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DISSERTATION

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Noncommutative Approximation: Smoothness, Approximation and Invertibility in Banach Algebras

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Deutsche Zusammenfassung

In dieser Arbeit werden Matrizen beschränkter linearer Operatoren mit Abklingverhalten der Nebendiagonalen behandelt, das sind Matrizen, deren Einträge mit dem Abstand zur Diagonale betragsmäßig abfallen.

Es wird untersucht, unter welchen Bedingungen Matrizen mit derartigem Abklingverhalten invers-abgeschlossene Teilalgebren der Algebra beschränkter Operatoren sind, das heißt, wann die Inverse einer als Operator invertierbaren Matrix mit Abklingverhalten wieder ein gleichartiges Abklingverhalten aufweist.

Derartige Fragen wurden bislang nur für gewisse Klassen von Matrizen behandelt (siehe dazu etwa die Arbeiten von Baskakov, Demko, Smith und Moss, Jaffard, oder Gröchenig und Leinert). In dieser Arbeit wird versucht, systematische Ansätze zur Behandlung des Abklingverhaltens von Matrizen zu geben, und auch die Invers-Abgeschlossenheit systematisch zu behandeln.

Zwei Konstruktionen werden eingeführt: Einerseits kann das Abklingverhalten von Matrizen mit Hilfe einer Glattheitstheorie von Banachräumen beschrieben werden, andererseits kann es auch durch die Güte der Approximation durch Bandmatrizen gemessen werden. Beide Konstruktionen liefern in systematischer Weise Klassen invers-abgeschlossener Teilalgebren zu einer gegebenen Banach-Algebra von Matrizen, und beide Konstruktionen lassen sich sinnvoll für größere Klassen von Banach-Algebren erklären. Auf die beschriebene Weise werden nicht nur bekannte Resultate über Matrizen mit Abklingverhalten wiedergewonnen, sondern auch neue invers-abgeschlossene Algebren von Matrizen mit Abklingverhalten konstruiert.

Der Zusammenhang zwischen beiden Konstruktionen – Abklingverhalten der Nebendiagonalen durch Approximation beziehungsweise durch Glattheit – wird wie im Fall der klassischen trigonometrischen Approximation durch Sätze vom Jackson-Bernstein-Typ vermittelt. Dies erlaubt eine konstruktive Beschreibung von Approximations- bzw. Glattheitsräumen durch Littlewood-Paley-Zerlegungen.

Schließlich wird versucht, die beschriebene Theorie für Matrizen mit Abklingverhalten jenseits der polynomialen Ordnung anzuwenden. Dazu werden Analoga zu ultradifferenzierbaren Funktionen für Operatoren konstruiert. Auch hier ist es wieder möglich, die Invers-Abgeschlossenheit der entstehenden Algebren in den beschränkten Operatoren nachzuweisen, und so etwa das klassische Resultat von Demko, Smith und Moss auf allgemeinere Formen des Abklingverhaltens auszudehnen.

Contents

CHAPTER 1

Introduction

1.1. Overview

In this thesis, we study matrices of bounded linear operators with entries that decay away from the diagonal in a certain way. We also study classes of operators that give raise to matrices with common decay properties. What can be said about the product of two of these operators?

More formally: If $\mathcal{A}$ is a subalgebra of matrices in $\mathcal{B}(\ell^2)$, the algebra of bounded operators on ℓ^2, defined by conditions on their off-diagonal decay, and $A \in \mathcal{A}$ has an inverse in $\mathcal{B}(\ell^2)$, is A^{-1} in $\mathcal{A}$? Using mathematical terminology: Is $\mathcal{A}$ inverse-closed in $\mathcal{B}(\ell^2)$?

We give a still more abstract formulation. Let us suppose that we are given a Banach algebra $\mathcal{B}$, which could be, for instance, an algebra of infinite matrices whose norm describes some form of off-diagonal decay. We try to find a systematic construction of subalgebras $\mathcal{A} \subseteq \mathcal{B}$ such that an element $a \in \mathcal{A}$ is invertible in $\mathcal{A}$ if and only if a is invertible in the larger algebra $\mathcal{B}$. In the context of matrices, we think of the smaller algebra as an algebra describing a stronger decay condition. While often the existence of an inverse-closed subalgebra is taken for granted, e.g., in non-commutative geometry [**18, 30**], our interest is in the systematic construction of inverse-closed subalgebras and their application to matrix algebras.

Questions of inverse-closedness in algebras of matrices with off-diagonal decay have been studied by several mathematicians. The *methods* that are used are quite diverse.

Demko [**35**] used refinements of a Neumann series argument to prove that the entries of the inverse of a banded matrix have exponential off-diagonal decay. In [**36**] the same result is obtained using spectral theory of $\mathcal{B}(\ell^2)$ and arguments from approximation theory.

Jaffard [**61**] proves that in case of polynomial decay of the matrix coefficients, the inverse has polynomial decay *of the same order*. His proof uses (implicitly) commutator estimates and some form of quotient rule for the inverse.

In [**44**] the method of *band extensions* is used to prove the inverse-closedness of the *matrix valued Wiener algebra* in $\mathcal{B}(\ell^2)$, that is, the operators $A \in \mathcal{B}(\ell^2)$, where $\sum \|\hat{A}(k)\|_{\mathcal{B}(\ell^2)} \leq \infty$, $\hat{A}(k)$ denoting the kth side diagonal of A.

Kurbatov [**68, 69**] introduces the concept of convolution dominated operators, which is essentially equivalent to that of the matrix valued Wiener algebra. An operator A on $\mathcal{B}(\ell^2)$ is convolution dominated, if $|Ax(k)| \leq h * |x|(k)$, where h is an ℓ^1- sequence, and $|x|$ denotes the vector with components $|x(k)|$. This notion offers the possibility to measure off-diagonal decay by using results on weighted convolution algebras.

Baskakov [**10, 11**], working on convolution dominated operators, applies the theorem of Bochner-Phillips, a Banach algebra version of Wiener's Lemma, to show the inverse-closedness of the algebra of weighted convolution dominated operators in $\mathcal{B}(\ell^2)$. The weight v is assumed to be submultiplicative and to satisfy the *Gelfand*

Raikov Shilov (*GRS*) condition, that is, a precise condition on the subexponential growth of v. It can be shown that this condition is in some sense optimal.

Gröchenig and Leinert [54] investigate "Schur algebras" of matrices, the norm of which is obtained by a weighted version of Schur's test. The weight is assumed to be log-concave and radial, to have minimal polynomial growth and to satisfy the GRS condition. Using a sequence of auxiliary weights that are constructed by a procedure similar to the Fenchel-Young conjugate of convex analysis and with the help of "Barnes' Lemma" [9, Lemma 4.6] they are able to show the inverse-closedness of the Schur algebras in $\mathcal{B}(\ell^2)$. An interpolation argument proves a similar statement for algebras of the type considered by Jaffard.

Sun [98] proves the inverse-closedness of algebras of generalized Schur and Jaffard type in $\mathcal{B}(\ell^2)$ for polynomial weights using (implicitly) Jackson-Bernstein conditions and a spectral radius estimate. This type of argument is extended in [99] to matrices indexed with spaces of homogeneous type and more general (anisotropic) weights.

The treatment in [56] unifies some approaches to off-diagonal decay of matrices by identification of a "natural class" of matrix algebras that includes the convolution dominated matrices and the matrices with off-diagonal decay in the sense of Jaffard. The proof of Baskakov [11] is adapted to this more general situation.

So far most of the research is done on three classes of matrices: The matrices with off-diagonal decay in the sense of Demko and Jaffard, the convolution-dominated operators (Kurbatov, Baskakov), and matrices in a (generalized) Schur class. We might conclude that, regardless of the important results discussed above, algebras of matrices with off-diagonal decay, that are inverse-closed in $\mathcal{B}(\ell^2)$ have not been investigated on a systematic level yet.

In this thesis we treat two *systematic constructions* of algebras of matrices with off-diagonal decay and investigate the question of their inverse-closedness. First, we identify off-diagonal decay of matrices with smoothness, using derivations and d-parameter automorphism groups. This is suggested by an analysis of Jaffard's proof, by Baskakov's use of operator valued Fourier series, or by a study of the relation between the decay of a convolution operator and the smoothness of its symbol. In the second approach we measure off-diagonal decay of matrices by the quality of approximation with banded matrices. The interplay of smoothness and approximation theory allows us to obtain Jackson-Bernstein Theorems that relate the off-diagonal decay of a matrix to the rate of approximation by banded matrices.

Both constructions can be carried out for general Banach algebras and lead to systematic constructions of inverse-closed subalgebras of a given Banach algebra. We are not only able to re-derive known results [36, 61] within a unified framework but we can also construct new forms of Banach algebras of matrices with off-diagonal decay.

Let us finally say a few words about applications and the relevance of the results in the main part of this work. Matrices with off-diagonal decay are abundant in applications. Without any claim to be exhaustive and focusing on the area of signal processing and applied Harmonic Analysis we mention the theory of localized frames [49] with its application to sampling theory, wavelet theory and time-frequency analysis, e.g. [2, 8, 20, 48, 53, 96]. Applications include modeling of time variant channels for mobile communications [97]. We also believe that a theory of approximation of matrices with off-diagonal decay by banded matrices might have some use in the numerical analysis of such matrices.

1.2. Organization of the Thesis

In the following we describe the main results of this thesis in more detail.

In Chapter 2 we present notation and background material on weights, and some results from the theory of Banach algebras, in particular on inverse-closedness. We introduce the *standard examples* of Banach algebras of matrices, and list the known results on off-diagonal decay and inverse-closedness in $\mathcal{B}(\ell^2)$ for them.

In Chapter 3 we review and adapt parts of the theory of generalized approximation spaces and algebras developed by Almira and Luther [**4, 5, 6, 73**]. This theory allows us to give constructive characterizations of the weighted approximation spaces of *solid* matrix algebras (i.e., matrix algebras that are ℓ^∞-modules with componentwise multiplication) of Littlewood-Paley type, see Proposition 3.13. For Banach algebras a result of Almira and Luther [**6**] shows that the approximation spaces $\mathcal{E}_r^p(\mathcal{A})$ of polynomial order are actually Banach algebras for an adapted approximation scheme. An estimate of their proof can be used to show that $\mathcal{E}_r^p(\mathcal{A})$ is inverse-closed in $\mathcal{A}$, if $\mathcal{A}$ is a *symmetric* Banach algebra (Theorem 3.26). For solid and unweighted matrix algebras indexed over $\mathbb{Z}$ we can go beyond polynomial approximations, and obtain inverse-closedness of weighted approximation spaces in $\mathcal{B}(\ell^2)$, where the weights may be subexponential (Proposition 3.29). The method of proof, if applied to solid matrix algebras over $\mathbb{Z}^d$, still gives an approximation result, but the hypotheses are too cumbersome to be useful.

The theory of smoothness in matrix and Banach algebras is developed in Chapter 4. The formal commutator $A \to [X, A] = XA - AX$ with the diagonal matrix X given by $X(k,l) = 2\pi i k \delta_{k,l}, k, l \in \mathbb{Z}$, has the entries

$$[X, A](k,l) = 2\pi i(k-l)A(k,l)$$

and defines a closed derivation on many matrix algebras of interest, however, the matrix algebras are neither C^*-algebras (they are only symmetric) nor is this derivation densely defined. We have to adapt the theory of derivations on C^*- and general Banach algebras $\mathcal{A}$ (e.g., [**23, 22, 64, 65**]) to show that the domain $\mathcal{D}(\mathcal{A})$ of a set of (commuting) derivations is inverse-closed in $\mathcal{A}$ (Proposition 4.8). For solid, symmetric matrix algebras this result implies the inverse-closedness of the subalgebras of integer polynomial weight in $\mathcal{B}(\ell^2)$ (Proposition 4.10). In turn, we can formulate an anisotropic version of Jaffard's theorem.

In order to describe smoothness and decay conditions of fractional order we assume the (bounded) action of a d-parameter automorphism group on (a closed subspace of) the Banach algebra $\mathcal{A}$. For matrix algebras the appropriate concept is related to *homogeneous matrix algebras* [**32, 33, 34**] and the matrix function

$$\chi_t(A)(r,s) = A(r,s)e^{2\pi i(r-s)t}$$

used by Baskakov [**10, 11**]. The theory of d-parameter (semi)groups on Banach spaces and the construction of the smoothness spaces of Besov and Bessel potential type are well-known (see [**27**] for a classic account, the subject is also treated briefly in the literature on Besov spaces or in the context of interpolation and approximation theory [**13, 15, 79, 102**]) and goes parallel to the corresponding theory for functions on $\mathbb{R}^d$ to a large extent. However, we want to present matters in a way

- to be useful for applications in approximation theory,
- to work for group actions on $\mathbb{R}^d$ that are only defined on a subspace,
- and to be suited to treat the questions of the algebra properties of the smoothness spaces.

We are able to prove the inverse-closedness of the algebra valued Besov spaces $\Lambda_r^p(\mathcal{A})$ and of the Bessel potential spaces $\mathcal{P}_r(\mathcal{A})$ in the Banach algebra $\mathcal{A}$ (Theorem 4.36, Theorem 4.47). An application of the latter result is Proposition 4.49. It states that, given a homogeneous matrix algebra $\mathcal{A}$, the weighted matrix spaces with correctly chosen polynomial weights are inverse-closed subalgebras of $\mathcal{A}$.

REMARK. A useful tool in the simplification of some proofs is a reiteration property of Besov spaces (Proposition 4.35). Though certainly well-known (and easy to prove using the reiteration theorem of interpolation thory), we have not found a reference in the literature, so a self-contained proof is included in the text.

Chapter 5 is dedicated to the proof of the Jackson-Bernstein Theorem 5.11 on the equivalence of approximation spaces and Besov spaces. We relate the concepts of smoothness and approximation developed in the two preceding chapters to each other and consider the approximation with bandlimited elements. A spectral characterization of bandlimited elements of a Banach space with automorphism group is in Proposition 5.6, and versions of the Weierstrass approximation theorem are given in Proposition 5.7 for periodic group actions, and in Corollary 5.16 in the general case. This theorem identifies the continuous elements of an algebra with the ones that can be approximated by bandlimited elements. With the Jackson-Bernstein theorem it is a routine procedure to construct a Littlewood-Paley decomposition for the Besov or approximation spaces. If $\mathcal{A}$ is a homogeneous matrix algebra, we obtain the following description of the approximation spaces of polynomial order (Proposition 5.21).

PROPOSITION. *If $\mathcal{A}$ is a homogeneous matrix algebra, and $\Phi = \{\varphi_k\}_{k\geq -1}$ a dyadic partition of unity, then*

$$\|A\|_{\mathcal{E}_r^p(\mathcal{A})} \asymp \Bigg(\sum_{k=0}^{\infty} 2^{kpr}\Big\| \sum_{\lfloor 2^{k-1}\rfloor \leq |l| < 2^{k+1}} \hat{\varphi}_k(l)\hat{A}(l)\Big\|_{\mathcal{A}}^p\Bigg)^{1/p}.$$

If $\mathcal{A}$ is solid, then the above expression simplifies to

$$\|A\|_{\mathcal{E}_r^p(\mathcal{A})} \asymp \Bigg(\sum_{k=-1}^{\infty} 2^{kpr}\Big\| \sum_{\lfloor 2^{k}\rfloor \leq |l| < 2^{k+1}} \hat{A}(l)\Big\|_{\mathcal{A}}^p\Bigg)^{1/p}.$$

This should be compared to Proposition 3.13. We do not need assumptions on the solidity of $\mathcal{A}$, the price to pay is that we consider only approximation of polynomial order.

In Chapter 6 we provide some results on smoothness and off-diagonal decay beyond polynomial order. We focus on conditions on the growth of derivations and investigate the properties of Denjoy-Carleman classes. We adapt a theorem of Malliavin [**74**] (see Siddiqi [**92**]) to the noncommutative setting and show the inverse-closedness of the Carleman classes under certain natural conditions (Proposition 6.17). In particular, the algebra of analytic elements of $\mathcal{A}$ is inverse-closed in $\mathcal{A}$. If $\mathcal{A}$ is a homogeneous matrix algebra, this result implies a version of the theorems of Demko, Smith, and Moss [**36**], and of Jaffard [**61**] on the inverses of matrices with exponential off-diagonal decay. We are able to prove a similar result for Carleman classes of matrices that satisfy the condition (M2') of Komatsu [**66**] .

While the Carleman classes are locally convex algebras, it is also possible to obtain results for Banach algebras with infinite smoothness. An interesting example is provided by the *Dales-Davie algebras* [**31**]. Again it is possible to adapt a scalar result [**1**] to show that a Dales-Davie algebra obtained from a Banach algebra $\mathcal{A}$ is inverse-closed in $\mathcal{A}$. However, in contrast to the result on the inverse-closedness of Carleman classes, this result is not optimal. In the special case of algebras of convolution operators we are able to obtain the optimal result using methods of approximation theory.

The results on ultradifferentiable classes are much more scattered than in some of the previous chapters. One has to admit that this is true to some extent for the theory of ultradifferentiable functions as well. There are many scales of ultradifferentiable functions besides Carleman classes (we mention the Beurling Björck approach), and the concept of Besov spaces does not generalize readily to the ultradifferentiable setting (see, however, [**81**]).

Remark. A word on the presentation of the material: Some of the proofs in Section 4.3 and 4.4 might be shorter using some interpolation theory. I have chosen a more "pedestrian" approach, which, if nothing else, might let appreciate a reader the compactness and elegance of the interpolation approach.

Remark. Parts of the results of this thesis will be published in *Constructive Approximation* (joint work with Karlheinz Gröchenig). A preprint can be found in [**52**].

CHAPTER 2

Preliminaries and Resources

2.1. Notation

2.1.1. Symbols, Sets, and Spaces. Constants will be denoted by C, C',C_1,c, etc.. The same symbol might denote different constants in each equation.

The cardinality of a finite set A is $|A|$. The natural numbers are $\mathbb{N} = \{1, 2, \ldots\}$, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For an integer $d \geq 1$ the sets $\mathbb{Z}^d$, $\mathbb{R}^d$, $\mathbb{C}^d$ denote d-tuples of integers, real and complex numbers, respectively, the symbol d will always have this meaning. The *d-dimensional torus* is $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ and will be often identified with the unit cube $[0, 1)^d$. Let $\mathbb{C}^d_* = \mathbb{C}^d \setminus \{0\}$, and $\mathbb{R}^d_* = \mathbb{R}^d \setminus \{0\}$. For a real number x, $\lfloor x \rfloor$ is the greatest integer smaller or equal to x.

A multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ is a d-tuple of nonnegative integers. We set $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, and $D^\alpha f(x) = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} f(x)$ is the partial derivative. The binomial coefficients are $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_1}{\beta_1}$, and the factorial is $\alpha! = \alpha_1! \cdots \alpha_d!$. The degree of x^α is $|\alpha| = \sum_{j=1}^d \alpha_j$, and $\beta \leq \alpha$ means that $\beta_j \leq \alpha_j$ for $j = 1, \ldots, d$. In Chapter 6 we need the relation $|\alpha|! < d^{|\alpha|}\alpha!$, it can be obtained from the multinomial theorem by setting all d summands to one.

If f and g are positive functions, $f \asymp g$ means that $C_1 f \leq g \leq C_2 f$ for positive constants C_1, C_2. If there are chains of inequalities of the form $f \leq C_1 g \leq C_2 h \cdots$ and the constants $C_1, C_2, \ldots$ are unimportant for our argument, we sometimes use the notation $f \lesssim g$ ($f \gtrsim g$) to express that there is a constant $C > 0$ such that $f \leq Cg$ ($f \geq Cg$).

For x in $\mathbb{C}^d$ and $1 \leq p \leq \infty$ let $|x|_p$ denote be the p-norm of x, $|x|$ will be used for the 1-norm. The vectors e_k, $1 \leq k \leq d$, are the standard basis of $\mathbb{C}^d$. The standard scalar product on $\mathbb{C}^d$ is $x \cdot y = \sum_{k=1}^d x_k \bar{y}_k$.

More generally, if Λ is an arbitrary set, the space $\ell^p(\Lambda)$, $1 \leq p \leq \infty$, consists of the sequences $(x_\lambda)_{\lambda \in \Lambda}$, for which the norm

$$\|x\|_{\ell^p(\Lambda)} = \begin{cases} \left(\sum_{\lambda \in \Lambda} |x|_\lambda^p\right)^{1/p}, & p < \infty, \\ \sup_{\lambda \in \Lambda} |x|_\lambda, & p = \infty \end{cases}$$

is finite. We will always use the symbol p' to denote the *conjugate exponent* to p, $1 \leq p \leq \infty$, that is $1/p' = 1 - 1/p$. If nothing else is said, the symbols p and q will always be used for ℓ^p spaces.

The standard basis in $\ell^p(\mathbb{Z}^d)$ is $e_k = (\delta_{jk})_{j \in \mathbb{Z}^d}$, $\langle x, y \rangle = \sum_{k \in \mathbb{Z}^d} x(k) y(k)$ is the standard dual pairing between $\ell^p(\mathbb{Z}^d)$ and its dual $\ell^{p'}(\mathbb{Z}^d)$. If $p = 2$, we define the scalar product as $\langle x, y \rangle = \sum_{k \in \mathbb{Z}^d} x(k) \overline{y(k)}$. This should not lead to confusion.

The *support* of a sequence $x = (x_\lambda)_{\lambda \in \Lambda}$ is the set of nonzero coordinates: $\operatorname{supp}(x) = \{\lambda \in \Lambda \colon x_\lambda \neq 0\}$.

Let $\mathscr{S}(\mathbb{R}^d)$ denote the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$. The Fourier transform of $f \in \mathscr{S}(\mathbb{R}^d)$ is $\mathcal{F}f(\omega) = \hat{f}(\omega) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i\omega\cdot x}\,dx$. This definition is extended by duality to $\mathscr{S}'(\mathbb{R}^d)$, the space of tempered distributions. The same symbol is also used for the Fourier transform on $\mathbb{Z}^d$ and $\mathbb{T}^d$.

The continuous embedding of the normed space X into the normed space Y is denoted as $X \hookrightarrow Y$. The operator norm of a bounded linear mapping $A : X \to Y$ between Banach spaces is denoted by $\|A\|_{X\to Y}$. In the special case of operators $A\colon \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ we write $\|A\|_{\mathcal{B}(\ell^2(\mathbb{Z}^d))} = \|A\|_{\ell^2(\mathbb{Z}^d)\to\ell^2(\mathbb{Z}^d)}$. If there is little chance of confusion, we write $\|A\|_{\mathcal{B}(\ell^2)}$.

We will consider Banach spaces with equivalent norms as equal.

2.1.2. Weights, Weighted Spaces, and Algebras. Decay conditions are often quantified by using weight functions. A weight w on a measure space X is a locally integrable, locally bounded, strictly positive function on X. We will use weights defined on $\mathbb{Z}^d$ to measure off-diagonal decay of matrices, and weights on $\mathbb{N}_0$ and occasionally on $\mathbb{R}_0^+$ to define approximation spaces. If nothing else is said we will always assume that

$$w(0) = 1\,.$$

We will further assume that all weights are defined on $\mathbb{Z}^d$ if nothing else is stated.

Special classes of weight functions are used to describe Banach algebra properties, module properties and spectral invariance of the underlying spaces. For a detailed discussion in the context of time frequency analysis see [**50**]. Feichtinger's fundamental paper [**40**] is an important source for weighted convolution algebras, see also [**39**].

A weight v on $\mathbb{Z}^d$ is *submultiplicative* if $v(x+y) \le v(x)v(y)$ for all $x, y \in \mathbb{Z}^d$. If $v(x+y) \le Cv(x)v(y)$ for some constant $C > 1$, then v is weakly submultiplicative. A weight w is *v-moderate*, if $w(x+y) \le Cw(x)v(y)$. The weight v is *subconvolutive* if $1/v \in \ell^1(\mathbb{Z}^d)$ and $1/v * 1/v \le C/v$. Note that subconvolutive weights are also (weakly) submultiplicative.

Definition 2.1. A weight w on $\mathbb{Z}^d$ satisfies the *Gelfand, Raikov , Shilov (GRS)-condition* if

$$\lim_{n\to\infty} w(n\,x)^{1/n} = 1 \quad \text{for all} \quad x \in \mathbb{Z}^d.$$

A discussion of the GRS condition can be found in [**50**], see also Section 2.2.1. A weight v on $\mathbb{Z}^d$ is *radial*, if $v(k) = v(|k|\,e_1)$ for all $k \in \mathbb{Z}^d$, v is *symmetric*, if $v(k) = v(-k)$ for all $k \in \mathbb{Z}^d$.

The standard polynomial weights on $\mathbb{Z}^d$ are denoted as $v_r(k) = (1+|k|)^r$, $r > 0$. If $r > d/p'$, then v_r is an algebra weight on $\ell^p(\mathbb{Z}^d)$.

Convention. We will often construct some "weighted object" $\mathcal{A}_v$ from a unweighted object A. If v is a polynomial weight, $v = v_r$ we often abbreviate $\mathcal{A}_r = \mathcal{A}_{v_r}$.

We call a weight v an *algebra weight* on $\ell^p(\mathbb{Z}^d)$ for $1 \le p \le \infty$, if $1/v \in \ell^{p'}(\mathbb{Z}^d)$, and

$$\Big(\sum_{k\in\mathbb{Z}^d} \frac{1}{\big(v(k)\,v(l-k)\big)^{p'}}\Big)^{1/p'} \le \frac{C}{v(l)} \quad \text{for all} \;\; l \in \mathbb{Z}^d\,,$$

with the obvious modification for $p = 1$. If $1 < p \le \infty$, then v is an algebra weight if and only if $v^{p'}$ is subconvolutive. A weight is an algebra weight for $\ell^1(\mathbb{Z}^d)$ if and only if it is weakly submultiplicative. Algebra weights are used to define Banach algebras of weighted ℓ^p-spaces over $\mathbb{Z}^d$. We write down the relevant definitions and results.

Let w be a weight on $\mathbb{Z}^d$. A sequence $x = (x_k)_{k\in\mathbb{Z}^d}$ is an element of the weighted space $\ell^p_w(\mathbb{Z}^d)$ if $\|x\|_{\ell^p_w(\mathbb{Z}^d)} = \|x\, w\|_{\ell^p(\mathbb{Z}^d)} < \infty$.

PROPOSITION 2.2 ([**40**, Satz 3.6]). *A sufficient condition for the space $\ell^p_w(\mathbb{Z}^d)$, $1 \le p \le \infty$ to be a Banach algebra under convolution is that w is an algebra weight ([**104**], see also [**39, 63**]). If $p = 1$ or $p = \infty$, this condition is also necessary.*

In [**39**, 3.2.8], [**78**, 7.1.5] simple additional conditions on the weight can be found that lead to a characterization of convolution algebras for $1 < p < \infty$. Note that $1/w \in \ell^{p'}(\mathbb{Z}^d)$ implies that $\ell^p_w(\mathbb{Z}^d) \subseteq \ell^1(\mathbb{Z}^d)$.

COROLLARY 2.3. *The space $\ell^p_r(\mathbb{Z}^d)$ is a convolution algebra if and only if $r > d/p'$.*

Let w be a *symmetric* algebra weight on $\ell^p(\mathbb{Z}^d)$. With the involution $x^*(k) = \overline{x(-k)}$, where $\overline{x}$ denotes the complex conjugate of x, the space $\ell^p_w(\mathbb{Z}^d)$ is a *Banach-$*$-algebra* (see Section 2.2).

Solid convolution algebras. The spaces $\ell^p_w(\mathbb{Z}^d)$ with w an algebra weight do not exhaust the class of sequence spaces that are Banach algebras under convolution. In Section 2.3 we need convolution algebras of *solid* sequence spaces.

DEFINITION 2.4. Let $\mathcal{X}$ be a Banach space of sequences $x = (x_\lambda)_{\lambda\in\Lambda}$, where Λ is an arbitrary index set. The space $\mathcal{X}$ is *solid* if

$$|x_\lambda| \le |y_\lambda| \text{ for all } \lambda \in \Lambda \text{ implies } \|x\|_{\mathcal{X}} \le \|y\|_{\mathcal{X}}$$

for all $x, y \in \mathcal{X}$.

Solid spaces of sequences and functions are often used, so there are many similar or equivalent concepts, including *Banach sequence* or *Banach function spaces*, *Riesz spaces*, or *Banach lattices* [**13, 106**].

PROPOSITION 2.5 ([**56**, Theorem 3.1]). *If $\mathcal{A}$ is a solid Banach-*-algebra of sequences under convolution on $\mathbb{Z}^d$, then $\|a\|_{\ell^1(\mathbb{Z}^d)} \le \|a\|_{\mathcal{A}}$. In particular, the space $\ell^1(\mathbb{Z}^d)$ is the maximal solid Banach-*-algebra on $\mathbb{Z}^d$.*

2.1.3. Matrices. An infinite matrix A (over $\mathbb{Z}^d$) is a function $A : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{C}$. If $A : \ell^p(\mathbb{Z}^d) \to \ell^q(\mathbb{Z}^d)$ is a bounded operator, we identify it with the matrix A having entries $A(k,l) = \langle Ae_l, e_k\rangle$.

We need to interpret the diagonals of a matrix both as matrices and as sequences.

DEFINITION 2.6. The mth side diagonal $\hat{A}(m)$ of the matrix A is the matrix

$$\hat{A}(m)(k,l) = \begin{cases} A(k,l), & k - l = m, \\ 0, & \text{otherwise}. \end{cases} \tag{2.1}$$

Furthermore let $A[m] = \big(A(k, k-m)\big)_{k\in\mathbb{Z}^d}$ be the *m-th diagonal sequence* associated to A.

The matrix A is *banded* with bandwidth N if

$$A = \sum_{|m|_\infty \le N} \hat{A}(m). \tag{2.2}$$

We denote the banded matrices with bandwidth $N - 1$ by $\mathcal{T}_N$.

REMARK. The choice of the infinity norm in the definition of banded matrices is justified in Corollary 5.9, where we identify banded matrices with bandlimited elements of a Banach algebra. The reader may check that results that do not depend on this identifications are valid for general choices of a norm on $\mathbb{Z}^d$.

Special Matrices and Operators. The adjoint of a matrix A is the matrix A^* with entries $A^*(k,l) = \overline{A(l,k)}$. The *translation operator* on $\ell^p_w(\mathbb{Z}^d)$ is given by $(T_k x)(l) = x(l-k)$.

More generally, if v is submultiplicative and w is v-moderate, the *convolution operator*

$$C_f\colon \ell^p_w(\mathbb{Z}^d) \to \ell^p_w(\mathbb{Z}^d), \quad (C_f x)(l) = \sum_{k\in\mathbb{Z}^d} f(l-k)x(k)$$

is well defined for every $f \in \ell^1_v(\mathbb{Z}^d)$ and satisfies

$$\|C_f x\|_{\ell^p_w(\mathbb{Z}^d)} \le \|f\|_{\ell^1_v(\mathbb{Z}^d)} \|x\|_{\ell^p_w(\mathbb{Z}^d)}$$

for every $x \in \ell^p_w(\mathbb{Z}^d)$. Note that $C_f^* = C_{f^*}$. If $\mathcal{A}$ is a solid convolution algebra, we define the algebra of convolution operators generated by $\mathcal{A}$ as

$$\mathrm{Conv}(\mathcal{A}) = \{C_f\colon f \in \mathcal{A}\} \subset \mathcal{B}(\ell^2)\,.$$

We will need the *modulation operator* $(M_t x)(k) = e^{2\pi i k\cdot t}x(k)$, $t \in \mathbb{R}^d$ to describe decay properties of matrices. Obviously it defines an isometry on any solid sequence space.

2.2. Concepts from the Theory of Banach Algebras

Many parts of our investigations on off-diagonal decay of matrices and their inverses can (and will) be carried out in the broader context of Banach algebras. Besides standard material we will use some less known concepts.

All Banach algebras are assumed to be *unital*. To verify that a Banach space $\mathcal{A}$ with norm $\|\ \|_{\mathcal{A}}$ is a Banach algebra we will often prove the weaker property $\|ab\|_{\mathcal{A}} \le C\|a\|_{\mathcal{A}}\|b\|_{\mathcal{A}}$ for some constant C. The norm $\|a\|'_{\mathcal{A}} = \sup_{\|b\|_{\mathcal{A}}=1}\|ab\|_{\mathcal{A}}$ is then an equivalent norm on $\mathcal{A}$ and satisfies $\|ab\|'_{\mathcal{A}} \le \|a\|'_{\mathcal{A}}\|b\|'_{\mathcal{A}}$.

2.2.1. Inverse Closed Subalgebras.

It is natural to ask if the off-diagonal decay of a matrix is preserved by inversion. To be more precise: Let $\mathcal{A}$ be a Banach algebra of matrices, and let $A \in \mathcal{A}$ be invertible in $\mathcal{B}(\ell^2)$. Is $A^{-1} \in \mathcal{A}$?

Definition 2.7 (Inverse-closedness). Let $\mathcal{A} \subseteq \mathcal{B}$ be a nested pair of (Banach) algebras with a common identity. The Banach algebra $\mathcal{A}$ is called *inverse-closed* in $\mathcal{B}$, if

$$a \in \mathcal{A} \text{ and } a^{-1} \in \mathcal{B} \quad \text{implies} \quad a^{-1} \in \mathcal{A}. \tag{2.3}$$

Inverse-closedness is equivalent to *spectral invariance*. This means that the spectrum $\sigma_{\mathcal{A}}(a) = \{\lambda \in \mathbb{C}\colon a-\lambda \text{ not invertible in } \mathcal{A}\}$ of an element $a \in \mathcal{A}$ does not depend on the algebra, and so

$$\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{B}}(a), \qquad \text{for all } a \in \mathcal{A}.$$

The relation of inverse-closedness is transitive: If $\mathcal{A}$ is inverse-closed in $\mathcal{B}$ and $\mathcal{B}$ is inverse-closed in $\mathcal{C}$, then $\mathcal{A}$ is inverse-closed in $\mathcal{C}$.

Inverse-closedness can be defined for general algebras. We will use it for locally convex algebras of matrices in Chapter 6.

Remark. The property of $\mathcal{A}$ being inverse-closed in $\mathcal{B}$ can be seen as generalization of *Wiener's Lemma*: Actually, Wiener's Lemma states precisely that the *Wiener algebra* $\mathcal{F}\ell^1(\mathbb{Z}^d)$ of absolutely convergent Fourier series is inverse-closed in $C(\mathbb{T}^d)$. See [**51**] for a concise overview of the importance of the concept of inverse-closedness.

This is a convenient place to address the relevance of GRS weights (Definition 2.1).

Proposition 2.8 ([**51**, Cor 3.4]). *Let v be a submultiplicative weight on $\mathbb{Z}^d$. The algebra*

$$\mathcal{F}\ell^1_v(\mathbb{Z}^d) = \{f \in C(\mathbb{T}^d) \colon \big(\hat f(k)\big)_{k\in\mathbb{Z}^d} \in \ell^1_v(\mathbb{Z}^d)\}$$

is inverse-closed in $C(\mathbb{T}^d)$ if and only if v satisfies the GRS condition.

Proposition 2.8 can be restated as a result on convolution operators.

Corollary 2.9. *Let v be a submultiplicative weight on $\mathbb{Z}^d$. The Banach algebra* $\mathrm{Conv}(\ell^1_v(\mathbb{Z}^d))$ *with the norm* $\|C_a\|_{\mathrm{Conv}(\ell^1_v(\mathbb{Z}^d))} = \|a\|_{\ell^1_v(\mathbb{Z}^d)}$ *is inverse-closed in $\mathcal{B}(\ell^2)$ if and only if v satisfies the GRS condition.*

It is easy to see this by taking the Fourier transform of both $\mathrm{Conv}(\ell^1_v(\mathbb{Z}^d))$ and $\mathcal{B}(\ell^2)$. The unweighted case is discussed in more detail in [**51**, 2.11].

If v does not satisfy the GRS condition, then there is a $k \in \mathbb{Z}^d$ and a constant $c > 0$ such that $v(n\,k) \geq e^{cn}$ for all n large enough. If T_k is the translation operator defined in Section 2.1.3, and if $0 < \delta < c$, the operator $A = \mathrm{id} - e^{-\delta} T_k$ is in $\mathcal{B}(\ell^2)$, and, using the geometric series,

$$A^{-1} = \sum_{l=0}^{\infty} e^{-\delta l} T_{l\,k}$$

is in $\mathcal{B}(\ell^2)$ as well. But $\|A^{-1}\|_{\mathrm{Conv}} = \sum_{l=0}^{\infty} e^{-\delta l} v(l\,k) = \infty$.

2.2.2. The Lemma of Hulanicki. The verification of inverse-closedness is often nontrivial. Under additional conditions this verification is sometimes possible by using an argument of Hulanicki [**60**], see [**41**] for a corrected proof.

We call the Banach algebra $\mathcal{A}$ a Banach $*$-algebra if it has an involution $*$ that is isometric, $\|a^*\|_{\mathcal{A}} = \|a\|_{\mathcal{A}}$ for all $a \in \mathcal{A}$.

The Banach $*$-algebra $\mathcal{A}$ is *symmetric*, if the spectrum of positive elements is non-negative,

$$\sigma_{\mathcal{A}}(a^* a) \subseteq [0, \infty)$$

for all $a \in \mathcal{A}$.

Denote the spectral radius of $a \in \mathcal{A}$ by $\rho_{\mathcal{A}}(a) = \sup\{|\lambda| : \lambda \in \sigma_{\mathcal{A}}(a)\}$.

Proposition 2.10 (Hulanicki's Lemma). *Let $\mathcal{B}$ be a symmetric Banach algebra, $\mathcal{A} \subseteq \mathcal{B}$ a $*$-subalgebra with a common involution and a common unit element. The following statements are equivalent.*

(1) *$\mathcal{A}$ is inverse-closed in $\mathcal{B}$.*
(2) *$\rho_{\mathcal{A}}(a) = \rho_{\mathcal{B}}(a)$ for all $a = a^*$ in $\mathcal{A}$.*

In particular, if $\mathcal{A}$ is a closed $*$-subalgebra of $\mathcal{B}$, then $\mathcal{A}$ is inverse-closed in $\mathcal{B}$.

Proof. Only the implication (2) $\Rightarrow$ (1) is nontrivial. We assume first that $a \in \mathcal{A}$ is invertible in $\mathcal{B}$ and satisfies

$$a = c^* c \geq 0, \text{ and } \|a\|_{\mathcal{B}} < 1\,. \tag{2.4}$$

Then $\sigma_{\mathcal{B}}(a)$ is contained in an interval $[\alpha, \beta]$ for some $0 < \alpha \leq \beta < 1$. So

$$\sigma_{\mathcal{B}}(1-a) \subset [1-\beta, 1-\alpha] \subset (0,1)\,.$$

This implies that the series $a^{-1} = \sum_{k=0}^{\infty} (1-a)^k$ converges in $\mathcal{B}$. As $1-a$ is hermitian, we have $\rho_{\mathcal{B}}(1-a) = \rho_{\mathcal{A}}(1-a) \leq 1-\alpha < 1$ by assumption; so there is a positive integer k_0 such that $\|(1-a)^{k_0}\|_{\mathcal{A}}^{1/k_0} < 1$. We rewrite the geometric series in the form

$$\sum_{k=0}^{\infty} (1-a)^k = \sum_{l=0}^{k_0-1} \sum_{m=0}^{\infty} (1-a)^{m\,k_0 + l},$$

which proves the convergence of the series in $\mathcal{A}$. So we have shown that $a^{-1} \in \mathcal{A}$.

The general situation of $a \in \mathcal{A}$, with $a^{-1} \in \mathcal{B}$ can be reduced to the case above by introducing

$$b = \frac{a^*a}{2\|a^*a\|_{\mathcal{B}}}.$$

The element b satisfies (2.4), so it has an inverse in $\mathcal{A}$. Writing

$$1 = b^{-1}b = \frac{b^{-1}a^*}{2\|a^*a\|_{\mathcal{B}}}a$$

verifies that a has a *left inverse* in $\mathcal{A}$. A similar procedure establishes the existence of a right inverse for a. □

2.2.3. Brandenburg's trick ([21]). This method is sometimes used to prove the equality of spectral radii. Let $\mathcal{A} \subseteq \mathcal{B}$ be two Banach algebras with the same identity and involution, and assume that $\mathcal{B}$ is symmetric. Assume that the norms satisfy

$$\|ab\|_{\mathcal{A}} \le C(\|a\|_{\mathcal{A}}\|b\|_{\mathcal{B}} + \|b\|_{\mathcal{A}}\|a\|_{\mathcal{B}}) \qquad \text{for all } a, b \in \mathcal{A}. \tag{2.5}$$

If we apply (2.5) with $a = b = c^n$, we obtain

$$\|c^{2n}\|_{\mathcal{A}} \le 2C\|c^n\|_{\mathcal{A}}\|c^n\|_{\mathcal{B}}.$$

Taking nth roots and the limit $n \to \infty$ yields $\rho_{\mathcal{A}}(c) \le \rho_{\mathcal{B}}(c)$. Since the reverse inequality is always true for $\mathcal{A} \subseteq \mathcal{B}$, we obtain the equality of spectral radii. By Proposition 2.10, $\mathcal{A}$ is inverse-closed in $\mathcal{B}$, .

Remark. Note that the inequality (2.5) is related to the concept of D_p subalgebras (if $p = 1$) considered in [65], see also [19].

2.3. Algebras of Matrices With Off Diagonal Decay

Off-diagonal decay of matrices is usually defined by weights. In this section we review examples of Banach spaces of matrices with off-diagonal decay used in the literature, and we define weighted matrix spaces and algebras.

2.3.1. Matrices Dominated by Convolution Operators.

Definition 2.11 ([56]). Let $\mathcal{A} \subseteq \ell^1(\mathbb{Z}^d)$ be a solid Banach space of sequences over $\mathbb{Z}^d$. A matrix A belongs to $\mathcal{C}_{\mathcal{A}}$, if $(\|A[k]\|_{\ell^\infty(\mathbb{Z}^d)})_{k\in\mathbb{Z}^d} \in \mathcal{A}$. The norm of A in $\mathcal{C}_{\mathcal{A}}$ is

$$\|A\|_{\mathcal{C}_{\mathcal{A}}} = \left\| \left(\|A[k]\|_{\ell^\infty(\mathbb{Z}^d)}\right)_{k\in\mathbb{Z}^d} \right\|_{\mathcal{A}}.$$

If $\mathcal{A} = \ell^p_w(\mathbb{Z}^d)$, we use the notation $\mathcal{C}^p_w$ for $\mathcal{C}_{\ell^p_w(\mathbb{Z}^d)}$. Explicitly the norm on $\mathcal{C}^p_w$ is

$$\|A\|_{\mathcal{C}^p_w} = \left(\sum_{k\in\mathbb{Z}^d} \|A[k]\|^p_{\ell^\infty(\mathbb{Z}^d)} w(k)^p \right)^{1/p}.$$

By definition the mapping $\mathcal{A} \to \mathcal{C}_{\mathcal{A}}$, $f \mapsto C_f$ is an isometric embedding. If $\mathcal{A}$ is a solid Banach-$*$-algebra under convolution over $\mathbb{Z}^d$, then Proposition 2.5 implies that $\mathcal{A} \subseteq \ell^1(\mathbb{Z}^d)$. The following result describes basic properties of $\mathcal{C}_{\mathcal{A}}$.

Proposition 2.12 ([56, Lemma 3.4]). *Let $\mathcal{A}$ be a solid Banach-$*$-algebra under convolution over $\mathbb{Z}^d$. Then*

1. *$\mathcal{C}_{\mathcal{A}}$ is a solid Banach-$*$-algebra of matrices.*
2. *If $\mathcal{Y}$ is a solid space of sequences over $\mathbb{Z}^d$, and if $\mathcal{A} * \mathcal{Y} \subseteq \mathcal{Y}$, then $\mathcal{C}_{\mathcal{A}} \cdot \mathcal{Y} \subseteq \mathcal{Y}$, and $\|Ax\|_{\mathcal{Y}} \le \|A\|_{\mathcal{C}_{\mathcal{A}}}\|y\|_{\mathcal{Y}}$ for all $A \in \mathcal{C}_{\mathcal{A}}$ and $y \in \mathcal{Y}$.*
3. *In particular, $\mathcal{C}_{\mathcal{A}} \subseteq \mathcal{B}(\ell^2)$ and $\|A\|_{\mathcal{B}(\ell^2)} \le \|A\|_{\mathcal{C}_{\mathcal{A}}}$.*

The spaces $\mathcal{C}_{\mathcal{A}}$ are important because they are inverse-closed in $\mathcal{B}(\ell^2)$.

PROPOSITION 2.13. *If $\mathcal{A}$ is a solid, Banach-$*$-algebra of sequences convolution over $\mathbb{Z}^d$, then the following are equivalent.*

(1) *The algebra $\mathcal{C}_{\mathcal{A}}$ is inverse-closed in $\mathcal{B}(\ell^2)$.*
(2) *The algebra $\mathrm{Conv}(\mathcal{A}) = \{C_x : x \in \mathcal{A}\}$ is inverse-closed in $\mathcal{B}(\ell^2)$.*
(3) *The weight $w(k) = \|e_k\|_{\mathcal{A}}$, $k \in \mathbb{Z}^d$, satisfies the GRS condition.*

Different versions of this important theorem have been proved by different methods, see the remarks below. A compact proof is in [56, Theorem 3.2].

REMARKS. Inverse-closedness of $\mathcal{C}^\infty_{v_s}$ in $\mathcal{B}(\ell^2)$ has been proved by Jaffard [61] and Baskakov [10, 11] for polynomial weights v_s, a very simple proof is due to Sun [98]. For a general subconvolutive weight a proof is by Baskakov [11]; Gröchenig and Leinert give a different proof in [55].

The inverse-closedness of $\mathcal{C}^1_{v_0}$ in $\mathcal{B}(\ell^2)$ was proved by Gohberg, Kaashoek and Woerdeman in [44], and by Sjöstrand in [93]. For submultiplicative GRS weights the proof is by Baskakov [11].

The matrices in $\mathcal{C}^1_v$ are also known as *convolution dominated operators*: $A \in \mathcal{C}^1_v$, v submultiplicative if and only if there is a $h \in \ell^1_v(\mathbb{Z}^d)$ such that $|Ax(k)| \le h * |x|(k)$, where $|x|$ denotes the vector with components $(|x(k)|)_{k\in\mathbb{Z}^d}$.

2.3.2. Schur Classes. A second scale of Banach algebras of matrices is given by the weighted Schur algebras.

DEFINITION 2.14 ([54], [98, 99] for general p). Let w be a weight, and $1 \le p \le \infty$. A matrix A is in the *Schur class* $\mathcal{S}^p_w$ if the norm

$$\|A\|_{\mathcal{S}^p_w} = \max\Big\{ \sup_{k\in\mathbb{Z}^d} \Big(\sum_{l\in\mathbb{Z}^d} A(k,l)^p w(k-l)^p\Big)^{1/p}, \quad \sup_{l\in\mathbb{Z}^d} \Big(\sum_{k\in\mathbb{Z}^d} A(k,l)^p w(k-l)^p\Big)^{1/p} \Big\}$$

is finite.

LEMMA 2.15. *If $1/w \in \ell^{p'}(\mathbb{Z}^d)$ then $\mathcal{S}^p_w \subseteq \mathcal{S}^1_0$.*

PROOF. This follows from the corresponding inclusion of weighted ℓ^p-spaces (see the remark after Proposition 2.2). □

PROPOSITION 2.16. *If $1 \le p \le \infty$ and w is an algebra weight for $\ell^p(\mathbb{Z}^d)$, then $\mathcal{S}^p_w$ is a Banach-$*$-algebra embedded in $\mathcal{B}(\ell^2)$.*

PROOF. The inclusion $\mathcal{S}^1_0 \subseteq \mathcal{B}(\ell^2)$ is the content of the *Schur test* for infinite matrices (see, e.g., [48, 6.2.1]). To prove that $\mathcal{S}^p_w$ is a Banach algebra we adapt the proof that $\ell^p_w(\mathbb{Z}^d)$ is a convolution algebra if w is an algebra weight [63, 104]. Let $A, B \in \mathcal{S}^p_w$ and define matrices A_w, B_w by $A_w(r,s) = A(r,s)w(r-s)$, $B_w(r,s) = B(r,s)w(r-s)$. Then $\|A_w\|_{\mathcal{S}^p_0} = \|A\|_{\mathcal{S}^p_w}$, likewise for B and B_w. We obtain

$$\begin{aligned} Q := \Big(\sum_{s\in\mathbb{Z}^d} |(AB)(r,s)|^p w(r-s)^p\Big)^{1/p} &\le \Big(\sum_{s\in\mathbb{Z}^d} \big(\sum_{u\in\mathbb{Z}^d} |A(r,u)B(u,s)|\big)^p w(r-s)^p\Big)^{1/p} \\ &= \Big(\sum_{s\in\mathbb{Z}^d} \big(\sum_{u\in\mathbb{Z}^d} \gamma_{r,s,u} |A_w(r,u)||B_w(u,s)|\big)^p\Big)^{1/p} \end{aligned}$$

with

$$\gamma_{r,s,u} = \frac{w(r-s)}{w(r-u)w(u-s)}\,.$$

Using Hölder's inequality we obtain the estimate

$$\sum_{u\in\mathbb{Z}^d} \gamma_{r,s,u}|A_w(r,u)||B_w(u,s)| \le \big(\sum_{u\in\mathbb{Z}^d} \gamma^{p'}_{r,s,u}\big)^{1/p'} \Big(\sum_{u\in\mathbb{Z}^d} |A_w(r,u)B_w(u,s)|^p\Big)^{1/p},$$

and so

$$\begin{aligned}Q^p &\le \sum_{s\in\mathbb{Z}^d}\Big(\sum_{u\in\mathbb{Z}^d}\gamma_{r,s,u}^{p'}\Big)^{p/p'}\Big(\sum_{u\in\mathbb{Z}^d}|A_w(r,u)B_w(u,s)|^p\Big)\\ &\le \sup_{s\in\mathbb{Z}^d}\big(\sum_{u\in\mathbb{Z}^d}\gamma_{r,s,u}^{p'}\big)^{p/p'}\sum_{s\in\mathbb{Z}^d}\sum_{u\in\mathbb{Z}^d}|A_w(r,u)|^p|B_w(u,s)|^p\\ &= C_r^p\sum_{u\in\mathbb{Z}^d}|A_w(r,u)|^p\sum_{s\in\mathbb{Z}^d}|B_w(u,s)|^p\end{aligned}$$

with

$$C_r = \sup_{s\in\mathbb{Z}^d}\Big(\sum_{u\in\mathbb{Z}^d}\gamma_{r,s,u}^{p'}\Big)^{1/p'}\,. \tag{2.6}$$

It is easy to verify that C_r is actually independent of r. Using Hölder's inequality again we obtain

$$Q^p \le C_r^p\big(\sum_{u\in\mathbb{Z}^d}|A_w(r,u)|^p\big)\sup_u\Big(\sum_{s\in\mathbb{Z}^d}|B_w(u,s)|^p\Big) \le C_r^p\|A_w\|_{\mathcal{S}_0^p}^p\|B_w\|_{\mathcal{S}_0^p}^p\,.$$

An easy computation verifies that (2.6) is actually equivalent to $1/w^{p'} * 1/w^{p'} \le C/w^{p'}$. This is the condition for w being an algebra weight and verifies that $C_r<\infty$, and finally proves the Banach algebra property. All other assertions are straightforward. □

Remarks.

