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DIPLOMARBEIT

Titel der Diplomarbeit

Extensions which do not have new large cardinals

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To Daniel and my parents

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Abstract

In this MA thesis we examine the following question regarding the relation between forcing and large cardinals: Which kinds of extensions do not create new large cardinals? For small forcing and measurable cardinals this question was settled in 1967 by a result of Lévy and Solovay, showing that measurable cardinals are neither created nor destroyed under small forcing. This result was then extended to other large cardinals, like strongly compact, supercompact and huge cardinals. As the proof for the case of strong and Woodin cardinals differs from the one of the Lévy-Solovay Theorem, we will present it separately. A much more general result is due to Joel Hamkins: He showed that for any extension $V \subseteq \bar{V}$, which satisfies the δ approximation and cover properties, every embedding $j : \bar{V} \to \bar{N}$ in $\bar{V}$ satisfying certain closure properties can be restricted to V such that $j \restriction V$ is amenable to V. For a large cardinal whose existence is witnessed by an elementary embedding, we can therefore show that it is not created in such an extension. Hamkins also showed that many important forcing extensions, like reverse Easton iterations, the Silver iteration, the Laver preperation and the canonical forcing of the GCH have the approximation and cover properties.

Zusammenfassung

In dieser Diplomarbeit untersuchen wir Erweiterungen, in denen keine neuen grossen Kardinalzahlen entstehen. 1967 wurde von Lévy und Solovay gezeigt, dass bei einem Forcing, das relativ zu der betrachteten Kardinalzahl klein ist, keine neuen messbaren Kardinalzahlen entstehen und auch keine verloren gehen. Dieses Resultat kann zu vielen grossen Kardinalzahlen verallgemeinert werden, wobei wir den Spezialfall der starken und Woodin Kardinalzahlen gesondert betrachten werden. Eine Verallgemeinerung für eine grosse Klasse an Forcings konnte allerdings erst von Joel Hamkins erziehlt werden. Wir untersuchen seinen Beweis, dass für eine Erweiterung $V \subseteq \bar{V}$, die die δ Approximierungs- und Überdeckungseigenschaften erfüllt, eine in geeigneter Form abgeschlossene elementare Einbettung $j : \bar{V} \to \bar{N}$ in $\bar{V}$ auf V beschränkt werden kann, so dass $j \restriction V$ "amenable"[1] zu V ist. Für

[1]Soweit ich weiss, existiert keine offizielle deutsche Bezeichnung für dieses Konzept.

grosse Kardinalzahen, die mit Hilfe von elementaren Einbettungen definierbar sind, heisst das, dass sie in solchen Erweiterungen nicht neu entstehen. Wir zeigen, welche Forcing Erweiterungen diese Approximierungs- und Überdeckungseigenschaften haben und wie sich dies auf bestimmt grosse Kardinalzahlen anwenden lässt.

Contents

Chapter 1

Basic Definitions and Facts

In this chapter we will give some basics from set theory and model theory. The goal is mainly to fix the notation and to refer to important results; the definitions, facts and proofs can be found in standard introductions to set theory, like [Jec03], [Kun80], [Kan09], [Lév02] and introductions to model theory like [Hod97].

1.1 Basics from Set Theory

As a preparation to the next chapter, we will introduce the concept of cardinal numbers and mention other relevant concepts like the structure of the universe V.

A binary relation $<$ on a set P is a *partial ordering* of P if:

(i) $p \nless p$ for any $p \in P$,

(ii) if $p < q$ and $q < r$, then $p < r$.

$(P, <)$ is called a *partially ordered set.* A partial ordering $<$ of P is a *linear ordering* if moreover

(iii) $p < q$ or $p = q$ or $q < p$ for all $p, q \in P$.

If $(P, <)$ and $(Q, <)$ are partially ordered sets and $f : P \to Q$, then f is *order-preserving* if $x < y$ implies $f(x) < f(y)$.

A one-to-one function of P onto Q is an *isomorphism* of P and Q if both f and f^{-1} are order-preserving.

Definition 1.1. *A linear ordering* $<$ *of a set* P *is a* well-ordering *if every nonempty subset of* P *has a least element.*

A set T is *transitive* if every element of T is a subset of T.

Definition 1.2. *A set is an* ordinal number *or* ordinal *if it is transitive and well ordered by* $\in$.

Usually, ordinals are denoted by lowercase Greek letters like α, β, γ and so on. It holds that $\beta < \alpha$ if and only if $\beta \in \alpha$. So for each ordinal α, $\alpha = \{\beta : \beta < \alpha\}$. For every α, $\alpha \cup \{\alpha\}$ is an ordinal and $\alpha \cup \{\alpha\} = inf\,\{\beta : \beta > \alpha\}$. $\alpha + 1 = \alpha \cup \{\alpha\}$ is the *successor* of α.

Definition 1.3. *If there is a* β *such that* $\alpha = \beta + 1$*, then* α *is called a* successor ordinal*. If* α *is not a successor ordinal, then* $\alpha = sup\,\{\beta : \beta < \alpha\} = \bigcup \alpha$*, and* α *is called a* limit ordinal*.*

Definition 1.4. *We denote the least nonzero limit ordinal* ω *or* $\mathbb{N}$*. The ordinals less then* ω*, the elements of* $\mathbb{N}$*, are* finite ordinals *or* natural numbers*:*

$$0 = \varnothing, \qquad 1 = 0 + 1 = \{\varnothing\}, \qquad 2 = 1 + 1 = \{\varnothing, \{\varnothing\}\}, \qquad etc.$$

A set X *is* finite *if there is a one-to-one mapping of* X *onto some* $n \in \mathbb{N}$*.* X *is* infinte *if it is not finite.*

We generalize the intuitive concept of "size of a set" as follows: The sets X and Y have the same *cardinality* $|X| = |Y|$ if there exists a one-to-one mapping of a set X onto a set Y. Then $|X| \leqslant |Y|$ if there is a one-to-one mapping of X into Y. A set X is finite, if $|X| = |n|$ for $n \in \mathbb{N}$.

Definition 1.5. *An ordinal* α *is called a* cardinal number *or a* cardinal *if* $|\alpha| \neq |\beta|$ *for all* $\beta < \alpha$*.*

Finite cardinals are natural numbers, i.e. $|n| = n$ for all $n \in \mathbb{N}$.

Theorem 1.6 (Cantor)**.** *For every set* X*,* $|X| < |P(X)|$*.*

Theorem 1.7 (Cantor-Bernstein)**.** *If* $|A| \leqslant |B|$ *and* $|B| \leqslant |A|$*, then* $|A| = |B|$*.*

The proofs of the last two theorems can be found in [Jec03], pp. 27–28.

Fact 1.8. *If* $|A| = \kappa$, *then* $|P(A)| = 2^\kappa$.

The ordinal ω is the least infinite cardinal. All infinite cardinals are limit ordinals.

Fact 1.9.

(i) *For every* α *let* α^+ *be the least cardinal number greater than* α. *Then* α^+ *is the* cardinal successor *of* α.

(ii) *If* X *is a set of cardinals, then* $sup\, X$ *is a cardinal.*

Definition 1.10. *The infinite ordinals which are cardinals are called* alephs. *We define their increasing enumeration in the following way:*

$$\aleph_0 = \omega_0 = \omega$$

$$\aleph_{\alpha+1} = \omega_{\alpha+1} = \aleph_\alpha^+$$

$$\aleph_\alpha = \omega_\alpha = sup\,\{\omega_\beta : \beta < \alpha\} \quad \textit{if } \alpha \textit{ is a limit ordinal}$$

Sets whose cardinality is $\aleph_0$ are called *countable*. Infinite sets that are not countable are *uncountable*.

A cardinal $\aleph_{\alpha+1}$ is a *successor cardinal*. A cardinal $\aleph_\alpha$ whose index is a limit ordinal is a *limit cardinal*.

Definition 1.11. *Let* α *be a limit ordinal.*

- *An increasing* β*-sequence* $\langle \alpha_\xi : \xi < \beta \rangle$, β *is a limit ordinal, is* cofinal *in* α *if* $lim_{\xi \to \beta}\, \alpha_\xi = \alpha$.
- $A \subset \alpha$ *is cofinal in* α *if* $\sup A = \alpha$.
- *The cofinality of* α *is defined as follows:*

$$\operatorname{cof} \alpha = \textit{the least limit ordinal } \beta \textit{ such that there is an increasing } \beta\textit{-sequence } \langle \alpha_\xi : \xi < \beta \rangle \textit{ with } lim_{\xi \to \beta}\, \alpha_\xi = \alpha$$

$\operatorname{cof} \alpha$ *is an infinite regular cardinal, and* $\operatorname{cof} \alpha \leqslant \alpha$.

Definition 1.12. *An infinite cardinal* $\aleph_\alpha$ *is* regular *if* $\operatorname{cof} \aleph_\alpha = \aleph_\alpha$. *It is* singular *if* $\operatorname{cof} \aleph_\alpha < \aleph_\alpha$.

In the proof of Laver's Theorem (see Chapter 5) we will work with a beth fixed point, so let us define the beth function and state its relation to fixed points:

Definition 1.13. *The beth function is defined by induction:*

$\beth_0 = \aleph_0$

$\beth_{\alpha+1} = 2^{\beth_\alpha}$

$\beth_\alpha = sup\,\{\beth_\beta : \beta < \alpha\}$ *if α is a limit ordinal.*

Definition 1.14. *A function $F : Ord \rightarrow Ord$ is called normal if it is strictly monotonic and continuous:*

1. *for all ordinals $\alpha < \beta$, $F(\alpha) < F(\beta)$.*
2. *for every infinite limit ordinal α, $F(\alpha) = sup\{F(\beta)|\beta < \alpha\}$.*

For a normal function it holds that for every ordinal α, $F(\alpha) \geqslant \alpha$ and for any non-empty set of ordinals S, $F(sup\ S) = sup\ F(S)$. From this follows the fixed-point lemma for normal functions:

Fact 1.15 (Fixed-point Lemma)**.** *Any normal function has unboundedly many fixed points, e.g. for every ordinal α there is a $\beta \geqslant \alpha$ such that $F(\beta) = \beta$.*

By definition the beth function is a normal function. So the fixed point Lemma applies to it and we know that there are arbitrarily large beth fixed points. For example the smallest fixed point of the beth sequence is the supremum of the following ω-sequence of cardinals:

$$\beth_0, \quad \beth_{\beth_0}, \quad \beth_{\beth_{\beth_0}}, \quad \dots$$

By transfinite induction we can define a hierarchy of sets:

$$V_0 = \varnothing, \qquad V_{\alpha+1} = P(V_\alpha)$$
$$V_\alpha = \bigcup_{\beta<\alpha} V_\beta \quad \text{if } \alpha \text{ is a limit ordinal}$$

The sets V_α have the following properties (by induction):

(i) Each V_α is transitive.

(ii) If $\alpha < \beta$, then $V_\alpha \subset V_\beta$.

(iii) $\alpha \subset V_\alpha$.

A word about the size of the V_α:

Lemma 1.16.

i) $\forall n \in \omega\ : |V_n| < \omega$

ii) $|V_\omega| = \omega$

iii) $|V_{\omega+\alpha}| = \beth_\alpha$

iv) If κ is inaccessible, then $|V_\kappa| = \kappa$.

The proof of $i) - iii)$ can be found in [Kun80], p. 97, the proof of $iv)$ in [Jec03], p. 70.

The axiom of regularity implies that every set is in some V_α.

Lemma 1.17. *For every x there is an α, such that $x \in V_\alpha$:*

$$\bigcup_{\alpha \in Ord} V_\alpha = V$$

The proof can be found in [Jec03], p. 64. This Lemma implies the the next definition:

Definition 1.18. *The* rank *of x is the least α such that $x \in V_{\alpha+1}$.*

Theorem 1.19 ($\in$-Induction)**.** *Let $\mathbf{T}$ be a transitive class, let φ be a property. Assume that:*

(i) $\varphi(\varnothing)$ and,

(ii) if $x \in \mathbf{T}$ and $\varphi(z)$ for every $z \in x$, then $\varphi(x)$.

Then every $x \in \mathbf{T}$ has property φ.

For a proof see [Jec03], p. 66.

Definition 1.20. *A class $\mathbf{R}$ is* well-founded *on a class $\mathbf{A}$ if and only if*

$$\forall X \subset \mathbf{A}\ (X \neq \varnothing \rightarrow \exists y \in X\ (\neg \exists z \in X\ (x\mathbf{R}y)))$$

Definition 1.21. *A class* $\mathbf{R}$ *is* set-like *on* $\mathbf{A}$ *if and only if for all* $x \in \mathbf{A}$*, the class* $\{y \in \mathbf{A} : y\mathbf{R}x\}$ *is a set.*

Definition 1.22.

1. *Let* $\mathbf{R}$ *be a well-founded and set-like relation on a class* $\mathbf{A}$*. We define the* Mostowski collapsing function $\mathbf{G}$ *for* $\mathbf{A}$ *and* $\mathbf{R}$ *by*

$$\mathbf{G}(x) = \{\mathbf{G}(y) : y \in \mathbf{A} \wedge y\mathbf{R}x\}.$$

2. *The* Mostowski collapse $\mathbf{M}$ *of* $\mathbf{A}$ *and* $\mathbf{R}$ *is the image of* $\mathbf{G}$*.*

Definition 1.23. $\mathbf{R}$ *is* extensional *on* $\mathbf{A}$ *if and only if*

$$\forall x, y \in \mathbf{A}\ (\forall z \in \mathbf{A}(z\mathbf{R}x \leftrightarrow z\mathbf{R}y) \rightarrow x = y).$$

Theorem 1.24 (Mostowski Collapsing Theorem)**.** *Suppose* $\mathbf{R}$ *is well-founded, set-like and extensional on* $\mathbf{A}$*. Then there exists a transitive class* $\mathbf{M}$ *and a one-to-one mapping* $\mathbf{G}$ *from* $\mathbf{A}$ *onto* $\mathbf{M}$ *such that* $\mathbf{G}$ *is an isomorphism between* $(\mathbf{A}, \mathbf{R})$ *and* $(\mathbf{M}, \in)$*. Furthermore,* $\mathbf{M}$ *and* $\mathbf{G}$ *are unique.*

For a proof see [Kun80], p. 106.

1.2 Basics from Model Theory

As the definition of large cardinals by means of elementary embeddings plays a central role, we will give a short model theoretic introduction to the subject. We start by defining models.[1]

Definition 1.25. *A* language *is a set of symbols consisting of relation symbols, function symbols and constant symbols:*

$$\mathcal{L} = \{P, \ldots, F, \ldots, c, \ldots\}. \tag{1.1}$$

where each P *is a* n*-ary relation symbol and each* F *is a* m*-ary function symbol for some integers* $n, m \geqslant 1$*.*

Starting from the language and the logical symbols $\neg$, $\rightarrow$, $=$, variables, punctuation signs and $\forall$, we define formulas in the following way:

[1] We will mainly follow the definitions from [Hod97].

Definition 1.26. Terms *of a language $\mathcal{L}$:*

1. *Every variable is a term of $\mathcal{L}$.*
2. *Every constant is a term of $\mathcal{L}$.*
3. *If $n > 1$, F is an n-ary function symbol of $\mathcal{L}$ and $t_1, \ldots, t_n$ are terms of $\mathcal{L}$, then the expression $F(t_1, \ldots, t_n)$ is a term of $\mathcal{L}$.*
4. *Nothing else is a term of $\mathcal{L}$.*

When we address the variables which occur in a term, we write t as $t(\bar{x})$, where $\bar{x} = (x_0, x_1, \ldots)$ is a sequence of distinct variables in which every variable of t is listed. If we substitute the variables $x_0, x_1, \ldots$ by terms $s_0, s_1, \ldots$, we write $t(\bar{s})$ with $\bar{s} = (s_0, s_1, \ldots)$.

Definition 1.27. Atomic formulas *of $\mathcal{L}$:*

1. *If s and t are terms of $\mathcal{L}$, then $s = t$ is an atomic formula of $\mathcal{L}$.*
2. *If $n > 1$, R is an n-ary relation symbol of $\mathcal{L}$ and $t_1, \ldots, t_n$ are terms of $\mathcal{L}$, then the expression $R(t_1, \ldots, t_n)$ is an atomic formula of $\mathcal{L}$.*

An atomic sentence is an atomic formula in which no variables occur.

