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Entanglement and Geometry in Multipartite Systems

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List of publications

Journal publications

P1 Beatrix C.Hiesmayr, Marcus Huber
Bohr's complementarity relation and the violation of the CP symmetry in high energy physics
Phys. Lett. A **372**, 3608, 2008

P2 Beatrix C. Hiesmayr, Marcus Huber
Multipartite entanglement measure for all discrete systems
Phys. Rev. A **78**, 012342 (2008).

P3 Beatrix C. Hiesmayr, Florian Hipp, Marcus Huber, Philipp Krammer, Christoph Spengler
A simplex of bound entangled multipartite qubit states
Phys. Rev. A **78**, 042327 (2008)

P4 Beatrix C. Hiesmayr, Marcus Huber, Philipp Krammer
Two computable sets of multipartite entanglement measures
Phys. Rev. A **79**, 062308 (2009)

E-prints and submissions

P5 Beatrix C. Hiesmayr, Marcus Huber
Two distinct classes of bound entanglement: PPT-bound and 'multi-particle'-bound
arXiv:quant-ph 0906.0238

P6 Christoph Spengler, Marcus Huber, Beatrix C. Hiesmayr
Optimization of Bell operators and visualization of the CGLMP-Bell inequality
arXiv:quant-ph 0907.0998

P7 Tsubasa Ichikawa, Marcus Huber, Philipp Krammer, Beatrix Hiesmayr
Mixed state entanglement measures for intermediate separability
arXiv:quant-ph 0911.4245

P8 Marcus Huber, Florian Mintert, Andreas Gabriel, Beatrix Hiesmayr
Detection of high-dimensional genuine multi-partite entanglement of mixed states
arXiv:quant-ph 0912.1870

P9 Nicolai Friis, Reinhold A. Bertlmann, Marcus Huber, Beatrix C. Hiesmayr
Relativistic entanglement of two massive particles
arXiv:quant-ph 0912.4863

Contents

1 Introduction

Entanglement is one of the most intriguing elements of quantum theory. It is one of the main features that distinguishes it from classical theories and has granted a deeper insight into the very nature of reality and locality, see e.g. Refs [1, 2]. It is a key resource in the emerging field of quantum information theory enabling computer algorithms that outperform any classical computer and opening the possibility for secure cryptography and state teleportation (for an overview see e.g. Refs [3, 4, 5])
It has been intensively studied for bipartite qubit systems and in the last few years even in multipartite and higher dimensional systems (for an overview see e.g. Refs [6, 7]). However many open problems still remain. There is still no general solution to the problem whether a given state is an entangled one or not. We will now give an overview over our recent contributions to that field:
Concerning the separability problem, i.e. deciding whether a given state is entangled, we have recently obtained a novel set of entanglement detection criteria, that in the multipartite scenario outperform many other criteria for certain classes of states. They not only provide a necessary condition for the separability of general states, but also allow to identify genuine multipartite entanglement. These can be found in the publication:

P8 Marcus Huber, Florian Mintert, Andreas Gabriel, Beatrix Hiesmayr
Detection of high-dimensional genuine multi-partite entanglement of mixed states
arXiv:quant-ph 0912.1870

Also if one takes entanglement seriously as a resource, one needs to not only be able to detect whether a state is entangled, but also to quantify the amount of entanglement present in a state. Here we have constructed a family of multipartite entanglement measures, which allow for easily computable lower bounds, even for multipartite states of arbitrary dimension. These results can be found in:

P4 Beatrix C. Hiesmayr, Marcus Huber, Philipp Krammer
Two computable sets of multipartite entanglement measures
Phys. Rev. A **79**, 062308 (2009)

P7 Tsubasa Ichikawa, Marcus Huber, Philipp Krammer, Beatrix Hiesmayr
Mixed state entanglement measures for intermediate separability
arXiv:quant-ph 0911.4245

There are also many open problems regarding procedures with which weakly entangled states may be distilled to higher entangled states using many copies.

These procedures are called distillation protocols. Surprisingly not every entangled state admits such a protocol, so there exist states which are entangled, yet even using infinitely many copies the entanglement can not be distilled. These states are called bound entangled and it still remains an open problem in general, whether a given entangled state is distillable. We have generalized the concept of the "magic" simplex, a special convex set of maximally entangled states, to a multipartite scenario in order to provide a testing ground for many concepts, such as entanglement detection criteria, entanglement measures and distillation procedures. With these tools we were able to construct a vast set of multipartite bound entangled states, which allow for distillation procedures if some parties cooperate. Such states are called unlockable bound entangled states and these results can be found in:

P3 Beatrix C. Hiesmayr, Florian Hipp, Marcus Huber, Philipp Krammer, Christoph Spengler
A simplex of bound entangled multipartite qubit states
Phys. Rev. A **78**, 042327 (2008)

P5 Beatrix C. Hiesmayr, Marcus Huber
Two distinct classes of bound entanglement: PPT-bound and 'multi-particle'-bound
arXiv:quant-ph 0906.0238

The following few sections will provide an introduction into the tools that were used to gain these results and show in more detail a connection between all the previously mentioned publications and even add a few minor unpublished results that have recently been obtained.

1.1 Basic definitions

1.1.1 Quantum states

A Hilbert space complex inner product space, which is also a metric space with respect to the metric induced by the inner product. Throughout this thesis we restrict ourselves to discrete Hilbert spaces of finite dimension d, so:

$$\mathcal{H}_d := \mathbb{C}^d \tag{1.1}$$

Observables in $\mathbb{C}^d$ are represented by self-adjoint operators $O = O^\dagger$. Their eigenvalues are the in principle measurable quantities. The state of a quantum system is described via a bounded normed linear functional on the algebra of observables O represented in $\mathbb{C}^d$ by an operator ρ defined as

$$\{\rho :\ \mathcal{H}_d \mapsto \mathcal{H}_d \,|\, \mathrm{Tr}(\rho) = 1 \,, \rho \geq 0 \,, \rho = \rho^\dagger \,\}. \tag{1.2}$$

A very important property of states, which we use, is their purity which may be defined as

$$P(\rho) := \frac{\mathrm{Tr}(\rho^2) - \frac{1}{d}}{1 - \frac{1}{d}} . \tag{1.3}$$

If $P(\rho) = 1$ the state is a pure state, if $P(\rho) < 1$ the state is called mixed and if $P(\rho) = 0$ the state is maximally mixed and its density matrix is given as $\rho = \frac{1}{d}\mathbb{1}_d$. Any pure state defines a projection of the form $\rho = |\psi\rangle\langle\psi|$, where $|\psi\rangle \in \mathcal{H}_d$. In general the purity is a convex function $P(\sum_i p_i \rho_i) \leq \sum_i p_i P(\rho_i)$. It is very useful to note that any state may be decomposed into pure states, so we define a pure state decomposition $\mathcal{D}(\rho)$ as an element of the set

$$\mathcal{S}_{\mathcal{D}} := \{\{p_i|\psi_i\rangle\}| \sum_i p_i \mid \psi_i\rangle\langle\psi_i| = \rho \wedge p_i > 0\} . \tag{1.4}$$

Note that there exist up to infinitely many pure state decompositions for a given mixed state, i.e. the set is possibly of infinite cardinality, which is one of the main problems throughout this thesis.

1.1.2 Quantum operations

If we want to consider all possible operations on arbitrary quantum systems we consider all possible maps $\Lambda(\rho) : \rho \mapsto \rho'$. Of course the resulting ρ' should also describe a physical system. Mathematically this yields the following description of a quantum operation

$$\{\Lambda_{CP} : \mathcal{H}_d \mapsto \mathcal{H}_{d'} \,|\, \Lambda_{CP}(\rho) \geq 0 \,, \Lambda_{CP} \otimes \mathbb{1}(\rho) \geq 0 \,\}. \tag{1.5}$$

The two requirements make the map completely positive, hence from now on we will refer to quantum operations as CP-maps. They include a variety of possible physical operations such as:

Unitary evolution:
A unitary evolution is a reversible process

$$\rho' = U \rho U^\dagger . \tag{1.6}$$

It can be realized through interaction in a closed system, following the Schroedinger equation, which for time independent Hamiltonians yields $\rho(t) = (e^{-\frac{i}{\hbar}\hat{H}t})\rho(0)(e^{-\frac{i}{\hbar}\hat{H}t})^\dagger$. Or it may also just be the mathematical representation of a basis change.

Measurements:
A measurement procedure is a probabilistic process

$$\rho' = \frac{M_i \rho M_i^\dagger}{\mathrm{Tr}(M_i \rho M_i^\dagger)} \tag{1.7}$$

where $\mathrm{Tr}(M_i \rho M_i^\dagger)$ is the probability that the measurement operation will result in the state $M_i \rho M_i^\dagger$. Therefore $\sum_i M_i M_i^\dagger = \mathbb{1}$ so that $\sum_i \mathrm{Tr}(M_i \rho M_i^\dagger) = 1$.

Ignoring parts of the system:
If one needs to consider subsystems of multipartite states one can choose to ignore the degrees of freedom from the remaining system. This will from now on be referred to as partial trace

$$\rho' = \mathrm{Tr}_{\{\alpha_j\}}(\rho) = \sum_{n_1, n_2, (\cdots), n_j} \langle n_1 n_2 (\cdots) n_j |_{\{\alpha_j\}} \rho | n_1 n_2 (\cdots) n_j \rangle_{\{\alpha_j\}} \,. \tag{1.8}$$

Here $\{\alpha_j\}$ denotes the set of subsystems over which the trace is performed and the $\{|n_i\rangle_{\alpha_i}\}$ constitute a complete orthonormal basis in subsystem α_i.

Addition of an ancilla: For many applications it may be useful to temporarily or permanently add another quantum system to the one you are considering. This can e.g. be done to then let the two systems interact for some time. Formally this operation is expressed as

$$\rho' = \rho \otimes \sigma \,. \tag{1.9}$$

LOCC
LOCC stands for Local Operations and Classical Communication and is a special class of CP-maps

$$\rho' = \frac{(L_1^i \otimes L_2^i \otimes (\cdots) \otimes L_n^i)\rho(L_1^i \otimes L_2^i \otimes (\cdots) \otimes L_n^i)^\dagger}{\mathrm{Tr}((L_1^i \otimes L_2^i \otimes (\cdots) \otimes L_n^i)\rho(L_1^i \otimes L_2^i \otimes (\cdots) \otimes L_n^i)^\dagger)} \,. \tag{1.10}$$

Of course

$$\sum_i (L_1^i \otimes L_2^i \otimes (\cdots) \otimes L_n^i)(L_1^i \otimes L_2^i \otimes (\cdots) \otimes L_n^i)^\dagger = \mathbb{1} \,. \tag{1.11}$$

A special class of LOCC are local unitary operations

$$U_{loc} := U_1 \otimes U_2 \otimes (\cdots) \otimes U_n \tag{1.12}$$

where

$$U_{loc}^\dagger = U_{loc}^{-1} \,. \tag{1.13}$$

1.1.3 Entropy

A fundamental concept of information theory is entropy. One possible entropy function is the linear entropy, which is closely related to the purity of a state

$$S_L(\rho) := \frac{d}{d-1}(1 - \mathrm{Tr}(\rho^2)) = 1 - P(\rho) \,. \tag{1.14}$$

This may of course be generalized to higher trace linear entropies

$$S_L^\alpha := \frac{d^{\alpha-1}}{d^{\alpha-1}-1}(1-\mathrm{Tr}(\rho^\alpha))\,. \tag{1.15}$$

As the linear entropy is not additive, we rather choose Renyi's entropies which are defined as

$$S_\alpha := \frac{1}{1-\alpha}\log_2(\mathrm{Tr}(\rho^\alpha)) = \frac{1}{1-\alpha}\log_2(1-\frac{d^{\alpha-1}}{d^{\alpha-1}-1}S_L^\alpha)\,. \tag{1.16}$$

For the limit $\alpha \to 1$ this is equal to von Neumann's entropy

$$S_N(\rho) = -\mathrm{Tr}(\rho\log_2(\rho))\,. \tag{1.17}$$

2 Entanglement for bipartite systems

2.1 Separability

A state ρ is separable if there exists a decomposition ($\mathcal{D}(\rho)$) satisfying

$$\rho = \sum_i p_i \rho_1^i \otimes \rho_2^i \,. \tag{2.1}$$

A state that is not separable is entangled. Maximally entangled states we define as pure states where $\mathrm{Tr}_{1\vee 2}(|\psi\rangle\langle\psi|) = \frac{1}{d}\mathbb{1}$. The set containing all separable bipartite states will henceforth be called $\mathcal{S}$.

2.2 Entanglement detection criteria

As $\mathcal{D}(\rho)$ is a set of infinite power, it is not trivial to decide whether a given density matrix is separable or not. In fact the problem is NP hard, see also Ref. [8]. There exist many entanglement detection criteria which help solving the problem.

2.2.1 Positive maps and partial transposition

Consider a positive map

$$\{\Lambda_P : \ \mathcal{H}_d \mapsto \mathcal{H}_{d'} \,|\, \Lambda_P(\rho) \geq 0\,\}. \tag{2.2}$$

For every separable state $\rho_{sep} \in \mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2}$ the following relation holds:

$$\Lambda_P \otimes \mathbb{1}_{d_2}(\rho_{sep}) \geq 0 \tag{2.3}$$

In fact if one extends this, one yields a necessary and sufficient criterion for detecting entanglement: Iff the state ρ is separable, then

$$\Lambda_P \otimes \mathbb{1}_{d_2}(\rho) \geq 0 \ \ \forall \Lambda_P \tag{2.4}$$

holds. However $\{\Lambda_P\}$ is an open set and therefore the problem has only been shifted and is not any easier to solve. There is a prominent application of this criterion, where Λ_P is the transposition. As separable states are always positive under partial transposition (PPT), it is an easily applicable detection criterion for entanglement also known as the Peres-Horodecki criterion (see Ref. [9, 10]). For 2×2 and 2×3 systems it is not only necessary but also sufficient, for further details see Ref. [10].

2.2.2 Matrix element inequalities

In (P8) we have recently developed inequalities for entanglement detection. They involve only the comparison of density matrix elements and are therefore easy to implement. The most general form for bipartite systems reads: Every separable state $\varrho_s \in \mathcal{S}$ satisfies

$$\begin{aligned} &\sqrt{\Re e(\langle\Phi|\varrho_s^{\otimes n}\Pi_c^A \otimes (\Pi_c^B)^{-1}|\Phi\rangle)} \leq \\ &\quad \sqrt{\langle\Phi|(\Pi_c^A \otimes \mathbb{1})\varrho_s^{\otimes n}(\Pi_c^A \otimes \mathbb{1})|\Phi\rangle} \end{aligned} \tag{2.5}$$

where Π_c is the cyclic permutation operator:

$$\Pi_c|\varphi_1\rangle \otimes |\varphi_2\rangle \otimes \ldots \otimes |\varphi_n\rangle = |\varphi_2\rangle \otimes |\varphi_3\rangle \otimes \ldots \otimes |\varphi_n\rangle \otimes |\varphi_1\rangle \ , \tag{2.6}$$

and $|\Phi\rangle$ is an arbitrary fully separable state:

$$|\Phi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes |\phi \otimes (\ldots) \otimes |\phi_{2n}\rangle \, . \tag{2.7}$$

The bilinear $n = 2$ case simplifies to

$$|\langle il|\rho|kj\rangle| - \sqrt{\langle ij|\rho|ij\rangle\langle kl|\rho|kl\rangle} \leq 0 \, . \tag{I}$$

where $|\Phi\rangle = |ilkj\rangle$ was chosen in the computational basis. A proof is given in (P8) in abbreviated form. The proof will be presented here to its full extent:
First consider the most general form of a bipartite density matrix in terms of its elements:

$$\rho := \sum_{i_1,i_3=0}^{d_1-1} \sum_{i_2,i_4=0}^{d_2-1} \rho_{i_1 i_2 i_3 i_4} \, |i_1 i_2\rangle \, \langle i_3 i_4| \tag{2.8}$$

We claim

$$\rho_{i_1 i_2 i_1 i_2}\rho_{i_3 i_4 i_3 i_4} \cdots \rho_{i_{2n-1} i_{2n} i_{2n-1} i_{2n}} \geq \Re e(\rho_{i_1 i_4 i_3 i_2}\rho_{i_3 i_6 i_5 i_4} \cdots \rho_{i_{2n-1} i_2 i_1 i_{2n}}) \, . \tag{2.9}$$

This is equivalent to inequality (2.5), only explicitly written in terms of density matrix elements. If $\rho \in \mathcal{S}$ then we can write it in the form

$$\rho_{i_1 i_2 i_3 i_4} = \sum_{\alpha_1} p_{\alpha_1} a_{i_1}^{\alpha_1} b_{i_2}^{\alpha_1} a_{i_3}^{*\alpha_1} b_{i_4}^{*\alpha_1} \, . \tag{2.10}$$

With that we get for the left side

$$\begin{aligned} &\rho_{i_1 i_2 i_1 i_2}\rho_{i_3 i_4 i_3 i_4} \cdots \rho_{i_{2n-1} i_{2n} i_{2n-1} i_{2n}} = \\ \sum_{\alpha_1,\ldots,\alpha_n} &\underbrace{p_{\alpha_1} \cdots p_{\alpha_n} a_{i_1}^{\alpha_1} b_{i_2}^{\alpha_1} a_{i_1}^{*\alpha_1} b_{i_2}^{*\alpha_1} a_{i_3}^{\alpha_2} b_{i_4}^{\alpha_2} a_{i_3}^{*\alpha_2} b_{i_4}^{*\alpha_2} \cdots a_{i_{2n-1}}^{\alpha_n} b_{i_{2n}}^{\alpha_n} a_{i_{2n-1}}^{*\alpha_n} b_{i_{2n}}^{*\alpha_n}}_{L_{\alpha_1 \alpha_2 \cdots \alpha_n}} \end{aligned}$$

and for the right side:

$$\rho_{i_1 i_4 i_3 i_2}\rho_{i_3 i_6 i_5 i_4}...\rho_{i_{2n-1} i_2 i_1 i_{2n}} =$$
$$\sum_{\alpha_1,...,\alpha_n} \underbrace{p_{\alpha_1}...p_{\alpha_n} a_{i_1}^{\alpha_1} b_{i_4}^{\alpha_1} a_{i_3}^{*\alpha_1} b_{i_2}^{*\alpha_1} a_{i_3}^{\alpha_2} b_{i_6}^{\alpha_2} a_{i_5}^{*\alpha_2} b_{i_4}^{*\alpha_2} ... a_{i_{2n-1}}^{\alpha_n} b_{i_2}^{\alpha_n} a_{i_1}^{*\alpha_n} b_{i_{2n}}^{*\alpha_n}}_{R_{\alpha_1\alpha_2...\alpha_n}} .$$

Now we can subtract all diagonal terms as

$$L_{kk...k} = \Re e(R_{kk...k})\forall k\,. \tag{2.11}$$

We now define:

$$C_{\alpha_1\alpha_2...\alpha_n} := a_{i_1}^{\alpha_1} b_{i_2}^{*\alpha_1} a_{i_3}^{\alpha_2} b_{i_4}^{*\alpha_2} ... a_{i_{2n-1}}^{\alpha_n} b_{i_{2n}}^{*\alpha_n} \tag{2.12}$$
$$LS := \frac{1}{p_{\alpha_1}...p_{\alpha_n}} \sum_{perm} L_{\alpha_1\alpha_2...\alpha_n} \tag{2.13}$$
$$RS := \frac{1}{p_{\alpha_1}...p_{\alpha_n}} \sum_{perm} R_{\alpha_1\alpha_2...\alpha_n} \tag{2.14}$$

With that definition follows obviously

$$LS = \sum_{perm} |C_{\alpha_1\alpha_2...\alpha_n}|^2 \tag{2.15}$$

and not so obviously

$$RS = \sum_{perm} C_{\alpha_1\alpha_2...\alpha_n} C^*_{\alpha_n\alpha_1...\alpha_{n-1}}\,. \tag{2.16}$$

Equation (2.16)is true since the product of the both outer factors of the factors of $R_{\alpha_1\alpha_2...\alpha_n}$, e.g. $\underline{a_{i_3}^{\alpha_2}} b_{i_6}^{\alpha_2} a_{i_5}^{*\alpha_2} \underline{b_{i_4}^{*\alpha_2}}$, is $C_{\alpha_1\alpha_2...\alpha_n}$ and the product of the both inner factors of the factors of $R_{\alpha_1\alpha_2...\alpha_n}$, e.g. $a_{i_3}^{\alpha_2} \underline{b_{i_6}^{\alpha_2} a_{i_5}^{*\alpha_2}} b_{i_4}^{*\alpha_2}$, is $C^*_{\alpha_n\alpha_1...\alpha_{n-1}}$.
Now if

$$LS \geq RS \tag{2.17}$$

then inequality (2.5) is proven.

So we bring the right side to the left and get

$$\frac{1}{2}\sum_{perm} |C_{\alpha_1\alpha_2...\alpha_n} - C_{\alpha_n\alpha_1...\alpha_{n-1}}|^2 \geq 0 \tag{2.18}$$

which is true and completes the proof. The presented criterion has so far only detected entangled states which are also detected by the Peres-Horodecki criterion, so except for simplicity its advantages lie in the multipartite case. As we will show in the section about multipartite entanglement the above inequality yields a possibility to construct criteria for the detection of genuine multipartite entanglement.

2.3 Entanglement measures

As many quantum informational applications, such as quantum key distribution (QKD) and many algorithms in quantum computing, rely heavily on entanglement as a resource (for an overview see e.g. Refs [3, 4, 5, 7]) it is crucial to not only detect entanglement, but also quantify the amount of entanglement present in a state.

2.3.1 Basic properties

An entanglement measure $E(\rho) \mapsto \mathbb{R}^+$ may be defined satisfying:

B1 $E(\rho) \ > 0 \ \ \forall\, \rho \notin \mathcal{S}$

B2 $E(\rho) \ = 0 \ \ \forall\, \rho \in \mathcal{S}$

B3 $E(\rho) = E(U_{loc}\rho U_{loc})$ (invariant under local unitaries)

B4 $E(\rho^{\otimes n}) = nE(\rho)$ (additive on copies)

B5 $E(\rho_{me}) = 2\log_2(d)$ (normalized by maximally entangled states ρ_{me})

B6 $E(\rho) \geq E(\Lambda_{LOCC}[\rho])$ (non increasing under LOCC)

B7 $E(\sum_i p_i\rho_i) \leq \sum_i p_i E(\rho_i)$ (convex)

2.3.2 Convex roof

It is quite common to define entanglement measures for pure states, as this proves to be very easy. Formally one can then always take the convex roof over all $\mathcal{D}(\rho)$ to extend them to mixed states also

$$E(\rho) := \inf_{\mathcal{D}(\rho)} \sum_i p_i E(|\psi_i\rangle\langle\psi_i|)\,. \tag{2.19}$$

2.3.3 Entanglement of Formation

Entanglement of Formation was first defined in Ref. [11]. It is the convex roof extension of the entropy of entanglement E_S, which follows from the simple observation that any separable pure state has pure reduced density matrices. So the entropy of the subsystems is not only a necessary and sufficient detection criterion for pure state entanglement, but also a good entanglement measure satisfying properties (B1-B5)

$$E_S(|\psi\rangle\langle\psi|) := S_N(\mathrm{Tr}_1(\rho))\,. \tag{2.20}$$

So the Entanglement of Formation is defined as

$$E_{OF} := \inf_{\mathcal{D}(\rho)} \sum_i p_i E_S(|\psi_i\rangle\langle\psi_i|)\,. \tag{2.21}$$

2.3.4 Concurrence

The concurrence was first introduced by Hill and Wootters in Ref. [12, 13] and was intended to be an auxiliary function, which helps to calculate the convex roof of entanglement of formation for bipartite qubits. It is defined for pure states as

$$C(|\psi\rangle\langle\psi|) := \sqrt{\mathrm{Tr}(|\psi\rangle\langle\psi|\sigma_y \otimes \sigma_y(|\psi\rangle\langle\psi|)^*\sigma_y \otimes \sigma_y)} = \sqrt{S_L(\rho_{1\vee 2})}\,. \tag{2.22}$$

Now considering the convex roof we can define

$$C(\rho) = \inf_{\mathcal{D}(\rho)} \sum_i p_i C(|\psi_i\rangle\langle\psi_i|)\,. \tag{2.23}$$

This convex roof may be computed via

$$C(\rho) = \max\left[0, 2\max[\{\lambda_i\}] - \sum_i \lambda_i\right]\,, \tag{2.24}$$

where the $\{\lambda_i\}$ are the square roots of the eigenvalues of $\rho\sigma_y \otimes \sigma_y\rho^*\sigma_y \otimes \sigma_y$. Entanglement of formation for bipartite qubits can now be computed purely algebraically via $E_{OF}(\rho) = h(C(\rho))$ with the function

$$h(C) := \sum_{\alpha=1,-1} \frac{1}{2}(1+\alpha\sqrt{1-C^2})\log_2(\frac{1}{2}(1+\alpha\sqrt{1-C^2}))\,. \tag{2.25}$$

2.3.5 The m-concurrence

In (P2) we first introduced the m-concurrence, which is an extension to Wootter's concurrence for states of arbitrary dimension. It is defined for bipartite pure states as

$$\begin{aligned} C_m(|\psi\rangle\langle\psi|) : &= \sqrt{\sum_{k_1\neq l_1, k_2\neq l_2} \mathrm{Tr}(|\psi\rangle\langle\psi|\sigma_{k_1 l_1} \otimes \sigma_{k_2 l_2}(|\psi\rangle\langle\psi|)^*\sigma_{k_1 l_2} \otimes \sigma_{k_1 l_2})} \\ &= \sqrt{S_L(\rho_1) + S_L(\rho_2)} \end{aligned} \tag{2.26}$$

where the σ_{kl} are the anti-symmetric Gell-Mann operators defined by

$$\sigma_{kl} = i|k\rangle\langle l| - i|l\rangle\langle k|\,. \tag{2.27}$$

The convex roof

$$C_m(\rho) = \inf_{\mathcal{D}(\rho)} \sum_i p_i C_m(|\psi_i\rangle\langle\psi_i|) \tag{2.28}$$

remains incomputable, however we have derived lower bounds $\widetilde{C}_m(\rho) \leq C_m(\rho)$ which are defined as

$$\widetilde{C}_m^2(\rho) := \sum_{k_1\neq l_1, k_2\neq l_2} \max[0, 2\max[\{\lambda^i_{k_1,l_1,k_2,l_2}\}] - \sum_i \lambda^i_{k_1,l_1,k_2,l_2}]^2 \tag{2.29}$$

where the $\{\lambda^i_{k_1,l_1,k_2,l_2}\}$ are the square roots of the eigenvalues of $\rho\sigma_{k_1l_1}\otimes\sigma_{k_2l_2}\rho^*\sigma_{k_1l_2}\otimes\sigma_{k_1l_2}$.
These results have also been published in:

- Beatrix C. Hiesmayr, Marcus Huber, Philipp Krammer
  Two computable sets of multipartite entanglement measures
  Phys. Rev. A **79**, 062308 (2009)

2.4 Distillation

2.4.1 Distillation procedures

Consider a scenario where one has many copies of a weakly entangled state available and can locally implement any operation on k-copies of the state. One can then try to use these copies to locally shift the entanglement into one desired pair, which should of course become more entangled in the process. Formally this requires a LOCC operation Λ_{LOCC} which with some non-vanishing probability $p_i = \mathrm{Tr}((A_i \otimes B_i)\rho^{\otimes k}(A_i^\dagger \otimes B_i^\dagger))$ produces the desired outcome. One can then repeat the process n times, always discarding the state if the outcome is different

$$\rho_n := \frac{(A_i \otimes B_i)\rho_{n-1}^{\otimes k}(A_i^\dagger \otimes B_i^\dagger)}{\mathrm{Tr}((A_i \otimes B_i)\rho_{n-1}^{\otimes k}(A_i^\dagger \otimes B_i^\dagger))}\,. \tag{2.30}$$

If there $\exists$ a set of operations $\{A_i, B_i\}$ for which

$$\lim_{n\to\infty} \rho_n = |\psi\rangle\langle\psi| \tag{2.31}$$

and $|\psi\rangle\langle\psi|$ is the desired maximally entangled state, then the state ρ is called k-copy-distillable. We know that every entangled two-qubit system is distillable, so multidimensional systems can be distilled if there exists a local projection from m-copies of the state to an entangled two-qubit subspace. If that is the case the state is called pseudo-m-copy-distillable, in short m-distillable. The set of all m-distillable states we denote as D_m. That this is a necessary and sufficient criterion for distillability is proven in Ref. [14].

2.4.2 Bound entanglement

A state that cannot be distilled does not necessarily have to be separable. It is possible for entangled states to be non-distillable, even for $k \to \infty$ and $n \to \infty$. These states are called bound entangled and it is still an open problem in general to identify them. One criterion, however, has proven extremely helpful in the identification of bound entangled states. The Peres-Horodecki criterion also enables one to identify non-distillable states. As proven in Ref. [14] any state being positive under partial transposition (PPT) remains so under operations of the form (2.30). This of course means that any PPT state identified as entangled by an independent criterion must be bound entangled. The question whether also bound entangled states which are negative under partial transposition (NPPT) exist, is still an open question and heavily debated.

3 Multipartite Entanglement

3.1 Separability

In the multipartite scenario the notion of separability cannot be unambiguously generalized as there are many partitions with respect to which the state may be separable. Furthermore for mixed states there is now the possibility that there exist pure state decompositions where the elements are separable with respect to different partitions. Such a state is called partially separable and is of course distinguished from genuine multipartite entangled states for which no such decomposition is possible. To reflect the complexity of the separability in multipartite systems we introduce several well defined definitions of multipartite separability in order to cope with many different physical situations.

3.1.1 The k-separability

A state $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes (\cdots) \otimes \mathcal{H}_n$ is m-separable iff there $\exists$ a $\mathcal{D}(\rho)$ satisfying

$$\rho = \sum_i p_i \rho_1^i \otimes \rho_2^i \otimes (\cdots) \otimes \rho_m^i \,. \tag{3.1}$$

As this is still ambiguous, a state is defined k-separable via

$$k = \max_{\mathcal{D}(\rho)}[m] \tag{3.2}$$

- if $k = n$ the state is fully separable
- if $k = 1$ the state is fully entangled
- if $\mathrm{Tr}_{\neg_s}(\rho) = \frac{1}{d}\mathbb{1}\ \forall\ s$ and $\rho^2 = \rho$ the state is maximally entangled (Here $\mathrm{Tr}_{\neg_s}$ means ignoring all subsystems except for the subsystem s)

The set containing all k-separable states will henceforth be called $\mathcal{S}_k$.

3.1.2 The γ_r-separability

To further classify the separability structure of multipartite states we have introduced the concept of γ_r-separability. We denote every irreducible subsystem by numbers from 1 to n. Then if there $\exists$ a $\mathcal{D}(\rho)$

$$\rho = \sum_i p_i \rho_{\{\beta_1\}}^i \otimes \rho_{\{\beta_2\}}^i \otimes (\cdots) \otimes \rho_{\{\beta_m\}}^i \,. \tag{3.3}$$

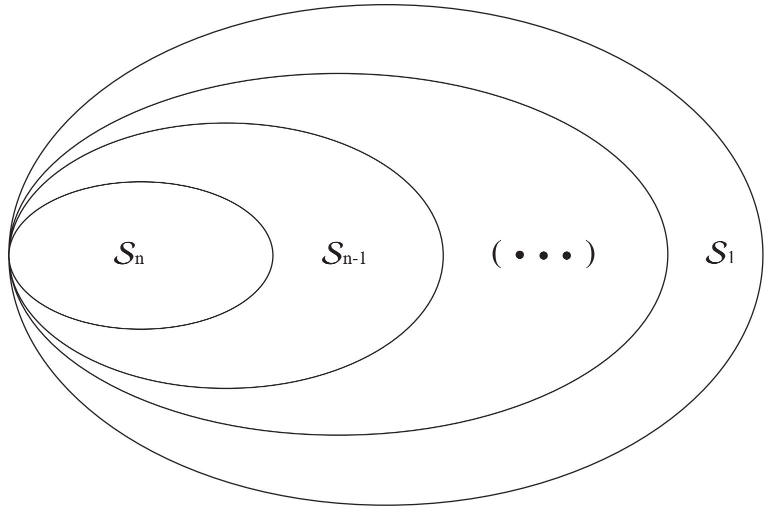

Figure 3.1: Here the convexity of *k–separability* is illustrated

where all subsystems ρ_j^i are $\in \mathcal{H}_j$, and

$$r = \max_{\mathcal{D}(\rho)}[m]\,. \tag{3.4}$$

We now define γ_r-separability as

$$\gamma_r := \{\{\beta_1\}|\{\beta_2\}|(\cdots)|\{\beta_r\}\}\,. \tag{3.5}$$

Surprisingly this definition is sometimes ambiguous. Take e.g. the Smolin state, Ref. [18], which in this notation allows for decompositions:

$$\begin{gathered}\{12|34\}\\ \{13|24\}\\ \{14|23\}\end{gathered} \tag{3.6}$$

Note that any state that allows for such ambiguous decompositions implies that the reduced density matrices are separable.

