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“Projective measure without projective Baire”

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Abstract

We prove that it is consistent relative to a Mahlo cardinal that all sets of reals definable from countable sequences of ordinals are Lebesgue measurable, but at the same time, there is a Δ_3^1 set without the Baire property. To this end, we introduce a notion of stratified forcing and stratified extension and prove an iteration theorem for these classes of forcings. Moreover we introduce a variant of Shelah's amalgamation technique that preserves stratification. The complexity of the set which provides a counterexample to the Baire property is optimal.

Für Gerda und Reinhold.

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—David Schrittemser, June 2010, Vienna

*We're all just part of some giant grand machine,
too big to really understand.*

—Electric President

One is too few, and two are too many.

—Donna Harraway

Chapter 1

Preliminaries

Forcing Facts

In this section we introduce the notions of *strong projection*, *strong sub-order* and *independence*, all of which we find very useful. The first two are practical in understanding how amalgamation is a stratified extension (see sections 3 and 4). The third is handy in proving lemma 5.4 (p. 95), via the notion of “remoteness” and lemma 3.33 (see below for further discussion, especially section 3.5).

After that we fix our terminology dealing with forcing iterations and terms such as “Cohen over V ”.

Strong projections and strong Sub-orders.

Definition 1.1. Let Q, P be forcing posets. We say $\pi: P \rightarrow Q$ is a *projection* if and only if

1. $p \leq p' : \pi(p) \leq \pi(p')$,
2. $\text{ran}(\pi) = Q$,
3. if $q \in Q$ and $q \leq \pi(p)$, there is $\bar{p} \in P$ such that $\pi(\bar{p}) \leq q$.

The following definition occurs, e.g., in [Abr00]. We call π a *strong projection* if and only if it satisfies the first two requirements above and the following strengthening of the third requirement:

- 3'. If $q \leq \pi(p)$, there is $\bar{p} \leq p$ such that
 - a. $\pi(\bar{p}) = q$,
 - b. for any $r \in P$, if $r \leq p$ and $\pi(r) \leq q$ then $r \leq \bar{p}$.

This uniquely determines \bar{p} , and we denote it by $q \cdot p$.

Remark 1.2. These definitions don't seem to be totally standardized, e.g. [Abr00] gives a definition of (ordinary) projection as above, but replaces 3. by the stronger: if $q \leq \pi(p)$, there is $\bar{p} \leq p$ such that $\pi(\bar{p}) = q$. This seems stronger than the notion of projection used here, but weaker than strong projection.

If $\pi: P \rightarrow Q$ is a projection, $\pi[G]$ generates a Q -generic Filter whenever G is a P -generic Filter. Thus $\text{r.o.}(Q)$ is a complete sub-algebra of $\text{r.o.}(P)$. If $\pi: P \rightarrow Q$ is a strong projection, the map i sending $q \in Q$ to $i(q) = q \cdot 1_P$ is a complete embedding and we can assume that Q is a subset of P . Observe that it follows from 3'b. that

$$\forall p \in P \quad p \leq i(\pi(p)). \quad (1.1)$$

Definition 1.3. Let Q be a complete sub-order of P . We say $q \in Q$ is a *reduction (to Q) of $p \in P$* if and only if for all $q' \in Q$, if $q' \leq q$ then q' and p are compatible.

Lemma 1.4. *Let Q be a complete sub-order of P and let π be the canonical projection $\pi: \text{r.o.}(P) \rightarrow \text{r.o.}(Q)$. Say $p \in P$ and $q \in Q$ is a reduction of p such that $q \geq p$; then $q = \pi(p)$. If $\bar{\pi}: P \rightarrow Q$ is a strong projection, then $\bar{\pi}$ coincides with the canonical projection on P .*

Proof. Firstly, say we have $q \in Q$, a reduction of p such that $q \geq p$. We immediately infer $q \leq \pi(p)$ (this is equivalent to q being a reduction of p). On the other hand, $\pi(p) \leq q$ holds as $p \leq q$ and by the definition of π . Secondly, let $\bar{\pi}: P \rightarrow Q$ be a strong projection. For every $p \in P$, $\bar{\pi}(p)$ is a reduction of p and $\bar{\pi}(p) \geq p$ by (1.1). By the previous, $\bar{\pi}$ and π must coincide. ☺

Observe that if $\pi: \text{r.o.}(P) \rightarrow \text{r.o.}(Q)$ is the canonical projection, then $\pi \upharpoonright P$ is a strong projection if and only if for every $p \in P$ and $q \in Q$ such that $q \leq \pi(p)$ we have $p \cdot q \in P$. All of the above gives us the following definition and lemma:

Definition 1.5. We say Q is a *strong sub-order of P* if and only if Q is a complete sub-order of P and for every $p \in P$ and $q \in Q$ such that $q \leq \pi(p)$, $q \cdot p \in P$.

Lemma 1.6. *The following are equivalent:*

- Q is a strong sub-order of P .

- There is a strong projection $\pi: P \rightarrow Q$.
- The restriction of the canonical projection $\pi: \text{r.o.}(P) \rightarrow \text{r.o.}(Q)$ to P is the unique strong projection from P to Q .

Independent sub-orders.

Lemma 1.7. *Say Q_0 and Q_1 are complete sub-orders of P and $\pi_0: P \rightarrow Q_0$ is a strong projection. The following are equivalent:*

1. $\forall (q_0, q_1) \in Q_0 \times Q_1, q_0 \cdot q_1 \neq 0$;
2. $\pi_0[Q_1] = \{1_P\}$.

Proof. This is obvious from the definition of the canonical projection:

$$\pi_0(q_1) = \sum \{q_0 \in Q_0 \mid \forall q'_0 \leq q_0 \quad q'_0 \cdot q_1 \neq 0\}$$

☺

Imagine an iteration $R = (Q_0 \times Q_1) * \dot{Q}_2$. Then in an extension by Q_0 , the pre-order Q_1 is a complete sub-order of the tail $R : Q_0 = Q_1 * \dot{Q}_2$. In general, if Q_0 and Q_1 are arbitrary complete sub-orders of a forcing R , it will not be the case that after forcing with Q_0 , the pre-order Q_1 is a complete sub-order of $R : Q_0$. In the next lemma, we give a handy sufficient condition for this to be the case.

Lemma 1.8. *Let Q and C be complete sub-orders of P and say $\pi_Q: P \rightarrow Q$ and $\pi_C: P \rightarrow C$ are strong projections. Assume for all $c \in C$ and $p \in P$ such that $c \leq \pi_C(p)$, we have $\pi_Q(p \cdot c) = \pi_Q(p)$. Then 1_Q forces that C is a complete sub-order of $P : Q$ and π_C is a strong projection.*

Proof. First observe that considering the assumption of the lemma for the special case $p = 1_P$ yields

$$\pi_Q[C] = \{1_Q\}, \tag{1.2}$$

and so $1_Q \Vdash \check{C} \subseteq P : Q$.

Let $q \in Q, p \in P$ such that $q \Vdash p \in P : Q$, i.e. $q \leq \pi_Q(p)$. We show $q \Vdash \pi_C(p)$ is a reduction of p . So let $c \in C$ and $q' \in Q$ be arbitrary such that $q' \leq q$ and $q' \Vdash_Q c \leq \pi_C(p)$ in $P : Q$ (by the first paragraph, $q' \Vdash_Q c \in P : Q$ holds for trivial reasons). In other words,

$$q' \cdot c \leq \pi_C(p). \tag{1.3}$$

We claim that $q' \leq \pi_Q(p \cdot c)$ and so $q' \Vdash p \cdot c \in P : Q$. As q' was arbitrary, we are done.

To this end, observe that (1.3) implies $\pi_C(q' \cdot c) \leq \pi_C(p)$. By (1.2) and by lemma 1.7, we have $\pi_C[Q] = \{1_C\}$ and so $\pi_C(q' \cdot c) = c \cdot \pi_C(q') = c$. We conclude $c \leq \pi_C(p)$. By assumption, it follows that $\pi_Q(p) = \pi_Q(p \cdot c)$, so as $q' \leq q \leq \pi_Q(p)$ by choice of q' , we finally obtain $q' \leq \pi_Q(p \cdot c)$. $\textcircled{\smile}$

For later reference, we shall give a name to this special relationship of Q and C described above:

Definition 1.9. Let Q and C be sub-orders of P with strong projections

$$\begin{aligned}\pi_Q: P &\rightarrow Q \\ \pi_C: P &\rightarrow C.\end{aligned}$$

We say C is independent over Q in P if and only if for all $c \in C$ and $p \in P$ such that $c \leq \pi_C(p)$, we have $\pi_Q(p \cdot c) = \pi_Q(p)$.

For a P -name \dot{C} , we say \dot{C} is independent in P over Q if and only if \dot{C} is a name for a generic of an independent complete sub-order of P ; i.e. there is a complete sub-order R_C of P (with a strong projection $\pi_C: P \rightarrow R_C$) such that R_C is dense in $\langle \dot{C} \rangle^{\text{r.o.}(P)}$ and R_C is independent in P over Q .

Lemma 1.10. If \dot{C} is a P -name which is independent over Q , then \dot{C} is not in V^Q .

Proof. This should be clear; for the skeptic, here's a proof: Fix a complete sub-order R_C of P such that R_C is dense in $\langle \dot{C} \rangle^{\text{r.o.}(P)}$ and R_C is remote in P over Q . Say G is generic for Q . By lemma 1.8, R_C is a complete sub-order of $P : Q$, which implies that R_C is dense in $\langle R_C \rangle^{\text{r.o.}(P:Q)}$ by the following argument:

For any antichain $X \subseteq R_C$ we can find X^* which is a maximal antichain in $P : Q$ such that $X \subseteq X^* \subseteq R_C$. Therefore $B = \{\sum X \mid X \subseteq R_C\}$ is closed under Boolean complement and thus is equal to $\text{r.o.}(R_C)$. But R_C is dense in B .

Now let $B = \langle \dot{C} \rangle^{\text{r.o.}(P)}$. By lemma 1.20, which we shall prove below,

$$\begin{aligned}Q * \langle R_C \rangle^{\text{r.o.}(P:Q)} &= \langle Q \cup R_C \rangle^{\text{r.o.}(P)} = \\ \langle Q \cup B \rangle^{\text{r.o.}(P)} &= Q * \langle B/\dot{C} \rangle^{\text{r.o.}(P:Q)}.\end{aligned}$$

Thus, in $V[G]$, the Boolean algebra $\langle B/G \rangle^{\text{r.o.}(P:Q)}$ has R_C as a dense subset. But if \dot{C} is in V^Q , we can assume $B/G = (\langle \dot{C} \rangle^{\text{r.o.}(P)})^V/G$ is the trivial Boolean algebra. This contradicts the assumption that R_C is a non-trivial forcing. $\textcircled{\smile}$

We can't resist and give a nice proof of the following fact:

Lemma 1.11. *Say Q is a complete sub-order of P and P is a complete sub-order of R . Then $\Vdash_Q P : Q$ is a complete sub-order of $R : Q$. Moreover, if $\pi_P : R \rightarrow P$ is a strong projection, Q forces $\pi_P \upharpoonright P : Q$ is a strong projection from $P : Q$ to $R : Q$.*

Proof. Let $\pi_Q : \text{r.o.}(R) \rightarrow \text{r.o.}(Q)$ and $\pi_P : \text{r.o.}(R) \rightarrow \text{r.o.}(P)$ denote the canonical projections. Show Q forces that for each $r \in R : Q$, the condition $\pi_P(r)$ is a reduction of r to P . It follows that $\Vdash_Q R : Q$ is a complete sub-order of $R : Q$. So let $r \in R$, $p \in P$ and $q \in Q$ be arbitrary such that $p \leq \pi_P(r)$ and $q \Vdash r \in R : Q$ and $p \in P : Q$, i.e. $q \leq \pi_Q(r) \cdot \pi_Q(p)$. Observe that $\pi_Q(p \cdot r) = \pi_Q(\pi_P(p \cdot r)) = \pi_Q(p \cdot \pi_P(r)) = \pi_Q(r)$. Thus $q \leq \pi_Q(p \cdot r)$, whence $q \Vdash r$ and p are compatible in $R : Q$. This proves that Q forces that $\pi_P(r)$ is a reduction of r . Moreover, if $p \cdot r \in R$, $q \Vdash p \cdot r \in R : Q$, so π_Q is forced to be a strong projection. $\textcircled{\smile}$

Iterations. Contrary to popular belief, the meaning of “iteration” is not the same to everyone. Below we specify how we understand this notion, fix some convenient terminology and state two trivial lemmas.

Definition 1.12. We say $\bar{Q}^\theta = (P_\iota, \dot{Q}_\iota)_{\iota < \theta}$ is an iteration (of length θ) if and only if for each $i < \theta$,

1. $\Vdash_{P_i} \dot{Q}_i$ is a pre-order
2. P_i consists of sequences p such that $\text{dom}(p) = \iota$ and for each $\nu < \iota$, $p(\nu)$ is a P_ν -name such that

$$1_{P_\nu} \Vdash p(\nu) \in \dot{Q}_\nu. \quad (1.4)$$

3. The ordering of P_i is given by:

$$r \leq p \iff \forall \nu < \iota \quad r \upharpoonright \nu \Vdash_{P_\nu} r(\nu) \leq_{\dot{Q}_\nu} p(\nu). \quad (1.5)$$

For the following, fix an iteration $\bar{Q}^{\theta+1}$.

Definition 1.13. 1. We call a sequence p with $\text{dom}(p) = \theta$ a *thread through (or in) \bar{Q}^θ* if and only if it satisfies (1.4). The set of threads through \bar{Q}^θ we shall sometimes denote by $\prod \bar{Q}^\theta$. It is endowed with the ordering given by (1.5) (for $r, p \in \prod \bar{Q}^\theta$).

2. For any pair $\iota \leq \bar{\iota} \leq \theta$ we denote the strong projection from $P_{\bar{\iota}}$ to P_ι by π_ι (in this case, it is allowed to identify maps with different domains).

3. We also use the term *thread* in a second, related sense: if $\bar{p} \in \prod_{\eta < \iota < \theta} P_\iota$ for some $\eta < \theta$ —i.e. $\bar{p} = (p_\iota)_{\iota \in (\eta, \theta)}$ and for each $\iota \in \text{dom}(\bar{p})$ we have $p_\iota \in P_\iota$ —we say \bar{p} forms or defines or simply *is a thread (through \bar{Q}^θ)* if and only if

$$\forall \iota, \bar{\iota} \in \text{dom}(\bar{p}) \quad \iota \leq \bar{\iota} : \pi_\iota(p_{\bar{\iota}}) = p_\iota. \quad (1.6)$$

Note that a sequence $\bar{p} \in \prod_{\eta < \iota < \theta} P_\iota$ is a thread in the sense of item 3 of 1.13 if and only if $\bigcup_{\eta < \iota < \theta} \bar{p}_\iota$ is a thread in the sense of item 1 of 1.13—i.e. if and only if $\bigcup_{\eta < \iota < \theta} \bar{p}_\iota \in \prod \bar{Q}^\theta$. In practice, we often identify \bar{p} and $\bigcup_{\eta < \iota < \theta} \bar{p}_\iota$. Conditions 2 and 3 of definition 1.12 just say that P_ι consists of threads. The following is rather trivial.

Lemma 1.14. *Say $\bar{p} = (p_\xi)_{\xi < \rho}$ is a sequence of threads through \bar{Q}^θ . Assume for each $\iota < \theta$, the sequence $\bar{p}^\iota = (\pi_\iota(p_\xi))_{\xi < \rho}$ has a greatest lower bound q^ι such that the sequence $q = (q^\iota)_{\iota < \theta}$ is a thread through \bar{Q}^θ (i.e. for $\iota < \bar{\iota} < \theta$, $\pi_\iota(q^{\bar{\iota}}) = q^\iota$). Then q is a greatest lower bound of \bar{p} in P_θ .*

Proof. Left to the reader. ☺

Lemma 1.15. *Say $\bar{p} = (p_\iota)_{\nu < \iota < \theta}$ is such that for $\iota \in (\nu, \theta)$, $p_\iota \in P_\iota$ and \bar{p} is a thread. Let $p_\theta = \bigcup \bar{p}$. Then $(p_\iota)_{\iota \leq \theta}$ is a thread in $\bar{Q}^{\theta+1}$ and in $\text{r.o.}(P_\theta)$ we have (the product sign denotes Boolean meet)*

$$p_\theta = \prod_{\iota < \theta} p_\iota.$$

Proof. Left to the reader. ☺

Unbounded, Random, Cohen... If c is a Borel-code, we write B_c for the Borel set coded by c . Of course given two models of set theory, both containing a Borel code c , it may be that c codes a different set in each model.

Definition 1.16. Say $\text{r.o.}(Q)$ is a complete sub-algebra of $\text{r.o.}(P)$. Let \dot{I} be a P -name for an ideal on the Borel sets in the extension via P . For a P -name \dot{r} and $p \in P$, we say p forces \dot{r} is \dot{I} -generic over V^Q , just if $p \Vdash_P \dot{r} \in \mathbb{R}$ and for every Q -name for a Borel code \dot{c} ,

$$p \Vdash_P B_{\dot{c}} \in \dot{I} : \dot{r} \notin B_{\dot{c}}.$$

We say p forces \dot{r} is *fully \dot{I} -generic* over V^Q if and only if p forces \dot{r} is \dot{I} -generic over V^Q and in addition, for every Q -name \dot{c} such that $\pi(p) \Vdash_Q \dot{c}$ is a Borel code,

$$p \Vdash_P \dot{r} \notin B_{\dot{c}} : p \Vdash_P B_{\dot{c}} \in \dot{I}.$$

In other words, p does not force anything non-trivial about \dot{r} . We say \dot{r} is (fully) I -generic just if 1_P forces \dot{r} is (fully) I -generic.

- If \dot{I} is a name for the ideal of Borel sets with measure zero, instead of “ \dot{I} -generic”, we say *Random over V^Q* .
- If \dot{I} is a name for the ideal of meager Borel sets, instead of “ \dot{I} -generic”, we say *Cohen over V^Q* .
- If \dot{I} is a name for $\mathcal{P}(\mathbb{R}^{V[G]})$, where G is Q -generic over V , we say $\dot{r} \notin V^Q$ or \dot{r} is not in V^Q instead of “ \dot{r} is \dot{I} -generic”.
- If \dot{I} is a name for the ideal of Borel sets which are bounded by a real in the ground model in the sense of eventual domination, we say *unbounded over V^Q* instead of “ \dot{I} -generic”.

The terms p forces \dot{r} is fully *Random over V^Q* and *fully Cohen over V^Q* are to be understood analogously.

Lemma 1.17. *Let P and Q be arbitrary partial orders and let \dot{r} be a P -name for a real. If \dot{r} is unbounded over V , viewing \dot{r} as a $P \times Q$ name via the natural embedding, \dot{r} is unbounded over V^Q .*

Proof. We take the proof from [JR93, lemma 3.3, p. 392]. Assume for a contradiction that \dot{r} is not unbounded over V^Q . Let \dot{s} be a Q -name for a real and $(p, q) \in P \times Q$ such that $(p, q) \Vdash \forall k \in \omega \ \dot{r}(k) \leq \dot{s}(k)$. For each $i \in \omega$, find $y(i) \in \omega$ and $q_i \in Q$ such that $q_i \leq q$ and $(p, q_i) \Vdash \dot{s}(i) = y(i)$. As $\Vdash_P \dot{r}$ is unbounded over V , we can find $n \in \omega$ and $p' \in P$ such that $p' \leq p$ and $p' \Vdash \dot{r}(n) > y(n)$. Thus $(p', q_n) \leq (p, q)$ forces both $\dot{r}(n) \leq \dot{s}(n) = y(n)$ and $\dot{r}(n) > y(n)$, contradiction. ☺

Boolean algebra facts

We revisit some Boolean algebra facts which are usually taken for granted. We do this in order to fix notation and in order to be able to prove lemma 5.4 much later on page 95, for which lemma 1.20 is a prerequisite (see below for further discussion of its role).

Let A be a complete Boolean algebra, B be a regular sub-algebra of A , and let $\pi: A \rightarrow B$ denote the canonical projection map,

$$\pi(a) = \prod \{b \in B \mid b \geq a\} = \|\forall b \in B \ b \cdot a \neq 0\| = \|[a]_{\dot{G}} \neq 0\|.$$

Let G be B -generic. In $V[G]$, consider A/G , the *quotient of A modulo G* , that is, A modulo the equivalence relation \equiv_G where

$$a \equiv_G a' \iff [(a \Delta a') \cdot b = 0 \text{ for some } b \in G].$$

As G is a filter, A/G is a Boolean algebra (with the operations inherited from A). The equivalence class of $a \in A$ modulo \equiv_G we denote by $[a]_G$. Setting

$$I_G = \{a \in A \mid \exists b \in G \ a \cdot b = 0\} = -G^\circ,$$

where G° denotes the upward closure of G , we also write A/I_G . Observe that I_G is an ideal. The generic G is complete with respect to the ground model, in the sense that whenever $(a_\nu)_{\nu \in I} \in V$, where $a_\nu \in G$ for each $\nu \in I$, then $\prod_{\nu \in I} a_\nu \in G$. A dual property holds for I_G : Let $\{a_\nu\}_{\nu \in I}$ be a set of elements of A , $a_\nu \in I_G$ for each $\nu \in I$. Then for each $\nu \in I$, there is $b \in G$ such that $a_\nu \cdot b = 0$, so $\pi(a_\nu) \cdot b = 0$ and thus $-\pi(a_\nu) \in G$; so

$$-\sum_{\nu \in I} \pi(a_\nu) = \prod_{\nu \in I} -\pi(a_\nu) \in G,$$

whence $\sum_{\nu \in I} \pi(a_\nu) \in I_G$. Observe, by the way,

$$\sum \{b \in B \mid b \leq a\} = \|[a]_G = 1\|.$$

We denote by $A : B$ a B -name for A/G , i.e.

$$\|A : B = A/\dot{G}\|^B = 1.$$

As usual, $A * A : B$ denotes the class of B -names \dot{x} such that

$$\|\dot{x} \in A : B\|^B = 1,$$

modulo the following equivalence relation: $\dot{x} \sim \dot{y}$ just if

$$\|\dot{x} = \dot{y}\|^B = 1.$$

There is no need to distinguish between \dot{x} and its equivalence class. Clearly, $A * A : B$ is a set (i.e. not a proper class). $A * A : B$ carries the Boolean algebra-operations given by the operations on $A : B$: for example, let $-\dot{x}$ be some \dot{y} such that

$$\|-\dot{x} = \dot{y}\|^B = 1;$$

similarly for the remaining operations.

Fact 1.18. A is isomorphic to $B * A : B$ and $A : B^G$ is a complete Boolean algebra in $V[G]$.

Proof. We begin with a lemma.

Lemma 1.19. *Let \dot{x} be a B -name such that $\|\dot{x} \in A : B\|^B = 1$. Then there is a uniquely determined $a \in A$ such that*

$$\|[a]_{\dot{G}} = \dot{x}\|^B = 1_B.$$

Proof of lemma. Let \dot{x} be given as above. We may assume that

$$\|\dot{x} = [\dot{a}]_{\dot{G}}\|^B = 1$$

for some B -name \dot{a} . Let K be an antichain in B deciding \dot{a} , that is, for each $b \in K$ there is a_b such that $b \Vdash \dot{a} = a_b$. Since $b \Vdash a_b \equiv_{\dot{G}} a_b \cdot b$, we may assume $a_b \leq b$.

Finally, let

$$a = \sum \{a_b \mid b \in K\}.$$

Then for $b \in K$, $b \cdot a = a_b$, whence $b \Vdash a \equiv_{\dot{G}} a_b$ and $b \Vdash \dot{a} = a_b$. As K was a maximal antichain,

$$\|a \equiv_{\dot{G}} \dot{a}\|^B = 1_B.$$

The following observation is not essential, but interesting. We can in fact pick K so that for each $b \in K$, either $a_b = 0$ or $\pi(a_b) = b$. For we can pick K such that for each $b \in K$, either

$$b \leq \|\dot{a} \equiv_{\dot{G}} 0\|$$

or

$$b \leq -\|\dot{a} \equiv_{\dot{G}} 0\|.$$

We may assume $a_b = 0_A$ whenever the first is the case. In the second case, $\pi(a_b) = b$: for say $\pi(a_b) < b$ and chose $b' \in A$ such that $b' \leq b$, $b' \cdot a_b = 0$. Then $b' \Vdash \dot{a} = a_b \equiv_{\dot{G}} 0$.

It remains to prove a is unique. Towards a contradiction say we have $a \neq a'$ such that both

$$\|[a]_{\dot{G}} = \dot{x}\|^B = 1_B$$

and

$$\|[a']_{\dot{G}} = \dot{x}\|^B = 1_B.$$

However this contradicts

$$\|a \neq_{\dot{G}} a'\|^B = \pi(a \Delta a') > 0.$$



For $a \in A$, let $i(a)$ be the (unique) element of $B * B : A$ such that

$$\|i(a) = [\check{a}]_{\dot{G}}\|^B = 1.$$

The map $i: A \rightarrow B * A : B$ is an isomorphism of Boolean algebras: in the previous lemma we showed that for every $x \in B * B : A$ there is exactly one $a \in A$ such that $i(a) = x$, so i is a bijection. We can now show that i preserves suprema, and at the same time we show

$$\|A : B \text{ is complete}\| = 1.$$

Say we have a B -name \dot{X} such that

$$\|\dot{X} \subseteq A : B\|^B = 1.$$

In V , we can pick B -names $\{\dot{x}_\nu\}_{\nu \in I}$ such that

$$\|\{\dot{x}_\nu\}_{\nu \in I} = \dot{X}\| = 1,$$

whence of course for each ν we have

$$\|\dot{x}_\nu \in A : B\|^B = 1,$$

i.e. $\dot{x}_\nu \in A * A : B$. By the lemma, we may find $a_\nu \in A$ such that $i(a_\nu) = \dot{x}_\nu$, for each $\nu \in I$. Let $a = \sum_{\nu \in I}^A a_\nu$.¹ We shall show that $\|[a]_G = \sum_{\nu \in I}^{A:B} \dot{X}\| = 1$. To this end, let G be B -generic and write $X = \dot{X}^G$. We have $X \subseteq A/G = A : B^G$; we must show $[a]_G$ is the least upper bound of X in A/G . As $a_\nu \leq a$ for each $\nu \in I$,

$$\dot{x}_\nu^G \leq [a]_G;$$

thus $[a]_G$ is an upper bound. Given any B -name \dot{x} such that $\dot{X}^G \leq \dot{x}^G$, assume without loss of generality that

$$\|\dot{x} \in A : B\|^B = 1,$$

and pick $x \in A$ such that $[x]_G = \dot{x}$. For every $\nu \in I$,

$$\|\dot{x}_\nu \leq [x]_{\dot{G}}\| \in G,$$

so $a_\nu - x \in I_G$, for each $\nu \in I$. By completeness with respect to V of I_G , $\sum_{\nu \in I} (a_\nu - x) = a - x \in I_G$, so

$$[a]_G \leq [x]_G = \dot{x}.$$

¹Of course, a does not depend on the choice of $\{\dot{x}_\nu\}_{\nu \in I}$: given $\{\dot{x}'_\nu\}_{\nu \in I'}$ and letting a' be obtained as above, we shall see $-(a \Delta a') = \|\sum_{\nu \in I'} \dot{x}'_\nu = \sum_{\nu \in I} \dot{x}_\nu\| = 1$.

So firstly, \dot{X}^G has a supremum in A/G , whence A/G is a complete Boolean algebra.

In fact, as G was arbitrary, we have just shown

$$\| [a]_{\dot{G}} = \sum_{A:B} \dot{X} = \sum_{\nu \in I} [a_\nu]_{\dot{G}} \| = 1.$$

In other words, $i(a) = \sum i(a_\nu)$ in $A * A : B$. Thus, secondly, i preserves suprema. \odot

Let $X \subseteq A$. We denote by $\langle X \rangle$ (or $\langle X \rangle^A$ when A is not clear from the context) the algebra generated by X in A , that is smallest complete subalgebra of A containing X as a subset. We obtain $\langle X \rangle$ by closing of under Boolean operations: Set $\sum_0^A(X) = X$ and define $\sum_\alpha^A(X)$ by induction on α , as the set of all elements a such that $a = \sum Y$ where for all $y \in Y$, either y or $-y$ is in $\sum_\beta^A(X)$, for some $\beta < \alpha$. Then

$$\langle X \rangle = \bigcup_{\alpha \in \text{On}} \sum_\alpha^A(X).$$

In fact it would be enough to take the union over all $\alpha < \text{sat}(A)$, where the latter denotes the least cardinal λ such that A has no antichain of size λ . Also, when $X \subseteq A$ we write $-X$ for $\{-x \mid x \in X\}$.

The following lemma was vital in understanding amalgamation (discussed in section 4), but it is explicitly used only in the proof of 1.10, dealing with *independence* (itself a rather minor point which nevertheless has a vital role in proving lemma 5.4 on page 95, mediated by the notion of “remoteness” and lemma 3.33).

Lemma 1.20. *Let $B \subseteq X \subseteq A$ and let \dot{C} be a B -name such that*

$$\| \dot{C} = \langle X/\dot{G} \rangle^{A:B} \| = 1,$$

where X/\dot{G} denotes $\{[x]_{\dot{G}} \mid x \in X\}$ (and \dot{G} is a name for a B -generic filter). Then $B * \dot{C}$ is isomorphic to $\langle X \rangle^A$. Letting i denote the isomorphism constructed in the proof of fact 1.18,

$$i[\langle X \rangle^A] = \langle i[X] \rangle^{B * \dot{C}} = B * \dot{C}.$$

In fact, $B * (\langle X \rangle^A : B)$ is the same as $B * \langle X/\dot{G} \rangle^{A:B}$.

Proof. Clearly, $B*\dot{C}$ is a complete sub-algebra of $A*A : B$. So $C = i^{-1}[B*\dot{C}]$ is a complete sub-algebra of A , and since $X \subseteq C$, $\langle X \rangle^A \subseteq C$. It remains to show that $C \subseteq \langle X \rangle^A$, or equivalently, $B*\dot{C} \subseteq i[\langle X \rangle^A]$. If \dot{c} is a B -name and

$$\|\dot{c} \in \langle X/\dot{G} \rangle^{A:B}\| = 1,$$

then for some α ,

$$\|\dot{c} \in \sum_{\alpha}^{A:B} (X/\dot{G})\|^B = 1, \quad (1.7)$$

So it suffices to prove by induction on α that for \dot{c} satisfying (1.7), we have $\dot{c} \in i[\langle X \rangle^A]$.

For a start, let $\alpha = 0$, and let \dot{c} be such that $\|\dot{c} \in X\|^B = 1$. From the previous lemma, $\dot{c} = i(a)$, for some $a \in A$, and in fact the proof showed $a \in \langle X \rangle^A$.

Now let $\alpha > 0$. We may find $\{\dot{x}_\nu\}_{\nu \in I}$ such that

$$1_B \Vdash \dot{c} = \sum_{\nu \in I} \dot{x}_\nu,$$

and for each $\nu \in I$

$$1_B \Vdash \dot{x}_\nu \in \bigcup_{\beta < \alpha} \left(\sum_{\beta} (X) \cup - \sum_{\beta} (X) \right).$$

In fact, we can assume that for each $\nu \in I$ there is $\beta < \alpha$ such that

$$1_B \Vdash \dot{x}_\nu \in \sum_{\beta} (X) \cup - \sum_{\beta} (X);$$

for we may always write each \dot{x}_ν as a sum of Boolean values who appear at a fixed stage β (from the viewpoint of the ground model). By induction hypothesis, each $\dot{x}_\nu \in i[\langle X \rangle^A]$; thus, as i preserves sums, $\dot{c} \in i[\langle X \rangle^A]$. \odot

Chapter 2

Stratified Forcing

In this section we assume $V = L[A]$ for some class A . We define *stratified partial orders*, show such orders preserve cofinalities, give some examples and show that stratification is preserved under composition. We also define diagonal support and state that iterations whose components are stratified are themselves stratified. The proof is left out, since we prove a slightly more general theorem in section 3 where we deal with iterations with stratified initial segments but where the components aren't necessarily stratified. Most of these definitions are heavily inspired by [Fri94]; see also [Fri00].

We present the definition of stratification in two parts: the first we dub *quasi-closure*. We treat this first part separately from the remaining axioms of stratification for the following reasons: firstly, the proofs that each of these two groups of axioms is preserved in iterations are not only different but virtually independent of each other.

Secondly, we hope that the reader will agree that quasi-closure is interesting in its own right. This view is in stark contrast to the fact that quasi-closure alone is not a very useful property—in fact, every partial order is quasi-closed. One should think of it as an incomplete notion, to which some other property has to be added in order to render it non-trivial. Stratification is one example of this, closely connected to the notion of centered forcing. There may be other examples, as well.

Before we define quasi-closure, we introduce pre-closure systems; analogously we will define pre-stratification systems. We can reuse these notions when we define *quasi-closed and stratified extension*; see section 3, p. 34.

Throughout, let $\langle R, \leq \rangle$ be a pre-order. We want to allow for R to collapse some cardinals, while preserving cofinalities greater than some fixed regular λ_0 . This explains the role of the otherwise superfluous parameter λ_0 in many of the following definitions; e.g. we talk about R being *stratified above* λ_0 .

2.1 Quasi-Closure

We now make a few convenient definitions that facilitate the treatment of quasi-closed partial orders, which we define afterward.

Definition 2.1. We say $\mathbf{s} = (\mathbf{F}, \preceq^\lambda)_{\lambda \geq \lambda_0}$ is a pre-closure system for R above λ_0 if and only if

$$\mathbf{F}: \text{Reg} \times V \times R \rightarrow R$$

is a function defined by a Δ_1^A formula and for every $\lambda \in \text{Reg} \setminus \lambda_0$,

- (C 1) The relation \preceq^λ is a preorder on R and $p \preceq^\lambda q: p \leq q$.
- (C 2) For $(\lambda, x, p) \in \text{dom}(\mathbf{F})$ we have $\mathbf{F}(\lambda, x, p) \preceq^\lambda p$.
- (C 3) If $p \leq q \leq r$ and $p \preceq^\lambda r$ then $p \preceq^\lambda q$.
- (C 4) If $\bar{\lambda} \in \text{Reg} \setminus \lambda$ then $q \preceq^{\bar{\lambda}} p: q \preceq^\lambda p$.

As a notational convenience, define \preceq^0 to mean \leq_R . Clause (C 3) can be dropped if one is not interested in iterations. Observe that by (C 3), \preceq^λ is well-defined with respect to equivalence modulo \approx (remember we say $p \approx q \iff p \leq q$ and $q \leq p$).

Think of each of the relations \preceq^λ as a notion of *direct extension*, as it is often called in the case of e.g. Prikry-like forcings. Intuitively, $p \preceq^\lambda q$ expresses that p extends q but some part “below λ ” is left unchanged. Think of \mathbf{F} as a kind of strategy. Together, this additional structure on R allows us to express that certain sequences have lower bounds in R . The missing ingredient and distinct flavor of quasi-closure is the condition that these sequences be *definable* in a sense.

For the next two definitions, fix R and a pre-closure system \mathbf{s} for R above λ_0 . All the notions in the next definition have their meaning *with respect to* \mathbf{s} .

Definition 2.2. 1. Let $\bar{p} = (p_\xi)_{\xi < \rho}$ be a sequence of conditions in R . We say \bar{p} is (λ, x) -*strategic* if and only if,

- (a) $\rho \leq \lambda$.
- (b) For each $\xi < \rho$, there is a regular cardinal λ' such that

$$p_{\xi+1} \leq \mathbf{F}(\lambda', x, p_\xi)$$

$$\text{and } p_{\xi+1} \preceq^{\lambda'} p_\xi.$$

- (c) For $\xi < \bar{\xi} < \rho$, we have $p_{\bar{\xi}} \preceq^\lambda p_\xi$.

- (d) For limit ordinals $\bar{\xi} < \rho$, $p_{\bar{\xi}}$ is a greatest lower bound of $(p_{\xi})_{\xi < \bar{\xi}}$.
2. If \bar{p} is (λ, x) -strategic and in addition, \bar{p} is Δ_1^A -definable with parameters from $\lambda \cup \{x\}$, we say \bar{p} is (λ, x) -adequate.
 3. If \bar{p} is (λ, x) -adequate for some x we say \bar{p} is λ -adequate.

Definition 2.3. We say $\langle R, \mathbf{s} \rangle$ is *quasi-closed above* λ_0 if and only if for each regular $\lambda \geq \lambda_0$

- (C I) If $\bar{\lambda}$ is regular and $p \preceq^{\bar{\lambda}} 1_R$, then $\mathbf{F}(\lambda, x, p) \preceq^{\bar{\lambda}} 1_R$.
- (C II) Every λ -adequate sequence $\bar{p} = (p_{\xi})_{\xi < \rho}$ in R has a greatest lower bound p in R and for all $\xi < \rho$, $p \preceq^{\lambda} p_{\xi}$. If $\bar{\lambda}$ is regular and for each $\xi < \rho$, $p_{\xi} \preceq^{\bar{\lambda}} 1_R$, then $p \preceq^{\bar{\lambda}} 1_R$.

We also use the expression *R is quasi-closed as witnessed by \mathbf{s}* . If we omit \mathbf{s} and no pre-closure system can be deduced from the context, we mean that there exists a pre-closure system \mathbf{s} such that $\langle R, \mathbf{s} \rangle$ is quasi-closed.

Clause (C I) and the last sentence of clause (C II) are useful regarding infinite iterations of quasi-closed forcings.

Remark 2.4. For arbitrary R , just define $p \preceq^{\lambda} q$ if and only if $p = q$ and $\mathbf{F}(\lambda, x, p) = p$ for all regular $\lambda \geq \lambda_0$ and all x . Then R is quasi-closed. Quasi-closure becomes non-trivial under the additional hypothesis that certain questions about the generic extension can be decided by strengthening a condition in the sense of \preceq^{λ} , for some λ . Stratified forcing satisfies such a hypothesis.

Remark 2.5. Say R is λ^+ -closed; then R trivially satisfies all the conditions of 2.3 for this one λ . The same is true if R is λ^+ -strategic: for if $\sigma: R \rightarrow R$ is a strategy for R , define $\mathbf{F}(\lambda, x, p) = \sigma(p)$. \mathbf{F} is clearly $\Delta_1^A(\{\sigma\})$. We can define \preceq^{λ} to be the same as \leq . Of course then every strategic (and thus every adequate) sequence has a greatest lower bound.

This is not vacuous. In fact, every sequence \bar{p} of length less than λ^+ which adheres to σ is λ -adequate: For fix \bar{p} of length less than λ^+ . By re-indexing, assume the length of \bar{p} is λ . Since \bar{p} is $\Delta_1^A(\{\bar{p}\})$, and since \mathbf{F} does not depend on x at all, \bar{p} is $(\lambda, \{\bar{p}\})$ -adequate.

These are our first examples of forcings which non-trivially satisfy the definition of quasi-closed (albeit for just one fixed λ), since any statement about the generic can be decided by extending in the sense of \preceq^{λ} .

2.2 A word about definability and set forcing

The concept of quasi-closure was devised in a class forcing context. Since we only apply it for set forcing, we can make do with restricting the realm of adequate sequences to those which are Δ_1^A (in certain parameters), as we have done above. We also circumvent any use of Π_n -uniformisation, which is necessary in a class context (see [Fri00], proof of theorem 8.17, p. 178, for details).

Think of x as a tuple of constants which can be used in in the definition of an adequate sequence \bar{p} . Observe that we can restrict the notion of λ -adequate sequences by demanding that they be (λ, x) -adequate for some x containing some given, fixed set of constants. We can, for example, freely assume that any parameters needed in the definition of \mathbf{F} are among those constants. We can and will assume that some large enough $L_\mu[A]$ is among the constants given by x , as well as predicates for the well-ordering of $L_\mu[A]$ and the cofinality and cardinality function restricted to $L_\mu[A]$. This allows us to bound quantifiers of certain statements and argue that they are $\Delta_1^A(\{x\} \cup \lambda)$.

Intuitively, this is analogous to the use of a large structure with predicates in the context of proper forcing. If you feel we are waving our hands to much, formally keep track of what parameters we use: augment the definition of pre-closure system so that $\mathbf{s} = (\mathbf{F}, \vec{c}, \preceq^\lambda)_{\lambda \geq \lambda_0}$ where \vec{c} is a tuple such that \mathbf{F} is $\Delta_1^A(\vec{c})$ -definable and in the definition of a (λ, x) -adequate sequence, demand that x be a tuple containing all the constants from \vec{c} . Clearly, if R is quasi-closed as witnessed by \mathbf{s} , R is also quasi-closed with respect to any pre-closure system \mathbf{s}' which is obtained from \mathbf{s} by adding some more constants to \vec{c} . We come back to these points when we discuss iterations.

2.3 Stratification

Definition 2.6. We say $\mathbf{S} = (\mathbf{F}, \preceq^\lambda, \preceq^\lambda, \mathbf{C}^\lambda)_{\lambda \geq \lambda_0}$ is a pre-stratification system for R above λ_0 if and only if $(\mathbf{F}, \preceq^\lambda)_{\lambda \geq \lambda_0}$ is a pre-closure system for R above λ_0 and for every $\lambda \in \text{Reg} \setminus \lambda_0$ the following conditions are met:

- (S 1) The binary relation \preceq^λ on R satisfies $p \leq q : p \preceq^\lambda q$.
- (S 2) If $p \leq q \preceq^\lambda r$ then $p \preceq^\lambda r$.¹

¹Note that we don't assume \preceq^λ to be transitive, since this does not seem to be preserved by composition. If \preceq^λ were transitive, condition (S 2) would follow from (S 1). We need (S 2) for lemma 2.9. We need that \preceq^λ is reflexive (i.e. $p \preceq^\lambda p$ for all p) for 3.18($\triangleleft_s 4$). In the context of (S 2), reflexivity is the same as the last part of (S 1).

- (S 3) If $\lambda \leq \bar{\lambda}$ and $\bar{\lambda} \in \text{Reg}$ then $p \preceq^\lambda q \preceq^{\bar{\lambda}} q$.
- (S 4) *Density*: $\mathbf{C}^\lambda \subseteq R \times \lambda$ is a binary relation such that $\text{dom}(\mathbf{C}^\lambda)$ is dense in R . Moreover, if $\lambda > \lambda_0$, for any regular $\lambda' \in [\lambda_0, \lambda)$ and $p \in R$, there is $q \preceq^{\lambda'} p$ such that $q \in \text{dom}(\mathbf{C}^\lambda)$.

The last part of condition (S 1), all of (S 3) and and the “moreover” part of (S 4) can be dropped if one is not interested in infinite iterations. Don’t think that \preceq^λ is a pre-order or well-defined on the separative quotient of R , although (S 2) guarantees some regularity with respect to \approx .

For the moment, fix R and a pre-stratification system \mathbf{S} for R above λ_0 . The following definition is relative to \mathbf{S} .

Definition 2.7. We say a pre-order $\langle R, \leq \rangle$ is *stratified above λ_0* if and only if $\langle R, \preceq^\lambda, \mathbf{F} \rangle_{\lambda \geq \lambda_0}$ is quasi-closed and for each $\lambda \in \text{Reg} \setminus \lambda_0$ the following conditions hold:

- (S I) *Continuity*: If λ' is regular such that $\lambda_0 \leq \lambda' < \lambda$ and p is a greatest lower bound of the λ' -adequate sequence $\bar{p} = (p_\xi)_{\xi < \rho}$ and for each $\xi < \rho$, $p_\xi \in \text{dom}(\mathbf{C}^\lambda)$, then $p \in \text{dom}(\mathbf{C}^\lambda)$. If in addition \bar{q} is another λ' -adequate sequence of length ρ and for each $\xi < \rho$, $\mathbf{C}^\lambda(p_\xi) \cap \mathbf{C}^\lambda(q_\xi) \neq \emptyset$, then for a greatest lower bound q of \bar{q} we have $\mathbf{C}^\lambda(p) \cap \mathbf{C}^\lambda(q) \neq \emptyset$.
- (S II) *Expansion*: If $p \preceq^\lambda d$ and $d \preceq^\lambda 1_R$, then in fact $p \leq d$.
- (S III) *Interpolation*: If $d \leq r$, there is $p \preceq^\lambda r$ such that $p \preceq^\lambda d$. In addition, whenever $\bar{\lambda}$ is regular and $d \preceq^{\bar{\lambda}} 1_R$, then also $p \preceq^{\bar{\lambda}} 1_R$.
- (S IV) *Centering*: If $p \preceq^\lambda d$ and $\mathbf{C}^\lambda(p) \cap \mathbf{C}^\lambda(d) \neq \emptyset$ then p and d are compatible. In fact, there is w such that for any regular $\lambda' \in [\lambda_0, \lambda)$, $w \preceq^{\lambda'} p$ and $w \preceq^{\lambda'} d$.

We also say R is *stratified as witnessed by \mathbf{S}* . If we omit \mathbf{S} and no pre-stratification system can be deduced from the context, we mean that there exists a pre-stratification system \mathbf{S} witnessing that R is stratified.

Conditions (S I) and (S II) are important to preserve stratification in (infinite) iterations. The second part of (S IV) was introduced to allow for amalgamation (see section 4), but is also useful to control the diagonal support in iterations (see below).

Finally, we can discuss preservation of cofinalities and the GCH.

²in our application, we could ask this for all regular λ' , not just those in the interval $[\lambda_0, \lambda)$.

Definition 2.8. In the following, we fix a regular cardinal $\lambda \geq \lambda_0$ and drop the superscripts on \mathbf{C} , \preceq and \prec .

1. Let $D, D^* \subseteq R$, and $r \in R$. We say r λ -reduces D to D^* (often, we don't mention the prefix λ) exactly if
 - (a) $D^* \subseteq \text{dom}(\mathbf{C})$ and $|D^*| \leq \lambda$;
 - (b) for each $d \in D^*$, $r \prec d$;
 - (c) for any $q \in \text{dom}(\mathbf{C}) \cap D$, if $q \leq r$, there is $d \in D^*$ such that $\mathbf{C}(q) \cap \mathbf{C}(d) \neq 0$.
2. Let $\dot{\alpha}$ be a name for an element in the ground model V , and let $r \in R$. We say $\dot{\alpha}$ is λ -chromatic below r just if there is a function H with $\text{dom } H \subseteq \lambda$ such that if $q \leq r$ decides $\dot{\alpha}$ and $q \in \text{dom}(\mathbf{C})$, then $\mathbf{C}(q) \cap \text{dom}(H) \neq 0$ and for all $\chi \in \mathbf{C}(q) \cap \text{dom}(H)$, $q \Vdash \dot{\alpha} = H(\chi)$ (to be pedantically precise, we mean the “standard name” for $H(\chi)$). We call such H a λ -spectrum (of $\dot{\alpha}$).
3. If \dot{s} is a name and $p \Vdash \dot{s}: \lambda \rightarrow V$, then we say \dot{s} is λ -chromatic (with λ -spectrum $(H_\xi)_{\xi < \lambda}$) below p if and only if for each $\xi < \lambda$, $\dot{s}(\check{\xi})$ is chromatic with spectrum H_ξ below p .

For notational convenience, we say \dot{x} is 0-chromatic below p if for some x , $p \Vdash_R \dot{x} = \check{x}$.

Observe that if for some ground model set x , $p \Vdash \dot{\alpha} = \check{x}$ (i.e. $\dot{\alpha}$ is 0-chromatic), then $\dot{\alpha}$ is in fact λ -chromatic for every regular λ , and the function with domain λ and constant value x is a λ -spectrum.

Lemma 2.9. *Say R is stratified above λ . For each $\xi < \lambda$, let D_ξ be an open dense subset of R . Let*

$$X = \{q \in R \mid \exists D^* \quad \forall \xi < \lambda \quad q \lambda\text{-reduces } D_\xi \text{ to } D^*\}.$$

Then X is dense in $\langle R, \preceq^\lambda \rangle$ and open in $\langle R, \leq \rangle$.

If \dot{s} is a name such that $p \Vdash \dot{s}: \lambda \rightarrow V$, the set of q such that \dot{s} is λ -chromatic below q is dense in $\langle R(\leq p), \preceq^\lambda \rangle$ and open.

Proof. We first show that the set of q such that for a single D , q reduces D is dense in $\langle R, \preceq^\lambda \rangle$: Given D open dense and $p \in R$, inductively build sequences of conditions $\bar{p} = (p_\xi)_{\xi < \lambda}$ and $D^* = \{d_\xi \mid \xi < \lambda\}$ in R , such that \bar{p} is λ -adequate and $p_0 \preceq^\lambda p$. In the end, p_λ will reduce D to D^* , where p_λ is a greatest lower bound of \bar{p} .

We now give a definition of \bar{p} from parameters in $\{x\} \cup \lambda$. Observe that we can chose x in such a way that this definition is Δ_1^A in $\lambda \cup \{x\}$: let

$$x = \langle p_0, \preceq^\lambda, \preceq^\lambda, \mathbf{C}^\lambda, \prec, R, D, y \rangle,$$

where \prec is a well-ordering on R (or on some large enough $L_\mu[A]$, if you prefer) and y contains any parameters needed in the definition of \mathbf{F} .

The definition of \bar{p} and D^* is by induction. Before we begin, choose $d^* \in R$ as a convenient default value when we cannot find an appropriate d_ξ : let d^* be the \prec -least element of $\text{dom}(\mathbf{C}^\lambda) \cap D$ such that $d^* \leq p_0$.

Firstly, let p_0 be the \prec -least conditions such that $p_0 \preceq^\lambda p$ and $p_0 \preceq^\lambda d^*$. Secondly, say ξ is a limit ordinal, and we have already constructed $(p_\nu)_{\nu < \xi}$. Let p_ξ be a greatest lower bound of $(p_\nu)_{\nu < \xi}$.

Finally, say p_ξ is already defined; we will now construct $p_{\xi+1}$ and d_ξ . Assume for the moment that there is $d \leq \mathbf{F}(\lambda, x, p_\xi)$ such that $\xi \in \mathbf{C}(d)$ and $d \in D$. Set d_ξ to be the \prec -least such d ; let $p_{\xi+1}$ be the \prec -least condition in R such that $p_{\xi+1} \preceq \mathbf{F}(\lambda, x, p_\xi)$ and $p_{\xi+1} \preceq d$. If such d cannot be found, set $d_\xi = d^*$ and $p_{\xi+1} = p_\xi$.

We show that p_λ reduces D to D^* . So say we are given $q \leq p_\lambda$ in $\text{dom}(\mathbf{C})$ which decides α and such that $\xi \in \mathbf{C}(q)$. As q witnesses that there is $w \leq \mathbf{F}(\lambda, x, p_\xi)$ such that $\xi \in \mathbf{C}(w)$ and $w \in D$, we have $d_\xi \in D$ such that $\xi \in \mathbf{C}(d_\xi)$ and $p_\lambda \leq p_{\xi+1} \preceq d_\xi$. By 2.7(S 2), $p_\lambda \preceq^\lambda d_\xi$, and so p_λ reduces D to D^* .