Definition 1.28. Formulas *of $\mathcal{L}$:*

1. *Every atomic formula is a formula of $\mathcal{L}$.*
2. *If φ is a formula of $\mathcal{L}$, then $(\neg\varphi)$ is a formula of $\mathcal{L}$.*
3. *If φ and ψ are formulas of $\mathcal{L}$, then $(\varphi \rightarrow \psi)$ is a formula of $\mathcal{L}$.*
4. *If φ is a formula of $\mathcal{L}$ and x a variable, then $\forall x\varphi$ is a formula of $\mathcal{L}$.*
5. *Nothing else is a formula of $\mathcal{L}$.*

Definition 1.29. *An $\mathcal{L}$-*structure *$\mathcal{M}$ is an object composed of the following:*

1. *A set M, which is called the domain or universe of $\mathcal{M}$.*
2. *A set of constant elements, i.e. for each c in $\mathcal{L}$ an element $c^{\mathcal{M}}$.*
3. *For each integer $n < 0$, a set of n-ary relations on M, i.e. for each n-ary relation symbol P in $\mathcal{L}$ an n-ary relation $P^{\mathcal{M}} \subseteq M^n$.*

4. *For each integer $n < 0$, a set of n-ary functions on M, i.e. for each n-ary function symbol F in $\mathcal{L}$ an n-ary function $F^{\mathcal{M}}$ on M.*

To give meaning to a term or formula, we will interpret them in a structure. For a term $t(\bar{x})$ this means that if $\mathcal{M}$ is an $\mathcal{L}$-structure and $\bar{a} = (a_0, a_1, \ldots)$ is a sequence of elements of M of at least the same length as $\bar{x}$, then the following holds:

1. if t is the variable x_i, then $t^{\mathcal{M}}[\bar{a}]$ is a_i.

2. if t is a constant c, then $t^{\mathcal{M}}[\bar{a}]$ is the element $c^{\mathcal{M}}$.

3. if t is of the form $F(s_1, \ldots, s_n)$ where each s_i is a term $s_i(\bar{x})$, then $t^{\mathcal{M}}[\bar{a}]$ is the element $F^{\mathcal{M}}(s_1^{\mathcal{M}}[\bar{a}], \ldots, s_n^{\mathcal{M}}[\bar{a}])$.

If no variables occur in t we write $t^{\mathcal{M}}$ for $t^{\mathcal{M}}[\bar{a}]$.
For formulas we define the relation $\models$:

Definition 1.30. *Let $\phi(\bar{x})$ and $\psi(\bar{x})$ be formulas of $\mathcal{L}$ with $\bar{x} = (x_0, x_1, \ldots)$. Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $\bar{a}$ a sequence $(a_0, a_1, \ldots)$ of elements of $\mathcal{M}$, where the length of $\bar{a}$ is greater or equal to the length of $\bar{x}$. We define "$\bar{a}$* satisfies ϕ in $\mathcal{M}$*", $\mathcal{M} \models \phi[\bar{a}]$, by induction on the complexity of the formula ϕ:*

1. *ϕ is atomic. Then*

   (a) *if ϕ is the formula $s = t$ where $s(\bar{x})$ and $t(\bar{x})$ are terms, then*

   $$\mathcal{M} \models \phi[\bar{a}] \quad \textit{if and only if} \quad s^{\mathcal{M}}[\bar{a}] = t^{\mathcal{M}}[\bar{a}].$$

   (b) *if ϕ is the formula $R(s_1, \ldots, s_n)$ where $s_1(\bar{x}), \ldots, s_n(\bar{x})$ are terms, then*

   $$\mathcal{M} \models \phi[\bar{a}] \quad \textit{if and only if} \quad (s_1^{\mathcal{M}}[\bar{a}], \ldots, s_n^{\mathcal{M}}[\bar{a}]) \in R^{\mathcal{M}}.$$

2. *$\mathcal{M} \models \neg\phi[\bar{a}]$ if ond only if it is not true that $\mathcal{M} \models \phi[\bar{a}]$*

3. *$\mathcal{M} \models \phi[\bar{a}] \wedge \psi[\bar{a}]$ if and only if $\mathcal{M} \models \phi[\bar{a}]$ and $\mathcal{M} \models \psi[\bar{a}]$*

4. *Suppose ϕ is $\forall y\, \psi$, where ψ is $\psi(y\bar{x})$. Then $\mathcal{M} \models \phi[\bar{a}]$ if and only if for all elements b of $\mathcal{M}$, $\mathcal{M} \models \psi[b, \bar{a}]$.*

5. *Suppose ϕ is $\exists y\, \psi$, where ψ is $\psi(y\bar{x})$. Then $\mathcal{M} \models \phi[\bar{a}]$ if and only if there is at least one elements b of $\mathcal{M}$, $\mathcal{M} \models \psi[b, \bar{a}]$.*

A formula is called a *sentence* if it contains no free variables, i.e. no variables that are not bound by a quantifier. So, for a sentence ϕ we will write $\mathcal{M} \models \phi$ instead of $\mathcal{M} \models \phi[\bar{a}]$. A *theory* is a class of sentences.

Definition 1.31. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure and ϕ a sentence. If $\mathcal{M} \models \phi$, we say that $\mathcal{M}$ is a* model *of ϕ.*
Let T be a theory. $\mathcal{M} \models T$ means that $\mathcal{M}$ is a model of every sentence of T. Than we say that $\mathcal{M}$ is a model *of T.*

In order to define elementary embeddings, some interrelations between structures are relevant: In the following, let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. An *embedding* $f{:}\mathcal{M} \to \mathcal{N}$ is a one-to-one function from M to N such that:

i) for each constant c of $\mathcal{L}$, $f(c^{\mathcal{M}}) = c^{\mathcal{N}}$;

ii) for each $n > 0$, each n-ary function symbol F of $\mathcal{L}$, and each n-tupel $\bar{a}$ from $\mathcal{M}$, $f(F^{\mathcal{M}})(\bar{a})) = \mathcal{F}^{\mathcal{N}}(f(\bar{a}))$.

iii) for each $n > 0$, each n-ary relation symbol R of $\mathcal{L}$, and each n-tupel $\bar{a}$ from $\mathcal{M}$, $\bar{a} \in R^{\mathcal{M}}$ if and only if $f(\bar{a}) \in R^{\mathcal{M}}$.

We say that $\mathcal{M}$ is a *substructure* of $\mathcal{N}$ (write $\mathcal{M} \subseteq \mathcal{N}$), if $M \subseteq N$ and the inclusion map $i : M \to N$ is an embedding.
$\mathcal{M}$ is an *elementary substructure* of $\mathcal{N}$ (write $\mathcal{M} \preceq \mathcal{N}$) if for every formula $\phi(x_1, \ldots, x_n)$ and all tuples $\bar{a}$ from M,

$$\mathcal{M} \models \phi[\bar{a}] \quad \text{if and only if} \quad \mathcal{N} \models \phi[\bar{a}]. \tag{1.2}$$

Theorem 1.32 (Tarski-Vaught Criterion for Elementary Substructures)**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures with $\mathcal{M} \subseteq \mathcal{N}$. Then the following are equivalent:*

1. *$\mathcal{M}$ is an elementary substructure of $\mathcal{N}$.*

2. *For every formula $\phi(\bar{x}, y)$ and all tuples $\bar{a}$ from $\mathcal{M}$, if $\mathcal{N} \models \exists y \phi(\bar{a}, y)$ then $\mathcal{N} \models \phi(\bar{a}, d)$ for some element d of $\mathcal{M}$.*

For a proof see [Hod97], p. 48.

Definition 1.33. *An* elementary embedding $j : \mathcal{M} \to \mathcal{N}$ *of an $\mathcal{L}$-structure $\mathcal{M}$ into an $\mathcal{L}$-structure $\mathcal{N}$ is an embedding, whose range is an elementary substructure.*

An elementary embedding preserves all first-order formulas.

As we work mainly with models of set theory, i.e. models for the language, which has one relation symbol $\in$, we present some facts about elementary embeddings of the type $j : V \to M$, where M is a transitive subclass of V:

Fact 1.34.

- *For all ordinals α, $j(\alpha) \geqslant \alpha$.*
- *If α is an ordinal, then $j(\alpha)$ is an ordinal.*
- *If $\alpha < \beta$, then $j(\alpha) < j(\beta)$.*

Definition 1.35. *An embedding $j : \bar{V} \to \bar{N}$ is amenable to $\bar{V}$ when $j \upharpoonright A \in \bar{V}$ for any $A \in \bar{V}$.*

Definition 1.36. *An elementary embedding $j : M \to N$ is cofinal if for every set $A \in N$ there is a set $B \in M$ such that $A \subseteq j(B)$.*

Chapter 2

Large Cardinals and Elementary Embeddings

2.1 Large Cardinals

Most large cardinals can be approached from different viewpoints, leading to several equivalent possibilities to define a large cardinal. A good example are the weakly compact cardinals, which can be defined, for example, by partitions or by an approach which involves the weak compactness theorem for infinitary languages and therefore explains the name "weakly compact". As we want to examine extensions which do not create new large cardinals, we will have to show that one of these large cardinal definitions holds in V. For that, we will use the Main Theorem from Chapter 4, which shows that if an elementary embedding j is amenable to the extension, then a restriction of j is amenable to the ground model. This restriction will be an elementary embedding witnessing the large cardinal property in question. So we will focus our attention on defining large cardinals by means of elementary embeddings.

2.2 Weakly Compact and Indescribable Cardinals

Definition 2.1. *A cardinal κ is* inaccessible *if*

(i) $\kappa > \aleph_0$,

(ii) κ *is regular,*

(iii) $2^\lambda < \kappa$ *for every* $\lambda < \kappa$.

Fact 2.2. *Inaccessibility is downwards absolute to any model.*

Definition 2.3. *A cardinal* κ *is* weakly compact *if it is uncountable and every partition* $F : [\kappa]^2 \to 2$ *is constant on* $[H]^2$ *for some* $H \subset \kappa$ *with* $|H| = \kappa$.

Fact 2.4. *The following are equivalent:*

1. κ *is weakly compact.*

2. κ *is inaccessible and for all* $A \subseteq \kappa$ *there is an elementary embedding* $j : M \to N$ *such that critical point of* j *is* κ, M *is of size* κ, M *and* N *are transitive,* $M^{<\kappa} \subseteq M$ *and* $A \in M$.

3. κ *is inaccessible and for every transitive structure* M *of size* κ *with* $\kappa \in M$ *there is an elementary embedding* $j : M \to N$ *into another transitive structure* N *with critical point* κ.

For the definition of indescribable cardinals we are going to introduce a hierarchy of formulas.

i) a formula is Σ^0_0 and Π^0_0 if all of its quantifiers are bounded, i.e. are of the form $\exists x < y$ and $\forall x < y$.

ii) a formula is Σ^0_{n+1} if it is of the form $\exists x\, \varphi$, where φ is a Π^0_n formula.

iii) a formula is Π^0_{n+1} if it is of the form $\forall x\, \varphi$, where φ is a Σ^0_n formula.

iv) a formula is Σ^m_n if it is a formula of order $n+1$ of the form $\exists X \forall Y \ldots \varphi$ (n quantifiers), where $X, Y, \ldots$ are $(m+1)$th order variables and φ is such that all quantified variables are of the order at most m.

v) a formula is Π^m_n if it is a formula of order $n+1$ of the form $\forall X \exists Y \ldots \varphi$ (n quantifiers), where $X, Y, \ldots$ are $(m+1)$th order variables and φ is such that all quantified variables are of the order at most m.[1]

For first-order formulas ($m = 0$) this is called the Lévy Hierarchy.

Definition 2.5. *A cardinal* κ *is* Π^m_n-indescribable *if whenever* $U \subset V_\kappa$ *and* σ *is a* Π^m_n *sentence such that* $(V_\kappa, \in, U) \models \sigma$, *then for some* $\alpha < \kappa$, $(V_\alpha, \in, U \cap V_\alpha) \models \sigma$.

[1] For details on iv) and v) see [Jec03], p. 295.

In [Hau91], Kai Hauser gives a definition of indescribable cardinals in terms of elementary embeddings:

Definition 2.6. *Let M be a transitive model of ZF^- with $\kappa \in M$, $m \geqslant 1$, $n \geqslant 1$. M is Σ^m_n correct for κ (in parameters from $V_{\kappa+m}$) if and only if $M^{|V_{\kappa+m-2}|} \subseteq M$ (for $m = 1$ we mean $M^{<\kappa} \subseteq M$), and for any Σ^m_n formula $\phi(A)$ in a parameter $A \in (M)_{\kappa+m}$ we have $M \models$ "$V_\kappa \models \phi(A)$" just in case $V_\kappa \models \phi(A)$. (Thus in case $n = 0$ the first clause implies the second.)*

The second clause of the definition means that M correctly computes the Σ^m_n facts over V_κ that hold in parameters from $M \cap V_{\kappa+m}$. Then the following theorem holds:

Theorem 2.7. *For natural numbers $m, n \geqslant 1$ an inaccessible cardinal κ is Π^m_n indescribable if and only if for any transitive model M of ZF^- of size κ with $M^{<\kappa} \subseteq M$ and $\kappa \in M$, there is a transitive set N and an elementary embedding $j : M \to N$ with critical point κ such that N is Σ^m_{n-1} correct for κ and $|N| = |V_{\kappa+m-1}|$.*[2]

Since any first order statement about $V_{\kappa+m}$ is Δ_0 in $V_{\kappa+m+1}$, using $V_{\kappa+m}$ as a parameter, it follows that Π^{m+1}_1 indescribability implies Π^m_n indescribability for any n.

Definition 2.8. *A cardinal κ is totally indescribable if it is Π^m_n indescribable for any $m, n \in \omega$, or equivalently, if it is Π^m_1 indescribable for every m.*

2.3 Ultrafilters and Elementary Embeddings

There are several large cardinals which can be defined by certain kinds of ultrafilters, amongst which the most prominent example are the measurable cardinals. We will define the relevant concepts of filters and ideals and then show how ultrafilters are related to elementary embeddings.

Definition 2.9. *A* filter *on a nonempty set S is a collection F of subsets of S such that*

1. *$S \in F$ and $\varnothing \notin F$,*
2. *if $X \in F$ and $Y \in F$, then $X \cap Y \in F$,*

[2] For a proof see [Hau91], pp. 444–445.

3. *if* $X, Y \subset S$, $X \in F$ *and* $X \subset Y$, *then* $Y \in F$.

An ideal *on a nonempty set* S *is a collection* I *of subsets of* S *such that*

1. $\varnothing \in I$ *and* $S \notin I$,

2. *if* $X \in I$ *and* $Y \in I$, *then* $X \cup Y \in I$,

3. *if* $X, Y \subset S$, $X \in I$ *and* $Y \subset X$, *then* $Y \in I$.

If F *is a filter on* S, *then the set* $I = \{S - X : X \in F\}$ *is an ideal on* S*; and conversely, if* I *is an ideal, then* $F = \{S - X : X \in I\}$ *is a filter. If this is the case we say that* F *and* I *are* dual *to each other.*

Definition 2.10. *A filter* U *on a set* S *is an* ultrafilter *if for every* $X \subset S$, *either* $X \in U$ *or* $S - X \in U$.
A ideal I *on a set* S *is a* prime ideal *if for every* $X \subset S$, *either* $X \in I$ *or* $S - X \in I$.
Note that if U *and* I *are dual to each other and* U *is an ultrafilter (equivalently* I *is a prime ideal), then* $I = P(S) - U$.

Definition 2.11. *A filter* F *on a nonempty set* S *is called* principal *if there is a nonempty subset* X_0 *of* S *such that* $F = \{X \subset S : X \supset X_0\}$.
An ultrafilter is nonprincipal *if it is not principal.*

Definition 2.12. *If* κ *is a regular uncountable cardinal, and* F *is a filter on* S, *then* F *is called* κ-complete *if whenever* $\{X_\alpha : \alpha < \gamma\}$ *is a family of subsets of* S, $\gamma < \kappa$, *and* $X_\alpha \in F$ *for every* $\alpha < \gamma$, *then*

$$\bigcap_{\alpha<\gamma} X_\alpha \in F.$$

A κ-complete *ideal* I *on* S *is such that if whenever* $\{X_\alpha : \alpha < \gamma\}$ *is a family of subsets of* S, $\gamma < \kappa$, *and* $X_\alpha \in I$ *for every* $\alpha < \gamma$, *then*

$$\bigcup_{\alpha<\gamma} X_\alpha \in I.$$

Fact 2.13. *An ultrafilter* U *is* κ*-complete if and only if for any* $\gamma < \kappa$ *and* $\bigcup\{Y_\alpha | \alpha < \gamma\} \in U$ *there is an* $\alpha < \gamma$ *such that* $Y_\alpha \in U$.

Definition 2.14. *Let* I *be a* κ*-complete ideal on* κ *containing all singletons.* I *is* λ-saturated *(*λ *is a cardinal) if there exists no collection* W *of size* λ *of* *subsets of κ such that $X \notin I$ for all $X \in W$ and $X \cap Y \in I$ whenever X and Y are distinct members of W.*

Lemma 2.15 (Tarski). *If I is a κ-complete, λ-saturated ideal over κ where $2^{<\lambda} < \kappa$, then κ is measurable.*

Proof. Let $I^+ = P(\kappa) - I$. Suppose that I has an atom, that is, a set $A \in I^+$ such that whenever $A = B \cup C$ is a disjoint union, either $B \in I$ or $C \in I$. Then the κ-complete ultrafilter $U = \{X \subseteq \kappa : X \cap A \in I^+\}$ witnesses that κ is measurable.

