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On some aspects of tsunami waves

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Abstract

The aim of this work is to discuss the mathematical modelling of surface water waves and currents with a special focus on tsunami waves. In particular, we investigate the possibility of incorporating a background flow field which is governed by the full Euler equations and models isolated regions of non-zero vorticity.

After briefly introducing the basic properties of tsunamis, a physical motivation of the governing equations for water waves is given. We discuss some important modelling assumptions and derive the equations of motion and boundary conditions for a two-dimensional setting. A stream function and the concept of vorticity is introduced in order to reformulate the governing equations in a simpler way. To get a better grasp of the subject, we review some important facts from the theory of ordinary differential equations. With the aid of a coercive functional, a dynamical systems approach enables us to overcome the difficulties due to the non-linear character of the vorticity function and the fact that the system is not autonomous. For a given non-linear vorticity distribution we find that the system of equations governing the background flow has a non-trivial, radially symmetric $\mathcal{C}^2$-solution with compact support, which models a background state with flat surface containing an isolated region of non-zero vorticity outside of which the water is still.

Contents

Chapter 1

Introduction

We will be concerned with understanding the motion of water in the ocean, describing surface water waves and underlying currents from a mathematical point of view. Some physical considerations will be necessary to justify modelling assumptions on the fluid, the types of boundaries and the physical laws of motion. We will focus our attention on a special type of surface water wave, namely tsunami waves.

In our context, the term “water wave” refers to the shape of the water surface resulting from a disturbance of the stable equilibrium state (a flat surface) under the force of gravity. As opposed to sound waves, which all move with the same speed, water waves propagate with different speeds according to their wavelengths. This fact can be observed in a simple experiment, throwing a big stone into a pond of still water. Waves will propagate outward from the source of the disturbance in circles and in this wave pattern, longer waves will always be further from the center than shorter waves. This phenomenon, that longer waves move faster than shorter waves, is called “wave dispersion” (see [Sto57] or [Lig03] for more details).

Note that when we talk about the propagation speed of a wave we do not refer to the velocity of the water or its particles, but to the speed at which the elevation of the water surface (the wave pattern) moves in a horizontal direction. Leonardo Da Vinci described this phenomenon by comparing the water surface to a cornfield exposed to strong wind where “one sees the waves running over the fields without the ears of corn changing their place” (cf. [WK08]).
Similarly, think of some small object floating on the water's surface. Instead of moving in the direction of the wave, it is lifted and then dropped again, following the vertical displacement of the water surface as the wave passes, before it returns (almost) to its initial position.
Waves whose wavelengths are long compared to the local water depth, so called

"long waves" or "shallow water waves", move with a speed of approximately

$$c = \sqrt{gh},$$

where g is the gravitational acceleration and h is the average local water depth. Tsunami waves, which have characteristic wavelengths much longer than the average ocean depth, are a prominent example of such waves (cf. [Seg04]).

The term "tsunami" is the Japanese word for "harbor wave" ("tsu" – harbor, "nami" – wave). It actually refers to a series of up to a half dozen separate waves, arriving at the coast in time intervals ranging from a few minutes to up to half an hour. Tsunamis have been documented for millenia, the earliest records dating back as far as 4000 years. The formation of a tsunami wave is quite a rare event, but it can cause major damage in the affected coastal areas. Tsunami waves are generated by a sudden vertical displacement of a body of water on a massive scale, caused by undersea earthquakes, landslides or volcanic eruptions. If the origin is volcanic, the body of water experiences a disruption by a collapse of the volcanic edifice, or landslides accompanying the eruption. If the origin of the displacement is due to an underwater earthquake, a large mass of earth on the bottom of the ocean drops or rises, thereby displacing the column of water directly above it. This frequently occurs along breaks in the earth's crust, so-called fault lines. There are three types of faults: a thrust fault is characterized by one plate moving up over an adjacent plate; in the case of a normal fault, one plate dips beneath another; strike-slip faults occur, when two plates slide past each other horizontally. The type of fault produced by the earthquake has significant influence on the generation of tsunami waves. Tectonic collisions in the form of strike-slip movement produce little vertical displacement and are therefore not likely to cause tsunamis. When the outcome of an undersea earthquake is a thrust (or normal) fault however, the ocean floor rises (or drops) by a few meters, causing the water directly above to rise (or fall) as well and thereby creating a positive (or negative) initial wave profile. Combinations of these types also occur[1]. Studies of these generation phenomena suggest a close correlation between the size and shape of tsunamis and the size of the earthquake, area and shape of the rupture zone, the amount of displacement of the earth's crust and the depth of water in the source area.

After the energy from the displacement of the tectonic plates or other landmasses is transferred to the water and the wave has obtained its initial profile, it spreads outwards from the source, travelling over thousands of kilometers at a speed of possibly more than $1000\,km/h$. The propagation of a tsunami is directional, depending on the geometry of the seafloor in the rupture zone. The source area is usually of elliptic shape, with the major axis up to $1300\,km$ long.

[1] An example is the December 2004 tsunami off the coast of Indonesia, where the earthquake generated an initial wave profile such that a positive wave was sent out travelling towards India and Sri Lanka, whereas a negative wave was sent out in the opposite direction and hit the coast of Thailand, where people observed a sudden drop of the water level prior to the arrival of the waves (cf. [Seg04]).

Most of the tsunami energy is transmitted at right angles to the rupture line and thus the direction of propagation of the wave is perpendicular to the major axis. This is why we can reasonably model the evolution of such waves in a two dimensional setting, assuming that the motion in the direction of the major axis of the rupture zone is uniform (cf. [LL89]).
Away from the shore, where the ocean can be assumed to have uniform depth over large distances, the evolution of the wave is governed essentially by the linear wave equation and the wave propagates at speed $\sqrt{gh}$. Since the amplitude of a tsunami wave out in the open sea is typically very small and the wavelength is hundreds of kilometers, ships in this water region usually do not even notice that a wave is passing by.
When the front part of a tsunami wave approaches the shore, it slows down, since the water depth and therefore also its speed decreases. The problem is that the rear of the wave might still be hundreds of kilometers out in the sea, travelling at much higher speed than the front. Thus, the back of the wave starts catching up with the front, causing the water to pile up vertically near the coast. Since the wave is so long, enormous amounts of water are involved in this process. This fact accounts for much of the devastating effects tsunami waves have in the coastal areas.

Before the arrival of the wave at the shore, the water in that region is unlikely to be still, as currents and other kinds of rotational disturbances in the body of water are common phenomena described in oceanography. There is an important concept in fluid dynamics called *vorticity*, which provides a measure for the local spin of fluid elements. Vorticity is present in most fluid bodies, even in your kitchen sink when you pull out the plug and the water flows out of the sink. You can visualize the rotational movement by throwing small soil particles into the water near the drain and observe that not only do they describe circular paths as they are washed down the sink, but they also rotate with angular velocity proportional to the vorticity. While Feynman [FLS63] suggests checking it out for a bucket of water on a turntable, I limited my efforts to soil particles in the kitchen sink. A famous quote describing why vorticity is a measure of the local spin of a fluid element is given in [MT68]:

> If a spherical element of the fluid were suddenly solidified and the surrounding fluid simultaneously annihilated, this solid element would rotate with [angular velocity half the vorticity].

A more detailed mathematical discussion of this phenomenon is given in Section 2.5.
With this in mind, it seems essential in a reasonable model for tsunami waves to allow for some kind of background flow field, which models the motion of water before the arrival of a tsunami. The wave may then be introduced as a perturbation of the background state and the interaction between a background flow and a tsunami wave can be studied.

A lot of work has been done on modelling the evolution of tsunami waves as they pass through the ocean and eventually approach the shoreline (see for example [CJ06], [Ham73], [Voi87], [ZYN09]). Most investigations are restricted to modelling tsunami waves on irrotational background flows (that is, flows with zero vorticity), which model background states of still water. The possibility of incorporating pre-existing vorticity was only recently investigated in [CJ08]. We will follow up this matter and try to explain and answer questions like:

- What are the physical laws of motion that govern the evolution of water waves and currents?
- How can we incorporate background flows in the modelling of water waves?
- What kind of vorticity distributions are allowed?

In [CJ08] various vorticity distributions were obtained in the shallow water regime. It was found that the requirement of a flat free surface is too restrictive, as it invalidates even the simple choice of constant non-zero vorticity throughout the flow field (as would be the case for example in tidal flows, cf. [dSP88]).

As opposed to passing to the long wave limit and studying approximations for the shallow water regime, the possibility of incorporating background flows with isolated regions of vorticity that allow for a flat free surface and are governed by the full Euler equations is even more recent. In [Con10] an example of a vorticity distribution such that the equations governing a two dimensional steady flow with vorticity admit a non-trivial solution that models an isolated region of vorticity near the shore is given. The aim of the present work is to discuss a generalization of this paper. The difficulty of finding appropriate vorticity distributions lies in the fact that for linear distributions and distributions of $\mathcal{C}^1$ regularity only trivial solutions are obtained. It is necessary to consider nonlinear vorticity functions which are continuous but not Lipschitz continuous.

We will proceed in the following way: In Chapter 2, we discuss some basic modelling assumptions for water waves. Furthermore we will motivate and derive the equations of motion and boundary conditions that govern the behavior of the water in the background state. In this setting we introduce the concept of a stream function, which allows us to reformulate the equations at hand in a simpler way. We will focus more closely on the aforementioned term vorticity and obtain a better understanding of this important concept in fluid dynamics. Furthermore, we shall see how vorticity can be related to the stream function to obtain yet another reformulation of our equations. In Chapter 3 we will review some important results of the theory of ordinary differential equations. The main result will then be presented in Chapter 4. It says essentially that our model allows for an isolated region of non-zero vorticity in the background flow field with a flat free surface.

Chapter 2

The Governing Equations

Before we discuss the equations that govern the motion and the conditions that specify the behavior at the boundary according to the given physical context, let us clarify some modelling assumptions on the properties of the fluid in question, water.

2.1 Modelling assumptions

In general fluid dynamics it is common to regard the fluid as a continuum. How can we justify this assumption? On a molecular level, water is not a uniform material, as the mass of its molecules for example is concentrated in the nuclei of the atoms. However, fluid dynamics is usually concerned with the behavior at a scale large compared to the distance between molecules. From this macroscopic point of view, we can suppose that the fluid behaves as if it were continuous in its structure (cf. [Bat67]). In the case of water this seems like a natural assumption, as it obviously behaves in a continuous way when observed with the naked eye or any of the usual measuring devices. This so called *continuum hypothesis* implies that we can give a definite meaning to the notion of a value "at a point" in the fluid. It is therefore reasonable to regard the quantities determining the state of the fluid (we will see in a moment which are the relevant ones for our purpose) as continuous functions of position $\boldsymbol{x}$ and time t, relative to Cartesian coordinate axes. On the basis of this Eulerian description of the fluid we are able to establish equations governing its motion, irrespective of the actual particle structure of the medium. It should be pointed out that there is an alternative description – the Lagrangian viewpoint – where the focus lies on the individual fluid particles and the time evolution of the particle paths. The independent variables then are the initial position a of a particle and time t. A particle path is obtained by integrating $\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{u}$, where the constant of integration is defined by $\boldsymbol{x}(t=0) = a$. These two descriptions are equivalent.

We focus our attention on models which are suitable for describing tsunami waves, where the main restoring force of the wave is gravity. This means that the formation of a wave is driven by a balance between the fluid's inertia and its tendency, under gravity, to return to a state of stable equilibrium, a flat surface (cf. [Lig03]). Such waves are therefore also called *gravity water waves* as opposed to capillary waves, were surface tension plays the role of the restoring force. We will, however, neglect surface tension as we focus our attention on waves and phenomena that are of some orders of magnitude larger than the effects of capillarity account for. We are also not concerned with how exactly a tsunami wave is generated – we start with the initial profile of the water surface and try to understand the evolution of the wave as it passes through the ocean and approaches the shore. In the introduction we already mentioned the fact that the behavior of tsunami waves is almost uniform in the direction perpendicular to the direction of propagation and we can therefore reasonably treat the problem in two dimensions only – a horizontal direction in which the wave spreads and a vertical direction that indicates the elevation of the water surface. Note that in this study the interaction between water and, for example, sand on the seabed is not taken into consideration. That is, we assume the bed to be impermeable, which means that water particles cannot penetrate the bed.

Another important modelling assumption for studying tsunami waves is that we regard the water to be inviscid. *Viscosity* is a measure of the resistance of the fluid towards shear stresses and one could think of it as something like the "thickness" of a fluid. Honey, for example, has higher viscosity than water and in this sense, honey is "thicker" than water. A slightly more physical formulation of this phenomenon is that viscosity describes the fluid's internal resistance to shear flow and measures a kind of internal friction between fluid particles. Inviscid fluids are sometimes also called ideal fluids. Although the study of water neglecting viscosity was described by John von Neumann as studying "dry water", this approximation is good enough for our purposes.

A further simplification of the model is obtained by imposing on the fluid the homogeneity assumption of water having constant density. Density is in fact a function of depth, temperature and salinity. However, variations of these properties of the water have negligible effects on density – changes in salinity account for variations of about 0.2%, temperature variations are responsible for up to 0.5% of density changes and a descent of $1km$ in depth increases the density by no more than 0.5% (cf. [AT81], [Lig03]).

After having discussed the main modelling assumptions, let us turn to the equations that govern the motion of water.

2.2 Equations of motion

How can we describe the motion of a fluid? One way to answer this question is to give the fluid's properties at every spacial point in any instant of time.

That is, to characterize the motion we would have to give the fluid's velocity, pressure, temperature, density and so on everywhere in the fluid. If we could find the equations which govern the time evolution of these properties, we would know how the water moves with time.

We already agreed on some simplifying assumptions and reduced the relevant properties to velocity and pressure. Now let us look at the equations determining the motion.

2.2.1 Mass conservation

The assumption of water being a homogeneous fluid is a good approximation for the phenomena we are concerned with and enables us to derive a version of the continuity equation for hydrodynamics. In physics, a continuity equation basically describes the transport of some kind of conserved quantity. In a fixed region the amount of this quantity can only change by the amount that enters or leaves the region through the boundary. In fluid dynamics this means that the rate at which water enters a fluid volume equals the rate at which it leaves the volume. The idea is that mass is neither generated nor destroyed anywhere in the fluid. That is, there are no sources or sinks where "new" water is added to the fluid body − if some amount of water flows away from one point, the amount left behind must decrease in kind. Now, let us derive an equation which formalizes these thoughts.

Imagine a test volume V with surface S in the fluid. The amount of mass which flows out of V through S in a unit time is

$$\int_S \rho\, \boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}S,$$

where $\boldsymbol{n}$ is the outward unit normal on S and $\boldsymbol{u}$ is the velocity vector of the water, so $\boldsymbol{u} \cdot \boldsymbol{n}$ is the velocity component perpendicular to the surface S. The change in time of the total amount of mass inside V is

$$\frac{d}{dt} \int_V \rho\, \mathrm{d}\boldsymbol{x}.$$

The principle of mass conservation says that this rate of change of mass is solely due to the rate of mass flowing into V across S, so

$$\frac{d}{dt} \int_V \rho\, \mathrm{d}\boldsymbol{x} = - \int_S \rho\, \boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}s.$$

In view of the divergence theorem and since V is a fixed region in space, we may write

$$\int_V \frac{\partial \rho}{\partial t}\, \mathrm{d}\boldsymbol{x} = - \int_V \nabla \cdot \rho\, \boldsymbol{u}\, \mathrm{d}\boldsymbol{x}.$$

Since we derived this equation for an arbitrary test volume V, this equation is true only when

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \boldsymbol{u} = 0, \tag{2.1}$$

which is the *continuity equation* for the conserved quantity ρ. Since we assumed the density ρ to be constant in the fluid we obtain the *equation of mass conservation* for homogenous fluids,

$$\nabla \cdot \boldsymbol{u} = 0.$$

For a two-dimensional vector field $\boldsymbol{u} = (u, v)$ and Cartesian coordinates $\boldsymbol{x} = (x, y)$ the equation of mass conservation simplifies to

$$u_x + v_y = 0. \tag{2.2}$$

At this point the reader should note the close connection between the notion of an incompressible fluid and the principle of mass conservation.
A fluid is called *incompressible* when the density of each element of the fluid is constant, that is, the rate of change of ρ following the motion of the fluid is zero,

$$\frac{D\rho}{Dt} = 0.$$

The expression $\frac{DQ}{Dt} = \frac{\partial Q}{\partial t} + \boldsymbol{u} \cdot \nabla Q$ is the material derivative of some quantity Q. It describes the rate of change of Q "at a point moving with the fluid's velocity u" or "following a fluid particle along its path". The material derivative consists of a differentiation with respect to time at a fixed point, $\frac{\partial}{\partial t}$, and a convective term $\boldsymbol{u} \cdot \nabla$ which takes account of the motion of the fluid particles.
We can expand (2.1) to

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\rho = 0,$$

and, using this new notation, rewrite the continuity equation in terms of the material derivative as

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \boldsymbol{u} = 0. \tag{2.3}$$

Thus, for incompressible fluids we recover the equation of mass conservation, since the material derivative vanishes and $\rho \nabla \cdot \boldsymbol{u} = 0$ only when $\nabla \cdot \boldsymbol{u} = 0$.
On the other hand, assuming that mass conservation $\nabla \cdot \boldsymbol{u} = 0$ holds, we have that $\frac{D\rho}{Dt} + \rho \underbrace{\nabla \cdot \boldsymbol{u}}_{=0} = \frac{D\rho}{Dt} = 0$, i.e. that the flow is incompressible.