(1) The scales $\mathcal{S}_w^p$ and $\mathcal{C}_w^p$ are identical at the endpoint $p=\infty$, i.e. $\mathcal{S}_w^\infty = \mathcal{C}_w^\infty$.
(2) It is easy to see that $\|A\|_{\mathcal{S}_w^p} \le \|A\|_{\mathcal{C}_w^p}$ for all $A\in\mathcal{C}_w^p$.

Again, the question, whether $\mathcal{S}_w^p$ is inverse-closed in $\mathcal{B}(\ell^2)$ is of interest.

Proposition 2.17 ([**54**]). *Let v be a weight of the form $v(k)=e^{\rho(|k|_2)}$, where ρ is a continuous concave function with $\rho(0)=0$ (such a weight is automatically submultiplicative). Assume further that v satisfies the GRS-condition and that v is* weakly growing*, that is,*

$$v(k)\ge C(1+|k|)^\delta \quad \textit{for some} \quad 0<\delta\le 1. \tag{2.7}$$

Then the Schur-algebra $\mathcal{S}_v^1$ is inverse-closed in $\mathcal{B}(\ell^2)$.

Sun [**98**] has obtained the following result for polynomial weights v_r.

Proposition 2.18. *Let $1\le p\le\infty$. If $r>d/p'$ then $\mathcal{S}_r^p$ is inverse-closed in $\mathcal{B}(\ell^2(\mathbb{Z}^d))$.*

In [**99**] more general (nonradial) weights are used. However, the methods do not allow to reproduce the result of [**54**].

2.3.3. A General Approach to Weighted Matrix Spaces. A *matrix space* $\mathcal{X}$ is a Banach space of matrices over $\mathbb{Z}^d$. We denote the banded matrices of bandwidth $n-1$ that are in $\mathcal{X}$ by $\mathcal{T}_n(\mathcal{X})$. If possible we drop the dependence on $\mathcal{X}$.
A *matrix algebra* $\mathcal{A}$ (over $\mathbb{Z}^d$) is a Banach algebra of matrices that is continuously embedded in $\mathcal{B}(\ell^2(\mathbb{Z}^d))$. We drop the reference to the index set $\mathbb{Z}^d$ whenever possible.
We gather some simple properties of matrix spaces and algebras.

Lemma 2.19.

(1) *If $\mathcal{A}$ is a matrix algebra, the selection of matrix elements is continuous: $|A(k,l)|\le C\|A\|_{\mathcal{A}}$.*

(2) *If $\mathcal{A}$ is a Banach-$*$-algebra of matrices, and $\mathcal{A} \subseteq \mathcal{B}(\ell^2)$, then $\|A\|_{\mathcal{B}(\ell^2)} \le \|A\|_{\mathcal{A}}$. In particular, $\mathcal{A}$ is a matrix algebra.*

Proof. (1) $|A(k,l)| = |\langle Ae_k, e_l\rangle| \le \|A\|_{\mathcal{B}(\ell^2)} \le C\|A\|_{\mathcal{A}}$.
(2) (see, e.g., [**87**]). Let $\rho_{\mathcal{A}}(A)$ denote the spectral radius of A as an element of the matrix algebra $\mathcal{A}$, and $\rho_{\mathcal{B}(\ell^2)}(A)$ the spectral radius of A in $\mathcal{B}(\ell^2)$. By known properties of the spectral radius we obtain $\|A\|^2_{\mathcal{B}(\ell^2)} = \|A^*A\|_{\mathcal{B}(\ell^2)} = \rho_{\mathcal{B}(\ell^2)}(A^*A) \le \rho_{\mathcal{A}}(A^*A) \le \|A^*A\|_{\mathcal{A}} \le \|A\|^2_{\mathcal{A}}$. □

Definition 2.20. Let $\mathcal{X}$ be a matrix space and w a weight on $\mathbb{Z}^d$. The weighted matrix space $\mathcal{X}_w$ consists of the matrices $A \in \mathcal{X}$ such that the matrix A_w with entries

$$A_w(k,l) = A(k,l)w(k-l)\,,$$

is in $\mathcal{X}$. The norm on $\mathcal{X}_w$ is $\|A\|_{\mathcal{X}_w} = \|A_w\|_{\mathcal{X}}$.

If $\mathcal{A}$ is a matrix algebra on $\mathbb{Z}^d$ and w a weight on $\mathbb{Z}^d$, then it is interesting to know if $\mathcal{A}_w$ is a matrix algebra again.

Proposition 2.21. *If $\mathcal{A}$ is a solid matrix algebra, i.e., $\mathcal{A}$ is solid as a sequence space, and w is a submultiplicative weight, then $\mathcal{A}_w$ is a solid matrix algebra.*

Proof. The only nontrivial part is to verify that the $\mathcal{A}_w$-norm is submultiplicative. Let A, B be in $\mathcal{A}_w$. We write $A_w(k,l) = A(k,l)w(k-l)$ as above, and $|A|$ for the matrix with entries $|A(k,l)|$. We obtain the following estimate for the entries of $|(AB)_w|$.

$$\begin{aligned}|(AB)_w|(k,l) &= \Big|\sum_m A(k,m)B(m,l)w(k-l)\Big| \\ &\le \sum_m |A(k,m)w(k-m)|\;|B(m,l)w(m-l)| = (|A_w||B_w|)(k,l)\,.\end{aligned}$$

Consequently,

$$\|AB\|_{\mathcal{A}_w} = \|(AB)_w\|_{\mathcal{A}} \le \|A_wB_w\|_{\mathcal{A}} \le \|A_w\|_{\mathcal{A}}\|B_w\|_{\mathcal{A}} = \|A\|_{\mathcal{A}_w}\|B\|_{\mathcal{A}_w}. \qquad \square$$

For a certain class of polynomial weights we will prove a similar theorem for more general non solid matrix algebras. In this case we will be able to show the inverse-closedness of $\mathcal{A}_w$ in $\mathcal{A}$ (see Section 4.4.3).

CHAPTER 3

Generalized Approximation Spaces and Algebras

As already stated in the introduction it is possible to measure the off-diagonal decay of matrices in terms of the speed of approximation with banded matrices. The precise description is by means of generalized approximation spaces. A theory of generalized approximation spaces has been developed by Almira and Luther (see [**4, 5, 73**], and, for approximation of Banach algebras [**6**]). As we need only special cases of this theory, we give a condensed and simplified survey in Section 3.2.

It is possible without any additional effort to develop the whole theory for *symmetric Banach algebras*, if the approximation scheme is compatible with the algebra multiplication. For approximation of polynomial order we obtain results on the algebra structure of approximation spaces in Section 3.3. For approximation orders beyond the polynomial order, we need additional properties of the Banach algebra. The most complete results can be achieved for *solid* matrix algebras over $\mathbb{Z}$ (see Section 3.3.3).

3.1. Definitions

Though our main interest lies in the theory of *approximation algebras*, we have to define terms and derive results valid in the more general context of approximation spaces.

DEFINITION 3.1. Let the index set Λ be either $\mathbb{R}_0^+$ or $\mathbb{N}_0$. A (linear) *approximation scheme* on the Banach space $\mathcal{X}$ is a family $(X_\lambda)_{\lambda\in\Lambda}$ of closed subspaces X_λ of $\mathcal{X}$ that fulfill the conditions

$$X_0 = \{0\} \quad \text{and} \quad X_\lambda \subseteq X_\mu \text{ for } \lambda \leq \mu. \tag{3.1}$$

REMARK. General (nonlinear) approximation schemes replace the condition that the X_λ are subspaces by conditions of the form $X_\lambda + X_\lambda \subseteq X_{\phi(\lambda)}$ for some bounded function ϕ, see [**5, 26, 73**]. We treat only linear approximation schemes.

The λ*-th approximation error* of $a \in \mathcal{X}$ by X_λ is

$$E_\lambda^{\mathcal{X}}(a) = E_\lambda(a) = \inf_{x\in X_\lambda} \|a - x\|_{\mathcal{X}}. \tag{3.2}$$

Note that the functionals E_λ define nested seminorms on $\mathcal{X}$, i.e.,

$$E_\lambda(a) \geq E_\mu(a) \quad \text{for } \lambda \leq \mu, \quad E_0(a) = \|a\|_{\mathcal{X}}.$$

REMARKS.

(1) Let $\mathcal{X}$ be a solid space of sequences over $\mathbb{Z}^d$. Assume that the approximation scheme on $\mathcal{X}$ can be described by a *nested support condition*

$$X_k = \{x \in \mathcal{X} \colon \operatorname{supp}(x) \subseteq \mathcal{N}_k\}$$

with $\mathcal{N}_j \subset \mathcal{N}_{j+1} \subseteq \mathbb{Z}^d$ for all $j \in \mathbb{N}_0$, then $E_k(x) = \|x - x\, c_{\mathcal{N}_k}\|_{\mathcal{X}}$, where $c_{\mathcal{N}_k}$ is the characteristic function of $\mathcal{N}_k$. In particular, if $\mathcal{X}$ is a solid matrix space, and $X_n = \mathcal{T}_n$, the $(n-1)$-banded matrices, then

$$E_n(A) = \|A - \sum_{|j|_\infty < n} \hat{A}(j)\|_{\mathcal{X}}. \tag{3.3}$$

(2) Observe that we do not assume that $\lim_{n\to\infty} E_n(a) = 0$ for all $a \in \mathcal{X}$. In general, the subspace of banded matrices is not dense in $\mathcal{X}$. As a matter of fact, there are "natural" matrices in matrix spaces that are not approximable by banded matrices. Define the anti-diagonal matrix Γ_r by

$$\Gamma_r(k,l) = \begin{cases} (1+|2k|)^{-r}, & l = -k \\ 0, & l \neq -k. \end{cases}$$

Then $\Gamma_r \in \mathcal{C}_r^\infty$ and $\Gamma_r \in \mathcal{S}_r^1$, and in fact $\|\Gamma_r\|_{\mathcal{C}_r^\infty} = \|\Gamma_r\|_{\mathcal{S}_r^1} = 1$. Likewise, Γ_0 is unitary in $\mathcal{B}(\ell^2)$. However, it is easy to see that

$$E_k^{\mathcal{S}_r^1}(\Gamma_r) = E_k^{\mathcal{C}_r^\infty}(\Gamma_r) = 1.$$

By a simple argument $E_k^{\mathcal{B}(\ell^2)}(\Gamma_0) = 1$ as well.

Specifying a rate of decay given by a weight w on Λ, we define a class of approximation spaces $\mathcal{E}_w^p(\mathcal{X})$ by the norm

$$\|a\|_{\mathcal{E}_w^p(\mathcal{X})}^p = \begin{cases} \|a\|_{\mathcal{X}}^p + \int_1^\infty E_\lambda(a)^p w(\lambda)^p \,\mathrm{d}\,\lambda, & \text{for } \Lambda = \mathbb{R}_0^+, \\ \sum_{k=0}^\infty E_k(a)^p w(k)^p, & \text{for } \Lambda = \mathbb{N}_0\,, \end{cases} \tag{3.4}$$

for $1 \leq p < \infty$ with the obvious change for $p = \infty$. Equivalent weights yield equivalent norms on the spaces $\mathcal{E}_w^p(\mathcal{X})$.

If the weight w on $\mathbb{R}_0^+$ satisfies a moderate regularity condition, then an approximation space with approximation scheme $(X_\lambda)_{\lambda\in\mathbb{R}_0^+}$ can be identified with its discretized version using the approximation scheme $(X_k)_{k\in\mathbb{N}_0}$. Assume that

$$\begin{aligned} w(k) &\asymp \Big(\int_k^{k+1} w(t)^p\, dt\Big)^{1/p}, \quad 1 \leq p < \infty \\ w(k) &\asymp \sup_{t\in[k,k+1)} w(t)\,, \quad p = \infty, \end{aligned} \tag{3.5}$$

then both norms in (3.4) are equivalent.

DEFINITION 3.2 (Classical approximation spaces). If $\mathcal{X}$ is a Banach space and $r \geq 0$ ($r > 0$, if $p = \infty$) the *classical approximation space* $\mathcal{E}_r^p(\mathcal{X})$ consists of those $x \in \mathcal{X}$, for which the norm

$$\|x\|_{\mathcal{E}_r^p(\mathcal{X})} = \Big(\sum_{k=0}^\infty \big(E_k(x)(1+k)^r\big)^p \frac{1}{1+k}\Big)^{1/p} \tag{3.6}$$

is finite (standard modification for $p = \infty$).

We introduce this normalization to be in accordance with the literature for approximation spaces with polynomial weights [**29, 37, 83**]. In particular, it is easy to formulate the relation of the classical approximation spaces to Besov spaces (see Chapter 5).

The reader should be warned that the normalization (3.6) might produce results that look confusing or inconsistent. If $w(\lambda) = v_{r-1/p}(\lambda) = (1+\lambda)^{r-1/p}$ with $r \geq 0$ we obtain

$$\mathcal{E}_r^p(\mathcal{X}) = \mathcal{E}_{v_{r-1/p}}^p(\mathcal{X}).$$

We define $\mathcal{E}^0(\mathcal{X})$ as the closure of $\cup_\lambda X_\lambda$ in $\mathcal{X}$. Obviously this is also the set of all elements $a \in \mathcal{X}$ for which $E_\lambda(a) \to 0$.

REMARK. If $\mathcal{X}$ is a matrix space and the approximation scheme is given by $X_n = \mathcal{T}_n$, the space $\mathcal{E}^0(\mathcal{X})$ is called the space of *band-dominated operators* in [**85, 86**].

Let us state some simple facts about approximation spaces.

PROPOSITION 3.3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces with $\mathcal{Y} \subseteq \mathcal{X}$, and with common approximation scheme $X_k = Y_k$ for $k \in \mathbb{N}_0$. If v and w are weights on $\mathbb{N}_0$, and $1 \le p \le \infty$, then the spaces $\mathcal{E}^p_w(\mathcal{X})$ and $\mathcal{E}^0(\mathcal{X})$ are Banach spaces. The following inclusion relations hold:*

$$\begin{aligned} \mathcal{X} \subseteq \mathcal{Y} \quad &\Rightarrow \quad \mathcal{E}^p_w(\mathcal{X}) \subseteq \mathcal{E}^p_w(\mathcal{Y})\,. \\ p \le q \quad &\Rightarrow \quad \mathcal{E}^p_w(\mathcal{X}) \subseteq \mathcal{E}^q_w(\mathcal{X})\,. \\ v \le C\,w \quad &\Rightarrow \quad \mathcal{E}^p_w(\mathcal{X}) \subseteq \mathcal{E}^p_v(\mathcal{X})\,. \end{aligned}$$

The proofs are straightforward.

In many arguments we need regularity conditions for the weights, or we have to know that an approximation space is a *proper* subspace of the underlying space. We put these conditions into a definition.

DEFINITION 3.4. A weight w on $\Lambda = \mathbb{N}_0$ or $\Lambda = \mathbb{R}^+_0$ is an *approximation weight*, if it satisfies the following conditions.

(1) $\|w(k)\|_{L^p(\Lambda)} = \infty$,
(2) $w(k+1) \le C_w w(k)$,

where C_w is the *growth constant* for w. Additionally, we assume that an approximation weight on $\mathbb{R}^+_0$ satisfies (3.5).

The first condition forces a minimal growth of the weights. Indeed, if $\|w\|_{\ell^p(\mathbb{N}_0)} < \infty$ then $\mathcal{E}^p_w(\mathcal{X}) = \mathcal{X}$. The second condition on an approximation weight is satisfied by every submultiplicative weight on $\mathbb{N}_0$.

LEMMA 3.5. *Let w be a weight on $\mathbb{N}_0$, and assume that $\|w\|_{\ell^p(\mathbb{N}_0)} = \infty$. If $a \in \mathcal{E}^p_w(\mathcal{X})$, then $\lim_{n\to\infty} E_n(a) = 0$. In particular, $\mathcal{E}^p_w(\mathcal{X}) \subseteq \mathcal{E}^0(\mathcal{X})$.*

PROOF ([**5, 73**]). Assume first that $p < \infty$. Let $\epsilon > 0$ and $a \in \mathcal{E}^p_w(\mathcal{X})$. Then there is an n_0 such that for all $m \ge n_0$

$$E^p_m(a) \sum_{k=n_0}^{m} w(k)^p \le \sum_{k=n_0}^{m} E^p_k(a) w(k)^p \le \epsilon.$$

For $m \to \infty$ this statement can be true only if $\lim_{k\to\infty} E^p_k(a) = 0$. The proof for $p = \infty$ is simpler. The definition of $\mathcal{E}^0(\mathcal{X})$ then implies that $\mathcal{E}^p_w(\mathcal{X}) \subseteq \mathcal{E}^0(\mathcal{X})$. □

Lemma 3.5 is needed at various places, e.g., in the proof of the "Equivalence Lemma" (Lemma 3.9), in Corollary 3.10, 3.11 and in the proof of Theorem 3.28. The Lemma assures that all elements of an approximation space can be approximated by elements of the approximation scheme.

REMARK. Note that for $p < \infty$ the constant weight $v_0 = 1$ is an approximation weight. In particular, the classical approximation spaces $\mathcal{E}^p_r(\mathcal{X})$ fulfill condition (1) if and only if $r \ge 0$ ($p < \infty$) or $r > 0$ ($p = \infty$).

For the matrix spaces $\mathcal{C}^p_v$ the norm in $\mathcal{E}^p_w(\mathcal{C}^p_v)$ can be computed explicitly.

PROPOSITION 3.6. *If v is a weight on $\mathbb{Z}^d$, and w is an approximation weight, then*

$$\mathcal{E}^p_w(\mathcal{C}^p_v) = \mathcal{C}^p_{v\,\widetilde{W}_p} \tag{3.7}$$

with $\widetilde{W}_p(k) = \left(\sum_{j\le|k|_\infty} w(j)^p\right)^{1/p}$, if $p < \infty$ and $\widetilde{W}_\infty(k) = \sup_{j\le|k|_\infty} w(j)$.

PROOF. Note that by (3.3) the approximation error is

$$E_j(A) = E_j^{\mathcal{C}_v^p}(A) = \Big(\sum_{|k|_\infty \geq j} \|A[k]\|_\infty^p v(k)^p\Big)^{1/p}.$$

With the function

$$u(j,k) = \begin{cases} 1, & |k|_\infty \geq j \\ 0, & \text{else} \end{cases}$$

we get

$$\begin{aligned}
\|A\|^p_{\mathcal{E}_w^p(\mathcal{C}_v^p)} &= \sum_{j=0}^\infty E_j^{\mathcal{C}_v^p}(A)^p w(j)^p \\
&= \sum_{j=0}^\infty \Big(\sum_{|k|_\infty \geq j} \|A[k]\|_\infty^p v(k)^p\Big) w(j)^p \\
&= \sum_{j=0}^\infty \sum_{k\in\mathbb{Z}^d} u(j,k)\|A[k]\|_\infty^p v(k)^p w(j)^p \\
&= \sum_{k\in\mathbb{Z}^d} \|A[k]\|_\infty^p v(k)^p \sum_{j=0}^\infty u(j,k)\, w(j)^p \\
&= \sum_{k\in\mathbb{Z}^d} \|A[k]\|_\infty^p v(k)^p \sum_{j\leq |k|_\infty} w(j)^p \\
&= \|A\|^p_{\mathcal{C}^p_{v\,\widetilde{W}_p}}.
\end{aligned}$$

□

REMARK. If $p = 1$, it is not difficult to show that $\widetilde{W}_1$ is weakly submultiplicative, if w is submultiplicative.

EXAMPLE 3.7 (Classical approximation spaces). In order to identify the classical approximation spaces $\mathcal{E}_r^p(\mathcal{C}_s^p)$ for $r, s \geq 0$, we need the following fact.

$$w(\lambda) = (1+\lambda)^{r-1/p}$$

$$\Rightarrow \widetilde{W}_p(k) \asymp \begin{cases} v_r(k), & r > 0 \text{ and } 1 \leq p \leq \infty, \\ \log(e + |k|_\infty)^{1/p}, & r = 0 \text{ and } 1 \leq p < \infty, \\ 1, & r = 0 \text{ and } p = \infty. \end{cases} \tag{3.8}$$

Then

$$\mathcal{E}_r^p(\mathcal{C}_s^p) = \begin{cases} \mathcal{C}^p_{r+s}, & r > 0 \text{ and } 1 \leq p \leq \infty, \\ \mathcal{C}^p_{v_s \log(e+|\cdot|_\infty)^{1/p}}, & r = 0 \text{ and } 1 \leq p < \infty, \\ \mathcal{C}^p_s, & r = 0 \text{ and } p = \infty. \end{cases} \tag{3.9}$$

The following result indicates the relationship between approximation with banded matrices and off-diagonal decay.

COROLLARY 3.8. *If $\mathcal{A}$ is a solid matrix algebra and w an approximation weight, then $\mathcal{E}_w^\infty(\mathcal{A}) \subseteq \mathcal{C}^\infty_{w(|\cdot|_\infty)}$, and $A \in \mathcal{E}_w^\infty(\mathcal{A})$ decays with the order $\mathcal{O}(1/w(|n|_\infty))$ off the diagonal.*

PROOF. If $\mathcal{A}$ is a solid matrix algebra, then for $A \in \mathcal{A}$

$$E_n(A) = \|A - \sum_{|k|_\infty < n} \hat{A}(k)\|_\mathcal{A}$$

by (3.3). If $A \in \mathcal{E}_w^\infty(\mathcal{A})$, the size of the lth diagonal is dominated by

$$\|\hat{A}(l)\|_{\mathcal{A}} \leq \|\sum_{|k|_\infty \geq |l|_\infty} \hat{A}(k)\|_{\mathcal{A}} \leq \|A\|_{\mathcal{E}_w^\infty(\mathcal{A})} w(|l|_\infty)^{-1} .$$

Since $\mathcal{A}$ is embedded into $\mathcal{B}(\ell^2)$, this implies that

$$\|\hat{A}(l)\|_{\mathcal{B}(\ell^2)} \leq \|\hat{A}(l)\|_{\mathcal{A}} \leq \|A\|_{\mathcal{E}_w^\infty(\mathcal{A})} w(|l|_\infty)^{-1} ,$$

and thus $A \in \mathcal{C}^\infty_{w(|\cdot|_\infty)}$. □

3.2. Equivalence, Representation and Interpolation Theorems

In this section we construct two equivalent norms for the approximation spaces $\mathcal{E}_w^p(\mathcal{X})$. The results are generalizations of the so-called *equivalence* and *representation* theorems for approximation of polynomial order (see [83] for an excellent exposition). With the help of these norms we give a constructive description of weighted approximation spaces for solid matrix spaces that is somewhat similar to the Littlewood-Paley-representation for Besov spaces.

We do not use interpolation theory in this text. However, it is of interest that under certain conditions an approximation space can be identified with an interpolation space. We sketch the relevant concepts (adapted from [73]).

The material presented in this section is inspired by the fundamental work of Almira and Luther on generalized approximation spaces [3, 4, 5, 6, 73]. We adapt their approach to our needs.

3.2.1. Equivalence and Representation Theorems.

We start with a simple geometric fact.

Lemma 3.9 ("Equivalence Lemma"). *(1) Let w be an approximation weight on $\mathbb{R}_0^+$, with $\int_0^\infty w(t)\,dt = \infty$. If $\kappa > 1$, then the sequence*

$$\varphi_0 = 0, \quad \varphi_j = \sup\{t \in \mathbb{R}_0^+ : \textstyle\int_0^t w \leq \kappa^{j-1}\},\ j \geq 1 \tag{3.10}$$

is well-defined, $\varphi_j < \infty$ for all $j \geq 0$, and $\lim_{j\to\infty} \varphi_j = \infty$. For all positive, nonincreasing functions f the following equivalence holds with constants C_1, C_2 independent of f.

$$C_1 \sum_{j=1}^\infty f(\varphi_j)\kappa^j \leq \int_0^\infty f(t)w(t)\,dt \leq C_2 \sum_{j=0}^\infty f(\varphi_j)\kappa^j . \tag{3.11}$$

(2) Let the weight w satisfy $\|w\|_\infty = \infty$, and set

$$\varphi_0 = 0, \quad \varphi_j = \sup\{t \in \mathbb{R}_0^+ : w(t) \leq \kappa^{j-1}\},\ j \geq 1 \tag{3.12}$$

For all positive, nonincreasing functions f the following equivalence holds with constants C_1, C_2 independent of f.

$$C_1 \sup_{j\geq 1} f(\varphi_j)\kappa^j \leq \sup_{t\geq 0} f(t)w(t) \leq C_2 \sup_{j\geq 0} f(\varphi_j)\kappa^j .$$

Proof. (1) As w is an approximation weight, Definition 3.4(1) implies that $\int_0^\infty w(t)\,dt = \infty$, and $\varphi_j < \infty$ follows. Definition 3.4(2) implies that $\varphi_j \to \infty$ for $j \to \infty$. We may assume that the function f is not identically zero. Since f is nonincreasing, estimating the integral by upper Riemann sums yields (see Figure 1)

$$\int_0^\infty f(t)w(t)\,dt \leq f(0)\kappa^0 + \sum_{j=1}^\infty f(\varphi_j)(\kappa^j - \kappa^{j-1}) \leq \sum_{j=0}^\infty f(\varphi_j)\kappa^j .$$

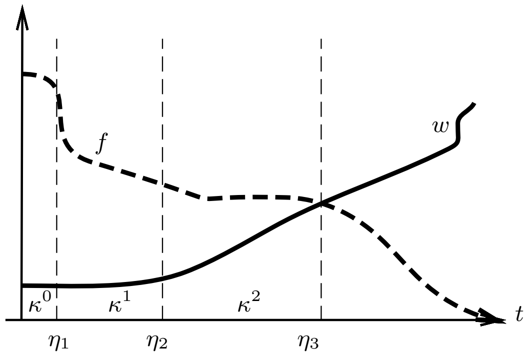

FIGURE 1

The lower Riemann sums give the inequality

$$f(\varphi_1) + \sum_{j=1}^{\infty} f(\varphi_{j+1})(\kappa^j - \kappa^{j-1}) \le \int_0^{\infty} f(t)w(t)\,dt\,.$$

This can be estimated further by

$$\sum_{j=0}^{\infty} f(\varphi_{j+1})\kappa^{j+1} \le C \int_0^{\infty} f(t)w(t)\,dt\,.$$

Shifting the index j by one gives the desired result.
(2) is proved in a similar manner. □

COROLLARY 3.10 (Equivalence Theorem). *Let $\mathcal{X}$ be a Banach space and $(X_\lambda)_{\lambda\in\mathbb{R}_0^+}$ an approximation scheme. Let w be an approximation weight for an exponent p, $1 \le p \le \infty$. Define the weight W_p by*

$$W_p(r) = \begin{cases} \left(\int_{t=0}^{r} w(t)^p\,dt\right)^{1/p}, & 1 \le p < \infty, \\ \sup_{0\le t\le r} w(t), & p = \infty. \end{cases} \tag{3.13}$$

Choose a constant $\kappa > 1$. With

$$\varphi_0 = 0, \quad \varphi_j = \sup\{t \in \mathbb{R}_0^+ : W_p(t) \le \kappa^{j-1}\} \quad \textit{for } j \ge 1 \tag{3.14}$$

the expression

$$\|a\|^*_{\mathcal{E}^p_w(\mathcal{X})} = \Big(\sum_{j=0}^{\infty} \kappa^{jp} E^p_{\varphi_j}(a)\Big)^{1/p}, \quad a \in \mathcal{E}^p_w(\mathcal{X})$$

is an equivalent norm for $\mathcal{E}^p_w(\mathcal{X})$ (with the obvious change for $p = \infty$).

PROOF. If $1 \le p < \infty$ the Equivalence Lemma, applied to $f(t) = E_t(a)^p$ with weight $w(t)^p$, gives

$$C_1 \sum_{j=1}^{\infty} E_{\varphi_j}(a)^p \kappa^{jp} \le \int_0^{\infty} E_\lambda(a)^p w(\lambda)^p\,d\lambda \le C_2 \sum_{j=0}^{\infty} E_{\varphi_j}(a)^p \kappa^{jp}\,.$$

Adding $E_0(a)^p$ yields the equivalence, details are left to the reader. The case $p = \infty$ is handled in a similar way. □

REMARK. The proof of Corollary 3.10 does not depend on any specific property of approximation spaces. It uses only that $E_\lambda(a)$ is nonincreasing in λ and that the norm is of the form (3.4). We shall use Corollary 3.10 to obtain estimates for the K-functional in Appendix C.

We can use the Equivalence Theorem also for approximation schemes defined on $\mathbb{N}_0$, if we define for $\lambda \in \mathbb{R}_0^+$

$$X_\lambda = X_{\lfloor \lambda \rfloor}, \quad w(\lambda) = w(\lfloor \lambda \rfloor), \quad E_\lambda(a) = E_{\lfloor \lambda \rfloor}(a), \tag{3.15}$$

where $\lfloor x \rfloor$ denotes the greatest integer smaller or equal x. Then the approximation spaces for this continuous approximation scheme coincide with the approximation spaces using the discrete approximation scheme.

More general versions of the Equivalence theorem can be found in [6].

Remark. If $w(k) = v_{r-1/p}(k) = (1+k)^{r-1/p}$, we can choose $\kappa = 2^r$ and obtain (standard change for $p = \infty$)

$$\|a\|_{\mathcal{E}_r^p(\mathcal{X})} \asymp \Big(E_0(a) + \sum_{k=0}^{\infty} 2^{jrp} (E_{2^j}(a))^p \Big)^{1/p}.$$

See, e.g., [83] for details.

Proposition 3.11 (Representation theorem [73]). *Let $\mathcal{X}$ be a Banach space with approximation scheme $(X_k)_{k\in\mathbb{N}_0}$, and let w be an approximation weight on $\ell^p(\mathbb{N}_0)$. Assume that $\kappa > 1$, and define φ_j as in (3.14). For an element $a \in \mathcal{X}$ the following are equivalent:*

(1) $a \in \mathcal{E}_w^p(\mathcal{X})$,
(2) $a = \sum_{j=1}^{\infty} a_j$ *with convergence in $\mathcal{X}$, $a_j \in X_{\varphi_j}$ and $(\kappa^j \|a_j\|_{\mathcal{X}})_{j\geq 1} \in \ell^p(\mathbb{N})$.*

An equivalent norm on $\mathcal{E}_w^p(\mathcal{X})$ is

$$\|a\|_{\widetilde{\mathcal{E}}_w^p(\mathcal{X})} = \inf\Big\{ \big(\sum_{j=1}^{\infty} \kappa^{jp} \|a_j\|_{\mathcal{X}}^p\big)^{1/p} : a = \sum_{j=1}^{\infty} a_j, a_j \in X_{\varphi_j} \Big\} \tag{3.16}$$

with the standard modification if $p = \infty$.

Proof. For $a \in \mathcal{E}_w^p(\mathcal{X})$ choose $b_j \in X_{\varphi_j}$ with $\|a - b_j\|_{\mathcal{X}} \leq E_{\varphi_j}(a) + \kappa^{-2j}\|a\|_{\mathcal{X}}$. Set $a_j = b_j - b_{j-1} \in X_{\varphi_j}$. Then $a = \sum_{k=0}^{\infty} a_j$ and $\|a_j\|_{\mathcal{X}} \leq 2(E_{\varphi_{j-1}}(a) + \kappa^{-2j}\|a\|_{\mathcal{X}})$. Summing up gives

$$\|(\kappa^j \|a_j\|_{\mathcal{X}})_{j\geq 0}\|_{\ell^p(\mathbb{N}_0)} \lesssim \|a\|_{\mathcal{X}} + \|(\kappa^j E_{\varphi_j}(a))_{j\geq 0}\|_{\ell^p(\mathbb{N}_0)} \asymp \|a\|_{\mathcal{E}_w^p(\mathcal{X})},$$

the last relation by the Equivalence theorem. For the reverse inequality let $a = \sum_{k=0}^{\infty} a_k$ be a representation as in (2). Then $E_{\varphi_j}(a) \leq \sum_{k=j}^{\infty} \|a_k\|_{\mathcal{X}}$, and so

$$\begin{aligned} \|a\|_{\mathcal{E}_w^p(\mathcal{X})} &\asymp \|(\kappa^j E_{\varphi_j}(a))_{j\geq 0}\|_{\ell^p(\mathbb{N})} \lesssim \|(\kappa^j \sum_{k=j}^{\infty} \|a_k\|_{\mathcal{X}})_{j\geq 0}\|_{\ell^p(\mathbb{N})} \\ &\lesssim \|(\kappa^j \|a_j\|)_{j\geq 0}\|. \end{aligned}$$

The last relation follows from the discrete Hardy inequality (Appendix A.2). – Now Equation (3.16) is an easy consequence of what has been proved. ☐

Again, a more general result can be found in [73]. If $w(k) = v_{r-1/p}(k)$ we obtain

Corollary 3.12 ([83]). *An element $a \in \mathcal{X}$ is in $\mathcal{E}_r^p(\mathcal{X})$ if and only if*

$$a = \sum_{j=1}^{\infty} a_j, \quad a_j \in X_{2^j} \text{ and } \sum_{j=0}^{j} 2^{rjp} \|a_j\|_{\mathcal{X}}^p < \infty$$

with convergence in $\mathcal{X}$ (standard modification for $p = \infty$). An equivalent norm on $\mathcal{E}_w^p(\mathcal{X})$ is

$$\|a\|_{\widetilde{\mathcal{E}}_w^p(\mathcal{X})} = \inf\Big\{ \big(\sum_{j=1}^{\infty} 2^{jrp} \|a_j\|_{\mathcal{X}}^p\big)^{1/p} : a = \sum_{j=1}^{\infty} a_j, a_j \in X_{2^j} \Big\} \tag{3.17}$$

with the standard modification for $p = \infty$.

3.2.2. Approximation in Solid Matrix Spaces. The Representation theorem can be used to obtain an explicit expression for the norm in an approximation space $\mathcal{E}^p_w(\mathcal{X})$, whenever $\mathcal{X}$ is a *solid* matrix space.

PROPOSITION 3.13. *Let* $\mathcal{X}$ *be a* solid *matrix space with the approximation scheme consisting of the banded matrices* $(\mathcal{T}_k)_{k\geq 0}$. *Let* w *be an approximation weight on* $\ell^p(\mathbb{N}_0)$, $1 \leq p \leq \infty$. *If* $\kappa > 1$, *and the quantities* φ_j *are defined as in* (3.14), *then for all* $A \in \mathcal{E}^p_w(\mathcal{X})$

$$\|A\|_{\mathcal{E}^p_w(\mathcal{X})} \asymp \Big(\sum_{j=0}^{\infty} \kappa^{jp} \Big\| \sum_{\varphi_j \leq |k|_\infty < \varphi_{j+1}} \hat{A}(k)\Big\|^p_{\mathcal{X}}\Big)^{1/p} \tag{3.18}$$

$$\asymp \Big(\sum_{j=0}^{\infty} \kappa^{jp} \Big\| \sum_{\lfloor \kappa^{j-1}\rfloor \leq W_p(|k|_\infty) < \kappa^j} \hat{A}(k)\Big\|^p_{\mathcal{X}}\Big)^{1/p} \tag{3.19}$$

$$\asymp \Big(\sum_{j=0}^{\infty} \Big\| \sum_{\lfloor \kappa^{j-1}\rfloor \leq W_p(|k|_\infty) < \kappa^j} \hat{A}(k) W_p(|k|)\Big\|^p_{\mathcal{X}}\Big)^{1/p} \tag{3.20}$$

with the standard modification for $p = \infty$.

PROOF. By the Representation Theorem we have the norm equivalence

$$\|A\|_{\mathcal{E}^p_w(\mathcal{X})} \asymp \inf \Big(\sum_{j=1}^{\infty} \kappa^{jp} \|B_j\|^p_{\mathcal{X}}\Big)^{1/p}, \tag{3.21}$$

where the infimum is taken over all $B_j \in \mathcal{T}_{\varphi_j}$ that satisfy $A = \sum_{j=0}^{\infty} B_j$ in the norm of $\mathcal{X}$.

Recall that $\mathcal{T}_{\varphi_j} = \{A \in \mathcal{X} \colon A = \sum_{|k|_\infty < \varphi_j} \hat{A}(k)\}$. The solidity of $\mathcal{X}$ implies that the infimum in (3.21) is attained for the choice

$$B_j = \sum_{\{k\colon\, \varphi_{j-1} \leq |k|_\infty < \varphi_j\}} \hat{A}(k) \quad j \geq 1,$$

where we set $\varphi_0 = 0$. Shifting the summation index by 1 gives (3.18). The equivalence (3.19) is just a different formulation of what has been just proved, and (3.20) follows from the equivalence

$$W_p(\varphi_j) \asymp \kappa^j \quad \text{for } j \geq 1\,, \tag{3.22}$$

which is a direct consequence of the definitions. □

Let us apply Proposition 3.13 to the standard examples of matrix spaces.

EXAMPLE 3.14. For the spaces $\mathcal{E}^p_w(\mathcal{C}^p_v)$ we have already obtained the characterization $\mathcal{E}^p_w(\mathcal{C}^p_v) = \mathcal{C}^p_{v\widetilde{W}_p}$ (Proposition 3.6). Now we can also obtain results for the more general approximation spaces $\mathcal{E}^q_w(\mathcal{C}^p_v)$. Actually, for $A \in \mathcal{E}^q_w(\mathcal{C}^p_v)$, Equation (3.19) implies the norm equivalences

$$\|A\|_{\mathcal{E}^q_w(\mathcal{C}^p_v)} \asymp \Big(\sum_{j=0}^{\infty} 2^{jq} \Big\| \sum_{W_p(|k|_\infty)\in[2^{j-1},2^j)} \hat{A}(k) v(k)\Big\|^q_{\mathcal{C}^p_0}\Big)^{1/q}$$

$$\asymp \Big(\sum_{j=0}^{\infty} 2^{jq} \Big(\sum_{W_p(|k|_\infty)\in[2^{j-1},2^j)} \|A[k]\|^p_{\ell^\infty(\mathbb{Z}^d)} v(k)^p\Big)^{q/p}\Big)^{1/q},$$

where we have chosen w.l.o.g. $\kappa = 2$. If we specialize to approximation of polynomial order and with the standard polynomial weights $w = v_{r-1/q}$, $v = v_s$ and choose $\kappa = 2^r$, we arrive at

(3.23)

$$\|A\|_{\mathcal{E}^q_r(\mathcal{C}^p_s)} \asymp \Big(\sum_{j=0}^{\infty} 2^{jqr}\big(\sum_{\lfloor 2^{j-1}\rfloor \le |k|_\infty < 2^j} \|A[k]\|^p_{\ell^\infty(\mathbb{Z}^d)}(1+|k|)^{sp}\big)^{q/p}\Big)^{1/q}$$
$$\asymp \Big(\sum_{j=0}^{\infty} 2^{jq(r+s)}\big(\sum_{\lfloor 2^{j-1}\rfloor \le |k|_\infty < 2^j} \|A[k]\|^p_{\ell^\infty(\mathbb{Z}^d)}\big)^{q/p}\Big)^{1/q} \asymp \|A\|_{\mathcal{E}^q_{r+s}(\mathcal{C}^p_0)} .$$

For the last equivalence we used $v_s(k) \asymp v_s(2k)$. In general, weights with $w(2k) \asymp w(k)$ are equivalent to polynomial weights, see [**40**].

We specialize to convolution matrices C_a for $a \in \ell^p_v(\mathbb{Z}^d)$. Note that the embedding $a \to C_a$ from $\ell^p_v(\mathbb{Z}^d)$ to $\mathcal{C}^p_v$ is isometric. That implies that, e.g., for $a \in \ell^p_s(\mathbb{Z}^d)$

$$\|a\|_{\mathcal{E}^q_r(\ell^p_s(\mathbb{Z}^d))} \asymp \Big(\sum_{j=0}^{\infty} 2^{jq(r+s)}\big(\sum_{\lfloor 2^{j-1}\rfloor \le |k|_\infty < 2^j} |a(k)|^p\big)^{q/p}\Big)^{1/q} .$$

The norm on the right side is the norm of the *discrete Besov space* $\mathfrak{b}^q_{r+s}(\ell^p(\mathbb{Z}^d))$, see Pietsch [**82**]). So the spaces $\mathcal{E}^q_w(\mathcal{C}^p_v)$ can be regarded as a noncommutative weighted generalization of discrete Besov spaces.

Remark. The equivalence $\mathcal{E}^q_r(\mathcal{C}^p_s) = \mathcal{E}^q_{r+s}(\mathcal{C}^p_0)$ in the above example depends on the equivalence $w(\varphi_{j+1}) \asymp w(\varphi_j)$. It is a special instance of a *reiteration theorem* for generalized approximation spaces. See [**5**] and [**73**, Section 6] for more details. We will obtain similar relations for approximation of polynomial order in Section 5.5.

3.2.3. Realization of Approximation Spaces as Interpolation Spaces. As for approximation with polynomial weights (see [**37, 83**]) it is possible to identify approximation spaces with *real interpolation spaces*, if Jackson and Bernstein inequalities are satisfied. We will only use some rudiments of interpolation theory in this treatment. A standard reference is [**15**].

Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces with $\mathcal{Y} \hookrightarrow \mathcal{X}$. For $a \in \mathcal{X}$ and $t > 0$ the K-functional is defined by (see, e.g., [**13, 15, 37**])

$$K(a,t) = \inf_{y\in\mathcal{Y}}\big(\|a-y\|_{\mathcal{X}} + t\|y\|_{\mathcal{Y}}\big) . \tag{3.24}$$

Let $0 < r < 1$. The interpolation space $(\mathcal{X}, \mathcal{Y})_{r,p}$ consists of all $a \in \mathcal{X}$ for which the norm

$$\|a\|_{r,p} = \left(\int_0^1 t^{-rp} K(a,t)^p \frac{dt}{t}\right)^{1/p} < \infty.$$

For every $C > 1$ an equivalent discrete version of this norm is [**37**, 6.7.6]

$$\|a\|_{r,p} \asymp \left(\sum_{j=1}^{\infty} C^{rpj} K(a, C^{-j})^p\right)^{1/p} . \tag{3.25}$$

Definition 3.15 (Jackson-Bernstein conditions). Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces, $\mathcal{Y} \hookrightarrow \mathcal{X}$, $(X_\lambda)_{\lambda\in\Lambda}$ an approximation scheme for $\mathcal{X}$ with $X_\lambda \subseteq Y$ for all $\lambda \in \Lambda$, and $(\omega_n)_{n\ge 0}$ an increasing weight function. The pair $(\mathcal{X}, \mathcal{Y})$ satisfies the JB-condition, if

$$E^{\mathcal{X}}_n(a) \le \frac{C}{\omega_n}\|a\|_{\mathcal{Y}} \quad \text{for all } a \in \mathcal{Y}, \qquad \text{(Jackson inequality)} \tag{3.26}$$

$$\|a_n\|_{\mathcal{Y}} \le C'\omega_n\|a_n\|_{\mathcal{X}} \quad \text{for all } a_n \in X_n \qquad \text{(Bernstein inequality)} \tag{3.27}$$

EXAMPLE 3.16. If $\mathcal{X}$ is a solid matrix space, $(\mathcal{T}_n)_{n\in\mathbb{N}_0}$ the approximation scheme consisting of the banded matrices in $\mathcal{X}$, $(\omega_n)_{n\geq 0}$ an increasing weight, and $\mathcal{X}_\omega$ the weighted matrix space as in Definition 2.20, then

$$E_n^{\mathcal{X}}(A) \leq \frac{1}{\omega_n}\|A\|_{\mathcal{X}_\omega} \quad \text{for all} \quad A \in \mathcal{X}_\omega \tag{3.28a}$$

$$\|A_n\|_{\mathcal{X}_w} \leq \omega_n \|A_n\|_{\mathcal{X}} \quad \text{for all} \quad A \in \mathcal{T}_n. \tag{3.28b}$$

The Jackson inequality follows from the solidity of $\mathcal{X}$.

We formulate the main result relating approximation spaces to interpolation spaces. The proof is in Appendix C.

PROPOSITION 3.17. *(Interpolation theorem, [**73**, 4.3, 4.5]) Let $\mathcal{X}, \mathcal{Y}$ and ω as in Definition 3.15. If v is an approximation weight with $V_p(m) \asymp \omega_m^r$ for some $0 < r < 1$, $1 \leq p \leq \infty$, then*

$$\mathcal{E}_v^p(\mathcal{X}) = (\mathcal{X}, \mathcal{Y})_{(r,p)}$$

with equivalent norms.

3.3. Approximation of Banach Algebras

In this section we treat approximation schemes that are *compatible* with the multiplication in a Banach algebra. These schemes include the trigonometric polynomials, and the banded matrices. For these schemes we show that the approximation space $\mathcal{E}_w^1(\mathcal{A})$ for a Banach algebra $\mathcal{A}$ is a Banach algebra itself.

If $\mathcal{A}$ is *symmetric*, we prove that approximable elements form an inverse-closed subalgebra of $\mathcal{A}$. Moreover, for approximation of polynomial order we obtain a similar result for the approximation spaces $\mathcal{E}_r^p(\mathcal{A})$, using again a result of Almira and Luther [**6**]. An application of this result to the Jaffard algebra allows for a shortcut in the proof of Jaffard's theorem [**61**].

For *solid* and *unweighted* matrix algebras over $\mathbb{Z}$ (see Proposition 3.29) we are able to prove the inverse-closedness of $\mathcal{E}_w^p(\mathcal{A})$ in $\mathcal{B}(\ell^2)$ if w is an algebra weight that satisfies the GRS condition.

3.3.1. Compatible Approximation Schemes and Approximation Algebras.

DEFINITION 3.18. Let $\mathcal{A}$ be a Banach algebra and $(X_\lambda)_{\lambda\in\Lambda}$ an approximation scheme (Definition 3.1). We call $(X_\lambda)_{\lambda\in\Lambda}$ *compatible (with multiplication)*, if

$$X_\lambda \cdot X_\mu \subseteq X_{\lambda+\mu}, \quad \text{for all} \quad \lambda, \mu \in \Lambda. \tag{3.29}$$

If $\mathcal{A}$ has an involution, we further assume that

$$\mathbf{1} \in X_1 \quad \text{and} \quad X_\lambda = X_\lambda^* \quad \text{for all } \lambda \in \Lambda. \tag{3.30}$$

If $\mathcal{A}$ is a Banach algebra with an approximation scheme $(X_\lambda)_{\lambda\in\Lambda}$, we assume that $(X_\lambda)_{\lambda\in\Lambda}$ is compatible with multiplication, if nothing else is said.

EXAMPLE 3.19.

(1) *Approximation with trigonometric polynomials.* Let $\mathcal{A} = L^\infty(\mathbb{T}^d)$ and choose the approximation scheme as

$$X_0 = \{0\}, \quad X_k = \operatorname{span}\{e^{2\pi i r\cdot t} \colon |r|_\infty < k\},\ k \geq 1.$$

Clearly the conditions (3.29) and (3.30) are fulfilled.

(2) *Approximation with banded matrices.* Let $\mathcal{A}$ be a matrix algebra and let $\mathcal{T}_N = \mathcal{T}_N(\mathcal{A})$ be the set of matrices in $\mathcal{A}$ with bandwidth smaller than N, then the sequence $(\mathcal{T}_k)_{k\geq 0}$ is an approximation scheme for $\mathcal{A}$. The closure of all banded matrices in $\mathcal{A}$ is the space of *band-dominated matrices* in $\mathcal{A}$ [**84, 86**].

Remark. The condition (3.30) is easy to verify. More generally, it is sufficient to verify that $\mathbf{1} \in \mathcal{E}^0(\mathcal{A}) = (\mathcal{E}^0(\mathcal{A}))^*$, as the proof of Proposition 3.25 will show.

The next proposition is proved by an estimate that is used again later.

Proposition 3.20. *If $\mathcal{A}$ is a Banach algebra with a compatible approximation scheme $(X_k)_{k\geq 0}$, then the set $\mathcal{E}^0(\mathcal{A})$ is a closed subalgebra of $\mathcal{A}$.*

Proof. The space $\mathcal{E}^0(\mathcal{A})$ is closed by definition. For the proof of the algebra property let $a, b \in \mathcal{A}$, and choose elements $a_k \in X_k$ and $b_l \in X_l$. Then

$$\begin{aligned} E_{k+l}(ab) &\leq \|ab - a_k b_l\|_{\mathcal{A}} \leq \|ab - a_k b\|_{\mathcal{A}} + \|a_k b - a_k b_l\|_{\mathcal{A}} \\ &\leq \|a - a_k\|_{\mathcal{A}} \|b\|_{\mathcal{A}} + \|a_k\|_{\mathcal{A}} \|b - b_l\|_{\mathcal{A}} \end{aligned}$$

Given $\epsilon > 0$ we can choose a_k and b_l such that $\|a - a_k\|_{\mathcal{A}} < (1+\epsilon)E_k(a)$, and likewise $\|b - b_l\|_{\mathcal{A}} < (1+\epsilon)E_l(b)$. So we arrive at

$$\begin{aligned} E_{k+l}(ab) &\leq (1+\epsilon)(E_k(a)\|b\|_{\mathcal{A}} + E_l(b)\|a_k\|_{\mathcal{A}}) \\ &\leq (1+\epsilon)(E_k(a)\|b\|_{\mathcal{A}} + 2E_l(b)\|a\|_{\mathcal{A}})\,, \end{aligned} \tag{3.31}$$

if a_k is close enough to a. If $a, b \in \mathcal{E}^0(\mathcal{A})$, this implies that $ab \in \mathcal{E}^0(\mathcal{A})$. □

We can obtain a similar result for approximation of polynomial order.

Proposition 3.21 ([**3, 6**]). *If $\mathcal{A}$ is a Banach algebra with a compatible approximation scheme $(X_\lambda)_{\lambda\in\Lambda}$, then for every $1 \leq p \leq \infty$ and $r \geq 0$ ($p < \infty$) or $r > 0$ ($p = \infty$) the approximation space $\mathcal{E}^p_r(\mathcal{A})$ is a Banach algebra and dense in $\mathcal{E}^0(\mathcal{A})$.*

Proof. We give the proof only for the index set $\Lambda = \mathbb{N}_0$. Choose $a_n, b_n \in X_n$ such that $\|a - a_n\| \leq 2E_n(a)$ and $\|b - b_n\| \leq 2E_n(b) \leq 2\|b\|_{\mathcal{A}}$. Then $\|b_n\| \leq \|b\| + \|b_n - b\| \leq 3\|b\|$ and

$$\begin{aligned} E_{2n+1}(ab) &\leq E_{2n}(ab) \leq \|ab - a_n b_n\|_{\mathcal{A}} \\ &\leq \|a\|_{\mathcal{A}} \, \|b - b_n\|_{\mathcal{A}} + \|b_n\|_{\mathcal{A}} \|a - a_n\|_{\mathcal{A}} \\ &\leq 2\|a\|_{\mathcal{A}} E_n(b) + 6\|b\|_{\mathcal{A}} E_n(a). \end{aligned} \tag{3.32}$$

Using the estimate (3.31) for $k = l = n$ and the equivalence $(1+n)^r \asymp (1+2n)^r$, we obtain

$$\|ab\|_{\mathcal{E}^p_r} \leq C \left(\|a\|_{\mathcal{E}^p_r} \|b\|_{\mathcal{A}} + \|b\|_{\mathcal{E}^p_r} \|a\|_{\mathcal{A}} \right). \tag{3.33}$$

The Banach algebra-property of $\mathcal{E}^p_r(\mathcal{A})$ now follows from (3.33). As $v_{r-1/p}$ is an approximation weight, the claimed density follows. □

For general submultiplicative weights we obtain the following result.