Assume to the contrary that I has no atoms. Then we build a tree T, which is ordered by $\supset$ and consists of sets which are indexed by some members of ${}^{<\kappa}2 = \bigcup_{\alpha<\kappa} {}^{\alpha}2$ in the following way:

1. Set $X_{\varnothing} = \kappa$.
2. For the successor step: If X_s has been defined for some $s \in {}^{<\kappa}2$, then define $X_{s^\frown\langle 0\rangle}$ and $X_{s^\frown\langle 1\rangle}$ exactly when $X_s \in I^+$ such that $X_{s^\frown\langle 0\rangle}, X_{s^\frown\langle 1\rangle} \in I^+$, $X_s = X_{s^\frown\langle 0\rangle} \cup X_{s^\frown\langle 1\rangle}$ and $X_{s^\frown\langle 0\rangle} \cap X_{s^\frown\langle 1\rangle} = \varnothing$.
3. If $\delta < \kappa$ is a limit and $s \in {}^{\delta}2$, define $X_s = \bigcap_{\alpha<\delta} X_{s\restriction\alpha}$ exactly when $X_{s\restriction\alpha}$ has been defined for each $\alpha < \delta$.

Note that the definition of the successor step is possible because we assumed that I has no atoms. So we get a tree which splits in every step and has the following property: If $\gamma \leqslant \kappa$ and $s \in {}^{\gamma}2$, then the collection of "offshoots"

$$W = \{X_{s\restriction\alpha^\frown\langle i\rangle} \;:\; \alpha < \gamma \;\wedge\; X_{s\restriction\alpha+1} \text{ is defined } \;\wedge\; s(\alpha) \neq i\}$$

is pairwise disjoint. Then T must have height at most λ, otherwise W would be a collection of size λ of subsets of κ such that $X \notin I$ for all $X \in W$ and $X \cup Y = \varnothing \in I$ whenever X and Y are distinct members of W, therefore contradicting the λ-saturation of I (see Definition 2.14). As all X_s's without tree successors are elements of I and κ is the union of these X_s, κ is the union of $2^{<\lambda}$ many sets in I. But, by assumption, $2^{<\lambda} < \kappa$ and therefore $\kappa \in I$, contradicting the κ-completeness of I.[3] □

Definition 2.16. *An uncountable cardinal κ is* measurable *if there exists a κ-complete nonprincipal ultrafilter U on κ.*

[3] In this proof we followed [Kan09], p. 212.

In 1961 Dana Scott introduced a technique to construct elementary embeddings from κ-complete nonprincipal ultrafilters via ultrapowers[4]. This technique provided a definition of measurable cardinals by elementary embeddings:

Theorem 2.17. *The following are equivalent:*

1. *There is a measurable cardinal.*
2. *There exists a nontrivial elementary embedding $j : V \to M$, where M is transitive.*

As the ideas which are developed in the proof of this theorem and the next lemma are important for the next chapters, we will show them in detail. In the proof we will mainly follow [Jec03], pp. 285–287

Proof.
(1 implies 2). First we will give a general description of ultrapowers: Let U be ultrafilter on a set S, let F be the class of all functions $f : S \to V$. We define

$$f =^* g \quad \text{if and only if} \quad \{x \in S : \ f(x) = g(x)\} \in U,$$

$$f \in^* g \quad \text{if and only if} \quad \{x \in S : \ f(x) \in g(x)\} \in U,$$

Since U is a filter, $=^*$ is a equivalence relation and for every $f \in F$ we get an equivalence class

$$[f] = \{g : \ f =^* g \text{ and } \forall h(h =^* f \ \to \ \operatorname{rank} g \leqslant \operatorname{rank} h)\}$$

Then the model $Ult_U = (Ult_U, \in^*)$ has as its universe the class of all $[f]$, where f is a function on S and $[f] \in^* [g]$ if and only if $f \in^* g$.

The following Lemma is crucial to see that the ultrapower is elementarily equivalent to the universe:

Lemma 2.18 (Łoś)**.** [5] *If $\varphi(x_1, \dots, x_n)$ is a formula and $[f_1], \dots, [f_n] \in Ult_U$, then*

$$Ult_U \models \varphi([f_1], \dots, [f_n]) \quad \textit{if and only if} \quad \{x \in S : \varphi(f_1(x), \dots, f_n(x))\} \in U$$

[4] see [Sco61].
[5] For a proof see [Jec03], pp.159–160.

We get the following elementary embedding of the universe V into Ult_U: Let $c_a : S \to V$ be the constant function with value a and define $j_U : V \to Ult_U$ as $j_U(a) = [c_a]$. Then for each formula $\varphi(x_1, \ldots, x_n)$ the following holds:

$$\varphi(x_1, \ldots, x_n) \quad \text{if and only if} \quad Ult_U \models \varphi(j_U(x_1), \ldots, j_U(x_n)).$$

Now we have by assumption a κ-complete, non-principal ultrafilter U on κ and therefore we get a model $(Ult_U, \in^*)$, where the relation $\in^*$ is well-founded, set-like and extensional on Ult_U[6]. So by the Mostowski Collapsing Theorem (see Theorem 1.24) there is an isomorphism π_U between $(Ult_U, \in^*)$ and a transitive model $(M_U, \in)$. To simplify the notation, $[f]$ will be identified with $\pi_U([f])$ and Ult_U with M_U. We say that the function f represents the element $[f]$ of M_U. M_U is a model of ZFC, which contains all the ordinals and therefore M_U is an inner model.

It remains to show that j_U is nontrivial, which means that j_U is not the identity. Since j is elementary, $j(\alpha)$ is an ordinal if α is an ordinal and the facts about elementary embeddings hold (see Fact 1.34).

By induction one can show that for all $\alpha < \kappa$, $j(\alpha) = \alpha$: Assume that $\beta = j(\beta)$ for all $\beta < \alpha (< \kappa)$. If $[f] < j(\alpha)$, then $\{\zeta < \kappa : f(\zeta) < \alpha\} \in U$ and by κ-completeness there is a $\beta < \alpha$ such that $\{\zeta : f(\zeta) = \beta\} \in U$ and therefore $[f] =^* j(\beta)$. So $j(\alpha) = [c_\alpha] = \alpha$.

But $\kappa < j(\kappa)$: Let d be the diagonal function on κ, $d(\alpha) = \alpha$ for $\alpha < \kappa$. Since U is κ-complete, it contains no bounded subsets of κ. Therefore for any $\gamma < \kappa$, $\{\alpha : \gamma < \alpha < \kappa\} \in U$. So we have $\alpha = j(\alpha) < [d]$, thus $\kappa \leqslant [d]$ and because $[d] < j(\kappa)$, $\kappa < j(\kappa)$.

(2 implies 1): As there exists an α such that $\alpha \neq j(\alpha)$, let κ be the least such; so κ is the critical point of j, $cp(j) = \kappa$. As $j(\omega) = \omega$, $\kappa > \omega$. To show that κ is a measurable cardinal, we will define a κ-complete nonprincipal ultrafilter D in the following way: Let D be the collection of subsets of κ defined by:

$$X \in D \quad \text{if and only if} \quad \kappa \in j(X) \qquad (X \subset \kappa) \tag{2.1}$$

D is a filter since $\kappa \in j(\kappa)$, $\kappa \in D$ and because $\varnothing \notin D$, $j(\varnothing) = \varnothing$. Also,

[6]For details see [Jec03], p.286.

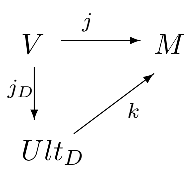
if $X, Y \in D$, then $\kappa \in j(X) \cap j(Y) = j(X \cap Y)$ and if $X \subset Y$, then $\kappa \in j(X) \subset j(Y)$.

D is an ultrafilter because $j(\kappa - X) = j(\kappa) - j(X)$ and D is nonprincipal because, for every $\alpha < \kappa$, $\{\alpha\} \notin D$ (as $j(\{\alpha\}) = \{j(\alpha)\} = \{\alpha\}$ and so $\kappa \notin j(\{\alpha\})$.

It remains to show that D is κ-complete: Let $\gamma < \kappa$ and $\mathcal{X} = \langle X_\alpha : \alpha < \gamma \rangle$ be a sequence of elements of D. Then $j(\mathcal{X})$ is a sequence of length $j(\gamma)$ of subsets of $j(\kappa)$, such that its $j(\alpha)$th term is $j(X_\alpha)$ for every $\alpha < \gamma$ and so $j(\mathcal{X}) = \rangle j(X_\alpha) : \alpha < \gamma \langle$. So $j(\bigcap_{\alpha<\gamma} X_\alpha) = \bigcap_{\alpha<\gamma} j(X_\alpha)$ and as $\kappa \in j(X_\alpha)$ for each $\alpha < \gamma$, $\bigcap_{\alpha<\gamma} X_\alpha \in D$. □

Lemma 2.19. *Let $j : V \to M$ be a nontrivial elemetary embedding with critical point κ and let D be the ultrafilter on κ defined in Definition 2.1. Let $j_D : V \to Ult_D$ be the canonical embedding of V in the ultrapower Ult_D. Then there is an elementary embedding k of Ult in M such that for all a:*

$$k(j_D(a)) = j(a)$$



Proof. To show that $k(j_D(a)) = j(a)$ for all a, we define for each $[f] \in Ult_D$: $k([f]) = (j(f))(\kappa)$, where f is a function on κ and $j(f)$ is a function on $j(\kappa)$. It can be shown that the definition is independent from the f chosen to represent $[f]$. Because of this and Definition 2.1, k is elementary.

From the ultrapower construction we know that $j_D(a) = [c_a]$, so, together with our definition from above, we have that $k(j_D(a)) = (j(c_a))(\kappa)$, where $j(c_a)$ is the constant function on $j(\kappa)$ with value $j(a)$ and so $(j(c_a))(\kappa) = j(a)$.[7] □

There are other large cardinals, for example strongly compact, supercompact and huge cardinals, which can be defined by means of ultrafilters and, because of the above construction, also by elementary embeddings.

[7] In this proof we follow [Jec03], pp. 288–289.

Definition 2.20. *A cardinal κ is* γ-compact *if for any set S, every κ-complete filter over S generated by at most $|\gamma|$ sets can be extended to a κ complete ultrafilter over S.*

Fact 2.21. *The following are equivalent:*

i) κ is γ-compact.

ii) There is an elementary embedding $j : V \to M$ with critical point κ such that: for any $X \subseteq M$ with $|X| \leqslant \gamma$, there is a $Y \in M$ such that $Y \supseteq X$ and $M \models |Y| < j(\kappa)$.

Definition 2.22. *A cardinals κ is* strongly compact *if and only if κ is γ-compact for every $\gamma \geqslant \kappa$.*

Definition 2.23. *Let A be a set of size at least κ, let F be the filter on $P_\kappa(A)$ generated by the sets $\bar{P} = \{Q \in P_\kappa(A) : P \subset Q\}$, $P \in P_\kappa(A)$:*

$$F = \{X \subset P_\kappa(A) : \ X \supset \bar{P} \text{ for some } P \in P_\kappa(A)\}$$

A κ-complete ultrafilter U on $P_\kappa(A)$ that extends F is called a fine measure. *A fine measure U on $P_{<\kappa}(A)$ is* normal *if whenever $f : P_\kappa(A) \to A$ is such that $j(P) \in P$ for all P in a set in U, then f is constant on a set in U. Equivalently U is normal if it is closed under diagonal intersections $\Delta_{a\in A} X_a = \{x \in P_\kappa(A) : x \in \bigcap_{a\in x} X_a\}$.*

Lemma 2.24. *The following are equivalent:*

1. κ is strongly compact.

2. For any A such that $|A| \geqslant \kappa$, there exists a fine measure on $P_\kappa(A)$.

The proof can be found in [Jec03], p. 366.

Definition 2.25. *An uncountable cardinal κ is* supercompact *if for every A such that $|A| \geqslant \kappa$ there exists a normal measure on $P_\kappa(A)$.*

The equivalent definition by elementary embeddings is:

Definition 2.26. *For $\kappa \leqslant \gamma$, a cardinal κ is* γ-supercompact *if and only if there is an elementary embedding $j : V \to M$ such that its critical point is κ, $\gamma < j(\kappa)$ and $M^\gamma \subseteq M$.*
An uncountable cardinal κ is supercompact *if and only if κ is γ-supercompact for every $\gamma \geqslant \kappa$.*

Definition 2.27. *A cardinal κ is* huge *if there exists an elementary embedding $j : V \to M$ with critical point κ such that $M^{j(\kappa)} \subset M$.*

Fact 2.28. *A cardinal κ is huge, with $j : V \to M$ and $j(\kappa) = \lambda$ if and only if there is a normal κ-complete ultrafilter U on $\{X \subset \lambda : ot(X) = \kappa\}$.*

2.4 Extenders

As Joel Hamkins and Hugh Woodin point out in [HW00], strong and Woodin cardinals have to be considered separately, because their embeddings are not simple ultrapowers, as in the case of measurable cardinals, but directed systems of them. These embeddings are constructed from extenders instead of ultrafilters. So we will first look at the relation between elementary embeddings and extenders and then give a definition for strong and Woodin cardinals.[8]

Let $j : V \to M$ be an elementary embedding with critical point κ and let $\kappa \leqslant \lambda \leqslant j(\kappa)$. Then the (κ, λ)-extender derived from j is the collection

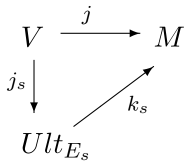
$$E = \{E_s : s \in [\lambda]^{<\omega}\}; \tag{2.2}$$

where, for every finite subset $s \subset \lambda$, E_s is the measure on $[\kappa]^{<\omega}$ defined as follows:

$$X \in E_s \quad \text{if and only if} \quad s \in j(X). \tag{2.3}$$

Note that E_s concentrates on $[\kappa]^{|s|}$, κ is the critical point of E and λ is the length of E.

As every measure E_s $[\kappa]^{<\omega}$ is κ-complete, we can build the corresponding ultrapower embedding similar to the proof of Theorem 2.17: So for every $s \in [\lambda]^{<\omega}$ let Ult_{E_s} denote the ultrapower of V by E_s and let $j_s : V \to Ult_{E_s}$ be the corresponding elementary embedding. If for each equivalence class $[h]$ of a function h on $[\kappa]^{<\omega}$ we let $k_s([h]) = j(h)(s)$, then k_s is an elementary embedding $k_s : Ult_{E_s} \to M$ and $k_s \circ j_s = j$:



[8] This section follows the presentation of extenders given in [Jec03], pp. 382–384.

Let $s \subset b$, where $b = \{\alpha_1, \ldots, \alpha_n\}$ with $\alpha_1 < \ldots < \alpha_n$. We show that there is an elementary embedding between the ultrapower built by E_s and the ultrapower built by E_b. First observe that the ultrafilters E_s, $s \in [\lambda]^{<\omega}$ are coherent in the following sense: Define $\pi_{b,s} : [\lambda]^{|b|} \to [\lambda]^{|s|}$ by

$$\pi_{b,s}(\{\xi_1, \ldots, \xi_n\}) = \{\xi_{i_1}, \ldots \xi_{i_m}\}, \qquad (\xi_1 < \ldots < \xi_n)$$

where $s = \{\alpha_{i_1}, \ldots, \alpha_{i_m}\}$, and

$$X \in E_s \quad \text{if and only if} \quad \{t : \pi_{b,s}(t) \in X\} \in E_b.$$

So we define $i_{s,b} : Ult_{E_s} \to Ult_{E_b}$ by

$$i_{s,b}([h]_{E_s}) = [h \circ \pi_{b,s}]_{E_b}$$

This is an elementary embedding, and

$$\{Ult_{E_s}, i_{s,b} : s \subset b \in [\lambda]^{<\omega}\} \tag{2.4}$$

is a directed system. The direct limit Ult_E of (2.4) is well founded: Note that the embeddings k_s have a direct limit $k : Ult_E \to M$ such that $k \circ j_E = j$, where j_E is the elementary embedding $j_E : V \to Ult_E$.

There is another description of Ult_E which resembles more closely the procedure given in the proof of Theorem 2.17. Instead of speaking about equivalence classes of functions, the elements of Ult_E are here the equivalence classes $[s, f]_E$ and (s, f) and (b, g) are equivalent if $\{t \in [\kappa]^{|s \cup b|} : \bar{f}(t) = \bar{g}(t)\} \in E_{s \cup b}$, where $\bar{f} = f \circ \pi_{s \cup b, s}$ and $\bar{g} = g \circ \pi_{s \cup b, b}$. The embedding $j_E : V \to Ult_E$ is then defined by $j_E(x) = [\varnothing, c_x]$, where c_x is the constant function with value x. The embedding $k : Ult_E \to M$ is defined by

$$k([s, f]) = j(f)(s). \tag{2.5}$$

It follows that $k \circ j_E = j$.