We saw that the assumption of constant density is actually not necessary to deduce the principle of mass conservation. Although it suffices to consider incompressible fluids to obtain mass conservation, we will maintain the assumption of constant density to simplify calculations. Furthermore, note that mass conservation implies incompressibility (and vice versa) but not necessarily constant density.

2.2.2 Euler equations

After establishing and formalizing the concept of mass conservation we now want to deduce an equation that governs the motion of water. The starting point for our deduction will be *Newton's second law of motion* which says that force equals mass times acceleration,

$$\boldsymbol{F} = m \cdot \boldsymbol{a}.$$

When we apply this equation to the fluid, the only quantity whose meaning is clear from the beginning is mass, which is equal to the fluid's density times some specified volume. We will proceed step by step and answer the following two questions:

1. What are the forces $\boldsymbol{F}$ acting on the fluid (at rest and in motion)?
2. What does acceleration $\boldsymbol{a}$ mean for a fluid?

In general fluid dynamics two types of forces are considered relevant in describing the motion of a fluid: external forces (or body forces, which have a similar effect on all fluid particles) and internal forces (which act between fluid particles). Among these we can furthermore distinguish between forces acting normal to any surface in the fluid and forces acting tangentially (shearing forces). We already mentioned that a fluid's viscosity gives a measure for how the fluid is able to resist shear. Since we assume that water is inviscid, we can neglect the shearing forces of viscosity. Therefore, the only relevant internal force is the normal force $\boldsymbol{P}$, whose magnitude is called *pressure*. As we are concerned with gravity water waves, the only external force playing a role for us is *gravity* $\boldsymbol{F} = (0, 0, -g)$, where g is the gravitational constant of acceleration. Gravity is a conservative force in the sense that the work W done by moving a particle from one point to another is independent of the path taken:

$$W = \int_C \boldsymbol{F} \, \mathrm{d}r = 0,$$

where the integral is taken over a closed contour C. This is equivalent to the fact that the force $\boldsymbol{F}$ can be written as the gradient of a potential ϕ,

$$\boldsymbol{F} = \nabla \phi.$$

Let us turn now to the internal pressure force and think for a moment about the fluid at rest. When fluids are at rest, there are no shear forces, not even in viscous fluids. This means that any stress (force per unit area) is always acting normal to any surface in the fluid. This normal force per unit area is called (hydrostatic) pressure and its magnitude is the same in all directions. This is also known as Pascal's Law. Although pressure is the same in all directions, it may vary from place to place. If we imagine a small cube of unit volume of water, the total force on it from pressure is $-\nabla P$, cf. [FLS63]. Even when the water is at rest, there is gravity acting upon it, with the force per unit volume in terms of the potential given by $\rho \nabla \phi$. These forces (gravity and pressure) must be balanced when the fluid is at rest, so that

$$-\nabla \boldsymbol{P} + \rho \nabla \phi = 0. \tag{2.4}$$

This equation is called the equation of hydrostatics. In our case of constant density, we can reformulate it to $\nabla(\boldsymbol{P} - \rho \phi) = 0$, which has a solution along any line where $\boldsymbol{P} - \rho \phi = const.$

Let us go back to Newton's law of motion, which tells us that the forces that are balanced when the fluid is at rest, behave according to $\boldsymbol{F} = m \cdot \boldsymbol{a}$ when things start to move. Now it is time to address the question about acceleration. The thrust lies in understanding the different meanings of differentiation from Lagragian and Eulerian viewpoints. The vector field $\boldsymbol{u}$ gives the velocity of a fluid particle at $(\boldsymbol{x}, t)$. A first guess for finding an expression for acceleration would probably be $\frac{\partial \boldsymbol{u}}{\partial t}$. However, this just gives the rate at which the velocity $\boldsymbol{u}$ changes in time at a fixed point in space. What we need is the rate at which the velocity of a particle changes as it moves through the fluid along its particle path. Think of $\boldsymbol{x}(t)$ as the path of a fluid particle as it moves with the fluid, that is, its velocity is equal to the fluid velocity, $\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{u}$. Then using the chain rule formula from vector calculus one gets for the rate of change $\frac{\partial}{\partial t}\boldsymbol{u}(\boldsymbol{x}(t), t) = \nabla \boldsymbol{u} \cdot \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} + \frac{\partial \boldsymbol{u}}{\partial t} = \frac{D\boldsymbol{u}}{Dt}$, which is the material derivative we discussed before.

Note that there can be an acceleration of particles although $\frac{\partial \boldsymbol{u}}{\partial t} = 0$. Just think of a river or creek as it bends around a big rock. The water flows steadily in the sense that there is (during the time of your observation) always the same amount of water flowing with the same speed in the same direction. So at every fixed location there is no observable change in the flow field, that is $\frac{\partial \boldsymbol{u}}{\partial t} = 0$. However, if you focus on one water particle and follow it with your eyes, as it approaches the rock you will find that it accelerates near the rock and after it has passed, slows down again. Thus, its material derivative $\frac{D\boldsymbol{u}}{Dt}$ will not be zero. From the point of view of a single particle, a change of velocity has occurred. This is exactly the type of acceleration that we need to take into into account.

Now we are ready to use Newton's second law of motion to balance the forces within the test volume V we had before: The total force acting on the water is equal to the water's mass times the total acceleration of all water particles in V,

$$\int_V -\nabla \boldsymbol{P} + \rho \nabla \phi \,\mathrm{d}\boldsymbol{x} = \int_V \rho \frac{D\boldsymbol{u}}{Dt} \,\mathrm{d}\boldsymbol{x}.$$

Let us set the constant density $\rho = 1$. Then the above equation yields

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla \boldsymbol{P} + \nabla \phi,$$

since we derived it for arbitrary V. Written separately for each component of $\boldsymbol{u} = (u, v)$ and using that the potential is simply given by $\phi = -gy$, where

$g = 9.81\,m/s$ is the gravitational acceleration, we obtain the two-dimensional *Euler equations*

$$u_t + uu_x + vu_y = -P_x \tag{2.5}$$

$$v_t + uv_x + vv_y = -P_y - g. \tag{2.6}$$

2.3 Boundary conditions

Before we discuss what conditions we have to impose at the boundary of the fluid domain, let us specify what the actual boundaries are in our setting. We consider water in the ocean with a so called free surface on top of the fluid which can be parametrized by a function $y = \mu(x,t)$. The free surface is to be determined and thus forms part of the problem (and the solution). On the bottom we have a fixed impermeable bed, which means that water particles cannot penetrate the seabed. In many cases the water is assumed to extend to infinity in either horizontal direction, and one is interested in the time evolution of the shape of the free surface as a wave passes through the fluid domain. In our case, we want the model to admit a shoreline. In cartesian coordinates (x, y), let the origin be the intersection of the flat free surface and the seabed at the shoreline $x = 0$. Let the horizontal x-axis be in the direction of the incoming right-running waves and the vertical y-axis pointing upwards. We assume the fluid to extend to $-\infty$ in the negative horizontal direction. The bed's topography is assumed to be given by the function $b(x)$ where $b(0) = 0$ and $b(x) < 0$ for $x < 0$. For a gently sloping beach we require $b'(0) > 0$. In the open sea we assume uniform depth h_0 such that $b(x) = h_0$ for x far away from the shoreline $x = 0$. We will denote the fluid domain by $D = \{(x, y) \in \mathbb{R}^2 : x < 0,\, b(x) < y < \mu(x,t)\}$.

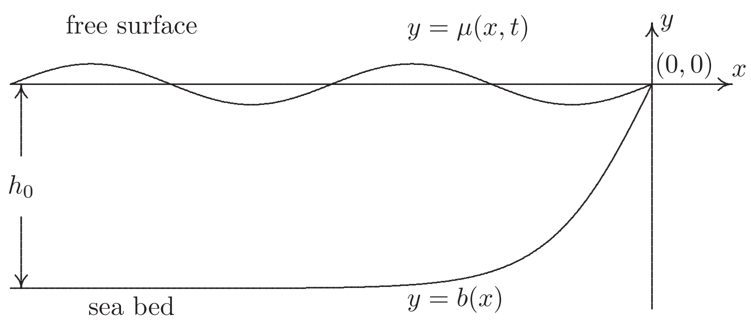

Figure 2.1: Fluid domain D

There are two types of boundary conditions, kinematic and dynamic. For the free surface a kinematic boundary condition can be derived using the fact that a surface $S(\boldsymbol{x}, t) = const.$ which moves with the fluid always contains the same fluid particles, that is, it must satisfy

$$\frac{DS}{Dt} = 0.$$

This also holds for our free surface $S(\boldsymbol{x}, t) = y - \mu(x, t) = 0$ and means that fluid particles that are on the surface initially remain there for all times. Writing out the material derivative for each component thus yields the kinematic boundary condition for the free surface

$$v = \mu_t + u\mu_x \qquad \text{on} \quad y = \mu(x, t).$$

Similarly for the bottom we have that $\frac{D}{Dt}(y - b(x)) = 0$, the difference being that b does not depend on time. The kinematic boundary condition for the bed is therefore given by

$$v = ub_x \qquad \text{on} \quad y = b(x).$$

The dynamic boundary condition solely involves pressure forces as we agreed on neglecting surface tension and viscous forces. We decouple the motion of air above the free surface from the motion of water via the condition

$$P = P_0 \qquad \text{on} \quad y = \mu(x, t),$$

where P_0 denotes the atmospheric pressure above sea level which we assume to be constant. This approach is reasonable since the density of air is very small compared to that of water. (If the fluid were viscous, we would also have to consider stresses that the atmosphere exerts on the fluid surface.)

The equations we have derived so far are valid for general water wave problems and can be used in various models describing the motion of water. We are, however, mainly concerned with applying them to velocity fields for background flows that describe the state of the water prior to the arrival of waves. In particular, we will be modelling a steady background flow, which means that at every point in the fluid domain the water moves in exactly the same way at all times. This does not mean that the water is still. The fluid particles move around and change their velocities all the time. It just expresses the fact that $\frac{\partial \boldsymbol{u}}{\partial t} = 0$ at each fixed point in the fluid. Thus, time dependency drops from our equations[1].

This more or less concludes our discussion of the physical laws of motion and the boundary conditions involved in the problem. We summarize by stating the

[1] In many settings (for example when we consider travelling waves which move at constant speed $c > 0$ without changing their shape), a steady flow means that the water flows at constant speed so that the $(\boldsymbol{x}, t)$-space-time dependence of the free surface, the pressure and the velocity field is of the form $(x - c, t)$. Then one can pass to a moving frame via the change of coordinates $(x - ct, y) \mapsto (x, y)$ and thereby eliminate time from the problem.

equations governing the background state of our flow field (u, v) in D:

$$uu_x + vu_y = -P_x \tag{2.7}$$
$$uv_x + vv_y = -P_y - g \tag{2.8}$$
$$u_x + v_y = 0 \tag{2.9}$$
$$P = P_0 \qquad \text{on } y = \mu(x) \tag{2.10}$$
$$v = u\mu_x \qquad \text{on } y = \mu(x) \tag{2.11}$$
$$v = ub_x(x) \qquad \text{on } y = b(x) \tag{2.12}$$

2.4 Stream function

For a two-dimensional velocity field (u, v) we obtain a simpler formulation of the problem by introducing a so-called *stream function* $\psi(x, y) : \mathbb{R}^2 \to \mathbb{R}$,

$$\psi_x = -v \tag{2.13}$$
$$\psi_y = u. \tag{2.14}$$

This definition is unique up to an additive constant since we may write

$$\psi(x, y) = \int_{b(x)}^{y} u(x, \xi)\, \mathrm{d}\xi + d(x).$$

Differentiating with respect to x in view of mass conservation (2.9) and the boundary condition (2.12) yields

$$\psi_x(x, y) = -b_x(x)u(x, b(x)) + \int_{b(x)}^{y} \underbrace{u_x(x, \xi)}_{-v_y(x,\xi)}\, \mathrm{d}\xi + d'(x) \tag{2.15}$$
$$= -v(x, b(x)) - v(x, y) + v(x, b(x)) + h'(x) = \psi_x(x, y) + d'(x), \tag{2.16}$$

so $d'(x) = 0$ and thus $d \equiv d_0$ is constant.
Existence of such a $\mathcal{C}^2$-function ψ is guaranteed, since in view of mass conservation (2.9) the vector field $(-v, u)$ is curl free, that is

$$\nabla \times (-v, u) = u_x + v_y = 0.$$

The fact that our fluid domain is simply connected thus ensures existence of a function ψ with $\nabla\psi = (-v, u)$, which is exactly how we defined the stream function. Because the gradient of ψ does not vanish, at least locally the level sets of ψ,

$$\psi_\alpha = \{(x, y) \in \mathbb{R}^2 : \psi(x, y) = \alpha\},$$

are curves with the same regularity as ψ. Furthermore $\psi_\alpha \perp \nabla\psi \perp (u, v)$ which means that the levelsets ψ_α are tangential to the velocity field. We call ψ_α *streamlines*. Note that although streamlines can be very useful to analyze the flow, they do not coincide with the particle trajectories unless the flow is steady.

We find that the stream function is constant on both the free surface and the bottom as
$$\frac{d}{dx}\psi(x,\mu(x)) = \psi_x + \psi_y\mu_x = -v + u\mu_x = 0$$
on $y = \mu(x)$ in view of (2.11), and
$$\frac{d}{dx}\psi(x,b(x)) = \psi_x + \psi_y b'(x) = -v + ub'(x) = 0$$
on $y = b(x)$ by the boundary condition (2.12). Thus the free surface as well as the sea bed are streamlines. In this terminology, existence of a shoreline at $x = 0$ means that these two streamlines intersect and $b(0) = \mu(0) = 0$. Therefore, the constant value of ψ on the two curves has to be the same and we may set it equal to zero. Thus, also $d_0 = 0$, which uniquely determines the stream function.

2.5 Vorticity

A heuristic explanation of this concept as a measure for the local spin of fluid particles was already given in the introduction. In the mathematical modelling of fluid motion *vorticity* is defined as the curl of the velocity field of the fluid,
$$\omega = \nabla \times \boldsymbol{u}. \tag{2.17}$$
To get a better understanding of this rotational movement from a mathematical point of view consider a three-dimensional velocity field $\boldsymbol{u}$. Imagine a small circular disc S in the fluid bounded by the contour C, centered at a point p, and oriented via the unit normal $\boldsymbol{n}(p)$. Recall *Stokes' Theorem*
$$\int_C \boldsymbol{u}\cdot\ \mathrm{d}l = \int_S \underbrace{(\nabla \times \boldsymbol{u}(x))}_{\boldsymbol{\omega}(x)} \cdot \boldsymbol{n}(x)\ \mathrm{d}S.$$
The expression on the left hand side is called *circulation* of $\boldsymbol{u}$ around C. This value is zero if the vector field $\boldsymbol{u}$ is aligned with $\boldsymbol{n}$, since then $\boldsymbol{u}$ has no components tangential to the disc. The circulation gets bigger the more $\boldsymbol{u}$ aligns with vectors tangential to the disc. When we take the disc S centered at p to be infinitely small, $\boldsymbol{\omega}(x)$ and $\boldsymbol{n}(x)$ can be reasonably approximated by $\boldsymbol{\omega}(p)$ and $\boldsymbol{n}(p)$. Their dot product will then be approximately the circular density of $\boldsymbol{u}$ at p,
$$\boldsymbol{\omega}(p)\cdot\boldsymbol{n}(p) \approx \frac{1}{S}\int_C \boldsymbol{u}\cdot\ \mathrm{d}l.$$
When $\boldsymbol{n}$ points in the direction of the vorticity vector $\boldsymbol{\omega}$, this value is greatest. We can therefore regard the vorticity as a measure for local rotation (or spin) on an infinitesimal disc perpendicular to the direction of the vorticity vector.

In the case of a two-dimensional velocity field, the vorticity is given by
$$\omega = v_x - u_y.$$

Note that this is in a sense abuse of notation, since the curl of a three dimensional vector field is by definition again a vector of three dimensions. What we are actually doing here is extending our two dimensional velocity field to $(u, v, 0)$. Taking the curl, we get a vector of the form $(0, 0, v_x - u_y)$. It points in the direction of the axis of rotation and its norm gives the magnitude of the infinitesimal rotation of the vector field. Thus, when we write $\omega = v_x - u_y$ we mean that there is vorticity present with magnitude $v_x - u_y$ and with the axis of rotation pointing in the z-direction, perpendicular to the (x, y)-plane.

Notice that vorticity does not describe global rotation of the fluid body, but local rotation of fluid particles. To illustrate why we have to be careful not to confuse the local spin of fluid elements with global rotation in a fluid domain, let us state two examples.

