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DIPLOMARBEIT

Relative Oscillation Theory for Dirac Operators

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Abstract

We investigate relative oscillation theory for one-dimensional Dirac operators H_0, H_1 with separated boundary conditions. Our main results read:

If $\lambda \in \mathbb{R}$ and $\lambda < \inf \sigma_{ess}(H_0)$, then the difference of the the dimensions of the spectral projections $P_{H_1}(-\infty, \lambda)$ and $P_{H_0}(-\infty, \lambda]$ equals the number of weighted sign flips of $W(\psi_{1,\mp}(\lambda_1), \psi_{0,\pm}(\lambda_1))$, where $\psi_{0,\pm}$ respectively $\psi_{1,\pm}$ denotes the corresponding Weyl solutions and W the Wronskian. Moreover, if $\lambda_0, \lambda_1 \in \mathbb{R}$ and $\sigma_{ess}(H_0) \cap [\lambda_0, \lambda_1] = \emptyset$, then the difference of the dimensions of the spectral projections $P_{H_1}[\lambda_0, \lambda_1)$ and $P_{H_0}(\lambda_0, \lambda_1]$ equals the number of weighted sign flips of $W(\psi_{1,\mp}(\lambda_1), \psi_{0,\pm}(\lambda_1))$ minus the number of weighted sign flips of $W(\psi_{1,\mp}(\lambda_0), \psi_{0,\pm}(\lambda_0))$. In an additional result the difference of the spectral projections is replaced by Krein's Spectral Shift function. Finally we derive relative oscillation criteria.

Zusammenfassung

Wir behandeln relative Oszillationstheorie für eindimensionale Dirac-Operatoren H_0, H_1 mit getrennten Randbedingungen. Unsere Hauptresultate lauten wie folgt:

Sei $\lambda \in \mathbb{R}$ und $\lambda < \inf \sigma_{ess}(H_0)$, dann ist die Differenz der Dimensionen der Spektral-Projektionen $P_{H_1}(-\infty, \lambda)$ und $P_{H_0}(-\infty, \lambda]$ gleich der Anzahl der gewichteten Vorzeichenwechsel von $W(\psi_{1,\mp}(\lambda_1), \psi_{0,\pm}(\lambda_1))$, wobei $\psi_{0,\pm}$ bzw. $\psi_{1,\pm}$ die entsprechenden Weyl-Lösungen sind und W die Wronski ist. Weiters, falls $\lambda_0, \lambda_1 \in \mathbb{R}$ und $\sigma_{ess}(H_0) \cap [\lambda_0, \lambda_1] = \emptyset$, dann ist die Differenz der Dimensionen der Spekral-Projektionen $P_{H_1}[\lambda_0, \lambda_1)$ und $P_{H_0}(\lambda_0, \lambda_1]$ gleich der Anzahl der gewichteten Vorzeichenwechsel von $W(\psi_{1,\mp}(\lambda_1), \psi_{0,\pm}(\lambda_1))$ minus der Anzahl der Vorzeichenwechsel von $W(\psi_{1,\mp}(\lambda_0), \psi_{0,\pm}(\lambda_0))$. Schließlich leiten wir noch relative Oszillationskriterien her.

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Chapter 0

Introduction

Our object of investigation is the Dirac differential expression

$$\tau = \frac{1}{\mathrm{i}} \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \frac{d}{dx} + \phi, \qquad (1)$$

where ϕ represents a potential. To be more precise, in ϕ the mass, the scalar potential, the electrostatic potential, and the anomalous magnetic moment are included (see (1.2)).

The corresponding Dirac differential equation is a relativistic quantum mechanical wave equation, which describes the characteristics of an electron (and other spin-1/2-particles). It is consistent with both the principles of quantum mechanics and the theory of special relativity. This theory was developed by the British physicist *Paul Dirac* in 1928.

The main objective of this thesis is to relate the number of eigenvalues of two different self-adjoint Dirac operators with the number of (weighted) zeros of the Wronskian. In terms of our notation this reads

$$\dim \operatorname{Ran} P_{H_1}(-\infty, \lambda) - \dim \operatorname{Ran} P_{H_0}(-\infty, \lambda] = \#(\psi_{1,\mp}(\lambda_0), \psi_{0,\pm}(\lambda_0)), \qquad (2)$$

where $\lambda < \inf \sigma_{ess}(H_0)$. Here P_H denotes the projection-valued measure of an selfadjoint operator H (cf. Notation 2.12) and $\#(u_0, u_1)$ counts the weighted sign flips of the Wronskian $W(u_0, u_1)$ (cf. Notation 2.4).

Furthermore, considering an essential spectral gap $[\lambda_0, \lambda_1] \cap \sigma_{ess}(H_0) = \emptyset$, we will derive the following result:

$$\begin{aligned} &\dim \operatorname{Ran} P_{H_1}[\lambda_0, \lambda_1) - \dim \operatorname{Ran} P_{H_0}(\lambda_0, \lambda_1] \\ &\quad = \#(\psi_{1,\mp}(\lambda_1), \psi_{0,\pm}(\lambda_1)) - \#(\psi_{1,\mp}(\lambda_0), \psi_{0,\pm}(\lambda_0)). \end{aligned} \qquad (3)$$

In Chapter 1 we will state some essential standard results and make some useful definitions such as e.g., regular respectively singular endpoints, limit point respectively limit circle endpoints and Wronskians. Furthermore, we fix boundary conditions (if necessary) to obtain a self-adjoint operator.

Chapter 2 is devoted to the Prüfer variables and weighted sign flips of Wronskians in order to finally obtain our main result at least for regular operators. To extend this result to the general case, more effort will be needed. For this purpose we will demonstrate two different approaches. This will be the topic of Chapter 4 and Chapter 5 respectively.

Chapter 3 discusses the definition of relative oscillation theory. At the end of this chapter we get a first estimate of our main result.

Chapter 4 illustrates the first approach using approximation of an operator by a sequence of regular operators. The standard approximation technique only implies strong convergence, which unfortunately is not sufficient for our purpose. Hence we will derive convergence of spectral projections in the trace norm for suitably chosen regular operators.

Chapter 5 provides an alternative approach, which connects our theory with Krein's spectral shift function. Firstly we prove our main result for a perturbation supported on a compact interval. In the final result the left hand side in (2) will be substituted by Krein's Spectral Shift function.

In Chapter 6 we derive some relative oscillation criteria.

Appendix A derives the main properties of Krein's Spectral Shift function. This appendix is adopted from [8] aside from a few modifications.

I wish to point out that my thesis is related to the paper *Relative oscillation theory, weighted zeros of the Wronskian and the spectral shift function* [8] by H. Krüger and G. Teschl, who developed an analogous theory for Sturm–Liouville operators. Also the paper *Renormalized oscillation theory for Dirac operators* [18] is related. In this paper the case of equal operators $H_0 = H_1$ but different spectral parameters $\lambda_0 \neq \lambda_1$ is covered. For related work see also [1].

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Chapter 1

Basic facts

1.1 The Dirac differential equation

We collect some basic facts from [9, 22, 18, 21, 25, 26].

A Dirac[1] differential expression is a differential expression of the form

$$\tau = \frac{1}{\mathrm{i}}\sigma_2 \frac{d}{dx} + \phi. \tag{1.1}$$

Here

$$\phi(x) := \phi_{\text{el}}(x)\mathbb{1} + \phi_{\text{am}}(x)\sigma_1 + (m + \phi_{\text{sc}}(x))\sigma_3, \tag{1.2}$$

where σ_1, σ_2, σ_3 denote the Pauli[2] matrices

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.3}$$

and m, ϕ_{sc}, ϕ_{el}, and ϕ_{am} are interpreted as mass, scalar potential, electrostatic potential, and anomalous magnetic moment. We require $m \in [0, \infty)$ and $\phi_{\text{sc}}, \phi_{\text{el}}, \phi_{\text{am}} \in L^1_{loc}(I, \mathbb{R})$ real-valued, $I := (a, b)$, with $-\infty \le a < b \le \infty$.

Explicitly we have

$$\tau f = \begin{pmatrix} \phi_{11} & -\frac{d}{dx} + \phi_{12} \\ \frac{d}{dx} + \phi_{12} & \phi_{22} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} -f_2' + \phi_{12} f_2 + \phi_{11} f_1 \\ f_1' + \phi_{12} f_1 + \phi_{22} f_2 \end{pmatrix}, \tag{1.4}$$

where primes denote derivatives with respect to x and $\phi_{11} := \phi_{\text{el}} + m + \phi_{\text{sc}}$, $\phi_{12} := \phi_{21} := \phi_{\text{am}}$, $\phi_{22} := \phi_{\text{el}} - m - \phi_{\text{sc}}$.

Definition 1.1.
A finite end point is called ***regular*** *if $\phi_{11}, \phi_{12}, \phi_{22}$ are integrable near this end point. In this case boundary values for all functions exist at this end point. In particular, τ is called regular if both end points are regular, that is, $a, b \in \mathbb{R}$ and $\phi_{11}, \phi_{12}, \phi_{22} \in L^1(I, \mathbb{R})$. Otherwise τ is called* ***singular****.*

[1] *Paul Dirac* (1902–1984)
[2] *Wolfgang Pauli* (1900–1958)

The maximal domain of definition for τ is given by

$$\mathfrak{D}(\tau) := \{f \in L^2(I, \mathbb{C}^2) | f \in AC_{loc}(I, \mathbb{C}^2),\ \tau f \in L^2(I, \mathbb{C}^2)\}. \tag{1.5}$$

We formulate the following important theorem which gives us some information when solutions of our corresponding Dirac equation

$$\tau u = \lambda u, \qquad \lambda \in \mathbb{C}, \tag{1.6}$$

exist.

Theorem 1.2 (cf. [27, Korollar 15.2])**.**
Let $g \in L^1_{loc}(I, \mathbb{C}^2)$. Then for all $z \in \mathbb{C}$, $x_0 \in I$ and $(y_1, y_2) \in \mathbb{C}^2$, there is a unique solution of the initial value problem

$$(\tau - z)u = g, \qquad u(x_0) = (y_1, y_2). \tag{1.7}$$

For all $x \in I$ the solution $u(z, x)$ is an entire function with respect to z.
Moreover, u is $\mathbb{R}^2$-valued if $(y_1, y_2) \in \mathbb{R}^2$.

For more information on Dirac operators see [21, 27].

1.2 Wronskians

Notation 1.3.
In $\mathbb{C}^2$ we define the scalar product via

$$\langle x, y\rangle := x_1^* y_1 + x_2^* y_2, \tag{1.8}$$

and the corresponding norm via

$$\|x\| := \sqrt{\langle x, x\rangle}. \tag{1.9}$$

Definition 1.4.
*The **Wronskian**[3] of two functions $f, g \in AC_{loc}(I, \mathbb{C}^2)$ is defined by*

$$W_x(f, g) := i\langle f(x)^*, \sigma_2 g(x)\rangle = f_1(x)g_2(x) - f_2(x)g_1(x), \qquad x \in (a, b). \tag{1.10}$$

Furthermore, we define $W_a(f, g) := \lim_{x\to a} W_x(f, g)$ and analogously for b provided this limit exists (this is the case e.g. if $f, g \in L^2$).

Differentiating the Wronskian of solutions of $\tau u = \lambda_0 u_0$ and $\tau v = \lambda_1 v_1$ gives

$$W_x'(u, v) = (\lambda_0 - \lambda_1)\langle u(x), v(x)\rangle. \tag{1.11}$$

So $W_x(u, v) = 0$ if $u(x)$ and $v(x)$ are parallel and $W_x'(u, v) = 0$ if $u(x)$ and $v(x)$ are orthogonal. In addition the Wronskian of two solutions can only have simple zeros unless $\lambda_0 = \lambda_1$ and $u = v$ or $u \equiv 0$ (resp. $v \equiv 0$) of course. In particular note that the Wronskian of two solutions of $\tau u = \lambda u$ is constant.

Once found a solution u, the following lemma shows how we can get a second solution v with $W(u, v) \equiv 1$.

[3] *Joseph Marie Wronski* (1778–1853)

Lemma 1.5 ([12, Lem. 1]).
Under the general hypothesis on $\phi_{\mathrm{sc}}, \phi_{\mathrm{am}}, \phi_{\mathrm{el}}$, let $u : (a,b) \to \mathbb{C}^2$ be a nontrivial solution of (1.6) and choose $x_0 \in (a,b)$. Then

$$v(x) := \left(2 \int_{x_0}^{x} \frac{\langle (m+\phi_{\mathrm{sc}})(t)\sigma_3 + \phi_{\mathrm{am}}(t)\sigma_1)u(t), u(t)\rangle}{\|u(t)\|^4} dt - i \frac{\sigma_2}{\|u(x)\|^2} \right) u(x), \tag{1.12}$$

$(x \in (a,b))$, is a second linearly independent solution. Moreover, the fundamental system (u,v) has Wronskian equal to 1.

Proof. Let $\breve{\phi} = \phi_{\mathrm{am}}\sigma_1 + (m+\phi_{\mathrm{sc}})\sigma_3$. Applying the commutator $[A,B] := AB - BA$, we see $[\sigma_2, \phi] = 2\sigma_2\breve{\phi}$ and obtain

$$v' = -i\sigma_2\phi v + \frac{2}{\|u\|^4}(-\langle u, \breve{\phi}u\rangle - \sigma_2\langle u, \sigma_2\phi u\rangle + \breve{\phi}\|u\|^2)u.$$

A calculation shows $(-\langle u, \breve{\phi}u\rangle - \sigma_2\langle u, \sigma_2\phi u\rangle + \breve{\phi}\|u\|^2)u = 0$, so $\tau v = \frac{1}{i}\sigma_2 v' + \phi v = \frac{1}{i}\sigma_2(-i\sigma_2\phi v) = 0$. $W(u,v) \equiv 1$ is straightforward, so v is a second linearly independent solution. □

1.3 Boundary conditions

We want to obtain a self-adjoint operator from τ. Using integration by parts a straightforward calculation shows the **Lagrange**[4] **identity**:

$$\int_a^b \langle g(t), (Hf)(t)\rangle dt = W_a(g^*, f) - W_b(g^*, f) + \int_a^b \langle (Hg)(t), f(t)\rangle dt. \tag{1.13}$$

So H is symmetric if and only if $W_a(g^*, f) = W_b(g^*, f)$ for all $g, f \in \mathfrak{D}(H)$.

Definition 1.6.
*We call τ **limit circle** at a if there is a $v \in \mathfrak{D}(\tau)$ with $W_a(v^*, v) = 0$ such that $W_a(v, f) \neq 0$ for one $f \in \mathfrak{D}(\tau)$. Otherwise τ is called **limit point** at a. Analogously for b.*

If τ is limit point at both endpoints a and b, then τ gives rise to a unique self-adjoint operator H when defined maximally (cf. e.g., [9], [25], [26]). Otherwise we fix a boundary condition at each endpoint where τ is limit circle:

$$\begin{aligned} BC_a(f) &:= W_a(v, f), \\ BC_b(f) &:= W_b(w, f), \end{aligned} \tag{1.14}$$

with v (resp. w) a function as in Definition 1.6 at a (resp. b).

Then H is given by

$$\begin{array}{rccl} H: & \mathfrak{D}(H) & \to & L^2(I, \mathbb{C}^2), \\ & f & \mapsto & \tau f, \end{array} \tag{1.15}$$

where

$$\begin{aligned} \mathfrak{D}(H) := \{ f \in L^2(I, \mathbb{C}^2) | f \in AC_{loc}(I, \mathbb{C}^2),\ \tau f \in L^2(I, \mathbb{C}^2), \\ BC_a(f) = BC_b(f) = 0 \text{ if } \tau \text{ is limit circle at } a \text{ resp. } b \}. \end{aligned} \tag{1.16}$$

[4] *Joseph-Louis de Lagrange* (1736–1813)

Notation 1.7.
From now on we will denote the associated self-adjoint operator of τ by H. By $\psi_\pm(\lambda, x)$ we will denote solutions of the differential equation $\tau u = \lambda u$, $\lambda \in \mathbb{C}$ (if they exist), which satisfy the following conditions:

(i) $\psi_\pm(\lambda, x) \in AC_{loc}(I, \mathbb{C}^2)$,

(ii) $\psi_+(\lambda, x)$ *(resp.* $\psi_-(\lambda, x)$*) is square integrable near b (resp. a) and fulfills the boundary condition (cf. (1.14)) of H at b (resp. a) if any (i.e., if τ is limit circle at b (resp. a)) and*

(iii) $\psi_\pm(\lambda, .) \not\equiv 0$.