Lemma 2.29.

i) $k(\alpha) = \alpha$ *for all* $\alpha < \lambda$.

ii) j_E *has critical point* κ *and* $j_E(\kappa) \geqslant \lambda$.

iii) $Ult_E = \{j_E(h)(s) : s \in [\lambda]^{<\omega}, h : [\kappa]^{<\omega} \to V\}$.

The Lemma can be proven from the following observations: For each $s \in [\lambda]^{<\omega}$, let $j_{s,\infty} : Ult_{E_s} \to Ult_E$ be the direct limit embedding such that $j_{s,\infty} \circ j_s = j_E$; then $k \circ j_{s,\infty} = k_s$. If $x \in Ult_E$ then $x = j_{s,\infty}([f])$ for some $[f] \in Ult_E$, and from (2.5) follows that

$$k(x) = k(j_{s,\infty}([f])) = k_s([f]) = j(f)(s).$$

Therefore

$$k''Ult_E = \{j(h)(s) : s \in [\lambda]^{<\omega}, h : [\kappa]^{<\omega} \to V\}. \tag{2.6}$$

Hence $j_E : V \to Ult_E$ is an elementary embedding with critical point κ. As $j = k \circ j_E$ and $k(s) = s$ for all $s \in [\lambda]^{<\omega}$, it follows that for all $X \in [\kappa]^{|s|}$, $s \in j_E(X)$ if and only if $s \in j(X)$. So E is the extender derived from j_E.

Definition 2.30. *A cardinal κ is a* strong cardinal *if for every set x there exists an elementary embedding $j : V \to M$ with critical point κ such that $x \in M$.*

*A cardinal κ is λ-*strong*, where $\lambda \geqslant \kappa$, if there exists some $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$ and $V_\lambda \subset M$. Such a j is also called a λ-strongness embedding. A cardinal κ is strong if and only if it is λ-strong for all $\lambda \geqslant \kappa$.*

Lemma 2.31. *A cardinal κ is strong if and only if for every $\lambda \geqslant \kappa$ there is a $(\kappa, |V_\lambda|^+)$-extender E such that $V_\lambda \subset Ult_E$ and $\lambda < j_E(\kappa)$.*

In the rest of the section we present some ideas from [HW00], regarding strongness embeddings.

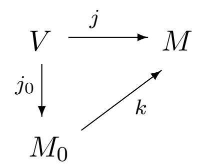
Definition 2.32. *A λ-strongness embedding $j : V \to M$ is* natural *when $M = \{j(h)(s) : h \in V \,\&\, s \in \gamma^{<\omega}\}$, where $\gamma = |V_\lambda|^M$.*

Equivalently, we could require that $M = \{j(h)(s) : h \in V \,\&\, s \in V_\lambda\}$ (see also (2.6)) .

Remark 2.33. *Every strongness embedding factors through a natural embedding.*

Proof. Let $j : V \to M$ be a λ strongness embedding, where $\lambda > \kappa$ and let $X = \{j(h)(s) : h \in V$ and $s \in \gamma^{<\omega}\}$. First we define the embedding

$j_0 : V \to M_0$: With the Tarski-Vaught Criterion (see Theorem 1.32) we can check that X is an elementary substructure of M covering $ran(j)$. Let $\pi : X \to M_0$ be the Mostowski Collapse of X. Then we get an embedding $j_0 : V \to M_0$ with M_0 transitive, by defining $j_0 = \pi \circ j$. Thus also $j = k \circ j_0$, where $k = \pi^{-1}$, and so j factors through j_0.



Since $V_\lambda \subseteq X$, it follows from the Mostowski Collapsing Theorem (see 1.24) that $\pi''V_\lambda = V_\lambda$ and $V_\lambda \subseteq M_0$, and, since $M_0 = ran(\pi)$ and $\pi \restriction \gamma = id$, it follows that $M_0 = \{j_0(h)(s) : h \in V \;\&\; s \in \gamma^{<\omega}\}$. Thus, j_0 is a natural λ-strongness embedding. □

Remark 2.34. *If $j : V \to M$ is a natural λ-strongness embedding with critical point κ and λ is either a successor ordinal or a limit ordinal with cofinality above κ, then M is closed under κ-sequences. Otherwise, M is closed under $< cof\lambda$-sequences.*

Proof. Suppose that $\lambda > \kappa$ and $M = \{j(h)(s) : h \in V \text{ and } s \in V_\lambda\}$. Then we have the following three cases:

Case 1: Suppose $\lambda = \xi + 1$ and $\langle j(h_\alpha)(s_\alpha) : \alpha < \kappa\rangle$ is a κ-sequence of elements from M, with each $s_\alpha \in V_{\xi+1}$. Since a κ-sequence of subsets of V_ξ can be coded with a single subset of V_ξ, it follows that $\langle s_\alpha : \alpha < \kappa\rangle$ is in M. Then, since the sequence $\langle j(h_\alpha) : \alpha < \kappa\rangle = j(\langle h_\alpha : \alpha < \kappa\rangle) \restriction \kappa$ is in M, it follows that $\langle j(h_\alpha)(s_\alpha) : \alpha < \kappa\rangle$ is in M.

Case 2: Suppose λ is a limit ordinal of cofinality larger than κ. Then on cofinality grounds the sequence $\langle s_\alpha : \alpha < \kappa\rangle$ is in V_λ, and hence in M, so again $\langle j(h_\alpha)(s_\alpha) : \alpha < \kappa\rangle$ is in M.

Case 3: Suppose λ is a limit ordinal, $\beta < cof(\lambda) \leqslant \kappa$ and $\langle j(h_\alpha)(s_\alpha) : \alpha < \beta\rangle$ is a sequence of elements of M. Then on cofinality grounds we know that $\langle s_\alpha : \alpha < \beta\rangle$ is in V_λ, and hence in M, and so $\langle j(h_\alpha)(s_\alpha) : \alpha < \beta\rangle$ is in M. □

So, while strongness embeddings in general need not satisfy any closure properties, natural strongness embeddings do satisfy certain closure properties.

Definition 2.35. *A cardinal* δ *is a* Woodin cardinal *if for all* $A \subset V_\delta$ *there are arbitrarily large* $\kappa < \delta$ *such that for all* $\lambda < \delta$ *there exists an elementary embedding* $j : V \to M$ *with critical point* κ*, such that* $j(\kappa) > \lambda$*,* $V_\lambda \subset M$*, and* $A \cap V_\lambda = j(A) \cap V_\lambda$*.*

Chapter 3

Small Forcing

One of the first results concerning the interrelation between forcing and large cardinals was a theorem by Azriel Lévy and Robert Solovay. In [LS67] they showed that if $ZFC+$ *"there is a measurable cardinal"* is consistent, then it is consistent with the Continuum Hypothesis and with its negation. They worked with a special kind of forcing, the so called "small forcing", which is characterized by $|P| < \kappa$, where $(P, <)$ is the notion of forcing and κ the measurable cardinal. Under small forcing the Continuum Hypothesis can be forced to hold or fail, so the authors of the above paper had to show that a measurable cardinal can neither be destroyed by small forcing (that means, if there is a measurable cardinal in the ground model, it is measurable in the extension) nor can it be created (so every measurable cardinals in the extension is measurable in the ground model). This result can be extended to various other large cardinals, for example strongly compact, supercompact and huge cardinals. Like measurable cardinals they can be defined by elementary embeddings which are ultrapower embeddings (see Theorem 2.17) and the proof for these cardinals resembles the Lévy-Solovay proof. Strong and Woodin cardinals on the other hand present a special case; as they are not defined by ultrapower embeddings but extender embeddings, they will be dealt with a separate section.

3.1 The Theorem of Lévy and Solovay

In [LS67], Lévy and Solovay proved both directions (from the ground model to the extension and vice versa) by showing that it is always possible to find a

κ-complete nonprincipal ultrafilter U witnessing that κ is measurable. But, as we also want to direct our attention to elementary embedding, we will present a proof from [Jec03][1] which uses embeddings for the direction from the ground model to the extension and for the other direction ultrafilters. The Main Theorem from Chapter 4 will give us then the direction from the extension to the ground model by elementary embeddings in a general way.

Theorem 3.1 (Lévy-Solovay)**.** *Let $(P, <)$ be a notion of forcing such that $|P| < \kappa$. Then κ is a measurable cardinal in the ground model if and only if κ is measurable in the generic extension.*

Proof. "$\Rightarrow$" Assume that κ is a measurable cardinal in V. So by Theorem 2.17 we have an elementary embedding $j : V \to M$, where M is transitive and κ is the critical point of j and we will extend j to an elementary embedding $\hat{j} : V[G] \to M[G]$ in the forcing extension, witnessing the measurability of κ in the forcing extension.

As $|P| < \kappa$, we assume that $P \in V_\kappa$ and, as κ is the critical point of j, $j(p) = p$ for all $p \in P$ and so $j(P) = P$. Let G be a generic ultrafilter on P and we work in $V[G]$. Since G is generic over V, G is generic over M. So the interpretation of P-names in M^P by G is the same wether computed in V or in M.

We extend j to $V[G]$ in the following way: For every $x \in V[G]$, let $\dot{x} \in V^P$ be its name, $x = \dot{x}^G$. Then $j(\dot{x}) \in M^P$ and so $(j(\dot{x}))^G \in M[G]$. So define $\hat{j}(x)$ for $x \in V[G]$ as follows:

$$\hat{j}(x) = (j(\dot{x}))^G \tag{3.1}$$

The choice of the name for x has no bearing on the definition in (3.1): Let $\dot{y}$ be another P-name for x. Then there is a condition $p \in G$ which forces the names to be equal: $p \Vdash \dot{x} = \dot{y}$. Under the embedding j we get in M that $j(p) \Vdash j(\dot{x}) = j(\dot{y})$. Now we use the fact that the forcing is small: Since $j(p) = p \in G$, it follows that $(j(\dot{x}))^G = (j(\dot{y}))^G$.

For $\hat{j}$ to witness the measurability of κ in the extension, $\hat{j}$ has to be elementary (note that $cp(\hat{j}) = cp(j)$).

Let $\varphi(x, \ldots)$ be a formula such that

$$V[G] \models \varphi(x, \ldots)$$

[1] See [Jec03], pp. 389–391.

Choosing names $\dot{x}, \ldots$ for the variables $x, \ldots$ such that $(\dot{x})^G = x, \ldots$, there is as before some $p \in G$ such that

$$p \Vdash \varphi(\dot{x}, \ldots)$$

Therefore we get in M by applying j:

$$p \Vdash \varphi(j(\dot{x}), \ldots)$$

and so

$$M[G] \models \varphi(\hat{j}(x), \ldots)$$

So, because φ was arbitrary, $\hat{j}$ is elementary.

"$\Leftarrow$" If κ is measurable in $V[G]$, let $U \in V[G]$ be a κ-complete nonprincipal ultrafilter on κ, let J be the dual prime ideal and let $\dot{J} \in V^P$ be its name. Without loss of generality we may assume that $[\![\dot{J}$ is a κ complete nonprincipal prime ideal$]\!] = 1$. From $\dot{J}$ we define the following κ-complete ideal containing all singletons:

$$I = \{X \subset \kappa : [\![X \in \dot{J}]\!] = 1\}$$

Then I is $|P|^+$-saturated (see Definition 2.14): If $p \Vdash \check{X} \notin \dot{J}$ and $p \Vdash \check{Y} \notin \dot{J}$, then $p \Vdash \check{X} \cap \check{Y} \notin \dot{J}$, because $\dot{J}$ is prime. So if X and Y are such that $X \notin I$, $Y \notin I$ and $X \cap Y \in I$, then $[\![X \cap Y \in \dot{J}]\!] = 1$. This is a contradiction and therefore I is $|P|^+$-saturated.

By Fact 2.2 we know that if κ is inaccessible in $V[G]$, then κ is inaccessible in V. Since I is ν-saturated for a $\nu < \kappa$ and κ is inaccessible, we know by Lemma 2.15 that κ is measurable in V. □

This result can be extended to other large cardinal properties:

Theorem 3.2. *Let κ be an infinite cardinal and let $(P, <)$ be a notion of forcing such that $|P| < \kappa$. Let G be a V-generic filter on P. Then κ is inaccessible (Mahlo, weakly compact, Ramsey, measurable, strongly compact, supercompact, huge) in V if and only if it is inaccessible (Mahlo, weakly compact, Ramsey, measurable, strongly compact, supercompact, huge) in $V[G]$.*

For large cardinals, which are defined like measurable cardinals by elementary embeddings or ultrafilters, the proof is quite similar to the one of the Lévy-Solovay Theorem: For the direction from V to $V[G]$ we define a $\hat{j}$ in $V[G]$ and prove the required properties be using j in the ground model. For the converse, we build an ideal I with certain properties, which gives us the desired large cardinal property of κ in V. Therefore we will only give the proof[2] for strongly compact cardinals as an example to show the similarities.[3]

Proof for strongly compact cardinals.

"$\Rightarrow$" Suppose that κ is strongly compact in V and let $\lambda \geqslant \kappa$. Because of Lemma 2.24, we show that there is a fine measure on $P_\kappa(\lambda)$. So let U be a fine measure on $P_\kappa(\lambda)$ in V. Then, similarly to the construction in the proof of Theorem 2.17, we can construct an elementary embedding $j = j_U : V \to Ult_U$ such that $X \in U$ if and only if $H \in j(X)$, where H is the set in Ult_U represented by the diagonal function $d(Z) = Z$ on $P_\kappa(\lambda)$. From the construction it follows that $H \supset j''\lambda$. Now we can follow the same steps as in the proof of Theorem 3.1: We extend j to $V[G]$ by defining $\hat{j}(x) = (j(\dot{x}))^G$, where $\dot{x}$ is a name for x. Since we can assume, without loss of generality, that $P \in V_\kappa$ and therefore $j(p) = p$ for all $p \in P$ and $j(P) = P$, the definition does not depend on the choice of the name. From this elementary embedding in $V[G]$ we can construct an ultrafilter W on $P_\kappa(\lambda)$ by the equation:

$$X \in W \qquad \text{if and only if} \qquad H \in \hat{j}(X)$$

Then W is a fine measure on $P_\kappa(\lambda)$: If Z_0 is an element of $P_\kappa(\lambda)$, then $j(Z_0) = \{j(\alpha) : \alpha \in Z_0\} \subset H$, because $H \supset j''\lambda$ and therefore $\{Z \in P_\kappa(\lambda) : Z \supset Z_0\} \in W$, as required in Definition 2.23.

"$\Leftarrow$" Suppose that κ is strongly compact in $V[G]$. Let S be a set in V and let F be a κ-complete filter on S in V. Following Definition 2.20, we will show that there is a κ-complete ultrafilter in V extending F. Since the forcing is small, every set $X \subset F$ with $|X|^{V[G]} < \kappa$ is included in some $Y \subset F$ with $|Y| < \kappa$ such that $Y \in V$. Therefore F generates a filter in $V[G]$ which is κ-complete and, because of the assumption, this filter can be included in a κ-complete ultrafilter U. Similar to the proof of the Lévy-Solovay Theorem,

[2] We follow the proof from [Jec03], pp. 399–400

[3] The proof for Mahlo, weakly compact and Ramsey cardinals can be found in [Jec03], p. 391.

let J be its dual prime ideal. Then $I = \{X \subset S : [\![X \in \dot{J}]\!] = 1\}$ is a κ-complete, $|P|^+$-saturated ideal and $X \in F$ implies $S - X \in I$. So, using Lemma 2.15, we know that I has an atom A and therefore if $X \in F$, then $X \cap A \notin I$. Hence $\{X \subset S : X \cap A \notin I\}$ is a κ-complete ultrafilter extending F. □

3.2 Strong and Woodin Cardinals

In Chapter 2 we defined strong and Woodin cardinals by elementary embeddings and then showed that there is an equivalent definition by extenders. In contrast to measurable cardinals, strong and Woodin cardinals are defined by the direct limit of a directed system of ultrapowers instead of a single ultrapower. For this reason, the proof of the Lévy-Solovay Theorem cannot simply be transferred. Specifically, the direction from the extension to the ground model requires another approach.

Here we will follow the paper [HJed] by Hugh Woodin and Joel Hamkins. The authors show a detailed Level-by-Level Version, which allows them to include also the case of partially strong cardinals.

Theorem 3.3. *Let $(P, <)$ be a notion of forcing, where $|P| < \kappa$ and κ is an uncountable cardinal. Suppose that $G \subseteq P$ is V-generic for P. Then κ is strong in V if and only if it is strong in $V[G]$.*

Proof. Assume $P \in V_\kappa$ and let $\delta = |P| < \kappa$.