Now for the second claim of lemma 2.9. If we have a sequence $\bar{D} = (D_\xi)_{\xi < \lambda}$ of dense open subsets of R , build a sequence as before: let

$$x = \langle p_0, \preceq^\lambda, \preceq^\lambda, \mathbf{C}^\lambda, \prec, R, \bar{D}, y \rangle.$$

At successor steps ξ , let p_ξ be the \prec -least p such that $p \preceq^\lambda \mathbf{F}(\lambda, x, p_{\xi-1})$ and such that we can pick D_ξ^* such that p_ξ reduces D_ξ to D_ξ^* . As before, a greatest lower bound p_λ exists and for each $\xi < \lambda$, p_λ reduces D_ξ to $\bigcup_{\xi < \lambda} D_\xi^*$.

Let $p \Vdash \dot{s}: \lambda \rightarrow V$. Let D_ξ be the set of conditions $p \in R$ which decide $\dot{f}(\xi)$. Find q reducing all D_ξ to D^* . We now find a spectrum for \dot{s} : For $\chi < \lambda$, if $w \leq q$ decides $\dot{s}(\xi)$ and $\chi \in \mathbf{C}^\lambda(w)$, there is also $d \in D^*$ which decides $\dot{s}(\xi)$ and such that $\chi \in \mathbf{C}^\lambda(d)$. Fix z such that $d \Vdash \dot{f}(\xi) = \check{z}$. Then we may set $H_\xi(\chi) = z$. It is easy to check that for each $\xi < \lambda$, H_ξ is a spectrum for $\dot{s}(\xi)$ (and thus $(H_\xi)_{\xi < \lambda}$ is a spectrum for \dot{s}): Say $w \leq q$ decides $\dot{s}(\xi)$ and fix some $\chi \in \mathbf{C}^\lambda(w)$. Then there is $d \in D^*$ with $\chi \in \mathbf{C}^\lambda(d)$ such that $d \Vdash \dot{s}(\xi) = H_\xi(\chi)$. As $d \in D^*$, $q \preceq^\lambda d$. So as $\chi \in \mathbf{C}^\lambda(w) \cap \mathbf{C}^\lambda(d)$, w and d are compatible and thus $w \Vdash \dot{s}(\xi) = H_\xi(\chi)$. $\textcircled{\smile}$

Corollary 2.10. *Cofinalities greater than λ remain greater than λ after forcing with R and $(2^\lambda)^V = (2^\lambda)^{V[G]}$ for any R -generic V .*

We illustrate definition 2.7 with some examples.

Example 2.11. A simple observation is that for any pre-order R , R is stratified above $|R|$. A little more generally, if R is λ_0 -centered, then R is stratified above λ_0 : for if $\lambda \geq \lambda_0$, we can simply define $p \preceq^\lambda q$ just if $p = q$. Similarly, $\mathbf{F}(\lambda, x, p) = p$ for all $p \in R$. Thus, quasi-closure and continuity become vacuous. Moreover, let $g : R \rightarrow \lambda_0$ be a function such that if $g(p) = g(q)$ then p and q are compatible. Set $\mathbf{C}^\lambda(p) = \{g(p)\}$ for any $p \in R$. Lastly, define $p \preceq^\lambda q$ to hold for *any* pair p, q . Then the only non-vacuous condition in the definition of stratification is *centering*, which holds for every $\lambda \geq \lambda_0$ since g witnessed that R was centered.

This example has a corollary:

Corollary 2.12. *If a pre-ordered set R is stratified, we can always assume that for $\lambda \geq |R|$, \mathbf{F} , \preceq^λ , \preceq^λ and \mathbf{C}^λ take the simple form discussed above in example 2.11.*

Observe that (C 4) and (S 3) remain valid if we modify a given pre-stratification system in such a way as to ensure that the above assumption holds. A more interesting example:

Example 2.13. Say $R = P * \dot{Q}$ where P is $(\lambda_0)^+$ -centered and $(\lambda_0)^+$ -closed and $\Vdash_P \dot{Q}$ is λ_0 -centered and λ_0 -closed. Then R is stratified—ignoring (S I). If the centering functions for P and \dot{Q}_ξ in the extension are continuous in the sense of (S I)—and it seems that for many centered forcings, this is the case— R is actually stratified.

Define \mathbf{F} as in the previous example. For $\lambda < \lambda_0$, define \preceq^λ to be identical to \leq_R . Define $p \preceq^\lambda q$ if and only if $p = q$ and $\mathbf{C}(p) = \lambda$ for every $p \in R$. Then *Interpolation* and *centering* hold at λ for trivial reasons, and quasi-closure at λ expresses the fact that R is closed under sequences of length at most λ . For $\lambda = \lambda_0$, fix a name for a centering function \dot{g} ; set $(p, \dot{q}) \preceq^\lambda (p', \dot{q}')$ if and only if $(p, \dot{q}) \leq (p', \dot{q}')$ and $\dot{q} = \dot{q}'$; set $(p, \dot{q}) \preceq^\lambda (p', \dot{q}')$ if and only if $p \leq_P p'$. Let $\chi \in \mathbf{C}^\lambda(p, \dot{q})$ if and only if $p \Vdash \dot{g}(\dot{q}) = \check{\chi}$. Lastly, R has a subset R' which is $(\lambda_0)^+$ centered and \preceq^{λ_0} -dense. This allows us to define a stratification above $(\lambda_0)^+$, in a similar way to the previous example.

2.4 Composition of stratified forcing

In the main theorem of this section, theorem 2.14 below, we show stratification is preserved by composition. In the proof, we use “guessing systems”, which we shall motivate now, before we state and prove the theorem.

Say P is stratified and \dot{Q} is forced by P to be stratified, and let λ be fixed. We know P has a centering function \mathbf{C} and \dot{Q} is forced to have a centering function $\dot{\mathbf{C}}$ in the extension. Similar to the proof that composition of centered forcing stays centered, we want to gain some control over $\dot{\mathbf{C}}$ in the ground model. If we ignore the requirements 2.6(S 4) *density* and 2.7(S I) *continuity*, we could define $\bar{\mathbf{C}}$ on $P * \dot{Q}$ in the following way:

$$(\chi, \xi) \in \bar{\mathbf{C}}(d, \dot{d}) \iff \chi \in \mathbf{C}(d) \text{ and } d \Vdash \xi \in \dot{\mathbf{C}}(\dot{d})$$

Then $\text{dom}(\bar{\mathbf{C}})$ is dense and 2.7(S IV) *centering* holds.

The following definition also satisfies 2.6(S 4) *density*: let

$$(\chi, X) \in \bar{\mathbf{C}}(d, \dot{d}) \iff \left(\chi \in \mathbf{C}(d) \text{ and for some } \dot{\xi} \text{ and } \lambda' \in \text{Reg} \cap [\lambda_0, \lambda), \right. \\ \left. X \text{ is a } \lambda'\text{-spectrum for } \dot{\xi} \text{ below } d \text{ and } d \Vdash \dot{\xi} \in \dot{\mathbf{C}}(\dot{d}) \right). \quad (2.1)$$

Let's check 2.6(S 4) *density* holds: Given a condition $\bar{p} = (p, \dot{p})$ and $\lambda' \in \text{Reg} \cap [\lambda_0, \lambda)$, we can find $\bar{d} = (d, \dot{d})$ such that $\bar{d} \preceq^{\lambda'} \bar{p}$, $d \in \text{dom}(\mathbf{C})$ and $d \Vdash \dot{d} \in \text{dom}(\dot{\mathbf{C}})$. Moreover, we can assume that for some name $\dot{\chi}$, $d \Vdash \dot{\chi} \in \dot{\mathbf{C}}(\dot{d})$ and $\dot{\chi}$ is λ' -chromatic. We have $\bar{d} \in \text{dom}(\bar{\mathbf{C}})$. Let's also check that 2.7(S IV) *centering* holds: say $\bar{d} \preceq^{\lambda} \bar{p}$ and $(\chi, X) \in \bar{\mathbf{C}}(\bar{p}) \cap \bar{\mathbf{C}}(\bar{d})$. First, observe that p and d are compatible. Fix $\dot{\chi}_0, \dot{\chi}_1$ such that both $p \Vdash \dot{\chi}_0 \in \dot{\mathbf{C}}(\dot{p})$ and $d \Vdash \dot{\chi}_1 \in \dot{\mathbf{C}}(\dot{d})$ and X is a spectrum for $\dot{\chi}_0$ below p and for $\dot{\chi}_1$ below d . As $p \cdot d \Vdash \dot{\chi}_0 = \dot{\chi}_1 \in \dot{\mathbf{C}}(\dot{d}) \cap \dot{\mathbf{C}}(\dot{p})$, by *centering* for \dot{Q} in the extension, $p \cdot d \Vdash \dot{d}$ and \dot{p} are compatible, whence \bar{d} and \bar{p} are compatible.

To show stratification is preserved at limits, we will have to use *Continuity* of the centering function; Unfortunately, the approach described above does not yield a *continuous* centering function in the sense of (S I). For say $\bar{d}_\xi = (d_\xi, \dot{d}_\xi)$ form a λ' -adequate sequence of length ρ , and for each $\xi < \rho$, $d_\xi \Vdash \dot{\chi}_\xi \in \dot{\mathbf{C}}(\dot{d}_\xi)$ and X_ξ is a λ_ξ -spectrum for $\dot{\chi}_\xi$ below d_ξ . By *Continuity* for the components of the forcing, if (d, \dot{d}) is a greatest lower bound, we know $d \Vdash \dot{d} \in \text{dom}(\dot{\mathbf{C}})$; but there is no reason to assume that there exists a P -name $\dot{\gamma}$, such that $d \Vdash \dot{\gamma} \in \dot{\mathbf{C}}(\dot{d})$ and $\dot{\gamma}$ is λ'' -chromatic for some $\lambda'' < \lambda$.

The solution to this problem is to allow a more general set of values for $\bar{\mathbf{C}}(\bar{d})$: in the situation described above, e.g. the sequence $(X_\xi)_{\xi < \rho}$ be used in much the same way as the single spectrum X . This leads to the notion of a guessing system, which will be precisely defined in 2.15.

Theorem 2.14. *Say P is stratified above λ_0 and \dot{Q} is forced by P to be stratified above λ_0 . Then $\bar{P} = P * \dot{Q}$ is stratified (above λ_0).*

Proof. Say stratification of P is witnessed by $\mathbf{F}, \mathbf{C}^\lambda, \preceq^\lambda, \preceq^\lambda$ for each regular $\lambda \geq \lambda_0$, and we have names $\dot{\mathbf{F}}$ and $\dot{\mathbf{C}}^\lambda, \dot{\preceq}^\lambda, \dot{\preceq}^\lambda$ for λ regular which are forced

by P to witness the stratification of \dot{Q} . We now define $\bar{\mathbf{F}}^\lambda, \bar{\mathbf{C}}^\lambda, \bar{\preceq}^\lambda$ and $\bar{\preceq}^\lambda$ for regular $\lambda \geq \lambda_0$ to witness stratification of $P * \dot{Q}$.

The auxiliary orderings

Let $\lambda \geq \lambda_0$ be regular. We say $(p, \dot{q}) \bar{\preceq}^\lambda (u, \dot{v})$ if and only if $p \preceq^\lambda u$ and $p \Vdash_P \dot{q} \bar{\preceq}^\lambda \dot{v}$. This defines a pre-order stronger than the natural ordering on $P * \dot{Q}$ (i.e. 2.1(C 1) holds). Define $\bar{p} \bar{\preceq}^\lambda \bar{q}$ if and only if $p \preceq^\lambda q$ and if $p \cdot q \neq 0$, $p \cdot q \Vdash_P \dot{p} \preceq^\lambda \dot{q}$.

The ordering axioms

Let $\bar{p} = (p, \dot{p})$, $\bar{q} = (q, \dot{q})$ and $\bar{r} = (r, \dot{r})$ be conditions in \bar{P} .

We check that 2.1(C 3) holds: Say $\bar{p} \leq_{\bar{P}} \bar{q} \leq_{\bar{P}} \bar{r}$ and $\bar{p} \bar{\preceq}^\lambda \bar{r}$. Then $p \preceq^\lambda r$ by 2.1(C 3) for P . Moreover, p forces 2.1(C 3) for \dot{Q} as well as $\dot{p} \leq \dot{q} \leq \dot{r}$ and $\dot{p} \preceq^\lambda \dot{r}$. So $p \Vdash_P \dot{p} \preceq^\lambda \dot{q}$, and we conclude $\bar{p} \bar{\preceq}^\lambda \bar{q}$.

Check that 2.6(S 2) holds: Say $\bar{p} \leq_{\bar{P}} \bar{q} \bar{\preceq}^\lambda \bar{r}$. By (S 2) for P , $p \preceq^\lambda r$. If $p \cdot r \neq 0$,

$$p \cdot r \Vdash_P \dot{p} \leq_{\dot{Q}} \dot{q} \preceq^\lambda \dot{r},$$

and so $p \cdot r \Vdash_P \dot{p} \preceq^\lambda \dot{r}$. Thus $\bar{p} \bar{\preceq}^\lambda \bar{r}$.

Next, check 2.6(S II). Say $\bar{p} \bar{\preceq}^\lambda \bar{q}$ and $\bar{q} \bar{\preceq}^\lambda 1_{\bar{P}}$. By (S II) for P , $p \leq q$. So $p \Vdash \dot{p} \preceq^\lambda q \preceq^\lambda 1_{\dot{Q}}$, so by (S II) applied in the extension, $p \Vdash \dot{p} \leq_{\dot{Q}} \dot{q}$, whence $\bar{p} \leq_{\bar{P}} \bar{q}$. We leave it to the reader to check 2.1(C 4) and 2.7(S 3).

Quasi-Closure

Define $\bar{\mathbf{F}}(\lambda, x, (p, \dot{q})) = (\mathbf{F}(\lambda, x, p), \dot{q}^*)$, where \dot{q}^* is the \prec -least P -name such that $1_P \Vdash_P \dot{q}^* = \dot{\mathbf{F}}(\lambda, x, \dot{q})$.

Why is this definable by a Δ_1^A formula? This involves an essential use of parameters, as discussed in 2.2. We assume that either P or some $L_\mu[A]$ for μ such that $P \subset L_\mu[A]$, as well as \prec restricted to P or $L_\mu[A]$ are among the constants in x .³ Then the following formula witnesses that $\bar{\mathbf{F}}$ is $\Delta_1^A(\lambda \cup \{x\})$:

$$\begin{aligned} (p^*, \dot{q}^*) = \bar{\mathbf{F}}(\lambda, x, (p, \dot{q})) \iff & \left[p^* = \mathbf{F}(\lambda, x, p) \text{ and} \right. \\ & 1_P \Vdash_P \dot{q}^* = \dot{\mathbf{F}}(\lambda, x, \dot{q}) \text{ and} \\ & \left. \forall \dot{q}' \in L_\mu[A] \quad (\dot{q}' \prec \dot{p}': \not\Vdash_P \dot{q}' = \dot{\mathbf{F}}(\lambda, x, \dot{q})) \right] \end{aligned}$$

³We could, but do not need to assume that \Vdash_P is available as a parameter, since this relation is Δ_1^A if restricted to Δ_1^A formulas of the forcing language.

Clearly, (C 2) is satisfied. Now say $(p_\xi, \dot{q}_\xi)_{\xi < \rho}$ is (λ, x) -adequate. We show this sequence has a greatest lower bound. We can immediately infer that $(p_\xi, \dot{q}_\xi)_{\xi < \rho}$ is (λ, x) -strategic. Fix a $\Delta_1^A(\lambda \cup \{x\})$ formula $\Phi(x, y)$ such that

$$\Phi(x, \xi) \Leftarrow: \left(\xi < \delta \text{ and } x = (p_\xi, \dot{q}_\xi) \right).$$

Then

$$\exists q \in L_\mu[A] \quad \Phi((p, q), \xi)$$

is a $\Delta_1^A(\lambda \cup \{x\})$ definition of $(p_\xi)_{\xi < \rho}$, if we assume that $P \subseteq L_\mu[A]$ and $L_\mu[A]$ is among the constants in x . So $(p_\xi)_{\xi < \rho}$ is (λ, x) -adequate and thus has a greatest lower bound p_ρ .

By a similar argument, $p_\rho \Vdash_P \text{“}(\dot{q}_\xi)_{\xi < \rho} \text{ is } (\lambda, x)\text{-adequate”}$. Here we also use $L_\mu[A]$ has a simple definition, as the $\Delta_1^A(\lambda \cup \{x\})$ definition of \bar{r} was relative to the ground model (alternatively, we could assume Δ_1 formulas are absolute for $L_\mu[A]$). So we can find \dot{q}_ρ such that $p_\rho \Vdash_P \text{“}\dot{q}_\rho \text{ is a greatest lower bound of } (\dot{q}_\xi)_{\xi < \rho}\text{”}$, whence (p_ρ, \dot{q}_ρ) is a greatest lower bound of the original sequence. Leaving the last sentence of (C II) to the reader, we conclude that $\langle P * \dot{Q}, \bar{\aleph}^\lambda, \bar{\mathbf{F}} \rangle$ is λ -quasi-closed above λ_0 .

To define $\bar{\mathbf{C}}^\lambda$, we first define the notion of a *guessing system*. Roughly speaking, a guessing system consists of conditions which are organized in levels; the conditions on the bottom have a $\bar{\mathbf{C}}^\lambda$ -value in the sense of (2.1). Conditions on higher levels are greatest lower bounds of conditions on the levels below, and we have some control over their $\bar{\mathbf{C}}^\lambda$ -value by *continuity* for \dot{Q} .

Definition 2.15. Say $(p, \dot{q}) \in P * \dot{Q}$ and λ is regular and uncountable. A λ -guessing system for \dot{q} below p is a quadruple $(T_g, H_g, \lambda_g, q_g)$ such that

1. T_g is a tree, $T_g \subseteq {}^{<\omega}\gamma$, where $\gamma = \text{width}(T) < \lambda$ and \triangleleft (initial segment) is reversely well founded on T_g . The root of T_g is \emptyset (i.e. the empty sequence).
2. For $s \in T_g$, $\rho_g(s) = \{\xi \mid s \hat{\ } \xi \in T_g\}$ is an ordinal. Write T_g^0 for the set of \triangleleft -maximal $s \in T_g$, i.e. $T_g^0 = \{s \in T_g \mid \rho_g(s) = 0\}$.
3. q_g is a function from T_g into the set of P -names for conditions in \dot{Q} and $\lambda_g: T_g \rightarrow \lambda \cap \text{Reg}$.
4. For $s \in T_g \setminus T_g^0$, $\{q_g(s \hat{\ } \xi)\}_{\xi < \rho_g(s)}$ is a $\lambda_g(s)$ -adequate sequence and p forces that $q_g(s)$ is a greatest lower bound of $\{q_g(s \hat{\ } \xi)\}_{\xi < \rho_g(s)}$.
5. $\text{dom}(H_g) = T_g^0$.

6. For $s \in T_g^0$, there is a P -name $\dot{\chi}$ such that $p \Vdash_P \dot{\chi} \in \mathbf{C}^\lambda(q_g(s))$ and $H_g(s)$ is a $\lambda_g(s)$ -spectrum of $\dot{\chi}$ below p .
7. $q_g(\emptyset) = \dot{q}$.

Now we are ready to define $\bar{\mathbf{C}}^\lambda$: let $s \in \bar{\mathbf{C}}^\lambda(p, \dot{q})$ if and only if *either*

- (a) $s \in \mathbf{C}^\lambda(p)$ and $p \Vdash \dot{q} \dot{\prec}^\lambda 1_{\dot{Q}}$ holds *or else*
- (b) if $\lambda > \lambda_0$, $s = (\chi, T_g, H_g, \lambda_g)$ where $\chi \in \mathbf{C}^\lambda(p)$ and for some q_g , $(T_g, H_g, \lambda_g, q_g)$ is a λ -guessing system for \dot{q} below p .
- (c) if $\lambda = \lambda_0$, $s = (\chi, \xi)$, where $\chi \in \mathbf{C}^\lambda(p)$ and $p \Vdash_P \check{\xi} \in \dot{\mathbf{C}}^\lambda(\dot{q})$.

It is straightforward to check that $\text{ran}(\bar{\mathbf{C}}^\lambda)$ has size at most λ . Thus we may assume $\bar{\mathbf{C}}^\lambda \subseteq (P * \dot{Q}) \times \lambda$, although this is not literally the case.

We have finally defined the stratification of $\bar{P} = P * \dot{Q}$. Let's check the remaining axioms.

Continuity

Say $\lambda' \in [\lambda_0, \lambda)$ is regular, both $\bar{p} = (p_\xi, \dot{p}_\xi)_\xi$ and $\bar{q} = (q_\xi, \dot{q}_\xi)_\xi$ are λ' -adequate sequences of length ρ and for each $\xi < \rho$, $\bar{\mathbf{C}}^\lambda(p_\xi, \dot{p}_\xi) \cap \bar{\mathbf{C}}^\lambda(q_\xi, \dot{q}_\xi) \neq \emptyset$. Moreover, let (p, \dot{p}) and (q, \dot{q}) denote greatest lower bounds of \bar{p} and \bar{q} , respectively.

First, by *Continuity* for P , we can find $\chi \in \mathbf{C}^\lambda(p) \cap \mathbf{C}^\lambda(q)$. For each $\xi < \rho$, fix $(T_g^\xi, H_g^\xi, \lambda_g^\xi)$ such that for some χ' ,

$$(\chi', T_g^\xi, H_g^\xi, \lambda_g^\xi) \in \bar{\mathbf{C}}^\lambda(p_\xi, \dot{p}_\xi) \cap \mathbf{C}^\lambda(q_\xi, \dot{q}_\xi).$$

and find p_g^ξ such that $(T_g^\xi, H_g^\xi, \lambda_g^\xi, p_g^\xi)$ is a guessing system for \dot{p}_ξ below p_ξ .

Now construct a guessing system $(T_g, H_g, \lambda_g, p_g)$ for \dot{p} below p , showing $(p, \dot{p}) \in \text{dom}(\bar{\mathbf{C}}^\lambda)$. It will be clear from the construction that T_g, H_g and λ_g do not depend on the sequence of p_g^ξ , $\xi < \rho$. Let $s \in T_g$ if and only if $s = \emptyset$ or $\xi \hat{\ } s \in T_g^\xi$. Let $\lambda_g(\emptyset) = \lambda'$, and of course $p_g(\emptyset) = \dot{p}$. Now let $s \in T_g \setminus \{\emptyset\}$ be given and define $\lambda_g(s)$, $p_g(s)$ and, in the case that $s \in T_g^0$, also define $H_g(s)$. Find s' such that $s = \xi \hat{\ } s'$. Let $\lambda_g(s) = \lambda_g^\xi(s')$ and let $H_g(s) = H_g^\xi(s')$ if $s \in T_g^0$ (or equivalently, if $s' \in (T_g^\xi)^0$). Let $p_g(s) = p_g^\xi(s')$.

To check that $(T_g, H_g, \lambda_g, p_g)$ is a guessing system, first observe that \triangleleft is reversely well-founded on T_g . Moreover, $\rho_g(\emptyset) = \rho$ is an ordinal and $\lambda_g(\emptyset) < \lambda$. Also, clause 4. holds for $s = \emptyset$, by construction. The rest of the conditions are straightforward to check; they hold by construction and because for each $\xi < \lambda'$, $(T_g^\xi, H_g^\xi, \lambda_g^\xi, p_g^\xi)$ is a guessing system.

If we carry out the same construction for (q, \dot{q}) , we obtain q_g such that $(T_g, H_g, \lambda_g, q_g)$ is a guessing system for \dot{q} below q . Thus,

$$(\chi, T_g, H_g, \lambda_g) \in \bar{\mathbf{C}}^\lambda(p, \dot{p}) \cap \bar{\mathbf{C}}^\lambda(q, \dot{q}).$$

Interpolation

Say $(d, \dot{d}) \leq_{\bar{P}} (r, \dot{r})$. First find $p \in P$ such that $p \preceq^\lambda d$ and $p \preceq^\lambda r$. If $p \cdot d \neq 0$, then $p \cdot d \Vdash_P \dot{d} \leq_{\dot{Q}} \dot{r}$, so we can find \dot{p} such that $p \cdot d \Vdash_P \dot{p} \preceq^\lambda \dot{d}$ and $p \Vdash_P \dot{p} \preceq^\lambda \dot{r}$.

Centering

Say $\bar{p} \preceq^\lambda \bar{d}$, where $\bar{p} = (p, \dot{p})$ and $\bar{d} = (d, \dot{d})$, and assume $\bar{\mathbf{C}}^\lambda(\bar{p}) \cap \bar{\mathbf{C}}^\lambda(\bar{d}) \neq \emptyset$. First assume we can find $(\chi, T_g, \lambda_g, H_g) \in \bar{\mathbf{C}}^\lambda(\bar{p}) \cap \bar{\mathbf{C}}^\lambda(\bar{d})$ (i.e. (b) holds in the definition of $\bar{\mathbf{C}}^\lambda$). As $\chi \in \mathbf{C}^\lambda(p) \cap \mathbf{C}^\lambda(d)$, by *centering* for P there exists w such that for all regular $\lambda' \in 0 \cup [\lambda_0, \lambda)$, both $w \preceq^{\lambda'} p$ and $w \preceq^{\lambda'} d$.

Now fix p_g and d_g such that $(T_g, \lambda_g, H_g, p_g)$ is a guessing system for \dot{p} below p and $(T_g, \lambda_g, H_g, d_g)$ is a guessing system for \dot{d} below d . We show by induction on the rank of s (in the sense of the reversed \triangleleft -order) that for each $s \in T_g$,

$$p \cdot d \Vdash_P \dot{\mathbf{C}}^\lambda(p_g(s)) \cap \dot{\mathbf{C}}^\lambda(d_g(s)) \neq 0. \quad (2.2)$$

First, let $s \in T_g^0$. By definition 2.15, 6. we can find P -names $\dot{\alpha}$ and $\dot{\beta}$ such that both have spectrum $H_g(s)$ below p and d , respectively, and moreover:

$$p \Vdash_P \dot{\alpha} \in \dot{\mathbf{C}}^\lambda(p_g(s))$$

and

$$d \Vdash_P \dot{\beta} \in \dot{\mathbf{C}}^\lambda(d_g(s)).$$

Thus, as $\dot{\alpha}$ and $\dot{\beta}$ have a common spectrum below $p \cdot d$, (2.2) holds.

For s of greater rank, we may assume by induction that for each $\xi < \rho_g(s)$,

$$p \cdot d \Vdash_P \dot{\mathbf{C}}^\lambda(p_g(s^\wedge \xi)) \cap \dot{\mathbf{C}}^\lambda(d_g(s^\wedge \xi)) \neq 0. \quad (2.3)$$

As p forces that

$$\{p_g(s^\wedge \xi)\}_{\xi < \rho_g(s)} \text{ is a } \lambda_g(s)\text{-adequate sequence and } p_g(s) \text{ is a} \quad (2.4)$$

greatest lower bound of $\{p_g(s^\wedge \xi)\}_{\xi < \rho_g(s)}$,

and as d forces the corresponding statement for $d_g(s)$ and $\{d_g(s^\wedge \xi)\}_{\xi < \rho_g(s)}$, *Continuity* for \dot{Q} in the extension allows us to infer (2.2) for this s . This finishes the inductive proof on the rank of s .

Finally, (2.2) holds for $s = \emptyset$, so as $p_g(\emptyset) = \dot{p}$ and $d_g(\emptyset) = \dot{d}$, by *centering* for \dot{Q} in the extension, $w \Vdash$ there exists \dot{w} such that for all regular $\lambda' \in 0 \cup [\lambda_0, \lambda)$, both $\dot{w} \preceq^{\lambda'} \dot{p}$ and $\dot{w} \preceq^{\lambda'} \dot{d}$. Then $\bar{w} = (w, \dot{w})$ is as desired.

Now secondly assume we have $\chi \in \bar{\mathbf{C}}^\lambda(\bar{p}) \cap \bar{\mathbf{C}}^\lambda(\bar{d})$ and (a) holds in the definition of $\bar{\mathbf{C}}^\lambda$. In this case $\chi \in \mathbf{C}^\lambda(p) \cap \mathbf{C}^\lambda(d)$. Let $w \in P$ such that

$w \preceq^{<\lambda} p$ and $w \preceq^{<\lambda} d$. By assumption, $w \Vdash \dot{d} \preceq^\lambda 1_{\dot{Q}}$ and $\dot{p} \preceq^\lambda \dot{d}$. By *expansion* (S II) for \dot{Q} , we conclude $w \Vdash \dot{p} \leq \dot{d}$. We claim $\bar{w} = (w, \dot{p})$ is the desired lower bound: $\bar{w} \preceq^{<\lambda} \bar{p}$ holds because \preceq^λ is a pre-order. We show $\bar{w} \preceq^{<\lambda} \bar{d}$: we have $w \Vdash \dot{p} \leq \dot{d} \leq 1_{\dot{Q}}$ and by assumption $w \Vdash \dot{p} \preceq^\lambda 1_{\dot{Q}}$. So by (C 3), we conclude $w \Vdash \dot{p} \preceq^{<\lambda} \dot{d}$ and are done.

Density

Let $(p_0, \dot{q}_0) \in \bar{R}$. First, assume $\lambda_0 < \lambda$ and fix a regular $\lambda' \in [\lambda_0, \lambda)$. By *Density* for \dot{Q} in the extension, we can find P -names $\dot{\chi}$ and \dot{q}_1 such that $\Vdash_P \dot{q}_1 \preceq^{\lambda'} \dot{q}_0$ and $\dot{\chi} \in \mathbf{C}^{\lambda'}(\dot{q}_1)$. By lemma 2.9, we can find $p_1 \preceq^{\lambda'} p_0$ such that $\dot{\chi}$ is λ' -chromatic below p_1 , and by *Density* for P we can find $p_2 \preceq^{\lambda'} p_1$ and ζ such that $\zeta \in \mathbf{C}^{\lambda'}(p_2)$.

Let $T_g = \{\emptyset\}$, $q_g(\emptyset) = \dot{q}_1$, $\lambda_g(\emptyset) = \lambda'$ and let $H_g(\emptyset)$ be a λ' -spectrum of $\dot{\chi}$ below p_2 . Thus $(T_g, \lambda_g, H_g, q_g)$ is a guessing system for \dot{q}_1 below p_2 —the only non-trivial clause is (6.), which holds as $H_g(\emptyset)$ is a λ' -spectrum of $\dot{\chi}$ below p_1 and $p_2 \leq_P p_1$. So $(\zeta, T_g, \lambda_g, H_g) \in \bar{\mathbf{C}}^{\lambda'}(p_2, \dot{q}_1)$, and $(p_2, \dot{q}_1) \preceq^{\lambda'} (p_0, \dot{q}_0)$.

It remains to show $\text{dom}(\mathbf{C}^\lambda)$ is dense in the case that $\lambda_0 = \lambda$. Find $(p_1, \dot{q}_1) \leq_{\bar{R}} (p_0, \dot{q}_0)$ such that for some ordinals $\zeta, \chi < \lambda$, $\zeta \in \mathbf{C}^\lambda(p)$ and $p_1 \Vdash_P \dot{\chi} \in \mathbf{C}^\lambda(\dot{q}_1)$. Then $(\zeta, \chi) \in \bar{\mathbf{C}}^\lambda(p_1, \dot{q}_1)$. ☺

2.5 Stratified iteration and diagonal support

We now proceed to show the notion of stratified forcing is iterable, if the right support is used. To this end, let's define stratified iteration with diagonal support.

To motivate this, imagine we want to take a product of forcings $P_\xi * \dot{Q}_\xi$, of the type of example 2.13. The present approach to showing these forcings preserve cofinalities makes use of the fact that P_ξ is closed under sequences of length λ_0 while \dot{Q}_ξ has a (strong form of) $(\lambda_0)^+$ -chain condition. If we want to preserve the latter, we should use support of size less than λ_0 ; if we want to preserve the first, our choice would be to use support of size λ_0 . This calls for a kind of mixed support: i.e. define

$$\Pi_{\xi < \lambda_0}^d (P_\xi * \dot{Q}_\xi)$$

to be the set of all sequences $(p(\xi), \dot{q}(\xi))_{\xi < \lambda_0} \in \Pi_{\xi < \lambda_0} (P_\xi * \dot{Q}_\xi)$ such that for all but less than λ_0 many ξ ,

$$p(\xi) \Vdash_P \dot{q}(\xi) = \dot{1}_\xi. \tag{2.5}$$

Using the stratification of $P * \dot{Q}$, (2.5) may be written as

$$(p(\xi), \dot{q}(\xi)) \preceq^{\lambda_0} 1_{P_\xi * \dot{Q}_\xi}.$$

The use of the term “diagonal” is motivated by the intuition that we allow large support on P_ξ , which we regard as the “upper” part, and small support on \dot{Q}_ξ , which we regard as the “lower” part of the forcing $P_\xi * \dot{Q}_\xi$.

Definition 2.16. 1. We say the iteration $\bar{Q}^\theta = \langle P_\nu; \dot{Q}_\nu, \leq_\nu, \dot{1}_\nu \rangle_{\nu < \theta}$ has *stratified components* if and only if for every $\nu < \theta$, \dot{Q}_ν is a P_ν -name and P_ν forces \dot{Q}_ν is a stratified partial order as witnessed by the system of P_ν -names

$$\bar{\mathbf{S}} = (\dot{\preceq}_\nu^\lambda, \dot{\succ}_\nu^\lambda, \dot{\mathbf{F}}_\nu, \dot{\mathbf{C}}_\nu)_{\lambda \in \text{Reg}, \nu < \theta}.$$

(which is called its *stratification*). Moreover, we demand that for all regular λ there is $\theta_\lambda < \lambda^+$ such that for all $\iota \in [\theta_\lambda, \theta)$ and all $p \in P_\theta$ we have $p \upharpoonright \iota \Vdash_{P_\iota} p(\iota) \dot{\preceq}_\iota^\lambda 1_{\dot{Q}_\iota}$.

2. P_θ is the *diagonal support limit* of the iteration with stratified components \bar{Q}^θ with stratification $\bar{\mathbf{S}}$ if and only if P_θ is the set of all threads through \bar{Q}^θ such that the following support condition is met: for each regular λ , $\text{supp}_\lambda(p)$ has size less than λ , where $\text{supp}_\lambda(p)$ is defined as

$$\text{supp}^\lambda(p) = \{\xi \mid p \upharpoonright \xi \not\Vdash_\xi p(\xi) \dot{\preceq}_\xi \dot{1}_\xi\}.$$

3. We say \bar{Q}_θ is an iteration with diagonal support if for all limit $\nu < \theta$, P_ν is the diagonal support limit of \bar{Q}_ν .

We omit the proof of the following theorem, since it will follow from theorem 3.23 and lemma 3.20 as corollary 3.26. In the proof of the main theorem, we will need to use these stronger lemmas, theorem 2.17 does not suffice.

Theorem 2.17. *Say $\bar{Q} = \langle P_\nu, \dot{Q}_\nu \rangle_{\nu < \theta}$ is an iteration with stratified components and diagonal support. Then P_θ is stratified.*

Chapter 3

Extension and iteration

The proof of the main result makes it necessary to consider iterations \bar{Q}^θ such that each initial segment P_ι is stratified above a regular cardinal λ_ι , but it is not forced that \dot{Q}_ι be stratified for all $\iota < \theta$. We deal with this difficulty by introducing the concept of $(P_\iota, P_{\iota+1})$ being a *stratified extension*. With the right support, this ensures that the initial segments are sufficiently coherent so that we can conclude that P_θ is stratified. This coherency provided by extension is vital: e.g. an iteration whose proper initial segments are all σ -strategically closed can add a real (see [KS10]).

To further complicate things, λ_ι is not the same fixed cardinal throughout the iteration.

We treat quasi-closed and stratified extension separately (sections 3.1 and 3.2). Each axiom of stratified (or quasi-closed) extension corresponds to an axiom of stratification (or quasi-closure)—in fact, interestingly, P is stratified if and only if $(\{1_P\}, P)$ is a stratified extension. To prove the iteration theorem, we also have to add some additional axioms concerning the interplay of the pre-stratification (pre-closure) systems on P_ι and $P_{\iota+1}$; see definitions 3.1 and 3.18.

In section 3.3 we show products of stratified forcings are stratified extensions. Finally, we introduce *the stable meet operator* in section 3.4 and *remote sub-orders* in section 3.5.

3.1 Quasi-closed extension and iteration

In this section, we show that composition of quasi-closed forcing is a special case of quasi-closed extension. We give a sufficient condition which makes sure that if (P_0, P_1) is a quasi-closed extension, then P_1 is quasi-closed. We prove that the relation of being a quasi-closed extension is transitive. Finally,

we formulate and prove an iteration theorem for quasi-closed forcing.

Let P_0 be a complete sub-order of P_1 and let $\pi: P_1 \rightarrow P_0$ be a strong projection. Moreover, assume we have a system $\mathbf{s}_i = (\mathbf{F}_i, \preceq_i^\lambda)_{\lambda \geq \lambda_0}$ for $i \in \{0, 1\}$ such that $\mathbf{F}_i: \text{Reg} \setminus \lambda_0 \times V \times P_i \rightarrow P_i$ is a (definable) function and for every $\lambda \geq \lambda_0$, \preceq_i^λ is a binary relation on P_i .

Definition 3.1. We write $\mathbf{s}_0 \triangleleft \mathbf{s}_1$ to mean

- (\triangleleft_c1) For all $p, q \in P_0$, $p \preceq_0^\lambda q \Rightarrow p \preceq_1^\lambda q$.
- (\triangleleft_c2) For all $p, q \in P_1$, $p \preceq_1^\lambda q \Rightarrow \pi(p) \preceq_0^\lambda \pi(q)$.
- (\triangleleft_c3) $\pi(\mathbf{F}_1(\lambda, x, p)) = \mathbf{F}_0(\lambda, x, \pi(p))$.

Observe that if $\mathbf{s}_0 \triangleleft \mathbf{s}_1$ we can drop the subscripts on $\preceq_0^\lambda, \preceq_1^\lambda$ and just write \preceq^λ without causing confusion. By the way, note that (\triangleleft_c3) has to be loosened in a class forcing context.

Definition 3.2. We say the pair (P_0, P_1) is a *quasi-closed extension above λ_0* , as witnessed by $(\mathbf{s}_0, \mathbf{s}_1)$ if and only if \mathbf{s}_0 witnesses that P_0 is quasi-closed above λ_0 , \mathbf{s}_1 is a pre-closure system on P_1 , $\mathbf{s}_0 \triangleleft \mathbf{s}_1$ and for $\lambda, \bar{\lambda} \in \text{Reg}$ such that $\lambda_0 \leq \lambda \leq \bar{\lambda}$, the following conditions hold:

- (E_cI) If $p \preceq^{\bar{\lambda}} \pi(p)$, then $\mathbf{F}_1(\lambda, x, p) \preceq^{\bar{\lambda}} \pi(\mathbf{F}_1(\lambda, x, p))$.
- (E_cII) If $\bar{p} = (p_\xi)_{\xi < \rho}$ is a sequence of conditions in P_1 such that for some $q \in P_0$
 - (a) q is a greatest lower bound of the sequence $(\pi(p_\xi))_{\xi < \rho}$ and for all $\xi < \rho$, $q \preceq^\lambda \pi(p_\xi)$,
 - (b) \bar{p} is (λ, x) -strategic and $\Delta_1^A(\{x\} \cup \bar{\lambda})$ -definable,
 - (c) either $\lambda = \bar{\lambda}$ or $p_\xi \preceq_1^{\bar{\lambda}} \pi(p_\xi)$ for each $\xi < \rho$,

then \bar{p} has a greatest lower bound p in P_1 such that for each $\xi < \rho$, $p \preceq^\lambda p_\xi$ and $\pi(p) = q$. Moreover, if $p_\xi \preceq_1^{\bar{\lambda}} \pi(p_\xi)$ for each $\xi < \rho$, then also $p \preceq_1^{\bar{\lambda}} \pi(p)$.

As before, if we say (P_0, P_1) is a quasi-closed extension and don't mention either of $\mathbf{s}_0, \mathbf{s}_1$ or λ_0 , the entity we forgot to mention is either clear from the context or we are claiming that one can find such an entity.

We will grow tired of repeating all the conditions \bar{p} has to satisfy in (E_cII), so we issue the following definition:

Definition 3.3. We say \bar{p} is $(\lambda, \bar{\lambda}, x)$ -adequate if and only if λ and $\bar{\lambda}$ are regular such that $\lambda_0 \leq \lambda \leq \bar{\lambda}$ and \bar{p} satisfies conditions (E_cIIb) and (E_cIIc) above. We say q is a π -bound if and only if (E_cIIa) holds.

Of course, the obvious example for quasi-closed extension is provided by composition of forcing notions:

Lemma 3.4. *If P is quasi-closed above λ_0 and $\Vdash_P \dot{Q}$ is quasi-closed above λ_0 , then $(P, P * \dot{Q})$ is a quasi-closed extension above λ_0 .*

*To be more precise, let \mathbf{s}_0 denote the pre-quasi-closure system witnessing that P is quasi-closed and let $\mathbf{s}_1 = (\bar{\mathbf{F}}, \bar{\preceq}^\lambda)_{\lambda \geq \lambda_0}$ be the pre-quasi-closure system constructed as in the proof of 2.14, where we showed that $P * \dot{Q}$ is stratified. Then $(\mathbf{s}_0, \mathbf{s}_1)$ witnesses that $(P, P * \dot{Q})$ is a quasi-closed extension above λ_0 .*

We give the proof after we prove the following simple lemma, which will be useful in several contexts.

Lemma 3.5. *Say R carries a pre-closure system \mathbf{s} above λ_0 and $\bar{p} = (p_\xi)_{\xi < \rho}$ is (λ, x) -strategic and $\Delta_1^A(\bar{\lambda} \cup \{x\})$ -definable. If for all $\xi < \rho$, $p_\xi \preceq^{\bar{\lambda}} 1_R$, then \bar{p} is in fact $(\bar{\lambda}, x)$ -adequate.*

Proof of lemma 3.5. For arbitrary $\xi < \bar{\xi} < \rho$, by 2.1(C 3), as $p_{\bar{\xi}} \leq p_\xi \leq 1_R$ and $p_{\bar{\xi}} \preceq^{\bar{\lambda}} 1_R$, we have $p_{\bar{\xi}} \preceq^{\bar{\lambda}} p_\xi$. Thus \bar{p} is $(\bar{\lambda}, x)$ -strategic. ☺

Proof of lemma 3.4. Just by looking at the definition of $\bar{\preceq}^\lambda$ and $\bar{\mathbf{F}}$, it is immediate that $\mathbf{s}_0 \triangleleft \mathbf{s}_1$ and that \mathbf{s}_1 is a pre-closure system. To check that \mathbf{s}_1 is a pre-closure system, observe 2.1(C 3) has already been checked in the proof of theorem 2.14. The other conditions we leave to the reader.

Condition 3.2(E_cI) holds since P forces 2.3(C I) for \dot{Q} : Say $(p, \dot{p}) \bar{\preceq}^{\bar{\lambda}} (p, 1_{\dot{Q}})$. That is,

$$p \Vdash_P \dot{p} \bar{\preceq}^{\bar{\lambda}} 1_{\dot{Q}}.$$

Then

$$p \Vdash \dot{\mathbf{F}}(\lambda, x, \dot{p}) \bar{\preceq}^{\bar{\lambda}} 1_{\dot{Q}},$$

and consequently,

$$\bar{\mathbf{F}}(\lambda, x, (p, \dot{p})) \bar{\preceq}^{\bar{\lambda}} (p, 1_{\dot{Q}}).$$

Now to the main point, that is 3.2(E_cII): Say $\bar{p} = (p_\xi, \dot{p}_\xi)_{\xi < \rho}$ is sequence of conditions in P_1 which is $(\lambda, \bar{\lambda}, x)$ -adequate—i.e. (E_cIIb) and (E_cIIc) hold—and assume $q \in P_0$ is a π -bound—i.e. (E_cIIa) holds.

Under suitable assumptions about x , q forces that $(\dot{p}_\xi)_{\xi < \rho}$ is $\Delta_1^A(\{x\} \cup \bar{\lambda})$ -definable (this is the same argument as in the proof of theorem 2.14). If $\lambda = \bar{\lambda}$, we may immediately conclude that q forces that $(\dot{p}_\xi)_{\xi < \rho}$ is λ -adequate in \dot{Q} . Thus we may pick a P -name \dot{q} such that q forces $\dot{q} \in \dot{Q}$ is the greatest lower bound of $(\dot{p}_\xi)_{\xi < \rho}$ in \dot{Q} . Then (q, \dot{q}) is the greatest lower bound of \bar{p} and we are done with the proof of 3.2(E_cII) in this case.

If on the other hand, $\lambda < \bar{\lambda}$, we may assume that for all $\xi < \rho$, $(p_\xi, \dot{p}_\xi) \preceq_1^{\bar{\lambda}} (p_\xi, 1_{\dot{Q}})$. We claim that q forces that $(\dot{p}_\xi)_{\xi < \rho}$ is $\bar{\lambda}$ -adequate in \dot{Q} . To this end, we first check that q forces $(\dot{p}_\xi)_{\xi < \rho}$ is (λ, x) -strategic. Let $\xi < \rho$ be given and fix λ' such that

$$(p_{\xi+1}, \dot{p}_{\xi+1}) \preceq^{\lambda'} (p_\xi, \dot{p}_\xi)$$

and

$$(p_{\xi+1}, \dot{p}_{\xi+1}) \leq \bar{\mathbf{F}}(\lambda', x, (p_\xi, \dot{p}_\xi)).$$

Clearly, q forces both $\dot{p}_{\xi+1} \preceq^{\lambda'} \dot{p}_\xi$ and $\dot{p}_{\xi+1} \leq \mathbf{F}(\lambda', x, \dot{p}_\xi)$. Thus, q forces $(\dot{p}_\xi)_{\xi < \rho}$ is (λ, x) -strategic. Observe that for each $\xi < \rho$, $q \Vdash_P \dot{p}_\xi \preceq^{\bar{\lambda}} 1_{\dot{Q}}$. Also, q forces that $(\dot{p}_\xi)_{\xi < \rho}$ is $\Delta_1^A(\{x\} \cup \bar{\lambda})$ -definable, whence by lemma 3.5 we have that q forces $(\dot{p}_\xi)_{\xi < \rho}$ is $\bar{\lambda}$ -adequate, as claimed. Thus q also forces that this sequence has a lower bound, for which we may fix a name \dot{q} . By quasi-closure for \dot{Q} in the extension and since for all $\xi < \rho$ we have

$$q \Vdash \dot{p}_\xi \preceq^{\bar{\lambda}} 1_{\dot{Q}},$$

we conclude that for any $\xi < \rho$ we have

$$q \Vdash \dot{q} \preceq^{\bar{\lambda}} \dot{q}_\xi.$$

Thus (q, \dot{q}) is a greatest lower bound of \bar{p} and

$$(q, \dot{q}) \preceq^{\bar{\lambda}} (q, 1_{\dot{Q}}).$$

☺

We now embark on a series of lemmas culminating in the insight that the second forcing of a quasi-closed extension (P_0, P_1) is itself quasi-closed. Thus, we obtain a second proof that $P * \dot{Q}$ is quasi-closed (under the assumptions of the previous lemma). This makes use of the fact that the projection map $\pi_0: P * \dot{Q} \rightarrow P$ is definable. In general, we shall see that we have to assume that the strong projection map from P_1 to P_0 is among the parameters \vec{c} , in the sense of 2.2.

Lemma 3.6. *Assume for $i \in \{0, 1\}$, P_i carries a pre-closure system \mathbf{s}_i above λ_0 and $\mathbf{s}_0 \triangleleft \mathbf{s}_1$. If $\bar{p} = (p_\xi)_{\xi < \rho}$ is a sequence of conditions in P_1 which is (λ, x) -strategic with respect to \mathbf{s}_1 , then $(\pi(p_\xi))_{\xi < \rho}$ is (λ, x) -strategic with respect to \mathbf{s}_0 .*

Proof. Suppose we are given \bar{p} as in the hypothesis. If $\xi < \bar{\xi} < \rho$, since $p_\xi \preceq_1^\lambda p_{\bar{\xi}}$, by 3.1($\triangleleft_c 2$), $\pi(p_\xi) \preceq_0^\lambda \pi(p_{\bar{\xi}})$. Let $\xi < \rho$ be arbitrary. Fix a regular λ' such that $p_{\xi+1} \preceq_1^{\lambda'} p_\xi$ and $p_{\xi+1} \leq \mathbf{F}_1(\lambda, x, p_\xi)$. By 3.1($\triangleleft_c 2$), $\pi(p_{\xi+1}) \preceq_0^{\lambda'} \pi(p_\xi)$ and by 3.1($\triangleleft_c 3$), $\pi(p_{\xi+1}) \leq \mathbf{F}_0(\lambda, x, \pi(p_\xi))$, finishing the proof. ☺

Lemma 3.7. *Assume (P_i, \mathbf{s}_i) , $i \in \{0, 1\}$ are as in lemma 3.6. Further, assume that the strong projection map $\pi: P_1 \rightarrow P_0$ is $\Delta_1^A(\lambda \cup \{x\})$ and that some large enough $L_\mu[A]$ is among the parameters in x , where $L_\mu[A] \supseteq P$. If $\bar{p} = (p_\xi)_{\xi < \rho}$ is a sequence of conditions in P_1 which is (λ, x) -adequate with respect to \mathbf{s}_1 , then $(\pi(p_\xi))_{\xi < \rho}$ is (λ, x) -adequate with respect to \mathbf{s}_0 .*

Proof. By the previous lemma, $(\pi(p_\xi))_{\xi < \rho}$ is (λ, x) -strategic. By assumption $(\pi(p_\xi))_{\xi < \rho}$ is $\Delta_1^A(\{x\} \cup \lambda)$, since having $L_\mu[A] \in x$ available as a parameter bounds the additional quantifier resulting from the projection from \bar{p} to $(\pi(p_\xi))_{\xi < \rho}$. Without it, the natural definition of $(\pi(p_\xi))_{\xi < \rho}$ would be $\Delta_2^A(\lambda \cup \{x\})$. $\textcircled{\smile}$

The following is useful e.g. when we show a condition has legal support. Here lies one of the reasons for asking (C 3).

Lemma 3.8. *Assume (P_i, \mathbf{s}_i) , for $i \in \{0, 1\}$ are as in lemma 3.6. For any $p \in P_1$ and any regular $\lambda \geq \lambda_0$ we have:*

$$(\exists q \in P_0 \quad p \preceq_1^\lambda q) \iff p \preceq_1^\lambda \pi(p) \quad (3.1)$$

Proof. One direction is clear, so say $p \preceq_1^\lambda q$ for some $q \in P_0$. Apply 2.1(C 3): As π is a strong projection, $p \leq \pi(p) \leq q$ and so $p \preceq_1^\lambda \pi(p)$. $\textcircled{\smile}$

The intuition behind definition 3.2 is that P_0 and P_1 are both quasi-closed, not independently of each other, but in a very coherent way. That P_1 is quasi-closed is almost implicit in definition 3.2—it depends on a further assumption about the definability of π (this is responsible for the distinct flavor of quasi-closure, setting it apart from the other axioms of stratification):

Lemma 3.9. *If (P_0, P_1) is a quasi-closed extension above λ_0 and π is $\Delta_1^A(\lambda_0)$, then P_1 is quasi-closed above λ_0 .*

Before we give the proof, note that this assumption on π is not entirely trivial: in an iteration, the canonical projection $\pi: P_\theta \rightarrow P_\iota$ is Δ_0 in the parameter ι ; it is not in general $\Delta_1^A(\lambda_0)$. Also we would like to note in passing that in fact P is quasi-closed exactly if $(\{1_P\}, P)$ is a quasi-closed extension; the same will be true for stratified forcing.

