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Causal Structure and Singularity Theory on Space-Times

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ABSTRACT

A large number of the more important Lorentzian manifolds that are used as models in general relativity fail to be geodesically complete. Our aim is to use these so called space-times to establish theorems which prove geodesical incompleteness and to develop the underlying causal structure. We first will establish the necessary tools to formulate the Jacobian tensor calculus which will be used to prove basic energy conditions. In the subsequent chapters we will introduce achronal and acausal sets and prove that every boundary of a future-set is an achronal closed topological hypersurface. The concept of achronality will directly lead us to Cauchy developments. We will show that the interior of the Cauchy development is a globally hyperbolic space-time, as long as the underlying subset is achronal. With these tools we will be able to formulate a basic singularity theorem, which will be used to prove further strong results. We will also show the existence of piecewise smooth future-inextendible timelike curves that are contained in the future Cauchy development of the future-horismos. This fundamental result will allow us to prove the famous Lemma of Hawking and Penrose, which may already be viewed as a singularity theorem. A famous result in the calculus of variations provides the necessary tool to formulate the final and strongest version of the singularity theorem of Hawking and Penrose. In the next section we will use Cauchy hypersurfaces to secure the global hyperbolicity of space-times. This result provides an elegant proof of the singularity theorem of Penrose. We will finish this thesis by uncovering the underlying similarities of the given results.

ZUSAMMENFASSUNG

In der Allgemeinen Relativitätstheorie werden Modelle von Raumzeiten durch abstrakte Lorentzmannigfaltigkeiten modelliert. Es stellt sich heraus, dass eine große Anzahl der wichtigsten Raumzeiten geodätisch unvollständig ist. Wir werden in dieser Arbeit die wichtigsten Existenztheoreme von geodätisch unvollständigen Raumzeiten formulieren und beweisen. Dabei werden wir die zu Grunde liegende kausale Struktur mitentwickeln. In den ersten Kapiteln definieren wir den Begriff der mathematischen Raumzeit und stellen die notwendigen Werkzeuge bereit um den Jacobi-Tensorkalkül zu formulieren, der es uns ermöglicht Aussagen über Energiebedingungen auf Raumzeiten zu treffen. In den späteren Kapiteln werden wir den Begriff der Achronalität bzw. der Akausalität einführen und beweisen, dass jeder Rand einer Zukunftsmenge eine abgeschlossene achronale topologische Hyperfläche ist. Das Konzept von achronalen Mengen führt uns notwendigerweise zu den Cauchy Entwicklungen. Wir werden zeigen, dass das Innere einer Cauchy Entwicklung stets eine global hyperbolische Raumzeit modelliert, sofern die zu Grunde liegende Menge achronal ist. Mit diesen Vorbereitungen werden wir ein erstes starkes Resultat in der Singularitätentheorie beweisen können. Mit großem Aufwand werden wir die Existenz von stückweise glatten zukunfts-nichterweiterbaren zeitartigen Kurven zeigen, die vollständig in der Zukunfts-Cauchy-Entwicklung des so genannten Zukunfts-Horismos liegen. Dieses fundamentale Resultat wird ausreichen um das berühmte Lemma von Hawking und Penrose zu beweisen, welches in der Literatur bereits häufig als Singularitätensatz angesehen wird. Ein bekanntes Resultat im Variation-skalkül liefert den Beweis des stärksten Singularitätentheorems - des Singularitätensatzes von Hawking und Penrose. Im nächsten Kapitel werden wir so genannte Cauchy Hyperflächen verwenden um die globale Hyperbolizität von Raumzeiten zu erzwingen. Dieses Resultat führt uns auf einen eleganten

Beweis des Singularitätensatzes von Penrose. Im letzten Kapitel analysieren wir die gemeinsame Grundlage der zentralen Resultate in der Singularitätentheorie.

CHAPTER 1

SPACE-TIMES

1.1 Basic concepts

In this chapter we develop some basic notions of space-times. In physics space-times are introduced in the special theory of relativity, where the underlying geometric structure is an affine space \mathbb{A} with $\dim(\mathbb{A}) = 4$ together with a scalar-product g of signature $(-+++)$. (\mathbb{A}, g) is called Minkowski space-time. It turns out that special relativity can only be expected to be a good description locally. A remedy for this restriction was found in the general theory of relativity. Here, every event in the universe has a neighbourhood whose geometrical and topological structure is well described by a Minkowski space-time, hence in general relativity space-times are locally Euclidean spaces. A rigorous formulation of this idea requires smooth manifolds. We will give a consistent definition for space-times later after some preparations. To establish certain global properties we will employ partitions of unity. To this end we will require that all manifolds are smooth and Hausdorff with a countable basis of the manifold topology. It is easy to show that every Hausdorff manifold with countable basis is paracompact, which will be needed in the course of the following chapters.

Let M be a manifold. Recall that a Lorentzian metric g on M is a smooth section of the $\binom{0}{2}$ -tensor bundle over TM such that for each $p \in M$, the $\binom{0}{2}$ -tensor $g_p \equiv g(p) : T_pM \times T_pM \rightarrow \mathbb{R}$ is symmetric, nondegenerate and its index equals 1. A Lorentzian manifold (M, g) is a manifold M as above together with a Lorentzian metric g .

Definition 1.1.1. Let $p \in M$ and $0 \neq v \in T_p M \subseteq TM$. v is said to be timelike (non spacelike, null, spacelike) if $g(p)(v, v) < 0$ ($g(p)(v, v) \leq 0$, $g(p)(v, v) = 0$, $g(p)(v, v) > 0$). We define the zero-vector to be spacelike. We will often abbreviate $g(p)(v, w)$ by $\langle v, w \rangle$.

With Definition 1.1.1 we can introduce the concept of time-orientation. We recall that a Lorentzian vector space V is a scalar product space of index 1 and dimension ≥ 2 . Let \mathcal{T} be the set of all timelike vectors in V , i.e.

$$\mathcal{T} := \{v \in V : \langle v, v \rangle < 0\} \subseteq V.$$

We choose an arbitrary $u \in \mathcal{T}$, i.e. u has to be timelike, and define the timecone of V containing u by

$$C(u) := \{v \in \mathcal{T} : \langle u, v \rangle < 0\} \subseteq \mathcal{T} \subseteq V.$$

We further define the opposite timecone by

$$C(-u) := \{v \in \mathcal{T} : \langle -u, v \rangle < 0\} = \{v \in \mathcal{T} : \langle u, v \rangle > 0\} = -C(u) \subseteq \mathcal{T} \subseteq V.$$

Lemma 1.1.2. Let V be a Lorentzian vector space and let z be a timelike vector in V , i.e. $z \in \mathcal{T}$. Then the subspace z^\perp of V is spacelike and $V = \mathbb{R}z \oplus z^\perp$, where $z^\perp := (\mathbb{R}z)^\perp$.

Proof. The vector space $\mathbb{R}z$ is non-degenerate with index 1, because it is negative definite. Hence we conclude that $V = \mathbb{R}z \oplus z^\perp$. Since $1 = \text{ind}(V) = \text{ind}(\mathbb{R}z) + \text{ind}(z^\perp)$ and $\text{ind}(\mathbb{R}z) = 1$ it follows that $\text{ind}(z^\perp) = 0$, i.e. z^\perp is spacelike. \square

Lemma 1.1.3. Two timelike vectors v and w in a Lorentzian vector space V , $v, w \in \mathcal{T} \subseteq V$, are in the same timecone if and only if $\langle v, w \rangle < 0$.

Proof. Let u be a timelike vector in V , i.e. $u \in \mathcal{T}$, and let $v \in C(u)$; in particular v is timelike, i.e. $v \in \mathcal{T}$. We show that if w is timelike, then $w \in C(u)$ if and only if $\langle v, w \rangle < 0$. We first note that

$$C(u) = C\left(\frac{u}{\|u\|}\right).$$

Therefore, without restriction, we can assume that $\|u\| = 1$. Now we define

$$au + \vec{v} := v \quad \text{and} \quad bu + \vec{w} := w,$$

where $a, b \in \mathbb{R}$ and $\vec{v}, \vec{w} \in u^\perp$ as in 1.1.2. Since v and w are timelike vectors, we can assume that $|a| > \|\vec{v}\|$ and $|b| > \|\vec{w}\|$. Since $\langle u, u \rangle = -1$ and $\vec{v}, \vec{w} \in u^\perp$ by assumption, we find

$$\langle v, w \rangle = \langle au + \vec{v}, bu + \vec{w} \rangle = -ab + \langle \vec{v}, \vec{w} \rangle.$$

By the Cauchy-Schwarz inequality we further get

$$|\langle \vec{v}, \vec{w} \rangle| \leq \|v\| \|w\| < |ab|.$$

By assumption $v \in C(u)$, hence $\langle u, v \rangle = -a < 0$, i.e. $a > 0$. Because of $\text{sgn}(\langle v, w \rangle) = \text{sgn}(-ab) = \text{sgn}(-b)$, the signum of $\langle v, w \rangle$ depends only on the signum of $-b$. This implies $\langle v, w \rangle < 0$ if and only if $b > 0$ if and only if $w \in C(u)$. \square

Remark 1.1.4. Let u and v be timelike vectors, i.e. $u, v \in \mathcal{T}$. Then $u \in C(v)$ if and only if $v \in C(u)$ if and only if $C(u) = C(v)$.