Proposition 3.22. *If $\mathcal{A}$ is a Banach algebra, $(X_k)_{k\in\mathbb{N}_0}$ is a compatible approximation scheme for $\mathcal{A}$, and w is a submultiplicative weight on $\mathbb{N}_0$, then the approximation space $\mathcal{E}^1_w(\mathcal{A})$ is a Banach algebra.*

Before proving the proposition we need two lemmas. The first states a property of the weights W_p.

Lemma 3.23. *If w is submultiplicative on $\mathbb{N}_0$, then W_p is weakly submultiplicative.*

Proof of the Lemma. Assume first that $p < \infty$.

$$\begin{aligned} W_p(m+n)^p &= \int_0^{m+n} w(j)^p\, dj \le \int_0^m w(j)^p\, dj + w(m)^p \int 0^n w(j)^p\, dj \\ &\le W_p(m)^p + w(m)^p W_p(n)^p \le W_p(m)^p + W_p(m)^p W_p(n)^p \\ &\le W_p(m)^p W_p(n)^p + W_p(m)^p W_p(n)^p = 2W_p(m)^p W_p(n)^p\,. \end{aligned} \tag{3.34}$$

If $p = \infty$ the proof is similar. □

Lemma 3.24. *If the integers φ_j are defined as in (3.14) and $\kappa > 1$, then*

$$\varphi_j + \varphi_k \le \varphi_{j+k} \text{ for all } j, k \in \mathbb{N}_0\,. \tag{3.35}$$

Proof. The definition of φ implies that $m \le \varphi_j$ is equivalent to $W_1(m) \le \kappa^{j-1}$. W.l.o.g we can assume that $\kappa \ge 2$. If $l \le \varphi_j + \varphi_k$ for an integer l, then we can write $l = m + n$, $m \le \varphi_j$, $n \le \varphi_k$, and so $W_1(m)W_1(n) \le \kappa^{j+k-2}$. As W_1 is weakly submultiplicative we obtain

$$W_1(l) \le 2W_1(m)W_1(n) \le 2\kappa^{j+k-2} \le \kappa^{j+k-1}\,,$$

and so $l \le \varphi_{j+k-1} \le \varphi_{j+k}$. □

Proof of Proposition 3.22. We use the representation theorem (Proposition 3.11). Let $\epsilon > 0$. Choose $a_j \in X_{\varphi_j}$ such that $a = \sum_{j=0}^{\infty} a_j$, and $\sum_{j=0}^{\infty} \kappa^j \|a_j\|_{\mathcal{A}} \le (1+\epsilon)\|a\|_{\widetilde{\mathcal{E}}^1_w(\mathcal{A})}$, where $\|a\|_{\widetilde{\mathcal{E}}^1_w(\mathcal{A})}$ denotes the norm defined in (3.16). For b we can assume a similar decomposition and norm relation. Then the sums $\sum_{j=0}^{\infty}\|a_j\|_{\mathcal{A}}$, $\sum_{k=0}^{\infty}\|b_k\|_{\mathcal{A}}$ are convergent, and so the Cauchy product

$$ab = \sum_{l=0}^{\infty} c_l, \quad c_l = \sum_{m=0}^{l} a_m b_{l-m}$$

is norm convergent in $\mathcal{A}$. Equation (3.35) implies that $c_l \in X_{\varphi_{l+1}}$, and with the representation theorem we obtain

$$\begin{aligned} \|ab\|_{\mathcal{E}^1_w(\mathcal{A})} &\lesssim \sum_{j=0}^{\infty} \kappa^{j+1}\|c_j\|_{\mathcal{A}} \\ &\le \sum_{j=0}^{\infty} \kappa^{j+1} \sum_{m=0}^{j} \|a_m\|_{\mathcal{A}}\|b_{j-m}\|_{\mathcal{A}} \\ &= \kappa \sum_{j=0}^{\infty} \sum_{m=0}^{j} \kappa^m \|a_m\|_{\mathcal{A}} \kappa^{j-m}\|b_{j-m}\|_{\mathcal{A}} \\ &\le (1+\epsilon)^2 \kappa \|a\|_{\widetilde{\mathcal{E}}^1_w(\mathcal{A})}\|b\|_{\widetilde{\mathcal{E}}^1_w(\mathcal{A})}\,, \end{aligned}$$

and this was to be shown. □

3.3.2. Approximation and Inverse Closedness in Symmetric Banach Algebras.

If the Banach algebra $\mathcal{A}$ is symmetric, it is possible to prove results on the inverse-closedness of the approximation spaces $\mathcal{E}^p_r(\mathcal{A})$ in $\mathcal{A}$.

Proposition 3.25. *If $\mathcal{A}$ is a* symmetric *Banach algebra with a compatible approximation scheme $(X_\lambda)_{\lambda\in\Lambda}$, then $\mathcal{E}^0(\mathcal{A})$ is inverse-closed in $\mathcal{A}$.*

Proof. Proposition 3.20 tells us that $\mathcal{E}^0(\mathcal{A})$ is a Banach algebra. Since $\mathcal{E}^0(\mathcal{A})$ is a closed $*$-subalgebra of the symmetric algebra $\mathcal{A}$, $\mathcal{E}^0(\mathcal{A})$ is inverse-closed in $\mathcal{A}$ (see Proposition 2.10). □

We now treat the inverse-closedness of approximation spaces.

THEOREM 3.26. *If $\mathcal{A}$ is a symmetric Banach algebra, and $(X_\lambda)_{\lambda\in\Lambda}$ is a compatible approximation scheme, then $\mathcal{E}_r^p(\mathcal{A})$ is inverse-closed in $\mathcal{A}$.*

PROOF. The norm inequality (3.33) is exactly the hypothesis for the application of Brandenburg's trick (Section 2.2.3), so (3.33) implies that

$$\rho_{\mathcal{E}_r^p}(a) = \rho_{\mathcal{A}}(a), \qquad \text{for all } a \in \mathcal{E}_r^p(\mathcal{A}).$$

Since $\mathcal{A}$ is symmetric, Proposition 2.10 shows that $\mathcal{E}_r^p(\mathcal{A})$ is inverse-closed in $\mathcal{A}$. □

REMARK. If $\mathcal{E}^0(\mathcal{A}) = \mathcal{A}$, then Proposition 3.26 follows from a result of Kissin and Shulman [**65**, Thm. 5] even without assuming that $\mathcal{A}$ is symmetric. However, in most of our examples $\mathcal{E}^0(\mathcal{A}) \neq \mathcal{A}$. In this situation the result of [**65**] implies only that $\mathcal{E}_r^p(\mathcal{A})$ is inverse-closed in $\mathcal{E}^0(\mathcal{A})$. For the inverse-closedness of $\mathcal{E}_0(\mathcal{A})$ in $\mathcal{A}$ we still need the symmetry assumption on $\mathcal{A}$. Our new proof has the advantage of being short and concise.

COROLLARY 3.27. *Let $\mathcal{A}$ be a symmetric matrix algebra.*

(1) *Then the band-dominated matrices in $\mathcal{A}$ form a closed and inverse-closed $*$-subalgebra of $\mathcal{A}$.*
(2) *Each approximation space $\mathcal{E}_r^p(\mathcal{A})$ is inverse-closed in $\mathcal{A}$.*

For the algebra of bounded operators on vector-valued ℓ^p-spaces special cases of (1) have been obtained in [**84, 86**].

Corollary 3.27 helps to simplify the proof of Jaffard's original theorem in [**61**]. Suppose we already know that $\mathcal{C}_{d+\epsilon}^\infty$ is inverse-closed in $\mathcal{B}(\ell^2(\mathbb{Z}^d))$ for $0 < \epsilon \leq \epsilon_0$. As $\mathcal{E}_s^\infty(\mathcal{C}_r^\infty) = \mathcal{C}_{r+s}^\infty$ by(3.9), Corollary 3.27 implies that $\mathcal{C}_s^\infty$, $s > d+\epsilon$, is inverse-closed in $\mathcal{C}_{d+\epsilon}^\infty$ and hence in $\mathcal{B}(\ell^2)$. Thus it suffices to prove Jaffard's result for the range $d < r < d + \epsilon_0$ for some small $\epsilon_0 > 0$.

REMARK. The same argument shows of course that $\mathcal{C}_{r+s}^p$ is inverse-closed in $\mathcal{C}_r^p$ for $s > 0$ and $r > d/p'$.

3.3.3. Approximation in Solid Matrix Algebras. In Section 3.3.2 we proved that the approximation space $\mathcal{E}_r^p(\mathcal{A})$ is inverse-closed in $\mathcal{A}$ for a polynomial weight and a general symmetric Banach algebra $\mathcal{A}$. If we restrict the class of algebras under consideration, we might expect to have more freedom in the choice of the weights. We can prove a theorem on the inverse-closedness of weighted approximation spaces of a solid matrix algebra $\mathcal{A}$ in $\mathcal{B}(\ell^2(\mathbb{Z}))$. We have to assume some mild regularity conditions on the weights and an invariance condition on the algebra giving a precise meaning to the idea of an *unweighted matrix algebra*. This condition will be discussed after the proof of the theorem.

THEOREM 3.28. *Let $\mathcal{A}$ be a solid matrix algebra with involution over $\mathbb{Z}$. Assume that*

$$\mathcal{C}_0^1 \hookrightarrow \mathcal{A} \tag{3.36}$$

for a value of p, $1 \leq p \leq \infty$. If v is a symmetric *algebra weight on $\ell^p(\mathbb{Z})$ (see Section 2.1.2) that satisfies the GRS condition, then $\mathcal{E}_v^p(\mathcal{A})$ is an inverse-closed subalgebra of $\mathcal{B}(\ell^2(\mathbb{Z}))$.*

PROOF. The idea of the proof is to show the inverse-closedness of $\mathcal{E}_v^p(\mathcal{A})$ in $\mathcal{C}_v^p$ using Brandenburg's trick (Section 2.2.3). For this to make sense we verify first that

$$\mathcal{E}_v^p(\mathcal{A}) \hookrightarrow \mathcal{C}_v^p. \tag{3.37}$$

Indeed, for $A \in \mathcal{E}_v^p(\mathcal{A})$ we have the chain of inequalities

$$\begin{aligned} \|A\|_{\mathcal{C}_v^p}^p &= \sum_{k\in\mathbb{Z}} \|\hat{A}(k)\|_{\mathcal{B}(\ell^2)}^p v(k)^p = \sum_{m=0}^{\infty} \sum_{|k|=m} \|\hat{A}(k)\|_{\mathcal{B}(\ell^2)}^p v(m)^p \\ &\le \sum_{m=0}^{\infty} \sum_{|k|=m} \|\hat{A}(k)\|_{\mathcal{A}}^p v(m)^p \le 2 \sum_{m=0}^{\infty} E_m^{\mathcal{A}}(A)^p v(m)^p, \end{aligned}$$

using $\|A\|_{\mathcal{B}(\ell^2)} \le \|A\|_{\mathcal{A}}$ and, for the last inequality, the solidity of $\mathcal{A}$.

In the next step we verify that the product AB of two elements of $\mathcal{E}_v^p(\mathcal{A})$ can be written as a Cauchy product that converges to AB in the norm of $\mathcal{A}$.

Claim. *If $A, B \in \mathcal{E}_v^p(\mathcal{A})$, then $A = \sum_{k\in\mathbb{Z}} \hat{A}(k)$ is absolutely convergent in $\mathcal{A}$, and AB can be written as an absolutely convergent Cauchy product*

$$AB = \sum_{k\in\mathbb{Z}} \hat{C}(k), \qquad \hat{C}(k) = \sum_{l\in\mathbb{Z}} \hat{A}(k-l)\hat{B}(l)\,.$$

The convergence is in the norm of $\mathcal{A}$.

We show that for $A \in \mathcal{E}_v^p(\mathcal{A})$ the sum $\sum_{k\in\mathbb{Z}} \|\hat{A}(k)\|_{\mathcal{A}}$ is convergent.

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \|\hat{A}(k)\|_{\mathcal{A}} &= \sum_{k\in\mathbb{Z}} \|\hat{A}(k)\|_{\mathcal{A}} \frac{v(k)}{v(k)} \\ &\le \|1/v\|_{\ell^{p'}(\mathbb{Z})} \Big(\sum_{l=0}^{\infty} (\|\hat{A}(l)\|_{\mathcal{A}}^p + \|\hat{A}(-l)\|_{\mathcal{A}}^p) v(l)^p \Big)^{1/p} \\ &\le 2\|1/v\|_{\ell^{p'}(\mathbb{Z})} \Big(\sum_{l=0}^{\infty} \big(E_l^{\mathcal{A}}(A) v(l)\big)^p \Big)^{1/p} \\ &\le 2\|1/v\|_{\ell^{p'}(\mathbb{Z})} \|A\|_{\mathcal{E}_v^p(\mathcal{A})}\,. \end{aligned}$$

We used again that $\|\hat{A}(k)\|_{\mathcal{A}} \le E_k^{\mathcal{A}}(A)$, if $\mathcal{A}$ is solid. The convergence of the Cauchy product is then a standard result from analysis. .

Now we can estimate the approximation error of the product AB.

$$\begin{aligned} E_n^{\mathcal{A}}(AB) &= \Big\| \sum_{|k|\ge n} \sum_{l\in\mathbb{Z}} \hat{A}(l)\hat{B}(k-l) \Big\|_{\mathcal{A}} \\ &\le \Big\| \sum_{|k|\ge n} \sum_{|l|<n} \hat{A}(l)\hat{B}(k-l) \Big\|_{\mathcal{A}} + \Big\| \sum_{|k|\ge n} \sum_{|l|>n} \hat{A}(l)\hat{B}(k-l) \Big\|_{\mathcal{A}} \\ &= I + II. \end{aligned} \tag{3.38}$$

For the further estimation of the first term observe that the inner sum is finite, and so we can exchange sums (using that $\sum_{k\in\mathbb{Z}} \|\hat{B}(k)\|_{\mathcal{A}} \le \infty$).

$$I = \Big\| \sum_{|l|<n} \sum_{|k|\ge n} \hat{A}(l)\hat{B}(k-l) \Big\|_{\mathcal{A}} \le \sum_{|l|<n} \Big(\|\hat{A}(l)\|_{\mathcal{A}} \Big\| \sum_{|k|\ge n} \hat{B}(k-l) \Big\|_{\mathcal{A}} \Big) \tag{3.39}$$

In the next step we show that

$$\| \sum_{|k|\ge n} \hat{B}(k-l) \|_{\mathcal{A}} \le E_{n-|l|}^{\mathcal{A}}(B) = \Big\| \sum_{|k|\ge n-|l|} \hat{B}(k) \Big\|_{\mathcal{A}} \quad \text{for} \quad |l| < n. \tag{3.40}$$

In fact, the left side of the inequality is $\|\sum_{|m+l|\ge n} \hat{B}(m)\|_{\mathcal{A}}$, and the right side equals $\|\sum_{|m|\ge n-|l|} \hat{B}(m)\|_{\mathcal{A}}$. As $|m+l| \ge n$ implies $|m| \ge n - |l|$, each term of the sum on left hand side of (3.40) appears also on the right hand side. The solidity of $\mathcal{A}$ then implies (3.40). So the first term in (3.38) can be estimated as

$$I \leq \sum_{|l|<n} \|\hat{A}(l)\|_{\mathcal{A}} E^{\mathcal{A}}_{n-|l|}(B). \tag{3.41}$$

This can be written in the form of a convolution on $\mathbb{Z}$. If we introduce the quantities

$$a_m = \sum_{|k|=m} \|\hat{A}(k)\|_{\mathcal{A}}, \qquad m \geq 0, \ \text{ and } \ a_m = 0, \ m < 0,$$
$$\gamma_m = E^{\mathcal{A}}_m(B), \qquad m \geq 0, \ \text{ and } \ \gamma_m = 0, \ m < 0$$

we can rewrite (3.41) as

$$I \leq \big(a * \gamma\big)(n-1). \tag{3.42}$$

For the estimation of the *second term* in (3.38) set $R_n(A) = \sum_{|k|\geq n} \hat{A}(k)$, so $\widehat{R_n(A)}(l) = \hat{A}(l)$ for $l \geq n$. Then

$$\begin{aligned} II = & \Big\| \sum_{|k|\geq n} \sum_{|l|>n} \widehat{R_n(A)}(l)\hat{B}(k-l) \Big\|_{\mathcal{A}} \leq \Big\| \sum_{k,l\in\mathbb{Z}} \widehat{R_n(A)}(l)\hat{B}(k-l) \Big\|_{\mathcal{A}} \\ = & \|R_n(A)B\|_{\mathcal{A}} \leq E^{\mathcal{A}}_n(A)\|B\|_{\mathcal{A}} \ . \end{aligned} \tag{3.43}$$

Putting the estimates for both terms together we obtain

$$\begin{aligned} E^{\mathcal{A}}_n(AB) &\leq \sum_{|l|<n} \|\hat{A}(l)\|_{\mathcal{A}} E^{\mathcal{A}}_{n-|l|}(B) + E^{\mathcal{A}}_n(A)\|B\|_{\mathcal{A}} \\ &\leq \sum_{|l|\leq n} \|\hat{A}(l)\|_{\mathcal{A}} E^{\mathcal{A}}_{n-|l|}(B) + E^{\mathcal{A}}_n(A)\|B\|_{\mathcal{A}} \\ &= (a*\gamma)(n-1) + E^{\mathcal{A}}_n(A)\|B\|_{\mathcal{A}}. \end{aligned} \tag{3.44}$$

Now we take $\ell^p_v(\mathbb{Z})$-norms on both sides of (3.44), using the assumption that v is an algebra weight for $\ell^p(\mathbb{Z})$. This leads to

$$\begin{aligned} \|AB\|_{\mathcal{E}^p_v(\mathcal{A})} &\leq \|a\|_{\ell^p_v(\mathbb{N}_0)} \|B\|_{\mathcal{E}^p_v(\mathcal{A})} + \|A\|_{\mathcal{E}^p_v(\mathcal{A})}\|B\|_{\mathcal{A}} \\ &\leq 2\Big(\sum_{k\in\mathbb{Z}} \|\hat{A}(k)\|^p_{\mathcal{A}} v(k)^p\Big)^{1/p} \|B\|_{\mathcal{E}^p_v(\mathcal{A})} + \|A\|_{\mathcal{E}^p_v(\mathcal{A})}\|B\|_{\mathcal{A}}. \end{aligned} \tag{3.45}$$

At this place we use the assumption that $\mathcal{C}^1_0 \hookrightarrow \mathcal{A}$. This implies that $\|\hat{A}(k)\|_{\mathcal{A}} \leq C\|\hat{A}(k)\|_{\mathcal{B}(\ell^2)}$. As $B \in \mathcal{E}^p_v(\mathcal{A}) \hookrightarrow \mathcal{C}^p_v \hookrightarrow \mathcal{C}^1_0 \hookrightarrow \mathcal{A}$ we can dominate $\|B\|_{\mathcal{A}}$ by $\|B\|_{\mathcal{C}^p_v}$. We finally obtain the estimate

$$\|AB\|_{\mathcal{E}^p_v(\mathcal{A})} \leq C\Big(\|A\|_{\mathcal{C}^p_v}\|B\|_{\mathcal{E}^p_v(\mathcal{A})} + \|A\|_{\mathcal{E}^p_v(\mathcal{A})}\|B\|_{\mathcal{C}^p_v}\Big) \tag{3.46}$$

Brandenburg's trick (Section 2.2.3) shows that $\mathcal{E}^p_v(\mathcal{A})$ is inverse-closed in $\mathcal{C}^p_v$. By Proposition 2.13 $\mathcal{C}^p_v$ is inverse-closed in $\mathcal{B}(\ell^2)$, if v is an algebra weight, so $\mathcal{E}^p_v(\mathcal{A})$ is inverse-closed in $\mathcal{B}(\ell^2)$. ☐

DISCUSSION. We want to shed some light on the condition (3.36). Actually it gives a precise meaning to the concept of an *unweighted* matrix algebra.

PROPOSITION 3.29. *Let $\mathcal{A}$ be a solid matrix algebra. The following are equivalent:*

(1) *For all $A \in \mathcal{C}^1_0$, $\|A\|_{\mathcal{A}} \leq C\|A\|_{\mathcal{C}^1_0}$.*
(2) *For all $A \in \mathcal{A}$, $\|\hat{A}(k)\|_{\mathcal{A}} \asymp \|\hat{A}(k)\|_{\mathcal{B}(\ell^2)} = \|A[k]\|_\infty$, uniformly in k.*
(3) *The translation operators on $\mathcal{A}$ are uniformly bounded: $\|T_k\|_{\mathcal{A}} \leq C$ for all $k \in \mathbb{Z}$.*

PROOF. (1) $\Rightarrow$ (2): If $A \in \mathcal{A}$ then $\hat{A}(k) \in \mathcal{A}$, and $\|\hat{A}(k)\|_{\mathcal{B}(\ell^2)} \leq \|\hat{A}(k)\|_{\mathcal{A}}$ by the definition of a matrix algebra. But $\|\hat{A}(k)\|_{\mathcal{B}(\ell^2)} = \|\hat{A}(k)\|_{\mathcal{C}_0^1}$, so $\hat{A}(k) \in \mathcal{C}_0^1$, and (2) follows.

(2) $\Rightarrow$ (1) is straightforward.

(1) $\Rightarrow$ (3): As $\|T_k\|_{\mathcal{C}_0^1} = 1$, we obtain $\|T_k\|_{\mathcal{A}} \leq C$, uniformly in k. On the other hand $\|T_k\|_{\mathcal{A}} \geq \|T_k\|_{\mathcal{B}(\ell^2)} = 1$.

(3) $\Rightarrow$ (1): Observe that the hypothesis implies that $T_k \in \mathcal{A}$. Let $A \in \mathcal{C}_0^1$. Then

$$\|A\|_{\mathcal{A}} \leq \sum_{k\in\mathbb{Z}} \|\hat{A}(k)\|_{\mathcal{A}} \leq \sum_{k\in\mathbb{Z}} \|A[k]\|_\infty \|T_k\|_{\mathcal{A}} \leq C\|A\|_{\mathcal{C}_0^1}\,,$$

the second inequality follows from the solidity of $\mathcal{A}$. □

Proposition 3.29 suggests to introduce the following concept.

DEFINITION 3.30. A matrix algebra $\mathcal{A}$ over $\mathbb{Z}$ is *unweighted*, if there is a constant $C > 0$ such that the translation operators are uniformly bounded: $\|T_k\|_{\mathcal{A}} \leq C$ for all $k \in \mathbb{Z}$.

REMARKS. (1) Theorem 3.28 gives an affirmative answer about algebra properties of weighted approximation spaces for *solid, unweighted* matrix algebras over $\mathbb{Z}$. Note that we need not assume that $\mathcal{A}$ is symmetric. We can consider the unweighted matrix algebras as a natural starting point to define off-diagonal decay by approximation. With this interpretation Proposition 3.29 gives a rather complete description of off-diagonal decay and inverse-closedness for the class of solid, unweighted matrix algebras over $\mathbb{Z}$.

(2) In particular, since $\mathcal{C}_0^1 \subseteq \mathcal{S}_0^1$, Theorem 3.28 is applicable, and the approximation space $\mathcal{E}_v^p(\mathcal{S}_0^1)$ for the Schur algebra $\mathcal{S}_0^1$ over $\mathbb{Z}$ is an inverse-closed subalgebra of $\mathcal{B}(\ell^2)$, if v is a symmetric algebra weight on $\ell^p(\mathbb{Z})$.

(3) Unfortunately the proof does not carry over to $\mathbb{Z}^d$ for $d > 1$. It is possible to mimic the method of proof, identifying matrices over $\mathbb{Z}^d$ with operator-valued matrices over $\mathbb{Z}$. We have to introduce an analogue of $\mathcal{C}_v^1$ as well, which can be shown to inverse-closed in $\mathcal{B}(\ell^2)$ by an easy adaption of the proof in [**11**]. However, we do not have a simple analogue of an unweighted algebra.

CHAPTER 4

Smoothness

Many results in the previous chapter have been verified only for solid matrix spaces. In this chapter we develop a concept of smoothness that is related to the off-diagonal decay of matrices.

We consider matrix algebras where the smoothness is defined by derivations (Section 4.1) or the operation of the translation group on $\mathbb{R}^d$. The smoothness spaces that can be constructed this way are well-known and include the operator-valued Triebel-Lizorkin spaces and the Besov spaces [**15, 27, 79, 102**], however, we treat only Besov and Bessel potential spaces besides the spaces $C^k(\mathcal{A})$ of k times differentiable elements.

It turns out that these smoothness spaces provide systematic constructions of inverse-closed algebras. In fact, this theory can be developed for general Banach algebras (sometimes symmetry has to be assumed), and in part also for Banach spaces with the action of the translation group or a set of commuting derivations. The standard literature (e.g. [**22, 24**]) is formulated for C^*-algebras and densely defined derivations, whereas we work mostly with Banach $*$-algebras and derivations without dense domain. We are therefore obliged to be especially careful before adopting a result for our purpose and provide the details of calculations.

The method of exposition is to develop concepts of smoothness and approximation in Banach algebras in analogy to the corresponding notions for real functions, and to apply them to matrix algebras afterwards.

4.1. Derivations

Off-diagonal decay of Convolution Operators. Let us begin with an example that sheds some light on the relation between smoothness and off-diagonal decay of matrices. Recall that the matrices in $\mathcal{C}_v^1$ defined in Section 2.3 are also known as *convolution dominated operators*: $A \in \mathcal{C}_v^1$ if and only if there is a $h \in \ell_v^1(\mathbb{Z}^d)$ such that $|Ax(k)| \leq h * |x|(k)$, where $|x|$ denotes the vector with components $|x(k)|$.

The off-diagonal decay of convolution operators C_f is closely related to the smoothness of their symbols f.

Let $f \in \ell^1(\mathbb{Z}^d)$ and $C_f \in \mathcal{B}(\ell^2)$ the corresponding convolution operator. Conjugation with the Fourier transform operator induces an isometric isomorphism Γ between $\mathcal{B}(\ell^2(\mathbb{Z}^d))$ and $\mathcal{B}(L^2(\mathbb{T}^d))$, given by $\Gamma(A) = \mathcal{F} A \mathcal{F}^*$ for $A \in \mathcal{B}(\ell^2(\mathbb{Z}^d))$. This isomorphism maps the convolution operators C_f into multiplication operators $M_{\mathcal{F}f} \colon g \mapsto (\mathcal{F}f)g$ for all $g \in L^2(\mathbb{T}^d)$, see the commutative diagram below.

$$\begin{array}{ccc} f \in \ell^1(\mathbb{Z}^d) & \xrightarrow{\;C_f\;} & \mathcal{B}(\ell^2(\mathbb{Z}^d)) \\ \downarrow{\scriptstyle \mathcal{F}} & & \downarrow{\scriptstyle \Gamma} \\ \mathcal{F}f \in L^\infty(\mathbb{T}^d) & \xrightarrow{\;M_{\mathcal{F}f}\;} & \mathcal{B}(L^2(\mathbb{T}^d)) \end{array}$$

By standard results of Fourier analysis the decay of the matrix C_f is related to the smoothness of $\mathcal{F}f$, e.g., $\mathcal{F}f \in C^k(\mathbb{T}^d)$ implies $\|\widehat{C_f}(m)\|_{\mathcal{B}(\ell^2)} = \mathcal{O}(|m|^{-k})$ for all $m \in \mathbb{Z}^d$.

We observe that the matrix $M_{\partial_j(\mathcal{F}f)}$ corresponds to a matrix $\delta_j C_f$ with $\widehat{\delta_j C_f}(k) = 2\pi i k_j \hat{C}_f(k)$.

Generalizing this observation, we define the formal operation

$$\delta_j \colon A \mapsto 2\pi i \sum_{k\in\mathbb{Z}^d} k_j \hat{A}(k)$$

for matrices over $\mathbb{Z}^d$. It turns out that δ_j is a *derivation*.

4.1.1. Definition and Basic Properties.

DEFINITION 4.1. A *derivation* δ on a Banach algebra $\mathcal{A}$ is a closed linear mapping $\delta\colon \mathcal{D} \to \mathcal{A}$, where the *domain* $\mathcal{D} = \mathcal{D}(\delta) = \mathcal{D}(\mathcal{A}) = \mathcal{D}(\delta, \mathcal{A})$ is a subspace of $\mathcal{A}$, and δ fulfills the Leibniz rule

$$\delta(ab) = a\delta(b) + \delta(a)b \qquad \text{for all } a, b \in \mathcal{D}. \tag{4.1}$$

If $\mathcal{A}$ possesses an involution, we assume that the derivation and the domain are symmetric, i.e., $\mathcal{D} = \mathcal{D}^*$ and $\delta(a^*) = \delta(a)^*$ for all $a \in \mathcal{D}$. The domain is normed with the graph norm $\|a\|_{\mathcal{D}} = \|a\|_{\mathcal{A}} + \|\delta(a)\|_{\mathcal{A}}$.

REMARK. The domain $\mathcal{D}(\mathcal{A})$ is not assumed to be closed or dense in $\mathcal{A}$.

Equation (4.1) implies that $\mathcal{D}(\mathcal{A})$ is a (not necessarily unital) Banach algebra, and the canonical mapping $\mathcal{D}(\mathcal{A}) \to \mathcal{A}$ is a continuous embedding.

EXAMPLE 4.2 (Derivations on L^∞). The classical derivative $\frac{d}{dx} : f \mapsto f'$ is a closed, symmetric derivation on the von Neumann-algebra $L^\infty(\mathbb{R})$. The domain of $\frac{d}{dx}$ in $L^\infty(\mathbb{R})$ consists of all Lipschitz functions with essentially bounded derivative. Clearly, $\mathcal{D}(\delta, L^\infty)$ is not dense in L^∞.

PROPOSITION 4.3 (Derivations on Matrix Algebras). *Let $\mathcal{A}$ be a matrix algebra over $\mathbb{Z}$. Define the diagonal matrix X by $X(k,k) = 2\pi i k$. Then the formal commutator*

$$\delta_X(A) = [X, A] = XA - AX \tag{4.2}$$

with domain $\mathcal{D}(\mathcal{A}) = \{A \in \mathcal{A} : \delta_X(A) \in \mathcal{A}\}$ has the entries

$$[X, A](k,l) = 2\pi i(k-l)A(k,l)$$

for $k, l \in \mathbb{Z}$, and δ_X defines a closed, symmetric derivation on $\mathcal{A}$.

PROOF. The Leibniz rule follows from the formal computation

$$A\delta_X(B) + \delta_X(A)B = A(XB - BX) + (XA - AX)B = XAB - ABX = \delta_X(AB).$$

Symmetry can be verified directly as well. For the closedness let us choose a sequence $A_n \to A$ in $\mathcal{A}$, and $\delta_X A_n \to B$ in $\mathcal{A}$. As $\mathcal{A} \hookrightarrow \mathcal{B}(\ell^2)$ the selection of coefficients is continuous, and so $\delta_X A_n(k,l) = 2\pi i\, A_n(k,l)(k-l) \to 2\pi i\, A(k,l)(k-l) = B(k,l)$ for all $k, l \in \mathbb{Z}$. This was to be proved. □

The operator δ_X is not necessarily densely defined. This will follow most easily from results proved later (see Section 4.2).

The derivation δ_X is closely related to matrix weights as defined in Section 2.3.3, at least in *solid* matrix algebras.

PROPOSITION 4.4. *If $\mathcal{A}$ is a solid matrix algebra over $\mathbb{Z}$, then $\mathcal{D}(\delta, \mathcal{A})$ is the weighted matrix algebra $\mathcal{A}_{v_1}$, where $v_1(k) = 1 + |k|$ is the polynomial standard weight, and the norms $\| \ \|_{\mathcal{D}(\mathcal{A})}$ and $\| \ \|_{\mathcal{A}_{v_1}}$ are equivalent.*

Proof. Set $\tilde{A}(k,l) = A(k,l)v_1(k-l) = A(k,l)(1+|k-l|)$. Since the norm of $A \in \mathcal{A}$ depends only on the absolute values of the entries of A, we obtain that

$$\|A\|_{\mathcal{A}_{v_1}} = \|\tilde{A}\|_{\mathcal{A}} \le \|A\|_{\mathcal{A}} + \|\,[X,A]\|_{\mathcal{A}} = \|A\|_{\mathcal{D}(\delta)} \le 2\cdot 2\pi \|A\|_{v_1}\,,$$

as claimed. □

Remark. For certain classes of non-solid matrix algebras a similar result can be obtained (see Section 4.4.4).

If δ is a densely defined $*$-derivation of a C^*-algebra $\mathcal{A}$, then by a result in [**23**] $\mathbf{1} \in \mathcal{D}(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$ is inverse-closed in $\mathcal{A}$. In [**64**] this result was extended to densely defined derivations on arbitrary Banach algebras without involution. We need an extension for derivations that are not necessarily densely defined.

Theorem 4.5. *Let $\mathcal{A}$ be a symmetric Banach algebra, and δ a symmetric derivation on $\mathcal{A}$. If $\mathbf{1} \in \mathcal{D}(\mathcal{A})$, then $\mathcal{D}(\mathcal{A})$ is inverse-closed in $\mathcal{A}$ and $\mathcal{D}(\mathcal{A})$ is a symmetric Banach algebra, and the quotient rule*

$$\delta(a^{-1}) = -a^{-1}\delta(a)a^{-1}$$

is valid, and yields the explicit norm estimate

$$\|a^{-1}\|_{\mathcal{D}(\delta)} \le \|a^{-1}\|^2_{\mathcal{A}}\|a\|_{\mathcal{D}(\delta)}. \tag{4.3}$$

Proof. The proof in [**23**] uses functional calculus and could be adapted to the setting of the theorem. We prefer a short conceptual argument based on Hulanicki's Lemma (Proposition 2.10). We show that $\rho_{\mathcal{D}(\mathcal{A})}(a) = \rho_{\mathcal{A}}(a)$ for any $a \in \mathcal{D}(\mathcal{A})$. The inequality

$$\|\delta(a^n)\|_{\mathcal{A}} \le n\|a\|^{n-1}_{\mathcal{A}}\|\delta(a)\|_{\mathcal{A}} \tag{4.4}$$

is established by induction. For $n = 1$ there is nothing to prove. Having established the inequality for $n-1$, we use the Leibniz rule for $\delta(a^n)$.

$$\delta(a^n) = \delta(a)a^{n-1} + a\delta(a^{n-1}).$$

Taking norms on both sides and using the induction hypotheses gives (4.4). Now we can estimate the norm of a^n by

$$\|a^n\|_{\mathcal{D}(\mathcal{A})} = \|a^n\|_{\mathcal{A}} + \|\delta(a^n)\|_{\mathcal{A}} \le \|a\|^n_{\mathcal{A}} + n\|a\|^{n-1}_{\mathcal{A}}\|\delta(a)\|_{\mathcal{A}}.$$

If we take nth roots on both sides and let n go to infinity, we obtain $\rho_{\mathcal{D}(\mathcal{A})}(a) \le \|a\|_{\mathcal{A}}$, and consequently $\rho_{\mathcal{D}(\mathcal{A})}(a) \le \rho_{\mathcal{A}}(a)$. The reverse inequality $\rho_{\mathcal{A}}(a) \le \rho_{\mathcal{D}(\mathcal{A})}(a)$ is always true for Banach algebras, since $\mathcal{D}(\mathcal{A}) \subseteq \mathcal{A}$, so Proposition 2.10 implies that $\mathcal{D}(\mathcal{A})$ is inverse-closed in $\mathcal{A}$. Consequently $\sigma_{\mathcal{D}(\mathcal{A})}(a^*a) = \sigma_{\mathcal{A}}(a^*a) \subseteq [0,\infty)$ for all $a \in \mathcal{D}(\mathcal{A})$, and thus $\mathcal{D}(\mathcal{A})$ is a symmetric Banach algebra.

Thus, if $a \in \mathcal{D}(\mathcal{A})$ and $a^{-1} \in \mathcal{A}$, then $a^{-1} \in \mathcal{D}(\mathcal{A})$ and so $\delta(a^{-1})$ is well-defined in $\mathcal{A}$. Therefore the quotient rule and the norm inequality follow from the Leibniz rule $0 = \delta(\mathbf{1}) = \delta(aa^{-1}) = \delta(a)a^{-1} + a\delta(a^{-1})$. □

Remarks. Theorem 4.5 is remarkable because it yields an explicit norm control of the inverse in the subalgebra $\mathcal{D}(\mathcal{A})$. Norms that satisfy (4.3) are called strong Leibniz norms in [**88**]. See, on the other hand, [**77**] for typical no-go results.

4.1.2. Commuting Derivations. The formulation of inverse-closedness results for matrices over $\mathbb{Z}^d$, and the definition of higher orders of smoothness require derivations for each "dimension" of the index set $\mathbb{Z}^d$.

Let $\{\delta_1, \cdots, \delta_d\}$ be a set of commuting derivations on the Banach algebra $\mathcal{A}$. Since products of unbounded operators and their domains are a subtle and rather technical subject with many pathologies, we will make the following assumptions and thus avoid many technicalities.

The domain of a finite product $\delta_{r_1}\delta_{r_2}\dots\delta_{r_n}$, $1 \le r_j \le d$ is defined by induction as

$$\mathcal{D}(\delta_{r_1}\delta_{r_2}\dots\delta_{r_n}) = \mathcal{D}(\delta_{r_1}, \mathcal{D}(\delta_{r_2}\dots\delta_{r_n}))\,.$$

We will assume throughout that the operator $\delta_{r_1}\delta_{r_2}\dots\delta_{r_n}$ and its domain $\mathcal{D}(\delta_{r_1}\delta_{r_2}\dots\delta_{r_n})$ are independent of the order of the factors δ_{r_j}.

Then for every multi-index α the operator $\delta^\alpha = \prod_{1\le k\le d}\delta_k^{\alpha_k}$ and its domain $\mathcal{D}(\delta^\alpha)$ are well defined. In analogy to $C^k(\mathbb{R}^d)$ we equip $\mathcal{D}(\delta^\alpha)$ with the norm

$$\|a\|_{\mathcal{D}(\delta^\alpha)} = \sum_{\beta\le\alpha}\|\delta^\beta(a)\|_{\mathcal{A}}\,.$$

Since δ_j is assumed to be a closed operator on $\mathcal{A}$, it follows that δ_j is a closed operator on $\mathcal{D}(\delta^\alpha)$.

The operator δ^α satisfies the general Leibniz rule

$$\delta^\alpha(ab) = \sum_{\beta\le\alpha}\binom{\alpha}{\beta}\delta^\beta(a)\delta^{\alpha-\beta}(b)\,. \tag{4.5}$$

Definition 4.6. Let $\mathcal{A}$ be a Banach algebra and k a nonnegative integer. The *derived space of order k* is

$$\mathcal{A}^{(k)} = \bigcap_{|\alpha|\le k}\mathcal{D}(\delta^\alpha), \qquad \text{and} \qquad \mathcal{A}^{(\infty)} = \bigcap_{k=0}^{\infty}\mathcal{A}^{(k)}.$$

We summarize the results on commuting derivations.

Lemma 4.7. *Let $\{\delta_k : 1 \le k \le d\}$ be a set of commuting derivations on the Banach algebra $\mathcal{A}$.*

(i) *Then $\mathcal{D}(\delta^\alpha)$ is a (not necessarily unital) subalgebra of $\mathcal{A}$ for every $\alpha \in \mathbb{N}_0^d$.*
(ii) *Let $\mathcal{R} \subseteq \mathbb{N}_0^d$ be an arbitrary finite index set and set*

$$\mathcal{D}_{\mathcal{R}}(\delta) = \bigcap_{\alpha\in\mathcal{R}}\mathcal{D}(\delta^\alpha)\,.$$

Then $\mathcal{D}_{\mathcal{R}}(\delta)$ is a Banach-subalgebra of $\mathcal{A}$ with the norm $\|a\|_{\mathcal{D}_{\mathcal{R}}(\delta)} = \sum_{\alpha\in\mathcal{R}}\|a\|_{\mathcal{D}(\delta^\alpha)}$. In particular $\mathcal{A}^{(k)}$ is a Banach-subalgebra of $\mathcal{A}$.

Proof. If $a, b \in \mathcal{D}(\delta^\alpha)$, i.e., $\delta^\beta(a), \delta^\beta(b) \in \mathcal{A}$ for $\beta \le \alpha$, then clearly $ab \in \mathcal{D}(\delta^\alpha)$ and the norm inequality $\|ab\|_{\mathcal{D}(\delta^\alpha)} \le C\|a\|_{\mathcal{D}(\delta^\alpha)}\|b\|_{\mathcal{D}(\delta^\alpha)}$ follows after taking norms in (4.5). Since the finite intersection of Banach algebras is a Banach algebra, $\mathcal{A}^{(k)}$ and $\mathcal{D}_{\mathcal{R}}(\delta)$ are Banach algebras. □

Proposition 4.8. *Assume that $\mathcal{A}$ is a symmetric Banach algebra with a set of commuting symmetric derivations $\{\delta_k : 1 \le k \le d\}$ satisfying $\mathbf{1} \in \mathcal{D}(\delta_k), 1 \le k \le d$. Then $\mathcal{D}(\delta^\alpha)$ is inverse-closed in $\mathcal{A}$. Furthermore, the Banach algebra $\mathcal{D}_{\mathcal{R}}(\delta)$ is inverse-closed in $\mathcal{A}$, and $\mathcal{A}^{(\infty)}$ is a Fréchet algebra that is inverse-closed in $\mathcal{A}$.*

Proof. Let $\delta^\alpha = \delta_{r_n}\cdots\delta_{r_1}$ with $n = |\alpha|$ and $1 \le r_j \le d$ for all j. By Theorem 4.5, $\mathcal{D}(\delta_1, \mathcal{A})$ is a symmetric Banach algebra and inverse-closed in $\mathcal{A}$. Now we argue by induction and assume that $\mathcal{D}(\delta_{r_j}\dots\delta_{r_1})$ is symmetric and inverse-closed in $\mathcal{A}$. Since by definition

$$\mathcal{D}(\delta_{r_{j+1}}\dots\delta_1) = \mathcal{D}(\delta_{r_{j+1}}, \mathcal{D}(\delta_{r_j}\dots\delta_{r_{r_1}}))$$

and $\delta_{r_{j+1}}$ is a closed derivation on the symmetric Banach algebra $\mathcal{D}(\delta_{r_j}\cdots\delta_{r_1})$, Theorem 4.5 asserts that $\mathcal{D}(\delta_{r_{j+1}}\dots\delta_{r_1})$ is symmetric and inverse-closed in $\mathcal{D}(\delta_{r_j}\dots\delta_{r_1})$ and thus inverse-closed in $\mathcal{A}$ by transitivity. We repeat this argument n times and find that $\mathcal{D}(\delta^\alpha) = \mathcal{D}(\delta_{r_n}\dots\delta_{r_1})$ is symmetric and inverse-closed in $\mathcal{A}$.

Finally, the finite or infinite intersection of inverse-closed subalgebras of $\mathcal{A}$ is again inverse-closed in $\mathcal{A}$. Specifically, if $a \in \mathcal{D}_{\mathcal{R}}(\delta) = \bigcap_{\alpha\in\mathcal{R}} \mathcal{D}(\delta^\alpha)$ and a is invertible, then the argument above shows that $a^{-1} \in \mathcal{D}(\delta^\alpha, \mathcal{A})$ for each $\alpha \in \mathcal{R}$, whence $a^{-1} \in \mathcal{D}_{\mathcal{R}}(\delta)$. The argument for $\mathcal{A}^{(\infty)}$ is the same. □

REMARK. The inverse-closedness of $\mathcal{A}^{(\infty)}$ in $\mathcal{A}$ is implicit in [**12**].

EXAMPLE 4.9 (Matrix algebras over $\mathbb{Z}^d$). If $\mathcal{A}$ is a matrix algebra over $\mathbb{Z}^d$, then we define the diagonal matrices X_j by $X_j(k,k) = 2\pi i k_j$ and the derivations $\delta_j(A)(k,l) = [X_j, A](k,l) = 2\pi i(k_j - l_j)A(k,l), 1 \le j \le d$. These derivations are symmetric and commute with each other, and $\mathbf{1} \in \mathcal{D}(\delta_j)$ for all j. An application of Proposition 4.8 gives that all spaces $\mathcal{D}_{\mathcal{R}}(\delta)$ are inverse-closed subalgebras of $\mathcal{A}$.

If $\mathcal{A}$ is solid, there is an immediate generalization of Proposition 4.4 to matrix algebras over the index set $\mathbb{Z}^d$.

PROPOSITION 4.10. *If $\mathcal{A}$ is a solid matrix algebra over $\mathbb{Z}^d$, then $\mathcal{A}^{(m)} = \mathcal{A}_{v_m}$. In particular, $\mathcal{A}_{v_m}$ is an inverse-closed subalgebra of $\mathcal{A}$.*

PROOF. The identity $\mathcal{A}^{(m)} = \mathcal{A}_{v_m}$ is proved as in Proposition 4.4. The inverse-closedness follows from Proposition 4.8. □

We apply Proposition 4.10 to the algebras $\mathcal{C}^p_v$ and $\mathcal{S}^1_v$.

COROLLARY 4.11. *Let $\mathcal{C}^p_v$ be defined as in Section 2.3.1. If v is an algebra weight, then for $k \in \mathbb{N}$ the algebra $\mathcal{C}^p_{v_k v}$ is inverse-closed in $\mathcal{C}_v$. Likewise, the weighted Schur algebra $\mathcal{S}^p_{v_k v}$ is inverse-closed in $\mathcal{S}^p_v$.*

The value of Proposition 4.8 lies in its potential to treat anisotropic decay conditions. As an example we state the following anisotropic generalization of Jaffard's theorem.

PROPOSITION 4.12. *Let A be a matrix over $\mathbb{Z}^d$, $r > d$, and $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$. If A is invertible on $\ell^2(\mathbb{Z}^d)$ and satisfies the anisotropic off-diagonal decay condition*

$$|A(k,l)| \le C(1+|k-l|)^{-r} \prod_{j=1}^d (1+|k_j - l_j|)^{-\alpha_j}, \qquad k,l \in \mathbb{Z}^d, \tag{4.6}$$

then the entries of the inverse matrix A^{-1} satisfy an estimate of the same type

$$|(A^{-1}(k,l)| \le C'(1+|k-l|)^{-r} \prod_{j=1}^d (1+|k_j - l_j|)^{-\alpha_j}, \qquad k,l \in \mathbb{Z}^d.$$

PROOF. The off-diagonal decay condition is equivalent to saying that the matrix $\tilde{A}$ with entries $\tilde{A}(k,l) = \prod_{j=1}^d (k_j - l_j)^{\alpha_j} A(k,l)$ is in the Jaffard algebra $\mathcal{J}_r$. But $\tilde{A}$ is just a multiple of $\prod_{j=1}^d \delta_j^{\alpha_j} A = \delta^\alpha A$, where $\delta_j(A)$ is defined in Example 4.9. Since $\mathcal{D}(\delta^\alpha, \mathcal{J}_r)$ is inverse-closed in $\mathcal{J}_r$ by Proposition 4.8 and $\mathcal{J}_r$ is inverse-closed in $\mathcal{B}(\ell^2)$, A^{-1} is again in $\mathcal{D}(\delta^\alpha, \mathcal{J}_r)$, which is nothing but the off-diagonal decay stated. □

REMARK. Proposition 4.12 could be also obtained from the conditions of [**99**, Thm 4.1].

4.2. Automorphism Groups and Continuity

Our next step is to treat the algebras $\mathcal{A}_{v_r}$ with non-integer parameter r in analogy to spaces with fractional smoothness. Two natural approaches are either fractional powers of the generators or automorphism groups and the associated Hölder-Zygmund continuity. In the next two sections we concentrate on the latter approach and introduce a new structure, namely automorphism groups. This choice is also motivated by the failure to distinguish between the spaces $\mathcal{D}(\frac{d}{dx}, L^\infty(\mathbb{T})) = \{f \in \mathrm{Lip}(\mathbb{T}) : f' \in L^\infty(\mathbb{T})\}$ and $\mathcal{D}(\frac{d}{dx}, C(\mathbb{T})) = C^1(\mathbb{T})$ by means of derivations alone. To explain this difference, we need to consider derivations that are generators of groups of automorphisms.

Parts of the theory of smoothness can be developed for automorphism groups acting on a Banach space. Though we are ultimately interested in the construction of Banach algebras that are inverse-closed in a given Banach algebra, we develop concepts in a natural framework.

Definition 4.13. An *automorphism group*, more precisely a d-parameter automorphism group acting on the Banach space $\mathcal{X}$ is a set of automorphisms $\Psi = \{\psi_t\}_{t\in\mathbb{R}^d}$ of $\mathcal{X}$ with the group properties

$$\psi_s\psi_t = \psi_{s+t} \quad \text{for all} \quad s,t \in \mathbb{R}^d. \tag{4.7}$$

In addition, we always assume that Ψ is a *uniformly bounded* automorphism group, that is,

$$M_\Psi = \sup_{t\in\mathbb{R}^d} \|\psi_t\|_{\mathcal{X}\to\mathcal{X}} < \infty\,.$$

If $\mathcal{A}$ is a Banach algebra, we assume that Ψ consists of Banach algebra automorphisms of $\mathcal{A}$, that is,

$$\psi_t(ab) = \psi_t(a)\psi_t(b)$$

for all $a, b \in \mathcal{A}$, and all $t \in \mathbb{R}^d$. If $\mathcal{A}$ is a $*$-algebra, we assume that Ψ consists of $*$-automorphisms.

This is all we need, but clearly the abstract theory works for much more general group actions [**57, 101**].

Definition 4.14. An element x of the Banach space $\mathcal{X}$ is *continuous*, if

$$\|\psi_t(x) - x\|_\mathcal{X} \to 0 \text{ for } t \to 0. \tag{4.8}$$

The set of continuous elements of $\mathcal{X}$ is denoted by $C(\mathcal{X})$.

We will use the following concept.

Definition 4.15. Let $P \in \mathbb{R}^d_+$. We call the action of Ψ periodic with period $P \geq 0$ or *P-periodic*, if $\psi_{t+P} = \psi_t$ for all $t \in \mathbb{R}^d$.