Proof. First check 2.3(C I): let $p \in P_1$ and say $p \preceq_1^{\bar{\lambda}} 1$. By lemma 3.1, $p \preceq_1^{\bar{\lambda}} \pi(p)$, so by 3.2(Ec1) and 3.1(\triangleleft_c 3)

$$\mathbf{F}_1(\lambda, x, p) \preceq_1^{\bar{\lambda}} \mathbf{F}_0(\lambda, x, \pi(p)).$$

By 2.1($\triangleleft_c 2$), we have $\pi(p) \preceq_0^{\bar{\lambda}} 1$, and so

$$\mathbf{F}_0(\lambda, x, \pi(p)) \preceq_0^{\bar{\lambda}} 1.$$

Using 3.1($\triangleleft_c 1$) and the fact that $\preceq_1^{\bar{\lambda}}$ is transitive, we finally conclude

$$\mathbf{F}_1(\lambda, x, p) \preceq_1^{\bar{\lambda}} 1,$$

finishing the proof of 2.3(C I).

It remains to check 2.3(C II), so say $\bar{p} = (p_\xi)_{\xi < \rho}$ is λ -adequate, as witnessed by x . By assumption, π is $\Delta_1^A(\lambda_0)$, so by lemma 3.7, $(\pi(p_\xi))_{\xi < \rho}$ is also (λ, x) -adequate. Since P_0 is quasi-closed, $(\pi(p_\xi))_{\xi < \rho}$ has a greatest lower bound q . Thus, applying 2.3(C II) for $\bar{\lambda} = \lambda$, we conclude that \bar{p} has a greatest lower bound. $\textcircled{\smile}$

The next lemma will be used in 3.12 when we show that if the initial segments of an iterations form a chain of quasi-closed extensions, then the limit is itself a quasi-closed extension. It says that the relation of being a quasi-closed extension is transitive. Let P_0, P_1 and P_2 be pre-orders such that for $i \in \{0, 1\}$, P_i is a strong sub-order of P_{i+1} .

Lemma 3.10. *Say $\pi_1: P_2 \rightarrow P_1$ and $\pi_0: P_2 \rightarrow P_0$ are strong projection maps and π_1 is $\Delta_1^A(\lambda_0)$. If both (P_0, P_1) and (P_1, P_2) are quasi-closed extensions above λ_0 , then (P_0, P_2) is also a quasi-closed extension above λ_0 .*

Proof. Let $(\mathbf{s}_0, \mathbf{s}_1)$ and $(\mathbf{s}_1, \mathbf{s}_2)$ witness that (P_0, P_1) and (P_1, P_2) are quasi-closed extensions.

We now check all the conditions of 3.2 for (P_0, P_2) and $(\mathbf{s}_0, \mathbf{s}_2)$. That \mathbf{s}_2 is a pre-closure system holds by assumption, and that $\mathbf{s}_0 \triangleleft \mathbf{s}_2$ is obvious.

Observe that by 3.1($\triangleleft_c 1$), we don't need to distinguish between $\preceq_0^{\bar{\lambda}}$, $\preceq_1^{\bar{\lambda}}$ and $\preceq_2^{\bar{\lambda}}$ and therefore we drop the subscripts in what follows.

We check 3.2(E_cI): Say $p \in P_2$ and

$$p \preceq^{\bar{\lambda}} \pi_0(p). \tag{3.2}$$

By lemma 3.8, it follows that $p \preceq^{\bar{\lambda}} \pi_1(p)$. Thus

$$\mathbf{F}_2(\lambda, x, p) \preceq^{\bar{\lambda}} \pi_1(\mathbf{F}_2(\lambda, x, p)). \tag{3.3}$$

By 3.1($\triangleleft_c 3$) for (P_1, P_2) , we have

$$\pi_1(\mathbf{F}_2(\lambda, x, p)) = \mathbf{F}_1(\lambda, x, \pi_1(p)), \tag{3.4}$$

Equation (3.2) and 3.1($\triangleleft_c 2$) imply $\pi_1(p) \preceq_1^{\bar{\lambda}} \pi_0(p)$, and so

$$\mathbf{F}_1(\lambda, x, \pi_1(p)) \preceq^{\bar{\lambda}} \pi_0(\mathbf{F}_1(\lambda, x, \pi_1(p))). \quad (3.5)$$

Equations (3.3), (3.4) and (3.5) yield

$$\mathbf{F}_2(\lambda, x, p) \preceq^{\bar{\lambda}} \pi_0(\mathbf{F}_1(\lambda, x, \pi_0(p))).$$

Using 3.1($\triangleleft_c 3$) twice, we have

$$\pi_0(\mathbf{F}_1(\lambda, x, \pi_1(p))) = \mathbf{F}_0(\lambda, x, \pi_0(p)) = \pi_0(\mathbf{F}_2(\lambda, x, p)).$$

So finally,

$$\mathbf{F}_2(\lambda, x, p) \preceq^{\bar{\lambda}} \pi_0(\mathbf{F}_2(\lambda, x, p)).$$

It remains to check 3.2(E_cII). So let $\bar{p} = (p_\xi)_{\xi < \rho}$ be a λ -strategic sequence of conditions in P_2 which is $\Delta_1^A(\bar{\lambda} \cup \{x\})$, and let q_0 be a greatest lower bound of $(\pi_0(p_\xi))_{\xi < \rho}$ as in the hypothesis. Since π_1 is $\Delta_1^A(\bar{\lambda} \cup \{x\})$, the sequence $\bar{q} = (\pi_1(p_\xi))_{\xi < \rho}$ is $\Delta_1^A(\bar{\lambda} \cup \{x\})$ -definable, and by lemma 3.6, it is λ -strategic with respect to \mathbf{s}_1 . Moreover, if it is the case that $\bar{\lambda} > \lambda$, then

$$\forall \xi < \rho \quad p_\xi \preceq^{\bar{\lambda}} \pi_0(p_\xi). \quad (3.6)$$

By 2.1($\triangleleft_c 2$), we have that

$$\forall \xi < \rho \quad \pi_1(p_\xi) \preceq^{\bar{\lambda}} \pi_0(p_\xi). \quad (3.7)$$

Thus \bar{q} satisfies the hypothesis of 3.2(E_cII) for (P_0, P_1) and we may find a greatest lower bound $q_1 \in P_1$ as in the conclusion of 3.2(E_cII) for (P_0, P_1) . In particular,

$$\pi_0(q_1) = q_0. \quad (3.8)$$

Thus \bar{p} satisfies the hypothesis of 3.2(E_cII) for (P_1, P_2) , and so we may find a greatest lower bound q as in the conclusion of 3.2(E_cII). In particular, $\pi_1(q) = q_1$ and so $\pi_0(q) = q_0$. If $\lambda < \bar{\lambda}$, lemma 3.8 and (3.6) yield

$$\forall \xi < \rho \quad p_\xi \preceq^{\bar{\lambda}} \pi_1(p_\xi). \quad (3.9)$$

So finally, as $q \preceq^\lambda \pi_1(q)$ by (3.9), and since $\pi_1(q) = q_1$ and $q_1 \preceq^{\bar{\lambda}} \pi_0(q_1)$ by (3.7), we conclude $q \preceq^{\bar{\lambda}} \pi_0(q)$ by (3.8). $\textcircled{\smile}$

Definition 3.11. Say θ is a limit ordinal, and \bar{Q}^θ is an iteration such that for each $\iota < \theta$, P_ι carries a pre-closure system \mathbf{s}_ι above λ_ι , where the sequence $\bar{\lambda} = (\lambda_\iota)_{\iota < \theta}$ is a non-decreasing sequence of regulars. All of the following definitions are relative to these pre-closure systems and to $\bar{\lambda}$.

1. For a thread p through \bar{Q}^θ , let

$$\text{supp}^\lambda(p) = \{\iota < \theta \mid \lambda < \lambda_\iota \text{ and } \pi_{\iota+1} \not\preceq_{\iota+1}^\lambda \pi_\iota(p)\},$$

and let $\sigma^\lambda(p)$ be the least ordinal σ such that $\text{supp}^\lambda(p) \subseteq \sigma$.

2. Let λ^* be regular such that $\lambda^* \geq \lambda_\iota$ for all $\iota < \theta$. We say P_θ is the λ^* -diagonal support limit of \bar{Q}^θ if and only if P_θ consists of all threads p through \bar{Q}^θ such that for each regular $\lambda \geq \lambda^*$, $\text{supp}^\lambda(p)$ has size less than λ and $\sigma^{\lambda^*}(p) < \theta$.
3. We define the *natural system of relations on P_θ* as follows
 - (a) $p \preceq_\theta^\lambda q \iff \forall \iota < \theta \quad \pi_\iota(p) \preceq_\iota^\lambda \pi_\iota(q)$;
 - (b) $\mathbf{F}_\theta(\lambda, x, p)$ is the thread $(\mathbf{F}_\iota(\lambda, x, \pi_\iota(p)))_{\iota < \theta}$.

We shall see in the proof of theorem 3.12 that under natural assumptions the natural system of relations is a pre-closure system.

4. We say \bar{Q}^θ is a $\bar{\lambda}$ -diagonal support iteration if and only if for any limit $\iota < \theta$, P_ι is the λ_ι -diagonal support limit of \bar{Q}^ι .

Theorem 3.12. *Let \bar{Q}^θ be an iteration such that for each $\iota < \theta$, P_ι carries a pre-closure system \mathbf{s}_ι above the regular cardinal λ_ι , where the sequence $\bar{\lambda} = (\lambda_\iota)_{\iota < \theta}$ is non-decreasing. Moreover, let $\lambda_\theta = \min(\text{Reg} \setminus \sup_{\iota < \theta} \lambda_\iota)$ and assume*

1. For all $\iota < \theta$, $(P_\iota, P_{\iota+1})$ is a quasi-closed extension above $\lambda_{\iota+1}$.
2. If $\bar{\iota} < \theta$ is limit, $\mathbf{s}_{\bar{\iota}}$ is the natural system of relations on $P_{\bar{\iota}}$ above $\lambda_{\bar{\iota}}$ and $\bar{Q}^{\bar{\iota}}$ is a $\bar{\lambda} \upharpoonright \bar{\iota}$ -diagonal support iteration.

Let P_θ be the λ_θ -diagonal support limit of \bar{Q}^θ . Then P_θ is quasi-closed above λ_θ .

In the proof of the theorem, we need the following lemmas 3.13–3.16, showing that the notion of λ -support behaves as we expect. So fix an iteration $\bar{Q}^{\theta+1}$ and pre-closure systems as in the hypothesis of the theorem. These lemmas are somewhat technical but straightforward to show.

Lemma 3.13. *For each regular $\lambda \geq \lambda_\theta$ and $p \in P_\theta$,*

$$\text{supp}^\lambda(p) = \bigcup_{\iota < \theta} \text{supp}^\lambda(\pi_\iota(p)).$$

Proof. First, prove \supseteq : Say ξ is a member of the set on the right. Thus there is some $\iota < \theta$ such that

$$\pi_{\xi+1}(\pi_\iota(p)) \not\preceq^\lambda \pi_\xi(\pi_\iota(p)). \quad (3.10)$$

We consider two cases: first, assume $\iota \leq \xi$. Then $\pi_{\xi+1}(\pi_\iota(p)) = \pi_\iota(p) = \pi_\xi(\pi_\iota(p))$, and so as \preceq^λ is a pre-order, (3.10) is false. Thus this case never occurs, and we can assume $\iota > \xi$. Then $\pi_{\xi+1}(\pi_\iota(p)) = \pi_{\xi+1}(p)$ and $\pi_\xi(\pi_\iota(p)) = \pi_\xi(p)$, so (3.10) is equivalent to $\pi_{\xi+1}(p) \not\preceq^\lambda \pi_\xi(p)$. We infer that $\xi \in \text{supp}^\lambda(p)$. All of the above inferences can be reversed, so \subseteq holds as well. $\textcircled{\smile}$

Lemma 3.14. *If $\lambda, \bar{\lambda}$ are regular such that $\lambda_\theta \leq \lambda \leq \bar{\lambda}$ and $p \preceq^{\bar{\lambda}} q$, then $\text{supp}^\lambda(p) \subseteq \text{supp}^\lambda(q)$.*

Proof. Left to the reader. $\textcircled{\smile}$

Observe though that \supseteq does not necessarily hold: in a two-step iteration $P * \dot{Q}$, we could have $(p, 1_{\dot{Q}}) \preceq^{\bar{\lambda}} (q, \dot{q})$ but $q \not\Vdash_P \dot{q} \preceq^{\bar{\lambda}} 1_{\dot{Q}}$ (say e.g. $p \Vdash \dot{q} = 1_{\dot{Q}}$). In this example we have $\text{supp}^\lambda(p, 1_{\dot{Q}}) = \{0\} \not\supseteq \text{supp}^\lambda(q, \dot{q}) = \{0, 1\}$.

Lemma 3.15. *Fix $\bar{\iota} < \theta$. If $p \in P_\theta$ and $\lambda, \bar{\lambda}$ are regular such that $\lambda_\theta \leq \lambda \leq \bar{\lambda}$ and $p \preceq^{\bar{\lambda}} \pi_{\bar{\iota}}(p)$, then*

$$\text{supp}^\lambda(p) = \text{supp}^\lambda(\pi_{\bar{\iota}}(p)).$$

Proof. A short proof: \supseteq holds by lemma 3.13 and \subseteq is a consequence of lemma 3.14. We also give a direct proof: If $\iota < \bar{\iota}$, $\pi_\iota(\pi_{\bar{\iota}}(p)) = \pi_\iota(p)$, so for such ι ,

$$\iota \in \text{supp}^\lambda(p) \iff \iota \in \text{supp}^\lambda(\pi_{\bar{\iota}}(p)).$$

If $\iota \geq \bar{\iota}$, we have $\pi_{\iota+1}(p) \leq \pi_\iota(p) \leq \pi_{\bar{\iota}}(p)$. By assumption and by 3.1($\triangleleft_c 2$) for $(P_{\iota+1}, P_\theta)$, we have $\pi_{\iota+1}(p) \preceq^{\bar{\lambda}} \pi_{\bar{\iota}}(p)$. So by lemma 3.8, $\pi_{\iota+1}(p) \preceq^\lambda \pi_\iota(p)$. We conclude that for $\iota \geq \bar{\iota}$, we have $\iota \notin \text{supp}^\lambda(p)$. $\textcircled{\smile}$

Lemma 3.16. *Let λ_1 be the maximum of $\text{cf}(\theta)$ and λ_θ . Say $q = (q^\iota)_{\iota < \theta}$ is a thread through \bar{Q}^θ and say there is $w \in P_\theta$ such that for all $\iota < \theta$, $q^\iota \preceq^{\lambda_1} w$. Then q has legal support, i.e. $q \in P_\theta$.*

Proof. Let λ be regular. First consider the case $\lambda_\theta \leq \lambda \leq \lambda_1$. As $q \preceq^{\lambda_1} w$, by lemma 3.14, $\text{supp}^\lambda(q) \subseteq \text{supp}^\lambda(w)$, which satisfies the requirement of diagonal support by assumption. Now say $\lambda > \lambda_1$ and fix a sequence $(\theta(\zeta))_{\zeta < \text{cf}(\theta)}$ which is cofinal in θ . By lemma 3.13

$$\text{supp}^\lambda(q) = \bigcup_{\zeta < \text{cf}(\theta)} \text{supp}^\lambda(q^{\theta(\zeta)}),$$

and by assumption the right hand side is a union over bounded subsets of λ . Thus $\text{supp}^\lambda(q)$ is a bounded subset of λ . $\textcircled{\smile}$

Finally we prove the theorem.

Proof of theorem 3.12. Observe we must make the assumption that $L_\mu[A]$ is large enough so that $\bar{Q}^\theta \in L_\mu[A]$ and x includes the parameters $L_\mu[A]$ and \prec , where \prec is a well-order of $L_\mu[A]$. Moreover, we assume that $\mu \geq \theta^+$ (to make sure we can talk about $\text{cf}(\theta)$ in $L_\mu[A]$).

We will show by induction on θ that for each pair $\iota < \bar{\iota} \leq \theta$, $(P_\iota, P_{\bar{\iota}})$ is a quasi-closed extension. Thus (P_0, P_θ) is a quasi-closed extension and so by lemma 3.9, P_θ is quasi-closed. The inductive hypothesis thus says that for each pair $\iota < \bar{\iota} < \theta$, $(P_\iota, P_{\bar{\iota}})$ is a quasi-closed extension as witnessed by $(\mathbf{s}_\iota, \mathbf{s}_{\bar{\iota}})$. We may assume θ is limit: For if θ is a successor ordinal, $\pi_{\theta-1}^\theta$ is a Δ_1 -definable function and thus by induction hypothesis and lemma 3.10, for any $\iota < \theta$, (P_ι, P_θ) is a quasi-closed extension.

So assume θ is limit and let \mathbf{s}_θ be the natural system of relations on the diagonal support limit P_θ . Fix an arbitrary $\iota^* < \theta$. We show that (P_{ι^*}, P_θ) is a quasi-closed extension witnessed by $(\mathbf{s}_{\iota^*}, \mathbf{s}_\theta)$. By definition of \mathbf{s}_θ , we have $\mathbf{s}_{\iota^*} \triangleleft \mathbf{s}_\theta$. It is straightforward to show that \mathbf{s}_θ is a pre-closure system (as defined in 2.1, p. 20). Take a moment to make sure \mathbf{F}_θ is Δ_1^A : Find a Δ_1^A formula Φ and $\bar{c} = (c_\iota)_{\iota < \theta}$ such that for each $\iota < \theta$ and each $p \in P_\iota$,

$$q = \mathbf{F}_\iota(\lambda, x, p) \iff \Phi(c_\iota, q, \lambda, x, p).$$

There is no need to assume that the \mathbf{F}_ι be uniformly definable in ι since we may always use a universal Δ_1^A -truth predicate¹ (or, in fact we could use a set fragment of each \mathbf{F}_ι as a parameter and recall corollary 2.12). Thus Φ and \bar{c} witness that \mathbf{F}_θ is $\Delta_1^A(\{\bar{c}\})$: for $q = \mathbf{F}_\theta(\lambda, x, p)$ is equivalent to

$$\forall \iota \in \text{dom}(p) \quad \Phi(c_\iota, \pi_\iota(q), \lambda, x, \pi_\iota(p)).$$

Note that $\text{dom}(p) = \theta$, which is not necessarily the same as the support of p . We finish the proof that \mathbf{s}_θ is a pre-closure system by proving 2.1(C 3), as the remaining conditions have similar proofs: Say $p \leq_\theta q \leq_\theta r$ and $p \preceq_\theta^\lambda r$. Fixing an arbitrary $\iota < \theta$, we have $\pi_\iota(p) \leq_\iota \pi_\iota(q) \leq_\iota \pi_\iota(r)$ and $\pi_\iota(p) \preceq_\iota^\lambda \pi_\iota(r)$. Thus, by 2.1(C 3) for P_ι , $\pi_\iota(p) \preceq_\iota^\lambda \pi_\iota(q)$. As $\iota < \theta$ was arbitrary, $p \preceq_\theta^\lambda q$ holds. So as mentioned earlier, the natural system of relations is a pre-closure system.

Being extremely careful, we check 3.2(EcI). Say $p \in P_\theta$ and $p \preceq_\theta^\lambda \pi_{\iota^*}(p)$. For arbitrary $\iota \in [\iota^*, \theta)$, by definition of \preceq_θ^λ —or by 3.1($\triangleleft_c 2$)—we have $\pi_\iota(p) \preceq_\iota^\lambda \pi_{\iota^*}(p)$; by 3.2(EcI) for (P_{ι^*}, P_ι) , it follows that $\mathbf{F}_\iota(\lambda, x, \pi_\iota(p)) \preceq_\iota^\lambda \pi_{\iota^*}(\mathbf{F}_\iota(\lambda, x, \pi_\iota(p)))$. Thus by definition of \mathbf{F}_θ and \preceq_θ^λ ,

$$\mathbf{F}_\theta(\lambda, x, p) \preceq_\theta^\lambda \pi_{\iota^*}(\mathbf{F}_\theta(\lambda, x, p)).$$

¹By the way, this remains true in a class-forcing context.

Now to 3.2(E_cII), the main point of the argument. Say $\bar{p} = (p_\xi)_{\xi < \rho}$ is a $(\lambda, \bar{\lambda}, x)$ -adequate sequence of conditions in P_θ ; then λ and $\bar{\lambda}$ are both regular, $\lambda_\theta \leq \lambda \leq \bar{\lambda}$ and \bar{p} is λ -strategic and $\Delta_1^A(\{x\} \cup \bar{\lambda})$ -definable. Moreover, let q^* be a greatest lower bound of the sequence $(\pi_{\iota^*}(p_\xi))_{\xi < \rho}$. Let

$$\sigma = \sup_{\xi < \rho} \sigma^{\text{cf}(\theta)}(p_\xi).$$

Our first goal is to find a greatest lower bound of $(\pi_\sigma(p_\xi))_{\xi < \rho}$ in P_σ . Say $\text{cf}(\theta) \leq \bar{\lambda}$ and $\lambda < \bar{\lambda}$. Then by assumption, for each $\xi < \rho$ we have $p_\xi \preceq^\lambda \pi_{\iota^*}(p_\xi)$. By lemma 3.15, $\text{supp}^{\text{cf}(\theta)}(p_\xi) \subseteq \iota^*$ for all $\xi < \rho$, and so $\sigma \leq \iota^*$. If, on the other hand $\text{cf}(\theta) \leq \bar{\lambda}$ and $\lambda = \bar{\lambda}$, since \bar{p} is λ -strategic we infer by lemma 3.14 that

$$\sigma \leq \sigma^{\text{cf}(\theta)}(p_0)$$

and so as $\sigma^{\text{cf}(\theta)}(p_0) < \text{cf}(\theta)$ we have $\sigma < \theta$. Thus we can use 3.2(E_cII) for (P_{ι^*}, P_σ) to get a lower bound q^0 of $(\pi_\sigma(p_\xi))_{\xi < \rho}$ such that $\pi_{\iota^*}(q^0) = q^*$. Finally, if $\bar{\lambda} < \text{cf}(\theta)$, as $\rho \leq \bar{\lambda}$ and $\sigma^{\text{cf}(\theta)}(p_\xi) < \text{cf}(\theta)$ for each $\xi < \rho$, we have

$$\sup_{\xi < \rho} \sigma^{\text{cf}(\theta)}(p_\xi) < \text{cf}(\theta)$$

and so $\sigma < \theta$ and thus once more we may use 3.2(E_cII) for (P_{ι^*}, P_σ) to get a lower bound q^0 as in the previous case.

In all three cases, we can assume we have ι' such that

$$\sigma = \sup_{\xi < \rho} \sigma^{\text{cf}(\theta)}(p_\xi) \leq \iota' \tag{3.11}$$

and there is a greatest lower bound $q^0 \in P_{\iota'}$ of $(\pi_{\iota'}(p_\xi))_{\xi < \rho}$. By (3.11) we have

$$\forall \xi < \rho \quad p_\xi \preceq^{\text{cf}(\theta)} \pi_{\iota'}(p_\xi). \tag{3.12}$$

Let $(\theta(\zeta))_{\zeta \leq \text{cf}(\theta)}$ be the \prec -least increasing continuous sequence such that $\theta(0) = \iota'$ and $\theta(\text{cf}(\zeta)) = \theta$. By induction on ζ , we now construct a lower bound $q^{\theta(\zeta)} \in P_{\theta(\zeta)}$ of the sequence $(\pi_{\theta(\zeta)}(p_\xi))_{\xi < \rho}$ for each $\zeta \leq \text{cf}(\theta)$. We have already constructed $q^{\theta(0)}$. Now assume we have $q^{\theta(\zeta)}$ and show how to find $q^{\theta(\zeta+1)}$. Firstly, letting $\bar{\lambda}_1$ denote the maximum of $\bar{\lambda}$ and $\text{cf}(\theta)$, notice $(\pi_{\theta(\zeta+1)}(p_\xi))_{\xi < \rho}$ is $\Delta_1^A(\bar{\lambda}_1 \cup \{x\})$ -definable (as usual, assuming some large enough $L_\mu[A]$ is among the parameters given by x). Observe that if $\lambda < \bar{\lambda}$ or $\bar{\lambda} \leq \text{cf}(\theta)$, by assumption and by (3.12) we have

$$\forall \xi < \rho \quad \pi_{\theta(\zeta+1)}(p_\xi) \preceq^{\lambda_1} \pi_{\theta(\zeta)}(p_\xi). \tag{3.13}$$

We also used ($\prec_c 2$) and lemma 3.8 here. So in this case, the sequence $\{\pi_{\theta(\zeta+1)}(p_\xi) \mid \xi < \rho\}$ is $(\lambda, \bar{\lambda}_1, x)$ -adequate in $(P_{\theta(\zeta+1)}, P_{\theta(\zeta)})$. In the other

case, when $\lambda = \bar{\lambda}$ and $\text{cf}(\theta) < \lambda$, this sequence is $\Delta_1^A(\lambda \cup \{x\})$ -definable and hence by assumption, it is (λ, λ, x) -adequate in $(P_{\theta(\zeta+1)}, P_{\theta(\zeta)})$. In any case, by 3.2(E_cII) we obtain a greatest lower bound $q^{\theta(\zeta+1)} \in P_{\theta(\zeta)}$.

Now let $\zeta \leq \theta$ be limit. By construction and by (E_cII), the $q^{\theta(\zeta')}$, for $\zeta' < \zeta$ form a thread. Define $q^{\theta(\zeta)}$ to be this thread. By construction, for each ζ , we have $q^{\theta(\zeta)} \preceq^{\text{cf}(\theta)} q^{\theta(0)}$. So lemma 3.16 allows us to infer that $q^{\theta(\zeta)}$ has legal support, and thus is a condition in $P_{\theta(\zeta)}$ and a $\pi_{\theta(\zeta)}$ -bound of \bar{p} (in the sense of definition 3.3; we tacitly used fact 1.14 here). The final condition $q^{\theta(\text{cf}(\theta))}$ is a greatest lower bound of $(p_\xi)_{\xi < \rho}$ and for all $\xi < \rho$, $q^{\theta(\text{cf}(\theta))} \preceq^\lambda p_\xi$. Lastly, consider the case that for all $\xi < \rho$, $p_\xi \preceq^{\bar{\lambda}} \pi_{\iota^*}(p_\xi)$. By 3.2(E_cII) and by induction, we have $q^{\theta(\zeta)} \preceq^{\bar{\lambda}} q^{\iota^*}$ for each $\zeta < \text{cf}(\theta)$, and so $q \preceq^{\bar{\lambda}} q^{\iota^*} = \pi_{\iota^*}(q)$. $\textcircled{\smile}$

We conclude this section with an observation about the support of a greatest lower bound of an adequate sequence.

Lemma 3.17. *Say $\bar{p} = (p_\xi)_{\xi < \rho}$ is a (λ, x) -adequate sequence with greatest lower bound p . Then for any regular $\bar{\lambda}$,*

$$\text{supp}^{\bar{\lambda}}(p) \subseteq \bigcup_{\xi < \rho} \text{supp}^{\bar{\lambda}}(p_\xi).$$

Proof. Assume $\iota < \theta$ and $\iota \notin \bigcup_{\xi < \rho} \text{supp}^{\bar{\lambda}}(p_\xi)$. We may assume $\iota < \bar{\lambda}$ (since p has diagonal support). Then as $\pi_{\iota+1}$ is $\Delta_1^A(\bar{\lambda})$, the sequence $(\pi_{\iota+1}(p_\xi))_{\xi < \rho}$ is $\Delta_1^A(\bar{\lambda} \cup \{x\})$ -definable, and for all $\xi < \rho$, $\pi_{\iota+1}(p_\xi) \preceq^{\bar{\lambda}} \pi_\iota(p_\xi)$. Therefore we can apply 3.2(E_cII) applied for $(P_\iota, P_{\iota+1})$ (see 3.2, p. 35). We conclude that $\pi_{\iota+1}(p) \preceq^{\bar{\lambda}} \pi_\iota(p)$ and so $\iota \notin \text{supp}^{\bar{\lambda}}(p)$. $\textcircled{\smile}$

3.2 Stratified extension and iteration

In this section, we show that composition of stratified forcing is a special case of stratified extension. We show that the second forcing in a stratified extension is stratified. Finally we prove an iteration theorem for stratified forcing.

Let P_0 be a complete sub-order of P_1 and let $\pi: P_1 \rightarrow P_0$ be a strong projection. Moreover, assume for $i \in \{0, 1\}$, we have a system

$$\mathbf{S}_i = (\mathbf{F}_i, \preceq_i^\lambda, \preceq_i^\lambda, \mathbf{C}_i^\lambda)_{\lambda \geq \lambda_0}$$

such that $\mathbf{F}: \text{Reg} \setminus \lambda_0 \times V \times P_i \rightarrow P_i$ is a (definable) function, and for every $\lambda \geq \lambda_0$, \preceq_i^λ and \preceq_i^λ are binary relations on P_i and $\mathbf{C}_i^\lambda \subseteq P_i \times \lambda$.

Definition 3.18. We write $\mathbf{S}_0 \triangleleft \mathbf{S}_1$ if and only if in addition to (\triangleleft_c1) , (\triangleleft_c2) and (\triangleleft_c3) (see 3.1, p. 35), the following hold:

- (\triangleleft_s1) If $q, q' \in P_0$ and $p, p' \in P_1$ are such that $q' \preceq_0^\lambda q \leq \pi(p')$ and $p' \preceq_1^\lambda p$, then $q' \cdot p' \preceq_1^\lambda q \cdot p$.
- (\triangleleft_s2) For all $p, q \in P_0$, $p \preceq_0^\lambda q : p \preceq_1^\lambda q$.
- (\triangleleft_s3) For all $p, q \in P_1$, $p \preceq_1^\lambda q : \pi(p) \preceq_0^\lambda \pi(q)$.
- (\triangleleft_s4) If $w \leq \pi(d), \pi(r)$ and $d \preceq^\lambda r$ then $w \cdot d \preceq^\lambda w \cdot r$.
- (\triangleleft_s5) If $\mathbf{C}_1^\lambda(p) \cap \mathbf{C}_1^\lambda(q) \neq 0$ then $\mathbf{C}_0^\lambda(\pi(p)) \cap \mathbf{C}_0^\lambda(\pi(q)) \neq 0$.

Observe that if $\mathbf{S}_0 \triangleleft \mathbf{S}_1$, we can drop the subscripts on $\preceq_0^\lambda, \preceq_1^\lambda$ and just write \preceq^λ without causing confusion. Observe also that by corollary 2.12, we can assume that $r \preceq^{|P_1|} p$ holds exactly if $p = r$. This implies²

$$\forall p \in P_1 (p \preceq^{|P_1|} \pi(p) \iff p \in P_0). \quad (3.14)$$

As 3.2(E_cI) together with (3.14) and 3.1(\triangleleft_c3) imply that $\mathbf{F}_0 = \mathbf{F}_1 \upharpoonright P_0$, we don't have to distinguish between \mathbf{F}_1 and \mathbf{F}_0 and we can just write \mathbf{F} without causing confusion. Moreover, we could also assume that $\mathbf{C}_0^\lambda = \mathbf{C}_1^\lambda \cap P_0 \times \lambda$. For if not, simply replace \mathbf{C}_1^λ by the following relation \mathbf{C}_*^λ : $s \in \mathbf{C}_*^\lambda(p)$ if and only if $s \in \leq^2 \lambda$ such that $s(0) \in \mathbf{C}_0^\lambda(\pi(p))$ and if $p \notin P_0$ then $1 \in \text{dom}(s)$ and $s(1) \in \mathbf{C}_1^\lambda(p)$ (now in fact we get $\mathbf{C}_0^\lambda(p) = \{s \upharpoonright 1 \mid s \in \mathbf{C}_1^\lambda(p)\}$ for $p \in P_0$). To sum up, we could in principle completely eliminate any mention of \mathbf{S}_0 from the definition of stratified extension.³

Replacing (\triangleleft_s1) by the following two conditions yields an equivalent version of the above definition:

- (\triangleleft_sA) $w \preceq_0^\lambda \pi(p) : w \cdot p \preceq_1^\lambda p$.
- (\triangleleft_sB) If $w \leq \pi(p')$ and $p' \preceq^\lambda p$ then $w \cdot p' \preceq^\lambda w \cdot p$.

Sometimes it is more convenient to check both of these rather than (\triangleleft_s1) , which is concise but cumbersome to show. Further notice that (\triangleleft_sB) implies

- (\triangleleft_sb) If $w \leq \pi(p)$ and $p \preceq^\lambda \pi(p)$ then $w \cdot p \preceq^\lambda w$.

²Interestingly, (3.14) also follows just from the assumption that for any $r, p \in P_1$, $r \preceq^{|P_1|} p$, together with (E_sII) *coherent expansion*

³If you want to generalize these notions to class forcing, this will not hold; in that context, \mathbf{F}_0 is not the restriction of \mathbf{F}_1 to P_0 .

This is weaker than $(\triangleleft_s B)$. We note in passing that we could do entirely with $(\triangleleft_s A)$ and $(\triangleleft_s b)$ and without $(\triangleleft_s B)$. Neither condition $(\triangleleft_s 1)$ nor any of its variants were included in 2.1, the definition of a pre-closure system simply because they are not needed to preserve quasi-closure in iterations—rather we need $(\triangleleft_s b)$ to preserve coherent centering, and $(\triangleleft_s A)$ helps to preserve density at limits; see below.

We fix some convenient notation: If $d \leq r$, we say p λ -interpolates d and r to mean that $p \preceq^\lambda d$ and $p \preceq^\lambda r$. We say $p \preceq^{<\lambda} q$ to mean that for all regular λ' such that $\lambda_0 \leq \lambda' < \lambda$, we have $p \preceq^{\lambda'} q$.

Definition 3.19. We say the pair (P_0, P_1) is a *stratified extension above λ_0* , as witnessed by $(\mathbf{S}_0, \mathbf{S}_1)$ if and only if \mathbf{S}_0 witnesses that P_0 is stratified above λ_0 , \mathbf{S}_1 is a pre-stratification system on P_1 and $\mathbf{S}_0 \triangleleft \mathbf{S}_1$; Moreover, for all $\lambda \in \text{Reg} \setminus \lambda_0$ we have that $(E_c I)$, $(E_c II)$ and all of the following conditions hold:

$(E_s I)$ *Coherent Continuity:* Let $\bar{p} = (p_\xi)_{\xi < \rho}$ and $\bar{q} = (q_\xi)_{\xi < \rho}$ be $(\lambda^*, \bar{\lambda}, x)$ -adequate sequences of conditions in P_1 and say $\bar{\lambda} < \lambda$. In other words, let $\lambda^*, \bar{\lambda}$ be regular cardinals such that $\lambda_0 \leq \lambda^* \leq \bar{\lambda} < \lambda$ and assume

- (a) \bar{p} is (λ^*, x) -strategic and $\Delta_1^A(\{x\} \cup \bar{\lambda})$ -definable,
- (b) either $\lambda^* = \bar{\lambda}$ or for each $\xi < \rho$, $p_\xi \preceq_1^{\bar{\lambda}} \pi(p_\xi)$.

Likewise for \bar{q} . Moreover, say \bar{p} has a greatest lower bound p and \bar{q} has a greatest lower bound q , and in case $\lambda^* < \lambda$, we have $\mathbf{C}_0^\lambda(\pi(p)) \cap \mathbf{C}_0^\lambda(\pi(q)) \neq \emptyset$. Then if for each $\xi < \rho$, $\mathbf{C}_1^\lambda(p_\xi) \cap \mathbf{C}_1^\lambda(q_\xi) \neq \emptyset$, we have both $p, q \in \text{dom}(\mathbf{C}_1^\lambda)$ and $\mathbf{C}_1^\lambda(p) \cap \mathbf{C}_1^\lambda(q) \neq \emptyset$.

$(E_s II)$ *Coherent Expansion:* For $p, d \in P_1$, if $p \preceq^\lambda d$, $d \preceq^\lambda \pi(d)$ and $\pi(p) \leq \pi(d)$, we have that $p \leq d$.

$(E_s III)$ *Coherent Interpolation:* Given $d, r \in P_1$ such that $d \leq r$ and $p_0 \in P_0$ such that p_0 λ -interpolates $\pi(d)$ and $\pi(r)$ we can find $p \in P_1$ which λ -interpolates d and r such that $\pi(p) = p_0$. If moreover $\bar{\lambda}$ is regular and $d \preceq^{<\bar{\lambda}} \pi(d)$, we can in addition assume $p \preceq^{<\bar{\lambda}} \pi(p) \cdot r$.

$(E_s IV)$ *Coherent Centering:* Say $d, p \in P_1$, $d \preceq^\lambda p$ and $\mathbf{C}_1^\lambda(d) \cap \mathbf{C}_1^\lambda(p) \neq \emptyset$. Given $w_0 \in P_0$ such that both $w_0 \preceq^{<\lambda} \pi(d)$ and $w_0 \preceq^{<\lambda} \pi(p)$, we can find $w \in P_1$ such that $w \preceq^{<\lambda} p, d$ and $\pi(w) = w_0$.

We find it relieving to notice that P is stratified exactly if $(\{1_P\}, P)$ is a stratified extension. Again, if we don't mention \mathbf{S}_0 , \mathbf{S}_1 or λ_0 we are either claiming that they can be appropriately defined or they can be inferred from the context.

Lemma 3.20. *If P is stratified above λ_0 and $\Vdash_P \dot{Q}$ is stratified above λ_0 , then $(P, P * \dot{Q})$ is a stratified extension above λ_0 .*

*To be more precise, let \mathbf{S}_0 denote the pre-stratification system witnessing that P is stratified and let $\mathbf{S}_1 = (\bar{\mathbf{F}}, \bar{\preceq}^\lambda, \bar{\succ}^\lambda, \bar{\mathbf{C}}^\lambda)_{\lambda \geq \lambda_0}$ be the pre-stratification system constructed as in the proof of 2.14, where we showed that $P * \dot{Q}$ is stratified. Then $(\mathbf{S}_0, \mathbf{S}_1)$ witnesses that $(P, P * \dot{Q})$ is a stratified extension above λ_0 .*

Proof. We have already checked (\triangleleft_c1) , (\triangleleft_c2) , (\triangleleft_c3) , (E_cI) and (E_cII) —i.e. that $(P, P * \dot{Q})$ is a *quasi-closed extension*—in lemma 3.4. We showed that \mathbf{S}_1 is a pre-stratification system when we proved theorem 2.14. It's technical but straightforward to check that $\mathbf{S}_0 \triangleleft \mathbf{S}_1$ (see definition 3.18, p. 45):

Fix $\bar{p} = (p, \dot{p}) \in P * \dot{Q}$ and $w \in P$, $w \leq p$. For (\triangleleft_sA) , say $w \preceq^\lambda p$. Then as $p \Vdash \dot{p} \preceq^\lambda p$, we have $(w, p) \bar{\preceq}^\lambda \bar{p}$. For (\triangleleft_sB) , fix another condition $\bar{q} = (q, \dot{q}) \in P * \dot{Q}$ such that $\bar{p} \bar{\preceq}^\lambda \bar{q}$. Then $p \Vdash \dot{p} \preceq^\lambda \dot{q}$, whence $w \Vdash \dot{p} \preceq^\lambda \dot{q}$ and so $(w, p) \bar{\preceq}^\lambda (w, 1_{\dot{Q}})$, done. For (E_sII) and (\triangleleft_s4) , let $\bar{r} = (r, \dot{r}) \in P * \dot{Q}$ and say $\bar{p} \bar{\succ}^\lambda \bar{r}$, i.e. $p \preceq^\lambda r$ and if $p \cdot r > 0$ then $p \cdot q \Vdash \dot{p} \preceq^\lambda \dot{r}$. To check (E_sII) *coherent expansion*, assume $\bar{r} \bar{\preceq}^\lambda (r, 1_{\dot{Q}})$ and $p \leq r$. Then $p \Vdash \dot{p} \preceq^\lambda \dot{r}$. As P forces *expansion* for \dot{Q} , $p \Vdash \dot{p} \leq \dot{q}$ and we are done with (E_sII) . To check (\triangleleft_s4) , say $w \leq p$. Then $w \cdot r \leq p \cdot r$, and so if $w \cdot r > 0$, it forces $\dot{p} \preceq^\lambda \dot{r}$. Since $w \preceq^\lambda w$, we infer that $(w, \dot{p}) \bar{\preceq}^\lambda \bar{r}$. The remaining (\triangleleft_s2) , (\triangleleft_s3) and (\triangleleft_s5) are immediate by the definition.

Now we check the conditions of 3.19 (see p. 47). For (E_sIII) *coherent interpolation*, just look at how we found an interpolant in the proof of theorem 2.14. Do the same for (E_sIV) *coherent centering*. It remains to check (E_sI) *coherent continuity*. So fix $\bar{p} = (p_\xi, \dot{p}_\xi)_{\xi < \rho}$ and $\bar{q} = (q_\xi, \dot{q}_\xi)_{\xi < \rho}$ with greatest lower bounds (p, \dot{p}) and (q, \dot{q}) respectively, satisfying the hypothesis of (E_sI) *coherent continuity*. Assume

$$\forall \xi < \rho \quad \bar{\mathbf{C}}^\lambda(p_\xi, \dot{p}_\xi) \cap \bar{\mathbf{C}}^\lambda(q_\xi, \dot{q}_\xi) \neq 0.$$

The case $\lambda^* = \bar{\lambda}$ reduces to ordinary continuity for $P * \dot{Q}$ and thus was treated in the proof of 2.14. So assume $\lambda^* < \bar{\lambda}$. By assumption, we have $\chi \in \mathbf{C}^\lambda(p) \cap \mathbf{C}^\lambda(q)$.

Look at the proof of theorem 3.4. On page 36, when we show 3.2(E_cII), we argue that p forces that $(\dot{p}_\xi)_{\xi < \rho}$ is $\bar{\lambda}$ -adequate. Now argue exactly as we did when we showed continuity for $P * \dot{Q}$ in theorem 2.14, p. 30: For each $\xi < \rho$, fix $(T_g^\xi, H_g^\xi, \lambda_g^\xi)$ such that for some χ' ,

$$(\chi', T_g^\xi, H_g^\xi, \lambda_g^\xi) \in \bar{\mathbf{C}}^\lambda(p_\xi, \dot{p}_\xi) \cap \mathbf{C}^\lambda(q_\xi, \dot{q}_\xi).$$

Exactly as we did in the proof just mentioned, we construct T_g, H_g and λ_g such that

$$(\chi, T_g, H_g, \lambda_g) \in \bar{\mathbf{C}}^\lambda(p, \dot{p}) \cap \bar{\mathbf{C}}^\lambda(q, \dot{q}),$$

finishing the proof. $\textcircled{\smile}$

The following is the analogue of 3.9 for quasi-closed extension:

Lemma 3.21. *If (P_0, P_1) is a stratified extension above λ_0 and π is $\Delta_1^A(\lambda_0)$, then P_1 is stratified above λ_0 .*

Proof. The proof is a straightforward consequence of the definition and lemma 3.9. We leave it to the reader. $\textcircled{\smile}$

Definition 3.22. Say \bar{Q}^θ is an iteration such that each initial segment P_ι carries a pre-stratification system \mathbf{S}_ι above λ_ι and let P_θ be its λ_θ -diagonal support limit, where $\lambda_\theta \geq \lambda_\iota$ for each $\iota < \theta$. We now add to the definition of the *natural system of relations on P_θ* . Let $\lambda \geq \lambda_\theta$. The relations \preceq^λ and \mathbf{F} are defined as in 3.11, p. 40. Let

1. $p \preceq_\theta^\lambda q \iff \forall \iota < \theta \quad \pi_\iota(p) \preceq_\iota^\lambda \pi_\iota(q)$;
2. $p \in \text{dom}(\mathbf{C}^\lambda)$ if and only if for all $\iota < \sigma^\lambda(p)$, $\pi_\iota(p) \in \text{dom}(\mathbf{C}_\iota^\lambda)$;
3. $s \in \mathbf{C}_\theta^\lambda(p)$ if and only if $s: \sigma^\lambda(p) \rightarrow \lambda$ and for all $\iota < \text{dom}(s)$, we have $s(\iota) \in \mathbf{C}_\iota^\lambda(p)$.

As before, the above yields a pre-stratification system under natural assumptions, as we shall see in the proof of theorem 3.23.

Theorem 3.23. *Let \bar{Q}^θ be an iteration such that for each $\iota < \theta$, P_ι carries a pre-stratification system \mathbf{S}_ι above λ_ι , where $\bar{\lambda} = (\lambda_\iota)_{\iota < \theta}$ is a non-decreasing sequence of regular cardinals. Moreover, let $\lambda_\theta = \min(\text{Reg} \setminus \sup_{\iota < \theta} \lambda_\iota)$ and assume*

1. *For all $\iota < \theta$, $(P_\iota, P_{\iota+1})$ is a stratified extension above λ_ι .*
2. *If $\bar{\iota} < \theta$ is limit, $\mathbf{S}_{\bar{\iota}}$ is the natural system of relations on $P_{\bar{\iota}}$ and $P_{\bar{\iota}}$ is the $\lambda_{\bar{\iota}}$ -diagonal support limit of $\bar{Q}^{\bar{\iota}}$.*
3. *For each regular $\lambda \geq \lambda_\theta$ there is $\iota < \lambda^+$ such that for all $p \in P_\theta$ we have $\text{supp}^\lambda(p) \subseteq \iota$.*

Let P_θ be the λ_θ -diagonal support limit of \bar{Q}^θ . Then P_θ is stratified above λ_θ .

Remark 3.24. In our particular application we will have that for each regular λ , there is $\iota < \lambda^+$ such that $\lambda < \lambda_\iota$. Observe that by the definition of $\text{supp}^\lambda(p)$, this implies that the last clause of the above is satisfied.

Of course, the following proof can be easily adapted to show that under the same hypothesis, for every $\iota < \theta$, (P_ι, P_θ) is a stratified extension above λ_θ ; while this approach facilitated the inductive proof in the case of quasi-closure, it would serve no purpose in the present context.

Proof of theorem 3.23. By lemma 3.21, we may assume θ is limit. That P_θ is stratified above λ_θ is witnessed by the natural system of relations \mathbf{S}_θ , as defined in 3.22. The proof of the following lemma is a straightforward induction, which we leave to the reader:

Lemma 3.25. *For any $\iota < \bar{\iota} \leq \theta$, $\mathbf{S}_\iota \triangleleft \mathbf{S}_{\bar{\iota}}$.*

Next, we check that \mathbf{S}_θ is a pre-stratification system (see 2.6 p. 22): Conditions (S 1), (S 2) and (S 3) are immediate by the definition of \preceq_θ^λ and the fact that for each $\iota < \theta$, \mathbf{S}_ι is a pre-stratification system. The proofs resemble that of (S II), see below.

The non-trivial condition is 2.6(S 4), *Density*. First we must check that $\text{ran}(\mathbf{C}_\theta^\lambda)$ has size at most λ : this is because by the last assumption of the theorem and by diagonal support, $\text{supp}^\lambda(p) \in [\iota]^{<\lambda}$ for some $\iota < \lambda^+$.

For the more interesting part of the argument, we use density and continuity for the initial segments P_ι , $\iota < \theta$ together with quasi-closure. Observe that by theorem 3.12, for any $\iota < \bar{\iota} \leq \theta$, (P_ι, P_θ) is a quasi-closed extension above λ_θ . Say we are given $p \in P_\theta$. Let $\sigma = \sigma^\lambda(p)$. We may assume that $\sigma = \theta$, for otherwise we can use induction and *Density* for P_σ and are done. Thus we can assume $\lambda_\theta < \lambda$, for otherwise, since P_θ is a diagonal support limit, $\text{supp}^\lambda(p)$ is bounded below θ .

So say we are given $\lambda' \in [\lambda_\theta, \lambda)$. We must find $q \preceq^{\lambda'} p$ such that $q \in \text{dom}(\mathbf{C}^\lambda)$. Let $\delta = \text{cf}(\theta)$ and assume without loss of generality $\lambda' \geq \delta$ (otherwise we may increase λ'). Fix a normal sequence $(\sigma(\xi))_{\xi < \delta}$ such that $\sigma(\delta) = \theta$. We inductively construct a λ' -adequate sequence $(p_\xi)_{\xi < \delta}$ such that $p_0 = p$ and for any ν, ξ such that $\nu < \xi < \delta$,

$$\pi_{\sigma(\nu)}(p_\xi) \in \text{dom}(\mathbf{C}_{\sigma(\nu)}^\lambda). \quad (3.15)$$

As always, first fix a parameter x such that the definition we are about to give is $\Delta_1^A(\lambda \cup \{x\})$. Let $p_0 = p$. Now assume we have p_ξ , we will show how to construct $p_{\xi+1}$. Find $q \in P_{\sigma(\xi)}$ such that $q \preceq^{\lambda'} \pi_{\sigma(\xi)}(\mathbf{F}(\lambda', x, p_\xi))$ and $q \in \text{dom}(\mathbf{C}_{\sigma(\xi)}^\lambda)$, using *Density* for $P_{\sigma(\xi)}$. Set $p_{\xi+1} = q \cdot \mathbf{F}(\lambda', x, p_\xi)$. Since

$\mathbf{S}_{\sigma(\xi)} \triangleleft \mathbf{S}_\theta$, by 3.18($\triangleleft_s A$), $p_{\xi+1} \preceq^{\lambda'} \mathbf{F}(\lambda', x, p_\xi)$. So of course also $p_{\xi+1} \preceq^{\lambda'} p_\xi$. Moreover, by 3.18($\triangleleft_s 5$), for any $\nu \leq \xi$,

$$\pi_{\sigma(\nu)}(p_{\xi+1}) \in \text{dom}(\mathbf{C}_{\sigma(\nu)}^\lambda).$$

At limit stages $\bar{\xi} \leq \delta$, let $p_{\bar{\xi}}$ be a greatest lower bound in P_θ of the sequence constructed so far. It exists by quasi-closure for P_θ . We show

$$\pi_{\sigma(\bar{\xi})}(p_{\bar{\xi}}) \in \text{dom}(\mathbf{C}_{\sigma(\bar{\xi})}^\lambda). \quad (3.16)$$

Let $\nu < \bar{\xi}$ be arbitrary. As $(P_{\sigma(\nu)}, P_\theta)$ satisfies (C II),

$$\pi_{\sigma(\nu)}(p_{\bar{\xi}}) = \prod_{\xi \in (\nu, \bar{\xi})} \pi_{\sigma(\nu)}(p_\xi). \quad (3.17)$$

We want to apply (S I) for $P_{\sigma(\nu)}$. Since we may assume $(\sigma(\xi))_{\xi \leq \delta}$ is $\Delta_1^A(\{x\} \cup \lambda')$, $(\pi_{\sigma(\nu)}(p_\xi))_{\xi \in (\nu, \bar{\xi})}$ is a λ' -adequate sequence; and so all the hypotheses of (S I) for $P_{\sigma(\nu)}$ are satisfied. Now by induction hypothesis, (3.15) holds for all $\xi \in (\nu, \bar{\xi})$ and so by (S I) we have $\pi_{\sigma(\nu)}(p_\xi) \in \text{dom}(\mathbf{C}_{\sigma(\nu)}^\lambda)$. As $\nu < \bar{\xi}$ was arbitrary and by definition of $\mathbf{C}_{\sigma(\bar{\xi})}^\lambda$ we conclude that (3.16) holds. In particular, for the last stage of our construction, we set $\bar{\xi} = \delta$ in (3.16) and conclude $p_\delta \in \text{dom}(\mathbf{C}_\theta^\lambda)$, finishing the proof of *Density*. So \mathbf{S}_θ is a pre-stratification system.