3.1.3 The γ_t-separability

You can further generalize the γ_r-separability concept via taking the unification of all ambiguous arrangements. This we call the γ_t-separability. It reflects a different physical aspect of separability. While γ_r-separability reflects how many parties must join to create this state via LOCC, the γ_t-separability reveals the amount of subsystems that are needed to access the entanglement of a state.

3.2 Entanglement detection criteria

3.2.1 Positive maps and partial transposition

Again we can use the concept of positive maps in the multipartite scenario. We can generalize statement (2.4) by

$$\left(\Lambda_P^{\{\alpha_j\}}, \mathbb{1}^{\{\bar{\alpha_j}\}}\right)(\rho) \geq 0 \forall \Lambda_P \Leftrightarrow \{\alpha_j\} \in \gamma_r \,. \tag{3.7}$$

Here $\{\bar{\alpha_j}\}$ is the complement of the set $\{\alpha_j\}$. Again we can use transposition $\Lambda_P = T$ as a special case of this statement to have a necessary but not sufficient criterion

$$\left(T^{\{\alpha_j\}}, \mathbb{1}^{\{\bar{\alpha_j}\}}\right)(\rho) \geq 0 \Leftarrow \{\alpha_j\} \in \gamma_r \,. \tag{3.8}$$

3.2.2 Detecting genuine multipartite entanglement

In (P8) we have introduced a method which not only provides necessary criteria for γ_r-separability, but also admits necessary conditions for k-separability. First note that for any multipartite state it is possible to consider a bipartition $B = \{\{\alpha_1\}|\{\alpha_2\}\}$ where $\{\alpha_1\}$ is a set containing the labels of all subsystems in the first part of the bipartition and analogously for $\{\alpha_2\}$. So we can employ inequality (I) for all possible bipartitions. It is also clear that for every 2-separable pure state the following equation holds

$$\sqrt{\langle\Phi|\rho^{\otimes 2}\Pi_c^{\otimes N}|\Phi\rangle} - \sum_B \sqrt{\langle\Phi|\Pi_c^{\{\alpha_1\}} \otimes \mathbb{1}^{\{\alpha_2\}}\rho^{\otimes 2}\Pi_c^{\{\alpha_1\}} \otimes \mathbb{1}^{\{\alpha_2\}}|\Phi\rangle} \leq 0 \,. \tag{II}$$

Notice that this inequality follows directly from ineq.(2.5) using the fact that $\Pi_c^{\{\alpha_1\}} \otimes \Pi_c^{\{\alpha_2\}} = \Pi_c^{\otimes N}$ for all bipartitions and $\Pi_c = (\Pi_c)^{-1}$ for the bilinear case. That the relation holds for mixed states also follows from the fact that $\sqrt{\langle\Phi|\rho^{\otimes 2}\Pi_c^{\otimes N}|\Phi\rangle}$ is the (convex) absolute value of certain off-diagonal elements of density matrices and the square root of a product of two diagonal density matrix elements $\sqrt{\langle\Phi|\Pi_c^{\{\alpha_1\}} \otimes \mathbb{1}^{\{\alpha_2\}}\rho^{\otimes 2}\Pi_c^{\{\alpha_1\}} \otimes \mathbb{1}^{\{\alpha_2\}}|\Phi\rangle}$ is concave, thus the whole expression is convex and holds for arbitrary biseparable mixed states also. Therefore, the above expression is a detection criterion for genuine multipartite entanglement.

For example in the case of three particles (A,B,C) the inequality is satisfied for all states which can be decomposed in the following way

$$\rho_{bisep} = \sum_j p_j \rho_{AB}^j \otimes \rho_C^j + \sum q_j \rho_{AC}^j \otimes \rho_B^j + \sum r_j \rho_{BC}^j \otimes \rho_A^j \,, \tag{3.9}$$

i.e. any violation proves that a certain state cannot be written in this form. Obviously, it is important property because genuine multipartite entanglement is

a necessary condition for distilling multipartite entangled pure states which are used e.g. in quantum secret sharing.
For a special choice of $|\Phi\rangle = |s_i\rangle \otimes |s_j\rangle$ with

$$|s_k\rangle := |x\rangle_{\overline{k}}^{\otimes N-1} \otimes |y\rangle_k = \quad |xx\ldots y\ldots x\rangle \tag{3.10}$$
$$12\ldots k\ldots N \tag{3.11}$$

one can derive a linear combination of the above expressions which also detects genuine multipartite entanglement:

$$\sum_{i\neq j} \sqrt{\langle s_i| \otimes \langle s_j|\rho^{\otimes 2}\Pi_c^{\otimes N}|s_i\rangle \otimes |s_j\rangle} - (N-2)\sum_{i,j} \sqrt{\langle s_i| \otimes \langle s_j|\Pi_c^i \otimes \mathbb{1}^{\bar{i}}\rho^{\otimes 2}\Pi_c^i \otimes \mathbb{1}^{\bar{i}}|s_i\rangle \otimes |s_j\rangle} \leq 0\,. \tag{III}$$

Here $\Pi_c^i \otimes \mathbb{1}^{\bar{i}}$ means cyclic permutation of subsystem i and identity operator in all other subsystems. Again we are dealing with a convex object which means that it is sufficient for pure biseparable states to fulfill inequality (III). This was also proven in (P8) but we will present a more detailed proof here:
To prove that inequality (III) holds for all biseparable pure states (and hence, due to convexity of the whole expression, for all biseparable states), firstly note that

$$\Pi|s_i\rangle|s_j\rangle = |s_j\rangle|s_i\rangle \tag{3.12}$$
$$\Pi_c^i \otimes \mathbb{1}^{\bar{i}}|s_k\rangle|s_k\rangle = |s_k\rangle|s_k\rangle \tag{3.13}$$
$$\Pi_c^i \otimes \mathbb{1}^{\bar{i}}|s_i\rangle|s_j\rangle = \Pi_c^j \otimes \mathbb{1}^{\bar{j}}|s_i\rangle|s_j\rangle\,, \tag{3.14}$$

It is clear that for any state the following relation follows from positivity of the density matrix:

$$|\langle s_i|\rho|s_j\rangle| \leq \frac{1}{2}\left(\langle s_i|\rho|s_i\rangle + \langle s_j|\rho|s_j\rangle\right) \forall i,j\,. \tag{3.15}$$

Furthermore, it follows from ineq. (2.5) that if a state ρ is separable w.r.t. a certain bipartition $A|B$, it satisfies

$$|\langle s_i|\rho|s_j\rangle| \leq \sqrt{\langle s_is_j|\Pi_c^{A_k\dagger} \otimes \mathbb{1}^{B_k}\rho^{\otimes 2}\Pi_c^{A_k} \otimes \mathbb{1}^{B_k}|s_is_j\rangle} = \sqrt{\langle s_is_j|\Pi_c^{i\dagger} \otimes \mathbb{1}^{\bar{i}}\rho^{\otimes 2}\Pi_c^i \otimes \mathbb{1}^{\bar{i}}|s_is_j\rangle}\,, \tag{3.16}$$

where the equality follows from eqs. (3.13) and (3.14). Note that inequality (3.16) is less general than inequality (3.15), but is also tighter (where valid). With these observations done, we can now rewrite inequality (III):

$$\begin{aligned}
0 \geq & \sum_{i\neq j} \underbrace{\sqrt{\langle s_i s_j|\rho^{\otimes 2}\Pi|s_i s_j\rangle}}_{x_{ij}} - \\
& (N-2)\sum_{i,j} \underbrace{\sqrt{\langle s_i s_j|\Pi_c^{i\dagger}\otimes \mathbb{1}^{\bar{i}}\rho^{\otimes 2}\Pi_c^{i}\otimes \mathbb{1}^{\bar{i}}|s_i s_j\rangle}}_{y_{ij}} = \\
& \sum_{i\neq j}(x_{ij}-(N-2)y_{ij}) - (N-2)\sum_i y_{ii} \qquad (3.17)
\end{aligned}$$

The two subsystems defined by the bipartition with respect to which $|\Psi\rangle$ is biseparable are called A and B and are comprised of N_A and N_B elementary subsystems. Now, we want to show

$$\sum_{i\neq j}(x_{ij}-(N-2)y_{ij}) - (N-2)\sum_i y_{ii} \leq 0\,. \qquad (3.18)$$

Now we can split the sum in terms, where i and j are either of the same or different partition $A|B$.

$$\begin{aligned}
& \underbrace{\sum_{i\neq j}(x_{ij}-(N-2)y_{ij})}_{same} + \underbrace{\sum_{i\neq j}(x_{ij}-(N-2)y_{ij})}_{different} \\
& \qquad -(N-2)\sum_i y_{ii} \leq 0\ , \qquad (3.19)
\end{aligned}$$

Now the statement

$$\underbrace{\sum_{i\neq j}(x_{ij}-(N-2)y_{ij})}_{different} \leq 0 \qquad (3.20)$$

is true due to the fact that in this case $x_{ij} \leq y_{ij}$ due to ineq. (II). Now what's left to prove is that

$$\underbrace{\sum_{i\neq j}(x_{ij}-(N-2)y_{ij})}_{same} -(N-2)\sum_i y_{ii} \leq 0\ , \qquad (3.21)$$

holds. We can even prove that

$$\begin{aligned}
& \underbrace{\sum_{i\neq j}(x_{ij})}_{same} -(N-2)\sum_i y_{ii} = \\
= & \underbrace{\sum_{i\neq j}(x_{ij})}_{same} - \frac{(N-2)}{2}(\sum_i y_{ii} + \sum_j y_{jj}) \leq 0\ , \qquad (3.22)
\end{aligned}$$

holds also. First note that after a bipartition the parts A or B can at most contain $N-1$ subsystems. So we conclude that

$$\underbrace{\sum_{i\neq j}(x_{ij})}_{same} - \frac{(N-2)}{2}(\sum_i y_{ii} + \sum_j y_{jj}) =$$
$$= \underbrace{\sum_{i\neq j}(x_{ij} - X\frac{1}{2}(y_{ii}+y_{jj}))}_{same} \leq 0\,. \tag{3.23}$$

The factor X is equal to $\frac{N-2}{N_{A/B}-1}$, where $N_{A/B}$ is the number of subsystems contained in A or B. So therefore

$$X \geq 1\,. \tag{3.24}$$

So in the end we have

$$\underbrace{\sum_{i\neq j}(x_{ij} - X\frac{1}{2}(y_{ii}+y_{jj}))}_{same} \leq \underbrace{\sum_{i\neq j}(x_{ij} - \frac{1}{2}(y_{ii}+y_{jj}))}_{same} \leq 0 \tag{3.25}$$

The last step we conclude as $x_{ij} \leq \frac{1}{2}(x_{ii}+x_{jj})$ holds for all i and j due to the positivity of a density matrix.

P8 Marcus Huber, Florian Mintert, Andreas Gabriel, Beatrix Hiesmayr
Detection of high-dimensional genuine multi-partite entanglement of mixed states
arXiv:quant-ph 0912.1870

3.3 Entanglement measures in multipartite systems

3.3.1 Basic properties

We can generalize the requirements for bipartite measures to multipartite measures in a straightforward way:

M1 $E(\rho)\ > 0\ \ \forall\, \rho \in \mathcal{S}_k \mid k < n$

M2 $E(\rho)\ = 0\ \ \forall\, \rho \in \mathcal{S}_n$

M3 $E(\rho) = E(U_{loc}\rho U_{loc})$ (invariant under local unitaries)

M4 $E(\rho^{\otimes n}) = nE(\rho)$ (additive on copies)

M5 $E(\rho\otimes\sigma) = E(\rho) + E(\sigma)$ (additive on tensor products)

M6 $E(\rho_{me}) = n\log_2(d)$ (normalized by maximally entangled states)

M7 $E(\rho) \geq E(\Lambda_{LOCC}[\rho])$ (non increasing under LOCC)

M8 $E(\sum_i p_i\rho_i) \leq \sum_i p_i E(\rho_i)$ (convex)

This is a list of properties that if such measure is found yields a good measure of how much entanglement is present within a state. However it would not answer the question of separability properly, i.e. which subsystems are separable from the rest and which subsystems share how much entanglement.

3.3.2 Separability measure

As no single measure can quantify entanglement and reveal γ_r-separability at the same time we have proposep a set of measures $E_{\{\alpha_j\}}$, which together form the separability measure $E_S(\rho) = \sum_{\{\alpha_j\}} E_{\{\alpha_j\}}$. We require that $E_S(\rho)$ fulfills all necessary requirements (M1-M8) and that all $E_{\{\alpha_j\}}$ individually fulfill:

S1 $E_{\{\alpha_j\}}(\rho) \; > 0 \quad \Leftrightarrow \quad \{\alpha_j\} \in \gamma_r$

S2 $E_{\{\alpha_j\}}(\rho) \; = 0 \quad \Leftrightarrow \quad \{\alpha_j\} \notin \gamma_r$

S3 $E_{\{\alpha_j\}}(\rho) = E_{\{\alpha_j\}}(U_{loc}\rho U_{loc})$

S4 $E_{\{\alpha_j\}}(\rho^{\otimes n}) = nE_{\{\alpha_j\}}(\rho)$

S5 $E(\rho) = |\{\alpha_j\}|\log_2(d) \; \forall \; \rho_{\{\alpha_j\}} = \rho_{me}$

We construct such a measure for pure states as

$$E_{\{\alpha_j\}} := \sum_{s\in\{\alpha_j\}} \left(S_\alpha(\rho_s) - \sum_{\{\beta_j\}\subset\{\alpha_j\}} E_{\{\beta_j\}} \right) \cdot \delta[S_\alpha(\rho_{\{\alpha_j\}}), 0] \tag{3.26}$$

where

$$\begin{aligned} \delta[S_\alpha(\rho_{\{\alpha_j\}}), 0] &= 1 \quad \text{if} \quad S_\alpha(\rho_{\{\alpha_j\}}) = 0 \\ \delta[S_\alpha(\rho_{\{\alpha_j\}}), 0] &= 0 \quad \text{if} \quad S(\rho_{\{\alpha_j\}}) > 0 \,. \end{aligned} \tag{3.27}$$

3.3.3 Physical measure

Sometimes we are interested in the structure of entanglement beyond separability. A famous example is the difference in entanglement between the GHZ (Greenberger-Horne-Zeilinger) state and the W state

$$|GHZ\rangle := \frac{1}{\sqrt{2}}(|00\cdots0\rangle + |11\cdots1\rangle) \tag{3.28}$$

$$|W\rangle := \frac{1}{\sqrt{n}}(|10\cdots0\rangle + \text{perm.})\,. \tag{3.29}$$

Both are fully entangled, but differ greatly in the way they may be exploited in quantum informational scenarios. We have therefore introduced another set of measures $\mathcal{E}_{\{\alpha_j\}}$ which together form the physical measure $\mathcal{E}_P := \sum_{\{\alpha_j\}} \mathcal{E}_{\{\alpha_j\}}$. This time we only require that $\mathcal{E}_P$ fulfills (M1-M5,M7,M8), in other words everything but to be normalized by maximally entangled states. The $\mathcal{E}_{\{\alpha_j\}}$ should individually fulfill:

Ph1 $\mathcal{E}_{\{\alpha_j\}}(\rho) \geq 0 \;\; \Leftrightarrow \; \{\alpha_j\} \subseteq \gamma_t$

Ph2 $\mathcal{E}_{\{\alpha_j\}}(\rho) = 0 \;\; \Leftrightarrow \; \{\alpha_j\} \supset \gamma_t$

Ph3 $\mathcal{E}_{\{\alpha_j\}}(\rho) = \mathcal{E}_{\{\alpha_j\}}(U_{loc}\rho U_{loc})$

Ph4 $\mathcal{E}_{\{\alpha_j\}}(\rho^{\otimes n}) = n\mathcal{E}_{\{\alpha_j\}}(\rho)$

Ph5 $\mathcal{E}(\rho) = |\{\alpha_j\}| \log_2(d) \; \forall \; \rho_{\{\alpha_j\}} = \rho_{me}$

We can now introduce a measure that fulfills all such properties except for the additivity on copies (M4,Ph4), which is only conjectured but not proven

$$\mathcal{E}_{\{\alpha_j\}}(\rho) := \max[(\inf_{\mathcal{D}(\rho_{\alpha_j})} \sum_i p_i \sum_{s\in\{\alpha_j\}} S_\alpha(\rho_s^i) - \sum_{\{\alpha_i\}\subset\{\alpha_j\}} \mathcal{E}_{\{\alpha_i\}}), 0] \,. \tag{3.30}$$

The construction of these measures has led to the publication:

- Beatrix C. Hiesmayr, Marcus Huber, Philipp Krammer
  Two computable sets of multipartite entanglement measures
  Phys. Rev. A **79**, 062308 (2009)

3.3.4 The m-concurrence in multipartite systems

The general m-concurrence is defined for multipartite states of arbitrary dimension. The easiest way of defining it is over the linear entropy of a subsystem ρ_s of a multipartite state ρ

$$\begin{aligned} S_L(\rho_s) &= \frac{d}{d-1}(1 - \mathrm{Tr}(\rho_s^2)) = C_m^2(|\psi\rangle\langle\psi|)) \\ &= \sum_\alpha C_{\mathbf{s}\alpha}^2 + \sum_\alpha \sum_\beta C_{\mathbf{s}\alpha\beta}^2 + (\ldots) + \\ &\quad + \sum_\alpha \sum_\beta \cdots \sum_\omega C_{\mathbf{s}\alpha\beta\cdots\omega}^2 \,. \end{aligned} \tag{3.31}$$

the concurrence functions are defined as

$$C_{\mathbf{s}\alpha\beta\cdots\omega}^2 := \sum_{O_C} \mathrm{Tr}(|\psi\rangle\langle\psi|(O_C + O_C^\dagger)|\psi^*\rangle\langle\psi^*|(O_C + O_C^\dagger)) \tag{3.32}$$

where

$$O_C = (A|\{i_n\}\rangle\langle\{i_n\}|\mathbb{1} - B|\{i_n\}\rangle\langle\{i_n\}|AB) \tag{3.33}$$

with

$$
\begin{aligned}
A &:= \left(\sigma_{k_K l_K}^{K\in\{\mathbf{s}\alpha\beta\cdots\omega\}}, \mathbb{1}^{K\notin\{\mathbf{s}\alpha\beta\cdots\omega\}}\right) \\
B &:= \left(\sigma_{k_K l_K}^{K=\mathbf{s}}, \mathbb{1}^{K\neq\mathbf{s}}\right)
\end{aligned}
\tag{3.34}
$$

and

$$
\sum_{O_C} := \sum_{k_K=0}^{d_K-1} \sum_{l_K>k_K} \sum_{\{i_n\}} . \tag{3.35}
$$

Here K denotes the respective subsystem and the flip operators are defined for a qudit system of dimension d in the following way

$$
\sigma_{kl}^{d\times d}\,|n\rangle \;=\; \delta_{nk}|l\rangle + \delta_{nl}|k\rangle\,. \tag{3.36}
$$

Note that these are the symmetric generalized Gell-Mann operators (see, e.g., Refs. [16, 17]). The generalized Gell-Mann operators are the SU(N) generators). For mixed states the concurrence may be defined via a convex roof

$$
C_m^2(\rho) := \inf_{p_i,|\psi_i\rangle} \sum_i p_i C_m^2(|\psi_i\rangle\langle\psi_i|)\,. \tag{3.37}
$$

As shown in our works (P2, P4) the advantage of rewriting the entropies by means of operators is that we can bounds. We present here a way analogous to the method introduced in Ref. [15].

$$
\begin{aligned}
&\inf_{p_i,|\psi_i\rangle} \sum_i p_i C_m^2(|\psi_i\rangle\langle\psi_i|) \geq \\
&\sum_\alpha \inf_{p_i,|\psi_i\rangle} C_{\mathbf{s}\alpha}^2 + \sum_\alpha\sum_\beta \inf_{p_i,|\psi_i\rangle} C_{\mathbf{s}\alpha\beta}^2 + (\ldots) + \sum_\alpha\sum_\beta\cdots\sum_\omega \inf_{p_i,|\psi_i\rangle} C_{\mathbf{s}\alpha\beta\cdots\omega}^2\,.
\end{aligned}
\tag{3.38}
$$

and

$$
\inf_{p_i,|\psi_i\rangle} C_{\mathbf{s}\alpha\beta\cdots\omega}^2 \geq \sum_{O_C} \inf_{p_i,|\psi_i\rangle} \mathrm{Tr}(|\psi_i\rangle\langle\psi_i|(O_C+O_C^\dagger)|\psi_i^*\rangle\langle\psi_i^*|(O_C+O_C^\dagger))
\tag{3.39}
$$

Now we define a flipped density matrix where the conjugation is taken in the computational basis

$$
\widetilde{\rho}_{O_C} := (O_C+O_C^\dagger)\;\rho^*\;(O_C+O_C^\dagger);\,, \tag{3.40}
$$

By calculating the square root of the eigenvalues of $\rho\widetilde{\rho}_{O_C}$ which we denote as $\{\lambda_{O_C}\}$ we can derive tight bounds for

$$
\begin{aligned}
&\inf_{p_i,|\psi_i\rangle} \mathrm{Tr}(|\psi_i\rangle\langle\psi_i|(O_C+O_C^\dagger)|\psi_i^*\rangle\langle\psi_i^*|(O_C+O_C^\dagger)) = \\
&\qquad\max[0, 2\max[\{\lambda_{O_C}\}] - \sum\{\lambda_{O_C}\}]^2
\end{aligned}
\tag{3.41}
$$

so that we obtain

$$C^2_{\mathbf{s}\alpha\beta\cdots\omega}(\rho) \geq \sum_{O_C} \max\left\{0, (2 \max_{\lambda_i^{O_C}}(\{\lambda_i\}^{O_C}) - \sum_i \lambda_i^{O_C})\right\}^2 . \tag{3.42}$$

So now we end up with computable lower bounds for C^2_m. Note that all $\rho(O_C + O_C^\dagger)\rho^*(O_C + O_C^\dagger)$ have only four non-zero eigenvalues, so the computation may be done very efficiently even for high dimensional multipartite systems.

3.3.4.1 Improving the bounds

While $C^2_m(\rho)$ is invariant under local unitary transformation, its individual constituents $\mathrm{Tr}(|\psi_i\rangle\langle\psi_i|(O_C + O_C^\dagger)|\psi_i^*\rangle\langle\psi_i^*|(O_C + O_C^\dagger))$ are not. This has grave consequences for the bounds also, as $\max[0, 2\max[\{\lambda_i\}] - \sum_i \lambda_i]^2$ is invariant under local unitaries only for $d = n = 2$. Instead of viewing this as a disadvantage of the lower bounds one can exploit this fact to improve them significantly

$$B_C := \max_{U_{\text{local}}}[\sum_{O_C} \max[0, 2\max[\{\lambda_{\widetilde{O}_C}\}] - \sum\{\lambda_{\widetilde{O}_C}\}]^2] , \tag{3.43}$$

where the $\{\widetilde{\lambda}_i\}$ are the square roots of the eigenvalues of $\rho\widetilde{O}_C\rho^*\widetilde{O}_C^\dagger$ and

$$\widetilde{O}_C := U^\dagger_{\text{local}}(O_C + O_C^\dagger)U^*_{\text{local}} . \tag{3.44}$$

Using this optimization also clarifies the power of the tool. As every $O_C + O_C^\dagger$ defines a projection into a two qubit subsystem we see, that for bipartite systems:

$$B_C \ > \ 0 \ \Leftrightarrow \ \rho \ \in \ D_1 \tag{3.45}$$

In the multipartite scenario it is no longer guaranteed that the projections onto two qubit subsystems are local, which proves the following theorem:
If $B_C \ > \ 0$ then ρ is either 1-distillable or unlockable bound entangled.

3.4 Multipartite distillation

3.4.1 Distillation procedures

We can generalize the notion of bipartite distillability in a straightforward way. Consider an LOCC operation Λ_{LOCC} which with some non-vanishing probability $p_i = \mathrm{Tr}((A_i \otimes B_i \otimes (\cdots) \otimes Z_i)\rho(A_i^\dagger \otimes B_i^\dagger \otimes (\cdots) \otimes Z_i^\dagger))$ produces a desired outcome. One can then repeat the process n times, always discarding the state if the outcome is different

$$\rho_n := \frac{(A_i \otimes B_i \otimes (\cdots) \otimes Z_i)\rho_{n-1}^{\otimes k}(A_i^\dagger \otimes B_i^\dagger \otimes (\cdots) \otimes Z_i^\dagger)}{\mathrm{Tr}((A_i \otimes B_i \otimes (\cdots) \otimes Z_i)\rho_{n-1}^{\otimes k}(A_i^\dagger \otimes B_i^\dagger \otimes (\cdots) \otimes Z_i^\dagger))} , \tag{3.46}$$

if there $\exists$ a set of operations $\{A_i, B_i, (\cdots), Z_i\}$ for which

$$\lim_{n\to\infty} \rho_n = |\psi\rangle\langle\psi| \tag{3.47}$$

and $|\psi\rangle\langle\psi|$ is a maximally entangled two qubit state, then the state ρ is called k-copy-distillable. Of course in the multipartite scenario we can extend this notion to other entangled states. The aim of generalized distillation procedures might be to create different multipartite entangled states. A straightforward extension would be for $|\psi\rangle\langle\psi|$ to be a maximally entangled m-partite state. However in the multipartite scenario there exist different possibilities for the separability of a maximally entangled state, therefore one could extend the definition of distillability.

3.4.2 Bound entanglement

States are bound entangled in the multipartite scenario if they are not distillable, even for $k \to \infty$ and $n \to \infty$ via operations of the form (3.46), although they are entangled. Just as in the bipartite scenario the partial transposition proves to be a versatile tool here also. If the state is positive under partial transposition (PPT) for all bipartitions, then the state is non-distillable. So any entangled state exhibiting this property is hence bound entangled.

3.4.3 Unlockable bound entanglement

A bound entangled state in the multipartite scenario may sometimes be distilled if a subset of cardinality c of the N parties cooperate. Of course this only makes sense for $c < N$. If that is the case, the c parties are allowed global operations on their joint part of the system. If this enables them to distill an otherwise bound entangled state, the state in question is called unlockable bound entangled. We have investigated such states in (P5).

4 The Simplex

4.1 A special simplex for two qubits

A single qubit state ω lives in a two dimensional Hilbert space, i.e. $\mathcal{H} \equiv \mathbb{C}^2$, and any state can be decomposed into the well known Pauli matrices σ_i

$$\omega \;=\; \frac{1}{2}\,(\mathbb{1}_2 + n_i\,\sigma_i)$$

with the Bloch vector components $\vec{n} \in \mathbb{R}^3$ and $\sum_{i=1}^3 n_i^2 = |\vec{n}|^2 \leq 1$. For $|\vec{n}|^2 < 1$ the state is mixed (corresponding to $\mathrm{Tr}\,\omega^2 < 1$) whereas for $|\vec{n}|^2 = 1$ the state is pure ($\mathrm{Tr}\,\omega^2 = 1$).

The density matrix of 2–qubits ρ on $\mathbb{C}^2 \otimes \mathbb{C}^2$ is usually obtained by calculating its elements in the standard product basis, i.e. $|00\rangle, |01\rangle, |10\rangle, |11\rangle$. Alternatively, we can write any 2–qubit density matrix in a basis of 4×4 matrices, the tensor products of the identity matrix $\mathbb{1}_2$ and the Pauli matrices,

$$\rho = \frac{1}{4}\,(\mathbb{1}_2 \otimes \mathbb{1}_2 + a_i\,\sigma_i \otimes \mathbb{1}_2 + b_i\,\mathbb{1}_2 \otimes \sigma_i + c_{ij}\,\sigma_i \otimes \sigma_j) \tag{4.1}$$

with $a_i, b_i, c_{ij} \in \mathbb{R}$. The parameters a_i, b_i are called *local* parameters as they determine the statistics of the reduced matrices, i.e. of Alice's or Bob's system. In order to obtain a geometrical picture one considers in the following only states where the local parameters are zero ($\vec{a} = \vec{b} = \vec{0}$), i.e., the set of all locally maximally mixed states, $Tr_A(\rho) = Tr_B(\rho) = \frac{1}{2}\mathbb{1}_2$ (see also Ref. [19, 20]).

As the property of separability does not change under local unitary transformations, the states under consideration can be written in the form [19]

$$\rho \;=\; \frac{1}{4}\,(\mathbb{1}_2 \otimes \mathbb{1}_2 + c_i\,\sigma_i \otimes \sigma_i)$$

where the c_i are three real parameters and can be considered as a vector $\vec{c}$ in Euclidean space. Differently stated, for any locally maximally mixed state ρ the action of two arbitrary unitary transformations $U_1 \otimes U_2$ can via the homomorphism of the groups $SU(2)$ and $SO(3)$ be related to unique rotations $O_1 \otimes O_2$. Thus the correlation matrix $c_{ij}\,\sigma_i \otimes \sigma_j$ can be chosen such that the matrix c_{ij} gets via singular value decomposition diagonal. Therefore, three real numbers combined to a vector $\vec{c}$ can be taken as an representative of the state itself.

In Fig. 4.1 we draw the 3–dimensional picture, where each point $\vec{c}$ corresponds to a locally maximally mixed state ρ. The origin $\vec{c} = \vec{0}$ corresponds to the totally mixed state, i.e. $\frac{1}{4}\mathbb{1}_2 \otimes \mathbb{1}_2$. The only pure states in the picture are given by

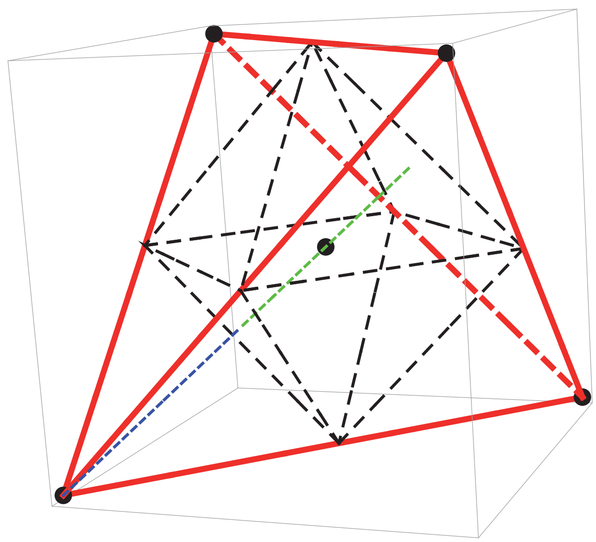

Figure 4.1: Here the geometry of the state space of two qubits is visualized. Each state is represent by a triple of three real numbers, $\vec{c}$, Eq. (4.1). The four black dots in the vertices of the cube represent four orthogonal "vertex" states. These are the four maximally entangled Bell states $\psi^{\pm}, \phi^{\pm}$. The positivity condition forms a tetrahedron (red) with the four "vertex" states and the totally mixed state in the origin (black dot in the middle). All separable states are represented by points inside and at the surface of the octahedron (dashed object).

$|\vec{c}|^2 = 3$ and represent the four maximally entangled Bell states $|\psi^{\pm}\rangle = \frac{1}{\sqrt{2}}\{|01\rangle \pm |10\rangle\}, |\phi^{\pm}\rangle = \frac{1}{\sqrt{2}}\{|00\rangle \pm |11\rangle\}$, which are located in vertices of the cube. The planes spanned by these four points are equivalent to the positivity criterion of the state ρ. Therefore, all points inside the tetrahedron represent the state space.

It is well known that density matrices which have at least one negative eigenvalue after partial transpose (PT) are entangled. The inversion of the argument is only true for systems with $2 \otimes 2$ and $2 \otimes 3$ degrees of freedom. PT corresponds to a reflection, i.e. $c_2 \to -c_2$ with all other components unchanged. Thus all points inside and at the surface of the octahedron represent all separable states in the set. Of course, one can always make the transformation $\vec{c} \longrightarrow -\vec{c}$, thus one obtains a mirrored tetrahedron, spanned by the four other vertices of the cube. Clearly, the intersection of the these two tetrahedrons contain all states which have positive eigenvalues after the action of PT.