If we speak of a periodic group action we usually mean a 1-periodic group action ($P_k = 1$ for all $k = 1,\dots,d$). It is easy to check that for each P-periodic group action Ψ we can define a new automorphism group $\overline{\Psi}$ by $\overline{\psi}_{(t_1,\dots,t_d)} = \psi_{(t_1/P_1,\dots,t_d/P_d)}$. The reader may verify that the smoothness spaces we will define in the sequel do not depend on this normalization.

Definition 4.16. If $\mathcal{X}$ is a Banach space, and Ψ an automorphism group acting on $\mathcal{A}$, the *generator* δ_t is defined for each $t \in \mathbb{R}^d \setminus \{0\}$ as

$$\delta_t(x) = \lim_{h\to 0} \frac{\psi_{ht}(x) - x}{h} \tag{4.9}$$

The domain of δ_t is the set of all $x \in \mathcal{X}$ for which this limit exists. The *canonical generators* of Ψ are δ_{e_k} and Ψ is called the *automorphism group generated by* $(\delta_{e_k})_{1\leq k\leq d}$.

If $\mathcal{A}$ is a Banach algebra, each generator δ_t, $t \in \mathbb{R}^d \setminus \{0\}$, is a closed derivation. If $\mathcal{A}$ is a Banach $*$-algebra, then δ_t is a $*$-derivation [**24**].

EXAMPLE 4.17. The classical example of an automorphism group acting on a Banach algebra is the translation group $\{T_x : x \in \mathbb{R}^d\}$, whose canonical generators are the partial derivatives $\partial_k, 1 \le k \le d$. If the translation group acts on $\mathcal{A} = L^\infty(\mathbb{R}^d)$, the continuous elements are the functions in $C(L^\infty(\mathbb{R}^d)) = C_u(\mathbb{R}^d)$, where $C_u(\mathbb{R}^d)$ denotes the space of bounded uniformly continuous functions on $\mathbb{R}^d$.

REMARKS.

(1) In a C^*-algebra all automorphisms are isometries. This is no longer true for symmetric algebras. Let $\widetilde{C}$ denote the C^1 functions on the line with norm $\|f\|_\sim := \|f\|_\infty + \|f'v_1\|_\infty < \infty$. Then $\widetilde{C}$ is a a symmetric algebra, and the natural action of the translation group on $\widetilde{C}$ is not uniformly bounded.

(2) In the theory of operator algebras it is usually assumed that Ψ is strongly continuous on all of $\mathcal{A}$, i.e. $\mathcal{A} = C(\mathcal{A})$. This is no longer true for most matrix algebras, and $C(\mathcal{A})$ is an interesting space in its own right.

DEFINITION 4.18 (Homogeneous matrix algebras). Let $M_t, t \in \mathbb{R}^d$, be the modulation operator $M_t x(k) = e^{2\pi i k\cdot t}x(k)$, $k \in \mathbb{Z}^d$. Then

$$\chi_t(A) = M_t A M_{-t},\ \chi_t(A)(k,l) = e^{2\pi i (k-l)\cdot t}A(k,l) \quad k,l \in \mathbb{Z}^d\,,$$

defines a group action on matrices. A matrix algebra $\mathcal{A}$ is called *homogeneous* (cf. [**33, 34**], see also [**90**, Chapter 9]), if the automorphism group $\chi = \{\chi_t\}_{t\in\mathbb{R}^d}$ is uniformly bounded on $\mathcal{A}$.

The canonical generators for the automorphism group χ are the derivations $\delta_k(A) = [X_k, A]$, $k = 1,\dots,d$, defined in Example 4.9. This automorphism group is uniformly bounded on every solid matrix algebra and on $\mathcal{B}(\ell^2)$.

The following proposition states the Banach algebra properties of $C(\mathcal{A})$.

PROPOSITION 4.19. *If $\mathcal{A}$ is a Banach algebra and Ψ a uniformly bounded automorphism group acting on $\mathcal{A}$, then $C(\mathcal{A})$ is a closed and inverse-closed subalgebra of $\mathcal{A}$. If $\mathcal{A}$ is a $*$-algebra, so is $C(\mathcal{A})$.*

PROOF. First we verify that $C(\mathcal{A})$ is an algebra. Let $a, b \in C(\mathcal{A})$. Then

$$\|\psi_t(ab) - ab\|_\mathcal{A} \le \|\psi_t(a)\|_\mathcal{A}\|\psi_t(b) - b\|_\mathcal{A} + \|\psi_t(a) - a\|_\mathcal{A}\|b\|_\mathcal{A}. \tag{4.10}$$

This expression tends to zero for $t \to 0$ as $\|\psi_t\|_{\mathcal{A}\to\mathcal{A}} \le M_\Psi$, so $ab \in C(\mathcal{A})$. For the completeness of $C(\mathcal{A})$ let $a_n \in C(\mathcal{A})$ for all n, and $a_n \to a$ in $\mathcal{A}$. Then

$$\|\psi_t(a) - a\|_\mathcal{A} \le \|\psi_t(a - a_n)\|_\mathcal{A} + \|\psi_t(a_n) - a_n\|_\mathcal{A} + \|a_n - a\|_\mathcal{A}.$$

The first and the third term can be made arbitrarily small by choosing n sufficiently large. Since $a_n \in C(\mathcal{A})$, the second term can be made small. Thus $a \in C(\mathcal{A})$.
To show the inverse-closedness, let $a \in C(\mathcal{A})$ and assume that a is invertible in $\mathcal{A}$. Then (as in the proof of the quotient rule) the algebraic identity

$$\psi_t(a^{-1}) - a^{-1} = \psi_t(a^{-1})(a - \psi_t(a))a^{-1} \tag{4.11}$$

yields that

$$\|\psi_t(a^{-1}) - a^{-1}\|_\mathcal{A} \le M_\Psi\, \|a^{-1}\|^2_\mathcal{A}\, \|a - \psi_t(a)\|_\mathcal{A} \to 0 \qquad \text{for } t \to 0,$$

and thus $a^{-1} \in C(\mathcal{A})$. □

Generators and Smoothness. Before defining the spaces $C^k(\mathcal{X})$, some technical preparations are needed, because generators commute only under some additional conditions (similar to partial derivatives).

PROPOSITION 4.20 ([**27, 58**]).

(1) *If δ is the generator of a one-parameter group, then the domain $\mathcal{D}(\delta)$ is dense in $C(\mathcal{X})$.*
(2) *If Ψ is a d-parameter automorphism group acting on $\mathcal{X}$, then Ψ and the generators commute, whenever defined, i.e.,*

$$\psi_s(\delta_t(x)) = \delta_t(\psi_s(x)) \text{ for } x \in \mathcal{D}(\delta_t, \mathcal{X}), s, t \in \mathbb{R}^d.$$

(3) *Derived spaces consist of continuous elements:*

$$\mathcal{X}^{(1)} = \bigcap_{k=1}^{d} \mathcal{D}(\delta_k, \mathcal{X}) \subseteq C(\mathcal{X}).$$

(4) *If $\mathcal{D}_{s,t} = \mathcal{D}(\delta_s, C(\mathcal{X})) \cap \mathcal{D}(\delta_t, C(\mathcal{X})) \cap \mathcal{D}(\delta_s\delta_t, C(\mathcal{X}))$, then for $s, t \neq 0$*

$$\mathcal{D}_{s,t} = \mathcal{D}_{t,s}, \text{ and } \delta_s\delta_t = \delta_t\delta_s \text{ on } \mathcal{D}_{s,t}.$$

DEFINITION 4.21. For $k \in \mathbb{N}_0$ the spaces $C^k(\mathcal{X})$ and $C^\infty(\mathcal{X})$ are defined as

$$C^k(\mathcal{X}) = \bigcap_{|\alpha|\leq k} \mathcal{D}(\delta^\alpha, C(\mathcal{X})) \quad \text{and} \quad C^\infty(\mathcal{X}) = \bigcap_{\alpha \geq 0} \mathcal{D}(\delta^\alpha, C(\mathcal{X}))\,.$$

The norm on $C^k(\mathcal{X})$ is $\|x\|_{C^k(\mathcal{X})} = \sum_{|\alpha|\leq k} \frac{1}{\alpha!}\|\delta^\alpha x\|_{\mathcal{X}}$. For $k = 0$ we set $C^0(\mathcal{X}) = C(\mathcal{X})$.

Proposition 4.20 shows that this definition does not depend on the ordering of the standard basis.

It is an important fact that the smoothness spaces consist of the continuous elements of the derived spaces, i.e.,

$$C(\mathcal{X}^{(k)}) = C^k(\mathcal{X}). \tag{4.12}$$

In fact, $x \in C(\mathcal{X}^{(k)})$ means that the norm $\sum_{|\beta|\leq k}\|\delta^\beta x\|_{\mathcal{X}}$ is finite, and, moreover, ψ_t acts continuously on x in this norm, so $\psi_t(x) \to x$ in $\mathcal{X}$, and $\delta^\beta\psi_t(x) = \psi_t\delta^\beta(x) \to \delta^\beta(x)$ in $\mathcal{X}$ for $t \to 0$ and all indices β with $|\beta| \leq k$. But this describes precisely the membership of x in $C^k(\mathcal{X})$.

Algebra properties and inverse-closedness of the spaces $C^k(\mathcal{A})$, $\mathcal{A}$ a Banach algebra, are summarized in the following proposition. Note that in contrast to Theorem 4.5 we do not need any further assumptions on $\mathcal{A}$.

PROPOSITION 4.22. *If $\mathcal{A}$ is a Banach algebra, each space $C^k(\mathcal{A})$ is an inverse-closed Banach subalgebra of $\mathcal{A}$. $C^\infty(\mathcal{A})$ is an inverse-closed Fréchet subalgebra of $\mathcal{A}$.*

PROOF. If $\mathcal{A}$ is symmetric, then $C^k(\mathcal{A})$ is inverse-closed in $C(\mathcal{A})$ already by Proposition 4.8. For general $\mathcal{A}$ let $a \in C(\mathcal{A})$ and $a \in \mathcal{D}(\delta, C(\mathcal{A}))$. Then we obtain directly from (4.11) that

$$\delta_{e_k}(a)^{-1} = \lim_{h\to 0} \frac{\psi_{he_k}(a^{-1}) - a^{-1}}{h} = \lim_{h\to 0} \psi_{he_k}(a^{-1})\frac{a - \psi_{he_k}(a)}{h}a^{-1} = -a\delta_{e_k}^{-1}(a)a^{-1}$$

and thus $a^{-1} \in \bigcap_{k=1}^d \mathcal{D}(\delta_{e_k}, C(\mathcal{A})) = C^1(\mathcal{A})$. Consequently $C^1(\mathcal{A})$ is inverse-closed in $C(\mathcal{A})$ and by induction $C^k(\mathcal{A})$ is inverse-closed in $C(\mathcal{A})$. Since $C(\mathcal{A})$ is inverse-closed in $\mathcal{A}$, $C^k(\mathcal{A})$ is inverse-closed in $\mathcal{A}$. If $a \in C^\infty(\mathcal{A}) \subseteq C^k(\mathcal{A})$, $k \geq 0$, is invertible in $\mathcal{A}$, then $a^{-1} \in C^k(\mathcal{A})$ for all $k \geq 0$ and thus $a^{-1} \in C^\infty(\mathcal{A})$. $\square$

We summarize the inclusion relations between the derived spaces $\mathcal{X}^{(k)}$ and the spaces $C^k(\mathcal{X})$.

$$\mathcal{X} \supseteq C(\mathcal{X}) \supseteq \mathcal{X}^{(1)} \supseteq C^1(\mathcal{X}) \supseteq \mathcal{X}^{(2)} \supseteq \cdots \supseteq C^\infty(\mathcal{X}) = \mathcal{X}^{(\infty)} . \tag{4.13}$$

In general, $C(\mathcal{X}^{(k)})$ is not dense in $\mathcal{X}^{(k)}$, but $C^\infty(\mathcal{X})$ is dense in $C(\mathcal{X})$. The inclusions follow from Proposition 4.20(3) and (4.12).

Weak Definition. If $\mathcal{X}$ is a Banach space, and $x \in C(\mathcal{X})$, $x' \in \mathcal{X}'$ (the dual of $\mathcal{X}$), we define

$$G_{x',x}(t) = \langle x', \psi_t(x) \rangle, \tag{4.14}$$

where $\langle \, , \rangle$ denotes the dual pairing of $\mathcal{X}' \times \mathcal{X}$.

Remark. Instead of the dual space $\mathcal{X}'$ we can take any norm fundamental set ([**13**, p.12]), i.e., a subspace $L \subseteq \mathcal{X}'$ with $\|x\|_\mathcal{X} = \sup\{\langle x', x\rangle \colon x' \in L, \|x'\|_{\mathcal{X}'} \le 1\}$; in particular the predual will be useful in many instances.

Lemma 4.23.

(1) *For every $x \in \mathcal{X}$ there holds the equivalence* $\|x\|_\mathcal{X} \asymp \sup_{\|x'\|_{\mathcal{X}'} \le 1} \|G_{x',x}\|_\infty$.
(2) $\|\delta^\alpha x\|_\mathcal{X} \asymp \sup_{\|x'\|_{\mathcal{X}'} \le 1} \|D^\alpha G_{x',x}\|_\infty$.

Proof. (1) follows from

$$\begin{aligned} \|x\|_\mathcal{X} = \sup_{\|x'\|_{\mathcal{X}'} \le 1} |\langle x', x\rangle| &\le \sup_{\|x'\|_{\mathcal{X}'} \le 1} \sup_{t \in \mathbb{R}^d} |\langle x', \psi_t x\rangle| \\ &= \sup_{\|x'\|_{\mathcal{X}'} \le 1} \|G_{x',x}\|_\infty \le M_\Psi \|x\|_\mathcal{X} . \end{aligned}$$

For the proof of the second statement it is sufficient to verify that

$$G_{x',\delta^\alpha x} = D^\alpha G_{x',x}. \tag{4.15}$$

If $|\alpha| = 1$, this is a direct consequence of the definition of a generator, the case $|\alpha| > 1$ follows by induction. □

Functional Calculus and Fourier Coefficients. Given $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $x \in C(\mathcal{X})$, the *action of* μ on x is defined by

$$\mu * x = \int_{\mathbb{R}^d} \psi_{-t}(x) d\mu(t). \tag{4.16}$$

This action generalizes the usual convolution and satisfies similar properties.

Proposition 4.24. *The space $C(\mathcal{X})$ is a $\mathcal{M}(\mathbb{R}^d)$- module that satisfies*

$$\|\mu * x\|_\mathcal{X} \le M_\Psi \|\mu\|_{\mathcal{M}(G)} \|x\|_\mathcal{X} \tag{4.17}$$

for $x \in C(\mathcal{X})$ and $\mu \in \mathcal{M}(\mathbb{R}^d)$.

If $f \in C^1(\mathbb{R}^d)$, then

$$\delta_j(f * x) = \partial_j(f) * x \in C(\mathcal{X}), \quad 1 \le j \le d. \tag{4.18}$$

If $\mu \in \mathcal{M}(\mathbb{R}^d)$, $x \in \mathcal{X}$, then for every $x' \in \mathcal{X}'$

$$G_{x',\mu * x} = \mu * G_{x',x}. \tag{4.19}$$

See, e.g., [**27**] for a proof of (4.17) and (4.18). The verification of (4.19) is straightforward.

Remarks. It should be noted that the action of $\mathcal{M}(\mathbb{R}^d)$ is only defined on $C(\mathcal{A})$; in this text a more general definition is not needed. Actually, a measurable group action is also strongly continuous [**38**]. However, for $\mathcal{B}(\ell^2)$ the continuity of $\Phi_{A,x}\colon t \mapsto \chi_t(A)x \in \ell^2(\mathbb{Z}^d)$ can be exploited (this approach follows [**33**]): Continuity implies that the integrals

$$\int_{\mathbb{T}^d} \chi_t(A)x\, d\mu(t)$$

are well defined, and

$$\Big\|\int_{\mathbb{T}^d} \chi_t(A)x\, d\mu(t)\Big\|_{\ell^2(\mathbb{Z}^d)} \le \|\chi_t(A)x\|_{\ell^2(\mathbb{Z}^d)}\|\mu\|_{\mathcal{M}(\mathbb{R}^d)} \le M_\Psi\|A\|_{\mathcal{B}(\ell^2)}\|x\|_{\ell^2(\mathbb{Z}^d)}\|\mu\|_{\mathcal{M}(\mathbb{R}^d)}\,.$$

So there exists a bounded operator

$$\int_{\mathbb{T}^d} \chi_t(A)\, d\mu(t)\colon x \mapsto \int_{\mathbb{T}^d} \chi_t(A)x\, d\mu(t)$$

such that (4.17) and (4.18) hold for *all* $A \in \mathcal{B}(\ell^2)$ (see also [**14**]).

Definition 4.25. If Ψ is a periodic automorphism group on $\mathcal{X}$ with period one in each variable, and $x \in C(\mathcal{X})$ we define the Fourier coefficients $\hat{x}(k)$, $k \in \mathbb{Z}^d$, by

$$\hat{x}(k) = \int_{\mathbb{T}^d} \psi_t(x)\, e^{-2\pi i k\cdot t}\, dt\,. \tag{4.20}$$

Proposition 4.26. *Let Ψ be periodic on $\mathcal{X}$. If*

$$\hat{x}(k) = 0 \quad \textit{for all} \quad k \in \mathbb{Z}^d \quad \textit{then } x = 0\,.$$

Proof. This follows from the uniqueness of scalar Fourier Series. As

$$\langle x', \hat{x}(k)\rangle = \widehat{G_{x',x}}(k) \tag{4.21}$$

for $x \in \mathcal{X}$, $x' \in \mathcal{X}'$, and $k \in \mathbb{Z}^d$, it follows that $\widehat{G_{x',x}}(k) = 0$ for all x' and all $k \in \mathbb{Z}^d$. By the uniqueness of scalar Fourier series $G_{x',x} = 0$ for all $x' \in \mathcal{X}'$, so $x = 0$. □

If the group action is periodic with period one, an application of the convolution theorem shows that the action of μ on x is

$$\widehat{\mu * x}(k) = \mathcal{F}(\mu)(k)\hat{x}(k)\,. \tag{4.22}$$

An application of (4.18) shows that for $x \in C^k(\mathcal{X})$ and $|\alpha| \le k$

$$\widehat{\delta^\alpha(x)}(l) = (2\pi i l)^\alpha \hat{x}(l)\,, \tag{4.23}$$

for all $l \in \mathbb{Z}^d$.

Remark. By an observation of Baskakov [**10**] for the action $\chi_t(A) = M_tAM_{-t}$ on a matrix A, the Fourier coefficient $\int_{\mathbb{T}^d}\chi_t(A)e^{-2\pi i k\cdot t}\, dt$ is exactly the kth side-diagonal $\hat{A}(k)$ of A. So there is no ambiguity in our notation. The identification of side-diagonals of matrices with Fourier coefficients is a key step in the proof of the inverse-closedness of $\mathcal{C}_v^1$ in $\mathcal{B}(\ell^2)$, see [**10**].

If the matrix algebra $\mathcal{A}$ is solid, the product $\mu * A$ can be defined entry-wise via $\mu * A(r,s) := \hat{\mu}(r-s)A(r,s)$, and solidity implies

$$\|\mu * A\|_{\mathcal{A}} \le \|\hat{\mu}\|_\infty \|A\|_{\mathcal{A}} \le \|\mu\|_{\mathcal{M}(\mathbb{R}^d)}\|A\|_{\mathcal{A}}.$$

The relations (4.17) and (4.18) are true in this case, too.

Smoothness in Matrix Spaces. We now identify the spaces $C(\mathcal{A})$ for the weighted matrix spaces $\mathcal{C}_v^p$ and $\mathcal{S}_v^1$ with respect to the automorphism group $\{\chi_t\}$.

PROPOSITION 4.27. *If v is a weight for $\mathcal{C}_v^p$, $1 \le p \le \infty$, then*

$$C(\mathcal{C}_v^p) = \mathcal{C}_v^p, \quad 1 \le p < \infty,$$
$$C(\mathcal{C}_v^\infty) = \{A \in \mathcal{C}_v^\infty \colon \lim_{|k|_\infty \to \infty} \|\hat{A}(k)\|_{\mathcal{C}_v^\infty} = \lim_{|k|_\infty \to \infty} \|A[k]\|_\infty v(k) = 0\}\,.$$

PROOF. We treat the case $1 \le p < \infty$ first and consider $A \in \mathcal{C}_v^p$ arbitrary. By definition, A is in $C(\mathcal{C}_v^p)$ if and only if

$$\begin{aligned} \|\chi_t(A) - A\|_{\mathcal{C}_v^p}^p &= \sum_{k \in \mathbb{Z}^d} \|\hat{A}(k)\|_{\mathcal{B}(\ell^2)}^p v(k)^p |1 - e^{2\pi i k\cdot t}|^p \\ &= 2^p \sum_{k \in \mathbb{Z}^d} \|\hat{A}(k)\|_{\mathcal{B}(\ell^2)}^p v(k)^p |\sin(\pi k \cdot t)|^p \to 0 \end{aligned} \tag{4.24}$$

if $t \to 0$. For any $\epsilon > 0$ there exists a $k_0 \in \mathbb{Z}^d$ such that

$$\sum_{|k|_\infty > k_0} \|\hat{A}(k)\|_{\mathcal{B}(\ell^2)}^p v(k)^p < \epsilon^p\,.$$

Now it is possible to find a $t_0 > 0$ such that

$$2^p \sum_{|k|_\infty \le k_0} \|\hat{A}(k)\|_{\mathcal{B}(\ell^2)}^p v(k)^p |\sin(\pi k \cdot t)|^p < \epsilon^p$$

for all t with $|t| < t_0$. This implies that $\|\chi_t(A) - A\|_{C(\mathcal{C}_v^p)}^p < (2^p + 1)\epsilon^p$, and, as ϵ was arbitrary, we have verified that every $A \in \mathcal{C}_v^p$ is continuous.

If $p = \infty$ we show first that $A \in C(\mathcal{C}_v^\infty)$ implies that $\lim_{k\to\infty} \|\hat{A}(k)\|_{\mathcal{C}_v^\infty} = 0$. Again, for every $\epsilon > 0$ there is a $t_0 > 0$ such that

$$\|\chi_t(A) - A\|_{\mathcal{C}_v^\infty} = 2 \sup_{k \in \mathbb{Z}^d} |\sin \pi k \cdot t|\, \|\hat{A}(k)\|_{\mathcal{C}_v^\infty} < \epsilon$$

for all t with $|t| < t_0$. If $|k|_2 > (2t_0)^{-1}$ and $t = \frac{k}{2|k|_2^2}$, then $\|\hat{A}(k)\|_{\mathcal{C}_v^\infty} < \epsilon$, and so $\lim_{k\to\infty} \|\hat{A}(k)\|_{\mathcal{C}_v^\infty} = 0$.

For the converse implication let us assume that $\lim_{k\to\infty} \|\hat{A}(k)\|_{\mathcal{C}_v^\infty} = 0$. The following estimate holds for every natural number N.

$$\|\chi_t(A) - A\|_{\mathcal{C}_v^\infty} \le \max_{|k|_\infty < N} \|\hat{A}(k)\|_{\mathcal{C}_v^\infty} |e^{2\pi i k\cdot t} - 1| + 2 \sup_{|k|_\infty \ge N} \|\hat{A}(k)\|_{\mathcal{C}_v^\infty}\,.$$

If we choose N sufficiently large, the second term of this expression becomes arbitrarily small. If t tends to zero, the first term becomes arbitrarily small as well. Consequently, $A \in C(\mathcal{C}_v^\infty)$. $\square$

Similarly, a matrix A is in $C(\mathcal{S}_0^1)$ if and only if

$$\lim_{N\to\infty} \sup_{k \in \mathbb{Z}^d} \sum_{|s|_\infty > N} |A(k, k-s)| = 0 \text{ and } \lim_{N\to\infty} \sup_{k \in \mathbb{Z}^d} \sum_{|s|_\infty > N} |A(k-s, k)| = 0. \tag{4.25}$$

This can be shown by a similar computation as above, however, it will follow immediately from Corollary 5.8.

Next we show that $C(\mathcal{B}(\ell^2)) \ne \mathcal{B}(\ell^2)$ by giving an explicit example. Define the anti-diagonal matrices Γ_v by

$$\Gamma_v(k,l) = \begin{cases} 1/v(k), & k = -l \\ 0, & k \ne -l\,. \end{cases}$$

If we choose $v_0 \equiv 1$, then $\Gamma = \Gamma_{v_0}$ is unitary in $\mathcal{B}(\ell^2)$. Now the matrix $\chi_t(\Gamma) - \Gamma$ has nonzero entries only on the anti-diagonal,

$$(\chi_t(\Gamma) - \Gamma)(k, -k) = 2|\sin(2\pi k \cdot t)| \, ,$$

and it is easy to see that

$$\limsup_{t \to 0} \|\chi_t(\Gamma) - \Gamma)\|_{\mathcal{B}(\ell^2)} = 2 \, .$$

In a similar manner we can use the matrix $\Gamma_v \in \mathcal{S}_v^1$ to conclude

$$\limsup_{t \to 0} \|\chi_t(\Gamma_v) - \Gamma_v)\|_{\mathcal{S}_v^1} = 2 \, .$$

We have shown that

$$C(\mathcal{B}(\ell^2)) \neq \mathcal{B}(\ell^2) \, ,$$
$$C(\mathcal{S}_v^1) \neq \mathcal{S}_v^1 \, .$$

Proposition 4.27 together with Proposition 4.10 and (4.12) gives the following characterization of C^k spaces

Corollary 4.28.

$$C^k(\mathcal{C}_v^p) = \mathcal{C}_{v\,v_k}^p, \quad 1 \le p < \infty,$$
$$C^k(\mathcal{C}_v^\infty) = C(\mathcal{C}_{v\,v_k}^\infty) \, ,$$
$$C^k(\mathcal{S}_v^1) = C(\mathcal{S}_{v\,v_k}^1) \, .$$

The continuous elements of the Jaffard and the Schur algebra are characterized by Proposition 4.27 and by (4.25).

4.3. Abstract Besov Spaces and Algebras

The theory of vector valued Besov spaces is well established. A classic treatment is [**27**], information on vector-valued Besov spaces can also be found in [**15, 79, 102**]. A treatment suited to the needs of approximation theory is in [**37**]. The purpose of this section is to introduce notation and to present some relevant results needed in the sequel. For proofs we refer to the literature cited above, some proofs are also included in Appendix D. However, I was not able to find a result on the "reiteration" of Besov spaces in the literature (Theorem 4.35), so a proof is included.

The main results of this section are the algebra properties of vector-valued Besov spaces derived from a given Banach algebra $\mathcal{A}$. Though possibly known, we were not able to find any references, so full proofs of the results are included.

Remark. There are many equivalent ways to define Besov spaces. The one chosen here is suited to the needs of approximation theory, cf. [**37, 103**].

4.3.1. Definition and Basic Properties.

Let $\mathcal{X}$ be a Banach space and Ψ a d-dimensional automorphism group acting on $\mathcal{X}$. For $t \in \mathbb{R}^d$ the kth difference of an element $x \in \mathcal{X}$ is given inductively as

$$\Delta_t(x) = \psi_t(x) - x, \qquad \Delta_t^k x = \Delta_t \Delta_t^{k-1} x, \quad k > 1.$$

The *modulus of continuity* of x for $h > 0$ is

$$\omega_h(x, \mathcal{X}) = \omega_h(x) = \sup_{|t| \le h} \|\Delta_t x\|_{\mathcal{X}}.$$

If $k > 1$, the kth *modulus of smoothness* of x for $h > 0$ is given as

$$\omega_h^k(x, \mathcal{X}) = \omega_h^k(x) := \sup_{|t| \le h} \|\Delta_t^k x\|_{\mathcal{X}}.$$

In the following Lemma we collect basic properties of moduli of smoothness. Proofs can be found, e.g., in [**13, 27, 37**].

LEMMA 4.29. *If $l, k \in \mathbb{N}$, $l \geq k$, $t \in \mathbb{R}^d$ and $h > 0$, then*

(1)
$$\|\Delta_t^l(x)\|_{\mathcal{X}} \leq (M_\Psi + 1)^k \|\Delta_t^{l-k}(x)\|_{\mathcal{X}} \text{ and } \omega_h^l(x) \leq (M_\Psi + 1)^k \omega_h^{l-k}(x)\,,$$

(2)
$$\|\Delta_{2t}^k(x)\|_{\mathcal{X}} \leq (M_\Psi + 1)^k \|\Delta_t^k(x)\|_{\mathcal{X}} \text{ and } \omega_{2h}^k(x) \leq (M_\Psi + 1)^k \omega_h^k(x)\,,$$

(3) *If $\lambda > 0$ then*
$$\|\omega_{\lambda t}^k(x)\|_{\mathcal{X}} \leq (M_\Psi + 1)^k (\lambda + 1)^k \|\Delta_t^k(x)\|_{\mathcal{X}}\,.$$

(4) *(Marchaud inequality)*
$$\omega_h^k(x) \leq C h^k \int_h^\infty \frac{\omega_u^l(x)}{u^k} \frac{du}{u}.$$

(5) *The averaged modulus of smoothness*
$$\overline{w}_h^k(x) = h^{-d} \int_{|t| \leq h} \|\Delta_t^k x\|_{\mathcal{X}}\, dt$$
is equivalent to the "standard" modulus of smoothness: $\overline{w}_h^k(x) \asymp \omega_h^k(x)$ [**37**, Lemma 6.5.1].

(6) *The modulus of smoothness is also equivalent to the iterated modulus of smoothness* [**13**, 5.4.11],
$$\omega_t^k(x) \asymp \sup_{\substack{|h_j| \leq t \\ 1 \leq j \leq k}} \|(\prod_{j=1}^k \Delta_{h_j}) x\|_{\mathcal{X}}.$$

(7) *If $x \in C^k(\mathcal{X})$, then*
$$\omega_h^{k+l}(x) \leq C \sup_{|\alpha| = k} \omega_h^l(\delta^\alpha(x))$$

For completeness the proof is included in Appendix D.

DEFINITION 4.30. Let $1 \leq p \leq \infty$, $r > 0$, $l = \lfloor r \rfloor + 1$. The (vector valued) *Besov space* $\Lambda_r^p(\mathcal{X})$ consists of all $x \in \mathcal{X}$ for which the norm

$$\|x\|_{\Lambda_r^p(\mathcal{X})} = \begin{cases} \|x\|_{\mathcal{X}} + \left(\int_{\mathbb{R}^+} (h^{-r}\omega_h^l(x))^p \frac{dh}{h}\right)^{1/p} = \|x\|_{\mathcal{X}} + |x|_{\Lambda_r^p(\mathcal{X})}\,, & 1 \leq p < \infty \\ \|x\|_{\mathcal{X}} + \sup_{h>0} h^{-r}\omega_h^l(x) = \|x\|_{\mathcal{X}} + |x|_{\Lambda_r^\infty(\mathcal{X})}\,, & p = \infty \end{cases}$$

is finite. The term $|x|_{\Lambda_r^p(\mathcal{X})}$ is the *Besov seminorm* of x, and r is the smoothness parameter.

It is well known that $\Lambda_r^p(\mathcal{X})$ is a Banach space for $1 \leq p \leq \infty$, see, e.g. [**27**]. For further properties of Besov spaces see, e.g., [**15, 79, 102**]. We will need some equivalent norms on the Besov spaces.

PROPOSITION 4.31. *If $x \in \mathcal{X}$, $r > 0$, then for any integer $k > \lfloor r \rfloor$ the following expressions define equivalent (semi)norms on $\Lambda_r^p(\mathcal{X})$.*

(1)
$$|x|_{\Lambda_r^p(\mathcal{X})} \asymp \left(\int_{\mathbb{R}^+} (h^{-r}\omega_h^k(x))^p \frac{dh}{h}\right)^{1/p}.$$

(2)
$$\|x\|_{\Lambda_r^p(\mathcal{X})} \asymp \|x\|_{\mathcal{X}} + \left(\int_{\mathbb{R}^d} (|t|^{-r}\|\Delta_t^k x\|_{\mathcal{X}})^p \frac{dt}{|t|^d}\right)^{1/p}.$$

(3)
$$\|x\|_{\Lambda^p_r(\mathcal{X})} \asymp \|x\|_{\mathcal{X}} + \Big(\sum_{l=0}^{\infty} \big(2^{rl}\omega^k_{2^{-l}}(x)\big)^p\Big)^{1/p},$$

(4) *If $l \in \mathbb{N}_0$, and $l \leq r$, then*
$$\|x\|_{\Lambda^p_r(\mathcal{X})} \asymp \|x\|_{\mathcal{X}} + \sum_{|\alpha|=l} \|\delta^\alpha(x)\|_{\Lambda^p_{r-l}(\mathcal{X})} .$$

The proof of the equivalences can be found in any of the references cited above. The modifications to cover the situation of group actions are minor. For completeness the proofs are included in Appendix D.2.

The Besov space $\Lambda^\infty_r(\mathcal{X})$ has a weak characterization of the norm, namely
$$\|x\|_{\Lambda^\infty_r(\mathcal{X})} \asymp \sup_{\|x'\|_{\mathcal{X}'}\leq 1} \|G_{x',x}\|_{\Lambda^\infty_r(\mathbb{R}^d)}.$$
As we do not need this relation we omit a proof, see [52].

We need to verify that the group action Ψ is bounded on $\Lambda^p_r(\mathcal{X})$.

Lemma 4.32. *If $\mathcal{X}$ is a Banach space with (bounded) automorphism group Ψ, then Ψ is a bounded automorphism group on $\Lambda^p_r(\mathcal{X})$ for every $1 \leq p \leq \infty$, and $r > 0$.*

Proof. Assume that $p < \infty$. If $x \in \Lambda^p_r(\mathcal{X})$, $k > \lfloor r \rfloor$ and $s \in \mathbb{R}^d$, then
$$\begin{aligned}\|\psi_s x\|_{\Lambda^p_r(\mathcal{X})} &= \|\psi_s x\|_{\mathcal{X}} + \Big(\int_{\mathbb{R}^d} (|t|^{-r}\|\Delta^k_t \psi_s x\|_{\mathcal{X}})^p \frac{dt}{|t|^d}\Big)^{1/p} \\ &\leq M_\Psi \|x\|_{\Lambda^p_r(\mathcal{X})} .\end{aligned}$$
The proof for $p = \infty$ is similar. □

For the continuous elements of Besov spaces we have

Proposition 4.33 ([27, Def. 3.1.5, Sec.3.4.3]). *Assume that $k > \lfloor r \rfloor$.*
$$C(\Lambda^p_r(\mathcal{X})) = \begin{cases} \Lambda^p_r(\mathcal{X}), & 1 \leq p < \infty, \\ \lambda^\infty_r(\mathcal{X}) = \{x \in \mathcal{X} \colon \lim_{h\to 0} h^{-r}\omega^k_h(x) = 0\}, & p = \infty. \end{cases}$$

4.3.2. Inclusion Relations. The proofs of the following result can be found in the standard literature [15, 37, 79].

Proposition 4.34. *If $1 \leq p, q \leq \infty$ and $0 < r < s$, then*
(1) $\Lambda^p_s(\mathcal{X}) \hookrightarrow \Lambda^q_r(\mathcal{X})$.
(2) *If $p < q$ then $\Lambda^p_r(\mathcal{X}) \hookrightarrow \Lambda^q_r(\mathcal{X})$.*

Does the iteration of the construction of Besov spaces yield refined smoothness spaces? Fortunately, this is not the case:

Theorem 4.35 (Reiteration theorem). *If $1 \leq p, q \leq \infty$ and $r, s > 0$ then*
$$\Lambda^q_s(\Lambda^p_r(\mathcal{X})) = \Lambda^q_{r+s}(\mathcal{X}) . \tag{4.26}$$

Remark. I was not able to find a reference for this result in the literature. With the “classical” notion of Besov spaces on $\mathbb{R}^d$ it is not even possible to formulate the result. We want to use (4.26) to simplify proofs of approximation results.

Proof. We assume first that x is in $\Lambda^q_{s+r}(\mathcal{X})$ and estimate $\|x\|_{\Lambda^q_s(\Lambda^p_r(\mathcal{X}))}$. As $\|x\|_{\Lambda^q_s(\Lambda^p_r(\mathcal{X}))} = \|x\|_{\Lambda^p_r(\mathcal{X})} + |x|_{\Lambda^q_s(\Lambda^p_r(\mathcal{X}))}$ and Proposition 4.34(1) implies that $\|x\|_{\Lambda^p_r(\mathcal{X})} \leq C\|x\|_{\Lambda^q_{r+s}(\mathcal{X})}$, it suffices to estimate $|x|_{\Lambda^q_s(\Lambda^p_r(\mathcal{X}))}$.

Assume that $\lfloor r \rfloor < m$ and $\lfloor s \rfloor < n$, $m, n \in \mathbb{N}$. Using Proposition 4.31(1) we can write

$$|x|_{\Lambda^q_s(\Lambda^p_r(\mathcal{X}))} \asymp \left\{ \int_{\mathbb{R}^+} \left[h^{-s} \omega_h^{n+m}(x, \Lambda^p_r(\mathcal{X})) \right]^q \frac{dh}{h} \right\}^{1/q} \tag{4.27}$$
$$= \| h^{-s} \omega_h^{n+m}(x, \Lambda^p_r(\mathcal{X})) \|_{L^q_*} ,$$

where $\|f\|_{L^q_*} = \left(\int_0^\infty f(t)^q \frac{dt}{t} \right)^{1/q}$. We can estimate the modulus of smoothness as

$$\begin{aligned} \omega_h^{n+m}(x, \Lambda^p_r(\mathcal{X})) &= \sup_{|u| \le h} \|\Delta_u^{n+m} x\|_{\Lambda^p_r(\mathcal{X})} \\ &\le \sup_{|u| \le h} \|\Delta_u^{n+m} x\|_{\mathcal{X}} + \sup_{|u| \le h} |\Delta_u^{n+m} x|_{\Lambda^p_r(\mathcal{X})} \\ &\lesssim \sup_{|u| \le h} \|\Delta_u^{n+m} x\|_{\mathcal{X}} + \sup_{|u| \le h} |\Delta_u^{n+m} x|_{\Lambda^1_r(\mathcal{X})} , \end{aligned} \tag{4.28}$$

where the last inequality uses the embedding $\Lambda^1_r(\mathcal{X}) \hookrightarrow \Lambda^p_r(\mathcal{X})$ for $p \ge 1$ (Proposition 4.34).

Inserting this estimate into (4.27) we obtain

$$|x|_{\Lambda^q_s(\Lambda^p_r(\mathcal{X}))} \lesssim |x|_{\Lambda^q_s(\mathcal{X})} + \| h^{-s} \sup_{|u| \le h} |\Delta_u^{n+m} x|_{\Lambda^1_r(\mathcal{X})} \|_{L^q_*} . \tag{4.29}$$

With

$$\phi(v, u) = \|\Delta_v^{n+m} \Delta_u^{n+m} x\|_{\mathcal{X}}$$

the L^q_*-norm in (4.29) can be rewritten as

(4.30)

$$\| h^{-s} \sup_{|u| \le h} \int_{\mathbb{R}^+} t^{-r} \sup_{|v| \le t} \phi(v, u) \frac{dt}{t} \|_{L^q_*}$$
$$\le \| h^{-s} \sup_{|u| \le h} \int_0^h t^{-r} \sup_{|v| \le t} \phi(v, u) \frac{dt}{t} \|_{L^q_*} + \| h^{-s} \sup_{|u| \le h} \int_h^\infty t^{-r} \sup_{|v| \le t} \phi(v, u) \frac{dt}{t} \|_{L^q_*}$$

$=: \mathrm{I} + \mathrm{II}.$

The first term can be estimated further using Hardy's inequality (Appendix A.1):

$$\begin{aligned} \mathrm{I}^q &= \int_0^\infty h^{-sq} \sup_{|u| \le h} \left[\int_0^h \sup_{|v| \le t} t^{-r} \phi(v, u) \frac{dt}{t} \right]^q \frac{dh}{h} \\ &\le \int_0^\infty h^{-sq} \left[\int_0^h \sup_{|v|, |u| \le h} t^{-r} \phi(v, u) \frac{dt}{t} \right]^q \frac{dh}{h} \\ &\overset{(*)}{\lesssim} \int_0^\infty \left(t^{-(r+s)} \sup_{|v|, |u| \le t} \phi(v, u) \right)^q \frac{dt}{t} \\ &\overset{(**)}{\lesssim} \int_0^\infty \left(t^{-(r+s)} \omega_t^{2(n+m)}(x, \mathcal{X}) \right)^q \frac{dt}{t} = |x|^q_{\Lambda^q_{r+s}(\mathcal{X})} , \end{aligned}$$

$(*)$ by Hardy's inequality, and $(**)$ using Lemma 4.29(6). For the second term we use (1) of Lemma 4.29 to get

$$\phi(v, u) = \|\Delta_v^{n+m} \Delta_u^{n+m} x\|_{\mathcal{X}} \lesssim \|\Delta_u^{n+m} x\|_{\mathcal{X}}.$$

Then $\sup_{|v| \le t} \phi(u, v)$ is independent of t, and

$$\begin{aligned} \mathrm{II}^q &\lesssim \int_0^\infty \left(h^{-s} \sup_{|u| \le h} \int_h^\infty t^{-r} \|\Delta_u^{n+m} x\|_{\mathcal{X}} \frac{dt}{t} \right)^q \frac{dh}{h} \\ &= \int_0^\infty h^{-(r+s)q} \sup_{|u| \le h} \|\Delta_u^{n+m} x\|^q_{\mathcal{X}} \frac{dh}{h} = |x|^q_{\Lambda^q_{r+s}(\mathcal{X})}. \end{aligned} \tag{4.31}$$

I and II together give the desired estimate.

For the converse assume that $x \in \Lambda^q_s(\Lambda^p_r(\mathcal{X}))$. Then, by Proposition 4.31 (2),

$$
\begin{aligned}
|x|^q_{\Lambda^q_{r+s}(\mathcal{X})} &\asymp \int_{\mathbb{R}^d} \Big(|t|^{-(r+s)} \|\Delta^{n+m}_t x\|_{\mathcal{X}} \Big)^q \frac{dt}{|t|^d} \\
&\asymp \int_{\mathbb{R}^d} |t|^{-rq} \Big(\int_{|\eta|\geq|t|} |\eta|^{-sp} \|\Delta^{n+m}_t x\|^p_{\mathcal{X}} \frac{d\eta}{|\eta|^d} \Big)^{q/p} \frac{dt}{|t|^d} \,,
\end{aligned} \tag{4.32}
$$

where we used $|t|^{-s} \asymp \big(\int_{|\eta|\geq|t|} |\eta|^{-sp} \frac{d\eta}{|\eta|^d}\big)^{1/p}$ for the last equivalence. As $\|\Delta^{n+m}_t x\|_{\mathcal{X}} \leq \sup_{|v|\leq|\eta|} \|\Delta^m_v \Delta^n_t\|_{\mathcal{X}}$ for $|\eta| \geq |t|$, we can dominate the right hand side of (4.32) by

$$
\begin{aligned}
&\int_{\mathbb{R}^d} |t|^{-rq} \sup_{|u|\leq|t|} \Big(\int_{|\eta|\geq|t|} |\eta|^{-sp} \sup_{|v|\leq|\eta|} \|\Delta^n_v \Delta^m_u x\|^p_{\mathcal{X}} \frac{d\eta}{|\eta|^d} \Big)^{q/p} \frac{dt}{|t|^d} \\
&\leq \int_{\mathbb{R}^d} |t|^{-rq} \sup_{|u|\leq|t|} \Big(\int_{\mathbb{R}^d} |\eta|^{-sp} \sup_{|v|\leq|\eta|} \|\Delta^n_v \Delta^m_u x\|^p_{\mathcal{X}} \frac{d\eta}{|\eta|^d} \Big)^{q/p} \frac{dt}{|t|^d} \\
&\leq \int_{\mathbb{R}^d} |t|^{-rq} \sup_{|u|\leq|t|} (|\Delta^m_u x|_{\Lambda^p_s(\mathcal{X})})^q \frac{dt}{|t|^d} \\
&\leq |x|^q_{\Lambda^q_r(\Lambda^p_s(\mathcal{X}))}.
\end{aligned}
$$

□

Remark. A different proof of the reiteration theorem can be obtained using the so-called reiteration theorems in interpolation theory (see, e.g. [**29**]) or in approximation theory [**83**].

4.3.3. Algebra Properties.

Theorem 4.36. *Let $\mathcal{A}$ be a Banach algebra, Ψ be a d-dimensional group of automorphisms acting on $\mathcal{A}$, $1 \leq p \leq \infty$ and $r > 0$. Then*

(1) *the Besov space $\Lambda^p_r(\mathcal{A})$ is a Banach subalgebra of $\mathcal{A}$, and*
(2) *the Besov space $\Lambda^p_r(\mathcal{A})$ is inverse-closed in $\mathcal{A}$.*

Proof. We treat the case $r < 1$ first. To show that $\Lambda^p_r(\mathcal{A})$ is a Banach algebra we use the identity

$$\Delta_t(ab) = \psi_t(a)\Delta_t(b) + \Delta_t(a)b \,. \tag{4.33}$$

Taking norms we obtain

$$
\begin{aligned}
\|\Delta_t(ab)\|_{\mathcal{A}} &\leq \|\psi_t(a)\|_{\mathcal{A}} \|\Delta_t(b)\|_{\mathcal{A}} + \|\Delta_t(a)\|_{\mathcal{A}} \|b\|_{\mathcal{A}} \\
&\leq M_\Psi \|a\|_{\mathcal{A}} \|\Delta_t(b)\|_{\mathcal{A}} + \|\Delta_t(a)\|_{\mathcal{A}} \|b\|_{\mathcal{A}} \\
&\leq M_\Psi \big(\|a\|_{\mathcal{A}} \|\Delta_t(b)\|_{\mathcal{A}} + \|\Delta_t(a)\|_{\mathcal{A}} \|b\|_{\mathcal{A}} \big) \,,
\end{aligned}
$$

where we used $M_\Psi \geq 1$ for the last inequality. This implies a similar relation for the Besov-seminorms, namely,

$$|ab|_{\Lambda^p_r(\mathcal{A})} \leq M_\Psi (\|a\|_{\mathcal{A}} |b|_{\Lambda^p_r(\mathcal{A})} + \|b\|_{\mathcal{A}} |a|_{\Lambda^p_r(\mathcal{A})}).$$

So

$$\|ab\|_{\Lambda^p_r(\mathcal{A})} = \|ab\|_{\mathcal{A}} + |ab|_{\Lambda^p_r(\mathcal{A})} \leq C \|a\|_{\Lambda^p_r(\mathcal{A})} \|b\|_{\Lambda^p_r(\mathcal{A})} \,,$$

and the assertion follows.

Next we show the inverse-closedness of $\Lambda^p_r(\mathcal{A})$ in $\mathcal{A}$. Assume that $a \in \Lambda^p_r(\mathcal{A})$ is invertible in $\mathcal{A}$. It is sufficient to verify that $|a^{-1}|_{\Lambda^p_r(\mathcal{A})}$ is finite. By a straightforward computation we obtain

$$\Delta_t(a^{-1}) = -\psi_t(a^{-1})\, \Delta_t(a)\, a^{-1} \,. \tag{4.34}$$

This implies that a^{-1} has a finite $\Lambda^p_r(\mathcal{A})$-norm.

In the general case we can use the reiteration theorem (Theorem 4.35) and the transitivity of inverse-closedness, and prove the statement by induction. Assume that the statement is proved for all smoothness parameters smaller than $s > 0$. If $s < r < s+1$ then $\Lambda^p_r(\mathcal{A}) = \Lambda^p_{r-s}(\Lambda^p_s(\mathcal{A}))$, and, by the preceding argument $\Lambda^p_{r-s}(\Lambda^p_s(\mathcal{A}))$ is inverse-closed in $\Lambda^p_s(\mathcal{A})$, which by hypotheses is inverse-closed in $\mathcal{A}$. The theorem is proved. □

4.4. Bessel Potential Spaces

4.4.1. Definitions, Basic Properties. Bessel potentials allow us to define an analogue of *polynomial* weights in Banach spaces with an automorphism group. We define the Bessel kernel $\mathcal{G}_r$ by its Fourier transform,

$$\mathcal{F}\mathcal{G}_r(\omega) = (1 + |2\pi\omega|_2^2)^{-r/2}, \quad r > 0.$$

Some properties of the Bessel kernel will be needed in the sequel.

Lemma 4.37 ([**95**, V.5]).
(1) $\mathcal{G}_r \in \Lambda^\infty_r(L^1(\mathbb{R}^d))$, $\|\mathcal{G}_r\|_{L^1(\mathbb{R}^d)} = 1$,
(2) $\mathcal{G}_r * \mathcal{G}_s = \mathcal{G}_{r+s}$ *for all* $r, s > 0$,
(3) $\mathcal{G}_r * \mathscr{S} = \{\mathcal{G}_r * \varphi \colon \varphi \in \mathscr{S}\} = \mathscr{S}$.

Definition 4.38. Let $\mathcal{X}$ be a Banach space and Ψ an automorphism group acting on $\mathcal{X}$. The *Bessel potential space* of order $r > 0$ is

$$\mathcal{P}_r(\mathcal{X}) = \mathcal{G}_r * \mathcal{X} = \{x \in \mathcal{X} \colon x = \mathcal{G}_r * y \text{ for some } y \in \mathcal{X}\}$$

with the norm

$$\|\mathcal{G}_r * y\|_{\mathcal{P}_r(\mathcal{X})} = \|y\|_{\mathcal{X}}.$$

We have to verify that the definition of the norm on $\mathcal{P}_r(\mathcal{X})$ is consistent, that is, we show that the convolution with $\mathcal{G}_r$ is injective on $\mathcal{X}$. We use a weak type argument.

Let $y \in \mathcal{X}$ with $\mathcal{G}_r * y = 0$. This is equivalent to

$$G_{x',\mathcal{G}_r * y}(t) = \mathcal{G}_r * G_{x',y}(t) = 0$$

for all $t \in \mathbb{R}^d$ and all $x' \in \mathcal{X}'$. Here $G_{x',y}(t)$ is the function defined in (4.23). Now we proceed as in [**95**, V.3.3]. We choose a test function $\varphi \in \mathscr{S}$ and obtain

$$\int_{\mathbb{R}^d} (\mathcal{G}_r * G_{x',y})(t)\varphi(t)\,dt = \int_{\mathbb{R}^d} G_{x',y}(t)(\mathcal{G}_r * \varphi)(t)\,dt = 0\,.$$

By Lemma 4.37 (3) the convolution with $\mathcal{G}_r$ is surjective on $\mathscr{S}$, and so it follows that $G_{x',y} = 0$ for all $x' \in \mathcal{X}'$, that is, $y = 0$.

An immediate consequence of Definition 4.38 is the embedding $\mathcal{P}_r(\mathcal{X}) \hookrightarrow \mathcal{X}$. If $x \in \mathcal{P}_r(\mathcal{X})$, then $x = \mathcal{G}_r * y$ for a $y \in \mathcal{X}$, and

$$\|x\|_{\mathcal{X}} \le \|\mathcal{G}_r\|_{L^1(\mathbb{R}^d)} \|y\|_{\mathcal{X}} = \|\mathcal{G}_r\|_{L^1(\mathbb{R}^d)} \|x\|_{\mathcal{P}_r(\mathcal{X})}. \tag{4.35}$$

As $\mathcal{G}_r * \mathcal{G}_s = \mathcal{G}_{r+s}$ for $r, s > 0$ we obtain a (trivial) reiteration property for the Bessel potential spaces.