Quasi-closure was shown in lemma 3.12. Now we check conditions (S I)–(S IV) of 2.7, stratification (see p. 23). We defer (S I) *Continuity* to the end. *Expansion* (S II) is trivial: If $d \preceq_\theta^\lambda r$ and $r \preceq_\theta^\lambda 1$, then for all $\iota < \theta$, $\pi_\iota(d) \preceq_\iota^\lambda \pi_\iota(r)$ and $\pi_\iota(r) \preceq_\iota^\lambda 1$. By induction, we may assume *expansion* holds for each P_ι , $\iota < \theta$. Thus $d \leq r$.

We show *interpolation* (S III) holds. So fix $d, r \in P_\theta$ such that $d \leq r$ holds. We construct the interpolant p by induction on its initial segments $p \upharpoonright \iota$, for $\iota < \theta$. Say we have already constructed $p \upharpoonright \iota$. Use coherent interpolation for $(P_\iota, P_{\iota+1})$ to obtain $p \upharpoonright \iota + 1$ interpolating $\pi_{\iota+1}(d)$ and $\pi_{\iota+1}(r)$: demand that

$$p \upharpoonright \iota + 1 \preceq^{<\bar{\lambda}(\iota)} \pi_{\iota+1}(r) \cdot p \upharpoonright \iota, \quad (3.18)$$

where $\bar{\lambda}(\iota)$ is the maximal $\bar{\lambda}$ with the property that $\pi_{\iota+1}(d) \preceq^{<\bar{\lambda}} \pi_\iota(d)$.⁴ We claim that for any $\gamma \in \text{Reg} \setminus \lambda_\theta$,

$$\iota \notin \text{supp}^\gamma(d) \cup \text{supp}^\gamma(r): p \upharpoonright \iota + 1 \preceq^\gamma p \upharpoonright \iota. \quad (3.19)$$

⁴actually, it would suffice to demand this whenever $\iota \geq \sigma^\lambda(d)$

So fix $\gamma \in \text{Reg}$ and assume the hypothesis of (3.19). As $d \upharpoonright \iota + 1 \preceq^\gamma d \upharpoonright \iota$, by 2.1(C 4) and by definition of $\bar{\lambda}(\iota)$, we have $\gamma < \bar{\lambda}(\iota)$. Thus, (3.18) yields

$$p \upharpoonright \iota + 1 \preceq^\gamma \pi_{\iota+1}(r) \cdot p \upharpoonright \iota. \quad (3.20)$$

Since $r \upharpoonright \iota + 1 \preceq^\gamma r \upharpoonright \iota$, by 3.18(\triangleleft_s b) we infer

$$\pi_{\iota+1}(r) \cdot p \upharpoonright \iota \preceq^\gamma p \upharpoonright \iota. \quad (3.21)$$

From (3.20) and (3.21) we get $p \upharpoonright \iota + 1 \preceq^\gamma p \upharpoonright \iota$.

At limit stages $\bar{\iota} \leq \theta$ of the construction of the interpolant p , (3.19) holds for all $\iota < \bar{\iota}$, and so $p \upharpoonright \bar{\iota}$ satisfies the support requirement. This completes the proof of interpolation.

Now for *centering* (S IV). Say $p \preceq^\lambda d$ and fix $s \in \mathbf{C}_\theta^\lambda(p) \cap \mathbf{C}_\theta^\lambda(d)$.

Write σ for $\text{dom}(s)$. By definition of $\mathbf{C}_\theta^\lambda$, $\sigma = \sigma^\lambda(p) = \sigma^\lambda(d)$. First, assume $\sigma = \theta$. In this case, we have $\lambda > \lambda_\theta$ by definition of diagonal support. We construct w by induction on its initial segments $w \upharpoonright \iota$, for $\iota < \sigma$. To start, use *centering* for P_1 to obtain $w \upharpoonright 1$. Assume we have $w \upharpoonright \iota$; just use *coherent centering* for $(P_\iota, P_{\iota+1})$ to obtain $w \upharpoonright \iota + 1$. At limits $\iota \leq \sigma$, use lemma 3.16 and the fact that $\text{cf}(\sigma) < \lambda$ and so $w_0 \upharpoonright \iota \preceq^{\text{cf}(\sigma)} \pi_\iota(d)$.

Secondly, if $\sigma < \theta$, we can use *centering* for P_σ to obtain a lower bound w_0 of $\pi_\sigma(p)$ and $\pi_\sigma(d)$ with the desired properties. We claim that $w = w_0 \cdot d$ is the desired condition, i.e. $w \preceq^{<\lambda} p, d$. The proof is of course by induction on $\iota \leq \theta$. For limit ι , just use the induction hypothesis and the definition of $\preceq_\iota^{<\gamma}$. For the successor case, write

$$\begin{aligned} d^* &= \pi_{\iota+1}(d), \\ p^* &= \pi_{\iota+1}(p), \\ w_0^* &= w_0 \cdot \pi_\iota(d) \end{aligned}$$

and let π denote π_ι . We may assume by induction that $w_0^* \preceq^{<\lambda} \pi(d^*), \pi(p^*)$. In the following, use that $\mathbf{S}_{\iota+1}$ is a pre-stratification system, $\mathbf{S}_\iota \triangleleft \mathbf{S}_{\iota+1}$ and 3.18(E_sII), *coherent expansion*.

Firstly, since $d^* \preceq^\lambda \pi(d^*)$ and $w_0^* \leq \pi(d^*)$, by 3.18(\triangleleft_s b), we have

$$w_0^* \cdot d^* \preceq^\lambda w_0^*. \quad (3.22)$$

In the same way, we can argue that

$$w_0^* \cdot p^* \preceq^\lambda w_0^*. \quad (3.23)$$

Equation (3.22) and $w_0^* \preceq^{<\lambda} \pi(d^*)$ give us $w_0^* \cdot d^* \preceq^{<\lambda} \pi(d^*)$, and together with $w_0^* \cdot d^* \leq d^* \leq \pi(d^*)$ and 2.1(C 3) we infer that $w_0^* \cdot d^* \preceq^{<\lambda} d^*$.

Since $d^* \preceq^\lambda p^*$ and $w_0^* \leq \pi(d^*), \pi(p^*)$, we may conclude by $\mathbf{S}_\iota \triangleleft \mathbf{S}_{\iota+1}$ and 3.18($\triangleleft_s 4$) that

$$w_0^* \cdot d^* \preceq^\lambda w_0^* \cdot p^*.$$

This together with (3.23), by *coherent expansion* 3.18(E_sII) yields

$$w_0^* \cdot d^* \leq w_0^* \cdot p^*.$$

Thus $w_0^* \cdot d^* \leq p^* \leq \pi(p^*)$ while at the same time $w_0^* \cdot d^* \preceq^\lambda w_0^* \preceq^{<\lambda} \pi(p^*)$. Another application of 2.1(C 3) yields $w_0^* \cdot d^* \preceq^{<\lambda} p^*$. This ends the successor step of the inductive proof that $w \preceq^{<\lambda} p, d$, and we are done with coherent centering. Finally, check (S I) *Continuity*: Fix regular λ^*, λ such that $\lambda_\theta \leq \lambda^* < \lambda$. Say \bar{p} and \bar{q} are (λ^*, x) -adequate sequences of length ρ with greatest lower bound p and q respectively. Further, say for each $\xi < \rho$, $\mathbf{C}_\theta^\lambda(p_\xi) \cap \mathbf{C}_\theta^\lambda(q_\xi) \neq \emptyset$. We show that $p \in \text{dom}(\mathbf{C}_\theta^\lambda)$ (and the same of course then holds for q); at the same time we show that $\mathbf{C}_\theta^\lambda(p) \cap \mathbf{C}_\theta^\lambda(q) \neq \emptyset$. Let $\sigma = \sigma^\lambda(p)$, and let $\delta = \text{cf}(\sigma)$.

The proof is simpler if we assume that $\sigma \leq \lambda^*$. Then for each $\iota < \sigma$, π_ι is $\Delta_1^A(\lambda^*)$ and so by lemma 3.7, $(\pi_\iota(p_\xi))_{\xi < \rho}$ is λ^* -adequate. Fix $\iota < \sigma$. For each $\xi < \rho$, we have $\pi_\iota(p_\xi), \pi_\iota(q_\xi) \in \text{dom}(\mathbf{C}_\iota^\lambda)$ and

$$\mathbf{C}_\iota^\lambda(\pi_\iota(p_\xi)) \cap \mathbf{C}_\iota^\lambda(\pi_\iota(q_\xi)) \neq \emptyset,$$

by definition of $\mathbf{C}_\theta^\lambda$ (or by 3.18($\triangleleft_s 5$) for (S_ι, S_θ)). Using *continuity* for P_ι (and the fact that (P_ι, P_θ) is a quasi-closed extension), we infer

$$\mathbf{C}_\iota^\lambda(\pi_\iota(p)) \cap \mathbf{C}_\iota^\lambda(\pi_\iota(q)) \neq \emptyset. \quad (3.24)$$

As ι was arbitrary, (3.24) holds for all $\iota < \sigma$. So by definition of $\mathbf{C}_\theta^\lambda$, (3.24) also holds for $\iota = \theta$. This finishes the proof under the assumption that $\sigma \leq \lambda^*$.

It is easy to generalize the above proof for the case $\delta = \text{cf}(\sigma) \leq \lambda^*$: At the beginning, setting $\delta = \text{cf}(\sigma)$, let $\sigma(\nu)_{\nu < \delta}$ be the \prec -least sequence which is cofinal in σ . Observe that assuming x contains a predicate for a large enough $L_\mu[A]$, we conclude that for each $\nu < \delta$, both $\sigma(\nu)$ and $\pi_{\sigma(\nu)}$ are $\Delta_1^A(\lambda^* \cup \{x\})$. By lemma 3.7, the sequence $(\pi_{\sigma(\nu)}(p_\xi))_{\xi < \rho}$ is (λ^*, x) -adequate for each $\nu < \delta$. Now argue as before.

To finally prove *continuity* (E_sI) in the full, that is without the restriction that $\text{cf}(\sigma) < \lambda^*$, we have to use (E_sI) for initial segments, of course. The argument is now a mixture of the last part of the proof of theorem 3.12 and the argument we just gave.

So now assume $\lambda^* < \text{cf}(\sigma)$ and write $\delta = \text{cf}(\sigma)$. Let $\sigma(\nu)_{\nu < \delta}$ be defined as before, except that we now ask $\sigma(0) = \sigma^\delta(p)$. Since $p \in P_\theta$ we have of course $\delta \leq \theta$, and so as $\sigma^\delta(p) < \delta$ we conclude $\sigma(0) < \theta$.

Fix ν such that $0 < \nu < \delta$. The sequences $(\pi_{\sigma(\nu)}(p_\xi))_{\xi < \rho}$ and $(\pi_{\sigma(\nu)}(q_\xi))_{\xi < \rho}$ satisfy the hypothesis of (E_sI), for $(P_{\sigma(0)}, P_{\sigma(\nu)})$: they are (λ^*, x) -strategic and $\Delta_1^A(\delta \cup \{x\})$ -definable, and for each $\xi < \rho$, both $\pi_{\sigma(\nu)}(p_\xi) \preceq^\delta \pi_\delta(p_\xi)$ and $\pi_{\sigma(\nu)}(p_\xi) \in \text{dom}(\mathbf{C}_{\sigma(\nu)}^\lambda)$. Analogously, for the projection to $P_{\sigma(\nu)}$ of the sequence \bar{q} . By (E_sI) for $(P_\sigma(0), P_{\sigma(\nu)})$, we conclude that (3.24) holds for $\iota = \sigma(\nu)$. As ν was arbitrary, (3.24) holds for cofinally many $\iota < \sigma$. As in the previous case, we conclude that (3.24) holds for $\iota = \theta$, and we are done with the proof of *continuity*. ☺

Corollary 3.26. *Theorem 2.17 holds.*

Proof. By theorem 3.23 and lemma 3.20. ☺

3.3 Products

So far, stratified extension has only given us an overly complicated proof that iterations with stratified components are stratified. Here is a first non-trivial application: as a consequence of the next lemma, one can mix composition and products of stratified forcing freely in iterations with diagonal support, and the resulting iteration will be stratified.

Lemma 3.27. *If P and Q are stratified above λ_0 , $(P, P \times Q)$ is a stratified extension (above λ_0).*

Proof. The proof is entirely as you expect. Fix pre-stratification systems $\mathbf{S}_P = (\mathbf{F}_P, \preceq_P^\lambda, \preceq_P^\lambda, \mathbf{C}_P^\lambda)_{\lambda \geq \lambda_0}$ and $\mathbf{S}_Q = (\mathbf{F}_Q, \preceq_Q^\lambda, \preceq_Q^\lambda, \mathbf{C}_Q^\lambda)_{\lambda \geq \lambda_0}$. We now define a stratification system $\bar{\mathbf{S}} = (\bar{\mathbf{F}}, \bar{\preceq}^\lambda, \bar{\preceq}^\lambda, \bar{\mathbf{C}}^\lambda)_{\lambda \geq \lambda_0}$ on $P \times Q$ in the most natural way: let $\bar{\mathbf{F}}(\lambda, x, (p, q)) = (\mathbf{F}_P(\lambda, x, p), \mathbf{F}_Q(\lambda, x, q))$ and let

$$\begin{aligned} (p, q) \bar{\preceq}^\lambda (\bar{p}, \bar{q}) &\Leftarrow: p \preceq_P^\lambda \bar{p} \text{ and } q \preceq_P^\lambda \bar{q} \\ (p, q) \bar{\preceq}^\lambda (\bar{p}, \bar{q}) &\Leftarrow: p \preceq_P^\lambda \bar{p} \text{ and } q \preceq_P^\lambda \bar{q} \\ s \in \bar{\mathbf{C}}^\lambda(p, q) &\Leftarrow: \left[s \in \mathbf{C}_P^\lambda(p) \text{ and } q \preceq^\lambda 1_Q \right] \text{ or } \left[s = (\chi, \zeta) \right. \\ &\quad \left. \text{where } \chi \in \mathbf{C}_P^\lambda(p) \text{ and } \zeta \in \mathbf{C}_Q^\lambda(q) \right]. \end{aligned}$$

That $\bar{\mathbf{S}}$ is a pre-stratification system requires but a glance at the definitions (see 2.1, p. 20 and 2.6, p. 22). The same holds for $(\triangleleft_c 1)$, $(\triangleleft_c 2)$ and $(\triangleleft_c 3)$ (see p. 45 for the definition of $\mathbf{S}_P \triangleleft \bar{\mathbf{S}}$, and see p. 35 for $(\triangleleft_c 1)$, $(\triangleleft_c 2)$ and $(\triangleleft_c 3)$). For the following, let $(p, q) \in P \times Q$, $w \in P$. For your entertainment,

we check 3.18(\triangleleft_s A) (see page 46). Say $w \preceq_p^\lambda p$. Then clearly $(w, q) \bar{\preceq}^\lambda (p, q)$, done. Now 3.18(\triangleleft_s b): say $w \leq p$ and $(p, q) \bar{\preceq}^\lambda (p, 1_Q)$. This means $q \preceq_Q^\lambda 1_Q$ and so $(w, q) \bar{\preceq}^\lambda (w, 1_Q)$, which is what we wanted to prove.

For the next two conditions, let $\bar{d} = (d, d^*), \bar{r} = (r, r^*) \in P \times Q$ satisfy $\bar{d} \bar{\preceq}^\lambda \bar{r}$. We jump ahead and check (E_sII) of 3.19 (see p. 47): say $d \leq r$ and $\bar{r} \bar{\preceq}^\lambda (r, 1_Q)$. Then $r^* \preceq_Q^\lambda 1_Q$ and $d^* \preceq_Q^\lambda r^*$ by assumption, so by 2.6(S II) for Q , $d^* \leq r^*$ and thus $\bar{d} \leq \bar{r}$.

Let's check (\triangleleft_s 4)). Say $w \leq d$ and $w \leq r$. By 2.6(S 1) for P , $w \preceq_P^\lambda w$ and so $(w, d^*) \bar{\preceq}^\lambda (w, r^*)$. We omit the rest of 3.18 and conclude that $\mathbf{S}_P \triangleleft \bar{\mathbf{S}}$.

The most interesting part of the present proof is that of *quasi-closed extension* (definition 3.2, see p. 21), of which we check (E_cII), leaving (E_cI) to the reader. So say $(p_\xi, q_\xi)_{\xi < \rho}$ is (λ, x) -strategic and $\Delta_1^A(\bar{\lambda} \cup \{x\})$, and $(p_\xi)_{\xi < \rho}$ has a greatest lower bound p . Firstly, since we can assume x contains a parameter X such that $P \subseteq X$, we conclude that $\bar{q} = (q_\xi)_{\xi < \rho}$ is $\Delta_1^A(\bar{\lambda} \cup \{x\})$ (by lemma 3.7). If $\lambda = \bar{\lambda}$, we are done as \bar{q} is λ -adequate and Q is quasi-closed. If on the other hand, $\lambda < \bar{\lambda}$, we have that for all $\xi < \rho$, $q_\xi \preceq_Q^{\bar{\lambda}} 1_Q$. By lemma 3.5, \bar{q} is $\bar{\lambda}$ -adequate. Moreover, if q is a greatest lower bound of \bar{q} , by quasi-closure for Q , we have $q \preceq_Q^{\bar{\lambda}} 1_Q$. So $(p, q) \bar{\preceq}^{\bar{\lambda}} (p, 1_Q)$ and we are done.

To conclude that $(P, P \times Q)$ is a stratified extension, we check the remaining conditions of 3.19 (see p. 47). *Coherent interpolation*, 3.19(E_sIII) and *Coherent centering*, 3.19(E_sIV) are identical to interpolation and centering for Q in this context. *Coherent continuity*, 3.19(E_sI) differs little from the proof of *quasi-closed extension* above, except for an application of continuity for P and Q at the end. ☺

3.4 Stable meets for strong sub-orders

In the next section, we introduce the operation of amalgamation and show that the amalgamation of a stratified forcing P is stratified. In that proof, we must show that a certain dense subset of P is closed under taking meets with conditions from a particular strong sub-order Q (see lemma 4.15, p. 72). This will be facilitated by the so-called Q -stable meet operation $p \wedge_Q r$, which we introduce in the present section. In boolean algebraic terms—and also in most iterations—this is a simple operation (see below). What makes it useful is the following: if r is “a direct extension on the tail $P : Q$ ” of a condition p , then $p \wedge_Q r$ is a *de-iure* direct extension of p , and moreover r can be obtained straightforwardly from $p \wedge_Q r$.

We now give a formal definition of such an operation, and then show that we can always define an operation \wedge on products and compositions. Then we

show how to define \wedge for infinite iterations. In the next section we shall see we also have a stable meet operator for amalgamation. We take this formal, inductive approach (rather than defining \wedge directly on the iteration used in the main theorem) since amalgamation necessarily introduces an element of recursion into the definition of this operation.

Let Q be a strong sub-order of P , and let $\pi: P \rightarrow Q$ be the strong projection. Say we have a tuple of relations $\mathbf{S} = (\dots, \preceq^\lambda, \dots)$ such that $\preceq^\lambda \subseteq P^2$, for $\lambda \in \text{Reg} \setminus \lambda_0$.

Definition 3.28. We call \wedge a *Q-stable meet operator on P* with respect to \mathbf{S} or a *stable meet on (Q, P)* if and only if

1. $\wedge: (p, r) \mapsto p \wedge r$ is a function with $\text{dom}(\wedge) \subseteq P^2$ and $\text{ran}(\wedge) \subseteq P$.
2. $\text{dom}(\wedge)$ is the set of pairs $(p, r) \in P^2$ such that $r \leq p$ and

$$\exists \lambda \in \text{Reg} \setminus \lambda_0 \quad r \preceq^\lambda \pi(r) \cdot p \quad (3.25)$$

3. Whenever $r \leq p$ and $r \preceq^\lambda \pi(r) \cdot p$, the following hold:

$$p \wedge r \preceq^\lambda p \quad (3.26)$$

$$\pi(p \wedge r) = \pi(p) \quad (3.27)$$

$$\pi(r) \cdot (p \wedge r) \approx r \quad (3.28)$$

As usual, if we don't mention \mathbf{S} —or just $(\preceq^\lambda)_{\lambda \in \text{Reg} \setminus \lambda_0}$ —then either it is to be inferred from the context or we mean that an appropriate \mathbf{S} exists.

A few remarks are in order to clarify this definition.

- We certainly don't have $p \wedge r = r \wedge p$.
- The gist of (3.25) is that we try to express that $\pi(r)$ forces that in $P : Q$, the “tail” of r is a direct extension (in the sense of \preceq^λ) of p ; (3.25) captures the essence of this even when $P : Q$ is not stratified.
- Observe that $r \leq p$ implies $\pi(r) \leq \pi(p)$ and so $\pi(r) \cdot p \in P$; thus (3.25) makes sense.
- By $\pi(r) \cdot (p \wedge r) \approx r$ we mean that $\pi(r) \cdot (p \wedge r) \leq r$ and $\pi(r) \cdot (p \wedge r) \geq r$. Admittedly, we are very careful here.

- Observe that there could be more than one map \wedge satisfying the definition. Intuitively, this is because (3.26) is not strong enough to fully determine $p \wedge r$ on $\pi(p) - \pi(r)$. If we add to the above the requirement that $-\pi(r) \cdot (p \wedge r) = p$ hold in $\text{r.o.}(P)$, this uniquely determines \wedge . In fact this entails

$$p \wedge r = r + (p - \pi(r)) \quad (3.29)$$

in $\text{r.o.}(P)$.⁵ For our purposes, this point is moot.

To understand the concept of stable meet operators, it is best to consider an instance of such an operator.

Lemma 3.29. *Say $\bar{P} = Q_0 \times Q_1$, and say for each regular $\lambda \geq \lambda_0$, $\bar{\preceq}^\lambda$ is obtained from \preceq_0^λ and \preceq_1^λ as in the proof of 3.27 (where of course $\preceq_i^\lambda \subseteq (Q_i)^2$). Then there is a stable meet operator on (Q_0, \bar{P}) with respect to $\bar{\preceq}^\lambda$.*

Proof. Let π denote the projection to the first coordinate. Define $\text{dom}(\wedge)$ to be the set of pairs prescribed in definition 3.28. Say $r = (r_0, r_1) \in Q_0 \times Q_1$ and $p = (p_0, p_1) \in Q_0 \times Q_1$ are such that $(r, p) \in \text{dom}(\wedge)$. Define

$$(p_0, p_1) \wedge (r_0, r_1) = (p_0, r_1).$$

As $(r, p) \in \text{dom}(\wedge)$, we can fix λ such that $(r_0, r_1) \bar{\preceq}^\lambda \pi(r) \cdot p = (r_0, p_1)$, and so $r_1 \preceq_1 p_1$. Thus $(p_0, r_1) \bar{\preceq}^\lambda (p_0, p_1)$. To check the other properties is left to the reader. ☺

Lemma 3.30. *Say $\bar{P} = Q * \dot{R}$, and say for each regular $\lambda \geq \lambda_0$, $\bar{\preceq}^\lambda$ is obtained from \preceq^λ and $\dot{\preceq}^\lambda$ as in the proof of 2.14. Then there is a stable meet \wedge on (Q_0, \bar{P}) with respect to $\bar{\preceq}^\lambda$.*

Proof. Let π denote the projection to the first coordinate. Again, define $\text{dom}(\wedge)$ to be the set of pairs prescribed in definition 3.28. Say $\bar{r} = (r, \dot{r})$ and $\bar{p} = (p, \dot{p})$ are such that $(\bar{r}, \bar{p}) \in \text{dom}(\wedge)$. Define $\bar{p} \wedge \bar{r} = (p, \dot{r}^*)$, where \dot{r}^* is such that $r \Vdash \dot{r}^* = \dot{r}$ and $-r \Vdash \dot{r}^* = \dot{p}$. Say we have $(r, \dot{r}) \bar{\preceq}^\lambda \pi(\bar{r}) \cdot \bar{p} = (r, \dot{p})$, and so $r \Vdash \dot{r} \dot{\preceq}^\lambda \dot{p}$. Then $(p, \dot{r}^*) \bar{\preceq}^\lambda (p, \dot{p})$, since $r \Vdash \dot{r}^* = \dot{r} \dot{\preceq}^\lambda \dot{p}$ and $p - r \Vdash \dot{r}^* = \dot{p} \dot{\preceq}^\lambda \dot{p}$. To check the other properties is left to the reader. ☺

The stable meet operator behaves very nicely in iterations:

Lemma 3.31. *Let $\bar{Q}^{\theta+1}$ be an iteration with diagonal support and say for each $\iota < \theta$, P_ι carries a pre-stratification system \mathbf{S}_ι above λ_0 and*

1. For all $\iota < \theta$, we have $\mathbf{S}_\iota \triangleleft \mathbf{S}_{\iota+1}$.

⁵In all the applications we have in mind, the natural definition of \wedge satisfies (3.29)—provided we work with the *separative quotient* of P .

2. If $\bar{\iota} \leq \theta$ is limit, $\mathbf{s}_{\bar{\iota}}$ is the natural system of relations on $P_{\bar{\iota}}$.

Moreover, say for each $\iota < \theta$, there is a stable meet operator $\wedge_{\iota}^{\iota+1}$ on $(P_{\iota}, P_{\iota+1})$ with respect to $\mathbf{S}_{\iota+1}$. Then for each $\iota < \theta$ such that $\iota > 0$ there is a P_{ι} -stable meet operator on P_{θ} .

Proof. By induction on θ , we show that for each pair ι, η such that $0 < \iota < \eta \leq \theta$, there is a stable meet operator \wedge_{ι}^{η} for (P_{ι}, P_{η}) . For ι, η as above and for $p, r \in P$ such that $r \leq p$ and (3.25) hold, define

$$p \wedge_{\iota}^{\eta} r = \prod_{\iota \leq \nu < \eta} \pi_{\nu+1}(p) \wedge_{\nu}^{\nu+1} \pi_{\nu+1}(r). \quad (3.30)$$

We prove by induction on θ that

1. For ι, η such that $0 < \iota < \eta \leq \theta$ and for $(p, r) \in \text{dom}(\wedge_{\iota}^{\theta})$,

$$\pi_{\eta}(p \wedge_{\iota}^{\theta} r) = \pi_{\eta}(p) \wedge_{\iota}^{\eta} \pi_{\eta}(r). \quad (3.31)$$

That is, the sequence of $\pi_{\nu}(p) \wedge_{\iota}^{\nu} \pi_{\nu}(r)$, for $\nu \in (\iota, \theta]$ determines a thread in P_{θ} , in the sense of definition 1.13.

2. For ι and η as above, \wedge_{ι}^{η} is a stable meet operator on (P_{ι}, P_{η}) .

Fix $\iota < \theta$. Let $(p, r) \in \text{dom}(\wedge_{\iota}^{\theta})$ be arbitrary and let λ be an arbitrary witness to (3.25). For the rest of the proof let t_{ν}^{ζ} denote $\pi_{\zeta}(p) \wedge_{\nu}^{\zeta} \pi_{\zeta}(r)$, for $\iota \leq \nu < \zeta \leq \theta$.

First assume θ is limit. By induction hypothesis, $(t_{\eta}^{\theta})_{\eta \in (\iota, \theta)}$ is a thread through \bar{Q}^{θ} ; let it be denoted by \bar{t} . By (3.30) and lemma 1.15, $\bar{t} = p \wedge_{\iota}^{\theta} r = t_{\iota}^{\theta}$. It follows immediately that $(t_{\eta}^{\theta})_{\eta \in (\iota, \theta]}$ is a thread through $\bar{Q}^{\theta+1}$ (see also lemma 1.15). We must show that \bar{t} has legal support. It suffices to show that for each $\gamma \in \text{Reg} \setminus \lambda_0$, $\text{supp}^{\gamma}(\bar{t}) \subseteq \text{supp}^{\gamma}(p) \cup \text{supp}^{\gamma}(r)$. So fix γ as above and $\xi < \theta$ such that we have

$$\pi_{\xi+1}(p) \preceq^{\gamma} \pi_{\xi}(p), \quad (3.32)$$

$$\pi_{\xi+1}(r) \preceq^{\gamma} \pi_{\xi}(r). \quad (3.33)$$

As $r \leq \pi_{\iota}(r) \cdot p \leq \pi(r)$ by assumption, by ($\triangleleft_c 2$) for π_{ξ} and by (C 3) in connection with (3.33) yields $\pi_{\xi+1}(r) \preceq^{\gamma} \pi_{\iota}(r) \cdot \pi_{\xi+1}(p)$. By definition of \bar{t} and since $\wedge_{\iota}^{\xi+1}$ is a stable meet operator, we have

$$\pi_{\xi+1}(\bar{t}) = \pi_{\xi+1}(p) \wedge_{\iota}^{\xi+1} \pi_{\xi+1}(r) \preceq^{\gamma} \pi_{\xi+1}(p).$$

By (3.32) and lemma 3.8, we conclude $\pi_{\xi+1}(\bar{t}) \preceq^{\gamma} \pi_{\xi}(\bar{t})$, i.e. $\xi \notin \text{supp}^{\gamma}(\bar{t})$, finishing the proof that \bar{t} has legal support. It is straightforward to prove

equations (3.26), (3.27) and (3.28) for $t_\iota^\theta = p \wedge_\iota^\theta r$, assuming by induction that for each $\eta < \theta$, \wedge_ι^η is a stable meet operator (t_ι^θ is a thread whose initial segments satisfy these equations). We leave this to the reader.

Now let $\theta = \eta + 1$. To see that $(t_\iota^\nu)_{\nu \leq \theta}$ is a thread, it suffices to show that $\pi_\eta(t_\iota^\theta) = t_\iota^\eta$. In order to show this, observe

$$\pi_\eta(t_\iota^\theta) = t_\iota^\eta \cdot \pi_\eta(t_\eta^\theta) = t_\iota^\eta \cdot \pi_\eta(p) = t_\iota^\eta,$$

where the last equation holds since by induction, $t_\iota^\eta \leq \pi_\eta(p)$. It follows by the induction hypothesis that $(t_\iota^\nu)_{\nu \leq \theta}$ is a thread.

It remains to show that \wedge_ι^θ is a P_ι -stable meet on P_θ , i.e we must show (3.26), (3.27) and (3.28). Firstly, by induction,

$$t_\iota^\eta \preceq^\lambda \pi_\eta(p),$$

and as \wedge_η^θ is a P_η -stable meet on P_θ ,

$$\pi_\eta(p) = \pi_\eta(t_\eta^{\eta+1}).$$

By 3.18($\triangleleft_s A$), this entails

$$t_\iota^\eta \cdot t_\eta^{\eta+1} \preceq^\lambda t_\eta^{\eta+1}.$$

As \wedge_η^θ is a P_η -stable meet on P_θ , we have $t_\eta^{\eta+1} \preceq^\lambda p$, whence $t_\iota^\theta = t_\iota^\eta \cdot t_\eta^{\eta+1} \preceq^\lambda p$, proving (3.26). Secondly,

$$\pi_\iota(t_\iota^\theta) = \pi_\iota(t_\iota^\eta \cdot \pi_\eta(t_\eta^{\eta+1})) = \pi_\iota(t_\iota^\eta) = \pi_\iota(p).$$

The first equality of (3.4) is trivial. The second holds since $t_\iota^\eta \leq \pi_\eta(p)$ by induction hypothesis and since by the assumption that $\wedge_\eta^{\eta+1}$ is a P_η -stable meet, we have $\pi_\eta(p) = \pi_\eta(t_\eta^{\eta+1})$. The last equality of holds by induction. Finally, we prove (3.28). We have

$$\pi_\iota(r) \cdot t_\iota^\theta = \pi_\iota(r) \cdot t_\iota^\eta \cdot t_\eta^{\eta+1} = \pi_\eta(r) \cdot t_\eta^{\eta+1} = r,$$

where the first equation holds by definition, the second by induction hypothesis, and the last one since $\wedge_\eta^{\eta+1}$ is a P_η -stable meet. We are done with the successor case of the induction, and thus with the inductive proof of the lemma. $\textcircled{\smile}$

By the lemma, if \bar{Q}^θ is an iteration as in the hypothesis of the lemma and $\iota < \eta < \theta$, the map \wedge_ι^η is the same as $\wedge_\iota^\theta \upharpoonright (P_\eta)^2$. So as we do for strong projections, we just write \wedge_ι and we speak of the P_ι -stable meet operator (without specifying the domain). Moreover, we can formally set $p \wedge_0 r = r$ and $p \wedge_\iota r = p$ for $\iota \geq \theta$.

3.5 How to obtain a complete sub-order of the tail

Let C be a complete sub-order of P , and say $\pi_C: P \rightarrow C$ is a strong projection. To avoid confusion let the strong projection $\pi: P \rightarrow Q$ be denoted by π_Q for the present discussion. We want to find a sufficient condition to ensure that C is a complete sub-order of $P:Q$, after forcing with Q . In our application C will just be κ -Cohen forcing of L , for κ the least Mahlo. Our iteration will be of the form $P = Q * (\dot{Q}_0 \times C) * \dot{Q}_1$, so after forcing with Q , C is a complete sub-order of $P:Q = (\dot{Q}_0 \times C) * \dot{Q}_1$. We want the same to hold for $\Phi[C]$ (where Φ is a member of a particular family of automorphisms of P which we construct using the technique of amalgamation); this helps to ensure “coding areas” don’t get mixed up by the automorphisms, see lemma 5.4 and lemma 6.2. So we have to introduce a property sufficient for C to be a complete sub-order of $P:Q$, in such a way that this condition is inherited by $\Phi[C]$. For this, we use of course the stratification of P .

Fix a pre-order P which is stratified above λ_0 . The following definition is, as usual, relative to a particular pre-stratification system.

Definition 3.32. We say C is *remote in P over Q (up to height κ)* if and only if for all $c \in C$ and $p \in P$ such that $c \leq \pi_C(p)$, we have

1. $p \cdot c \preceq^\lambda p$ for every $\lambda \in [\lambda_0, \kappa)$;
2. $\pi_Q(p \cdot c) = \pi_Q(p)$.

Observe that if we drop the first clause, this just says that C is independent in P over Q (see definition 1.8).

For a P -name \dot{C} , we say \dot{C} is *remote in P over Q* if and only if it is a name for a generic of a remote complete sub-order of P ; i.e. there is a complete sub-order R_C of P (with a strong projection $\pi_C: P \rightarrow R_C$) such that R_C is dense in $\langle \dot{C} \rangle^{\text{r.o.}(P)}$ and R_C is remote in P over Q .

Lemma 3.33. *If \dot{C} is a P -name which is remote over Q , then \dot{C} is not in V^Q .*

Proof. An immediate consequence of lemma 1.10



Chapter 4

Amalgamation

Amalgamation is a technique to build iterations which admit a homomorphism. We need two types of amalgamations: using type-1 amalgamation, we make sure a stage of our iteration has an automorphism extending an isomorphism of two complete sub-algebras B_0, B_1 of the previous stage of the iteration. Using type-2 amalgamation, we take care that we can extend automorphisms of initial segments (e.g. those created by type-1 amalgamation). The technique presented here differs substantially from that of [She84] (described also in [JR93]) in two important (and related) aspects: firstly, it has a “full support” flavour rather than a “finite support” flavour; secondly, additional fine tuning was needed to allow for amalgamation to preserve stratification (most instances are discussed in detail below).

In 4.1, we define the forcing $P_f^{\mathbb{Z}}$ which will be put to use when we define either type of amalgamation. Before issuing this definition, we pause to analyze $P_f^{\mathbb{Z}}$ and find that it can be decomposed as a product after forcing with B_0 (section 4.2). In section 4.3 we define type-1 amalgamation (denoted by \mathbf{Am}_1) and show it preserves stratification, and in section 4.4 we do the same for type-2 amalgamation (denoted by \mathbf{Am}_2). In the last section, we construct a stable meet operator for amalgamation and discuss remote suborders.

4.1 Basic amalgamation

Let P be a forcing, Q a complete sub-order of P such that $\pi: P \rightarrow Q$ is a strong projection (see 1.6, p. 8 and the preceding discussion). For $i \in \{0, 1\}$, let \dot{B}_i be a Q -name such that $\Vdash_Q \dot{B}_i$ is a complete sub-algebra of $P : Q$. Moreover, say we have a Q -name \dot{f} such that $\Vdash_Q \dot{f}: \dot{B}_0 \rightarrow \dot{B}_1$ is an isomorphism of Boolean algebras.

Our task is to find P' containing P as a complete sub-order, carrying an automorphism $\Phi: P' \rightarrow P'$ which extends the isomorphism of \dot{B}_0 and \dot{B}_1 (in the extension by Q) and which is trivial on Q . Moreover, we want to preserve stratification: if (Q, P) is a stratified extension above λ_0 , we want λ_1 (possibly strictly) greater than λ_0 such that (P', P) is a stratified extension above λ_1 .

We first make some observations: Let $\text{r.o.}(Q) * \dot{B}_i$ be denoted by B_i . This is a complete sub-algebra of $B = \text{r.o.}(P)$, consisting Q -names (or if you prefer, $\text{r.o.}(Q)$ -names) \dot{b} such that $1_Q \Vdash_Q \dot{b} \in \dot{B}_i$. Keep in mind that we can canonically identify the partial order $Q * (\dot{B}_i \setminus \{0\})$ with the set of $b \in B_0$ such that $\pi_Q(b) \in Q$. Also, don't confuse this with the set of $b \in B_0$ such that $\pi_Q(b) = 1$ —or, equivalently, $1_Q \Vdash_Q [b]_{\dot{G}} > 0$, which is called the term-forcing, usually denoted by $(\dot{B}_i \setminus \{0\})^Q$.

Let π_i denote the canonical projection from P to B_i . Then π_i coincides with π on Q (by 1.4). Moreover, \dot{f} can be viewed as an isomorphism f of B_0 and B_1 (mapping names to names). We have

$$\pi \circ \dot{f} = f \circ \pi = \pi. \quad (4.1)$$

In fact, for any pair of sub-algebras B_0, B_1 of $\text{r.o.}(P)$ such that $Q \subseteq B_0 \cap B_1$ and an isomorphism $f: B_0 \rightarrow B_1$, equation (4.1) holds if and only if f generates an isomorphism of the pair $Q * (B_i : Q)$, $i \in \{0, 1\}$. Thus instead of starting with \dot{f} and \dot{B}_0, \dot{B}_1 as in the first paragraph, we could also have started with f, B_0 and B_1 as above, satisfying (4.1).

In a first step, we define $P_f^{\mathbb{Z}}$, the amalgamation of P over f . $P_f^{\mathbb{Z}}$ contains P as a complete sub-order and has an automorphism Φ extending f .

Remark 4.1. If we want to preserve stratification of P , we have to be more careful: we must carefully pick a dense subset \mathbf{D} of P , such that $P' = \mathbf{D}_f^{\mathbb{Z}}$ is stratified. The partial order $\mathbf{D}_f^{\mathbb{Z}}$ is in general not equivalent to $P_f^{\mathbb{Z}}$, but solves the problem described in the first paragraph. Finally, we will define a forcing \mathbf{Am}_1 which is equivalent to $\mathbf{D}_f^{\mathbb{Z}}$, and moreover (P, \mathbf{Am}_1) is a stratified extension. Let's postpone these complications, and first look at $P_f^{\mathbb{Z}}$.

Amalgamation is not a canonical operation. Firstly, if D is a dense subset of P , we cannot infer that $D_f^{\mathbb{Z}}$ is dense in $P_f^{\mathbb{Z}}$. This in combination with the fact that stratification is also not canonical is the main obstacle in this proof. Secondly, even the weaker statement fails: if $\text{r.o.}(P) = \text{r.o.}(R)$, we cannot conclude $\text{r.o.}(P_f^{\mathbb{Z}}) = \text{r.o.}(R_f^{\mathbb{Z}})$.

Without precautions, we cannot even preclude $P_f^{\mathbb{Z}} = \emptyset$, although this pathology does not arise if we ask $B_0 \cup B_1 \subseteq P$. On the other hand, we cannot simply work with $\text{r.o.}(P)$; for although $\text{r.o.}(P)$ has a dense stratified

subset (namely P), this doesn't mean that $\text{r.o.}(P)_f^{\mathbb{Z}}$ will have a dense stratified subset. Therefore, we want to stick as closely to P as possible, but still have $B_0, B_1 \subseteq P$, so we define a ‘‘hybrid’’:

Definition 4.2. Consider the set $P \times B_0 \times B_1$, i.e. the set of triples $(p, \dot{b}^0, \dot{b}^1)$ where $p \in P$ and $\Vdash_Q \dot{b}^i \in \dot{B}_i$ for $i \leq 2$. Order this set by $(p, \dot{b}^0, \dot{b}^1) \leq (p', \dot{b}'^0, \dot{b}'^1)$ if and only if $p \leq p'$ and $p \cdot b_0 \cdot b_1 \leq p' \cdot b'_0 \cdot b'_1$ in $\text{r.o.}(P)$. We call $\widehat{P} = \widehat{P}(Q, f)$ the set of $(p, \dot{b}^0, \dot{b}^1) \in P \times B_0 \times B_1$ such that

$$\pi(p) \Vdash p \cdot \dot{b}^0 \cdot \dot{b}^1 \neq 0, \quad (4.2)$$

or equivalently,

$$\pi(p \cdot \dot{b}^0 \cdot \dot{b}^1) = \pi(p). \quad (4.3)$$

For $\hat{p} \in \widehat{P}$, when we refer to the components of \hat{p} , we use the notation $\hat{p} = (\hat{p}^P, \hat{p}^0, \hat{p}^1)$. When appropriate, we identify \hat{p} with $\hat{p}^P \cdot \hat{p}^0 \cdot \hat{p}^1$, i.e. the meet of the components in $\text{r.o.}(P)$. In particular, if g is a function such that $\text{dom}(g) = \text{r.o.}(P)$, we write $g(\hat{p})$ for $g(\hat{p}^P \cdot \hat{p}^0 \cdot \hat{p}^1)$.

Clearly, P is isomorphic to the subset of \widehat{P} where the two latter components are equal to $1_{\text{r.o.}(P)}$, and this set is in turn dense in \widehat{P} . So P can be considered a dense subset of \widehat{P} . Thus, the separative quotient of \widehat{P} is the completion under \cdot of $P \cup B_0 \cup B_1$ in $\text{r.o.}(P)$ (leaving aside the 0 element). Observe, moreover, that if $D \subseteq P$ is dense in P , then $\{\hat{p} \in \widehat{P} \mid \hat{p}^P \in D\}$ is the same as \widehat{D} , and we shall often use this fact tacitly. Lastly, observe that

$$\hat{p} \leq \hat{q} \iff [\hat{p}^P \leq \hat{q}^P \text{ and } \pi_j(\hat{p}) \leq \pi_j(\hat{q}) \text{ for } j \in \{0, 1\}] \quad (4.4)$$

and $\hat{p} \approx (\hat{p}^P, \pi_0(\hat{p}), \pi_1(\hat{p}))$.¹ These two observations together would make for an equivalent, more strict definition of \widehat{P} , yielding separative \widehat{P} provided P is separative. Notwithstanding, we find the current definition more convenient—if less elegant. In the following, we identify P with $\{\hat{p} \in \widehat{P} \mid \hat{p}^0 = \hat{p}^1 = 1\}$.

Much of the following would work if we replace (4.3) by the weaker $p \cdot \dot{b}_0 \cdot \dot{b}_1 \neq 0$. The advantage of asking (4.3) is that it makes the projection $\bar{\pi}: P_f^{\mathbb{Z}} \rightarrow P$ take a simple form. Also, when we show the amalgamation of a stratified forcing is stratified, we need to apply \mathbf{F} to every coordinate; if we allow $\pi(\bar{p}(i)) < \pi(\bar{p}(i)^P)$, we don't know if $\mathbf{F}(\dots, \bar{p}(i)^P)$ and $\pi(\bar{p}(i))$ are even compatible; this seems to make it impossible to define an operator analogous to \mathbf{F} on the amalgamation.

¹We may regard $(\hat{p}^P, \pi_0(\hat{p}), \pi_1(\hat{p}))$ the canonical representative of \hat{p} if P is separative.

Definition 4.3. We define $P_f^{\mathbb{Z}}$ to consist of all sequences $\bar{p} : \mathbb{Z} \rightarrow \widehat{P}$ that satisfy

$$f(\pi_0(\bar{p}(k+1)^P \cdot \bar{p}(k+1)^0 \cdot \bar{p}(k+1)^1)) = \pi_1(\bar{p}(k)^P \cdot \bar{p}(k)^0 \cdot \bar{p}(k)^1),$$

or, simply

$$f(\pi_0(\bar{p}(k+1))) = \pi_1(\bar{p}(k)). \quad (4.5)$$

for all $k \in \mathbb{Z}$. The ordering on $P_f^{\mathbb{Z}}$ is given by $\bar{r} \leq \bar{p}$ if and only if for all k , $\bar{r}(k) \leq \bar{p}(k)$ in \widehat{P} . We define a map $\Phi : P_f^{\mathbb{Z}} \rightarrow P_f^{\mathbb{Z}}$ by:

$$\Phi(\bar{p})(i) = \bar{p}(i+1) \text{ for } i \in \mathbb{Z}.$$

Obviously, Φ is one-to-one and onto, and $\Phi(\bar{p}) \leq \Phi(\bar{q}) \iff \bar{p} \leq \bar{q}$.

Observe that (4.1) together with (4.5) and (4.3) imply that for all $i \in \mathbb{Z}$,

$$\pi(\bar{p}(i)) = \pi(\bar{p}(0)) = \pi(\bar{p}(0)^P). \quad (4.6)$$

Let $F : \widehat{P} \rightarrow B_1$ be defined by $F(x) = f(\pi_0(x))$ and let $G : \widehat{P} \rightarrow B_0$ be defined by $G(x) = f^{-1}(\pi_1(x))$.

It may seem more natural to replace (4.5) by the weaker requirement that $f(\pi_0(p(k+1)))$ and $\pi_1(p(k))$ be compatible; however, I'm not sure how to show P is a complete sub-order in this case. Moreover, we need (4.6) to be able to even define $\bar{\mathbf{F}}$ witnessing that the amalgamation is quasi-closed when P is (see the discussion preceding definition 4.9).

We now define a complete embedding $e : \widehat{P} \rightarrow P_f^{\mathbb{Z}}$ and a strong projection $\bar{\pi} : P_f^{\mathbb{Z}} \rightarrow \widehat{P}$. For $\hat{u} \in \widehat{P}$ define $e(\hat{u}) : \mathbb{Z} \rightarrow \widehat{P}$ by

$$e(\hat{u})(i) = \begin{cases} (\pi(\hat{u}^P), G^i(\hat{u}), 1) & \text{for } i > 0, \\ \hat{u} & \text{for } i = 0, \\ (\pi(\hat{u}^P), 1, F^i(\hat{u})) & \text{for } i < 0. \end{cases}$$

For $\bar{p} \in P_f^{\mathbb{Z}}$, define $\bar{\pi}(\bar{p}) \in \widehat{P}$ by $\bar{\pi}(\bar{p}) = \bar{p}(0)$.

Lemma 4.4. *The map $\bar{\pi}$ is a strong projection, that is: if $\hat{w} \leq \bar{\pi}(\bar{q})$ in \widehat{P} , we may find $e(\hat{w}) \cdot \bar{q} \in P_f^{\mathbb{Z}}$.*

Proof. Let $\hat{w} \leq \bar{\pi}(\bar{p})$. We define \bar{w} by induction, as follows:

$$\bar{w}(0) = \hat{w}$$

Assume $\bar{w}(i) \in \widehat{P}$ has already been defined. We know $\pi(\bar{w}(i)) = \pi(\bar{w}(i)^P)$. Assume by induction that $\pi(\bar{w}(i)^P) = \pi(\hat{w}^P)$. Also, assume by induction

that $\bar{w}(i) \leq \bar{p}(i)$ and $\bar{w}(i) \leq e(\hat{w})(i)$ in \widehat{P} . To inductively define \bar{w} on the positive integers, assume $i \geq 0$ and define:

$$\bar{w}(i+1) = (\pi(\hat{w}) \cdot \bar{p}(i+1)^P, \bar{p}(i+1)^0, \bar{p}(i+1)^1 \cdot F(\bar{w}(i))).$$

The definition of \bar{w} on the negative integers is also by induction. Assuming $i \leq 0$, we set:

$$\bar{w}(i-1) = (\pi(\hat{w}) \cdot \bar{p}(i-1)^P, \bar{p}(i-1)^0 \cdot G(\bar{w}(i)), \bar{p}(i-1)^1)$$

For $i \geq 0$, as $\bar{w}(i) \leq \bar{p}(i)$, we have

$$\begin{aligned} f(\pi_0(\bar{w}(i))) &= F(\bar{w}(i)) \cdot f(\pi_0(\bar{p}(i))) \\ &= F(\bar{w}(i)) \cdot \pi_1(\bar{p}(i+1)) \\ &= \pi_1(\pi(\hat{w}^P) \cdot \bar{p}(i+1) \cdot F(\bar{w}(i))) \end{aligned}$$

where the second equation holds as (4.5) holds for \bar{p} , and the last equation follows from $F(\bar{w}(i)) \leq \pi(\bar{w}(i)) = \pi(\hat{w}^P)$. We conclude, by definition of $\bar{w}(i+1)$, that

$$f(\pi_0(\bar{w}(i))) = \pi_1(\bar{w}(i+1)). \quad (4.7)$$

Applying π to (4.7), we see $\pi(\bar{w}(i+1)) = \pi(\bar{w}(i))$, and so

$$\begin{aligned} \pi(\bar{w}(i+1)) &= \pi(\hat{w}^P) = \\ \pi(\pi(\hat{w}^P) \cdot \bar{p}(i+1)^P) &= \pi(\bar{w}(i+1)^P), \end{aligned}$$

where the first equation follows from the induction hypothesis and the second follows from

$$\pi(\hat{w}^P) \leq \pi(\bar{p}(0)^P) = \pi(\bar{p}(i+1)^P).$$

Thus, $\bar{w}(i+1) \in \widehat{P}$, $\pi(\bar{w}(i)) = \pi(\hat{w}^P)$ and by construction, both $\bar{w}(i+1) \leq \bar{p}(i+1)$ and $\bar{w}(i+1) \leq e(\hat{w}^P)(i+1)$ hold.