Example 1 In a torus, consider the velocity field defined by

$$u = -\frac{y}{x^2 + y^2}, \qquad v = \frac{x}{x^2 + y^2}.$$

A short calculation shows that $\omega = v_x - u_y = 0$, so there is no local spin. On the other hand, if we look at the particle trajectories

$$x'(t) = -\frac{y}{x^2 + y^2}, \qquad y'(t) = \frac{x}{x^2 + y^2},$$

with initial positions $x(0) = x_0$, $y(0) = y_0$ we can solve this system of ordinary differential equations by rewriting it in polar coordinates

$$x = r\cos(\theta),\ y = r\sin(\theta).$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t} r^2(t) = \frac{\mathrm{d}}{\mathrm{d}t}(x^2(t) + y^2(t)) = xx' + yy' = 0,$$

we have for the radius that $r(t) = r_0$. We find that $\theta(t) = t/r_0^2 + \theta_0$ and thus

$$x(t) = r_0 \cos(\theta_0 + t/r_0^2), \qquad y(t) = r_0 \sin(\theta_0 + t/r_0^2).$$

This means that the particles move in circles around the torus. Therefore there is a global rotation with angular velocity $1/r_0^2$ although the flow is irrotational.

Example 2 In contrast, consider in the same torus the velocity field defined by

$$u = -y, \qquad v = x.$$

Then the vorticity of the flow is $\omega = v_x - u_y = 2$ and again the particle trajectories show global rotation with angular velocity 1. Hence in this case we get both global and local rotation.

Vorticity can in some cases even be expressed as a function of ψ, the stream function. To see this take the curl of the Euler equations (2.5)

$$\begin{aligned}
0 &= \partial_x[uv_x + v_v y + P_y + g] - \partial_y[uu_x + vu_y + P_x] \\
&= u_x v_x + uv_{xx} + v_x v_y + vv_{yx} + P_{yx} \\
&- u_y u_x - uu_{xy} - v_y u_y - vu_{yy} - P_{xy} \\
&\overset{(2.9)}{=} u(v_{xx} - u_{xy}) - v(u_{yy} - v_{yx}) \\
&= \psi_y\,\omega_x - \psi_x\,\omega_y,
\end{aligned} \tag{2.18}$$

where we have used the condition of mass conservation and the definitions of ω and ψ. $\psi_y\,\omega_x - \psi_x\,\omega_y = 0$ is equivalent to $(u, v) \cdot \nabla\omega = 0$, which means that the velocity field (u, v) is perpendicular to the gradient of ω. We already know that the streamlines ψ_α are tangential to (u, v), so $\nabla\omega \perp \psi_\alpha$. Furthermore, the gradient of ω is perpendicular to the level sets of ω, $\nabla\omega \perp \omega_\alpha$. Summing up, we have that ψ and ω have the same levelsets, so that we may specify ω by means of a *vorticity function* γ with

$$\omega = \gamma(\psi). \tag{2.19}$$

2.6 Reformulation of the system

To obtain a more convenient formulation of the governing equations (2.7)-(2.12) in terms of the stream function ψ and the vorticity distribution γ, we introduce the energy functional

$$E(x, y) = \frac{|\psi|^2}{2} + P + g\,y + \Gamma(\psi), \tag{2.20}$$

where $\Gamma(\psi) = \int_0^\psi \gamma(s)\ \mathrm{d}s$. It is easy to see that $E(x, y) \equiv E_0$ is constant throughout the fluid domain D, whenever the Euler equations hold:

$$\begin{aligned}
\partial_x E(x, y) &= \psi_x\psi_{xx} + \psi_y\psi_{yx} + P_x + \psi_x \underbrace{\gamma(\psi)}_{=\omega} \\
&= vv_x + uu_x + P_x - v(v_x - u_y) \\
&= uu_x + vu_y + P_x = 0,
\end{aligned}$$

and

$$\begin{aligned}
\partial_y E(x, y) &= \psi_x\psi_{xy} + \psi_y\psi_{yy} + P_y + g + \psi_y \underbrace{\gamma(\psi)}_{=\omega} \\
&= vv_y + uu_y + P_y + g - u(v_x - u_y) \\
&= uv_x + vv_y + P_y + g = 0.
\end{aligned}$$

In the expression (2.20), the term $\frac{|\psi|^2}{2}$ is the kinetic energy, gy is the gravitational potential energy, P is the energy due to the fluid pressure that is exerted upon each particle by the fluid around it, and $\Gamma(\psi)$ can be interpreted as the rotational energy. Notice that

$$\begin{aligned}\Delta\psi &= \partial_x \psi_x + \partial_y \psi_y = -v_x + u_y = -\omega \\ &= -\gamma(\psi) \qquad \text{in } D.\end{aligned}$$

In terms of the stream function, the kinematic boundary conditions (2.11) and (2.12) read

$$\begin{aligned}\psi_x &= -\psi_y \mu_x \qquad \text{on} \quad y = \mu(x) \\ \psi_x &= -\psi_y b_x \qquad \text{on} \quad y = b(x).\end{aligned}$$

To express the remaining dynamic boundary condition involving P we consider the above expression E on $y = \mu(x)$, which is equivalent to

$$2(E_0 - P_0 - gy) = \psi_y^2(\mu_x^2 + 1) \qquad \text{on } y = \mu(x). \tag{2.21}$$

At this point, recall that we want to model background states where the water surface is flat in the absence of waves. That is, we are dealing with the special situation of a flat free surface $y = \mu(x) = 0$. This simplifies the boundary conditions even further. We already know that $\psi = 0$ on both boundaries, so in particular $\psi_x(x, 0) = 0$. Also, since we assumed $b_x(0) \neq 0$, the kinematic boundary condition on the bottom yields $\psi_x(0, 0) = -\psi_y(0, 0)b_x(0) = 0$, so $\psi_y(0, 0) = 0$. But evaluating E at $(0, 0)$ yields $E_0 = P_0$ and therefore, in view of (2.21), $\psi_y = 0$ on $y = 0$.

We can finally formulate the equations governing the background state with a flat surface in terms of the stream function ψ as

$$\begin{cases} \Delta\psi = -\gamma(\psi) & \text{in } D \\ \psi = \psi_y = 0 & \text{on } y = 0 \\ \psi = 0 & \text{on } y = b(x) \end{cases} \tag{2.22}$$

given a vorticity distribution γ and the bottom profile b in the fluid domain $D = \{(x, y) \in \mathbb{R}^2 : x < 0,\, b(x) < y < 0\}$.

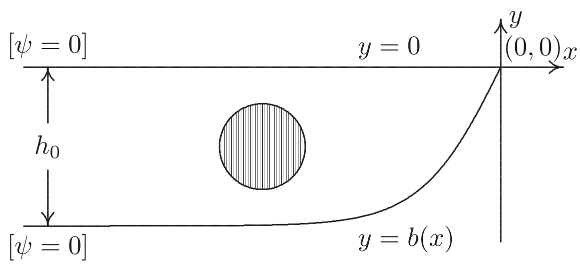

Figure 2.2: Fluid domain D with a flat free surface and an isolated region of non-zero vorticity

Our aim is to show existence of an isolated region of non-zero vorticity in D (see Figure 2.2). That is, we have to find a suitable vorticity function γ and prove that (2.22) has a non-trivial solution with compact support.

In (2.22) we have boundary conditions for the solution ψ on top and on the bottom, as well as a boundary condition on top for its derivative. Consequently, we are dealing with an over-determined boundary value problem and it is expected that a non-trivial solution will only exist for certain classes of functions γ. We shall see in Section 4.4.2 that for a linear vorticity function $\gamma(\psi) = a\psi + b$, system (2.22) admits only trivial solutions. Furthermore we will prove that choosing $\gamma \in \mathcal{C}^1$ precludes radially symmetric solutions with compact support in the fluid domain. The details will be discussed in Chapter 4.

Chapter 3

Basic Theory of Ordinary Differential Equations

In the discussion of the main result in Chapter 4, we will be essentially concerned with solving a system of two first order ordinary differential equations subject to initial conditions. Problems in solving this initial value problem arise due to the fact that it fails to be locally Lipschitz in the dependent variable, and even displays a discontinuity in the independent variable. In this chapter we will give a quick overview of some important results in the theory of ordinary differential equations (cf. [CL55], [Cop65]). It will then become clear in the next chapter why we cannot apply standard results in our case, which makes the analysis of the problem at hand somewhat harder.

3.1 Existence

Consider a domain $D = \{(t, x) \subset \mathbb{R}^2 : t \in I$ an open interval on the real line, $|x| < \infty\}$ and $f \in \mathcal{C}(D)$ a real valued continuous function in D. We denote the derivative with respect to the independent variable t by $' = \frac{\mathrm{d}}{\mathrm{d}t}$.

Initial value Problem (IVP): Find an interval I and a solution φ of the differential equation

$$x' = f(t, x), \qquad t \in I, \tag{3.1}$$

which satisfies the initial condition

$$x(t_0) = \xi_0, \qquad t_0 \in I. \tag{3.2}$$

A function φ is called a *solution* of (3.1) if it is defined and differentiable on I such that

(i) $(t, \varphi(t)) \in D$, and it satisfies

(ii) $\varphi'(t) = f(t, \varphi(t))$ for $t \in I$.

Clearly, if φ is a solution to (3.1) then $\varphi \in \mathcal{C}^1$ on I, on account of (ii), since f is continuous.

We will state all results for first order ODE and $x \in \mathbb{R}$, so that f takes values in the (x,t)-plane. This can be easily generalized to higher dimensions and higher order ODE or systems of ODE, regarding the dependent variables x_i as coordinates of a single vector variable $x = (x_1, \ldots, x_n)$ and $f = (f_1, \ldots, f_n)$, thus treating (3.1) as a vector differential equation.

There are several methods establishing existence of solutions, and all depend essentially on the fact that the differential equation (3.1) can be replaced by an integral equation:

Lemma 3.1.1 *The (IVP)of finding an interval I and a solution φ to (3.1) with (3.2) is equivalent to finding all continuous functions φ on I satisfying the integral equation*

$$\varphi(t) = \xi_0 + \int_{t_0}^{t} f(s, \varphi(s))\, \mathrm{d}s.$$

Local existence of solutions to the (IVP) follows from

Theorem 3.1.2 (Cauchy – Peano) *Suppose $f \in \mathcal{C}(D)$ and $(t_0, \xi_0) \in D$. Then there exists a solution $\varphi \in \mathcal{C}^1$ of (3.1) satisfying (3.2) on some t interval containing t_0 in its interior.*

The proof is based on the construction of a sequence of approximate solutions φ_n of the (IVP), which is uniformly bounded and equicontinuous. Ascoli's theorem then guarantees existence of a subsequence converging uniformly to a limit function satisfying (3.1.1).
If the solution is known to be unique, then every sequence of the polygon paths φ_n converges uniformly to this solution. Thus, the proof is constructive in this case.

Existence of solutions of the (IVP) can also be proved via an extension of Brouwer's fixed point theorem to function spaces:

Theorem 3.1.3 (Schauder – Tychonoff) *Consider the set of continuous functions $F = \{x(t) \in \mathcal{C}(I) : |x(t)| \le \mu(t)$ for all $t \in I\}$, where $\mu(t)$ is a fixed positive continuous function. Let $T : F \to F$ such that*

(i) T is continuous, in the sense that if $x_n \in F$ and $x_n \to x$ uniformly on every compact subinterval of J, then $Tx_n \to Tx$ uniformly on every compact subinterval of J,

(ii) the functions $Tx \in TF$ in the image set of T are equicontinuous and bounded at every point of J.

Then the mapping T has at least one fixed point in F.

Its application to the initial value problem follows immediately. Let $f(t,x)$ be continuous and bounded on the strip $t \in I = [t_0, t_1]$, $|x| < \infty$ and let T be the continuous mapping of $\mathcal{C}(I)$ into itself defined by

$$(Tx)(t) = \xi_0 + \int_{t_0}^{t} f(s, x(s))\,\mathrm{d}s.$$

If $|f(t,x)| \leq M$, then

$$|(Tx)(t) - (Tx)(\tilde{t})| \leq \int_{\tilde{t}}^{t} |f(s, x(s))|\,\mathrm{d}s \leq M|t - \tilde{t}|,$$

so the function T is equicontinuous and bounded by $|\xi_0| + M(t_1 - t_0)$. Taking $\mu(t) = |\xi_0| + M(t_1 - t_0)$, all conditions of the Schauder–Tychonoff theorem are satisfied and hence the mapping T has at least one fixed point in F, that is, there exists $\hat{x} \in F$ such that $\hat{x} = T\hat{x}$. By the definition of T, $\hat{x}$ is a solution to (3.1.1),

$$\hat{x}(t) = T\hat{x}(t) = \xi_0 + \int_{t_0}^{t} f(s, \hat{x}(s))\,\mathrm{d}s,$$

and thus a solution to the (IVP) in view of Lemma 3.1.1.

3.2 Uniqueness

An essential concept in establishing uniqueness results is that of Lipschitz continuity. A function f defined in the domain D of the (t,x)-plane is said to satisfy the *Lipschitz condition* with respect to x in D, if there exists a constant $L > 0$ such that

$$|f(t, x_1) - f(t, x_2)| \leq L\,|x_1 - x_2|, \tag{3.3}$$

for every $(t, x_1), (t, x_2)$ in D. This condition is trivially satisfied for functions of class $\mathcal{C}^1$. A function is called locally Lipschitz in D, if for every point (t,x) in D there exists a neighborhood where the function satisfies the Lipschitz condition. Notice that if f is Lipschitz in D, it is in particular uniformly continuous in x for fixed t, although nothing is implied about the continuity of f in t.

Lemma 3.2.1 (Gronwall) *Let $\lambda(t)$ be a real continuous function and $\mu(t)$ a non-negative continuous function on an interval $[a,b]$. If a continuous function $y(t)$ has the property that*

$$y(t) \leq \lambda(t) + \int_{a}^{t} \mu(s)\,\mathrm{d}s,$$

for $a \leq t \leq b$, then on the same interval

$$y(t) \leq \lambda(t) + \int_{a}^{t} \lambda(t)\mu(s) \exp\left(\int_{s}^{t} \mu(r)\,\mathrm{d}r\right)\mathrm{d}s.$$

With this lemma, a straightforward proof of another important inequality can be given, from which existence and uniqueness as well as results on the continuity of solutions can be deduced.

Theorem 3.2.2 *Let $x_1(t), x_2(t)$ be differentiable functions such that*

$$|x_1(a) - x_2(a)| \leq \delta,$$

and

$$|x_i'(t) - f(t, x_i(t))| \leq \varepsilon_i, \quad (i = 1, 2)$$

for $t \in [a, b]$. If the function f is Lipschitz in x with constant L, then

$$|x_1(t) - x_2(t)| \leq \delta e^{L(t-a)} + (\varepsilon_1 + \varepsilon_2)[e^{L(t-a)} - 1]/L$$

for $t \in [a, b]$.

When $\delta = \varepsilon_1 = \varepsilon_2 = 0$, x_1, x_2 are two actual solutions of (3.1) passing through the same point at $t = t_0$. Then $x_1(t) = x_2(t)$ everywhere in $[a, b]$, which proves the following uniqueness result:

Theorem 3.2.3 *Suppose $f(t, x)$ is Lipschitz in x and continuous in t. If φ_1, φ_2 are two solutions of (3.1) on some interval I containing t_0 such that $\varphi_1(t_0) = \varphi_2(t_0) = \xi_0$, then $\varphi_1 \equiv \varphi_2$ everywhere on I.*

Existence can also be proved from Theorem 3.2.2 and we will see another application in the next chapter on continuity of solutions.

Another way of obtaining existence and uniqueness of solutions of the (IVP) in one go is based on the method of successive approximations, originally used by Peano for linear ODE and later for non-linear equations by Picard (it is sometimes referred to as the theorem of Picard – Lindelöf). In a more abstract setting, it can be formulated using the notion of a contraction principle.
On a Banach space X a mapping $T : F \subset X \to X$ is called a *contraction* iff there exists a real constant $0 < K < 1$ such that

$$|Tx - Ty| \leq K|x - y|, \qquad x, y \in F.$$

Consider Banach's contraction principle in the following form:

For $F \subset X$ a closed subspace of a Banach space X, any contraction $T : F \to F$ has a unique fixed point.

Any contraction has at most one fixed point, the contraction principle states that this is the only one. The proof is based on the fact that we can recursively define a sequence $\{x_n\}$ by setting $x_n = Tx_{n-1}, n = 1, 2, \ldots$ which is Cauchy and therefore, as F is closed, converges in F.
The application to an (IVP) is stated in the following

Theorem 3.2.4 *If $f(t,x) \in \mathcal{C}(D)$ is Lipschitz in x with constant L, then the (IVP) has a unique solution on I.*

For the proof, take the Banach space X of continuous functions on $I = [t_0, t_1]$ with norm $\|x\| = \sup_{\{t_0 \le t \le t_1\}}\{e^{-K(t-t_0)}|x(t)|\}$, where K is some constant greater than L, and define the mapping

$$T : X \to X, \quad (Tx)(t) = \xi_0 + \int_{t_0}^{t} f(s, x(s))\, \mathrm{d}s.$$

After proving that T is a contraction, the theorem follows by the contraction principle. We already saw that a fixed point of T is a solution to the (IVP) in the discussion after Theorem 3.1.3.