The functions $\psi_\pm(\lambda, x)$ are called **Weyl**[5] **solutions**. In Lemma 1.16 we provide a condition on their existence.

Remark 1.8.
Looking at equation (1.13) we infer that $W_a(g^*, f) = W_b(g^*, f)$ for all $g, f \in \mathfrak{D}(\tau)$ if τ is limit point. Otherwise this is ensured by the additional boundary conditions.

Lemma 1.9 (cf. [22, Thm. A.4])**.**

(i) A regular endpoint is limit circle.

(ii) If $b = \infty$ then b is limit point. Analogously, if $a = -\infty$ then a is limit point.

Theorem 1.10 (Weyl alternative, cf. [27, Satz 15.15])**.**
The operator τ is limit circle at a $\Leftrightarrow$ $\forall\ \lambda \in \mathbb{C}$ all solutions u of $\tau u = \lambda u$ are square integrable near a. Similary for b.

Corollary 1.11.
All eigenvalues of H are simple.

Proof. If τ is limit point at a then there is at most one (linearly independent) solution of $\tau u = \lambda u$ which is square integrable near a. If τ is limit circle at a then the Wronskian of two solution, which satisfy the same boundary condition, vanishes. So they are linearly dependent. Similary for b. □

1.4 The resolvent

Cf. [18, Sec. 1].

Notation 1.12.
Let $u, v \in \mathbb{R}^2$. We define

$$u \otimes v := \begin{pmatrix} u_1 v_1 & u_1 v_2 \\ u_2 v_1 & u_2 v_2 \end{pmatrix}. \tag{1.17}$$

[5] *Hermann Weyl* (1885–1955)

The resolvent $R_H(z) := (H-z)^{-1}$ of H can be expressed in terms of $\psi_\pm(z,.)$ as follows:

$$R_H(z)f(x) = \int_a^b G(z,x,y)f(y)dy, \quad z \in \rho(H), \tag{1.18}$$

where

$$G(z,x,y) = \frac{\psi_\pm(z,x) \otimes \psi_\mp(z,y)}{W(\psi_+(z), \psi_-(z))}, \quad \pm(x-y) > 0, \tag{1.19}$$

denotes the **Green**[6] **Function** of H. Recall that $W_x(\psi_+(z), \psi_-(z))$ is independent of x (cf. (1.11)). In addition, we set $G(z,x,x) = \lim_{\varepsilon\to 0}(G(z,x+\varepsilon,x) + G(z,x-\varepsilon,x))/2$.

Remark 1.13.
Note that $W_x(\psi_+(z), \psi_-(z))$ only vanishes if $\psi_+(z,.)$ and $\psi_-(z,.)$ are linearly depended. But then $\psi_\pm(z,.)$ satisfy both boundary conditions, that is $z \in \sigma(H)$.

Lemma 1.14 (First resolvent identity, [19, Equation (2.81)])**.**
If $z, z' \in \rho(A)$, we have the first resolvent formula:

$$R_A(z) - R_A(z') = (z-z')R_A(z)R_A(z') = (z-z')R_A(z')R_A(z). \tag{1.20}$$

Proof. We have

$$(A-z)^{-1} - (z-z')(A-z)^{-1}(A-z')^{-1} =$$
$$(A-z)^{-1}(1-(z-A+A-z')(A-z')^{-1}) = (A-z')^{-1}$$

which proves the first equality. The second follows after interchanging z and z'. □

Lemma 1.15 (Second resolvent identity, [19, Lem. 6.4])**.**
Suppose A and B are closed and $\mathfrak{D}(A) \subseteq \mathfrak{D}(B)$. Then we have the second resolvent formula

$$R_{A+B}(z) - R_A(z) = -R_A(z)BR_{A+B}(z) = -R_{A+B}(z)BR_A(z) \tag{1.21}$$

for $z \in \rho(A) \cap \rho(A+B)$.

Proof. We compute

$$\begin{aligned} R_{A+B}(z) + R_A(z)BR_{A+B}(z) &= (\mathbb{1} + R_A(z)B)R_{A+B}(z) = \\ &(A-z)^{-1}(A+B-z)R_{A+B}(z) = R_A(z). \end{aligned} \tag{1.22}$$

The second identity is similar. □

Denote by $H_{x,-}$ (resp. $H_{x,+}$), $x \in I$, self-adjoint operators associated with τ on $L^2((a,x),\mathbb{C}^2)$ (resp. $L^2((x,b),\mathbb{C}^2)$) obtained from H by imposing the additional boundary condition $f_1(x) = 0$. Then $H_{x,-} \oplus H_{x,+}$ is a rank one resolvent perturbation of H and hence $\sigma_{ess}(H) = \sigma_{ess}(H_{x,-}) \cup \sigma_{ess}(H_{x,+})$ (cf. [28, Korollar 6.2]). If $G_{x,\pm}(z,.,..)$ denotes the resolvent kernel of $H_{x,\pm}$ we define the **Weyl *m*-functions** $m_\pm(z,x)$ (w.r.t. the base point x) by

$$G_{x,\pm}(z,x,x) =: \begin{pmatrix} 0 & \pm\frac{1}{2} \\ \pm\frac{1}{2} & m_{x,\pm}(z) \end{pmatrix}. \tag{1.23}$$

[6] *George Green* (1793–1841)

The first resolvent identity shows that $m_\pm(z,x)$ are Herglotz[7] functions (cf., e.g., [22]).

We have not said anything about the existence of the Weyl solutions $\psi_\pm(z,x)$ yet. The next lemma shows that they exist if we are away from the essential spectrum.

Lemma 1.16 (cf. [18, Lem. 1.1]).
The solutions $\psi_\pm(z,x)$ exist for $z \in \mathbb{C}\backslash\sigma_{ess}(H_{x_0,\pm})$. They can be assumed analytic with respect to $z \in \mathbb{C}\backslash\sigma(H_{x_0,\pm})$ and $\psi_\pm(z,x)^* = \psi_\pm(z^*,x)$ holds. In addition, we can include a finite number of isolated eigenvalues in the domain of holomorphy of $\psi_\pm(z,x)$ by removing the corresponding poles.

Proof. If $U(z,x,x_0)$, $z \in \mathbb{C}$, is a fundamental matrix solution for $\tau u = zu$ (i.e., $U(z,x_0,x_0) = \mathbb{1}$, $x_0 \in I$) and $m_\pm(z,x_0)$ are the Weyl m-functions with respect to the base point x_0. Then we can choose

$$\psi_\pm(z,x) = U(z,x,x_0)\begin{pmatrix} 1 \\ \pm m_\pm(z,x_0) \end{pmatrix}. \tag{1.24}$$

By removing the corresponding poles of $m_{x_0,\pm}(z)$ we can include a finite number of isolated eigenvalues in the domain of holomorphy of $\psi_\pm(z,x)$. □

Note that there are also Weyl solutions $\psi_\pm(z,x)$ if $z \in \sigma_{ess}(H)$ is an eigenvalue.

Theorem 1.17 (cf. [25, Satz 15.16]).
Let τ be limit circle at both endpoints. Then the resolvent is a Hilbert[8]-Schmidt[9] operator. In particular the spectrum of any self-adjoint extension is purely discrete.

[7] *Gustav Herglotz* (1881–1953)
[8] *David Hilbert* (1862–1943)
[9] *Erhard Schmidt* (1876–1959)

Chapter 2

Prüfer variables and the case of regular operators

2.1 Prüfer variables

We introduce Prüfer[1] variables ρ_u and θ_u for $u \in C(I, \mathbb{R}^2)$, $I = (a, b)$, defined by

$$u_1(x) =: \rho_u(x)\sin(\theta_u(x)), \qquad u_2(x) =: \rho_u(x)\cos(\theta_u(x)). \tag{2.1}$$

If u is never $(0, 0)$ and u is continuous, then the **Prüfer radius**

$$\rho_u(x) = \sqrt{u_1^2(x) + u_2^2(x)} > 0, \tag{2.2}$$

and the **Prüfer angle** θ_u is uniquely determined once a value of $\theta_u(x_0)$, $x_0 \in I$, is chosen by the requirement $\theta_u \in C(I, \mathbb{R})$.

Note that for solutions u of $\tau u = \lambda u$ the case $u(x) = (0, 0)$ cannot occur unless $u \equiv 0$ of course.

If $f, g \in C(I, \mathbb{R}^2)$, linking the Prüfer angles with the definition of Wronskians, (1.10) also reads

$$W_x(f, g) = -\rho_f(x)\rho_g(x)\sin(\theta_g(x) - \theta_f(x)). \tag{2.3}$$

2.2 Weighted zeros of Wronskians

In this section we will establish the concept of weighted zeros of Wronskians. For this purpose the PrŸfer variables defined in the previous section will be adjuvant. See [18, Chap. 2] for the Sturm[2]-Liouville[3] case.

In the following we will investigate solutions u_i of $\tau_i u_i = \lambda_i u_i$, $i \in \{0, 1\}$. Since we can replace $\phi \mapsto \phi - \lambda\mathbb{1}$ we will assume $\lambda_0 = \lambda_1 = 0$ without restriction.

[1] *Heinz Prüfer* (1896–1934)
[2] *Charles-François Sturm* (1803–1855)
[3] *Joseph Liouville* (1809–1882)

Notation 2.1.
For a matrix A we will write $A > 0$ (resp. $A \geq 0$) if it satisfies $\langle \psi, A\psi \rangle > 0$, (resp. $\langle \psi, A\psi \rangle \geq 0$) $\forall\ \psi \neq 0$. Similary, we will write $A < 0$ and $A \leq 0$. By $A > B$ we mean $A - B > 0$, similary for $\geq, \leq, <$ of course. Furthermore we denote $A_+ := P_A[0, \infty)A$ and $A_- = P_A(-\infty, 0)A$.

So we have $A_+ \geq 0$ and $A_- \leq 0$.

The following generalization of (1.11) is true:

Lemma 2.2.
Suppose u_i satisfies $\tau_i u_i = 0$, $i \in \{0, 1\}$, then

$$W'_x(u_0, u_1) = \langle u_0(x), (\phi_1 - \phi_0)u_1(x) \rangle. \tag{2.4}$$

Proof. The result follows by a straightforward calculation. □

Lemma 2.3 (Weighted zeros, cf. [18, Lem. 2.1])**.**
Let u_i solve $\tau_j u_j = 0$, $i \in \{0, 1\}$, abbreviate $\Delta_{1,0}(x) = \theta_{u_1}(x) - \theta_{u_0}(x)$ and suppose $\Delta_{1,0}(x_0) \equiv 0 \mod \pi$. If $\phi_0(x) - \phi_1(x)$ is (i) negative, (ii) zero, or (iii) positive for a.e. $x \in (x_0, x_0 + \varepsilon)$ respectively for a.e. $x \in (x_0 - \varepsilon, x_0)$ for some $\varepsilon > 0$, then the same is true for $(\Delta_{1,0}(x) - \Delta_{1,0}(x_0))/(x - x_0)$.

Proof. By (2.4) we have

$$\begin{aligned} W_x(u_0, u_1) &= -\rho_0(x)\rho_1(x)\sin(\Delta_{1,0}(x)) \\ &= -\int_{x_0}^{x} \langle u_0(t), (\phi_0(t) - \phi_1(t))u_1(t) \rangle dt. \end{aligned}$$

If we have $\Delta_{1,0}(x_0) \equiv \pi \mod 2\pi$ then $\sin(x)$ behaves like $x \mapsto -x$ at x_0 and $(\sin(\theta_{u_0}(x_0)), \cos(\theta_{u_0}(x_0))) = -(\sin(\theta_{u_1}(x_0)), \cos(\theta_{u_1}(x_0)))$, therefore

$$\begin{aligned} &\operatorname{sgn}(\langle u_0(x_0), (\phi_0(x_0) - \phi_1(x_0))u_1(x_0) \rangle) = \\ &\operatorname{sgn}\left(\left\langle \begin{pmatrix} \sin(\theta_{u_0}(x_0)) \\ \cos(\theta_{u_0}(x_0)) \end{pmatrix}, (\phi_0(x_0) - \phi_1(x_0)) \begin{pmatrix} \sin(\theta_{u_1}(x_0)) \\ \cos(\theta_{u_1}(x_0)) \end{pmatrix} \right\rangle \right) = \\ &-\operatorname{sgn}(\phi_0(x_0) - \phi_1(x_0)), \end{aligned}$$

and the claim follows. In the case $\Delta_{1,0}(x_0) \equiv 0 \mod 2\pi$ the claim follows analogously by removing the minus. □

Notation 2.4.
We will denote

$$\begin{aligned} &\#_{(c,d)}(u_0, u_1) := \lceil \Delta_{1,0}(d)/\pi \rceil - \lfloor \Delta_{1,0}(c)/\pi \rfloor - 1, \\ &\textit{for } (c, d) \subseteq (a, b) \textit{ if } \tau_i \textit{ are regular and } (c, d) \subset (a, b) \textit{ else.} \end{aligned} \tag{2.5}$$

So by Lemma 2.3 $\#_{(c,d)}(u_0, u_1)$ counts the weighted sign flips of the Wronskian $W_x(u_0, u_1)$ inside of (c, d), where a sign flip is counted as $+1$ if $\phi_0 - \phi_1$ is positive in a neighborhood of the sign flip, it is counted as -1 if $\phi_0 - \phi_1$ is negative in a neighborhood of the sign flip. If $\phi_0 - \phi_1$ changes sign (i.e., it is positive on one side and negative on the other) the Wronskian will not change its sign. In particular, we obtain:

Corollary 2.5 (cf. [18, Lem. 2.2]).
Let u_i solve $\tau_i u_i = 0$, $i \in \{0, 1\}$, and $\phi_0 - \phi_1 \geq 0$, then $\#_{(c,d)}(u_0, u_1)$ equals the number of sign flips of $W(u_0, u_1)$ inside the interval (c, d).

Remark 2.6.
In the case $\phi_0 - \phi_1 \leq 0$ we get of course the corresponding negative number except for the fact that zeros at the boundary points are counted as well since $\lfloor -x \rfloor = -\lceil x \rceil$. That is, if $\phi_0 - \phi_1 > 0$, then $\#_{(c,d)}(u_0, u_1)$ equals the number of zeros of the Wronskian in (c, d) while if $\phi_0 - \phi_1 < 0$, it equals minus the number of zeros in $[c, d]$. In addition, note that $\#(u, u) = -1$.

2.3 More on Prüfer angles and the case of regular operators

Our presentation in this section follows [8, Sec. 5].

We will derive our main result for regular operators τ_i, $i \in \{0, 1\}$. At first we require an analysis of the behaviour of the PrŸfer angles when using linear interpolation between τ_0 and τ_1.

Notation 2.7.
For functions y_0, y_1 we define $y_\varepsilon := (1 - \varepsilon)y_0 + \varepsilon y_1$ to denote the linear interpolation between y_0 and y_1.

In the regular case the resolvent of H is Hilbert-Schmidt and hence the spectrum is purely discrete (i.e., $\sigma_{ess}(H) = \emptyset$, cf. Theorem 1.17). In this case the boundary conditions (1.14) also read

$$\begin{aligned} BC_a(f) &= \cos(\alpha) f_1(a) - \sin(\alpha) f_2(a), \\ BC_b(f) &= \cos(\beta) f_1(b) - \sin(\beta) f_1(b), \end{aligned} \tag{2.6}$$

cf. [25, Satz 15.12] for example. Note that α (resp. β) depends on v (resp. w) in (1.14) (cf. [19, Sec. 9.2] for the Sturm-Liouville case).

Hence we can choose $\psi_\pm(\lambda, x)$ such that $\psi_-(\lambda, a) = (\sin(\alpha), \cos(\alpha))$, respectively $\psi_+(\lambda, b) = (\sin(\beta), \cos(\beta))$. In particular, we may choose

$$\theta_-(\lambda, a) = \alpha \in [0, \pi), \quad -\theta_+(\lambda, b) = \pi - \beta \in [0, \pi), \tag{2.7}$$

where we have abbreviated $\theta_{\psi_\pm(\lambda)}(x)$ by $\theta_\pm(\lambda, x)$.