Replacing F by G in the above, we obtain a similar argument for the inductive step from $i \leq 0$ to $i-1$; we leave the details to the reader. Finally we have that $\bar{w}(i) \in \widehat{P}$ and (4.7) holds for all $i \in \mathbb{Z}$, whence $\bar{w} \in P_f^{\mathbb{Z}}$. We have already shown $\bar{w} \leq \bar{p}$ and $\bar{w} \leq e(\hat{w})$.

We now show $\bar{w} \geq e(\hat{w}) \cdot \bar{p}$: Say $\bar{r} \in P_f^{\mathbb{Z}}$ such that $\bar{r} \leq e(\hat{w}) \cdot \bar{p}$. Clearly $\bar{r}(0) \leq \bar{w}(0) = w$. Now assume by induction that $\bar{r}(i) \leq \bar{w}(i)$. Then by (4.5),

$$\bar{r}(i+1) \leq \pi_1(\bar{r}(i+1)) \leq F(\bar{w}(i))$$

so as $\bar{r}(i+1) \leq \bar{p}(i+1)$, we have $\bar{r}(i+1) \leq \bar{w}(i+1)$.

A similar argument shows $\bar{r}(i-1) \leq \bar{w}(i-1)$, so we've shown by induction that $\bar{r} \leq \bar{w}$. So finally, $\bar{w} = e(\hat{w}) \cdot \bar{p}$. $\textcircled{\smile}$

For $i \in \mathbb{Z}$, we write e_i for $\Phi^i \circ e$ and $\bar{\pi}_i$ for $\bar{\pi} \circ \Phi^i$.

Corollary 4.5. *For each $i \in \mathbb{Z}$, the map e_i is a complete embedding of \widehat{P} into $P_f^{\mathbb{Z}}$. It is well-defined and injective on the separative quotient of \widehat{P} . The map $\bar{\pi}_i: P_f^{\mathbb{Z}} \rightarrow \widehat{P}$ is a strong projection. The map $e_i \upharpoonright P$ is a complete embedding of P into $P_f^{\mathbb{Z}}$. Letting $R = \{\bar{p} \in P_f^{\mathbb{Z}} \mid \bar{p}(i)^0 = \bar{p}(i)^1 = 1\}$, R is dense in $P_f^{\mathbb{Z}}$, we have $e_i[P] \subseteq R$ and $\bar{\pi}_i \upharpoonright R: R \rightarrow P$ is a strong projection.*

Proof. The first claim is an obvious corollary of the lemma. The rest follows straightforwardly from elementary properties of e and $\bar{\pi}$. \smile

From now on, we identify \widehat{P} with $e[\widehat{P}]$ and accordingly P with $\{e(p, 1, 1) \mid p \in P\}$.

Corollary 4.6. *Φ is an automorphism of $P_f^{\mathbb{Z}}$ extending f .*

Proof. Let $b \in B_0$. We may assume $\pi(b) \in Q$ (this holds for a dense set of conditions in B_0). Thus $b \in \widehat{P}$ (to be precise, we should write $(\pi(b), b, 1)$ instead of b). Now as $F^n(f(b)) = F^{n+1}(b)$ and $G^{n+1}(f(b)) = G^n(b)$,

$$\begin{aligned} \Phi(e(b)) &= \Phi((\dots, G^2(b), G(b), \overset{0}{\downarrow} b, f(b), F^2(b), \dots)) = \\ &(\dots, G^2(b), G(b), b, \overset{0}{\downarrow} f(b), F^2(b), \dots) = e(f(b)) \end{aligned}$$

So since Φ and f agree on a dense set of conditions in B_0 , they are equal on B_0 . \smile

4.2 Factoring the amalgamation

Interestingly, we can factor the amalgamation over a generic for B_0 . We will put this to use when we investigate the tail $\mathbf{Am}_1: P$. In particular, it enables us to show that if \dot{r} is a P -name which is unbounded over V^Q , $\Phi(\dot{r})$ will be unbounded not just over V^Q but over V^P . This will play a crucial role in the proof of the main theorem, ensuring that when we make the set without the Baire property definable, the coding (ensuring its definability) doesn't conflict with the homogeneity afforded by the automorphisms. The main point of the present section is lemma 4.8; it is used in section 5.3 on p. 95, to prove lemma 5.4. This is in turn used in section 6 to prove the crucial lemma 6.2.

For an interval $I \subseteq \mathbb{Z}$, let P_f^I be the set of $\bar{p}: I \rightarrow \widehat{P}$ such that whenever both $k \in I$ and $k+1 \in I$, (4.5) holds. In other words

$$P_f^I = \{\bar{p} \upharpoonright I \mid \bar{p} \in P_f^{\mathbb{Z}}\}.$$

It is clear that for each $k \in I$, the map $e_k^I: \widehat{P} \rightarrow P_f^I$, defined by $e_k^I(p) = e_k(p) \upharpoonright I$ is a complete embedding. Similarly, there is a strong projection $\pi_k^I: P_f^I \rightarrow \widehat{P}$.

Lemma 4.7. *Let $G_0 = G_Q * H_0$ be $Q * \dot{B}_0$ -generic. Then in $V[G_0]$, there is a dense embedding of $P_f^{\mathbb{Z}}: G_0$ into*

$$[P_f^{(-\infty, 0]} : (G_Q * H_0)] \times [P_f^{[1, \infty)} : (G_Q * f[H_0])]$$

and another one into

$$[P_f^{(-\infty, -1]} : (G_Q * f^{-1}[H_0])] \times [P_f^{[0, \infty)} : (G_Q * H_0)].$$

Proof. We only show how to construct the first embedding; the second part of the proof is only different in notation. Let R_0 denote $P_f^{(-\infty, 0]}$ and R_1 denote $P_f^{[1, \infty)}$, let $H_1 = f[H_0]$ and $G_1 = G_Q * H_1$. In V , let S denote the obvious map $S: P_f^{\mathbb{Z}} \rightarrow R_0 \times R_1$: $S(\bar{p})_0 = \bar{p} \upharpoonright (-\infty, 0]$ and $S(\bar{p})_1 = \bar{p} \upharpoonright [1, \infty)$.

Let $S^* = S \upharpoonright (P_f^{\mathbb{Z}} : G_0)$. We show that the range of S^* is dense in $(R_0 : G_0) \times (R_1 : G_1)$. Since $S(\bar{p}) \leq S(\bar{q}) \iff \bar{p} \leq \bar{q}$, this implies that S^* is injective on the separative quotient of its domain and thus is a dense embedding.

To show that $\text{ran}(S^*)$ is dense, let \bar{p}_0, \bar{p}_1 be given such that $\bar{p}_i \in R_i : G_i$, for $i \in \{0, 1\}$. Fix $i \in \{0, 1\}$ for the moment. Without loss of generality, $\bar{p}_i(i) \in P$ (and not just in \widehat{P}). Let $b_i = \pi_i(\bar{p}_i(i)) \in \dot{B}_i^{G_Q}$. Then as $\bar{p}_i \in R_i : G_i$, $b_i \in H_i$. Find $q \in G_Q$ and a Q -name \dot{b} such that $q \leq \pi_Q(\bar{p}_0), \pi_Q(\bar{p}_1)$ and q forces that

$$\dot{b} = b_0 \cdot f^{-1}(b_1) > 0 \quad \text{in } (\dot{B}_0)^{G_Q}. \quad (4.8)$$

We have that $q \Vdash \dot{b} \cdot \bar{p}_0(0) \neq 0$ and $f(\dot{b}) \cdot \bar{p}_1(1) \neq 0$, or in other words, $q \cdot \dot{b} \in \widehat{P}$, $q \cdot \dot{b} \leq \bar{\pi}(\bar{p}_0)$ and $q \cdot f(\dot{b}) \leq \bar{\pi}(\bar{p}_1)$. So we can define $\bar{p}_0^* = e(q \cdot \dot{b}) \cdot \bar{p}_0$ and $\bar{p}_1^* = e_1(q \cdot f(\dot{b})) \cdot \bar{p}_1$. As

$$q \in G_Q \text{ and } \dot{b}^{G_Q} \in H_0, \quad (4.9)$$

we have $\bar{p}_0^* \in R_0 : G_0$ and $\bar{p}_1^* \in R_1 : G_1$. Define \bar{p}^* :

$$\bar{p}^* = (\dots, \bar{p}_0^*(-1), \bar{p}_0^*(0), \bar{p}_1^*(1), \bar{p}_1^*(2), \dots)$$

Then $\pi_Q(\bar{p}^*) = q$ and by (4.8), q forces that the following hold in $(\dot{B}_0)^{G_Q}$:

$$\begin{aligned} f(\pi_0(\bar{p}_0^*(0))) &= f(\pi_0(q \cdot \bar{p}_0(0) \cdot \dot{b})) = f(\dot{b}) \\ \pi_1(\bar{p}_1^*(1)) &= \pi_1(q \cdot \bar{p}_1(1) \cdot f(\dot{b})) = f(\dot{b}). \end{aligned}$$

Thus $\bar{p}^* \in D_f^{\mathbb{Z}}$, and again by (4.9), $\bar{p}^* \in D_f^{\mathbb{Z}} : G_0$. As $S^*(\bar{p}^*) = (\bar{p}_0^*, \bar{p}_1^*) \leq (\bar{p}_0, \bar{p}_1)$, we are done. \smile

Let \dot{R}_i be a $Q * \dot{B}_0$ -name for R_i , for each $i \in \{0, 1\}$. We just showed that $Q * \dot{B}_0$ forces that there is a dense embedding from $P_f^{\mathbb{Z}} : G_0$ into $\dot{R}_0 \times \dot{R}_1$. So there is a dense embedding of $P_f^{\mathbb{Z}}$ into $Q * \dot{B}_0 * (\dot{R}_0 \times \dot{R}_1)$. Since the latter is equivalent to $P_f^{(-\infty, 0]} * \dot{R}$ for some \dot{R} , we find that $P_f^{(-\infty, 0]}$ is a complete sub-order of $P_f^{\mathbb{Z}}$. The same is true for $P_f^{[0, \infty)}$ (or more generally, for P_f^I , where I is any interval in \mathbb{Z}). In fact, it's easy to show that the natural embedding and projection witness this.

The previous lemma affords insight concerning the action of the automorphism Φ . E.g. it enables us to show that if \dot{x} is a P -name which is not in V^{B_0} (and hence also not in V^{B_1}), then for all $i \in \mathbb{Z} \setminus \{0\}$, $\Phi(\dot{x}) \notin V^P$. In fact, for the proof of the main theorem, we shall need something a bit more specific:

Lemma 4.8. *Assume that \dot{r}_0, \dot{r}_1 be P -names for reals random over V^Q , and assume $\Vdash_Q \dot{B}_i = \langle \dot{r}_i \rangle^{P:Q}$ (as is the case in our application). If \dot{r} is a P -name for a real such that \dot{r} is unbounded over V^Q , then for any $i \in \mathbb{Z} \setminus \{0\}$, $\Phi^i(\dot{r})$ unbounded over V^P .*

Proof. Firstly, \dot{r} is unbounded over V^{B_i} , for each $i \in \{0, 1\}$, since the random algebra does not add unbounded reals. For a start, let's assume $i = 1$.

Let $G_1 = G_Q * f[H_0]$ be $Q * \dot{B}_1$ -generic and work in $W = V[G_1]$. We have that \dot{r} is a $P : G_1$ name for a real which is unbounded over W (in the sense of definition 1.16—in any $P : G_1$ -generic extension of $V[G_1]$, the interpretation of \dot{r} will be unbounded over $V[G_1]$). Let R_0, R_1 be defined as in the previous proof, i.e.

$$\begin{aligned} R_1 &= P_f^{[1, \infty)} : G_1, \\ R_0 &= P_f^{(-\infty, 0]} : (G_Q * H_0), \end{aligned}$$

let \dot{R}_i be a $Q * \dot{B}_0$ -name for R_i , for each $i \in \{0, 1\}$, and let $I = [1, \infty)$. As $P : G_1$ is a complete sub-order of R_1 (the skeptic is referred to lemma 1.11, p. 11), $e_1^I(\dot{r})$ is an R_1 -name which is unbounded over W . By lemma 1.17, viewing $e_1^I(\dot{r})$ as a $R_0 \times R_1$ -name, it is unbounded over W^{R_0} . As G_1 was

arbitrary, $e_1^I(\dot{r})$ is a $Q * \dot{B}_0 * (\dot{R}_1 \times \dot{R}_0)$ -name unbounded over $V^{Q*\dot{B}_0*\dot{R}_0}$. By the previous theorem this means that $e_1(\dot{r})$ is a $P_f^{\mathbb{Z}}$ -name unbounded over $V^{Q*\dot{B}_0*\dot{R}_0} = V^{P_f^{(-\infty, 0]}}$ and hence over V^P , since $S \circ e_0 = e_0^I$ shows that P is a complete sub-order of $Q * \dot{B}_0 * \dot{R}_0$.

For arbitrary $i \in \mathbb{Z}$ such that $i > 0$: We just showed that $e_1(\dot{r})$ is a $P_f^{\mathbb{Z}}$ -name unbounded over $V^{P_f^{(-\infty, 0]}}$. Since $e_{-i+1}^I[P]$ is a complete sub-order of $P_f^{(-\infty, 0]}$, we know $e_1(\dot{r})$ is unbounded over $V^{e_{-i+1}^I[P]}$. Apply Φ^{i-1} to see $\Phi^i(\dot{r})$ is unbounded over $V^{e_0^I[P]}$, as $e_0 = \Phi^{i-1} \circ e_{-i+1}^I$. For $i < 0$, argue exactly as above but use the second dense embedding mentioned in lemma 4.7. $\textcircled{\smile}$

4.3 Stratified type-1 amalgamation

We now turn to the matter of stratification. Assume (Q, P) is a stratified extension above λ_0 , as witnessed by $\mathbf{S}_Q = (\mathbf{F}_Q, \preceq_Q^\lambda, \prec_Q^\lambda, \mathbf{C}_Q^\lambda)_{\lambda \geq \lambda_0}$ and $\mathbf{S}_P = (\mathbf{F}, \preceq^\lambda, \prec^\lambda, \mathbf{C}^\lambda)_{\lambda \geq \lambda_0}$. We never need to mention $\preceq_Q^\lambda, \prec_Q^\lambda, \mathbf{C}_Q^\lambda$ and \mathbf{F}_Q as we can always use the corresponding relation from \mathbf{S}_P (see the remark following definition 3.18, p. 45). Moreover, assume $\Vdash_Q |\dot{B}_0| \leq \lambda_0$.

The main problem with stratification and amalgamation is quasi-closure: Firstly, if p and q are compatible, $\mathbf{F}(p)$ and $\mathbf{F}(q)$ needn't be (in fact, it's easy to show there has to be a counterexample if \mathbf{F} is non-trivial). This is why we asked the somewhat strict (4.6) in the definition of amalgamation and ($\triangleleft_c 3$) in the definition of stratified extension—although there might be more subtle ways to circumvent this issue.

Secondly, consider two sequences $(p_\xi)_{\xi < \rho}$ and $(q_\xi)_{\xi < \rho}$ such that p_ξ and q_ξ are compatible for every $\xi < \rho$, with greatest lower bounds p and q respectively. In general, p and q don't have to be compatible. A similar problem occurs with regard to the defining equation (4.5) of amalgamation: say we have a sequence of conditions $\bar{p}_\xi \in \mathbf{Am}_1$ and for each $i \in \mathbb{Z}$, $\bar{p}(i)$ is a greatest lower bound of $(\bar{p}_\xi(i))_\xi$. Even though (4.5) holds for every \bar{p}_ξ , it could fail for \bar{p} .

The solution to this problem is to thin out to a dense subset of P where π_i is stable with respect to “direct extension”, before we amalgamate. That is, on this dense subset, π_i doesn't change (in a strong sense) when conditions are extended in the sense of \preceq^λ , for $\lambda \geq \lambda_0$.

Definition 4.9. Let $\mathbf{D} = \mathbf{D}(Q, P, f, \lambda_0)$ be the set of $p \in P$ such that for all $q \in P$, if $q \preceq^{\lambda_0} p$ we have

$$\forall (b_0, b_1) \in B_0 \times B_1 \quad (\pi(q) \cdot p \cdot b_0 \cdot b_1 \neq 0) : (q \cdot b_0 \cdot b_1 \neq 0) \quad (4.10)$$

Observe that (4.10) is equivalent to:

$$\forall j \in \{0, 1\} \quad \pi(q) \Vdash_Q \forall b \in \dot{B}_{1-j} \quad \pi_j(q \cdot b) = \pi_j(p \cdot b), \quad (4.11)$$

and also to the following:

$$\forall j \in \{0, 1\} \quad \forall b \in B_{1-j} \quad \pi_j(q \cdot b) = \pi(q) \cdot \pi_j(p \cdot b). \quad (4.12)$$

Lemma 4.10. \mathbf{D} is open dense in $\langle P, \preceq^{\lambda_0} \rangle$.

Proof. Let p_0 be given. We inductively construct an adequate sequence of p_ξ , $0 < \xi \leq \lambda_0$ with $p_{\lambda_0} \in \mathbf{D}$. First fix x such that the following definition is Δ_1^A in parameters from x . Fix Q -names \dot{b}_j such that $\Vdash_Q \dot{b}_j: \lambda_0 \rightarrow \dot{B}_j$ is onto, for $j = 0, 1$, and let $\xi \mapsto (\alpha_\xi, \beta_\xi, \zeta_\xi)$ be a surjection from λ_0 onto $(\lambda_0)^3$.

For limit ξ , let p_ξ be the greatest lower bound of the sequence constructed so far. Say we have constructed p_ξ , we shall define $p_{\xi+1}$. Let's first assume there are p^*, \bar{p} such that $\bar{p} \preceq^{\lambda_0} \mathbf{F}(\lambda_0, x, p_\xi)$, $p^* \leq \bar{p}$ and

1. $\pi(p^*) \Vdash \bar{p} \cdot \dot{b}_0(\alpha_\xi) \cdot \dot{b}_1(\beta_\xi) = 0$,
2. $\zeta_\xi \in \mathbf{C}^{\lambda_0}(p^*)$.

In this case pick $p_{\xi+1}$ such that $p_{\xi+1} \preceq^{\lambda_0} \bar{p}$ and $p_{\xi+1} \preceq^{\lambda_0} p^*$ (using interpolation). If, on the other hand, no such \bar{p}, p^* exist, let $p_{\xi+1} = p_\xi$.

We now show (4.11) holds for the final condition p_{λ_0} : say, to the contrary, we can find $j \in \{0, 1\}$ and $\dot{b} \in B_{1-j}$ together with $\bar{q} \preceq^{\lambda_0} p_{\lambda_0}$ such that

$$\pi(\bar{q}) \not\Vdash_Q \pi_j(\bar{q} \cdot \dot{b}) = \pi_j(p_{\lambda_0} \cdot \dot{b}).$$

Without loss of generality say $j = 0$. We can find $q^* \leq \bar{q}$ such that for some $\alpha, \beta < \lambda_0$

- (i) $\pi(q^*) \Vdash \pi_0(p_{\lambda_0} \cdot \dot{b}) - \pi_0(\bar{q} \cdot \dot{b}) = \dot{b}_0(\alpha) \neq 0$,
- (ii) $\pi(q^*) \Vdash \dot{b} = \dot{b}_1(\beta)$,
- (iii) $q^* \in \text{dom}(\mathbf{C}^{\lambda_0})$.

Find $\xi < \lambda_0$ so that $\alpha = \alpha_\xi$, $\beta = \beta_\xi$ and $\zeta_\xi \in \mathbf{C}^{\lambda_0}(q^*)$. By construction, at stage ξ of our construction we had \bar{p} and p^* satisfying (1) and (2). As $\mathbf{C}^{\lambda_0}(p^*) \cap \mathbf{C}^{\lambda_0}(q^*) \neq 0$ and $q^* \leq p_{\xi+1} \preceq^{\lambda_0} p^*$, we can find $w \leq p^*, q^*$. But by (i), $\pi(w) \Vdash p_{\lambda_0} \cdot \dot{b}_0(\alpha) \cdot \dot{b}_1(\beta) \neq 0$ while since $w \leq p^*$, $\pi(w) \Vdash p_{\lambda_0} \cdot \dot{b}_0(\alpha) \cdot \dot{b}_1(\beta) = 0$, contradiction.

Now we show \mathbf{D} is open: For any $r \preceq^{\lambda_0} q$, $j = 0, 1$ and $\dot{b} \in B_{1-j}$, since $r \preceq^{\lambda_0} p$, we have $\pi(r) \Vdash \pi_j(r \cdot \dot{b}) = \pi_j(p \cdot \dot{b})$. Since $\pi(q) \Vdash \pi_j(q \cdot \dot{b}) = \pi_j(p \cdot \dot{b})$ and $r \leq q$, $\pi(r) \Vdash \pi_j(r \cdot \dot{b}) = \pi_j(q \cdot \dot{b})$. So $q \in \mathbf{D}$. $\textcircled{\smile}$

Having $Q \subseteq \mathbf{D}$ helps in many circumstances, in particular we like to have $1_P \in \mathbf{D}$. To this end we introduce the notion of B_0, B_1 being λ_0 -reduced.

Definition 4.11. We say the pair B_0, B_1 is λ -reduced over Q if and only if whenever $p \in P$, $p \preceq^\lambda q$ for some $q \in Q$ and $b \in B_j$ for $j = 0$ or $j = 1$, we have

$$\pi_{1-j}(p \cdot b) = \pi(p) \cdot \pi(b).$$

Henceforth assume B_0, B_1 is a λ_0 -reduced pair. We will later see that this is a very mild assumption, see lemmas 4.13 and 5.3.

Lemma 4.12. *If $p \preceq^{\lambda_0} q$ for some $q \in Q$ and $j \in \{0, 1\}$ we have*

$$\pi(p) \Vdash \forall b \in \dot{B}_{1-j} \setminus \{0\} \quad \pi_j(p \cdot b) = 1,$$

and moreover, $p \in \mathbf{D}$. In particular, we have $Q \subseteq \mathbf{D}$.

Proof. Fix p as in the hypothesis. Say $r \in Q$, $r \leq \pi(p)$ and $b \in B_0$ such that $r \Vdash b \in \dot{B}_0 \setminus \{0\}$. Then $r \leq \pi(b)$. So as B_0, B_1 is λ_0 -reduced, $r \leq \pi_1(p \cdot b)$, whence $r \Vdash \pi_1(p \cdot b) = 1$. This proves the first statement for $j = 1$, and in the other case the proof is the same.

We now show $p \in \mathbf{D}$: Say $p' \preceq^{\lambda_0} p$. Since also $p' \preceq^{\lambda_0} q$, we have

$$\pi(p') \Vdash \forall b \in \dot{B}_{1-j} \setminus \{0\} \quad \pi_j(p' \cdot b) = 1 = \pi_j(p \cdot b),$$

and thus $p \in \mathbf{D}$. ☺

In fact, the first statement of lemma 4.12 is equivalent to B_0, B_1 being a reduced pair (this is really just a slight variation of lemma 4.13).

The following provides a hint as to how we can assume that B_0, B_1 is λ_0 -reduced:

Fact 4.13. Assume that \dot{r}_0, \dot{r}_1 are P -names for reals random over V^Q , and assume $\Vdash_Q \dot{B}_i = \langle \dot{r}_i \rangle^{P:Q}$ (as is the case in our application). Say $j = 0$ or $j = 1$. The following are equivalent (interestingly, in (2), there is no mention of j):

1. Whenever $p \in P$, $p \preceq^\lambda q$ for some $q \in Q$ and $b \in B_j$, we have

$$\pi_{1-j}(p \cdot b) = \pi(p) \cdot \pi(b).$$

2. Whenever $p \in P$, $p \preceq^\lambda q$ for some $q \in Q$ and b_0, b_1 are Q -names for Borel sets such that for some $w \leq \pi(p)$, $w \Vdash_Q$ “both b_0 and b_1 are not null”, there is $p' \leq p$ such that $p' \Vdash_P \dot{r}_0 \in b_0$ and $\dot{r}_1 \in b_1$.

Proof. First, assume (2). We carry out the proof for $j = 0$ (the other case is exactly the same). Let $p \in P$ such that for some $q \in Q$, $p \preceq^{\lambda_0} q$ and let $b_0 \in B_0$. As for any $r \in \text{r.o.}(P)$, $r \leq \pi_j(r) \leq \pi(r)$ holds, we have $\pi_j(p \cdot b_0) \leq \pi(p) \cdot \pi(b_0)$. We now show $\pi_j(p \cdot b_0) \geq \pi(p) \cdot \pi(b_0)$. It suffices to show that whenever $b_1 \in B_1$ is compatible with $\pi(p) \cdot \pi(b_0)$, it is compatible with $p \cdot b_0$. So fix $b_1 \in B_1$. We have $\pi(b_1) \cdot \pi(b_0) \cdot \pi(p) \neq 0$, so we may pick $w \leq \pi(b_1) \cdot \pi(b_0) \cdot \pi(p)$. For $j = 0, 1$, let \dot{b}_j be a Q -name for a Borel set such that $b_j = \|\dot{r}_j \in \dot{b}_j\|^{\text{r.o.}(P)}$. The last inequality means $w \Vdash \dot{b}_0$ and \dot{b}_1 are not null. So by assumption, we can find p' forcing $\dot{r}_j \in \dot{b}_j$ for both $j = 0, 1$. In other words, $p' \leq p \cdot b_0 \cdot b_1$, whence b_1 is compatible with $p \cdot b_0$.

For the other direction, assume (1) and again assume $j = 0$, fix p as above, and say \dot{b}_0, \dot{b}_1 are Q -names such that $w \Vdash \dot{b}_0, \dot{b}_1 \in \mathbf{Borel}^+$ for some $w \leq \pi(p)$. Let $b_j = \|\dot{r}_j \in \dot{b}_j\|^{\text{r.o.}(P)}$. As $\pi(b_0) \cdot \pi(b_1) \cdot \pi(p) \neq 0$, b_1 is compatible with $\pi(b_0) \cdot \pi(p) = \pi_1(p \cdot b_0)$. Thus b_1 is compatible with $p \cdot b_1$. So we may pick $p' \in P$, $p' \leq p \cdot b_0 \cdot b_1$. $\textcircled{\smile}$

Definition 4.14. Under the assumptions of the previous lemma, we also say the pair \dot{r}_0, \dot{r}_1 is λ -reduced.

We shall need the next lemma to show that P completely embeds into \mathbf{Am}_1 (see 4.17). Observe that the next lemma does not make the assumption that B_0, B_1 is a λ_0 -reduced pair obsolete, i.e. by itself the lemma does not imply $Q \subseteq \mathbf{D}$.

Lemma 4.15. *Assume that there exists a Q -stable meet operator \wedge_Q on P with respect to \mathbf{S} . Then $Q \cdot \mathbf{D} \subseteq \mathbf{D}$. More precisely, if $p \in \mathbf{D}$ and $q \in Q$ are such that $q \leq \pi(p)$, we have $q \cdot p \in \mathbf{D}$. Moreover, if $(p, r) \in \text{dom}(\wedge_Q)$ and $p \in \mathbf{D}$, for any $j \in \{0, 1\}$ and $b \in B_{1-j}$ we have $\pi_j((p \wedge_Q r) \cdot b) = \pi_j(p \cdot b)$.*

Proof. Let $p \in P$, $q \in Q$ and $q \leq \pi(p)$. We check that $q \cdot p \in \mathbf{D}$. So let

$$r \preceq^{\lambda_0} q \cdot p, \quad (4.13)$$

and fix $j \in \{0, 1\}$ and $b \in B_{1-j}$. To prove that $q \cdot p \in \mathbf{D}$, it suffices to show

$$\pi_j(r \cdot b) = \pi(r) \cdot \pi_j(q \cdot p \cdot b). \quad (4.14)$$

Observe that (4.13) implies that $r \preceq^{\lambda_0} \pi(r) \cdot p$ —for by 3.1($\triangleleft_c 2$), $\pi(r) \preceq^\lambda \pi(q \cdot r) = q$; now use 3.18($\triangleleft_s A$). Thus $(p, r) \in \text{dom}(\wedge_Q)$ and $p \wedge_Q r \preceq^{\lambda_0} p$. Thus $p \wedge_Q r \in \mathbf{D}$ and

$$\pi_j((p \wedge_Q r) \cdot b) = \pi(p \wedge_Q r) \cdot \pi_j(p \cdot b) = \pi_j(p \cdot b),$$

where the last equation holds because $\pi(p \wedge_Q r) = \pi(p) \geq \pi_j(p \cdot b)$. Note in passing that this proves of the “moreover” clause of the lemma. We continue

with the proof of the remaining part of the lemma. By the previous, as $r = \pi(r) \cdot (p \wedge_Q r)$,

$$\pi_j(r \cdot b) = \pi(r) \cdot \pi_j((p \wedge_Q r) \cdot b) = \pi(r) \cdot \pi_j(p \cdot b) = \pi(r) \cdot \pi_j(q \cdot p \cdot b).$$

The last equation holds as $\pi(r) \leq q$. This finishes the proof of the lemma. $\textcircled{\smile}$

From now on, assume we have a Q -stable meet \wedge_Q on P .

While it is true that $(\widehat{\mathbf{D}}, \mathbf{D}_f^{\mathbb{Z}})$ is a stratified extension, this is not quite the partial order we use in the main theorem: for this construction would require to repeatedly thin out to a dense set. As a consequence, we would need the main iteration theorem 3.23 not just for iterations but rather for sequences $(D_\xi)_{\xi < \theta}$ where $(D_\xi, D_{\xi+1})$ is a stratified extension, but we do not have strong projections from $D_{\bar{\xi}}$ to D_ξ for $\xi < \bar{\xi} \leq \theta$. Moreover, we would need to prove that the limits in this directed system of partial orders are what we expect them to be (in particular, that each D_ξ is embedded in this limit as a complete sub-order).² Instead, we have a much simpler solution.

Definition 4.16 (Type-1 amalgamation). Let $\mathbf{Am}_1 = \mathbf{Am}_1(Q, P, f, \lambda)$ be the set of $\bar{p}: \mathbb{Z} \rightarrow \widehat{P}$ such that the following conditions are met.

1. For all $i \in \mathbb{Z}$, $\pi(\bar{p}(i)^P) = \pi(\bar{p}(0)^P)$.
2. For all $i \in \mathbb{Z} \setminus \{-1, 0\}$, $f(\pi_0(\bar{p}(i))) = \pi_1(\bar{p}(i+1))$ — that is, (4.5) holds.
3. $\bar{p}(0) \in P$, i.e. $\bar{p}^0(0) = \bar{p}^1(0) = 1$ and

$$f(\pi_0(\bar{p}(-1))) \geq \pi_1(\bar{p}(0)), \quad (4.15)$$

$$f(\pi_0(\bar{p}(0))) \leq \pi_1(\bar{p}(1)). \quad (4.16)$$

4. For $i \in \mathbb{Z} \setminus \{0\}$, $\bar{p}(i)^P \in \mathbf{D}(Q, P, f, \lambda)$.

Observe we can replace (4.15) and (4.16) by

$$\bar{p}(0)^P \leq f(\pi_0(\bar{p}(-1))) \cdot f^{-1}(\pi_1(\bar{p}(1))). \quad (4.17)$$

and obtain an equivalent definition. Thus, $\bar{p} \in \mathbf{Am}_1$ if and only if the following conditions are met:

1. $\bar{p}(0) \in P$,

²It seems plausible that theorem 3.23 would go through in this broader case. This provided, it is possible, but lengthy to show that limits contain the D_ξ 's.

2. $\bar{p} \upharpoonright [1, \infty) \in \mathbf{D}_f^{[1, \infty)}$ and $\bar{p} \upharpoonright (-\infty, -1] \in \mathbf{D}_f^{(-\infty, -1]}$
3. for both $j \in \{-1, 1\}$ we have $\pi(\bar{p}(0)) = \pi(\bar{p}(j))$ and (4.17) holds.

Let $a: P \rightarrow \mathbf{Am}_1$ be defined by $a(p)(0) = (p, 1, 1)$ and $a(p)(i) = (\pi(p), 1, 1)$ for all $i \in \mathbb{Z} \setminus \{0\}$. As before, let $\bar{\pi}(\bar{p}) = \bar{p}(0)^P$ (we see no problem in using the same designation as for the projection from $\mathbf{D}_f^{\mathbb{Z}}$ to \mathbf{D} —see the remark after the next lemma).

Lemma 4.17. *The map $a: P \rightarrow \mathbf{Am}_1$ is a complete embedding and*

$$\bar{\pi}: \mathbf{Am}_1 \rightarrow P$$

is a strong projection.

Proof. Let $\bar{p} \in \mathbf{Am}_1$, $w \in P$, and $w \leq \bar{p}(0)$. Define \bar{p}' by

$$\bar{p}'(i) = \begin{cases} w & \text{for } i = 0, \\ (\pi(w) \cdot \bar{p}(i)^P, \bar{p}(i)^0, \bar{p}(i)^1) & \text{for } i \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

Clearly $\bar{p}' \in \mathbf{Am}_1$, $\bar{p}' \leq a(w)$ and $\bar{p}' \leq \bar{p}$. Moreover, for arbitrary $\bar{q} \in \mathbf{Am}_1$, if $\bar{q} \leq a(w)$ and $\bar{q} \leq \bar{p}$, clearly $\bar{q} \leq \bar{p}'$; so $\bar{p}' = a(w) \cdot \bar{p}$. This shows that $\bar{\pi}$ is a strong projection and accordingly, a is a complete embedding. $\textcircled{\smile}$

In what follows, we identify P and $a[P]$ —except when we feel this would hide the point of the argument. Next we show that in fact, \mathbf{Am}_1 and $\mathbf{D}_f^{\mathbb{Z}}$ are presentations of the same forcing.

Lemma 4.18. *The set $D^* = \{\bar{p} \in \mathbf{D}_f^{\mathbb{Z}} \mid \bar{p}(0)^0 = \bar{p}(0)^1 = 1\}$ is dense in both $\mathbf{D}_f^{\mathbb{Z}}$ and \mathbf{Am}_1 .*

Proof. First, we notice that $D^* \subseteq \mathbf{Am}_1$ and that the ordering of $\mathbf{D}_f^{\mathbb{Z}}$ and that of \mathbf{Am}_1 coincide on D^* . Observe that since we identify \mathbf{D} with a subset of $\widehat{\mathbf{D}}$, we may write $\bar{\pi}^{-1}[\mathbf{D}] = D^*$ (this holds no matter if we consider $\bar{\pi}$ to have domain \mathbf{Am}_1 or $\mathbf{D}_f^{\mathbb{Z}}$). As $\widehat{\mathbf{D}}$ is a complete sub-order of $\mathbf{D}_f^{\mathbb{Z}}$ and \mathbf{D} is dense in $\widehat{\mathbf{D}}$, $D^* = \bar{\pi}^{-1}[\mathbf{D}]$ is dense in $\mathbf{D}_f^{\mathbb{Z}}$. In other words, given $\bar{p} \in \mathbf{D}_f^{\mathbb{Z}}$, find $d \in \mathbf{D}$ such that $d \leq \bar{p}(0)^P \cdot p(0)^0 \cdot p(0)^1$; clearly, $d \cdot \bar{p} \in D^*$.

Now let $\bar{p} \in \mathbf{Am}_1$. We find $\bar{w} \leq \bar{p}$, such that $\bar{w} \in D^*$. Find $d \in \mathbf{D}$ such that $d \leq \bar{p}(0)$. First let $I = (-\infty, 0]$ and construct $\bar{w}^- = \bar{w} \upharpoonright I$. Let $b_0 = f(\pi_0(\bar{p}(-1)))$ and define $\bar{p}^- \in \mathbf{D}_f^I$ by

$$\bar{p}^- = (\dots, \bar{p}(i), \dots, \bar{p}(-1), b_0),$$

where of course we identify b_0 and $(\pi(b_0), 1, b_0) \in \widehat{\mathbf{D}}$. Since $d \leq \bar{p}(0) \leq b_0$ and $b_0 = \bar{\pi}_0^I(\bar{p}^-)$, we can let $\bar{w}^- = d \cdot \bar{p}^- \in \mathbf{D}_f^I$. Observe that $\bar{\pi}_0^I(\bar{w}^-) = d$.

Now let $I = [0, \infty)$. In an analogous fashion, define $\bar{w}^+ \in \mathbf{D}_f^I$ such that $\bar{w}^+ \leq \bar{p} \upharpoonright I$ and $\bar{\pi}_0^I(\bar{w}^+) = d$. Letting

$$\bar{w}(i) = \begin{cases} \bar{w}^-(i) & \text{for } i < 0, \\ \bar{w}^+(i) & \text{for } i \geq 0, \end{cases}$$

we conclude $\bar{w} \in \mathbf{Am}_1$. Moreover, $\bar{\pi}(\bar{w}) = d \in \mathbf{D}$ whence $\bar{w} \in D^*$, and $\bar{w} \leq \bar{p}$ in \mathbf{Am}_1 . ☺

Thus, although Φ is not an automorphism of \mathbf{Am}_1 , since it is an automorphism of $\mathbf{D}_f^{\mathbb{Z}}$, it gives rise to an automorphism of the associated Boolean algebra. We call Φ the *automorphism resulting from the amalgamation*, and we refer to Q as the *base of the amalgamation* or, interchangeably, *the base of Φ* .

That $\text{r.o.}(\mathbf{Am}_1) = \text{r.o.}(\mathbf{D}_f^{\mathbb{Z}})$ justifies that we use the same notation for the strong projections $\bar{\pi}: \mathbf{Am}_1 \rightarrow P$ and $\bar{\pi}: \mathbf{D}_f^{\mathbb{Z}} \rightarrow \mathbf{D}$ —as we know a strong projection coincides with the canonical projection on (the separative quotient of) its domain. The next lemma clarifies the role of \mathbf{D} .

Lemma 4.19. *Let $\bar{p} \in \mathbf{Am}_1$ and say $\bar{q}: \mathbb{Z} \rightarrow P \times B_0 \times B_1$ satisfies the following conditions:*

1. for each $i \in \mathbb{Z}$, $\pi(\bar{q}(i)^P) = \pi(\bar{q}(0)^P)$.
2. $\bar{q}(0)^0 = \bar{q}(0)^1 = 1$.
3. $\forall i \in \mathbb{Z} \setminus \{0\} \quad \bar{q}(i)^P \preceq^{\lambda_0} \bar{p}(i)^P$.
4. $\forall i \in \mathbb{Z} \setminus \{0\} \quad \pi_j(\bar{q}(i)) = \pi(\bar{q}(i)^P) \cdot \pi_j(\bar{p}(i))$

Then $\bar{q} \in \mathbf{Am}_1$.

Proof. First, let $I = [1, \infty)$ and show $\bar{q} \upharpoonright I \in \mathbf{D}_f^I$. Let $i \in I$ be arbitrary. By 4 above, we have

$$\pi_j(\bar{q}(i)) = \pi(\bar{q}(i)^P) \cdot \pi_j(\bar{p}(i)) \tag{4.18}$$

for $j \in \{0, 1\}$. Since by 1 we have $\pi(\bar{q}(i)^P) = \pi(\bar{q}(0)^P) \leq \pi(\bar{p}(0)^P) = \pi(\bar{p}(i))$, applying π to (4.18) yields

$$\pi(\bar{q}(i)) = \pi(\bar{q}(i)^P) \cdot \pi(\bar{p}(i)) = \pi(\bar{q}(i)^P), \tag{4.19}$$

which means

$$\bar{q}(i) \in \widehat{P}. \tag{4.20}$$

Since $\bar{p} \in \mathbf{Am}_1$ and since (4.18) holds, we have

$$f(\pi_0(\bar{q}(i))) = \pi(\bar{q}(0)) \cdot f(\pi_0(\bar{p}(i))) = \pi(\bar{q}(0)) \cdot \pi_1(\bar{p}(i+1)) = \pi_1(\bar{q}(i))$$

Thus $\bar{q} \upharpoonright I \in \mathbf{D}_f^I$. Repeat the argument above to show $\bar{p} \upharpoonright (-\infty, -1] \in \mathbf{D}_f^{(-\infty, -1]}$. As $\bar{q}(0)^0 = \bar{q}(0)^1 = 1$ by assumption, (4.20) holds for $i = 0$. Let $b = f(\pi_0(\bar{p}(-1))) \cdot f^{-1}(\pi_1(\bar{p}(1)))$. As $\bar{q}(0) \leq \bar{p}(0) \leq b$, clearly

$$\bar{q}(0) \leq \pi(\bar{q}(0)) \cdot b = f(\pi_0(\bar{q}(-1))) \cdot f^{-1}(\pi_1(\bar{q}(1))).$$

Thus, finally $\bar{q} \in \mathbf{Am}_1$. ☺

Finally, we are ready to state and prove the main theorem of this section:

Theorem 4.20. *(P, \mathbf{Am}_1) is a stratified extension above $(\lambda_0)^+$.*

Proof. We proceed to define a stratification of \mathbf{Am}_1 . \mathbf{Am}_1 is going to be stratified above $(\lambda_0)^+$, but in general not above λ_0 . For notational convenience, we define $\bar{q} \overset{\sim}{\preceq}^\lambda \bar{p}$ for arbitrary \mathbb{Z} -sequences $\bar{q}, \bar{p} \in \mathbb{Z}(P \times B_0 \times B_1)$ and for $\lambda \geq \lambda_0$: $\bar{q} \overset{\sim}{\preceq}^\lambda \bar{p}$ exactly if for every $i \in \mathbb{Z}$, $\bar{q}(i)^P \preceq^\lambda \bar{p}(i)^P$ and for every $i \in \mathbb{Z} \setminus \{0\}$ we have $\pi(\bar{q}(i)^P) \Vdash_Q \pi_j(\bar{q}(i)) = \pi_j(\bar{p}(i))$ —or equivalently,

$$\pi_j(\bar{q}(i)) = \pi(\bar{q}(i)^P) \cdot \pi_j(\bar{p}(i)) \quad (4.21)$$

for both $j \in \{0, 1\}$.

Corollary 4.21. *Using this notation we can state lemma 4.19 in the following way: If for some regular $\lambda \geq \lambda_0$, $\bar{p} \in \mathbf{Am}_1$ and $\bar{q}: \mathbb{Z} \rightarrow P \times B_0 \times B_1$ satisfy $\bar{q} \overset{\sim}{\preceq}^\lambda \bar{p}$ and moreover $\bar{q}(0) \in P$ and for all $i \in \mathbb{Z}$, $\pi(\bar{q}(i)^P) = \pi(\bar{q}(0)^P)$ holds, then $\bar{q} \in \mathbf{Am}_1$.*

Lemma 4.22. *Observe that if $\bar{q}: \mathbb{Z} \rightarrow P \times B_0 \times B_1$ and $\bar{p} \in \mathbf{Am}_1$ satisfy $\bar{q}(i)^P \preceq^\lambda \bar{p}(i)^P$ for all $i \in \mathbb{Z}$ and $\bar{q}(i)^j = \bar{p}(i)^j$ for all $i \in \mathbb{Z} \setminus \{0\}$ and $j \in \{0, 1\}$, then $\bar{q} \overset{\sim}{\preceq}^\lambda \bar{p}$.*

Proof. For $i \in \mathbb{Z} \setminus \{0\}$ and $j \in \{0, 1\}$, we have

$$\begin{aligned} \pi_j(\bar{q}(i)) &= \bar{p}(i)^j \cdot \pi_j(\bar{q}(i)^P) \cdot \bar{p}(i)^{1-j} \\ &= \bar{p}(i)^j \cdot \pi(\bar{q}(i)^P) \cdot \pi_j(\bar{p}(i)^P) \cdot \bar{p}(i)^{1-j} = \pi(\bar{q}(i)^P) \cdot \pi_j(\bar{p}(i)). \end{aligned}$$

where the second line is equal to the first as $\bar{p}(i)^P \in \mathbf{D}$ and $\bar{q}(i)^P \preceq^\lambda \bar{p}(i)^P$. Thus, $\bar{q} \overset{\sim}{\preceq}^\lambda \bar{p}$. ☺

Now let $\bar{p}, \bar{q} \in \mathbf{Am}_1$ and say λ is regular and $\lambda > \lambda_0$. Define

$$\bar{\mathbf{F}}(\lambda, x, \bar{p})(i) = (\mathbf{F}(\lambda, x, \bar{p}^P(i)), \bar{p}(i)^0, \bar{p}(i)^1).$$

We say $\bar{q} \bar{\preceq}^\lambda \bar{p}$ exactly if

$$\forall i \in \mathbb{Z} \quad \bar{q}(i)^P \preceq^\lambda \bar{p}(i)^P.$$

Next we define $\bar{\mathbf{C}}^\lambda$. Fix a name $\dot{\mathbf{B}}$ such that

$$\Vdash_P \dot{\mathbf{B}}: \dot{B}_0 \cup \dot{B}_1 \rightarrow \check{\lambda}_0 \text{ is a bijection.}$$

Let $\text{dom}(\bar{\mathbf{C}}^\lambda)$ be the set of all $\bar{p} \in \mathbf{Am}_1$ such that for each $i \in \mathbb{Z}$, we have $\bar{p}(i)^P \in \text{dom}(\mathbf{C}^\lambda)$ and if $i \neq 0$, there is $\lambda' < \lambda$ such that for $j \in \{0, 1\}$ we have that $\dot{\mathbf{B}}(\pi_j(\bar{p}(i)))$ is λ' -chromatic below $\pi(\bar{p}(i)^P)$. If $\bar{p} \in \text{dom}(\bar{\mathbf{C}}^\lambda)$, we define $\bar{\mathbf{C}}^\lambda(\bar{p})$ to be the set of all $(c(i), \lambda'(i), H^0(i), H^1(i))_{i \in \mathbb{Z}}$ such that for all $i \in \mathbb{Z}$, $c(i) \in \mathbf{C}^\lambda(\bar{p}(i))$ and for all $i \in \mathbb{Z} \setminus \{0\}$ and $j \in \{0, 1\}$, $H^j(i)$ is a $\lambda'(i)$ -spectrum of $\dot{\mathbf{B}}(\pi_j(\bar{p}(i)))$ below $\pi(\bar{p}(i)^P)$. Observe that $\lambda'(0)$, $H^0(0)$ and $H^1(0)$ can be chosen arbitrarily—they merely serve as place-holders to facilitate notation. This finishes the definition of the stratification of $\mathbf{D}_f^\mathbb{Z}$.

First we check that $\bar{\mathbf{F}}$ and $(\bar{\preceq}^\lambda)_{\lambda \in \text{Reg} \setminus \lambda_0}$ give us a pre-closure system, see 2.1, p. 20. That $\bar{\mathbf{F}}$ is Δ_1^A is immediate (without any further assumptions on the parameter x). For the following, let $\bar{p}, \bar{q}, \bar{r} \in \mathbf{Am}_1$, $\lambda \in \text{Reg} \setminus \lambda_0$ and x be arbitrary.

Observe that $\bar{w} = \bar{\mathbf{F}}(\lambda, x, \bar{p})$ satisfies all the requirements of 4.19, and so $\bar{w} \in \mathbf{Am}_1$. For (C 1), we must prove transitivity, so say $\bar{p} \bar{\preceq}^\lambda \bar{q} \bar{\preceq}^\lambda \bar{r}$ and show $\bar{p} \bar{\preceq}^\lambda \bar{r}$. Fix $i \in \mathbb{Z}$ and $j \in \{0, 1\}$. Clearly, $\bar{p}(i)^P \preceq^\lambda \bar{r}(i)^P$. As $\pi(\bar{p}(i)^P) \Vdash_Q \pi_j(\bar{p}(i)) = \pi_j(\bar{q}(i))$ and $\pi_j(\bar{q}(i)) = \pi_j(\bar{r}(i))$, we get $\pi(\bar{p}(i)^P) \Vdash_Q \pi_j(\bar{p}(i)) = \pi_j(\bar{r}(i))$ and so as i, j were arbitrary, $\bar{p} \bar{\preceq}^\lambda \bar{r}$. It remains to show that $\bar{p} \bar{\preceq}^\lambda \bar{q}: \bar{p} \leq \bar{q}$. So assume $\bar{p} \bar{\preceq}^\lambda \bar{q}$ and fix $i \in \mathbb{Z}$. Firstly, $\bar{p}(i)^P \leq \bar{q}(i)^P$; moreover, (4.21) implies $\pi_j(\bar{p}(i)) \leq \pi_j(\bar{q}(i))$ for $j \in \{0, 1\}$, and so as $i \in \mathbb{Z}$ was arbitrary and by (4.4), we infer $\bar{p} \leq \bar{q}$. For (C 2), simply observe that $\bar{\mathbf{F}}(\lambda, x, \bar{p}) \bar{\preceq}^\lambda \bar{p}$ holds by definition and by remark 4.22. (C 3): Say $\bar{p} \leq \bar{q} \leq \bar{r}$ and $\bar{p} \bar{\preceq}^\lambda \bar{r}$. Let $i \in \mathbb{Z}$ be arbitrary; clearly $\bar{p}(i)^P \preceq^\lambda \bar{q}(i)^P$. Let $j \in \{0, 1\}$ be arbitrary; as

$$\pi_j(\bar{p}(i)) \leq \pi(\bar{p}(i)^P) \cdot \pi_j(\bar{q}(i)) \leq \pi(\bar{p}(i)^P) \cdot \pi_j(\bar{r}(i))$$

and the terms on the sides of the equation are equal, we conclude $\bar{p} \bar{\preceq}^\lambda \bar{q}$. Condition (C 4) is left to the reader (it requires only a glance at the definition).

We continue by checking the remaining conditions of 2.6, i.e. that we have a pre-stratification system on \mathbf{Am}_1 . The first, (S 1) is immediate by the definition. The conditions (S 2) and (S 3) are immediate by definition, and we leave checking them to the reader. Finally, we prove (S 4):

Lemma 4.23. *Density holds; i.e for $\bar{p} \in \mathbf{Am}_1$ and $\lambda' \in [\lambda_0, \lambda)$ there is $\bar{q} \in \mathbf{Am}_1$ such that $\bar{q} \in \text{dom}(\bar{\mathbf{C}}^\lambda)$ and $\bar{q} \preceq^{\lambda'} \bar{p}$.*

Proof. First, look through the following definition and find a set of parameters x such that it is Δ_1^A in parameters from x . We define conditions $p_i^n \in P$ for $n \in \mathbb{N}$ and $i \in \mathbb{Z}$ and $q^n \in Q$ for $n \in \mathbb{N}$. We do so by induction on n , in each step using induction on i . First, as Q is stratified we can find $q^0 \in Q$ such that $q^0 \preceq^{\lambda'} \pi(\bar{p}(0)^P)$ and for all $i \in \mathbb{Z}$ and both $j \in \{0, 1\}$, $\pi_j(\bar{p}(i))$ is λ' -chromatic below q^0 .

Set $p_i^0 = \bar{p}(i)^P$, for $i \in \mathbb{Z}$.