Proof. By definition we have that $u \in C(v)$ if and only if $\langle u, v \rangle < 0$ if and only if $v \in C(u)$, hence the first equivalence is proved. Now let $C(u) = C(v)$, then it follows that $v \in C(v)$ and hence $v \in C(u)$. Now let $u \in C(v)$ and $w \in C(u)$. We show that $w \in C(v)$. Therefore we proceed as just above. By 1.1.2 we can write $dv + \vec{w} := w$ and $cv + \vec{u} := u$, where $c, d \in \mathbb{R}$ and $\vec{u}, \vec{w} \in v^\perp$. Again we have $-1 = \text{sgn}(\langle u, w \rangle) = -\text{sgn}(d)$ and we conclude that $d > 0$, hence $\langle v, w \rangle = -d < 0$ and by definition of the timecone we have $w \in C(v)$. So we have $C(u) \subseteq C(v)$ and hence, by symmetry, $C(u) = C(v)$. \square

Remark 1.1.5. Let V be a Lorentzian vector space. We show that there are precisely two timecones in V . To this end we choose an arbitrary timelike u_0 , i.e. $u_0 \in \mathcal{T} \subseteq V$. Now let $v \in \mathcal{T}$. We first assume that $\langle u_0, v \rangle < 0$. By Lemma 1.1.3 the two timelike vectors u_0 and v are in the same timecone, that is $u_0 \in C(v)$ and $v \in C(u_0)$ and by Remark 1.1.4 it follows that $C(u_0) = C(v)$. Now let $\langle u_0, v \rangle > 0$, then $\langle -u_0, v \rangle < 0$ and the two timelike vectors $-u_0$ and v are in the same timecone, i.e. $-u_0 \in C(v)$ and $v \in C(-u_0)$. Again, by Remark 1.1.4 it follows that $C(-u_0) = C(v)$. Lemma 1.1.2 shows that the case $\langle u_0, v \rangle = 0$ cannot occur, because u_0 is timelike and its orthogonal complement is spacelike, but v is timelike.

Remark 1.1.6. Let V be a Lorentzian vector space. We choose an arbitrary timelike vector $u_0 \in V$ and find that the disjoint union of $C(u_0)$ and $C(-u_0)$ is the set of all timelike vectors in V , i.e. we have

$$\mathcal{T} = C(u_0) \sqcup C(-u_0).$$

Proof. We assume that the intersection of $C(u)$ and $C(-u_0)$ is nonempty, hence there exists some $w \in C(u_0) \cap C(-u_0)$. By Remark 1.1.4 it follows that

$$C(u_0) = C(w) = C(-u_0).$$

Therefore u_0 and $-u_0$ are in the same timecone and by Lemma 1.1.3 we find $\langle u_0, -u_0 \rangle < 0$, contradicting the fact that u_0 is timelike. \square

Timecones are of utmost importance for studying causality. The following definition is based on this concept.

Definition 1.1.7. *Let (M, g) be a Lorentzian manifold. A time orientation of M is a map*

$$\tau : M \rightarrow \mathcal{P}(TM),$$

where $\mathcal{P}(TM)$ is the power set of the tangent bundle, such that τ maps every $p \in M$ to a timecone τ_p in T_pM and for all $p \in M$ there exists a neighbourhood $U \subseteq M$ of p and a smooth vector field $X \in \mathfrak{X}(U)$ with $X_q \in \tau_q$ for all $q \in U$. If M possesses a time-orientation, we say that M is time-orientable.

Proposition 1.1.8. *A Lorentzian manifold M is time-orientable if and only if there exists a smooth timelike vector field $X \in \mathfrak{X}(M)$.*

Proof. (\Rightarrow) We assume that M is time-orientable. Let τ be a time-orientation for M . Since τ is smooth, for each $p \in M$ we can find a neighbourhood U_p of p in M and a smooth vector field X_{U_p} defined on U_p , i.e. $X_{U_p} \in \mathfrak{X}(U_p)$, with $X_{U_p}(q) \in \tau_q$ for all $q \in U_p \subseteq M$. We choose a smooth partition of unity $\{\chi_m : m \in \mathbb{N}\}$ subordinated to the set $\{U_p : p \in M\}$. Since χ_m are positive for all $m \in \mathbb{N}$ and all timecones are convex, we see that the smooth vector field $X := \sum_{m=1}^{\infty} \chi_m X_{U_{p_m}}$ is timelike, where $\text{supp}\chi_m \subseteq U_{p_m}$ for all $m \in \mathbb{N}$.

(\Leftarrow) If there exists a smooth timelike vector field $X \in \mathfrak{X}(M)$, then we may take for τ the map that assigns to each $p \in M$ the timecone that contains X_p . \square

Remark 1.1.9. We recall that a Riemannian manifold M is a smooth manifold together with a Riemannian metric tensor field, i.e. a smooth map $\tilde{g} : M \rightarrow T_0^2M$ with $\pi_0^2 \circ \tilde{g} = id_M$ such that for each $p \in M$, the $\binom{0}{2}$ -tensor $\tilde{g}_p \equiv \tilde{g}(p) : T_pM \times T_pM \rightarrow \mathbb{R}$ defines an inner product on each tangent space.

Remark 1.1.10. Let U and X be smooth vector fields on a semi-Riemannian manifold (M, g) . Recall that the map

$$\Phi : \mathfrak{X}(M) \rightarrow \Omega^1(M)$$

$$\Phi : U \mapsto U^*, U^*(X) := \langle U, X \rangle = g(U, X)$$

defines a $\mathcal{C}^\infty(M)$ -linear isomorphism between $\mathfrak{X}(M)$ and $\Omega^1(M)$. For further considerations it is useful to mention that this result holds for vector and covector fields along smooth mappings. A proof can be found in [15], Chapter 3, Proposition 10.

Proposition 1.1.11. *Let U be a smooth unit vector field on a Riemannian manifold M with metric tensor \tilde{g} . Then*

$$g := \tilde{g} - 2U^* \otimes U^*$$

is a Lorentzian metric for M , i.e. we can induce a Lorentzian metric tensor by tensorizing the metric-equivalent 1-forms of U . Furthermore, U is timelike w.r.t. \tilde{g} and so the resulting Lorentzian manifold is time-orientable.

Proof. The $\binom{0}{2}$ -tensor field g is symmetric, since

$$\begin{aligned} g(V, W) &= \tilde{g}(V, W) - 2\langle U, V \rangle \langle U, W \rangle \\ &= g(W, V). \end{aligned}$$

We show that the index of M equals 1. For all $p \in M$ we can locally find smooth vector fields E_j ($j = 2, 3, \dots, n$) such that $\{U, E_2, \dots, E_n\}$ is a frame field relative to \tilde{g} . We first show that $g(p)(E_i(p), E_j(p)) = \delta_{ij}$ for all $p \in M$. We choose an arbitrary $p \in M$ and see that by definition of the tensor product of tensor fields

$$\begin{aligned} (U^*(p) \otimes U^*(p))(E_i(p), E_j(p)) &= (U^*(p)(E_i(p)))(U^*(p)(E_j(p))) \\ &= (\langle U(p), E_i(p) \rangle)(\langle U(p), E_j(p) \rangle) \\ &= 0 \end{aligned}$$

since $\{U, E_2, \dots, E_n\}$ is a frame field relative to \tilde{g} . We now calculate that

$$\begin{aligned} g(p)(E_i(p), E_j(p)) &= \tilde{g}(p)(E_i(p), E_j(p)) - 2(U^*(p) \otimes U^*(p))(E_i(p), E_j(p)) \\ &= \tilde{g}(p)(E_i(p), E_j(p)) \\ &= \delta_{ij} \end{aligned}$$

for all $p \in M$. Further we have $g(p)(U(p), E_j(p)) = 0$ for $\{j = 2, 3, \dots, n\}$, because

$$\begin{aligned} g(p)(U(p), E_j(p)) &= \tilde{g}(p)(U(p), E_j(p)) - 2(U^*(p) \otimes U^*(p))(U(p), E_j(p)) \\ &= \tilde{g}(p)(U(p), E_j(p)) \end{aligned}$$

and $\tilde{g}(p)(U(p), E_j(p)) = 0$ for $\{j = 2, 3, \dots, n\}$ since $\{U, E_2, \dots, E_n\}$ is a frame field relative to \tilde{g} . Here we have used that

$$\begin{aligned} (U^*(p) \otimes U^*(p))(U(p), E_j(p)) &= (U^*(p)(U(p)))(U^*(p)(E_j(p))) \\ &= (\langle U(p), U(p) \rangle)(\langle U(p), E_j(p) \rangle) \\ &= 0 \end{aligned}$$

for $\{j = 2, 3, \dots, n\}$. Finally $g(p)(U(p), U(p)) = -1$ since $g(p)(U(p), U(p)) = \tilde{g}(p)(U(p), U(p)) - 2(U^*(p)U(p)U^*(p)U(p)) = -1$. Remains to show that \tilde{g} is non-degenerate. To this end we assume that $\tilde{g}(V, W) = g(V, W) - 2U^* \otimes U^*(V, W) = 0$ for all $W \in \mathfrak{X}(M)$. We have to show that V is the zero vector field. We choose a local frame as above and set $W := U$, hence we obtain that $0 = g(V, U) - 2U^* \otimes U^*(V, U)$. Thus we have

$$\begin{aligned} g(V, U) &= 2U^* \otimes U^*(V, U) = 2U^*(V)U^*(U) \\ &= 2U^*(V) = 2g(U, V). \end{aligned}$$

Since g is symmetric, we obtain that $g(U, V) = 0$ and if we now set $W := V$ we have that $g(V, V) = 2(g(U, V))^2 = 0$. Since g is a Riemannian tensor field, it is non-degenerate and we finally see that V must be the null vector field. Proposition 1.1.8 guaranties that M has a time orientation, i.e. M is time-orientable. \square

Time-oriented Lorentzian manifolds are called spacetimes. More precisely, we have

Definition 1.1.12. *Let (M, g) be a connected Lorentzian manifold of dimension ≥ 2 . (M, g) is called a space-time if it is time-orientable.*

We recall the following

Definition 1.1.13. *Let M and \tilde{M} be smooth manifolds. A smooth map $\kappa : \tilde{M} \rightarrow M$ onto M is said to be a covering map provided each point $p \in M$ has a connected neighbourhood \mathcal{U} that is evenly covered by κ , i.e. κ maps each component of $\kappa^{-1}(\mathcal{U})$ diffeomorphically onto \mathcal{U} . The set $F_p := \kappa^{-1}(p)$ for $p \in M$ is called the fiber over p . A double covering is a covering map κ such that each fiber over p contains precisely two elements.*

We also recall the following Proposition (details can be found in [15], Chapter 7).