"$\Rightarrow$" This case is analog to the proof of the Lévy-Solovay Theorem: Any λ-strongness embedding $j : V \to M$ in V lifts to an embedding $\hat{j} : V[G] \to M[G]$ in $V[G]$ by defining $\hat{j}(x)$ for $x \in V[G]$ as follows: Let $\dot{x} \in V^P$ be a name for x, $x = \dot{x}^G$. Let

$$\hat{j}(x) = (j(\dot{x}))^G \tag{3.2}$$

This embedding witnesses the λ-strongness of κ in $V[G]$ because $V[G]_\lambda = V_\lambda[G]$ and as $V_\lambda \subseteq M$, $V[G]_\lambda \subseteq M[G]$.

"$\Leftarrow$" Here we will prove a slightly different version of the theorem above. *Level-by-level Version*: If $G \subseteq P$ is V-generic for the small forcing P, then for every ordinal λ (except possibly the limit ordinals with $cof\lambda \leqslant |P|^+$) every natural λ-strongness embedding $j : V[G] \to M[G]$ in the extension lifts a λ-strongness embedding $j \upharpoonright V : V \to M$ definable in the ground model.

Let κ be λ-strong in $V[G]$ with natural embedding $j : V[G] \to M[G]$ (see Definition 2.32), where $\lambda > \kappa$ is either a successor ordinal or a limit ordinal with $cof(\lambda) > \delta^+$. Since j is natural, we know that $M[G] = \{j(h)(s) : h \in V[G] \text{ and } s \in \gamma^{<\omega}\}$, where $\gamma = \beth_\lambda^M$.

We will define $j{\restriction}V$ by an extender E (Claim 1) and then show that E is in V (Claim 2). Moreover we will prove that $j{\restriction}V$ is a λ-strongness embedding (Claim 3).

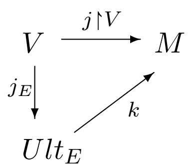
First observe that $V_\kappa = (V_\kappa)^M$ because $cp(j) = \kappa$ and therefore, sets with rank below the κ are fixed by j. Also $V_{\kappa+1} = (V_{\kappa+1})^M$: If $A \subseteq \kappa$ in V, then A in $V[G]$, $j(A)$ in $M[G]$, $A = j(A) \cap \kappa$ in M and so $P[\kappa]^V \subseteq M$. Conversely if $A \subseteq \kappa$ in M, then every initial segment of A is in V, but because $|P| < \kappa$ we have only less than κ many conditions which decide that these inital segments are in V, so there is a condition p which decides it for cofinally many initial segments and then this p decides A. So we have that A is in V.

Let E be the following induced V-extender (see also (2.3) and (2.2)):

$$E = \{\langle A, s\rangle : s \in \gamma^{<\omega} \;\&\; A \in V_{\kappa+1} \;\&\; s \in j(A)\} \tag{3.3}$$

This extender defines $j{\restriction}V$:

Claim 1. The restricted embedding $j \restriction V : V \to M$ is precisely the embedding induced by the extender E



Proof of Claim 1. Since $j \restriction V = j_E \circ k$, we have to show that $j \restriction V = j_E$ and therefore that $k : \langle h, s\rangle \mapsto j(h)(s)$ is an isomorphism. For this it suffices to show that

$$M = \{j(h)(s) : h \in V \text{ and } s \in \gamma^{<\omega}\} = k''Ult_E \tag{3.4}$$

Since j is natural in $V[G]$, any set a in M has the form $j(h)(s)$ for some function $h : [\kappa]^n \to V$ in $V[G]$ and some $s \in \gamma^n$ and $n \in \omega$. To show (3.4), we need to find such a function in V and we will define it by a name for the corresponding function h in $V[G]$: Let $\dot{h}$ be a name for a function h (i.e. $\dot{h}_G = h$) such that $j(\dot{h})_G(s) = a$ in $M[G]$. Then there must be a condition $p \in G$ such that we have in M: $p \Vdash j(\dot{h})(\check{s}) = \check{a}$. Then let $\bar{h}$ be the following function in V: $\bar{h}(x) = y$ when p forces that $\dot{h}(\check{x}) = \check{y}$. It follows that $j(\bar{h})(s) = a$ and so a has the desired form. $\square$

As we have established $j \upharpoonright V$ to be the above extender embedding, the question of $j \upharpoonright V$ being definable in V reduces to the problem if E is in V:

Claim 2. E lies in V.

Proof of Claim 2. First we prove the following statement:

$$\text{If } F \subseteq E \text{ and } |F| = \delta = |P|, \text{ then there is } F^* \in V \text{ such that } F \subseteq F^* \subseteq E \tag{3.5}$$

Using a fixed F with the above properties we will build an κ-complete non-principal ultrafilter in V from which we can compute F^* to be a restriction of the extender E.

Let σ be the set of ordinals appearing in any s which appears in F (σ is the set of generators which appear in F). Since $|F| = \delta$ and $s \in \gamma^{<\omega}$ we have that $\sigma \subseteq \gamma$ and $|\sigma| \leqslant \delta$. Because of the closure properties for the natural strongness embedding j from Remark 2.34, we know that $M[G]$ is closed under κ-sequences and by small forcing we have that $\delta < \kappa$ (the needed assumptions on λ are given by the hypothesis of the Level-by-level Version). So, σ is in $M[G]$ and therefore the set σ has names both in M and V. We use such names from V and M to build an increasing δ^+-sequence of sets $\vec{\sigma} = \langle \sigma_\alpha : \alpha < \delta^+ \rangle$ such that every σ_α has cardinality δ. We start with $\sigma_0 = \sigma$, take a name for σ in V and built from it a σ_1 in V covering σ; then we go to M and use a name for σ to get a σ_2 covering σ_1 and so on. Then we have that for cofinally many α the set σ_α is in V, and that for cofinally many α it is in M. Again, by the closure properties of the embedding, $\vec{\sigma} \in M[G]$ has names in both M and V.

Let $\tau = \bigcup_\alpha \sigma_\alpha$. Then τ is in V: As $\vec{\sigma}$ has a name in V, there are conditions in P which force the various σ_α to appear. Since $\vec{\sigma}$ is a $\delta^+ = |P|^+$-sequence and δ^+ is regular there must be a condition p which decides unboundedly many of the $\dot{\sigma}_\alpha$ in $\dot{\vec{\sigma}}$. This p then decides τ and so $\tau \in V$. As $\vec{\sigma}$ has also a name in M, it holds by the same argument that $\tau \in M$.

Let $\mu = \{A \in V_{\kappa+1} : \tau \in j(A)\}$. This is a κ-complete nonprincipal V-ultrafilter on κ in $V[G]$ and by the Lévy-Solovay Theorem 3.1 for measurable cardinals it lies in V. From μ and τ we build F^* as the restriction of E to τ.

$$F^* = \{\langle A, s\rangle : s \in \tau^{<\omega} \,\&\, A \in V_{\kappa+1} \,\&\, s \in j(A)\} = E \restriction \tau \tag{3.6}$$

That F^* is in V follows from the fact that any s from F^* corresponds to a projection of τ: If s consists of the $\alpha_0^{\text{th}}, \ldots, \alpha_n^{\text{th}}$ elements of τ, then $\langle A, s\rangle \in F^*$ exactly when $\pi^{-1}A \in M$ (because $\langle A, s\rangle \in F^*$ iff $s \in j(A)$ iff $\pi(\tau) \in j(A)$ iff $\tau \in j(\pi^{-1}A)$ iff $\pi^{-1}A \in \mu$) where $\pi(t)$ restricts t to its $\alpha_0^{\text{th}}, \ldots, \alpha_n^{\text{th}}$ elements. So F^* is in V.

To prove Claim 2, we will show that E in intersection with a certain elementary substructure of V is in V: As $E \in V[G]$, it has a name $\dot{E} \in V$. Let $X \prec V_\eta$ be an elementary substructure of size δ which contains P, F^*, $\dot{E}$ and every element of P with $\eta \gg \gamma$ (recall that $\gamma = \beth_\lambda^M$). Then G is X-generic, $X[G] \prec V_\lambda$ and $E \in X[G]$. Let $F = E \cap X$. As we have proven above, there is a set $F^* \in V$ such that $F \subseteq F^* \subseteq E$. Then also F is in V because by $F = F \cap X \subseteq F^* \cap X \subseteq E \cap X = F$ it follows that $F = F^* \cap X$. So $E \cap X$ is in V and this means that there is a condition $p \in G$ which decides "$\langle A, s\rangle \in \dot{E}$" for all $\langle A, s\rangle$ in X. Since X is an elementary substructure of V_η, p decides "$\langle A, s\rangle \in \dot{E}$" for all $\langle A, s\rangle$ in V. Hence E is in V. □

With Claim 1 and Claim 2 we have defined $j \restriction V$ and proven that it is definable from E in V. Now it remains to show that $j \restriction V$ is a λ-strongness embedding and therefore witnesses that κ is λ-strong in V. For this it suffices to show that the following claim holds:

Claim 3: $V_\lambda \subseteq M$.

Proof of Claim 3. Since $j : V[G] \to M[G]$ is λ-strong, we know that $V_\lambda \subseteq M[G]$. Let $a \in V_\lambda$. Then there is a condition $p \in G$ which decides a in V, but since we have the same generic G in $M[G]$, p has to decide a in the same way in M. So p forces over V that $\check{a} = \dot{a}$ (where $\dot{a}$ is a name in M) and using p and $\dot{a}$ in M it follows that $a \in M$. So, $V_\lambda \subset M$. □

□

With the techniques from above we can prove the same result for Woodin cardinals.

Corollary 3.4. *Let $(P, <)$ be a notion of forcing, where $|P| < \kappa$ and κ is a cardinal. Suppose that $G \subseteq P$ is V-generic for P. Then κ is Woodin in V if and only if it is Woodin in $V[G]$.*

Proof. "⇒" This is analog to Theorem 3.3 and the respective direction in the Lévy-Solovay Theorem.

"⇐" Let δ be a Woodin cardinal in $V[G]$ and $A \subseteq \delta$ in V. By definition of a Woodin cardinal (Definiton 2.35), there is a cardinal $\kappa < \delta$ such that for every $\lambda < \delta$ there is a natural λ-strongness embedding $j : V[G] \to M[G]$ with critical point κ such that $j(A) \cap \lambda = A \cap \lambda$. By the Level-by-level Version of Theorem 3.3 we know that $j \upharpoonright V$ has the same properties in V if λ is a successor ordinal. But then it follows that δ is Woodin in V because the successor ordinals are unbounded in δ.

□

Chapter 4

The Approximation and Cover Properties

In the previous chapter we showed that under small forcing large cardinals are neither destroyed nor created. But what can we say about forcings of size greater or equal than κ? Here the results for the direction from the ground model to the extension and for the converses direction can be very different. For example, an inaccessible cardinal in the ground model becomes a successor cardinal in the extension by the Lévy Collapse, but for the other direction one can show that inaccessible cardinals, like Mahlo cardinals, are downwards absolute to any model.

Furthermore, there are extensions which have new large cardinals: For example, Joel Hamkins mentions in [Ham03] a result by Kunen which showed that "a non-weakly compact cardinal κ can become measurable or more after adding a branch to a κ-Suslin tree"[1].

However, Hamkins succeeds in showing that for a large class of forcing notions new large cardinals are not created. In fact he identifies two properties, the δ approximation property and δ cover property, and shows that in every extension satisfying these properties the following holds: If there is an elementary embedding (with certain closure properties) in the extension, then its restriction to the ground model is amenable to the ground model. As we have seen in Chapter 2, certain embeddings witness the existence of certain large cardinals and therefore Hamkins can answer the above question for the direction from the extension to the ground model.

[1] [Ham03], p. 257. Hamkins refers to [Kun78].

So in this chapter we present the main results from Joel Hamkins' paper [Ham03].

4.1 The Main Theorem

We will follow the notation of [Ham03], where a model of set theory means a model of the Σ_{100} fragment of ZFC. Unlike in his paper, we will first prove the general case of the Main Theorem and then show the modifications for the less complex Central Case.

Definition 4.1. *A pair of transitive classes $M \subseteq N$ satisfies the δ* approximation property *(with $\delta \in Card^N$) if whenever $A \subseteq M$ is a set in N and $A \cap a \in M$ for any $a \in M$ of size less than δ in M, then $A \in M$.*

For models of set theory equipped with classes, the pair $M \subseteq N$ satisfies the δ approximation property for classes *if whenever $A \subseteq M$ is a class of N and $A \cap a \in M$ for any a of size less than δ in M, then A is a class of M.*

We will refer to the sets $A \cap a$ where a has size less than δ in M as the δ approximations to A over M.

Definition 4.2. *The pair $M \subseteq N$ satisfies the δ* cover property *if for every set A in N with $A \subseteq M$ and $|A|^N < \delta$, there is a set $B \in M$ with $A \subseteq B$ and $|B|^M < \delta$.*

Theorem 4.3 (Main Theorem)**.** *Suppose that $V \subseteq \bar{V}$ satisfies the δ approximation and cover properties, δ is regular in $\bar{V}$, $\bar{M}$ is a transitive submodel of $\bar{V}$ such that $M = \bar{M} \cap V$ is also a model of set theory, and $j : \bar{M} \to \bar{N}$ is a (possibly external) cofinal elementary embedding of $\bar{M}$ into a transitive class $\bar{N} \subseteq \bar{V}$.*

Suppose further that $\delta < cp(j)$, $P(\delta)^{\bar{V}} \subseteq \bar{M}$ and that $\bar{M}^{<\delta} \subseteq \bar{M}$ and $\bar{N}^{\delta} \subseteq \bar{N}$ in $\bar{V}$. Let $N = \bigcup j''M$ so that $j \upharpoonright M : M \to N$. Then:

1. *If $\bar{M}$ is a set in $\bar{V}$, then M is a set in V.*

2. *$N \subseteq V$; indeed, $N = \bar{N} \cap V$*

3. *If j is amenable to $\bar{V}$, then $j \upharpoonright M$ is amenable to V. In particular, if j is a set in $\bar{V}$, then the restricted embedding $j \upharpoonright M$ is a set in V.*

4. *If j and M are classes in $\bar{V}$ and $V \subseteq \bar{V}$ satisfies the δ approximation property for classes, then $j \upharpoonright M$ is a class of V. If $V \subseteq \bar{V}$ satisfies the δ approximation property for classes, then $j \upharpoonright M$ is definable in V.*

The situation is described in the following picture:

$\bar{V} \supseteq \bar{M} \xrightarrow{j} \bar{N}$

$\cup\!\!|$

$V \supseteq M \xrightarrow{j \restriction M} N = \bigcup j''M$

Proof.

Lemma 4.4. *$M \subseteq \bar{M}$ satisfies the δ approximation property and δ cover property.*

Proof. First we show the δ approximation property: Suppose that $A \in \bar{M}$, $A \subseteq M$ and, for any $a \in M$ of size less than δ in M, it holds that $A \cap a \in M$. Since $M = \bar{M} \cap V$, we only have to show that $A \in V$ and for this we will use the δ approximation property for $V \subseteq \bar{V}$: We know that $A \subseteq V$ and $A \in \bar{V}$, because $\bar{M}$ is an elementary submodel of $\bar{V}$. Let $a = \sigma \cap M$ for some fixed σ of size less than δ in V. For sufficiently large β this is the same as $\sigma \cap (V_\beta)^M$ and therefore $a \in V$. Since $a \subseteq M$ has size less than δ, it follows by the closure properties of $\bar{M}$ that a is in $\bar{M}$ and hence in $M = \bar{M} \cap V$. So $A \cap a \in M$. Since $A \cap \sigma = A \cap a$ and $M \subseteq V$, this means that all δ approximations to A over V are in V, and so $A \in V$. Hence $A \in \bar{M} \cap V = M$.

For the δ cover property we will use the respective property for $V \subseteq \bar{V}$: Let $a \in \bar{M}$, $A \subseteq M$ and A has size less than δ in $\bar{M}$. Since $A \subseteq V$, $A \in \bar{V}$ and A has size less than δ in $\bar{V}$, there is a set B_0 of size less than δ in V with $A \subseteq B_0$ (δ cover property for $V \subseteq \bar{V}$). For a sufficiently large $(V_\beta)^M$ there is a set $B_1 \in M \subseteq V$ with $A \subseteq B_1$. The set $B_0 \cap B_1$ is a subset of M (because $B_1 \in M$), it has size less than δ in V (bec. of B_0) and $A \subseteq B_0 \cap B_1$. Because of the closure properties of $\bar{M}$, $B_0 \cap B_1$ is in $\bar{M}$. So $B_0 \cap B_1$ is in $\bar{M} \cap V = M$ and it has size less than δ in M because any bijection witnessing that this set has size less than δ in V will be in $\bar{M}$ So $B_0 \cap B_1$ in M as well. □

With Lemma 4.4 we can prove the first claim of the Main Theorem: Assume that $\bar{M}$ is a set in $\bar{V}$. We prove that M is a set in V with the δ approximation property. Let a be a set in V with size less than δ in V, then $M \cap a \subseteq B$ for some $B \in M \subseteq V$. Then $M \cap a \subseteq B \cap a$ because $M \cap a \subseteq B$ and conversely $B \cap a \subseteq M \cap a$ because $B \in M$ and therefore

$M \cap a = B \cap a \in V$. So all the δ approximations to M over V are in V and hence we have that $M \in V$.