Now say we have already defined a \mathbb{Z} -sequence $(p_i^n)_{i \in \mathbb{Z}}$ of conditions in P and $q^n \in Q$. We first find $p_i^{n+1} \in P$ for $i \in \mathbb{Z}$, by induction on i . Find $p_0^{n+1} \preceq^{\lambda'} \mathbf{F}(\lambda', x, q^n \cdot \bar{p}_0^n)$ such that $p_0^{n+1} \in \text{dom}(\mathbf{C}^\lambda)$. Assume by induction that for all $i \in \mathbb{Z}$, $q^n \preceq^{\lambda'} \pi(p_i^n)$, whence also $\pi(p_0^{n+1}) \preceq^{\lambda'} q^n \preceq^{\lambda'} \pi(p_i^n)$. Continue by induction, choosing, for each $i \in \mathbb{N} \setminus \{0\}$, a condition p_i^{n+1} such that

$$p_i^{n+1} \preceq^{\lambda'} \mathbf{F}(\lambda', x, \pi(p_{i-1}^{n+1}) \cdot p_i^n) \quad (4.22)$$

and $p_i^{n+1} \in \text{dom}(\mathbf{C}^\lambda)$. By induction hypothesis, $\pi(p_{i-1}^{n+1}) \preceq^{\lambda'} q^n \preceq^{\lambda'} \pi(p_i^n)$, so $\pi(p_{i-1}^{n+1}) \cdot p_i^n$ is a well defined condition in P and $\pi(p_{i-1}^{n+1}) \cdot p_i^n \preceq^{\lambda'} p_i^n$. Thus we have defined p_i^{n+1} for $i \geq 0$. Before we consider the case $i < 0$, observe that for any $i \in \mathbb{N} \setminus \{0\}$, by (4.22), 3.1($\triangleleft_c 2$) and ($\triangleleft_c 3$), we have

$$\pi(p_i^{n+1}) \leq \pi(\mathbf{F}(\lambda', x, \pi(p_{i-1}^{n+1}) \cdot p_i^n)) = \mathbf{F}(\lambda', x, \pi(p_{i-1}^{n+1})).$$

For the second equality we also use that by construction, $\pi(p_{i-1}^{n+1}) \leq q^{n+1} \leq \pi(p_i^n)$. Thus, $(\pi(p_i^{n+1}))_{i \in \mathbb{N}}$ is (λ', x) -strategic and we may assume by choice of x that it is (λ', x) -adequate. Let q^* be a greatest lower bound for $(\pi(p_i^{n+1}))_{i \in \mathbb{N}}$. Now we define p_i^{n+1} , by induction on i for $i < 0$: Find $p_{-1}^{n+1} \in P$ such that

$$p_{-1}^{n+1} \preceq^{\lambda'} \mathbf{F}(\lambda', x, q^* \cdot p_{-1}^n)$$

such that $p_{-1}^{n+1} \in \text{dom}(\mathbf{C}^\lambda)$. Again, continue by induction, choosing for each $i \in \mathbb{N}$, $i > 1$ a condition $p_{-i}^{n+1} \preceq^{\lambda'} \mathbf{F}(\lambda', x, p_{-i}^n \cdot \pi(p_{-i+1}^{n+1}))$ such that $p_{-i}^{n+1} \in \text{dom}(\mathbf{C}^\lambda)$. Finally, let q^{n+1} be a greatest lower bound of $(\pi(p_{-i}^{n+1}))_{i \in \mathbb{N}}$.

For each $i \in \mathbb{Z}$, $(p_i^n)_{n \in \mathbb{N}}$ is a λ' -adequate sequence and thus has a greatest lower bound which we call $\bar{q}(i)^P$. By lemma 3.7 and by choice of x , $\{\pi(p_i^n)\}_{n \in \mathbb{N}}$ is a λ' -adequate sequence in Q . By quasi-closure for Q , $\pi(\bar{q}(i)^P)$

is a greatest lower bound of this sequence. As for each $n \in \mathbb{N}$, $q^{n+1} \preceq^{\lambda'} \pi(p_i^n) \preceq^{\lambda'} q^n$, $(q_n)_{n \in \mathbb{N}}$ also has greatest lower bound $\pi(\bar{q}(i)^P)$, whence for all $i \in \mathbb{Z}$, $\pi(\bar{q}(i)^P) = \pi(\bar{q}(0)^P)$. Set $\bar{q}(i)^j = \bar{p}(i)^j$ for $j = 0, 1$ and observe that $\bar{q} \preceq^{\lambda'} \bar{p}$. Thus as $\lambda' \geq \lambda_0$, we see \bar{q} satisfies the hypothesis of lemma 4.19 and thus $\bar{q} \in \mathbf{Am}_1$. Lastly, as $\bar{q}(i)^P$ is a greatest lower bound of $\{p_i^n\}_{n \in \mathbb{N}}$, we conclude $\bar{q}(i)^P \in \text{dom}(\mathbf{C}^\lambda)$. For each $i \in \mathbb{Z}$, fix $c(i) \in \mathbf{C}^\lambda(\bar{q}(i)^P)$.

Fix $i \in \mathbb{Z} \setminus \{0\}$ and $j \in \{0\}$. At the beginning, we chose q_0 such that $\pi_j(\bar{p}(i))$ is λ' -chromatic below q_0 . So we may fix a λ' -spectrum $H^j(i)$ of $\pi_j(\bar{p}(i))$ below q_0 —and hence also below $\pi(\bar{q}(0)^P) \leq q_0$. As $\bar{q} \preceq^{\lambda'} \bar{p}$ we have $\pi_j(\bar{q}(i)) = \pi(\bar{q}(i)^P) \cdot \pi_j(\bar{p}(i))$. Thus, as $i \in \mathbb{Z} \setminus \{0\}$ and $j \in \{0, 1\}$ were arbitrary,

$$(c(i), \lambda', H^0(i), H^1(i))_{i \in \mathbb{Z}} \in \bar{\mathbf{C}}^\lambda(\bar{q}).$$

☺

Now we check that the pre-stratification system on \mathbf{Am}_1 extends that of P . We start with 3.18, i.e. the conditions necessary for the preservation of quasi-closure. Clearly, if $p, q \in P$ and $q \preceq^\lambda p$, by $(\triangleleft_c 2)$ for (Q, P) we have $a(q) \preceq^\lambda a(p)$. So $(\triangleleft_c 1)$ holds. If $\bar{p} \preceq^\lambda \bar{q}$, of course $\bar{p}(0)^P \preceq^\lambda \bar{q}(0)^P$, so $(\triangleleft_c 2)$ holds. By the definition of $\bar{\mathbf{F}}$, $\bar{\pi}(\bar{\mathbf{F}}(\lambda, x, \bar{p})) = \mathbf{F}(\lambda, x, \bar{p}(0)^P)$, i.e. $(\triangleleft_c 3)$ holds. We continue with the conditions of 3.18. For $(\triangleleft_s 1)$, it suffices to check $(\triangleleft_s A)$ and $(\triangleleft_s B)$.

$(\triangleleft_s A)$: Say $q \in P$ and \bar{p} are such that $q \preceq^\lambda \bar{\pi}(\bar{p})$. Let $i \in \mathbb{Z} \setminus \{0\}$. By $(\triangleleft_s A)$ for (Q, P) we have $\pi(q) \cdot \bar{p}(i)^P \preceq^\lambda \bar{p}(i)^P$. Moreover, $\pi_j(\bar{p} \cdot q(i)) = \pi(q) \cdot \pi_j(\bar{p}(i))$, so as $\bar{p} \cdot q(0) = q \preceq^\lambda \bar{p}(0)^P$, we conclude $\bar{p} \cdot q \preceq^\lambda \bar{p}$.

$(\triangleleft_s B)$: Say $q \in P$ and $\bar{p}, \bar{r} \in \mathbf{Am}_1$ are such that $q \leq \bar{\pi}(\bar{r})$ and $\bar{r} \preceq^{\lambda'} \bar{p}$. Let $i \in \mathbb{Z} \setminus \{0\}$. By $(\triangleleft_s B)$ for (Q, P) we have $\pi(q) \cdot \bar{r}(i)^P \preceq^\lambda \pi(q) \cdot \bar{p}(i)^P$. Moreover,

$$\begin{aligned} \pi_j(\bar{r} \cdot q(i)) &= \pi(q) \cdot \pi_j(\bar{r}(i)) \\ &= \pi(q) \cdot \pi(\bar{p}(0)^P) \cdot \pi_j(\bar{p}(i)) = \pi(q) \cdot \pi_j(\bar{p}(i)) = \pi_j(\bar{p} \cdot q(i)), \end{aligned}$$

so as $\bar{r} \cdot q(0) = q = \bar{p} \cdot q(0)$, we conclude $\bar{r} \cdot q \preceq^{\lambda'} \bar{p} \cdot q$. Conditions $(\triangleleft_s 2)$, $(\triangleleft_s 3)$ and $(\triangleleft_s 5)$ are left to the reader. Being cautious, we check $(\triangleleft_s 4)$. Say $w \in P$, $\bar{d}, \bar{r} \in \mathbf{Am}_1$ and $w \leq \bar{\pi}(\bar{d})$ while $\bar{d} \preceq^{\lambda'} \bar{r}$. By $(\triangleleft_s 4)$ for (Q, P) , we have $w \cdot \bar{d}(i)^P \preceq^\lambda w \cdot \bar{r}(i)^P$. As $\bar{d} \cdot w(0) = w = \bar{r} \cdot w(0)$, we conclude that $w \cdot \bar{d} \preceq^\lambda w \cdot \bar{r}$.

We check 3.2, i.e. that (P, \mathbf{Am}_1) is a quasi-closed extension. $(E_c I)$: Say $\bar{p} \preceq^{\lambda'} \bar{\pi}(\bar{p})$, that is, $\bar{p} \preceq^{\lambda'} a(\bar{p}(0)^P)$. Write $p_0 = \bar{p}(0)^P$, and $p'_0 = \mathbf{F}(\lambda, x, p_0)^P$.

Fix $i \in \mathbb{Z} \setminus \{0\}$ and $j \in \{0, 1\}$. We have

$$\bar{p}(i)^P \preceq^{\bar{\lambda}} \pi(p_0) \quad (4.23)$$

$$\pi_j(\bar{p}(i)) = \pi(p_0) \cdot \pi_j(a(p_0)(i)). \quad (4.24)$$

Since $\pi(p'_0) \leq \pi(p_0)$ and $\mathbf{F}(\lambda, x, \bar{p}(i)^P) \preceq^\lambda \bar{p}(i)^P \in \mathbf{D}$ and by (4.24), we infer

$$\begin{aligned} \pi_j(\mathbf{F}(\lambda, x, \bar{p})(i)) &= \pi_j(\mathbf{F}(\lambda, x, \bar{p}(i)^P) \cdot \bar{p}(i)^0 \cdot \bar{p}(i)^1) \\ &= \pi(\mathbf{F}(\lambda, x, \bar{p}(i)^P)) \cdot \pi_j(\bar{p}(i)) \\ &= \pi(p'_0) = \pi(p'_0) \cdot \pi_j(a(p'_0)(i)). \end{aligned} \quad (4.25)$$

The next to last equation holds as $\pi_j(a(p_0)(i)) = \pi(p_0)$ and the last equation holds as $\pi_j(a(p'_0)(i)) = \pi(p'_0)$. By (4.23) we infer $\mathbf{F}(\lambda, x, \bar{p}(i)^P) \preceq^{\bar{\lambda}} \pi(\mathbf{F}(\lambda, x, \bar{p}(i)^P))$. Finally, as $i \in \mathbb{Z} \setminus \{0\}$ and $j \in \{0, 1\}$ were arbitrary, this together with (4.25) allows us to infer $\bar{\mathbf{F}}(\lambda, x, \bar{p}) \preceq^{\bar{\lambda}} a(p'_0)$ and we are done.

We prove (E_cII): So say $\bar{\lambda} < \lambda$ and $\bar{p} = (\bar{p})_{\xi < \rho}$ is a $(\lambda^*, \bar{\lambda}, x)$ -adequate sequence in \mathbf{Am}_1 with a $\bar{\pi}$ -bound $p \in P$. Fix $i \in \mathbb{Z} \setminus \{0\}$. By definition of $\bar{\mathbf{F}}$ and $\preceq^{\bar{\lambda}}$, the sequence $\{\bar{p}_\xi(i)^P\}_{\xi < \rho}$ is $(\lambda^*, \bar{\lambda}, x)$ -adequate in P . Since $\{\pi(\bar{p}_\xi(0)^P)\}_{\xi < \rho}$ is the same as $\{\pi(\bar{p}_\xi(i)^P)\}_{\xi < \rho}$, the condition $\pi(p(0)^P) = \pi(p(i)^P) \in Q$ is a π -bound of the latter sequence. Thus by (E_cII) for (Q, P) , the sequence $\{\bar{p}(i)^P\}_{\xi < \rho}$ has a greatest lower bound $p_i \in P$ such that for all $\xi < \rho$, $p_i \preceq^{\lambda^*} \bar{p}_\xi(i)^P$ and $\pi(p_i) = \pi(\bar{p}(0)^P)$. Moreover, if $\lambda^* < \bar{\lambda}$, we have $p_i \preceq^{\bar{\lambda}} \pi(p_i)$. For each $i \in \mathbb{Z}$, let $\bar{p}(i)^P = p_i$ and for $j \in \{0, 1\}$ let $\bar{p}(i)^j = \bar{p}_0(i)^j$. By corollary 4.21, $\bar{p} \in \mathbf{Am}_1$ and $\bar{p} \preceq^{\lambda^*} \bar{p}_0$. We must check that for all $\xi < \rho$, $\bar{p} \preceq^{\bar{\lambda}^*} \bar{p}_\xi$. This is clear as for every $i \in \mathbb{Z}$ we have $\bar{p}(i)^P \preceq^{\lambda^*} \bar{p}_\xi(i)^P$ by construction, and for every $i \in \mathbb{Z} \setminus \{0\}$ and $j \in \{0, 1\}$ we have

$$\pi_j(\bar{p}(i)) = \pi(\bar{p}(i)^P) \cdot \pi_j(\bar{p}_0(i)) = \pi(\bar{p}(i)^P) \cdot \pi_j(\bar{p}_\xi(i)),$$

where the first equation holds since $\bar{p} \preceq^{\lambda^*} \bar{p}_0$ second equation holds since

$$\pi(\bar{p}(i)^P) = \pi(p) \leq \pi(\bar{p}_\xi(i)^P)$$

and $\bar{p}_\xi \preceq^{\lambda^*} \bar{p}_0$ gives us

$$\pi_j(\bar{p}_\xi(i)) = \pi(\bar{p}_\xi(i)^P) \cdot \pi_j(\bar{p}_0(i)).$$

We check the remaining conditions of 3.19, showing that (P, \mathbf{Am}_1) is a stratified extension above $(\lambda_0)^+$.

(E_sI): Say $\bar{p} = (\bar{p})_{\xi < \rho}$ and $\bar{q} = (\bar{q})_{\xi < \rho}$ are both $(\lambda^*, \bar{\lambda}, x)$ -adequate for $\bar{\lambda} < \lambda$, such that

$$\forall \xi < \rho \quad \bar{\mathbf{C}}^\lambda(\bar{p}_\xi) \cap \bar{\mathbf{C}}^\lambda(\bar{q}_\xi) \neq \emptyset. \quad (4.26)$$

Say the sequence $\bar{p} = (\bar{p})_{\xi < \rho}$ has a greatest lower bound \bar{p} , the sequence $\bar{q} = (\bar{q})_{\xi < \rho}$ has a greatest lower bound \bar{q} and say

$$\mathbf{C}^\lambda(\bar{p}(0)^P) \cap \mathbf{C}^\lambda(\bar{q}(0)^P) \neq \emptyset. \quad (4.27)$$

We show

$$\bar{\mathbf{C}}^\lambda(\bar{p}) \cap \bar{\mathbf{C}}^\lambda(\bar{q}) \neq \emptyset. \quad (4.28)$$

Fix $c(0) \in \mathbf{C}^\lambda(\bar{p}(0)^P) \cap \mathbf{C}^\lambda(\bar{q}(0)^P)$, and let $\lambda'(0)$, $H^0(0)$ and $H^1(0)$ be arbitrary. Fix $i \in \mathbb{Z} \setminus \{0\}$. By (4.27) and as $\pi(\bar{p}(i)^P) = \pi(\bar{p}(0)^P)$ (analogously for \bar{q}) we have

$$\mathbf{C}^\lambda(\pi(\bar{p}(i)^P)) \cap \mathbf{C}^\lambda(\pi(\bar{q}(i)^P)) \neq \emptyset. \quad (4.29)$$

Moreover, as in the previous proof, $\{\bar{p}_\xi(i)^P\}_{\xi < \rho}$ and $\{\bar{q}_\xi(i)^P\}_{\xi < \rho}$ are $(\lambda^*, \bar{\lambda}, x)$ -adequate and so by (E_sI) for (Q, P) we can find $c(i) \in \mathbf{C}^\lambda(\bar{p}(i)^P) \cap \mathbf{C}^\lambda(\bar{q}(i)^P)$. We have now constructed $c(i)$ for $i \in \mathbb{Z}$ which together with appropriate $\lambda'(i)$, $H^0(i)$, $H^1(i)$ will witness (4.28). Fix $(c_0(i), \lambda'_0(i), H_0^0(i), H_0^1(i))_{i \in \mathbb{Z}} \in \bar{\mathbf{C}}^\lambda(\bar{p}_0) \cap \bar{\mathbf{C}}^\lambda(\bar{q}_0)$. We shall now check that

$$(c(i), \lambda'_0(i), H_0^0(i), H_0^1(i))_{i \in \mathbb{Z}} \in \bar{\mathbf{C}}^\lambda(\bar{p}) \cap \bar{\mathbf{C}}^\lambda(\bar{q}). \quad (4.30)$$

This is clear by definition: fix $i \in \mathbb{Z} \setminus \{0\}$ and $j \in \{0, 1\}$. Firstly, $\bar{p} \preceq^\lambda \bar{p}_0$ and so

$$\pi_j(\bar{p}(i)) = \pi(\bar{p}(i)^P) \cdot \pi_j(\bar{p}_0(i)).$$

Moreover, by choice of $\lambda'_0(i)$ and $H_0^j(i)$ there is $b^j \in B_j$ such that we have

$$\pi_j(\bar{p}_0(i)) = \pi(\bar{p}_0(i)^P) \cdot b^j.$$

and H_0^j is a $\lambda'_0(i)$ -spectrum for b^j below $\pi(\bar{p}(i)^P)$. The last two equations together yield

$$\pi_j(\bar{p}(i)) = \pi(\bar{p}(i)^P) \cdot b^j,$$

and so (4.30) holds. This finishes the proof of (E_sI).

(E_sII), *coherent expansion*: Assume $\bar{q} \preceq^\lambda \bar{p}$ and $\bar{p} \preceq^\lambda \bar{a}(\bar{p}(0))$. Moreover, assume $\bar{q}(0) \leq \bar{p}(0)$. We show $\bar{q} \leq \bar{p}$. Let $i \in \mathbb{Z} \setminus \{0\}$ be arbitrary. As $\bar{q}(i)^P \preceq^\lambda \bar{p}(i)^P$ and $\bar{p}(i)^P \preceq^\lambda a(\bar{p}(0)^P)(i)^P = \pi(\bar{p}(i)^P)$, and as $\pi(\bar{q}(0)^P) \leq \pi(\bar{p}(0))$, we have $\bar{p}(i)^P \leq \bar{q}(i)^P$ by (E_sII) for (Q, P) . Say $i \neq 0$ and $j \in \{0, 1\}$. Then

$$\pi_j(\bar{p}(i)) \leq \pi(\bar{p}(i)) = \pi(\bar{p}(0)^P) \leq \pi(\bar{q}(0)^P) = \pi(\bar{q}(i)) = \pi_j(\bar{q}(i)),$$

where the last equality holds as $\bar{q}(i)^P \preceq^\lambda \pi(\bar{q}(0)^P) \in Q \subseteq \mathbf{D}$. Thus we have $\bar{q}(i)^X \leq \bar{p}(i)^X$ for $X \in \{0, 1, P\}$ and each $i \in \mathbb{Z}$, whence by (4.4), $\bar{q} \leq \bar{p}$, and we are done.

We show *coherent interpolation* (E_sIII): Let $\bar{d}, \bar{r} \in \mathbf{Am}_1$ be such that $\bar{d} \preceq^\lambda \bar{r}$, and say $p \in P$ interpolates $\bar{\pi}(\bar{d})$ and $\bar{\pi}(\bar{r})$. We find $\bar{p} \in \mathbf{Am}_1$ such that $\bar{p} \preceq^\lambda \bar{r}$ and $\bar{p} \preceq^\lambda \bar{d}$ and moreover $\bar{\pi}(\bar{p}) = p$. For $i \in \mathbb{Z} \setminus \{0\}$, use coherent interpolation for (Q, P) to find $p_i \in P$ such that $p_i \preceq^\lambda \bar{r}(i)^P$ and $p_i \preceq^\lambda \bar{d}(i)^P$ and moreover $\pi(p_i) = \pi(p)$. Now we define a sequence $\bar{p}: \mathbb{Z} \rightarrow P \times B_0 \times B_1$. Set $\bar{p}(0) = (p, 1, 1)$ and set $\bar{p}(i) = (p_i, \bar{r}(i)^0, \bar{r}(i)^1)$ for $i \in \mathbb{Z} \setminus \{0\}$. Clearly, $\bar{p} \preceq^\lambda \bar{r}$ and so $\bar{p} \in \mathbf{Am}_1$. By construction, $\bar{p} \preceq^\lambda \bar{d}$ and $\bar{\pi}(\bar{p}) = \bar{p}(0)^P = p$.

It remains to demonstrate (E_sIV):

Lemma 4.24. *Coherent centering holds: Say $\lambda > \lambda_0$, $\bar{p} \preceq^\lambda \bar{d}$ and $\bar{\mathbf{C}}^\lambda(\bar{p}) \cap \bar{\mathbf{C}}^\lambda(\bar{d}) \neq \emptyset$. Say further we have $w_0 \in P$ such that $w_0 \preceq^{<\lambda} \bar{p}(0)^P$ and $w_0 \preceq^{<\lambda} \bar{d}(0)^P$. Then there is $\bar{w} \in \mathbf{Am}_1$ such that $\bar{\pi}(\bar{w}) = w_0$ and both $\bar{w} \preceq^{<\lambda} \bar{p}$ and $\bar{w} \preceq^{<\lambda} \bar{d}$.*

Proof. Fix \bar{p}, \bar{d} and w_0 as in the hypothesis. Fix $i \in \mathbb{Z} \setminus \{0\}$ for the moment. Observe we have Since $\bar{\mathbf{C}}^\lambda(\bar{p}(i)^P) \cap \bar{\mathbf{C}}^\lambda(\bar{d}(i)^P) \neq \emptyset$, by coherent centering for (Q, P) we can find $w_i \in P$ such that $\pi(w_i) = \pi(w_0)$. If the additional assumption at the end of the lemma holds, we may assume $w_i \preceq^{<\lambda} \bar{p}(i)^P$ and $w_i \preceq^{<\lambda} \bar{d}(i)^P$. For $i \in \mathbb{Z}$, set

$$\bar{w}(i) = (w_i, \bar{p}(i)^0, \bar{p}(i)^1).$$

Since $\bar{w} \preceq^{\lambda_0} \bar{p}$ and $\pi(\bar{w}(i)^P) = \pi(w_0)$ for each $i \in \mathbb{Z}$, by lemma 4.19, $\bar{w} \in \mathbf{Am}_1$.

Now say the additional assumption holds. By construction, $\bar{w}(i)^P \preceq^{<\lambda} \bar{p}$ for each $\lambda' \in [\lambda_0, \lambda)$. Fix $i \in \mathbb{Z}$. Since $\bar{\mathbf{C}}^\lambda(\bar{p}) \cap \bar{\mathbf{C}}^\lambda(\bar{d}) \neq \emptyset$, $\bar{p}(i)^j$ and $\bar{d}(i)^j$ have a common λ -spectrum below $\pi(w_0)$, and so

$$\pi(w_0) \Vdash_Q \bar{p}(i)^j = \bar{d}(i)^j. \quad (4.31)$$

Thus for each $i \in \mathbb{Z}$,

$$\bar{w}(i) = w(i)^P \cdot d(i)^0 \cdot d(i)^1 \leq \bar{d}(i)$$

whence $\bar{w} \leq \bar{d}$. In fact, as $\bar{w}(i)^P \preceq^{\lambda'} \bar{d}(i)^P$ and (4.31) holds, $\bar{w} \preceq^{\lambda'} \bar{d}$ for each $\lambda' \in [\lambda_0, \lambda)$. ☺

☺

Corollary 4.25. *(P, \mathbf{Am}_1) is a stratified extension above λ_0^+ .*

4.4 Stratified type-2 amalgamation

We now consider the simpler case when we want to extend an automorphism already defined on an initial segment of the iteration. Let P be a forcing, Q a complete sub-order, f an automorphism of Q and $\pi: P \rightarrow Q$ a strong projection. Assume λ_0 is regular and (Q, P) is a stratified extension above λ_0 . We denote the stratification on Q by $\preceq_Q^\lambda, \prec_Q^\lambda, \dots$ and write $\preceq^\lambda, \prec^\lambda, \dots$ for the stratification of P .

Further we assume that for each regular λ and $q, r \in Q$,

1. $q \preceq_Q^\lambda r \iff f(q) \preceq_Q^\lambda f(r)$;
2. $q \prec_Q^\lambda r \iff f(q) \prec_Q^\lambda f(r)$;
3. for each k , $f(\mathbf{F}_Q(\lambda, x, q)) = \mathbf{F}(\lambda, f(q))$;
4. $\mathbf{C}_Q^\lambda(q) \cap \mathbf{C}_Q^\lambda(r) \neq \emptyset \iff \mathbf{C}_Q^\lambda(f(q)) \cap \mathbf{C}_Q^\lambda(f(r)) \neq \emptyset$.

We define the type-2 amalgamation $\mathbf{Am}_2(Q, P, f)$ (or just \mathbf{Am}_2 where the context allows) as the set of all $\bar{p}: \mathbb{Z} \rightarrow P$ such that for all $i \in \mathbb{Z}$,

$$f(\pi(\bar{p}(i))) = \pi(\bar{p}(i+1)). \quad (4.32)$$

The ordering is $\bar{p} \leq \bar{q}$ if and only if for each $i \in \mathbb{Z}$, $\bar{p}(i) \leq \bar{q}(i)$.

Define $\bar{\pi}: \mathbf{Am}_2 \rightarrow P$ by $\bar{\pi}(\bar{p}) = \bar{p}(0)$. The map $\bar{e}: P \rightarrow \mathbf{Am}_2$ is defined by $\bar{e}(p)(0) = p$ and $\bar{e}(p)(i) = \pi(p)$ for all $i \in \mathbb{Z}$, $i \neq 0$.

It is straightforward to check that \bar{e} is a complete embedding and $\bar{\pi}$ is the restriction of the canonical projection from $\text{r.o.}(\mathbf{Am}_2)$ to $\text{r.o.}(\bar{e}[P])$. Moreover, if $q \in P$, $\bar{p} \in \mathbf{Am}_2$ and $q \leq \bar{\pi}(\bar{p})$, then $q \cdot \bar{p} \in \mathbf{Am}_2$.

We now define the stratification of \mathbf{Am}_2 , consisting of $\bar{\mathbf{C}}^\lambda, \bar{\mathbf{F}}^\lambda, \bar{\preceq}^\lambda, \bar{\prec}^\lambda$ for each regular $\lambda \geq \lambda_0$. We say $\bar{q} \bar{\preceq}^\lambda \bar{p}$ exactly if for every $i \in \mathbb{Z}$, $\bar{q}(i) \preceq^\lambda \bar{p}(i)$, and $\bar{q} \bar{\prec}^\lambda \bar{p}$ exactly if for every $i \in \mathbb{Z}$, $\bar{q}(i) \prec^\lambda \bar{p}(i)$. Define $\bar{\mathbf{F}}^\lambda(\bar{p}, k)(i) = \mathbf{F}(\lambda, x, \bar{p}(i))$. For \bar{p} such that for each $i \in \mathbb{Z}$, $\bar{p}(i) \in \text{dom}(\mathbf{C}^\lambda)$, we define $\bar{\mathbf{C}}^\lambda(\bar{p})$ to be the set of all $c: \mathbb{Z} \rightarrow \lambda$ such that for each $i \in \mathbb{Z}$, $c(i) \in \mathbf{C}^\lambda(\bar{p}(i))$.

Lemma 4.26. (P, \mathbf{Am}_2) is a stratified extension above λ_0 .

Proof. The proof is a slight modification of the argument for type-1 amalgamation. Therefore, we only point out the main points, and leave the rest to the reader.

Lemma 4.27. (P, \mathbf{Am}_2) is a quasi-closed extension above λ_0 .

Proof. Let $(\bar{p}_\xi)_{\xi < \theta}$ be λ -adequate. Fix $i \in \mathbb{Z}$ and let $\bar{p}(i)$ be the greatest lower bound of the λ -adequate sequence $(\bar{p}_\xi(i))_{\xi < \theta}$. By coherency, $(\pi(\bar{p}_\xi(i)))_{\xi < \theta}$ is also adequate and its greatest lower bound is $\pi(\bar{p}(i))$. As f is an automorphism, for each $i \in \mathbb{Z}$, $f(\pi(\bar{p}(i)))$ is a greatest lower bound of $(q_\xi(i))_{\xi < \theta}$, where $\bar{q}_\xi(i) = f(\pi(\bar{p}_\xi(i)))$. As the latter is equal to $\pi(\bar{p}_\xi(i-1))$, we obtain (4.32) for \bar{p} . So $\bar{p} \in \mathbf{Am}_2$; it is straightforward to check it is a greatest lower bound of $(\bar{p}_\xi)_{\xi < \theta}$. $\textcircled{\smile}$

Let $\lambda > \lambda_0$.

Lemma 4.28. *Coherent interpolation holds, i.e. whenever $\bar{r}, \bar{d} \in \mathbf{Am}_2$, $\bar{d} \leq \bar{r}$ and $p_0 \in P$ such that $p_0 \preceq^\lambda \bar{\pi}(\bar{r})$ and $p_0 \preceq^\lambda \bar{\pi}(\bar{d})$, there is $\bar{p} \in \mathbf{Am}_2$ such that $\bar{p} \preceq^\lambda \bar{r}$, $\bar{p} \preceq^\lambda \bar{d}$ and $\bar{\pi}(\bar{p}) = p_0$.*

Proof. Given \bar{r}, \bar{d} and p_0 as above, first set $\bar{p}(0) = p_0$. As $\pi(\bar{r}(i)) = \pi(\bar{r}(0))$ and $\pi(\bar{d}(i)) = \pi(\bar{d}(0))$, $p_0 \preceq^\lambda \pi(\bar{d}(i))$ and $p_0 \preceq^\lambda \pi(\bar{r}(i))$, for all $i \in \mathbb{Z}$. Coherent interpolation for (Q, P, π) allows us to find, for each $i \in \mathbb{Z}$, $i \neq 0$ a condition $\bar{p}(i) \in P$ such that $\pi(\bar{p}(i)) = \pi(p_0)$, $p_0 \preceq^\lambda \bar{d}(i)$ and $p_0 \preceq^\lambda \bar{r}(i)$. As for each $i \in \mathbb{Z}$, $\pi(\bar{p}(i)) = \pi(p_0)$, $\bar{p} \in \mathbf{Am}_2$. $\textcircled{\smile}$

Lemma 4.29. *Coherent centering holds. That is: Say $\bar{p} \preceq^\lambda \bar{d}$ and either of the following holds: $\mathbf{C}^\lambda(\bar{p}) \cap \mathbf{C}^\lambda(\bar{d}) \neq \emptyset$ or for some $q \in Q$, $\bar{p} \preceq^\lambda q$ or $\bar{d} \preceq^\lambda q$. Say further $w_0 \in \mathbf{D}$ such that for each regular $\lambda' \in [\lambda_0, \lambda)$, $w_0 \preceq^{\lambda'} \bar{\pi}(\bar{p})$ and $w_0 \preceq^{\lambda'} \bar{\pi}(\bar{d})$. Then there is $\bar{w} \in \mathbf{Am}_2$ such that for each regular $\lambda' \in [\lambda_0, \lambda)$, $\bar{w} \preceq^{\lambda'} \bar{p}$, $\bar{w} \preceq^{\lambda'} \bar{d}$ and $\bar{\pi}(\bar{w}) = w_0$.*

Lemma 4.30. *If $\bar{p} \in \mathbf{Am}_2$ and $\lambda' \in [\lambda_0, \lambda)$ there is $\bar{q} \in \mathbf{Am}_2$ such that $\bar{q} \in \text{dom}(\bar{\mathbf{C}}^{\lambda'})$ and $\bar{q} \preceq^{\lambda'} \bar{p}$.*

Proof. We define a sequence $(q_i)_{i \in \mathbb{Z}}$ of conditions in P , by induction on i . As usual, read through the following definition and pick x such that it is Δ_1^A with parameters in x . First, find $q_0 \preceq^{\lambda'} \bar{p}(0)^P$ such that $q_0 \in \text{dom}(\mathbf{C}^{\lambda'})$. Continue by induction, choosing, for each $n \in \mathbb{N} \setminus \{0\}$, a condition $q_n \preceq^{\lambda'} \bar{p}(n) \cdot \mathbf{F}(\lambda, x, \pi(q_{n-1}))$ such that $q_n \in \text{dom}(\mathbf{C}^{\lambda'})$. Let q_1^* be a greatest lower bound for $(\pi(q_k))_{k \in \mathbb{N}}$; it exists by quasi-closure for Q . Find $q_1 \preceq^{\lambda'} q_1^* \cdot \bar{p}(1)$ such that $q_1 \in \text{dom}(\mathbf{C}^{\lambda'})$. Again, continue by induction, choosing for each $n \in \mathbb{N} \setminus \{0, 1\}$, a condition $q_{-n} \preceq^{\lambda'} \bar{p}(-n) \cdot \mathbf{F}(\lambda, x, \pi(q_{-n+1}))$ such that $q_{-n} \in \text{dom}(\mathbf{C}^{\lambda'})$. Finally, let q be a greatest lower bound of $(\pi(q_{-k}))_{k \in \mathbb{N}}$. For each $i \in \mathbb{Z}$, $q \preceq^{\lambda'} \pi(q_i)$, so $q \cdot q_i \preceq^{\lambda'} q_i \preceq^{\lambda'} \bar{p}(i)$. Observe that by coherent stratification, $q \cdot q_i \in \text{dom}(\mathbf{C}^{\lambda'})$ for each $i \in \mathbb{Z}$. Setting $\bar{q}(i) = q \cdot q_i$, we have $\pi(\bar{q}(i)) = q$, for all $i \in \mathbb{Z}$. Thus $\bar{q} \in \mathbf{Am}_2$, $\bar{q} \preceq^{\lambda'} \bar{p}$ and $\bar{q} \in \text{dom}(\bar{\mathbf{C}}^{\lambda'})$. $\textcircled{\smile}$

We leave the rest of the proof that (P, \mathbf{Am}_2) is a stratified extension above λ_0 to the reader. $\textcircled{\smile}$

4.5 Remote sub-orders, and a stable meet operator

The following lemma helps to ensure “coding areas” don’t get mixed up by the automorphisms, as we shall see in lemmas 5.4 and 6.2. Also see the discussion at the beginning of section 3.5.

Lemma 4.31. *Say C is remote in P over Q (up to some height κ , where $\kappa \geq \lambda_0$). Then $\Phi^k[C]$ is remote in \mathbf{Am}_1 over P (up to the same height) for any $k \in \mathbb{Z} \setminus \{0\}$.*

Proof. Let $D^* = \bar{\pi}^{-1}[\mathbf{D}] \subseteq \mathbf{Am}_1 \cap \mathbf{D}_f^{\mathbb{Z}}$ as in lemma 4.18. Let $j \in \{0, 1\}$ arbitrary. If $\hat{p} \in \hat{\mathbf{D}}$, $c \in C$ and $c \leq \pi_C(\hat{p}^P)$, by the definition of \mathbf{D} ,

$$\pi_Q(\hat{p} \cdot c) \Vdash \pi_j(\hat{p} \cdot c) = \pi_j(\hat{p}),$$

that is, $\pi_j(\hat{p} \cdot c) = \pi_j(\hat{p}) \cdot \pi_Q(\hat{p}^P \cdot c)$, so as C is independent over Q and thus $\pi_Q(\hat{p}^P \cdot c) = \pi_Q(\hat{p}^P)$, we have $\pi_j(\hat{p} \cdot c) = \pi_j(\hat{p})$. In fact, if we have $\hat{p}^P \in \mathbf{D}$, we have $c \cdot \hat{p} \in \hat{\mathbf{D}}$. Observe further that for any $c \in C$, $\pi_j(c) = 1$, and moreover, $C \subseteq \mathbf{D}$. Thence, $C \subseteq D^* \subseteq \text{dom}(\Phi^k)$. Moreover, $\Phi^k(c)(0) = e_k(c)(0) = (1, 1, 1)$ and so $\Phi^k(c) \in D^* \subseteq \mathbf{Am}_1$.

We now show $\Phi^k[C] = e_k[C]$ is independent in \mathbf{Am}_1 over P : Let $c \in C$, $\bar{p} \in \mathbf{Am}_1$ and say $c \leq (\bar{\pi}_k \circ \pi_C)(\bar{p}) = \pi_C(\bar{p}(k))$.

Since $\pi_j(c \cdot \bar{p}(k)) = \pi_j(\bar{p}(k))$, for every $i \in \mathbb{Z}$,

$$i \neq k: e_k(c) \cdot \bar{p}(i) = \bar{p}(i). \quad (4.33)$$

Thus $e_k(c) \cdot \bar{p} \in \mathbf{Am}_1$. This firstly shows that $\bar{\pi}_k \circ \pi_C$ is a strong projection from \mathbf{Am}_1 to C . Moreover $\bar{\pi}(\bar{p} \cdot e_k(c)) = \bar{p}(0) = \bar{\pi}(\bar{p})$, and we are done with the proof that $\Phi^k[C] = e_k[C]$ is independent in \mathbf{Am}_1 over P .

It follows that $\Phi^k[C]$ is remote in \mathbf{Am}_1 over P , by definition of $\bar{\pi}^\lambda$: let $\lambda \in [\lambda(P), \kappa)$. Say $c \leq \bar{\pi}_k(\bar{p})$. Then $e_k(c) \cdot \bar{p}(k) = (\bar{p}(k)^P \cdot c, \bar{p}(k)^0, \bar{p}(k)^1)$ and $\bar{p}(k)^P \cdot c \preceq^\lambda p(k)^P$, by clause (1) of definition 3.32. So by (4.33), $e_k(c) \cdot \bar{p} \bar{\preceq}^\lambda \bar{p}$, and we are done. $\textcircled{\smile}$

The last lemma of this section is the counterpart of lemmas 3.29 and 3.30. Together these lemmas make sure that in the iteration used in our application, we have stable meet operators for every initial segment.

Lemma 4.32. *There is a P -stable meet operator \wedge_P on \mathbf{Am}_1 .*

Proof. Of course we set

$$\text{dom}(\wedge_P) = \{(\bar{p}, \bar{r}) \mid \exists \lambda \in \text{Reg} \setminus \lambda_0 \quad \bar{r} \preceq^\lambda \bar{\pi}(\bar{r}) \cdot \bar{p}\}.$$

Say we have $\bar{p}, \bar{r} \in \mathbf{Am}_1$ such that $(\bar{p}, \bar{r}) \in \text{dom}(\wedge_P)$. This means we can fix a regular $\lambda \geq \lambda_0$ such that for each $i \in \mathbb{Z} \setminus \{0\}$, $\bar{r}(i)^P \preceq^\lambda \pi(\bar{r}(i)^P) \cdot \bar{p}(i)^P$. Let $w_i = \bar{p}(i)^P \wedge \bar{r}(i)^P$ for $i \in \mathbb{Z} \setminus \{0\}$ and set

$$\bar{p} \wedge_P \bar{r} = \begin{cases} (w_i, \bar{p}(i)^0, \bar{p}(i)^1) & \text{for } i \in \mathbb{Z} \setminus \{0\} \\ \bar{p}(0) & \text{for } i = 0. \end{cases}$$

Let $\bar{p} \wedge_P \bar{r}$ be denoted by \bar{w} . By lemma 4.15, for $i \in \mathbb{Z} \setminus \{0\}$ and $j \in \{0, 1\}$ we have

$$\begin{aligned} \pi_j(\bar{w}(i)) &= \bar{p}(i)^j \cdot \pi_j(w_i \cdot \bar{p}(i)^{1-j}) \\ &= \bar{p}(i)^j \cdot \pi_j(\bar{p}(i)^P \cdot \bar{p}(i)^{1-j}) = \pi_j(\bar{p}(i)). \end{aligned} \quad (4.34)$$

In particular, as $w_i \preceq^\lambda \bar{p}(i)^P$ and i was arbitrary, we have


$$\bar{p} \wedge_P \bar{r} \preceq^\lambda \bar{p}. \quad (4.35)$$

Moreover, $\pi(w_i) = \pi(\bar{p}(i)^P) = \pi(\bar{p}(0)) = \pi(\bar{w}(0))$. So \bar{w} satisfies the hypothesis of lemma 4.19 and therefore $\bar{w} \in \mathbf{Am}_1$. Clearly, $\bar{\pi}(\bar{p} \wedge_P \bar{r}) = \bar{p}(0)$. It remains to see that $\bar{\pi}(\bar{r}) \cdot (\bar{p} \wedge_P \bar{r}) \approx \bar{r}$; we have

$$\bar{\pi}(\bar{r}) \cdot \bar{w} = \begin{cases} (\pi(\bar{r}(0)^P) \cdot w_i, \bar{p}(i)^0, \bar{p}(i)^1) & \text{for } i \in \mathbb{Z} \setminus \{0\} \\ \bar{r}(0) & \text{for } i = 0. \end{cases}$$

Write $\bar{u} = \bar{\pi}(\bar{r}) \cdot \bar{w}$ and write $\bar{v} = \bar{\pi}(\bar{r}) \cdot \bar{p}$. For arbitrary $i \in \mathbb{Z} \setminus \{0\}$ and $j \in \{0, 1\}$ we have

$$\begin{aligned} \pi_j(\bar{u}(i)) &= \pi(\bar{r}(0)^P) \cdot \pi_j(\bar{w}(i)) = \pi(\bar{r}(0)^P) \cdot \pi_j(\bar{p}(i)) && \text{by (4.34),} \\ &= \pi(\bar{r}(0)^P) \cdot \pi_j(\bar{v}(i)) = \pi_j(\bar{r}(i)) && \text{as } \bar{r} \preceq^\lambda \bar{v}. \end{aligned}$$

Thus by (4.4), $\bar{u}(i) \approx \bar{r}(i)$. As $i \in \mathbb{Z} \setminus \{0\}$ was arbitrary and as $\bar{u}(0) = \bar{r}(0)$, we conclude $\bar{u} \approx \bar{r}$, finishing the proof that \wedge_P is a P -stable meet on \mathbf{Am}_1 . 

Chapter 5

Projective measure without Baire

We begin with the assumption $V = L$ and fix κ , the least Mahlo. The first step is to force with $\bar{T} = \prod_{\xi < \kappa}^{\leq \kappa} T(\xi)$, the product with supports of size less than κ of κ -many independent, κ -closed κ^+ -Suslin trees. In fact, \bar{T} is itself a κ^+ -Suslin tree (with the product order). This adds a sequence of branches $\bar{B} = (B(\xi))_{\xi < \kappa}$, where $B(\xi)$ denotes the branch through $T(\xi)$. As a notational convenience, we often assume the sequence of trees (resp. branches) is indexed by elements of $J = {}^{<\kappa}2 \times \omega \times 2 \times 2$ rather than by ordinals in κ , that is as $B(s, n, i, j)$ and $T(s, n, i, j)$ for $s \in {}^{<\kappa}2$, $n \in \mathbb{N}$ and $i, j \in \{0, 1\}$.

We now work in $W = L[\bar{B}]$. Since \bar{T} is κ -closed and has the κ^+ -chain condition, W has the same cardinals and the same subsets of κ as L and the GCH still holds in W . In particular, L and W have the same reals.

We now define an iteration $\bar{Q}^{\kappa+1} = (P_\xi)_{\xi \leq \kappa}$, by induction on ξ . We construct this iteration to deal with the following tasks:

Task 1 Add a set of reals Γ^0 such that P_κ forces that the Baire-property fails for Γ^0 ;

Task 2 For each real r added by P_κ , make sure that P_κ forces $r \in \Gamma^0 \iff \Psi(r, 0) \iff \neg \Psi(r, 1)$, where $\Psi(x, y)$ is Σ_3^1 .

Task 3 Make sure every projective set of reals is Lebesgue-measurable in the extension by P_κ .

Task 4 To make the construction more uniform, we force with a Levy-collapse at certain stages.

We force with the Levy-collapse for two reasons: firstly, when we amalgamate, whether we collapse the continuum depends on factors beyond our control. So we always make sure we collapse the current continuum at the next stage

after amalgamation. Secondly, for the purpose of task 2 (which involves Jensen coding), we want to make sure CH holds all the time.

Task 2 requires the sophisticated technique of Jensen coding, which made its first appearance in [BJW82] and has since undergone a long development culminating in [Fri00]. In a little more detail: to tackle task 2, we will make the real r (along with information about its membership in Γ) definable by coding a subset of our set of branches \bar{B} by a real s , where s is generic for Jensen coding. Say we have iterated for ξ steps and are in $L[\bar{B}][G_\xi]$. Call the set of branches we “code” at the $\xi + 1$ -th step $\bar{B}^+ = \{B(\xi) \mid \xi \in I\}$, where $I \subseteq \kappa$.

Why do we use a subset of size κ ? Since a real carries only a countable amount of information, one would think that a countable set of branches would suffice. The point here is that the automorphisms that arise from amalgamation (task 3) will make any such coding “unreadable” (see section 6). This is not surprising since by [She84], a definable well-ordering of a set of reals of length ω_1 yields a definable non-measurable set. In fact, if the present construction is altered so that each real is coded using a block of trees of length ω , we must fail since this would add such a well-order (since the set of trees is of course well-ordered). It is also easy to see how such a coding is made unreadable: if the trivial condition forces that the real \dot{r} is coded using the branches indexed by the block $[\xi, \xi + \omega)$, for any automorphism Φ , also $\Phi(\dot{r})$ will be coded on the same block. See section 6 for the solution.

Now pick a set $A \subseteq \kappa^+$ such that $H_\kappa = L_\kappa[A \cap \kappa]$ and $\{B(\xi) \mid \xi \in I\}$ is definable in some simple recursive fashion from A . We should force to obtain a real s such that $A \in L[s]$ and moreover the following is true in the extension:

$$\begin{aligned} &\text{for all } \alpha, \beta < \kappa \text{ if } L_\beta[s] \text{ is a model of } ZF^- \text{ and of “} \alpha \text{ is the} \\ &\text{least Mahlo and } \alpha^+ \text{ exists” then:} \\ &I \cap \alpha \in L_\beta \text{ and } L_\beta[s] \models \text{“} \forall \xi \in I \cap \alpha T^\alpha(\xi) \text{ has a branch,} \end{aligned} \tag{5.1}$$

where T^α denotes the outcome of the construction of \bar{T} carried out in L_β .

This can be done using the forcing described in [Fri99] or [Fri00][section 6.2, pp. 129], a variant of Jensen coding using the so-called “David’s trick”, devised in [Dav82]. We shall use the notation from [Fri00][6.2] and [Fri00][4.2] when we speak about this forcing from now on. Observe that (5.1) already holds for $\alpha = \kappa$, so we don’t have to restrict S_κ as in section 6.2 of [Fri00], leaving the forcing *at* κ looking more like the one in section 4.2 of [Fri00] (a version of Jensen coding without using David’s trick). Also observe that since the cardinals in L and $L[\bar{B}][G_\xi]$ are the same above $(\omega_1)^{L[\bar{B}][G_\xi]}$, we don’t have to worry about reshaping at all.

Yet further care has to be taken: in order to make lemma 5.7 below go through, we want that for any inaccessible $\alpha < \kappa$, the set of $\beta < \alpha$ such that $p_\beta \neq \emptyset$ has size less than α . That is, we require “Easton support”. It is a historical coincidence that Jensen’s earliest version of the forcing to code the universe by a real used this kind of support; unfortunately no published account of this early work is available. An Easton supported variant of Jensen coding was devised in [SS95] in a much more general setting (that is, in purely combinatorial terms, without mentioning the constructible hierarchy). Lastly, we would like to be able to “speed up the coding” in the sense that we want to be able to promise that extensions of a condition will only use restraints of the form $b^s \setminus \eta$ which are subsets of a club specified by this promise. This essential but unproblematic prerequisite for the proof of lemma 5.7 is achieved using a diamond in L on the set of inaccessibles below κ .

For the sake of the present argument we will simply use the following:

Fact 5.1. There is a forcing $J(A)$ which adds a real s such that $A \in L[s]$, (5.1) holds and $J(A)$ is stratified above ω_1 (of $L[\bar{B}][G_\xi]$). Moreover, $J(A)$ is Easton supported; i.e. the set of $\beta < \alpha$ such that $p_\beta \neq \emptyset$ has size less than α for any inaccessible $\alpha < \kappa$.

There are 4 types of forcing involved, so we fix a simple and convenient partition E^0, \dots, E^3 of κ : let E^n , for $0 \leq n \leq 3$, denote the set of ordinals $\xi < \kappa$ such that for some limit ordinal η and $k \in \omega$, $\xi = \eta + k$ and $k \equiv n \pmod{4}$. For an ordinal $\xi < \kappa$, let $E^n(\xi)$ denote the ξ -th element of E^n . Also fix, for each $\rho < \kappa$, a sequence $\bar{\alpha}_\rho = (\alpha_\rho^\zeta)_{\zeta < \kappa}$ of ordinals $> \rho$ cofinal in κ ; e.g. let $\alpha_\rho^\zeta = G(\zeta, \rho)$, where G is the Gödel pairing function.

As we have to tackle certain tasks for every real of the extension, our definition will make use of two book-keeping devices, $\bar{s} = (\dot{s}_\xi)_{\xi < \kappa}$ and $\bar{r} = (\bar{i}(\xi), \dot{r}_\xi^0, \dot{r}_\xi^1)_{\xi < \kappa}$. We define \bar{s} to list all reals which end up in the complement of Γ^0 , in order to handle task 2 for each of these. To make sure all projective sets of reals are measurable (task 3) we shall define \bar{r} to list all reals added by P_κ which are random over an initial segment—random over $W^{P_{\bar{i}(\xi)}}$, to be precise. These book-keeping devices should be defined inductively, simultaneously with the P_ξ ; nonetheless, we shall first proceed with the definition of the iteration, and after that argue that a book-keeping with the requisite properties can be defined at the same time.

We define a sequence $\bar{\lambda} = (\lambda_\xi)_{\xi \leq \kappa}$ by induction:

$$\lambda_\xi = \begin{cases} \min(\text{Reg} \setminus \bigcup_{\nu < \xi} \lambda_\nu) & \text{if } \xi \text{ is limit,} \\ (\lambda_{\xi-1})^+ & \text{if } \exists \rho \text{ s.t. } \xi - 1 = E^3(\alpha_\rho^0), \\ \lambda_{\xi-1} & \text{otherwise.} \end{cases} \quad (5.2)$$

When we define P_ξ , we will also define some auxiliary sets D_ξ . At stages where we do type-1 amalgamation, they are similar to \mathbf{D} of section 4.3; at other stages $D_\xi = P_\xi$. At all limit stages $\xi \leq \kappa$, we define P_ξ to be the $\bar{\lambda}$ -diagonal support limit of the iteration up to that point.