Before moving on to the next topic of continuity of solutions with respect to the initial condition, we briefly address the issue of continuation of solutions. The previously stated results on existence and uniqueness of solutions are all valid only locally around a starting point (t_0, ξ_0). However, if f remains bounded solutions can be continued over the entire real line, that is, solutions are global unless blow-up occurs.

Theorem 3.2.5 *Let $f \in \mathcal{C}(D)$ and φ a solution of (3.1) on an interval (a, b). If f is bounded on D, then the limits $\varphi(a^+) = \lim_{t\to a^+} \varphi(t)$ and $\varphi(b^-) = \lim_{t\to b^-} \varphi(t)$ exist. If furthermore $(t, \varphi(a^+))$, $(t, \varphi(b^-))$ lie in D, then the solution φ may be continued to the left of a, or right of b.*

3.3 Continuous dependence on initial conditions

We can consider a solution of a differential equation not only as a function of the independent "time" variable t, but also as a function depending on the coordinates of an initial point through which the solutions passes. That is, we consider a solution $\varphi = \varphi(t, t_0, \xi_0)$ of the (IVP) and ask ourselves under what assumptions φ is a continuous function of (t, t_0, ξ_0).
Based on Theorem 3.2.2 from the previous section, we can say that if f is Lipschitz, then a solution to the (IVP) depends continuously on initial conditions.

Corollary 3.3.1 *Suppose f is Lipschitz with constant L and consider two solutions $\varphi_1 = \varphi_1(t, a, \xi_1)$ and $\varphi_2 = \varphi_2(t, a, \xi_2)$ of (3.1) with initial conditions $\varphi_1(a) = \xi_1$, $\varphi_2(a) = \xi_2$ such that $|\xi_1 - \xi_2| < \delta$. Then for $t \in [a, b]$ they satisfy the inequality*

$$|\varphi_1(t) - \varphi_2(t)| \le \delta e^{L(t-a)}.$$

The requirement of f being Lipschitz continuous is actually more than we need; its consequence, uniqueness of solutions, suffices to prove continuous dependence of solutions on initial conditions.

Theorem 3.3.2 *Consider a domain D in (t, x)-space and let f be continuous and bounded by a constant on D. Suppose the (IVP) has a unique solution φ_0 on $[a, b]$, with $t_0 \in [a, b]$. Then there exists $\delta > 0$ such that for all (τ, μ) fixed satisfying*

$$|\tau - t_0| + |\mu - \xi_0| < \delta$$

all solutions $\varphi = \varphi(t, \tau, \mu)$ of

$$x' = f(t, x), \quad x(\tau) = \mu$$

exist over the entire interval $[a, b]$ and as $(\tau, \mu) \to (t_0, \xi_0)$,

$$\varphi(t, \tau, \mu) \to \varphi_0 = \varphi(t, t_0, \xi_0)$$

uniformly over $[a, b]$.

Chapter 4

Background States with Isolated Regions of Vorticity

In Chapter 2 we deduced a model for a background flow with flat free surface in the fluid domain D in terms of the stream function ψ, given a vorticity distribution γ and a function b determining the shape of the seabed. We found that the motion is governed by the equation

$$\Delta\psi = -\gamma(\psi) \qquad \text{in } D, \tag{4.1}$$

subject to the boundary conditions

$$\begin{array}{ll} \psi = \psi_y = 0 & \text{on } y = 0 \\ \psi = 0 & \text{on } y = b(x). \end{array} \tag{4.2}$$

Consider the vorticity function

$$\gamma(\psi) = \begin{cases} \psi - \psi|\psi|^{-\alpha} & \text{for } \psi \neq 0, \\ 0 & \text{for } \psi = 0, \end{cases} \qquad \alpha \in (0,1). \tag{4.3}$$

Theorem 4.1 *System (4.1)-(4.2) has a non-trivial, radially symmetric $\mathcal{C}^2$-solution ψ with compact support in the fluid domain D. This models a background state with flat surface containing an isolated region of non-zero vorticity outside of which the water is still.*

Note that for solutions with compact support the boundary conditions (4.2) will be trivially satisfied, as $\psi \equiv 0$ outside some compact region. The restriction to radially symmetric solutions allows us to use the Ansatz

$$\psi(x,z) = \psi(r) \text{ with } r = \sqrt{(x-x_0)^2 + (y-y_0)^2} \text{ for } (x_0, y_0) \in D.$$

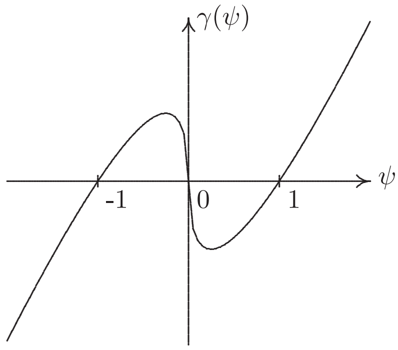

Figure 4.1: The vorticity function γ for the value of $\alpha = \frac{1}{2}$

Equation (4.1) thus simplifies to the second order ordinary differential equation

$$\psi'' + \frac{1}{r}\psi' = -\gamma(\psi), \qquad r > 0, \tag{4.4}$$

where $'$ denotes the derivative with respect to r. To be able to uniquely determine a solution to (4.4), we have to specify initial values for ψ and ψ' at $r = 0$, say $\psi(0) = a_1$, $\psi'(0) = a_2$.
We claim the existence of $a > 0$, such that (4.4) has a unique, non-trivial solution with compact support on $[0, \infty)$ with initial values

$$(\psi(0), \psi'(0)) = (a, 0). \tag{4.5}$$

Instead of solving this second order initial value problem, consider the equivalent system of first order ordinary differential equations

$$\begin{cases} \psi' = \beta, \\ \beta' = -\frac{1}{r}\beta - \psi + \psi|\psi|^{-\alpha}, \end{cases} \qquad r > 0, \tag{4.6}$$

with initial values

$$\psi(0) = a, \qquad \beta(0) = 0. \tag{4.7}$$

Define

$$a_\alpha := (\frac{2}{2-\alpha})^{\frac{1}{\alpha}} > 1 \quad \text{and} \quad M_\alpha := \begin{cases} a_\alpha^{\frac{2}{\alpha^4}} & \text{for } 0 < \alpha \leq \frac{1}{2}, \\ a_\alpha^{\frac{8}{(1-\alpha)\alpha^2}} & \text{for } \frac{1}{2} \leq \alpha < 1. \end{cases} \tag{4.8}$$

The following two results will be crucial in proving Theorem 4.1. Their proofs will be the subject of Sections 4.1 and 4.2, respectively.

Proposition 4.2 *For all $a > a_\alpha$ there exists a unique $\mathcal{C}^2$-solution (ψ, β) to (4.6)-(4.7) which depends continuously on the initial data $(a, 0)$ on any compact interval on which $\psi^2(r) + \beta^2(r) > 0$. Furthermore the solution satisfies $\psi > 1$ for $r \in [0, 1]$.*

Proposition 4.3 *There exists $a > M_\alpha > a_\alpha$ such that for the corresponding solution (ψ, β) of (4.6)-(4.7) there is a finite value $T > 0$ with $\psi(T) = \beta(T) = 0$.*

Proof of Theorem 4.1 As we are looking for radially symmetric solutions to (4.1), it suffices to consider the initial value problem (4.4)-(4.5) which in turn is equivalent to the system (4.6)-(4.7). By virtue of Propositions 4.2 and 4.3 there exists a value of $a > M_\alpha > a_\alpha$ such that for the corresponding uniquely defined $\mathcal{C}^2$-solution to (4.6)-(4.7), we can find $T > 0$ such that $\psi(T) = \beta(T) = 0$. Then by setting $\psi(r) = 0$ for $r \geq T$ we obtain a compactly supported solution of (4.4) defined for all $r \geq 0$. By Proposition 4.2 the solution is strictly greater than 1 on $[0, 1]$. This is the radially symmetric compactly supported non-trivial $\mathcal{C}^2$-solution of (4.1) we were looking for.
Recall from (2.19) that we set the vorticity $\omega = \gamma(\psi)$. Since ψ has compact support with $\psi > 1$ on a ball of unit radius in the fluid domain D, we obtain an isolated region of non-zero vorticity which contains a ball of unit radius where $\omega = \gamma(\psi) > 0$. $\square$

4.1 Uniqueness and continuous dependence of solutions

The aim of this section is to prove Proposition 4.2. We claim that for any $a > a_\alpha$ there exists a unique $\mathcal{C}^2$-solution (ψ, β) to (4.6)-(4.7) which depends continuously on the initial data $(a, 0)$ on any compact interval on which $\psi^2(r)+\beta^2(r) > 0$ and for which $\psi > 1$ for $r \in [0, 1]$.

This is not immediately clear for two reasons:

- the right hand side of (4.6) displays a discontinuity at $r = 0$, so the system is not a classical initial value problem.
- since the vorticity function $\gamma(\psi)$ fails to be locally Lipschitz when $\psi = 0$ the right hand side of (4.6) is not locally Lipschitz and we cannot apriori expect uniqueness of solutions or continuous dependence on initial data from the standard theory discussed in Chapter 3.

In the first part of the proof, summed up in Lemma 4.1.1, we consider the system in the vicinity of the discontinuity at $r = 0$. In this case, it turns out to be more convenient to work with formulation (4.4) . By a simple change of variables we overcome the problem of the discontinuity and find that locally ψ stays away from zero. We then solve this equivalent system using an integral Ansatz and Banach's fixed point theorem. After that we ensure continuous dependence of the solution on the initial data $(a, 0)$. In Lemma 4.1.2 we introduce an important function which will be helpful in deriving results throughout the proofs of both Proposition 4.2 and 4.3. In Lemma 4.1.3 we tackle the second part of the proof as we treat the system (4.6) away from the discontinuity. By rewriting it in polar coordinates we get existence and uniqueness of solutions as well as continuous dependence on initial data via standard results whenever the right hand side is $\mathcal{C}^1$. We overcome the problem of a lack of local Lipschitz continuity in one of the dependent variables by a simple application of the inverse function theorem. This yields yet another formulation of the system which guarantees local uniqueness and continuous dependence at points where the right hand side fails to be Lipschitz.

Lemma 4.1.1 *For $r \in [0,1]$ the system (4.6) can be equivalently written as*

$$v'' + e^{-2s}(v - v|v|^{-\alpha}) = 0, \qquad s \geq 0, \tag{4.9}$$

where the initial values (4.7) are described by the limits

$$v(s) \to a \quad \text{and} \quad v'(s)e^s \to 0 \quad \text{for } s \to \infty. \tag{4.10}$$

(4.9) has a unique $\mathcal{C}^2$-solution which depends continuously on the parameter a and is strictly greater than 1.

Proof We perform the change of variables

$$s = -\ln r, \qquad \psi(r) = v(s), \tag{4.11}$$

and find that (4.6) is equivalent to

$$v'' + e^{-2s}(v - v|v|^{-\alpha}) = 0, \qquad s \in \mathbb{R},$$

since (4.6) is equivalent to (4.4) and

$$\psi'(r) = \frac{\partial v(s)}{\partial s}\frac{\partial s}{\partial r} = -v'(s)\frac{1}{r} \quad \text{and} \quad \psi''(r) = v''(s)\frac{1}{r^2} + v'(r)\frac{1}{r^2}.$$

Then

$$\psi'' + \frac{1}{r}\psi' + \psi - \psi|\psi|^{-\alpha} = v''\frac{1}{r^2} + v'\frac{1}{r^2} - \frac{1}{r^2}v' + v - v|v|^{\alpha} = 0$$

yields, in view of (4.11),

$$v'' + r^2(v - v|v|^{\alpha}) = v'' + e^{-2s}(v - v|v|^{\alpha}) = 0.$$

The restriction $0 \leq r \leq 1$ is equivalent to $s \geq 0$ in the new variable.
Let an arbitrary $a > a_\alpha$ be fixed. We can deal with local existence and uniqueness issues of a solution to (4.6)-(4.7) by considering the integral equation

$$v(s) = a - \int_s^\infty (\tau - s)e^{-2\tau}\underbrace{(v(\tau) - v(\tau)|v(\tau)|^{-\alpha})}_{\gamma(v(\tau))}\, d\tau, \qquad s \geq 0, \tag{4.12}$$

with

$$v'(s) = \int_s^\infty e^{-2\tau}\gamma(v(\tau))\, d\tau, \qquad s \geq 0, \tag{4.13}$$

ensuring the corresponding asymptotic behavior (4.10), since

$$\lim_{s\to\infty} v(s) = \lim_{s\to\infty} a - \int_s^\infty (\tau - s)e^{-2\tau}\gamma(v(\tau))d\tau = a$$

and

$$\begin{aligned}\lim_{s\to\infty} v'(s)e^s &= \lim_{s\to\infty} \frac{\int_s^\infty e^{-2\tau}\gamma(v(\tau))d\tau}{e^{-s}} = \lim_{s\to\infty} \frac{e^{-2s}\gamma(v(s))}{e^{-s}} \\ &= \lim_{s\to\infty} e^{-s} \underbrace{\gamma(v(s))}_{\to a-a|a|^{-\alpha}} = 0,\end{aligned}$$

where we used the rule of de l'Hospital in the second equality.
As long as $v(s) \geq 1$ we have that v is non-decreasing, since $\gamma(v) \geq 0$ which in view of (4.13) gives $v'(s) \geq 0$. We even have

$$v(s) > a^{1-\alpha} > 1 \quad \text{ for } s \geq 0. \tag{4.14}$$

Indeed, if this were not so, define $s_1 := \sup\{s \geq 0 : v(s) = a^{1-\alpha}\}$. Then for all $s \geq s_1$ we have $1 < a^{1-\alpha} \leq v(s) \leq a$, which in view of (4.12) yields a contradiction as

$$\begin{aligned}0 < a - a^{1-\alpha} = a - v(s_1) &= \int_{s_1}^\infty (\tau - s_1)e^{-2\tau}\Big(v(\tau) - v(\tau)^{1-\alpha}\Big)d\tau \\ &\leq (a - a^{1-\alpha}) \int_{s_1}^\infty (\tau - s_1)e^{-2\tau}d\tau = (a - a^{1-\alpha})\frac{e^{-2s_1}}{4} \\ &\leq \frac{(a - a^{1-\alpha})}{4},\end{aligned}$$

in view of the fact that $\gamma(v)$ is strictly increasing for $v \in [a^{1-\alpha}, a]$. So for $s \geq 0$ we have that $v(s) > a^{1-\alpha} > 1$ is non-decreasing.

These considerations allow us to view the solution of the integral equation (4.12) as the unique fixed point of the contraction T_a defined by

$$T_a(v)(s) := a - \int_s^\infty (\tau - s)e^{-2\tau}(v(\tau) - v(\tau)|v(\tau)|^{-\alpha})d\tau, \qquad s \geq 0, \tag{4.15}$$

on the closed subspace $X_a := \{v \in X : a^{1-\alpha} \leq v(s) \leq a,\ s \geq 0\}$ of the Banach space X of bounded continuous functions on $[0, \infty)$ endowed with the supremum norm $\|v\| = \sup_{s\geq 0}\{|v(s)|\}$.

Recall Banach's contraction principle from Chapter 3:
For $F \subset X$ a closed subspace of a Banach space X, any contraction $T : F \to F$ has a unique fixed point.

To be able to apply this principle to (4.15) and subsequently to the integral equation (4.12), we have to check the hypothesis.

X_a is clearly a closed subspace of X. Notice that for $v \in X_a$ we have $v \geq 1$, since $a > a_\alpha$ and thus $a^{1-\alpha} > (\frac{2}{2-\alpha})^{\frac{1-\alpha}{\alpha}} > 1$ for $0 < \alpha < 1$. Let us check that $T_a(v) \in X_a$, i.e.

$$a^{1-\alpha} \leq a - \int_s^\infty (\tau - s)e^{-2\tau}\gamma(v(\tau))d\tau \leq a, \qquad s \geq 0.$$

The upper bound follows from the fact that the integral is positive, since for $v \in X_a$, $v \geq 1$ and thus $\gamma(v) \geq 0$. For the lower bound, we use the same reasoning as in the proof of (4.14).
Now we show that T_a as defined above is a contraction.
Since the vorticity function γ defined in (4.3) is $\mathcal{C}^1$ on $[1, \infty)$, by the mean value theorem (cf. [Die69]) there exists $\xi \in (v, w)$ for $v, w \geq 1$ such that $\gamma(v) - \gamma(w) = f'(\xi)(v - w)$. This yields

$$|\gamma(v) - \gamma(w)| \leq |v - w| \qquad \text{for } v, w, \geq 1, \tag{4.16}$$

since for $\xi \geq 1$, $f'(\xi) = 1 - \xi^{-\alpha} + \alpha\xi^{-\alpha} \leq 1$ is true.
Then for $s \geq 0$ we have

$$\begin{aligned}\Big|\int_s^\infty (\tau-s)e^{-2\tau}[\gamma(v(\tau)) - \gamma(w(\tau))]d\tau\Big| &\leq \int_s^\infty (\tau-s)e^{-2\tau}\Big|\gamma(v(\tau)) - \gamma(w(\tau))\Big|d\tau \\ &\leq \int_s^\infty (\tau-s)e^{-2\tau}|v(\tau) - w(\tau)|d\tau \\ &\leq \|v-w\| \int_s^\infty (\tau-s)e^{-2\tau}d\tau = \frac{1}{4}\|v-w\|,\end{aligned}$$

whenever $v, w \in X_a$. Thus

$$\begin{aligned}\|T_a(v) - T_a(w)\| &\leq \Big|\int_s^\infty (\tau - s)e^{-2\tau}\Big(\gamma(v(\tau)) - \gamma(w(\tau))\Big)d\tau\Big| \\ &\leq \frac{1}{4}\|v - w\| \qquad \text{for } v, w \in X_a, s \geq 0,\end{aligned}$$

which shows that T_a is a contraction on X_a with contraction constant $K \leq \frac{1}{4}$.
Therefore, according to Banach's contraction principle, T_a has a unique fixed point, i.e. the integral equation (4.12) has a unique solution $v \in X_a$. To confirm that the fixed point $v = T_a(v)$ is really a solution of the differential equation (4.9), observe that

$$\partial_s \int_s^\infty (\tau - s)e^{-2\tau}\gamma(v(\tau))d\tau = -\int_s^\infty e^{-2\tau}\gamma(v(\tau))d\tau$$

So $v'(s) = \int_s^\infty e^{-2\tau}\gamma(v(\tau))d\tau$ and the fundamental theorem of calculus (cf. [Die69]) yields (4.9), since

$$v''(s) = \partial_s \int_s^\infty e^{-2\tau}\gamma(v(\tau))d\tau = -e^{-2s}\gamma(s).$$

The solution is of class $\mathcal{C}^2$ since $v''(s) = -e^{-2s}(v(s) - v(s)^{1-\alpha})$ is continuous for $s \geq 0$.