If u_ε solves $\tau_\varepsilon u_\varepsilon = 0$, then the corresponding Prüfer angles satisfy

$$\dot{\theta}_\varepsilon(x) = -\frac{W_x(u_\varepsilon, \dot{u}_\varepsilon)}{\rho_\varepsilon^2(x)}, \tag{2.8}$$

where the dot denotes a derivative with respect to ε.

Lemma 2.8.
For two Dirac operators τ_i, $i \in \{0, 1\}$, we have

$$W_x(\psi_{\varepsilon,\pm}, \dot{\psi}_{\varepsilon,\pm}) = \begin{cases} \int_x^b \langle \psi_{\varepsilon,+}(t), (\phi_0(t) - \phi_1(t))\psi_{\varepsilon,+}(t) \rangle dt, \\ -\int_a^x \langle \psi_{\varepsilon,-}(t), (\phi_0(t) - \phi_1(t))\psi_{\varepsilon,-}(t) \rangle dt, \end{cases} \tag{2.9}$$

where the dot denotes a derivative with respect to ε and $\psi_{\varepsilon,\pm}(x) = \psi_{\varepsilon,\pm}(0, x)$.

Proof. Integrating (2.4) we obtain

$$W_x(\psi_{\varepsilon,\pm}, \psi_{\tilde{\varepsilon},\pm}) = (\tilde{\varepsilon} - \varepsilon) \begin{cases} \int_x^b \langle \psi_{\varepsilon,+}(t), (\phi_0(t) - \phi_1(t))\psi_{\tilde{\varepsilon},+}(t)\rangle dt, \\ -\int_a^x \langle \psi_{\varepsilon,-}(t), (\phi_0(t) - \phi_1(t))\psi_{\tilde{\varepsilon},-}(t)\rangle dt. \end{cases}$$

Now use this to evaluate the limit

$$\lim_{\tilde{\varepsilon} \to \varepsilon} W_x\Big(\psi_{\varepsilon,\pm}, \frac{\psi_{\pm,\varepsilon} - \psi_{\tilde{\varepsilon},\pm}}{\varepsilon - \tilde{\varepsilon}}\Big).$$

□

Corollary 2.9.
Denoting the Prüfer angles of $\psi_{\varepsilon,\pm}(x) = \psi_{\varepsilon,\pm}(0,x)$ *by* $\theta_{\varepsilon,+}(x) = \theta_{\varepsilon,+}(0,x)$*, for* $\phi_0 - \phi_1 \geq 0$*, the previous lemma implies*

$$\dot{\theta}_{\varepsilon,+}(x) = -\frac{\int_x^b \langle \psi_{\varepsilon,+}(t), (\phi_0(t) - \phi_1(t))\psi_{\varepsilon,+}(t)\rangle dt}{\rho_{\varepsilon,+}(x)^2} \leq 0,$$
$$\dot{\theta}_{\varepsilon,-}(x) = \frac{\int_a^x \langle \psi_{\varepsilon,-}(t), (\phi_0(t) - \phi_1(t))\psi_{\varepsilon,-}(t)\rangle dt}{\rho_{\varepsilon,-}(x)^2} \geq 0, \tag{2.10}$$

with strict inequalities if $\phi_0 - \phi_1 > 0$ *on a subset of positive Lebesgue measure of* (x,b)*, respectively* (a,x)*.*

Corollary 2.10.

$$\frac{\partial \theta_{\varepsilon,+}}{\partial \lambda}(x) = -\frac{\int_x^b \|\psi_{\varepsilon,+}(t)\|^2 dt}{\rho_{\varepsilon,+}(x)^2} \leq 0,$$
$$\frac{\partial \theta_{\varepsilon,-}}{\partial \lambda}(x) = \frac{\int_x^b \|\psi_{\varepsilon,-}(t)\|^2 dt}{\rho_{\varepsilon,-}(x)^2} \geq 0. \tag{2.11}$$

Proof. Consider $\tau_1 = \tau_0 - \lambda$. □

Now we are ready to investigate the associated operators H_0 and H_1. In addition, we will choose the same boundary conditions for H_ε as for H_0 and H_1.

Lemma 2.11.
Suppose $\phi_0 - \phi_1 \geq 0$ *(resp.* $\phi_0 - \phi_1 \leq 0$*). Then the eigenvalues of* H_ε *are decreasing (resp. increasing) with respect to* ε*.*

Proof. First of all the Prüfer angles $\theta_{\varepsilon,\pm}(\lambda,x)$ are analytic with respect to ε since τ_ε is by a well-known result from ordinary differential equations (see e.g., [23, Thm. 13.III]). Moreover, $\lambda \in \sigma(H_\varepsilon)$ is equivalent to $\theta_{\varepsilon,+}(\lambda,a) \equiv \alpha \mod \pi$ (resp. $\theta_{\varepsilon,-}(\lambda,b) \equiv \beta \mod \pi$), where α (respectively β) generates the boundary condition. By Corollary 2.9 and Corollary 2.10 the implicit function theorem implies:

$$\frac{\partial \lambda(\varepsilon)}{\partial \varepsilon} = -\frac{\frac{\partial \theta_+}{\partial \varepsilon}}{\frac{\partial \theta_+}{\partial \lambda}} \leq 0.$$

Note that the same inequality is true if we replace θ_+ by θ_-. □

Notation 2.12.
Let H *be a self-adjoint operator. We denote the corresponding projection-valued measure by* P_H*. Cf. [19, Thm. 3.7] for example.*

Remark 2.13.
In particular, Lemma 2.11 implies that $\dim \operatorname{Ran} P_{H_\varepsilon}(-\infty, 0)$ *is continuous from below (resp. above) in* ε *if* $\phi_0 - \phi_1 \geq 0$ *(resp.* $\phi_0 - \phi_1 \leq 0$*).*

After this preparations we are ready for the announced main result of this chapter.

Theorem 2.14.
Let H_i*,* $i \in \{0, 1\}$*, be regular operators associated with the same boundary conditions at* a *and* b*. Then the following equation holds:*

$$\dim \operatorname{Ran}\, P_{H_1}(-\infty, \lambda_1) - \dim \operatorname{Ran}\, P_{H_0}(-\infty, \lambda_0] = \#_{(a,b)}(\psi_{0,\pm}(\lambda_0), \psi_{1,\mp}(\lambda_1)). \tag{2.12}$$

Proof. It suffices to prove the result for $\lambda_0 = \lambda_1 = 0$ and $\#(\psi_{0,+}(0,.), \psi_{\varepsilon,-}(0,.))$. We consider τ_i, $i \in \{0, 1\}$, and split $\phi_0 - \phi_1$ according to

$$\phi_0 - \phi_1 = \phi_+ - \phi_-, \qquad \phi_+, \phi_- \geq 0,$$

and introduce the operator $\tau_- = \tau_0 - \phi_-$. Then τ_- is a negative perturbation of τ_0 and τ_1 is a positive perturbation of τ_-.

Furthermore define τ_ε by

$$\tau_\varepsilon = \begin{cases} \tau_0 + 2\varepsilon\phi_-, & \varepsilon \in [0, 1/2], \\ \tau_- + 2(\varepsilon - 1/2)\phi_+, & \varepsilon \in [1/2, 1]. \end{cases}$$

Let us look at

$$N(\varepsilon) = \#_{(a,b)}(\psi_{0,+}, \psi_{\varepsilon,-}) = \lceil \Delta_\varepsilon(b)/\pi \rceil - \lfloor \Delta_\varepsilon(a)/\pi \rfloor - 1, \quad \Delta_\varepsilon(x) = \Delta_{\psi_{\varepsilon,-}, \psi_{0,+}}(x),$$

with $\psi_{\varepsilon,-} = \psi_{\varepsilon,-}(0,.)$ and consider $\varepsilon \in [0, 1/2]$. At the left boundary $\Delta_\varepsilon(a)$ remains constant whereas at the right boundary $\Delta_\varepsilon(b)$ is increasing by Corollary 2.9. Moreover, it hits a multiple of π whenever $0 \in \sigma(H_\varepsilon)$. So $N(\varepsilon)$ is a piecewise constant function which is continuous from below and jumps by one whenever $0 \in \sigma(H_\varepsilon)$. By Lemma 2.11 the same is true for

$$P(\varepsilon) = \dim \operatorname{Ran}\, P_{H_\varepsilon}(-\infty, 0) - \dim \operatorname{Ran}\, P_{H_0}(-\infty, 0]$$

and since we have $N(0) = P(0)$, we conclude $N(\varepsilon) = P(\varepsilon)$ for all $\varepsilon \in [0, 1/2]$. To see the remaining case $\varepsilon = (1/2, 1]$, simply replace increasing by decreasing and continuous from below by continuous from above. □

Chapter 3

Relative oscillation theory

Our presentation in this chapter follows [8, Sec. 3].

We will now shed some light on the title of this thesis. That is we determine the meaning of two operators being relatively oscillating.

Definition 3.1.
For τ_0, τ_1 possibly singular Dirac operators as in (1.1) on $I = (a, b)$ and two solutions u_i of $\tau_j u_j = \lambda_j u_j$, $j \in \{0, 1\}$, we define

$$\underline{\#}(u_0, u_1) := \liminf_{d\uparrow b,\, c\downarrow a} \#_{(c,d)}(u_0, u_1) \quad \text{and} \quad \overline{\#}(u_0, u_1) := \limsup_{d\uparrow b,\, c\downarrow a} \#_{(c,d)}(u_0, u_1). \tag{3.1}$$

If $\overline{\#}(u_0, u_1) = \underline{\#}(u_0, u_1)$, we define

$$\#(u_0, u_1) := \overline{\#}(u_0, u_1) = \underline{\#}(u_0, u_1). \tag{3.2}$$

Remark 3.2.
If $\phi_0 - \phi_1$ has the same definite sign near the endpoints a and b, we infer by Lemma 2.3 that $\#(u_0, u_1)$ exists. On the other hand, note that $\#(u_0, u_1)$ might not exist even if both a and b are regular, since the difference of Prüfer angles might oscillate around a multiple of π near an endpoint. In addition, even if it exists, it is not sure whether $\#(u_0, u_1) = \#_{(a,b)}(u_0, u_1)$, except the cases when there are no zeros at the endpoints or if $\phi_0 - \phi_1 \geq 0$ at least near the endpoints.

Theorem 3.3 (Comparison theorem for Wronskians)**.**
Suppose u_j satisfies $\tau_j u_j = \lambda_j u_j$, $j \in \{0, 1, 2\}$, where $\lambda_0 - \phi_0 \leq \lambda_1 - \phi_1 \leq \lambda_2 - \phi_2$.

If $c < d$ are two zeros of $W_x(u_0, u_1)$ such that $W_x(u_0, u_1)$ does not vanish identically, then there is at least one sign flip of $W_x(u_0, u_2)$ in (c, d). Similarly, if $c < d$ are two zeros of $W_x(u_1, u_2)$ such that $W_x(u_1, u_2)$ does not vanish identically, then there is at least one sign flip of $W_x(u_0, u_2)$ in (c, d).

Proof. We assume $\lambda_0 = \lambda_1 = \lambda_2 = 0$ w.l.o.g. Let c, d be two consecutive zeros of $W_x(u_0, u_1)$. We consider $\tau_\varepsilon = (2 - \varepsilon)\tau_1 + (\varepsilon - 1)\tau_2$, $\varepsilon \in [1, 2]$, restricted to (c, d) with boundary condition generated by the Prüfer angle of u_0 at c. For $\psi_{\varepsilon,-}(x) = \psi_{\varepsilon,-}(0, x)$ we have $\Delta_{u_0,\psi_{\varepsilon,-}}(c) \equiv 0 \mod \pi$ for all ε and hence we can even assume $\Delta_{u_0,\psi_{\varepsilon,-}}(c) = 0$. By (2.10) $\Delta_{u_0,\psi_{\varepsilon,-}}(d)$ is increasing, implying that $W_x(u_0, \psi_{\varepsilon,-})$ has at least one sign flip in (c, d) for $\varepsilon > 1$.

Since $W_x(\psi_{2,-}, u_2)$ is constant, we can assume $0 < \Delta_{u_2,\psi_{2,-}}(x) < \pi$. This implies $\Delta_{u_2,u_0}(c) = \Delta_{u_2,\psi_{2,-}}(c) < \pi$ and $\Delta_{u_2,u_0}(d) = \Delta_{u_2,\psi_{2,-}}(d) + \Delta_{\psi_{2,-},u_0}(d) > \pi$. Consequently $W_x(u_0, u_2)$ also has at least one sign flip in (c, d).

The second claim is proven analogously. □

Theorem 3.4 (Triangle inequality for Wronskians)**.**
Suppose u_i, $i \in \{0, 1, 2\}$ satisfy $\tau_i u_i = 0$. Then

$$\underline{\#}(u_0, u_1) + \underline{\#}(u_1, u_2) - 1 \leq \underline{\#}(u_0, u_2) \leq \underline{\#}(u_0, u_1) + \underline{\#}(u_1, u_2) + 1, \tag{3.3}$$

and similarly for $\underline{\#}$ replaced by $\overline{\#}$.

Proof. Take $a < c < d < b$. By definition

$$\#_{(c,d)}(u_0, u_2) = \lceil \Delta_{2,0}(d)/\pi \rceil - \lfloor \Delta_{2,0}(c)/\pi \rfloor - 1,$$

and using $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1$ respectively $\lceil x \rceil + \lceil y \rceil - 1 \leq \lceil x + y \rceil \leq \lceil x \rceil + \lceil y \rceil$ and $\Delta_{2,0} = \Delta_{2,1} + \Delta_{1,0}$, we obtain

$$\#_{(c,d)}(u_0, u_2) \leq \#_{(c,d)}(u_0, u_1) + \#_{(c,d)}(u_1, u_2) + 1.$$

Thus the result follows by taking the limits $c \downarrow a$ and $d \uparrow b$. □

Corollary 3.5.
Let $\tau_j u_j = \tau_j v_j = 0$, $j \in \{0, 1\}$. Then

$$|\underline{\#}(u_0, u_1) - \underline{\#}(v_0, v_1)| \leq 3, \quad |\overline{\#}(u_0, u_1) - \overline{\#}(v_0, v_1)| \leq 3. \tag{3.4}$$

Proof. Using Theorem 3.4 we obtain

$$\underline{\#}(u_0, u_1) - \underline{\#}(v_0, v_1) \leq \underline{\#}(u_0, v_1) + \underline{\#}(v_1, u_1) + 1 - \underline{\#}(v_0, u_0) - \underline{\#}(u_0, v_1) + 1 =$$

$$\underline{\#}(v_1, u_1) - \underline{\#}(v_0, u_0) + 2 = \begin{cases} 1 \text{ if } & \underline{\#}(v_0, u_0) = 0,\ \underline{\#}(v_1, u_1) = -1, \\ 3 \text{ if } & \underline{\#}(v_0, u_0) = -1,\ \underline{\#}(v_1, u_1) = 0, \\ 2 & \text{else.} \end{cases}$$

The first claims follows using this inequality and replacing u_i by v_i. The second claim is proven analogously by replacing $\underline{\#}$ by $\overline{\#}$. □

Definition 3.6.

*(i) An operator τ_1 is called **relatively nonoscillatory** with respect to τ_0 if $\underline{\#}(u_0, u_1)$ and $\overline{\#}(u_0, u_1)$ are finite for all solutions $\tau_j u_j = 0$, $j \in \{0, 1\}$.*

*(ii) An operator τ_1 is called **relatively oscillatory** with respect to τ_0, if $\underline{\#}(u_0, u_1)$ or $\overline{\#}(u_0, u_1)$ is infinite for two (and hence for all (cf. Corollary 3.5)) solutions $\tau_j u_j = 0$, $j \in \{0, 1\}$.*

This definition of course induces a symmetric relation. So we can also say that *two operators τ_j, $j \in \{0, 1\}$, are relatively nonoscillatory (resp. relatively oscillatory).*

In the following theorem we will connect operators' property of being relatively nonoscillatory with the spectra of self-adjoint operators.

Theorem 3.7.
Let H_j be self-adjoint operators associated with τ_j, $j \in \{0, 1\}$, and $\lambda_0 < \lambda_1$. Then

(i) *$\tau_0 - \lambda_0$ is relatively nonoscillatory with respect to $\tau_0 - \lambda_1$ if and only if* $\dim \operatorname{Ran} P_{H_0}(\lambda_0, \lambda_1) < \infty$.