Proposition 1.1.14. *Let $\kappa : \Sigma \rightarrow M$ be a two-to-one map of a set Σ onto a manifold M . Let Λ be a collection of functions $\lambda : \mathcal{U} \rightarrow \Sigma$, where \mathcal{U} is an open subset of M , such that the following conditions hold.*

- (i) $\kappa \circ \lambda = id_{\mathcal{U}}$ for all $\lambda \in \Lambda$.
- (ii) If $\lambda(p) = \mu(p)$ for some $p \in M$ and $\lambda, \mu \in \Lambda$, then $\lambda = \mu$.
- (iii) Every point in Σ is the image of some $\lambda \in \Lambda$.

Then there is a unique way to make Σ a manifold such that $\kappa : \Sigma \rightarrow M$ is a double covering map.

Remark 1.1.15. We denote by M^T the set of all timecones in the tangent bundle of M . Obviously there is a natural two-to-one map $\kappa : M^T \rightarrow M$. Since all conditions of Proposition 1.1.14 are satisfied, $\kappa : M^T \rightarrow M$ becomes a double covering map. The pulled-back metric tensor on M^T makes this a Lorentzian covering, called the time-orientation covering of M .

Now we can show a fundamental result.

Proposition 1.1.16. *Let M be a smooth manifold. Then the following are equivalent.*

- (i) *There exists a Lorentzian metric on M .*
- (ii) *There exists a time-orientable Lorentzian metric on M .*
- (iii) *There is a nonvanishing vector field on M .*
- (iv) *Either M is noncompact, or M is compact and has Euler number $\chi(M) = 0$.*

Proof. The equivalence between (iii) and (iv) is a Corollary of the theorem of Poincaré and Hopf, which can be found in [22], Chapter 6, Theorem 6.27. Obviously (ii) implies (i). We assume that (iii) holds. Since every smooth manifold admits a Riemannian metric tensor, we can apply Proposition 1.1.11 to the unit vector field $\frac{X}{|X|}$ and obtain a Lorentzian metric tensor. We assume that (ii) holds. Then Proposition 1.1.8 implies the existence of a nonvanishing (even timelike) vector field on M . Now let condition (i) be satisfied. If M is time-orientable, the preceding results prove (iv). Hence let M be not time-orientable. By Remark 1.1.15 M has a double-covering Lorentzian manifold \tilde{M} that is time-orientable, thus \tilde{M} is either noncompact or has $\chi(\tilde{M}) = 0$. We show that \tilde{M} is compact if and only if M is compact. To this end let $\{\mathcal{U}_i\}_{i \in I}$ be an open covering of \tilde{M} . We show that we can choose a finite subcover. Let $p \in M$. Then $F_p := \kappa^{-1}(p)$ is finite, i.e. there is a finite subset $J \subseteq I$ such that F_p is covered by $\{\mathcal{U}_j\}_{j \in J}$. We set $\mathcal{O}_p := \cup_{j \in J} \mathcal{U}_j$. We show that for each \mathcal{O}_p there is an open neighbourhood \mathcal{V}_p of p such that $\kappa^{-1}(\mathcal{V}_p) \subseteq \mathcal{O}_p$. Since κ is a covering, we only have to set $\mathcal{V}_p := \kappa(\cup_{i=1}^n \mathcal{U}_i \cap \mathcal{O}_p)$. Obviously the set $\{\mathcal{V}_p : p \in M\}$ is an open covering for M and since M is compact, there are $\mathcal{V}_{p_1}, \dots, \mathcal{V}_{p_n}$ that cover M . Now the corresponding \mathcal{U}_j ($j \in J_{p_i}$ with $i = 1, \dots, n$) of \mathcal{V}_{p_i} form a finite subcover. Hence M is compact if and only if \tilde{M} is compact, since κ is continuous. In the compact case we obtain that $\chi(M) = \frac{\chi(\tilde{M})}{2} = 0$. \square

CHAPTER 2

JACOBIAN TENSOR FIELDS

2.1 Basics

Let (M, g) be a space-time of dimension ≥ 2 as in Definition 1.1.12. For $a, b \in \mathbb{R}$ we consider a timelike geodesic

$$c : [a, b] \subseteq \mathbb{R} \rightarrow M.$$

For all $s \in [a, b]$ we define $N(c(s))$ to be the set of all tangent vectors orthogonal to $c'(s) \in T_{c(s)}M$, i.e.

$$N(c(s)) := \{v \in T_{c(s)}M : \langle v, c'(s) \rangle = 0\}.$$

$N(c(s))$ is a subspace of the \mathbb{R} -vector space $T_{c(s)}M$ for all $s \in [a, b]$.

Remark 2.1.1. Let $n := \dim(M) \geq 2$ be the dimension of the space-time (M, g) . Then for all $s \in [a, b]$ the \mathbb{R} -vector space $N(c(s))$ is $(n - 1)$ dimensional. Furthermore, since $c'(s) \in T_{c(s)}M$ is timelike by assumption, $N(c(s))$ consists of spacelike tangent vectors and thus $\{v \in N(c(s)) : \langle v, v \rangle \leq 1\}$ is a compact subset of $N(c(s)) \subseteq T_{c(s)}M$ for all $s \in [a, b]$.

Proof. All these statements follow directly from Lemma 1.1.2. \square

Remark 2.1.2. Let $[a, b] \subseteq \mathbb{R}$ and let (M, g) be an arbitrary spacetime of dimension ≥ 2 . For a timelike geodesic $c : [a, b] \rightarrow M$ we define $V^\perp(c)$ as the set of all piecewise smooth vector fields Y along c with $\langle Y(s), c'(s) \rangle = 0$ for all $s \in [a, b]$. $V^\perp(c)$ is an \mathbb{R} -subspace of the \mathbb{R} -vector space of all piecewise smooth vector fields along the geodesic c . Furthermore we set $V_0^\perp(c) := \{Y \in V^\perp(c) : Y(a) = 0 \text{ and } Y(b) = 0\}$.

Remark 2.1.3. Let E_1, \dots, E_k and E be finite dimensional vector spaces over the field \mathbb{R} . Recall that we defined $L^k(E_1, \dots, E_k; F)$ to be the \mathbb{R} -vector space of all multilinear maps $T : E_1 \times \dots \times E_k \rightarrow F$. Furthermore we defined the \mathbb{R} -vector space $T_s^r E := L^{r+s}(E^* \times \dots \times E^* \times E \times \dots \times E; \mathbb{R})$ of r -times contra- and s -times covariant tensors, or, for short, $\binom{r}{s}$ -tensors. Here, E^* occurs r times and E occurs s times. Now we consider a bijective map $\varphi \in L(E; F)$ and define the map

$$T_s^r(\varphi) \equiv \varphi_s^r \in L(T_s^r E; T_s^r F)$$

$$(\varphi_s^r T)(\beta^1, \dots, \beta^r, f_1, \dots, f_s) := T(\varphi^*(\beta^1), \dots, \varphi^*(\beta^r), \varphi^{-1}(f_1), \dots, \varphi^{-1}(f_s)),$$

where $T \in T_s^r E$, $\beta^1, \dots, \beta^r \in F^*$, $f_1, \dots, f_s \in F$. Here, φ^* denotes the adjoint map of φ .

We also recall the following

Proposition 2.1.4. *Let E, F and G be finite dimensional \mathbb{R} -vector spaces. For linear maps $\varphi : E \rightarrow F$ and $\psi : F \rightarrow G$ we have*

$$(i) \quad (\varphi \circ \psi)_s^r = \psi_s^r \circ \varphi_s^r.$$

$$(ii) \quad (id_E)_s^r = id_{T_s^r E}.$$

$$(iii) \quad \varphi_s^r : T_s^r E \rightarrow T_s^r F \text{ is a linear isomorphism and } (\varphi_s^r)^{-1} = (\varphi^{-1})_s^r.$$

$$(iv) \quad \text{If } t_1 \in T_{s_1}^{r_1} E, t_2 \in T_{s_2}^{r_2} E, \text{ then } \varphi_{s_1+s_2}^{r_1+r_2}(t_1 \otimes t_2) = \varphi_{s_1}^{r_1}(t_1) \otimes \varphi_{s_2}^{r_2}(t_2).$$

Proof. A proof can be found in [7], Chapter 2, Proposition 2.6.6. □

2.2 Tensor fields along smooth mappings

It is standard in differential geometry texts to introduce tensor fields as smooth sections of tensor bundles. Furthermore the following theorem holds.

Theorem 2.2.1. *Let M be a smooth manifold. We denote the $C^\infty(M)$ -module of all $\binom{r}{s}$ -tensor fields by $\mathcal{T}_s^r(M)$. Furthermore let*

$$L_{C^\infty(M)}^{r+s} := L_{C^\infty(M)}^{r+s} \underbrace{(\Omega^1(M) \times \dots \times \Omega^1(M))}_{r\text{-times}} \times \underbrace{(\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M))}_{s\text{-times}}; C^\infty(M)$$

be the $C^\infty(M)$ -module of all $C^\infty(M)$ -multilinear maps $t : \Omega^1(M) \times \dots \times \Omega^1(M) \times \mathfrak{X}(M) \times \dots \times \mathfrak{X}(M) \rightarrow C^\infty(M)$. Then there is a $C^\infty(M)$ -linear isomorphism

$$A : \mathcal{T}_s^r(M) \rightarrow L_{C^\infty(M)}^{r+s}.$$

Proof. A proof can be found in [7], Chapter 2, Theorem 2.6.19. \square

We recall that we have introduced an induced covariant derivation for vector fields along a smooth curve $c : [a, b] \rightarrow M$. This concept has to be generalized for our considerations concerning Jacobian tensor fields. We start by defining $\binom{r}{s}$ -tensor fields along a smooth map.

Definition 2.2.2. *Let M and N be smooth manifolds and let $f \in \mathcal{C}^\infty(N, M)$. An $\binom{r}{s}$ -tensor field along f is a smooth map $A : N \rightarrow T_s^r M$ such that $\pi_s^r \circ A = f$, i.e. a map $A : N \ni p \mapsto A(p) : (T_{f(p)}M)^* \times \dots \times (T_{f(p)}M)^* \times (T_{f(p)}M) \times \dots \times (T_{f(p)}M) \rightarrow \mathbb{R}$ (where $(T_{f(p)}M)^*$ occurs r times and $T_{f(p)}M$ occurs s times in the cartesian product), hence $A(p) \in L^{r+s}(T_{f(p)}M)$. For $s = r = 0$ we define a $\binom{0}{0}$ -tensor field along f to be a smooth map $A : N \rightarrow \mathbb{R}$, hence $A \in \mathcal{C}^\infty(N)$. We will denote the space of all $\binom{r}{s}$ -tensor fields along f by $\Gamma(N, T_s^r M, f)$.*

Remark 2.2.3. For $r = 1, s = 0$, Definition 2.2.2 reduces to the definition of vector fields along smooth maps.