Lemma 4.5. *$N \subseteq \bar{N}$ satisfies the δ approximation and cover properties.*

Proof. We prove the δ approximation property of $N \subseteq \bar{N}$ by using the elementarity of j and the δ approximation property for $M \subseteq \bar{M}$.

So by Defintion 4.1 let $A \in \bar{N}$, $A \subseteq N$ and $A \cap a \in N$ for any $a \in N$ of size less than δ in N. We show that $A \in N$: By the cofinality of j (see Definition 1.36) there is some $B \in \bar{M}$ with $A \subseteq j(B)$. By the δ approximation property we know that if $A' \subseteq B$, $A' \in P(B)^{\bar{M}}$ such that $A' \cap a \in P(B)^M$ for any $a \in P(B)^M$ of size less than δ in $P(B)^M$, then $A' \in P(B)^M$. So any subset of B having all δ approximations over $P(B)^M$ in $P(B)^M$ is in $P(B)^M$. By elementarity of j we can transfer these facts to $\bar{N}$: Any subset of $j(B)$ having all $j(\delta) = \delta$ approximations over $j(P(B)^M) = P(j(B))^N$ (as $N = \bigcup j''M$) in $P(j(B))^N$ is in $P(j(B))^N$. Since we know that A is a subset of $j(B)$ and we know by assumption that A has all its δ approximations over N in N, we conclude that $A \in N$ (because $A \in P(j(B))^N$).

Similarly we prove the δ cover property by using the elementarity of j and the cover property of $M \subseteq \bar{M}$: Let $A \in \bar{N}$, $A \subseteq N$, $|A|^{\bar{N}} < \delta$. Since j is cofinal, there is a $B \in \bar{M}$ with $A \subseteq j(B)$. In M any subset of B can be covered by an $B' \in M$ with the right properties. Since j is elementary, the corresponding fact is true about subsets of $j(B)$ and $j(\delta) = \delta$ guaranties that the size of the cover is still $< \delta$. □

Lemma 4.6. *If $A \subset ORD^N$ is a set of size less than δ in $\bar{V}$, then there is a set $B \in V \cap N$ of size at most δ with $A \subseteq B$.*

Proof. Starting from A, we build B as the union of a sequence of sets A_α of size less than δ such that unboundedly many of the A_α are in V and unboundedly many are in N.

So let $A = A_0 \subseteq ORD^N$, $|A|^{\bar{V}} < \delta$. Then, because of the closure properties of $\bar{N}$, $A_0 \in \bar{N}$. Using the δ cover property of Lemma 4.5, there is a set of ordinals $A_1 \in N$ of size less than δ with $A_0 \subseteq A_1$. Since $A \in N \subseteq \bar{N} \subseteq \bar{V}$, A_1 is in $\bar{V}$ and because of the cover property for $V \subseteq \bar{V}$ there is a set $A_2 \in V$ of size less than δ with $A_1 \subseteq A_2$.

We can repeat this procedure by alternately using the δ cover property for $N \subseteq \bar{N}$ and $V \subseteq \bar{V}$. At limit stages we make use of the regularity of δ. So we have constructed a sequence $\langle A_\alpha | \alpha < \delta \rangle$ in $\bar{V}$ such that $\alpha < \beta \Rightarrow A_\alpha \subseteq A_\beta$, all A_α are subsets of ORD^N having size less than δ and unboundedly often $A_\alpha \in V$ and unboundedly often $A_\alpha \in N$.

Let $B = \bigcup_{\alpha<\delta} A_\alpha$ (therefore $B \subseteq ORD^N$). Since B is an union of δ many sets of size less than δ, B has size at most δ in $\bar{V}$ and as $\bar{N}^\delta \subseteq \bar{N}$, it follows that B is in $\bar{N}$ and has size at most δ there. To show that B is in V and N, we use the respective δ approximation properties: Let a be a set of ordinals of size less than δ in V. Then $B \cap a = A_\alpha \cap a$ for sufficiently large α, and so $B \cap a \in V$. Therefore all the δ approximations to B over V are in V and so $B \in V$. By the same argument, all the δ approximations to B over N are in N, and so $B \in N$. We conclude that $B \in V \cap N$. □

Lemma 4.7. *V and N have the same subsets of ORD^N of size less than δ.*

Proof. Assume that $A \subseteq ORD^N$ and $|A|^{\bar{V}} < \delta$. Then by Lemma 4.6 there is a set $B \in V \cap N$ of size at most δ in $\bar{V}$ with $A \subseteq B$. We enumerate B in the natural order as $B = \{\beta_\alpha | \alpha < \bar{\delta}\}$, where $\bar{\delta} = ot(B) < \delta^+$. Let $a = \{\alpha < \bar{\delta} | \beta_\alpha \in A\}$. We show that a is in V if and only if a is in N: Suppose that $a \in V$. Then, by assumption, $a \in \mathcal{P}(\delta)^{\bar{V}} \subseteq \bar{M}$, so $a \in \bar{M}$ and therefore $a \in M = \bar{M} \cap V$. Equivalently, $a \in \mathcal{P}(\bar{\delta})^M$ if and only if $j(a) \in j(\mathcal{P}(\bar{\delta})^M)$, which holds if and only if $a \in \mathcal{P}(\bar{\delta})^N$ because $j(a) = a(\bar{\delta} < cp(j))$, and this is equivalent to $a \in N$. But a is in V (respectively in N) if A is in V (respectively in N) because a is constructible from A and B. So if A is in either V or N, then a is in both of them. But then A is in both V and N as well, because A is constructible from B and a. □

With these lemmata we can now prove the second claim of the Main Theorem:

Lemma 4.8. *$N \subseteq V$. Indeed, $N = \bar{N} \cap V$.*

Proof.

"$N \subseteq \bar{N} \cap V$": We have to show that every set in N is in V. But as V and N are models of set theory, every set can be coded with a set of ordinals by the axiom of choice and therefore it suffices to show that every set of ordinals in N is in V. So let $A \subseteq ORD^N$ and $A \in N$. We use the δ approximation property to show that $A \in V$. So let a be a fixed set in V of size less than δ. Because of the δ approximation property, we are interested in the intersection $A \cap a$, and so we can assume that $a \subseteq ORD^N$. So, by Lemma 4.7 it follows that $a \in N$ and hence also $A \cap a \in N$. By Lemma 4.7 again, $A \cap a$ is in V, and so all the δ approximations to A over V are in V. Hence $A \in V$.

"$N \supseteq \bar{N} \cap V$": For the converse direction we work with sets in $\bar{N} \cap V$. Here we can consider ordinals only as a special case and then we will prove the Lemma for arbitrary sets. So, let $A \in \bar{N} \cap V$ be a set of ordinals. Then $a \in N$ by the δ approximation property of $N \subseteq \bar{N}$: Let $a \subseteq ORD$ be of size less than δ in N. By Lemma 4.7, it follows that $a \in V$ and so $A \cap a \in V$. Therefore, using Lemma 4.7 again, $A \cap a \in N$ and so all the δ approximations to A over N are in N. Hence, $A \in N$.

For the general case suppose that A is any set in $\bar{N} \cap V$. We prove that A is in N by $\in$-induction (see Theorem 1.19): Suppose that every element of A is in N. So there is some set $B \in N$ such that $A \subseteq B$. We enumerate $B = \{b_\alpha | \alpha < \beta\}$ in $N \subseteq V$ in $N \subseteq V$ and consider the following set of ordinals: $A_0 = \{\alpha < \beta | b_\alpha \in A\}$. A_0 is constructible from A and B and $B \in N \subseteq \bar{N} \cap V$, as we have proven above. So A_0 is in $\bar{N}$ and V. Since A_0 is a set of ordinals, it is in N and so also $A \in N$, as A is constructible from A_0 and the enumeration of B. □

Lemma 4.9. *If j is amenable to $\bar{V}$, then $j \restriction M$ is amenable to V.*

Proof. Assume that j is amenable to $\bar{V}$ (see Definition 1.35). We show that $j \restriction A \in V$ for any $A \in M$. So suppose that $A \in M$. By Lemma 4.8, it follows that $N = \bigcup j''M \subseteq V$ and so $j \restriction A \subseteq V$. So we can show that $j \restriction A$ is in V by showing that $j \restriction A$ has all its δ approximations in V, and for this it suffices to show that for any a of size less than δ in V, $j \restriction a \in V$. So let a be any set in V, $|a|^V < \delta$, and enumerate a as $\vec{a} = \langle a_\alpha | \alpha < \beta \rangle$. Since $\beta < \delta < cp(j)$, it follows that $j(\vec{a}) = \langle j(a_\alpha) | \alpha < \beta \rangle$. As $\mathcal{P}(\delta)^{\bar{V}} \subseteq \bar{M}$, a is in $M = \bar{M} \cap V$ and therefore $j(\vec{a}) \in N \subseteq V$. So we can construct $j \restriction a = \{\langle a_\alpha, j(a_\alpha) \rangle | \alpha < \beta\}$ from $\vec{a}$ and $j(\vec{a})$ in V.

□

The same argument as in the proof of Lemma 4.9 shows that if j and M are classes in $\bar{V}$ and one has the δ approximation property for classes, then $j \restriction M$ is a class in V. So if j is a set in $\bar{V}$, then $j \restriction M$ is a set in V. Now we have completed the proof of the Main Theorem.

□

Corollary 4.10. *Under the hypothesis of the Main Theorem, for any* λ*,*

1. *If* $\bar{N}^\lambda \subseteq \bar{N}$ *in* $\bar{V}$*, then* $N^\lambda \subseteq N$ *in* V
2. *If* $V_\lambda \subseteq \bar{N}$*, then* $V_\lambda \subseteq N$*.*

Proof. For 1. Any λ sequence over N in V is in $\bar{N}$ because of the assumption and in V and therefore in $\bar{N} \cap V = N$.

For 2. If $V_\lambda \subseteq \bar{N}$, then, again by $\bar{N} \cap V = N$, V_λ is a subset of N.

□

As we have seen in Chapter 2, many large cardinals like measurable or supercompact cardinals are defined by elementary embeddings from V to a transitive N. So we will consider the case where $\bar{M} = \bar{V}$ and hence $M = V$ as a special case of the Main Theorem.

Theorem 4.11 (The Central Case)**.** *Suppose that* $V \subseteq \bar{V}$ *satisfies the* δ *approximation and cover properties,* δ *is regular in* $\bar{V}$ *and* $j : \bar{V} \to \bar{N}$ *is a (possibly external) cofinal elementary embedding into a transitive class* $\bar{N} \subseteq \bar{V}$*. Suppose further that* $\delta < cp(j)$ *and* $\bar{N}^\delta \subseteq \bar{N}$ *(*$\bar{N}$ *is closed under* δ *sequences) in* $\bar{V}$*. Let* $N = \bigcup j''V$ *so that* $j \restriction V : V \to N$*. Then:*

1. $N \subseteq V$*; indeed,* $N = \bar{N} \cap V$
2. *If* j *is amenable to* $\bar{V}$*, then* $j \restriction V$ *is amenable to* V*.*
3. *If* j *and* V *are classes in* $\bar{V}$ *and* $V \subseteq \bar{V}$ *satisfies the* δ *approximation property for classes, then* $j \restriction V$ *is a class of* V*.*

So our picture from above simply becomes the following:

$$\begin{array}{c} j : \bar{V} \longrightarrow \bar{N} \subseteq \bar{V} \\ \cup\!\!| \\ j \restriction V : V \longrightarrow N \end{array}$$

Proof. First note that the restricted embedding $j\restriction V : V \to N$ is elementary. To prove this, we have to show that for every formula φ, and every $a_1, \ldots, a_n$ in the universe of $j(V)$, $j(V) \models \varphi[a_1, \ldots, a_n]$ if and only if $N \models \varphi[a_1, \ldots, a_n]$. But, because $N = \bigcup j''V$, this can be done by induction on formulas.

Now we can prove the central case with the same succession of lemmas as in the proof of the Main Theorem:

Lemma 4.12. *$N \subseteq \bar{N}$ satisfies the δ approximation and cover properties.*

Lemma 4.13. *If $A \subset ORD^N$ is a set of size less than δ in $\bar{V}$, then there is a set $B \in V \cap N$ of size at most δ with $A \subseteq B$.*

Lemma 4.14. *V and N have the same subsets of ORD^N of size less than δ.*

Lemma 4.15. *$N \subseteq V$. Indeed, $N = \bar{N} \cap V$*

Lemma 4.16. *If j is amenable to $\bar{V}$, then $j\restriction V$ is amenable to V.*

The proof of Lemma 4.12 differs from the one of Lemma 4.5 only by using the δ approximation and cover properties for $V \subseteq \bar{V}$ instead of $M \subseteq \bar{M}$.

The proof of Lemma 4.13 doesn't change and the proof of Lemma 4.14 simplifies the proof of Lemma 4.7: We prove that $a \in V$ if and only if $a \in N$. We need the fact that $j(P(\bar{\delta})^V) = P(\bar{\delta})^N$ and we get the equivalences: $a \in V \leftrightarrow a \in P(\bar{\delta})^V \leftrightarrow j(a) \in j(P(\bar{\delta})^V) \leftrightarrow a \in P(\bar{\delta})^N \leftrightarrow a \in N$.

The proof of Lemma 4.15 is the same as the one of Lemma 4.8; Lemma 4.16 is proven like Lemma 4.9 and the third claim then follows easily. □

We can drop the property of j being cofinal by adding the assumption that $N = \bar{N} \cap V$ (we only needed the cofinality to prove Lemma 4.12). By speaking directly about classes, we get the following summary of the Central Case:

Corollary 4.17. *Suppose that $V \subseteq \bar{V}$ satisfies the δ approximation and cover properties for classes. If V is a class in $\bar{V}$ and $j : \bar{V} \to \bar{N}$ is a class embedding in $\bar{V}$ with $\delta < cp(j)$ and $\bar{N}^\delta \subseteq \bar{N}$ in $\bar{V}$, then the restriction $j\restriction V : V \to N$, where $N = \bar{N} \cap V$, is a class elementary embedding in the ground model.*

4.2 Closure Point Forcing

Of course, forcing will be the principal example for the extensions of the Main Theorem. We get a first result regarding forcing extensions $V[G]$ if we take the assumptions of Corollary 4.17 and additionally assume that the models involved are only equipped with their classes definable from parameters.

Lemma 4.18. *Suppose that $V \subseteq V[G]$ is a set forcing extension satisfying the δ approximation property (for sets). If the models are equipped with only their definable classes, allowing a predicate for V in $V[G]$, then $V \subseteq V[G]$ also satisfies the δ approximation property for classes.*

Proof. Suppose that $A \subseteq V$ is a class in $V[G]$ such that $A \cap a \in V$ for any set a of size less than δ in V. We have to show that A is a class in V. First let $A_\eta = A \cap V_\eta$ for any ordinal η and consider $A_\eta \cap a$ for some $a \in V$ of size less than δ in V. Then $A_\eta \cap a = (A \cap V_\eta) \cap a = (A \cap a) \cap V_\eta$ and the right part of the equation is the intersection of two sets in V, and therefore $A_\eta \cap a$ is in V. As $A_\eta \cap a$ are the δ approximations to A_η over V, it follows by the δ approximation property for sets that $A_\eta \in V$ for all η.

Since we have assumed that A is definable in $V[G]$, there is a formula φ (allowing a predicate for the ground model) and parameter z such that

$$V[G] \models x \in A \quad \Longleftrightarrow \quad \varphi(x, z). \tag{4.1}$$

From the argument above we know that $A \cap V_\eta = A_\eta \in V$, so there is some condition $p_\eta \in G$ such that

$$x \in A_\eta \quad \Longleftrightarrow \quad p_\eta \Vdash \varphi(\check{x}, \dot{z}). \tag{4.2}$$

where $\dot{z}$ is a name for z. Since G is a set in $V[G]$ and the mapping $\eta \mapsto p_\eta$ exists in $V[G]$, the value of p_η must be the same for unboundedly many η. Let p^* be this common value. Then we can use p^* for any p_η and so we have for any η that

$$x \in A_\eta \quad \Longleftrightarrow \quad p^* \Vdash \varphi(\check{x}, \dot{z}). \tag{4.3}$$

Therefore, by using the parameters $\dot{z}$, p^* and the forcing poset, the formula $x \in A \Leftrightarrow p^* \Vdash \varphi(\check{x}, \dot{z})$ provides a definition of A as a class of V. □

So, in set forcing extensions we don't have to consider classes explicitly and we get the following corollary:

Corollary 4.19. *If $V \subseteq V[G]$ is a set forcing extension with the δ approximation and cover properties and $j : V[G] \to \bar{N}$ is a definable embedding in $V[G|$ with $\bar{N}^\delta \subseteq \bar{N}$ and $\delta < cp(j)$, then the restriction $j \restriction V : V \to N$, where $N = \bar{N} \cap V$ is an elementary embedding definable in the ground model.*

Now, the most interesting question is: Which kind of forcing extensions actually satisfy the approximation and cover properties? To answer this question we will introduce the notion of a closure point forcing and show that such a forcing satisfies the δ^+ approximation and cover properties.