(ii) *Suppose* $\dim \operatorname{Ran} P_{H_0}(\lambda_0, \lambda_1) < \infty$ *and $\tau_1 - \lambda$ is relatively nonoscillatory with respect to $\tau_0 - \lambda$ for one $\lambda \in [\lambda_0, \lambda_1]$. Then we have: $\tau_1 - \lambda$ is relatively nonoscillatory with respect to $\tau_0 - \lambda$ for all $\lambda \in [\lambda_0, \lambda_1]$* $\Leftrightarrow \dim \operatorname{Ran} P_{H_1}(\lambda_0, \lambda_1) < \infty$.

Proof. (i) This is [18, Thm. 4.5].
(ii) Let $\lambda, \tilde{\lambda} \in [\lambda_0, \lambda_1]$, $\tau_j u_j(\lambda) = \lambda u_j(\lambda)$, $\tau_j u_j(\tilde{\lambda}) = \tilde{\lambda} u_j(\tilde{\lambda})$, $j \in \{0, 1\}$ and $\tau_j - \lambda$, relatively nonoscillatory. Then applying our triangle inequality twice implies

$$\overline{\#}(u_0(\tilde{\lambda}), u_0(\lambda)) + \overline{\#}(u_0(\lambda), u_1(\lambda)) + \overline{\#}(u_1(\lambda), u_1(\tilde{\lambda})) - 2 \leq$$
$$\overline{\#}(u_0(\tilde{\lambda}), u_1(\tilde{\lambda})) \leq$$
$$\overline{\#}(u_0(\tilde{\lambda}), u_0(\lambda)) + \overline{\#}(u_0(\lambda), u_1(\lambda)) + \overline{\#}(u_1(\lambda), u_1(\tilde{\lambda})) + 2$$

and similar estimates with $\overline{\#}$ replaced by $\underline{\#}$. Hence due to (i), "$\Rightarrow$" follows by the first inequality and "$\Leftarrow$" by the second. □

The case of equal operators $H_0 = H_1$ but different spectral parameters $\lambda_0 \neq \lambda_1$, as in item (i) in the above theorem, is called renormalized oscillation theory by Gesztesy, Simon, and Teschl (see [3]).

Remark 3.8.
We remark that $A \geq \alpha\mathbb{1}$ and $B \geq \beta\mathbb{1}$ imply that $A + B \geq (\alpha + \beta)\mathbb{1}$ because $\langle \phi, ((A+B) - (\alpha+\beta))\phi \rangle = \langle \phi, (A-\alpha)\phi \rangle + \langle \phi, (B-\beta)\phi \rangle \geq 0+0 = 0$. Furthermore we point out that $\|A\| \leq \alpha$ implies $-\alpha\mathbb{1} \leq A \leq \alpha\mathbb{1}$.

For a practical use of the previous theorem we require criteria when $\tau_1 - \lambda$ is relatively nonoscillatory with respect to $\tau_0 - \lambda$ for λ inside an essential spectral gap. Therefore we establish the following lemma:

Lemma 3.9.
Let $\lim_{x \to a} \|\phi_0(x) - \phi_1(x)\| = 0$ if τ_0 or τ_1 is not regular at a and similarly, $\lim_{x \to b} \|\phi_0(x) - \phi_1(x)\| = 0$ if τ_0 or τ_1 is not regular at b. Then $\sigma_{ess}(H_0) = \sigma_{ess}(H_1)$ and $\tau_1 - \lambda$ is relatively nonoscillatory with respect to $\tau_0 - \lambda$ for $\lambda \in \mathbb{R} \backslash \sigma_{ess}(H_0)$.

Proof. We can write τ_1 as $\tau_1 = \tau_0 + \tilde{\phi}_0 + \tilde{\phi}_1$, where $\tilde{\phi}_0$ has compact support near singular endpoints and $\|\tilde{\phi}_1\| < \varepsilon$, for arbitrarily small $\varepsilon > 0$. Using [24, Satz 9.9] and Theorem 5.4 we infer that $R_{H_1}(z) - R_{H_0}(z)$ is the norm limit of compact operators. Thus $R_{H_1}(z) - R_{H_0}(z)$ is compact and hence $\sigma_{ess}(H_0) = \sigma_{ess}(H_1)$.

Let $\delta > 0$ be the distance of λ to the essential spectrum and choose $a < c < d < b$, such that

$$\|(\phi_1(x) - \phi_0(x))\| \leq \delta/2, \qquad x \notin (c, d).$$

Clearly $\#_{(c,d)}(u_0,u_1)<\infty$, since both operators are regular on (c,d). Moreover, observe that

$$\phi_0-\lambda_+\mathbb{1}\le\phi_1-\lambda\mathbb{1}\le\phi_0-\lambda_-\mathbb{1},\qquad \lambda_\pm=\lambda\pm\delta/2,$$

on $I=(a,c)$ or $I=(d,b)$. Then Theorem 3.7 (i) implies $\overline{\#}(u_0(\lambda_-),u_0(\lambda_+))<\infty$ and invoking Theorem 3.3 shows $\overline{\#}(u_0(\lambda_\pm),u_1(\lambda))<\infty$. From Theorem 3.4 and Theorem 3.7 (i) we infer

$$\overline{\#}(u_0(\lambda),u_1(\lambda))\le\overline{\#}(u_0(\lambda),u_0(\lambda_+))+\overline{\#}(u_0(\lambda_+),u_1(\lambda))+1<\infty,$$

and similarly for $\underline{\#}(u_0(\lambda),u_1(\lambda))$. This shows that $\tau_1-\lambda$ is relatively nonoscillatory with respect to τ_0. □

Our next goal is to relate the number of weighted sign flips with the spectra of H_1 and H_0. The special case $H_0=H_1$ is covered by [18]:

Theorem 3.10 (cf. [18, Thm. 4.5])**.**
Let H be a self-adjoint operator associated with τ and suppose $(\psi_+(\lambda_0),\psi_-(\lambda_1))$ or $(\psi_-(\lambda_0),\psi_+(\lambda_1))$ exist.
Then

$$\dim\operatorname{Ran}P_{(\lambda_0,\lambda_1)}(H)=\#(\psi_\mp(\lambda_0),\psi_\pm(\lambda_1)).\tag{3.5}$$

If H is limit point at a (resp. b) we can replace $\psi_-(\lambda_i)$ (resp. $\psi_+(\lambda_i)$), $i\in\{0,1\}$, by any solution of $\tau u=\lambda_i u$.

Remark 3.11.
Note that both sides in (3.5) equal ∞ if $(\lambda_0,\lambda_1)\cap\sigma_{ess}(H_0)\neq\emptyset$, which follows from Theorem 3.7 (i).

Combining this theorem with the triangle inequality (3.3) we obtain a first estimate:

Lemma 3.12.
Let H_i, $i\in\{0,1\}$ be a self-adjoint operator associated with τ_i and separated boundary conditions. Suppose that $(\lambda_0,\lambda_1)\subseteq\mathbb{R}\backslash(\sigma_{ess}(H_0)\cup\sigma_{ess}(H_1))$, then

$$\begin{aligned}\dim\operatorname{Ran}P_{H_1}(\lambda_0,\lambda_1)&-\dim\operatorname{Ran}P_{H_0}(\lambda_0,\lambda_1)\\&\le\underline{\#}(\psi_{1,\mp}(\lambda_1),\psi_{0,\pm}(\lambda_1))-\overline{\#}(\psi_{1,\mp}(\lambda_0),\psi_{0,\pm}(\lambda_0))+2,\end{aligned}\tag{3.6}$$

respectively,

$$\begin{aligned}\dim\operatorname{Ran}P_{H_1}(\lambda_0,\lambda_1)&-\dim\operatorname{Ran}P_{H_0}(\lambda_0,\lambda_1)\\&\ge\overline{\#}(\psi_{1,\mp}(\lambda_1),\psi_{0,\pm}(\lambda_1))-\underline{\#}(\psi_{1,\mp}(\lambda_0),\psi_{0,\pm}(\lambda_0))-2.\end{aligned}\tag{3.7}$$

Proof. By the triangle inequality (cf. Theorem 3.4) we have

$$\begin{aligned}&\#_{(c,d)}(\psi_{1,-}(\lambda_1),\psi_{1,+}(\lambda_0))-\#_{(c,d)}(\psi_{0,-}(\lambda_1),\psi_{0,+}(\lambda_0))\\&\quad\le\#_{(c,d)}(\psi_{1,-}(\lambda_1),\psi_{0,+}(\lambda_1))+\#_{(c,d)}(\psi_{1,-}(\lambda_1),\psi_{0,+}(\lambda_1))+2.\end{aligned}$$

The result now follows by taking limits using that

$$\lim_{c\downarrow a, d\uparrow b} \#_{(c,d)}(\psi_{1,-}(\lambda_1), \psi_{1,+}(\lambda_0)) = \dim \operatorname{Ran} P_{H_1}(\lambda_0, \lambda_1)$$

and

$$\lim_{c\downarrow a, d\uparrow b} \#_{(c,d)}(\psi_{0,-}(\lambda_0), \psi_{0,+}(\lambda_1)) = -\dim \operatorname{Ran} P_{H_0}(\lambda_0, \lambda_1)$$

by the previous theorem. The second claim follows similarly. □

Chapter 4

Approximation by regular operators

Our presentation in this chapter follows [8, Sec. 6]. See also [26, Chap. 14].

In this chapter we want to extend some of our results for regular operators to the general case. The key to achieve this is the approximation of an operator by a sequence of regular operators.

We abbreviate in the following $L^2((c,d),\mathbb{C}^2)$ as $L^2(c,d)$. Fix functions $u, v \in \mathfrak{D}(\tau)$ and pick sequences $(c_n)_n \downarrow a$, $(d_n)_n \uparrow b$. We define

$$\tilde{H}_n : \begin{array}{lcl} \mathfrak{D}(\tilde{H}_n) & \rightarrow & L^2(c_n, d_n) \\ f & \mapsto & \tau f \end{array} , \tag{4.1}$$

where

$$\mathfrak{D}(\tilde{H}_n) = \{f \in L^2(c_n, d_n) |\ f, pf' \in AC((c_n, d_n), \mathbb{C}^2),\ \tau f \in L^2(c_n, d_n), \\ W_{c_n}(u, f) = W_{d_n}(v, f) = 0\}. \tag{4.2}$$

Take $H_n = \alpha\mathbb{1} \oplus \tilde{H}_n \oplus \alpha\mathbb{1}$ on $L^2(a,b) = L^2(a, c_n) \oplus L^2(c_n, d_n) \oplus L^2(d_n, b)$, where α is a fixed real constant. Then we have the following result:

Lemma 4.1.
Suppose that either H is limit point or that u fulfills the boundary condition at a and similarly, that either H is limit point or v fulfills the boundary condition at b. Then H_n converges to H in strong resolvent sense as $n \to \infty$.

To estimate the dimension of the eigenspaces of H_n the following result will be valuable in the sequel of this chapter.

Lemma 4.2 (Estimating eigenspaces, cf. [19, Thm. 4.12(ii)])**.**
Suppose A is a self-adjoint operator and $\psi_j \in \mathfrak{D}(A)$, $1 \leq j \leq k$, are linearly independent. Let $\lambda_0 < \lambda_1$.
If

$$\|(A - \frac{\lambda_1 + \lambda_0}{2})\psi\| < \frac{\lambda_1 - \lambda_0}{2}\|\psi\| \tag{4.3}$$

for any $0 \neq \psi \in \operatorname{span}\{\psi_j\}_{j=1}^k$*, than*

$$\dim \operatorname{Ran}(P_A(\lambda_0, \lambda_1)) \geq k. \tag{4.4}$$

Remark 4.3.
We remark that for a self-adjoint projector P *we have*

$$\dim \operatorname{Ran}(P) = \operatorname{tr}(P) = \|P\|_{\mathcal{J}^1}, \tag{4.5}$$

where $\|.\|_{\mathcal{J}^1}$ *denotes the trace class norm. If* P *is not finite-rank, all three numbers equal* ∞*.*

The standard approximation technique only implies strong convergence (Cf. Lemma 4.1), which unfortunately is not sufficient for our purpose. Hence our argument is based on a refinement of a method by Stolz and Weidmann [16] which will provide convergence of spectral projections in the trace norm for suitably chosen regular operators (see [27] for a nice overview).

Lemma 4.4 (cf. [17, Lem. 2], see also [16])**.**
Let $A_n \to A$ *in strong resolvent sense and* $\operatorname{tr}(P_{A_n}(\lambda_0, \lambda_1)) \leq \operatorname{tr}(P_A(\lambda_0, \lambda_1))$*.*
Then,

$$\lim_{n\to\infty} \operatorname{tr}(P_{A_n}(\lambda_0, \lambda_1)) = \operatorname{tr}(P_A(\lambda_0, \lambda_1)), \tag{4.6}$$

and if $\operatorname{tr}(P_A(\lambda_0, \lambda_1)) < \infty$*, we have*

$$\lim_{n\to\infty} \| \operatorname{tr}(P_{A_n}(\lambda_0, \lambda_1)) - \operatorname{tr}(P_A(\lambda_0, \lambda_1))\|_{\mathcal{J}^1} = 0. \tag{4.7}$$

Proof. This follows from (see e.g. [3, Lem. 5.2])

$$\operatorname{tr}(P_{(\lambda_0,\lambda_1)}(A)) \leq \liminf_{n\to\infty} \operatorname{tr}(P_{(\lambda_0,\lambda_1)}(A_n)), \tag{4.8}$$

together with Grümm's theorem ([14, Thm. 2.19]). □

Lemma 4.5 ([16])**.**
Suppose $[\lambda_0, \lambda_1] \cap \sigma_{ess}(H) = \emptyset$ *and let* H_n *be defined as in (4.1) with* $\alpha \notin [\lambda_0, \lambda_1]$*,* $u = \psi_-(\lambda_-)$*,* $v = \psi_+(\lambda_+)$ *and* $\lambda_\pm \in [\lambda_0, \lambda_1]$*.*
Then,

$$\operatorname{tr}(P_{\tilde{H}_n}(\lambda_0, \lambda_1)) \leq \operatorname{tr}(P_H(\lambda_0, \lambda_1)). \tag{4.9}$$

Furthermore, if H *is limit point at* a *(resp.* b*), we can replace* u *(resp.* v*) by any solution of* $\tau u = \lambda_- u$ *(resp.* $\tau u = \lambda_+ u$*).*

Proof. Abbreviate $P = P_H(\lambda_0, \lambda_1)$, $P_n = P_{\tilde{H}_n}(\lambda_0, \lambda_1)$. For $\tilde{\psi}_1, \dots, \tilde{\psi}_k$ being eigenfunctions of $\tilde{H}_n$ corresponding to eigenvalues in (λ_0, λ_1), construct

$$\psi_j(x) = \begin{cases} \eta_j u(x), & x < c_n, \\ \tilde{\psi}_j(x), & c_n \leq x \leq d_n, \\ \nu_j v(x), & x > d_n, \end{cases}$$

where η_j, ν_j are chosen such that ψ_j is continuous. According to Lemma 4.2 we have to investigate

$$\|(H - \frac{\lambda_1 + \lambda_0}{2})\psi\|,$$

where $0 \neq \psi \in \operatorname{span}\{\psi_j\}_{j=1}^k$.

Therefore, we split its square into three parts:

$$\underbrace{\int_a^{c_n} \|(H - \frac{\lambda_1+\lambda_0}{2})\psi\|^2}_{=:(1)} + \underbrace{\int_{c_n}^{d_n} \|(H - \frac{\lambda_1+\lambda_0}{2})\psi\|^2}_{=:(2)} + \underbrace{\int_{d_n}^{b} \|(H - \frac{\lambda_1+\lambda_0}{2})\psi\|^2}_{=:(3)}$$

For the first term we have

$$(1) = \left|\lambda_- - \frac{\lambda_1+\lambda_0}{2}\right|^2 \int_a^{c_n} \|\psi\|^2 \leq \left(\frac{\lambda_1-\lambda_0}{2}\right)^2 \int_a^{c_n} \|\psi\|^2$$

and similary for the third.