Remark 2.2.4. Recall that a module over a commutative ring (R, \oplus, \odot) with unit element $I \in (R, \oplus, \odot)$ is an abelian group $(M, +)$ together with a map (the scalar multiplication)

$$\begin{aligned} \cdot : (R, \oplus, \odot) \times (M, +) &\rightarrow (M, +) \\ (r, m) &\mapsto r \cdot m \end{aligned}$$

such that

- (i) $r_1 \cdot (r_2 \cdot m) = (r_1 \odot r_2) \cdot m$,
- (ii) $(r_1 \oplus r_2) \cdot m = r_1 \cdot m + r_2 \cdot m$ and
- (iii) $r \cdot (m_1 + m_2) = r \cdot m_1 + r \cdot m_2$

for all $r, r_1, r_2 \in (R, \oplus, \odot)$ and for all $m, m_1, m_2 \in (M, +)$. It is clear that $\Gamma(N, T_s^r(M), f)$ with the operations

$$(A_1 + A_2)(p) := A_1(p) + A_2(p) \in T_{f(p)}M \quad \text{and} \quad (fA)(p) := f(p)A(p),$$

is a $\mathcal{C}^\infty(N)$ -module.

Now let $\phi \in L^2(E_1, E_2; E)$, i.e. $\phi : E_1 \times E_2 \rightarrow E$ is a bilinear map. For all $(e_1, e_2) \in E_1 \times E_2$ we define two maps

$$r_\phi : E_2 \rightarrow L^1(E_1; E)$$

$$r_\phi(e_2)(e_1) := \phi(e_1, e_2)$$

and

$$l_\phi : E_1 \rightarrow L^1(E_2; E)$$

$$l_\phi(e_1)(e_2) := \phi(e_1, e_2)$$

and see that r_ϕ and l_ϕ are \mathbb{R} -linear. Now we consider the maps

$$r : L^2(E_1, E_2; E) \rightarrow L^1(E_2; L^1(E_1; E))$$

$$r(\phi) := r_\phi$$

and

$$l : L^2(E_1, E_2; E) \rightarrow L^1(E_1; L^1(E_2; E))$$

$$l(\phi) := l_\phi$$

and show the following

Proposition 2.2.5. *Let E_1, E_2 and E be \mathbb{R} -vector spaces. Then the maps r and l define \mathbb{R} -isomorphisms.*

Proof. It suffices to show the first statement. Let $\alpha \in \mathbb{R}$ and $\phi, \psi \in L^2(E_1, E_2; E)$. Then, for $(e_1, e_2) \in E_1 \times E_2$ we have

$$\begin{aligned} r(\alpha\phi + \psi)(e_2)(e_1) &= r_{\alpha\phi + \psi}(e_2)(e_1) = (\alpha\phi + \psi)(e_1, e_2) \\ &= \alpha\phi(e_1, e_2) + \psi(e_1, e_2) = \alpha r_\phi(e_2)(e_1) + r_\psi(e_2)(e_1) \\ &= \alpha r(\phi)(e_2)(e_1) + r(\psi)(e_2)(e_1) = (\alpha r(\phi) + r(\psi))(e_2)(e_1), \end{aligned}$$

hence r is \mathbb{R} -linear. Now let $r(\phi) = 0$. Then we have for all $(e_1, e_2) \in E_1 \times E_2$

$$\begin{aligned} \phi(e_1, e_2) &= r_\phi(e_2)(e_1) \\ &= r(\phi)(e_2)(e_1) = 0, \end{aligned}$$

hence $\phi = 0$ and r is one-to-one. Now let $u \in L^1(E_2; L^1(E_1; E))$. Then we set $\phi : E_1 \times E_2 \rightarrow E$, $\phi(e_1, e_2) := u(e_2)(e_1)$, and see that ϕ is bilinear, hence $\phi \in L^2(E_1, E_2; E)$. Now we have

$$\begin{aligned} r(\phi)(e_2)(e_1) &= \phi(e_1, e_2) \\ &= u(e_2)(e_1), \end{aligned}$$

i.e. $r(\phi) = u$, so r is onto. □

Definition 2.2.6. Let $c : [a, b] \rightarrow M$ be a timelike geodesic. A $\binom{1}{1}$ -tensor field A on $V^\perp(c)$ is a $\binom{1}{1}$ -tensor field along c such that the $\binom{1}{1}$ -tensor $A(s)$ is defined on the restriction $N(c(s))$ of $T_{c(s)}M$, i.e. we have a smooth map $A : [a, b] \ni s \mapsto A(s) : (N(c(s)))^* \times N(c(s)) \rightarrow \mathbb{R}$ such that $A(s)$ is a $\binom{1}{1}$ -tensor for all $s \in [a, b]$.

Remark 2.2.7. For our further considerations it is very useful that we can write the $\binom{1}{1}$ -tensor $A(s)$ as a linear map $N(c(s)) \rightarrow N(c(s))$ for all $s \in [a, b]$ (cf. Proposition 2.2.5).

Remark 2.2.8. Let A and B be two $\binom{1}{1}$ -tensor fields on $V^\perp(c)$. Since $A(s)$ and $B(s)$ are linear maps for all $s \in [a, b]$, their matrix product $A(s)B(s)$ is defined.

Remark 2.2.9. Let (M, g) be a semi-Riemannian manifold with Levi-Civita-connection ∇ . Recall that the Riemannian curvature tensor R is the $\binom{1}{3}$ -tensor field

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$$

defined by

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

It should be noted that some authors use the sign convention

$$R(X, Y)Z := -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X, Y]} Z$$

for the curvature R .

Definition 2.2.10. Let A be a $\binom{1}{1}$ -tensor field. We define a composite endomorphism $RA(s) : N(c(s)) \rightarrow N(c(s))$ by

$$RA(s)(v) := R(A(s)(v), c'(s))c'(s) = R_{A(s)(v), c'(s)}c'(s)$$

for each $s \in [a, b]$. Here, R denotes the Riemannian curvature tensor. Since $RA(s)$ is linear, it again defines a $\binom{1}{1}$ -tensor on $N(c(s))$ by Proposition 2.2.5.

Remark 2.2.11. In Definition 2.2.10 R denotes the curvature-tensor $R = R_{c'(s_0), c'(s_0)} : T_{c(s_0)}M \rightarrow T_{c(s_0)}M$ restricted to the subspace $N(c(s_0))$ of $T_{c(s_0)}M$. We find that $\langle R(v, c'(s_0))c'(s_0), c'(s_0) \rangle = \langle R(c'(s_0), c'(s_0))v, c'(s_0) \rangle = 0$, where we have used anti-symmetry and pair-symmetry of the curvature-operator, hence we obtain that indeed $R(v, c'(s_0))c'(s_0) \in N(c(s_0))$.

We now define the concept of the adjoint tensor field A^* of A . To this end we give the following

Definition 2.2.12. Let A and A^* be $\binom{1}{1}$ -tensor fields on $V^\perp(c)$ (cf. Definition 2.2.6) such that the condition $g(c(s))(A(s)(w), v) = g(c(s))(A^*(s)(v), w)$ holds for all $s \in [a, b]$. Then A^* is said to be an adjoint of A . We see that A^* is uniquely determined by the above equation, i.e. we can write "the" adjoint of A . We will not distinguish between $A^*(s)$ and $(A(s))^*$.

2.3 Covariant derivations

Our next aim is to define Jacobian tensor fields. Our definition will use induced covariant derivations for tensor fields along smooth maps. We start with the following

Remark 2.3.1. Let $(U, \varphi = (x^1, \dots, x^n))$ be a chart for M . Recall that for all $t \in \Gamma(M, T_s^r M)$ we have the local representation

$$t|_U = t_{j_1, \dots, j_s}^{i_1, \dots, i_r} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s},$$

hence for all $A \in \Gamma(I, T_s^r M, c)$ and for all $s \in J$, where $J \subseteq I$ is such that $c(J) \subseteq U$, we find that

$$A(s) = A_{j_1, \dots, j_s}^{i_1, \dots, i_r}(s) \left(\frac{\partial}{\partial x^{i_1}} \circ c \right)(s) \otimes \dots \otimes \left(\frac{\partial}{\partial x^{i_r}} \circ c \right)(s) \otimes (dx^{j_1} \circ c)(s) \otimes \dots \otimes (dx^{j_s} \circ c)(s).$$

Convention 2.3.2. To simplify notations we will write ∂_{x^i} instead of $\frac{\partial}{\partial x^i}$. Furthermore we write $\partial_{x^{i_1}} \otimes \dots \otimes dx^{j_s}$ for $\frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s}$ and $(\partial_{x^{i_1}} \circ c)(s) \otimes \dots \otimes (dx^{j_s} \circ c)(s)$ for $(\frac{\partial}{\partial x^{i_1}} \circ c)(s) \otimes \dots \otimes (\frac{\partial}{\partial x^{i_r}} \circ c)(s) \otimes (dx^{j_1} \circ c)(s) \otimes \dots \otimes (dx^{j_s} \circ c)(s)$.

Proposition 2.3.3. Let M be a smooth manifold and let $c \in \mathcal{C}^\infty(I, M)$. We consider the map $A : I \ni s \mapsto A(s) : (T_{c(s)}M)^* \times \dots \times (T_{c(s)}M)^* \times (T_{c(s)}M) \times \dots \times (T_{c(s)}M) \rightarrow \mathbb{R}$. Then the following are equivalent:

- (1) A is smooth, i.e. $A \in \Gamma(I, T_s^r M, c)$.
- (2) In every chart representation all coefficient functions $A_{j_1, \dots, j_s}^{i_1, \dots, i_r}$ of A are smooth (cf. Remark 2.3.1).