In the proof of Lemma 4.23 we will follow [HJed], as this version is easier than the proof originally given in [Ham03].

Lemma 4.20. *Let P be a forcing poset, $A \subseteq P$ an antichain and τ_a be a P-name for each $a \in A$. Then there is a P-name τ such that $\forall a \in A$ it holds that $a \Vdash \tau = \tau_a$.*[2]

Definition 4.21. *A poset $\mathbb{P} = \langle \mathbb{P}, \leqslant_{\mathbb{P}} \rangle$ is said to be κ-strategically closed if Player II has a winning strategy in the following game:*

$$\begin{array}{r|cccccccccl} \text{Player I} & p_0 & & p_2 & \cdots & & & p_{\delta+1} & \cdots & & \\ & \downarrow & \nearrow & \downarrow & & \downarrow & \nearrow & \downarrow & & & \delta \text{ limit.} \\ \text{Player II} & p_1 & & \cdots & & p_\delta & & p_{\delta+2} & \cdots & & \end{array}$$

That means that Player I plays odd stages and Player II even and limit stages. Here $\langle p_\alpha : \alpha < \eta \leqslant \kappa \rangle$ should be a descending chain in $\mathbb{P}$ with respect to $\leqslant_{\mathbb{P}}$. Player II wins iff for all $\alpha < \kappa$ p_α can be played. $\mathbb{P}$ is $< \kappa$-strategically closed iff $\mathbb{P}$ is λ-strategically closed for all $\lambda < \kappa$.

A notion of forcing is called nontrivial if it necessarily adds a new set.

Definition 4.22. *A forcing notion has a closure point at δ when it factors as $\mathbb{P} * \dot{\mathbb{Q}}$, where $\mathbb{P}$ is nontrivial, $|\mathbb{P}| \leqslant \delta$ and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is $\leqslant \delta$-strategically closed.*

Lemma 4.23 (Main Lemma)**.** *Forcing with a closure point at δ satisfies the δ^+ approximation and cover properties.*

Proof. Suppose that $V \subseteq V[g][G]$ is a forcing extension by the forcing $g * G \subseteq \mathbb{P} * \dot{\mathbb{Q}}$, where $\mathbb{P}$ is nontrivial, $|\mathbb{P}| \leqslant \delta$ and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is $\leqslant \delta$-strategically closed.

[2] For a proof of this Lemma see [Kun80], p. 226.

The δ^+ cover property holds because if we consider $V \subseteq V[g] \subseteq V[g][G]$, each step of the extension holds because of the properties of $\mathbb{P}$ and $\mathbb{Q}$.

For the δ^+ approximation property suppose that $A \in V[g][G]$ and $A \subseteq V$, but $A \notin V$. Let $\dot{A}$ be a $\mathbb{P} * \dot{\mathbb{Q}}$-name for A, $\dot{A}^{g*G} = A$, for which

$$\mathbb{1} \Vdash \dot{A} \subseteq V \text{ and } \mathbb{1} \Vdash \dot{A} \notin V \tag{4.4}$$

(4.4) is justified: Assume that $\dot{A}'$ is $\dot{A}$ if (4.4) and $\dot{g}(\mathbb{P})$ otherwise. Then $\mathbb{1} \Vdash \dot{g}(\mathbb{P}) \notin V$ because $\mathbb{P}$ is nontrivial.

Since $\dot{A}$ is forced not to be in V, there cannot exist a condition which decides for every $b \in V$ if $\check{b} \in \dot{A}$, because this condition would decide the whole $\dot{A}$. So for every condition $(p, \dot{q})$ there is $b \in V$ such that $(p, \dot{q})$ does not decide whether $\check{b} \in \dot{A}$. Then there are two conditions $(p_0, \dot{q}_0)$ and $(p_1, \dot{q}_1)$ below $(p, \dot{q})$ such that $(p_0, \dot{q}_0) \Vdash \check{b} \notin \dot{A}$ and $(p_1, \dot{q}_1) \Vdash \check{b} \in \dot{A}$. Since $\mathbb{P}$ is nontrivial, we will assume without loss of generality that p_0 and p_1 are incompatible. By Lemma 4.20 it follows that there is a name $\dot{q}'$ such that p_0 forces $\dot{q}' = \dot{q}_0$ and p_1 forces $\dot{q}' = \dot{q}_1$. Indeed, we may assume that $\mathbb{1}$ forces $\dot{q}' < \dot{q}$.

So, in summary, instead of taking $(p_0, \dot{q}_0)$ and $(p_1, \dot{q}_1)$ as above, we could have used the conditions $(p_0, \dot{q}')$ and $(p_1, \dot{q}')$ with the same second coordinate, such that $\mathbb{1} \Vdash_{\mathbb{P}} \dot{q}' < \dot{q}$.

Now enumerate $\mathbb{P} = \{p_\beta | \beta < \delta\}$ and let τ be a $\mathbb{P}$-name for the strategy witnessing that $\mathbb{1} \Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is $\leqslant \delta$-strategically closed. Our goal is to build a sequence of $\mathbb{P}$-names $\dot{q}_\beta$ for elements of $\dot{\mathbb{Q}}$ such that $\alpha < \beta$ implies $(\mathbb{1}, \dot{q}_\beta) < (\mathbb{1}, \dot{q}_\alpha)$, and there are p^0_β and p^1_β below p_β and an element b_β such that $(p^0_\beta, \dot{q}_\beta) \Vdash \check{b}_\beta \notin \dot{A}$ and $(p^1_\beta, \dot{q}_\beta) \Vdash \check{b}_\beta \in \dot{A}$. For successor steps it would be enough to use the argument from above iteratively. But, as we want to continue the construction through limits, we have to combine it with an application of the strategy τ, so that the construction accords with τ.

Finally we get the following: For any $(p_\beta, \dot{t}_\beta)$, there are p^0_β and p^1_β below p_β, a name $\dot{q}_\beta$ and an element b_β such that $(\mathbb{1}, \dot{q}_\beta) < (\mathbb{1}, \dot{t}_\beta)$ and $(p^0_\beta, \dot{q}_\beta) \Vdash \check{b}_\beta \notin \dot{A}$ and $(p^1_\beta, \dot{q}_\beta) \Vdash \check{b}_\beta \in \dot{A}$. Furthermore, since $\mathbb{1}$ forces that the $\dot{q}_\beta$ are descending and conform with the strategy, there is $\dot{q}_\delta$ such that $(\mathbb{1}, \dot{q}_\delta) < (\mathbb{1}, \dot{q}_\beta)$ for all $\beta < \delta$. But, if $(p, \dot{r})$ is a condition which is stronger than $(\mathbb{1}, \dot{q}_\delta)$, there is a β such that $p = p_\beta$ from the enumeration of P and therefore $(p, \dot{r})$ cannot decide $\dot{A}$ on $\{b_\beta | \beta < \delta\}$. So not all approximations to $\dot{A}$ are in V. $\square$

Closure point forcings include a large class of forcings: For trivial reasons every small forcing has a closure point and the Lemma also applies, for example, to the Silver iteration, the Laver preparation, the lottery preparation and reverse Easton iterations.

Chapter 5

Applications of the Main Theorem

5.1 Application to Large Cardinal Properties

In this section we are going to examine how the Main Theorem can be applied to several large cardinals[1]. The procedure will essentially be the following: We will use the results of Chapter 2 to get an elementary embedding j witnessing the existence of a large cardinal in the extension and then apply Claim 3, therefore showing that $j \restriction V$ is amenable to V. Then it only remains to show that $j \restriction V$ witnesses the same large cardinals property in V. As this procedure can be seen more clearly in the simplified version of Theorem 4.11, we will start with some applications of the Central Case.

As we have seen in Lemma 4.23, the cardinal δ will depend on the forcing in question. For now let δ be an arbitrary regular cardinal.

Corollary 5.1. *Suppose $V \subseteq \bar{V}$ satisfies the δ approximation and cover properties. Then every measurable cardinal above δ in $\bar{V}$ is measurable in V.*

Proof. Since κ is measurable in $\bar{V}$, we can construct an elementary embedding $j : \bar{V} \to \bar{N}$ with critical point κ via ultrapowers (see Proof of Theorem 2.17). Since $\bar{N}^\kappa \subseteq \bar{N}$, the Central Case implies that the restricted embedding $j \restriction V : V \to N$ is amenable to V. Then, as in the Proof of Theorem 2.17,

[1] All results presented here can be found in [Ham03], pp. 266–272.

we can construct a normal V-ultrafilter μ on κ from $j \restriction P(\kappa)^V$ by defining $X \in \mu \Leftrightarrow \kappa \in j(X)$. □

Corollary 5.2. *Suppose $V \subseteq \bar{V}$ satisfies the δ approximation and cover properties. Then every strong cardinal above δ in $\bar{V}$ is strong in V.*

Proof. Suppose that κ is θ strong in $\bar{V}$ and θ is either a successor ordinal or has cofinality above δ. Then by Section 2.4 we know that in $\bar{V}$ there is a θ strongness extender embedding $j : \bar{V} \to \bar{N}$ with $cp(j) = \kappa$, $\bar{V}_\theta \subseteq \bar{N}$ and $\bar{N}^\delta \subseteq \bar{N}$. By the Central Case the restricted embedding $j \restriction V : V \to N$ is amenable to V, and by Corollary 4.10 we know $V_\theta \subseteq N$. So, in V, let $E = j \restriction P(\kappa)^V$ be an extender and let $j_E : V \to M_E$ be the corresponding extender embedding. Then $j_E \restriction P(\kappa)^V = j \restriction P(\kappa)^V$, and consequently $V_\theta \subseteq M_E$. So j_E witnesses that κ is θ strong in V. □

Corollary 5.3. *Suppose $V \subseteq \bar{V}$ satisfies the δ approximation and cover properties. Then every Woodin cardinal above δ in $\bar{V}$ is Woodin in V.*

Proof. Suppose that κ is Woodin in $\bar{V}$. Then by Definition 2.35 for every $A \subseteq \kappa$ in V there is $\gamma \in (\delta, \kappa)$ such that for arbitrarily large $\lambda < \kappa$ there is an extender embedding $j : \bar{V} \to \bar{N}$ such that $cp(j) = \gamma$ and $j(A) \cap \lambda = A \cap \lambda$. We may also assume $\bar{N}^\gamma \subseteq \bar{N}$. It follows from the Central Case that the restriction $j \restriction V : V \to N$ is amenable to V and still satisfies $j(A) \cap \lambda = A \cap \lambda$. Since $j \restriction P(\kappa)^V \in V$ by amenability, the induced extender embeddings therefore witness that κ is a Woodin cardinal in V. □

Corollary 5.4. *Suppose $V \subseteq \bar{V}$ satisfies the δ approximation and cover properties. Then every supercompact cardinal above δ in $\bar{V}$ is supercompact in V. Indeed, for any γ, every γ supercompact cardinal above δ in $\bar{V}$ is γ supercompact in V.*

Proof. Suppose that κ is γ-supercompact in $\bar{V}$. Then by Definition 2.26 there is an elementary embedding $j : \bar{V} \to \bar{N}$ in $\bar{V}$ such that its critical point is κ, $\gamma < j(\kappa)$ and $\bar{N}^\gamma \subseteq \bar{N}$. Then by the Central Case the restriction $j \restriction V : V \to N$ is amenable to V and $N = \bar{N} \cap V$. By Corollary 4.10 we know that $N^\gamma \subseteq N$ in V, and so $j''\gamma \in N$. Thus, from $j \restriction P(P_\kappa\gamma)^V$ we may in V construct the induced normal fine measure μ on $P_\kappa\gamma$ by defining $X \in \mu \Leftrightarrow j''\gamma \in j(X)$. So κ is γ supercompact in V. □

For the application of the Main Theorem, we first have to show two slightly technical lemmata:

Lemma 5.5. *Suppose that* $V \subseteq \bar{V}$ *satisfies the* δ *approximation and cover properties. If* $\bar{X}^{<\delta} \subseteq \bar{X}$ *in* $\bar{V}$ *and* $\bar{X} \prec \bar{V}_\theta$ *in the language with a predicate for* V*, so that* $\langle \bar{X}, X, \in \rangle \prec \langle \bar{V}_\theta, V_\theta, \in \rangle$*, where* $X = \bar{X} \cap V$*, then* $X \in V$*. Further, if* $\bar{M}$ *is the Mostowski collapse of* $\bar{X}$*, then the Mostowski collapse of* X *is the same as* $\bar{M} \cap V$*.*

Proof. We will use the δ approximation property to show that $X \in V$: Therefore, if we take an $a \in V$ that has size less than δ in V, it only remains to show that $X \cap a \in V$. Since $X \cap a \subseteq X$ is of size less than δ in $\bar{V}$, it is in $\bar{X}$ and it is an element of $\bar{V}_\theta$ of size less than δ. By the δ cover property there is an element $b \in V_\theta$ which has size less than δ in V_θ and covers $X \cap a$. By elementarity there is such a b in X. Since b has size less than δ and δ is a subset of X, b is also a subset of X. So we have shown that $X \cap a \subseteq b \subseteq X$ and therefore $X \cap a = b \cap a$. As $b \in V$, we have that $X \cap a$ is in V and therefore all the δ approximations to X over V are in V. By the δ approximation property it follows that $X \in V$.

Our second task is to show that the Mostowski collapse of X is the same as $\bar{M} \cap V$. So let $\langle \bar{M}, M, \in \rangle$ be the Mostowski collapse of $\langle \bar{X}, X, \in \rangle$. We show that the Mostowski Collapse of X is the same as the image of X under the Mostowski Collapse of $\bar{X}$: As $\bar{V}_\theta$ knows that V_θ is transitive, every element of $\bar{X}$ which is an element of an element of X is itself in X. So M is the Mostowski Collapse of X and that means that $M \in V$. Therefore $M \subseteq \bar{M} \cap V$.