4.2 A "magic" simplex for two qudits

For bipartite qudits a similar construction is possible: the vertex states $P_{i,j}$ of the "magic" simplex $\mathcal{W}$ are the maximally entangled states in d dimensions (Refs. [21, 22]):

$$|\Phi^+\rangle : \;=\; \sum_{i=0}^{d-1} |ii\rangle \;, \quad P_{0,0} \;:=\; |\Phi^+\rangle\langle\Phi^+| \tag{4.2}$$

$$P_{k,l} : \;=\; \mathbb{1}_d \otimes \mathcal{W}_{k,l}\, P_{0,0}\, \mathbb{1}_d \otimes \mathcal{W}_{k,l}^{\dagger} \tag{4.3}$$

where the $W_{k,l}$ are the Weyl operators defined by

$$W_{k,l} := \sum_{n=0}^{d-1} e^{\frac{(2\pi i)(kn)}{d}} |n\rangle\langle n+l| \;. \tag{4.4}$$

The magic simplex $\mathcal{W}$ is the convex combination of all vertex states

$$\mathcal{W} := \left\{ \sum_{k,l=0}^{d^2-1} c_{k,l}\, P_{k,l} \mid c_{i,j} \geq 0,\; \sum_{k,l=0}^{d^2-1} c_{k,l} = 1 \right\} \;. \tag{4.5}$$

One main property of this class of states forming a $d^2 - 1$ dimensional simplex is that any partial trace results in maximally mixed state.

4.3 A set of simplices for multipartite qudits

The generalization to multipartite systems is not so straightforward. This is because maximally entangled states for multipartite systems can exhibit different k-separability properties and there exist many different classes of entangled states in multipartite systems. For this reason we first choose to construct a special simplex $\mathcal{W}_n$ out of fully and maximally entangled states. The vertex states are therefore the maximally and fully entangled states in d dimensions:

$$|\Phi^+\rangle : \; = \; \sum_{i=0}^{d-1} |i\rangle^{\otimes n}\,, \quad P_{0,0} \; := \; |\Phi^+\rangle\langle\Phi^+| \tag{4.6}$$

$$P_{k,l} : \; = \; \mathbb{1}_d \otimes \mathcal{W}_{k,l}\, P_{0,0}\, \mathbb{1}_d \otimes \mathcal{W}_{k,l}^\dagger \tag{4.7}$$

where the $W_{k,l}$ are the Weyl operators defined by

$$W_{k,l} := \sum_{n=0}^{d^{n-1}-1} e^{\frac{(2\pi i)(kn)}{d^{n-1}}} |n\rangle\langle n+l|\,. \tag{4.8}$$

The magic simplex $\mathcal{W}_n$ is the convex combination of all vertex states

$$\mathcal{W}_\backslash := \left\{ \sum_{k,l=0}^{d^n-1} c_{k,l}\, P_{k,l} \mid c_{i,j} \geq 0,\; \sum_{k,l=0}^{d^n-1} c_{k,l} = 1 \right\}\,. \tag{4.9}$$

Again one main property of this class of states forming a d^n-1 dimensional simplex is that the reduction to single subsystems results in maximally mixed state.

4.3.1 A special subset of $\mathcal{W}_n$ for qubits

Consider qubit states of the form:

$$\rho = \frac{1}{2^n}\left(\mathbb{1}_{2^n} + \sum_i c_i \sigma_i^{\otimes n}\right) \tag{4.10}$$

They are a large subset of $\mathcal{W}_n$ and exhibit interesting properties. As proven in or work (P3) this subset of W_n contains only fully separable states for odd n, but for even n all entangled states within this simplex are bound entangled. Moreover concerning the geometry of separability, positivity and PPT this simplex is for even n exactly equivalent to the special simplex for two qubits. Also for $n=4$ the vertex states are local unitary equivalents of the Smolin state [18], which was the first multipartite bound entangled state that was found and which is known to be biseparable. This also proves that a convex combination of genuinely multipartite entangled can result in bound entangled biseparable states.
These results can be found in:

P3 Beatrix C. Hiesmayr, Florian Hipp, Marcus Huber, Philipp Krammer, Christoph Spengler
A simplex of bound entangled multipartite qubit states
Phys. Rev. A **78**, 042327 (2008)

A similar construction can be employed for multipartite qudits, which has been intensively investigated in:

P5 Beatrix C. Hiesmayr, Marcus Huber
Two distinct classes of bound entanglement: PPT-bound and 'multi-particle'-bound
arXiv:quant-ph 0906.0238

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A simplex of bound entangled multipartite qubit states

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We construct a simplex for multipartite qubit states of even number n of qubits, which has the same geometry concerning separability, mixedness, kind of entanglement, amount of entanglement and nonlocality as the bipartite qubit states. We derive the entanglement of the class of states which can be described by only three real parameters with the help of a multipartite measure for all discrete systems. We prove that the bounds on this measure are optimal for the whole class of states and that it reveals that the states possess only n–partite entanglement and not e.g. bipartite entanglement. We then show that this n–partite entanglement can be increased by stochastic local operations and classical communication to the purest maximal entangled states. However, pure n–partite entanglement cannot be distilled, consequently all entangled states in the simplex are n–partite bound entangled. We study also Bell inequalities and find the same geometry as for bipartite qubits. Moreover, we show how the (hidden) nonlocality for all n–partite bound entangled states can be revealed.

PACS numbers: 03.67.Mn
Keywords: multiparticle systems, bound entanglement, distillation, Bell inequality, entanglement measure

I. INTRODUCTION

Entanglement is at the heart of the quantum theory. It is the source of several new applications as quantum cryptography or a possible quantum computer. In recent years by studying higher dimensional quantum systems and/or multipartite systems one realizes that different aspects of the entanglement feature arise. They may have new applications such as multiparty cryptography.

In this paper we contribute to the classification of entanglement in a twofold way, i.e. which kind of entanglement a certain class of multipartite qubit states possesses by using the multipartite measure proposed in Ref. [1] and whether this kind of entanglement can be distilled. Our results suggest that one can distinguish for multipartite systems between different possibilities.

The class of states we analyze are a generalization of the class of states which form the well–known simplex for bipartite qubits (Sec. II), i.e. all locally maximally mixed states [2, 3]. We make an obvious generalization and find for states composed of an even number of qubits n an analogous simplex, i.e. this class of states shows the same geometry concerning positivity, mixedness, separability and entanglement (Sec. III). Further the used multipartite measure [1] reveals that the kind of entanglement possessed is only a n–partite entanglement where n is the number of qubits involved. The vertex states of the simplex are represented in the bipartite case by the well known Bell states, for $n > 2$ they are equivalent to the generalized smolin states proposed by Ref. [4, 6, 7, 8].

Then we discuss the distillability of the entangled states and find states for which the n–partite entanglement can be increased by a protocol only based on copy states and stochastic local operations and classical communications (LOCC). We show that the state is not distillable for any subset of parties and hence bound entangled, however, the n–partite entanglement can be enhanced to reach the maximal possible purity and n–partite entanglement within the class of states under investigation, i.e. the vertex states. For a subset of these states it has been shown that they allow for quantum information concentration ,e.g. Ref. [4, 5], so we suggest that it might still be advantageous to enhance the n–partite bound entangled states for some applications.

Last but not least, in Sec. VI we address to the question which of the simplex states violate the generalized Bell inequality which was shown to be optimal in this case and draw its geometrical picture, Fig. 4.

II. THE SIMPLEX FOR BIPARTITE QUBITS

A single qubit state ω lives in a two dimensional Hilbert space, i.e. $\mathcal{H} \equiv \mathbb{C}^2$, and any state can be decomposed into the well known Pauli matrices σ_i

$$\omega \;=\; \frac{1}{2}\left(\mathbb{1}_2 + n_i\,\sigma_i\right)$$

with the Bloch vector components $\vec{n} \in \mathbb{R}^3$ and $\sum_{i=1}^3 n_i^2 = |\vec{n}|^2 \leq 1$. For $|\vec{n}|^2 < 1$ the state is mixed (corresponding to $\mathrm{Tr}\,\omega^2 < 1$) whereas for $|\vec{n}|^2 = 1$ the state is pure ($\mathrm{Tr}\,\omega^2 = 1$).

The density matrix of 2–qubits ρ on $\mathbb{C}^2 \otimes \mathbb{C}^2$ is usually obtained by calculating its elements in the standard product basis, i.e. $|00\rangle, |01\rangle, |10\rangle, |11\rangle$. Alternatively, we can write any 2–qubit density matrix in a basis of 4×4 matrices, the tensor products of the identity matrix $\mathbb{1}_2$ and the Pauli matrices,

$$\rho = \frac{1}{4}\left(\mathbb{1}_2 \otimes \mathbb{1}_2 + a_i\,\sigma_i \otimes \mathbb{1}_2 + b_i\,\mathbb{1}_2 \otimes \sigma_i + c_{ij}\,\sigma_i \otimes \sigma_j\right)$$

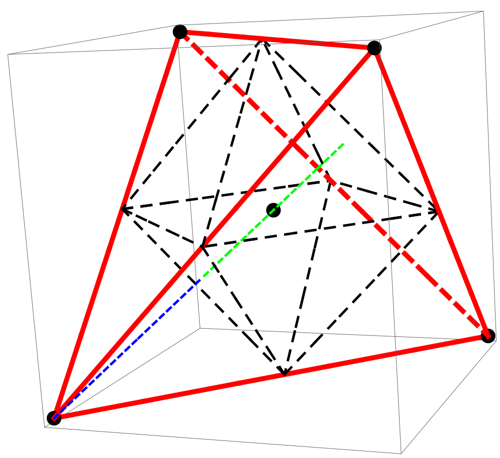

FIG. 1: (Color online) Here the geometry of the state space of even number of qubits is visualized. Each state is represent by a triple of three real numbers, $\vec{c}$, Eq. (1). The four black dots in the vertices of the cube represent four orthogonal "vertex" states. In the case of two qubits these are the four maximally entangled Bell states $\psi^\pm, \phi^\pm$ and for higher n they are equally mixtures of $2^n/4$ GHZ–states. The positivity condition forms a tetrahedron (red) with the four "vertex" states and the totally mixed state in the origin (black dot in the middle). All separable states are represented by points inside and at the surface of the octahedron (dashed object). The dashed line represents for $n=2$ the Werner states and for $n>2$ the generalized Smolin states (becoming separable when blue changes into green).

with $a_i, b_i, c_{ij} \in \mathbb{R}$. The parameters a_i, b_i are called *local* parameters as they determine the statistics of the reduced matrices, i.e. of Alice's or Bob's system. In order to obtain a geometrical picture one considers in the following only states where the local parameters are zero ($\vec{a} = \vec{b} = \vec{0}$), i.e., the set of all locally maximally mixed states, $Tr_A(\rho) = Tr_B(\rho) = \frac{1}{2}\mathbb{1}_2$ (see also Ref. [2, 3]).

A state is called separable if and only if it can be written in the form $\sum_i p_i\, \rho_i^A \otimes \rho_i^B$ with $p_i \geq 0, \sum p_i = 1$, otherwise it is entangled. As the property of separability does not change under local unitary transformation and classical communication (LOCC) the states under consideration can be written in the form [2]

$$\rho \;=\; \frac{1}{4}\left(\mathbb{1}_2 \otimes \mathbb{1}_2 + c_i\, \sigma_i \otimes \sigma_i\right)\,,$$

where the c_i are three real parameters and can be considered as a vector $\vec{c}$ in Euclidean space. Differently stated, for any locally maximally mixed state ρ the action of two arbitrary unitary transformations $U_1 \otimes U_2$ can via the homomorphism of the groups $SU(2)$ and $SO(3)$ be related to unique rotations $O_1 \otimes O_2$. Thus the correlation matrix $c_{ij}\, \sigma_i \otimes \sigma_j$ can be chosen such that the matrix c_{ij} gets via singular value decomposition diagonal. Therefore, three real numbers combined to a vector $\vec{c}$ can be taken as an representative of the state itself.

In Fig. 1 we draw the 3–dimensional picture, where each point $\vec{c}$ corresponds to a locally maximally mixed state ρ. The origin $\vec{c} = \vec{0}$ corresponds to the totally mixed state, i.e. $\frac{1}{4}\mathbb{1}_2 \otimes \mathbb{1}_2$. The only pure states in the picture are given by $|\vec{c}|^2 = 3$ and represent the four maximally entangled Bell states $|\psi^\pm\rangle = \frac{1}{\sqrt{2}}\{|01\rangle \pm |10\rangle\}, |\phi^\pm\rangle = \frac{1}{\sqrt{2}}\{|00\rangle \pm |11\rangle\}$, which are located in vertices of the cube. The planes spanned by these four points are equivalent to the positivity criterion of the state ρ. Therefore, all points inside the tetrahedron represent the state space.

It is well known that density matrices which have at least one negative eigenvalue after partial transpose (PT) are entangled. The inversion of the argument is only true for systems with $2 \otimes 2$ and $2 \otimes 3$ degrees of freedom. PT corresponds to a reflection, i.e. $c_2 \rightarrow -c_2$ with all other components unchanged. Thus all points inside and at the surface of the octahedron represent all separable states in the set. Of course, one can always make the transformation $\vec{c} \longrightarrow -\vec{c}$, thus one obtains a mirrored tetrahedron, spanned by the four other vertices of the cube. Clearly, the intersection of the these two tetrahedrons contain all states which have positive eigenvalues after the action of PT.

In Ref. [9, 10, 11] a generalization to higher dimensional bipartite states is considered and a so called magic simplex for qudits is obtained. Here the class of all locally maximally mixed states have to be reduced in order to obtain this generalized simplex. Already for bipartite qutrits many new symmetries arise and regions of bound entanglement can be found (see also Refs. [12, 13, 14, 15, 16]).

We also want to generalize the simplex of bipartite qubits, however, in our case we increase the number of qubits.

III. A SIMPLEX FOR n–PARTITE QUBIT STATES

Assume we have n qubits. Then a generalization can be written as

$$\begin{aligned} \rho \;&=\; \frac{1}{2^n}\left(\mathbb{1} + \sum c_i\, \sigma_i \otimes \sigma_i \otimes \cdots \otimes \sigma_i\right) \\ &:=\; \frac{1}{2^n}\left(\mathbb{1} + \sum c_i\, \sigma_i^{\otimes n}\right) \end{aligned} \qquad (1)$$

Obviously, for this generalization we follow the strategy to set the local parameters of all subsystems j, $Tr_{1,2,\ldots,j-1,j+1,\ldots,n}(\rho)$, to zero, as well as the parameters shared by two parties j,k, $Tr_{1,2,\ldots,j-1,j+1,\ldots,k-1,k+1,\ldots,n}(\rho)$, zero and so on until $n-1$ zero.

Again the state can be represented by a three dimensional vector $\vec{c}$. For $n=3$ the positivity condition $\rho \geq 0$ requires

$$|\vec{c}|^2 \leq 1\,. \qquad (2)$$

This turns out to be the case for all odd numbers of qubits involved.

For even numbers of qubits the positivity condition $\rho \geq 0$ requires that the vector is within the following four planes [30]:

$$1 + \vec{c} \cdot \vec{n}^{\,(i)} \;\geq\; 0 \tag{3}$$
$$\text{with} \quad \vec{n}^{\,(i)} \;=\; \begin{pmatrix} -1 \\ +1 \\ +1 \end{pmatrix}, \begin{pmatrix} +1 \\ -1 \\ +1 \end{pmatrix}, \begin{pmatrix} +1 \\ +1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

These conditions are exactly the same ones as for the two qubit case $n = 2$, i.e. the four above planes form the magic tetrahedron.

The purity $Tr(\rho^2)$ gives $\frac{1}{2^n}(1 + |\vec{c}|^2)$, thus the states with $|\vec{c}|^2 = 3$ are the purest states of the class of states under investigation and are located in the vertices of the tetrahedron. Note with increasing n the percentage of purity decreases, i.e. only for $n = 2$ the vertices present pure states. Further analysis of these vertex states follows later.

Now we want to investigate if the separability condition also for $n > 2$ corresponds to the octahedron. The partial transpose of one qubit ($PT_{\text{one qubit}}$) changes the sign in front of the $\sigma_2^{\otimes n}$ matrix, i.e. the y–component of the vector $\vec{c}$ changes sign. Therefore the states under investigation are entangled by the necessary but not sufficient (one qubit) Peres criterion

$$\begin{aligned} \text{n=3,5,\ldots:} &\quad |\vec{c}|^2 \leq 1 \\ \text{n=2,4,\ldots:} &\quad 1 - \vec{c} \cdot \vec{n}^{\,(i)} \;\leq\; 0\,. \end{aligned} \tag{4}$$

Taking the partial transpose of two, four, …qubits changes two, four,…times the sign and consequently one obtains the positivity criterion (3). Taking the partial transpose of odd qubits is equivalent to $PT_{\text{one qubit}}$.

For even number of qubits the above Peres criterion implies a mirrored tetrahedron, analogously to the bipartite case, however, we do not know if the intersection, the octahedron, contains only separable states. For odd numbers of qubits the situation is different and we will not investigate it further.

Now two questions arise, firstly, are all states represented by the octahedron separable and, secondly, what kind of entanglement does this class of states possess?

Let us tackle the second question first. To analyze our generalized states ρ further we use the multipartite entanglement measure for all discrete systems introduced by Ref. [1]. The main idea is that the information content of any n–partite quantum system of arbitrary dimension can be separated in the following form:

$$\underbrace{I(\rho) + R(\rho)}_{\text{single property}} + \underbrace{E(\rho)}_{\text{entanglement}} \;=\; n \tag{5}$$

where

$$I(\rho) \;:=\; \sum_{s=1}^{n} \underbrace{\mathcal{S}_s^2(\rho)}_{\text{single property of subsystem s}} \tag{6}$$

contains all locally obtainable information (i.e. obtainable information a party can measure on its particle) and $E(\rho)$ contains all information encoded in entanglement and $R(\rho)$ is the complementing missing information, it is due to a classical lack of knowledge about the quantum state. The total amount of entanglement $E(\rho)$ can be separated into m–flip concurrences by rewriting the linear entropy of all subsystems in an operator sum, thus one obtains

$$\begin{aligned} E(\rho) \;:=\; & \underbrace{\mathbf{C}^2_{(2)}(\rho)}_{\text{two flip concurrence}} + \underbrace{\mathbf{C}^2_{(3)}(\rho)}_{\text{three flip concurrence}} + (\ldots) \\ & + \underbrace{\mathbf{C}^2_{(n)}(\rho)}_{\text{n-flip concurrence}}\,. \end{aligned} \tag{7}$$

These m–flip concurrences are useful for two reason: firstly, one can obtain bounds on the operators and thus handle mixed states and secondly, the authors of Ref. [1] showed (for three qubits) that the m–flip concurrences can be reordered such that they give the m–partite entanglement, which in addition coincides with the m–separability [17].

Here we extend their result for the states under investigation. Due to high symmetry of the class of states under investigation the bounds of the m–partite entanglement can be computed and herewith we can reveal the following substructure of total entanglement $E(\rho)$

$$\begin{aligned} E(\rho) \;=\; & \underbrace{E_{(2)}(\rho)}_{\text{bipartite entanglement}} + \underbrace{E_{(3)}(\rho)}_{\text{tripartite entanglement}} \\ & + \cdots + \underbrace{E_{(n)}(\rho)}_{n\text{–partite entanglement}} \end{aligned} \tag{8}$$

with the sub–substructure

$$\begin{aligned} E_{(2)}(\rho) \;&=\; E_{(12)}(\rho) + E_{(13)}(\rho) + \cdots + E_{(1n)}(\rho) \\ &\quad + E_{(23)}(\rho) + \cdots + E_{(2n)}(\rho) + \cdots + E_{(n-1,n)}(\rho) \\ E_{(3)}(\rho) \;&=\; E_{(123)}(\rho) + \cdots + E_{(n-2,n-1,n)}(\rho) \\ \ldots \;&=\; \ldots \\ E_{(n)}(\rho) \;&=\; E_{(12\ldots n)}(\rho)\,. \end{aligned} \tag{9}$$

We find that for the states under investigation the only non–vanishing entanglement is the n–partite entanglement and it derives to (for details next Sec. IV)

$$\begin{aligned} E_{(n)} \;=\; E_{12\ldots n} \;=\; & X \max\Big[0,\; \frac{1}{2} \max\Big[-1 + \vec{c} \cdot \vec{n}^{\,(1)}, \\ & -1 + \vec{c} \cdot \vec{n}^{\,(2)}, -1 + \vec{c} \cdot \vec{n}^{\,(3)}, -1 + \vec{c} \cdot \vec{n}^{\,(4)}\,\Big]\Big]^2\,, \end{aligned} \tag{10}$$

where $X = 1$ except for bipartite qubits then it is $X = 2$ (the reason of this difference is explained later). Hence, we find the same condition for being entangled as given by the one qubit Peres criterion.

Now, if these bounds are exact also for $n > 2$, then all states represented by the octahedron are separable. Indeed, it turns out that this is the case. We give the proof of separability separately in the appendix.

In summary, we have found for even number of qubits the same geometry as in the case of bipartite qubits, also depicted by Fig. 1. Moreover, we have shown that the multipartite entanglement measure proposed by Ref. [1] works tightly as the bounds are exact and it reveals only n–partite entanglement. Let us discuss this result more carefully.

For the purest states, $|\vec{c}|^2 = 3$, located in the vertices of the tetrahedron, the maximal n–partite entanglement derives to $E_{(n)} = 1$ except for $n = 2$ it is $E_{(n)} = 2$. Thus the amount of entanglement for $n > 2$ is independent of the number of qubits involved. The reason for the difference can be found in the information content of a multipartite system, Eq. (5). The maximal entanglement of a n–partite state is n. This is the case if and only if the local obtainable information of all subsystems is zero and the classical lack of knowledge of the quantum state is also zero, i.e. the total state is pure. For bipartite qubits, $n = 2$, the vertex states are the Bell states, which have maximal entanglement 2 whereas there locally obtainable information S is zero as well as the lack of classical knowledge about the quantum state $R = 0$.

By construction for $n > 2$ we set the locally obtainable information S of all subsystems zero, however, also all possible locally obtainable information shared by two, three, $\ldots$, $n-1$ parties is set to zero; obviously this is not compatible with being maximally entangled. The information content for $n > 2$ is given by

$$n \;=\; E_n + R \;=\; 1 + R\,, \qquad (11)$$

and consequently the lack of classical knowledge is nonzero, i.e. $R = n - 1$. Differently stated for $n = 4$, any party has the trace state as well as any two parties and any three parties share the trace state, therefore $R = 3$.
Remark: The local information $S_s(\rho)$ of one subsystem s is nothing else than Bohr's quantified complementarity relation [18, 19, 20], with its well known physical interpretation in terms of predictability and visibility (coherence). One can extend this concept for two parties sharing a state, then their (bi–)local information of total multipartite system can be defined in similar way and is complemented by the mixedness of the shared bipartite system. Again this (bi–)local information is only obtainable if and only if the state is not the trace state.

Coming back to the simplex geometry we see that the closer we get to the origin the more the amount of entanglement reduces by increasing the amount of classical uncertainty R only.

For bipartite qubits the vertex states $|\vec{c}|^2 = 3$ are the four Bell states. For n qubits we find for $|\vec{c}|^2 = 3$ also four unitary equivalent states, however, they are no longer pure. For $n = 4$ the state is a equally weighted mixture of four $|GHZ\rangle$ states: Starting with one GHZ–state, e.g.

$$|GHZ\rangle \;=\; \frac{1}{\sqrt{2}}\big\{|0000\rangle + |1111\rangle\big\} \qquad (12)$$

one obtains another representation by applying two flips, i.e. $\mathbb{1}\otimes\mathbb{1}\otimes\sigma_x\otimes\sigma_x$, then applying on the new GHZ–state representation the operator $\mathbb{1}\otimes\sigma_x\otimes\sigma_x\otimes\mathbb{1}$ and onto that new GHZ–state representation the operator $\sigma_x\otimes\sigma_x\otimes\mathbb{1}\otimes\mathbb{1}$ gives the last GHZ–state representation. The other three vertex states are obtained by applying only one Pauli matrix. For $n = 6$ we have 2^6 GHZ–states where $2^6/4$ GHZ–states equally mix for one vertex state.
Remark: The same symmetry we find for the bipartite qubit case, one Bell states is mapped into another by one Pauli matrix, however, applying two Pauli matrices maps a Bell state onto itself, therefore we have no mixture of different maximally entangled states.

In the next section we give the detailed calculation of the measure and in the following section we investigate the question whether the entangled states are bound entangled and if in what sense their entanglement is bound. In particular we discuss what it means that the substructure revealed by the measure shows only n–partite entanglement.

IV. DERIVATION OF THE MULTIPARTITE MEASURE FOR THE SIMPLEX STATES

In Ref. [1] a multipartite measure for multidimensional systems as a kind of generalization of Bohr's complementarity relation was derived. Here, we give explicitly the results for $n = 2$ and $n = 4$ expressed in the familiar Pauli matrix representation

It is well known that to compute concurrence introduced by Hill and Wootters [21] one has to consider

$$\rho\;(\sigma_y\otimes\sigma_y)\;\rho^*\;(\sigma_y\otimes\sigma_y) \qquad (13)$$

where the complex conjugation is taken in computational basis. The concurrence is then given by the formula

$$\mathcal{C} \;=\; \max\big\{0, 2\max\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\} - (\lambda_1+\lambda_2+\lambda_3+\lambda_4)\big\} \qquad (14)$$

where the λ_i's are the square roots of the eigenvalues of the above matrix. To obtain the information content we have to multiply this measure by two.

The first observation in Ref. [1] is that the linear entropy, $M(\rho) = \frac{2}{3}\big(1 - Tr(\rho^2)\big)$, can be rewritten by operators. This means e.g. for any pure 4 qubit state

$$|\psi\rangle \;=\; \sum_{i,j,k,l=0}^{1} a_{ijkl}\;|ijkl\rangle\,, \qquad (15)$$

the linear entropy of one subsystem can be written as

$$
\begin{aligned}
&M^2(Tr_{234}|\psi\rangle\langle\psi|) = M^2(\rho_1) = \\
&\sum_{k,l=0}^{1}\sum_{\{i_1\neq i_1'\};\{i_2\neq i_2'\}} \left|\langle\psi|(\sigma_x\otimes\sigma_x\otimes\mathbb{1}\otimes\mathbb{1})\,(|i_1\,i_2\,k\,l\rangle\langle i_1\,i_2\,k\,l| - |i_1'\,i_2'\,k\,l\rangle\langle i_1'\,i_2'\,k\,l|)\,|\psi^*\rangle\right|^2 \\
&+\sum_{k,l=0}^{1}\sum_{\{i_1\neq i_1'\};\{i_3\neq i_3'\}} \left|\langle\psi|(\sigma_x\otimes\mathbb{1}\otimes\sigma_x\otimes\mathbb{1})\,(|i_1\,k\,i_3\,l\rangle\langle i_1\,k\,i_3\,l| - |i_1'\,k\,i_3'\,l\rangle\langle i_1'\,k\,i_3'\,l|)\,|\psi^*\rangle\right|^2 \\
&+\sum_{k,l=0}^{1}\sum_{\{i_1\neq i_1'\};\{i_3\neq i_3'\}} \left|\langle\psi|(\sigma_x\otimes\mathbb{1}\otimes\mathbb{1}\otimes\sigma_x)\,(|i_1\,k\,l\,i_4\rangle\langle i_1\,k\,l\,i_4| - |i_1'\,k\,l\,i_3'\rangle\langle i_1'\,k\,l\,i_4'|)\,|\psi^*\rangle\right|^2 \\
&+\sum_{k,l=0}^{1}\sum_{\{i_2\neq i_2'\};\{i_3\neq i_3'\}} \left|\langle\psi|(\mathbb{1}\otimes\sigma_x\otimes\sigma_x\otimes\mathbb{1})\,(|k\,i_2\,i_3\,l\rangle\langle k\,i_2\,i_3\,l| - |k\,i_2'\,i_3'\,l\rangle\langle k\,i_2'\,i_3'\,l|)\,|\psi^*\rangle\right|^2 \\
&+\sum_{k,l=0}^{1}\sum_{\{i_2\neq i_2'\};\{i_4\neq i_4'\}} \left|\langle\psi|(\mathbb{1}\otimes\sigma_x\otimes\mathbb{1}\otimes\sigma_x)\,(|k\,i_2\,l\,i_4\rangle\langle k\,i_2\,l\,i_4| - |k\,i_2'\,l\,i_4'\rangle\langle k\,i_2'\,l\,i_4'|)\,|\psi^*\rangle\right|^2 \\
&+\sum_{k,l=0}^{1}\sum_{\{i_3\neq i_3'\};\{i_4\neq i_4'\}} \left|\langle\psi|(\mathbb{1}\otimes\mathbb{1}\otimes\sigma_x\otimes\sigma_x)\,(|k\,l\,i_3\,i_4\rangle\langle k\,l\,i_3\,i_4| - |k\,l\,i_3'\,i_4'\rangle\langle k\,l\,i_3'\,i_4'|)\,|\psi^*\rangle\right|^2 \\
&+\sum_{k}^{1}\sum_{\{i_1\neq i_1'\};\{i_2\neq i_2'\};\{i_3\neq i_3'\}} \left|\langle\psi|(\sigma_x\otimes\sigma_x\otimes\sigma_x\otimes\mathbb{1})\,(|i_1\,i_2\,i_3\,k\rangle\langle i_1\,i_2\,i_3\,k| - |i_1'\,i_2'\,i_3'\,k\rangle\langle i_1'\,i_2'\,i_3'\,k|)\,|\psi^*\rangle\right|^2 \\
&+\sum_{k=0}^{1}\sum_{\{i_1\neq i_1'\};\{i_2\neq i_2'\};\{i_4\neq i_4'\}} \left|\langle\psi|(\sigma_x\otimes\sigma_x\otimes\mathbb{1}\otimes\sigma_x)\,(|i_1\,i_2\,k\,i_4\rangle\langle i_1\,i_2\,k\,i_4| - |i_1'\,i_2'\,k\,i_4'\rangle\langle i_1\,i_2'\,k\,i_4'|)\,|\psi^*\rangle\right|^2 \\
&+\sum_{k=0}^{1}\sum_{\{i_1\neq i_1'\};\{i_3\neq i_3'\};\{i_4\neq i_4'\}} \left|\langle\psi|(\sigma_x\otimes\mathbb{1}\otimes\sigma_x\otimes\sigma_x)\,(|i_1\,k\,i_3\,i_4\rangle\langle i_1\,k\,i_3\,i_4| - |i_1'\,k\,i_3'\,i_4'\rangle\langle i_1'\,k\,i_3'\,i_4'|)\,|\psi^*\rangle\right|^2 \\
&+\sum_{k=0}^{1}\sum_{\{i_2\neq i_2'\};\{i_3\neq i_3'\};\{i_4\neq i_4'\}} \left|\langle\psi|(\mathbb{1}\otimes\sigma_x\otimes\sigma_x\otimes\sigma_x)\,(|k\,i_2\,i_3\,i_4\rangle\langle k\,i_2\,i_3\,i_4| - |k\,i_2'\,i_3'\,i_4'\rangle\langle k\,i_2'\,i_3'\,i_4'|)\,|\psi^*\rangle\right|^2 \\
&+\sum_{\{i_1\neq i_1'\};\{i_2\neq i_2'\};\{i_3\neq i_3'\};\{i_4\neq i_4'\}} \left|\langle\psi|(\sigma_x\otimes\sigma_x\otimes\sigma_x\otimes\sigma_x)\,(|i_1\,i_2\,i_3\,i_4\rangle\langle i_1\,i_2\,i_3\,i_4| - |i_1'\,i_2'\,i_3'\,i_4'\rangle\langle i_1'\,i_2'\,i_3'\,i_4'|)\,|\psi^*\rangle\right|^2
\end{aligned}
\tag{16}
$$

where e.g. $\{i_1\}\neq\{i_1'\},\{i_2\}\neq\{i_2'\}$ means that the set of indexes are not the same, i.e. the sum is taken over

$$
\begin{aligned}
\{i_1,i_2;i_1',i_2'\} \;=\; &\{0,1;0,0\},\{0,0;0,1\},\{0,1;1,0\},\{0,0;1,1\},\{1,1;0,0\},\{1,0;0,1\},\{1,1;0,0\},\{1,0;0,1\},\\
&\{0,0;1,0\},\{1,0;0,0\},\{0,0;1,1\},\{1,0;0,1\},\{0,1;1,0\},\{1,1;0,0\},\{0,1;1,1\},\{1,1;0,1\}\,.
\end{aligned}
\tag{17}
$$

Likewise the linear entropies for the other subsystem can be derived, i.e. separated in terms where the flip operator σ_x is applied two, three or four times. It is well known that for pure states the sum over the entropies of all reduced density matrices is an entanglement measure, therefore using the linear entropy we get the following entanglement measure

$$
E(|\psi\rangle) := \sum_{s=1}^{4} M^2(\rho_s) \;=\; \sum_{m=2}^{4}\left(C^m(\psi)\right)^2\,, \tag{18}
$$

where $(C^m)^2$ is the sum of all terms of all reduced matrices which contain m–flip operators. These quantities where called (squared) m–concurrences, because they play a similar role as Wootters concurrence.

For mixed states ρ the infimum of all possible decompositions is an entanglement measure

$$
E(\rho) \;=\; \inf_{p_i,|\psi_i\rangle} \sum_{p_i,|\psi_i\rangle} p_i E(|\psi_i\rangle)\,. \tag{19}
$$

The problem of the whole entanglement theory is that this infimum can in general not be calculated. Now we bring the operator representation of the linear entropy into the game, because for operators upper bounds can be obtained.