To show continuous dependence of the solution on the parameter a, let $v_1 \in X_{a_1}, v_2 \in X_{a_2}$. Then the integral equation (4.12) yields, in view of (4.16), that for $s \geq 0$

$$\begin{aligned} |v_1(s) - v_2(s)| &\leq |a_1 - a_2| + \int_s^\infty (\tau - s)e^{-2\tau}\Big|\gamma(v_1(\tau)) - \gamma(v_2(\tau))\Big|d\tau \\ &\leq |a_1 - a_2| + \int_s^\infty (\tau - s)e^{-2\tau}|v_1(\tau) - v_2(\tau)|d\tau \qquad (4.17) \\ &\leq |a_1 - a_2| + \frac{e^{-2s}}{4}\|v_1 - v_2\| \leq |a_1 - a_2| + \frac{1}{4}\|v_1 - v_2\|, \end{aligned}$$

and therefore

$$\|v_1 - v_2\| \leq \frac{4}{3}|a_1 - a_2|. \qquad (4.18)$$

This tells us that v depends continuously on the initial data a for $s \geq 0$. (We actually even obtain that the solution is stable, cf. [Cop65]). □

Remark In view of the formulation (4.6) of the problem, Lemma 4.1.1 tells us that for $r \leq 1$, (4.6)-(4.7) has a unique $\mathcal{C}^2$-solution (ψ, β) which depends continuously on a for $r \in [0, 1]$ and is such that $\psi > 1$.

Before we proceed to the case where $r \geq 1$, we prove the following useful

Lemma 4.1.2 *The function*

$$E(r) = E(\psi, \beta) = \frac{1}{2}\beta^2 + \frac{1}{2}\psi^2 - \frac{1}{2-\alpha}|\psi|^{2-\alpha} \qquad (4.19)$$

satisfies

$$E'(r) = -\frac{1}{r}\beta^2, \qquad r > 0, \qquad (4.20)$$

as long as solutions to (4.6)-(4.7) *exist and remains bounded for all* $r > 0$. *Furthermore,* $E(r)$ *is strictly decreasing whenever* $(\psi(r), \beta(r)) \notin \{(0, 0), (\pm 1, 0)\}$. *We can conclude that solutions to* (4.6)-(4.7) *are defined for all* $r \geq 0$ *and that* ψ *and* β *are bounded functions of* r.

Proof As long as a solution to (4.6)-(4.7) exists, we have $E'(r) = -\frac{1}{r}\beta^2$, since the derivative with respect to r of the function $E(r)$ given by (4.19) can be computed as

$$\begin{aligned} E'(r) &= \psi\psi' - |\psi|^{1-\alpha} sgn(\psi)\psi' + \beta\beta' \\ &= \psi\beta - \beta|\psi|^{-\alpha}\psi + \beta(-\frac{1}{r}\beta - \psi + \psi|\psi|^{-\alpha}) \\ &= -\frac{1}{r}\beta^2. \end{aligned}$$

Notice that E attains its minimum $E_{min} = \frac{\alpha}{2(\alpha-2)} < 0$ at $(\psi, \beta) = (\pm 1, 0)$, so that (4.20) ensures that E remains bounded. Furthermore,

$$\inf_{\psi \in \mathbb{R}} \left\{ \frac{\psi^2}{2} - \frac{1}{2-\alpha} |\psi|^{2-\alpha} \right\} = \frac{\alpha}{2(\alpha-2)} \quad \text{and} \quad \lim_{|\psi| \to \infty} \left\{ \frac{\psi^2}{2} - \frac{1}{2-\alpha} |\psi|^{2-\alpha} \right\} = \infty.$$

Therefore ψ and β remain bounded as long as solutions exist, since otherwise E would become unbounded. We can conclude that the solutions to (4.6)-(4.7) are defined for all $r \geq 0$.
In view of (4.20) we see that $E(r)$ is strictly decreasing whenever $(\psi(r), \beta(r)) \notin \{(0,0), (\pm 1, 0)\}$. Otherwise, for $r_2 > r_1 > 0$ with $E(r_2) = E(r_1)$, we would have

$$0 = E(r_2) - E(r_1) = \int_{r_1}^{r_2} E'(r) dr = -\int_{r_1}^{r_2} \frac{\beta^2(r)}{r} dr.$$

This implies $\beta(r) = 0$ on $[r_1, r_2]$ and consequently from (4.6) we have that $\psi'(r) = 0$ and $\beta'(r) = -\psi(r) + \psi(r)|\psi(r)|^{-\alpha} = 0$ for all $r \in [r_1, r_2]$. Thus, $\psi(r) = \psi(r_1)$ is constant in $[r_1, r_2]$ and $\psi(r_1) = \psi(r_1)|\psi(r_1)|^{-\alpha}$ so that $\psi(r) \in \{0, \pm 1\}$, a contradiction. □

Now we consider the system away from the discontinuity at $r = 0$ and prove existence, uniqueness and continuous dependence of solutions on the parameter a as long as $\psi^2 + \beta^2 > 0$.

Lemma 4.1.3 *For $r \geq 1$ system (4.6) can be equivalently reformulated as*

$$\begin{cases} \theta'(r) = -\dfrac{1}{2r} \sin(2\theta) - 1 + R^{-\alpha} |\cos(\theta)|^{2-\alpha}, \\[2ex] R'(r) = -\dfrac{1}{r} R \sin^2(\theta) + R^{1-\alpha} \sin(\theta) \dfrac{\cos(\theta)}{|\cos(\theta)|^{\alpha}}, \end{cases} \quad r \geq 1. \tag{4.21}$$

As long as $R > 0$ this system of first order differential equations has a unique $\mathcal{C}^2$-solution which depends continuously on the initial data $(\theta(1), R(1))$, which in turn depends continuously on the parameter a.

Proof We introduce polar coordinates

$$\psi = R\cos(\theta), \qquad \beta = R\sin(\theta), \tag{4.22}$$

to show that (4.21) is yet another equivalent formulation of (4.6):

$$\begin{aligned} \theta'(r) &= \frac{d}{dr} \arctan\left(\frac{\beta(r)}{\psi(r)}\right) = \frac{\beta'\psi - \beta\psi'}{\psi^2 + \beta^2} \overset{(4.6)}{=} \frac{-\frac{1}{r}\beta\psi - \psi^2 + |\psi|^{2-\alpha} - \beta^2}{\psi^2 + \beta^2} \\ &\overset{(4.22)}{=} \frac{-\frac{1}{r} R^2 \sin(\theta)\cos(\theta) - R^2\cos^2(\theta) + R^{2-\alpha}|\cos(\theta)|^{2-\alpha} - R^2\sin^2(\theta)}{R^2} \\ &= -\frac{1}{2r}\sin(2\theta) + R^{-\alpha}|\cos(\theta)|^{2-\alpha} - 1, \end{aligned}$$

and

$$\begin{aligned}
R'(r) &= \frac{\psi' \cos(\theta) + \psi \sin(\theta)\theta'}{\cos^2(\theta)} \\
&= \frac{R\sin(\theta)\cos(\theta) + R\cos(\theta)\sin(\theta)(-\frac{1}{r}\sin(\theta)\cos(\theta) + R^{-\alpha}|\cos(\theta)|^{2-\alpha} - 1)}{\cos^2(\theta)} \\
&= -\frac{1}{r}\sin^2(\theta)R + R^{1-\alpha}\sin(\theta)\cos(\theta)|\cos(\theta)|^{-\alpha}.
\end{aligned}$$

The initial data $(\theta(1), R(1))$ is specified after solving the integral equation (4.12) on $[0, \infty)$.
To show continuous dependence of $(\theta(1), R(1))$ on a, notice that (4.13) in view of (4.16) and (4.18) yields for $s \geq 0$

$$\begin{aligned}
|v_1'(s) - v_2'(s)| &\leq \int_s^\infty e^{-2\tau}|\gamma(v_1(\tau)) - \gamma(v_2(\tau))|d\tau \\
&\leq \int_s^\infty e^{-2\tau}|v_1(\tau) - v_2(\tau)|d\tau \\
&\leq \frac{1}{4}\|v_1 - v_2\| \leq \frac{1}{3}|a_1 - a_2|. \qquad (4.23)
\end{aligned}$$

Evaluating inequalities (4.17) and (4.23) at $s = 0$ together with (4.18) yields

$$|v_1(0) - v_2(0)| + |v_1'(0) - v_2'(0)| \leq \frac{5}{3}|a_1 - a_2|. \qquad (4.24)$$

This means that if the initial data a varies little, both v and v' vary little at $s = 0$. In view of the formulation (4.21) of the initial value problem (4.6)-(4.7) this means that ψ and $\psi' = \beta$ vary little at $r = 1$. Thus, $\theta(1) = \arctan\left(\frac{\beta(1)}{\psi(1)}\right)$ and $R(1) = \sqrt{\psi^2(1) + \beta^2(1)}$ depend continuously on a. The considerations we made in Lemma 4.1.1 show that $\psi(1) > a^{1-\alpha} > 0$ and $R(1) = \sqrt{\psi^2(1) + \beta^2(1)} > 0$, so $\cos(\theta(1)) = \frac{\psi(1)}{R(1)} > 0$. As long as $R > 0$ and $\cos(\theta) > 0$ the right hand side of (4.21) is $\mathcal{C}^1$. Thus we get local existence and uniqueness as well as continuous dependence on initial data $(\theta(1), R(1))$ for a solution to (4.21) by the standard results discussed in Chapter 3.

What we have to show is that as long as $R > 0$ this continues to hold even if $\cos(\theta(r)) = 0$, that is, when a solution intersects the vertical axis in the (ψ, β)-phase plane. At such points, the right hand side of (4.21) is still continuous, but fails to be locally Lipschitz. Thus we can still rely on local existence of solutions from Theorem 3.1.2 (Cauchy – Peano), since we know that the right hand side of (4.21) is continuous and bounded by virtue of Lemma 4.1.2. Uniqueness and continuous dependence on initial data on the other hand are no longer guaranteed. We can overcome this problem by transforming the system in a neighborhood of such values of r, taking advantage of its local structure.

Denote by r_0 the smallest value of $r > 1$ where $\cos(\theta(r_0)) = 0$, say $\theta(r_0) = -\frac{\pi}{2}$. Since for $r \in (1, r_0)$ the right hand side of (4.21) is $\mathcal{C}^1$, the solution is unique and depends continuously on the initial data $(\theta(1), R(1))$ up to r_0. We then select one of the possible continuations of the solution across $r = r_0$, cf. 3.2.5, and show that this selection is unique and depends continuously on $(\theta(1), R(1))$ close to $r = r_0$.
Since $\cos(\theta(r_0)) = 0$ and $\theta'(r_0) = -1$, the inverse function theorem (cf. [Die69]) guarantees the existence of neighborhoods $(r_0 - \varepsilon, r_0 + \varepsilon)$ of r_0 and $(-\delta, \delta^+)$ of 0 for sufficiently small $\varepsilon > 0$ and $\delta, \delta^+ > 0$, as well as a uniquely determined $\mathcal{C}^1$-function $\varphi(\tau) = r$ such that $\varphi(0) = r_0$, $\varphi(-\delta) = r_0 - \varepsilon$ and $\varphi(\delta^+) = r_0 + \varepsilon$ which allows us to locally set

$$\cos(\theta(r)) = -\tau.$$

Notice that this transformation preserves the monotonicity of the respective independent variables, since $\theta' < 0$ and $\cos(\theta)$ is increasing in a neighborhood of $-\frac{\pi}{2}$. Then $r_0 - \varepsilon < r_0 < r_0 + \varepsilon$ implies $\cos(\theta(r_0 - \varepsilon)) > \cos(\theta(r_0)) > \cos(\theta(r_0 + \varepsilon))$, or, equivalently, $\cos(\varphi(-\delta)) > 0 > \cos(\varphi(\delta^+))$. This implies $-\delta < 0 < \delta^+$, since $\cos(\theta(\varphi(\tau))) = -\tau$ in a δ-neighborhood of 0. Differentiating the equation $\varphi(\tau) = \varphi(-\cos(\theta(r))) = r$ with respect to $r \in (r_0 - \varepsilon, r_0 + \varepsilon)$ yields

$$\varphi'(\tau) = \frac{1}{\theta'(r)\sin(\theta(r))}, \qquad \tau \in (-\delta, \delta^+).$$

Setting

$$\rho(\tau) = R(r)$$

yields

$$\rho'(\tau) = R'(r)\,\varphi'(\tau), \qquad \text{for } r \in (r_0 - \varepsilon, r_0 + \varepsilon),\ \tau \in (-\delta, \delta^+).$$

Now we transfer (4.21) for $r \in (r_0 - \varepsilon, r_0 + \varepsilon)$ and $\tau \in (-\delta, \delta^+)$ into the system

$$\begin{cases} \varphi'(\tau) = \dfrac{1}{\sqrt{1-\tau^2} + \frac{\tau(1-\tau^2)}{\varphi(\tau)} - \rho(\tau)^{-\alpha}|\tau|^{2-\alpha}\sqrt{1-\tau^2}}, \\[2ex] \rho'(\tau) = -\dfrac{\frac{1}{\varphi(\tau)}\rho(\tau)\sqrt{1-\tau^2} - \rho(\tau)^{1-\alpha}\frac{\tau}{|\tau|^{\alpha}}}{-1 - \frac{\tau\sqrt{1-\tau^2}}{\varphi(\tau)} + \rho^{-\alpha}(\tau)|\tau|^{2-\alpha}}. \end{cases} \tag{4.25}$$

A straightforward calculation and the fact that

$$\sin(\theta) = -\sqrt{1-\cos^2(\theta)} = -\sqrt{1-\tau^2}$$

and

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\tau\sqrt{1-\tau^2}$$

for $\varepsilon > 0$ small enough shows that (4.25) and (4.21) are equivalent.

The advantage of the system (4.25) with respect to (4.21) is that the lack of $\mathcal{C}^1$-regularity in θ was shifted into a lack of $\mathcal{C}^1$-regularity in τ. Consequently, the new system is $\mathcal{C}^1$ in the unknown variables $(\varphi, \rho) \in (1, \infty) \times (0, \infty)$ and continuous in the independent variable τ. This is enough to ensure uniqueness and continuous dependence on initial data $(\varphi(-\delta), \rho(-\delta))$ of the solutions to (4.25), cf. Sections 3.2 and 3.3. Furthermore, $(\varphi(-\delta), \rho(-\delta))$ depends continuously on $(\theta(r_0 - \varepsilon), R(r_0 - \varepsilon))$ via the $\mathcal{C}^1$-function $\varphi(\tau) = r$ and we already mentioned the continuous dependence of solutions $(\theta(r), R(r))$ on $(\theta(1), R(1))$ and thus on the parameter a for $r \in (1, r_0)$. We can therefore deduce that uniqueness and continuous dependence on a of the solution to (4.21) holds also in a neighborhood of r_0.