For the inductive definition of the P_ξ etc. to make sense, we need to prove the following crucial facts, by induction on ξ (simultaneously with our definition of the iteration).

- Lemma 5.2.** *1. For all $\xi < \kappa$, $(P_\xi, P_{\xi+1})$ is a stratified extension above $\lambda_{\xi+1}$ (witnessed by the pre-stratification systems stemming from the constructions in lemmas 3.20, 3.27 and theorem 4.20 of course). Moreover, if ξ is not of the form $E^3(\alpha_\rho^0)$ for some ρ , we have that $P_{\xi+1} \Vdash |\lambda_{\xi+1}| = \omega$.*
- 2. For all $\xi < \kappa$, P_ξ is stratified above λ_ξ .*
- 3. For each $\xi < \kappa$, $P_\xi \Vdash \text{GCH}$.*
- 4. If $\nu < \xi$, $\xi \in \kappa \cap E^1$ and $p \in P_\xi$, there is a P_ξ -name \dot{r} such that p forces \dot{r} is fully random over W^{P_ν} .*

We now define P_ξ , by induction. We generally write $B_\xi = \text{r.o.}(P_\xi)$ for $\xi \leq \kappa$. In the inductive definition of the iteration, we also define

1. A sequence $\bar{c} = (\dot{c}_\xi)_{\xi < \kappa}$ of names for reals where each \dot{c}_ξ is Cohen over W^{P_ξ} .
2. A sequence $(C_\xi)_{\xi < \kappa}$ of so-called coding areas, where each $C_\xi \in {}^\kappa 2$ is generic over W^{P_ξ} but has constructible initial segments.
3. Maps Φ_ρ^ζ , for $\rho, \zeta < \kappa$, where $\Phi_\rho^{\bar{\zeta}}$ extends Φ_ρ^ζ for $\zeta < \bar{\zeta}$. Finally, $\bigcup_{\zeta < \kappa} \Phi_\rho^\zeta$ uniquely determines an automorphism Φ_ρ of $\text{r.o.}(P_\kappa)$ such that $\Phi_\rho(\dot{r}_\rho^0) = \dot{r}_\rho^1$ and $\Phi_\rho \upharpoonright P_{\bar{\iota}(\rho)}$ is the identity. For a more uniform notation, we also write Φ_ρ^κ for Φ_ρ . We call any stage of the iteration $P_{\xi+1}$ such that $\xi = E^3(\alpha_\rho^\zeta)$ for some $\zeta < \kappa$ and thus such that Φ_ρ^κ extends Φ_ρ^ζ , associated to Φ_ρ .
4. The sequence $(D_\xi)_{\xi < \kappa}$.

5.1 The successor stage of the iteration

For the successor stage, assume by induction that we have already defined P_ξ for $\nu < \xi$ and $\dot{r}_\nu^0, \dot{r}_\nu^1, \dot{s}_\nu$ for $\nu \leq \xi$. Fix k such that $\xi \in E^k$ and η such

that $\xi = E^k(\eta)$. We now define $P_{\xi+1}$. (We also define D_ξ as well as \dot{c}_ν, C_ν . In case $\xi \in E^3(\alpha_\rho^\zeta)$, we also define Φ_ρ^ζ .) In any case except when $\xi \in E^3(\alpha_\rho^0)$, for some $\rho < \kappa$ —that is, ξ is a stage where we do type-1 amalgamation—we let $D_\xi = P_\xi$. Let G_ξ denote a generic for P_ξ .

$k = 0$ At this stage we make sure that $\omega_1 = (\omega_{\xi+1}^L)$ and the GCH holds (task 4). This needn't be the case: in a previous stage, amalgamation may or may not have collapsed ω_1 , depending happenstantially on the partial isomorphism that we wanted to extend at that stage. Also, when ξ is limit, it is not clear if P_ξ collapses $\min(\text{Reg} \setminus \bigcup_{\nu < \xi} \lambda_\nu)$. Let

$$P_{\xi+1} = P_\xi * \dot{Q}_\xi,$$

where

$$\Vdash_\xi \dot{Q}_\xi = \text{Coll}(\omega, \lambda_\xi).$$

Since the Cohen algebra completely embeds into $\text{Coll}(\omega, \gamma)$, we can pick a $P_{\xi+1}$ -name which is fully Cohen over P_ξ , and define \dot{c}_η to be this name.

$k = 1$ Let $P_{\xi+1} = P_\xi \times (\text{Add}(\kappa))^L$. We denote by C_η the characteristic function of the set added by $\text{Add}(\kappa)^L$ over $W[G_\xi]$ (and let \dot{C}_η denote its canonical $P_{\xi+1}$ -name). This will be the generic “coding area” used in the next step.

$k = 2$ We take care of task 2, making sure $\Psi(c, j)$, holds for some real c given to us by book-keeping ($j = 0, 1$ indicates whether $c \in \Gamma^0$). Let $P_{\xi+1} = P_\xi * \dot{Q}_\xi$ where \dot{Q}_ξ is defined in the extension:

If η is a limit or $\eta = 0$, let c denote $\dot{c}_\eta^{G_\xi}$ (the Cohen real defined at stage $E^0(\eta)$), and let $j = 0$ (indicating that c will be in Γ^0). If η is a successor, let c denote $\dot{s}_{\eta-1}^{G_\xi}$, and let $j = 1$ (indicating that c will not be in Γ^0).

We wish to code a branch through $T(s, n, i, j)$ if and only if s is an initial segment of $C = C_\eta$ (the coding area from the previous step) and $c(n) = i$. That is, we let

$$B(C, c, j) = \{B(s, n, i, j) \mid s \triangleleft C, c(n) = i\}$$

be the set of branches to code, and represent it in a Δ_1 way as a subset of κ :

$$B^\#(C, c, j) = \{\#(s, n, i, j, t) \mid s \triangleleft C, c(n) = i, t \in B(s, n, i, j)\}$$

where $\#x$ denotes the constructible code for x . Finally, we define

$$\dot{Q}_\xi = J(B^\#(C, c, j)),$$

the forcing from fact 5.1 to code $B^\#(C, c, j)$ by a real.

$k = 3$ Say $\eta = \alpha_\rho^\zeta$. We first treat the case where $\zeta = 0$: By induction the book-keeping device \bar{r} gives us $\bar{r}(\rho) = (\bar{v}(\rho), \dot{r}_\rho^0, \dot{r}_\rho^1)$, where $\bar{v}(\rho) < \eta$ and the pair of names reals $\dot{r}_\rho^0, \dot{r}_\rho^1$ is fully random over $W^{P_{\bar{v}(\rho)}}$ and λ_ξ -reduced over P_ξ (over $P_{\bar{v}(\rho)}$ would suffice).

Let f be the automorphism of the complete Boolean algebras generated by \dot{r}_ρ^0 and \dot{r}_ρ^1 in B_ξ and let $P_{\xi+1}$ be the type-1 amalgamation of P_ξ over f and $P_{\bar{v}(\rho)}$:

$$P_{\xi+1} = \mathbf{Am}_1(P_{\bar{v}(\rho)}, P_\xi, f, \lambda_\xi).$$

Set $D_\xi = \mathbf{D}(P_{\bar{v}(\rho)}, P_\xi, f, \lambda_\xi)$. The resulting automorphism of $P_{\xi+1}$ we denote by Φ_ρ^0 .

Observe that, in general, this automorphism need not extend to an automorphism of B_κ . Also observe that by induction and theorem 4.20 ($P_\xi, \mathbf{Am}_1(P_{\bar{v}(\rho)}, P_\xi, f, \lambda_\xi)$) is a stratified extension above $\lambda_{\xi+1}$.

In the second case, when $\eta = \alpha_\rho^\zeta$ and $\zeta > 0$, we make sure Φ_ρ^0 is extended by an automorphism of $P_{\xi+1}$. So we let

$$P_{\xi+1} = \mathbf{Am}_2(\text{dom}(\Phi), P_\xi, \Phi),$$


where Φ is (an extension of) Φ_ρ^0 , constructed at an earlier stage of the iteration:

If ζ is a successor ordinal, at a previous stage $E^3(\alpha_\rho^{\zeta-1})$, we defined $\Phi_\rho^{\zeta-1}$ extending Φ_ρ^0 . Set $\Phi = \Phi_\rho^{\zeta-1}$.

If ζ is a limit, we have a sequence $(\Phi_\rho^\nu)_{\nu < \zeta}$, forming an increasing chain, and all extending Φ_ρ^0 . Letting $\delta = \bigcup_{\nu < \zeta} \alpha_\rho^\nu < \xi$, there is a unique automorphism of P_δ , extending each of them. Let Φ be this automorphism.

The resulting automorphism of $P_{\xi+1}$ we denote by Φ_ρ^ζ . In both cases we say $\xi + 1$ or $P_{\xi+1}$ is an amalgamation stage associated to Φ_ρ^ζ .

It's now easy to show lemma 5.2.

Proof of lemma 5.2. The first item holds by induction and lemmas 3.20, 3.27 and theorem 4.20; and since we force with $\text{Coll}(\omega, \lambda_\xi)$ after limit and type-1 amalgamation stages. The second item holds by theorem 3.23. The third one is a corollary of the previous ones and lemma 2.9. The last one is again true since we collapse λ_ξ at the right stage. 

5.2 A word about book-keeping

We give a recipe for cooking up a definition of $\bar{r} = (\bar{r}(\rho), \dot{r}_\rho^0, \dot{r}_\rho^1)_{\rho < \kappa}$. The definition is given by induction “on blocks”. For a moment, fix a pair $\iota < \xi$, assuming $\bar{r} \upharpoonright \xi$ has been defined (or, for the induction start, assume $\xi = \iota = 0$). We shall now define \bar{r} and \bar{r} on $[\xi, (\lambda_\xi)^+)$ —the “next block”. We can assume by induction that $\xi \in E^0$ is a limit ordinal.

Let $\beta = (\lambda_\xi)^+$. We can enumerate all the reals in W^{P_ξ} which are random over V^{P_ι} in order type β . In other words, find names P_ξ -names $(\dot{x}_\nu^t)_{\nu < \beta}$ such that

$$P_\xi \Vdash \mathbb{R} \setminus \dot{N} = \{\dot{x}_\nu^t\}_{\nu < \beta},$$

where \dot{N} is a name for the union of the Borel null sets with code in W^{P_ι} .

Observe that as P_ξ is stratified above λ_ξ and $P_{\xi+1}$ collapses λ_ξ , we have

$$P_{\xi+1} \Vdash |\mathbb{R} \cap W[G_\xi]| = \omega. \quad (5.3)$$

For each $\nu, \nu' < \beta$, apply the lemma 5.3 below, with $\dot{x}^0 = \dot{x}_\nu^t$ and $\dot{x}^1 = \dot{x}_{\nu'}^t$, and with $\theta = \beta$ (you can let $\theta = (\lambda_\nu)^+$ if you want; it doesn't matter) and $\lambda = \lambda_\xi$. You obtain a set $Y = Y(\nu, \nu', \iota)$ of size β consisting of pairs which are λ_ξ -reduced over P_ξ . If there are no reals in W^{P_ξ} which are random over W^{P_ι} , let Y be any set of pairs of random reals which are λ_ξ -reduced over P_ξ (such a set always exist—if in doubt, look at the proof of lemma 5.3).

Now define $\bar{r} \upharpoonright \beta$ and $\bar{r} \upharpoonright \beta$ (using a bijection of β with $\xi \times \beta^3$) in such a way that all pairs obtained in this way are listed, i.e. for each $\iota < \xi$, each pair and $\nu, \nu' < \beta$ and each $y \in Y(\nu, \nu', \iota)$ there is $\rho \in [\xi, \beta)$ such that $\bar{r}(\rho) = \iota$ and $(\dot{r}_\rho^0, \dot{r}_\rho^1) = y$.

Note that our construction relies on lemma 5.7, which will allow us to conclude that we catch our tail and \bar{r} enumerates all the pairs of random reals of the final model $W[G]$ (see lemma 5.13).

Lemma 5.3. *Let $\iota < \xi$, where $\xi \in E^0$, assume $P_{\xi+1}$ is stratified above λ and $P_{\xi+1}$ collapses the continuum of $W[G_\xi]$ to ω . Say 1_{P_ξ} forces \dot{x}^0, \dot{x}^1 are P_ξ -names random over W^{P_ι} . Then there is a set $Y = \{(\dot{y}_\nu^0, \dot{y}_\nu^1)\}_{\nu < \theta}$ such that*

$$1 \Vdash (\dot{x}^0, \dot{x}^1) \in \{(\dot{y}_\nu^0, \dot{y}_\nu^1)\}_{\nu < \theta}$$

and each pair in Y is λ -reduced over P_ξ .

Proof. Find $\{\dot{q}_\zeta\}_{\zeta < \theta}$ such that $\Vdash_\xi \{\dot{q}_\zeta\}_{\zeta < \theta}$ is a maximal antichain in $\dot{Q}_\xi = \text{Coll}(\omega, \theta)$. Note that $\{(1_{P_\xi}, \dot{q}_\zeta)\}_{\zeta < \theta}$ is maximal antichain in $P_{\xi+1}$. Fix a map

$$b: \zeta \mapsto (\dot{b}_0(\zeta), \dot{b}_1(\zeta))$$

such that $\Vdash_{P_\xi} b: \theta \rightarrow (\mathbf{Borel}^+)^2$ is onto (\mathbf{Borel}^+ denotes the set of Borel sets with positive measure). For each $\zeta < \theta$ and $j = 0, 1$ pick R_ζ^0, R_ζ^1 such that $(1_{P_\xi}, \dot{q}_\zeta)$ forces R_ζ^j is random over W^{P_ξ} and $R_\zeta^j \in \dot{b}_j(\zeta)$ for both $j = 0, 1$. This is possible by (5.3). Fix $\nu < \theta$ for the moment, in order to define $\dot{y}_\nu^0, \dot{y}_\nu^1$: for both $j = 0, 1$, pick \dot{y}_ν^j such that $(1_{P_\xi}, \dot{q}_\nu) \Vdash \dot{y}_\nu^j = \dot{x}^j$ and for each $\zeta \in \theta \setminus \{\nu\}$ we have $(1_{P_\xi}, \dot{q}_\zeta) \Vdash \dot{y}_\nu^j = \dot{R}_\zeta^j$.

As $\{(1_{P_\xi}, \dot{q}_\zeta)\}_{\zeta < \theta}$ is maximal, 1_{P_ξ} forces \dot{r}_j is random over W^{P_ξ} . For each $\nu < \theta$, the pair $\dot{y}_\nu^0, \dot{y}_\nu^1$ is λ -reduced over P_ξ : Let $p \preceq^\lambda q \in P_\xi$, let \dot{b}_0, \dot{b}_1 be P_ξ -names and fix $w \leq \pi_\xi(p)$ such that $w \Vdash \dot{b}_0$ and \dot{b}_1 are positive Borel sets. Find $w' \in P_\xi$ and $\zeta < \theta$, such that $w' \leq w$, $\zeta \neq \nu$ and

$$w' \Vdash \dot{b}_j(\zeta) \subseteq \dot{b}_j \text{ for } j = 0, 1. \quad (5.4)$$

We can ask $\zeta \neq \nu$ because we are content with \subset instead of $=$ in (5.4). As $w' \Vdash p(\xi) \preceq^\lambda 1$, $w' \cdot p$ is compatible with $(1_{P_\xi}, \dot{q}_\zeta)$. If $p' \leq w' \cdot p$ and $p' \leq (1_{P_\xi}, \dot{q}_\zeta)$, we have $p' \Vdash \dot{y}_\nu^j \in \dot{b}_j(\zeta) \subseteq \dot{b}_j$. Lastly, as $\{(1_{P_\xi}, \dot{q}_\zeta)\}_{\zeta < \theta}$ is maximal and $(1_{P_\xi}, \dot{q}_\nu) \Vdash \dot{x}^j = \dot{y}_\nu^j$,

$$1 \Vdash (\dot{x}^0, \dot{x}^1) \in \{(\dot{y}_\nu^0, \dot{y}_\nu^1)\}_{\nu < \lambda}.$$

☺

We now define Γ_ξ^0 , an approximation of Γ^0 at stage $\xi < \kappa$ of the iteration. Let Γ_ξ^0 be the smallest superset of $\{\dot{c}_\eta \mid E^0(\eta) < \xi \text{ and } \eta \text{ is limit or } \eta = 0\}$ (for limit η of course $E^0(\eta) = \eta$, but never mind) closed under all of the functions $F = \Phi_\rho^\zeta, (\Phi_\rho^\zeta)^{-1}$ such that $\text{dom } F \subseteq P_\xi$, i.e. closed under functions in

$$\{\Phi_\rho^\zeta, (\Phi_\rho^\zeta)^{-1} \mid E^3(\alpha_\rho^\zeta) \leq \xi\}$$

Let

$$\dot{\Gamma}_\xi^0 = \{(1_{P_\xi}, \dot{c}) \mid \dot{c} \in \Gamma_\xi^0\},$$

that is, $\dot{\Gamma}_\xi^0$ is the canonical choice for a name whose interpretation consists of the interpretations of the elements of Γ_ξ^0 .

When defining \bar{s} at stage ξ , we need to make sure that all P_ξ -names for reals \dot{s} which have the following property are listed (in the course of the iteration) by \bar{s} : for any $\dot{r} \in \dot{\Gamma}_\xi^0$, $\Vdash_{P_\xi} \dot{r} \neq \dot{s}$. We can easily make sure this is the case using arguments as above. As P_ξ forces $|\mathbb{R}| < \kappa$ (in fact $\leq \kappa$ would suffice), we can find \dot{f}_ξ such that

$$\Vdash_{P_\xi} \dot{f}_\xi: \kappa \rightarrow \mathbb{R} \setminus \dot{\Gamma}_\xi^0 \text{ is onto.}$$

We may assume (by induction hypothesis) we have such \dot{f}_ν for $\nu < \xi$. Pick \dot{s}_ξ such that for $\xi = G(\eta, \zeta)$, $\Vdash_{P_\xi} \dot{s}_\xi = \dot{f}_\eta(\zeta)$.

Later (see lemma 5.5), we show that \bar{s} lists exactly the reals of the final model $W[G]$ which are not in Γ^0 (which we are about to define). This concludes the definition of $(P_\xi)_{\xi \leq \kappa}$, \bar{c} , $(C_\xi)_{\xi < \kappa}$, Φ_ρ^ζ for $\zeta \leq \kappa$ and $\rho < \kappa$, \bar{r} , and \bar{s} , as well as that of Γ_ξ^0 and $\dot{\Gamma}_\xi^0$.

5.3 Cohen reals, coding areas and the sets Γ^0 and Γ^1

Let Γ^0 be the least superset of

$$\{\dot{c}_\xi \mid \xi < \kappa, \xi \text{ limit ordinal or } \xi = 0\}.$$

closed under all functions $\Phi_\xi, (\Phi_\xi)^{-1}$, $\xi < \kappa$ and let $\dot{\Gamma}^0$ be the P_κ -name $\Gamma^0 \times \{1_{P_\kappa}\}$. Recalling Γ_ξ^0 from the previous subsection (defined in the discussion of the coding device \bar{s}), note that $\bigcup_{\xi < \kappa} \Gamma_\xi^0 \subseteq \Gamma^0$; that the two sets are in fact equal, if not clear, follows from the next lemma. The lemma also helps to see that Γ^0 and Γ^1 (defined below) give rise to disjoint sets in the extension, as intended. Lastly, the lemma also is important to show that the coding areas C_ν behave in the same way as do the reals c_ν , and this will be used in 6.2 to show that the coding does not conflict with the automorphisms coming from amalgamation.

Let $\Gamma^1 = \{\dot{s}_\xi \mid \xi < \kappa\}$ and let $\dot{\Gamma}^1$ be the P_κ -name $\Gamma^1 \times \{1_{P_\kappa}\}$. Let \dot{x}_ν denote either \dot{c}_ν or \dot{C}_ν . Say \dot{y} is of the following form:

$$\dot{y} = (\Phi_{\xi_m})^{k_m} \circ \dots \circ (\Phi_{\xi_1})^{k_1}(\dot{x}_\nu)$$

where $\nu, \xi_1 \dots \xi_m < \kappa$ and $k_i \in \mathbb{Z}$ for $1 \leq i \leq m$. For $1 \leq i \leq m$, write

$$\dot{y}_i = (\Phi_{\xi_i})^{k_i} \circ \dots \circ (\Phi_{\xi_1})^{k_1}(\dot{x}_\nu),$$

and write \dot{y}_0 for \dot{x}_ν . Note that we can trivially assume that $\xi_{i+1} \neq \xi_i$, for i such that $1 \leq i < m$. We can also assume $\not\vdash_{P_\kappa} \dot{y}_{i+1} = \dot{y}_i$ for such i . We then call $\nu, \xi_0, \dots, \xi_m, k_0, \dots, k_m$ an *index sequence of \dot{y}* . Observe that every $\dot{y} \in \Gamma^0$ can be written in the form above.

Lemma 5.4. *There are $\rho_0, \dots, \rho_m < \kappa$ such that*

1. $\nu < \rho_0 < \dots < \rho_m$,
2. if $0 < i \leq m$, $P_{\rho_{i+1}}$ is an amalgamation stage associated to Φ_{ξ_i} ,
3. if $0 \leq i \leq m$, \dot{y}_i is a $P_{\rho_{i+1}}$ -name not in $W^{P_{\rho_i}}$. Moreover, \dot{y}_i is either unbounded over $W^{P_{\rho_i}}$ (if $\dot{y}_0 = \dot{c}_\nu$) or remote over P_{ρ_i} up to height κ (if $\dot{y}_0 = \dot{C}_\nu$).

Moreover, for $\dot{y}, \dot{y}' \in \Gamma^0$, either $\Vdash_{P_\kappa} \dot{y} = \dot{y}'$ or $\Vdash_{P_\kappa} \dot{y} \neq \dot{y}'$. If \dot{y} and \dot{y}' have different index sequences, the latter holds.

Proof. By induction on m . For $m = 0$, since $\dot{y}_0 = \dot{x}_\nu$, we pick ρ_0 so that \dot{Q}_{ρ_0} adds \dot{x}_ν (over $W[G_{\rho_0}]$). Then all of the above holds.

Now assume by induction the above holds for $i \leq m$. Let ρ_{m+1} be the least $\rho < \kappa$ such that $P_{\rho+1}$ is an amalgamation stage associated to $\Phi_{\xi_{m+1}}$, and \dot{y}_m is a $P_{\rho+1}$ -name. Since by induction, \dot{y}_m is forced to be different from any element of $W^{P_{\rho_m}}$, $\rho_m \leq \rho_{m+1}$. Moreover, $\rho_m < \rho_{m+1}$, for otherwise, $\xi_m = \xi_{m+1}$, contrary to assumption.

We have that either $P_{\rho_{m+1}+1} = \mathbf{Am}_1(P_\zeta, D, \dot{r}_0, \dot{r}_1)$ (for some ζ, D a dense subset of $P_{\rho_{m+1}}$, and some \dot{r}_0, \dot{r}_1), or $P_{\rho_{m+1}+1} = \mathbf{Am}_2(P_\zeta, P_{\rho_{m+1}}, \Phi)$ (for some Φ and ζ). We prove the lemma assuming the first holds; very similar arguments work for the other alternative, which we leave to the reader.

Observe that \dot{y}_m is not a P_ζ -name, as otherwise, contrary to assumption,

$$\Vdash_{P_\kappa} \dot{y}_{m+1} = \Phi_{\xi_{m+1}}(\dot{y}_m) = \dot{y}_m.$$

We have to consider two cases: If $\dot{x}_\nu = \dot{c}_\nu$, we can assume by induction that \dot{y}_m is unbounded over W^{P_ζ} and thus also over $W^{P_\zeta * \dot{B}(\dot{r}_i)}$ for $i = 0, 1$. Thus by lemma 4.8 applied for $P = D$, \dot{y}_{m+1} is unbounded over $W^{P_{\rho_{m+1}}}$ and we are done. If on the other hand, $\dot{x}_\nu = \dot{C}_\nu$, we can assume by induction that \dot{y}_m is remote over P_ζ up to height κ and $\kappa > \lambda_{\xi_{m+1}}$. So by lemma 4.31, \dot{y}_{m+1} is remote over $P_{\rho_{m+1}}$. In either sub-case, we conclude that \dot{y}_{m+1} is not in $W^{P_{\rho_{m+1}}}$ (for the second case, using lemma 3.33).

Lastly, say $\nu, \xi_0, \dots, \xi_m, k_0, \dots, k_m$ is an index sequence of \dot{y} and say $\nu', \xi'_0, \dots, \xi'_m, k'_0, \dots, k'_m$ is an index sequence of \dot{y}' . Assume $\not\Vdash_{P_\kappa} \dot{y} = \dot{y}'$; we show $\Vdash_{P_\kappa} \dot{y} \neq \dot{y}'$. Let ρ_0, \dots, ρ_m and ρ'_0, \dots, ρ'_m be obtained as above for \dot{y} and \dot{y}' respectively. Let l be least such $\xi_l \neq \xi'_l k$, if such exists. Then also $\rho_l \neq \rho'_l$, whence

$$\Vdash_{P_\kappa} \dot{y}_k \neq \dot{y}'_k.$$

Apply $\Phi_{\xi_m}^{k_m} \circ \dots \circ \Phi_{\xi_{l+1}}^{k_{l+1}}$ to this to obtain

$$\Vdash_{P_\kappa} \dot{y} \neq \dot{y}'.$$

If no l as above exists, the index sequences for \dot{y} and \dot{y}' are identical except possibly in the first coordinate. Now observe that $\Vdash_{P_\kappa} \dot{y}_0 = \dot{y}'_0$ if $\nu = \nu'$ and $\Vdash_{P_\kappa} \dot{y}_0 \neq \dot{y}'_0$ if $\nu \neq \nu'$. Apply $\Phi_{\xi_m}^{k_m} \circ \dots \circ \Phi_{\xi_0}^{k_0}$ and we're done. ☺

Lemma 5.5. $\Vdash_{P_\kappa} \dot{\Gamma}^0 = \mathbb{R} \setminus \dot{\Gamma}^1$.

It would be easier to show this in the following way: Show by induction on the number of applications of automorphisms that all names \dot{c} in $\Gamma^0 \setminus \Gamma_\xi^0$ have the following property: there is $\rho \geq \xi$ such that \dot{c} is in $W^{P_{\rho+1}}$ but not in W^{P_ρ} . This would make the slightly more complicated proof of lemma 5.4 unnecessary.

Proof. First we show $\Vdash_{P_\kappa} \dot{\Gamma}^0 \cup \dot{\Gamma}^1 = \mathbb{R}$. Let $r \notin (\dot{\Gamma}^0)^G$. Find $\xi < \kappa$ such that $r \in W[G_\xi]$. As $r \notin (\dot{\Gamma}_\xi^0)^{G_\xi}$, \bar{s} was defined to list a name for \dot{r} , so $r \in (\dot{\Gamma}^1)^G$.

Now let $\dot{c} \in \Gamma^0$ and $\dot{s} \in \Gamma^1$, and show $\Vdash_{P_\kappa} \dot{s} \neq \dot{c}$. Fix $\xi < \kappa$ so that \dot{s} is a P_ξ -name and $\Vdash_{P_\xi} \dot{s} \notin \Gamma_\xi^0$. Let $\nu, \rho_1, \dots, \rho_n$ be obtained as in the previous lemma from an index sequence for \dot{c} and write $\rho = \rho_n$. By the last lemma \dot{c} is a $P_{\rho+1}$ name not in W^{P_ρ} . If $\rho+1 \leq \xi$, we are clearly done, for then $\dot{c} \in \Gamma_\xi^0$. Otherwise, if $\rho \geq \xi$, \dot{c} is not in $W^{P_\rho} \supseteq W^{P_\xi}$, so in any case, $\Vdash_{P_\kappa} \dot{c} \neq \dot{s}$. \smile

5.4 The κ -stage of the iteration

The iteration preserves cardinals and cofinalities greater than κ :

Lemma 5.6. *P_κ is stratified above κ .*

Proof. This is a consequence of theorem 3.23 and lemma 5.2. \smile

Lemma 5.7. *In W , let $\theta \leq \kappa$, let $(\dot{\alpha}_\xi)_{\xi < \kappa}$ be a sequence of P_θ -names for ordinals below κ and let $p \in P_\theta$. Then for any $\beta_0 < \kappa$ there is an inaccessible $\alpha \in (\beta_0, \kappa)$ and a condition $p' \leq p$ such that for all $\xi < \alpha$, $p' \Vdash \dot{\alpha}_\xi < \alpha$. Moreover if $\theta = \kappa$, there is a sequence of P_α -names $(\dot{\alpha}'_\xi)_{\xi < \alpha}$ such that for each $\xi < \alpha$, $p' \Vdash \dot{\alpha}_\xi = \dot{\alpha}'_\xi$.*

The “moreover” clause is of course meaningless if $\theta < \kappa$. Before we treat the lemma, we draw two corollaries.

Corollary 5.8. *1. If $r \in L[\bar{B}][G_\kappa]$ is a real, there is $\alpha < \kappa$ such that $r \in L[\bar{B}][G_\alpha]$. In particular, κ remains uncountable in $L[\bar{B}][G_\kappa]$ (i.e. $\kappa = \omega_1$ in the final model).*

2. If $\theta < \kappa$, κ remains Mahlo in $L[\bar{B}][G_\theta]$.

Proof. For the first corollary, fix a real $r \in L[\bar{B}][G_\kappa]$ and working in $W = L[\bar{B}]$, let $\dot{\alpha}_n$ be a P_κ -name for $r(n)$, for each $n \in \mathbb{N}$. The lemma shows we can find $p' \in G_\kappa$, $\alpha < \kappa$ and a sequence of P_α -names $\{\dot{\alpha}'_n \mid n \in \mathbb{N}\}$, where $\alpha < \kappa$, such that for each $n \in \mathbb{N}$, $p' \Vdash \dot{\alpha}_n = \dot{\alpha}'_n$. Obviously, $r \in L[\bar{B}][G_\alpha]$.

For the second, say $\theta < \kappa$ and fix a P_θ -name \dot{C} for a closed unbounded subset of κ . Let $\dot{\alpha}_\xi$ be a name for the least element of \dot{C} above ξ . By the

lemma, we may find an inaccessible $\alpha \in (\lambda_\theta, \kappa)$ and $p' \in G_\theta$ such that for each $\xi < \alpha$, $p' \Vdash \dot{\alpha}_\xi < \alpha$. Thus, $p' \Vdash \dot{\alpha} \in \dot{C}$. Now observe that as P_θ is stratified above λ_θ , α is inaccessible in $L[\bar{B}][G_\theta]$. $\textcircled{\smile}$

In order to prove lemma 5.7, we define some dense subsets of P_κ , dubbed D_α^* and D_α^Σ for each regular $\alpha \leq \kappa$. In fact, D_α^Σ is a concrete presentation of a variant of $\text{dom}(\mathbf{C}^\alpha)$; for we shall need a more detailed account than the relatively abstract treatment of \mathbf{C}^α given in previous chapters affords. For notational convenience, we shall also define operators \uparrow , K_α , R_δ on P_κ , for regular $\alpha < \kappa$ and $\delta < \kappa$ and a pre-order \leq^* on P_κ .

Let $\alpha \leq \kappa$ be regular and let $\sigma < \alpha$. First, define $D_\alpha^\sigma \subseteq P_\kappa$, by induction on the length of a condition. For the successor step, say $p \in P_{\nu+1}$. We let $p \in D_\alpha^\sigma$ if and only if $\pi_\nu(p) \in D_\alpha^\sigma$ and the following hold:

1. in case $\nu \in E^0$ (i.e. \dot{Q}_ν is $\text{Coll}(\omega, \lambda_\nu)$), we require that $\pi_\nu(p) \Vdash \text{ran}(p(\nu)) \subseteq \sigma$,
2. in case $\nu \in E^2$ (i.e. \dot{Q}_ν is Jensen coding), we require that $\pi_\nu(p) \Vdash$ for all $\delta \in \text{supp}(p(\nu)) \cap \alpha$, $|p(\nu)_\delta| < \sigma$.
3. in case $\nu \in E^3$ (i.e. $P_{\nu+1}$ is an amalgamation), we require that for all $i \in \mathbb{Z} \setminus \{0\}$ we have $p(i)^P \in D_\alpha^\sigma$ (it would be redundant to require this also for $i = 0$).

For $p \in P_\nu$ where $\nu \leq \kappa$ is a limit ordinal, let $p \in D_\alpha^\sigma$ if and only if for all $\nu' < \nu$, $\pi_{\nu'}(p) \in D_\alpha^\sigma$. Finally, define $p \in D_\alpha^\Sigma$ if and only if there is $\sigma < \alpha$ such that $p \in D_\alpha^\sigma$. Also, for any $p \in D_\alpha^\Sigma$, let $\sigma_\alpha^2(p)$ be the least $\sigma < \alpha$ such that $p \in D_\alpha^\sigma$ and $\text{supp}^\alpha(p) \cap \alpha \subseteq \sigma$.

The sets D_α^* are defined in a similar fashion. Formally, they are binary relations,

$$D_\alpha^* \subseteq \{ \text{sequences of length } \leq \kappa \} \times P_\kappa;$$

that is, for any such sequence H , $D_\alpha^*(H) \subseteq P_\kappa$.

So let $\alpha \leq \kappa$ be regular. The definition of $D_\alpha^*(H)$ is by induction on the length of conditions: for the successor step, assume we have already defined D_α^* on

$$\{ \text{sequences of length } \leq \nu \} \times P_\nu.$$

Fix an arbitrary sequence H . Assume $p \in P_{\nu+1}$ and let $p \in D_\alpha^*(H)$ if and only if $\pi_\nu(p) \in D_\alpha^*(H \upharpoonright \nu)$ and either $p \preceq^\alpha \pi_\nu(p)$ or the following hold:

1. $H = (H(\xi))_{\xi < \nu+1}$ is a sequence of length $\nu + 1$,
2. $p \in D_\alpha^\Sigma$,

3. in case $\nu \in E^0$, we require that $p(\nu)$ (a collapsing condition) is β -chromatic below $\pi_\nu(p)$, for some $\beta < \alpha$, with spectrum $H(\nu)$,
4. in case $\nu \in E^2$, we require that $H(\nu)(\delta)$ is defined for each cardinal $\delta < \sigma_\alpha^2(p(\nu))$ and for each such δ , $p(\nu)(\delta)$ is β -chromatic below $\pi_\nu(p)$, for some $\beta < \alpha$, with spectrum $H(\nu)(\delta)$.
5. in case $\nu \in E^3$, we require that $H(\nu) = (\bar{H}_i^P, \bar{H}_i^0, \bar{H}_i^1)_{i \in \mathbb{Z} \setminus \{0\}}$ and for all $i \in \mathbb{Z} \setminus \{0\}$, $p(i)^P \in D_\alpha^*(\bar{H}_i^P)$ and for $j \in \{0, 1\}$, $p(i)^j$ is β -chromatic with spectrum \bar{H}_i^j below $\pi_\iota(p(0)^P)$ for some $\beta < \alpha$ —where ι is chosen so that P_ι is the base of the amalgamation $P_{\nu+1}$ (see p. 75 for the definition of *base*).

Observe in the case $\nu \in E^2$, we lightheartedly wrote “ $p(\nu)(\delta)$ is β -chromatic”; we mean here to identify in some convenient way those constituents of the tuple $p(\nu)(\delta) = (p(\nu)_\delta, \dots)$ which are in \mathbf{H}_δ with a single function $f: [\delta, \zeta) \rightarrow 2$, for some $\zeta < \delta^+$.

For $p \in P_\nu$ where $\nu \leq \kappa$ is a limit ordinal, let $p \in D_\alpha^*(H)$ if and only if for all $\nu' < \nu$, $\pi_{\nu'}(p) \in D_\alpha^*(H \upharpoonright \nu')$. Finally, define $p \in D_\alpha^*$ if any only if there is H such that $p \in D_\alpha^*(H)$; any such H we call an α -spectrum of p .

Observe that by definition, we may assume without loss of generality that if $p \in D_\alpha^*(H)$ and $\nu \notin \text{supp}^\alpha(p)$, $H(\nu)$ is either undefined or equal to \emptyset . We shall always make this assumption from now on. Thus, if $p \in D_\alpha^*(H)$, we can assume that H is a sequence of length $\sup\{\delta + 1 \mid \delta \in \text{supp}^\alpha(p) \cap \alpha\}$. It is straightforward to check that this also implies $H \in \mathbf{H}_\alpha$. Also observe that since $\text{supp}^\alpha(p) \cap \alpha$ is a bounded subset of α , if $p \in D_\alpha^*(H)$, and α is inaccessible, then there is $\beta < \alpha$ such that for each $\gamma \in [\beta, \alpha]$, we have $p \in D_\gamma^*(H)$.

Lemma 5.9. *For $\theta \leq \kappa$ and λ, α regular such that $\lambda < \alpha \leq \kappa$, both D_α^Σ and D_α^* are dense in $\langle P_\theta, \preceq^\lambda \rangle$. If $p \in D_\alpha^*$ and $q \preceq^\alpha p$, we have $q \in D_\alpha^*$; likewise for D_α^Σ .*

Proof. The proof is by induction; the successor case should be clear. The limit case works exactly as the proof that $\text{dom}(\mathbf{C}^\alpha)$ is dense, see the proof of 3.23, p. 50. \odot

For every $p \in P_\kappa$, we define a condition $p^\uparrow \in P_\kappa$, which one should think of as “the upper part of p with respect to κ .” Again, the definition is by induction on the length of p , so assume we have defined q^\uparrow for all $q \in P_\nu$ and for all such q , we have $q^\uparrow \preceq^{<\kappa} 1_{P_\nu}$. Let $p \in P_{\nu+1}$ be given. If $\nu \in E^3$ (an amalgamation stage), define p^\uparrow to be the sequence $i \mapsto (p_i^*, 1, 1)$, for $i \in \mathbb{Z}$, where $p_i^* = (p(i)^P)^\uparrow$. Observe that since we always amalgamate over

κ -reduced reals, $\pi_0(p_i^*) = \pi_1(p_i^*) = 1$ and so this defines a condition in the amalgamation $P_{\nu+1}$.

For $\nu \notin E^3$, set $\pi_\nu(p^\uparrow) = (p \upharpoonright \nu)^\uparrow$ and find $p^\uparrow(\nu)$ as follows: If $\nu \in E^0$ (i.e. \dot{Q}_ν is a collapse), we let $p^\uparrow(\nu) = 1_{\dot{Q}_\nu}$. For $\nu \in E^1$ (i.e. Q_ν is κ -Cohen forcing of L), let $p^\uparrow(\nu) = p(\nu)$. Now assume $\nu \in E^2$ (a Jensen coding stage of the iteration). Let $p^\uparrow(\nu)$ be a P_ν -name q for a function on $\text{Card} \cap \kappa + 1$ such that

$$1_{P_\nu} \Vdash_\nu q(\kappa) = p(\nu)(\kappa)$$

and

$$\forall \alpha \in \kappa \cap \text{Card} \quad 1_{P_\nu} \Vdash_\nu q(\alpha) = \emptyset.$$

Observe that this is (forced by P_ν to be) a condition in Jensen coding. This concludes the definition of p^\uparrow . The following fact is a straightforward consequence of the definition: Simply put, it says that if a condition $r \in P_\theta$ is trivial below κ after some stage δ of the iteration, then r is equivalent to r^\uparrow after δ .

Fact 5.10. Say $p^*, r \in P_\kappa$ and $w \in P_\delta$ are such that $r \Vdash p^* = r^\uparrow$, $w \leq \pi_\delta(r)$, $\pi_\delta(p^*)$ and $r \preceq^{<\kappa} \pi_\delta(r)$. Then $w \cdot r \approx w \cdot p^*$.

An explanation is in order regarding what is meant by “ $r \Vdash p^* = r^\uparrow$ ”. We would like to express that *whenever appropriate*, p^* and r should be considered as names. That is, whenever $\nu \in E^0 \cup E^2$ (where the iteration is given by composition), $\pi_\nu(r) \Vdash p^*(\nu) = r^\uparrow(\nu)$; when $\nu \in E^1$, $p^*(\nu) = r^\uparrow(\nu)$ (where the iteration is given by a product); and when $\nu \in E^3$ (amalgamation), for each $i \in \mathbb{Z} \setminus \{0\}$, $r(i)^P \Vdash p^*(i)^P = r^\uparrow(i)^P$ — in the sense of an inductive definition. Nevertheless, we find this choice of notation both adequate and intuitive and refrain from a formal, inductive definition of, say, a new relation on P_κ .

Proof of fact 5.10. By induction on $\nu \in [\delta, \kappa)$. Say we already know

$$\pi_\nu(w \cdot r) \approx \pi_\nu(w \cdot p^*).$$

If $\nu \in E^0$, clearly $\pi_\nu(r) \Vdash p^*(\nu) = \emptyset$. As $r \preceq^{<\kappa} \pi_\delta(r)$, also $\pi_\nu(r) \Vdash r(\nu) = \emptyset$. So $\pi_\nu(w \cdot r) \Vdash p^*(\nu) = r(\nu)$, and by induction, so does $\pi_\nu(w \cdot p^*)$. If $\nu \in E^1$, by the assumption that $r \Vdash p^* = r^\uparrow$ and by convention, we have $r(\nu) = p^*(\nu)$. If $\nu \in E^2$, $\pi_\nu(r) \Vdash r(\nu) \upharpoonright \kappa = \emptyset$ and thus $\pi_\nu(r)$ forces $p^*(\nu) = r^\uparrow(\nu) = r(\nu)$. In all three cases we have $w \cdot \pi_{\nu+1}(r) \approx w \cdot \pi_{\nu+1}(p^*)$. Now say $\nu \in E^3$ and $P_{\nu+1}$ is a type-1 amalgamation (we leave the other case to the reader, as it is similar). Write $\bar{p} = \pi_{\nu+1}(p^*)$, $\bar{r} = \pi_{\nu+1}(r)$ and let P_ι be the base of the amalgamation. By convention $r \Vdash p^* = r^\uparrow$ means that for each $i \in \mathbb{Z}$,

$$\bar{r}(i)^P \Vdash \bar{p}(i)^P = (\bar{r}(i)^P)^\uparrow. \quad (5.5)$$

As $\bar{r} \preceq^{<\kappa} e_0(\pi_\delta(r))$, we have

$$\bar{r}(i)^P \preceq^{<\kappa} \pi_\iota(\pi_\delta(r)) \quad (5.6)$$

and thus $\bar{r}(i)^P \preceq^{<\kappa} \pi_\iota(\bar{r}(i)^P)$ by lemma 3.8. For $v = \pi_\iota(w \cdot r)$ we have $v \leq \pi_\iota(\bar{r}(i)^P), \pi_\iota(\bar{p}(i)^P)$, so by induction, $v \cdot \bar{r}(i)^P \approx v \cdot \bar{p}(i)^P$ and so

$$\pi_\iota(w) \cdot \bar{r}(i)^P = v \cdot \bar{r}(i)^P \approx v \cdot \bar{p}(i)^P \approx \pi_\iota(w) \cdot \bar{p}(i)^P.$$

The last part holds as by induction, $\pi_\iota(w \cdot r) \approx \pi_\iota(w \cdot p^*)$. Observe that $\bar{r}(i)^j = 1$ for $j \in \{0, 1\}$. So we conclude $w \cdot \bar{r} \approx w \cdot \bar{p}$, finishing the inductive proof. \odot

Let p be a condition in Jensen coding, $\delta < \kappa$, and let $\alpha < \kappa$ be regular. Let $R_\delta(p)$ denote the condition obtained from p by increasing the ordinal in \dot{p}_κ to ensure that in all further extension, restraints will start above δ . Let $K_\alpha(p)$ be the condition extending p such that all further extensions will have restraints starting above α at all inaccessible cardinals larger than α (i.e. put α into the support of p).

We now define operators on P_κ , also denoted by R_δ and K_α (this will not cause confusion), by induction on the length of a condition. Assume $\nu < \kappa$ and $p \in P_{\nu+1}$. If $\nu \in E^2$, i.e. at a coding stage, let $R_\delta(p) \upharpoonright \nu = R_\delta(\pi_\nu(p))$ and choose $R_\delta(p)(\nu)$ so that $\Vdash_\nu R_\delta(p)(\nu) = R_\delta(p(\nu))$. If $\nu \in E^3$, i.e. at an amalgamation stage, let $R_\delta(p)$ be the sequence

$$i \mapsto (R_\delta(p(i)^P), p(i)^0, p(i)^1).$$

At all other cases, let $\pi_\nu(R_\delta(p)) = R_\delta(\pi_\nu(p))$ and let $R_\delta(p)(\nu) = p(\nu)$. The operator $K_\alpha: P_\kappa \rightarrow P_\kappa$ is defined analogously.

For a condition in Jensen coding p , we write $p_{<\alpha}$ for $\bigcup_{\delta \in \text{Card} \cap \alpha} p_\delta$. For two arbitrary conditions $q \leq p$ in P_κ we write $q \leq^* p$ if and only if

$$\forall \nu < \theta \quad 1_{P_\nu} \Vdash q(\nu) \leq_{\dot{Q}_\nu} p(\nu)$$

Proof of lemma 5.7. For the first part of the proof, we must work in L and show $1_{\bar{T}}$ forces that the statement holds (this is because \bar{T} introduces a new subset of κ , see below for an explanation). So fix $\beta_0 < \kappa$, let $t_0 \in \bar{T}$ arbitrary, let $\theta \leq \kappa$, let $(\dot{\alpha}_\zeta)_{\zeta < \kappa}$ be $\bar{T} * P_\theta$ -names and $(t_0, p_0) \in \bar{T} * P_\theta$ so that t_0 forces the hypothesis of the lemma holds. Let the map $S: \xi \mapsto \langle (\xi)_1, (\xi)_2 \rangle$ be such that for any cardinal $\alpha \leq \kappa$ and every $\vec{x} \in \alpha^2$ we have

$$|\{\xi < \alpha \mid S(\xi) = \vec{x}\}| = \alpha.$$

Also, fix an enumeration $(h(\xi))_{\xi < \kappa}$ of \mathbf{H}_κ . We inductively construct a sequence $(t_\xi, p_\xi)_{\xi < \kappa}$, starting with (t_0, p_0) . At the same time, we shall define a sequence of ordinals $(\alpha_\xi)_{\xi < \kappa}$ and another sequence of conditions $(q_\xi)_{\xi < \kappa}$. Assume we already have (t_ξ, p_ξ) . First, assume there is $(t, q) \leq (t_\xi, p_\xi)$ such that for some $\sigma < \xi^+$,

1. $t \Vdash_{\bar{T}} \sigma_2^\kappa(q) = \check{\sigma}$,
2. $t \Vdash_{\bar{T}} q \in D_{\xi^+}^*(\check{h}((\xi)_1))$.
3. (t, q) decides $\dot{\alpha}_{(\xi)_2}$.

We let $\sigma_2^\kappa(q)$ denote σ in the following, slightly abusing notation. Let α_ξ be the ordinal such that $(t, q) \Vdash \dot{\alpha}_{(\xi)_2} = \check{\alpha}_\xi$ and let $q_\xi = q$. To define $(t_{\xi+1}, p_{\xi+1})$, first find a \bar{T} name for a condition $p^* \in P_\theta$ such that

- i. $(t, q_\xi) \Vdash p^* = q_\xi^\uparrow$,
- ii. $t \Vdash_{\bar{T}} p^* \leq^* p_\xi$,
- iii. $t \Vdash_{\bar{T}} p^* \preceq^{<\kappa} 1_{P_\theta}$.

It should be clear how to find p^* , as we may inductively assume that $t \Vdash_{\bar{T}} p_\xi \preceq^{<\kappa} 1_{P_\theta}$ and so $(t, q_\xi) \Vdash q_\xi^\uparrow \leq p_\xi$. Let $p_{\xi+1}$ be a name such that $t \Vdash_{\bar{T}} p_{\xi+1} = R_\tau(p^*)$, where $\tau = \sup_{\zeta \leq \xi} (\text{supp}(q_\zeta) \cap \kappa)$ and let $t_{\xi+1} = t$. Thus if (t', q') extends $(t, p_{\xi+1})$,

$$\begin{aligned} &\text{for every } \nu \in \kappa \cap E^2, (t, \pi_\nu(q)) \text{ forces that restraints for the} \\ &\text{coding from } \kappa^+ \text{ into } \kappa \text{ in } q'(\nu)_{\kappa} \setminus p_{\xi+1}(\nu)_{\kappa} \text{ are forced by } \pi_\nu(q) \quad (5.7) \\ &\text{to start above } \sup_{\zeta \leq \xi} (\text{supp}(q_\zeta) \cap \kappa). \end{aligned}$$

If on the other hand, (t, q) as above does not exist, simply let $(t_{\xi+1}, p_{\xi+1}) = (t_\xi, p_\xi)$ (in this case, we leave q_ξ and α_ξ undefined).

At limits ξ , we can set $p(\xi)$ to be the greatest lower bound of the $p(\nu)$, for $\nu < \xi$. This is because we don't have to do David's trick at κ and since we can always take the union of less than κ many restraints at κ . Moreover, κ -Cohen forcing of L at every stage as well as \bar{T} are κ -closed.

Let $C \subseteq \kappa$ be a club consisting of cardinals such that for any $\alpha \in C$ and any $\xi < \alpha$:

1. $\mathbf{H}_\alpha = \{h(\nu) \mid \nu < \alpha\}$,
2. $\xi^+ < \alpha$,
3. $\sigma_2^\kappa(q_\xi) < \alpha$ and $\alpha_\xi < \alpha$.

Find an inaccessible $\alpha < \kappa$ such that $\alpha \in C$ and so that

$$\forall \xi \in C \cap \alpha \forall s \in S_\alpha \quad [\xi^+, \xi^{+C}] \cap b^s = \emptyset \quad (5.8)$$

Observe that there are unboundedly (in fact, stationarily) many such α , so we can assume $\alpha > \beta_0$. This is where we implicitly use that we are working in L , since inaccessible restraints are part of a diamond sequence of L and not $L[\bar{B}]$ —for if we use a diamond of $L[\bar{B}]$, it is not clear to me how to decode. Moreover, in the definition of our coding forcing, we don't want to use all of \bar{B} —see section 6 dealing with the preservation of the coding.¹

Let p' be a \bar{T} -name such that $t_\alpha \Vdash p' = K_\alpha(p_\alpha)$. Thus t_α forces that for all $q \leq p'$

$$\begin{aligned} &\text{for all inaccessible cardinals } \beta \in (\alpha, \kappa] \text{ and all } \nu \in \kappa \cap E^2, \\ &\pi_\nu(q) \text{ forces that restraints for the coding from } \beta^+ \text{ into } \beta \text{ in } \\ &q(\nu)_\beta \text{ start above } \alpha. \end{aligned} \quad (5.9)$$

We claim that (t_α, p') is the condition we are looking for. Fix $\zeta < \alpha$.

Claim 5.11. *The condition t_α forces that $\{q_\xi \mid \xi < \alpha, (\xi)_2 = \zeta, q_\xi \text{ is defined}\}$ is pre-dense in P_θ below p' .*

By construction, $(\xi)_2 = \zeta$ implies that $(t_\alpha, q_\xi) \Vdash_{\bar{T} * P_\theta} \dot{\alpha}_\zeta = \check{\alpha}_\xi$ in L . Moreover, as $\alpha \in C$, $\alpha_\xi < \alpha$. So this proves that $p' \Vdash_{P_\theta} \dot{\alpha}_\zeta < \check{\alpha}$ in $L[\bar{B}]$, the first part of the lemma.

