and $\omega(x)$ cannot be empty.*

Definition A.8. (Stability and Asymptotic Stability)
A rest point z of an ODE $\dot{x} = f(x)$ is said to be *stable*, if for any neighborhood U of z, there exists a neighborhood W of z such that any orbit through W remains in U for all $t > 0$.
It is said to be *asymptotically stable* if, in addition, such orbits converge to z, (i.e. $x(t) \to z \quad \forall x \in W$).
The set of points x with $x(t) \to z$ as $t \to +\infty$ is called the *basin of attraction* of z. If z is asymptotically stable then it is an open invariant set. If it is the whole state space, or at least its interior, then z is said to be *globally asymptotically stable*.

Definition A.9. (Liapunov's Theorem for ω-limits) Let $\dot{x} = f(x)$ be a time-independent ODE defined on some subset G of $\mathbf{R}^n$. Let $V : G \to \mathbf{R}$ be continuously differentiable. If for some solution $x(t)$ the derivative $\dot{V}$ of the map $t \to V(x(t))$ satisfies the inequality $\dot{V} \leq 0$ (or $\dot{V} \geq 0$) for all $t \geq 0$, then $\omega(x) \bigcap G$ is contained in the set $\left\{x \in G : \dot{V}(x) = 0\right\}$.

Definition A.10. (Reachability) An equilibrium point $E = (\bar{x}, \bar{y})$ is *reachable* under a given dynamics if there exists an interior orbit $(x(t), y(t))$ of the dynamics with $x(t) \to \bar{x}$ and $y(t) \to \bar{y}$ for $t \to \infty$.

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List of Figures

All figures and phase portraits done with pplane 7 for MATLAB.

Curriculum Vitae

I was born on June 4th, 1982 in Vienna to Mr. Abdolmajid Rahimi and Mrs. Shahpar Rahimi M.D.. After spending 5 years in Teheran between 1982 and 1987, I moved back to Vienna at the age of 5 and did my Matura in 2000 at the De La Salle Schule Strebersdorf. After 1 year in the academic wilderness, I started mathematics at the University of Vienna in 2001. Between 2004 and 2005, I had to commit to the alternative military service at the Donauklinikum Gugging and put my studies on hold.
My leisure activities are centered around football, basketball, photography, traveling, music and cinema.