Lets start with the calculation of the 4-flip concurrence $C^{(4)}$, which is the sum of all terms containing 4-flips of the entropies of all reduced matrices, i.e.

$$\left(C^{(4)}(\rho)\right)^2 = \inf_{p_i,|\psi_i\rangle} \sum_{p_i,|\psi_i\rangle} p_i \left(C^{(4)}(\psi_i)\right)^2 \qquad (20)$$

As shown in Ref. [1] one can derive bounds on the above expression for any m–flip concurrence by defining, in an analogous way to Hill and Wootters flip density matrix [21], the m–flip density matrix:

$$\begin{aligned}\widetilde{\rho}_s^m = O_s(|\{i_n\}\rangle\langle\{i_n\}| - |\{i'_n\}\rangle\langle\{i'_n\}|)\, \rho^* \cdot \\ \cdot\, O_s(|\{i_n\}\rangle\langle\{i_n\}| - |\{i'_n\}\rangle\langle\{i'_n\}|) \qquad (21)\end{aligned}$$

and calculating the λ_m^s's which are the squared roots of the eigenvalues of $\widetilde{\rho}_s^m \rho$. The bound $B^{(m)}$ of the m-flip concurrence $C^{(m)}$ is then given by

$$\begin{aligned} B^m(\rho) :=& \\ &\left(\sum_s \mathbf{max}\left[0, 2\mathbf{max}\left(\{\lambda_m^s\}\right) - \sum\{\lambda_m^s\}\right]^2\right)^{\frac{1}{2}} \end{aligned} \qquad (22)$$

From Eq. (16) we see that for the 4–flip concurrence of subsystem ρ_1 four different operators occur, thus we have in total 16 different operators listed in the appendix VII.

Inserting our class of states we find that for each operator $\mathcal{O}^s$ the eigenvalues are the same, i.e. one obtains 8 zeros and the remaining four eigenvalues are exactly equivalent to the Peres criterion Eq. (4).

The same procedure has to be applied to calculate the 3–flip concurrence and the 2–flip concurrence. As can be seen from Eq. (16) here the unity and σ_z matrix is involved which lead to no contribution for the states under investigation. Remember, that they are mixtures of the vertex states, which are equal mixtures of such GHZ–states which differ by two flips.

Therefore, the total entanglement is given by the $C^{(4)}$ concurrence only and is a 4–partite entanglement. For $n = 6, 8, \dots$ the scenario is the same, because of the same underlying symmetry.

In the Appendix we show that all states not detected by the measure are separable, thus the bounds are optimal and therefore the measure detects all bound entangled states.

V. ARE THE ENTANGLED STATES BOUND ENTANGLED?

In Refs. [4, 6, 7, 8] the special states $c = c_1 = -c_2 = c_3$ for $n > 2$, which were named generalized Smolin states (for $n = 2$ these states are the Werner states), are investigated and they show that for $1 \geq c > \frac{1}{3}$ these states are bound entangled. In particular, the authors

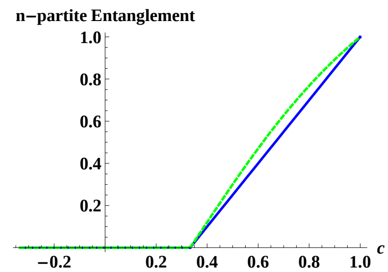

FIG. 2: (Color online) Here the n–partite entanglement of the Werner states $n = 2$ (here the y–axis has to be multiplied by two) or the generalized Smolin states $n > 2$ before and after the application of the introduced protocol (upper dashed green curve) is plotted. Note that the vertex states are mapped onto itself by the given protocol.

argued that these states are bound entangled, because the states are separable against bipartite symmetric cuts like $12|34\dots, 14|23\dots, \dots$ and therefore no Bell state between any two subsystem can be distilled. This is obviously also the case for the whole class of states under investigation.

As the considered measure of entanglement revealed only n–partite entanglement and e.g. not any m–partite entanglement ($m < n$), it may not seem directly obvious that Bell states (bipartite entanglement) cannot be distilled, because the class of states do not possess any bipartite entanglement. Thus the question could be refined to ask whether n–partite pure entanglement can be distilled.

For the n–partite class of states under investigation we consider a similar distillation protocol as the recurrence protocol by Bennett et al. [22]. For that we generalize it such that each party gets a copy onto which a unitary bilateral XOR operation is performed and afterwards a measurement in say z-direction is performed. Only states are kept where all parties found their copy qubit in say up–direction. This protocol favours as all protocols do one state, in our case for $n = 2$ it is the Φ^+ state and for $n > 2$ its equivalents.

In detail it goes like the following: We consider one state and its copy

$$\rho^{\otimes 2} = \left(\frac{1}{2^n}\left\{\mathbb{1}^{\otimes n} + c_i\, \sigma_i^{\otimes n}\right\}\right)^{\otimes 2} \qquad (23)$$

and all parties get a copy state. Therefore, we reorder the state by a unitary transformation such that the first term and second term in the tensor product belongs to Alice and the third and forth term to Bob and so on:

$$\begin{aligned}\rho^{\otimes 2} \longrightarrow (\frac{1}{2^n})^2 \Big\{ &(\mathbb{1}\otimes\mathbb{1})^{\otimes n} + c_i\, (\mathbb{1}\otimes\sigma_i)^{\otimes n} \\ &+ c_i\, (\sigma_i\otimes\mathbb{1})^{\otimes n} +\, c_i c_j\, (\sigma_i\otimes\sigma_j)^{\otimes n}\Big\} \end{aligned} \qquad (24)$$

Now each party perform on its two subsystems a unitary

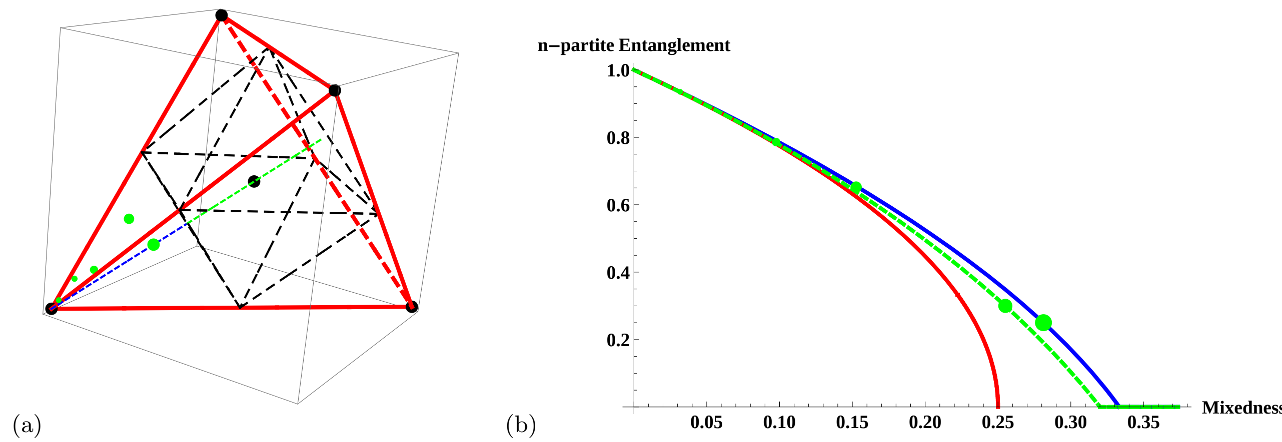

FIG. 3: (Color online) Fig. (a) shows the final states after each step of the introduced protocol of an initial Werner or Smolin state $c = 0.5$, where each (green) point represents the obtained state after one step of the protocol. Fig. (b) shows a mixedness, $\frac{2^n}{2^n-1}(1 - Tr(\rho^2))$, versus n–partite entanglement diagram (for $n = 2$ the y–axis has to be multiplied by 2), where the (blue) curve corresponds to the Werner or Smolin state whereas the (red) curve is the state connecting two vertices. All states of the simplex have their mixedness–entanglement ratio between these two curves. The middle (dashed, green) curve corresponds to the final states of a distilled Werner or Smolin state. And the (green) points represent the final states after each step of an initial Werner or Smolin state $c = 0.5$.

XOR operation

$$U_{XOR} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{25}$$

and then projects on the copy–subsystem with $P = \frac{1}{2}(\mathbb{1} + \sigma_z)$. This gives again a state in the class of states under investigation, i.e. one finds

$$\vec{c} = \begin{pmatrix} c_x \\ c_y \\ c_z \end{pmatrix} \longrightarrow \vec{c}_{\text{dis}} = \begin{pmatrix} \frac{c_x^2 + c_y^2}{1+c_z^2} \\ \frac{2c_x c_y}{1+c_z^2} \\ \frac{2c_z}{1+c_z^2} \end{pmatrix}. \tag{26}$$

Comparing with the separability condition and with the positivity condition, one verifies that only separable states are mapped into separable states.

Let us consider the Werner states and the generalized Smolin states ($c = c_x = c_y = c_z$), for which we derive that the n–partite entanglement is always increased after the above protocol, see Fig. 2. For $-\frac{1}{\sqrt{3}} \leq c \leq \frac{1}{3}$ the measure before and after the protocol is zero and for $c = 1$ the state is mapped onto itself. For $\frac{1}{3} < c < 1$ the entanglement of the distilled state is increased compared to the input state. In Fig. 3 (a) we give the 3–dimensional picture of how the initial state $c = 0.5$ moves after each step towards the vertex state. Note that the states are no longer in the set of the generalized Smolin sets, another advantage of considered set of states as no random bilateral rotation to regain the rotational symmetry is needed. In Fig. 3 (b) we show the mixedness–entanglement relation of this example. Note that all states of the simplex are within the two curves and the middle curve is the result for the generalized Smolin state after one step of the protocol.

Remark: Not all states of the simplex are mapped into more entangled states by this protocol. For example, the mixture of two vertex states ($\vec{c}^T = (0, 0, c)$ with $c \neq 1$) is left invariant.

In summary, we have found a protocol that increases the amount of entanglement with local operations and classical communication only and the final states are always within the class of states. Only for $n = 2$ the final state is pure and maximal entangled and therefore the above protocol is a distillation protocol, i.e. pure maximally entangled states can be obtained. However, for $n > 2$ the final state is no longer pure, but has the maximal n–partite entanglement of the class of states under investigation.

Thus the next logical step is to search for a distillation protocol which distills the vertex states into pure maximally entangled states, i.e. GHZ–states. However this is not possible for the following reasons: In general, any equally weighted mixture of two maximally entangled states cannot be distilled by mainly two observations. As for all maximally entangled states ρ_i obviously the entanglement can only be reduced by any completely positive map $\Lambda : \rho_i \mapsto \rho_i'$, i.e. $E(\rho_i') \leq E(\rho_i)\ \forall\ \Lambda$. And as the entanglement $E(\rho)$ is convex, i.e. $E(\rho_i') + E(\rho_j') \leq 2E(\rho_i')$, we conclude that at least one ρ_i must be mapped unitary onto itself or another maximally entangled state. Because all maximally entangled states are equivalent by local unitaries, such a map consequently maps also the other maximally entangled state of the mixture into a (different) maximally entangled state. Hence, for no equally mixture of maximally entangled states a maximally entangled state can be distilled. Note that in the case of bipartite qubits this is trivially true, because any equally mixture of Bell states is separable, however, for multipartite states this is not necessarily the case (e.g. our vertex states).

Thus we find that we can increase the amount of the n–partite entanglement until the vertex state, but not furthermore and therefore all entangled states are bound entangled, i.e. no pure n–partite entanglement can be distilled among any subset of parties using stochastic LOCC. The common definition of distillation is that no pure maximally entanglement among any subset of parties using LOCC can be obtained, see e.g. [23, 24]. A different way to prove that the entangled states are bound is given in Ref. [25], where they show that if no singlets can be distilled also no GHZ—state can be obtained. Therefore for the class of states under investigation we can also not distill any bipartite entanglement.

VI. THE GEOMETRY OF THE STATES VIOLATING THE CHSH—BELL INEQUALITY

Analog to the bipartite qubit state one can derive a CHSH–Bell type inequality for n qubit states [26]. Here $n-1$ parties measure their qubit in direction $\vec{a}$ or $\vec{a}'$ and the nth party in direction $\vec{b}$ or $\vec{b}'$, then one obtains the following Bell inequality

$$Tr(\mathcal{B}_{\text{Bell-CHSH}}\rho) \;\leq\; 2 \tag{27}$$

with

$$\begin{aligned}\mathcal{B}_{\text{Bell-CHSH}} &= \underbrace{\vec{a}\vec{\sigma}\otimes\vec{a}\vec{\sigma}\otimes\cdots\otimes\vec{a}\vec{\sigma}}_{n-1}\otimes(\vec{b}+\vec{b}')\vec{\sigma}\\ &+\underbrace{\vec{a}'\vec{\sigma}\otimes\vec{a}'\vec{\sigma}\otimes\cdots\otimes\vec{a}'\vec{\sigma}}_{n-1}\otimes(\vec{b}-\vec{b}')\vec{\sigma} \end{aligned}\tag{28}$$

where $\vec{a},\vec{a}',\vec{b},\vec{b}'$ are real unit vectors and the value 2 is the upper bound on any local realistic theory.

It is known that for $n=2$ the maximal violation by quantum mechanics can simply be derived by the state ρ itself [27]. A matrix ρ violates the Bell–CHSH inequality if and only if $\mathcal{M}(\rho)\geq 1$, where $\mathcal{M}(\rho)$ is the sum of the two largest eigenvalues of the Hermitian matrix $C^\dagger C$ with $(C)_{ij}=Tr(\sigma_i\otimes\sigma_j\rho)$. A generalization for n qubits is simple, because the matrix C is diagonal for the states under investigation, thus the same proof works.

In our case $\mathcal{M}(\rho)$ is simply the sum of the two largest squared vector components. In particular, if c_1 and c_2 are greater than c_3 we obtain the following Bell inequality

$$c_1^2+c_2^2\leq 1\,. \tag{29}$$

This gives a simple geometric interpretation of all states violating the Bell inequality. All possible saturated Bell inequalities give three different cylinders in the picture representing the state space, see Fig.4. All states outside of these three cylinders violate the Bell inequality.

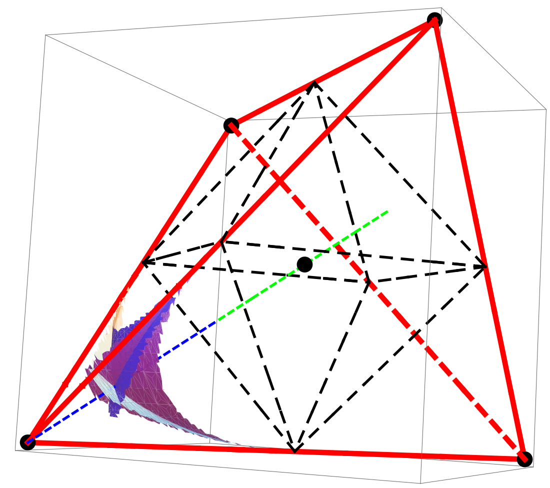

FIG. 4: (Color online) The three cylinders show the saturation of the Bell inequality. All states outside these cylinders violate the Bell inequality. The vertex states violate the Bell inequality maximal, i.e. by $2\sqrt{2}$.

Furthermore, this result shows that an entangled state not violating the Bell inequality (27), can be transformed via the introduced protocol into a state violating the Bell inequality, leading to the conclusion that all entangled states of the picture have nonlocal features. Moreover, in agreement with Ref. [28] the possibility to construct realistic local models or not is no criterion for being bound entangled or not.

Let us also remark that Werner states ($n=2$) violate the Bell inequality for $c>\frac{1}{\sqrt{2}}$ whereas successful teleportation requires only $c>\frac{1}{2}$.

VII. SUMMARY AND DISCUSSION

We generalized the magic simplex for locally maximally mixed bipartite qubit states such that we add even numbers n of qubits and set all partial traces equal to the maximally mixed states, i.e. no local information obtainable by any subset of parties is available. This class of states can be described by three real numbers which enables us to draw a three dimensional picture. Interestingly, we find the same geometry concerning separability, mixedness, kind of entanglement, amount of entanglement and nonlocality for all even numbers of qubits, see also Fig. 1 and Fig. 4.

For $n>2$ the purest states, located in the vertices of the simplex, are not pure except in the case of bipartite qubits ($n=2$). We show how to derive a recently proposed measure for all discrete multipartite systems [1] in this case. For mixed states only bounds exist, however, we show that they are for the class of states optimal by proving that all states not detected by the measure are separable.

The measure reveals that these states only possess n–partite entanglement and no other kind of entanglement, e.g. bipartite entanglement. The information content of the states can be quantified by the generalized Bohr's complementarity relation for $n>2$

$$n \;=\; \mathcal{S}+E_n+R \;=\; 1+R\,, \tag{30}$$

where R lack of classical knowledge and $\mathcal{S}=0$ the local information obtainable by any party.

Then we investigated the question whether the n–partite entanglement can be distilled. We find a protocol using only local operation and classical communication (LOCC) which increases the n–partite entanglement to the maximal entanglement of the class of states under investigation. These states are the vertex states of the simplex, for $n=2$ they are the Bell states and for $n>2$ they are equal mixtures of such GHZ–states which are obtained by applying only two flips, σ_x.

For bipartite qubits $n=2$ this protocol is a distillation protocol, i.e. pure maximally entangled states are obtained. For $n>2$ the vertex states are not pure, therefore we search for a distillation protocol that leaves the class of states under investigation to obtain a pure n–partite maximally entangled state, i.e. the GHZ–states. Indeed, we argue that such a protocol cannot be found, more precisely, any equal mixture of GHZ–states cannot be distilled. Thus for the class of states under investigation all entangled states are bound entangled and herewith we found a simplex where all states are either separable or bound entangled.

In detail, we show how an initial state moves after each step of the protocol increasing the entanglement in the simplex, see Fig. 2. Moreover, we find that the states violating the CHSH–Bell like inequality, which was shown to be optimal in this case, have for all even numbers of qubits the same geometry, see Fig. 4. These two results taken together mean that one can enhance the n–partite bound entanglement by only using LOCC until the Bell inequality is violated. Therefore, for all n–partite bound entangled states its (hidden) nonlocality is revealed and in agreement with Ref. [28] a possibility whether a local realistic theory can be constructed is not a criterion for distillability and likewise whether its entanglement can be increased by LOCC is also no criterion.

Our results suggest also that one can distinguish between bound states for which a certain entanglement measure cannot be increased by LOCC (in our case the vertex states) and states for which the entanglement can be increased by LOCC, which may be denoted by "quasi" bound entangled states (all bound entangled states of the class except the vertex states). The introduced (distillation) protocol distills maximally entangled states within the set of states which are, however, not pure, but the purest of the set of states.

Last but not least we want to remark that a subset of the class of states was considered in literature, e.g. [4, 6, 7, 8], the so called Smolin states. For which it was shown that no Bell states may be distilled. The theorem in Ref. [25] states that if and only if bipartite entanglement can be distilled then also GHZ-states —in our terminology n–partite entanglement— can be distilled.

In summary, we have shown in this paper explicitly that the multipartite measure proposed by [1] detects all bound entanglement in the class of states and that the states do not possess bipartite entanglement and how the n–partite entanglement can be increased to a certain value.

These results do not only help to reveal the mysteries of bound entanglement by refining its kind of entanglement, but they may also help to construct quantum communication scenarios where bound entangled states actually help to perform a certain process [29]. This is clearly important, when one has future application in mind, e.g. a multipartite cryptography scenario.

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Appendix: Proof that all states represented by the octahedron are separable.

To prove that all states represented by the octahedron are separable, we show that this is the case for the following points in the octahedron

$$\vec{c}=\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\;. \tag{31}$$

As any convex combination of separable states have to be also separable, we have finalized the proof. We start with $n=2$ and show how this construction generalizes for $n=4,6,\ldots$.

Suppose Alice prepares her qubits in the following two states:

$$\omega^A_{i,\pm} \;=\; \frac{1}{\sqrt{2}}(\mathbb{1}_2\pm r^A_i\sigma_i)\,, \tag{32}$$

where r_i is a Bloch vector pointing in i–direction and is given by any number in $[-1,1]$. Bob does prepares his qubits in the very same way. If Alices chooses the positive i–axis and Bob does the same, if Alice chooses the negative sign, Bob does the same, thus they share the following separable state if the preparation is done randomly with the same probability:

$$\begin{aligned}\rho^{AB}_{i,+} \;&=\; \frac{1}{2}\,\omega^A_{i,+}\otimes\omega^B_{i,+}+\frac{1}{2}\,\omega^A_{i,-}\otimes\omega^B_{i,-}\\ &=\; \frac{1}{4}(\mathbb{1}_4+r^A_i\cdot r^B_i\;\sigma_i\otimes\sigma_i)\,.\end{aligned} \tag{33}$$

These states represent three vertices of the octahedron, thus the proof is finalized for $n=2$.

Explicitly, we find that for the generalized Smolin state ($c_1=c_2=c_3=c$), the following state derives

$$\rho_c \;=\; \sum_i\frac{1}{3}\,\rho^{AB}_{i,+} \;=\; \frac{1}{4}(\mathbb{1}_4+\sum_i\frac{r^A_i\cdot r^B_i}{3}\;\sigma_i\otimes\sigma_i)\,. \tag{34}$$

therefore as $r_i^A \cdot r_i^B \in [-1,1]$ the generalized Smolin state is separable for $p \in [-\frac{1}{3}, \frac{1}{3}]$.

For $n=4$ we remark that with the combination

$$\begin{aligned}\rho_{i,-}^{AB} &= \frac{1}{2}\omega_{i,+}^A \otimes \omega_{i,-}^B + \frac{1}{2}\omega_{i,-}^A \otimes \omega_{i,+}^B \\ &= \frac{1}{4}(\mathbb{1}_4 - r_i^A \cdot r_i^B\ \sigma_i \otimes \sigma_i)\end{aligned} \tag{35}$$

one obtains the minus sign, and for the very same construction Alice, Bob, Charly and Daisy obtain the following separable states

$$\begin{aligned}\rho_{i,+}^{AB} &= \frac{1}{2}\ \rho_{i,+}^{AB} \otimes \rho_{i,+}^{CD} + \frac{1}{2}\ \rho_{i,-}^{AB} \otimes \rho_{i,-}^{CD} \\ &= \frac{1}{4}(\mathbb{1}_4 + r_i^A \cdot r_i^B \cdot r_i^C \cdot r_i^D\ \sigma_i \otimes \sigma_i \otimes \sigma_i \otimes \sigma_i)\end{aligned} \tag{36}$$

As the combination $+-, -+$ gives again the minus sign this proof generalizes for any even n.

Appendix: All 4–flip operators for $n=4$: For convenience of the reader we list all 4–flip operators in the Pauli–matrix representation:

$$\begin{aligned}
\mathcal{O}^1 = \frac{1}{4}\{ &\quad \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \\
&- \sigma_y \otimes \sigma_y \otimes \sigma_x \otimes \sigma_x \\
&- \sigma_y \otimes \sigma_x \otimes \sigma_y \otimes \sigma_x \\
&- \sigma_y \otimes \sigma_x \otimes \sigma_x \otimes \sigma_y\} \\
\mathcal{O}^2 = \frac{1}{4}\{ &\quad \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \\
&- \sigma_y \otimes \sigma_y \otimes \sigma_x \otimes \sigma_x \\
&+ \sigma_y \otimes \sigma_x \otimes \sigma_y \otimes \sigma_x \\
&+ \sigma_y \otimes \sigma_x \otimes \sigma_x \otimes \sigma_y\} \\
\mathcal{O}^3 = \frac{1}{4}\{ &\quad \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \\
&+ \sigma_y \otimes \sigma_y \otimes \sigma_x \otimes \sigma_x \\
&- \sigma_y \otimes \sigma_x \otimes \sigma_y \otimes \sigma_x \\
&+ \sigma_y \otimes \sigma_x \otimes \sigma_x \otimes \sigma_y\} \\
\mathcal{O}^4 = \frac{1}{4}\{ &\quad \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \\
&+ \sigma_y \otimes \sigma_y \otimes \sigma_x \otimes \sigma_x \\
&+ \sigma_y \otimes \sigma_x \otimes \sigma_y \otimes \sigma_x \\
&- \sigma_y \otimes \sigma_x \otimes \sigma_x \otimes \sigma_y\}
\end{aligned} \tag{37}$$

$$\begin{aligned}
\mathcal{O}^5 = \frac{1}{4}\{ &\quad \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \\
&- \sigma_x \otimes \sigma_y \otimes \sigma_x \otimes \sigma_y \\
&- \sigma_x \otimes \sigma_y \otimes \sigma_y \otimes \sigma_x \\
&- \sigma_y \otimes \sigma_y \otimes \sigma_x \otimes \sigma_x\} \\
\mathcal{O}^6 = \frac{1}{4}\{ &\quad \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \\
&- \sigma_x \otimes \sigma_y \otimes \sigma_x \otimes \sigma_y \\
&+ \sigma_x \otimes \sigma_y \otimes \sigma_y \otimes \sigma_x \\
&+ \sigma_y \otimes \sigma_y \otimes \sigma_x \otimes \sigma_x\} \\
\mathcal{O}^7 = \frac{1}{4}\{ &\quad \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \\
&+ \sigma_x \otimes \sigma_y \otimes \sigma_x \otimes \sigma_y \\
&- \sigma_x \otimes \sigma_y \otimes \sigma_y \otimes \sigma_x \\
&+ \sigma_y \otimes \sigma_y \otimes \sigma_x \otimes \sigma_x\} \\
\mathcal{O}^8 = \frac{1}{4}\{ &\quad \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \\
&+ \sigma_x \otimes \sigma_y \otimes \sigma_x \otimes \sigma_y \\
&+ \sigma_x \otimes \sigma_y \otimes \sigma_y \otimes \sigma_x \\
&- \sigma_y \otimes \sigma_y \otimes \sigma_x \otimes \sigma_x\}
\end{aligned} \tag{38}$$

$$\begin{aligned}
\mathcal{O}^9 = \frac{1}{4}\{ &\quad \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \\
&- \sigma_x \otimes \sigma_x \otimes \sigma_y \otimes \sigma_y \\
&- \sigma_x \otimes \sigma_y \otimes \sigma_y \otimes \sigma_x \\
&- \sigma_y \otimes \sigma_x \otimes \sigma_y \otimes \sigma_x\} \\
\mathcal{O}^{10} = \frac{1}{4}\{ &\quad \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \\
&- \sigma_x \otimes \sigma_x \otimes \sigma_y \otimes \sigma_y \\
&+ \sigma_x \otimes \sigma_y \otimes \sigma_y \otimes \sigma_x \\
&+ \sigma_y \otimes \sigma_x \otimes \sigma_y \otimes \sigma_x\} \\
\mathcal{O}^{11} = \frac{1}{4}\{ &\quad \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \\
&+ \sigma_x \otimes \sigma_x \otimes \sigma_y \otimes \sigma_y \\
&- \sigma_x \otimes \sigma_y \otimes \sigma_y \otimes \sigma_x \\
&+ \sigma_y \otimes \sigma_x \otimes \sigma_y \otimes \sigma_x\} \\
\mathcal{O}^{12} = \frac{1}{4}\{ &\quad \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \\
&+ \sigma_x \otimes \sigma_x \otimes \sigma_y \otimes \sigma_y \\
&+ \sigma_x \otimes \sigma_y \otimes \sigma_y \otimes \sigma_x \\
&- \sigma_y \otimes \sigma_x \otimes \sigma_y \otimes \sigma_x\}
\end{aligned} \tag{39}$$

$$
\begin{aligned}
\mathcal{O}^{13} \;=\; \frac{1}{4}\{ \quad & \sigma_y\otimes\sigma_y\otimes\sigma_y\otimes\sigma_y \\
- \; & \sigma_x\otimes\sigma_x\otimes\sigma_y\otimes\sigma_y \\
- \; & \sigma_x\otimes\sigma_y\otimes\sigma_x\otimes\sigma_y \\
- \; & \sigma_y\otimes\sigma_x\otimes\sigma_x\otimes\sigma_y\} \\
\mathcal{O}^{14} \;=\; \frac{1}{4}\{ \quad & \sigma_y\otimes\sigma_y\otimes\sigma_y\otimes\sigma_y \\
- \; & \sigma_x\otimes\sigma_x\otimes\sigma_y\otimes\sigma_y \\
+ \; & \sigma_x\otimes\sigma_y\otimes\sigma_x\otimes\sigma_y \\
+ \; & \sigma_y\otimes\sigma_x\otimes\sigma_x\otimes\sigma_y\} \\
\mathcal{O}^{15} \;=\; \frac{1}{4}\{ \quad & \sigma_y\otimes\sigma_y\otimes\sigma_y\otimes\sigma_y \\
+ \; & \sigma_x\otimes\sigma_x\otimes\sigma_y\otimes\sigma_y \\
- \; & \sigma_x\otimes\sigma_y\otimes\sigma_x\otimes\sigma_y \\
+ \; & \sigma_y\otimes\sigma_x\otimes\sigma_x\otimes\sigma_y\} \\
\mathcal{O}^{16} \;=\; \frac{1}{4}\{ \quad & \sigma_y\otimes\sigma_y\otimes\sigma_y\otimes\sigma_y \\
+ \; & \sigma_x\otimes\sigma_x\otimes\sigma_y\otimes\sigma_y \\
+ \; & \sigma_x\otimes\sigma_y\otimes\sigma_x\otimes\sigma_y \\
- \; & \sigma_y\otimes\sigma_x\otimes\sigma_x\otimes\sigma_y\} \qquad (40)
\end{aligned}
$$

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[30] Note that for $n = 4, 8, \ldots$ the mirrored tetrahedron ($\vec{c} \longrightarrow -\vec{c}$) is obtained.

Two computable sets of multipartite entanglement measures

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We present two sets of computable entanglement measures for multipartite systems where each subsystem can have different degrees of freedom (so-called qudits). One set, called "separability" measure, reveals which of the subsystems are separable/entangled. For that we have to extend the concept of k–separability for multipartite systems to a novel unambiguous separability concept which we call γ_k–separability. The second set of entanglement measures reveals the "kind" of entanglement, i.e. if it is bipartite, tripartite, ..., n-partite entangled and is denoted as the "physical" measure. We show how lower bounds on both sets of measures can be obtained by the observation that any entropy may be rewritten via operational expressions known as m–concurrences. Moreover, for different classes of bipartite or multipartite qudit systems we compute the bounds explicitly and discover that they are often tight or equivalent to positive partial transposition (PPT).

Keywords: entanglement measure, multipartite qudit system, separability
PACS: 03.67.Mn

I. INTRODUCTION

Quantum entanglement is a fascinating property of quantum states that has many important consequences for modern physics. It exhibits aspects that are counter-intuitive to classical physics, like the incompatibility with local realistic theories [1, 2]. For example it turned out that a symmetry violation in particle physics, the CP violation in mixing ($C\ldots$charge conjugation, $P\ldots$parity), is incompatible with any local realistic theory [3, 4].

Furthermore entanglement is a highly useful resource for quantum information tasks. Thus it makes quantum information theory a conceptually different theory than classical information theory (for an overview see, e.g., Refs. [5, 6, 7]). The characterization of entanglement is, however, a nontrivial mathematical task and not at all completed (for an overview see, e.g., Refs. [8, 9]). The first concepts were derived for bipartite systems, which are the simplest systems that can contain entanglement. Here many important results were obtained, like the detection and quantification of entanglement for (pure and mixed) bipartite qubits, which can be conclusively performed for any states of such systems. In a finite dimensional Hilbert space the most general quantum states one can think of are multipartite arbitrary dimensional states, i.e. states that describe systems of n subsystems, where each subsystem is ascribed a finite dimensional Hilbert space. Already the classification of entanglement according to possible reversible quantum operations is a nontrivial task, see e.g. Refs. [10, 11, 12] in this context.

There are different approaches to the quantification of multipartite entanglement. A common method is to describe the same state with different entanglement measures, e.g. in terms of bipartite cuts [13] or with different entanglement measures according to invariance classes under statistical local operations and classical communication (SLOCC) [12]. Another way is to determine a global entanglement measure for the whole state [14, 15, 16, 17, 18, 19, 20, 21]. Our approach picks up a conception of entanglement that on the one hand differentiates between possible entanglement between any parties sharing the state and on the other hand sums up to a total global entanglement. In this way we can both quantify the entanglement that any parties share with each other, and the "whole" entanglement present in the state.

This would also provide advantages for the description of quantum communication protocols with multipartite entangled states (see, e.g., Refs. [22, 23, 24, 25]). Of course this simple concept already calls for more than one entanglement measure - for different tasks different entanglement measures seem to be appropriate. We want to present two of such possibilities that in our opinion seem to be good candidates, i.e. one revealing the separability property, "the separability measure", and the other one revealing different kinds of entanglement, "the physical measure". Further approaches to quantify multipartite entanglement can be found, e.g., in Refs. [14, 21, 26] and higher dimensional generalizations of bipartite entanglement measures in [27, 28, 29, 30].