Proof. For every chart (ψ, V) we found vector bundle charts $(T\psi)_s^r$ of the tensor bundle $T_s^r M$. Let $J \subseteq I$ such that $c(J) \subseteq V$. For $s \in J$ we have

$$\begin{aligned} ((T\psi)_s^r \circ A(s)) &= (T\psi)_s^r(A_{j_1, \dots, j_s}^{i_1, \dots, i_r}(s)(\partial_{x^{i_1}} \circ c)(s) \otimes \dots \otimes (dx^{j_s} \circ c)(s)) \\ &= (s, (T_p\psi)_s^r(A_{j_1, \dots, j_s}^{i_1, \dots, i_r}(s)(\partial_{x^{i_1}} \circ c)(s) \otimes \dots \otimes (dx^{j_s} \circ c)(s))) \\ &= (s, A_{j_1, \dots, j_s}^{i_1, \dots, i_r}(s)(T_p\psi)_0^1(\partial_{x^{i_1}} \circ c)(s) \otimes \dots \otimes (T_p\psi)_1^0(dx^{j_s} \circ c)(s)) \\ &= (s, A_{j_1, \dots, j_s}^{i_1, \dots, i_r}(s))(e_{i_1} \otimes \dots \otimes \alpha^{j_s}). \end{aligned}$$

□

Proposition 2.3.4. *Let M be a semi-Riemannian manifold and let $c : \mathbb{R} \supseteq I := [a, b] \rightarrow M$ be a smooth curve in M . Then there is a unique map*

$$\frac{\nabla}{ds} : \Gamma(I, T_s^r M, c) \rightarrow \Gamma(I, T_s^r M, c),$$

$$A \mapsto \frac{\nabla A}{ds} \equiv \frac{\nabla}{ds}(A),$$

the induced covariant derivation, such that

- (i) $\frac{\nabla}{ds}(A_1 + \lambda A_2) = \frac{\nabla}{ds}(A_1) + \lambda \frac{\nabla}{ds}(A_2)$ for all $A_1, A_2 \in \Gamma(I, T_s^r M, c)$ and $\lambda \in \mathbb{R}$.
- (ii) $\frac{\nabla}{ds}(hA) = \frac{dh}{ds}A + h \frac{\nabla}{ds}(A)$ for all smooth mappings $\mathcal{C}^\infty(I) \ni h : s \mapsto h(s)$ and for all $A \in \Gamma(I, T_s^r M, c)$.
- (iii) $\frac{\nabla}{ds}(\delta \circ c)(s) = (\nabla_{c'(s)}\delta)(s)$ for all $\delta \in \Gamma(M, T_s^r M)$ and $s \in I$.

Proof. We first show local uniqueness, i.e. uniqueness in a chart $(\varphi = (x^1, \dots, x^n), U)$. Suppose there exists an induced covariant derivation $\frac{\nabla}{ds} : \Gamma(I, T_s^r M, c) \rightarrow \Gamma(I, T_s^r M, c)$, $A \mapsto \frac{\nabla A}{ds}$ with the above properties (i), (ii) and (iii). Since A is a smooth map, all coefficient functions $A_{j_1, \dots, j_s}^{i_1, \dots, i_r}$ are smooth (cf. Proposition 2.3.3) and we find

$$\begin{aligned} \frac{\nabla A}{ds}(s) &= \frac{dA_{j_1, \dots, j_s}^{i_1, \dots, i_r}}{ds}(s) (\partial_{x^{i_1}} \circ c)(s) \otimes \dots \otimes (dx^{j_s} \circ c)(s) \\ &\quad + A_{j_1, \dots, j_s}^{i_1, \dots, i_r}(s) \frac{\nabla}{ds}((\partial_{x^{i_1}} \circ c)(s) \otimes \dots \otimes (dx^{j_s} \circ c)(s)) \\ &= \frac{dA_{j_1, \dots, j_s}^{i_1, \dots, i_r}}{ds}(s) (\partial_{x^{i_1}} \circ c)(s) \otimes \dots \otimes (dx^{j_s} \circ c)(s) \\ &\quad + A_{j_1, \dots, j_s}^{i_1, \dots, i_r}(s) \nabla_{c'(s)}(\partial_{x^{i_1}} \otimes \dots \otimes dx^{j_s}) \end{aligned}$$

Hence $\frac{\nabla}{ds}$ is locally uniquely determined. To prove existence, we define on each $J \subseteq I$ with $c(J) \subseteq U$ the map $\frac{\nabla}{ds} : A \mapsto \frac{\nabla A}{ds}$ by the local representation

$$\begin{aligned} \frac{\nabla A}{ds}(s) &= \frac{dA_{j_1, \dots, j_s}^{i_1, \dots, i_r}}{ds}(s) (\partial_{x^{i_1}} \circ c)(s) \otimes \dots \otimes (dx^{j_s} \circ c)(s) \\ &\quad + A_{j_1, \dots, j_s}^{i_1, \dots, i_r}(s) \nabla_{c'(s)}(\partial_{x^{i_1}} \otimes \dots \otimes dx^{j_s}) \end{aligned}$$

and show (i) – (iii):

- (i) Let $A_1, A_2 \in \Gamma(I, T_s^r M, c)$. Then we find the following local representation for $A_1 + \lambda A_2$ with respect to the chart $(\varphi = (x^1, \dots, x^n), U)$.

$$(A_1 + \lambda A_2)|_J = (A_{j_1, \dots, j_s}^{i_1, \dots, i_r} + \lambda A_{j_1', \dots, j_s'}^{i_1', \dots, i_r'}) (\partial_{x^{i_1}} \circ c)(s) \otimes \dots \otimes (dx^{j_s} \circ c)(s).$$

By using the local representation we find that

$$\begin{aligned} & \frac{\nabla}{ds} (A_1 + \lambda A_2)(s) \\ &= \frac{d(A_{j_1, \dots, j_s}^{i_1, \dots, i_r} + \lambda A_{j_1, \dots, j_s}^{i_1, \dots, i_r})(s)}{ds} (\partial_{x^{i_1}} \circ c)(s) \otimes \dots \otimes (dx^{j_s} \circ c)(s) \\ &+ ((A_{j_1, \dots, j_s}^{i_1, \dots, i_r} + \lambda A_{j_1, \dots, j_s}^{i_1, \dots, i_r})(s)) \nabla_{c'(s)} (\partial_{x^{i_1}} \otimes \dots \otimes dx^{j_s}) \\ &= \frac{d(A_{j_1, \dots, j_s}^{i_1, \dots, i_r})(s)}{ds} (\partial_{x^{i_1}} \circ c)(s) \otimes \dots \otimes (dx^{j_s} \circ c)(s) \\ &+ ((A_{j_1, \dots, j_s}^{i_1, \dots, i_r})(s)) \nabla_{c'(s)} (\partial_{x^{i_1}} \otimes \dots \otimes dx^{j_s}) \\ &+ \lambda \frac{d(A_{j_1, \dots, j_s}^{i_1, \dots, i_r})(s)}{ds} (\partial_{x^{i_1}} \circ c)(s) \otimes \dots \otimes (dx^{j_s} \circ c)(s) \\ &+ \lambda (A_{j_1, \dots, j_s}^{i_1, \dots, i_r})(s) \nabla_{c'(s)} (\partial_{x^{i_1}} \otimes \dots \otimes dx^{j_s}) \\ &= \frac{\nabla}{ds} (A_1)(s) + \lambda \frac{\nabla}{ds} (A_2)(s). \end{aligned}$$

- (ii) Again, by using the local representation, we find

$$\begin{aligned} \frac{\nabla}{ds} (hA)(s) &= \frac{d(hA_{j_1, \dots, j_s}^{i_1, \dots, i_r})(s)}{ds} (\partial_{x^{i_1}} \circ c)(s) \otimes \dots \otimes (dx^{j_s} \circ c)(s) \\ &+ ((hA_{j_1, \dots, j_s}^{i_1, \dots, i_r})(s)) \nabla_{c'(s)} (\partial_{x^{i_1}} \otimes \dots \otimes dx^{j_s}) \\ &= \frac{dh}{ds}(s) A(s) + h(s) \frac{\nabla}{ds} (A)(s) \end{aligned}$$

- (iii) We first recall the identity

$$\frac{d}{ds} (\delta_{j_1, \dots, j_s}^{i_1, \dots, i_r} \circ c)(s) = T_s c \left(\frac{d}{ds} (\delta_{j_1, \dots, j_s}^{i_1, \dots, i_r}) \right) = c'(s) (\delta_{j_1, \dots, j_s}^{i_1, \dots, i_r})$$

for all $s \in J$ and find

$$\begin{aligned} \nabla_{c'(s)} (\delta) &= \nabla_{c'(s)} ((\delta_{j_1, \dots, j_s}^{i_1, \dots, i_r}) (\partial_{x^{i_1}} \otimes \dots \otimes dx^{j_s})) \\ &= c'(s) (\delta_{j_1, \dots, j_s}^{i_1, \dots, i_r}) (\partial_{x^{i_1}} \otimes \dots \otimes dx^{j_s}) + \delta_{j_1, \dots, j_s}^{i_1, \dots, i_r} (c(s)) \nabla_{c'(s)} (\partial_{x^{i_1}} \otimes \dots \otimes dx^{j_s}) \\ &= \frac{d}{ds} (\delta_{j_1, \dots, j_s}^{i_1, \dots, i_r} (c(s))) (\partial_{x^{i_1}} \otimes \dots \otimes dx^{j_s}) + \delta_{j_1, \dots, j_s}^{i_1, \dots, i_r} (c(s)) \nabla_{c'(s)} (\partial_{x^{i_1}} \otimes \dots \otimes dx^{j_s}) \\ &= \left(\frac{\nabla}{ds} \right) (\delta \circ c)(s) \end{aligned}$$

Now let $J_1, J_2 \subseteq I$ such that $c(J_i) \subseteq U$ for $i = 1, 2$ and let F_1 and F_2 be the corresponding induced covariant derivations $s \mapsto \frac{\nabla A}{ds}$. Then both mappings, F_1 and F_2 , satisfy (1) – (3) for all $s \in J_1 \cap J_2$. Due to the uniqueness proof, they coincide on $J_1 \cap J_2$. Therefore, $\frac{\nabla}{ds} : \Gamma(I, T_s^r M, c) \rightarrow \Gamma(I, T_s^r M, c)$ is a well defined map. Hence all statements have been proved. \square

Remark 2.3.5. Let A be a $\binom{0}{0}$ -tensor field along c (cf. Definition 2.2.2). Then the induced covariant derivation coincides with the "usual" derivation for scalar-valued maps.