So it remains to show that $\bar{M} \cap V \subseteq M$. Let $\pi : \bar{X} \cong \bar{M}$ be the Mostowski Collapse of $\bar{X}$ and suppose that $\pi(A) \in \bar{M} \cap V$, where $A \in \bar{X}$. We show that $\pi(A) \in M$. By induction we assume that every element of $\pi(A)$ is in M. Therefore $A \cap \bar{X} \subseteq X$ and so by elementarity $A \subseteq V$. To show that $A \in V$, we will use the δ approximation property: Let a be an element of X having size less than δ there. Then $A \cap a$ is an element of $\bar{X}$ of size less than δ in $\bar{X}$ and a subset of X. By the δ cover property there is some $b \in X$ of size less than δ in X such that $A \cap a \subseteq b$. We enumerate $b = \{b_\alpha \mid \alpha < \beta\}$ in V, where $\beta < \delta$ and let $A_0 = \{\alpha \mid b_\alpha \in A \cap a\}$. Then $\alpha \in \pi(A_0) = A_0$ if and only if $\pi(b_\alpha) \in \pi(A) \cap \pi(a)$ because π fixes all ordinals below δ and all subsets of δ. Since $\pi(b_\alpha)$, $\pi(A)$, $\pi(a)$ and the sequence $\langle \pi(b_\alpha) | \alpha < \beta \rangle$ are all in V, it follows that $A_0 \in V$ and therefore $A \cap a \in V$. As $X = \bar{X} \cap V$ and $A \in \bar{X}$, we have that $A \cap a \in X$ and so all δ approximations to A using $a \in X$ are in X. By elementarity it follows that all δ approximations to A over V are in V and so we conclude that $A \in V$. Thus $\pi(A) \in M$. □

The next Lemma makes use of the following Reflection Principle of Montague and Lévy[2]: For any formula $\varphi(v_1, \ldots, v_n)$ and any β, there is a limit ordinal $\alpha > \beta$ such that for any $x_1, \ldots, x_n \in V_\alpha$,

$$\varphi[x_1, \ldots, x_n] \text{ iff } \varphi^{V_\alpha}[x_1, \ldots, x_n]$$

Lemma 5.6. *Suppose that $V \subseteq \bar{V}$ satisfies the δ approximation and cover properties and $\kappa \geqslant \delta$ is an inaccessible cardinal. If $A \subseteq \kappa$ is any set in $\bar{V}$, then there is a transitive model of set theory $\bar{M}$, $|\bar{M}| = \kappa$ such that $A \in \bar{M}$, $\bar{M}^{<\kappa} \subseteq \bar{M}$ and $M = \bar{M} \cap V \in V$ is a model of set theory.*

Proof. Suppose ZFC^* is a fixed finite fragment of ZFC used to define the models of set theory. The Reflection Principle from above shows that there is an ordinal θ above κ such that every formula appearing in ZFC^* reflects from the structure $\langle \bar{V}, V, \in \rangle$ to $\langle \bar{V}_\theta, V_\theta, \in \rangle$. In particular, both $\bar{V}_\theta$ and V_θ are models of set theory. In $\bar{V}$, let $\bar{X} \prec \bar{V}_\theta$ be an elementary substructure of size κ in the language with a predicate for V, so that $\langle \bar{X}, X, \in \rangle \prec \langle \bar{V}_\theta, V_\theta, \in \rangle$, where $X = \bar{X} \cap V$, such that $\bar{X}^{<\kappa} \subseteq \bar{X}$ and $A \in \bar{X}$. So it follows by Lemma 5.5 that the collapse $\bar{M}$ of $\bar{X}$ has the property that $M = \bar{M} \cap V$ is in V. And since M is the collapse of X, it is a model of set theory. □

Corollary 5.7. *Suppose $V \subseteq \bar{V}$ satisfies the δ approximation and cover properties. Then every weakly compact cardinal above δ in $\bar{V}$ is weakly compact in V.*

Proof. Suppose κ is weakly compact in $\bar{V}$. By Lemma 5.6 there is for any subset $A \subseteq \kappa$ in V a model of set theory $\bar{M}$ in $\bar{V}$ such that $A \in \bar{M}$, $|\bar{M}| = \kappa$, $\bar{M}^{<\kappa} \subseteq \bar{M}$ and $M = \bar{M} \cap V$ is a model of set theory in V. That κ is weakly compact in $\bar{V}$ implies by Fact 2.4 that there is an elementary embedding $j : \bar{M} \to \bar{N}$ in $\bar{V}$ with critical point κ. In order to use the Main Theorem, it remains to show that j is cofinal, $\bar{M}^{<\delta} \subseteq \bar{M}$ in $\bar{V}$ and $\bar{N}^\delta \subseteq \bar{N}$ in $\bar{V}$.

That $\bar{M}^{<\delta} \subseteq \bar{M}$ in $\bar{V}$ is clear because $\bar{M}^{<\kappa} \subseteq \bar{M}$ and $\delta < \kappa$. Similarly to the proof of Theorem 2.17 we can construct the induced normal $\bar{M}$-measure.

[2]For a reference see [Kan09], p. 58.

Then j is cofinal because if $x \in \bar{M}_\mu$ then $x = j(f)(\kappa)$ and $X \in ran\, j(f)$. So choose $y = ran\, f = \{f(\xi) : \xi < \kappa\}$ and therefore $x = j(f)(\kappa) \in j(y) = ran\, j(f)$. Furthermore $\bar{N}^\delta \subseteq \bar{N}$ in $\bar{V}$, because if $\vec{s} \in \bar{N}^{<\kappa}$, then $\vec{s} = \langle s_\xi : \xi < \alpha\rangle$ for some $\alpha < \kappa$ and each $s_\xi = j(f_\xi)(\kappa)$ for some $f_\xi \in \bar{M}$, $f_\xi : \kappa \to \bar{M}$, $\vec{f} = \langle f_\xi : \xi < \alpha\rangle \in \bar{M}$. Then $j(\vec{f}) = \langle j(f_\xi) : \xi < \alpha\rangle$ and $\vec{s}$ is constructible from $j(\vec{f})$ and κ, so $\vec{s} \in \bar{N}$.

So by the Main Theorem $j \restriction M : M \to N$ is an embedding in V. Since this restricted embedding still has critical point κ and $A \in M$, it follows from Fact 2.4 that κ is weakly compact in V. □

By a theorem of Hanf and Scott[3] κ is weakly compact if and only if κ is Π^1_1-indescribable. So we will give a generalization of Corollary 5.7 by proving that indescribable cardinals are not created in extensions with the δ approximation and cover properties. We will use the definition of indescribable cardinals by elementary embeddings as we gave it in Theorem 2.7 and Definition 2.6. Note that the statement about the size of N in Theorem 2.7 is only needed in the part of the proof where we show that if κ is Π^m_n indescribable, then there is an elementary embedding.[4]

Corollary 5.8. *Suppose $V \subseteq \bar{V}$ satisfies the δ approximation and cover property. Then every totally indescribable cardinal above δ in $\bar{V}$ is totally indescribable in V. Indeed, for $m \geqslant 1$ every Π^m_1 indescribable cardinal above δ in $\bar{V}$ is Π^m_1 indescribable in V.*

Proof. Suppose that κ is Π^m_1 indescribable in $\bar{V}$, and let M_0 be any transitive model of set theory in V of size less than κ with $M_0^{<\kappa} \subseteq M_0$ and $\kappa \in M_0$. By Lemma 5.6 we know that there is a model of set theory $\bar{M}$ in $\bar{V}$ with $\bar{M}^{<\kappa} \subseteq \bar{M}$ in $\bar{V}$ and $M_0 \in \bar{M}$ such that $M = \bar{M} \cap V$ is also a model of set theory. Since κ is Π^m_1 indescribabale in $\bar{V}$, there is an elementary embedding $j : \bar{M} \to \bar{N}$ such that $\bar{N}$ is Σ^m_0 correct in $\bar{V}$. By applying the Main Theorem it follows that the restricted embedding $j \restriction M : M \to N$ lies in V. Then we can restrict the j further down to M_0 and obtain the embedding $j_0 = j \restriction M_0 : M_0 \to N_0$, where $N_0 = j(M_0)$.

To show that j_0 witnesses the indescribability of κ in V, it remains to show that N_0 is Σ^m_0 correct, because, as we noted above, we don't need the statement about the size of N for this direction. Since $V_{\kappa+m-2} \subseteq \bar{V}_{\kappa+m-2}$

[3]See [HS61].

[4]For details see [Hau91].

and $\bar{N}^{|\bar{V}_{\kappa+m-2}|} \subseteq \bar{N}$ in $\bar{V}$, we know by Corollary 4.10 that $N^{|V_{\kappa+m-2}|} \subseteq N$ in V, and consequently $|V_{\kappa+m-2}|^V < (|\bar{V}_{\kappa+m-2}|^+)^{\bar{V}}$. The fact that M knows that $M_0^{<\kappa} \subseteq M_0$ transfers by elementarity to the fact that N knows that $N^{<j(\kappa)} \subseteq N_0$. Since N has all the sequences over N_0 of length up to $|V_{\kappa+m-2}|$, which is less than $j(\kappa)$, it follows that $N_0^{|V_{\kappa+m-2}} \subseteq N_0$ in V. So the first clause of Definition 2.6 is fulfilled. Furthermore, because $(N_0)_{\kappa+m}$ is a transitive subset of $V_{\kappa+m}$, it follows that Σ_0 truth is preserved. So, as the second clause means that M correctly computes the Σ_n^m facts over V_κ that hold in parameters from $M \cap V_{\kappa+m}$, the embedding $j_0 : M_0 \to N_0$ is Σ_0^m correct. □

5.2 Definability of V in $V[G]$

Here we present a result from Laver which uses Hamkins' work on the δ approximation and cover properties. In [Lav07] Theorem 5.10 is stated for models of ZFC. But, as Laver uses it to proof Theorem 5.13 where we only have models for $ZFC - Replacement$, we will follow an approach by Jonas Reitz.[5]

Definition 5.9. *Let ZFC_δ be the theory consisting of Zermelo set theory, Choice and $\leqslant \delta$-Replacement (that is, Replacement holds for functions with domain δ, which is a regular cardinal), together with the axiom*

$$(*) \qquad \forall A\, \exists \alpha \in ORD\, \exists E \subseteq \alpha \times \alpha \quad \langle \alpha, E \rangle \simeq \langle TC(\{A\}), \in \rangle$$

which asserts "every set is coded by a set of ordinals".

Theorem 5.10. *Suppose V, V' and W are transitive models of ZFC_δ, δ is a regular cardinal in W, the extensions $V \subseteq W$ and $V' \subseteq W$ have the δ cover and δ approximation properties, $\mathcal{P}(\delta)^V = \mathcal{P}(\delta)^{V'}$ and $(\delta^+)^V (= (\delta^+)^{V'}) = (\delta^+)^W$. Then $V = V'$.*

Proof. Because of axiom $(*)$, we have to show that V and V' have the same sets of ordinals. If $A \subseteq ORD$, $A \in W$ then the statements "$|A| < \delta$" and "$|A| = \delta$" means the same in all models, because of the δ cover properties and because $(\delta^+)^V = (\delta^+)^{V'} = (\delta^+)^W$ respectively.

[5] See [Rei07].

We will now adjust the proof of Lemma 4.6 and Lemma 4.7 to this situation and show where the $\leqslant \delta$-Replacement is needed.

Lemma 5.11. *If $A \subset ORD$, $A \in W$, $|A| < \delta$, then there is a set $B \in V \cap V'$, $B \leqslant \delta$ with $A \subseteq B$.*

Proof. Let $A = A_0 \subseteq \alpha$ be as above and fix in W a well-ordering $\preceq$ of $(\alpha)^W$. As before we construct a sequence $\langle A_\xi | \xi < \delta \rangle$ of subsets of α, each of size less than δ. If $A_\xi \in V$, then let $A_{\xi+1} \in V'$ be the $\preceq$-least subset of α such that $A_{\xi+1} \supset A_\xi$ and $|A_{\xi+1}| < \delta$. Such a set exists because of the δ cover property of $V' \subseteq W$. If $A_\xi \notin V$, then let $A_{\xi+1} \in V$ to be the $\preceq$-least subset of α such that $A_{\xi+1} \supset A_\xi$ and $|A_{\xi+1}| < \delta$. For limit ξ, let $A_\xi = \bigcup_{\beta<\xi} A_\beta$. Here we need $\leqslant \delta$-Replacement for A_ξ to be in W. $|A_\xi| < \delta$, as δ is regular and each A_β has size less than δ. So we get a sequence for every $\xi < \delta$ and $A_\xi \in V$ for cofinally many ξ and $A_\xi \in V'$ for cofinally many ξ as well.

Let $B = \bigcup_{\xi<\delta} A_\xi$. Then $B \in W$ and $|B| < \delta$. If $a \in V$ is any set of size less than δ, then (by regularity of δ) $B \cap a = A_\alpha \cap a$ for all sufficiently large α, and so $B \cap a \in V$. Thus, all the δ approximations to B over V are in V and so $B \in V$. Similarly for V'. Therefore $B \in V \cap V'$. □

Lemma 5.12. *V and V' have the same sets of ordinals of size less than δ.*

Proof. Suppose that $A \subseteq ORD$, $|A| < \delta$, $A \in V$. By Lemma 5.11 there is a set $B \in V \cap V'$ of size at most δ with $A \subseteq B$. Then $ot(B) < \delta^+$, so pick a well ordering $w \in V$ of δ, or possibly of a subset of δ of length $ot(B)$ which lies in $V \cap V'$. Since $w \subset \delta \times \delta$ and by assumption $\mathcal{P}(\delta)^V = \mathcal{P}(\delta)^{V'}$, w must be in V'. The ordering induces an enumeration $B = \{b_\alpha | \alpha < \delta'\}$, for some $\delta' \leqslant \delta$, which, by $\leqslant \delta$-Replacement, lies in $V \cap V'$. Then $a = \{\alpha < \delta' | b_\alpha \in A\} \in V$, and it follows again from $\mathcal{P}(\delta)^V = \mathcal{P}(\delta)^{V'}$ that $a \in V'$. Since A is constructible from a, B and w, A is in V'. By the same argument every set of ordinals in V' of size less than δ is in V. □

By the δ approximation property we can now show that every set of ordinals in V is in V' and vice versa: Let A be a set of ordinals in V (therefore in W). Let $a \in V'$ be a set of ordinals with $|a| < \delta$. Then it follows by Lemma 5.12 that $a \in V$, and therefore $a \cap A \in V$. So, by Lemma 5.12 again, $a \cap A \in V'$. □

Theorem 5.13. *Suppose V is a model of ZFC, $\mathcal{P} \in V$, and $V[G]$ is a $\mathcal{P}$-generic extension of V. Then, in $V[G]$, V is definable from parameter $V_{\delta+1}$, for $\delta = |\mathcal{P}|^+$.*

Proof. In $V[G]$, let $\gamma > \delta$ be a beth fixed point of cofinality $> \delta$. Because of Lemma 1.15 such a fixed point does always exist. Then V_γ is definable as the unique transitive model $\mathcal{M}$ of height γ such that:

(i) $\mathcal{M}$ satisfies Zermelo set theory, Choice and $\leqslant \delta$-Replacement.

(ii) the extension $\mathcal{M} \subseteq (V[G])_\gamma$ has the δ approximation and cover properties, and

(iii) $\mathcal{M}_{\delta+1} = V_{\delta+1}$

We show that, living in $V[G]$, $\mathcal{M} = V_\gamma$ satisfies (i)-(iii):

About (i): For any limit ordinal γ, V_γ is a model of Zermelo set theory and Choice. But for V_γ to be a model of replacement, γ would have to be regular and $\forall\lambda < \gamma$: $2^\lambda < \gamma$. So we don't have full Replacement, but only $\leqslant \delta$-Replacement. This holds because of the cofinality of γ: For any function $f : a \to V_\gamma$ with $|a| = \delta$ has its range contained in some V_β for $\beta < \gamma$ and so $ran(f) \in V_{\beta+1}$.

About (ii): Note that the forcing from above (small forcing) is a closure point forcing with closure point at $|\mathcal{P}|$. So $V \subseteq V[G]$ has the $|\mathcal{P}|^+ = \delta$ approximation and cover properties. It holds further that $V_\gamma[G] = (V[G])_\gamma$: Let $\sigma^G \in (V[G])_\gamma$, so $rank(\sigma^G) < \gamma$. Since $|\mathcal{P}|$ is less than γ, there is a name σ' in V, with $\sigma'^G = \sigma^G$ and $rank(\sigma') < \gamma$ and because $V_\gamma = V \cap V_\gamma[G]$, $\sigma' \in V_\gamma[G]$. Then also $V_\gamma \subseteq (V[G])_\gamma = V_\gamma[G]$ has the δ approximation and cover properties: Let $A \in (V[G])_\gamma$, $A \subseteq V_\gamma$ and $|A|^{(V[G])_\gamma} < \delta$, then $A \in (V[G])_\beta$ for some $\beta < \gamma$. By the δ cover property of $V \subseteq V[G]$ there is a $B \in V$ with size less than δ which covers A. Then $B \cap V_\beta$ covers A in V_γ and has size less than δ there. For the δ approximation property assume that $A \in (V[G])_\gamma$, $A \subseteq V_\gamma$ and $A \cap a \in V_\gamma$ for any $a \in V_\gamma$ of size less than δ. Then, by the δ approximation property for $V \subseteq V[G]$, $A \in V$ and because of the above argument $A \in V_\gamma$.

It remains to show that any $\mathcal{M}$, which is transitive, satisfies (i)-(iii) and is of height γ, equals V_γ. Such an $\mathcal{M}$ is a model of ZFC_δ and has every set codeable in an absolute way by a set of ordinals: In general it holds that if $\mathcal{M}$ is a transitive model of ZFC, then we can code all sets in $\mathcal{M}$ by sets of ordinals in $\mathcal{M}$, that means if $x \in \mathcal{M}$, where x is transitive, there is an isomorphism:

$$(x, \in) \simeq (Card(x), R)$$

where R is a binary relation on $Card(x)$. If x is not transitive, we work with the transitive closure of x, $TC(x)$. The point here is that the coding must happen in $\mathcal{M}$, so it must not happen that the cardinality of $TC(x)$ is not an element of $\mathcal{M}$ (as it could be the case, for example, in $V_{\omega+\omega}$). In our case $\mathcal{M}$ is not a full model of ZFC but it has height γ, which is a beth fixed point. Therefore the transitive closure of x is of size less than γ.
Furthermore $(\delta^+)^{\mathcal{M}}(= (\delta^+)^{V_\gamma}) = (\delta^+)^{(V[G])_\gamma}$, because $(V[G])_\gamma$ is a forcing extension by forcing of size $< \delta$.
Thus Theorem 5.10 applies to the extensions $\mathcal{M} \subseteq (V[G])_\gamma$ and $V_\gamma \subseteq (V[G])_\gamma$, and so $V_\gamma = \mathcal{M}$. □

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