To prove the claim, we work in $L[\bar{B}']$, where \bar{B}' is \bar{T} generic and $t_\alpha \in \bar{B}'$. It then suffices to show that the following set is pre-dense in P_θ below p' :

$$\{q_\xi \mid \xi < \alpha, (\xi)_2 = \zeta, q_\xi \text{ defined} \}$$

So let $q \leq p'$ be arbitrary. We may assume that q decides $\dot{\alpha}_\zeta$. We can also assume that $q \in D_\alpha^* \cap D_\kappa^\Sigma$ (this set is dense by the previous lemma). Let H be an α -spectrum for q . Find $\xi < \alpha$ such that

$$\xi > \sigma_2^\alpha(q) \quad (5.10)$$

and H is a ξ^+ -spectrum of q . Without loss of generality, $(\xi)_2 = \zeta$ and $h((\xi)_1) = H$ (we can assume the latter since $H \in \mathbf{H}_\alpha$ and $\alpha \in C$). Observe that $q \in D_{\xi^+}^*(H)$ and so witnesses that at stage ξ of the construction, we found t_ξ and $q_\xi \in D_{\xi^+}^*(H)$ so that $(t_\xi, q_\xi) \Vdash_{\bar{T} * P_\theta} \dot{\alpha}_\zeta = \alpha_\xi$, whence of course q_ξ forces the same in $L[\bar{B}]$. Moreover observe that

$$\sigma_2^\kappa(q_\xi) < \xi^{+C}. \quad (5.11)$$

¹alternatively, we could have used trees on κ^{++} . These trees do not add a subset to κ and there is no difference between a diamond of L and a diamond of W below κ .

We claim that q_ξ and q are compatible, by induction on the length of these conditions. For the sake of the inductive argument, we prove the following claim. We will apply this claim for $r = q_\xi$; we introduce an additional parameter w to carry out the induction at amalgamation stages, where we have to “go back in construction” for $i \neq 0$ and start again at the length of the base of the amalgamation (see below).

Claim 5.12. *Let $r, q \in P_\theta$, $u \in P_\delta$ and H be given such that $u \leq \pi_\delta(q), \pi_\delta(r)$ and $\pi_\zeta(u) \preceq^\alpha \pi_\zeta(r)$, where $\zeta = \min(\alpha, \delta)$, and moreover*

- (a) $\sigma_\kappa^2(r) < \xi^{+C}$ (let $\sigma = \sigma_\kappa^2(r)$ in the following),
- (b) H is a ξ^+ -spectrum of r ,
- (c) either H is a ξ^+ -spectrum of q and $\sigma_2^\alpha(q) < \xi$, or $q \preceq^\alpha \pi_\delta(q)$,
- (d) there is a condition p^* such that $q \leq K_\alpha(R_\sigma(p^*))$, $p^* \preceq^{<\kappa} 1_{P_\theta}$ and $r \Vdash p^* = r^\uparrow$;

then there is $w \in P_\theta$ such that $\pi_\delta(w) = u$, $w \leq q, r$ and moreover, $\pi_\zeta(w) \preceq^\alpha \pi_\zeta(r)$.

As all of the hypotheses of claim 5.12 are satisfied when we set $r = q_\xi$, $\delta = 0$ and $u = 1_P$ this suffices to see that q and q_ξ are compatible. Note that the second possibility of item (c) is necessary to carry the induction through amalgamation stages $\nu \notin \text{supp}^\alpha(q)$, where for $i \neq 0$ we are faced with conditions of length ν which have no spectrum at all. We proof the claim by induction on $\nu > \delta$, assuming we have found $\pi_\nu(w)$ such that $\pi_\nu(w) \leq \pi_\nu(q)$ and if $\nu \leq \alpha$, $\pi_\nu(w) \preceq^\alpha \pi_\nu(q_\xi)$. We split into cases:

Collapsing stages: Assume $\nu \in E^0$. If both r and q are in $D_{\xi^+}^*(H)$, $r(\nu)$ and $q(\nu)$ have a common spectrum below $\pi_\nu(w)$ whenever $\nu < \xi$. Thus $\pi_\nu(w) \Vdash_\nu r(\nu) = q(\nu)$ in this case. If $\nu \in [\xi, \alpha)$ or if H is not a spectrum of q and hence, by (c), $q \preceq^\alpha \pi_\nu(q)$ holds, we have $\pi_\nu(w)$ forces that $q(\nu) = \emptyset$ by (5.10). In both cases, we can set $w(\nu) = r(\nu)$. Observe $\pi_{\nu+1}(w) \preceq^\alpha \pi_{\nu+1}(r)$. If $\nu \geq \alpha$, $\pi_\nu(w) \Vdash r(\nu) = \emptyset$, so we can set $w(\nu) = q(\nu)$.

Stages adding a κ -Cohen: If $\nu \in E^1$, $P_{\nu+1} = P_\nu \times Q_\nu$, where Q is κ -Cohen forcing of L . As $q \leq p^*$ and $p^*(\nu) = r^\uparrow(\nu)$ by (d) and by convention, and as $r^\uparrow(\nu) = r(\nu)$ by definition of \uparrow , we have that $q(\nu) \leq r(\nu)$. So we can set $w(\nu) = q(\nu)$. Observe that if $\nu < \alpha$, the induction hypothesis gives us $\pi_{\nu+1}(w) \preceq^\alpha \pi_{\nu+1}(r)$ (as Q_ν lies entirely in the “upper part” with respect to α).

Coding stages: Assume $\nu \in E^2$. If $\nu < \alpha$, let $w(\nu)$ be a P_ν -name such that $w(\nu) \upharpoonright \alpha = r(\nu) \upharpoonright \alpha$ and $w(\nu) \upharpoonright [\alpha, \kappa] = q(\nu) \upharpoonright [\alpha, \kappa]$. We shall show that $w(\nu)$ is forced by $\pi_\nu(w)$ to be a condition in Jensen coding.

First observe that by (b) and (c) we have that one of the following holds:

- For every cardinal $\delta < \xi^+$, $r(\nu)(\delta)$ and $q(\nu)(\delta)$ have a common spectrum below $\pi_\nu(w)$, so $\pi_\nu(w) \Vdash r(\nu) \upharpoonright \xi^+ = q(\nu) \upharpoonright \xi^+$. Observe also that $\pi_\nu(w)$ forces $q(\nu)_\delta = \emptyset$ for $\delta \in [\xi^+, \alpha)$, since $\sigma_2^\alpha(q) < \xi^+$.
- $\pi_\nu(w) \Vdash q(\nu)_\delta = \emptyset$ for $\delta \in \alpha \cap \text{Card}$.

Thus, in either case $\pi_\nu(w)$ forces that $q(\nu) \upharpoonright \alpha$ is an initial segment of $r(\nu) \upharpoonright \alpha$. Moreover, by (a) and (5.8), $\pi_\nu(w)$ forces that $r(\nu)_{<\alpha}$ does not violate any of the restraints in $q(\nu)_\alpha$.

It is also forced by $\pi_\nu(w)$ that $r(\nu)_{<\kappa}$ does not violate any restraints from any $(q(\nu))_\delta$ for cardinals $\delta \in (\alpha, \kappa)$: for by (d), as $p^* \preceq^{<\kappa}$ and $q \leq K_\alpha(p^*)$, all such restraints must start above α .

Lastly, $\pi_\nu(w)$ forces that $r(\nu)_{<\kappa}$ does not violate any of the restraints in $q(\nu)_\kappa$: As $q \leq R_\tau(p^*)$ for some $\tau > \sigma_2^\kappa(r)$, restraints in $q(\nu)_\kappa \setminus p^*(\nu)_\kappa$ start above $\sigma_2^\kappa(r)$. Moreover, as $\pi_\nu(w) \Vdash p^*(\nu)_\kappa = r(\nu)_\kappa$ (by (d) and the definition of the \upharpoonright -operator)—observe that we defined inaccessible coding so that $r(\nu)_{<\kappa}$ obeys all restraints in $r(\nu)_\kappa$ — $r(\nu)_{<\kappa}$ does not violate any restraints in $p^*(\nu)_\kappa$.

So we conclude that $\pi_\nu(w)$ forces that $r(\nu) \upharpoonright \alpha$ end-extends $q(\nu) \upharpoonright \alpha$ and $(r)_{<\alpha}$ does not violate any restraints in $q(\nu)$ or $r(\nu)$. Since $q \leq p^*$, and since by (d) and the definition of the \upharpoonright -operator we have $\pi_\nu(w) \Vdash_\nu p^*(\nu)(\kappa) = r(\nu)(\kappa)$, we see that $\pi_\nu(w) \Vdash_\nu q(\nu)(\kappa) \leq r(\nu)(\kappa)$. Finally, we conclude that $w(\nu)$ is forced by $\pi_\nu(w)$ to be a condition in Jensen coding, we have $\pi_\nu(w)$ forces that $w(\nu) \leq q(\nu), r(\nu)$, and moreover, $\pi_{\nu+1}(w) \preceq^\alpha \pi_{\nu+1}(r)$. This finishes the case $\nu < \alpha$.

If $\nu \geq \alpha$, $\pi_\nu(w) \Vdash r(\nu) \upharpoonright \kappa = \emptyset$ and $q(\nu)(\kappa) \leq r(\nu)(\kappa)$. So in this case, we may set $w(\nu) = q(\nu)$.

Amalgamation stages: If $\nu \in E^3$, write \bar{q} for $\pi_{\nu+1}(q)$ and \bar{r} for $\pi_{\nu+1}(r)$ and let P_ν be the base of the amalgamation. Again, we assume $P_{\nu+1}$ is a type-1 amalgamation and leave the case of type-2 amalgamation to the reader.

First, assume that $\nu < \alpha$ and the second alternative of (c) obtains, i.e. we have $q \preceq^\alpha \pi_\delta(q)$. Then by lemma 3.8 we have $q \leq^\alpha \pi_\nu(q)$, whence $\bar{q} \preceq^\alpha e_0(\pi_\nu(q))$ and thus for all $i \in \mathbb{Z} \setminus \{0\}$, $\bar{q}(i)^P \preceq^\alpha \pi_\nu(\bar{q}(i)^P)$ and for $j \in \{0, 1\}$, $\bar{q}(i)^j = 1$ (observe $\lambda_\nu < \alpha$ as α is inaccessible). Use induction with $u = \pi_\nu(w)$ to find $w_i \in P_\nu$ so that $w_i \leq \bar{q}(i)^P$, $w_i \preceq^\alpha \bar{r}(i)^P$ and of course

$\pi_\iota(w_i) = \pi_\iota(w)$. Setting $\bar{w}(i) = (w_i, \bar{r}(i)^0, \bar{r}(i)^1)$, we see $\bar{w} \preceq^\alpha \bar{r}$ and so by corollary 4.21, \bar{w} is a condition in the amalgamation; clearly, $\bar{w} \leq \bar{q}$. Set $w_{\nu+1} = \bar{w}$.

Next, consider the case $\nu < \alpha$ and the first alternative of (c) obtains. Note we may assume that $\nu < \sigma_2^\alpha(q) = \sigma_2^{\xi^+}(q)$ (otherwise, $q \preceq^\alpha \pi_\nu(q)$, which is handled by the previous case). It is straightforward to check that for each $i \in \mathbb{Z} \setminus \{0\}$, $\bar{r}(i)^P$ and $\bar{q}(i)^P$ satisfy the induction hypotheses, when we set $u = \pi_\iota(w)$, so we may find w_i as in the previous case. Also define \bar{w} as in the previous case, again observing that $\bar{w} \preceq^\alpha \bar{r}$. As by assumption, for $i \in \mathbb{Z} \setminus \{0\}$ and $j \in \{0, 1\}$, $\bar{q}(i)^j$ and $\bar{r}(i)^j$ have a common ξ^+ -spectrum below $\pi_\iota(w)$, we conclude that $\bar{w} \leq \bar{q}$.

Lastly, assume $\nu \geq \alpha$. By (a), $\bar{r} \preceq^{<\kappa} \pi_\nu(\bar{r})$. Moreover, by convention and by (d), $\bar{r} \Vdash \bar{p}^* = \bar{r}^\uparrow$, where $\bar{p}^* = \pi_{\nu+1}(p^*)$. Finally, $\pi_\nu(w) \leq \pi_\nu(\bar{p})$ and $\pi_\nu(w) \leq \pi_\nu(r)$, so by fact 5.10, $\pi_\nu(w) \cdot \bar{r} \approx \pi_\nu(w) \cdot \bar{p}^*$. Thus, as $q \leq p^*$ holds by (d), we have $\pi_\nu(w) \cdot \bar{q} \leq \pi_\nu(w) \cdot \bar{p}^* \approx \pi_\nu(w) \cdot \bar{p}$, and we can set $\pi_{\nu+1}(w) = \pi_\nu(w) \cdot \bar{q}$. This finishes the proof of claim 5.12 and thus the proof that $p' \Vdash \dot{\alpha}_\zeta < \alpha$.

It remains to find a $\bar{T} * P_\alpha$ -name $\dot{\alpha}'_\zeta$. Still working in $L[\bar{B}']$, find $\dot{\alpha}'_\zeta$ so that:

$$\text{for all } \xi \text{ such that } (\xi)_2 = \zeta \text{ and } q_\xi \text{ is defined, } \pi_\alpha(q_\xi) \Vdash \dot{\alpha}'_\zeta = \check{\alpha}_\xi.$$

By claim 5.11, this defines a name below p' provided we can show that if $\alpha_\xi \neq \alpha_{\xi'}$ and $(\xi)_2 = \zeta$, $\pi_\alpha(q_\xi)$ and $\pi_\alpha(q_{\xi'})$ are incompatible. Assume otherwise and let $w \leq \pi_\alpha(q_\xi), \pi_\alpha(q_{\xi'})$. Assume $\xi < \xi'$; we claim that $w \cdot q_{\xi'} \leq w \cdot q_\xi$, a contradiction, as q_ξ and $q_{\xi'}$ force different values for $\dot{\alpha}_\zeta$. To prove the claim, it suffices to observe that by construction, there is p^* such that $q_\xi \Vdash p^* = (q_\xi)^\uparrow$ and $q_{\xi'} \leq p^*$. By construction, $\sigma_2^\kappa(q_\xi) < \alpha$ and so $q \preceq^\kappa \pi_\alpha(q)$. By fact 5.10, we conclude that $w \cdot p^* \approx w \cdot q_\xi$ and so $w \cdot q_{\xi'} \leq w \cdot q_\xi$. We have thus proved that $\dot{\alpha}_\zeta$ is a well-defined P_α -name. It is clear that $(t_\alpha, p') \Vdash \dot{\alpha}_\zeta = \dot{\alpha}'_\zeta$, and we are done with the proof of lemma 5.7 $\textcircled{\smile}$

It is crucial that by lemma 5.7, the book-keeping devices \bar{r} and \bar{s} “catch” all the relevant reals in the final extension by P_κ :

Lemma 5.13. *If $\iota < \kappa$, \dot{r}^0, \dot{r}^1 are P_κ -names for reals and $p \in P_\kappa$ forces \dot{r}^0, \dot{r}^1 are random over W^{P_ι} , there is $q \leq p$ and $\nu < \kappa$ such that $\bar{r}(\nu) = \iota$*

$$q \Vdash \dot{r}^j = \dot{r}_\nu^j.$$

If \dot{s} is a P_κ -name for a real and $p \in P_\kappa$, there is $q \leq p$ and $\xi < \kappa$ such that either $q \Vdash \dot{s} = \dot{s}_\xi$, or $q \Vdash \dot{s} \in \Gamma_\xi$.

Proof. By lemma 5.7 there is $q \leq p$, $\xi < \kappa$ and P_ξ -names \dot{x}^0, \dot{x}^1 such that $q \Vdash \dot{r}^j = \dot{x}^j$. As P_κ adds a random real below every condition (lemma 5.2, 4), we may assume 1_{P_κ} forces \dot{x}^j is random over W^{P_ν} . Using the notation from lemma 5.3, find ν, ν' and $q' \leq q$ such that $q' \Vdash \dot{x}^0 = \dot{x}_\nu$ and $q' \Vdash \dot{x}^1 = \dot{x}_{\nu'}$, and find $q'' \leq q'$ and $y \in Y(\nu, \nu')$ such that $q'' \Vdash (\dot{x}^0, \dot{x}^1) = y$. Thus, by construction of \bar{r} , we may find ν such that $\bar{\iota}(\nu) = \iota$ and $y = (\dot{r}_\nu^0, \dot{r}_\nu^1)$, and we have $q'' \Vdash \dot{r}^0 = \dot{r}_\nu^0$ and $\dot{r}^1 = \dot{r}_\nu^1$.

The second claim follows immediately from lemmas 5.7, 5.5 and the definition of \bar{s} . $\textcircled{\smile}$

5.5 Every projective set of reals is measurable

Let G be a generic for P_κ (and let G_ξ be the resulting generic on P_ξ , for $\xi < \kappa$).

Lemma 5.14. *For any $\nu < \kappa$, $\bigcup N_\nu^*$ is a null set, where*

$$N_\nu^* = \{N \in W[G_\nu] \mid W[G_\nu] \Vdash N \subset \mathbb{R} \text{ has measure zero}\}$$

Proof. Every null set $N \in W[G_\nu]$ is covered by a null Borel set whose Borel code is also in $W[G_\nu]$. The set C^* of Borel codes for null sets in $W[G_\nu]$ is countable in $W[G]$, so $\bigcup N_\nu^*$, which is equal to the union of all the Borel sets with code in C^* , is a countable union of null sets in $W[G]$. $\textcircled{\smile}$

The following, together with the last lemma, suffices to show that in the extension by P_κ , every projective set of reals is measurable.

Lemma 5.15. *Let $\nu < \kappa$. There is a name \dot{r}_* which is fully random over W^{P_ν} such that the following hold:*

1. *Let $\dot{B}(\dot{r}_*)$ be a P_ν -name for the complete sub-algebra of $B_\kappa : P_\nu$ generated by \dot{r}_* in $W[G_\nu]$ and let $B^0 = P_\nu * \dot{B}(\dot{r}_*)$. For any $b \in B_\kappa \setminus B^0$, there is an automorphism Φ of B_κ such that $\Phi(b) \neq b$ and $\Phi \upharpoonright B^0 = \text{id}$.*
2. *For any P_κ -name \dot{r} which is random over W^{P_ν} and any $p \in P_\kappa$ there is $q \leq p$ and an automorphism Φ of B_κ such that $q \Vdash \dot{r} = \Phi(\dot{r}_*)$ and $\pi_\nu \circ \Phi = \Phi \circ \pi_\nu = \pi_\nu$.*

Proof. For \dot{r}_* we may use any \dot{r}_η^0 (from our list \bar{r}) such that $\bar{\iota}(\eta) \geq \nu$ (i.e. its fully random over W^{P_ν}).

Let π be the canonical projection $\pi: B_\kappa \rightarrow B^0$, where B^0 is as in the hypothesis of item 1 of the lemma. Pick $\xi < \kappa$ such that

1. $\pi_\xi(b) \notin B^0$; this holds for large enough ξ since $b \notin B^0$;

2. r^* is a P_ξ -name, i.e. B^0 is a complete sub-algebra of B_ξ .
3. $P_{\xi+1} = \mathbf{Am}_1(P_\iota, D_\xi, r_*, r_*)$, where $\iota \geq \nu$.

Let b_0 denote $\pi_\xi(b)$. Clearly, there is $p \in D_\xi$, $p \leq b_0$ such that $\pi(p) \not\leq b_0$: for otherwise, the set

$$X = \{d \in B^0 \mid d \leq b_0\}$$

would be predense in P_ξ below b_0 , and thus $b_0 = \sum X \in B^0$, contradiction.

So pick p as above and let $q \in D_\xi$, $q \leq \pi(p)$ such that $q \cdot b_0 = 0$. Let $b_1 = \pi(q)$. Look at the condition $\bar{p} \in (D_\xi)_f^{\mathbb{Z}}$ such that $\bar{p}(-1) = q$, $\bar{p}(0) = b_1 \cdot p$ and for $i \in \mathbb{Z} \setminus \{-1, 0\}$, $\bar{p}(i) = b_1$. Then we have $\bar{p} \in (D_\xi)_f^{\mathbb{Z}}$, $\bar{p} \leq p \leq b_0$. Letting Φ denote the automorphism of B_κ resulting from $P_{\xi+1}$, we have $\Phi(\bar{p}) \leq q$ whence $\Phi(\bar{p}) \cdot b_0 = 0$. So as $\bar{p} \leq \pi_\xi(b)$ and $\Phi(\bar{p}) \cdot b = 0$, it follows that $\Phi(b) \neq b$; for otherwise since $\bar{p} \leq \pi_\xi(b)$, we have $\bar{p} \cdot b \neq 0$ but $\Phi(\bar{p} \cdot b) = \Phi(\bar{p}) \cdot b = 0$.

The second claim is clear from the construction, as $\Phi_\rho(\dot{r}_\rho^0) = \dot{r}_\rho^1$ for each $\rho < \kappa$.

☺

Finally, we show in $W[G]$:

Lemma 5.16. *Say $s \in [\text{On}]^\omega$, ϕ a formula. If $X = \{r \in \mathbb{R} \mid \phi(r, s)\}$, X is measurable.*

Proof. Let X, s be as above, and say $s = \dot{s}^G$. Without loss of generality, \dot{s} is a P_ν -name, where $\nu < \kappa$ and $\Vdash_\nu \dot{s} \in [\text{On}]^\omega$ (by lemma 5.7). Fix \dot{r}_* as in the previous lemma. Let $\dot{B}(\dot{r}_*)$ be a P_ν -name for the complete sub-algebra of $B_\kappa : P_\nu$ generated by \dot{r}_* in $W[G_\nu]$.

Claim. $\|\phi(\dot{r}_*, \dot{s})\|^{B_\kappa} \in \text{r.o.}(P_\nu) * \dot{B}(\dot{r}_*)$.

Proof of Claim. Write $B^0 = P_\nu * \dot{B}(\dot{r}_*)$ and $b = \|\phi(\dot{r}_*, \dot{s})\|^{B_\kappa}$. Towards a contradiction, assume $b \notin B^0$. By lemma 5.15, (1), there is an automorphism Φ of B_κ such that $\Phi(b) \neq b$ while $\Phi(\dot{s}) = \dot{s}$ and $\Phi(\dot{r}_*) = \dot{r}_*$. This is a contradiction, as we infer

$$b = \|\phi(\dot{r}_*, \dot{s})\|^{B_\kappa} = \|\phi(\Phi(\dot{r}_*), \Phi(\dot{s}))\|^{B_\kappa} = \Phi(b).$$

☺

Let N^* denote

$$\bigcup \{N \in W[G_\nu] \mid W[G_\nu] \models N \text{ has measure zero}\},$$

and let \dot{N}^* be a P_ν -name for this set. N^* is null in $W[G]$.

We find a Borel set B such that for arbitrary $r \notin N^*$, we have $r \in X \iff r \in B$. Then $X \setminus N^* = B \setminus N^*$ is measurable, finishing the proof. We may regard $B(\dot{r}_*)$ as identical to the Random algebra in $W[G_\nu]$, so we may write $\|\phi(\dot{r}_*, \dot{s})\|^{B_\kappa : B_\nu} = [B]_{\mathfrak{n}}$ for a Borel set B .

To show B is the Borel set we were looking for, let $r \notin N^*$ be arbitrary. Find \dot{r} and $p \in G$ such that $\dot{r}^G = r$ and $p \Vdash \dot{r} \notin \dot{N}^*$, i.e. p forces \dot{r} is random over W^{P_ν} . By 2 of the previous lemma, there is an automorphism Φ of P_κ and $q \in G$ such that $q \Vdash \Phi(\dot{r}_*) = \dot{r}$, and thus $\Phi(\dot{r}_*)^G = \dot{r}_*^{\Phi^{-1}[G]} = \dot{r}^G$.

Work in $W[G_\nu]$. Since $\pi_\nu \circ \Phi = \Phi \circ \pi_\nu = \pi_\nu$, Φ generates an automorphism $\bar{\Phi}$ of $B_\kappa : B_\nu$, and $\dot{r}_*^{\bar{\Phi}^{-1}[G]} = \dot{r}^G$. We have

$$\begin{aligned} \phi(\dot{r}^G, \dot{s}^G) \iff \|\phi(\dot{r}, \dot{s})\| \in G &\iff: \\ &\iff: \bar{\Phi}^{-1}(\|\phi(\dot{r}, \dot{s})\|) \in \bar{\Phi}^{-1}[G] \iff: \\ &\iff: \|\phi(\bar{\Phi}^{-1}(\dot{r}), \dot{s})\| \in \bar{\Phi}^{-1}[G] \iff: \\ &\iff: \|\phi(\dot{r}_*, \dot{s})\| \in \bar{\Phi}^{-1}[G] \cap B(\dot{r}_*) \iff: \dot{r}_*^{\bar{\Phi}^{-1}[G]} \in B \end{aligned}$$

As $\dot{r}_*^{\bar{\Phi}^{-1}[G]} = \dot{r}^G$, we are done. ☺

Chapter 6

The set Γ^0 is Δ_3^1

We now check that Γ^0 is in fact Δ_3^1 . By [Bar84], this is optimal, since under the assumption that all Σ_2^1 sets are Lebesgue-measurable, all Σ_2^1 sets do have the property of Baire.

Let $\Theta(r, s, \alpha, \beta)$ be the sentence

$L_\beta[r, s]$ is a model of ZF^- and of “ α is the least Mahlo and α^+ exists”.

Definition 6.1. For an ordinal α and $C \in {}^\alpha 2$, write $\sigma \triangleleft C$ to express σ is an initial segment of C , i.e. for some $\rho < \alpha$, $\sigma = C \upharpoonright \rho$. Let $\phi(x)$ be a formula. When we write $\forall^* \sigma \triangleleft C \phi(\sigma)$, we mean there exists $\zeta < \alpha$ such that for all $\rho \in (\zeta, \alpha)$, $\phi(C \upharpoonright \rho)$ holds. In other words, *for almost all initial segments σ of C , $\phi(\sigma)$ holds.*

For $j \in \{0, 1\}$, let $\Psi(r, j)$ denote the formula

$\exists s \in {}^\omega 2 \ \forall \alpha, \beta < \kappa$ if $\theta(r, s, \alpha, \beta)$ then:
 $L_\beta[r, s] \models \exists C \in {}^\alpha 2 \ \forall^* \sigma \triangleleft C \ \forall (n, i) \in \omega \times 2$
 $(r(n) = i) : T^\alpha(\sigma, n, i, j)$ has a branch.”

Lemma 6.2. For $j \in \{0, 1\}$ and any real r ,

$$r \in (\dot{\Gamma}^j)^G \iff \Psi(r, j).$$

Proof. For $\xi \leq \kappa$, let \mathcal{F}_ξ be the smallest set closed under (relational) composition and containing all functions $F = \Phi_\rho^\xi, (\Phi_\rho^\xi)^{-1}$ such that $\text{dom } F \subseteq P_\xi$. In other words, \mathcal{F}_ξ is the closure under relational composition of

$$\{\Phi_\rho^\xi, (\Phi_\rho^\xi)^{-1} \mid E^3(\alpha_\rho^\xi) < \xi\}.$$

First assume $r \in (\dot{\Gamma}^j)^G$ and show $\Psi(r, j)$ holds. If $j = 0$, by definition of $\dot{\Gamma}^0$ we can find $\eta < \kappa$ and $\Phi \in \mathcal{F}_\kappa$ such that $r = (\Phi(\dot{c}_\eta))^G$. If $j = 1$, we fix $\eta < \kappa$ such that $r = (\dot{s}_{\eta-1})^G$. Let \dot{r}_0 denote \dot{c}_η if $j = 0$ and let \dot{r}_0 denote $\dot{s}_{\eta-1}$ if $j = 1$ and let $r_0 = (\dot{r}_0)^G$.

In either case, at stage $\xi = E^2(\eta)$ we force with Jensen coding, adding a real s_0 such that

for all $\alpha, \beta < \kappa$, if $\theta(r_0, s_0, \alpha, \beta)$ then $C_\eta \upharpoonright \alpha, r_0 \in L_\beta[s_0]$ and $L_\beta[s_0] \models \text{“}\forall \sigma \text{ such that } \sigma \triangleleft C_\eta \upharpoonright \alpha \text{ and for all } n, i \text{ such that } r_0(n) = i, T^\alpha(\sigma, n, i, j) \text{ has a branch”}$.

So

$$1_{P_\kappa} \Vdash \Psi(\dot{r}_0, j),$$

which completes the proof in case $j = 1$. For $j = 0$, apply Φ to get

$$1_{P_\kappa} \Vdash \Psi(\Phi(\dot{r}_0), j),$$

and we are done as $(\Phi(\dot{r}_0))^G = r$.

Now assume $\Psi(r, j)$ and show $r \in (\dot{\Gamma}^j)^G$: Fix s to witness $\Psi(r, j)$. It must be the case that

$$\begin{aligned} L[r, s] \models \exists C \in {}^\kappa 2 \quad \forall^* \sigma \triangleleft C \quad \forall (n, i) \in \omega \times 2 \\ (r(n) = i): T(\sigma, n, i, j) \text{ has a branch.} \end{aligned} \quad (6.1)$$

For let $L_\beta[r, s]$ be isomorphic to an elementary sub-model of $L_{\kappa^{++}}[r, s]$ which contains r and s , and let α be the least Mahlo in $L_\beta[r, s]$. Then as $\theta(r, s, \alpha, \beta)$ holds, by $\Psi(r, j)$,

$$\begin{aligned} L_\beta[r, s] \models \exists C \in {}^\alpha 2 \quad \forall^* \sigma \triangleleft C \quad \forall (n, i) \in \omega \times 2 \\ (r(n) = i): T^\alpha(\sigma, n, i, j) \text{ has a branch.} \end{aligned}$$

So by elementarity, (6.1) holds.

Fix $\xi < \kappa$ such that $r, s \in W[G_\xi]$ and fix C witnessing (6.1). Pick $\zeta < \kappa$ such that

$$\begin{aligned} \text{for } \Phi \in \mathcal{F}_\xi \text{ and } \nu, \nu' < \xi \text{ such that } \Phi \neq \text{id and } \nu \neq \nu' \text{ we have} \\ \Phi(C_\nu) \upharpoonright \zeta \neq C_{\nu'} \upharpoonright \zeta. \end{aligned} \quad (6.2)$$

This is possible by lemma 5.4. As (6.1) holds, we can also assume ζ to be large enough so that whenever $r(n) = i$, and $\rho \geq \zeta$, $T(C \upharpoonright \rho, n, i, j)$ has a branch in $L[r, s]$.

Since $r, s \in W[G_\xi]$, lemma 6.3 below gives us: for any n and i , if the tree $T(C \upharpoonright \zeta, n, i, j)$ has a branch in $L[r, s]$ then there is $\Phi \in \mathcal{F}_\xi$ and $\eta < \xi$ such that $C \upharpoonright \zeta \triangleleft \Phi(C_\eta)$. By (6.2), Φ and η are unique and do not depend

on n and i , so let Φ and η be fixed. By the way, it follows that $C = \Phi(C_\eta)$ (which we do not use in the following). More importantly, whenever $r(n) = i$, $T(\Phi(C_\eta) \upharpoonright \zeta, n, i, j)$ has a branch in $L[r, s]$.

Moreover, lemma 6.3 yields that whenever $T(C \upharpoonright \rho, n, i, j)$ has a branch in $L[r, s]$,

1. if $j = 0$ then η is a limit and $(\Phi(\dot{c}_\eta))^G(n) = i$
2. if $j = 1$ then η is a successor and $(\Phi(\dot{s}_{\eta-1}))^G(n) = i$.

Thus, in the first case, $r = \Phi(c_\eta)$ and so $r \in (\dot{\Gamma}^0)^G$. In the second case, $r = (\Phi(\dot{s}_{\eta-1}))^G$. As $(\dot{s}_{\eta-1})^G \in (\dot{\Gamma}^1)^G$ and $(\dot{\Gamma}^1)^G$ is closed under all the automorphisms $\{\Phi_\rho^* \mid \rho < \kappa\}$ (by lemma 5.5), $r \in (\dot{\Gamma}^1)^G$. \odot

For $\xi \leq \kappa$, let I_ξ be the set of triples (σ, n, i, j) such that for some $\eta < \xi$ and $\Phi \in \mathcal{F}_\xi$, $\sigma \triangleleft \Phi(C_\eta)$ and

1. if η is limit ordinal, $\Phi(c_\eta)(n) = i$ and $j = 0$
2. if η is a successor ordinal, $\Phi(s_{\eta-1})(n) = i$ and $j = 1$.

Lemma 6.3. *Say $\xi < \kappa$ and let u an arbitrary real in $L[\bar{B}][G_\xi]$. Then $T(\sigma, n, i, j)$ has a branch in $L[u]$ only if $(\sigma, n, i, j) \in I_\xi$.*

Proof. Fix $\nu \in I$, $\xi_0 < \kappa$ and $p_0 \in P_{\xi_0}$ such that $p_0 \Vdash \check{\nu} \notin \dot{I}_{\xi_0}$ in $L[\bar{B}]$. Let \bar{B}^- denote $\{\bar{B}(\xi)\}_{\xi \in I \setminus \{\nu\}}$.

We will show in a moment that $P_{\xi_0}(\leq p_0)$ is equivalent to a forcing $P_{\xi_0}^* \in L[\bar{B}^-]$, whence $\bar{T} * P_{\xi_0}(\leq p_0)$ is equivalent to

$$\left[\left(\prod_{\zeta \in I \setminus \{\nu\}}^{\leq \kappa} T(\zeta) \right) * P_{\xi_0}^*(\leq p_0) \right] \times T(\nu).$$

Assuming this for the moment, we can prove the lemma thus: As $T(\nu)$ doesn't add reals, any real $u \in L[\bar{B}][G_{\xi_0}]$ is actually an element of $L[\bar{B}^-][G_{\xi_0}]$, and as $T(\nu)$ is Suslin in $L[\bar{T}^-]$ and $P_{\xi_0}^*$ is κ -centered, $T(\nu)$ remains Suslin in $L[\bar{B}^-][G_{\xi_0}]$ and thus in $L[u]$. It remains to see that $P_{\xi_0}(\leq p_0)$ is equivalent to a forcing which is an element of $L[\bar{B}^-]$. For the purpose of carrying out the inductive proof, we prove a stronger statement, in the claim below. First, note that for $\xi < \bar{\xi} \leq \xi_0$, as $I_\xi \subseteq I_{\bar{\xi}}$,

$$\|\nu \notin \dot{I}_{\bar{\xi}}\| \leq \|\nu \notin \dot{I}_\xi\|,$$

and so

$$\pi_\xi(\|\nu \notin \dot{I}_{\bar{\xi}}\|) \leq \|\nu \notin \dot{I}_\xi\|$$

Let b_ξ denote $\|\nu \notin \dot{I}_\xi\|^{B_\xi}$, for $\xi < \xi_0$.

Claim 6.4. *For each $\xi \leq \xi_0$, there is an isomorphism¹*

$$j_\xi: P_\xi(\leq b_\xi) \rightarrow P_\xi^*,$$

such that

$$\text{for } \xi < \bar{\xi} \leq \xi_0 \text{ and } p \in P_{\bar{\xi}}(\leq b_{\bar{\xi}}), j_{\bar{\xi}}(\pi_{\bar{\xi}}(p)) = \pi_{\bar{\xi}}(j_{\bar{\xi}}(p)). \quad (6.3)$$

Moreover, $P_{\bar{\xi}}^* \in L[\bar{B}^-]$ and $P_{\bar{\xi}}^* = P_{\bar{\xi}}(\leq b_{\bar{\xi}}) \cap L_{\kappa^+}[B^-]$.

Remark 6.5. 1. There is no need to distinguish between G_ξ and $j_\xi[G_\xi]$, we write G_ξ for either one.

2. The argument is slightly more elegant if we work with trees on κ^{++} , as then \bar{T} is κ^{++} -distributive, and this entails $\mathbf{H}(\kappa^{++})^{L[\bar{B}]} = \mathbf{H}(\kappa^{++})^L$. Our \bar{T} , a (sequence of) tree(s) on κ^+ , is not κ^+ -distributive, but $T(\nu)$ is κ^+ -distributive. So we have

$$H(\kappa^+)^{L[\bar{B}]} = H(\kappa^+)^{L[\bar{B}^-]} = L_{\kappa^+}[\bar{B}^-]. \quad (6.4)$$

At heart, the claim is a consequence of this simple observation:

Fact 6.6. If P has the κ -chain condition and $p \Vdash \dot{x} \in H(\kappa^+)$, there is $\dot{x}' \in H(\kappa^+)$ such that $p \Vdash \dot{x} = \dot{x}'$.

Proof. Use nice names. ☺

The induction splits into cases. For the successor case, assume j_ξ is already defined and define $j_{\xi+1}$. Observe that by induction, $\bar{T} * P_\xi$ is equivalent to

$$\left[\left(\prod_{\zeta \in I \setminus \{\nu\}}^{< \kappa} T(\zeta) \right) * P_\xi^* \right] \times T(\nu),$$

and since $T(\nu)$ is κ^+ -distributive in $L[\bar{B}^-][G_\xi]$ (because it is still Suslin in that model),

$$H(\kappa^+)^{L[\bar{B}][G_\xi]} \subseteq L[\bar{B}^-][G_\xi], \quad (6.5)$$

¹By isomorphism, we mean of course j_ξ is injective on the separative quotient of its domain.

Easiest Case: As a warm up, assume $\xi \in E^1$. Thus $P_{\xi+1} = P_\xi \times Q$ for $Q \in L$. Of course, $P_\xi^* \times Q \in L[\bar{B}^-]$. We can set $j_{\xi+1}(p, q) = (j_\xi(p), q)$.

Observe that the claim asks for an isomorphism of $P_{\xi+1}^*$ with $P_\xi^* \dot{Q}_\xi(\leq b_\xi)$, not $P_\xi(\leq b_\xi) * \dot{Q}_\xi$. So to formally satisfy the claim – and to make the induction work in the next step – restrict j_ξ to $P_{\xi+1}(\leq b_{\xi+1})$. We should check $P_\xi^* \times Q(\leq b_{\xi+1}) \in L[\bar{B}^-]$, though:

Lightheartedly identify $P_{\xi+1}$ names and $P_\xi^* \times Q$ -names and assume (by fact 6.6 and (6.4)) that $\dot{I}_\xi \in L[\bar{B}^-]$. Then for $p \in P_\xi^* \times Q$,

$$p \Vdash \nu \in \dot{I}_{\xi+1}$$

is absolute between $L[\bar{B}]$ and $L[\bar{B}^-]$, so $P_\xi^* \times Q(\leq b_{\xi+1}) \in L[\bar{B}^-]$. So set $P_{\xi+1}^* = P_\xi^* \times Q(\leq b_{\xi+1})$.

Jensen Coding (and another easy case): Now, assume $\xi \in E^2$, i.e. $P_{\xi+1} = P_\xi * \dot{Q}_\xi$ where \dot{Q}_ξ is a name for Jensen coding. Now it is crucial that we work below $b_\xi = \|\nu \notin \dot{I}_\xi\|^{B_\xi}$: Work in $L[\bar{B}][G_\xi]$ for now, where G_ξ is P_ξ^* -generic over $L[\bar{B}]$. Then $\nu \notin (\dot{I}_\xi)^{G_\xi}$, so the set of branches we code at this stage does not contain $B(\nu)$. Thus $\dot{Q}_\xi^{G_\xi}$ is a subset of $H(\kappa^+)$ (of the extension), which is definable over $\langle H(\kappa^+), \bar{B}^- \rangle$. By (6.5), $\dot{Q}_\xi^{G_\xi} \in L[\bar{B}^-][G_\xi]$. This immediately implies that $P_{\xi+1}$ is equivalent to a forcing which is an element of $L[\bar{B}^-]$, but in order to carry out the inductive proof at limits, we need (6.3). For this, let $\phi(x)$ be a formula defining membership in $\dot{Q}_\xi^{G_\xi}$ over $\langle H(\kappa^+), \bar{B}^- \rangle$ in $L[\bar{B}][G_\xi]$. Set

$$P_{\xi+1}^* = \{(p, \dot{q}) \mid p \in P_\xi^*, \dot{q} \in H(\kappa^+) \text{ is a } P_\xi^* \text{-name, } P_\xi^* \Vdash \dot{\phi}(q)^{H(\kappa^+)}\} \quad (6.6)$$

As P_ξ^* has the κ^+ -chain condition, any $x \in H(\kappa^+)^{L[\bar{B}][G_\xi]}$ has a P_ξ^* -name in $H(\kappa^+)^{L[\bar{B}]}$. Therefore, by (6.4),

$$P_\xi^* \Vdash \dot{\phi}(q)^{H(\kappa^+)}$$

is absolute between $L[\bar{B}]$ and $L[\bar{B}^-]$ and thus (6.6) witnesses that $P_{\xi+1}^* \in L[\bar{B}^-]$. For $(p, \dot{q}) \in P_{\xi+1}$, we can now define $j_{\xi+1}(p, \dot{q})$. Since

$$P_\xi^* \Vdash j_\xi(\dot{q}) \in j_\xi(\dot{Q}_\xi) \subseteq H(\kappa^+)$$

using fact 6.6, we can find a P_ξ^* -name $\dot{q}' \in H(\kappa^+)$ such that $P_\xi^* \Vdash j_\xi(\dot{q}) = \dot{q}'$. Let $j_{\xi+1}(p, \dot{q}) = (j_\xi(p), \dot{q}')$.

Clearly, $j_\xi(p, \dot{q}) \in P_{\xi+1}^*$. It is straightforward to check that $j_{\xi+1}$ preserves the ordering and is onto. It is injective on the separative quotient of $P_{\xi+1}$.

Again, as in the previous case, restrict j_ξ to $P_{\xi+1}(\leq b_{\xi+1})$ to formally satisfy the claim. The case $\xi \in E^0$ can be treated in an analogous – but simpler – way.

Remark 6.7. Observe, by the way that for any P_ξ^* name \dot{Q} such that $P_\xi^* \Vdash \dot{Q} = j_\xi(\dot{Q}_\xi)$,

$$P_{\xi+1}^* = (P_\xi^* * \dot{Q}) \cap H(\kappa^+).$$

In particular, it follows by induction that $P_{\xi+1}^* = P_{\xi+1} \cap H(\kappa)$. Moreover, if $\dot{Q} \in L[\bar{B}^-]$ then

$$P_{\xi+1}^* = ((P_\xi^* * \dot{Q}) \cap H(\kappa^+))^{L[\bar{B}^-]}.$$

If \dot{Q}_ξ is chosen reasonably (e.g. in the most obvious way), in fact $j(\dot{Q}_\xi) \in L[\bar{B}^-]$.² This means that we could find P_ξ^* by interpreting the definition of P_ξ in $L[\bar{B}^-]$ if we commit to using only *names which have size at most κ* .

Amalgamation: Now let $\xi \in E^3$ and say $\xi = E^3(\alpha_\rho^0)$, i.e. $P_{\xi+1} = \mathbf{Am}_1(P_\xi, P_\xi, f, \lambda_\xi)$, where f is the isomorphism of the algebras generated by some P_ξ -names \dot{r}_0 and \dot{r}_1 , and let π_i denote the canonical projection from P_ξ to the domain and range of f . Let Φ be the resulting automorphism. Let R denote the set of $\bar{p} \in P_{\xi+1}$ such that for all $i \neq 0$, $\bar{p}(i) \in \widehat{D}_\xi(\leq b_\xi)$ and $\bar{p}(0) \in P_\xi(\leq b_\xi)$. We show $P_{\xi+1}(\leq b_{\xi+1}) \subseteq R$ and that (the separative quotient of) R is in $L[\bar{B}^-]$.

For the first, it is crucial that $\dot{I}_{\xi+1}$ is closed under Φ . Say $\bar{p} \in P_{\xi+1}(\leq b_{\xi+1})$, that is, $\bar{p} \Vdash \nu \notin \dot{I}_{\xi+1}$. By the definition of $\dot{I}_{\xi+1}$, for each $i \in \mathbb{Z}$,

$$\Phi^i(\bar{p}) \Vdash_{P_{\xi+1}} \nu \notin \dot{I}_{\xi+1},$$

and so

$$\bar{p}(i) = \bar{\pi}(\Phi^i(\bar{p})) \Vdash_{\widehat{P}_\xi} \nu \notin \dot{I}_\xi,$$

where $\bar{\pi}$ denotes the canonical projection from $P_{\xi+1}$ to \widehat{P}_ξ . Thus, $\bar{p} \in R$.

By induction, $P_\xi(\leq b_\xi)$ is isomorphic to P_ξ^* . A little care is needed to see $\widehat{D}_\xi(\leq b_\xi)$ (or, to be precise, its separative quotient) is in $L[\bar{B}^-]$: $\widehat{D}_\xi(\leq b_\xi)$ is not the same as $\widehat{D}_\xi(\leq b_\xi)$ in general. The two orderings are equivalent, but once more, this doesn't mean that we can use them interchangeably in the definition of R . At the same time, $(p, \dot{b}_0, \dot{b}_1) \in \widehat{D}_\xi(\leq b_\xi)$ *does not imply* that $p \leq b_\xi$, and so p needn't be in the domain of j_ξ .

So we have to check that in fact, $B_\xi^* = \text{r.o.}(P_\xi^*) \in L[\bar{B}^-]$. This is because we may regard B_ξ^* the collection of regular open cuts which are given by

²It would be tempting to define $j_{\xi+1}(p, \dot{q}) = (j_\xi(p), j_\xi(\dot{q}))$, but we do not know if $j_\xi(\dot{q}) \in L[\bar{B}^-]$.

antichains in P_ξ^* . As P_ξ^* has the κ^+ -chain condition all such antichains and hence all regular cuts are in $L[\bar{B}^-]$ (once more by (6.4)). So $B_\xi^* \in L[\bar{B}^-]$ and j_ξ can be viewed as an isomorphism of B_ξ with B_ξ^* . Thus, as we have assumed \dot{r}_0, \dot{r}_1 are in $L[\bar{B}^-]$, we can define $B(\dot{r}_i)^{B_\xi^*}$ and canonical projections from B_ξ^* to $P_\iota^* * (\langle \dot{r}_i \rangle^{B_\xi^*: P_\iota^*})$ in $L[\bar{B}^-]$. In fact,

$$P_\iota^* * (\langle \dot{r}_i \rangle^{B_\xi^*: P_\iota^*}) = j[C],$$

where C is the algebra obtained from $P_\iota * (\langle \dot{r}_i \rangle^{B_\xi: P_\iota})$ by factoring through the ideal of elements below $-b_\xi$. Thus also $D_\xi^* = j_\xi[\widehat{D}_\xi(\leq b_\xi)] \in L[\bar{B}^-]$ (it is a subset of $B_{\xi+1}^*$ with a sufficiently absolute definition). We leave it to the reader to check that this suffices to find an isomorphic copy R^* of R in $L[\bar{B}^-]$. Finally, let $P_{\xi+1} = R^*(\leq b_{\xi+1})$ and let $j_{\xi+1}$ be defined by $j_{\xi+1}(\bar{p})(i) = j_\xi(\bar{p}(i))$. A very similar but simpler argument works if $\xi = E^3(\alpha_\rho^\zeta)$ for $\zeta > 0$ and $P_{\xi+1} = \mathbf{Am}_2(\text{dom}(\Phi), P_\xi, \Phi)$ for some Φ . This completes the successor cases.

For ξ limit, check that the λ_ξ -diagonal limit is absolute between $L[\bar{B}^-]$ and $L[\bar{B}]$. So let P_ξ^* be the λ_ξ -diagonal limit of the sequence constructed so far, inside $L[\bar{B}^-]$, restricted to conditions below b_ξ . By (6.3), the isomorphisms constructed at earlier stages can be glued together to form j_ξ . This finishes the proof of the claim. $\textcircled{\smile}$

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Zusammenfassung

Wir zeigen: unter der Annahme der Konsistenzstärke einer Mahlo-Kardinalzahl ist es konsistent, dass alle projektiven Mengen Lebesgue-messbar sind, jedoch eine Δ_3^1 -Menge ohne die Baire-Eigenschaft existiert. Damit ist das Thema der vorliegenden Arbeit die Frage, wie unabhängig die Struktur des Ideals der Nullmengen von der Struktur des Ideals der mageren Mengen ist. Die Frage nach ihrer Unabhängigkeit wird im Hinblick darauf untersucht, auf welcher Stufe der projektiven Hierarchie die erste irreguläre Menge auftritt; irregulär heißt dabei eine Menge die nicht Borel modulo des jeweils betrachteten Ideals ist.

Klassische Arbeiten von Gödel und Solovay haben gezeigt, dass in Bezug auf jedes dieser beiden Ideale eine irreguläre Menge schon auf sehr niedriger Ebene der projektiven Hierarchie auftreten kann, dass andererseits aber auch alle projektiven Mengen regulär sein können. Auch mithilfe von Woodin-Kardinalzahlen lassen sich ähnliche Resultate zeigen. Dabei treten jedoch in all den erwähnten Modellen die irregulären Mengen in beiden Idealen auf der selben Stufe auf, weshalb diese Modelle nicht dazu geeignet sind, Unabhängigkeit dieser Ideale nachzuweisen. Tatsächlich gibt es überraschenderweise auf niedrigen Stufen der projektiven Hierarchie auch keine Unabhängigkeit (siehe [Bar84]).

In [She84] wurde gezeigt, dass es möglich ist, dass sehr einfache Mengen nicht-messbar sind, während alle projektiven Mengen die Baire-Eigenschaft besitzen. Komplementär dazu zeigt [She85], dass es möglich ist, dass alle projektiven Mengen messbar sind, während gleichzeitig eine Menge ohne Baire-Eigenschaft existiert; in letzterem Modell ist die irreguläre Menge jedoch nicht projektiv.

Der Beweis, dass diese Menge auch projektiv sein kann stützt sich auf die Kombination einer Weiterentwicklung der Amalgamations-Technik aus [She85] mit Jensens Forcing, welches das Universum durch eine reelle Zahl kodiert. Letzteres wurde in [Dav82] schon verwendet, um eine Menge projektiv zu machen. Wesentlich für diese Zusammenführung ist der hier entwickelte Begriff von *stratified forcing*, der erlaubt zu zeigen, dass nicht unerwün-

schterweise Kardinalzahlen kollabieren. Gleichzeitig muss die Amalgamation so beschaffen sein, dass sie die Eigenschaft, *stratified* zu sein, erhält. Dass im letzten Limeschritt der Forcing-Iteration keine Kardinalzahl kollabiert wird, verwendet wesentlich die Existenz einer Mahlo-Kardinalzahl. Bis jetzt ist nur bekannt, dass das vorliegende Resultat eine unerreichbare Kardinalzahl voraussetzt (siehe [She84]); ob die Annahme der Konsistenz einer Mahlo-Kardinalzahl notwendig ist, bleibt offen. Die Komplexität der Menge ohne die Baire Eigenschaft ist nach [Bar84] optimal.

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