Remark 2.3.6. Let M be a smooth manifold. Remember that there is a unique $\mathcal{C}^\infty(M)$ -linear map, the $\binom{1}{1}$ -contraction

$$\mathcal{C} : \mathcal{T}_1^1(M) \rightarrow \mathcal{C}^\infty(M)$$

such that $\mathcal{C}(X \otimes \omega) = \omega(X)$ holds for all smooth vector fields $X \in \mathfrak{X}(M)$ and for all one-forms $\omega \in \Omega^1(M)$, where $\mathcal{T}_1^1(M) := \Gamma(M, T_1^1(M))$ denotes the $\mathcal{C}^\infty(M)$ -module of all smooth sections of the $\binom{1}{1}$ -tensor bundle. Let $A \in \mathcal{T}_1^1(M)$ and let $(\psi = (x^1, \dots, x^n), V)$ be a chart of M . Then we obtain locally that

$$\mathcal{C}(A) = \sum_{i=1}^n A_i^i = A(dx^i, \partial_{x^i}).$$

The same results with analogous proofs hold if we write $\Gamma(I, T_1^1(M), c)$ instead of $\mathcal{T}_1^1(M)$ for $I \subseteq \mathbb{R}$ and $c : I \rightarrow M$ a smooth curve in M . In the next Lemma we will derive a product-rule for tensor fields along smooth curves. Furthermore we will show that the induced covariant derivation (cf. Proposition 2.3.4) commutes with contraction.

Lemma 2.3.7. *Let M be a semi-Riemannian manifold and let $c : \mathbb{R} \supseteq I \rightarrow M$ be a smooth curve in M . Let $A \in \Gamma(I, T_s^r(M), c)$, $B \in \Gamma(I, T_{s'}^{r'}(M), c)$ and $C \in \Gamma(I, T_1^1(M), c)$. Then we have*

$$(i) \quad \frac{\nabla}{ds}(A \otimes B) = \left(\frac{\nabla}{ds}(A)\right) \otimes B + A \otimes \frac{\nabla}{ds}(B) \text{ and}$$

$$(ii) \quad \frac{\nabla}{ds}(\mathcal{C}(C)) = \mathcal{C}\left(\frac{\nabla}{ds}C\right)$$

Proof. (i) By Remark 2.3.1 we have local representations

$$A(s) = (A_{j_1, \dots, j_s}^{i_1, \dots, i_r})(s)(\partial_{x^{i_1}} \circ c)(s) \otimes \dots \otimes (dx^{j_s} \circ c)(s)$$

and

$$B(s) = (B_{j'_1, \dots, j'_{s'}}^{i'_1, \dots, i'_{r'}})(s)(\partial_{x^{i'_1}} \circ c)(s) \otimes \dots \otimes (dx^{j'_{s'}} \circ c)(s).$$

By Proposition 2.3.3 all coefficient-functions are smooth. Then

$$\begin{aligned} & \frac{\nabla}{ds} ((A_{j_1, \dots, j_s}^{i_1, \dots, i_r} (\partial_{x^{i_1}} \circ c) \otimes \dots \otimes (dx^{j_s} \circ c)) \otimes \\ & (B_{j'_1, \dots, j'_{s'}}^{i'_1, \dots, i'_{r'}} (\partial_{x^{i'_1}} \circ c) \otimes \dots \otimes (dx^{j'_{s'}} \circ c))) \\ & = \frac{\nabla}{ds} (A_{j_1, \dots, j_s}^{i_1, \dots, i_r} B_{j'_1, \dots, j'_{s'}}^{i'_1, \dots, i'_{r'}} (\partial_{x^{i_1}} \circ c) \otimes \dots \otimes (dx^{j'_{s'}} \circ c)). \end{aligned}$$

The last expression becomes

$$\begin{aligned} & \left(\frac{d(A_{j_1, \dots, j_s}^{i_1, \dots, i_r} B_{j'_1, \dots, j'_{s'}}^{i'_1, \dots, i'_{r'}})}{ds} (\partial_{x^{i_1}} \circ c) \otimes \dots \otimes (dx^{j'_{s'}} \circ c) \right) \\ & + ((A_{j_1, \dots, j_s}^{i_1, \dots, i_r}) (B_{j'_1, \dots, j'_{s'}}^{i'_1, \dots, i'_{r'}}) \frac{\nabla}{ds} ((\partial_{x^{i_1}} \circ c) \otimes \dots \otimes (dx^{j'_{s'}} \circ c))) \end{aligned}$$

and we obtain that this equals

$$\begin{aligned} & \left(\frac{dA_{j_1, \dots, j_s}^{i_1, \dots, i_r}}{ds} B_{j'_1, \dots, j'_{s'}}^{i'_1, \dots, i'_{r'}} (\partial_{x^{i_1}} \circ c) \otimes \dots \otimes (dx^{j'_{s'}} \circ c) \right) \\ & + (A_{j_1, \dots, j_s}^{i_1, \dots, i_r} \frac{dB_{j'_1, \dots, j'_{s'}}^{i'_1, \dots, i'_{r'}}}{ds} (\partial_{x^{i_1}} \circ c) \otimes \dots \otimes (dx^{j'_{s'}} \circ c)) \\ & + (A_{j_1, \dots, j_s}^{i_1, \dots, i_r} B_{j'_1, \dots, j'_{s'}}^{i'_1, \dots, i'_{r'}} \nabla_{c'(s)} (\partial_{x^{i_1}} \otimes \dots \otimes dx^{j'_{s'}})), \end{aligned}$$

where we have used Proposition 2.3.4 (i), (ii) and (iii). The last term can be split since $\nabla_{c'(s)}$ is a tensor derivation. We finally obtain

$$\begin{aligned} & \left(\frac{dA_{j_1, \dots, j_s}^{i_1, \dots, i_r}}{ds} B_{j'_1, \dots, j'_{s'}}^{i'_1, \dots, i'_{r'}} (\partial_{x^{i_1}} \circ c) \otimes \dots \otimes (dx^{j'_{s'}} \circ c) \right) \\ & + (A_{j_1, \dots, j_s}^{i_1, \dots, i_r} \frac{dB_{j'_1, \dots, j'_{s'}}^{i'_1, \dots, i'_{r'}}}{ds} (\partial_{x^{i_1}} \circ c) \otimes \dots \otimes (dx^{j'_{s'}} \circ c)) \\ & + (A_{j_1, \dots, j_s}^{i_1, \dots, i_r} B_{j'_1, \dots, j'_{s'}}^{i'_1, \dots, i'_{r'}} (\nabla_{c'(s)} (\partial_{x^{i_1}} \otimes \dots \otimes dx^{j'_{s'}})) \otimes (\partial_{x^{i'_1}} \circ c \otimes \dots \otimes dx^{j'_{s'}} \circ c)) \\ & + (A_{j_1, \dots, j_s}^{i_1, \dots, i_r} B_{j'_1, \dots, j'_{s'}}^{i'_1, \dots, i'_{r'}} (\partial_{x^{i_1}} \circ c \otimes \dots \otimes dx^{j'_{s'}} \circ c) \otimes (\nabla_{c'(s)} (\partial_{x^{i'_1}} \otimes \dots \otimes dx^{j'_{s'}}))), \end{aligned}$$

and we are done.

(ii) By Remark 2.3.6 we find that $\mathcal{C}(C) = \sum_{i=1}^n C_i^i$ and obtain

$$\begin{aligned} \frac{\nabla}{ds} (\mathcal{C}(C)) & = \frac{\nabla}{ds} (\sum_{i=1}^n C_i^i) = \frac{d}{ds} (\sum_{i=1}^n C_i^i) \\ & = \sum_{i=1}^n \frac{d}{ds} (C_i^i) = \mathcal{C} \left(\frac{\nabla}{ds} C \right), \end{aligned}$$

where we have used the local representation in the proof of Proposition 2.3.4 (ii). \square

Definition 2.3.8. Let A be an $\binom{r}{s}$ -tensor field along a smooth curve $c : I \rightarrow M$. Since $\frac{\nabla A}{ds} \in \Gamma(I, T_s^r M, c)$, we can use Proposition 2.3.4 to define the second induced covariant derivation of A by

$$A'' := \frac{\nabla}{ds} \frac{\nabla A}{ds}.$$

2.4 Jacobian tensor fields

Definition 2.4.1. A $\binom{1}{1}$ -tensor field A on $V^\perp(c)$ is said to be a Jacobian tensor field along c , if

- (i) $A''(s) + RA(s) = 0$ for all $s \in I$ (cf. Definition 2.2.10, Proposition 2.3.4 and Definition 2.3.8) and
- (ii) $\ker(A(s)) \cap \ker(\frac{\nabla A}{ds}(s)) = \{0\}$ for all $s \in I$ (cf. Proposition 2.3.4).

Definition 2.4.2. Let A be a Jacobian tensor field along c . Then A is said to be a Lagrangian tensor field, if

$$\left(\frac{\nabla A}{ds}\right)^*(s)A(s) - (A)^*(s)\frac{\nabla A}{ds}(s) = 0$$

for all $s \in [a, b]$.

Definition 2.4.3. Let A_1 and A_2 be Jacobian tensor fields along c . Then their Wronskian $W(A_1, A_2)$ is the $\binom{1}{1}$ -tensor field on $V^\perp(c)$ given by

$$W(A_1, A_2)(s) := \left(\frac{\nabla A_1}{ds}\right)^*(s)A_2(s) - (A_1)^*(s)\frac{\nabla A_2}{ds}(s)$$

for all $s \in [a, b]$ (cf. Definition 2.2.12).