The paper is organized as follows: In the first section we define separability of multipartite systems and list the requirements for bipartite entanglement measures. In Sec. III we discuss entropies and introduce the m–concurrence which enables computation of bounds on entanglement of mixed states. The next Sec. IV introduces two measures, one for the partial separability and one for what kind of entanglement is present. Then follows a section with further instructive examples to which we applied the two measures. In the appendix we give all proofs of the requirements for these two measures.

II. BASIC DEFINITIONS

A. A definition of *partial* separability and the γ_k–separability

In multipartite systems the notion of separability can be extended in order to answer the question which particles are joint inseparably. Throughout the paper we assume that partial traces of the multipartite quantum system are only taken over physical subsystems, i.e. over one or more particles. It means that possible information which may result by tracing over certain degrees of freedom of a certain particle/qudit is not taken into account.

A pure multipartite state $|\psi\rangle$ is called *k–separable* if it can be written as [9]

$$|\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_k\rangle\,, \quad k \leq n\,, \tag{1}$$

where n is the total number of particles. It is called fully separable iff $k = n$, this is the natural generalization of the separability of bipartite systems. We call a state 1–separable or fully entangled iff $k = 1$. This notation of full separability or entanglement can be generalized to mixed states in a straight forward way. If a pure state is not fully separable or fully entangled, it is called partially separable or k–separable.

The definition of partial separability for mixed states is more involved. One obvious possibility is the following: A mixed state is called k–separable if there exists a decomposition that satisfies [9]

$$\sigma_{k-sep} = \sum_i \; p_i \; \rho_i^1 \otimes \rho_i^2 \otimes \cdots \otimes \rho_i^k\,, \qquad \text{with} \quad p_i \geq 0,\; \sum_i p_i = 1\,, \tag{2}$$

where the ρ_i^js are states of some number of subsystems and can always be chosen to be pure. The terms in Eq. (2) all have the same k, but it is in general not fixed which subsystems are contained in the states ρ_i^j.

For the argumentation in this paper we are interested to fix the subsystems involved in the states ρ_i^j and therefore extend the k–separability definition to the so-called γ_k–separability. For this we introduce the following notation:

$$\gamma_k := \{\{\beta_1\}|\{\beta_2\}|\cdots|\{\beta_k\}\}\,. \tag{3}$$

Here the sets $\{\beta_j\}$ represent subsystems, i.e. particles, which are inseparably joined.

Instructive example: $|\psi\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes |\phi^+\rangle_{34}$ with $|\phi^+\rangle = \frac{1}{\sqrt{2}}\{|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle\}$. Here the number of particles is $n = 4$ and the separability is a 3–separability with the substructure $\gamma_3 = \{1|2|34\}$.
This state is obviously equivalent to $\frac{1}{\sqrt{2}}\{|0000\rangle + |1010\rangle\}$ with the substructure $\gamma_3 = \{2|4|13\}$, here just the role of the first and second subsystems are interchanged. Therefore, it is convenient to reorder the subsystems of the state if necessary.

Note that there is a difference between fully entangled and being maximally entangled, which we define as a pure state where all subsystems representing particles are in the maximally mixed state. For example the state $|\phi^+\rangle \otimes |\phi^+\rangle$ is not fully entangled, but according to the above definition maximally entangled.

The extension of the γ_k separability to mixed states is not straightforward as an ambiguity can happen as we explain later in an example.

Definition of γ_k–separability:
To every ρ we associate a separability property, the set γ_k, which is made up of $\{\beta_j\}$, i.e. sets of numbers representing subsystems. A state ρ is called γ_k–separable iff there exists an unambiguous decomposition with maximal k into:

$$\sigma_{\gamma_k-sep} = \sum_i p_i\, \rho_i^{\{\beta_1\}} \otimes \rho_i^{\{\beta_2\}} \otimes \cdots \otimes \rho_i^{\{\beta_k\}}\,, \qquad \text{with} \quad p_i \geq 0,\ \sum_i p_i = 1\,. \tag{4}$$

The following instructive example shows the difference of the k–separability and the γ_k–separability.

Consider the generalized Smolin state [31, 32, 33]

$$\rho_{Smolin} = \frac{1}{2^n}\left(\mathbb{1} + \sum c_i\, \sigma_i^{\otimes n}\right)\,, \tag{5}$$

where n is an even number, σ_i are the Pauli matrices and c_i are real numbers (see also the instructive example in Sect. IV A). This state can be decomposed into bipartite pure states, i.e. the Bell states. For $n = 4$ this would correspond to $\gamma_k = \{12, 34\}$. This however is not the proper γ_k separability as any other bipartite cut is also valid, i.e. $\gamma_k = \{13, 24\}$, $\gamma_k = \{14, 23\}$. So the only unambigous set of subsystems is: $\gamma_k = \{1234\}$. So in the notion of γ_k–separability, the generalized Smolin states are always completely inseparable (γ_1-separable), whereas in the notion of k-separability they are $\frac{n}{2}$-separable.

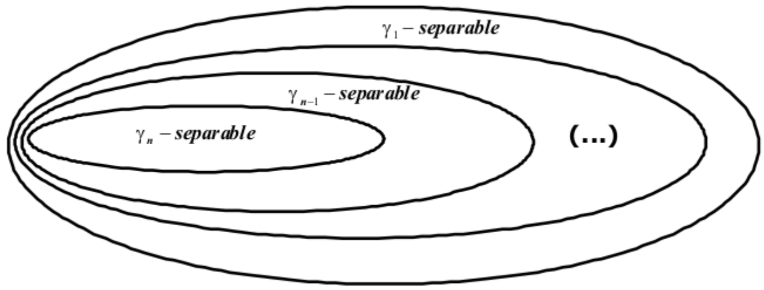

FIG. 1: Here the convexity of γ_k-separability is visualized, i.e. any convex mixture of two γ_k–separable states, e.g. γ_{k_1} and γ_{k_2}, is either γ_{k_1}– or γ_{k_2}– or γ_k–separable with $k_1, k_2 < k$.

Both views are in a way justified: The γ_k-separability reflects the fact that any further reduction (partial trace) of the state yields a fully separable state (which is independent of notion) and therefore the useful entanglement properties can only be extracted if one uses all contained subsystems. Whereas the k-separability reflects how many parties need to join together in order to prepare the state using LOCC. Note again that for pure states the k in k-separability and the k in γ_k are identical.

To sum up, γ_k-separability for pure states is an extension of k-separability, it captures which subsystems are involved, and for mixed states it captures an essential novel feature (cf. the above example of the Smolin states) that would be missed by only considering k-separability. Another important feature of the γ_k–separability is the convexity in the sense that the mixture of two γ_k–separable states, e.g. γ_{k_1} and γ_{k_2}, is either γ_{k_1}– or γ_{k_2}– or γ_k–separable with $k_1, k_2 < k$. This is visualized in Fig. 1.

One aim of this paper is to quantify entanglement and classify the γ_k–separability of a given state which is done in Sect. IV A.

B. Proper properties for being entangled

Now we investigate the question what properties a proper entanglement measure should have. Let us first summarize the conditions which are required for bipartite entanglement measures $E(\rho)$ (Sep is the set of all separable states) [34, 35, 36, 37]:

B1: $E(\rho) > 0 \quad \forall \quad \rho \notin Sep$

B2: $E(\rho) = 0 \quad \forall \quad \rho \in Sep$

B3: $E(\rho^{\otimes n}) = n\, E(\rho)$ (Additivity)

B4: $E(\lambda\, \rho_1 + (1-\lambda)\, \rho_2) \;\leq\; \lambda\, E(\rho_1) + (1-\lambda)\, E(\rho_2)$ (Convexity)

B5: $E\left(U^A \otimes U^B \rho \left(U^A \otimes U^B\right)^\dagger\right) = E(\rho)$ (Invariance under local unitary operations)

B6: $\sum_i \mathrm{Tr}\left(V_i \rho V_i^\dagger\right) E\left(\frac{V_i \rho V_i^\dagger}{\mathrm{Tr}\left(V_i \rho V_i^\dagger\right)}\right) \leq E(\rho)$ (Non-increasing on average under LOCC), where V_i is a separable operator, i.e. of the local form $V_i := A_i \otimes B_i$.

For multipartite systems we claim that there cannot be only a single entanglement measure, since it could not correctly quantify the substructure of the k–separability or the γ_k–separability and simultaneously reveal which parts of the system are entangled in which way with other parts.

Therefore we propose for multipartite systems a set of entanglement measures $E_{\{\alpha_j\}}$ where the set $\{\alpha_j\}$ denotes subsystems of the whole. As any bipartite system can be seen as a substructure of a bigger system, clearly the same requirements as for bipartite systems applies also to bipartite $E_{\{\alpha_j\}}$. The same should hold true for any tripartite, four-partite, ... and so on, entanglement. The sum over the whole set should constitute the total entanglement

$$E_{tot}(\rho) = \sum_{j=2}^{n} \sum_{\{\alpha_j\}} E_{\{\alpha_j\}}(\rho) \ . \tag{6}$$

It is well known that the entanglement of a pure state can easily be quantified by the entropy of its subsystems [35]. Possible entropy measures are, e.g., the quantum version of Renyi's α–entropies [38]:

$$S^q_\alpha := \frac{1}{1-\alpha} \log_q \mathrm{Tr}(\rho^\alpha) \tag{7}$$

which for $\alpha \to 1$ equals the famous von Neumann entropy. The logarithmic entropies have the advantage that they imply additivity, for the general cases of probability distributions, for which they were originally intended, as well as for entanglement measures constructed out if it.

Another possibility are the linear entropies

$$S_r(\rho) := \frac{d^{r-1}}{d^{r-1}-1}(1 - \mathrm{Tr}(\rho^r)) \ , \tag{8}$$

where d is the dimension of ρ.

For any multipartite pure state $\rho = |\psi\rangle\langle\psi|$ one can quantify the total entanglement by

$$E_{tot}(\rho) := \sum_{s=1}^{n} S(\rho_s) \ , \tag{9}$$

where $\rho_s := \mathrm{Tr}_{\neg s}\rho$ denotes the reduced density matrix of the respective subsystem s and S is any entropy function. A standard method to generalized this measure for mixed states $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ is by constructing the convex roof [39]

$$E_{tot}(\rho) := \inf_{\{p_i,\psi_i\}} \sum_i p_i \sum_{s=1}^{n} S(\rho^i_s) \ . \tag{10}$$

The ψ_i are throughout the paper considered as normalized. In general it is not known how to find the infimum, we will show in the next section how with a simple algebraic trick operators can be constructed which allow to compute bounds on the entanglement which turn out to be tight in many cases.

III. ENTROPY AND m–CONCURRENCE

The m–concurrence was introduced in Ref. [40]. It can be efficiently used to compute bounds for the convex roof extension of the entanglement measures for mixed states. For pure multipartite states it is a simple algebraic rewriting of the entropies of the subsystems in terms of such operators. For the generally mixed subsystems states one can via these operators obtain bounds on the entanglement.

The definition we present here will be slightly modified to the above cited works in oder to yield a simpler and more compact notation. The linear entropy $S_{r=2}$, Eq. (8), of any subsystem s can be rewritten as a sum of terms named concurrences in analogy to Hill and Wootters concurrence [41, 42] and concurrences defined for bipartite systems of arbitrary dimension [28, 29]:

$$\begin{aligned} S_2(\rho_s) &= \frac{d}{d-1}(1-\mathrm{Tr}(\rho_s^2)) \\ &= \sum_\alpha C^2_{\mathbf{s}\alpha} + \sum_\alpha\sum_\beta C^2_{\mathbf{s}\alpha\beta} + (\ldots) + \sum_\alpha\sum_\beta\cdots\sum_\omega C^2_{\mathbf{s}\alpha\beta\cdots\omega}\,. \end{aligned} \tag{11}$$

This m–concurrences $C^2_{\{\alpha_m\}}$ containing m–indices are obtained using m–flip operators in the following way:

$$C^2_{\mathbf{s}\alpha\beta\cdots\omega} := \sum_{O_C} \left| \langle\psi|\, \underbrace{(A|\{i_n\}\rangle\langle\{i_n\}|\mathbb{1} - B|\{i_n\}\rangle\langle\{i_n\}|AB)}_{O_C}\, |\psi^*\rangle \right|^2 \tag{12}$$

where

$$\begin{aligned} A &:= \left(\sigma^{K\in\{\mathbf{s}\alpha\beta\cdots\omega\}}_{k_K l_K}, \mathbb{1}^{K\notin\{\mathbf{s}\alpha\beta\cdots\omega\}}\right) \\ B &:= \left(\sigma^{K=\mathbf{s}}_{k_K l_K}, \mathbb{1}^{K\neq\mathbf{s}}\right) \end{aligned} \tag{13}$$

and

$$\sum_{O_C} := \sum_{k_K=0}^{d_K-1} \sum_{l_K>k_K} \sum_{\{i_n\}}\,. \tag{14}$$

Here K denotes the respective subsystem and the flip operators are defined for a qudit system of dimension d in the following way:

$$\sigma^{d\times d}_{kl}\,|k\rangle \;=\; |l\rangle, \quad \sigma^{d\times d}_{kl}\,|l\rangle \;=\; |k\rangle \quad \text{and} \quad \sigma^{d\times d}_{kl}\,|t\rangle \;=\; 0 \quad \forall\, t\neq k,l\,. \tag{15}$$

Note that these are the symmetric generalized Gell-Mann operators (see, e.g., Refs. [43, 44]; generalized Gell-Mann operators are the SU(N) generators). In order to obtain Renyi's entropy we use the relationship between this entropy (7) and the linear entropy (8)

$$S^q_\alpha(\rho) = \frac{1}{1-\alpha}\mathbf{log}_q(\mathrm{Tr}(\rho^\alpha)) = \frac{1}{1-\alpha}\mathbf{log}_q(1-\frac{d^{\alpha-1}-1}{d^{\alpha-1}}\;S_\alpha(\rho))\,. \tag{16}$$

Note that one can also obtain the von Neumann entropy by means of the m-concurrence. Obviously, that requires computation of all S_k from $\alpha=2$ to $\alpha=d$. We choose Renyi's entropy with $\alpha=2$ and $q=2$. In the following we write S for S^2_2.

As shown in [31, 40], the advantage of rewriting the entropies by means of operators is that it is known how to derive bounds. We present here a way analogous to the method introduced for the concurrence for bipartite systems in Ref. [30]. For that we define a flipped density matrix

$$\widetilde{\rho}_{O_C} := (O_C + O_C^\dagger)\ \rho^*\ (O_C + O_C^\dagger)\ , \tag{17}$$

where the conjugation is taken in the computational basis. By calculating the square root of the eigenvalues of $\rho\widetilde{\rho}_{O_C}$, which we denote as $\lambda_i^{O_C}$, the bounds for the concurrence are given by:

$$C_{\mathbf{s}\alpha\beta\cdots\omega}(\rho) \geq \max\Big\{0, \sum_{O_C}(2 \max_{\lambda_i^{O_C}}(\{\lambda_i^{O_C}\}) - \sum_i\ \lambda_i^{O_C})\Big\}\ . \tag{18}$$

IV. MULTIPARTITE ENTANGLEMENT MEASURES

In this section we propose the two sets of multipartite entanglement measures. First, we introduce the *separability measure* that is based on the γ_k–separability, and second, the *physical measure* that reveals the "kind" of entanglement between subsystems (bipartite, tripartite,. . . entanglement).

A. Separability measure

In the following we assume that the total state ρ is pure. For the generalized multipartite set of entanglement measures there are a few alternatives, we propose the following generalization:

S1a: $E_{tot}(\rho) = \sum_{s=1}^n S(\rho_s) := \sum_{\{\alpha_j\}} E_{\{\alpha_j\}}\ >\ 0 \qquad \forall\ \rho \quad \text{with } k < n$

S1b: $E_{tot}(\rho)\ =\ 0 \qquad \forall\ \rho \quad \text{with } k = n$

S2: $E_{\{\alpha_j\}}(\rho)\ >\ 0 \qquad \forall \quad \{\alpha_j\} \in \gamma_k \quad \text{and} \quad |\{\alpha_j\}|\ \geq\ 2$

S3: $E_{\{\alpha_j\}}(\rho)\ =\ 0 \qquad \forall \quad \{\alpha_j\} \notin \gamma_k \quad \text{or} \quad |\{\alpha_j\}|\ =\ 1$

S4: $E_{\{\alpha_j\}}(\rho^{\otimes n})\ =\ n\ E_{\{\alpha_j\}}(\rho)$ (additivity on copies of the same state)

S5: $E_{\{\alpha_j\}}\big(U_{\alpha_1} \otimes U_{\alpha_2} \otimes \cdots \otimes U_{\alpha_j}\, \rho\, (U_{\alpha_1} \otimes U_{\alpha_2} \otimes \cdots \otimes U_{\alpha_j})^\dagger\big)\ =\ E_{\{\alpha_j\}}(\rho)$ (invariance under local unitary operations)

S6: $E_{tot}(\rho_1 \otimes \rho_2)\ =\ E_{tot}(\rho_1) + E_{tot}(\rho_2)$ (additivity on tensor products of arbitrary states)

With a measure that fulfills all this requirements one obtains the γ_k–separability and, moreover, the quantified information content of a given state.

According to our notation of γ_k–separability a pure state of three qubits can be entangled in four different ways, $\{1|23\}; \{12|3\}; \{13|2\}; \{123\}$, hence we have four different entanglement measures, which we define in an intuitive way by

$$E_{12} :\ =\ \{S(\rho_1) + S(\rho_2)\} \cdot \delta[S(\rho_{12}), 0] \tag{19}$$
$$E_{13} :\ =\ \{S(\rho_1) + S(\rho_3)\} \cdot \delta[S(\rho_{13}), 0] \tag{20}$$
$$E_{23} :\ =\ \{S(\rho_2) + S(\rho_3)\} \cdot \delta[S(\rho_{23}), 0] \tag{21}$$
$$E_{123} :\ =\ S(\rho_1) + S(\rho_2) + S(\rho_3) - E_{12} - E_{13} - E_{23} \tag{22}$$

with

$$\begin{aligned} \delta[S(\rho_{\{\alpha_j\}}),0] &= 1 \quad \text{if} \quad S(\rho_{\{\alpha_j\}}) = 0 \\ \delta[S(\rho_{\{\alpha_j\}}),0] &= 0 \quad \text{if} \quad S(\rho_{\{\alpha_j\}}) > 0 \,. \end{aligned} \tag{23}$$

Instructive example for three qubits: For the state

$$|\tilde{\phi}\rangle \;=\; \frac{1}{N}\{p\,|GHZ\rangle + (1-p)\,|\phi^+\rangle \otimes \{\cos\alpha|0\rangle + \sin\alpha|1\rangle\}\} \tag{24}$$

with $|GHZ\rangle = \frac{1}{\sqrt{2}}\{|000\rangle + |111\rangle\}$ we expect for $p = 0$ that it is 2–separable with $\gamma_2 = \{12|3\}$ (in detail $E_{12} = 2, E_{13} = E_{23} = E_{123} = 0$) and for $p = 1$ it is 1–separable (=fully entangled) ($E_{123} = 1, E_{12} = E_{13} = E_{23} = 0$). For values of $p \in \{0,1\}$ it depends on α. With the separability measure this can be easily calculated:

(a) If we want E_{12} to vanish, then $S(\rho_{12})$ has to be zero, this can be obtained for $p \leq 0.58$ and $\cos(\pm\alpha) = \frac{p^2-p\pm\sqrt{p^4-6p^3+11p^2-8p+2}}{2(p^2-2p+1)}$. The state is fully separable, except for $p = 0.5$ where $S(\rho_{12}) = 0$, but also $S(\rho_1), S(\rho_2), S(\rho_3) = 0$.

(b) No α and p exist such that $S(\rho_{13})$ or $S(\rho_{23})$ vanish, thus $E_{13} = E_{23} = 0$.

(c) As E_{13} and E_{23} are always zero, the state is either $\gamma_1 = \{123\}$–separable ($E_{123} > 0$) or $\gamma_2 = \{12|3\}$–separable ($E_{12} > 0$); except for $p = 0.5$ and the above α then the state is fully separable $\gamma_3 = \{1|2|3\}$.

(d) If the GHZ state is interchanged with a W state the state is for all p and α 3–separable except for $p = 0$, where it is clearly γ_2–separable.

The separability measure can be generalized for multipartite qudit systems as

$$E_{\{\alpha_j\}} := \sum_{s\in\{\alpha_j\}} \left(S(\rho_s) - \sum_{\{\beta_j\}\subset\{\alpha_j\}} E_{\{\beta_j\}} \right) \cdot \delta[S(\rho_{\{\alpha_j\}}),0]\,, \tag{25}$$

where

$$\sum_{\{\alpha_j\}} E_{\{\alpha_j\}} = E_{tot}(\rho)\,. \tag{26}$$

This is an important feature as any violation of this necessary requirement would imply either neglecting or over-quantifying of the information content. As is proven in the Appendix VIII A this proposed set of measures meets all requirements S1-S6.

The separability measure provides a set of entanglement measures for pure states. In principle it can be extended to mixed states using the convex roof method,

$$E_{\{\alpha_j\}}(\rho) \;:=\; \inf \sum_i p_i\; E_{\{\alpha_j\}}(|\psi_i\rangle\langle\psi_i|)\,. \tag{27}$$

Since there still is no method to calculate the convex roof for arbitrary states, the proposed measure is computable only for pure states or mixed states for which the bound of the $m-$concurrences are exact or for states where we can know if all involved entropies vanish or not. Clearly, if one cannot execute the defined Kronecker δ's exactly, the computation of the measure may fail. Moreover, when applying the separability measure for mixed states, one does in general not obtain e.g. whether the state at all is entangled as the the following example shows:

> **Instructive example for even number of qubits:** Consider the generalized Smolin state $\rho_{Smolin} = \frac{1}{2^n}\left(\mathbb{1} + \sum c_i\, \sigma_i^{\otimes n}\right)$, where n is a even number. This state is mixed (except for $n = 2$ and $|\vec{c}| = 3$) and the states of all subsystems are maximally mixed. Therefore all $E_{12\ldots j}$ with $j < n$ derive to zero. Thus for the Smolin state only $E_{12\ldots n}$ can be nonzero. To show for which parameters the state is entangled, we need another measure which we introduce in the next section and apply in Sect. V D to the Smolin state.

B. Physical measure

As we have mentioned, there are many different aspects of multipartite entanglement one might be interested in. The separability measure quantifies the total information content in entanglement and yields an answer to the γ_k–separability in a multipartite state. From a physical point of view, however, we can also present another approach to quantify multipartite entanglement. The basic motivation is to reveal structures of quantum states that go beyond separability. Take for instance the instructive example of a W state $|W\rangle$ and the Greenberger-Horne-Zeilinger state $|GHZ\rangle$ in the three qubit system, where $|W\rangle = 1/\sqrt{3}\,(|001\rangle + |010\rangle + |100\rangle)$ and $|GHZ\rangle$ as defined in Eq. (24). Both are completely inseparable and thus it is not possible to distinguish them by the separability measure. From a physical point of view the most obvious difference between these two states is the following: Ignoring an arbitrary subsystem will, in case of the GHZ state, yield a mixed separable state, whereas in case of the W state, will yield a mixed entangled state. Any set of entanglement measures that is designed to capture this difference will need a modification in requirements (S2) and (S3) and for mixed density matrices we need the additional requirements (P4) and (P5) as defined below. Thus the properties we propose are the following:

S1a: $\mathcal{E}_{tot}(\rho) = \sum_{s=1}^{n} S(\rho_s) \;>\; 0 \qquad \forall\, \rho \quad \text{with } k < n$

S1b: $\mathcal{E}_{tot}(\rho) \;=\; 0 \qquad \forall\, \rho \quad \text{with } k = n$

P2: $\mathcal{E}_{\{\alpha_j\}}(\rho) \;\geq\; 0 \quad \forall \quad \{\alpha_j\} \subseteq \{\beta_i\} \in \gamma_k \quad \text{and} \quad |\{\alpha_j\}| \geq 2$

P3: $\mathcal{E}_{\{\alpha_j\}}(\rho) \;=\; 0 \quad \forall \quad \{\alpha_j\} \supset \{\beta_i\} \in \gamma_k \quad \text{or} \quad |\{\alpha_j\}| = 1$

P4: $\mathcal{E}_{\{\alpha_j\}}(\lambda\, \rho_1 + (1-\lambda)\, \rho_2) \;\leq\; \lambda\, \mathcal{E}_{\{\alpha_j\}}(\rho_1) + (1-\lambda)\, \mathcal{E}_{\{\alpha_j\}}(\rho_2) \qquad \text{(convexity)}$

P5: $\sum_i \mathrm{Tr}\left(V_i \rho V_i^\dagger\right) \mathcal{E}_{tot}\left(\frac{V_i \rho V_i^\dagger}{\mathrm{Tr}\left(V_i \rho V_i^\dagger\right)}\right) \leq \mathcal{E}_{tot}(\rho)$ (non-increasing on average under LOCC), where V_i is a separable operator, i.e. of the local form $V_i := V_i^1 \otimes V_i^2 \otimes \ldots \otimes V_i^n$.

Of course capturing this essential difference needs computation of entanglement of all possible subsystems, which are in general mixed. Therefore we propose a set of measures which contain a convex roof extension already for the subsystems of pure multipartite states.

For that let us first define the following useful quantity for any density matrix $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$:

$$P(\rho) := \inf_{p_i,\psi_i,\gamma_k} \sum_i p_i \; (\sum_s S(\mathrm{Tr}_{\neg s}|\psi_i\rangle\langle\psi_i|)) \; . \tag{28}$$

Here the sum over all subsystems s is taken over the entropy of pure states ψ_i, thus is the "correct" entanglement content of this certain state ψ_i. Note that we take the infimum also over γ_k. All examples we have considered show that the infimum over γ_k has not separably to be claimed, because the infimum over all decompositions was always achieved in the correct γ_k-separable decomposition. Moreover, for physical reasons it would be surprising if this was not the case. It would mean that there exist e.g. a partially separable state for which the infimum of the subsystem's entropies was realized for a completely inseparable decomposition. However, we were yet not able to prove that rigorously and therefore we have to conjecture that for any state with the following γ_k separability

$$\gamma_k := \{\{\beta_1\}|\{\beta_2\}|\cdots|\{\beta_k\}\} \tag{29}$$

the equality

$$P(\rho_{\gamma_k}) = P(\rho_{\{\beta_1\}}) + P(\rho_{\{\beta_2\}}) + (\cdots) + P(\rho_{\{\beta_k\}}) \; . \tag{30}$$

holds. Note that in this way we trivially obtain the additivity property we proposed for the physical measure (see Appendix).

For convenience, we start to define the set of measures for four particles by

$$\begin{aligned}
\text{two–particle entanglement:} \quad & \mathcal{E}_{12} = P(\rho_{12}) \; , \quad \mathcal{E}_{13} = P(\rho_{13}) \; , && (31)\\
& \mathcal{E}_{14} = P(\rho_{14}) \; , \quad \mathcal{E}_{23} = P(\rho_{23}) \; , && (32)\\
& \mathcal{E}_{24} = P(\rho_{23}) \; , \quad \mathcal{E}_{34} = P(\rho_{34}) \; , && (33)\\
\text{three–particle entanglement:} \quad & \mathcal{E}_{123} = \max\big[0, P(\rho_{123}) - \mathcal{E}_{12} - \mathcal{E}_{13} - \mathcal{E}_{23}\big] \; , && (34)\\
& \mathcal{E}_{124} = \max\big[0, P(\rho_{124}) - \mathcal{E}_{12} - \mathcal{E}_{14} - \mathcal{E}_{24}\big] \; , && (35)\\
& \mathcal{E}_{134} = \max\big[0, P(\rho_{134}) - \mathcal{E}_{13} - \mathcal{E}_{14} - \mathcal{E}_{34}\big] \; , && (36)\\
& \mathcal{E}_{234} = \max\big[0, P(\rho_{234}) - \mathcal{E}_{23} - \mathcal{E}_{24} - \mathcal{E}_{34}\big] \; , && (37)\\
\text{four–particle entanglement:} \quad & \mathcal{E}_{1234} = \max\big[0, P(\rho_{1234}) - \mathcal{E}_{123} - \mathcal{E}_{124} - \mathcal{E}_{134} - \mathcal{E}_{234} && (38)\\
& \qquad - \mathcal{E}_{12} - \mathcal{E}_{13} - \mathcal{E}_{14} - \mathcal{E}_{23} - \mathcal{E}_{24} - \mathcal{E}_{34}\big] && (39)
\end{aligned}$$

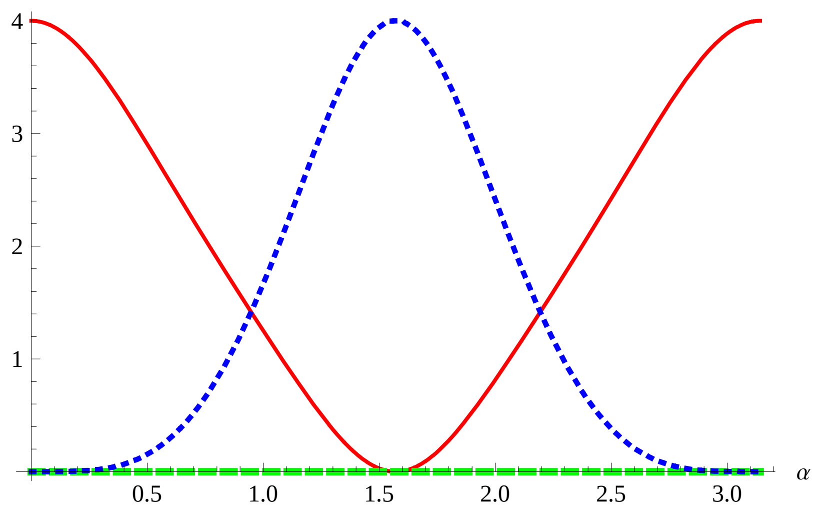

FIG. 2: (Color online) The graphes show the set of the physical measure of the mixture of the GHZ state and the $EPR \otimes EPR$ state, Eq. (40). The solid (red) curve shows the four–partite entanglement $\mathcal{E}_4 = \mathcal{E}_{1234}$, the dashed (green) curve shows the three–partite entanglement $\mathcal{E}_3 = \mathcal{E}_{123} + \mathcal{E}_{124} + \mathcal{E}_{134} + \mathcal{E}_{234}$ and the dotted (blue) curve shows the two–partite entanglement $\mathcal{E}_2 = \mathcal{E}_{12} + \mathcal{E}_{13} + \mathcal{E}_{14} + \mathcal{E}_{23} + \mathcal{E}_{24} + \mathcal{E}_{34}$ in dependence of α. The amount of the total entanglement is for the GHZ state and the $EPR \otimes EPR$ state 4, however, in the first case it due to four–partite entanglement whereas in the other case the bipartite entanglement maximizes. The separability measure reveals that the state is $\gamma_1 = \{1234\}$–separable ($E_{1234} = 4$, all others zero) except for $\alpha = \frac{\pi}{2}$ then the state is $\gamma_2 = \{12|34\}$–separable ($E_{tot} = 4, E_{12} = E_{34} = 2$ all others zero).

Instructive example for 4 qubits: Consider the mixture of the GHZ state and a pair of EPR-states state:

$$\rho \;=\; \cos^2(\alpha)\;|GHZ\rangle\langle GHZ| + \sin^2(\alpha)\;|\Phi^+\rangle\langle\Phi^+| \otimes |\Phi^+\rangle\langle\Phi^+| \tag{40}$$

with

$$\begin{aligned} |GHZ\rangle \;&=\; \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle) \\ |\Phi^+\rangle \;&=\; \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\;. \end{aligned} \tag{41}$$

The set of the physical measure is visualized in Fig. 2.

The generalization for any multipartite qudit system is straight forward:

$$\mathcal{E}_{\{\alpha_j\}} = \max\big[0, P(\rho_{\{\alpha_j\}}) - \sum_{\{\beta_j\}\subset\{\alpha_j\}} \mathcal{E}_{\{\beta_j\}}\big]\;. \tag{42}$$

Note that in case of the physical measure $\sum_{\{\alpha_j\}} \mathcal{E}_{\{\alpha_j\}} = \mathcal{E}_{tot}(\rho)$ is no longer a requirement. Indeed there exist states which even violate this condition. This is due to the fact that the physical measure quantifies the entanglement of subsystems of a larger systems with respect to possible applications or distillation. In case of overlapping indices of subsystems, e.g. ρ_{123} and ρ_{124} the possibility arises that both share the same entanglement, e.g. in subsystem 1 and 2, and thus in sum overquantify the actual total entanglement. However, there is no contradiction to possible experiments as one would have to decide, which subsystems to use, e.g. ρ_{123} or ρ_{124}, as their entanglement properties cannot be exploited simultaneously.

With the help of the m–concurrence, Eq. (12), bounds for every $P(\rho)$ can be computed, see Eq. (18), and thus for the whole set of entanglement measures. In the next section we give further examples and explicit formulae.