Proposition 4.39.

$$\mathcal{P}_r(\mathcal{P}_s(\mathcal{X})) = \mathcal{P}_{r+s}(\mathcal{X}), \quad r, s > 0.$$

The following inclusion will be complemented in Proposition 4.44.

Lemma 4.40.

$$\mathcal{P}_r(\mathcal{X}) \hookrightarrow \Lambda^\infty_r(\mathcal{X}).$$

PROOF. Let $x \in \mathcal{P}_r(\mathcal{X})$ with $x = \mathcal{G}_r * y$, $y \in \mathcal{X}$. Then $|x|_{\Lambda_r^\infty(\mathcal{X})}$ can be estimated for $k > \lfloor r \rfloor$ as

$$\begin{aligned}|x|_{\Lambda_r^\infty(\mathcal{X})} &= \sup_{|t|\neq 0} \frac{\|\Delta_t^k(\mathcal{G}_r * y)\|_{\mathcal{X}}}{|t|^r} = \sup_{|t|\neq 0} \|\frac{\Delta_t^k(\mathcal{G}_r)}{|t|^r} * y\|_{\mathcal{X}} \\ &\leq \|G_r\|_{\Lambda_r^\infty(L^1)} \|y\|_{\mathcal{X}} = \|G_r\|_{\Lambda_r^\infty(L^1)} \|x\|_{\mathcal{P}_r(\mathcal{X})} ,\end{aligned}$$

and this is the desired embedding. □

4.4.2. Characterization by Hypersingular Integrals.

LEMMA 4.41. *If $x \in \mathcal{P}_r(\mathcal{X})$, then $\|x\|_{\mathcal{P}_r(\mathcal{X})} \asymp \sup_{\|x'\|_{\mathcal{X}'} \leq 1} \|G_{x',x}\|_{\mathcal{P}_r(L^\infty)}$, where the dual pairing in $G_{x',x}$ is the one of $\mathcal{X}' \times \mathcal{X}$.*

PROOF. Let $x = \mathcal{G}_r * y$. Then

$$\begin{aligned}\|x\|_{\mathcal{P}_r(\mathcal{X})} =& \|y\|_{\mathcal{X}} \asymp \sup_{\|x'\|_{\mathcal{X}'} \leq 1} \|G_{x',y}\|_\infty \\ &= \sup_{\|x'\|_{\mathcal{X}'} \leq 1} \|\mathcal{G}_r * G_{x',y}\|_{\mathcal{P}_r(L^\infty)} = \sup_{\|x'\|_{\mathcal{X}'} \leq 1} \|G_{x',\mathcal{G}_r * y}\|_{\mathcal{P}_r(L^\infty)} .\end{aligned}$$ □

We state a special case of a result by Wheeden [**105**] (see also [**94**],[**95**, V.6.10]).

THEOREM 4.42. *Let $0 < r < 2$. A function f is an element of $\mathcal{P}_r(L^\infty(\mathbb{R}^d))$ if and only if $f \in L^\infty(\mathbb{R}^d)$ and*

$$\sup_{\epsilon > 0} \Big\| \int_{|t|\geq \epsilon} |t|^{-r} \Delta_t(f) \frac{dt}{|t|^d} \Big\|_{L^\infty(\mathbb{R}^d)} < \infty. \tag{4.36}$$

If (4.36) holds,

$$\|f\|_{L^\infty(\mathbb{R}^d)} + \sup_{\epsilon > 0} \Big\| \int_{|t|\geq \epsilon} |t|^{-r} \Delta_t(f) \frac{dt}{|t|^d} \Big\|_{L^\infty(\mathbb{R}^d)} < \infty \tag{4.37}$$

defines an equivalent norm on $\mathcal{P}_r(L^\infty(\mathbb{R}^d))$.

An application of Lemma 4.41 allows us to state an equivalent result for vector-valued Bessel potential spaces.

THEOREM 4.43. *Let $\mathcal{X}$ be a Banach space and Ψ an automorphism group acting on it. For $0 < r < 2$ the norm $\|x\|_{\mathcal{P}_r(\mathcal{X})}$ is equivalent to*

$$\|x\|_{\mathcal{X}} + \sup_{\epsilon > 0} \Big\| \int_{|t|\geq \epsilon} |t|^{-r} \Delta_t(x) \frac{dt}{|t|^d} \Big\|_{\mathcal{X}}. \tag{4.38}$$

PROOF. The proof is a direct calculation using the scalar result and Lemma 4.41. □

In the following result we obtain a comparison of Bessel potential spaces and Besov spaces.

PROPOSITION 4.44.

$$\Lambda_r^1(\mathcal{X}) \hookrightarrow \mathcal{P}_r(\mathcal{X}) \hookrightarrow \Lambda_r^\infty(\mathcal{X}) \qquad \text{if } r > 0.$$

PROOF. The embedding $\mathcal{P}_r(\mathcal{X}) \hookrightarrow \Lambda_r^\infty(\mathcal{X})$ has been proved in Lemma 4.40. We have to verify only the first inclusion. Assume first that $0 < r < 1$. By Theorem 4.43, for an $x \in \mathcal{P}_r(\mathcal{X})$

$$\begin{aligned}\|x\|_{\mathcal{P}_r(\mathcal{X})} &\asymp \|x\|_{\mathcal{X}} + \sup_{\epsilon > 0} \Big\| \int_{|t|\geq \epsilon} |t|^{-r} \Delta_t(x) \frac{dt}{|t|^d} \Big\|_{\mathcal{X}} \\ &\leq \|x\|_{\mathcal{X}} + \int_{|t|\geq 0} |t|^{-r} \|\Delta_t(x)\|_{\mathcal{X}} \frac{dt}{|t|^d} \\ &= \|x\|_{\Lambda_r^1(\mathcal{X})} .\end{aligned}$$

In the general case we proceed by induction. Let the statement be true for all positive values up to $s > 0$, and $s < r < s+1$. Then

$$\Lambda^1_r(\mathcal{X}) = \Lambda^1_{r-s}(\Lambda^1_s(\mathcal{X})) \subseteq \mathcal{P}_{r-s}(\Lambda^1_s(\mathcal{X})) \subseteq \mathcal{P}_{r-s}(\mathcal{P}_s(\mathcal{X})) = \mathcal{P}_r(\mathcal{X}),$$

where we have used the reiteration theorems for the Bessel and the Besov spaces. □

Remark. In order for this proof to be useful it is necessary to know that we "play fair" here: For the proof of Theorem 4.42 only the embedding of Lemma 4.40 is needed.

Another application of the reiteration theorem and the representation of the norm of $\mathcal{P}_r(\mathcal{X})$ by the hypersingular integral (4.38) yields the following result.

Proposition 4.45. *If $r, s > 0$, then*

$$\mathcal{P}_r(\Lambda^p_s(\mathcal{X})) = \Lambda^p_s(\mathcal{P}_r(\mathcal{X})) = \Lambda^p_{r+s}(\mathcal{X}). \tag{4.39}$$

Proof. Using the reiteration theorems for Bessel potential spaces and Besov spaces, it suffices to prove the proposition only for $0 < r, s < 1$.

We show first that $\mathcal{P}_r(\Lambda^p_s(\mathcal{X})) \hookrightarrow \Lambda^p_s(\mathcal{P}_r(\mathcal{X}))$. Now $y \in \Lambda^p_s(\mathcal{X})$ if and only if $x = \mathcal{G}_r * y \in \mathcal{P}_r(\Lambda^p_s(\mathcal{X}))$. We obtain the following estimate.

$$\begin{aligned}
\|\mathcal{G}_r * y\|^p_{\Lambda^p_s(\mathcal{P}_r(\mathcal{X}))} &= \int_{\mathbb{R}^d} \frac{\|\Delta_t(\mathcal{G}_r * y)\|^p_{\mathcal{X}}}{|t|^{sp}} \frac{dt}{|t|^d} \\
&= \int_{\mathbb{R}^d} \frac{\|\mathcal{G}_r * \Delta_t(y)\|^p_{\mathcal{X}}}{|t|^{sp}} \frac{dt}{|t|^d} \\
&\leq \int_{\mathbb{R}^d} \frac{\|\mathcal{G}_r\|^p_{L^1(\mathbb{R}^d)} \|\Delta_t(y)\|^p_{\mathcal{X}}}{|t|^{sp}} \frac{dt}{|t|^d} \\
&= \|y\|^p_{\Lambda^p_s(\mathcal{X})} = \|\mathcal{G}_r * y\|^p_{\mathcal{P}_r(\Lambda^p_s(\mathcal{X}))}.
\end{aligned}$$

Now let $x = \mathcal{G}_r * y \in \mathcal{P}_r(\Lambda^p_s(\mathcal{X}))$. Then

$$\begin{aligned}
\|x\|^p_{\mathcal{P}_r(\Lambda^p_s(\mathcal{X}))} &= \|\mathcal{G}_r * y\|^p_{\mathcal{P}_r(\Lambda^p_s(\mathcal{X}))} = \|y\|^p_{\Lambda^p_s(\mathcal{X})} = \int_{\mathbb{R}^d} \frac{\|\Delta_t(y)\|^p_{\mathcal{X}}}{|t|^{sp}} \frac{dt}{|t|^d} \\
&= \int_{\mathbb{R}^d} \frac{\|\Delta_t(\mathcal{G}_r * y)\|^p_{\mathcal{P}_r(\mathcal{X})}}{|t|^{sp}} \frac{dt}{|t|^d} \\
&= \|\mathcal{G}_r * y\|_{\Lambda^p_s(\mathcal{P}_r(\mathcal{X}))} = \|x\|_{\Lambda^p_s(\mathcal{P}_r(\mathcal{X}))}.
\end{aligned}$$

Proposition 4.44 implies that

$$\Lambda^p_s(\Lambda^1_r(\mathcal{X})) \hookrightarrow \Lambda^p_s(\mathcal{P}_r(\mathcal{X})) \hookrightarrow \Lambda^p_s(\Lambda^\infty_r(\mathcal{X})),$$

and the first and last space in this chain equal $\Lambda^p_{r+s}(\mathcal{X})$ by the reiteration theorem for Besov spaces. □

4.4.3. Algebra Properties. If $\mathcal{A}$ is a Banach algebra, the characterization of Bessel potential spaces by a hypersingular integral in Theorem 4.43 yields the Banach algebra properties of $\mathcal{P}_r(\mathcal{A})$. For the proof we need one more norm equivalence.

Lemma 4.46. *If $r > 0$, then*

$$\|x\|_{\mathcal{X}} + \sup_{\epsilon>0} \Big\| \int_{|t|\geq\epsilon} \frac{\Delta_t(x)}{|t|^r} \frac{dt}{|t|^d} \Big\|_{\mathcal{X}} \asymp \|x\|_{\mathcal{X}} + \sup_{\epsilon>0} \Big\| \int_{\epsilon\leq|t|\leq 1} \frac{\Delta_t(x)}{|t|^r} \frac{dt}{|t|^d} \Big\|_{\mathcal{X}}.$$

Proof. For the nontrivial part of the Lemma observe that

$$\Big\|\int_{\epsilon\le|t|}\frac{\Delta_t(x)}{|t|^r}\frac{dt}{|t|^d}\Big\|_{\mathcal X}\le\Big\|\int_{\epsilon\le|t|\le1}\frac{\Delta_t(x)}{|t|^r}\frac{dt}{|t|^d}\Big\|_{\mathcal X}+\Big\|\int_{|t|\ge1}\frac{\Delta_t(x)}{|t|^r}\frac{dt}{|t|^d}\Big\|_{\mathcal X}$$
$$\le\Big\|\int_{\epsilon\le|t|\le1}\frac{\Delta_t(x)}{|t|^r}\frac{dt}{|t|^d}\Big\|_{\mathcal X}+(1+M_\Psi)\|x\|_{\mathcal X}\int_{|t|\ge1}|t|^{-r}\frac{dt}{|t|^d}$$

□

Theorem 4.47. *Let $\mathcal A$ be a Banach algebra and Ψ be a d-dimensional group of automorphisms acting on $\mathcal A$.*

(1) *For each $r>0$ the Bessel potential space $\mathcal P_r(\mathcal A)$ is a Banach subalgebra of $\mathcal A$.*
(2) *$\mathcal P_r(\mathcal A)$ is inverse-closed in $\mathcal A$.*

Proof. We treat the case $r<1$ first. Let $a,b\in\mathcal P_r(\mathcal A)$. Using

$$\Delta_t(ab)=\Delta_t(a)\Delta_t(b)+a\Delta_t(b)+\Delta_t(a)b$$

we obtain

$$\Big\|\int_{\epsilon\le|t|\le1}\frac{\Delta_t(ab)}{|t|^r}\frac{dt}{|t|^d}\Big\|_{\mathcal A}\le\Big\|\int_{\epsilon\le|t|\le1}\frac{\Delta_t(a)\Delta_t(b)}{|t|^r}\frac{dt}{|t|^d}\Big\|_{\mathcal A}$$
$$+\Big\|a\int_{\epsilon\le|t|\le1}\frac{\Delta_t(b)}{|t|^r}\frac{dt}{|t|^d}\Big\|_{\mathcal A}+\Big\|\Big(\int_{\epsilon\le|t|\le1}\frac{\Delta_t(a)}{|t|^r}\frac{dt}{|t|^d}\Big)b\Big\|_{\mathcal A}. \tag{4.40}$$

The second and third term of the expression on the right hand side of the inequality are dominated by

$$\|a\|_{\mathcal A}\|b\|_{\mathcal P_r(\mathcal A)}+\|a\|_{\mathcal P_r(\mathcal A)}\|b\|_{\mathcal A}\lesssim\|a\|_{\mathcal P_r(\mathcal A)}\|b\|_{\mathcal P_r(\mathcal A)}\,.$$

For the estimation of the first term in (4.40) we use the embedding $\mathcal P_r(\mathcal A)\hookrightarrow\Lambda_r^\infty(\mathcal A)$ (Proposition 4.44), so $\|\Delta_t a\|_{\mathcal A}\lesssim|t|^r\|a\|_{\mathcal P_r(\mathcal A)}$, with a similar estimate for b. Therefore

$$\Big\|\int_{\epsilon\le|t|\le1}\frac{\Delta_t(a)\Delta_t(b)}{|t|^r}\frac{dt}{|t|^d}\Big\|_{\mathcal A}\lesssim\|a\|_{\mathcal P_r(\mathcal A)}\|b\|_{\mathcal P_r(\mathcal A)}\int_{0\le|t|\le1}|t|^r\frac{dt}{|t|^d}\le C_r\|a\|_{\mathcal P_r(\mathcal A)}\|b\|_{\mathcal P_r(\mathcal A)}\,,$$

and C_r does not depend on ϵ. Combining the estimates and using Lemma 4.46 we have proved that

$$\|ab\|_{\mathcal P_r(\mathcal A)}\lesssim\|a\|_{\mathcal P_r(\mathcal A)}\|b\|_{\mathcal P_r(\mathcal A)}.$$

For the verification of the inverse-closedness of $\mathcal P_r(\mathcal A)$ in $\mathcal A$ we use a similar argument: Expand the identity (4.34) to obtain

$$\Delta_t(a^{-1})=-\Delta_t(a^{-1})\Delta_t(a)a^{-1}-a^{-1}\Delta_t(a)a^{-1}.$$

So

$$\Big\|\int_{\epsilon\le|t|\le1}\frac{\Delta_t(a^{-1})}{|t|^r}\frac{dt}{|t|^d}\Big\|_{\mathcal A}\le\Big\|\int_{\epsilon\le|t|\le1}\frac{\Delta_t(a^{-1})\Delta_t(a)a^{-1}}{|t|^r}\frac{dt}{|t|^d}\Big\|_{\mathcal A}$$
$$+\Big\|\int_{\epsilon\le|t|\le1}\frac{a^{-1}\Delta_t(a^{-1})a^{-1}}{|t|^r}\frac{dt}{|t|^d}\Big\|_{\mathcal A}. \tag{4.41}$$

As $a\in\Lambda_r^\infty(\mathcal A)$, we know that

$$\|\Delta_t(a)\|_{\mathcal A}\lesssim|t|^r\|a\|_{\Lambda_r^\infty(\mathcal A)},$$
$$\|\Delta_t(a^{-1})\|_{\mathcal A}\lesssim|t|^r\|a^{-1}\|_{\Lambda_r^\infty(\mathcal A)}\lesssim|t|^r\|a^{-1}\|_{\mathcal A}^2\|a\|_{\Lambda_r^\infty(\mathcal A)},$$

the last inequality follows by taking norms in (4.34). The first term on the right hand side of (4.41) can be dominated by

$$\int_{\epsilon\le|t|\le 1} \frac{\|\Delta_t(a^{-1})\|_{\mathcal{A}}\|\Delta_t a\|_{\mathcal{A}}\|a^{-1}\|_{\mathcal{A}}}{|t|^r}\frac{dt}{|t|^d} \lesssim \|a^{-1}\|_{\mathcal{A}}^3\|a\|^2_{\Lambda_r^\infty(\mathcal{A})} \lesssim \|a^{-1}\|_{\mathcal{A}}^3\|a\|^2_{\mathcal{P}_r(\mathcal{A})}\,.$$

For the second term we get

$$\Big\|\int_{\epsilon\le|t|\le 1} \frac{a^{-1}\Delta_t(a)a^{-1}}{|t|^r}\frac{dt}{|t|^d}\Big\|_{\mathcal{A}} = \Big\|a^{-1}\Big(\int_{\epsilon\le|t|\le 1}\frac{\Delta_t(a)}{|t|^r}\frac{dt}{|t|^d}\Big)a^{-1}\Big\|_{\mathcal{A}}$$
$$\lesssim \|a^{-1}\|^2_{\mathcal{A}}\|a\|_{\mathcal{P}_r(\mathcal{A})}.$$

Putting both estimates together we get $a^{-1}\in\mathcal{P}_r(\mathcal{A})$, that is, the inverse-closedness of $\mathcal{P}_r(\mathcal{A})$ in $\mathcal{A}$.

If $r\ge 1$ we can proceed by induction. Let $\mathcal{P}_s(\mathcal{A})$ be inverse-closed in $\mathcal{A}$, and $s<r<s+1$. By what we have just proved $\mathcal{P}_r(\mathcal{A})=\mathcal{P}_{r-s}(\mathcal{P}_s(\mathcal{A}))$ is inverse-closed in $\mathcal{P}_s(\mathcal{A})$. As $\mathcal{P}_s(\mathcal{A})$ is inverse-closed in $\mathcal{A}$ by hypotheses we are done. □

4.4.4. Application to Weighted Matrix Algebras. We call a weight of the form $v_r^*(k)=(1+|2\pi k|^2)^{r/2}$ for $r>0$ the *Bessel weight* of order r.

PROPOSITION 4.48. *If $\mathcal{X}$ be a homogeneous matrix space, then*

$$\mathcal{X}_{v_r^*}=\mathcal{P}_r(\mathcal{X})\,.$$

PROOF. By definition A is in $\mathcal{P}_r(\mathcal{X})$, if there is a $A_0\in\mathcal{X}$ such that $A=\mathcal{G}_r*A_0$. This is equivalent to

$$\hat{A}(k)=(1+|2\pi k|^2)^{-r/2}\hat{A}_0(k)\,,$$

or $\hat{A}_0(k)=(1+|2\pi k|^2)^{r/2}\hat{A}(k)$, and therefore

$$\|A\|_{\mathcal{P}_r(\mathcal{X})}=\|A_0\|_{\mathcal{X}}=\|A\|_{\mathcal{X}_{v_r^*}}$$

by (4.22), i.e., $A\in\mathcal{X}_{v_r^*}$. □

If $\mathcal{A}$ is a homogeneous matrix algebra, the weighted matrix spaces $\mathcal{A}_{v_r^*}$ are inverse-closed matrix algebras.

PROPOSITION 4.49. *If $\mathcal{A}$ is a homogeneous matrix algebra, and v_r^*, $r>0$, is a Bessel weight, then $\mathcal{A}_{v_r^*}=\mathcal{P}_r(\mathcal{A})$ is a matrix algebra. This algebra is inverse-closed in $\mathcal{A}$.*

PROOF. This is an application of Theorem 4.47. □

For solid matrix algebras the standard polynomial weights v_r can be taken instead of v_r^*.

COROLLARY 4.50. *If $\mathcal{A}$ is a solid matrix algebra, then $\mathcal{A}_{v_r}$ is an inverse-closed subalgebra of $\mathcal{A}$.*

This result should be compared with Proposition 2.21 and Proposition 4.10. For later use we state the results of Proposition 4.44 and Proposition 4.45 for weighted matrix algebras.

PROPOSITION 4.51. *If $\mathcal{A}$ is a homogeneous matrix algebra, and $r,s>0$, then*

$$\Lambda_r^1(\mathcal{A})\hookrightarrow\mathcal{A}_{v_r^*}\hookrightarrow\Lambda_\infty^r(\mathcal{A}),$$
$$\Lambda_r^p(\mathcal{A}_{v_s^*})=(\Lambda_r^p(\mathcal{A}))_{v_s^*}=\Lambda_{r+s}^p(\mathcal{A}).$$

Remark. We have not identified the Besov spaces related to a matrix algebra yet. Though this is not difficult, we postpone it to Section 5.5. However, from the inclusion relations in Prop 4.34 and the first relation in Proposition 4.51 we conclude that for $\epsilon > 0$ and $r > 0$

$$\mathcal{A}_{v^*_{r+\epsilon}} \hookrightarrow \Lambda^1_r(\mathcal{A}) \hookrightarrow \mathcal{A}_{v^*_r} \hookrightarrow \Lambda^\infty_r(\mathcal{A}) \hookrightarrow \mathcal{A}_{v^*_{r-\epsilon}}\,,$$

so Besov spaces are related to off-diagonal decay. The last embedding in the relation above follows directly from the characterization of the norm of $\mathcal{P}_{r-\epsilon}(\mathcal{A})$ by the hypersingular integral (4.38).

CHAPTER 5

Smoothness and Approximation with Bandlimited Elements

So far the two constructions of inverse-closed subalgebras are based on different structural features of Banach algebras, namely, derivations or commutative automorphism groups, and approximation schemes. Again classical approximation theory indicates how to relate smoothness properties to approximation properties. The prototype of such a connection is the Jackson-Bernstein theorem for polynomial approximation of periodic functions.

In this section we develop a similar theory for Banach spaces with an automorphism group Ψ. The application to matrix algebras then supports once more the insight that "smoothness of matrices" amounts to their off-diagonal decay.

Throughout this chapter we assume that $\mathcal{X}$ is a Banach space with automorphism group Ψ. The letter $\mathcal{A}$ indicates a Banach algebra.

5.1. Bandlimited Elements and Their Spectral Characterization

We need an analogue of the trigonometric polynomials in the context of a Banach space with an automorphism group.

Definition 5.1. An element $x \in \mathcal{X}$ is *σ-bandlimited* for $\sigma > 0$, if there is a constant $C > 0$ such that

$$\|\delta^\alpha(x)\|_{\mathcal{X}} \leq C(2\pi\sigma)^{|\alpha|} \tag{5.1}$$

for every multi-index α. An element is *bandlimited*, if it is σ-bandlimited for some $\sigma > 0$. Inequality (5.1) is a generalized Bernstein inequality.

Example 5.2. In $C(\mathbb{T})$ the N-bandlimited elements are exactly the trigonometric polynomials of degree $N \in \mathbb{N}_0$. If f is a trigonometric polynomial of degree N, then, by the classical Bernstein inequality, we have $\|f'\|_\infty \leq 2\pi N \|f\|_\infty$. This implies (5.1).

Conversely, if $f \in C(\mathbb{T})$ is N-bandlimited in the sense of (5.1), then

$$C(2\pi N)^k \geq \|D^k f\|_{L^\infty(\mathbb{T})} \geq \|D^k f\|_{L^2(\mathbb{T})} = \|((2\pi i l)^k \hat{f}(l))_{l\in\mathbb{Z}}\|_{\ell^2} \geq (2\pi|m|)^k |\hat{f}(m)|$$

for all $m \in \mathbb{Z}$. This is true for all $k \geq 0$, whence $\hat{f}(m) = 0$ for $|m| > N$. See [**103**, 3.4.2] for related statements.

Lemma 5.3. *Let $\mathcal{A}$ be a Banach algebra, and $a, b \in \mathcal{A}$. If a is σ-bandlimited, and b is τ-bandlimited, then ab is $\sigma + \tau$-bandlimited.*

Proof. Taking norms in the iterated Leibniz rule for $\delta^\alpha(ab)$ and using that $\|\delta^\alpha(a)\|_{\mathcal{A}} \leq C_a(2\pi\sigma)^{|\alpha|}$, $\|\delta^\alpha(b)\|_{\mathcal{A}} \leq C_b(2\pi\tau)^{|\alpha|}$ for all multiindices α, we obtain

$$\|\delta^\alpha(ab)\|_{\mathcal{A}} \leq \sum_{k=0}^{|\alpha|} \binom{|\alpha|}{k} C_a C_b (2\pi\sigma)^k (2\pi\tau)^{|\alpha|-k} = C_a C_b (2\pi(\sigma+\tau))^{|\alpha|}.$$

□

We next generalize Fourier arguments to obtain an alternative characterization of bandlimited elements in a Banach space. To avoid vector-valued distributions, we need some technical preparation.

DEFINITION 5.4 ([**22**, Def. 2.2.5],[**7**]). Let $\mathcal{X}$ be a Banach space with automorphism group Ψ. For $x \in C(\mathcal{X})$ let $\mathfrak{J}(x) = \{f \in L^1(\mathbb{R}^d)\colon f * x = 0\}$. Then the *spectrum* of x is

$$\operatorname{spec}(x) = \{\omega \in \mathbb{R}^d\colon \hat{f}(\omega) = 0 \text{ for all } f \in \mathfrak{J}(x)\}. \tag{5.2}$$

This condition can also be written in the form

$$\operatorname{spec}(x) = \bigcap_{f \in \mathfrak{J}(x)} \hat{f}^{-1}(0), \tag{5.3}$$

which shows immediately that $\operatorname{spec}(x)$ is a closed subset of $\mathbb{R}^d$.
For basic properties of the spectrum, see e.g. [**22**].

REMARK. The spectrum of an element of a Banach space can be defined whenever the action $f * x$ of $f \in L^1(\mathbb{R}^d)$ on $x \in \mathcal{X}$ is well-defined, see the discussion in Section 4.2. An important example is $L^\infty(\mathbb{R}^d)$ with the translation group, see [**62**]. For completeness, we note that for $h \in L^\infty(\mathbb{R}^d)$ a distributional description of the spectrum can be given [**62**, 6.6.1] as

$$\operatorname{spec}(h) = \operatorname{supp} \hat{h}, \tag{5.4}$$

where $\operatorname{supp} \hat{h}$ is the distributional support of the pseudo measure $\hat{h}$.

We need a similar ("weak type") characterization in the general case.

PROPOSITION 5.5. *If $x \in C(\mathcal{X})$, then*

$$\operatorname{spec}(x) = \overline{\bigcup_{x' \in \mathcal{X}'} \operatorname{supp}\, \mathcal{F}(G_{x',x})} = \overline{\bigcup_{x' \in \mathcal{X}'} \operatorname{spec}(G_{x',x})}\,, \tag{5.5}$$

where the Fourier transform $\mathcal{F}$ is used in the distributional sense and $G_{x'x,}(t) = \langle x', \psi_t x\rangle$.

PROOF. Assume first that $\omega \in \operatorname{supp} \mathcal{F}(G_{x',x})$ for some $x' \in \mathcal{X}'$. If $f \in \mathfrak{J}(X)$, then $f * G_{x',x} = G_{x',f*x} = 0$, the first equality by (4.19). Taking the Fourier transform we obtain $\mathcal{F}(f)\mathcal{F}(G_{x',x}) = 0$ (the product exists in the distributional sense, see [**62**, VI.4.10]). As $\omega \in \operatorname{supp} \mathcal{F}(G_{x',x})$, it follows that $\hat{f}(\omega)$ must be 0. This implies that

$$\bigcup_{x' \in \mathcal{X}'} \operatorname{supp}\, \mathcal{F}(G_{x',x}) \subseteq \operatorname{spec}(x)\,.$$

Taking closures on both sides of this relation we obtain

$$\overline{\bigcup_{x' \in \mathcal{X}'} \operatorname{supp}\, \mathcal{F}(G_{x',x})} \subseteq \operatorname{spec}(x)\,.$$

On the other hand, if $\omega \notin \overline{\bigcup_{x' \in \mathcal{X}'} \operatorname{supp}\, \mathcal{F}(G_{x',x})}$, then there exists an open ball $B_\epsilon(\omega)$ around ω with

$$B_\epsilon(\omega) \cap \overline{\bigcup_{x' \in \mathcal{X}'} \operatorname{supp}\, \mathcal{F}(G_{x',x})} = \emptyset.$$

Now for every $f \in L^1(\mathbb{R}^d)$ with $\operatorname{supp} f \subset B_\epsilon(\omega)$ and $\hat{f}(\omega) \neq 0$ the product $\hat{f}\, \widehat{G}_{x',x}$ is zero for every $x' \in \mathcal{X}'$, so $f * x = 0$, and $f \in \mathfrak{J}(x)$. As $\hat{f}(\omega) \neq 0$ we conclude that $\omega \notin \operatorname{spec}(x)$. So we have shown that

$$\operatorname{spec}(x) \subseteq \overline{\bigcup_{x' \in \mathcal{X}'} \operatorname{supp}\, \mathcal{F}(G_{x',x})}\,.$$

The last equality in (5.5) follows from (5.4). ☐

Here is a spectral characterization of σ-bandlimited elements in $\mathcal{X}$.

PROPOSITION 5.6. *An element $x \in C(\mathcal{X})$ is σ-bandlimited if and only if*

$$\operatorname{spec}(x) \subseteq [-\sigma, \sigma]^d .$$

PROOF. Assume first that $\operatorname{spec}(x) \subseteq [-\sigma, \sigma]^d$. Then by Proposition 5.5

$$\operatorname{supp} \mathcal{F}(G_{x',x}) \subseteq [-\sigma, \sigma]^d \qquad \text{for all } x' \in \mathcal{X}'.$$

By the Paley-Wiener-Schwartz theorem [**103**, 3.4.9], [**59**, 7.3.1] the bandlimited function $G_{x',x}$ can be extended to an entire function of exponential type σ, i.e., for every $\epsilon > 0$ there is a constant $A = A(\epsilon)$ such that

$$|G_{x',x}(t + iy)| \le A e^{(\sigma+\epsilon)|y|} \qquad \text{for } t, y \in \mathbb{R}^d .$$

Since $G_{x',x} = \langle x', \psi_t(x)\rangle$ is holomorphic for all $x' \in \mathcal{X}'$, the mapping $t \mapsto \psi_t(x)$ is holomorphic. This implies the existence of $\delta^\alpha(x) \in \mathcal{X}$ for each multi-index α. To deduce (5.1) we use the Bernstein inequality for entire functions [**103**, 3.4.8],

$$\|D^\alpha G_{x',x}\|_\infty \le (2\pi\sigma)^{|\alpha|} \|G_{x',x}\|_\infty \tag{5.6}$$

for all $x' \in \mathcal{X}'$. In particular, the weak type characterization of $\|\delta^\alpha(x)\|_{\mathcal{X}}$ in Lemma 4.23 implies that

$$\begin{aligned} \|\delta^\alpha(x)\|_{\mathcal{X}} &\asymp \sup_{\|x'\|_{\mathcal{X}'} \le 1} \|D^\alpha G_{x',x}\|_\infty \\ &\le (2\pi\sigma)^{|\alpha|} \sup_{\|x'\|_{\mathcal{X}'} \le 1} \|G_{x',x}\|_\infty \le M_\Psi (2\pi\sigma)^{|\alpha|} \|x\|_{\mathcal{X}}. \end{aligned} \tag{5.7}$$

Therefore x is σ-bandlimited.

Conversely, assume that x is bandlimited with bandwidth σ. For arbitrary $t_0 \in \mathbb{R}^d$ and $x' \in \mathcal{X}'$ the weak-type argument representation of $\|\delta^\alpha(x)\|_{\mathcal{X}}$ used above implies that

$$|D^\alpha G_{x',x}(t_0)| \le \|x'\|_{\mathcal{X}'} \, \|\delta^\alpha(x)\|_{\mathcal{X}} \le C(2\pi\sigma)^{|\alpha|} . \tag{5.8}$$

Consequently the Taylor series of $G_{x',x}$ at t_0 converges uniformly on $\mathbb{R}^d$ and can be extended to an entire function

$$G_{x',x}(z) = \sum_{\alpha \ge 0} \frac{D^\alpha G_{x',x}(t_0)}{\alpha!} (z - t_0)^\alpha \qquad \text{for } z \in \mathbb{C}^d .$$

The extension of $G_{x',x}$ is clearly independent of the base point t_0 and satisfies the growth estimate

$$|G_{x',x}(z)| \le C \sum_{\alpha \ge 0} \frac{(2\pi\sigma)^{|\alpha|}}{\alpha!} |z - t_0|^{|\alpha|} \le C e^{2\pi\sigma|z - t_0|} .$$

(Recall that $|\alpha| = |\alpha|_1$.) For $z = t_0 + iy$, $y \in \mathbb{R}^d$, we obtain $|G_{x',x}(t_0 + iy)| \le C e^{2\pi\sigma|y|}$, and thus $G_{x',x}$ is an entire functions of exponential type σ [**100**, 4.8.3] for every $x' \in \mathcal{X}'$. Once again, the Paley-Wiener-Schwartz theorem implies that $\operatorname{supp} \mathcal{F}(G_{x',x}) \subseteq [-\sigma, \sigma]^d$ for all $x' \in \mathcal{X}$. We conclude that $\operatorname{spec}(x)$ is contained in $[-\sigma, \sigma]^d$. ☐

5.2. Periodic Group Actions

If the automorphism group Ψ on $\mathcal{X}$ is *periodic*, the bandlimited elements can be described more explicitly by means of a Banach algebra-valued Fourier series. Without loss of generality we will assume that $P = (1, \dots, 1)$. Remember that the kth Fourier coefficient of $x \in C(\mathcal{X})$ is

$$\hat{x}(k) = \int_{\mathbb{T}^d} \psi_t(x)\, e^{-2\pi i k\cdot t}\, dt. \tag{5.9}$$

If A is a matrix and $\chi_t(A) = M_t A M_{-t}$, the Fourier coefficient $\int_{\mathbb{T}^d} \chi_t(A) e^{-2\pi i k\cdot t}\, dt$ is exactly the kth side-diagonal $\hat{A}(k)$ of A. The formal series $\sum_{k\in\mathbb{Z}^d} \hat{a}(k) e^{2\pi i k\cdot t}$ is the Fourier series of a. See deLeeuw's work [**32**, **33**, **34**] for first developments of operator-valued Fourier series.

Proposition 5.7 ([**33**, Prop. 3.4]). *Let Ψ be a periodic automorphism group on $\mathcal{X}$. The following statements are equivalent for $x \in \mathcal{X}$.*

(1) *$a \in C(\mathcal{X})$.*
(2) *The Fejer-means of the Fourier series of $x \in \mathcal{X}$ converge in norm:*

$$\psi_t(x) = \lim_{n\to\infty} \sum_{|k|_\infty \le n} \prod_{j=1}^{d} \Big(1 - \frac{|k_j|}{n+1}\Big) \hat{x}(k) e^{2\pi i k\cdot t}.$$

(3) *The $C1$-means of the Fourier coefficients converge in norm to x:*

$$x = \lim_{n\to\infty} \sum_{|k|_\infty \le n} \prod_{j=1}^{d} \Big(1 - \frac{|k_j|}{n+1}\Big) \hat{x}(k).$$

DeLeeuw considers only the algebra $\mathcal{B}(\ell^2)$, but the proof for general $\mathcal{X}$ is identical. See also [**62**, 2.12]. An immediate consequence of Proposition 5.7 is a Weierstrass-type density theorem for periodic group actions.

Corollary 5.8.

(1) *The set of bandlimited elements is dense in $C(\mathcal{X})$.*
(2) *$C^k(\mathcal{X})$ is dense in $C(\mathcal{X})$.*
(3) *An element $x \in \mathcal{X}$ is σ-bandlimited if and only if $\psi_t(x)$ is the trigonometric polynomial of the form*

$$\psi_t(x) = \sum_{|k|_\infty \le \sigma} \hat{x}(k) e^{2\pi i k\cdot t} \tag{5.10}$$

We single out a characterization of bandlimited elements of matrix algebras.

Corollary 5.9. *A matrix A is banded with bandwidth N in the matrix algebra $\mathcal{A}$ if and only it is N-bandlimited with respect to the group action $\{\chi_t\}$.*

Remark. Corollary 5.8(1), when applied to $C(\mathcal{S}_0^1)$ gives Equation (4.25). To be more precise, $A \in C(\mathcal{S}_0^1)$ if and only if A is in the closure of the band matrices in $\mathcal{S}_0^1$. As $\mathcal{S}_0^1$ is solid, this is equivalent to $A = \lim_{n\to\infty} \sum_{|k|_\infty \le n} \hat{A}(k)$ in the norm of $\mathcal{S}_0^1$, or, written differently, to

$$\| \sum_{|k|_\infty > n} \hat{A}(k) \|_{\mathcal{S}_0^1} \to 0 \text{ for } n \to \infty\,.$$

Using the definition of the norm of $\mathcal{S}_0^1$, we obtain (4.25).

5.3. Characterization of Smoothness by Approximation

When working with an automorphism group on $\mathcal{X}$, the subspaces of bandlimited elements of given bandwidth provide a natural approximation scheme for $\mathcal{X}$. For this case, we show that the Besov spaces defined in Chapter 4 can be characterized as approximation spaces. Put differently, we state and prove a general version of the Jackson-Bernstein theorem. Although our proofs are similar to the classical ones in [**37, 100, 103**], we gain a new insight from the generalization to Banach spaces. In particular, we need a theory of smoothness based on the action of an automorphism group, and a spectral characterization of bandlimited elements (Section 5.1). Related results were obtained independently in [**57, 101**].

It is important to realize that for a Banach algebra $\mathcal{A}$ the approximation scheme of bandlimited elements is compatible with multiplication (Definition 3.18).

Lemma 5.10. *Let $\mathcal{A}$ be a Banach algebra with automorphism group Ψ, and set*

$$X_0 = \{0\}, \quad X_\sigma = \{a \in \mathcal{A} \colon \operatorname{spec}(a) \subseteq [-\sigma,\sigma]^d\}, \quad \sigma > 0. \tag{5.11}$$

Then $\{X_\sigma \colon \sigma \geq 0\}$ is an approximation scheme for $\mathcal{A}$ consisting of the bandlimited elements.

Proof. We have to show that $X_\sigma X_\tau \subseteq X_{\sigma+\tau}$. But this is the content of Lemma 5.3. □

Remark. From now on we use the approximation scheme of bandlimited elements $\{X_\sigma \colon \sigma \geq 0\}$ without further notice.

Next we formulate a theorem of Jackson-Bernstein type for Banach spaces.

Theorem 5.11. *Let $\mathcal{X}$ be a Banach space with automorphism group Ψ and assume that $r > 0$ and $1 \leq p \leq \infty$. If $\{X_\sigma : \sigma \geq 0\}$ is the approximation scheme of bandlimited elements, then*

$$\Lambda^p_r(\mathcal{X}) = \mathcal{E}^p_r(\mathcal{X})\,. \tag{5.12}$$

We will split the proof into several statements. One of the main tools will be smooth approximating units in $\mathcal{X}$, which we will review next.

Proposition 5.12. *Taylor's formula: if $x \in C^k(\mathcal{X})$, then*

$$\psi_t(x) = \sum_{|\alpha|\leq k} \frac{\delta^\alpha(x)}{\alpha!} t^\alpha + R_k(t,x)\,, \tag{5.13}$$

$$R_k(t,x) = k \sum_{|\alpha|=k} \frac{t^\alpha}{\alpha!} \int_0^1 (1-u)^{k-1} \Delta_{ut}(\delta^\alpha(x))\,du \tag{5.14}$$

$$= (k+1) \sum_{|\alpha|=k+1} \frac{t^\alpha}{\alpha!} \int_0^1 (1-u)^k \psi_{ut}(\delta^\alpha(x))\,du \tag{5.15}$$

Proof. We use a weak-type argument and prove the remainder estimate in the form (5.14). Recall that $G_{x',x}(t) = \langle x', \psi_t(x)\rangle$ for $x' \in \mathcal{X}'$. A version of Taylor's theorem [**43**] yields

$$G_{x',x}(t) = \sum_{|\alpha|\leq k} \frac{D^\alpha G_{x',x}(0)}{\alpha!} t^\alpha + k \sum_{|\alpha|=k} \frac{t^\alpha}{\alpha!} \int_0^1 (1-u)^{k-1} (D^\alpha G_{x',x}(ut) - D^\alpha G_{x',x}(0))\,,$$

or, written explicitly,

$$\langle x', \psi_t(x)\rangle = \langle x', \sum_{|\alpha|\leq k} \frac{\delta^\alpha(x)}{\alpha!} t^\alpha + k \sum_{|\alpha|=k} \frac{t^\alpha}{\alpha!} \int_0^1 (1-u)^{k-1} \Delta_{ut}(\delta^\alpha(x))\,du\rangle\,.$$

As $x' \in \mathcal{X}'$ is arbitrary the proof is complete. The proof of the remainder estimate (5.15) is similar. □

For the construction of approximating units let $f_\rho(t) = \rho^{-d} f(\rho^{-1}t)$, $\rho > 0$, be the dilation of $f \in L^1(\mathbb{R}^d)$. Then

$$f_\rho * x = \int_{\mathbb{R}^n} \psi_{-\rho u}(x) f(u)\, du\,. \tag{5.16}$$

LEMMA 5.13. *Let $\mathcal{X}$ be a Banach space with an automorphism group Ψ acting on it, $x \in C(\mathcal{X})$, and $\kappa \in L^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \kappa(t)\, dt = 1$.*

(1) *If $\kappa \in L^1_{v_1}(\mathbb{R}^d)$, where $v_1(k) = (1 + |k|)$, then*
$$\|x - \kappa_\rho * x\|_{\mathcal{X}} \le C\omega_\rho(x).$$
(2) *If $\kappa \in L^1_{v_{k+1}}(\mathbb{R}^d)$ for some $k \in \mathbb{N}$, and if the moments $\int_{\mathbb{R}^d} t^\alpha \kappa(t)\, dt = 0$ for $1 \le |\alpha| \le k$, then for every $x \in C^k(\mathcal{X})$*
$$\|x - \kappa_\rho * x\|_{\mathcal{X}} \le C\rho^k \sum_{|\beta|=k} \omega_\rho(\delta^\beta(x))\,.$$
(3) *If $\kappa \in L^1_{v_{k+2}}(\mathbb{R}^d)$ for some $k \in \mathbb{N}$, if $\int_{\mathbb{R}^d} t^\alpha \kappa(t)\, dt = 0$ for $1 \le |\alpha| \le k+1$, and if $\kappa(-t) = \kappa(t)$ for all $k \in \mathbb{R}^d$, then for every $x \in C^k(\mathcal{X})$*
$$\|x - \kappa_\rho * x\|_{\mathcal{X}} \le C\rho^k \sum_{|\beta|=k} \omega^2_\rho(\delta^\beta(x))$$

The constants C depend on k.

PROOF. The proof is similar to standard approximation results for $C(\mathbb{T})$ or $C_u(\mathbb{R}^d)$. We want to estimate the identity

$$x - \kappa_\rho * x = \int_{\mathbb{R}^d} \kappa(t)\big(x - \psi_{-\rho t}(x)\big)\, dt\,.$$

Part (1) follows from

$$\|x - \kappa_\rho * x\|_{\mathcal{X}} \le \int_{\mathbb{R}^n} |\kappa(u)|\, \|x - \psi_{-\rho u}(x)\|_{\mathcal{X}}\, du \le \int_{\mathbb{R}^d} |\kappa(u)|\, \omega_{\rho|u|}(x)\, du,$$

and the property

$$\omega_{\rho|u|}(x) \le M_\Psi\,(1 + |u|)\omega_\rho(x)\,,$$

see Lemma 4.29.

The proof of (2) uses Taylor's formula (5.13) for $\psi_{\rho t}(x)$. As $\int_{\mathbb{R}^d} \kappa(t) t^\alpha\, dt = 0$ for $1 \le |\alpha| \le k$, only the remainder term $R_k(t, x)$ does not cancel in the integral (5.3). We obtain the norm estimate

(5.17)

$$\begin{aligned}
\|x - \kappa_\rho * x\|_{\mathcal{X}} &\le k\Big\| \sum_{|\alpha|=k} \frac{1}{\alpha!} \int_{\mathbb{R}^d} \kappa(t)(-\rho t)^\alpha \int_0^1 (1-u)^{k-1} \Delta_{-u\rho t}(\delta^\alpha x)\, du\, dt \Big\|_{\mathcal{X}} \\
&\le k\rho^k \sum_{|\alpha|=k} \frac{1}{\alpha!} \int_{\mathbb{R}^d} |\kappa(t)||t|^{k-1} \int_0^1 (1-u)^{k-1} \|\Delta_{-u\rho t}(\delta^\alpha x)\|_{\mathcal{X}}\, du\, dt \\
&\le \rho^k \sum_{|\alpha|=k} \frac{1}{\alpha!} \int_{\mathbb{R}^d} |\kappa(t)||t|^{k-1} \omega_{\rho|t|}(\delta^\alpha x)\, dt \\
&\le \rho^k \sum_{|\alpha|=k} \frac{1}{\alpha!} \omega_\rho(\delta^\alpha x) \int_{\mathbb{R}^d} |\kappa(t)||t|^{k-1}(1 + |t|)\, dt \\
&= C\rho^k \sum_{|\alpha|=k} \omega_\rho(\delta^\alpha x)\,.
\end{aligned}$$

For the proof of (3) we use the fact that κ is an even function and therefore

$$x - \kappa_\rho * x = \frac{1}{2}\int_{\mathbb{R}^d} \kappa(t)(2x - \psi_{-\rho t}(x) - \psi_{\rho t}(x))\, dt = \frac{1}{2}\int_{\mathbb{R}^d} \kappa(t)\psi_{-\rho t}\big(\Delta^2_{\rho t}(x)\big)\, dt\,.$$

We proceed as in the proof of (2), but we use the remainder estimate (5.15) for $R_{k-1}(-\rho t, x)$ and obtain

$$x - \kappa_\rho * x = \frac{k+1}{2}\sum_{|\alpha|=k}\frac{1}{\alpha!}\int_{\mathbb{R}^d}\kappa(t)(-\rho t)^\alpha \int_0^1 (1-u)^{k-1}\Delta^2_{\rho t}\big(\psi_{-u\rho t}(\delta^\alpha x)\big)\, du\, dt\,.$$

This yields

$$\begin{aligned}
\|x - \kappa_\rho * x\|_{\mathcal{X}} &\le \frac{k+1}{2}\rho^k \sum_{|\alpha|=k+1}\frac{1}{\alpha!}\int_{\mathbb{R}^d}|\kappa(t)||t|^k\int_0^1(1-u)^{k-1}\|\psi_{-ut}\big(\Delta^2_{u\rho t}(\delta^\alpha x)\big)\|_{\mathcal{X}}\, du\, dt\\
&\le C\rho^k\sum_{|\alpha|=k}\frac{1}{\alpha!}\int_{\mathbb{R}^d}|\kappa(t)||t|^k\omega^2_{\rho|t|}(\delta^\alpha x)\, dt\\
&\le C\rho^k\sum_{|\alpha|=k}\frac{1}{\alpha!}\omega^2_\rho(\delta^\alpha x)\int_{\mathbb{R}^d}|\kappa(t)||t|^k(1+|t|)^2\, dt\\
&\le C\rho^k\sum_{|\alpha|=k}\omega^2_\rho(\delta^\alpha x)\,.
\end{aligned}$$

□

We need another property of the spectrum.

LEMMA 5.14. *If $x \in C(\mathcal{X})$ and $f \in L^1(\mathbb{R}^d)$, then*

$$\operatorname{spec}(f * x) \subseteq \operatorname{supp}(\mathcal{F}f) \cap \operatorname{spec}(x). \tag{5.18}$$

PROOF.

$$\begin{aligned}
\operatorname{spec}(f*x) &= \overline{\bigcup_{x'\in\mathcal{X}'}\operatorname{supp}(\mathcal{F}G_{x',f*x})}\\
&= \overline{\bigcup_{x'\in\mathcal{X}'}\operatorname{supp}\big(\mathcal{F}(f*G_{x',x})\big)}\\
&= \overline{\bigcup_{x'\in\mathcal{X}'}\operatorname{supp}\big(\mathcal{F}f\,\mathcal{F}G_{x',x}\big)}\\
&\subseteq \overline{\bigcup_{x'\in\mathcal{X}'}\operatorname{supp}(\mathcal{F}f)\cap\operatorname{supp}(\mathcal{F}G_{x',x})}\\
&= \overline{\operatorname{supp}(\mathcal{F}f)\cap\bigcup_{x'\in\mathcal{X}'}\operatorname{supp}(\mathcal{F}G_{x',x})}\\
&\subseteq \operatorname{supp}(\mathcal{F}f)\cap\overline{\bigcup_{x'\in\mathcal{X}'}\operatorname{supp}(\mathcal{F}G_{x',x})}\\
&= \operatorname{supp}(\mathcal{F}f)\cap\operatorname{spec}(x)\,,
\end{aligned}$$

where we have used that $\operatorname{supp}(\mathcal{F}f)$ is closed, and that $\operatorname{supp}((\mathcal{F}f)G) \subseteq \operatorname{supp}(\mathcal{F}f)\cap\operatorname{supp}(G)$ for $f \in L^1(\mathbb{R}^d)$ and $G \in \mathcal{F}(L^\infty(\mathbb{R}^d))$ ([**62**, 6.4.10]). □

With the existence of approximating kernels we can now state a Jackson-type theorem for automorphism groups.

PROPOSITION 5.15. *Let $x \in \mathcal{X}$ and $\sigma > 0$.*

(1) *There is a σ-bandlimited element $x_\sigma \in C(\mathcal{X})$ such that*

$$\|x - x_\sigma\|_{\mathcal{X}} \le C\omega_{1/\sigma}(x)$$

with C independent of σ and x.

(2) *If $x \in C^k(\mathcal{X})$, then there exists a σ-bandlimited element $x_\sigma \in \mathcal{X}$ such that*

$$\|x - x_\sigma\|_{\mathcal{X}} \leq C\sigma^{-k} \sum_{|\alpha|=k} \omega^2_{1/\sigma}(\delta^\alpha x)\,.$$

Proof. (1) We follow [**103**, 4.4.3]. Let $\kappa \in S(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} \kappa = 1$, supp $\mathcal{F}\kappa \subseteq [-1,1]^d$. By Lemma 5.13(1)

$$\|x - \kappa_{1/\sigma} * x\|_{\mathcal{X}} \leq C\omega_{1/\sigma}(x).$$

Since supp $\mathcal{F}(\kappa_{1/\sigma}) \subseteq [-\sigma,\sigma]^d$, Lemma 5.14 implies that $\kappa_{1/\sigma} * x$ is σ-bandlimited, and we can take $x_\sigma = \kappa_{1/\sigma} * x$.