This procedure can be repeated in almost the same way for the next value of $r > r_0$ where $\cos(\theta(r)) = 0$ as long as $R > 0$, i.e. where the solution intersects the vertical axis in the upper half plane. Denote by r_1 the value of $r > r_0$ such that $\theta(r_1) = \frac{\pi}{2}$. Then again, since $\theta'(r_1) = -1$, the inverse function theorem (cf. [Die69]) guarantees the existence of neighborhoods $(r_1 - \varepsilon, r_1 + \varepsilon)$ of r_1 and $(-\delta, \delta^+)$ of 0 for sufficiently small $\varepsilon > 0$ and $\delta, \delta^+ > 0$, as well as a uniquely determined $\mathcal{C}^1$-function $\varphi(\tau) = r$ such that $\varphi(0) = r_1$, $\varphi(-\delta) = r_1 - \varepsilon$ and $\varphi(\delta^+) = r_1 + \varepsilon$ which allows us to locally set

$$\cos(\theta(r)) = \tau.$$

Notice that this time we choose τ instead of $-\tau$ to preserves monotonicity of the respective independent variables, since $\cos(\theta)$ is now decreasing in a neighborhood of $\frac{\pi}{2}$. So $\cos(\theta(r_1 - \varepsilon)) < \cos(\theta(r_1)) < \cos(\theta(r_1 + \varepsilon))$, or, equivalently, $\cos(\varphi(-\delta)) < 0 < \cos(\varphi(\delta^+))$. This implies $-\delta < 0 < \delta^+$ since $\cos(\theta(\varphi(\tau))) = \tau$ in a δ-neighborhood of 0. Differentiating the equation $\varphi(\tau) = \varphi(\cos(\theta(r))) = r$ with respect to $r \in (r_1 - \varepsilon, r_1 + \varepsilon)$ yields

$$\varphi'(\tau) = \frac{-1}{\theta'(r)\sin(\theta(r))}, \qquad \tau \in (-\delta, \delta^+).$$

As before, setting

$$\rho(\tau) = R(r),$$

yields

$$\rho'(\tau) = R'(r)\,\varphi'(\tau), \qquad \text{for } r \in (r_1 - \varepsilon, r_1 + \varepsilon),\ \tau \in (-\delta, \delta^+).$$

Now we transfer (4.21) for $r \in (r_1 - \varepsilon, r_1 + \varepsilon)$ and $\tau \in (-\delta, \delta^+)$ into the system

$$\begin{cases} \varphi'(\tau) = \dfrac{1}{\sqrt{1-\tau^2} + \frac{\tau(1-\tau^2)}{\varphi(\tau)} - \rho(\tau)^{-\alpha}|\tau|^{2-\alpha}\sqrt{1-\tau^2}}, \\[2ex] \rho'(\tau) = \dfrac{\frac{1}{\varphi(\tau)}\rho(\tau)\sqrt{1-\tau^2} - \rho(\tau)^{1-\alpha}\frac{\tau}{|\tau|^{\alpha}}}{-1 - \frac{\tau\sqrt{1-\tau^2}}{\varphi(\tau)} + \rho^{-\alpha}(\tau)|\tau|^{2-\alpha}}. \end{cases} \tag{4.26}$$

Again, the fact that $\sin(\theta) = \sqrt{1 - \cos^2(\theta)} = \sqrt{1 - \tau^2}$ and

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\tau\sqrt{1 - \tau^2}$$

for $\varepsilon > 0$ small enough ensures that (4.26) and (4.21) are equivalent. (4.26) differs from (4.25) only by a change of sign in the second equation. Thus by the same reasoning as above we deduce that uniqueness and continuous dependence on a of the solution to (4.21) also holds in a neighborhood of r_1.

Summing up, we can say that for values of r where $\cos(\theta(r)) = 0$, that is, where $\theta(r) = -\frac{\pi}{2} + 2k\pi$ or $\theta(r) = \frac{\pi}{2} + 2k\pi$ for $k \in \mathbb{Z}$, the above local transformations guarantee uniqueness and continuous dependence on a of the solution to (4.21) also in neighborhoods of such values as long as $R > 0$. In between these values of r, the right hand side of (4.21) is $\mathcal{C}^1$ and everything follows from standard results. $\square$

This concludes the proof of Proposition 4.2, as we have seen that for any $a > a_\alpha$ there exists a unique $\mathcal{C}^2$-solution (ψ, β) to (4.6)-(4.7) for which $\psi > 1$ on $[0, 1]$ by virtue of (4.14) and which depends continuously on the initial data $(a, 0)$ on any compact interval on which $\psi^2(r) + \beta^2(r) > 0$.

4.2 Existence of a compactly supported solution

In this section, we prove Proposition 4.3, that is, we show that there exists a value of $a > a_\alpha$ such that for the corresponding solution to (4.6)-(4.7) we can find some $0 < T < \infty$ with $\psi(T) = \beta(T) = 0$.

The idea is to perform a detailed qualitative analysis for the system (4.6)-(4.7), similar to the phase-plane analysis of autonomous systems. We introduce two sets $\Omega_\pm$ defined by the solution sets of the equation $E(\psi, \beta) = 0$, where E is the function defined in Section 4.1. In Lemma 4.2.1 we are going to show that for initial data a large enough the solution can enter the region $\Omega_\pm$ only for values of $r > \frac{2}{\alpha}$. We will also see that the value of r at which the solution can enter $\Omega_\pm$ will tend to infinity as $a \to \infty$. After that, Lemma 4.2.2 ensures that there exists an initial value a_+ such that the corresponding solution stays outside $\Omega_- \cup \Omega_+$ for all $r \geq 0$. Finally, in Lemma 4.2.3, we prove that for solutions corresponding to such initial data a_+ there exists a finite value $T > 0$ such that $E(T) = 0$, and therefore also $\psi(T) = \beta(T) = 0$.

Let us start with defining the sets $\Omega_\pm$. From (4.19) in Lemma 4.1.2 we have that

$$E(\psi, \beta) = 0 \quad \text{if and only if} \quad \beta^2 = \frac{2}{2 - \alpha}|\psi|^{2-\alpha} - \psi^2.$$

In the plane (ψ, β) the set where $E < 0$ consists of the interiors $\Omega_\pm$ of the closed curves representing the solution set of the above equation. These curves are symmetrical with respect to the vertical and the horizontal axis and are

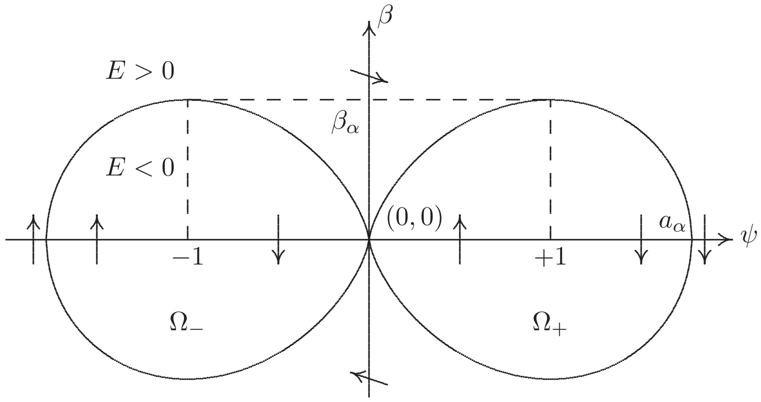

Figure 4.2: The solution set of $E = 0$ in the phase plane (ψ, β) with arrows indicating the dynamics of the system

tangential to one another and to the vertical axis at the origin. Note from (4.8) that the curves reach their maximum $\beta_\alpha = (\frac{\alpha}{2-\alpha})^{\frac{1}{2}}$ at $\psi = \pm 1$ and they intersect the horizontal axis at the points $\psi = 0$ and $\psi = \pm(\frac{2}{2-\alpha})^{\frac{1}{\alpha}} = \pm a_\alpha$.
To get a better understanding of the dynamics of the system (4.6), consider the right half plane, where $\psi > 0$. At $\beta = 0$ we have $\psi' = \beta = 0$ and $\beta' = -\frac{1}{r}\beta - \psi + \psi|\psi|^{-\alpha} > 0$ when $\psi|\psi|^{-\alpha} > \psi$ which is true for $0 < \psi < 1$, whereas $\beta' < 0$ for $\psi > 1$. In the left half plane, we have exactly the opposite situation. Therefore, solutions intersect the horizontal axis perpendicularly from the upper to the lower half plane for $\psi > 1$ and for $-1 < \psi < 0$. On the complement of these sets, they intersect the axis in the opposite direction.
For $\psi = 0$ and $\beta > 0$ we have that $\psi' > 0$ and $\beta' < 0$, which means that solutions intersect the vertical axis from left to right in the upper half plane. In the lower half plane, the opposite is true (see Figure 4.2).

By Lemma 4.1.2, E is strictly decreasing as long as $(\psi, \beta) \notin \{(0,0), (\pm 1, 0)\}$. Therefore, once a solution reaches the boundary of $\Omega_\pm$ at a point other than $(0,0)$ it will enter $\Omega_\pm$. Once inside, a solution will stay in either Ω_+ or Ω_- for all subsequent times, as E is strictly decreasing.

Recall from (4.8) that

$$M_\alpha := \begin{cases} a_\alpha^{\frac{2}{\alpha^4}} & \text{for } 0 < \alpha \leq \frac{1}{2}, \\ a_\alpha^{\frac{8}{(1-\alpha)\alpha^2}} & \text{for } \frac{1}{2} \leq \alpha < 1. \end{cases}$$

For certain initial data, solutions stay outside of $\Omega_\pm$ for some time:

Lemma 4.2.1 *For $a > M_\alpha$ we have that $E(r) > 0$ as long as $r \in [0, \frac{2}{\alpha}]$. This means that a solution to (4.6)-(4.7) with $a > M_\alpha$ can enter $\Omega_\pm$ only for values of $r > \frac{2}{\alpha}$. Additionally, we find that the value of r such that a solution can enter $\Omega_\pm$ tends to infinity as $a \to \infty$.*

Proof Let $a > a_\alpha^{\frac{2}{1-\alpha}}$. We know from the results in Lemma 4.1.1 that $R(r) = \frac{\psi(r)}{\cos(\theta(r))} > a^{1-\alpha} > a_\alpha$ for $r \in [0,1]$, since (4.14) implies $\psi(r) > a^{1-\alpha}$ for $r \leq 1$. If a solution with initial data a enters the region $\Omega_+ \cup \Omega_-$, then for some value of $r^* > 1$ we will have $R(r^*) = a_\alpha < a^{\frac{1-\alpha}{2}}$. We can therefore define

$$\begin{aligned} r_0 &= \inf\{r > 0 : R(r) = a_\alpha\} \\ r_1 &= \sup\{0 < r < r_0 : R(r) = a^{1-\alpha}\} > 1 \\ r_2 &= \sup\{r_1 < r < r_0 : R(r) = a^{\frac{1-\alpha}{2}}\} \end{aligned} \tag{4.27}$$

such that

$$a^{1-\alpha} = R(r_1) \geq R(r) \geq R(r_2) = a^{\frac{1-\alpha}{2}} \qquad \text{for } r \in [r_1, r_2].$$

The argument leading to the desired result requires us to consider separately the case where $\alpha \in (0, \frac{1}{2}]$ and the case where $\alpha \in [\frac{1}{2}, 1)$. Some inequalities involving functions of α in the exponent will be denoted by (a)-(c) and will be shown at the end of the proof of this Lemma.
Let $\alpha \in (0, \frac{1}{2}]$. We claim that

$$r_2 \geq a^{\alpha^3}. \tag{4.28}$$

Indeed, assume to the contrary that $r_2 < a^{\alpha^3}$, then from (4.21) we infer that

$$R'(r) = -\frac{1}{r}R\underbrace{\sin^2(\theta)}_{\leq 1} + R^{1-\alpha}\underbrace{\sin(\theta)\frac{\cos(\theta}{|\cos(\theta)|^\alpha}}_{\geq -1} \geq -\frac{1}{r}R - R^{1-\alpha} > -\frac{2}{r}R.$$

In the last inequality we have used that $R(r) \geq a^{\frac{1-\alpha}{2}}$, thus for $\alpha \in (0, \frac{1}{2}]$ we have that $R^\alpha \geq a^{\frac{(1-\alpha)\alpha}{2}} \overset{(a)}{\geq} a^{\alpha^3} > r_2 \geq r$ for $r \in [r_1, r_2]$, which yields $R > rR^{1-\alpha}$ or, equivalently, $-R^{1-\alpha} > -\frac{1}{r}R$. Integrating the differential inequality

$$\frac{R'(r)}{R(r)} > -\frac{2}{r}, \qquad r \in [r_1, r_2]$$

with respect to r on $[r_1, r_2]$ yields

$$\ln R(r_2) - \ln R(r_1) > -2(\ln r_2 - \ln r_1).$$

Then

$$\ln r_2 > \ln r_1 + \frac{1}{2}\ln\left(\frac{R(r_1)}{R(r_2)}\right) = \ln r_1 + \frac{1}{2}\ln\left(\frac{a^{1-\alpha}}{a^{\frac{1-\alpha}{2}}}\right) = \ln r_1 + \ln(a^{\frac{1-\alpha}{4}}),$$

gives

$$r_2 > r_1\, a^{\frac{1-\alpha}{4}} > a^{\frac{1-\alpha}{4}} \overset{(a)}{\geq} a^{\alpha^3} \qquad \text{for } \alpha \in (0, \frac{1}{2}].$$

This last argument yields a contradiction and we are done proving the claim that $r_2 \geq a^{\alpha^3}$.

Note that this also means that $r_0 > r_2 \geq a^{\alpha^3}$ and the smallest value of r such that a solution can enter Ω_- or Ω_+ is r_0. We can therefore deduce that solutions corresponding to initial data $a > a_\alpha^{\frac{2}{\alpha^4}}$ will stay outside $\Omega_- \cup \Omega_+$ at least for values of $r \in [0, \frac{2}{\alpha}]$, since $a > a_\alpha^{\frac{2}{\alpha^4}}$ implies $a^{\alpha^3} > a_\alpha^{\frac{2}{\alpha}} = (\frac{2}{2-\alpha})^{\frac{2}{\alpha^2}} \overset{(c)}{>} \frac{2}{\alpha}$ and thus $r_0 > a^{\alpha^3} > \frac{2}{\alpha}$.
Now let $\alpha \in [\frac{1}{2}, 1)$. We claim that in this case

$$r_2 \geq a^{\frac{1-\alpha}{4}}. \tag{4.29}$$

If we assume to the contrary that $r_2 < a^{\frac{1-\alpha}{4}}$, then from (4.21) we infer again that

$$R'(r) \geq -\frac{1}{r}R - R^{1-\alpha} > -\frac{2}{r}R,$$

since $R(r) \geq a^{\frac{1-\alpha}{2}}$ and thus $R^\alpha \geq a^{\frac{(1-\alpha)\alpha}{2}} \overset{(b)}{\geq} a^{\frac{1-\alpha}{4}} > r_2 \geq r$ for $r \in [r_1, r_2]$ and $\alpha \in [\frac{1}{2}, 1)$, which yields $-R^{1-\alpha} > -\frac{1}{r}R$.
Integrating the differential inequality

$$\frac{R'(r)}{R(r)} > -\frac{2}{r}, \qquad r \in [r_1, r_2],$$

with respect to r on $[r_1, r_2]$ now yields

$$r_2 > r_1\, a^{\frac{1-\alpha}{4}} > a^{\frac{1-\alpha}{4}}, \qquad \alpha \in [\frac{1}{2}, 1).$$

This last argument again yields a contradiction and we are done proving the claim that $r_2 \geq a^{\frac{1-\alpha}{4}}$.
Note that this also means that $r_0 > r_2 \geq a^{\frac{1-\alpha}{4}}$ and as before the smallest value of r such that a solution can enter Ω_- or Ω_+ is r_0. We can therefore deduce that solutions corresponding to initial data $a > a_\alpha^{\frac{8}{(1-\alpha)\alpha^2}}$ will stay outside $\Omega_- \cup \Omega_+$ at least for values of $r \in [0, \frac{2}{\alpha}]$, since $a^{\frac{(1-\alpha)\alpha}{4}} > a_\alpha^{\frac{2}{\alpha}} = (\frac{2}{2-\alpha})^{\frac{2}{\alpha^2}} \overset{(c)}{>} \frac{2}{\alpha}$ and thus $r_0 > a^{\frac{1-\alpha}{4}} > a_\alpha^{\frac{2}{\alpha^2}} \geq a_\alpha^{\frac{2}{\alpha}} > \frac{2}{\alpha}$.
Summing up, we can state that for $a > M_\alpha$ and $r \leq \frac{2}{\alpha}$ we have $R(r) > a_\alpha$ and it follows that

$$E(r) > 0 \quad \text{ for } r \in [0, \frac{2}{\alpha}], \quad \text{ if } a > M_\alpha, \tag{4.30}$$

since

$$E(r) = \frac{1}{2}(\beta^2 + \psi^2) - \frac{1}{2-\alpha}|\psi|^{2-\alpha} = \frac{1}{2}R^2 - \frac{1}{2-\alpha}R^{2-\alpha}|\cos\theta|^{2-\alpha} \tag{4.31}$$

$$\geq R^{2-\alpha}\left(\frac{1}{2}R^\alpha - \frac{1}{2-\alpha}\right) > a_\alpha^{2-\alpha}\underbrace{\left(\frac{1}{2}a_\alpha^\alpha - \frac{1}{2-\alpha}\right)}_{=\frac{1}{2}(\frac{2}{2-\alpha})^{\frac{\alpha}{\alpha}} - \frac{1}{2-\alpha} = 0} = 0. \tag{4.32}$$

This means that a solution with $a > M_\alpha$ can enter Ω_- or Ω_+ only for a value of $r > \frac{2}{\alpha}$. Moreover, (4.28) and (4.29) show that as $a \to \infty$ the value of $r > 0$ at which a solution enters the region $\Omega_- \cup \Omega_+$ also approaches infinity.