Moreover,

$$(2) < \left(\frac{\lambda_1-\lambda_0}{2}\right)^2 \int_{c_n}^{d_n} \|\psi\|^2$$

since the $\{\psi_j\}_{j=1}^k$ correspond to eigenvalues in (λ_0, λ_1). Altogether we have the required inequality (4.3). □

Remark 4.6.
The requirement $[\lambda_0, \lambda_1] \cap \sigma_{ess}(H) = \emptyset$ on the one hand ensures the existence of the Weyl solutions $\psi_-(\lambda_-)$ and $\psi_+(\lambda_+)$. On the other hand for this reason we of course have $\operatorname{tr}(P_H(\lambda_0, \lambda_1)) < \infty$.

Corollary 4.7.
Suppose $[\lambda_0, \lambda_1] \cap \sigma_{ess}(H) = \emptyset$ and let H_n be defined as in (4.1) with $\alpha \notin [\lambda_0, \lambda_1]$, $u = \psi_-(\lambda_-)$, $v = \psi_+(\lambda_+)$ and $\lambda_\pm \in [\lambda_0, \lambda_1]$.

Then,

$$\lim_{n\to\infty} \operatorname{tr}(P_{\tilde{H}_n}(\lambda_0, \lambda_1)) = \operatorname{tr}(P_H(\lambda_0, \lambda_1)). \tag{4.10}$$

Furthermore, if H is limit point at a (resp. b), we can replace u (resp. v) by any solution of $\tau u = \lambda_- u$ (resp. $\tau u = \lambda_+ u$).

Proof. This follows by the last three lemmas. □

Lemma 4.8 ([16])**.**
Suppose $[\lambda_0, \lambda_1] \cap \sigma_{ess}(H) = \emptyset$. Let $\hat{\tau} = \tau + \hat{\phi}$, where $\|\hat{\phi}(x)\|$ is bounded, and pick the same boundary conditions for $\hat{H}$ as for H (if any). Abbreviate

$$\begin{aligned} Q_a &:= [l_a, r_a] := [\liminf_{x\to a} E_<(\hat{\phi}(x)), \limsup_{x\to a} E_>(\hat{\phi}(x))], \\ Q_b &:= [l_b, r_b] := [\liminf_{x\to b} E_<(\hat{\phi}(x)), \limsup_{x\to b} E_>(\hat{\phi}(x))], \end{aligned} \tag{4.11}$$

where $E_<(\hat{\phi}(x)) \leq E_>(\hat{\phi}(x))$ are the eigenvalues of $\hat{\phi}(x)$. Choose λ_- such that one of following conditions holds:

(i) *$\lambda_- - Q_a \subseteq (\lambda_0, \lambda_1)$, or*

(ii) *$\lambda_- - Q_a \subseteq [\lambda_0, \lambda_1)$ and $E_>(\hat{\phi}(x)) \leq r_a$ near a, or*

(iii) $\lambda_- - Q_a \subseteq (\lambda_0, \lambda_1]$ *and* $E_<(\hat{\phi}(x)) \geq l_a$ *near* a.

Similarly, choose λ_+ *to satisfy one of these conditions with* a *replaced by* b.

Then, H_n *defined as in (4.1) with* $\alpha \notin [\lambda_0, \lambda_1]$, $u = \hat{\psi}_-(\lambda_-)$, $v = \hat{\psi}_+(\lambda_+)$, *satisfies*

$$\limsup_{n\to\infty} \operatorname{tr}(P_{(\lambda_0,\lambda_1)}(\tilde{H}_n)) \leq \operatorname{tr}(P_{(\lambda_0,\lambda_1)}(H)). \tag{4.12}$$

Again u *(resp.* v*) can be replaced by any solution of* $\hat{\tau}u = \lambda_- u$ *(resp.* $\hat{\tau}u = \lambda_+ u$*) if* $\hat{\tau}$ *is limit point at* a *(resp.* b*).*

Proof. Any of our three conditions implies $\|\lambda_- \mathbb{1} - \hat{\phi}(x) - \frac{(\lambda_1+\lambda_0)}{2}\mathbb{1}\| \leq \frac{(\lambda_1-\lambda_0)}{2}$ for x sufficiently close to a, respectively $\|\lambda_+ \mathbb{1} - \hat{\phi}(x) - \frac{(\lambda_1+\lambda_0)}{2}\mathbb{1}\| \leq \frac{(\lambda_1-\lambda_0)}{2}$ for x sufficiently close to b.

We again consider a splitting as in the proof of the previous lemma and obtain:

$$(1) = \int_a^{c_n} \|(\lambda_- - \hat{\phi} - \frac{\lambda_1+\lambda_0}{2})\psi\|^2 \leq \left(\frac{\lambda_1-\lambda_0}{2}\right)^2 \int_a^{c_n} \|\psi\|^2$$

and similary for (3). The strict inequality for the second term resembles that one of the previous lemma and hence the second claim follows. □

Since our results involve projections to half-open intervals, we need one further step.

Corollary 4.9.
Assume the same requirements as in the previous lemma, apart from the conditions

$$\lambda_- - Q_a \subseteq (\lambda_0, \lambda_1], \quad \lambda_+ - Q_b \subseteq (\lambda_0, \lambda_1] \text{ and } E_<(\hat{\phi}(x)) \leq l_a, l_b \text{ near } a \text{ and } b,$$

then

$$\limsup_{n\to\infty} \operatorname{tr}(P_{\tilde{H}}(\lambda_0, \lambda_1]) \leq \operatorname{tr}(P_H(\lambda_0, \lambda_1]). \tag{4.13}$$

And similarly, if

$$\lambda_- - Q_a \subseteq [\lambda_0, \lambda_1), \quad \lambda_+ - Q_b \subseteq [\lambda_0, \lambda_1) \text{ and } E_>(\hat{\phi}(x)) \geq r_a, r_b \text{ near } a \text{ and } b,$$

then

$$\limsup_{n\to\infty} \operatorname{tr}(P_{\tilde{H}_n}[\lambda_0, \lambda_1)) \leq \operatorname{tr}(P_H[\lambda_0, \lambda_1)). \tag{4.14}$$

Proof. We just prove the first claim. The second is similar.

Choose $\varepsilon > 0$ so small that we have $\operatorname{tr}(P_H(\lambda_1, \lambda_1+\varepsilon)) = 0$ (which is possible because the eigenvalues of H cannot accumulate at $\lambda_1 \notin \sigma_{ess}(H)$). Thus by the previous lemma

$$\limsup_{n\to\infty} \operatorname{tr}(P_{\tilde{H}_n}(\lambda_0, \lambda_1+\varepsilon)) \leq \operatorname{tr}(P_H(\lambda_0, \lambda_1+\varepsilon))$$

and

$$\limsup_{n\to\infty} \operatorname{tr}(P_{\tilde{H}_n}(\lambda_1, \lambda_1+\varepsilon)) \leq \operatorname{tr}(P_H(\lambda_1, \lambda_1+\varepsilon)) = 0.$$

So the $\limsup_{n\to\infty}$ can be replaced by $\lim_{n\to\infty}$ in the second inequality.

Hence the result follows from $P_{(\lambda_0,\lambda_1]} = P_{(\lambda_0,\lambda_1+\varepsilon)} - P_{(\lambda_1,\lambda_1+\varepsilon)}$. □

Corollary 4.10.
Suppose $[\lambda_0, \lambda_1] \cap \sigma_{ess}(H) = \emptyset$. Let $\hat{\tau} = \tau + \hat{\phi}$, where $\lim_{x\to a} \hat{\phi}(x) = 0$ and $\lim_{x\to b} \hat{\phi}(x) = 0$. Furthermore, we pick the same boundary conditions for $\hat{H}$ as for H (if any).

Define H_n as in (4.1) with $\alpha \notin [\lambda_0, \lambda_1]$, $u = \hat{\psi}_-(\lambda)$ and $v = \hat{\psi}_+(\lambda)$, $\lambda \in \{\lambda_0, \lambda_1\}$. If $\lambda = \lambda_1$ and $E_<(\hat{\phi}(x)) \le 0$ (near a and b), then

$$\limsup_{n\to\infty} \operatorname{tr}(P_{\tilde{H}_n}(\lambda_0, \lambda_1]) \le \operatorname{tr}(P_H(\lambda_0, \lambda_1]), \tag{4.15}$$

and if $\lambda = \lambda_0$ and $E_>(\hat{\phi}(x)) \ge 0$ (near a and b), then

$$\limsup_{n\to\infty} \operatorname{tr}(P_{\tilde{H}_n}[\lambda_0, \lambda_1)) \le \operatorname{tr}(P_H[\lambda_0, \lambda_1)). \tag{4.16}$$

Proof. This follows immediately from Corollary 4.9. □

In the next theorem this corollary allows eigenvalues at the boundary of the spectral intervals in the essential spectral gaps. But before this we state the following lemma:

Lemma 4.11.
Suppose $\tau_j u_j = \lambda_j u_j$, $j = 0, 1$, with $\phi_1 - \lambda_1 - \phi_0 + \lambda_0 \le 0$ near singular endpoints. If $\tau_j u_{j,n} = \lambda_{j,n} u_{j,n}$, where $\lambda_{j,n} \to \lambda_j$ and $u_{j,n} \to u_j$, uniformly on compact sets $[c, d] \subseteq (a, b)$, then

$$\liminf_{n\to\infty} \#(u_{0,n}, u_{1,n}) \ge \#(u_0, u_1). \tag{4.17}$$

Proof. Let $N \in \mathbb{N}_0$ be any finite number with $N \le \#(u_0, u_1)$. Choose a compact set $[c, d]$ containing N sign flips of $W(u_0, u_1)$. Then, for n sufficiently large, $W(u_{0,n}, u_{1,n})$ has N sign flips in $[c, d]$. Hence $\#(u_{0,n}, u_{1,n}) \ge \#_{(c,d)}(u_{0,n}, u_{1,n}) \ge N$ and the claim follows. □

After this preparations we are ready for

Theorem 4.12.
Let H_0, H_1 be self-adjoint operators associated with τ_0, τ_1, respectively, and separated boundary conditions. Suppose

(i) *$\phi_1 - \phi_0 \le 0$ near singular endpoints,*

(ii) *$\lim_{x\to a} \|\phi_0(x) - \phi_1(x)\| = 0$ if a is singular and $\lim_{x\to b} \|\phi_0(x) - \phi_1(x)\| = 0$ if b is singular,*

(iii) *H_0 and H_1 are associated with the same boundary conditions near a and b, that is, $\psi_{0,-}(\lambda)$ satisfies the boundary condition of H_1 at a (if any) and $\psi_{1,+}(\lambda)$ satisfies the boundary condition of H_0 at b (if any).*

Suppose $\lambda_0 < \inf \sigma_{ess}(H_0)$. Then

$$\dim \operatorname{Ran} P_{H_1}(-\infty, \lambda_0) - \dim \operatorname{Ran} P_{H_0}(-\infty, \lambda_0] = \#(\psi_{1,\mp}(\lambda_0), \psi_{0,\pm}(\lambda_0)). \tag{4.18}$$

Suppose $\sigma_{ess}(H_0) \cap [\lambda_0, \lambda_1] = \emptyset$. Then $\tau_1 - \lambda_0$ is nonoscillatory with respect to $\tau_0 - \lambda_0$ and

$$\begin{aligned} &\dim \operatorname{Ran} P_{H_1}[\lambda_0, \lambda_1) - \dim \operatorname{Ran} P_{H_0}(\lambda_0, \lambda_1] \\ &\qquad = \#(\psi_{1,\mp}(\lambda_1), \psi_{0,\pm}(\lambda_1)) - \#(\psi_{1,\mp}(\lambda_0), \psi_{0,\pm}(\lambda_0)). \end{aligned} \tag{4.19}$$

Proof. It suffices to show the $\#(\psi_{1,-}(\lambda_j), \psi_{0,+}(\lambda_j))$ case. Define $\tilde{H}_{j,n}$, $j = 0, 1$, as in (4.1) with $u = \psi_{1,-}(\lambda_0)$ and $v = \psi_{0,+}(\lambda_0)$.

Denote by $\psi^n_{j,\pm}(\lambda)$, $j \in \{0, 1\}$, the solutions of the approximating problems. Then, by Theorem 2.14,

$$\mathrm{tr}(P_{\tilde{H}_{1,n}})(-\infty, \lambda_0) - \mathrm{tr}(P_{\tilde{H}_{0,n}}(-\infty, \lambda_0]) = \#_{(c_n,d_n)}(\psi^n_{1,-}(\lambda_0), \psi^n_{0,+}(\lambda_0))$$

and we need to investigate the limits as $n \to \infty$.

First of all $\psi^n_{1,-}(\lambda_0, x) = \psi_{1,-}(\lambda_0, x)$, $\psi^n_{0,+}(\lambda_0, x) = \psi_{0,+}(\lambda_0, x)$ for $x \in (c_n, d_n)$ implies

$$\begin{aligned}\lim_{n\to\infty} \#_{(c_n,d_n)}(\psi^n_{1,-}(\lambda_0), \psi^n_{0,+}(\lambda_0)) &= \lim_{n\to\infty} \#_{(c_n,d_n)}(\psi_{1,-}(\lambda_0), \psi_{0,+}(\lambda_0)) \\ &= \#(\psi_{1,-}(\lambda_0), \psi_{0,+}(\lambda_0)).\end{aligned}$$

This takes care of the number of sign flips and it remains to look at the spectral projections. Let $\lambda_0 < \inf \sigma_{ess}(H_0)$, that is H_0 and hence also H_1 are bounded from below (cf. [24, Korollar 8.25]). Replacing $P_{H_j}(-\infty, \lambda_0)$ by $P_{H_j}(\lambda, \lambda_0)$ with some λ below the spectrum of both H_0 and H_1 we infer from Corollary 4.10

$$\lim_{n\to\infty} \mathrm{tr}(P_{(-\infty,\lambda_0)}(\tilde{H}_{1,n})) = \mathrm{tr}(P_{(-\infty,\lambda_0)}(H_1))$$

and

$$\lim_{n\to\infty} \mathrm{tr}(P_{(-\infty,\lambda_0]}(\tilde{H}_{0,n})) = \mathrm{tr}(P_{(-\infty,\lambda_0]}(H_0)).$$

This settles the first claim (4.18), where $\lambda_0 < \sigma_{ess}(H_0)$.

For the second claim (4.19), we first note that $\tau_1 - \lambda_0$ is relatively nonoscillatory with respect to $\tau_0 - \lambda_0$ by Lemma 3.9. Next note that $\psi^n_{0,+}(\lambda_1, .) \to \psi_{0,+}(\lambda_1, .)$ pointwise, since we have

$$u^n_{0,+}(\lambda, x) = U(\lambda, x, x_0) \begin{pmatrix} 1 \\ m^n_{0,+}(\lambda, x_0) \end{pmatrix}, \tag{4.20}$$

by (1.24), where $U(z, x, x_0)$ is a fundamental system of solutions for $\tau_0 - \lambda$, and $m^n_{0,+}(\lambda)$ are the corresponding Weyl m-functions. Next, strong resolvent convergence implies convergence of the Weyl m-function and hence uniform convergence of $\psi^n_{0,+}(\lambda_1, x) \to \psi_{0,+}(\lambda_1, x)$ on compact sets. Clearly the same applies to $\psi^n_{1,-}(\lambda_1, x) \to \psi_{1,-}(\lambda_1, x)$. Thus, by Lemma 4.4, Corollary 4.10, and Lemma 4.11,

$$\begin{aligned}&\mathrm{tr}(P_{[\lambda_0,\lambda_1)}(H_1)) - \mathrm{tr}(P_{(\lambda_0,\lambda_1]}(H_0)) \\ &\qquad \geq \#(\psi_{1,-}(\lambda_1), \psi_{0,+}(\lambda_1)) - \#(\psi_{1,-}(\lambda_0), \psi_{0,+}(\lambda_0)).\end{aligned} \tag{4.21}$$

Repeating the argument with $u = \psi_{1,-}(\lambda_1)$ and $v = \psi_{0,+}(\lambda_1)$ shows that

$$\begin{aligned}&\mathrm{tr}(P_{[\lambda_0,\lambda_1)}(H_1)) - \mathrm{tr}(P_{(\lambda_0,\lambda_1]}(H_0)) \\ &\qquad \leq \#(\psi_{1,-}(\lambda_1), \psi_{0,+}(\lambda_1)) - \#(\psi_{1,-}(\lambda_0), \psi_{0,+}(\lambda_0)).\end{aligned} \tag{4.22}$$

This proves the second claim. □

Chapter 5

Approximation in trace norm

Our presentation in this chapters generalizes [8, Sec. 7] to the case of Dirac operators.