V. FURTHER INSTRUCTIVE EXAMPLES

In this section we apply our two sets of entanglement measures to certain classes of states and show explicitly how to derive the desired quantities.

A. Two-qubit states

In the case of pure bipartite qubit states obviously both measures coincide:

$$\begin{aligned} E(\rho_{12}) \;=\; E_{12} \;=\; \mathcal{E}_{12} \;&=\; S(\rho_1)+S(\rho_2) \\ &=\; -\log_2(\mathrm{Tr}(\rho_1^2)) - \log_2(\mathrm{Tr}(\rho_2^2)) \qquad (43) \\ &=\; -\;\log_2(1-\frac{1}{2}\mathbf{C}_{12}^2) -\; \log_2(1-\frac{1}{2}\mathbf{C}_{12}^2) \\ &=\; -2\;\log_2(1-\frac{1}{2}\mathbf{C}_{12}^2) \qquad (44) \end{aligned}$$

where the concurrence $\mathbf{C}_{12}$ is twice the Hill–Wootters concurrence [41]. hgbz76 For mixed states $\rho_{12}=\sum_i p_i|\psi_i\rangle\langle\psi_i|$, we obtain the physical measure by

$$\begin{aligned} \mathcal{E}_{12}(\rho_{12}) \;&=\; P(\rho_{12}) \;=\; \inf_{p_i,\psi_i}\sum p_i\;\{S(\mathrm{Tr}_2(\rho_i))+S(\mathrm{Tr}_1(\rho_i))\} \\ &=\; 2\inf_{p_i,\psi_i}\sum p_i\; S(\mathrm{Tr}_2(\rho_i)) \;=\; -2\inf_{p_i,\psi_i}\sum p_i \log_2(\mathrm{Tr}\{(\mathrm{Tr}_2(\rho_i))^2\}) \\ &=\; -2\inf_{p_i,\psi_i}\sum p_i \log_2(1-\frac{1}{2}\mathbf{C}_{12}^2(\psi_i)) \\ &\geq\; -2\inf_{p_i,\psi_i}\log_2(1-\frac{1}{2}\sum p_i\mathbf{C}_{12}^2(\psi_i)) \;=\; -2\;\log_2(1-\frac{1}{2}\mathbf{C}_{12}^2(\rho_{12}))\,, \qquad (45) \end{aligned}$$

where the concurrence can be derived operationally via

$$\mathbf{C}_{12}(\rho_{12}) = \max\left\{0, 2\max_{\lambda_i^{O_C}}(\{\lambda_i^{O_C}\}) - \sum_i\;\lambda_i^{O_C}\right\}\;, \qquad (46)$$

where the $\lambda_i^{O_C}$ are the square roots of the eigenvalues of $\rho_{12}\;\tilde\rho_{12}$ and $\tilde\rho_{12}=(O_C+O_C^\dagger)\;\rho_{12}^*\;(O_C+O_C^\dagger)$ with $O_C+O_C^\dagger=\sigma_y\otimes\sigma_y$. For bipartite qubits it is known that there always exists a decomposition such that all concurrences of the pure states $|\psi_i\rangle$ are equal [41], therefore the inequality is in fact an equality and the bounds are also known to be exact.

B. Two-qutrit states

In the case of qutrits the linear entropies can be written by only six different operators which are all possible tensor products of the three symmetric Gell-Mann matrices $\sigma^{(i)} = |j\rangle\langle k| + |k\rangle\langle j|$ with $0 \leq j < k \leq 2$:

$$\begin{aligned} S(\rho_1) &= -\log_2(1 - \frac{1}{2}(\sum_{ij} \mathbf{C}_{\mathbf{12}}^{\sigma^{(i)}\otimes\sigma^{(j)}})) \\ S(\rho_2) &= -\log_2(1 - \frac{1}{2}(\sum_{ij} \mathbf{C}_{\mathbf{12}}^{\sigma^{(i)}\otimes\sigma^{(j)}})) \end{aligned} \tag{47}$$

Lower bounds on $\mathcal{E}_{12}(\rho_{12}) = P(\rho_{12})$ are then obtained by calculating the squared eigenvalues of all operators $O^{\sigma^{(i)}\otimes\sigma^{(j)}}$ and adding them.

Consider the class of qutrit states which are composed of any two generalized Bell states denoted by P_{00}, P_{01} and the totally mixed state (for an review on the geometry of that class of states see [45, 46, 47])

$$\rho(\alpha, \beta) = \frac{1-\alpha-\beta}{9}\mathbb{1} + \alpha P_{00} + \beta P_{01} \; . \tag{48}$$

Here $P_{k,l} = |\Omega_{k,l}\rangle\langle\Omega_{k,l}|$ are obtained by choosing one maximally entangled state, e.g. $\Omega_{00} = \frac{1}{\sqrt{d}} \sum_{s=0}^{d-1} |s\rangle \otimes |s\rangle$, and applying Wely-Operators $W_{k,l}|s\rangle = w^{k(s-l)}|s-l\rangle$ onto one subsystem, i.e. $\Omega_{k,l} = W_{k,l} \otimes \mathbb{1}_d \, \Omega_{0,0}$.

The result of the physical measure is visualized in Fig. 3 (a). If α or β is zero, then we obtain the famous isotropic states, for which we now that concurrence increases linearly with $\alpha(\beta)$. If α and β are both positive, we observe that not all states negative under partial transpose (NPT) are detected, thus the bounds are not exact. If either α or β is zero, the derived bounds detect all NPT states, however, as shown in [45] in this case bound entangled states exist, therefore the bounds are not exact. Recently, by using bounds obtained by an operator acting globally on two copies of a state, these bound entangled states could be detected [48], however, in the region $\alpha, \beta > 0$ the bounds did not detect all NPT states either, but they were tighter than the bounds introduced in this work.

Let us consider the class of states

$$\rho(\alpha, \beta) = \frac{1-\alpha-\beta}{9}\mathbb{1} + \alpha P_{00} + \frac{\beta}{2}(P_{01} + P_{02}) \; , \tag{49}$$

where the third Bell states is obtained by applying the same Weyl operator, which transforms P_{00} to P_{01}, to P_{01}. This class of states are visualized in Fig. 3 (b). Here more symmetries are involved, therefore no bound entangled states can be found and the optimal entanglement witnesses, $\mathcal{K}_1, \mathcal{K}_2$, correspond to lines. It turns out that the bounds are only exact for $\alpha > \frac{1}{4}$. The requirements for the bounds to be tight is left for further investigation.

C. Three-qubit states

Let us consider the most general tripartite pure qubit state,

$$|\psi\rangle = a|000\rangle + b|001\rangle + c|010\rangle + d|011\rangle + e|100\rangle + f|101\rangle + g|110\rangle + h|111\rangle \; . \tag{50}$$

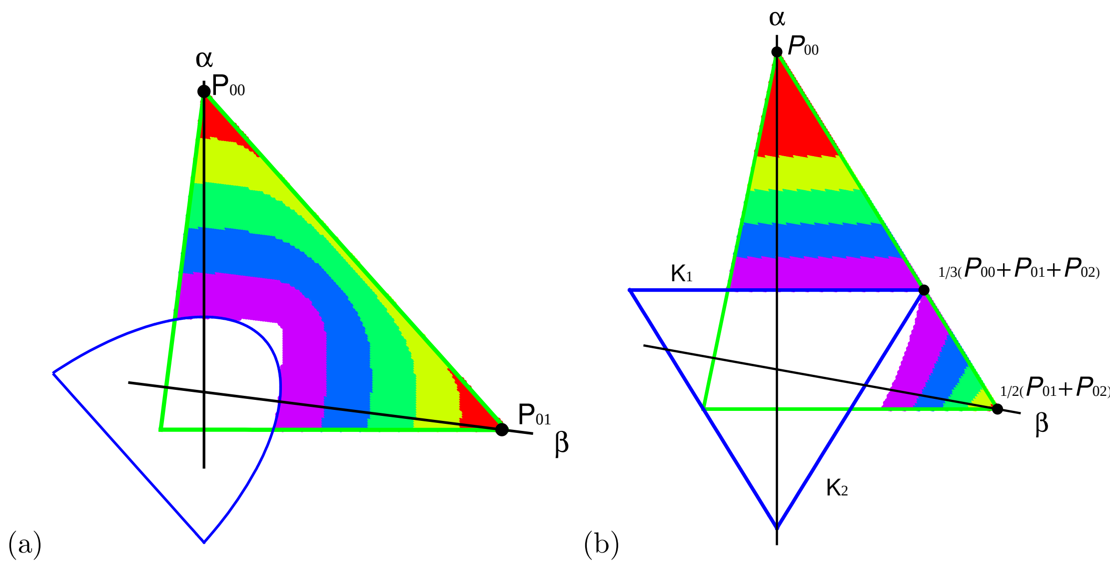

FIG. 3: (Color online) Here two slices through the class of "line" states [47], Eq. (48) and Eq. (49), are shown. The green triangle visualizes the parameter space for which the states are positive, the blue triangle/ellipse the parameter space for which the states are positive under partial transpose (PPT). The colored areas denote the regions where the bounds on the physical measure are nonzero (red: $1 \geq \mathbf{C} > 0.8$; yellow: $0.8 \geq \mathbf{C} > 0.6$; green: $0.6 \geq \mathbf{C} > 0.4$; blue: $0.4 \geq \mathbf{C} > 0.2$; purple: $0.2 \geq \mathbf{C} > 0$). Note that not all states negative under partial transpose are detected. In Fig. (a) for $\alpha < 0$ or $\beta < 0$ the bound is equivalent to the boundary by PPT, however, as was shown in Ref. [47] a small region of bound entangled states exist in this case. Only for the class of states visualized in Fig. (b) for $\alpha \geq \frac{1}{4}$ the bounds are tight.

The linear entropies of all three subsystems can be rewritten in terms of m–concurrences

$$S(\rho_1) = -\mathbf{log}_2(1 - \frac{1}{2}(\mathbf{C}_{12}^2 + \mathbf{C}_{13}^2 + \mathbf{C}_{123}^2)), \tag{51}$$

$$S(\rho_2) = -\mathbf{log}_2(1 - \frac{1}{2}(\mathbf{C}_{12}^2 + \mathbf{C}_{23}^2 + \mathbf{C}_{123}^2)), \tag{52}$$

$$S(\rho_3) = -\mathbf{log}_2(1 - \frac{1}{2}(\mathbf{C}_{23}^2 + \mathbf{C}_{13}^2 + \mathbf{C}_{123}^2)) \ . \tag{53}$$

Also the entropies of the partially reduced subsystems can be rewritten into the m–concurrences

$$S(\rho_{12}) = -\mathbf{log}_2(1 - \frac{1}{2}(\mathbf{C}_{23}^2 + \mathbf{C}_{13}^2 + \mathbf{C}_{123}^2)), \tag{54}$$

$$S(\rho_{13}) = -\mathbf{log}_2(1 - \frac{1}{2}(\mathbf{C}_{12}^2 + \mathbf{C}_{23}^2 + \mathbf{C}_{123}^2)), \tag{55}$$

$$S(\rho_{23}) = -\mathbf{log}_2(1 - \frac{1}{2}(\mathbf{C}_{12}^2 + \mathbf{C}_{13}^2 + \mathbf{C}_{123}^2)) \ . \tag{56}$$

Note that for pure tripartite qubit systems there is an intuitive relation between the entropies of the subsystems, $S(\mathbf{Tr}_k|\psi\rangle\langle\psi|) = S(\mathbf{Tr}_{\neg k}|\psi\rangle\langle\psi|)$. For explicit examples see Ref. [40].

D. The generalized Smolin states

As shown in Ref. [31] for the generalized Smolin state $\rho_{Smolin} = \frac{1}{2^n}\left(\mathbb{1} + \sum c_i\, \sigma_i^{\otimes n}\right)$ (n even) only the n–flip concurrence $\mathbf{C}_{12\ldots n}$ is nonzero and the bounds turn out to be tight. In detail one obtains the n–partite entanglement ($X = 1$ for $n \geq 4$ and $X = 2$ for $n = 2$)

$$\begin{aligned} E_{12\ldots n} \;=\; \mathcal{E}_{12\ldots n} \;&= -4\log_2\Big\{\frac{1}{4}\,X \max\big[0,\;\frac{1}{2}\max\{-1+\vec{c}\cdot\vec{n}^{\,(1)}, -1+\vec{c}\cdot\vec{n}^{\,(2)}, -1+\vec{c}\cdot\vec{n}^{\,(3)}, \\ &\qquad -1+\vec{c}\cdot\vec{n}^{\,(4)}\,\}\big]^2\Big\} \end{aligned} \qquad (57)$$

with

$$\{\vec{n}^{\,(0)}, \vec{n}^{\,(1)}, \vec{n}^{\,(2)}, \vec{n}^{\,(3)}\} \;=\; \{\begin{pmatrix}-1\\-1\\-1\end{pmatrix}, \begin{pmatrix}+1\\+1\\-1\end{pmatrix}, \begin{pmatrix}-1\\+1\\+1\end{pmatrix}, \begin{pmatrix}+1\\-1\\+1\end{pmatrix}\}\,.$$

Therefore, the state is fully or n–separable for $-1+\vec{c}\cdot\vec{n}^{\,(i)} \leq 0$ for all i and n–partite entangled else.

VI. CONCLUSION

In this paper we first extend the notion of k–separability to the γ_k–separability which includes the knowledge which subsystems are inseparable joint. We have pointed out that for mixed states the extension is not straightforward as an ambiguity could arise, however, we could overcome this problem by an appropriate definition, i.e. by a maximization over k. Moreover, this novel γ_k–separability concept shows also the desired convexity property for mixtures of different γ_k states, as visualized in Fig. 1.

Based on this extended concept we could define two different sets of entanglement measures, the first one reveals the γ_k separability property, the second one reveals the structural, physical properties, e.g. the kind of entanglement.

Both measures are based on the convex roof extension which in general cannot be computed. We use the method of Ref. [40] based on the observation that any entropy can be rewritten by m–flip concurrences, i.e. in an operator form. This includes any qudit system. Therefore bounds on the set of measures can be obtained and we show their usefulness in several examples.

For certain applications, such as quantum cryptography scenarios, one is rather interested in the structure of entanglement. For that we have defined a set of measures revealing the two–partite (bipartite), three–partite (tripartite), …, n–partite entanglement, which we denote as a "physical" measure. It captures for example the different entanglement features of e.g. the GHZ states, the W states or of the $EPR \otimes EPR$ states.

In the last section we gave more instructive examples with explicit formulae to compute lower bounds of the entanglement measures. We show cases where the bounds are surprisingly tight and cases were they are not. Further investigations have to be performed in order to understand in which cases the bounds are equivalent to the infimum of the convex roof.

In summary, we have pointed out that all entanglement features in multipartite systems cannot be revealed by a single set of measures. We defined two sets of measures for multipartite qudit systems and demonstrated its usefulness and computability. Herewith we believe one may find novel application exploiting the entanglement of multipartite systems, which is —as this work shows— at least mathematically considerably different to bipartite qubit entanglement.

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VIII. APPENDIX

In this appendix we give the proofs that the proposed set of separability measures and the set of the physical measures fulfill the proposed requirements.

A. Proofs for the separability measure

Here we prove the proposed requirements S2-S6 for the separability measure, S1 is a definition.

Property:

$$S2: \quad E_{\{\alpha_j\}}(\rho) \; > \; 0 \qquad \forall \quad \{\alpha_j\} \in \gamma_k \quad \text{and} \quad |\{\alpha_j\}| \; \geq \; 2$$

with $E_{\{\alpha_j\}} := \sum_{s\in\{\alpha_j\}}\left(S(\rho_s) - \sum_{\{\beta_j\}\subset\{\alpha_j\}} E_{\{\beta_j\}}\right) \cdot \delta[S(\rho_{\{\alpha_j\}}), 0]$.

Proof. Consider a n-partite pure state ρ with the separability property $\gamma_k := \{\{\varepsilon_1\}|\{\varepsilon_2\}|\cdots|\{\varepsilon_k\}\}$. Now if $\{\alpha_j\} \in \gamma_k$ then it follows that $\delta[S(\rho_{\{\alpha_j\}}), 0] = 1$ and consequently $\forall \; \{\beta_j\} \subset \{\alpha_j\}$ and $\forall \; \{\beta_j\} \supset \{\alpha_j\}$ is $\delta[S(\rho_{\{\beta_j\}}), 0] = 0$. Therefore $E_{\{\alpha_j\}} = \sum_{s\in\alpha_j} S(\rho_s) > 0$ as required. □

Property:

$$S3: \quad E_{\{\alpha_j\}}(\rho) \; = \; 0 \qquad \forall \quad \{\alpha_j\} \notin \gamma_k \quad \text{or} \quad |\{\alpha_j\}| \; = \; 1$$

Proof. Again consider a n-partite pure state ρ with the separability property $\gamma_k := \{\{\varepsilon_1\}|\{\varepsilon_2\}|\cdots|\{\varepsilon_k\}\}$. Now if $\{\alpha_j\} \notin \gamma_k$ then it follows that $\delta[S(\rho_{\{\alpha_j\}}), 0] = 0$ and therefore $E_{\{\alpha_j\}} = 0$. If $\{\alpha_j\} \in \gamma_k$ but $|\{\alpha_j\}| \; = \; 1$ then $E_{\{\alpha_j\}} = S(\rho_{\alpha_j})\; \delta[S(\rho_{\alpha_j}), 0] = 0 \cdot 1 = 0$. □

Property:

$$S4: E_{\{\alpha_j\}}(\rho^{\otimes n}) \; = \; n \; E_{\{\alpha_j\}}(\rho)$$

Proof. Again consider a n-partite pure state ρ with the following separability property $\gamma_k := \{\{\varepsilon_1\}|\{\varepsilon_2\}|\cdots|\{\varepsilon_k\}\}$. $\rho^{\otimes n}$ must have the same γ_k property. Thus any nonzero $E_{\{\alpha_j\}}$ will be of the form $E_{\{\alpha_j\}} = \sum_{s=\alpha_1}^{\alpha_j} S(\rho_s^{\otimes n})$. Hence it sufficient to prove that

$$S(\rho_s^{\otimes n}) = n\, S(\rho_s) \tag{58}$$

where $S(\rho_s) := \log_2(\mathrm{Tr}(\rho_s^2))$. This is the case as $\mathrm{Tr}((\rho_s^{\otimes n})^2) = (\mathrm{Tr}((\rho_s)^2))^n$. □

Property:

$$S5 : E_{\{\alpha_j\}}\big(U_{\alpha_1} \otimes U_{\alpha_2} \otimes \cdots \otimes U_{\alpha_j}\, \rho\, (U_{\alpha_1} \otimes U_{\alpha_2} \otimes \cdots \otimes U_{\alpha_j})^\dagger\big) \;=\; E_{\{\alpha_j\}}(\rho)\ .$$

Proof. This proof is trivial as every underlying property $S(\rho) := \log_2(\mathrm{Tr}(\rho^2))$ is clearly invariant under local unitary transformations, i.e.

$$\mathrm{Tr}(U\rho \underbrace{U^\dagger U}_{\mathbb{1}} \rho U^\dagger)) = \mathrm{Tr}(\rho^2 \underbrace{U^\dagger U}_{\mathbb{1}}) = \mathrm{Tr}(\rho^2)$$

□

Property:

$$S6 : E_{tot}(\rho_1 \otimes \rho_2) \;=\; E_{tot}(\rho_1) + E_{tot}(\rho_2) \quad \text{with} \quad \sum_{\{\alpha_j\}} E_{\{\alpha_j\}} = E_{tot} = \sum_{s=1}^{n} S(\rho_s)\ .$$

Proof. Consider now a n_1–partite pure state ρ_1 with the separability property $\gamma_{k_1} := \{\{\varepsilon_1\}|\{\varepsilon_2\}|\cdots|\{\varepsilon_{k_1}\}\}$ and another n_2–partite pure state ρ_2 with the separability property $\gamma_{k_2} := \{\{\kappa_1\}|\{\kappa_2\}|\cdots|\{\kappa_{k_2}\}\}$. The tensor product of those two states have the separability property

$$\gamma_{k_3} := \{\{\varepsilon_1\}|\{\varepsilon_2\}|\cdots|\{\varepsilon_{k_1}\}|\{\kappa_1\}|\{\kappa_2\}|\cdots|\{\kappa_{k_2}\}\}\ .$$

In this notation the counting of the subsystems of the second system starts with n_1+1. For every n-partite pure state the total entanglement is

$$E_{tot}(\rho) = \sum_{s=1}^{n} S(\rho_s)$$

and hence

$$E_{tot}(\rho_1 \otimes \rho_2) = \sum_{s=1}^{n_1+n_2} S(\rho_s) = \underbrace{\sum_{s=1}^{n_1} S(\rho_s)}_{E_{tot}(\rho_1)} + \underbrace{\sum_{s=n_1+1}^{n_2} S(\rho_s)}_{E_{tot}(\rho_2)}\ .$$

□

B. Proofs for the physical measure

Property:

P1: $\mathcal{E}_{\{\alpha_j\}}(\rho) \;\geq\; 0 \qquad \forall \quad \{\alpha_j\} \subseteq \gamma_k \wedge |\{\alpha_j\}| \;\geq\; 2$

P2: $\mathcal{E}_{\{\alpha_j\}}(\rho) \;=\; 0 \qquad \forall \quad \{\alpha_j\} \supset \gamma_k \vee |\{\alpha_j\}| \;=\; 1$

Proof. Consider first that

$$P(\rho_{\{\alpha_j\}}) = \sum_{\{\beta_m\}\in\{\alpha_j\}} P(\rho_{\{\beta_m\}}) + P(\mathrm{Tr}_{\{\beta_m\}\in\{\alpha_j\}}\rho) \tag{59}$$

and as well

$$\sum_{m=2}^{j-1} \sum_{\{\epsilon_m\}\subset\{\alpha_j\}} \mathcal{E}(\rho_{\{\epsilon_m\}}) \;\geq\; \sum_{\{\beta_m\}\in\{\alpha_j\}} P(\rho_{\{\beta_m\}}) + P(\mathrm{Tr}_{\{\beta_m\}\in\{\alpha_j\}}\rho) \tag{60}$$

$$\text{iff} \quad \exists\, \{\beta_m\} \in \{\alpha_j\} \tag{61}$$

such that the difference derives to

$$P(\rho_{\{\alpha_j\}}) - \sum_{m=2}^{j-1} \sum_{\{\epsilon_m\}\subset\{\alpha_j\}} \mathcal{E}_{\{\epsilon_m\}}(\rho) \;\leq\; 0 \tag{62}$$

$$\text{iff} \quad \exists\, \{\beta_m\} \in \{\alpha_j\} \tag{63}$$

from which consequently follows

$$\mathcal{E}_{\{\alpha_j\}}(\rho) \;=\; 0 \qquad \forall \quad \{\alpha_j\} \supset \gamma_k \quad \text{or} \quad |\{\alpha_j\}| \;=\; 1 \tag{64}$$

□

Property:

P3: $\mathcal{E}_{\{\alpha_j\}}(\rho^{\otimes n}) \;=\; n\, \mathcal{E}_{\{\alpha_j\}}(\rho)$ (additivity on copies of the same state)

Proof. Additivity follows directly if the conjecture is valid, i.e.

$$\mathcal{E}_{\{\alpha_j\}}(\rho^{\otimes n}) = \mathcal{E}_{\{\alpha_j\}}(\rho') \tag{65}$$

where the separability property of ρ' is

$$\gamma'_k = \{\{\gamma_k\}|\{\gamma_k\}|(\cdots)|\{\gamma_k\}\} \tag{66}$$

so the infimum is achieved in the appropriate decomposition such that

$$\mathcal{E}_{\{\alpha_j\}}(\rho^{\otimes n}) \;=\; n\, \mathcal{E}_{\{\alpha_j\}}(\rho)\ . \tag{67}$$

□

Property:

P4: $\mathcal{E}_{\{\alpha_j\}}(\lambda\, \rho_1 + (1-\lambda)\, \rho_2) \;\leq\; \lambda\, \mathcal{E}_{\{\alpha_j\}}(\rho_1) + (1-\lambda)\, \mathcal{E}_{\{\alpha_j\}}(\rho_2)$ (convexity)

Proof. Invariance under local unitaries is easy to prove as the constituting functions $P(\rho)$ are themselves invariant under local unitary transformations

$$P(\rho) := \inf_{p_i,\psi_i} \sum_i p_i (\sum_s S(\mathrm{Tr}_{\neg s}|\psi_i\rangle\langle\psi_i|)) \tag{68}$$

and therefore

$$\mathrm{Tr}_{\neg s}(U_{\alpha_1}\otimes U_{\alpha_2}\otimes\cdots\otimes U_{\alpha_j}|\psi_i\rangle\langle\psi_i|(U_{\alpha_1}\otimes U_{\alpha_2}\otimes\cdots\otimes U_{\alpha_j})^\dagger) \tag{69}$$

$$= \mathrm{Tr}_{\neg s}(\underbrace{(U_{\alpha_1}\otimes U_{\alpha_2}\otimes\cdots\otimes U_{\alpha_j})^\dagger U_{\alpha_1}\otimes U_{\alpha_2}\otimes\cdots\otimes U_{\alpha_j}}_{\mathbb{1}}|\psi_i\rangle\langle\psi_i|) \tag{70}$$

such that

$$P(U_{\alpha_1}\otimes U_{\alpha_2}\otimes\cdots\otimes U_{\alpha_j}\,\rho\,(U_{\alpha_1}\otimes U_{\alpha_2}\otimes\cdots\otimes U_{\alpha_j})^\dagger) = P(\rho)\ . \tag{71}$$

□

Property:

P5: $\mathcal{E}_{\{\alpha_j\}}(\lambda\ \rho_1 + (1-\lambda)\ \rho_2)\ \leq\ \lambda\ \mathcal{E}_{\{\alpha_j\}}(\rho_1) + (1-\lambda)\ \mathcal{E}_{\{\alpha_j\}}(\rho_2)$ (Convexity)

Proof. To prove that the total entanglement $E_{tot}(\rho)$ is convex one needs to prove that $P(\rho)$ is convex, which is trivial, as:

$$\begin{aligned}
& P(\lambda\rho_1 + (1-\lambda)\rho_2) \\
= & \inf_{p_i,r_j,\psi_i,\phi_j,\gamma_n,\tau_m} (\sum_i p_i \lambda \sum_s S(\mathrm{Tr}_{\neg_s}|\psi_i\rangle\langle\psi_i|) + \sum_j r_j(1-\lambda)\sum_s S(\mathrm{Tr}_{\neg_s}|\phi_j\rangle\langle\phi_j|)) \\
\leq & \underbrace{\inf_{p_i,\psi_i,\gamma_n} (\sum_i p_i\lambda\sum_s S(\mathrm{Tr}_{\neg_s}|\psi_i\rangle\langle\psi_i|))}_{\lambda P(\rho_1)} + \underbrace{\inf_{r_j,\phi_j,\tau_m}(\sum_j r_j(1-\lambda)\sum_s S(\mathrm{Tr}_{\neg_s}|\phi_j\rangle\langle\phi_j|))}_{(1-\lambda)P(\rho_2)}
\end{aligned} \tag{72}$$

Note that the constituting elements $E_{\{\alpha_j\}}(\rho)$ are only convex under local combinations of the form

$$\begin{aligned}
\rho \underbrace{+}_{local} \sigma :\ = &\ \lambda \sum_i p_i(\rho^i_{\{\beta_1\}} \otimes \rho^i_{\{\beta_2\}} \otimes (\cdots)\rho^i_{\{\beta_k\}}) + (1-\lambda)\sum_i p_i(\sigma^i_{\{\beta_1\}} \otimes \rho^i_{\{\beta_2\}} \otimes (\cdots)\rho^i_{\{\beta_k\}}) \\
= &\ \sum_i p_i((\lambda\rho^i_{\{\beta_1\}} + (1-\lambda)\sigma^i_{\{\beta_1\}}) \otimes \rho^i_{\{\beta_2\}} \otimes (\cdots)\rho^i_{\{\beta_k\}})
\end{aligned} \tag{73}$$

□

Property:

P6: $\sum_i \mathrm{Tr}\left(V_i\rho V_i^\dagger\right)\mathcal{E}_{tot}\left(\frac{V_i\rho V_i^\dagger}{\mathrm{Tr}\left(V_i\rho V_i^\dagger\right)}\right) \leq \mathcal{E}_{tot}(\rho)$ (non-increasing on average under LOCC),

Proof. This has already been proven, see e.g. Ref. [49]. □

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Two distinct classes of bound entanglement: PPT–bound and "multi-particle"–bound

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We introduce systematically with the help of Weyl operators novel classes of multipartite and multidimensional states which are all bound entangled for arbitrary dimension. We find that the entanglement is bound due to different reasons: unlockable due to the multi–particle nature and some states are in addition bound due the fact being positive under partial transposition (PPT). By a general construction ($\mathcal{W}$ simplices) we obtain classes of states which have the same geometry concerning separability and entanglement independent of the number of involved particle pairs. Moreover, we introduce a distillation protocol and demonstrate for $d=3$ that for a certain set of states the entanglement can be increased only up to a certain amount.

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Quantum entanglement is a key feature of quantum theory with many important consequences for modern physics. It has become a highly valuable resource for novel applications, such as cryptography and a formidable quantum computer. However, the mathematical and/or physical characterization of all types of entanglement and their implementations are far from being fully explored. E.g. the quantification or even the classification of entanglement of multipartite systems is still an open problem.

This paper will analyze the nature of at least two distinct classes of bound entanglement, i.e. entanglement which cannot be distilled by local operations and classical communication (LOCC) into pure maximally entangled states, when each local observer posses only one particle. This in return means that there should exists different applications for these states due to the different nature of their entanglement.

We first review a huge class of bipartite qudit states. A qudit is a quantum systems with d degrees of freedom. With the help of group theoretical methods which allows for considerable simplifications a geometrical picture of the state space can be drawn, i.e. the properties separability, *bound* entanglement or PPT entanglement (PPT = positive under partial transposition) and NPT entanglement (NPT = negativ under partial transposition) can be characterized. For bipartite qudits this state space was called "magic" simplex $\mathcal{W}$ in Ref. [1] and extensively discussed in Refs. [2, 3] in different contexts. The construction of a simplex of states with maximally mixed subsystems has so far proven to be a powerful tool in analyzing bipartite qubits and qutrits (e.g. Ref. [4, 5, 6]) and recently even for multipartite qubits [7]. It provides a deep insight into the structure of entangled states and helps in constructing entanglement witnesses and exploring entanglement measures.

We will extend the simplex of bipartite qudits, i.e. one pair of qudits, to n pairs of qudits where n is any natural number. We will prove that interestingly this extended class has the same properties concerning separability, bound entanglement and NPT–entanglement by proving that the optimal entanglement witnesses reduces to the same mathematical conditions (Theorem 2). Therefore, results for bipartite qudits become automatically true for any n pairs of bipartite qudits, which may otherwise due to the high computational effort would not be obtainable.

This extended class of states shows due to their multiparticle nature a feature which was called unlockable–bound entanglement [8, 10]. In detail NPT–entangled states can be distilled to certain extremal states, the so called "vertex" states of the simplex $\mathcal{W}^{\otimes n}$, however, not into pure maximally entangled states: this novel class of states are bound to their own class. For multipartite qubits this was shown in Refs. [7]. We prove in this paper that this is a general feature of such multipartite simplex states and, moreover, the fact that PPT–bound entangled states exist for dimensions $d \geq 3$ implies that there are two different kinds of bound entanglement. Explicitly, we give a multidimensional distillation protocol for $d=3$ which distills certain states within the simplex to the vertex states, which are themselves bound entangled.