(2) The proof is similar. We choose an even kernel $\kappa \in S(\mathbb{R}^d)$ that satisfies $\int_{\mathbb{R}^d} \kappa(t)\,dt = 1$ and $\int_{\mathbb{R}^d} t^\alpha \kappa(t)\,dt = 0$ for $1 \leq |\alpha| \leq k+1$. Now we use part (3) of Lemma 5.13 instead of part (1). □

We draw two consequences of Proposition 5.15. The first one is a density result in the style of Weierstrass' theorem, the second one is a Jackson-type theorem that proves one half of the fundamental Theorem 5.11.

Corollary 5.16 (Weierstrass). *The set of bandlimited elements is dense in $C(\mathcal{X})$. Since $C^k(\mathcal{X})$ contains the bandlimited elements, $C^k(\mathcal{X})$ is also dense in $C(\mathcal{X})$.*

If the group action Ψ is periodic, we obtain again Corollary 5.8.

Corollary 5.17. *If $x \in \Lambda^p_r(\mathcal{X})$ for $r > 0$, then $x \in \mathcal{E}^p_r(\mathcal{X})$.*

Proof. We use the integral version of the norm for an approximation space in (3.4) and assume that $1 \leq p < \infty$. The proof for $p = \infty$ is simpler.

Assume first that $0 < r < 1$. Then, by Proposition 5.15(1),

$$\int_1^\infty \big(E_\sigma(x)\sigma^r\big)^p \frac{d\sigma}{\sigma} \leq C \int_0^1 \big(\omega_\tau(x)\tau^{-r}\big)^p \frac{d\tau}{\tau} \leq C|x|^p_{\Lambda^p_r(\mathcal{X})}\,,$$

and so the approximation norm is dominated by the Besov norm.

If $r = k + \eta, 0 < \eta \leq 1$, and $k \in \mathbb{N}$, we use Proposition 5.15(2), and get

$$\int_1^\infty \big(E_\sigma(x)\sigma^r\big)^p \frac{d\sigma}{\sigma} \leq C \sum_{|\alpha|=k} \int_0^1 \big(\omega^2_\tau(\delta^\alpha(x))\tau^{-\eta}\big)^p \frac{d\tau}{\tau}$$

and so $\|x\|_{\mathcal{E}^p_r(\mathcal{X})}$ is dominated by the Besov norm (see Proposition 4.31). □

Before proving the converse implication in Theorem 5.11, i.e., the Bernstein-type result, we need a mean-value property of automorphism groups.

Lemma 5.18. *If x is σ-bandlimited, then*

$$\|\Delta_t x\|_{\mathcal{X}} \leq C\sigma\,|t|\,\|x\|_{\mathcal{X}}\,. \tag{5.19}$$

Proof. We use a weak-type argument.

$$\begin{aligned}\|\Delta_t x\|_{\mathcal{X}} &= \sup_{\|x'\|\leq 1} |\langle x', \psi_t(x) - x\rangle| = \sup_{\|x'\|\leq 1} \Big|\int_0^1 \nabla G_{x',x}(\lambda t)\cdot t\,d\lambda\Big| \\ &\leq \sup_{\|x'\|\leq 1} C|t|_2 \|\,|\nabla G_{x',x}|_2\|_\infty\,.\end{aligned}$$

Since $G_{x',x}$ is bandlimited, Bernstein's inequality for scalar functions yields that $\|\,|\nabla G_{x',x}|_2\|_\infty \leq C\sigma\|G_{x',x}\|_\infty$. We may continue the estimate by

$$\begin{aligned}\|\Delta_t x\|_{\mathcal{X}} &\lesssim |t|_2 \sup_{\|x'\|\le 1} \||\nabla G_{x',x}|_2\|_\infty \lesssim |t|_2\,\sigma \sup_{\|x'\|\le 1} \|G_{x',x}\|_\infty \\ &\lesssim |t|_2\,\sigma\,\|x\|_{\mathcal{X}} \lesssim \sigma\,|t|\,\|x\|_{\mathcal{X}}\,.\end{aligned}$$

□

Proposition 5.19. *Let $x \in \mathcal{X}$, and $r > 0$, $1 \le p \le \infty$. If $x \in \mathcal{E}^p_r(\mathcal{X})$, then $x \in \Lambda^p_r(\mathcal{X})$.*

Proof. We adapt a standard proof (e.g.,[**28**]) and verify the statement for $p < \infty$. We will work with the discrete Besov norm (see Proposition 4.31)

$$\|x\|_{\Lambda^p_r(\mathcal{X})} \asymp \|x\|_{\mathcal{X}} + \Big(\sum_{l=0}^{\infty} \big(2^{lr}\omega^m_{2^{-l}}(x)\big)^p\Big)^{1/p},$$

where $m > \lfloor r \rfloor$. If $x \in \mathcal{E}^p_r(\mathcal{X})$, the Corollary of the Representation Theorem (Corollary 3.12 implies that

$$x = \sum_{k=0}^{\infty} x_k, \quad \text{with } x_k \in X_{2^k} \quad \text{and } \sum_{k=0}^{\infty} 2^{krp}\|x_k\|^p_{\mathcal{X}} < \infty\,, \tag{5.20}$$

where $(X_\sigma)_{\sigma\ge 0}$ is the approximation scheme of bandlimited elements. By Corollary 3.12 or a direct application of Hölders inequality $\sum_{k=0}^{\infty} x_k$ is convergent in $\mathcal{X}$. Note that (5.20) implies that

$$\|x_k\|_{\mathcal{X}} \le C 2^{-kr} \tag{5.21}$$

for all $k \in \mathbb{N}_0$.

We assume first that $0 < r < 1$. We need an estimate for the norm of $\Delta_t x$.

$$\begin{aligned}\|\Delta_t x\|_{\mathcal{X}} &\le \sum_{k=0}^{M} \|\Delta_t x_k\|_{\mathcal{X}} + \sum_{k=M+1}^{\infty} \|\Delta_t x_k\|_{\mathcal{X}} \\ &\le \sum_{k=0}^{M} \|\Delta_t x_k\|_{\mathcal{X}} + (M_\Psi + 1) \sum_{k=M+1}^{\infty} \|x_k\|_{\mathcal{X}}\,,\end{aligned} \tag{5.22}$$

where the value of M will be chosen later.

Lemma 5.18 implies that

$$\|\Delta_t x_k\|_{\mathcal{X}} \le C 2^k |t|\,\|x_k\|_{\mathcal{X}}$$

for all $k \in \mathbb{N}$. Substituting back into (5.22) yields

$$\|\Delta_t x\|_{\mathcal{X}} \le C\Big(\sum_{k=0}^{M} 2^k |t| \|x_k\|_{\mathcal{X}} + \sum_{k=M+1}^{\infty} \|x_k\|_{\mathcal{X}}\Big)\,. \tag{5.23}$$

We use this relation for the estimation of the Besov seminorm.

$$\begin{aligned}|x|_{\Lambda^p_r(\mathcal{X})} &\asymp \Big(\sum_{l=0}^{\infty} \big(2^{lr}\omega_{2^{-l}}(x)\big)^p\Big)^{1/p} \\ &\lesssim \Big(\sum_{l=0}^{\infty} 2^{lrp}\Big(\sum_{k=0}^{M} 2^k 2^{-l}\|x_k\|_{\mathcal{X}} + \sum_{k=M+1}^{\infty} \|x_k\|_{\mathcal{X}}\Big)^p\Big)^{1/p}\,.\end{aligned}$$

We split this expression into two parts and assume that $M = l$ in the inner sums.

$$|x|_{\Lambda^p_r(\mathcal{X})} \lesssim \Big(\sum_{l=0}^{\infty} 2^{l(r-1)p}\Big(\sum_{k=0}^{l} 2^k \|x_k\|_{\mathcal{X}}\Big)^p\Big)^{1/p} + \Big(\sum_{l=0}^{\infty} 2^{lrp}\Big(\sum_{k=l+1}^{\infty} \|x_k\|_{\mathcal{X}}\Big)^p\Big)^{1/p}.$$

We apply Hardy's inequalities (Appendix A) to both terms on the right hand side and obtain

$$|x|_{\Lambda^p_r(\mathcal{X})} \lesssim \Big(\sum_{l=0}^{\infty} 2^{l(r-1)p}2^{lp}\|x_l\|^p_{\mathcal{X}}\Big)^{1/p} + \Big(\sum_{l=0}^{\infty} 2^{lrp}\|x_l\|^p_{\mathcal{X}}\Big)^{1/p} = 2\Big(\sum_{l=0}^{\infty} 2^{lrp}\|x_l\|^p_{\mathcal{X}}\Big)^{1/p}.$$

After taking the infimum over the representations $x = \sum_{k=0}^{\infty} x_k$ as in (3.16) we conclude that $|x|_{\Lambda^p_r(\mathcal{X})} \lesssim \|x\|_{\widetilde{\mathcal{E}}^p_r(\mathcal{X})}$. Next consider the case $r = m + \eta$ for $m \in \mathbb{N}_0$ and $0 < \eta < 1$. By (5.7) we have

$$\|\delta^\alpha(x_k)\|_{\mathcal{X}} \le C(2\pi 2^k)^{|\alpha|}\|x_k\|_{\mathcal{X}}$$

for all $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^d$. Consequently the series $\sum_{k=0}^{\infty} \delta^\alpha x_k$ converges in $\mathcal{X}$ for all α with $|\alpha| \le m$ and its sum must be $\delta^\alpha(x)$ (because each δ_j is closed on $\mathcal{D}(\delta^\alpha)$). We now apply the above estimates (5.22) and (5.23) with $\delta^\alpha(x)$ instead of x and deduce that $\delta^\alpha(a)$ must be in $\Lambda^p_\eta(\mathcal{X})$ for $|\alpha| \le k$ by Proposition 4.31. Thus $x \in \Lambda^p_r(\mathcal{X})$.

If r is an integer, then we have to use second order differences and a corresponding version of the mean value theorem. The argument is almost the same as above (see [**103**] for details in the scalar case). ☐

Combining Propositions 5.17 and 5.19, we have completed the proof of Theorem 5.11.

5.4. Littlewood-Paley Decomposition

The various equivalent norms for Besov spaces given Proposition 4.31 are not easily computable. We construct another explicit norm for these spaces by means of a Littlewood-Paley decomposition.

The procedure is well-known. We include the derivation of the relevant results to keep the presentation self-contained. We follow [**15**], but we use approximation arguments where feasible.

Dyadic scaling functions. Let $\varphi \in \mathscr{S}(\mathbb{R}^d)$ with

$$\begin{aligned} \operatorname{supp}\hat\varphi &\subseteq \{\omega \in \mathbb{R}^d \colon 2^{-1} \le |\omega|_\infty \le 2\}, \\ \hat\varphi(\omega) &> 0 \quad \text{for} \quad 2^{-1} < |\omega|_\infty < 2, \\ \sum_{k\in\mathbb{Z}} \hat\varphi(2^{-k}\omega) &= 1 \quad \text{for all} \quad \omega \in \mathbb{R}^d \setminus \{0\} \end{aligned} \tag{5.24}$$

Set $\hat\varphi_k(\omega) = \hat\varphi(2^{-k}\omega), k \in \mathbb{N}_0$, so $\varphi_k(x) = 2^{kd}\varphi_0(2^k x)$, and let $\hat\varphi_{-1} = 1 - \sum_{k=0}^{\infty}\hat\varphi_k$. We call $\{\hat\varphi_k\}_{k\ge -1}$ a dyadic partition of unity.

Obviously, $\operatorname{supp}\hat\varphi_k = 2^k \operatorname{supp}\hat\varphi \subseteq \{\omega\colon 2^{k-1} \le |\omega|_\infty \le 2^{k+1}\}$ for $k \ge 0$, and $\operatorname{supp}\varphi_{-1} \subseteq \{\omega\colon |\omega|_\infty \le 1\}$. As the intersection of $\operatorname{supp}(\hat\varphi_k)$ with $\operatorname{supp}(\hat\varphi_l)$ is nonempty only for $l \in \{k-1, k, k+1\}$ we obtain that

$$\begin{aligned} \varphi_k &= \varphi_k * (\varphi_{k-1} + \varphi_k + \varphi_{k+1}) \quad k \geq 0, \\ \varphi_{-1} &= \varphi_{-1} * (\varphi_{-1} + \varphi_0)\,. \end{aligned} \tag{5.25}$$

REMARKS.

(1) The existence of dyadic partitions of unity is elementary. A common construction works as follows: Choose a function $f \in \mathscr{S}(\mathbb{R}^d)$ that satisfies the first two conditions above, and set $F(\omega) = \sum_{k=-\infty}^{\infty} f(2^{-k}\omega)$. Then $\hat{\varphi} = f/F$ has the desired properties.

(2) The condition that $\varphi \in \mathscr{S}(\mathbb{R}^d)$ can be weakened considerably [**102**].

Littlewood-Paley decomposition. For $x \in C(\mathcal{X})$ each element $\varphi_k * x$ is well defined in $\mathcal{X}$. As $\sum_{k=-1}^{\infty} \varphi_k = \delta$ in $\mathscr{S}'$, it is natural to ask about the convergence of the *Littlewood-Paley decomposition*

$$x \sim \sum_{k=0}^{\infty} \varphi_k * x\,. \tag{5.26}$$

PROPOSITION 5.20. *Let $x \in \mathcal{X}$ and let $\{\hat{\varphi}_k\}_{k \geq -1}$ be a dyadic partition of unity, and $r > 0$. An element $x \in \mathcal{X}$ is in $\Lambda_r^p(\mathcal{X})$, if and only if*

$$\left(\sum_{k=-1}^{\infty} 2^{rkp} \|\varphi_k * x\|_{\mathcal{X}}^p \right)^{1/p} < \infty\,. \tag{5.27}$$

The expression (5.27) defines an equivalent norm on $\Lambda_r^p(\mathcal{X})$. Moreover the Littlewood-Paley decomposition (5.26) is convergent in the norm of $\mathcal{X}$.

If $p = \infty$, we can prove the statement by a weak type argument and use the corresponding results for functions (see [**52**]). This approach does not work for $p < \infty$, so we adapt the proof given in [**15**].

PROOF. Assume first that (5.27) holds. Then $\|\varphi_k * x\|_{\mathcal{X}} \leq C 2^{-rk}$, and so the series $\sum_{k=-1}^{\infty} \varphi_k * x$ is norm convergent in $\mathcal{X}$. We use a distributional argument to show that the limit is actually x. Actually, $y = \sum_{k=-1}^{\infty} \varphi_k * x$ implies $G_{x',y} = \sum_{k=-1}^{\infty} \varphi_k * G_{x',x}$ with convergence in the norm of $L^\infty(\mathbb{R}^d)$ for all $x' \in \mathcal{X}'$. Choose a test function $\upsilon \in \mathscr{S}$, then

$$\langle G_{x',y}, \upsilon \rangle = \langle \sum_{k=-1}^{\infty} \varphi_k * G_{x',x}, \upsilon \rangle = \langle \sum_{k=-1}^{\infty} \hat{\varphi}_k \hat{G}_{x',x}, \hat{\upsilon} \rangle\,.$$

where $\langle \;,\; \rangle$ denotes the dual pairing between $\mathscr{S}'$ and $\mathscr{S}$. As $\sum_{k=-1}^{\infty} \hat{\varphi}_k = 1$ in $\mathscr{S}'$, we conclude

$$\langle \sum_{k=-1}^{\infty} \hat{\varphi}_k \hat{G}_{x',x}, \hat{\upsilon} \rangle = \langle \hat{G}_{x',x}, \hat{\upsilon} \rangle = \langle G_{x',x}, \upsilon \rangle\,.$$

As the identity

$$\langle \widehat{G}_{x',y}, \widehat{\upsilon} \rangle = \langle G_{x',x}, \upsilon \rangle$$

is valid for all $\upsilon \in \mathscr{S}$, we conclude that $G_{x',x} = G_{x',y}$ for all $x' \in \mathcal{X}'$, and this implies $y = x$.

For $x \in \Lambda_r^p(\mathcal{X})$ and $m > \lfloor r \rfloor$ we use the norm equivalence

$$\|x\|_{\Lambda_r^p(\mathcal{X})} \asymp \|x\|_{\mathcal{X}} + \left(\sum_{k=0}^{\infty} \left(2^{rk} \omega_{2^{-k}}^m(x) \right)^p \right)^{1/p}\,.$$

As $\|\Delta_t^m(\varphi_k * x)\|_{\mathcal{X}} \leq C_m\|\varphi_k * x\|_{\mathcal{X}}$ by Lemma 4.29 (1), and $\|\Delta_t^m(\varphi_k * x)\|_{\mathcal{X}} \leq C|t|^m 2^{mk}\|\varphi_k * x\|_{\mathcal{X}}$ by repeated application of Lemma 5.18 we conclude that

$$\|\Delta_t^m(\varphi_k * x)\|_{\mathcal{X}} \leq C\min(1, |t|^m 2^{mk})\|\varphi_k * x\|_{\mathcal{X}}\,. \tag{5.28}$$

As an immediate consequence we obtain

$$\omega_{|t|}^m(x) \leq C\sum_{k=-1}^{\infty}\min(1, t^m 2^{mk})\|\varphi_k * x\|_{\mathcal{X}}\,, \tag{5.29}$$

and so

$$2^{rj}\omega_{2^{-j}}^m(x) \leq C\sum_{k=1}^{\infty}2^{(j-k)r}2^{kr}\min(1, 2^{-(j-k)m})\|\varphi_k * x\|_{\mathcal{X}}\,. \tag{5.30}$$

The right hand side of this relation can be written as a convolution. Set

$$u(l) = \min(1, 2^{-lm})2^{lr} \quad \text{for} \quad l \in \mathbb{Z}\,,$$

$$v(l) = \begin{cases} 2^{lr}\|\varphi_l * x\|_{\mathcal{X}} & , \quad l > -1 \\ 0 & , \quad l < 0 \end{cases}$$

then u and v are sequences in $\ell^1(\mathbb{Z})$, and the right hand side of (5.30) is just $(u * v)(j)$.

So

$$\|\left(2^{rj}\omega_{2^{-j}}^m(x)\right)_{j\in\mathbb{N}}\|_{\ell^p(\mathbb{N})} \leq C\|u\|_{\ell^1(\mathbb{Z})}\|v\|_{\ell^p(\mathbb{Z})}\,,$$

and this means that

$$\|x\|_{\Lambda_r^p(\mathcal{X})} \leq C\Big(\sum_{k=-1}^{\infty}2^{rkp}\|\varphi_k * x\|_{\mathcal{X}}^p\Big)^{1/p}\,, \tag{5.31}$$

so (5.27) implies that $x \in \Lambda_r^p(\mathcal{X})$.

For the other inequality we use

$$\|x\|_{\Lambda_r^p(\mathcal{X})} \asymp \|x\|_{C^m(\mathcal{X})} + \sum_{|\alpha|=m}\|\delta^\alpha(x)\|_{\Lambda_{r-m}^p(\mathcal{X})}$$

with $m < r \leq m+1$.

First we show the following inequalities.

$$\|\varphi_k * x\|_{\mathcal{X}} \leq C2^{-mk}\|\varphi_k * \delta^\alpha(x)\|_{\mathcal{X}}\,, \quad m = |\alpha| \tag{5.32}$$

and

$$\|\varphi_k * \delta^\alpha(x)\|_{\mathcal{X}} \leq C\omega_{2^{-k}}^2(\delta^\alpha x). \tag{5.33}$$

For the proof of these relations choose $\Phi \in S(\mathbb{R}^d)$ such that

$$\hat{\Phi} \equiv 1 \quad \text{on } \operatorname{supp}\hat{\varphi}_0,$$
$$\hat{\Phi} \equiv 0 \quad \text{in a neighbourhood of } 0$$

and $\hat{\Phi}(\omega) = \hat{\Phi}(-\omega)$ for all $\omega \in \mathbb{R}^d$, so $\hat{\Phi}$ and Φ are even functions. Set $\Phi_k(t) = 2^{kd}\Phi(2^k t)$, then $\|\Phi_k\|_1 = \|\Phi\|_1$ and $\Phi_k * \varphi_k = \varphi_k$.

The function $\eta^{(\alpha)}$ defined by

$$\hat{\eta}^{(\alpha)}(\omega) = \frac{\hat{\Phi}(\omega)}{(2\pi i\omega)^\alpha}$$

is an element of $\mathscr{S}$. Again, if we set $\eta_k^{(\alpha)}(t) = 2^{kd}\eta^{(\alpha)}(2^k t)$, then $\|\eta_k^{(\alpha)}\|_1 = \|\eta^{(\alpha)}\|_1$. Then

$$\hat{\Phi}_k(\omega) = \hat{\Phi}(2^{-k}\omega) = 2^{-k|\alpha|}(2\pi i\omega)^\alpha\hat{\eta}_k^{(\alpha)}(w)\,,$$

and so, assuming that $|\alpha| = m$,

$$\hat{\varphi}_k(\omega) = 2^{-km}\hat{\eta}_k^{(\alpha)}(\omega)(2\pi i\omega)^\alpha\hat{\varphi}_k(\omega)$$

for all $\omega \in \mathbb{R}^d$, which implies

$$\varphi_k * x = 2^{-k\,m}\eta_k^{(\alpha)} * \delta^\alpha(\varphi_k * x) = 2^{-k\,m}\eta_k^{(\alpha)} * \varphi_k * \delta^\alpha(x),$$

the last equality by (4.18). Now (5.32) follows immediately.

For the proof of (5.33) set $y = \delta^\alpha(x)$ and $y_k = \varphi_k * y = \Phi_k * \varphi_k * y = \Phi_k * y_k$. We obtain

$$\begin{aligned}\varphi_k * y = \Phi_k * y_k &= \int_{\mathbb{R}^d} \Phi_k(t)\psi_{-t}(y_k)\,dt\\ &= \tfrac{1}{2}\int_{\mathbb{R}^d} \Phi_k(t)\big\{\psi_{-t}(y_k) - 2y_k + \psi_t(y_k)\big\}\,dt\\ &= \tfrac{1}{2}\int_{\mathbb{R}^d} \Phi_k(t)\psi_{-t}\Delta_t^2(y_k)\,dt\,,\end{aligned}$$

as $\int_{\mathbb{R}^d}\Phi_k = 0$ and $\Phi_k(-t) = \Phi_k(t)$. Changing variables we obtain

$$\varphi_k * y = \tfrac{1}{2}\int_{\mathbb{R}^d} \Phi(u)\psi_{-2^{-k}u}\Delta^2_{2^{-k}u}(y_k)\,dt = \tfrac{1}{2}\int_{\mathbb{R}^d} \Phi(u)\psi_{-2^{-k}u}\big(\varphi_k * \Delta^2_{2^{-k}u}(y)\big)\,dt.$$

Taking norms we get

$$\begin{aligned}\|\varphi_k * y\|_{\mathcal{X}} &\le \tfrac{M_\psi}{2}\int_{\mathbb{R}^d}|\Phi(u)|\|\varphi_k\|_1\omega^2_{2^{-k}|u|}(y)\,dt.\\ &\le \tfrac{M_\psi}{2}\|\varphi_0\|_1\int_{\mathbb{R}^d}|\Phi(u)|(1+|u|^2)\omega^2_{2^{-k}}(y)\,dt\\ &\le C\omega^2_{2^{-k}}(y)\,,\end{aligned}$$

where the estimate for $\omega^2_{2^{-k}|u|}(y)$ follows from Lemma 4.29. This is what we wanted to show.

The proof of the reverse inclusion now follows by putting (5.32) and (5.33) together.

$$2^{rk}\|\varphi_k * x\|_{\mathcal{X}} \le C2^{(r-m)k}\|\varphi_k * \delta^\alpha(x)\|_{\mathcal{X}} \le C2^{(r-m)k}\omega^2_{2^{-k}}(\delta^\alpha(x))\,,$$

and so

$$\begin{aligned}\sum_{k=-1}^{\infty} 2^{rpk}\|\varphi_k * x\|^p_{\mathcal{X}} &\le C\big(\|x\|^p_{\mathcal{X}} + \sum_{k=0}^{\infty}2^{(r-m)pk}\omega^2_{2^{-k}}(\delta^\alpha(x))^p\big)\\ &\le C'(\|x\|^p_{\mathcal{X}} + |\delta^\alpha(x)|^p_{\Lambda^p_{r-m}(\mathcal{X})})\\ &\le C"\|x\|^p_{\Lambda^p_{r-m}(\mathcal{X})}\,.\end{aligned} \tag{5.34}$$

We have shown that $x \in \Lambda^p_r(\mathcal{X})$ implies (5.27). The norm equivalence follows from (5.31) and (5.34). ☐

5.5. Approximation of Polynomial Order in Homogeneous Matrix Spaces

If the action of Ψ on $\mathcal{X}$ is periodic, then for $x \in C(\mathcal{X})$, and $\{\varphi_k\}_{k\ge -1}$ a dyadic partition of unity,

$$\varphi_k * x = \sum_{\lfloor 2^{k-1}\rfloor \le |l|_\infty < 2^{k+1}} \hat{\varphi}_k(l)\hat{x}(l)\,. \tag{5.35}$$

PROOF. Let $\varphi_k^\Pi(t) = \sum_{l\in\mathbb{Z}^d} \varphi_k(t+l)$ denote the periodization of φ_k. Then

$$\varphi_k^\Pi(t) = \sum_{\lfloor 2^{k-1}\rfloor \le |l|_\infty < 2^{k+1}} \hat{\varphi}_k(l) e^{2\pi i l\cdot t}$$

by Poisson's summation formula. As

$$\varphi_k * x = \int_{\mathbb{R}^d} \psi_{-t}(x)\varphi_k(t)\, dt = \int_{\mathbb{T}^d} \psi_{-t}(x)\varphi_k^\Pi(t)\, dt\,,$$

Equation (5.35) follows. □

With this result, and with the Jackson-Bernstein Theorem of the previous section we get a constructive characterization of the approximation spaces for homogeneous matrix algebras.

PROPOSITION 5.21. *Let $\mathcal{A}$ be a homogeneous matrix algebra, $r > 0$, and $\Phi = \{\varphi_k\}_{k\ge -1}$ a dyadic partition of unity. Then the norm on the approximation space $\mathcal{E}_r^p(\mathcal{A}) = \Lambda_r^p(\mathcal{A})$ is equivalent to*

$$\|A\|_{\mathcal{E}_r^p(\mathcal{A})} \asymp \Big(\sum_{k=0}^\infty 2^{kpr} \Big\| \sum_{\lfloor 2^{k-1}\rfloor \le |l|_\infty < 2^{k+1}} \hat{\varphi}_k(l)\hat{A}(l)\Big\|_{\mathcal{A}}^p\Big)^{1/p}. \tag{5.36}$$

If $v_s^*(k) = 1 + (2\pi|k|)^2)^{s/2}$, $s > 0$ is a Bessel weight, then

$$\mathcal{E}_r^p(\mathcal{A}_{v_s^*}) = \mathcal{E}_{r+s}^p(\mathcal{A})\,.$$

The spaces $\mathcal{E}_r^p(\mathcal{A})$ and $\mathcal{A}_{v_s^*}$ are inverse-closed subalgebras of $\mathcal{A}$.

If $\mathcal{A}$ is solid, then the statements simplify to

$$\|A\|_{\mathcal{E}_r^p(\mathcal{A})} \asymp \Big(\sum_{k=-1}^\infty 2^{kpr} \Big\| \sum_{\lfloor 2^{k}\rfloor \le |l|_\infty < 2^{k+1}} \hat{A}(l)\Big\|_{\mathcal{A}}^p\Big)^{1/p}. \tag{5.37}$$

and

$$\mathcal{E}_r^p(\mathcal{A}_{v_s}) = \mathcal{E}_{r+s}^p(\mathcal{A})\,. \tag{5.38}$$

PROOF. The results for general homogeneous matrix algebras follow from the Jackson Bernstein Theorem (Theorem 5.11), the Littlewood-Paley decomposition, and the results of Section 4.4.4. We still have to prove the representation (5.37). Set $C_k = \|\sum_{2^k\le|l|<2^{k+1}} \hat{A}(l)\|_{\mathcal{A}}$. The solidity of $\mathcal{A}$ implies that, for $k \ge -1$,

$$B_k = \|\varphi_k * A\|_{\mathcal{A}} \le \| \sum_{2^{k-1}\le |l|_\infty < 2^{k+1}} \hat{A}(l)\|_{\mathcal{A}} = C_{k-1} + C_k$$

On the other hand, since $\phi_{k-1} + \phi_k + \phi_{k+1} \equiv 1$ on $\{\xi\colon 2^{k-1} \le |\xi|_2 \le 2^{k+1}\}$, we obtain $C_k \le B_{k-1} + B_k + B_{k+1}$.

The identity (5.38) follows from Proposition 4.51 together with $\mathcal{A}_{v_s} = \mathcal{A}_{v_s^*}$, which is valid in solid matrix algebras. □

For the standard matrix algebras $\mathcal{C}_r^p$ we obtain

$$\mathcal{E}_s^q(\mathcal{C}_r^p) = \mathcal{E}_{s+r}^q(\mathcal{C}_0^p) \tag{5.39}$$

The same result has been obtained in (3.23). One might argue that now the "reason" for this result is more transparent: $\mathcal{E}_s^q(\mathcal{C}_r^p) = \Lambda_s^q(\Lambda_r^p(\mathcal{C}_0^p))$, and the reiteration theorem for Besov spaces together with the characterization of Besov spaces as approximation spaces yields the result. However, this is a matter of taste. There are reiteration theorems for approximation spaces [**73, 83**] that can produce this result as well.

For the Schur algebras $\mathcal{S}_r^p$ we obtain a similar result.

$$\mathcal{E}_s^q(\mathcal{S}_r^p) = \mathcal{E}_{s+r}^q(\mathcal{S}_0^p)\,.$$

In this case we can argue as follows: $\mathcal{E}^q_s(\mathcal{S}^p_r) = \mathcal{E}^q_s(\mathcal{P}_r(\mathcal{S}^p_0)) = \mathcal{E}^q_{s+r}(\mathcal{S}^p_0)$ by Proposition 4.51. A direct argument using the Littlewood-Paley-like representation of Proposition 3.13 is also possible.

If $\mathcal{A}$ is not solid, the results of Proposition 5.21 are new. In particular, if $\mathcal{A} = \mathcal{B}(\ell^2)$, Proposition 5.21 states a result that cannot be obtained with the methods of Chapter 3 alone.

CHAPTER 6

Smooth and Ultradifferentiable Classes

In the final chapter we present some applications of the theory developed so far that go beyond smoothness and approximation of polynomial order. In general we restrict the discussion to periodic group actions, and sometimes we assume that the action is over $\mathbb{R}$, rather than $\mathbb{R}^d$.

In the first section we treat some important special cases, namely the classes C^∞ and the analytic elements of a Banach algebra with automorphism group.

Guided by the results for real functions and for operators on Hilbert spaces [**45, 46**] we introduce Carleman classes (of Roumieu type) $C_M(\mathcal{A})$ related to a Banach algebra $\mathcal{A}$ with automorphism group. These classes are defined by a growth condition on the norms of the derivations. Some analogues of familiar results for Carleman classes of real functions can also be obtained in this general setup. In particular, the Carleman classes are algebras, and a criterion of Malliavin [**74**] (see also [**92**]) leads to a sufficient condition for the inverse-closedness of $C_M(\mathcal{A})$ in $\mathcal{A}$. As a corollary, we derive a new proof of the result of Demko, Smith and Moss [**36**] (and a similar result of Jaffard [**61**]) on the inverse-closedness of matrices with exponential off-diagonal decay in $\mathcal{B}(\ell^2)$. We can extend this result to some sorts of sub-exponential decay by clarifying the relation between the spaces $C_M(\mathcal{A})$ and weighted matrix algebras.

The Carleman classes are locally convex algebras. For functions in the complex domain Dales and Davie [**31**] introduced a class of Banach algebras that is also defined by growth conditions on the derivatives. We introduce a similar concept for general Banach algebras. Following [**1**] we call these algebras *Dales-Davie algebras*. Some results of the classical theory can be transferred to the context of Banach algebras. In particular, we generalize a result of [**1**] on the inverse-closedness of Dales-Davie algebras.

Carleman classes and Dales-Davie algebras for matrices cannot in general be realized as weighted matrix algebras, so we obtain new inverse-closed classes of matrices with off-diagonal decay. However, in some cases we can identify Carleman classes with unions of weighted matrix algebras, and we give inclusion relations.

As said above, the results in this chapter allow alternative versions. Other approaches to ultradifferentiable functions, in particular the Beurling Björck approach [**16, 25**] could have served as starting point for our generalizations.

6.1. Smooth and Analytic Classes

In the following proposition we characterize the smooth elements of $\mathcal{A}$ in terms of their approximation properties and, if the group action is periodic, in terms of the decay of the Fourier coefficients.

PROPOSITION 6.1. *Let $\mathcal{A}$ be a Banach algebra with automorphism group Ψ, and $C^\infty(\mathcal{A}) = \bigcap_{k\geq 0} C^k(\mathcal{A})$ the inverse-closed Fréchet subalgebra of elements with derivations of all order (see Proposition 4.22).*

If $(X_\sigma)_{\sigma\geq 0}$ is the approximation scheme that consists of the bandlimited elements of $\mathcal{A}$, then $a \in C^\infty(\mathcal{A})$ if and only if for all $r > 0$

$$\lim_{k\to\infty} E_k(a)k^r = 0\,. \tag{6.1}$$

If the action of Ψ is periodic, this is equivalent to

$$\lim_{|k|\to\infty} \|\hat{a}(k)\|_{\mathcal{A}} |k|^r = 0 \tag{6.2}$$

for all $r > 0$.

PROOF. The proof works as for the scalar case. If $a \in C^\infty(\mathcal{A})$, then for any $r > 0$ we get $a \in \Lambda^\infty_{r+1}(\mathcal{A}) = \mathcal{E}^\infty_{r+1}(\mathcal{A})$, so $E_k(a)k^{r+1} \leq C$, and $E_k(a)k^r \to 0$ for $k \to \infty$. For the other inclusion observe that (6.1) implies $a \in \Lambda^\infty_r(\mathcal{A})$, and further $a \in C^{\lfloor r \rfloor}(\mathcal{A})$.

If the action of Ψ is periodic, we use the following property of Fourier coefficients. For all $b \in X_{|k|}$ the following identity holds.

$$\hat{a}(k) = \int_{\mathbb{T}^d} (\psi_t(a) - \psi_t(b))e^{-2\pi i k\cdot t}\, dt\,. \tag{6.3}$$

If we take the infimum of the norm over all $b \in X_{|k|_\infty}$, we obtain

$$\|\hat{a}(k)\|_{\mathcal{A}} \leq C E_{|k|_\infty}(a)\,, \tag{6.4}$$

and (6.2) follows. If we assume (6.2) then $\sum_{k\in\mathbb{Z}^d}(2\pi i)^k \hat{a}(k)$ converges in the norm of $\mathcal{A}$ for all multi-indices α, and the limit is $\delta^\alpha(a)$ (because each δ_j is closed in $\mathcal{D}(\delta^\alpha)$). □

COROLLARY 6.2. *If $\mathcal{A}$ is a homogeneous matrix algebra and $A \in \mathcal{A}$, the following are equivalent:*

$$\|\hat{A}(k)\|_{\mathcal{A}} = \mathcal{O}(|k|^{-r}) \quad \textit{for all } r > 0,$$
$$E_l(A) = \mathcal{O}(l^{-r}) \quad \textit{for all } r > 0\,.$$

If $A^{-1} \in \mathcal{A}$ then both conditions are equivalent to

$$\|\widehat{A^{-1}}(k)\|_{\mathcal{A}} = \mathcal{O}(|k|^{-r}) \quad \textit{for all } r > 0\,.$$

Analytic elements. An element $a \in \mathcal{A}$ is *analytic*, if the series

$$\sum_{\alpha\in\mathbb{N}_0^d} \frac{\delta^\alpha(a)}{\alpha!} t^\alpha \tag{6.5}$$

converges in a ball $B_\rho(0)$ around 0. Let $\mathrm{Hol}(\mathcal{A})$ denote the analytic elements of $\mathcal{A}$. As ψ_t and δ^α commute, it is obvious that $\psi_t(a)$ is analytic, if a is analytic.

REMARK. If $a \in \mathcal{A}$ is analytic, then $\psi_t(a)$ can be extended to an analytic function in $B_\rho(0)$, and the power series expansion of $\psi_t(a)$ coincides with (6.5).

LEMMA 6.3. *An element $a \in \mathcal{A}$ is in $\mathrm{Hol}(\mathcal{A})$ if and only if there are constants $C, m > 0$ such that*

$$\|\delta^\alpha(a)\|_{\mathcal{A}} \leq C m^{|\alpha|} |\alpha|! \tag{6.6}$$

for all $\alpha \in \mathbb{N}_0^d$.

PROOF. If a is analytic, then $\sum_{\alpha\in\mathbb{N}_0^d} \frac{\delta^\alpha(a)}{\alpha!} t^\alpha$ converges absolutely for small $|t|$, so $\|\delta^\alpha(a)\|_{\mathcal{A}} \leq C(1/|t|)^{|\alpha|}\alpha!$.

If (6.6) holds, we obtain, using that $|\alpha|! < d^{|\alpha|}\alpha!$, the estimate $\|\delta^\alpha(a)\|_{\mathcal{A}} < C^{|\alpha|}$. This implies that the series (6.5) converges absolutely for small values of $|t|$. □

Proposition 6.4. *If $\mathcal{A}$ is a Banach algebra with periodic automorphism group Ψ, then the following are equivalent:*

(1) $a \in \mathrm{Hol}(\mathcal{A})$,
(2) *There is a constant $\gamma > 0$ such that $\|\hat{a}(k)\| \lesssim e^{-\gamma|k|}$ for all $k \in \mathbb{Z}^d$,*
(3) *There is a constant $\gamma' > 0$ such that $E_l(a) \lesssim e^{-\gamma' l}$ for all $l \in \mathbb{N}$.*

Proof. Again, this result is well-known in the approximation theory of functions [**100**]. We sketch a proof.

(1) ⇒ (2). We want to obtain an estimate for $\|\hat{a}(k)\|_{\mathcal{A}}$. W.l.o.g. we can assume that $|k_1| = |k|_\infty$. Then for fixed $t_2, \cdots, t_d$ the function.

$$\eta(t) = \psi_{(t,t_2,\cdots,t_d)}(a)e^{2\pi i k_1 t}$$

is analytic on $\mathbb{T}$ and 1-periodic. So there is a $\gamma_1 > 0$ such that η is analytic in the strip $\mathbb{R} + i[-\gamma, \gamma]$, and the rectangular path integral $\int_\Gamma \eta(t)\, dt$ vanishes, Γ the rectangle with corners $0, 1, 1 + i\gamma, i\gamma$, if $k_1 > 0$ (otherwise use the rectangle below the real axis). The periodicity of η implies that the contributions of the vertical paths cancel. We obtain

$$\Big|\int_0^1 \eta(t)dt\Big| \le M_\Psi \|a\|_{\mathcal{A}} \int_0^1 e^{-2\pi\gamma t} \le Ce^{-2\pi\gamma}\,.$$

Inserting this in the definition of $\hat{a}(k)$ we obtain

$$\|\hat{a}(k)\|_{\mathcal{A}} \le Ce^{-\gamma'|k|_\infty} \le Ce^{-\tilde{\gamma}|k|}\,.$$

This is true as proved only for the indices $k \in \mathbb{Z}^d$ that satisfy $|k_1| = |k|_\infty$. For the other choices different values $\gamma_2, \cdots \gamma_d$ have to be considered. If we take $\gamma = \min_{1\le i\le d}(\gamma_i)$, we obtain an uniform estimate.

(2) ⇒ (3).

$$\begin{aligned} E_l(a) \le \sum_{|k|_\infty \ge l} \|\hat{a}(k)\|_{\mathcal{A}} &\lesssim \sum_{|k|_\infty \ge l} e^{-\gamma k} \\ &\lesssim \int_{|t|_2 \ge l} e^{-\gamma |t|_2}\, dt \\ &\lesssim \int_l^\infty e^{-\gamma r} r^{d-1}\, dr\,. \end{aligned}$$

The second line in the chain of estimates above follows from the relation between the ∞- and 2-norms on $\mathbb{R}^d$. Now choose $0 < \beta < \gamma$. Then $r^{d-1} < e^{\beta r}$ for r large enough. If we insert this estimate in the last inequality we obtain

$$E_l(a) \lesssim e^{-(\gamma-\beta)dl}$$

for all $l > 0$.

(3) ⇒ (2) follows from (6.4).

(2) ⇒ (1). If $\|\hat{a}(k)\|_{\mathcal{A}} \le Ce^{-\gamma|k|}$ the series $\sum_{k\in\mathbb{Z}^d} (2\pi i k)^\alpha \hat{a}(k)$ converges to $\delta^\alpha(a)$ for all $\alpha \in \mathbb{N}_0^d$. Arguing as above we obtain

$$\begin{aligned} \|\delta^\alpha(a)\| \le \sum_{k\in\mathbb{Z}^d} (2\pi|k|)^{|\alpha|} e^{-\gamma|k|} &\le C^{|\alpha|} \int_0^\infty e^{-\gamma r} r^{|\alpha|+d-1} \\ &\lesssim C'^{|\alpha|}\Gamma(|\alpha| + d) \\ &\lesssim C'' m^{|\alpha|} |a|! \end{aligned}$$

for some constants C'' and m. □

A description of the algebra properties of $\mathrm{Hol}(\mathcal{A})$ follows in the next section.

6.2. Carleman Classes

For the description of smoothness between C^∞ and analytic and the related off-diagonal decay of matrices we need additional concepts.

DEFINITION 6.5. Let $\mathcal{A}$ be a Banach algebra with commuting derivations $\delta_1, \cdots, \delta_d$, and the *defining sequence* $M = \{M_k\}_{k\in\mathbb{N}_0}$ a sequence of positive numbers with $M_0 = 1$. For each $m > 0$ we define the Banach space

$$C_{m,M}(\mathcal{A}) = \{a \in \mathcal{A} \colon \exists C > 0, \|\delta^\alpha a\|_{\mathcal{A}} \le C m^{|\alpha|} M_{|\alpha|} \text{ for all } \alpha \in \mathbb{N}_0^d\}$$

with the norm

$$\|a\|_{C_{m,M}(\mathcal{A})} = \sup_{\alpha\in\mathbb{N}_0^d} \frac{\|\delta^\alpha(a)\|_{\mathcal{A}}}{m^{|\alpha|}M_{|\alpha|}}\,.$$

The *Carleman* Class $C_M(\mathcal{A})$ is the union of the spaces $C_{m,M}(\mathcal{A})$,

$$C_M(\mathcal{A}) = \bigcup_{m>0} C_{m.M}(\mathcal{A})$$

with the inductive limit topology.

We call $C_M(\mathcal{A})$ *trivial*, if for each $a \in \mathcal{A}$ the relation $\psi_t(a) = a$ holds for all $t \in \mathbb{R}^d$, otherwise $C_M(\mathcal{A})$ is *nontrivial*.

REMARK. If $C_M(\mathcal{A})$ is a matrix algebra, then $C_M(\mathcal{A})$ is trivial if and only if it consists only of diagonal matrices.

If the derivations $\delta_1, \ldots, \delta_d$ are generators of the automorphism group Ψ acting on $\mathcal{A}$, we can give a weak type characterization of $C_{m,M}(\mathcal{A})$ using the functions $G_{a,a'}$.

LEMMA 6.6. *An element $a \in \mathcal{A}$ is in $C_{m,M}(\mathcal{A})$ if and only if $G_{a',a}$ is in $C_{m,M}(L^\infty(\mathbb{R}^d)) = C_{m,M}(\mathbb{R}^d)$ for all $a' \in \mathcal{A}'$. In this case*

$$\|a\|_{C_{m,M}(\mathcal{A})} \le \sup_{\|a'\|_{\mathcal{A}'}\le 1} \|G_{a',a}\|_{C_{m,M}(\mathbb{R}^d)} \le M_\Psi \|a\|_{C_{m,M}(\mathcal{A})}\,.$$

PROOF.

$$\begin{aligned}
\sup_{\alpha\in\mathbb{N}_0^d} \frac{\|\delta^\alpha a\|_{\mathcal{A}}}{m^{|\alpha|}M_{|\alpha|}} &\le \sup_{\alpha\in\mathbb{N}_0^d}\ \sup_{\|a'\|_{\mathcal{A}'}\le 1} \frac{\|G_{a',\delta^\alpha a}\|_{L^\infty(\mathbb{R}^d)}}{m^{|\alpha|}M_{|\alpha|}} \\
&= \sup_{\|a'\|_{\mathcal{A}'}\le 1}\ \sup_{\alpha\in\mathbb{N}_0^d} \frac{\|D^\alpha G_{a',a}\|_{L^\infty(\mathbb{R}^d)}}{m^{|\alpha|}M_{|\alpha|}} \\
&\le M_\Psi \|a\|_{C_{m,M}(\mathcal{A})}\,.
\end{aligned}$$

□

From now on we assume that the derivations $\delta_1, \ldots, \delta_d$ are the generators of a periodic group action Ψ on $\mathcal{A}$.

EXAMPLE 6.7. If $M_k = 1$ for all k, then $C_{2\pi m,M}(\mathcal{A})$ consists of the m-bandlimited elements of $\mathcal{A}$.

If $M_k = k!$ for all k, then $C_M(\mathcal{A}) = \mathrm{Hol}(\mathcal{A})$ by Lemma 6.3.

The *Gevrey-class* $\mathcal{J}_r(\mathcal{A})$ is the space $C_M(\mathcal{A})$ for $M_k = k!^r$, $r > 0$. If $r \le 1$ then $\mathcal{J}_r(\mathcal{A})$ consists only of analytic elements and is a subspace of $\mathrm{Hol}(\mathcal{A})$.

The example $N_k = c^k M_k$ shows that different defining sequences give raise to the same Carleman class. More generally, if

$$c^k N_k \le M_k \le C^k N_k$$

for all indices k, then $C_M(\mathcal{A}) = C_N(\mathcal{A})$. For the Gevrey classes this implies (using Stirling's formula) that

$$N_k = k^{rk} \tag{6.7}$$

is a defining sequence for $\mathcal{J}_r$.

In order to state the necessary and sufficient conditions for two defining sequences M_k, N_K to generate the same Carleman class we introduce weights associated to the sequence $(M_k)_{k\geq 0}$.

Definition 6.8. Let M be a defining sequence. The *associated function* to M is

$$T_M(r) = \sup_{k\geq 0} \frac{r^k}{M_k} \quad \text{for } r > 0\,. \tag{6.8}$$

By definition, T_M is increasing. If for all $C > 0$ $M_k \gtrsim C^k$ for $k > k(C)$, that is, M_k grows faster than any power, then T_M is finite valued. If $N_k = C^k M_k$ then $T_N(r) = T_M(r/C)$. We call T_N and T_M equivalent and write $T_N \sim T_M$, if there are positive constants c, C such that $T_N(r/c) \leq T_M(r) \leq T_N(r/C)$ for all $r > 0$.

Lemma 6.9. *$\log T_M(r)$ is convex in $\log r$.*

Proof. The logarithm of T_M is the supremum of a set of affine functions in $\rho = \log r$,

$$\tau_M(\rho) = \log T_M(r) = \sup_{k\geq 0}(k \log r - \log M_k)\,, \tag{6.9}$$

and the supremum of a set of affine functions is convex (see, e.g.[**89**]). □

Remarks. The associated function T_M is related by (6.9) to a concept of convex analysis, the *Fenchel-Young conjugate* of a real valued function. Actually, if $f\colon \mathbb{R} \to \mathbb{R}$ is continuous then $f^*(y) = \sup_{x\in\mathbb{R}}(xy - f(x))$ is the Fenchel - Young conjugate of f, and the Fenchel-Young duality expresses that

$$f^{**} = (f^*)^* \tag{6.10}$$

is the largest convex minorant of f, that is, the largest convex function smaller than f. If f is convex, it follows that $f^{**} = f$. See, e.g. [**89**]. It can be shown ([**75, 76**], see also [**67**]) that τ_M is the Fenchel-Young conjugate of the function $\nu(x)$ that is constructed by piecewise linear interpolation of $\log M_k$ for $x \geq 0$, and ∞ elsewhere. Restating the Fenchel-Young duality (6.10) for T_M we get

Proposition 6.10 ([**75, 76, 66**]). *The* log-convex regularization M^c *of the sequence $M = (M_k)_{k\in\mathbb{N}_0}$ is defined as*

$$M_k^c = \sup_{r>0} \frac{r^k}{T_M(r)}\,. \tag{6.11}$$

It is the largest logarithmically convex sequence smaller than M. Moreover,

$$T_{M^c} = T_M \quad \textit{and} \quad M^{cc} = M^c\,.$$

We will also need the following simple facts about log-convex sequences.

Lemma 6.11. [**67, 76**]

(1) *For all $k, l \in \mathbb{N}_0$ the sequence M satisfies $M_k^c M_l^c \leq M_{k+l}^c$*
(2) *The sequence $(M_k^c)^{1/k}$ is increasing.*

Example 6.12. An elementary calculation shows that the function $T_M(r)$ associated to $M_k = k^{ak}$ is

$$T_M(r) \asymp \exp(\frac{a}{e} r^{1/a})\,. \tag{6.12}$$

The notions just introduced allow us to classify the sequences M_k.

Proposition 6.13 ([**47, 76**]). *Let M be a defining sequence for the Carleman class $C_M(\mathcal{A})$.*

(1) *If $\liminf_{k\to\infty} M_k^{1/k} = 0$, then $C_M(\mathcal{A})$ is trivial.*
(2) *If $0 < \liminf_{k\to\infty} M_k^{1/k} < \infty$, then $C_M(X)$ is the class of bandlimited elements.*
(3) *If $\lim_{k\to\infty} M_k^{1/k} = \infty$, and*

$$(M_k^c)^{1/k} \asymp (N_k^c)^{1/k}\,, \tag{6.13}$$

then $C_M(\mathcal{A}) = C_N(\mathcal{A})$. Moreover, condition (6.13) is equivalent to $T_M \sim T_N$.

Sketch of proof. We do not give a detailed proof of this result but we indicate how it follows from the existing literature by straightforward adaptions. As $a \in C_M(\mathcal{A})$ if and only if $G_{a',a} \in C_M(\mathbb{R}^d)$ for all $a' \in \mathcal{A}'$, the conditions follow from [**76**, 6.5.III] by a weak type argument. The statement given there is for functions on the real line, but it remains true for functions on $\mathbb{R}^d$. In the proof one has to replace the Kolmogorov inequality [**75**, 6.3.III] by the Cartan-Gorny estimates [**76**, (6.4.5)]. They can be verified for functions on $\mathbb{R}^d$ as well (see [**67**, IV.E.,Problem 7]).