Now we begin with an alternative approach toward singular differential operators by proving the case where $\phi_1 - \phi_0$ has compact support. The next lemma would follow from Theorem 4.12, but to demonstrate that this approach is independent of the last, we will provide an alternative proof.

Lemma 5.1.
Let H_j, $j \in \{0,1\}$, be Sturm–Liouville operators on (a,b) associated with τ_j, and suppose that $\phi_1 - \phi_0$ has support in a bounded interval $(c,d) \subseteq (a,b)$, where $a < c$ if a is singular and $d < b$ if b is singular. Moreover, suppose H_0 and H_1 have the same boundary conditions (if any).

Suppose $\lambda_0 < \inf \sigma_{ess}(H_0)$. Then,

$$\dim \operatorname{Ran} P_{H_1}(-\infty, \lambda_0) - \dim \operatorname{Ran} P_{H_0}(-\infty, \lambda_0] = \#(\psi_{1,\mp}(\lambda_0), \psi_{0,\pm}(\lambda_0)). \quad (5.1)$$

Suppose $\sigma_{ess}(H_0) \cap [\lambda_0, \lambda_1] = \emptyset$. Then,

$$\begin{aligned} &\dim \operatorname{Ran} P_{H_1}[\lambda_0, \lambda_1) - \dim \operatorname{Ran} P_{H_0}(\lambda_0, \lambda_1] \\ &\qquad = \#(\psi_{1,\mp}(\lambda_1), \psi_{0,\pm}(\lambda_1)) - \#(\psi_{1,\mp}(\lambda_0), \psi_{0,\pm}(\lambda_0)). \end{aligned} \quad (5.2)$$

Proof. By splitting $\phi_1 - \phi_0$ into a positive and negative part as in the proof of the regular case (Theorem 2.14), we can reduce it to the case where $\phi_1 - \phi_0$ is of one sign, say $\phi_1 - \phi_0 \geq 0$. Define $H_\varepsilon = \varepsilon H_1 + (1-\varepsilon) H_0$ and observe that $\psi_{\varepsilon,-}(\lambda, x) = \psi_{0,-}(\lambda, x)$ for $x \leq c$, respectively, $\psi_{\varepsilon,+}(\lambda, x) = \psi_{0,+}(\lambda, x)$ for $x \geq d$. Furthermore, $\psi_{\varepsilon,\pm}(\lambda, x)$ is analytic with respect to ε and $\lambda \in \sigma_p(H_\varepsilon)$ if and only if $W_d(\psi_{0,+}(\lambda), \psi_{\varepsilon,-}(\lambda)) = 0$. Now the proof can be done as in the regular case. □

Definition 5.2.
Let A be a densely defined closed linear operator. The self-adjoint operator $|A|$ is defined (as usual) by

$$|A| := \sqrt{S^* S} \geq 0. \quad (5.3)$$

Lemma 5.3 ([2, Thm. 2.7]).
Let $T : \mathfrak{D}(T) \to H_2$, $\mathfrak{D}(T) \subseteq H_1$, be a densely defined closed operator with polar decomposition

$$T = U|T| = |T^*|U = UT^*U \text{ on } \mathfrak{D}(T) = \mathfrak{D}(|T|). \tag{5.4}$$

In addition, assume that ϕ and ψ are Borel functions on $\mathbb{R}$ such that $\phi(\lambda)\psi(\lambda) = \lambda$, $\lambda \in \mathbb{R}$, and $\mathfrak{D}(|T|) \subseteq \mathfrak{D}(\psi(|T|))$. Then T has the representation

$$T = \phi(|T^*|)U\psi(|T|) \text{ on } \mathfrak{D}(T) = \mathfrak{D}(|T|), \tag{5.5}$$

with U a partial isometry.
In particular, for each $\alpha \in [0, 1]$,

$$T = |T^*|^\alpha U|T|^{1-\alpha} \text{ on } \mathfrak{D}(T) = \mathfrak{D}(|T|). \tag{5.6}$$

Lemma 5.4.
Suppose H_ε are defined as in the previous Lemma 5.1 and satisfy the same assumptions. Then,

$$\|\sqrt{|\phi_1 - \phi_0|}R_{H_\varepsilon}(z)\|_{\mathcal{J}_2} \le C(z), \qquad \varepsilon \in [0, 1]. \tag{5.7}$$

In particular, the resolvent difference of H_0 and H_1 is trace class and

$$\xi(\lambda, H_1, H_0) = \#(\psi_{1,\mp}(\lambda), \psi_{0,\pm}(\lambda)) \tag{5.8}$$

for every $\lambda \in \mathbb{R} \cap \rho(H_0) \cap \rho(H_1)$. Here $\xi(H_1, H_0)$ is assumed to be constructed such that $\varepsilon \mapsto \xi(H_\varepsilon, H_0)$ is a continuous mapping from $[0, 1] \to L^1(\mathbb{R}, (\lambda^2 + 1)^{-1}d\lambda)$.

Proof. Denote by

$$G_\varepsilon(z, x, y) = \frac{\psi_{\varepsilon,-}(z, x_<) \otimes \psi_{\varepsilon,+}(z, y_>)}{W(\psi_{\varepsilon,-}(z), \psi_{\varepsilon,+}(z))},$$

where $x_< = \min(x, y)$, $y_> = \max(x, y)$, the Green function of H_ε. As pointed out in the proof of the Lemma 5.1, $\psi_{\varepsilon,\pm}(z, x)$ is analytic (i.e. in particular continuous) with respect to ε. Thus we have the following estimate:

$$\int_a^b \int_a^b \|G_\varepsilon(z, x, y)\|^2 \|\phi_1(y) - \phi_0(y)\| dx\, dy \le C(z), \quad \varepsilon \in [0, 1],$$

which establishes the first claim.

Furthermore the second resolvent formula (1.21) (with $A = H_\varepsilon(z)$ and $B = (\varepsilon' - \varepsilon)(\phi_1 - \phi_0)$) implies

$$G_{\varepsilon'}(z, x, y) = G_\varepsilon(z, x, y) + (\varepsilon - \varepsilon') \int_a^b G_{\varepsilon'}(z, x, t)(\phi_1(t) - \phi_0(t))G_\varepsilon(z, t, y)dt.$$

Hence, using Lemma 5.3, we can carry out the following calculation:

$$\frac{1}{\varepsilon - \varepsilon'}(R_{H_{\varepsilon'}}(z) - R_{H_\varepsilon}(z))f(x) =$$
$$\int_a^b \int_a^b G_{\varepsilon'}(z,x,t)(\phi_1(t) - \phi_0(t))G_\varepsilon(z,t,y)\,dt\,f(y)\,dy =$$
$$\int_a^b \int_a^b G_{\varepsilon'}(z,x,y)\sqrt{|\phi_1(y) - \phi_0(y)|}U(y)\sqrt{|\phi_1(y) - \phi_0(y)|}G_\varepsilon(z,y,t)f(t)\,dt\,dy =$$
$$\int_a^b G_{\varepsilon'}(z,x,y)\sqrt{|\phi_1(y) - \phi_0(y)|}U(y)\int_a^b \sqrt{\phi_1(y) - \phi_0(y)}G_\varepsilon(z,y,t)f(t)\,dt\,dy =$$
$$R_{H_{\varepsilon'}}\sqrt{|\phi_1 - \phi_0|}U\int_a^b \sqrt{|\phi_1(x) - \phi_0(x)|}G_\varepsilon(z,x,t)f(t)\,dt =$$
$$R_{H_{\varepsilon'}}(z)\sqrt{|\phi_1 - \phi_0|}U\sqrt{|\phi_1 - \phi_0|}R_{H_\varepsilon}(z)f(x),$$

i.e., $R_{H_{\varepsilon'}}(z) - R_{H_\varepsilon}(z)$ can be written as the product of two Hilbert–Schmidt operators and a partial isometry. Now we can estimate its norm by the first claim:

$$\|R_{H_{\varepsilon'}}(z) - R_{H_\varepsilon}(z)\|_{\mathcal{J}_1} \leq |\varepsilon' - \varepsilon| C(z)^2. \tag{5.9}$$

Thus $\varepsilon \mapsto \xi(H_\varepsilon, H_0)$ is continuous by Lemma A.5. The rest follows from (A.4). □

Remark 5.5.
Compared to the case of Sturm–Liouville operators the proof of Lemma 5.4 is a bit more delicate because we had to state Lemma 5.3 to manage to extract a root of a matrix.

Hypothesis 5.6.
Suppose H_0 and V are self-adjoint such that:

(i) V is relatively bounded with respect to H_0 with H_0-bound less than one

(ii) $|V|^{1/2}R_{H_0}(z)$ is Hilbert–Schmidt for one (and hence for all) $z \in \rho(H_0)$.

We recall that (i) means that $\mathfrak{D}(V) \supseteq \mathfrak{D}(H_0)$ and for some $a < 1$, $b \geq 0$,

$$\|V\psi\| \leq a\|H_0\psi\| + b\|\psi\|, \quad \forall\, \psi \in \mathfrak{D}(H_0). \tag{5.10}$$

It will be shown in Appendix A that these conditions ensure that we can interpolate between H_0 and H_1 using operators H_ε, $\varepsilon \in [0,1]$, such that the resolvent difference of H_0 and H_ε is continuous in ε with respect to the trace norm. Hence we can fix $\xi(\lambda, H_1, H_0)$ by requiring $\varepsilon \mapsto \xi(\lambda, H_\varepsilon, H_0)$ to be continuous in $L^1(\mathbb{R}, (\lambda^2+1)^{-1}d\lambda)$, where we of course set $\xi(\lambda, H_0, H_0) = 0$ (see Lemma A.7). While ξ is only defined a.e., it is constant on the intersection of the resolvent sets $\mathbb{R} \cap \rho(H_0) \cap \rho(H_1)$, and we will require it to be continuous there. In particular, note that by Weyl's theorem the essential spectra of H_0 and H_1 are equal, $\sigma_{ess}(H_0) = \sigma_{ess}(H_1)$. Now we are ready for:

Theorem 5.7.
Let H_0, H_1 be self-adjoint operators such that H_0 and $V := \phi_0 - \phi_1$ satisfy Hypothesis 5.6. Then for every $\lambda \in \mathbb{R} \cap \rho(H_0) \cap \rho(H_1)$ we have

$$\xi(\lambda, H_1, H_0) = \#(\psi_{0,\pm}(\lambda), \psi_{1,\mp}(\lambda)). \tag{5.11}$$

Proof. We first assume that we have compact support near one endpoint, say a. Define by K_ε the operator of multiplication by $\chi_{(a,b_\varepsilon]}\mathbb{1}$ with $b_\varepsilon \uparrow b$ as $\varepsilon \uparrow 1$. Then K_ε satisfies the assumptions of Lemma A.7. Introduce $H_\varepsilon = H_0 - K_\varepsilon V$, and denote by $\psi_{\varepsilon,-}(\lambda, x)$ the corresponding solutions satisfying the boundary condition at a.

By Lemma A.7 we have $\xi(., H_\varepsilon, H_0) \to \xi(., H_1, H_0)$ as $\varepsilon \uparrow 1$ in $L^1(\mathbb{R}, (\lambda^2 + 1)^{-1} d\lambda)$. Moreover, $H_\varepsilon \to H_1$ in (trace) norm resolvent sense and hence $\lambda \in \rho(H_1)$ implies $\lambda \in \rho(H_\varepsilon)$ for ε sufficiently close to 1. Since $\xi(\lambda, H_\varepsilon, H_0) \in \mathbb{Z}$ is constant near every $\lambda \in \mathbb{R} \cap \rho(H_0) \cap \rho(H_\varepsilon)$, we must have $\xi(\lambda, H_\varepsilon, H_0) = \xi(\lambda, H_1, H_0)$ for $\varepsilon \geq \varepsilon_0$ with some ε_0 sufficiently close to 1.

Now let us turn to the Wronskians. We first prove the $\#(\psi_{1,-}(\lambda), \psi_{0,+}(\lambda))$ case. By Lemma 5.4 we know

$$\xi(\lambda, H_\varepsilon, H_0) = \#(\psi_{\varepsilon,-}(\lambda), \psi_{0,+}(\lambda)$$

for every $\varepsilon < 1$. Concerning the right-hand side observe that

$$W_x(\psi_{\varepsilon,-}(\lambda), \psi_{0,+}(\lambda)) = W_x(\psi_{1,-}(\lambda), \psi_{0,+}(\lambda))$$

for $x \leq b_\varepsilon$ and that $W_x(\psi_{\varepsilon,-}(\lambda), \psi_{0,+}(\lambda))$ is constant for $x \geq b_\varepsilon$. This implies that for $\varepsilon \geq \varepsilon_0$ we have

$$\begin{aligned}\xi(\lambda, H_1, H_0) &= \xi(\lambda, H_\varepsilon, H_0) = \#(\psi_{\varepsilon,-}(\lambda), \psi_{0,+}(\lambda)) \\ &= \#_{(a,b_\varepsilon)}(\psi_{\varepsilon,-}(\lambda), \psi_{0,+}(\lambda)) = \#_{(a,b_\varepsilon)}(\psi_{1,-}(\lambda), \psi_{0,+}(\lambda)).\end{aligned}$$

In particular, the last item $\#_{(a,b_\varepsilon)}(\psi_{1,-}(\lambda), \psi_{0,+}(\lambda))$ is eventually constant and thus has a limit which, by Definition 3.1, is $\#(\psi_{1,-}(\lambda), \psi_{0,+}(\lambda))$.

For the corresponding $\#(\psi_{1,+}(\lambda), \psi_{0,-}(\lambda))$ case one simply exchanges the roles of H_0 and H_1.

Hence the result holds if the perturbation has compact support near one endpoint. Now one repeats the argument to remove the compact support assumption near the other endpoint as well. □

Corollary 5.8.
Under the assumptions of Theorem 5.7 we have that $\tau_1 - \lambda$ is relatively nonoscillatory with respect to $\tau_0 - \lambda$ for every λ in an essential spectral gap.

Chapter 6

Relative oscillation criteria

Our presentation in this chapter uses results from [7, 12].

Now we want to apply relative oscillation theory to obtain criteria for when an edge of an essential spectral gap is an accumulation point of eigenvalues for Dirac operators.

We will assume that $a \in \mathbb{R}$ and that $b = \infty$. Furthermore, we always assume the usual local integrability assumptions on the coefficients (see Section 1.1).

Let H_0 be a given background operator associated with

$$\phi_0(x) = \phi_{\mathrm{el}}(x)\mathbb{1} + \phi_{\mathrm{am}}(x)\sigma_1 + (m + \phi_{\mathrm{sc}}(x))\sigma_3 \tag{6.1}$$

and suppose that E is a boundary point of the essential spectrum of H_0 (which is not an accumulation point of eigenvalues). Then we want to know when a perturbation

$$\phi_1(x) = (\phi_{\mathrm{el}}(x)+\tilde{\phi}_{\mathrm{el}}(x))\mathbb{1}+(\phi_{\mathrm{am}}(x)+\tilde{\phi}_{\mathrm{am}}(x))\sigma_1+(m+\phi_{\mathrm{sc}}(x)+\tilde{\phi}_{\mathrm{sc}}(x))\sigma_3 \tag{6.2}$$

gives rise to an infinite number of eigenvalues accumulating at E. By Theorem 3.7(ii), this question reduces to the question of when a given operator $\tau_1 - E$ is relatively oscillatory with respect to $\tau_0 - E$. That is, we have to investigate if the difference of Prüfer angles $\Delta_{1,0} = \theta_1 - \theta_0$ is bounded or not.

Hence the first step is to derive an ordinary differential equation for $\Delta_{1,0}$. While this can easily be done, the result turns out to be not very effective for our purpose. However, since the number of weighted sign flips $\#_{(c,d)}(u_0, u_1)$ is all we are eventually interested in, any *other* Prüfer angle which gives the same result will be as good:

Definition 6.1.
We will call a continuous function ψ a ***Prüfer angle*** *for the Wronskian $W(u_0, u_1)$, if $\#_{(c,d)}(u_0, u_1) = \lceil \psi(d)/\pi \rceil - \lfloor \psi(c)/\pi \rfloor - 1$ for any $c, d \in (a, b)$.*

Hence we will try to find a more effective Prüfer angle ψ than $\Delta_{1,0}$ for the Wronskian of two solutions.