Lemma 2.4.4. Let A and B be $\binom{1}{1}$ -tensor fields on $V^\perp(c)$. Then

- (i) $\frac{\nabla}{ds}(AB) = \left(\frac{\nabla}{ds}A\right)B + A\left(\frac{\nabla}{ds}B\right)$, cf. Remark 2.2.8,
- (ii) $R = R^*$ for the Riemannian curvature operator,
- (iii) $(AB)^* = B^*A^*$, in particular we obtain that $(RA)^* = A^*R$ for the composite endomorphism (cf. Definition 2.2.10),
- (iv) if A is invertible for all $s \in I$, then $\frac{\nabla}{ds}A^{-1} = -A^{-1}\left(\frac{\nabla A}{ds}\right)A^{-1}$,

(v) $\frac{\nabla}{ds}(A^*) = (\frac{\nabla}{ds}A)^*$ and

(vi) $\frac{\nabla}{ds}W(A, B) = 0$ for all Jacobian tensor fields A and B .

Proof. (i) By Remark A.3.7 we find a $\binom{1}{1}$ -frame field along c on M . For $A(s) = A_j^i(s)E_i(s) \otimes E^{*j}(s)$ and $B(s) = B_l^j(s)E_j(s) \otimes E^{*l}(s)$ we obtain that $A(s)B(s) = A_j^i(s)B_l^j(s)E_i(s) \otimes E^{*l}(s)$. Since E_i and E^{*j} are parallel vector fields and one-forms we can use Proposition 2.3.4 (i), (ii), (iii) and Lemma 2.3.7 (i) to derive

$$\begin{aligned} \frac{\nabla}{ds}(A(s)B(s)) &= \frac{d(A_j^i(s)B_l^j(s))}{ds}E_i(s) \otimes E^{*l}(s) \\ &\quad + A_j^i(s)B_l^j(s)\nabla_{c'(s)}(E_i \otimes E^{*l}) \\ &= \frac{dA_j^i(s)}{ds}B_l^j(s)E_i(s) \otimes E^{*l}(s) + A_j^i(s)\frac{dB_l^j(s)}{ds}E_i(s) \otimes E^{*l}(s) \\ &= (\frac{\nabla A}{ds}(s))(B(s)) + A(s)(\frac{\nabla B}{ds}(s)). \end{aligned}$$

(ii) By Definition 2.2.12 we find

$$\begin{aligned} 0 &= g(c(s))(R(s)(z), v) - g(c(s))(R^*(s)(v), z) \\ &= g(c(s))(R(s)(v), z) - g(c(s))(R^*(s)(v), z). \end{aligned}$$

Thus $R(s) = R^*(s)$.

(iii) This is clear by definition of the adjoint tensor field.

(iv) We have to show that

$$A(s)\frac{\nabla}{ds}(A^{-1}(s)) = -\frac{\nabla}{ds}(A(s))A^{-1}(s).$$

By applying (i) to $A(s)A^{-1}(s) = I(s)$, where $I(s)$ denotes the identity-operator, we obtain

$$0 = \frac{\nabla}{ds}(A(s))(A^{-1}(s)) + A(s)\frac{\nabla}{ds}(A^{-1}(s)).$$

(v) By Remark A.3.7 we find a $\binom{1}{1}$ -frame field along c on M . In terms of this frame we may write

$$A(s) = A_j^i(s)E_i(s) \otimes E^{*j}(s),$$

where all $A_j^i : I \rightarrow \mathbb{R}$ are smooth. By Proposition 2.3.4 its induced covariant derivation is

$$\frac{\nabla}{ds}A(s) = \frac{d(A_j^i)}{ds}(s)E_i(s) \otimes E^{*j}(s) + A_j^i(s)\nabla_{c^j(s)}(E_i \otimes E^{*j}),$$

where the last term vanishes. Since we operate with frame fields, we find

$$A^*(s) = \sum_{i,j=1}^n A_i^j(s)E_i(s) \otimes E^{*j}(s)$$

for the adjoint tensor field A^* of A . We obtain that

$$\left(\frac{\nabla}{ds}A(s)\right)^* = \sum_{i,j=1}^n \frac{d(A_i^j)}{ds}(s)E_i(s) \otimes E^{*j}(s) = \left(\frac{\nabla}{ds}A^*(s)\right).$$

(vi) We use Proposition 2.3.4 (i) and find that

$$\begin{aligned} \frac{\nabla}{ds}W(A, B)(s) &= \frac{\nabla}{ds}\left(\left(\frac{\nabla}{ds}A(s)\right)^*B(s) - A(s)^*\frac{\nabla}{ds}B(s)\right) \\ &= \frac{\nabla}{ds}\left(\left(\frac{\nabla}{ds}A(s)\right)^*B(s)\right) - \frac{\nabla}{ds}\left(A(s)^*\frac{\nabla}{ds}B(s)\right). \end{aligned}$$

Using (i) we conclude that

$$\begin{aligned} \frac{\nabla}{ds}W(A, B)(s) &= \frac{\nabla}{ds}\left(\frac{\nabla}{ds}\left(\frac{\nabla}{ds}A(s)\right)^*B(s)\right) + \left(\frac{\nabla}{ds}\left(\frac{\nabla}{ds}A(s)\right)\right)^*\frac{\nabla}{ds}B(s) \\ &\quad - \frac{\nabla}{ds}\left(A(s)^*\right)\frac{\nabla}{ds}B(s) - A(s)^*B''(s). \end{aligned}$$

By (v) we obtain that

$$\begin{aligned} \frac{\nabla}{ds}W(A, B)(s) &= \frac{\nabla}{ds}\left(\frac{\nabla}{ds}\left(\frac{\nabla}{ds}A(s)\right)^*B(s)\right) - A(s)^*B''(s) \\ &= A''^*(s)B(s) - A^*(s)B''(s). \end{aligned}$$

Since $A^*(s)(B''(s) + R(s)B(s)) = 0$ we see that

$$\begin{aligned} A''^*(s)B(s) - A^*(s)B''(s) &= A''^*(s)B(s) + A^*(s)B''(s) + A^*(s)RB(s) - A^*(s)B''(s) \\ &= A''^*(s)B(s) + A^*(s)RB(s) \\ &= (A''(s) + RA(s))^*B(s) = 0 \end{aligned}$$

□

Proposition 2.4.5. *Let A be a Jacobian tensor field. If $A(s_0) = 0$ for some $s_0 \in [a, b]$, then A is a Lagrangian tensor field, i.e.*

$$\left(\frac{\nabla A}{ds}\right)^*(s)A(s) = (A)^*(s)\frac{\nabla A}{ds}(s)$$

for all $s \in [a, b]$.

Proof. By Lemma 2.4.4 (vi) $W(A, A)$ is a parallel $\binom{1}{1}$ -tensor field on $V^\perp(c)$. By assumption there exists an $s_0 \in [a, b]$ such that the linear map

$$A(s_0) : N(c(s_0)) \rightarrow N(c(s_0)), v \mapsto A(s_0)(v)$$

vanishes (cf. Definition 2.2.6 and Remark 2.2.7). Therefore

$$W(A, A)(s_0) = \left(\frac{\nabla A}{ds}\right)^*(s_0)A(s_0) - (A)^*(s_0)\frac{\nabla A}{ds}(s_0) = 0,$$

hence

$$W(A, A)(s) = 0$$

for all $s \in [a, b]$. □

2.5 The Raychaudhuri equation

We now derive the Raychaudhuri equation for Jacobian tensor fields along timelike geodesics. We start with the definition of the expansion, vorticity and shear of a Jacobian tensor field A . I will denote the $\binom{1}{1}$ -tensor field along $V^\perp(c)$ such that $I(s) = id : N(c(s)) \rightarrow N(c(s))$ for each $s \in [a, b]$.

Definition 2.5.1. *Let A be a Jacobian tensor field along a timelike geodesic $c : [a, b] \rightarrow M$, where $\dim(M) = n$. We assume that $A(s)^{-1}$ exists for all $s \in I$ and set $B(s) := \left(\frac{\nabla A}{ds}\right)(s)A^{-1}(s)$. Then we define*

(i) the expansion $\theta(s) := \text{tr}(B(s))$,

(ii) the vorticity tensor $\omega(s) := \frac{1}{2}(B(s) - B(s)^*)$ and

(iii) the shear tensor $\sigma(s) := \frac{1}{2}(B(s) + B(s)^*) - \frac{\theta(s)}{n-1}I(s)$

for all $s \in [a, b]$

Proposition 2.5.2. *Let A be a Jacobian tensor field along a timelike geodesic $c : [a, b] \rightarrow M$ with $\dim M := n$. Then the Raychaudhuri equation*

$$\frac{d\theta}{ds}(s) = -\text{Ric}(c'(s), c'(s)) - \text{tr}((\omega(s))^2) - \text{tr}((\sigma(s))^2) - \frac{(\theta(s))^2}{n-1}$$

holds on $[a, b]$.

Proof. Let $B(s) := (\frac{\nabla A}{ds})(s)A^{-1}(s)$. By Lemma 2.4.4 (i) and (iv) we find that

$$\frac{\nabla B}{ds}(s) = -R(s) - (B(s))^2.$$

A straightforward calculation shows that

$$\text{tr}(\omega(s)^2 + \sigma(s)^2) + \frac{\theta(s)^2}{(n-1)^2}I(s) = \text{tr}(B(s)^2).$$

Therefore we obtain

$$\begin{aligned} \frac{d\theta}{ds}(s) &= \text{tr}\left(\frac{\nabla B}{ds}(s)\right) \\ &= -\text{tr}(R(s)) - \text{tr}((\omega(s))^2 + (\sigma(s))^2) + \frac{(\theta(s))^2}{(n-1)^2}I(s) \\ &= -\text{tr}(R(s)) - \text{tr}(\omega(s)^2) - \text{tr}(\sigma(s)^2) - \frac{(\theta(s))^2}{n-1} \\ &= -\text{Ric}(c'(s), c'(s)) - \text{tr}(\omega(s)^2) - \text{tr}(\sigma(s)^2) - \frac{(\theta(s))^2}{n-1} \end{aligned}$$

□

Proposition 2.5.3. *Let A be a Lagrangian tensor field, then*

$$B(s) := \frac{\nabla A}{ds}(s)A(s)^{-1} = A(s)^{-1}\left(\frac{\nabla A}{ds}\right)^*(s) = B(s)^*.$$

Proof. By Definition 2.4.2,

$$\left(\frac{\nabla A}{ds}\right)^*(s)A(s) = (A)^*(s)\frac{\nabla A}{ds}(s)$$

for all $s \in [a, b]$. A right multiplication with $(A(s))^{-1}$ and a left multiplication with $((A(s))^*)^{-1}$ yields

$$B(s) = \left(\frac{\nabla A}{ds}\right)^*(s)(A(s))^{-1} = (A(s)^*)^{-1}\frac{\nabla A^*}{ds}(s) = B^*(s)$$

for all $s \in [a, b]$

□

Remark 2.5.4. Let $B(s) := (\frac{\nabla A}{ds})(s)A^{-1}(s)$. The shear tensor field (cf. Definition 2.5.1 (iii)) is self-adjoint with respect to the Lorentzian metric g for arbitrary Jacobian tensor fields, since both $(B + B^*)$ and I are obviously self-adjoint.