The equivalence between condition (6.13) and $T_m \sim T_N$ follows directly from the definition of equivalent associated functions. □

Remark. If for all $k \in \mathbb{Z}^d$ there are elements $a_k \in \mathcal{A}$ such that $\hat{a}_k(k) \neq 0$ (weaker statements are possible) then an argument of Siddiqi [**91**, Theorem B] shows that the equality of the classes $C_M(\mathcal{A})$ and $C_N(\mathcal{A})$ implies that $(M_k^c)^{1/k} \asymp (N_k^c)^{1/k}$.

Proposition 6.14 (see, e.g., [**66**]). *Each Carleman class $C_M(\mathcal{A})$ is an algebra.*

Proof. The cases $C_0(\mathcal{A})$ (C_M is trivial) and $C_1(\mathcal{A})$ (bandlimited elements) are straightforward, so we may assume that $M = M^c$. Let $a_1, a_2 \in C_M(\mathcal{A})$, so there are constants $C_1, C_2 > 0$, and $m_1, m_2 > 0$ such that for all indices $\alpha \in \mathbb{N}_0^d$ $\|\delta^\alpha a_j\|_\mathcal{A} \le C_j m_j^{|\alpha|} M_{|\alpha|}$, $j = 1, 2$. Then

$$\begin{aligned}
\|\delta^\alpha(a_1a_2)\|_\mathcal{A} &\le \sum_{\beta\le\alpha}\binom{\alpha}{\beta}\|\delta^\beta a_1\|_\mathcal{A}\|\delta^{\alpha-\beta}a_2\|_\mathcal{A}\\
&\le C_1C_2\sum_{\beta\le\alpha}\binom{\alpha}{\beta}m_1^{|\beta|}m_2^{|\alpha-\beta|}M_{|\beta|}M_{|\alpha-\beta|}\\
&\le C_1C_2(m_1+m_2)^{|\alpha|}M_{|\alpha|}\,,
\end{aligned}$$

where we used $M_{k-l}M_l \le M_k$, see Lemma 6.11. □

Definition 6.15. A sequence $(u_k)_{k\in\mathbb{N}_0}$ of positive numbers is almost increasing, if $u_k < Cu_l$ for all $k < l$ and a constant $C > 0$.

Lemma 6.16. *Assume that the defining sequence M satisfies $M = M^c$. The sequence $(M_k/k!)^{1/k}$ is almost increasing if and only if there is a $C > 0$ such that for all $l \in \mathbb{N}$ and all indices $j_k, k = 1, \dots, l$*

$$\prod_{k=1}^{l}\frac{M_{j_k}}{j_k!} \le C^{\sum_{k=1}^{l} j_k}\frac{M_{\sum_{k=1}^{l} j_k}}{(\sum_{k=1}^{l} j_k)!}\,. \tag{6.14}$$

PROOF. By definition,

$$\frac{M_{j_k}}{j_k!} \le C^k \Big(\frac{M_{\sum_{k=1}^l j_k}}{(\sum_{k=1}^l j_k)!} \Big)^{\frac{k}{\sum_{k=1}^l j_k}} ,$$

and the "if" part follows by multiplying these estimates. For the other direction observe first that Stirling's formula implies that $(M_k/k!)^{1/k}$ is almost increasing if and only if

$$\frac{M_k^{1/k}}{k} \le C \frac{M_l^{1/l}}{l} \quad \text{for all } k < l.$$

If $l = rk$, for an integer r then (6.14) implies

$$\frac{M_k^{1/k}}{k} \le C \frac{M_{rk}^{1/rk}}{rk} .$$

If $rk < l < (r+1)k$, we use an interpolation argument. By Lemma 6.11 the sequence $M_k^{1/k}$ is increasing in k, so

$$\frac{M_l^{1/l}}{l} \ge \frac{M_{kr}^{1/kr}}{kr} \frac{kr}{l} \ge \frac{kr}{l} \frac{1}{C} \frac{M_k^{1/k}}{k}$$

by what has been just proved. So

$$\frac{M_k^{1/k}}{k} \le C \frac{l}{kr} \frac{M_l^{1/l}}{l} \le 2C \frac{M_l^{1/l}}{l} .$$

□

REMARK. Note that for the proof of the direct implication we did not need the hypothesis that $M = M^c$.

THEOREM 6.17 ([**74**, **92**]). *Assume that $C_M(\mathcal{A})$ is nontrivial, and its defining sequence satisfies $\lim_{k\to\infty} M_k^{1/k} = \infty$. If $\left(\frac{M_k}{k!}\right)^{1/k}$ is almost increasing, then $C_M(\mathcal{A})$ is inverse-closed in $\mathcal{A}$.*

We adapt the proof in [**92**] to the noncommutative situation. We need the following form of the iterated quotient rule whose proof can be found in Appendix B.

LEMMA 6.18. *Let $E = \{1, \dots, d\}$ and $\delta_1, \dots, \delta_d$ be derivations satisfying the quotient rule:*

$$\delta_j(a^{-1}) = -a^{-1}\delta_j(a)a^{-1} \quad \textit{for all } j \in E.$$

For every $k \in \mathbb{N}$ and every tuple $B = (b_1, \dots, b_k) \in E^k$ set $|B| = k$, and

$$\delta_B(a) = \delta_{b_1} \dots \delta_{b_k}(a) .$$

Define the partitions of B into m nonempty subtuples as

$$P(B,m) = \{(B_1, \cdots, B_m) \colon B = (B_1, \cdots, B_m), B_i \neq \emptyset \textit{ for all } i\} .$$

Then

$$\delta_B(a^{-1}) = \sum_{m=1}^{|B|} (-1)^m \sum_{(B_i)_{1\le i\le m} \in P(B,m)} \Big(\prod_{j=1}^m a^{-1} \delta_{B_i}(a) \Big) a^{-1} . \tag{6.15}$$

PROOF OF THEOREM 6.17. Assume that $|\alpha| = k$. With the notation of Lemma 6.18 there is a set B with $|B| = k$ such that $\delta^\alpha = \delta_B$. Observe that the number of (nonempty) partitions of B in sets $(B_i)_{1\le i\le m} \in P(B,m)$ of cardinality k_i is

$$\binom{k}{k_1, \dots, k_m} .$$

As $a \in C_M(\mathcal{A})$ we know that $\|\delta_{B_i}(a)\| \le Ah^{|B_i|}M_{|B_i|}$ for some constants $A, h > 0$. We obtain the norm estimate

$$\begin{aligned}\|\delta^\alpha(a^{-1})\|_{\mathcal{A}} &\le \sum_{m=1}^{k} \|a^{-1}\|_{\mathcal{A}}^{m+1} \sum_{\substack{k_1+\cdots k_m=k\\ k_j\ge 1}} \binom{k}{k_1,\dots,k_m}\Big(\prod_{j=1}^{m} Ah^{k_i}M_{k_i}\Big)\\ &= h^k \sum_{m=1}^{k} \|a^{-1}\|_{\mathcal{A}}^{m+1} A^m \sum_{\substack{k_1+\cdots k_m=k\\ k_j\ge 1}} \binom{k}{k_1,\dots,k_m}\Big(\prod_{j=1}^{m} M_{k_i}\Big)\end{aligned} \tag{6.16}$$

As $(M_k/k!)^{1/k}$ is almost increasing we obtain

$$\prod_{i=1}^{m} M_{k_i} \le C^k \frac{k_1!\cdots k_m!}{k!} M_k\,, \tag{6.17}$$

and so

$$\begin{aligned}\|\delta^\alpha(a^{-1})\|_{\mathcal{A}} &\le h^k C^k M_k \sum_{m=1}^{k} \|a^{-1}\|_{\mathcal{A}}^{m+1} A^m \sum_{\substack{k_1+\cdots k_m=k\\ k_j\ge 1}} 1\\ &= h^k C^k M_k \sum_{m=1}^{k} \|a^{-1}\|_{\mathcal{A}}^{m+1} A^m \binom{k-1}{m-1}\\ &\le Ah^k C^k M_k \|a^{-1}\|_{\mathcal{A}}^2 \Big(1 + A\|a^{-1}\|_{\mathcal{A}}\Big)^{k-1}\\ &\le \|a^{-1}\|_{\mathcal{A}}^2 \Big(hC(1 + A\|a^{-1}\|_{\mathcal{A}})\Big)^{k} M_k\,,\end{aligned}$$

and this is what we wanted to show. □

COROLLARY 6.19. *The Gevrey classes $\mathcal{J}_r(\mathcal{A})$ are inverse-closed in $\mathcal{A}$, if $r \ge 1$. In particular,* Hol $\mathcal{A}$ *is inverse-closed in $\mathcal{A}$.*

SKETCH OF PROOF. By Stirling's formula,

$$(\frac{M_k}{k!})^{1/k} = (\frac{k!^r}{k!})^{1/k} \asymp (\frac{k}{e})^{r-1}\,,$$

and so $(M_k/k!)^{1/k}$ is almost increasing. □

If $\mathcal{A} = \mathcal{B}(\ell^2)$ and $M_k = k!$ we obtain the following result by using the characterization of analytic elements in Proposition 6.4

COROLLARY 6.20 (cf. [**36, 61**]). *If $A \in \mathcal{B}(\ell^2)$ with $|A(r,s)| \le Ce^{-\gamma|k|}$ for constants $C, \gamma > 0$ and all $r, s, \in \mathbb{Z}^d$. If $A^{-1} \in \mathcal{B}(\ell^2)$, then there exist $C', \gamma' > 0$ such that*

$$|A^{-1}|(r,s) \le C'e^{-\gamma'|r-s|} \quad \textit{for all } r, s \in \mathbb{Z}^d.$$

6.3. Relation to Weighted Spaces

If $\mathcal{A}$ is a matrix algebra, the relation between the Carleman classes $C_M(\mathcal{A})$ and the class of weighted spaces $\mathcal{A}_w$ is of interest.

DEFINITION 6.21. Let $\mathcal{A}$ be a Banach algebra with periodic automorphism group Ψ. Let $1 \le p \le \infty$, and v a weight for $\ell^p(\mathbb{Z}^d)$. In analogy to Definition 2.11 we introduce the spaces

$$\mathcal{C}_v^p(\mathcal{A}) = \{a \in A \colon \|a\|_{\mathcal{C}_v^p(\mathcal{A})} = \Big(\sum_{k\in\mathbb{Z}^d} \|\hat{a}(k)\|_{\mathcal{A}}^p v(k)^p\Big)^{1/p} < \infty\}$$

with the obvious modification for $p = \infty$.

REMARK. If v is an algebra weight for $\ell^p(\mathbb{Z}^d)$, then $\mathcal{C}^p_v(\mathcal{A})$ is an inverse-closed subalgebra of $\mathcal{A}$. The proof is a straightforward adaption of [**56**, Theorem 3.2].

LEMMA 6.22. *If M is a defining sequence for $C_M(\mathcal{A})$, $m>0$, and $T_{m,M}(k) = T_M(\frac{2\pi|k|_\infty}{m})$, then*

$$C^1_{T_{m,M}}(\mathcal{A}) \subseteq C_{m,M}(\mathcal{A}) \subseteq C^\infty_{T_{m,M}}(\mathcal{A})\,. \tag{6.18}$$

PROOF. Assume first that $a \in C_{m,M}(\mathcal{A})$. We want to obtain an estimate for $\|\hat{a}(k)\|_\mathcal{A}$. Let j be an index such that $|k_j| = |k|_\infty$. Then

$$\hat{a}(k) = \int_{\mathbb{T}^d} \psi_t(a) e^{-2\pi i k\cdot t}\, dt = \frac{1}{(2\pi i k_j)^l} \int_{\mathbb{T}^d} \psi_t(\delta^l_{e_j} a) e^{-2\pi i k\cdot t}\, dt\,,$$

the second equality by l-fold partial integration (see (4.18)). Taking norms we obtain

$$\|\hat{a}(k)\|_\mathcal{A} \le \frac{1}{|2\pi k_j|^l} \|\int_{\mathbb{T}^d} \psi_t(\delta^l_{e_j} a) e^{-2\pi i k\cdot t}\, dt\|_\mathcal{A} \le C\frac{m^l M_l}{(2\pi|k|_\infty)^l}\,.$$

This relation is valid for all $l \in \mathbb{N}_0$, and therefore also for the infimum, which yields

$$\|\hat{a}(k)\|_\mathcal{A} \le C/T_{m,M}(k)\,, \tag{6.19}$$

or $a \in \mathcal{C}^\infty_{T_{m,M}}(\mathcal{A})$.

For the other inclusion assume that $\|a\|_{\mathcal{C}^1_{T_{m,M}}(\mathcal{A})} \le \infty$ for an $m > 0$. For $\alpha \in \mathbb{N}_0^d$ we can estimate the norm of $\delta^\alpha(a)$ by

$$\begin{aligned}\|\delta^\alpha(a)\|_\mathcal{A} &\le \sum_{k\in\mathbb{Z}^d} \|\delta^\alpha(\hat{a}(k))\|_\mathcal{A} \le \sum_{k\in\mathbb{Z}^d} (2\pi|k|_\infty)^{|\alpha|}\|\hat{a}(k)\|_\mathcal{A} \\ &\le \|a\|_{\mathcal{C}^1_{T_{m,M}}(\mathcal{A})} \sup_{k\in\mathbb{Z}^d} \frac{(2\pi|k|_\infty)^{|\alpha|}}{T_{m,M}(k)} \le \|a\|_{\mathcal{C}^1_{T_{m,M}}(\mathcal{A})} \sup_{r>0} \frac{r^{|\alpha|}}{T_M(r/m)} \\ &= \|a\|_{\mathcal{C}^1_{T_{m,M}}(\mathcal{A})} m^{|\alpha|} M^c_{|\alpha|}\,,\end{aligned} \tag{6.20}$$

the last equality by (6.11), and so $a \in C_{m,M}(\mathcal{A})$. □

REMARK. In general the weight $T_{m,M}$ is not submultiplicative.

COROLLARY 6.23. *With the notation of Lemma 6.22,*

$$\bigcup_{m>0} \mathcal{C}^1_{T_{m,M}}(\mathcal{A}) \hookrightarrow C_M(\mathcal{A}) \hookrightarrow \bigcup_{m>0} \mathcal{C}^\infty_{T_{m,M}}(\mathcal{A})\,,$$

where the spaces $\bigcup_{m>0} \mathcal{C}^1_{T_{m,M}}(\mathcal{A})$ and $\bigcup_{m>0} \mathcal{C}^\infty_{T_{m,M}}(\mathcal{A})$ are equipped with their natural inductive limit topologies.

PROOF. The only thing to verify is the continuity of the embeddings. By the properties of inductive limits (see, e.g.[**42**]) it is sufficient to show that $\mathcal{C}^1_{T_{m,M}}(\mathcal{A}) \hookrightarrow C_M(\mathcal{A})$ for every $m > 0$, and that $C_{m,M}(\mathcal{A}) \hookrightarrow \bigcup_{m>0} \mathcal{C}^\infty_{T_{m,M}}(\mathcal{A})$. This, however, follows from (6.19) and (6.20). □

A sufficient condition for the equality of the spaces in Corollary 6.23 is the condition (M2') of Komatsu [**66**] for the defining sequence M.

(M2') There exist constants $c > 0$, $h > 1$ such that for all $k \in \mathbb{N}$

$$M_{k+1} \le ch^k M_k\,.$$

(M2') is equivalent to the following conditions on T_M.

LEMMA 6.24. *If M is a defining sequence, the following are equivalent:*

(1) *M satisfies (M2') with constants c and h.*
(2) *$T_M(hr) \ge CrT_M(r)$ for all $r > 0$.*

(3) $\frac{T_M(\lambda r)}{T_M(r)} \geq \exp\big(\log(r/c)\log\lambda/\log h\big)$ *for all* $r, \lambda > 0$.

For a proof see [**80**] (cf. also [**66**, Proposition 3.4]). A related condition on weights can be found in [**45**, 2.4].

EXAMPLE 6.25. The Gevrey-classes $\mathcal{J}_r$, $r > 0$ satisfy (M2').

PROPOSITION 6.26. *If the defining sequence satisfies (M2'), then*

$$\bigcup_{m>0} \mathcal{C}^1_{T_{m,M}}(\mathcal{A}) = C_M(\mathcal{A}) = \bigcup_{m>0} \mathcal{C}^\infty_{T_{m,M}}(\mathcal{A})\,,$$

and this equality is also an isomorphism of topological vector spaces.

PROOF. (see, e.g.,[**70**]) We can estimate the norm in $\mathcal{C}^1_{T_{m,M}}(\mathcal{A})$ by the norm in $\mathcal{C}^\infty_{T_{m,M}}(\mathcal{A})$ using Lemma 6.24, (3).

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d} \|\hat{a}(k)\|_{\mathcal{A}} T_{m,M}(|k|_\infty) &\leq \sum_{k\in\mathbb{Z}^d} \|\hat{a}(k)\|_{\mathcal{A}} T_{m,M}(\lambda|k|_\infty)\exp\big(-\log(\frac{|k|_\infty}{c})\frac{\log\lambda}{\log h}\big)\\ &\leq \sup_{k\in\mathbb{Z}^d} \|\hat{a}(k)\|_{\mathcal{A}} T_{m,M}(\lambda|k|_\infty)\sum_{k\in\mathbb{Z}^d}\big(\frac{|k|_\infty}{c}\big)^{-\log\lambda/\log h}.\end{aligned}$$

If we choose λ such that $\log\lambda/\log h > d$, the sum on the right hand side of the inequality is convergent. Then $\mathcal{C}^\infty_{T_{\lambda m,M}}(\mathcal{A}) \subseteq \mathcal{C}^1_{T_{m,M}}(\mathcal{A})$, so taking unions over $m > 0$ gives the equality of the involved spaces. Equation (6.3) implies also the topological embedding

$$\bigcup_{m>0} \mathcal{C}^\infty_{T_{m,M}}(\mathcal{A}) \hookrightarrow \bigcup_{m>0} \mathcal{C}^1_{T_{m,M}}(\mathcal{A})\,. \qquad \square$$

If M satisfies (M2') we can obtain a characterization of $C_M(\mathcal{A})$ that is a generalization of Proposition 6.4.

PROPOSITION 6.27. *If $\mathcal{A}$ is a Banach algebra with periodic automorphism group Ψ, and M a defining sequence for $C_M(\mathcal{A})$ that satisfies (M2'), then the following are equivalent:*

(1) $a \in C_M(\mathcal{A})$,
(2) *There are constants* $C, m > 0$ *such that* $\|\hat{a}(k)\| < C/T_M(|k|/m)$ *for all* $k \in \mathbb{Z}^d$,
(3) *There are constants* $C', m' > 0$ *such that* $E_l(a) < C'/T_M(l/m')$ *for all* $l \in \mathbb{N}$.

PROOF. (1) $\Leftrightarrow$ (2) is an immediate consequence of Proposition 6.26. (3) $\Rightarrow$ (2) follows from (6.4).
(2) $\Rightarrow$ (3). As (M2') holds for $T_{m,M}$ with different constants c', h', we assume w.l.o.g that $m = 1$. The approximation error can be estimated by

$$E_l(a) \leq \sum_{|k|_\infty\geq l} \|\hat{a}(k)\| \lesssim \sum_{|k|_\infty\geq l} T_M^{-1}(|k|)\,.$$

As $T_M(r)$ is increasing in r, we can replace the sum by an integral. We assume that l is so large that $\frac{\log(l/c)}{\log h} > 2d$, and obtain

$$
\begin{aligned}
\sum_{|k|_\infty \geq l} T_M^{-1}(|k|) &\leq \int_{|k|\infty \geq l} T_M^{-1}(|k|)\, dk \lesssim \int_l^\infty \frac{1}{T_M(u)} u^{d-1}\, du \\
&= l^d \int_1^\infty \frac{1}{T_M(lv)} v^{d-1}\, dv \\
&\leq \frac{l^d}{T_M(l)} \int_1^\infty v^{d-1} e^{-\frac{\log(l/c)\log v}{\log h}}\, dv \\
&= \frac{l^d}{T_M(l)} \int_1^\infty v^{d-1-\frac{\log(l/c)}{\log h}}\, d\eta \\
&= \frac{l^d}{T_M(l)} \frac{1}{\frac{\log(l/c)}{\log h} - d} \leq \frac{l^d}{T_M(l)} d^{-1},
\end{aligned}
\tag{6.21}
$$

using again (3) of Lemma 6.24. Observe that by (2) of Lemma 6.24 $T_M(l) \geq C l^d T(l/h^d)$ with a constant C independent of l. Substituting this estimate for $T_M(l)$ in the last line of (6.21) we finally get

$$E_l(a) \leq C/T_M(l/h^d)\,,$$

and the constant C is independent of l. □

A somewhat similar approximation result can be found in [**17**], see also [**81**].

6.4. Dales-Davie Algebras

For this section we assume that Ψ is a *one* parameter automorphism group acting on the Banach algebra $\mathcal{A}$.

It is possible to construct *Banach algebras* related to $\mathcal{A}$ that are determined by growth conditions on the sequence $(\|\delta^k(a)\|_\mathcal{A})_{k\in\mathbb{N}_0}$. We adapt some notions introduced in [**31**] for scalar functions in the complex plane.

Definition 6.28. Let $M = (M_k)_{k\geq 0}$ be an *algebra sequence*, that is, a sequence of positive numbers with $M_0 = 1$ and

$$\frac{M_{k+l}}{(k+l)!} \geq \frac{M_k}{k!}\frac{M_l}{l!} \quad \text{for all } k, l \in \mathbb{N}_0. \tag{6.22}$$

The *Dales-Davie algebra* $D^1_M(\mathcal{A})$ consists of the elements $a \in \mathcal{A}$ with finite norm

$$\|a\|_{D^1_M(\mathcal{A})} = \sum_{k=0}^\infty \frac{\|\delta^k(a)\|_\mathcal{A}}{M_k}\,.$$

The space $D^1_M(\mathcal{A})$ is indeed a Banach algebra. This will be proved in Proposition 6.31.

Example 6.29. (1) Let $\mathcal{A} = \mathcal{C}^1_{v_0}$ the unweighted Baskakov algebra. Then

$$\|A\|_{D^1_M(\mathcal{A})} = \sum_{k=0}^\infty M_k^{-1} \sum_{l\in\mathbb{Z}} \|A[l]\|_\infty (2\pi|l|)^k = \sum_{l\in\mathbb{Z}} \|A[l]\|_\infty \sum_{k=0}^\infty \frac{(2\pi|l|)^k}{M_k}\,.$$

If we set

$$v_M(l) = \sum_{k=0}^\infty \frac{(2\pi|l|)^k}{M_k} \tag{6.23}$$

we obtain

$$D^1_M(\mathcal{C}^1_{v_0}) = \mathcal{C}^1_{v_M}\,.$$

At least in this situation we have established a relation between the growth of derivatives and weights.

We call the function v_M defined in (6.23) the *weight associated* to the sequence M.

In the following lemma some basic properties of v_M are collected. We need two notions from complex analysis (see, e.g. [**71**]). For an entire function f let $M_f(r) = \sup_{|x|\le r}|f(x)|$.

The *order* of f is

$$\rho_f = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}\,.$$

If f has finite order ρ_f, the *type* of f is

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\rho_f}}\,.$$

If $\sigma_f = 0$, we say that f has minimal type.

LEMMA 6.30.

(1) *If M is an algebra sequence, then $v_M(|k|)$ is submultiplicative.*
(2) *The weight v_M associated to M can be extended from the positive real axis to an entire function if and only if $\lim_{k\to\infty} M_k^{1/k} = \infty$.*
(3) *The weight v_M associated to M satisfies the GRS condition (see Definition 2.1) if and only if*

$$\lim_{k\to\infty}\big(\frac{M_k}{k!}\big)^{1/k} = e\lim_{k\to\infty}\frac{M_k^{1/k}}{k} = \infty\,.$$

This is further equivalent to the analytic continuation of v_M being an entire function of order $\rho_{v_M}\le 1$, and, if $\rho_{v_M} = 1$ then v_M is of minimal type.

PROOF. (1) Let $r, s \ge 0$. Then

$$\begin{aligned} v_M(r+s) &= \sum_{k=0}^{\infty}\frac{(2\pi)^k}{M_k}\sum_{l=0}^{k}\frac{k!}{l!(k-l)!}r^l s^{k-l} \le \sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{(2\pi r)^l}{M_l}\frac{(2\pi s)^{k-l}}{M_{k-l}} \\ &\le v_M(r)v_M(s)\,. \end{aligned}$$

As v_M is increasing on $\mathbb{R}_0^+$, $v_M(|r+s|) \le v_M(|r|+|s|)$, and this proves (1) for all values of $r, s \ge 0$.

(2) This follows from the formula for the convergence radius R of a power series $\sum_{k=0}^{\infty} a_k x^k$, $1/R = \limsup_{k\to\infty}|a_k|^{1/k}$. So

$$\limsup_{k\to\infty}|a_k|^{1/k} = 0 \text{ if and only if } \lim_{k\to\infty} M_k^{1/k} = \infty\,.$$

(3) We use the following formulas for order and type of the entire function $f(x) = \sum_{k=0}^{\infty} a_k x^k$ [**71**, Theorem 1.2].

$$\rho_f = \limsup_{n\to\infty}\frac{n\log n}{\log(1/|a_n|)}, \tag{6.24}$$

$$\sigma_f = \frac{1}{\rho_f\, e}\limsup_{n\to\infty} n|a_n|^{\rho_f/n}\,. \tag{6.25}$$

If v_M satisfies the GRS condition then for all $\epsilon > 0$ there is a $r(\epsilon)$ such that

$$1 \le v_M(r) \le (1+\epsilon)^r \tag{6.26}$$

for all $r > r(\epsilon)$. This implies that $v_M(r) \le \exp(\log(1+\epsilon)r)$, if $r > r(\epsilon)$, and this means that $\rho_{v_M} \le 1$, and if $\rho_{v_M} = 1$, then v_M is of minimal type. The relation

$\lim_{k\to\infty}(M_k/k!)^{1/k} = \infty$ then follows from (6.24) and (6.25) by a straightforward calculation, using Stirling's formula.

Set $\widetilde{M_K} = M_k/(2\pi)^k$. Assume first that $\rho_{v_m} < 1$ and choose $\epsilon > 0$ so small that $\rho_{v_m} + \epsilon < 1$. Then (6.24) implies that

$$k \log k \leq (\rho_{v_M} + \epsilon) \log \widetilde{M_k}$$

for $k > k(\epsilon)$, and so

$$k^k < \widetilde{M_k}^{\rho_{v_M}+\epsilon}.$$

It follows that

$$\widetilde{M_k} > (k^k)^{\frac{1}{\rho_{v_m}+\epsilon}} = k^{k(1+\delta)}$$

for a $\delta > 0$, and therefore

$$\frac{\widetilde{M}_k^{1/k}}{k} > k^\delta \to \infty \tag{6.27}$$

for $k \to \infty$.

If $\rho_{v_M} = 1$, then v_M is of minimal type, and (6.25) implies that

$$0 = \lim_{k\to\infty} \frac{k}{\widetilde{M}_k^{1/k}}\,, \tag{6.28}$$

and that is what we wanted to show.

If we assume that $\lim_{k\to\infty}(M_k/k!)^{1/k} = \infty$, then the relations (6.27) and (6.28) together with (6.24) and (6.25) imply that v_m is of order ≤ 1, and, if order one, of minimal type, so for all $\epsilon > 0$ there is $r(\epsilon)$ such that $v_M(r) \leq (1+\epsilon)^r$ for all $r > r(\epsilon)$. This implies that v_M is a GRS weight. □

PROPOSITION 6.31. *If $\mathcal{A}$ is a Banach algebra with a one-parameter group of automorphisms Ψ generated by δ acting on $\mathcal{A}$, and M is an algebra sequence, then $D^1_M(\mathcal{A})$ is a Banach algebra.*

PROOF. The algebra property is straightforward, using Lemma 6.30(1). For the completeness let a_n be a Cauchy sequence in $D^1_M(\mathcal{A})$. This implies that $\delta^k a_n$ is a Cauchy sequence in $\mathcal{A}$ for all indices k. As δ is a closed operator and $\mathcal{A}$ is complete it follows that there is an element $a \in \mathcal{A}$ such that for all $k \geq 0$ the sequence $\delta^k a_n$ converges to $\delta^k a$ in $\mathcal{A}$. By standard arguments this implies that $a_n \to a$ in $D^1_M(\mathcal{A})$. □

PROPOSITION 6.32. *If $\mathcal{A}$ is a Banach algebrawith a one-parameter group of automorphisms Ψ acting on $\mathcal{A}$, and M an algebra sequence, then the action of ψ is continuous on the whole of $D^1_M(\mathcal{A})$, and so*

$$C(D^1_M(\mathcal{A})) = D^1_M(\mathcal{A}).$$

PROOF. For $a \in D^1_M(\mathcal{A})$,

$$\|\psi_t(a) - a\|_{D^1_M(\mathcal{A})} \leq \sum_{k=0}^{M} \frac{\|\psi_t(\delta^k(a)) - \delta^k(a)\|_{\mathcal{A}}}{M_k} + (M_\Psi + 1) \sum_{k=M+1}^{\infty} \frac{\|\delta^k(a)\|_{\mathcal{A}}}{M_k}\,.$$

For $\epsilon > 0$ given we can choose M such that the second sum in the expansion above is smaller than ϵ. As $\delta^k(a) \in C(\mathcal{A})$ for all $k \in \mathbb{N}_0$ by Proposition 4.20 the first sum can be made small by choosing t small enough. □

An application of Weierstrass' theorem (Corollary 5.16) yields

COROLLARY 6.33. *The bandlimited elements are dense in $D^1_M(\mathcal{A})$.*

PROPOSITION 6.34.

$$D^1_M(\Lambda^1_r(\mathcal{A})) = \Lambda^1_r(D^1_M(\mathcal{A})) \quad \textit{for } r > 0.$$

PROOF. Let $a \in D^1_M(\Lambda^1_r(\mathcal{A}))$, and assume that $l > \lfloor r \rfloor$. Then

$$\begin{aligned}\|a\|_{D^1_M(\Lambda^1_r(\mathcal{A}))} &= \sum_{k=0}^{\infty} M_k^{-1} \|\delta^k a\|_{\Lambda^1_r(\mathcal{A})} \\ &= \sum_{k=0}^{\infty} M_k^{-1} \int_0^{\infty} \frac{\|\Delta_t^l \delta^k a\|_{\mathcal{A}}}{|t|^r} \frac{dt}{|t|^d} \\ &= \int_0^{\infty} \frac{\|\Delta_t^l \delta^k a\|_{D^1_M(\mathcal{A})}}{|t|^r} \frac{dt}{|t|^d} = \|a\|_{\Lambda^1_r(D^1_M(\mathcal{A}))}\,. \end{aligned}$$ □

As $\mathcal{E}^1_r(D^1_M(\mathcal{A}))$=$\Lambda^1_r(D^1_M(\mathcal{A}))$ by (5.12), Proposition 6.34 identifies the approximation spaces $\mathcal{E}^1_r(D^1_M(\mathcal{A}))$ with the Dales-Davie algebras over $\Lambda^1_r(\mathcal{A})$.

It would be natural to assume that $D^1_M(\mathcal{A})$ is inverse-closed in $\mathcal{A}$ if and only if v_M is A GRS weight. However we can only prove the following.

THEOREM 6.35. *Let $\mathcal{A}$ be a* symmetric *Banach algebra, and M an algebra sequence. Set $P_k = M_k/k!$. If*

$$A_m = \sup\Big\{ {P_k}^{-1} \prod_{j=1}^{m} P_{l_j} : l_j \geq 1 \text{ for } 1 \leq j \leq m, \sum_{j=1}^{m} l_j = k \Big\} \tag{6.29}$$

satisfies $\lim_{m\to\infty} A_m^{1/m} = 0$, *then $D^1_M(\mathcal{A})$ is inverse-closed in $\mathcal{A}$.*

PROOF. We adapt [1, Theorem 3.3]. If $a \in D^1_M(\mathcal{A})$, then for every positive integer p,

$$\|a^p\|_{D^1_M(\mathcal{A})} = \sum_{k=0}^{\infty} \frac{\|\delta^k(a^p)\|_{\mathcal{A}}}{M_k} \leq \sum_{k=0}^{\infty} \frac{1}{M_k} \sum_{l_1+\dots l_p = k} \frac{k!}{l_1! \cdots l_p!} \prod_{j=1}^{p} \|\delta^{l_j} a\|_{\mathcal{A}}\,. \tag{6.30}$$

We want to isolate terms with $l_j = 0$. If $C_{p,k}$ denotes the partitions of $\{1, \dots, k\}$ in p possibly empty subsets,

$$\begin{aligned} C_{p,k} &= \{(l_1, \dots, l_p) \colon l_j \geq 0 \text{ for all } 1 \leq j \leq p, \sum_{j=1}^{p} l_j = k\} \\ |C_{p,k}| &= \frac{k!}{l_1! \cdots l_p!} \end{aligned}$$

and

$$C^*_{m,k} = \{(l_1, \dots, l_m) \colon l_j \geq 1 \text{ for all } 1 \leq j \leq m, \sum_{j=1}^{m} l_j = k\}$$

then

$$\begin{aligned} |C_{p,k}| &= \sum_{m=1}^{p} \binom{p}{m} |C^*_{m,k}|\,, \\ \frac{k!}{l_1! \cdots l_p!} &= \sum_{m=1}^{p} \binom{p}{m} \sum_{\substack{l_1+\dots l_m = k \\ l_j \geq 1}} \frac{k!}{l_1! \cdots l_m!} \end{aligned}$$

Using this relation we can further simplify (6.30).

$$\begin{aligned}
\|a^p\|_{D^1_M(\mathcal{A})} &\le \sum_{k=0}^{\infty} \frac{1}{M_k} \sum_{m=0}^{\min\{p,k\}} \binom{p}{m} \|a\|_{\mathcal{A}}^{p-m} \sum_{\substack{l_1+\dots l_m=k \\ l_j\ge 1}} \frac{k!}{l_1!\cdots l_m!} \prod_{j=1}^{m} \|\delta^{l_j} a\|_{\mathcal{A}} \\
&= \sum_{m=0}^{p} \binom{p}{m} \|a\|_{\mathcal{A}}^{p-m} \sum_{k=m}^{\infty} \frac{k!}{M_k} \sum_{\substack{l_1+\dots l_m=k \\ l_j\ge 1}} \prod_{j=1}^{m} \frac{\|\delta^{l_j} a\|_{\mathcal{A}}}{l_j!} \\
&= \sum_{m=0}^{p} \binom{p}{m} \|a\|_{\mathcal{A}}^{p-m} \sum_{k=m}^{\infty} \sum_{\substack{l_1+\dots l_m=k \\ l_j\ge 1}} \Big(\frac{1}{P_k} \prod_{j=1}^{m} P_{l_j}\Big) \prod_{j=1}^{m} \frac{\|\delta^{l_j} a\|_{\mathcal{A}}}{M_{l_j}} \\
&\le \sum_{m=0}^{p} \binom{p}{m} \|a\|_{\mathcal{A}}^{p-m} A_m \sum_{k=m}^{\infty} \sum_{\substack{l_1+\dots l_m=k \\ l_j\ge 1}} \prod_{j=1}^{m} \frac{\|\delta^{l_j} a\|_{\mathcal{A}}}{M_{l_j}} \\
&= \sum_{m=0}^{p} \binom{p}{m} \|a\|_{\mathcal{A}}^{p-m} A_m (\|a\|_{D^1_M(\mathcal{A})} - \|a\|_{\mathcal{A}})^m .
\end{aligned}$$

We need the following result [1, Lemma 3.1].

LEMMA 6.36. *If $K > 0$ and $(\epsilon_m)_{m\ge 0}$ is a sequence of positive numbers with $\lim_{m\to\infty} \epsilon_m = 0$, then*

$$\limsup_{p\to\infty} \Big(\sum_{m=0}^{p} \binom{p}{m} \epsilon_m^m K^{p-m}\Big)^{1/p} \le K .$$

If we set $\epsilon_m = A_m^{1/m}(\|a\|_{D^1_M(\mathcal{A})} - \|a\|_{\mathcal{A}})$ we obtain

$$\|a^p\|_{D^1_M(\mathcal{A})} \le \sum_{m=0}^{p} \binom{p}{m} \|a\|_{\mathcal{A}}^{p-m} \epsilon_m^m ,$$

and Lemma 6.36 implies that

$$\rho_{D^1_M(\mathcal{A})}(a) \le \limsup_{p\to\infty} \|a^p\|_{D^1_M(\mathcal{A})}^{1/p} \le \|a\|_{\mathcal{A}}.$$

As $\mathcal{A}$ is symmetric, Hulanickis Lemma (Proposition 2.10) implies that $D^1_M(\mathcal{A})$ is inverse-closed in $\mathcal{A}$. □

The condition (6.29) is not easy to verify. In [1] sufficient conditions on the sequence $P_k = M_k/k!$ are identified that guarantee (6.29).

PROPOSITION 6.37. *The following conditions on the sequence $P_k = M_k/k!$ imply that $\lim_{m\to\infty} A_m^{1/m} = 0$ and that $D^1_M(\mathcal{A})$ is inverse-closed in the symmetric algebra $\mathcal{A}$.*

(1) *$P_j P_k \le C P_{j+k-1}$ for a constant $C > 0$ and all $j, k \in \mathbb{N}_0$,*
(2) *$\max_{k\le n-1} P_n^{-1}(P_k P_{n-k}) \to 0$ for $n \to \infty$,*
(3) *$P_k^2 \le P_{k-1} P_{k+1}$ for all $k \in \mathbb{N}$, i.e. P_k is log-convex.*

Proofs of these statements can be found in [1]. Condition (3) is sometimes referred to as "M_k is strongly log-convex". Unfortunately we did not find equivalent descriptions for the weights v_M.

EXAMPLE 6.38. The Gevrey sequence $M_k = k!^r$ is an algebra sequence that satisfies (3) of Proposition 6.37, if $r > 1$. In particular, if $\mathcal{A} = \mathcal{B}(\ell^2)$ we obtain the following result.

COROLLARY 6.39. *If $A \in \mathcal{B}(\ell^2(\mathbb{Z}))$ satisfies*

$$\sum_{k=0}^{\infty}(k!)^{-r}\|\delta^k A\|_{\mathcal{B}(\ell^2)(\mathbb{Z})} < \infty$$

for a $r > 1$, and A has an inverse $A^{-1} \in \mathcal{B}(\ell^2(\mathbb{Z}))$, then this inverse satisfies the same estimate.

If the algebra $\mathcal{A}$ is *commutative*, we can do better by adapting a proof of Hulanicki [60].

PROPOSITION 6.40. *If $\mathcal{A}$ is a commutative, symmetric Banach algebra, Ψ a periodic one-parameter group of automorphisms acting on $\mathcal{A}$, and M a weight sequence that satisfies $\lim_{k\to\infty}(M_k/k!)^{1/k} = \infty$ (equivalently, v_M is a GRS weight), then $D^1_M(\mathcal{A})$ is inverse-closed in $\mathcal{A}$.*

PROOF. Assume an $\epsilon > 0$, and decompose $a \in \mathcal{A}$ in

$$a = a_\sigma + r\,,$$

where $\|r\|_{D^1_M(\mathcal{A})} < \epsilon$, and a_σ is σ-bandlimited for a $\sigma > 0$ that clearly depends on ϵ. Bernstein's inequality for bandlimited elements (Equation (5.7)) implies that

$$\|a_\sigma\|_{D^1_M(\mathcal{A})} = \sum_{k=0}^{\infty}\frac{\|\delta^k(a_\sigma)\|_{\mathcal{A}}}{M_k} \le \sum_{k=0}^{\infty}\frac{(2\pi\sigma)^k}{M_k}\|a_\sigma\|_{\mathcal{A}} = v_M(\sigma)\|a_\sigma\|_{\mathcal{A}}\,. \tag{6.31}$$

Then

$$\begin{aligned}
\|a^p\|_{D^1_M(\mathcal{A})} &\le \sum_{l=0}^{p}\binom{p}{l}\|a^l_\sigma\|_{D^1_M(\mathcal{A})}\epsilon^{p-l} \\
&\le C\sum_{l=0}^{p}\binom{p}{l}v_M(l\sigma)\|a_\sigma\|^l_{\mathcal{A}}\epsilon^{p-l} \\
&\le Cv_M(p\sigma)\sum_{l=0}^{p}\binom{p}{l}\|a_\sigma\|^l_{\mathcal{A}}\epsilon^{p-l} \\
&= Cv_M(p\sigma)(\|a_\sigma\|_{\mathcal{A}} + \epsilon)^p \\
&\le Cv_M(p\sigma)(\|a\|_{\mathcal{A}} + 2\epsilon)^p\,,
\end{aligned}$$

the estimate for v_M in the second line uses that σ^l is $l\sigma$-bandlimited. So

$$\rho_{D^1_M(\mathcal{A})}(a) = \lim_{p\to\infty}\|a^p\|^{1/p}_{D^1_M(\mathcal{A})} \le \|a\|_{\mathcal{A}} + 2\epsilon\,.$$

The Lemma of Hulanicki (Lemma 2.10) shows that $D^1_M(\mathcal{A})$ is inverse-closed in $\mathcal{A}$. □

REMARK. Let $\mathcal{B}$ be a solid convolution algebra and $\mathcal{C}_{\mathcal{B}}$ as in Definition 2.11. It is an open question whether $D^1_M(\mathcal{C}_{\mathcal{B}})$ is inverse-closed in $\mathcal{C}_{\mathcal{B}}$, if v_M is a GRS weight.

APPENDIX A

Hardy's Inequality

A.1. Integral Version

PROPOSITION A.1 ([37, 2.3.1]). *Let $r > 0$, $1 \le p < \infty$. Then for ϕ a positive measurable function on $\mathbb{R}^+$*

$$\int_0^\infty \Big[t^{-r}\int_0^t \phi(u)\frac{du}{u}\Big]^q \frac{dt}{t} \le r^{-p}\int_0^\infty t^{-rq}\phi(t)^q \frac{dt}{t},$$
$$\int_0^\infty \Big[t^{r}\int_t^\infty \phi(u)\frac{du}{u}\Big]^q \frac{dt}{t} \le r^{-p}\int_0^\infty t^{rq}\phi(t)^q \frac{dt}{t}.$$

For $p = \infty$ we obtain

$$\sup_{t>0}\Big[t^{-r}\int_0^t \phi(u)\frac{du}{u}\Big] \le \frac{1}{r}\|t^{-r}\phi(t)\|_\infty,$$
$$\sup_{t>0}\Big[t^{r}\int_t^\infty \phi(u)\frac{du}{u}\Big] \le \frac{1}{r}\|t^{r}\phi(t)\|_\infty.$$

A.2. Discrete Version

We also need the following discrete version of Hardy's inequality [73, Lemma 2.1], [37, Lemma 2.3.4]

PROPOSITION A.2. *Let $r > 0$, $1 \le p, q \le \infty$. Then for $C > 1$ and all sequences of complex numbers $(a_i)_{i\ge 0}$*

$$\sum_{l=0}^\infty C^{rlq}(\sum_{k=l}^\infty |a_k|^p)^{q/p} \lesssim \sum_{l=0}^\infty C^{rlq}|a_l|^q,$$
$$\sum_{l=0}^\infty C^{-rlq}(\sum_{k=0}^l |a_k|^p)^{q/p} \lesssim \sum_{l=0}^\infty C^{-rlq}|a_l|^q$$

with the obvious interpretation for $p, q = \infty$.

APPENDIX B

A General Quotient Rule

We provide a proof of Lemma 6.18.

Lemma. *Let $E = \{1, \ldots, d\}$ and $\delta_1, \ldots, \delta_d$ be derivations satisfying the quotient rule:*

$$\delta_j(a^{-1}) = -a^{-1}\delta_j(a)a^{-1} \quad \textit{for all } j \in E.$$

For every $k \in \mathbb{N}$ and every tuple $B = (b_1, \ldots, b_k) \in E^k$ set $|B| = k$, and

$$\delta_B(a) = \delta_{b_1} \ldots \delta_{b_k}(a)\,.$$

Define the partitions of B into m nonempty subtuples as

$$P(B, m) = \{(B_1, \cdots, B_m)\colon B = (B_1, \cdots, B_m), B_i \neq \emptyset \textit{ for all } i\}\,.$$

Then

$$\delta_B(a^{-1}) = \sum_{m=1}^{|B|} (-1)^m \sum_{(B_i)_{1\le i\le m} \in P(B,m)} \Big(\prod_{j=1}^{m} a^{-1}\delta_{B_i}(a)\Big)a^{-1}\,. \tag{B.1}$$

The proof is by induction over $|B|$. If $|B| = 1$ there is nothing to prove. Assume that the statement is true for $|B| < k$, and assume $|B| = k$. The Leibniz rule for $\delta_B(a^{-1}a)$ yields

$$\delta_B(a^{-1}a) = 0 = \sum_{(B_1,B_2)\in P(B,2)} \delta_{B_1}(a^{-1})\delta_{B_2}(a) + a^{-1}\delta_B(a) + \delta_B(a^{-1})a\,, \tag{B.2}$$

So

$$\delta_B(a^{-1}) = -a^{-1}\delta_B(a)a^{-1} - \sum_{(B_1,B_2)\in P(B,2)} \delta_{B_1}(a^{-1})\delta_{B_2}(a)a^{-1}\,. \tag{B.3}$$

As $|B_1| < k$ we can apply the induction hypothesis.

$$\begin{aligned}\delta_B(a^{-1}) = &- a^{-1}\delta_B(a)a^{-1} \\ &- \sum_{(B_1,B_2)\in P(B,2)} \sum_{m=1}^{|B_1|} (-1)^m \sum_{(D_i)_{1\le i\le m}\in P(B_1,m)} \Big(\prod_{j=1}^{m} a^{-1}\delta_{D_i}(a)\Big)a^{-1}\,\delta_{B_2}(a)a^{-1}\,.\end{aligned}$$

Interchanging the first two summations we obtain

$$\begin{aligned}\delta_B(a^{-1}) = &- a^{-1}\delta_B(a)a^{-1} \\ &- \sum_{m=1}^{k-1} (-1)^m \sum_{\substack{(B_1,B_2)\in P(B,2) \\ |B_1|\ge m}} \sum_{(D_i)_{1\le i\le m}\in P(B_1,m)} \Big(\prod_{j=1}^{m} a^{-1}\delta_{D_i}(a)\Big)a^{-1}\,\delta_{B_2}(a)a^{-1}\,.\end{aligned}$$

Now observe that in this expression $(D_1, \cdots, D_m, B_2)$ varies over all partitions of B, The condition $(D_i)_{1\le i\le m} \in P(B_1, m)$ already implies that $|B_1| \ge m$, so we can set $D_{m+1} = B_2$ and obtain

$$\delta_B(a^{-1}) = -\, a^{-1}\delta_B(a)a^{-1} - \sum_{m=1}^{k-1}(-1)^m \sum_{(D_i)_{1\le i\le m+1}\in P(B,m+1)} \Big(\prod_{j=1}^{m+1} a^{-1}\delta_{D_i}(a)\Big)a^{-1}\,.$$

We change the summation index.

$$\delta_B(a^{-1}) = -\, a^{-1}\delta_B(a)a^{-1} + \sum_{l=2}^{k}(-1)^m \sum_{(D_i)_{1\le i\le l}\in P(B,l)} \Big(\prod_{j=1}^{l} a^{-1}\delta_{D_i}(a)\Big)a^{-1}\,.$$

The term $-a^{-1}\delta_B(a)a^{-1}$ can included into the sum for $l = 1$. We obtain

$$\delta_B(a^{-1}) = \sum_{l=1}^{k}(-1)^m \sum_{(D_i)_{1\le i\le l}\in P(B,l)} \Big(\prod_{j=1}^{l} a^{-1}\delta_{D_i}(a)\Big)a^{-1}\,,$$

and this is (B.1).

APPENDIX C

The Interpolation Theorem

We give a proof of Proposition 3.17.

PROPOSITION (Interpolation theorem, [**73**, 4.3, 4.5]). *Let $\mathcal{X}, \mathcal{Y}$ and ω as in Definition 3.15. Let v be an approximation weight with $V_p(m) \asymp \omega_m^r$ for some $0 < r < 1$, $1 \leq p \leq \infty$. Then*

$$\mathcal{E}_v^p(\mathcal{X}) = (\mathcal{X}, \mathcal{Y})_{(r,p)}$$

with equivalent norms.

We need the following auxiliary result.

LEMMA C.1 ([**73**, Lemma 4.2]). *Assume that the pair $(\mathcal{X}, \mathcal{Y})$ satisfies the JB-condition for an approximation scheme $(X_n)_{n\geq 0}$ and with a weight $(\omega_n)_{n\geq 0}$. If $1 = n_0 \leq n_1 \leq \cdots \leq n_j$ is a sequence of natural numbers, then for every $a \in \mathcal{X}$ and every $n \in \mathbb{N}_0$*

$$E_n^{\mathcal{X}}(a) \leq K(a, 1/\omega_n), \tag{C.1}$$

and

$$K(a, 1/\omega_{n_j}) \leq C \frac{1}{\omega_{n_j}} \sum_{l=0}^{j} \omega_{n_l} E_{n_{l-2}}^{\mathcal{X}}(a) \tag{C.2}$$

using the convention $n_{-1} = n_{-2} = 0$.

PROOF. Choose elements $y \in \mathcal{Y}$, and $x_n \in X_n$, then $E_n^{\mathcal{X}}(a) \leq \|a-y\|_{\mathcal{X}} + \|y - x_n\|_{\mathcal{X}}$. If x_n converges to the best approximation to y in X_n, we obtain with the help of Jackson's inequality $E_n^{\mathcal{X}}(a) \leq \|a-y\|_{\mathcal{X}} + E_n^{\mathcal{X}}(y) \lesssim \|a-y\|_{\mathcal{X}} + \frac{1}{\omega_n}\|y\|_{\mathcal{Y}}$. Taking the infimum over $y \in \mathcal{Y}$ we get (C.1).