The magic simplex $\mathcal{W}$ for bipartite qudits: For bipartite qudits the vertex states $P_{i,j}$ of the "magic" simplex $\mathcal{W}$ are the maximally entangled states in d dimensions (Refs. [1, 2]):

$$|\Phi^+\rangle :\; = \sum_{i=0}^{d-1} |ii\rangle\,, \quad P_{0,0} \;:=\; |\Phi^+\rangle\langle\Phi^+| \qquad (1)$$

$$P_{k,l} :\; = \; \mathbb{1}_d \otimes \mathrm{W}_{k,l}\; P_{0,0}\; \mathbb{1}_d \otimes \mathrm{W}_{k,l}^\dagger \qquad (2)$$

where the $\mathrm{W}_{k,l}$ are the Weyl operators defined by

$$\mathrm{W}_{k,l}|s\rangle \;=\; w^{k(s-l)}\,|s-l\rangle \qquad \text{with} \quad w \;=\; e^{2\pi i/d}\,. \quad (3)$$

The magic simplex $\mathcal{W}$ is the convex combination of all vertex states

$$\mathcal{W} := \left\{ \sum_{k,l=0}^{d-1} c_{k,l}\; P_{k,l} \mid c_{k,l} \geq 0,\; \sum_{k,l=0}^{d-1} c_{k,l} = 1 \right\}\,. \quad (4)$$

One main property of this class of states forming a d^2-1 dimensional simplex is that any trace of one particle

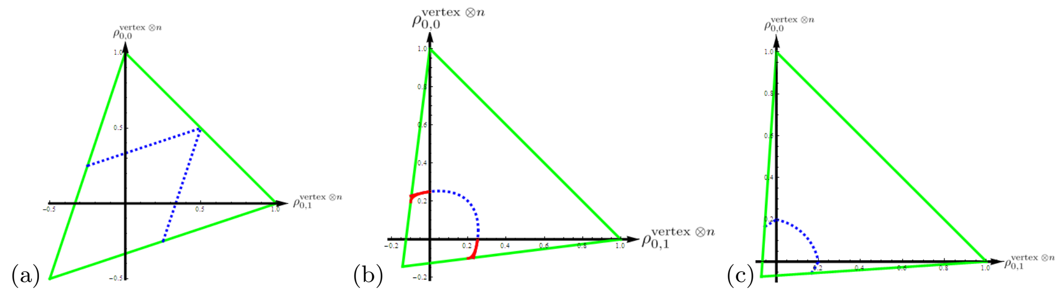

FIG. 1: (Color online) Slices via the simplices for the states which are mixtures of any two vertex states and the maximally mixed state, Eq. (13), for the dimensions (a) $d = 2$, (b) $d = 3$, (c) $d = 4$. The (green) triangles are given by the positivity condition, the dotted (blue) lines/curves represents the PPT condition. For $d = 3$ one finds a whole region of PPT bound entanglement if either α or β is negative (filled (red) region). As expected the region of separable states shrinks with increasing dimension d.

results in a maximally mixed state. We want to conserve this property for the multipartite scenario, i.e. any trace of one or more particles should result into a maximally mixed state:

$$\rho_{0,0}^{\text{vertex}\otimes n} := \frac{1}{d^2}\sum_{i,j=0}^{d-1} P_{i,j}\otimes P_{i,j}\otimes\cdots\otimes P_{i,j} = \frac{1}{d^2}\sum_{i,j=0}^{d-1} P_{i,j}^{\otimes n}\;. \tag{5}$$

For $d = 2$ this state was investigated by Smolin [8] and has proven to be an interesting state, exhibiting many counter-intuitive properties: such that it is biseparable under any bipartite cut, but ignorance of any arbitrary number of subsystems will render this state useless for quantum informational tasks, Refs. [9, 10], though it violates a Bell inequality (see Refs. [7, 9, 11]). Moreover, applying the two sets of multipartite entanglement measures proposed in Ref. [12] it turns out that n paired LOCCs are needed to prepare the state, whereas $2n$ parties are needed to cooperate locally to perform quantum informational tasks with that state.

We prove now that the above state has unlockable entanglement and then generalize to a whole class of states with all that features.

Theorem 1: The state, Eq. (5), is (multipartite) bound entangled for any dimension d, because no locally working of all involved parties can by LOCC distill a pure maximally entangled state.

Proof. As for every state that exhibits a partial separability like $\rho = \sum_i p_i\;\rho_i^A\otimes\rho_i^B$ will remain A-B separable under every LOCC of the form:

$$\Lambda_{LOCC}[\rho] = \frac{\sum_k A_k\otimes B_k\;\rho\;A_k^\dagger\otimes B_k^\dagger}{\mathrm{Tr}(\sum_k A_k\otimes B_k\;\rho\;A_k^\dagger\otimes B_k^\dagger)} \tag{6}$$

and the state in question allows a biseparable decomposition even if two subsystems are arbitrarily exchanged, this special property is preserved under LOCC. No maximally entangled pure state can exhibit this property, hence the state is bound entangled. □

The magic simplex $\mathcal{W}^{\otimes n}$ for n pairs of qudits: A certain vertex states of any n pairs of qudits can be defined by

$$n = 1: \qquad \rho_{0,0}^{\text{vertex}\otimes 1} := P_{0,0}$$
$$n \geq 2: \qquad \rho_{0,0}^{\text{vertex}\otimes n} := \frac{1}{d^2}\sum_{i,j=0}^{d-1} P_{i,j}^{\otimes n}\;. \tag{7}$$

By applying in one subsystem a Weyl operator $W_{k,l} := \mathbb{1}_d\otimes \mathrm{W}_{k,l}$ one obtains as before the remaining d^2-1 vertex states

$$\rho_{k,l}^{\text{vertex}\otimes n} = \mathbb{1}_{d^2}^{\otimes(n-1)}\otimes W_{k,l}\;\rho_{0,0}^{\text{vertex}\,\otimes n}\;\mathbb{1}_{d^2}^{\otimes(n-1)}\otimes W_{k,l}^\dagger = \frac{1}{d^2}\sum_{i,j=0}^{d-1} P_{i,j}^{\otimes n-1}\otimes W_{k,l}P_{i,j}W_{k,l}^\dagger\;. \tag{8}$$

Note that if the Weyl operator is applied on a different subsystem we obtain an equivalent simplex, however, with different labeling (all states and partial states have for any n same eigenvalues).

Now we can define a huge class of states which have the same geometry concerning separability and entanglement for a given d, the "magic" n pair qudit simplex $\mathcal{W}^{\otimes n}$:

$$\mathcal{W}^{\otimes n} := \left\{\sum_{k,l=0}^{d-1} c_{k,l}\;\rho_{k,l}^{\text{vertex}\,\otimes n}\;|\;c_{k,l}\geq 0,\;\sum_{k,l=0}^{d-1} c_{k,l} = 1\right\}\;.$$

These states have the same properties as the vertex states, i.e. all subsystems are maximally mixed, all states have n–separable decompositions, where always any two subsystems can be grouped together and single subsystems may arbitrarily be interchanged. The mixedness of any vertex state, $M := \frac{d}{d-1}(1-\mathrm{Tr}(\rho_{k,l}^{\text{vertex}\,\otimes n}\rho_{k,l}^{\text{vertex}\,\otimes n}))$, for $n \geq 2$ is $\frac{1-d^{-2}}{1-d^{-n}}$, thus gets less mixed with increasing n and/or d.

We prove now that the structure of separability is for any n equivalent by the powerful tool of witnesses, then we proceed to discuss the feature of bound entanglement and unlockable–bound entanglement.

Optimal witnesses in the simplex $\mathcal{W}^{\otimes n}$: An entanglement witness EW_ρ is a criterion to "witness" for an certain state ρ that it is not in the set of separable states SEP. Knowing that SEP is convex it can be completely characterized by the tangential hyperplanes, thus we search for tangential or optimal witnesses on the surface of SEP, i.e.

$$\begin{aligned} EW_\rho^{opt} &= \{K = K^\dagger \neq 0 | \forall\, \rho_{sep} \in SEP : \\ & Tr(K\, \rho_{sep}) < 0 \quad \text{and} \quad Tr(K\, \rho) = 0\}\,. \end{aligned} \tag{9}$$

As proven in Ref. [1] any witness operator for states within the simplex $\mathcal{W}$ can only be of the form $K = \sum_{k,l} \kappa_{k,l}\; P_{k,l}$. As $\mathcal{W}$ and $\mathcal{W}^{\otimes n}$ have the same group symmetries by their construction via the Weyl operators (see Theorem 6 in Ref. [1]) any witness operator within $\mathcal{W}^{\otimes n}$ has to have the form $K_n = \sum_{k,l} \kappa_{k,l} \rho_{k,l}^{\text{vertex} \otimes n}$.

Theorem 2: The operator $K_n = \sum_{k,l} \kappa_{k,l} \rho_{k,l}^{\text{vertex} \otimes n}$ is an optimal entanglement witness if $\det M_\Phi = 0$ with $M_\Phi = \sum_{k,l} \kappa_{k,l} W_{k,l} |\Phi\rangle\langle\Phi| W_{k,l}^\dagger \geq 0 \quad \forall \quad \Phi \in \mathbb{C}^d$. This means that the set of separable, PPT–entangled and NPT–entangled states have for any d and all n the same geometry because the $d \times d$ matrix M_Φ is identical.

Proof. Any separable state ρ_{sep} can be written as a convex combination of pure product states and therefore $Tr(K_n\, \rho_{sep}) \geq 0 \quad \forall \quad \rho_{sep} \in SEP$ implies that

$$\langle K_n\rangle := \langle \eta_1, \chi_1| \otimes \langle \eta_2, \chi_2| \otimes \ldots \langle \eta_n, \chi_n|\; K_n\; |\eta_1, \chi_1\rangle \otimes |\eta_2, \chi_2\rangle \otimes \ldots |\eta_n, \chi_n\rangle \geq 0 \qquad \forall \quad \eta_1, \chi_1, \eta_2, \chi_2 \cdots \eta_n, \chi_n, \in \mathbb{C}^d\,.$$

By the observation that $P_{k,l} = \frac{1}{d}\sum_{s,t=0}^{d-1} W_{k,l}\; |ss\rangle\langle t,t|\; W_{k,l}^\dagger = \frac{1}{d}\sum_{s,t} \mathrm{W}_{k,l} \otimes \mathbb{1}_d\; |ss\rangle\langle t,t|\; \mathrm{W}_{k,l}^\dagger \otimes \mathbb{1}_d$ follows

$$\langle \eta_i, \chi_i|\; P_{k,l}\; |\eta_i, \chi_i\rangle = \frac{1}{d}\sum_{s,t} \langle \eta_i|\mathrm{W}_{k,l}|s\rangle\langle\chi_i|s\rangle\langle t|\chi_i\rangle\langle t|\mathrm{W}_{k,l}^\dagger|\eta_i\rangle = \frac{1}{d}\langle\eta_i|\mathrm{W}_{k,l}|\phi_i\rangle\langle\phi_i|\mathrm{W}_{k,l}^\dagger|\eta_i\rangle\,, \tag{10}$$

where we defined all $\phi_i \in \mathbb{C}^d$ as $|\phi_i\rangle = \sum_s \langle\chi_i|s\rangle\, |s\rangle$. Therefore, $P_{k,l}$ is obviously an entanglement witness, because

$$\frac{1}{d}\langle\eta_i|\mathrm{W}_{k,l}|\phi_i\rangle\langle\phi_i|\mathrm{W}_{k,l}^\dagger|\eta_i\rangle = \frac{1}{d}|\langle\eta_i|\tilde{\phi}_i\rangle|^2 \geq 0\;, \forall\, \eta_i, \tilde{\phi}_i \in \mathbb{C}^d\,. \tag{11}$$

The expectation value of the witness operator K_n in $(d \times d)^n$ reduces to an expectation value of $d \times d$ operators

$$\begin{aligned} \langle K_n\rangle &= \frac{1}{d^2}\frac{1}{d^n}\sum_{k,l}\kappa_{k,l}\sum_{g,h}\langle\eta_1|\mathrm{W}_{g,h}|\phi_1\rangle\langle\phi_1|\mathrm{W}_{g,h}^\dagger|\eta_1\rangle \cdot\; \ldots\; \cdot \langle\eta_{n-1}|\mathrm{W}_{g,h}|\phi_{n-1}\rangle\langle\phi_{n-1}|\mathrm{W}_{g,h}^\dagger|\eta_{n-1}\rangle \\ & \cdot\langle\eta_n|\mathrm{W}_{k,l}\mathrm{W}_{g,h}|\phi_n\rangle\langle\phi_n|\mathrm{W}_{g,h}^\dagger\mathrm{W}_{k,l}^\dagger|\eta_n\rangle = \frac{d^2}{d^2}\cdot C \cdot \frac{1}{d^n}\sum_{k,l}\kappa_{kl}\langle\eta_n|\mathrm{W}_{k,l}|\tilde{\phi}_n\rangle\langle\tilde{\phi}_n|\mathrm{W}_{k,l}^\dagger|\eta_n\rangle \end{aligned} \tag{12}$$

with $C \geq 0$. Therefore, K_n is an entanglement witness if the operator $M_\phi = \sum_{k,l}\kappa_{k,l}\mathrm{W}_{k,l}|\phi\rangle\langle\phi|\mathrm{W}_{k,l}^\dagger$ is not negative for all $\phi \in \mathbb{C}^d$ and it is optimal if $\det M_\phi = 0$. □

Example showing the geometry of separability and PPT–bound entanglement for different dimensions: Let us consider any two vertex states mixed with the totally mixed state, i.e.

$$\rho = \frac{1-\alpha-\beta}{d^2}\mathbb{1}_d^{\otimes 2n} + \alpha\; \rho_{0,0}^{\text{vertex} \otimes n} + \beta\; \rho_{0,1}^{\text{vertex} \otimes n}\,. \tag{13}$$

The positivity condition of the density matrix on the parameters α, β give three lines which form a triangle. Likewise we obtain the parameter region for the states which are PPT entangled. This is visualized in Fig. 1 for dimension $d = 2, 3, 4$. The authors of Ref. [1] found by optimizing the witness operator for bipartite qutrits PPT–bound entanglement if either α or β is negative. By Theorem 2 this means that we found a whole region of PPT–bound entanglement for any number of qutrit pairs n.

Distilling bound entanglement: While the basic geometric structure of separable and PPT–bound entangled and entangled states remains unchanged with n, the properties of the states in the simplex change drastically. They are bound entangled as the vertices states cannot

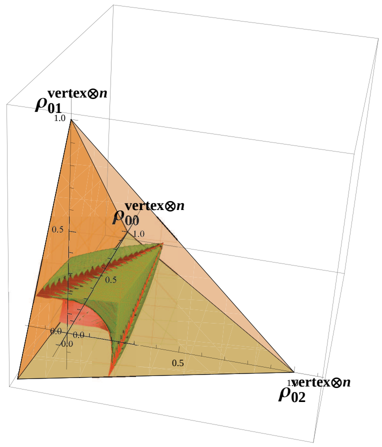

FIG. 2: Three dimensional slice through the eight dimensional simplex for $d = 3$, given by Eq. (14). The (transparent yellow) tetrahedron is given by the positivity condition, the (red) cone represents the PPT condition. Inside the PPT cone there are also bound entangled states, which cannot be distilled at all. The (green) Christmas tree shaped area is obtained via application of the distillation protocol and shows states that can not be distilled to one of the edge states. This area was obtained numerically, yet it is most intriguing that it, up to numerical precision, coincides with the states not detected by entanglement measures derived from the m–concurrence [12].

be distilled (theorem 1). However, as we prove in the following for $d = 3$ some states inside $\mathcal{W}^{\otimes n}$ can be distilled by a certain protocol to the vertices states. Let's consider the following distillation protocol:

1. Take a copy of the state: $\rho^{\otimes 2}$, the first dit will be regarded as source, the second as target dit.

2. Apply the unitary gate U_m in all subsystems: $\rho_T = U_m^{\otimes n} \rho^{\otimes 2} U_m^{\otimes n}$ with $U_m := (1 - \delta_{ij})\, |ij\rangle\langle ij| + \delta_{ij}\, (|ij\rangle\langle im| + |im\rangle\langle ij|)$.

3. Project onto $|m\rangle\langle m|$ in all target systems: $\mathbb{1}_d \otimes |m\rangle\langle m|\, \rho_T\, \mathbb{1}_d \otimes |m\rangle\langle m|$

4. Discard target dits.

With this protocol it is possible to "distill" many NPT-entangled states in the simplex into a vertex state. Consider e.g. the following state

$$\rho = \frac{1-\alpha-\beta-\gamma}{9} \mathbb{1}_3^{\otimes 2n} + \alpha\, \rho_{0,0}^{\text{vertex}\,\otimes n} + \beta\, \rho_{0,1}^{\text{vertex}\,\otimes n} + \gamma\, \rho_{0,2}^{\text{vertex}\,\otimes n} . \qquad (14)$$

This is an example of a so called "line" state, where the same Weyl operator connects all vertex states. This is visualized in Fig. 2. Surprisingly, the "distillable" states are the ones which are detected by the bounds on the multipartite qudit measure introduced in Ref. [12].

Clearly, for $n = 1$ the vertex states are pure and therefore it is a genuine distillation protocol, however, for $n \geq 2$ the vertex states are no longer pure, the protocol distills up to a certain degree of entanglement and purity. Note, that for $d = 2$ and $n = 2$ this has already proven to be very useful, as the vertex states can be used to reduce communication complexity and for remote information concentration for $2n$ parties [10].

Conclusion: We have introduced a whole new class of bound entangled states for arbitrary n pairs of qudits (d degrees of freedom), the extended simplex $\mathcal{W}^{\otimes n}$, and proven that all states are non–distillable. The very nature of their bound entanglement stems from the multipartite construction and may be unlocked if two parties work together. Inside the simplex ($d \geq 3$) there also exist states which cannot be distilled, because they are nonseparable PPT–states. Thus in the multipartite and multidimensional scenario there exist at least two classes of bound entangled states: those which may be unlocked via multipartite cooperation and those which cannot be distilled even if two or more parties cooperate. One could also say the PPT–bound states for any $n \geq 2$ are *bound–bound* entangled, i.e. PPT–bound and multi-particle–bound. Moreover, this feature is given for arbitrary dimensions d. In Fig. 1 we showed how the geometry of separability, PPT and entanglement changes with increasing dimension d. Last but not least our distillation protocol for $d = 3$ shows that almost all NPT–entangled states are two copy distillable to the vertex states and, consequently, the states are noise resistant. All these special features of these state spaces may help to develop novel applications and novel schemes for multipartite quantum communication.

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Mixed State Entanglement Measures for Intermediate Separability

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To determine whether a given multipartite quantum state is separable with respect to some partition we construct a family of entanglement measures $\{R_m(\rho)\}$. This is done utilizing generalized concurrences as building blocks which are defined by flipping of M constituents and indicate states that are separable with regard to bipartitions when vanishing. Further, we provide an analytically computable lower bound for $\{R_m(\rho)\}$ via a simple ordering relation of the convex roof extension. Using the derived lower bound, we illustrate the effect of the isotropic noise on a family of four-qubit mixed states for each intermediate separability.

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I. INTRODUCTION

Quantum entanglement plays a crucial role in foundations of quantum physics and is an indispensable ingredient for quantum information processing tasks [1]. In recent years it has become important to quantify the entanglement of quantum states, since not all entangled states are equally useful for quantum protocols. In particular, for multipartite systems it is of high interest to detect and quantify the entanglement not only of the whole system, but also between various constituting subsystems.

There are different approaches to multipartite entanglement quantification. A proposed measure, introduced in Ref. [2] by Meyer and Wallach, quantifies the global entanglement of the multipartite system, and vanishes for fully separable states only. Another approach, introduced by Love et. al. in Ref. [3], quantifies the amount of genuine multipartite entanglement and vanishes for any partially separable states. A different approach defines families of entanglement measures that quantifies the amount of entanglement also for intermediate or partial separability, as proposed in Refs. [4, 5, 6]. In Ref. [6], a family of entanglement measures for intermediate separability of pure states of n qubits, called R_m measures, has been introduced. This family includes the Meyer-Wallach measure and the Love measure as elements of the family. It manifests its usefulness by exhibiting a clear difference between the well-known multipartite GHZ and W states for systems of up to fifty qubits. To generalize the entanglement measures to mixed states, one uses the well-established convex roof extension. The measure is then defined as the weighted measure for pure states of the mixed states' decomposition, where one has to take the infimum over all possible decompositions. However, it is in general hard to calculate the convex roof, since mixed states allow infinitely many decompositions into pure states.

The aim of this paper is to derive lower bounds for the R_m measures for mixed states of n qudits. Lower bounds guarantee at least a certain value of entanglement that has to be present in the system. To do so, we utilize a method introduced in Ref. [4] and similarly in Ref. [7], and applied in Refs. [5, 8]. This method decomposes entanglement measures into sums of generalized concurrences (so-called M-concurrences), where for each M-concurrence a lower bound can be easily computed, and thus it can also be achieved for the entanglement measure. In the following we give the necessary definitions, show how to decompose the R_m measures into M-concurrences, and thus are able to derive a formula for a lower bound of the R_m measures for mixed states. We illustrate these results by instructive examples of four-partite states.

II. MEASURES AND THEIR LOWER BOUNDS

We consider an n-qudit system $\mathcal{H} = \bigotimes_{i=1}^{n} \mathcal{H}_i$ with constituent systems $\mathcal{H}_i = \mathbb{C}^d$ for all i. To specify how to focus on the total system, let us introduce the partition set $\Gamma := \{\gamma_j\}_{j=1}^{m}$, whose elements satisfy

$$\bigcup_{j=1}^{m} \gamma_j = \mathcal{N}, \quad \text{and} \quad \gamma_j \cap \gamma_k = \emptyset \quad \text{for} \quad j \neq k, \tag{1}$$

where $\mathcal{N} := \{1, 2, \cdots, n\}$ is the set of the labels of the constituents, and m is the total number of subsystems. We denote the complement of γ_j with regard to $\mathcal{N}$ by $\bar{\gamma}_j$ and the number of the elements of the (sub)set γ by $|\gamma|$ (see FIG. 1).

Let us start by defining generalized concurrences for multipartite states (for bipartite pure qudit states, related generalized concurrences were introduced in [9, 10]). For that purpose, let us introduce flip operators σ^i_{kl} with $k, l = 0, 1, \cdots, d-1$ that act on the i-th qudit and are defined as

$$\sigma^i_{kl} := |k\rangle\langle l| + |l\rangle\langle k| \,. \tag{2}$$

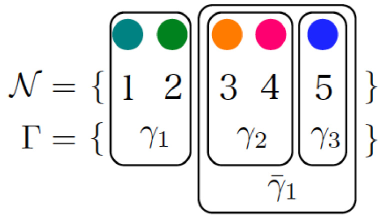

FIG. 1: Schematic diagram for the partition with $n = |\mathcal{N}| = 5, m = 3$, $|\gamma_1| = |\gamma_2| = 2$ and $|\gamma_3| = 1$.

Flip operators $f_\delta(\{k_i, l_i\}_{i\in\delta})$ with respect to the set of subsystems $\delta \subseteq \mathcal{N}$ are given by

$$f_\delta(\{k_i, l_i\}_{i\in\delta}) := \bigotimes_{i\in\delta} \sigma^i_{k_i l_i} \otimes \bigotimes_{j\in\bar{\delta}} \mathbb{1}^j, \qquad (3)$$

where $\mathbb{1}^i$ is the identity operator on $\mathcal{H}_i$. Since it is evident that the indices in the argument of f_δ run only within the elements of δ, hereafter we use the abbreviated notation $f_\delta(\{k_i, l_i\})$ for the flip operators. Using the notation $|\{j\}\rangle := |j_1\rangle \otimes |j_2\rangle \otimes \ldots \otimes |j_n\rangle$ with $\langle k|l\rangle = \delta_{kl}$ and $k, l = 0, 1, \cdots, d-1$ for states of the computational basis, we construct an operator

$$\begin{aligned} O_{\gamma,\delta}(\{k_i, l_i\}; \{j\}) &:= f_\delta(\{k_i, l_i\})|\{j\}\rangle\langle\{j\}| \\ &- f_\gamma(\{k_i, l_i\})|\{j\}\rangle\langle\{j\}|f_{\bar{\gamma}\cap\delta}(\{k_i, l_i\}) \end{aligned} \qquad (4)$$

by the help of two subsets γ, δ of $\mathcal{N}$ satisfying $\delta \subseteq \mathcal{N}$ and $\delta \cap \gamma = \emptyset$. In the following we use the abbreviated notation $O_{\gamma,\delta}$ if there is no possibility to cause confusions. The generalized (squared) concurrences (also called M-concurrences) of pure states $|\psi\rangle \in \mathcal{H}$ for the two subsets γ, δ are defined as

$$C^2_{\gamma,\delta}(\psi) := \sum_{i\in\delta} \sum_{k_i<l_i} \sum_{|\{j\}\rangle} |\langle\psi|O_{\gamma,\delta}|\psi^*\rangle|^2, \qquad (5)$$

which vanishes if and only if the state $|\psi\rangle$ is separable with respect to the bipartition $\{\gamma, \bar{\gamma}\}$. Here, $|\psi^*\rangle$ denotes the complex conjugated state to $|\psi\rangle$. Note that the Hill-Wootters concurrence [11] for pure states is reproduced for $n = 2$ and $d = 2$ (two qubits).

We define the generalized (squared) concurrences for mixed states via the convex roof,

$$C^2_{\gamma,\delta}(\rho) := \inf_{\{p_\alpha, \psi_\alpha\}} \sum_\alpha p_\alpha C^2_{\gamma,\delta}(\psi_\alpha)\,, \qquad (6)$$

where the infimum is taken over all possible decompositions of the given density matrix $\rho = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|$ into a probability distribution $\{p_\alpha\}$ and pure states $|\psi_\alpha\rangle$. Although it is in general hard to evaluate the convex roof extension, we can explicitly determine a lower bound for the generalized concurrences by

$$C^2_{\gamma,\delta}(\rho) \geq \Lambda^2_{\gamma,\delta}(\rho) \qquad (7)$$

with

$$\Lambda_{\gamma,\delta}(\rho) := \max\left\{0, \sum_{O_{\gamma,\delta}} \left(2\lambda(O_{\gamma,\delta}) - \mathrm{Tr}\sqrt{\rho\tilde{\rho}(O_{\gamma,\delta})}\right)\right\}, \qquad (8)$$

where the summation is taken under the same condition as in Eq. (5) and $\lambda(O_{\gamma,\delta})$ is the largest eigenvalue of $\sqrt{\rho\tilde{\rho}(O_{\gamma,\delta})}$ with

$$\tilde{\rho}(O_{\gamma,\delta}) := (O_{\gamma,\delta} + O^\dagger_{\gamma,\delta})\rho^*(O_{\gamma,\delta} + O^\dagger_{\gamma,\delta}). \qquad (9)$$

This lower bound will be helpful in the following derivation of the lower bound for the R_m measures.

Summing up all generalized concurrences for subsets δ, we can define an entanglement measure for the set γ of constituents,

$$\eta_\gamma(\psi) := \sum_{\delta\subseteq\mathcal{N}} C^2_{\gamma,\delta}(\psi), \qquad (10)$$

where the sum is restricted by $\delta \cap \gamma = \emptyset$. The relation to the linear entropy of the reduced density matrix $\rho_\gamma := \mathrm{Tr}_{\bar{\gamma}}|\psi\rangle\langle\psi|$ of the set γ of the constituents has been established in [4, 5]:

$$\eta_\gamma(\psi) = N(|\gamma|)\left(1 - \mathrm{Tr}\rho^2_\gamma\right), \qquad (11)$$

where $N(|\gamma|) := d^{|\gamma|}/(d^{|\gamma|} - 1)$ is a normalization factor in order to obtain $\eta_\gamma(\psi) = 1$ if ρ_γ is the maximally mixed state.

Let us generalize the measure $\eta_\gamma(\psi)$ (10) to measures for particular partitions of the n-qudit system. To do so, we rewrite $\gamma \subseteq \mathcal{N}$ as γ_i, such that it can be regarded as an element of general partition $\Gamma = \{\gamma_i\}^m_{i=1}$. Taking the arithmetic average of $\eta_{\gamma_i}(\psi)$ for all i, we define the following entanglement measures with regard to a particular partition Γ:

$$\xi_\Gamma(\psi) := \frac{1}{m}\sum_{i=1}^m \eta_{\gamma_i}(\psi). \qquad (12)$$

Furthermore, taking the geometric average of ξ_Γ for all possible partitions under the condition that the number of the elements of the partitions m is fixed, we obtain the family of entanglement measures $R_m(\psi)$ for intermediate separability,

$$R_m(\psi) := \left(\prod_{|\Gamma|=m} \xi_\Gamma(\psi)\right)^{1/S(n,m)}, \qquad (13)$$

where

$$S(n,m) := \sum_{k=1}^m \frac{(-)^{m-k}k^{n-1}}{(k-1)!(m-k)!} \qquad (14)$$

is the Stirling number in the second kind, representing the number of subsystem combinations that result in m partitions.

In order to generalize $R_m(\psi)$ to mixed states, we take the convex roof of $\xi_\gamma(\psi)$,

$$\xi_\Gamma(\rho) := \inf_{\{p_\alpha,\psi_\alpha\}} \sum_\alpha p_\alpha \xi_\Gamma(\psi_\alpha), \tag{15}$$

and define $R_m(\rho)$ for mixed states as a quantity obtained by taking the geometric average of $\xi_\Gamma(\rho)$ for all possible partitions with the fixed number of subsystems.

For that purpose, we have to prove the following lemma:

Lemma 1 *Suppose that there exist pure state entanglement measures $\mu_s(\psi)$, labeled by the index s. Then, the convex roof of the sum of them is no less than the sum of the convex roofs of each measure, i.e.*

$$\mu(\rho) \geq \sum_s \mu_s(\rho) \tag{16}$$

where

$$\mu(\rho) := \inf_{\{p_\alpha,\psi_\alpha\}} \sum_\alpha p_\alpha \sum_s \mu_s(\psi_\alpha) \tag{17}$$

and

$$\mu_s(\rho) := \inf_{\{p_\alpha,\psi_\alpha\}} \sum_\alpha p_\alpha \mu_s(\psi_\alpha). \tag{18}$$

Proof. Suppose that the decomposition of the given mixed state ρ which yields $\mu(\rho)$ is given by $\rho = \sum_\alpha p'_\alpha |\psi'_\alpha\rangle\langle\psi'_\alpha|$. Then, starting from Eq. (17), we get

$$\mu(\rho) = \sum_{s,\alpha} p'_\alpha \mu_s(\psi'_\alpha) = \sum_s \sum_\alpha p'_\alpha \mu(\psi'_\alpha) \geq \sum_s \mu_s(\rho) \tag{19}$$

where we have to use "$\geq$" since the decomposition $\{p'_\alpha, |\psi'_\alpha\rangle\}$ does not necessarily yield the infimum of $\sum_\alpha p_\alpha \mu_s(\psi_\alpha)$ for all s. ■

Applying Lemma 1 to Eq. (15), we obtain a lower bound of $\xi_\Gamma(\rho)$:

$$\xi_\Gamma(\rho) \geq \frac{1}{m} \sum_{i=1}^m \eta_{\gamma_i}(\rho), \tag{20}$$

where

$$\eta_{\gamma_i}(\rho) := \inf_{\{p_\alpha,\psi_\alpha\}} \sum_\alpha p_\alpha \eta_{\gamma_i}(\psi_\alpha). \tag{21}$$

Furthermore, utilizing (10), Lemma 1, (6), and Ineq. (7) for Eq. (21), we find

$$\begin{aligned} \eta_{\gamma_i}(\rho) &= \inf_{\{p_\alpha,\psi_\alpha\}} \sum_\alpha p_\alpha \eta_{\gamma_i}(\psi_\alpha) \\ &= \inf_{\{p_\alpha,\psi_\alpha\}} \sum_\alpha p_\alpha \sum_\delta C^2_{\gamma_i,\delta}(\psi_\alpha) \\ &\geq \sum_\delta C^2_{\gamma_i,\delta}(\rho) \geq \sum_\delta \Lambda^2_{\gamma_i,\delta}(\rho). \end{aligned} \tag{22}$$

Representative partition	Equivalent partitions
$\{\{1\},\{2,3,4\}\}$	$\{\{2\},\{1,3,4\}\}$
$\{\{3\},\{1,2,4\}\}$	$\{\{4\},\{1,2,3\}\}$
$\{\{1,3\},\{2,4\}\}$	$\{\{1,4\},\{2,3\}\}$
$\{\{1,2\},\{3,4\}\}$	

Representative partition	Equivalent partitions
$\{\{1\},\{2\},\{3,4\}\}$	
$\{\{3\},\{4\},\{1,2\}\}$	
$\{\{1\},\{3\},\{2,4\}\}$	$\{\{1\},\{4\},\{2,3\}\}$
	$\{\{2\},\{3\},\{1,4\}\}$
	$\{\{2\},\{4\},\{1,3\}\}$

TABLE I: Classification of the partitions of four-partite systems. The equivalent partitions can be mapped into the representative partition in the same line by the actions of the elements of V. (Above) The classification of the bipartitions. (Below) The classification of the tripartitions.

Thus, a computable lower bound of $R_m(\rho)$ is given by

$$R_m(\rho) \geq \tilde{R}_m(\rho)\,, \tag{23}$$

where

$$\tilde{R}_m(\rho) := \frac{1}{m} \left(\prod_{|\Gamma|=m} \sum_{i=1}^m \sum_{\delta \subseteq \mathcal{N}} \Lambda^2_{\gamma_i,\delta}(\rho) \right)^{1/S(n,m)}, \tag{24}$$

and the second sum is again conditioned by $\delta \cap \gamma_i = \emptyset$ for each i.