Notation 6.2.
For a function $A : \mathbb{R} \to \mathbb{R}$ *we will denote* $\liminf_{x\to\infty} A(x)$ *by* A_- *and* $\limsup_{x\to\infty} A(x)$ *by* A_+.

In the following theorem we denote the Dirac operator associated with (6.1) by τ_0 and the Dirac operator associated with (6.2) by τ_1.

Theorem 6.3.
Let $a \in \mathbb{R}$ *and* $b = \infty$. *Assume that*

$$\tilde{\phi}_{\rm sc}(x) \sim \frac{\hat{\phi}_{\rm sc}}{x^2}, \quad \tilde{\phi}_{\rm am}(x) \sim \frac{\hat{\phi}_{\rm am}}{x^2}, \quad \tilde{\phi}_{\rm el}(x) \sim \frac{\hat{\phi}_{\rm el}}{x^2}, \quad (x \to \infty), \tag{6.3}$$

with constants $\hat{\phi}_{\rm sc}, \hat{\phi}_{\rm am}, \hat{\phi}_{\rm el} \in \mathbb{R}$.

Let E *be a boundary point of the essential spectrum such that there is a solution* u *of the unperturbed Dirac equation* $\tau_0 u = Eu$ *satisfying*

$$\|u\| = O(1) \quad \text{and} \quad \|u\|^{-1} = O(1). \tag{6.4}$$

Let $\ell > 0$, *then we define*

$$\begin{aligned} A(x) :=& -\frac{2}{\ell}\int_\ell^{\ell+x} \frac{\langle ((m+\phi_{\rm sc}(t))\sigma_3 + \phi_{\rm am}(t)\sigma_1)u(t), u(t)\rangle}{\|u(t)\|^4} dt, \\ B(x) :=& \frac{1}{\ell}\int_\ell^{\ell+x} \langle u(t), \hat{\phi}u(t)\rangle dt, \end{aligned} \tag{6.5}$$

where $\hat{\phi} := \hat{\phi}_{\rm el}\mathbb{1} + \hat{\phi}_{\rm am}\sigma_1 + \hat{\phi}_{\rm sc}\sigma_3$.

Then $\tau_0 - E$ *is relatively non-oscillatory with respect to* $\tau_1 - E$ *at* ∞ *if*

(i) $0 < A_- \le A_+ < \infty$ *and* $4A_+B_+ < 1$ *or*

(i)' $-\infty < A_- \le A_+ < 0$ *and* $4A_-B_- < 1$,

and relatively oscillatory if

(ii) $0 < A_- \le A_+ < \infty$ and $4A_-B_- > 1$ or

(ii)' $-\infty < A_- \le A_+ < 0$ and $4A_+B_+ > 1$.

The special case of periodic Dirac operators is treated by Karl Michael Schmidt in [12]. In this case we can choose the (anti-)periodic solution for u which clearly satisfies (6.4) and for ℓ the period length which implies that $A(x)$ and $B(x)$ are constant functions.

The proof will be given at the end of the chapter. Firstly we state the following Lemmata.

Notation 6.4.
Let $\ell > 0$. *We denote by*

$$\overline{g}(x) = \frac{1}{\ell}\int_x^{x+\ell} g(t)dt \tag{6.6}$$

the average of g *over an interval of length* ℓ.

Lemma 6.5 ([7, Lem. 5.3]).
Let φ obey the equation

$$\varphi'(x) = \rho(x)f(x) + o(\rho(x)), \qquad x \in (a, \infty), \tag{6.7}$$

where $f(x)$ is bounded. If

$$\frac{1}{\ell}\int_0^\ell |\rho(x+t) - \rho(x)|\, dt = o(\rho(x)) \tag{6.8}$$

then

$$\overline{\varphi}'(x) = \rho(x)\overline{f}(x) + o(\rho(x)). \tag{6.9}$$

Moreover, suppose $\rho(x) = o(1)$. If $f(x) = A(x)g(\varphi(x))$, where $A(x)$ is bounded and $g(x)$ is bounded and Lipschitz continuous, then

$$\overline{f}(x) = \overline{A}(x)g(\overline{\varphi}) + o(1). \tag{6.10}$$

Proof. To show the first statement observe

$$\begin{aligned}\overline{\varphi}'(x) = \frac{\varphi(x+\ell) - \varphi(x)}{\ell} &= \frac{1}{\ell}\int_x^{x+\ell} \rho(t)f(t)dt + o(\rho(x)) \\ &= \rho(x)\overline{f}(x) + \frac{1}{\ell}\int_x^{x+\ell}(\rho(t) - \rho(x))f(t)dt + o(\rho(x)).\end{aligned}$$

Now the first claim follows from (6.8) since f is bounded. Note that (6.8) implies that the $o(\rho)$ property is preserved under averaging.

To see the second, we use

$$\begin{aligned}\overline{f}(x) &= \frac{1}{\ell}\int_x^{x+\ell} A(t)g(\varphi(t))dt \\ &= \overline{A}(x)g(\overline{\varphi}(x)) + \frac{1}{\ell}\int_x^{x+\ell} A(t)(g(\varphi(t)) - g(\overline{\varphi}(x)))dt.\end{aligned}$$

Since g is Lipschitz we can use the mean value theorem together with

$$|\varphi(x+t)) - \overline{\varphi}(x)| \le C \sup_{0\le s\le \ell} \rho(x+s)$$

to finish the proof. □

Condition (6.8) is a strong version of saying that $\overline{\rho}(x) = \rho(x)(1 + o(1))$ (it is equivalent to the latter if ρ is monotone). It will be typically fulfilled if ρ decreases (or increases) polynomially (but not exponentially). For example, the condition holds if $\sup_{t\in[0,1]} \frac{\rho'(x+t)}{\rho(x)} \to 0$.

Corollary 6.6 ([7, Cor. 5.4]).
Let φ obey the equation

$$\varphi'(x) = \rho(x)\Big(A(x)\sin^2(\varphi(x)) + \sin(\varphi(x))\cos(\varphi(x)) + B(x)\cos^2(\varphi(x))\Big) + o(\rho(x)) \tag{6.11}$$

with A, B bounded functions and assume that $\rho = o(1)$ satisfies (6.8).

Then the averaged function $\overline{\varphi}$ *obeys the equation*

$$\overline{\varphi}'(x) = \rho(x)\Big(\overline{A}(x)\sin^2(\overline{\varphi}(x)) + \sin(\overline{\varphi}(x))\cos(\overline{\varphi}(x)) + \overline{B}(x)\cos^2(\overline{\varphi}(x))\Big) + o(\rho(x)). \tag{6.12}$$

Lemma 6.7 ([7, Lem. 5.1])**.**
Let $A, B \in \mathbb{R}$, $a > 0$ *and* $\varphi : (a, \infty) \to \mathbb{R}$. *The equation*

$$\varphi'(x) = \frac{1}{x}\Big(A\sin^2(\varphi(x)) + \cos(\varphi(x))\sin(\varphi(x)) + B\cos^2(\varphi(x))\Big) + o\Big(\frac{1}{x}\Big) \tag{6.13}$$

has only unbounded solutions at ∞ *if* $4AB > 1$ *and only bounded solutions if* $4AB < 1$.

Proof. By a straightforward computation we have

$$A\sin^2(\varphi) + \sin(\varphi)\cos(\varphi) + B\cos^2(\varphi) = \frac{A+B}{2} + \frac{\sqrt{1+(A-B)^2}}{2}\cos(2(\varphi - \varphi_0)).$$

for some constant $\varphi_0 = \varphi_0(A, B)$. Hence $\psi(x) = \varphi(x) - \varphi_0$ satisfies

$$\psi'(x) = \rho(x)\Big(\frac{A+B}{2} + \frac{\sqrt{1+(A-B)^2}}{2}\cos(2\psi(x))\Big) + o(\rho(x)) \tag{6.14}$$

If $4AB < 1$, we have $|A + B| < \sqrt{1+(A-B)^2}$ from which it follows that the right-hand side of our differential equation is strictly negative for $\psi(x) \pmod \pi$ close to $\pi/2$ and strictly positive if $\psi(x) \pmod \pi$ close to 0. Hence any solution remains in such a strip.

If $4AB > 1$, we have $|A + B| > \sqrt{1+(A-B)^2}$ and thus the right-hand side is always positive, $\psi'(x) \geq C\rho(x)$, if $A, B > 0$ and always negative, $\psi'(x) \leq -C\rho(x)$, if $A, B < 0$. Since ρ is not integrable by assumption, ψ is unbounded.

In order to derive the asymptotics, rewrite (6.14) as

$$\psi'(x) = \rho(x)\left(\frac{C+D}{2}\cos^2(\psi(x)) + \frac{C-D}{2}\sin^2(\psi(x))\right) + o(\rho(x)),$$

where $C = A + B$ and $D = \sqrt{1+(A-B)^2}$. Now, introduce

$$\tilde{\psi}(x) = \arctan\left(\sqrt{\frac{C-D}{C+D}}\tan(\psi(x))\right)$$

and observe $|\psi - \tilde{\psi}| < \pi$. Moreover,

$$\tilde{\psi}'(x) = \frac{\rho(x)}{2}\operatorname{sgn}(C+D)\sqrt{C^2-D^2} + o(\rho(x)).$$

Hence the claim follows since by assumption $4AB > 1$, which implies $\operatorname{sgn}(C + D) = \operatorname{sgn}(A)$. □

Corollary 6.8 ([7, Cor. 5.2])**.**
Let $a > 0$ and $\varphi : (a, \infty) \to \mathbb{R}$. Suppose $0 < A_- \leq A_+ < \infty$ and $-\infty < B_- \leq B_+ < \infty$. All solutions of the equation

$$\varphi'(x) = \frac{1}{x}\Big(A(x)\sin^2(\varphi(x)) + \cos(\varphi(x))\sin(\varphi(x)) + B(x)\cos^2(\varphi(x))\Big) + o\Big(\frac{1}{x}\Big) \tag{6.15}$$

are unbounded at ∞ if

$$4A_-B_- > 1 \tag{6.16}$$

and bounded if

$$4A_+B_+ < 1 \tag{6.17}$$

Proof. If $4A_-B_- > 1$, there is a $\varepsilon > 0$ such that $(A_- - \varepsilon)(B_- - \varepsilon) > 1$. Choose $\tilde{a} > a$ such that $A(x) \geq A_- - \varepsilon$ and $B(x) \geq B_- - \varepsilon$ for $x \in (\tilde{a}, \infty)$ holds. The solution $\varphi_-(x)$ of

$$\begin{aligned}\varphi'_-(x) =& \frac{1}{x}\Big((A_- - \varepsilon)\sin^2(\varphi_-(x)) + \cos(\varphi_-(x))\sin(\varphi_-(x)) + \\ & (B_- - \varepsilon))\cos^2(\varphi_-(x))\Big) + o\Big(\frac{1}{x}\Big)\end{aligned}$$

with $\varphi_-(\tilde{a}) < \varphi(a)$, is an unbounded subsolution of φ. Hence by [20, Lemma 1.1], $\varphi(x) \geq \varphi_-(x)$ for $x \geq \tilde{a}$. So $\varphi(x)$ is unbounded.

If $4A_+B_+ < 1$, then there is a $\varepsilon > 0$ such that $(A_+ + \varepsilon)(B_+ + \varepsilon) < 1$. Choose $\tilde{a} > a$ such that $A(x) \leq A_+ + \varepsilon$ and $B(x) \leq B_+ + \varepsilon$ for $x \in (\tilde{a}, \infty)$. Now the solution $\varphi_+(x)$ of

$$\begin{aligned}\varphi'_+(x) =& \frac{1}{x}\Big((A_+ + \varepsilon)\sin^2(\varphi_+(x)) + \cos(\varphi_+(x))\sin(\varphi_+(x)) + \\ & (B_+ + \varepsilon))\cos^2(\varphi_+(x))\Big) + o\Big(\frac{1}{x}\Big)\end{aligned}$$

with $\varphi_+(\tilde{a}) > \varphi(a)$, is a bounded supersolution of $\varphi(x)$. Hence $\varphi(x) \leq \varphi_+(x) \leq C_+$ for $x \geq \tilde{a}$. So $\varphi(x)$ is bounded from above.

Similary, there are $\varepsilon > 0$ and $\tilde{a} > a$ such that $A(x) \geq A_- - \varepsilon$, $B(x) \geq B_- - \varepsilon$ for $x \geq \tilde{a}$ and $\underbrace{(A_- - \varepsilon)}_{>0}\underbrace{(B_- - \varepsilon)}_{>0} \leq A_-B_- \leq A_+B_+ < 1$,
if $B_- > 0$ (otherwise trivial). So we can find a bounded subsolution $\varphi_-(x)$ with $\varphi(x) \geq \varphi_-(x) \geq C_-$ for $x \geq \tilde{a}$. So $\varphi(x)$ remains bounded from below. □

Remark 6.9.
The case $A_+ < 0$ can be reduced to $A_- > 0$ by the transformation $(\varphi(x), A(x), B(x)) \mapsto (-\varphi(x), -A(x), -B(x))$.

Now we turn to the proof of our main theorem in this chapter.

Proof of Theorem 6.3. Let u be a $\mathbb{R}^2$-valued solution of the unperturbed Dirac system with spectral parameter E_0 and v given by Lemma 1.5. Furthermore, let w be a $\mathbb{R}^2$-valued solution of the perturbed system. Then, denoting the PrŸfer angles of u, v and w by θ_1, θ_2 and θ and by ρ_1, ρ_2 and ρ the corresponding PrŸfer radii.

Trough the ansatz $w(x) =: a(u(x)\sin(\gamma(x)) - v(x)\cos(\gamma(x)))$, i.e.

$$\rho\begin{pmatrix}\sin(\theta(x))\\ \cos(\theta(x)))\end{pmatrix} =: \begin{pmatrix}\rho_1\sin(\theta_1(x)) & \rho_2\sin(\theta_2(x))\\ \rho_1\cos(\theta_1(x)) & \rho_2\cos(\theta_2(x))\end{pmatrix}\begin{pmatrix}a\sin(\gamma(x))\\ -a\cos(\gamma(x))\end{pmatrix}$$

we find

$$\begin{pmatrix}\rho(x)\sin(\theta(x)) & \rho_1(x)\sin(\theta_1(x))\\ \rho(x)\cos(\theta(x)) & \rho_1(x)\cos(\theta_1(x))\end{pmatrix} = \begin{pmatrix}\rho_1(x)\sin(\theta_1(x)) & \rho_2(x)\sin(\theta_2(x))\\ \rho_1(x)\cos(\theta_1(x)) & \rho_2(x)\cos(\theta_2(x))\end{pmatrix}\begin{pmatrix}a\sin(\gamma(x)) & 1\\ -a\cos(\gamma(x)) & 0\end{pmatrix}.$$

Taking determinants shows $W_x(w, u) = a\cos(\theta(x))$. Similarly we obtain $W_x(w, v) = a\sin(\theta(x))$. We follow

$$a\begin{pmatrix}\sin(\gamma(x))\\ -\cos(\gamma(x))\end{pmatrix} = \rho(x)\begin{pmatrix}-\rho_2(x)\sin(\theta(x) - \theta_2(x))\\ \rho_1(x)\sin(\theta(x) - \theta_1(x))\end{pmatrix}.$$

Here γ is the PrŸfer angle for the Wronskian. Through further calculation we find

$$\begin{aligned}\tan(\gamma(x)) =& \frac{\rho_2(x)}{\rho_1(x)}\cdot\frac{\sin(\theta(x) - \theta_2(x))}{\sin(\theta(x) - \theta_1(x))} = \\ & \frac{\rho_2(x)}{\rho_1(x)}\sin(\theta_2(x) - \theta_1(x))\frac{\cos(\theta(x) - \theta_2(x) + \frac{\pi}{2})}{\sin(\theta(x) - \theta_1(x))\sin(\theta_2(x) - \theta_1(x))} = \\ & \frac{\rho_2(x)}{\rho_1(x)}\sin(\theta_2(x) - \theta_1(x))\frac{\cos(\theta(x) - \theta_1(x) + \frac{\pi}{2} - \theta_2(x) + \theta_1(x))}{\cos(\theta(x) - \theta_1(x) + \frac{\pi}{2})\sin(\theta_2(x) - \theta_1(x))} = \\ & \frac{\rho_2(x)}{\rho_1(x)}\sin(\theta_2(x) - \theta_1(x))\left(\frac{\cos(\theta_2(x) - \theta_1(x))\cos(\theta(x) - \theta_1(x) + \frac{\pi}{2})}{\cos(\theta(x) - \theta_1(x) + \frac{\pi}{2})\sin(\theta_2(x) - \theta_1(x))} + \right.\\ & \left.\frac{\sin(\theta(x) - \theta_1(x) + \frac{\pi}{2})\sin(\theta_2(x) - \theta_1(x))}{\cos(\theta(x) - \theta_1(x) + \frac{\pi}{2})\sin(\theta_2(x) - \theta_1(x))}\right) = \\ & \frac{\rho_2(x)}{\rho_1(x)}\sin(\theta_2(x) - \theta_1(x))\left(\tan\left(\theta(x) - \theta_1(x) + \frac{\pi}{2}\right) + \cot(\theta_2(x) - \theta_1(x))\right).\end{aligned}$$

As the Wronskian $W(u, v) = -\rho_1\rho_2\sin(\theta_2 - \theta_1) = 1$, and thus $\cot(\theta_2 - \theta_1)$ is locally bounded

$$\theta(x) = \theta_1(x) + \gamma(x) + O(1), \qquad (x \to \infty).$$

Therefore (6.2) is relatively oscillatory at ∞ if $|\gamma|$ tends to ∞, and it is relatively non-oscillatory is γ remains bounded.