For the second inequality let a_j be an arbitrary element of $X_{n_{j-1}}$. Then

$$\begin{aligned} K(a, 1/\omega_{n_j}) &\leq \|a - a_j\|_{\mathcal{X}} + \frac{1}{\omega_{n_j}} \|\sum_{l=0}^{j} (a_l - a_{l-1})\|_{\mathcal{Y}} \\ &\lesssim \|a - a_j\|_{\mathcal{X}} + \frac{1}{\omega_{n_j}} \sum_{l=0}^{j} \omega_{n_{l-1}} \|a_l - a_{l-1}\|_{\mathcal{X}} \\ &\lesssim \|a - a_j\|_{\mathcal{X}} + \frac{1}{\omega_{n_j}} \sum_{l=0}^{j} \omega_{n_l} (\|a - a_l\|_{\mathcal{X}} + \|a - a_{l-1}\|_{\mathcal{X}}). \end{aligned}$$

Taking infima over the a_l we obtain

$$K(a, 1/\omega_{n_j}) \lesssim E_{n_{j-1}}^{\mathcal{X}}(a) + \frac{1}{\omega_{n_j}} \sum_{l=0}^{j} \omega_{n_l} E_{n_{l-2}}^{\mathcal{X}}(a). \tag{C.3}$$

As $E^{\mathcal{X}}_{n_{j-1}}(a) \le E^{\mathcal{X}}_{n_{l-2}}(a)$ for all $l \le j$, and trivially $\omega_{n_j} \le \sum_{l=0}^{j} \omega_{n_l}$, we can control $E^{\mathcal{X}}_{n_{j-1}}(a)$ by

$$E^{\mathcal{X}}_{n_{j-1}}(a) \le \frac{1}{\omega_{n_j}} \sum_{l=0}^{j} \omega_{n_l} E^{\mathcal{X}}_{n_{j-1}}(a) \le \frac{1}{\omega_{n_j}} \sum_{l=0}^{j} \omega_{n_l} E^{\mathcal{X}}_{n_{l-2}}(a) \,.$$

Inserting this in (C.3) we obtain (C.2). □

The plan of the proof of Proposition 3.17 is as follows: First we derive an equivalent norm for $(\mathcal{X},\mathcal{Y})_{(r,p)}$. Using this characterization, we verify the embeddings $\mathcal{E}^p_v(\mathcal{X}) \hookrightarrow (\mathcal{X},\mathcal{Y})_{(r,p)}$ and $(\mathcal{X},\mathcal{Y})_{(r,p)} \hookrightarrow \mathcal{E}^p_v(\mathcal{X})$.

Claim. *Under the conditions of Proposition 3.17,*

$$\|a\|_{(r,p)} \asymp \|(K(a,1/\omega_n))_{n\ge 0}\|_{\ell^p_v(\mathbb{N})} \,.$$

Proof of the claim. We show first that

$$(\text{C.4}) \qquad (h_n)_{n\ge0} \asymp (\eta_n)_{n\ge0} \quad \text{implies} \quad (K(a,h_n))_{n\ge0} \asymp (K(a,\eta_n))_{n\ge0} \,.$$

Indeed, if $(h_n) \asymp (\eta_n)$ then $h_n \le C\eta_n$, and w.l.o.g we can assume that $C \ge 1$. Trivially, for $C \ge 1$ and $t \ge 0$,

$$\|a-y\|_{\mathcal{X}} + Ct\|y\|_{\mathcal{Y}} \le C(\|a-y\|_{\mathcal{X}} + t\|y\|_{\mathcal{Y}}),$$

which shows that $K(a,Ct) \le CK(a,t)$ for $C \ge 1$, and so $K(a,h_n) \le K(a,C\eta_n) \le CK(a,\eta_n)$. Reversing the roles of η_n and h_n proves the other direction of the inequality. The assumptions on V_p imply that

$$(\text{C.5}) \qquad 1/\omega_n \asymp V_p(n)^{-1/r}.$$

So, with φ_j defined as in (3.14) we get

$$(\text{C.6}) \qquad \begin{aligned} \|(K(a,1/\omega_n))_{n\ge0}\|_{\ell^p_v(\mathbb{N}_0)} &\asymp \|(K(a,V_p(n)^{-1/r}))_{n\ge0}\|_{\ell^p_v(\mathbb{N}_0)} \\ &\asymp \|(\kappa^j K(a,V_p(\varphi_j)^{-1/r})_{j\ge0}\|_{\ell^p(\mathbb{N}_0)} \\ &\asymp \|(\kappa^j K(a,\kappa^{-j/r})_{j\ge0}\|_{\ell^p(\mathbb{N}_0)} \,. \end{aligned}$$

In the above chain the second line follows from the remark follwoing the Equivalence Theorem 3.10. Indeed, $\|a\|_{(r,p)} \asymp \|a\|_{\mathcal{X}} + \int_1^\infty \big(t^r K(a,1/t)\big)^p \frac{dt}{t}$, and we can apply Corollary 3.10 with $E_\lambda(a) = K(a,1/\lambda)$. The third equivalence in (C.6) results from (3.22). The last expression is an equivalent discrete norm on $(\mathcal{X},\mathcal{Y})_{(r,p)}$, see (3.25). □

Proof of Proposition 3.17. To verify the embedding $\mathcal{E}^p_v(\mathcal{X}) \hookrightarrow (\mathcal{X},\mathcal{Y})_{(r,p)}$ Let $a \in (\mathcal{X},\mathcal{Y})_{(r,p)}$. By (C.4) and using $\omega_{\varphi_j} \asymp \kappa^{j/r}$ we obtain

$$\|a\|_{(r,p)} \asymp \|\big(\kappa^j K(a,\kappa^{-j/r})\big)_{j\ge0}\|_{\ell^p(\mathbb{N}_0)} \asymp \|\big(\kappa^j K(a,1/\omega_{\varphi_j})\big)\|_{\ell^p(\mathbb{N}_0)} \,.$$

With (C.3) we arrive at

$$\begin{aligned} \|a\|_{(r,p)} &\lesssim \|\Big(\kappa^j \frac{1}{\omega_{\varphi_j}} \sum_{i=0}^{j} \omega_{\varphi_i} E_{\varphi_{i-2}}(a)\Big)_{j\ge0}\|_{\ell^p(\mathbb{N}_0)} \\ &\asymp \|\Big(\kappa^{j(1-1/r)} \sum_{i=0}^{j} \kappa^{i/r} E_{\varphi_{i-2}}(a)\Big)_{j\ge0}\|_{\ell^p(\mathbb{N}_0)} \,. \end{aligned}$$

Hardy's inequality (Appendix A.2) implies that the last norm can be dominated by

$$\|\big(\kappa^j E_{\varphi_{j-2}}(a)\big)_{j\ge0}\|_{\ell^p(\mathbb{N})0} \lesssim \|\big(\kappa^j E_{\varphi_j}(a)\big)_{j\ge0}\|_{\ell^p(\mathbb{N}_0)} \asymp \|a\|_{\mathcal{E}^p_w(\mathcal{X})} \,,$$

and so $\|a\|_{(r,p)} \lesssim \|a\|_{\mathcal{E}^p_v(\mathcal{X})}$.

For the proof of the embedding $(\mathcal{X},\mathcal{Y})_{(r,p)} \hookrightarrow \mathcal{E}_v^p(\mathcal{X})$ set

$$\begin{aligned}\|a\|_{\mathcal{E}_w^p(\mathcal{X})} &\asymp \|a\|_{\mathcal{X}} + \|(v_k E_k(a))_{k\geq 1}\|_{\ell^p(\mathbb{N})} \\ &\lesssim \|a\|_{\mathcal{X}} + \|(v_k K(a, 1/\omega_k))_{k\geq 1}\|_{\ell^p(\mathbb{N})}\,,\end{aligned}$$

using (C.1). As $\|a\|_{\mathcal{X}} \leq \|a-g\|_{\mathcal{X}} + \|g\|_{\mathcal{X}} \leq \|a-g\|_{\mathcal{X}} + \|g\|_{\mathcal{Y}}$ for all $g \in \mathcal{Y}$ we obtain that $\|a\|_{\mathcal{X}} \leq K(a,1) = K(a,1/\omega_0)$, and actually $\|a\|_{\mathcal{E}_v^p(\mathcal{X})} \lesssim \|a\|_{(r,p)}$. This was to be proved. □

APPENDIX D

Smoothness in Banach Algebras

D.1. Moduli of Smoothness

We give a proof of Lemma 4.29.

LEMMA. *If $l, k \in \mathbb{N}$, $l \geq k$, $t \in \mathbb{R}^d$ and $h > 0$, then*

(1)
$$\|\Delta_t^l(x)\|_{\mathcal{X}} \leq (M_\Psi + 1)^k \|\Delta_t^{l-k}(x)\|_{\mathcal{X}} \text{ and } \omega_h^l(x) \leq (M_\Psi + 1)^k \omega_h^{l-k}(x)\,,$$

(2)
$$\|\Delta_{2t}^k(x)\|_{\mathcal{X}} \leq (M_\Psi + 1)^k \|\Delta_t^k(x)\|_{\mathcal{X}} \text{ and } \omega_{2h}^k(x) \leq (M_\Psi + 1)^k \omega_h^k(x)\,,$$

(3) *If $\lambda > 0$ then*
$$\|\omega_{\lambda t}^k(x)\|_{\mathcal{X}} \leq (M_\Psi + 1)^k (\lambda + 1)^k \|\Delta_t^k(x)\|_{\mathcal{X}}\,.$$

(4) *(Marchaud inequality)*
$$\omega_h^k(x) \leq C h^k \int_h^\infty \frac{\omega_u^l(x)}{u^k}\frac{du}{u}.$$

(5) *The averaged modulus of smoothness*
$$\overline{w}_h^k(x) = h^{-d} \int_{|t| \leq h} \|\Delta_t^k x\|_{\mathcal{X}}\, dt$$
is equivalent to the "standard" modulus of smoothness: $\overline{w}_h^k(x) \asymp \omega_h^k(x)$ [**37**, Lemma 6.5.1].

(6) *The modulus of smoothness is also equivalent to the iterated modulus of smoothness* [**13**, 5.4.11],
$$\omega_t^k(x) \asymp \sup_{\substack{|h_j| \leq t \\ 1 \leq j \leq k}} \|(\prod_{j=1}^k \Delta_{h_j}) x\|_{\mathcal{X}}.$$

(7) *If $a \in C^k(\mathcal{X})$, then*
$$\omega_h^{k+l}(x) \leq C \sup_{|\alpha| = k} \omega_h^l(\delta^\alpha(x))$$

PROOF. The proofs of (1), (2) and (3) are easy calculations in complete analogy to the corresponding properties of the moduli of smoothness for functions. See, e.g., [**37**].

PROOF OF 4. The proof of the Marchaud inequality is in many textbooks [**37, 13, 100**]. We reproduce the proof of [**37, 13**].

CLAIM. *For all $k \geq 0$ and $t \in \mathbb{R}^d$ the identity*

$$\Delta_{2t}^k = \sum_{j=1}^k \binom{k}{j} \sum_{m=0}^{j-1} \Delta_t^{k+1} \psi_{mt} + 2^k \Delta_t^k \tag{D.1}$$

is valid [**13**, 5.4.8].

Using this algebraic identity we obtain the norm inequality

$$\begin{aligned}\|\Delta_t^k(x)\|_{\mathcal{X}} &\le 2^{-k}\left(\|\Delta_{2t}^k(x)\|_{\mathcal{X}} + \sum_{j=1}^{k}\binom{k}{j}\sum_{m=0}^{j-1}\|\Delta_t^{k+1}\psi_{mt}(x)\|_{\mathcal{X}}\right)\\ &\le 2^{-k}\left(\|\Delta_{2t}^k(x)\|_{\mathcal{X}} + M_\Psi\sum_{j=1}^{k}\binom{k}{j} j\|\Delta_t^{k+1}(x)\|_{\mathcal{X}}\right)\\ &\le 2^{-k}\|\Delta_{2t}^k(x)\|_{\mathcal{X}} + 2^{-k}M_\Psi 2^{k-1}\, k\|\Delta_t^{k+1}(x)\|_{\mathcal{X}},\end{aligned}$$

and so we obtain

$$\|\Delta_t^k(x)\|_{\mathcal{X}} \le 2^{-k}\|\Delta_{2t}^k(x)\|_{\mathcal{X}} + \frac{k}{2}M_\Psi\|\Delta_t^{k+1}(x)\|_{\mathcal{X}}. \tag{D.2}$$

Iterating this relation and taking suprema over $|t| \le h$ gives

$$\omega_h^k(x) \le \frac{k}{2}M_\Psi\sum_{j=0}^{n} 2^{-jk}\omega_{2^j h}^{k+1}(x) + 2^{-(n+1)k}\omega_{2^n h}^k(x), \tag{D.3}$$

which can be considered as a discrete form of the Marchaud inequality for $l = k+1$. The continuous form follows using the identity

$$(2^j t)^{-k} = k(1-2^{-k})^{-1}\int_{2^j t}^{2^{j+1}t} s^{-k-1}\, ds$$

in (D.3). Then, as $(1-2^{-k})^{-1} \le 2$

$$\begin{aligned}\omega_h^k(x) &\le k^2 M_\Psi t^k \sum_{j=0}^{n}\int_{2^j t}^{2^{j+1}t} s^{-k-1}\omega_{2^j h}^{k+1}(x)\, ds + 2^{-(n+1)k}\omega_{2^n h}^k(x)\\ &\le k^2 M_\Psi t^k\int_t^{2^{n+1}t} s^{-k}\omega_s^{k+1}(x)\,\frac{ds}{s} + 2^{-(n+1)k}\omega_{2^n h}^k(x),\end{aligned} \tag{D.4}$$

and for $n \to \infty$ this is the Marchaud inequality for $l = k+1$. The general case follows by induction: If

$$\omega_h^k(x) \le Ct^k\int_t^\infty s^{-k}\omega_s^l(x)\,\frac{ds}{s}$$

for some $l > k$, then by (D.4)

$$\omega_h^k(x) \le Cl^2 M_\Psi t^k\int_t^\infty s^{-k}\, s^k\int_s^\infty u^{-l}\omega_u^{l+1}(x)\frac{du}{u}\frac{ds}{s}.$$

Changing the order of integration the Marchaud inequality follows for $l+1$.

Proof of the claim.

$$\Delta_{2t}^k = (\psi_{2t} - \mathrm{id})^k = (\psi_t^2 - \mathrm{id})^k = (\psi_t + \mathrm{id})^k(\psi_t - \mathrm{id})^k\,.$$

But

$$\begin{aligned}(\psi_t + \mathrm{id})^k &= \sum_{j=0}^{k}\binom{k}{j}\psi_{jt}\,,\\ (\psi_t + \mathrm{id})^k - 2^k\,\mathrm{id} &= \sum_{j=1}^{k}\binom{k}{j}(\psi_{jt} - \mathrm{id})\\ &= \sum_{j=1}^{k}\binom{k}{j}\sum_{m=0}^{j-1}\psi_{mt}\Delta_t\end{aligned}$$

Putting these identities together yields (D.1). □

□

Proof of (5). We adapt a proof of [37]. The relation $|h|^{-d}\int_{|t|<h}\|\Delta_t^k x\|_{\mathcal{X}}\,dt \lesssim \omega_h^k(x)$ is immediate. For the other inequality we need the identity

$$\Delta_h^k = \sum_{j=1}^{k}\binom{k}{j}(-1)^j\Big(\Delta_{js}^k\psi_{jh} - \Delta_{h+js}^k\Big), \tag{D.5}$$

which is valid for all $h, s \in \mathbb{R}^d$. Then

$$\|\Delta_h^k(x)\|_{\mathcal{X}} \lesssim \sum_{j=1}^{k}\binom{k}{j}\big(\|\Delta_{js}^k(x)\|_{\mathcal{X}} + \|\Delta_{h+js}^k(x)\|_{\mathcal{X}}\big)$$

Integrating yields

$$\|\Delta_h^k(x)\|_{\mathcal{X}}\,\mathrm{Vol}(\{s : |s| \le 1\})|h|^d \lesssim \sum_{j=1}^{k}\binom{k}{j}\big(\int_{|s|\le|h|}\|\Delta_{js}^k(x)\|_{\mathcal{X}}\,ds + \int_{|s|\le|h|}\|\Delta_{h+js}^k(x)\|_{\mathcal{X}}\,ds\big),$$

so

$$\|\Delta_h^k\|_{\mathcal{X}} \lesssim |h|^{-d}\sum_{j=1}^{k}\binom{k}{j}\big(\int_{|s|\le|h|}\|\Delta_{js}^k\|_{\mathcal{X}}\,ds + \int_{|s|\le|h|}\|\Delta_{h+js}^k\|_{\mathcal{X}}\,ds\big).$$

A change of variables gives

$$\|\Delta_h^k\|_{\mathcal{X}} \lesssim |h|^{-d}\int_{|s|\le(k+1)|h|}\|\Delta_s^k\|_{\mathcal{X}}\,ds = \overline{w}_{(k+1)h}^k.$$

So we have shown that $\omega_h^k \lesssim \overline{w}_{(k+1)h}^k \lesssim \omega_{(k+1)h}^k$. As $\omega_{\lambda h}^k \asymp \omega_h^k$ we are finished, if (D.5) is verified. For this,

$$\begin{aligned}\sum_{j=0}^{k}(-1)^{j+k}\binom{k}{j}\Delta_{h+js}^k &= \sum_{j=0}^{k}(-1)^{j+k}\binom{k}{j}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\psi_{l(h+js)}\\ &= \sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\sum_{j=0}^{k}(-1)^{j+k}\binom{k}{j}\psi_{lh}\psi_{ljs}\\ &= \sum_{l=1}^{k}\binom{k}{l}(-1)^{k+l}\Delta_{ls}^k\psi_{lh}\,.\end{aligned}$$

Isolating the term $j = 0$ on the left and reshuffling gives (D.5). □

Proof of 6. This is [13, Lemma 5.4.11]. We reproduce the proof given there. The relation $\omega_t^k(x) \le \sup_{\substack{|h_j|\le t\\1\le j\le k}}\|\big(\prod_{j=1}^k \Delta_{h_j}\big)x\|_{\mathcal{X}}$ is a consequence of the definition of $\omega_t^k(x)$. To prove the other inequality we need the following fact.

$$\prod_{j=1}^{k}\Delta_{h_j} = \sum_{D\subseteq\{1,\dots,k\}}(-1)^{|D|}\psi_{h_D^*}\Delta_{h_D}^k\,, \tag{D.6}$$

where the sum is over all subsets D of $\{1,\dots,k\}$, and $h_D^* = \sum_{j\in D}h_j$, $h_D = -\sum_{j\in D}\frac{h_j}{j}$.

Using this relation and the observation that $|h_D| \le kt$ we obtain

$$\sup_{\substack{|h_j|\le t\\1\le j\le k}}\|\big(\prod_{j=1}^{k}\Delta_{h_j}\big)x\|_{\mathcal{X}} \le C\omega_{kt}^k(x) \le C'\omega_t^k(x)\,,$$

an that is what we wanted to show.

The proof of (D.6) is purely algebraic. For each integer l with $0 \le l \le k$

$$\begin{aligned}\prod_{j=1}^{k} \Delta_{(j-l)h_j} &= \prod_{j=1}^{k} (\psi_{(j-l)h_j} - \mathrm{id}) = \sum_{D \subseteq \{1,\dots,k\}} (-1)^{k-|D|} \prod_{j\in D} \psi_{(j-l)h_j} \\ &= \sum_{D \subseteq \{1,\dots,k\}} (-1)^{k-|D|} \psi_{(\sum_{j\in D} jh_j)} \psi^l_{(-\sum_{j\in D} h_j)} .\end{aligned} \tag{D.7}$$

Obviously, $\prod_{j=1}^{k} \Delta_{(j-l)h_j} = 0$, if $l > 0$. So

$$\begin{aligned}\sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} \prod_{j=1}^{k} \Delta_{(j-l)h_j} &= (-1)^k \prod_{j=1}^{k} \Delta_{(j-l)h_j} \\ &= \sum_{D \subseteq \{1,\dots,k\}} (-1)^{k-|D|} \psi_{(\sum_{j\in D} jh_j)} \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} \psi^l_{(-\sum_{j\in D} h_j)} \\ &= \sum_{D \subseteq \{1,\dots,k\}} (-1)^{k-|D|} \psi_{(\sum_{j\in D} jh_j)} \Delta^l_{(-\sum_{j\in D} h_j)} .\end{aligned}$$

If we replace h_j by h_j/j, we obtain (D.6). □

Proof of 7. Let $M_1 = c_{[0,1]}$, $M_j = M_1 * M_{j-1}$, $j > 1$ the B-splines of order j. Then, as we will show later, for each $x \in C^k(\mathcal{X})$

$$\Delta^k_h(x) = \int_{-\infty}^{\infty} M_k(\xi) \sum_{|\alpha|=k} \frac{k!}{\alpha!} \psi_{\xi h}(\delta^\alpha x) h^\alpha \, d\xi. \tag{D.8}$$

So,

$$\Delta^{k+l}_h(x) = \int_{-\infty}^{\infty} M_k(\xi) \sum_{|\alpha|=k} \frac{k!}{\alpha!} \psi_{\xi h}(\delta^\alpha \Delta^l_h x) h^\alpha \, d\xi.$$

Taking norms we obtain

$$\|\Delta^{k+l}_h(x)\|_{\mathcal{X}} \le M_\psi \sum_{|\alpha|=k} \frac{k!}{\alpha!} \|\Delta^l_h \delta^\alpha x\| h^k \lesssim h^k \sup_{|\alpha|=k} \omega^l_h(\delta^\alpha x) ,$$

and this was to prove. Equation (D.8) follows from the one-dimensional identity

$$\Delta^k_h(x) = \int_{-\infty}^{\infty} \psi_{\xi h}(\delta^k x) h^k M_k(\xi) \, d\xi \tag{D.9}$$

for a one-parameter group ψ_t and reducing to the one-dimensional case as usual. Equation (D.9) itself is proved by induction. See [**13**, 5.4.7-8] for details. □

□

D.2. Equivalent Norms on Besov Spaces

In this section we give a prove of Proposition 4.31.

Proposition. *Let $x \in \mathcal{X}$, $r > 0$. Then for every integer $k > \lfloor r \rfloor$ the following expressions define equivalent (semi)norms on $\Lambda^p_r(\mathcal{X})$.*

(1)

$$|x|_{\Lambda^p_r(\mathcal{X})} \asymp \left[\int_{\mathbb{R}^+} (h^{-r} \omega^k_h(x))^p \frac{dh}{h} \right]^{1/p} .$$

(2)

$$\|x\|_{\Lambda^p_r(X)} \asymp \|x\|_{\mathcal{X}} + \left[\int_{\mathbb{R}^d} (|t|^{-r} \|\Delta^k_t x\|_{\mathcal{X}})^p \frac{dt}{|t|^d} \right]^{1/p} .$$

(3)
$$\|x\|_{\Lambda^p_r(X)} \asymp \|x\|_{\mathcal{X}} + \left(\sum_{l=0}^{\infty} \left(2^{rl}\omega^k_{2^{-l}}(x)\right)^p\right)^{1/p},$$

(4) *If l is an integer, and $l \le r$, then*
$$\|x\|_{\Lambda^p_r(\mathcal{X})} \asymp \|x\|_{\mathcal{X}} + \sum_{|\alpha|=l} \|\delta^\alpha(x)\|_{\Lambda^p_{r-l}(\mathcal{X})}.$$

PROOF. Part (1) follows from the Marchaud inequality, see [**37**, 2.10.1]. For the proof of (2) observe that
$$\int_{\mathbb{R}^d} \left(|t|^{-r}\|\Delta^k_t x\|_{\mathcal{X}}\right)^p \frac{dt}{|t|^d} \le \int_{\mathbb{R}^d} \left(|t|^{-r} \sup_{|s|\le|t|} \|\Delta^k_t x\|_{\mathcal{X}}\right)^p \frac{dt}{|t|^d}$$
$$\le C \int_{\mathbb{R}^+} \left(h^{-r}\omega^k_h(x)\right)^p \frac{dh}{|h|}.$$

For the proof of the converse we need the averaged modulus of smoothness defined in Lemma 4.29.
$$\int_{\mathbb{R}^+} ((h^{-r}\omega^k_h(x))^p \frac{dh}{h} \le C \int_{\mathbb{R}^+} h^{-rp} \left[\frac{1}{h^d}\int_{|t|\le h} \|\Delta^k_t x\|_{\mathcal{X}}\, dt\right]^p dh$$
$$\le C \int_{\mathbb{R}^+} h^{-rp} \left[\int_{|t|\le h} \frac{\|\Delta^k_t x\|_{\mathcal{X}}}{|t|^d}\, dt\right]^p dh.$$

We use the the surface measure σ on the unit sphere $\mathbf{S}^{d-1}$ and define $\Omega(r) = \int_{\mathbf{S}^{d-1}} \|\Delta^k_{rt'} x\|_{\mathcal{X}}\, d\sigma(t')$ (t' a unit vector in $\mathbb{R}^d$). Then we get
$$\int_{\mathbb{R}^+} h^{-rp} \left[\int_{|t|\le h} \frac{\|\Delta^k_t x\|_{\mathcal{X}}}{|t|^d}\, dt\right]^p dh = \int_{\mathbb{R}^+} h^{-rp}\left[\int_0^h \Omega(r)\frac{dr}{r}\right]^p dh$$
$$\le C' \int_{\mathbb{R}^+} (r^{-r}\Omega(r))^p \frac{dr}{r}$$

For the last relation we use Hardy's inequality (Appendix A). The right hand side is equal to
$$C'\left[\int_{\mathbb{R}^d} (|t|^{-r}\|\Delta^k_t x\|_{\mathcal{X}})^p \frac{dt}{|t|^d}\right]^{1/p},$$
as desired.

The relation (3) can be obtained from (1) by discretizing, see [**37**, 2.10.5].

For the proof of (4) we verify first
$$\|x\|_{\Lambda^p_r(\mathcal{X})} \le C\|x\|_{C^l(\mathcal{X})} + \sum_{|\alpha|=l} \|\delta^\alpha(x)\|_{\Lambda^p_{r-l}(\mathcal{X})},$$
which follows directly from Lemma 4.29(7). Indeed, for $k > r$
$$\int_0^\infty \left(h^{-r}\omega^k_h(x)\right)^p \frac{dh}{h} \le C \sum_{|\alpha|=l} \int_0^\infty [h^{-r+l}\omega^{k-l}_h(\delta^\alpha(x))]^p \frac{dh}{h},$$
which proves the assertion. For the other inequality we need the important relation
$$\text{(D.10)} \qquad (-\delta_{e_j})(x) = \frac{1}{C_k}\int_0^\infty \frac{\Delta^k_{he_j}(x)}{h}\frac{dh}{h}, \qquad C_k = \int_0^\infty \frac{(e^{-h}-1)^k}{h}\frac{dh}{h}$$
for all $x \in \Lambda^1_1(\mathcal{X})$ and integers $k > 1$, which is proved in Appendix D.3.

We prove

$$\|x\|_{\Lambda_r^p(\mathcal{X})} \geq C\|x\|_{\mathcal{X}} + \sum_{|\alpha|=l} \|\delta^\alpha(x)\|_{\Lambda_{r-l}^p(\mathcal{X})}$$

by induction on l. If $l = 1$ choose $k > r$ and estimate

$$\begin{aligned}\|\Delta_t^k \delta_{e_j}(x)\|_{\mathcal{X}} &\lesssim \int_0^\infty \frac{\|\Delta_t^k \Delta_{he_j}^k(x)\|_{\mathcal{X}}}{h}\frac{dh}{h} \\ &= \int_0^{|t|} \frac{\|\Delta_t^k \Delta_{he_j}^k(x)\|_{\mathcal{X}}}{h}\frac{dh}{h} + \int_{|t|}^\infty \frac{\|\Delta_t^k \Delta_{he_j}^k(x)\|_{\mathcal{X}}}{h}\frac{dh}{h} = I + II\end{aligned}$$

Then

$$\begin{aligned}I &\leq (M_\psi + 1)^k \int_0^{|t|} \frac{\|\Delta_{he_j}^k(x)\|_{\mathcal{X}}}{h}\frac{dh}{h}, \\ II &\leq (M_\psi + 1)^k \frac{\|\Delta_t^k(x)\|_{\mathcal{X}}}{|t|},\end{aligned}$$

both inequalities using Lemma 4.29(1). With this estimates we obtain for the Λ_r^p-seminorm of $\delta_{e_j}(x)$

$$\begin{aligned}\int_{\mathbb{R}^d\setminus\{0\}} \left[|t|^{-r-1}\int_0^{|t|} \frac{\|\Delta_{he_j}^k(x)\|_{\mathcal{X}}}{h}\frac{dh}{h}\right]^p \frac{dt}{|t|^d} &\lesssim \int_0^\infty \left[u^{-r-1}\int_0^u \frac{\|\Delta_{he_j}^k(x)\|_{\mathcal{X}}}{h}\frac{dh}{h}\right]^p \frac{du}{u} \\ &\lesssim \int_0^\infty \left[u^{-r}\|\Delta_{ue_j}^k(x)\|_{\mathcal{X}}\right]^p \frac{du}{u} \\ &\lesssim \int_0^\infty \left[u^{-r}\omega_u^k(x)\right]^p \frac{du}{u},\end{aligned}$$

using Hardy's inequality for the estimation of the double integral. The second term yields the same estimate, so

$$|\delta_{e_j}(x)|_{\Lambda_{r-1}^p(\mathcal{X})} \lesssim |x|_{\Lambda_r^p(\mathcal{X})}$$

If $x \in \Lambda_r^p(\mathcal{X})$, $r > 1$, then $x \in \Lambda_1^1(\mathcal{X})$ by Proposition 4.34. The integral representation (D.10) implies that

$$\|\delta_{e_j}(x)\|_{\mathcal{X}} \lesssim \|\delta_{e_j}(x)\|_{\Lambda_1^1(\mathcal{X})} \lesssim \|\delta_{e_j}(x)\|_{\Lambda_r^p(\mathcal{X})},$$

and this finally proofs the assertion for $l = 1$. The case $l > 1$ follows by induction.

It remains to verify the important relation (D.10). For completeness we reproduce a proof in appendix D.3. □

D.3. Integral Representation of Derivations

PROPOSITION D.1. *Let $\mathcal{X}$ be a Banach space with a d-parameter automorphism group Ψ. If x in $\Lambda_1^1(\mathcal{X})$, and $h \in \mathbb{R}^d$, then*

$$-\delta_h(x) = \frac{1}{C_k}\int_0^\infty \frac{(\psi_{th} - \mathrm{id})^k(x)}{t}\frac{dt}{t}$$

for any integer $k > 1$, where $C_k = \int_0^\infty \frac{(e^{-t}-1)^k}{t}\frac{dt}{t}$.

The statement is proved in [**72**, 2.3] or in [**27**, Prop.3.4.5]. We reproduce the proof of [**27**].

PROOF. We need the following identities

$$\Delta_t^k = \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} \Delta_{jt}, \tag{D.11}$$

$$\sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^l = \begin{cases} 0, & l < k \\ k!, & l = k \end{cases} \tag{D.12}$$

The first of these follows from the binomial theorem for Δ_t^k and $(\mathrm{id} - \mathrm{id})^k$, the second from expanding $(e^t - 1)^k$ and differentiating l times at $t = 0$. An immediate consequence of (D.12) is the relation

$$\sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^l \int_{j\epsilon}^{k\epsilon} t^{-(l-n)} \frac{dt}{t} = 0 \quad \text{for } 1 \le n < l < k. \tag{D.13}$$

Now to the proof of the proposition! As $x \in \Lambda_1^1(\mathcal{X})$ the integral $\int_0^\infty \frac{\|\Delta_{th}^k(x)\|_{\mathcal{X}}}{t} \frac{dt}{t}$ exists, and so does the Bochner integral

$$\int_0^\infty \frac{\Delta_{th}^k(x)}{t} \frac{dt}{t}$$

consequently. So the elements

$$x_\epsilon = \int_\epsilon^\infty \frac{\Delta_{th}^k(x)}{t} \frac{dt}{t}$$

are well defined. We rewrite this expression somewhat.

$$x_\epsilon \overset{(D.11)}{=} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} \int_\epsilon^\infty t^{-1} \Delta_{jth}(x) \frac{dt}{t} = \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j \int_{j\epsilon}^\infty t^{-1} \Delta_{th}(x) \frac{dt}{t}$$

As $\sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j \int_{k\epsilon}^\infty t^{-1} \Delta_{th}(x) \frac{dt}{t} = 0$ by (D.13), we get

$$x_\epsilon = \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j \int_{j\epsilon}^{k\epsilon} t^{-1} \Delta_{th}(x) \frac{dt}{t}$$

The following algebraic identity is easily verified:

$$\int_0^s \psi_{\sigma h}(\Delta_{th}(x))\, d\sigma = \int_0^t \psi_{\sigma h}(\Delta_{sh}(x))\, d\sigma$$

Then

$$\begin{aligned} \int_0^s \psi_{\sigma h}(x_\epsilon)\, d\sigma &= \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j \int_{j\epsilon}^{k\epsilon} t^{-1} \int_0^s \psi_{\sigma h}(\Delta_{th}(x))\, d\sigma \frac{dt}{t} \\ &= \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j \int_{j\epsilon}^{k\epsilon} t^{-1} \int_0^t \psi_{\sigma h}(\Delta_{sh}(x))\, d\sigma \frac{dt}{t}\,. \end{aligned}$$

As $t^{-1} \int_0^t \psi_{\sigma h}(\Delta_{sh}(x))\, d\sigma \to \Delta_{sh}(x)$ for $t \to 0$ we get

$$\int_0^s \psi_{\sigma h}(x_0)\, d\sigma = \lim_{\epsilon \to 0} \int_0^s \psi_{\sigma h}(x_\epsilon)\, d\sigma = \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j \Delta_{sh}(x) \log(\frac{k}{j}),$$

and so we infer that $x_0 \in \mathcal{D}(\delta_h)$, and

$$x_0 = \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j \log(\frac{k}{j}) \delta_h(x).$$

It remains to verify the value of the constant C_k. As the preceding derivation is true for any semigroup action it is also valid for the action $\psi_t(x) = e^{-t}x$. Using this we obtain the desired value of the constant. □

Remark: $C_2 = 2 \log 2$.

Bibliography

[1] M. Abtahi and T. G. Honary. On the maximal ideal space of Dales-Davie algebras of infinitely differentiable functions. *Bull. Lond. Math. Soc.*, 39(6):940–948, 2007.

[2] A. Aldroubi and K. Gröchenig. Nonuniform sampling and reconstruction in shift-invariant spaces. *SIAM Rev.*, 43(4):585–620 (electronic), 2001.

[3] J. M. Almira and U. Luther. Approximation algebras and applications. In *Trends in approximation theory (Nashville, TN, 2000)*, Innov. Appl. Math., pages 1–10. Vanderbilt Univ. Press, Nashville, TN, 2001.

[4] J. M. Almira and U. Luther. Compactness and generalized approximation spaces. *Numer. Funct. Anal. Optim.*, 23(1-2):1–38, 2002.

[5] J. M. Almira and U. Luther. Generalized approximation spaces and applications. *Math. Nachr.*, 263/264:3–35, 2004.

[6] J. M. Almira and U. Luther. Inverse closedness of approximation algebras. *J. Math. Anal. Appl.*, 314(1):30–44, 2006.

[7] W. Arveson. The harmonic analysis of automorphism groups. In *Operator algebras and applications, Part I (Kingston, Ont., 1980)*, volume 38 of *Proc. Sympos. Pure Math.*, pages 199–269. Amer. Math. Soc., Providence, R.I., 1982.

[8] N. Atreas, J. J. Benedetto, and C. Karanikas. Local sampling for regular wavelet and Gabor expansions. *Sampl. Theory Signal Image Process.*, 2(1):1–24, 2003.

[9] B. A. Barnes. The spectrum of integral operators on Lebesgue spaces. *J. Operator Theory*, 18(1):115–132, 1987.

[10] A. G. Baskakov. Wiener's theorem and asymptotic estimates for elements of inverse matrices. *Funktsional. Anal. i Prilozhen.*, 24(3):64–65, 1990.

[11] A. G. Baskakov. Estimates for the elements of inverse matrices, and the spectral analysis of linear operators. *Izv. Ross. Akad. Nauk Ser. Mat.*, 61(6):3–26, 1997.

[12] R. Beals. Characterization of pseudodifferential operators and applications. *Duke Math. J.*, 44(1):45–57, 1977.

[13] C. Bennett and R. Sharpley. *Interpolation of operators*, volume 129 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1988.

[14] G. Bennett. Schur multipliers. *Duke Math. J.*, 44(3):603–639, 1977.

[15] J. Bergh and J. Löfström. *Interpolation Spaces. An Introduction.* Springer-Verlag, Berlin, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[16] G. Björck. Linear partial differential operators and generalized distributions. *Ark. Mat.*, 6:351–407 (1966), 1966.

[17] G. Björck. Beurling distributions and linear partial differential equations. In *Symposia Mathematica, Vol. VII (Convegno sulle Equazioni Ipoellittiche e Spazi Funzionali, INDAM, Rome, 1970)*, pages 367–379. Academic Press, London, 1971.

[18] B. Blackadar. *K-theory for operator algebras*, volume 5 of *Mathematical Sciences Research Institute Publications*. Cambridge University Press, Cambridge, second edition, 1998.

[19] B. E. Blackadar and J. Cuntz. The structure of stable algebraically simple C^*-algebras. *Amer. J. Math.*, 104(4):813–822, 1982.

[20] H. Bölcskei and A. J. E. M. Janssen. Gabor frames, unimodularity, and window decay. *J. Fourier Anal. Appl.*, 6(3):255–276, 2000.

[21] L. Brandenburg. On identifying the maximal ideals in Banach algebras. *J. Math. Anal. Appl.*, 50:489–510, 1975.

[22] O. Bratteli. *Derivations, Dissipations and Group Actions on C^*-algebras*, volume 1229 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986.

[23] O. Bratteli and D. W. Robinson. Unbounded derivations of C^*-algebras. *Comm. Math. Phys.*, 42:253–268, 1975.

[24] O. Bratteli and D. W. Robinson. *Operator algebras and quantum statistical mechanics. 1.* Texts and Monographs in Physics. Springer-Verlag, New York, second edition, 1987. C^*- and W^*-algebras, symmetry groups, decomposition of states.

[25] R. W. Braun, R. Meise, and B. A. Taylor. Ultradifferentiable functions and Fourier analysis. *Results Math.*, 17(3-4):206–237, 1990.

[26] Y. A. Brudnyĭ and N. Y. Krugljak. *Interpolation functors and interpolation spaces. Vol. I*, volume 47 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1991. Translated from the Russian by Natalie Wadhwa, With a preface by Jaak Peetre.

[27] P. L. Butzer and H. Berens. *Semi-groups of operators and approximation.* Die Grundlehren der mathematischen Wissenschaften, Band 145. Springer-Verlag New York Inc., New York, 1967.

[28] P. L. Butzer and R. J. Nessel. *Fourier analysis and approximation.* Academic Press, New York, 1971. Volume 1: One-dimensional theory, Pure and Applied Mathematics, Vol. 40.

[29] P. L. Butzer and K. Scherer. *Approximationsprozesse und Interpolationsmethoden.* B. I. Hochschulskripten 826/826a. Bibligraphisches Institut, Mannheim, 1968.

[30] A. Connes. *Noncommutative geometry.* Academic Press Inc., San Diego, CA, 1994.

[31] H. G. Dales and A. M. Davie. Quasianalytic Banach function algebras. *J. Functional Analysis*, 13:28–50, 1973.

[32] K. DeLeeuw. Fourier series of operators and an extension of the F. and M. Riesz theorem. *Bull. Amer. Math. Soc.*, 79:342–344, 1973.

[33] K. DeLeeuw. An harmonic analysis for operators. I: Formal properties. *Ill. J. Math.*, 19:593–606, 1975.

[34] K. DeLeeuw. An harmonic analysis for operators. II: Operators on Hilbert space and analytic operators. *Ill. J. Math.*, 21:164–175, 1977.

[35] S. Demko. Inverses of band matrices and local convergence of spline projections. *SIAM J. Numer. Anal.*, 14(4):616–619, 1977.

[36] S. Demko, W. F. Moss, and P. W. Smith. Decay rates for inverses of band matrices. *Math. Comp.*, 43(168):491–499, 1984.

[37] R. A. DeVore and G. G. Lorentz. *Constructive approximation*, volume 303 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1993.

[38] N. Dunford. On one parameter groups of linear transformations. *Ann. of Math. (2)*, 39(3):569–573, 1938.

[39] O. Èl′-Falla, N. K. Nikol′skiĭ, and M. Zarrabi. Estimates for resolvents in Beurling-Sobolev algebras. *Algebra i Analiz*, 10(6):1–92, 1998.

[40] H. G. Feichtinger. Gewichtsfunktionen auf lokalkompakten Gruppen. *Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II*, 188(8-10):451–471, 1979.

[41] G. Fendler, K. Gröchenig, and M. Leinert. Symmetry of weighted L^1-algebras and the GRS-condition. *Bull. London Math. Soc.*, 38(4):625–635, 2006.

[42] K. Floret and J. Wloka. *Einführung in die Theorie der lokalkonvexen Räume.* Lecture Notes in Mathematics, No. 56. Springer-Verlag, Berlin, 1968.

[43] G. B. Folland. Remainder estimates in Taylor's theorem. *Amer. Math. Monthly*, 97(3):233–235, 1990.

[44] I. Gohberg, M. A. Kaashoek, and H. J. Woerdeman. The band method for positive and strictly contractive extension problems: an alternative version and new applications. *Integral Equations Operator Theory*, 12(3):343–382, 1989.

[45] V. I. Gorbachuk and M. L. Gorbachuk. *Boundary value problems for operator differential equations*, volume 48 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1991. Translated and revised from the 1984 Russian original.

[46] V. I. Gorbachuk and A. V. Knyazyuk. Boundary values of solutions of operator-differential equations. *Uspekhi Mat. Nauk*, 44(3(267)):55–91, 208, 1989.

[47] A. Gorny. Contribution à l'étude des fonctions dérivables d'une variable réelle. *Acta Math.*, 71:317–358, 1939.

[48] K. Gröchenig. *Foundations of time-frequency analysis.* Applied and Numerical Harmonic Analysis. Birkhäuser Boston Inc., Boston, MA, 2001.

[49] K. Gröchenig. Localization of frames, Banach frames, and the invertibility of the frame operator. *J. Fourier Anal. Appl.*, 10(2):105–132, 2004.

[50] K. Gröchenig. Weight functions in time-frequency analysis. In *Pseudo-differential operators: partial differential equations and time-frequency analysis*, volume 52 of *Fields Inst. Commun.*, pages 343–366. Amer. Math. Soc., Providence, RI, 2007.

[51] K. Gröchenig. *Wiener's Lemma: Theme and Variations. An Introduction to Spectral Invariance.* Applied and Numerical Harmonic Analysis. Birkhäuser, Boston, 2009. Inzell Lectures on Harmonic Analysis.

[52] K. Gröchenig and A. Klotz. Noncommutative approximation: Inverse-closed subalgebras and off-diagonal decay of matrices. arXiv:0904.0386, 2009.

[53] K. Gröchenig and M. Leinert. Wiener's lemma for twisted convolution and Gabor frames. *J. Amer. Math. Soc.*, 17(1):1–18 (electronic), 2004.

[54] K. Gröchenig and M. Leinert. Symmetry and inverse-closedness of matrix algebras and functional calculus for infinite matrices. *Trans. Amer. Math. Soc.*, 358(6):2695–2711 (electronic), 2006.

[55] K. Gröchenig and M. Leinert. Symmetry and inverse-closedness of matrix algebras and functional calculus for infinite matrices. *Trans. Amer. Math. Soc.*, 358(6):2695–2711 (electronic), 2006.

[56] K. Gröchenig and Z. Rzeszotnik. Banach algebras of pseudodifferential operators and their almost diagonalization. *Ann. Inst. Fourier (Grenoble)*, 58(7):2279–2314, 2008.

[57] Y. Grushka and S. Torba. Direct theorems in the theory of approximation of vectors in a Banach space with exponential type entire vectors. *Methods Funct. Anal. Topology*, 13(3):267–278, 2007.

[58] E. Hille and R. S. Phillips. *Functional analysis and semi-groups.* American Mathematical Society Colloquium Publications, vol. 31. American Mathematical Society, Providence, R. I., 1957. rev. ed.

[59] L. Hörmander. *The analysis of linear partial differential operators. I*, volume 256 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1990. Distribution theory and Fourier analysis.

[60] A. Hulanicki. On the spectrum of convolution operators on groups with polynomial growth. *Invent. Math.*, 17:135–142, 1972.

[61] S. Jaffard. Propriétés des matrices "bien localisées" près de leur diagonale et quelques applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 7(5):461–476, 1990.

[62] Y. Katznelson. *An introduction to harmonic analysis.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, third edition, 2004.

[63] R. Kerman and E. Sawyer. Convolution algebras with weighted rearrangement-invariant norm. *Studia Math.*, 108(2):103–126, 1994.

[64] E. Kissin and V. S. Shul′man. Dense Q-subalgebras of Banach and C^*-algebras and unbounded derivations of Banach and C^*-algebras. *Proc. Edinburgh Math. Soc. (2)*, 36(2):261–276, 1993.

[65] E. Kissin and V. S. Shul′man. Differential properties of some dense subalgebras of C^*-algebras. *Proc. Edinburgh Math. Soc. (2)*, 37(3):399–422, 1994.

[66] H. Komatsu. Ultradistributions. I. Structure theorems and a characterization. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 20:25–105, 1973.

[67] P. Koosis. *The logarithmic integral. I*, volume 12 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1988.

[68] V. G. Kurbatov. Algebras of difference and integral operators. *Funktsional. Anal. i Prilozhen.*, 24(2):87–88, 1990.

[69] V. G. Kurbatov. Some algebras of operators majorized by a convolution. *Funct. Differ. Equ.*, 8(3-4):323–333, 2001. International Conference on Differential and Functional Differential Equations (Moscow, 1999).

[70] M. Langenbruch. Ultradifferentiable functions on compact intervals. *Math. Nachr.*, 140:109–126, 1989.

[71] B. Y. Levin. *Lectures on entire functions*, volume 150 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1996. In collaboration with and with a preface by Yu. Lyubarskii, M. Sodin and V. Tkachenko, Translated from the Russian manuscript by Tkachenko.

[72] J.-L. Lions and J. Peetre. Sur une classe d'espaces d'interpolation. *Inst. Hautes Études Sci. Publ. Math.*, (19):5–68, 1964.

[73] U. Luther. Representation, interpolation, and reiteration theorems for generalized approximation spaces. *Ann. Mat. Pura Appl. (4)*, 182(2):161–200, 2003.

[74] P. Malliavin. Calcul symbolique et sous-algèbres de $L_1(G)$. I, II. *Bull. Soc. Math. France*, 87:181–186, 187–190, 1959.

[75] S. Mandelbrojt. Analytic functions and classes of infinitely differentiable functions. *Rice Inst. Pamphlet*, 29(1):142, 1942.

[76] S. Mandelbrojt. *Séries adhérentes, régularisation des suites, applications.* Gauthier-Villars, Paris, 1952.

[77] N. Nikolski. In search of the invisible spectrum. *Ann. Inst. Fourier (Grenoble)*, 49(6):1925–1998, 1999.

[78] N. Nikolski. The problem of efficient inversions and Bezout equations. In *Twentieth century harmonic analysis—a celebration (Il Ciocco, 2000)*, volume 33 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 235–269. Kluwer Acad. Publ., Dordrecht, 2001.

[79] J. Peetre. *New thoughts on Besov spaces*. Mathematics Department, Duke University, Durham, N.C., 1976. Duke University Mathematics Series, No. 1.

[80] H.-J. Petzsche. Die Nuklearität der Ultradistributionsräume und der Satz vom Kern. I. *Manuscripta Math.*, 24(2):133–171, 1978.

[81] H.-J. Petzsche. Approximation of ultradifferentiable functions by polynomials and entire functions. *Manuscripta Math.*, 48(1-3):227–250, 1984.

[82] A. Pietsch. Eigenvalues of integral operators. I. *Math. Ann.*, 247(2):169–178, 1980.

[83] A. Pietsch. Approximation spaces. *J. Approx. Theory*, 32(2):115–134, 1981.

[84] V. S. Rabinovich, S. Roch, and B. Silbermann. Fredholm theory and finite section method for band-dominated operators. *Integral Equations Operator Theory*, 30(4):452–495, 1998. Dedicated to the memory of Mark Grigorievich Krein (1907–1989).

[85] V. S. Rabinovich, S. Roch, and B. Silbermann. Band-dominated operators with operator-valued coefficients, their Fredholm properties and finite sections. *Integral Equations Operator Theory*, 40(3):342–381, 2001.

[86] V. S. Rabinovich, S. Roch, and B. Silbermann. *Limit operators and their applications in operator theory*, volume 150 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 2004.

[87] C. E. Rickart. *General theory of Banach algebras*. The University Series in Higher Mathematics. D. van Nostrand Co., Inc., Princeton, N.J.-Toronto-London-New York, 1960.

[88] M. A. Rieffel. Leibniz seminorms for "matrix algebras converge to the sphere". arXiv:0707.3229v3 [math.OA], 2010.

[89] R. T. Rockafellar. *Convex analysis*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1997. Reprint of the 1970 original, Princeton Paperbacks.

[90] H. S. Shapiro. *Topics in approximation theory*. Springer-Verlag, Berlin, 1971. With appendices by Jan Boman and Torbjörn Hedberg, Lecture Notes in Math., Vol. 187.

[91] J. A. Siddiqi. On the equivalence of classes of infinitely differentiable functions. *Izv. Akad. Nauk Armyan. SSR Ser. Mat.*, 19(1):19–30, 1984.

[92] J. A. Siddiqi. Inverse-closed Carleman algebras of infinitely differentiable functions. *Proc. Amer. Math. Soc.*, 109(2):357–367, 1990.

[93] J. Sjöstrand. Wiener type algebras of pseudodifferential operators. In *Séminaire sur les Équations aux Dérivées Partielles, 1994–1995*, pages Exp. No. IV, 21. École Polytech., Palaiseau, 1995.

[94] E. M. Stein. The characterization of functions arising as potentials. *Bull. Amer. Math. Soc.*, 67:102–104, 1961.

[95] E. M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[96] T. Strohmer. Approximation of dual Gabor frames, window decay, and wireless communications. *Appl. Comput. Harmon. Anal.*, 11(2):243–262, 2001.

[97] T. Strohmer. Pseudodifferential operators and Banach algebras in mobile communications. *Appl. Comput. Harmon. Anal.*, 20(2):237–249, March 2006.

[98] Q. Sun. Wiener's lemma for infinite matrices with polynomial off-diagonal decay. *C. R. Math. Acad. Sci. Paris*, 340(8):567–570, 2005.

[99] Q. Sun. Wiener's lemma for infinite matrices. *Trans. Amer. Math. Soc.*, 359(7):3099–3123 (electronic), 2007.

[100] A. F. Timan. *Theory of approximation of functions of a real variable*. Translated from the Russian by J. Berry. English translation edited and editorial preface by J. Cossar. International Series of Monographs in Pure and Applied Mathematics, Vol. 34. A Pergamon Press Book. The Macmillan Co., New York, 1963.

[101] S. Torba. Inverse theorems in the theory of approximation of vectors in a Banach space with exponential type entire vectors. arXiv:0809.4030v1 [math.FA], 2008.

[102] H. Triebel. *Theory of function spaces. II*, volume 84 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1992.

[103] R. M. Trigub and E. S. Bellinsky. *Fourier analysis and approximation of functions*. Kluwer Academic Publishers, Dordrecht, 2004.

[104] J. Wermer. On a class of normed rings. *Ark. Mat.*, 2:537–551, 1954.

[105] R. L. Wheeden. On hypersingular integrals and Lebesgue spaces of differentiable functions. *Trans. Amer. Math. Soc.*, 134:421–435, 1968.

[106] A. C. Zaanen. *Integration*. Completely revised edition of An introduction to the theory of integration. North-Holland Publishing Co., Amsterdam, 1967.

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