III. EXAMPLE

As an example of an explicit calculation of the lower bound formula, let us consider a family of four-qubit mixed states on $(\mathbb{C}^2)^{\otimes 4}$

$$\rho = p_1 P^+_{12} \otimes P^+_{34} + p_2 P^{\mathrm{GHZ}} + \frac{1-p_1-p_2}{16} \bigotimes_{i=1}^4 \mathbb{1}^i. \tag{25}$$

Here, P^+_{ij} is that onto $|\phi^+\rangle_{ij}$, one of Bell bases spanning $\mathcal{H}_i \otimes \mathcal{H}_j$, that is, $|\phi^+\rangle_{ij} := (|00\rangle_{ij} + |11\rangle_{ij})/\sqrt{2}$, P^{GHZ} is the projector onto the GHZ state $|\mathrm{GHZ}\rangle := (|0000\rangle + |1111\rangle)/\sqrt{2}$, and $0 \leq p_1 + p_2 \leq 1$, with $p_1, p_2 \geq 0$. Note that the first and second terms in Eq. (25) can be produced from the second order non-linear effect of a $\beta - \mathrm{BaB_3O_6}$ (BBO) crystal [12, 13], respectively. Since the third term can be regarded as the isotropic noise, we expect that due to the quantification of entanglement of the state, we can see not only a variety of entanglement produced by the BBO crystal, but also how much the noise affects the entanglement in the system.

The state in Eq. (25) clearly lacks the symmetry under actions of S_4 for the labels of the constituents, which

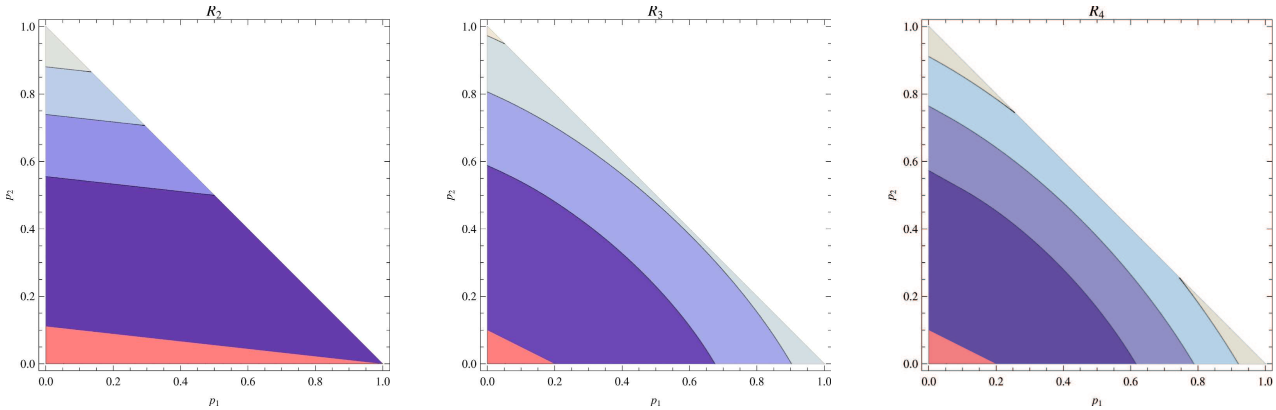

FIG. 2: Contour plots of the values of the lower bounds $\tilde{R}_m$ of R_m measures for the state (25). (Left) R_2 measure. (Center) R_3 measure. (Right) R_4 measure. Each colored area denotes the region where the lower bound of the measure has the specific range (red: $\tilde{R}_m = 0$, dark purple: $0 < \tilde{R}_m \leq 0.25$, bright purple: $0.25 < \tilde{R}_m \leq 0.5$, blue: $0.5 < \tilde{R}_m \leq 0.75$, ash: $0.75 < \tilde{R}_m \leq 1$).

is due to the first term in the summand. However, we easily see that it still holds a symmetry under actions of four elements of S_4, i.e. the identity operation e, two transpositions $(1,2)$, $(3,4)$ and their consecutive operation $(1,2)(3,4)$. We can find that these four elements constitute a subgroup of S_4, which has been known as Vierergruppe V [14]. Hence, ρ is invariant under the actions of V. Such symmetry has a relevant role for the reduction of computational complexity. For example, the number of the bipartitions of four-partite systems, $S(4,2) = 7$, effectively reduces to 4. By the same way, that of tripartitions, $S(4,3) = 6$, reduces to 3 (see TABLE I).

The amount of entanglement in the state (25) is visualized in FIG. 2. Notice that the area with $R_m = 0$ with larger m is included in the same area with smaller m. This reflects the fact that the lower bound $\tilde{R}_m(\rho)$ captures the property that a m-separable state can be regarded as a m'-separable state with $m \geq m'$. To analyze these graphs in more detail, it is convenient to introduce two variables

$$q := 1 - p_1 - p_2 \quad \text{and} \quad r := \frac{p_2}{p_1}. \tag{26}$$

The former variable q corresponds to the degree of the noise, while the latter r characterizes the original noiseless state which has been altered into the state specified by the coordinates (p_1, p_2) due to the presence of noise.

Keeping q fixed and varying r, let us observe the variety of entanglement under the fixed noise situation. We can immediately see that the R_2 measure decreases monotonically as r decreases, while the others behave differently. Since the smaller value of r implies that the ratio of the bi-separable state $P^+_{12} \otimes P^+_{34}$ in ρ becomes larger, the preceding observation means that by the addition of the biseparable state, the state approaches the biseparable state monotonically, while the state does not approach the tri-separable or four-separable state. This comes from the fact that $P^+_{12} \otimes P^+_{34}$ in ρ is a genuinely biseparable state, and cannot be regarded as a tri-separable or four-separable state. On the other hand, varying q and fixing r, we see that all graphs share a common behavior: the monotonic approach to $R_m = 0$ for all m by the addition of the noise. This is due to the fact that the noise $\left(\bigotimes_i \mathbb{1}^i\right)/16$ can be interpreted as a separable state for any partition. From these observations, we may conclude that the lower bound $\tilde{R}_m(\rho)$ derived in this letter captures the natural behavior of the multipartite entanglement suitably.

IV. SUMMARY

In this letter, starting from the m-concurrences, we systematically derived the computable lower bound of the family of the entanglement measures $\{R_m(\rho)\}_{m=2}^n$ by utilizing Lemma 1, which manifests the non-commutativity of the convex roof extention and summations of entanglement measures. As a testing ground of the derived lower bound, we examined the amount of the entanglement of the state (25) and showed that the resultant graphs are explained by the natural behavior of the system in question. Thus, this example confirms the consistency of the lower bound and is useful for a finer analysis of entanglement.

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Detection of high-dimensional genuine multi-partite entanglement of mixed states

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We derive a general framework to identify genuinely multipartite entangled mixed quantum states in arbitrary-dimensional systems and show on exemplary cases that the constructed criteria are stronger than previously known ones. Our criteria are simple functions of the given quantum state and detect genuine multi-partite entanglement that had not been identified so far. They are easily computable as no optimization or eigenvalue evaluation is needed.

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Many-particle entanglement is a striking feature of quantum many-body systems. Entanglement was first recognized as a curiosity of quantum mechanics because it gives rise to seemingly non-local correlations of measurement results of distant observers. Whereas the central role of many-body entanglement for various applications of quantum information processing (e.g. [1]) is undoubted, its role in *e.g.* quantum phase transitions (e.g. [2]) or ionization processes is still debated (e.g. [3]), and questions concerning *e.g.* its potential assistance to the astonishing transport efficiency of biological compounds (e.g. [4]) are still essentially open.

To answer such questions we need reliable techniques to characterize entanglement properties of general quantum states. However, even the conceptually rather simple question 'Is a given quantum state entangled or not?' is in general unanswered so far. It is usually addressed by means of separability criteria, which work very well in many cases, but are far from perfect [5]. Even more challenging is the detection of genuine multipartite entanglement, which has already been intensely studied (see for example [6, 7, 8, 9]), but still has not yielded satisfying results. Vast areas of the considered state-spaces are still widely unexplored due to the lack of suitable tools for detecting and characterizing entanglement. The central difficulty arises from the complicated structure of multipartite entangled states: even states that do not separate into blocks of subsystems that are *not* entangled with each other are not necessarily genuinely n-body entangled. Recently, inequalities to identify genuinely n-body entangled states have been proposed based on non-linear functions of matrix-elements [10]. Although these new criteria are promising in the sense that they allow to characterize states as entangled that can not be detected with the standard criteria, it is also evident that the characterization of entangled states will not be facilitated by a huge set of separability criteria unless we have a systematic way to construct and understand these criteria. Here, we present a very general, systematic approach to construct such criteria, and show that our newly constructed criteria are stronger than all formerly known ones. In particular, all these criteria apply to systems of arbitrarily many subsystems of arbitrary finite dimensions. In more detail, we derive

- a m-linear inequality (3) and its bilinear version ineq.(I) to detect bipartite entanglement. Based on this, we derive
- a general framework to obtain bilinear inequalities (II) which characterize genuine multipartite entanglement and
- construct a particularly strong criterion, i.e. ineq. (III), for which the efficiency is demonstrated in the consecutive examples.

A pure n-partite state $|\Psi\rangle$ is called *k–separable* if it can be written as a product [5]

$$|\Psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_k\rangle \ , \qquad (1)$$

of k states $|\phi_i\rangle$ each of which corresponds to a single subsystem or a group of subsystems. If there is no such form with at least two factors, then $|\Psi\rangle$ is considered genuinely n-partite entangled. On the level of pure states the question of k-separability can be answered in a straight forward fashion by means of separability criteria for bipartite systems, simply by considering all segmentations of the k-partite system into two parts. However, the same question becomes significantly more difficult to answer for mixed states ϱ: here, a state is considered genuinely k-partite entangled if *any* decomposition into pure states

$$\varrho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \ , \qquad (2)$$

with probabilities $p_i > 0$ contains at least one genuinely k-partite entangled component. Therefore, a mixed state can still be partially separable, even if the k subsystems can not be split into two groups that are not entangled with each other. Consider for instance the tri-partite state

$$\rho_{bisep} = \sum_j p_j \rho^j_{AB} \otimes \rho^j_C + \sum_j q_j \rho^j_{AC} \otimes \rho^j_B + \sum_j r_j \rho^j_{BC} \otimes \rho^j_A \ ,$$

Here the two-body states ρ^j_{AB}, ρ^j_{BC} and ρ^j_{AC} describe entangled states. Even though there is no bipartite splitting with respect to which the state ρ is separable, it is considered biseparable since it can be prepared through a statistical mixture of bipartite entangled states.

To be certain that some state is really genuinely n-body entangled, we thus have to make sure that there is no pure state decomposition with only at least partially entangled components. Since this reduces to the problem of deciding whether each of such pure state components is at least biseparable, let us first introduce a suitable criterion for biseparability, which then will turn out to be the central building block for the subsequent generalization to genuine many-body entanglement. What we employ here, are m-linear functions of a quantum state ϱ on $\mathcal{H}_A \otimes \mathcal{H}_B$ that can be expressed in terms of the m-fold tensor product $\varrho^{\otimes m}$ of the density matrix ϱ acting on the m-fold tensor product space $(\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes m}$. As it is shown at the end of our letter, any separable state ϱ_s satisfies

$$\sqrt{\Re e(\langle\Phi|(\mathbb{1} \otimes \Pi_B)^\dagger \varrho_s^{\otimes m} (\Pi_A \otimes \mathbb{1})|\Phi\rangle)} \leq \sqrt{\langle\Phi|\varrho_s^{\otimes m}|\Phi\rangle} \ , \quad (3)$$

for any positive integer m, where $|\Phi\rangle$ is any fully separable state of the m-tupled system, *i.e.* $|\Phi\rangle$ factorizes into $2m$ single-body states. Π_A is the cyclic permutation operator acting on $\mathcal{H}_A^{\otimes m}$, *i.e.*

$$\Pi_A|\varphi_1\rangle \otimes |\varphi_2\rangle \otimes \ldots \otimes |\varphi_m\rangle = |\varphi_2\rangle \otimes |\varphi_3\rangle \otimes \ldots \otimes |\varphi_m\rangle \otimes |\varphi_1\rangle \ , \quad (4)$$

and Π_B is defined analogously for subsystem B. In our following extension to multipartite systems, we will content ourselves with the bilinear case $m = 2$, as it is already very powerful in detecting entanglement and ineq. (3) takes the rather simple form

$$|\langle il|\rho|kj\rangle| - \sqrt{\langle ij|\rho|ij\rangle\langle kl|\rho|kl\rangle} \leq 0 \ , \quad \text{(I)}$$

which corresponds to the choice $|\Phi\rangle = |ijkl\rangle$.
For our following generalization of ineq. (3) to the multipartite case we will consider all $(2^{n-1} - 1)$ different partitions of an n-partite systems into two subsystems, because a mixed state is biseparable exactly if there is a decomposition into pure states each of which is separable with respect to some partition. The fictitious subsystems will be labeled A_i and B_i ($i = 1, \ldots, 2^{n-1} - 1$) in the following. Introducing the *global* permutation operator $\mathbf{\Pi}$ which performs simultaneous permutations on all subsystems, we can formulate now the generalization of ineq. (3) to multipartite systems:

$$\sqrt{\langle\Phi|\rho^{\otimes 2}\mathbf{\Pi}|\Phi\rangle} - \sum_i \sqrt{\langle\Phi|\mathcal{P}_i^\dagger \rho^{\otimes 2} \mathcal{P}_i|\Phi\rangle} \leq 0 \ , \quad \text{(II)}$$

with $\mathcal{P}_i = \Pi_{A_i} \otimes \mathbb{1}_{B_i}$, and where the sum runs over all inequivalent bipartitions.
To convince ourselves that ineq. (II) is indeed satisfied by all at least partially separable states ρ, let us first verify that this holds for any *pure* state $\rho_\Psi = |\Psi_{bs}\rangle\langle\Psi_{bs}|$ that is biseparable with respect some partition labeled i_0. Just like any duplicated state $|\Psi\rangle^{\otimes 2}$ is invariant under the global permutation $\mathbf{\Pi}$, the duplicated state $|\Psi_{bs}\rangle^{\otimes 2}$ is invariant under $\Pi_{A_{i_0}} \otimes \mathbb{1}_{B_{i_0}}$. Therefore, the first term in ineq. (II) cancels with the $i = i_0$ term in the summation. All remaining terms are expectation values of positive operators, and given the negative sign in front of the sum, the left-hand-side is indeed non-positive. Hence, ineq. (II) is satisfied for any pure state that is not genuinely multipartite entangled.

The generalization of ineq. (II) to mixed states is a direct consequence of its convexity which we can see in the following, where we will use that the state $|\Phi\rangle$ is completely separable. That is, independently of which decomposition of the Hilbert space into two subspaces we take, we can always write it as a direct product of two states $|\Phi_1\rangle$ and $|\Phi_2\rangle$ of the respective subspaces. The first term in ineq. (II) is the absolute value of the matrix element $\langle\Phi_1|\rho|\Phi_2\rangle$:

$$\sqrt{\langle\Phi|\rho^{\otimes 2}\mathbf{\Pi}|\Phi\rangle} = |\langle\Phi_1|\rho|\Phi_2\rangle| \ , \quad (5)$$

since $\mathbf{\Pi}$ simply permutes $|\Phi_1\rangle$ and $|\Phi_2\rangle$, *i.e.* $\mathbf{\Pi}|\Phi_1\rangle \otimes |\Phi_2\rangle = |\Phi_2\rangle \otimes |\Phi_1\rangle$. And the absolute value is convex, *i.e.* $|a + b| \leq |a| + |b|$ for arbitrary complex numbers a and b. Each summand $\mathcal{K}_i = \sqrt{\langle\Phi|\mathcal{P}_i^\dagger \rho^{\otimes 2} \mathcal{P}_i|\Phi\rangle}$ in the second term of ineq. (II) is the square root of a product of two diagonal density matrix elements, *i.e.* non-negative numbers

$$\mathcal{K}_i = \sqrt{\langle\widetilde{\Phi}_1|\rho|\widetilde{\Phi}_1\rangle\langle\widetilde{\Phi}_2|\rho|\widetilde{\Phi}_2\rangle} \ , \quad (6)$$

with $|\widetilde{\Phi}_1\rangle \otimes |\widetilde{\Phi}_2\rangle = \Pi_{A_i} \otimes \mathbb{1}^{B_i}|\Phi\rangle$. Now, Cauchy-Schwarz's inequality $\sum_j p_j q_j \geq \sqrt{\sum_j p_j^2}\sqrt{\sum_j q_j^2}$ with $p_j = \sqrt{\langle\widetilde{\Phi}_1|\rho_j|\widetilde{\Phi}_1\rangle}$ and $q_j = \sqrt{\langle\widetilde{\Phi}_2|\rho_j|\widetilde{\Phi}_2\rangle}$ yields

$$\mathcal{K}_i \leq \sum_j \sqrt{\langle\widetilde{\Phi}_1|\rho_j|\widetilde{\Phi}_1\rangle\langle\widetilde{\Phi}_2|\rho_j|\widetilde{\Phi}_2\rangle} \ , \quad (7)$$

for any set of positive operators ρ_j satisfying $\rho = \sum_j \rho_j$. Therefore, eq. (6) is a concave quantity, so that ineq. (II) is indeed convex. Since, as shown above, it is satisfied for all biseparable *pure* states, this implies the same also for *mixed* states.

Ineq. (II) is valid for any choice of a completely separable pure state-vector $|\Phi\rangle$, but the potential to detect the genuine multipartite character of a given entangled state will depend on a suitable choice of $|\Phi\rangle$. To ensure that two different states that are connected to each other by local unitary transformations, and, therefore,

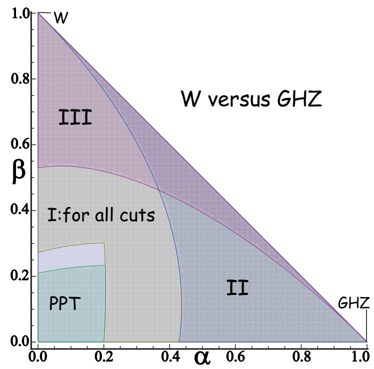

FIG. 1: Here the detection quality of the bilinear inequalities (I), (II) and (III) is shown for the state $\rho = \frac{1-\alpha-\beta}{8}\mathbb{1} + \alpha\rho_{GHZ} + \beta\rho_W$. Area II contains genuine multipartite entanglement detected by (II). Area III contains genuine multipartite entanglement detected by (III). Area I is not biseparable w.r.t. any bipartition, since it violates inequality (I) for all partitions. The area labeled PPT constitutes all states not detected by the Peres-Horodecki criterion [5].

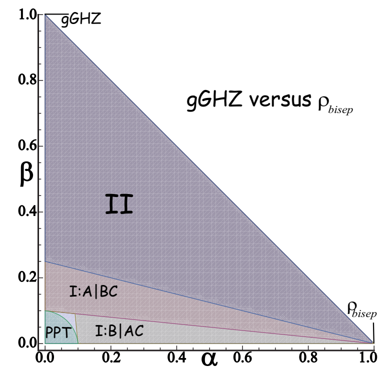

FIG. 2: Here the detection quality of the bilinear inequalities (I) and (II) is shown for the tripartite qutrit state (with subsystems labeled ABC) $\rho = \frac{1-\alpha-\beta}{27}\mathbb{1} + \alpha\rho_{bisep} + \beta\rho_{gGHZ}$. Area II contains genuine multipartite entanglement, since it violates inequality (II), area I:$B|AC$ is not biseparable w.r.t. $B|AC$, since it violates inequality (I) for this partition (the result for $AB|C$ is equivalent). Area I:$A|BC$ contains states that violate inequality (I) for bipartition $A|BC$. The area labeled PPT constitutes all states not detected by the Peres-Horodecki criterion [5].

have equivalent entanglement properties, are characterized equivalently by ineq. (II) it is desirable find the state-vector $|\Phi\rangle$ for any given density matrix ρ that yields the maximum violation of ineq. (II). Whereas, such an optimization can be rather intricate in general, the method derived in Ref. [12] facilitates such a task significantly. Besides such an optimization, one can combine different choices of states $|\Phi\rangle$ to tailor criteria that are suited particularly well for a specific class of states, as we demonstrate here with the exemplary choice of $|\Phi_{ij}\rangle = |s_i\rangle\otimes|s_j\rangle$ with $|s_i\rangle = |x\dots xyx\dots x\rangle$ in terms of two single-particle states $|x\rangle$ and $|y\rangle$, and $|y\rangle$ is chosen exactly for the i-th entry of $|s_i\rangle$. Taking linear combinations of ineq. (II) for these choices we arrive at

$$\sum_{i\neq j}\sqrt{\langle\Phi_{ij}|\rho^{\otimes 2}\mathbf{\Pi}|\Phi_{ij}\rangle} - (n-2)\sum_{ij}\sqrt{\langle\Phi_{ij}|\mathcal{P}_i^\dagger\rho^{\otimes 2}\mathcal{P}_i|\Phi_{ij}\rangle} \leq 0 \tag{III}$$

where $\mathcal{P}_i = \Pi_{A_i}\otimes\mathbb{1}_{B_i}$ is defined analogously to above. However, in contrast to above, not all bipartitions are taken into account, but A_i is the duplicated Hilbert space of the i-th subsystem and B_i the rest. Exactly as in ineq. (II) also the left-hand-side in ineq. (III) is convex, so that the inequality is proven for biseparable *mixed* states, since it is proven for biseparable *pure* states in the end of the paper. Now that we have derived our criteria let us introduce a few examples.

Example 1. First consider the three qubit state $\rho = \frac{1-\alpha-\beta}{8}\mathbb{1} + \alpha\,\rho_{GHZ} + \beta\,\rho_W$ where $\rho_{GHZ} = \frac{1}{2}(|000\rangle + |111\rangle)(\langle 000| + \langle 111|)$ and $\rho_W = \frac{1}{3}(|001\rangle + |010\rangle + |100\rangle)(\langle 001| + \langle 010| + \langle 100|)$. It is a mixture between the GHZ-state and the W-state dampened by isotropic noise. In Fig. 1 the detection parameter spaces of the inequalities (I), (II) and (III) are illustrated. In the case of genuine multipartite entanglement detection for qubits, these criteria work as well as the best known method so far. For example in Ref. [13] the above state for ($\alpha = 0$ and $\beta = 1-p$) was found to be genuinely multipartite entangled by means of entanglement witnesses up to a threshold of $p < 8/19$. This bound was then improved to $p < 8/17$ [14], which is also our result. For qudits, our criteria are the first detection criteria known so far.

Example 2. Consider the three qutrit state $\rho = \frac{1-\alpha-\beta}{27}\mathbb{1} + \alpha\rho_{bisep} + \beta\rho_{gGHZ}$ where $\rho_{gGHZ} = \frac{1}{3}(|000\rangle + |111\rangle + |222\rangle)(\langle 000| + \langle 111| + \langle 222|)$ and $\rho_{bisep} = \frac{1}{2}(|0\rangle\langle 0|\otimes(|00\rangle + |11\rangle + |22\rangle)(\langle 00| + \langle 11| + \langle 22|)$. It is a mixture between a generalized GHZ-state for qutrits and a biseparable qutrit state dampened by isotropic noise. In Fig. 2 the detection parameter spaces of the violation of the inequalities (I) and (II) are illustrated.

Example 3. Now consider the following four qudit state:

$$\rho_S = \frac{1-\alpha-\beta}{d^4}\mathbb{1} + \alpha(\sum_i \frac{1}{d}\rho^i_{gGHZ1}) + \beta(\sum_i \frac{1}{d}\rho^i_{gGHZ2}) \tag{8}$$

where $\rho^i_{gGHZx} := |gGHZx(i)\rangle\langle gGHZx(i)|$ with $|gGHZx(i)\rangle := \sum_k \frac{1}{\sqrt{d}}|k\rangle|k\oplus x\rangle|k\oplus i\rangle|k\oplus i\oplus x\rangle$. Where $\oplus$ is the addition modulo d. For $d=2$ and $\alpha=\beta$ this is the bound entangled Smolin state (see Ref. [15]) dampened by isotropic noise. Also in this case our criteria work well. Ineq. (3) detects all states in the region $1-(d^2+1)\alpha-\beta < 0$ and $1-\alpha-(d^2+1)\beta < 0$ which for $d=2$ constitutes all entangled states and for higher dimensions is as good as any other known criterion. Moreover ineq. (II) shows that the state only becomes biseparable outside the region not detected by ineq. (3) for $\alpha = \beta > 1/(d^2+2)$, i.e. all entangled states in this region except for the line $\alpha = \beta$ are definitely multipartite entangled. This is in good correspondence to the fact that the Smolin state is biseparable.

Example 4 Consider the most general states, which maximize multipartite entanglement for n-partite systems in d-dimensions. They can be written explicitly as:

$$|\psi_{mme}\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle^{\otimes n} \tag{9}$$

If we consider them dampened by isotropic (white) noise:

$$\rho = p\rho_{mme} + (1-p)\frac{1}{d^n}\mathbb{1} \tag{10}$$

By application of inequality (II) we can show analytically that these states are genuinely multipartite entangled for $p > \frac{3}{d^{n-1}+3}$, which shows that even in systems involving many parties and being very high dimensional these criteria work very well.

In conclusion ineq. (III) is only one specific of many possible criteria derived from ineq. (II) and the versatility of our approach allows to tailor many criteria suited for specific classes of entangled states. Given the extremely simple algebraic form of all the present criteria, they can be employed for far more applications than mere entanglement detection, as for example dynamical control of entanglement, and also the construction of quantitative estimates of genuine many-body entanglement based on our presented estimates is not out of reach.

Finally, let us prove ineqs. (3) and (III). For the former, we have to show

$$\langle\Phi|\varrho_s^{\otimes m}|\Phi\rangle \geq \frac{1}{2}(\langle\Phi|\mathcal{P}_A^\dagger \varrho_s^{\otimes m}\mathcal{P}_B|\Phi\rangle + \langle\Phi|\mathcal{P}_B^\dagger \varrho_s^{\otimes m}\mathcal{P}_A|\Phi\rangle)$$

for any separable mixed state $\varrho_s = \sum_i |\varphi_i\rangle\langle\varphi_i| \otimes |\chi_i\rangle\langle\chi_i|$ and any completely separable state-vector $|\Phi\rangle = \bigotimes_{i=1}^m |\alpha_i\rangle \otimes \bigotimes_{i=1}^m |\beta_i\rangle$. This amounts to showing

$$\vec{X}^*\vec{X} - \frac{1}{2}\left(\Pi_A\vec{X}\right)^*\left(\Pi_B\vec{X}\right) - \frac{1}{2}\left(\Pi_B\vec{X}\right)^*\left(\Pi_A\vec{X}\right) \geq 0 \ , \tag{11}$$

with $[\vec{X}]_{p_1\ldots p_m q_1\ldots q_n} = \prod_{i=1}^m \langle\alpha_i|\varphi_{p_i}\rangle \prod_{i=1}^m \langle\beta_i|\chi_{q_i}\rangle$, $[\Pi_A\vec{X}]_{p_1\ldots p_m q_1\ldots q_n} = \prod_{i=1}^m \langle\alpha_i|\varphi_{p_{i+1}\text{mod}_m}\rangle \prod_{i=1}^m \langle\beta_i|\chi_{q_i}\rangle$, $[\Pi_B\vec{X}]_{p_1\ldots p_m q_1\ldots q_n} = \prod_{i=1}^m \langle\alpha_i|\varphi_{p_i}\rangle \prod_{i=1}^m \langle\beta_i|\chi_{q_{i+1}\text{mod}_m}\rangle$. Since $\left(\Pi_{A/B}\vec{X}\right)^*\left(\Pi_{A/B}\vec{X}\right) = \vec{X}^*\vec{X}$, ineq. (11) simplifies to $\frac{1}{2}\left|\Pi_A\vec{X} - \Pi_B\vec{X}\right|^2 \geq 0$, which proves ineq.(3).

To prove ineq. (III) we have to verify that it is satisfied for all biseparable pure states $|\Psi\rangle$, since it is convex as shown for ineq. (II). With the short hand notation $x_{ij} = \sqrt{\langle\Phi_{ij}|\rho_\Psi^{\otimes 2}\mathbf{\Pi}|\Phi_{ij}\rangle}$ and $y_{ij} = \sqrt{\langle\Phi_{ij}|\mathcal{P}_i^\dagger \rho_\Psi^{\otimes 2}\mathcal{P}_i|\Phi_{ij}\rangle}$, ineq. (III) reads $\sum_{i\neq j} x_{ij} - (n-2)\sum_{ij} y_{ij}$. We will have to distinguish between the cases in which both indices i and j correspond to different, or the same parts A and B in the bipartition with respect to which $|\Psi\rangle$ (without loss of generality we assume i to correspond to A). The former contributions to ineq. (III) we denote as $B_d = \sum_{i,j\in B}(x_{ij} - (n-2)y_{ij})$, the latter as $B_s = \sum_{i\neq j\in A}(x_{ij} - (n-2)y_{ij}) - (n-2)\sum_i y_{ii}$, so that ineq. (III) reads $B_s + B_d \leq 0$. B_d is non-positive since $x_{ij} \leq y_{ij}$ as shown for ineq. (II). Since the y_{ij} are non-negative, we obtain $B_s \leq \sum_{i\neq j\in s} x_{ij} - (n-2)\sum_i y_{ii} = \sum_{i\neq j\in s}(x_{ij} - z_i y_{ii}) \leq \sum_{i\neq j\in s}(x_{ij} - y_{ii})$ with $z_i = (n-2)/(n_i-1)$, where n_i is the number of subsystems in A, where $z_i \geq 1$ since A can comprise at maximum $n-1$ subsystems. Now, we can symmetrize the last term in $\sum_{i\neq j\in s}(x_{ij} - y_{ii})$, *i.e.* rewrite it as $\sum_{i\neq j\in s}(x_{ij} - 1/2(y_{ii} + y_{jj}))$. Since $y_{ii} = \langle s_i|\varrho_\Psi|s_i\rangle$ (due to the relation $\mathcal{P}_i|\Phi_{ii}\rangle = |\Phi_{ii}\rangle$), we can conclude $x_{ij} = |\langle s_i|\varrho_\Psi|s_i\rangle| \leq 1/2(y_{ii} + y_{jj})$), such that B_s is non-negative, what finishes the proof of ineq. (III).

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Abstract

Entanglement is at the heart of quantum theory. It has no classical counterpart and has therefore been subject of a controversial debate for almost a century. What has remained a purely philosophical debate for many decades has now sparked a whole new field in physics. Quantum information theory takes the concept of entanglement in nature seriously and has since offered novel applications such as e.g. quantum cryptography and quantum computing. For these practical applications entanglement is a crucial resource, which enables the quantum informational tasks to outperform any classical counterpart. However it is far from well understood. Even seemingly simple problems, such as the mathematical distinction between entangled states and separable ones, remain unsolved to date.

In this work we intensively investigate the mathematical theory of entanglement. We were able to develop techniques to further improve the detection of entangled states, especially in multipartite systems of arbitrary dimension. Also we were able to provide general sets of entanglement measures, which not only quantify bipartite entanglement, but also give a consistent quantification of entanglement in multipartite systems. Furthermore we have derived computable lower bounds on these measures and have shown that they are not only computed very efficiently via four dimensional matrices but also very tight in most cases. Finally we have developed and improved techniques that serve as a testing ground for newly developed measures and distillation procedures.

Zusammenfassung

Verschränkung ist eines der zentralsten Themen der Quantentheorie. Sie besitzt kein klassisches Äquivalent und war daher Thema kontroversieller Debatten seit fast einem Jahrhundert. Was über lange Zeit eine rein philosophische Debatte blieb, hat Heute ein neues Gebiet der Physik initiiert. Die Quanteninformationstheorie nimmt das Konzept der Verschränkung in der Natur ernst und hat seither neue Anwendungen, wie z.B. Quantenkryptographie oder den Quantencomputer, gefunden. Für diese praktischen Anwendungen ist Verschränkung die wichtigste Ressource, die es ermöglicht, dass die quanteninformationstheoretischen Anwendungen jedes klassische Gegenstück übertreffen. Allerdings ist diese Eigenschaft keineswegs gut verstanden. Selbst scheinbar einfache Probleme, wie die mathematische Unterscheidung zwischen verschränkten und separablen Zustanden, sind nach wie vor ungelöst.
In dieser Arbeit behandeln wir intensiv die mathematische Theorie der Verschränkung. Wir waren in der Lage Techniken zu entwickeln, die die Detektion von Verschränkten Zuständen weiter verbessern, besonders in Vielteilchen-Systemen beliebiger Dimension. Außerdem waren wir in der Lage allgemeine Mengen von Verschränkungsmaßen zu entwickeln, die nicht nur Zweiteilchenverschränkung quantifizieren, sondern auch eine konsistente Quantifikation von Verschränkung in Vielteilchen-Systemen ermöglichen. Weiters haben wir berechenbare untere Schranken an diese Maße hergeleitet und gezeigt, dass diese nicht nur sehr effizient mittels vier dimensionaler Matrizen berechnet werden können, sondern auch in den meisten Fällen sehr knapp am wahren Wert liegen. Schlussendlich haben wir noch Techniken entwickelt und verbessert mit denen sich neu entwickelte Maße und Distillationsprozeduren gut testen lassen.

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Teaching:

2008-2009 – theoretical physics for teachers
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Conferences & Seminars:

Since 2006 – “non-local seminar” with the Research Centre for Quantum Information in Bratislava
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