A straightforward calculation shows that

$$\gamma'(x) = \langle u(x)\sin(\gamma(x)) - v(x)\cos(\gamma(x)), \tilde{\phi}(x)(u(x)\sin(\gamma(x)) - v(x)\cos(\gamma(x)))\rangle.$$

Expressing v in terms of u according to Lemma 1.5 and applying the Kepler transformation

$$\varphi(x) = \arctan\left(\frac{1}{x}\left(\tan(\gamma(x)) - 2\int_a^x \frac{\langle((m+\phi_{\mathrm{sc}}(t))\sigma_3 + \phi_{\mathrm{am}}(t)\sigma_1)u(t), u(t)\rangle}{\|u(t)\|^4}dt\right)\right),$$

we obtain

$$\begin{aligned}\varphi'(x) = & -\frac{1}{x}\sin(\varphi(x))\cos(\varphi(x)) - \\ & \frac{2}{x}\frac{\langle((m+\phi_{\mathrm{sc}}(x))\sigma_3 + \phi_{\mathrm{am}}(x)\sigma_1)u(x), u(x)\rangle}{\|u(x)\|^4}\cos^2(\varphi(x)) + \\ & \frac{\cos^2(\varphi(x))}{x}\Big\langle u(x)x\tan(\varphi(x)) - i\sigma_2\frac{u(x)}{\|u(x)\|^2}, \\ & (\tilde{\phi}_{\mathrm{sc}}(x)\sigma_3 + \tilde{\phi}_{\mathrm{am}}(x)\sigma_1 + \tilde{\phi}_{\mathrm{el}}(x))\left(u(x)x\tan(\varphi(x)) - i\sigma_2\frac{u(x)}{\|u(x)\|^2}\right)\Big\rangle \\ = & \frac{1}{x}\Big(\mathcal{A}(x)\cos^2(\varphi(x)) - \sin(\varphi(x))\cos(\varphi(x)) + \mathcal{B}(x)\sin^2(\varphi(x))\Big) + \\ & \frac{Q(x)}{x}\sin^2(\varphi(x)) + O\left(\frac{1}{x^2}\right), \qquad (x \to \infty),\end{aligned}$$

where

$$\mathcal{A}(x) := -2\frac{(\langle(m+\phi_{\mathrm{sc}}(x))\sigma_3 + \phi_{\mathrm{am}}(x)\sigma_1)u(x), u(x)\rangle}{\|u(x)\|^4} \quad \text{and}$$
$$\mathcal{B}(x) := \langle u(x), \hat{\phi}(x)u(x)\rangle$$

are locally integrable and

$$Q(x) := \langle u(x), (\tilde{\phi}(x)x^2 - \hat{\phi})u(x)\rangle = o(1), \qquad (x \to \infty).$$

Introducing the averages $A := \bar{\mathcal{A}}$ and $B := \bar{\mathcal{B}}$, the claim now follows by applying Corollary 6.6 and Corollary 6.8. □

Appendix A

Krein's Spectral Shift

This appendix is adopted from [8, Sec. 8] apart from only a few slight modifications.

In what follows we collect some facts on Krein's[1] spectral shift function which are of relevance to us. Most results are taken from [29] (see also [13] for an easy introduction). For historical purposes also see the paper by Krein [6].

Definition A.1.
*Two operators H_0 and H_1 are called **resolvent comparable**, if*

$$R_{H_1}(z) - R_{H_0}(z) \tag{A.1}$$

is trace class for one $z \in \rho(H_1) \cap \rho(H_0)$. By the first resolvent identity (1.20) then (A.1) holds for all $z \in \rho(H_1) \cap \rho(H_0)$.

Theorem A.2 (Krein [6])**.**
Let H_1 and H_0 be two resolvent comparable self-adjoint operators, then there exists a function

$$\xi(\lambda, H_1, H_0) \in L^1(\mathbb{R}, (\lambda^2+1)^{-1}d\lambda) \tag{A.2}$$

such that

$$\mathrm{tr}(f(H_1) - f(H_0)) = \int_{-\infty}^{\infty} \xi(\lambda, H_1, H_0) f'(\lambda) d\lambda \tag{A.3}$$

for every smooth function f with compact support.

Remark A.3.
Equation (A.3) holds in fact for a much larger class of functions f. See [29, Thm. 8.7.1] for this and a proof of the last theorem.

The function $\xi(\lambda) = \xi(\lambda, H_1, H_0)$ is called Krein's spectral shift function and is unique up to an additive constant. Moreover, $\xi(\lambda)$ is constant on every interval $(\lambda_0, \lambda_1) \subset \rho(H_0) \cap \rho(H_1)$. Hence, if $\dim \mathrm{Ran}\, P_{(\lambda_0,\lambda_1)}(H_j) < \infty$, $j = 0, 1$, then $\xi(\lambda)$ is a step function and

$$\dim \mathrm{Ran}\, P_{(\lambda_0,\lambda_1)}(H_1) - \dim \mathrm{Ran}\, P_{(\lambda_0,\lambda_1)}(H_0) = \lim_{\varepsilon \downarrow 0} \Big(\xi(\lambda_1 - \varepsilon) - \xi(\lambda_0 + \varepsilon) \Big). \tag{A.4}$$

[1] *Mark Grigorjewitsch Krein* (1907–1989)

This formula explains the name spectral shift function.

Before investigating further the properties of the SSF, we will recall a few things about trace ideals (see for example [14]). First, for $1 \le p < \infty$ denote by $\mathcal{J}^p$ the Schatten p-class, and by $\|.\|_{\mathcal{J}^p}$ its norm. We will use $\|.\|$ for the usual operator norm. Using $\|A\|_{\mathcal{J}^p} = \infty$ if $A \notin \mathcal{J}^p$, we have the following inequalities for all operators:

$$\|AB\|_{\mathcal{J}^p} \le \|A\|\|B\|_{\mathcal{J}^p}, \quad \|AB\|_{\mathcal{J}^1} \le \|A\|_{\mathcal{J}^2}\|B\|_{\mathcal{J}^2}. \tag{A.5}$$

Furthermore, we will use the notation of $\mathcal{J}^p$-converges to denote convergence in the respective $\|.\|_{\mathcal{J}^p}$-norm.

The following result from [4, Thm. IV.11.3] will be needed.

Lemma A.4 ([8, Lem. 8.2]).
Let $p > 0$, $A \in \mathcal{J}^p$, $T_n \xrightarrow{s} T$, $S_n \xrightarrow{s} S$ be sequences of strongly convergent bounded linear operators in some separable Hilbert space, then

$$\|T_n A S_n^* - T A S^*\|_{\mathcal{J}^p} \to 0. \tag{A.6}$$

We will also need the following continuity result for ξ. It will also allow us to fix the unknown constant.

Lemma A.5 ([8, Lem. 8.3]).
Suppose H_ε, $\varepsilon \in [0,1]$, is a family of self-adjoint operators, which is continuous in the metric

$$\rho(A,B) = \|R_A(z_0) - R_B(z_0)\|_{\mathcal{J}^1}, \tag{A.7}$$

for some fixed $z_0 \in \mathbb{C}\backslash\mathbb{R}$ and abbreviate $\xi_\varepsilon = \xi(H_\varepsilon, H_0)$. Then there exists a unique choice of ξ_ε such that $\varepsilon \mapsto \xi_\varepsilon$ is a continuous map $[0,1] \to L^1(\mathbb{R}, (\lambda^2 + 1)^{-1} d\lambda)$ with $\xi_0 = 0$.

If $H_\varepsilon \ge \lambda_0$ is bounded from below, we can also allow $z = \lambda \in (-\infty, \lambda_0)$.

Proof. The first statement can be found in [29, Lem. 8.7.5]. To see the second statement, let $\lambda < \lambda_0$ and $|\lambda - z| < \lambda_0 - \lambda$ for some $z \in \mathbb{C}\backslash\mathbb{R}$. Abbreviate $R_\varepsilon(z) = R_{H_\varepsilon}(z)$. Now using the first resolvent identity (1.20) gives

$$\begin{aligned}\|R_\varepsilon(z) - R_{\varepsilon'}(z)\|_{\mathcal{J}^1} \le & \|R_\varepsilon(\lambda) - R_{\varepsilon'}(\lambda)\|_{\mathcal{J}^1} \\ & + |z-\lambda|\|R_\varepsilon(z)\|\|R_\varepsilon(\lambda) - R_{\varepsilon'}(\lambda)\|_{\mathcal{J}^1} \\ & + |z-\lambda|\|R_{\varepsilon'}(\lambda)\|\|R_\varepsilon(z) - R_{\varepsilon'}(z)\|_{\mathcal{J}^1}\end{aligned}$$

and our conditions imply

$$|z-\lambda|\|R_{\varepsilon'}(\lambda)\| \le \frac{|z-\lambda|}{\lambda_0 - \lambda} < 1$$

and thus

$$\|R_\varepsilon(z) - R_{\varepsilon'}(z)\|_{\mathcal{J}^1} \le \frac{1 + \frac{|z-\lambda|}{|\mathrm{Im}(z)|}}{1 - \frac{|z-\lambda|}{|\lambda_0 - \lambda|}} \|R_\varepsilon(\lambda) - R_{\varepsilon'}(\lambda)\|_{\mathcal{J}^1},$$

from which the statement follows. ∎

Definition A.6.
*A family $\{\Phi(t),\ t \in [0,\infty]\}$, of operators on a Banach space B is called **strongly continuous** if it fulfills*

$$\forall\, x \in B : \lim_{t \to t_0} \Phi(t)x = \Phi(t_0)x.$$

Our final aim is to find some conditions which allow us to verify the assumptions of this lemma. To do this, we derive some properties of relatively bounded operators multiplied by strongly continuous families of operators. The key example for these operators will be multiplication operators by characteristic functions strongly converging to the identity operator. Now we take Hypotheses 5.6 into our considerations as announced in Chapter 5.

Lemma A.7 ([8, Lem. 8.5])**.**
Let $\varepsilon \ni [0,1] \to K_\varepsilon$ be a strongly continuous family of bounded self-adjoint operators which commute with V and $0 = K_0 \leq K_\varepsilon \leq K_1 = \mathbb{1}$.
Assume Hypothesis 5.6. Then

$$H_\varepsilon = H_0 + K_\varepsilon V \tag{A.8}$$

are self-adjoint operators such that the assumptions of Lemma A.5 hold.

Proof. We will abbreviate $V_\varepsilon = K_\varepsilon V$ and $R_\varepsilon(z) = R_{H_\varepsilon}(z)$.

By the Kato[2]–Rellich[3] Theorem ([10, Thm. X.12]) H_ε is well-defined and self-adjoint. Moreover, there is a z with $\mathrm{Im}(z) \neq 0$ such that $\|VR_0(z)\| \leq a < 1$. Hence $\|V_\varepsilon R_0(z)\| \leq a$ and a straightforward calculation using the second resolvent identity 1.21,

$$VR_\varepsilon(z) = VR_0(z)(1 + V_\varepsilon R_0(z))^{-1},$$

shows that

$$\|VR_\varepsilon(z)\| \leq \frac{a}{1-a}.$$

Furthermore, again using the second resolvent identity, we have

$$|V|^{1/2} R_\varepsilon(z) = |V|^{1/2} R_0(z)(1 - V_\varepsilon R_\varepsilon(z)),$$

which shows that

$$\||V|^{1/2} R_\varepsilon(z)\|_{\mathcal{J}_2} \leq \frac{1}{1-a} \||V|^{1/2} R_0(z)\|_{\mathcal{J}_2}.$$

To show $\mathcal{J}^1$-continuity at some fixed $\varepsilon \in [0,1]$ observe

$$R_{\varepsilon'}(z) - R_\varepsilon(z) = R_{\varepsilon'}(z)|V|^{1/2}\Big((K_{\varepsilon'} - K_\varepsilon)\,\mathrm{sgn}(V)|V|^{1/2} R_\varepsilon\Big),$$

where the first term $R_{\varepsilon'}(z)|V|^{1/2} \subseteq (|V|^{1/2} R_{\varepsilon'}(z^*))^*$ is uniformly $\mathcal{J}^2$-bounded in ε' by our previous argument and the second term $\mathcal{J}^2$-converges to 0 as ε' to ε by Lemma A.4. □

[2] *Tosio Kato* (1917–1999)
[3] *Franz Rellich* (1906–1955)

Appendix B

Notation

$AC_{loc}(I, \mathbb{C}^2)$	the set of all functions $I \to \mathbb{C}^2$, which are locally absolutely continuous
$\lvert A \rvert$	cf. Definition 5.2
A^*	the adjoint of A
$\chi_M(x)$	the characteristic function of the set M
$\mathbb{C}$	the complex numbers
$C(X, Y)$	the continuous functions $f : X \to Y$
z^*	the complex conjugate of z
$\mathfrak{D}(\tau)$	the domain of τ
$E_<, E_>$	the eigenvalues of a matrix of size two
f'	the derivative of f in x
$\#_{(c,d)}(u_0, u_1)$	the weighted sign flips of the Wronskian $W_x(u_0, u_1)$ inside of (c, d)
$G(z, x, y)$	the Green Function, cf. Notation 1.4
id_B	the identity operator on B
$\mathbb{N}$	the positive integers
$\mathbb{N}_0$	the non-negative integers
$\oplus$	orthogonal sum of linear spaces or operators
$\overline{\#}(u_0, u_1)$, $\underline{\#}(u_0, u_1)$ and $\#(u_0, u_1)$	cf. Notation 3.1
$\lfloor\ \rfloor$, $\lceil\ \rceil$	the Gaussian brackets
$\mathbb{1}$	the identity matrix of size two
L^1	the space of all integrable functions
$L^1_{loc}(I, \mathbb{C}^2)$	the set of all functions $I \to \mathbb{C}^2$, which are locally integrable
L^2	the space of all square integrable functions
$\mathrm{L}(B)$	the linear and bounded operators from B to itself
$\lVert B \rVert$	the operator norm of an operator B
$\psi_\pm(\lambda, x)$	the solutions of $\tau u = \lambda u$, $\lambda \in \mathbb{C}$, satisfying the boundary conditions, cf. Notation 1.7
$\otimes$	cf. Notation 1.12
P_H	the projection-valued measure of H
$\mathbb{R}$	the real numbers
$R_H(z)$	the resolvent of H

$\rho(A)$	the resolvent set of A
$\mathcal{J}^p$	the Schatten-p-class
sgn(x)	the sign of $x \in \mathbb{R}$
$\sigma_{ess}(A)$	the essential spectrum of A
σ_1, σ_2, σ_3	the Pauli matrices
span(V)	the linear span of a set V of vectors
θ_u	the Prüfer angle of u
tr(A)	the trace of an operator A
$W(u,v)$	the Wronskian
$\sim$	asymptotic similar

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