What is still missing in the proof of this Lemma is to show that certain inequalities hold, namely

(a) $a^{\frac{1-\alpha}{4}} > a^{\frac{(1-\alpha)\alpha}{2}} > a^{\alpha^3}$, for $\alpha \in (0, \frac{1}{2})$

(b) $a^{\frac{(1-\alpha)\alpha}{2}} > a^{\frac{1-\alpha}{4}}$, for $\alpha \in (\frac{1}{2}, 1)$

(c) $(\frac{2}{2-\alpha})^{\frac{2}{\alpha^2}} > \frac{2}{\alpha}$, for $\alpha \in (0,1)$.

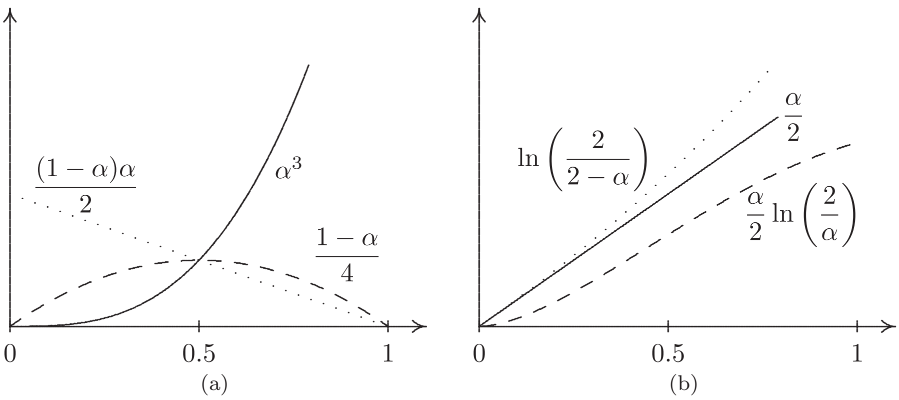

Figure 4.3

Since $a > 1$, in (a) and (b) it suffices to consider the exponent functions of α in and compare them in size (cf. Figure 4.3a). Inequality (c) is a bit more tricky and can be shown in the following way (cf. Figure 4.3b).
First note that the statement is equivalent to

$$\ln \frac{2}{2-\alpha} > \frac{\alpha^2}{2} \ln \frac{2}{\alpha} \qquad \text{for } \alpha \in (0,1).$$

We are going to show that this is true, since for all $\alpha \in (0,1)$ we have on the one hand $\ln \frac{2}{2-\alpha} > \frac{\alpha}{2}$ and on the other hand $\frac{\alpha}{2} > \frac{\alpha^2}{2} \ln \frac{2}{\alpha}$.
Let us start by proving the first claim. Define $f(\alpha) = \frac{\alpha}{2}, g(\alpha) = \ln \frac{2}{2-\alpha}$ for $\alpha \in (0,1)$. Both functions are monotone increasing and equal to 0 at $\alpha = 0$. Furthermore $g'(\alpha) = \frac{1}{2-\alpha} > \frac{1}{2} = f'(\alpha)$, so $f(\alpha) < g(\alpha)$ for all $\alpha \in (0,1)$.
For the proof of the second claim define $h(\alpha) = \alpha \ln \frac{2}{\alpha}$. Then $h(\alpha) \to 0$ as $\alpha \to 0$ and $h(2) = 0$. It is easy to check that $h(\alpha)$ has its only maximum at $\alpha = \frac{2}{e}$ with $h(\frac{2}{e}) = \frac{2}{e} < 1$ which shows that $h(\alpha) = \alpha \ln \frac{2}{\alpha} < 1$. Thus $\frac{\alpha}{2} h(\alpha) = \frac{\alpha^2}{2} \ln \frac{2}{\alpha} < \frac{\alpha}{2}$.
□

Now we know that the solution to certain initial conditions $a > M_\alpha$ can only enter the region $\Omega_\pm$ for a value of $r > \frac{2}{\alpha}$. Furthermore, this value of r increases if we let the starting point a tend to infinity. The question that arises is: will all solutions with initial data $a > M_\alpha$ enter $\Omega_\pm$? The answer is no, which is the result of the next

Lemma 4.2.2 *There exists initial data $a_+ > M_\alpha$ such that the corresponding solution to (4.6)-(4.7) stays outside of $\Omega_+ \cup \Omega_-$ for all $r \geq 0$.*

Proof Outside of $\Omega_- \cup \Omega_+$ we have $E > 0$ so that $\psi^2 + \beta^2 > \frac{2}{2-\alpha}|\psi|^{2-\alpha}$. Passing to polar coordinates (4.22) we find

$$-1 - \frac{1}{2r} \leq \theta'(r) \leq -\frac{\alpha}{2} + \frac{1}{2r}, \tag{4.33}$$

for the values of $r > 0$ where $E(r) > 0$. This is easy to see, as the derivative with respect to r of $\theta(r)$ in view of (4.6) is given by

$$\begin{aligned}\theta'(r) &= \frac{d}{dr}\arctan\left(\frac{\beta(r)}{\psi(r)}\right) = \frac{\beta'\psi - \beta\psi'}{\psi^2+\beta^2} \overset{(4.6)}{=} \frac{-\frac{1}{r}\beta\psi - \psi^2 + |\psi|^{2-\alpha} - \beta^2}{\psi^2+\beta^2} \\ &= -1 - \underbrace{\frac{\beta\psi}{r(\psi^2+\beta^2)}}_{=:A} + \underbrace{\frac{|\psi|^{2-\alpha}}{\psi^2+\beta^2}}_{=:B}.\end{aligned}$$

Since $r > 0$ we have $|A| = \frac{|\beta\psi|}{r(\psi^2+\beta^2)} \leq \frac{1}{2r}$, and since $E > 0$ we have $B < \frac{2-\alpha}{2}$. Thus

$$\begin{cases}\theta'(r) = -1 - A + B \geq -1 - A \geq -1 - \frac{1}{2r}, \\ \theta'(r) = -1 - A + B \leq -1 + \frac{1}{2r} + \frac{2-\alpha}{2} = -\frac{\alpha}{2} + \frac{1}{2r}.\end{cases}$$

This estimate on $\theta'(r)$ gives an upper and lower bound on the angular velocity of the solution for all values of $r > 0$ where $E(r) > 0$. In the previous lemma we showed that $E(r) > 0$ at least for $r < \frac{2}{\alpha}$ if $a > M_\alpha$. Now, let us consider values of $r > \frac{2}{\alpha}$ for which $E(r)$ is still positive. For such values, (4.33) reads

$$-1 - \frac{\alpha}{4} < \theta'(r) < -\frac{\alpha}{4}, \tag{4.34}$$

since

$$\theta'(r) \geq -1 - \frac{1}{2r} > -1 - \frac{\alpha}{4} \text{ and } \theta'(r) \leq -\frac{\alpha}{2} + \frac{1}{2r} < -\frac{\alpha}{2} + \frac{\alpha}{4} = -\frac{\alpha}{4}.$$

Denote by D_+, D_- as the sets of points $\{(a,0) : a > M_\alpha\}$ such that a solution (ψ, β) to (4.6)-(4.7) with initial data $(a,0)$ will enter Ω_+, Ω_-, respectively, for some finite value of r. Both sets D_+ and D_- are open by continuous dependence of the solution on initial data. To see this, consider a point $(a^*, 0) \in D_+$ whose corresponding solution enters the region Ω_+ at time $r = r^*$. Then by continuous dependence on the initial data, a solution whose starting point $(a,0)$ lies sufficiently close to $(a^*,0)$ will also enter Ω_+ at some time close to r^* and

in view of (4.20) stay there for all times.
In the beginning of the discussion of the proof of Proposition 4.2 we analyzed the dynamics of the system and found that in the plane (ψ, β), outside of $\Omega_- \cup \Omega_+$ a solution to (4.6)-(4.7) intersects the horizontal axis from the upper to the lower half-plane on the right of the origin, and in the other direction on the left of the origin. Notice also that a solution intersects the horizontal axis a finite number of times as it winds around the region $\Omega_- \cup \Omega_+$ before entering it. Denote by $D_N \subseteq D_- \cup D_+$ the set of initial data such that corresponding solutions intersect the positive horizontal axis exactly N times prior to entering Ω_- or Ω_+. Since these intersections are transversal (that is, not tangential to the axis), they are stable under small perturbations (cf. [GP74]). Thus, again by continuous dependence on initial data, for any N these sets D_N are open. In view of the fact that solutions are unique once we specify the initial condition $(a, 0)$, they are disjoint. We can therefore write $D_- \cup D_+ = \dot{\bigcup}_N D_N$.
Assume for a moment that all solutions will at one point enter Ω_+ or Ω_-. (4.28) and (4.29) in Lemma 4.2.1 show that as $a \to \infty$ the value of $r > 0$ at which a solution can enter the region where $E < 0$ approaches infinity. In view of the above inequality (4.34) and since E is strictly decreasing whenever $(\psi, \beta) \notin \{(0,0), (\pm 1, 0)\}$ by virtue of (4.20), this means that for $r > \frac{2}{\alpha}$ a solution to (4.6)-(4.7) with $a > M_\alpha$ keeps winding around the region $\Omega_- \cup \Omega_+$ before entering it as $a \to \infty$. We deduce that there exist infinitely many open, non-empty sets D_N with $N \to \infty$ as a tends to infinity. By assumption, $D_- \cup D_+ = (a_\alpha, \infty)$. But this is an open interval in $\mathbb{R}$ which cannot be written as the disjoint union of open non-empty sets D_N. Hence, there exists $a_+ > M_\alpha$ such that $(a_+, 0) \notin D_+ \cup D_-$. Solutions to such initial data will therefore not enter the region $\Omega_- \cup \Omega_+$. $\square$

Lemma 4.2.3 *For solutions to (4.6)-(4.7) corresponding to initial data $a > M_\alpha$ such that they stay outside of $\Omega_- \cup \Omega_+$ for all $r \geq 0$, there exist $0 < T < \infty$ such that $E(T) = 0$ and $\psi(T) = \beta(T) = 0$.*

Proof Let us assume that $E(r) > 0$ for all $r \geq 0$ and show that this leads us to a contradiction.
Recall (4.34) from the previous lemma. Under the assumption that $E > 0$ for all $r \geq 0$, this bound on θ' holds in particular for all $r > \frac{2}{\alpha}$. Consequently, a solution to (4.6)-(4.7) with $a > M_\alpha$ and $r > \frac{2}{\alpha}$ would surround the region $\Omega_- \cup \Omega_+$ with angular velocity between $1 + \frac{\alpha}{4}$ and $\frac{\alpha}{4}$ in clockwise direction. Thus we can construct an increasing sequence $\{r_n\}_{n\geq 1}$ with $r_1 > \frac{2}{\alpha}$ such that $\theta(r_n) = \frac{\pi}{6} + 2(n-k)\pi$ where $k \in \mathbb{N}$ is fixed. From (4.34) we infer that

$$\frac{8\pi}{\alpha + 4} < r_{n+1} - r_n < \frac{8\pi}{\alpha} \qquad \text{for } r_1 > \frac{2}{\alpha}, n \geq 1. \tag{4.35}$$

This is true since the angular velocity of θ varies between $1 + \frac{\alpha}{4}$ and $\frac{\alpha}{4}$ and thus $2\pi = \theta(r_{n+1}) - \theta(r_n) < (1 + \frac{\alpha}{4})(r_{n+1} - r_n)$ or, equivalently, $r_{n+1} - r_n > \frac{8\pi}{\alpha+4}$ and $2\pi = \theta(r_{n+1}) - \theta(r_n) > \frac{\alpha}{4}(r_{n+1} - r_n)$, so $r_{n+1} - r_n < \frac{8\pi}{\alpha}$.
This shows that independent of the number of cycles n the solution has completed, the "time" it takes the solution to return to the ray $\theta(r_n) = \frac{\pi}{6}$ is bounded

from above and below by constants. Now consider the region

$$A := \Big\{(\psi, \beta) : E > 0, \frac{\pi}{6}\Big(1 - \frac{1}{3}\alpha\Big) < \theta < \frac{\pi}{6}\Big\}. \tag{4.36}$$

From the dynamics of the system (4.6) we infer that the solutions enter the region A crossing the ray $\theta = \frac{\pi}{6}$ at some time $r = r_n$ and leave it crossing the ray $\theta = \frac{\pi}{6}(1 - \frac{1}{3}\alpha)$ at some bigger value of $r = r_n^+$. This value r_n^+ satisfies

$$r_n + \frac{2\pi}{9}\frac{\alpha}{\alpha+4} < r_n^+ < r_n + \frac{2\pi}{9}, \tag{4.37}$$

since from (4.34)

$$(-1 - \frac{\alpha}{4})(r_n^+ - r_n) < \theta(r_n^+) - \theta(r_n) = \frac{\pi}{6}(1 - \frac{1}{3}\alpha) - \frac{\pi}{6} = -\alpha\frac{\pi}{18}$$

$$\text{and } -\frac{\alpha}{4}(r_n^+ - r_n) > \theta(r_n^+) - \theta(r_n) = -\alpha\frac{\pi}{18}.$$

Passing to polar coordinates (4.22), we find that

$$R > \left(\frac{\sqrt{3}}{2}\right)^{\frac{2-\alpha}{\alpha}}\left(\frac{2}{2-\alpha}\right)^{\frac{1}{\alpha}} \text{ in A.} \tag{4.38}$$

To see this, note that for $\theta < \frac{\pi}{6}$ we have $\cos\theta > \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and thus $E > 0$ whenever $\psi^2 + \beta^2 > \frac{2}{2-\alpha}|\psi|^{2-\alpha}$ or, equivalently,

$$R^2 > \frac{2}{2-\alpha}R^{2-\alpha}|\cos\theta|^{2-\alpha} \geq \frac{2}{2-\alpha}R^{2-\alpha}\left(\frac{\sqrt{3}}{2}\right)^{2-\alpha},$$

so $R^\alpha > \left(\frac{\sqrt{3}}{2}\right)^{2-\alpha}\frac{2}{2-\alpha}$.
Consequently in view of (4.22)

$$\beta^2(r) > \sin^2(\frac{\pi}{9})\left(\frac{\sqrt{3}}{2}\right)^{\frac{2(2-\alpha)}{\alpha}} =: K_\alpha \qquad \text{for } r \in (r_n, r_n^+), \tag{4.39}$$

since $\frac{\pi}{6}\Big(1 - \frac{1}{3}\alpha\Big) > \frac{\pi}{6}\Big(1 - \frac{1}{3}\Big) = \frac{\pi}{9}$ for $\alpha \in (0,1)$ and

$$\sin(\theta) > \sin\Big(\frac{\pi}{6}(1 - \frac{1}{3}\alpha)\Big) \text{ in A.}$$

So in view of (4.38) we have that

$$\begin{aligned}\beta^2(r) = R^2\sin^2(\theta) &> \left(\frac{\sqrt{3}}{2}\right)^{\frac{2(2-\alpha)}{\alpha}}\left(\frac{2}{2-\alpha}\right)^{\frac{2}{\alpha}}\sin^2(\frac{\pi}{9}) \\ &> \sin^2(\frac{\pi}{9})\left(\frac{\sqrt{3}}{2}\right)^{\frac{2(2-\alpha)}{\alpha}} \qquad \text{in } A.\end{aligned}$$

Furthermore, by virtue of (4.35) and (4.37),

$$r_n + \frac{2\pi}{9}\frac{\alpha}{\alpha+4} < r_n^+ < r_{n+1}, \tag{4.40}$$

since $r_n^+ < r_n + \frac{2\pi}{9} < r_n + \frac{8\pi}{\alpha+4} < r_{n+1}$ for all $\alpha \in (0,1)$. But now from (4.20), together with (4.39), (4.37) and (4.40), we get a contradiction:

$$\begin{aligned} E\left(\frac{2}{\alpha}\right) - E(\infty) &= -\int_{2/\alpha}^{\infty} E'(r)dr = \int_{\frac{2}{\alpha}}^{\infty} \frac{\beta^2(r)}{r}dr \\ &\geq K_\alpha \sum_{n\geq 1} \int_{r_n}^{r_n^+} \frac{1}{r}dr \geq K_\alpha \sum_{n\geq 1} \frac{1}{r_n^+}(r_n^+ - r_n) \\ &> K_\alpha \frac{2\pi}{9}\frac{\alpha}{\alpha+4} \sum_{n\geq 1} \frac{1}{r_n + \frac{2\pi}{9}} = \infty. \end{aligned}$$

The series is divergent since $r_n < \frac{8\pi}{\alpha}(n-1) + r_1$ in view of (4.35), and $E(\infty)$ is some finite number, as E is bounded by virtue of Lemma 4.1.2. Recall that we assumed $E(r)$ for all $r \geq 0$, which lead to the above contradiction. Thus there exists a finite value of $T > 0$ such that $E(T) = 0$. Notice that for such values of T we also have $\psi(T) = \beta(T) = 0$. If this were not the case, the dynamics of the system would force the solution to enter Ω_+ or Ω_- at this point, which is contradictory to the assumption of this Lemma. □

This completes the proof of Proposition 4.3, as we have seen that there exists a sufficiently large such that the solution to (4.6)-(4.7) corresponding to such initial data $(a, 0)$, which is uniquely defined and $\mathcal{C}^2$ by Proposition 4.2, goes to zero in finite time.

Remark Notice that solutions to (4.6)-(4.7) corresponding to initial data $(a_0, 0)$ with $a_0 \in (0, a_\alpha]$ will stay inside Ω_+ for all $r \geq 0$. Such solutions clearly do not have compact support.

4.3 Limiting cases of the parameter α

The case $\alpha = 1$

In this case the vorticity function γ simplifies to

$$\gamma(\psi) = \psi - \frac{\psi}{|\psi|} \tag{4.41}$$

and has a point of discontinuity at $\psi = 0$, since

$$\lim_{\psi\to 0^-} \gamma(\psi) = 1 \text{ whereas } \lim_{\psi\to 0^+} \gamma(\psi) = -1 \text{ but } \gamma(0) = 0,$$

by definition of the vorticity function. As we are only interested in classical solutions, we will not consider this case.

The case $\alpha = 0$

In this case we simply have

$$\gamma(\psi) \equiv 0. \tag{4.42}$$

Thus, equation (4.5)-(4.4), for which we seek compactly supported $\mathcal{C}^2$-solutions, reads

$$\psi'' + \frac{1}{r}\psi' = 0, \qquad r > 0, \tag{4.43}$$

with boundary conditions

$$\psi(0) = a, \qquad \psi'(0) = 0. \tag{4.44}$$

If we denote $\psi'(r) = \beta(r)$, then (4.43) reads

$$\beta'(r) + \frac{1}{r}\beta(r) = 0,$$

which we can solve easily, obtaining $\beta(r) = \frac{C_1}{r}$ for some constant $C_1 \in \mathbb{R}$. This in turn yields

$$\psi(r) = C_1 \ln(r) + C_2,$$

for some constants $C_1, C_2 \in \mathbb{R}$.
Since ψ is continuous and $\psi(0) = a$ we conclude

$$C_1 = 0 \text{ and } C_2 = a,$$

and we get that $\psi(r) \equiv a$ is constant for all $r \geq 0$.

In the setting of ψ being the stream function on the fluid domain D discussed in Chapter 2, this means that $\psi \equiv a$ is constant throughout the flow field. The boundary conditions $\psi = \psi_y = 0$ on the flat free surface require this constant to be zero. So $\psi \equiv 0$ and the water is still throughout the fluid domain, which is why we do not consider this case.

4.4 Types of vorticity functions

4.4.1 Why not choose a vorticity function $\gamma \in \mathcal{C}^1$?

The proof discussed in the previous sections to show that system (2.22) has a radially symmetric $\mathcal{C}^2$-solution with compact support turned out to be quite elaborate. The difficulties were mainly due to the fact that we could not rely upon classical existence and uniqueness theorems from the theory of ordinary differential equations, because the vorticity function γ failed to be locally Lipschitz. The natural question that arises is therefore - why did we pick such an "ugly" function, why not simply take, for example, $\gamma \in \mathcal{C}^1$?

To shed some light on this issue, assume that ψ is a radially symmetric solution to (2.22) with compact support in the fluid domain D and take some vorticity function γ to be of class $\mathcal{C}^1$. Since ψ is radially symmetric, we can make the Ansatz $\psi(x,y) = \psi(r)$ where $r = \sqrt{(x-x_0)^2 + (y-y_0)^2}$ for some $(x_0, y_0) \in D$ and again rewrite the system in the equivalent form of a two-dimensional system of non-linear first order ordinary differential equations

$$\begin{cases} \psi' = \beta, \\ \beta' = -\frac{1}{r}\beta - \gamma(\psi), \end{cases} \qquad r > 0, \tag{4.45}$$

with initial values

$$\psi(0) = a, \qquad \beta(0) = 0. \tag{4.46}$$

From the classical theory of ordinary differential equations discussed in Chapter 3, we know that this initial value problem has a unique solution defined for r in some interval (r_0, r_1), if the right hand side is continuous in r and Lipschitz in ψ. In our case, for r away from zero and $\gamma \in \mathcal{C}^1$ this is certainly true. Since we assumed the solution ψ to the above system to have compact support, we can find a value $T \in (r_0, r_1)$ such that $\psi(T) = \psi'(T) = 0$, that is $(\psi(T), \beta(T)) = (0,0)$. Thus, if we "go backwards in time"[1], by uniqueness of solutions we would have $(\psi(r), \beta(r)) \equiv (0,0)$ for all values of $r > 0$.Therefore, if we choose a continuously differentiable vorticity function γ, the only possible radially symmetric solution with compact support is the trivial solution.

4.4.2 Linear vorticity functions

The aim of this section is to discuss a result in [CJ08] concerning linear vorticity functions. It turns out also that in this case the system only admits trivial solutions:

Proposition 4.4.1 *If ψ is a $\mathcal{C}^2$-solution of problem (2.22) for a linear vorticity function γ, then $\psi \equiv 0$ and $\gamma \equiv 0$.*

Proof Assume that ψ is a solution of (2.22) for a linear vorticity function $\gamma(s) = as + b,\ s \in \mathbb{R},\ a, b$ real constants. We will first prove that $b = 0$ and $a = 0$, so $f \equiv 0$. Then ψ is a harmonic function in D and we infer from the weak maximum principle (cf. [Fra00]) that $\max_{\bar{D}} \psi = \max_{\partial D} \psi = 0$ in view of the boundary conditions in (2.22). Thus, $\psi \equiv 0$ in D.
Assume $b > 0$. Since γ is linear and because of $\mathcal{C}^2$-regularity of ψ up to the boundary, on $y = 0$ the partial differential equation $\psi_{xx} + \psi_{yy} = -\gamma(\psi)$ simplifies to $\psi_{yy} = -b < 0$. In view of the boundary condition $\psi = \psi_y = 0$ this means that ψ has a local maximum at $y = 0$. Thus, for $y < 0$ sufficiently close to the surface, we have $\psi < 0$ and $\psi_y > 0$. Suppose $a > 0$. We can find an open bounded domain $D_0 \subset D$, with upper boundary $y = 0$ and lower boundary a semicircle centered at some point $(x_0, 0)$ with $x_0 < 0$, such that $0 < -a\psi(x,y) < b$ in D_0 (that is, we can make the radius of the semicircle small enough to ensure $\psi < 0$ and furthermore such that the absolute value of ψ in D_0 is small

[1] This can be made explicit by performing a change of variables $r \mapsto -r$ and taking as initial condition $(\psi(T), \beta(T)) = (0,0)$.

enough to ensure $-a\psi < b$). Then in D_0 we have $-a\psi - b < 0$. Suppose on the other hand that $a < 0$. Then $a\psi > 0$ in D_0, and thus $-a\psi - b < 0$. So for any $a \in \mathbb{R}$, we have that $\psi_{xx} + \psi_{yy} = -a\psi - b < 0$ in D_0. Furthermore, $0 = \psi(x_0, 0) = \sup_{D_0} \psi(x, y)$. Since D_0 satisfies an interior sphere condition in $(x_0, 0)$, a version of the strong maximum principle (cf. [Fra00]) implies that the outer normal derivative $\frac{\partial\psi}{\partial\nu}(x_0, 0) = \psi_y(x_0, 0) < 0$, unless $\psi \equiv 0$. This is a contradiction to the boundary condition on $y = 0$. The case for $b < 0$ can be dealt with in a similar way. This proves that $b = 0$.
Next we will try to produce a contradiction assuming $a \neq 0$, knowing $b = 0$.
Consider the operator $T := -(\partial_x^2 + \partial_y^2)$ acting on $L^2(D)$. Then in view of the partial differential equation in (2.22), we have $T\psi = a\psi$ and ψ is an eigenfunction of the operator T with corresponding eigenvalue a. The essential Dirichlet spectrum of T is $[0, \infty)$ for some $s > 0$, and the number of Dirichlet eigenvalues λ_j in $(0, s)$ is finite, (cf. [EE87]). We want to rule out the possibility of ψ being a Dirichlet eigenfunction corresponding to the eigenvalue a.
Consider the connected components of the complements of the set where $\psi = 0$, so called nodal domains. From [MP85] we infer that the number of nodal domains is at most j. Furthermore, in the neighborhood of a point (x, y) in D where $\psi(x, y) = 0$, the set $\psi = 0$ is either a smooth curve or an intersection of up to j smooth curves at equal angles (cf. [CH53]). In order to be able to apply these results, we consider the set

$$D_s = \{(x, y) \in \mathbb{R}^2 : x < 0, b(x) < y < -b(x)\}$$

and symmetrically extend ψ to D_s setting

$$\psi(x, y) = \psi(x, -y), \qquad \text{for } (x, y) \in D_s,\ y > 0.$$

In view of the boundary condition for ψ on $y = 0$, $\psi \in \mathcal{C}^2(Ds)$. Thus, we can view ψ as a Dirichlet eigenfunction for the operator T acting on $L^2(Ds)$ with nodal line $y = 0$. We want to find an open subset D_0 of D_s such that ψ has no zeros in D_0, whereas the boundary ∂D_0 contains a subset of the line $y = 0$ of the form $\{(x, 0) : a < x < b\}$ for some $a < b < 0$. Existence of such a set D_0 is guaranteed by the above reasoning on the number of eigenvalues and nodal domains and in view of the structure of the set $\{\psi = 0\} \cap D_s$. Hence we can apply the strong maximum principle (cf. [Fra00]) to the operator T acting on $L^2(D_0)$ and obtain a contradiction, since $\psi = 0$ on $y = 0$ and in view of the boundary condition $\psi_y(x, 0) = 0$ for some $x \in (a, b)$.

4.4.3 Necessary condition on γ for existence of solutions

In this section we present another general result (cf. [CJ08]).

Proposition 4.4.2 *The vorticity function γ must satisfy $\gamma(0) = 0$ for problem* (2.22) *to have a solution.*

Proof We have already seen in the beginning of the proof of Proposition 4.4.1 that if $\gamma \equiv 0$ then $\psi \equiv 0$ is a solution. On the other hand, if (2.22) has a solution, then $\gamma(0) = 0$ has to hold. To see this, assume to the contrary that $\gamma(0) > 0$. Then, because on the boundary condition on $y = 0$, $\psi = \psi_y = 0$ and we have $\psi_{yy} = -\gamma(0) < 0$ on $y = 0$. Just as in the proof of (4.4.1) we can find a bounded open domain $D_0 \subset D$ such that $\psi < 0$ and $\psi_y > 0$ in D_0, which guarantees that $0 = \psi(x_0, 0) = \sup_{D_0} \psi(x, y)$ in D_0. Again, the strong maximum principle implies a contradiction to the boundary condition $\psi_y(x, 0) = 0$. The case where $\gamma(0) < 0$ is similar. $\square$

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Deutsche Einleitung und Zusammenfassung

In dieser Arbeit geht es um die mathematische Beschreibung von Wasserwellen und Strömungen im Ozean, insbesondere um Tsunamis.
Unter dem Begriff "Wasserwelle" verstehen wir Oberflächenwellen an der Grenze zwischen Luft und Wasser die das Resultat einer Auslenkung aus der Gleichgewichtslage (flache Oberfläche) unter Einwirkung von Gravitation als entgegengerichtete Rückstellkraft ist. Im Unterschied zu Schallwellen, die sich alle mit derselben Geschwindigkeit ausbreiten, hängt die Ausbreitungsgeschwindigkeit von Wasserwellen von deren Wellenlänge ab. Dieses Phänomen nennt man "Dispersion von Wellen".
Mit Ausbreitungsgeschwindigkeit meinen wir nicht die Geschwindigkeit des Wassers selbst, sondern die Geschwindigkeit mit der sich das Wellenprofil in horizontaler Richtung ausbreitet. Leonardo Da Vinci hat dies beschrieben, indem er die Wasseroberfläche mit einem Ährenfeld verglich: "Die Ähnlichkeit der Wellen ist groß mit den Wellen, die der Wind in einem Kornfeld hervorbringt, welche man über das Feld hineilen sieht, ohne dass das Getreide sich vom Platz bewegt".

Wir betrachten in dieser Arbeit vornehmlich Wellen, deren Wellenlänge groß ist im Vergleich zu der Wassertiefe. Ein prominentes Beispiel solcher Wellen sind Tsunamiwellen. Der Begriff "Tsunami" kommt aus dem Japanischen und bedeutet "Hafenwelle". Tsunamiwellen werden meist durch Seebeben hervorgerufen, bei denen die Erdkruste am Meeresboden abrupt in vertikaler Richtung verschoben wird. Dadurch hebt bzw. senkt sich auch die gesamte Wassersäule direkt darüber und definiert dadurch das anfängliche Wellenprofil des Tsunamis. Die so entstandene Welle breitet sich mit einer Geschwindigkeit von bis zu $1000km/h$ aus und legt dabei enorme Distanzen zurück. Die Ausbreitungsrichtung steht im Wesentlichen senkrecht auf die Hauptachse der meist elliptischen Störungszone des Erdbebens, wobei man davon ausgeht dass die Bewegung parallel zu dieser Achse einheitlich ist. Aus diesem Grund kann man sich bei Modellen, welche die Evolution solcher Wellen beschreiben, auf zwei Dimensionen beschränken – die horizontale Ausbreitungsrichtung und die Auslenkung in vertikaler Richtung.
Auf offener See kann die Wellenbewegung im wesentlichen durch lineare Theorie beschrieben werden und die Ausbreitungsgeschwindigkeit mit $\sqrt{gh}$ approximiert werden, wobei g die Gravitationskonstante und h die typische Wassertiefe bezeichnet. Die Amplitude eines Tsunami ist im Vergleich zur Wellenlänge so klein, dass Schiffe am offenen Meer meist gar nicht merken, wenn sich gerade eine Tsunamiwelle unter ihnen bewegt. Erst wenn der vordere Teil der Welle in die Nähe der Küste kommt, wo die Wassertiefe und damit auch die Geschwindigkeit geringer ist, baut sich das Wasser zu riesigen Wellen auf, da der hintere Teil der Welle ja immer noch im offenen Meer liegt und mit viel größerer Geschwindigkeit auf die Küste zusteuert. Da die Welle so lang ist, werden bei diesem Vorgang enorme Mengen an Wasser transportiert, die sich in Küstennähe zu verheerenden Wellen auftürmen und mitunter enorme Schäden anrichten.

Bevor eine Tsunamiwelle die Küste erreicht, ist das Wasser in dieser Region meist nicht in Ruhe, sondern unterliegt Strömungen und anderen wirbelartigen Störungen. Um ein aussagekräftiges Modell für die Tsunamiausbreitung zu erhalten, versucht man daher diesen Hintergrundzustand zu berücksichtigen, um dann die Interaktion zwischen einer Tsunamiwelle und etwaigen Hintergrundströmungen studieren zu können. Ein wichtiges Konzept in diesem Zusammenhang ist die sogenannte "Vortizität" oder "Wirbelstärke" (engl. vorticity). Sie ist ein Maß für die lokale Rotation von Flüssigkeitsteilchen bzw. für die lokale Scherung einer Strömung.

In dieser Arbeit soll untersucht werden, welche Art von Wirbelverteilung für Hintergrundströmungen in einem Modell für Wasserwellen zulässig ist. Dazu ziehen wir die zweidimensionalen Eulergleichungen heran und zeigen an einem konkreten Beispiel einer Wirbelverteilung, dass die Bewegungsgleichungen einer derartigen stationären Strömung eine nicht triviale Lösung zulassen, die eine isolierte Wirbelregion in Küstennähe modelliert.
Die Schwierigkeit eine passende Wirbelverteilung zu finden besteht darin, dass man für lineare und stetig differenzierbare Verteilungen nur die triviale Lösung erhält, was der Modellierung von stillem Wasser entspricht. Es ist daher notwendig, mit nicht linearen Wirbelverteilungen zu arbeiten, die stetig, jedoch nicht Lipschitz stetig sind.
Für den Beweis werden die Eulergleichungen zunächst mit Hilfe einer Strömungsfunktion umgeformt und dann über einen radialsymmetrischen Lösungsansatz in ein System nicht linearer gewöhnlicher Differenzialgleichungen transformiert. Da dieses System eine Unstetigkeitsstelle bei Null aufweißt und außerdem nicht lokal Lipschitz stetig ist, betrachtet man das Problem zunächst in einer Umgebung der Unstetigkeitsstelle und führt den Existenz- und Eindeutigkeitsbeweis mittels eines Kontraktionsarguments. Außerhalb dieses kritischen Punktes wird ein koerzives Funktional definiert, mit Hilfe dessen und einer weiteren Transformation gezeigt wird, dass Lösungen global existieren und eindeutig sind, sowie stetig von den Anfangsdaten abhängen. Im zweiten Teil des Beweises werden Eigenschaften dieses Funktionals und die Theorie dynamischer Systeme herangezogen um zu zeigen, dass das System eine Lösung mit kompaktem Träger besitzt, welche die gesuchte isolierte Wirbelregion modelliert.

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