Now let E_1, \dots, E_n be an orthonormal frame field at $c(s)$ with $E_n(s) = c'(s)$. Then we can write $\sigma(s)$ as a symmetric matrix $(\sigma_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}$. Furthermore,

$$\begin{aligned} \text{tr}((\sigma(s))^2) &= \text{tr}\left(\sum_{k=1}^{n-1} \sigma_{ik}(s)\sigma_{kj}(s)\right) \\ &= \sum_{i=1}^{n-1} \sum_{k=1}^{n-1} \sigma_{ik}(s)\sigma_{ki}(s) \\ &= \sum_{i=1}^{n-1} \sum_{k=1}^{n-1} (\sigma_{ik}(s))^2 \geq 0, \end{aligned}$$

hence $\text{tr}((\sigma(s))^2) = 0$ if and only if $\sigma(s) = 0$.

Corollary 2.5.5. *Let A be a Lagrangian tensor field. Then the vorticity*

$$\omega(s) = \frac{1}{2}(B(s) - (B(s))^*)$$

vanishes along c .

Proof. Immediate from Proposition 2.5.3. □

We now obtain the following version of the Raychaudhuri equation

Proposition 2.5.6. *Let A be a Lagrangian tensor field along a timelike geodesic $c : [a, b] \rightarrow M$, where $\dim M = n$. Then the vorticity free Raychaudhuri equation*

$$\frac{d\theta}{ds}(s) = -\text{Ric}(c'(s), c'(s)) - \text{tr}((\sigma(s))^2) - \frac{(\theta(s))^2}{n-1}$$

holds for all $s \in [a, b]$.

Proof. This follows from Proposition 2.5.2 and Lemma 2.5.5. □

CHAPTER 3

CAUSAL STRUCTURE

3.1 Causality conditions

Let V be a Lorentzian vector space, i.e. a scalar product space of index 1 and dimension ≥ 2 . Let \mathcal{T} be the set of all timelike vectors in V , i.e.

$$\mathcal{T} := \{v \in V : \langle v, v \rangle < 0\} \subseteq V$$

and let $\overline{\mathcal{T}}$ be the set of all causal vectors in V , i.e.

$$\overline{\mathcal{T}} := \{v \in V : \langle v, v \rangle \leq 0\} \subseteq V.$$

Let $v \in \mathcal{T}$. Then we define the causal cone of V containing v to be the set

$$\overline{\mathcal{C}}(v) := \{w \in \overline{\mathcal{T}} : \langle v, w \rangle < 0\}.$$

The opposite causal cone of V is defined to be the set

$$\overline{\mathcal{C}}(-v) := \{w \in \overline{\mathcal{T}} : \langle v, w \rangle > 0\} = -\overline{\mathcal{C}}(v).$$

As in Remark 1.1.5 it can be shown that there are precisely two causal cones in each Lorentzian vector space V , the so called future and past causal cone, whose disjoint union is the set of all causal vectors in V . Now let (M, g) be a space-time, $p \in M$ and $v \in T_p M$ timelike. v is said to be future-directed (or future-pointing), if $v \in \tau_p$ and past-directed (or past-pointing) if v is contained in the opposite timecone of τ_p . A causal vector $v \in T_p M$ is called future-directed (or future-pointing), if v is contained in the future causal cone of $T_p M$ and past-directed (or past-pointing), if v is contained in the past causal cone of $T_p M$. A timelike (resp. causal) curve is future-directed if all its tangent vectors are future-directed. By a curve we always mean a piecewise smooth curve.

Definition 3.1.1. For $p, q \in M$ we define

- (i) $p \ll q$ if there exists a piecewise smooth future-directed timelike curve from p to q ,
- (ii) $p < q$ if there is a piecewise smooth future-directed causal curve from p to q ,
- (iii) $p \leq q$ if either $p = q$ or there is a piecewise smooth future-directed causal curve from p to q ,
- (iv) $p \gg q$ or $q \ll p$ if there is a piecewise smooth past-directed timelike curve from p to q ,
- (v) $p > q$ if there is a piecewise smooth past-directed causal curve from p to q and
- (vi) $p \geq q$ or $q \leq p$ if either $p = q$ or there is a piecewise smooth past-directed causal curve from p to q .

Definition 3.1.2. The chronological future $I^+(p)$ of p is the set

$$I^+(p) := \{q \in M : p \ll q\}$$

and the chronological past $I^-(p)$ of p is the set

$$I^-(p) := \{q \in M : q \ll p\}.$$

The causal future $J^+(p)$ of p is the set

$$J^+(p) := \{q \in M : p \leq q\}$$

and the causal past $J^-(p)$ is the set

$$J^-(p) := \{q \in M : q \leq p\}.$$

It is often necessary to generalize Definition 3.1.2 and consider the chronological future etc. of a subset A of M . Therefore we give the following

Definition 3.1.3. Let A be an arbitrary subset of M . The chronological future $I^+(A)$ of A is the set

$$I^+(A) := \{q \in M : \text{there is a } p \in A \text{ with } p \ll q\}$$

and the chronological past $I^-(A)$ of A is the set

$$I^-(A) := \{q \in M : \text{there is a } p \in A \text{ with } q \ll p\}.$$

The causal future $J^+(A)$ of A is the set

$$J^+(A) := \{q \in M : \text{there is a } p \in A \text{ with } p \leq q\}$$

and the causal past $J^-(A)$ is the set

$$J^-(A) := \{q \in M : \text{there is a } p \in A \text{ with } q \leq p\}.$$

Remark 3.1.4. By Definition 3.1.3 we find that

$$J^+(A) \supseteq A \cup I^+(A)$$

and

$$J^-(A) \supseteq A \cup I^-(A).$$

Furthermore we see that Definition 3.1.3 is indeed a generalization of Definition 3.1.2. In particular we find that

$$I^+(A) = \bigcup_{p \in A} I^+(p) \quad \text{and} \quad J^+(A) = \bigcup_{p \in A} J^+(p)$$

and

$$I^-(A) = \bigcup_{p \in A} I^-(p) \quad \text{and} \quad J^-(A) = \bigcup_{p \in A} J^-(p).$$

Remark 3.1.5. The concatenation of two piecewise smooth curves is still piecewise smooth. Since the causal character does not change by linking together two piecewise smooth curves, we obtain that the relations \ll and \leq are transitive; for p_1, p_2 and q we find

$$p_1 \ll q \quad \text{and} \quad q \ll p_2 \quad \text{implies} \quad p_1 \ll p_2$$

and

$$p_1 \leq q \quad \text{and} \quad q \leq p_2 \quad \text{implies} \quad p_1 \leq p_2$$

For the next Proposition we need the following

Lemma 3.1.6. *Let M be a Lorentzian manifold and let α be a piecewise smooth causal curve from p_1 to p_2 that is not a null pregeodesic. Then there is a timelike curve from p_1 to p_2 arbitrarily close to α .*

Proof. See [15], Chapter 10, Proposition 46. □

Proposition 3.1.7. *Let $p_1 \ll q$ and $q \leq p_2$ or $p_1 \leq q$ and $q \ll p_2$. Then we find*

$$p_1 \ll p_2.$$

Proof. We prove the first statement. The second statement is proven analogously. By assumption we find a future-directed timelike piecewise smooth curve α_1 from p_1 to q and a future-directed causal piecewise smooth curve α_2 from q to p_2 . By Remark 3.1.5 we see that the concatenation α of α_1 and α_2 is causal (not necessary timelike) and connects p_1 and p_2 . Since α is not a null geodesic, the statement follows from Lemma 3.1.6. \square

Definition 3.1.8. Let $A \subseteq M$ and let U be an open subset of M with $A \subseteq U$. U is a space-time, too and we define the chronological future of A relative to U by

$$I^+(A, U) := \{p \in M : \text{there is a future-directed, timelike curve } \alpha \subseteq U \text{ from } A \text{ to } p\}$$

and the chronological past of A relative to U by

$$I^-(A, U) := \{p \in M : \text{there is a past-directed, timelike curve } \alpha \subseteq U \text{ from } A \text{ to } p\}$$

Furthermore we define the causal future of A relative to U by

$$J^+(A, U) := \{p \in M : p \in A \text{ or there is a future-directed, causal curve } \alpha \subseteq U \text{ from } A \text{ to } p\}$$

and the causal past of A relative to U by

$$J^-(A, U) := \{p \in M : p \in A \text{ or there is a past-directed, causal curve } \alpha \subseteq U \text{ from } A \text{ to } p\}$$

Proposition 3.1.9. Let A be a subset of M . Then we find (cf. Definition 3.1.3)

$$I^+(A) = I^+(I^+(A)) = I^+(J^+(A)) = J^+(I^+(A)) \subseteq J^+(J^+(A)) = J^+(A)$$

and

$$I^-(A) = I^-(I^-(A)) = I^-(J^-(A)) = J^-(I^-(A)) \subseteq J^-(J^-(A)) = J^-(A).$$

Proof. We first show that $I^+(A) = I^+(I^+(A))$. To this end let $p \in I^+(A)$, i.e. there is a piecewise smooth future-directed timelike curve c that connects some $a \in A$ with p . We choose an arbitrary element $a_1 \in \{c\}$ and obtain that $a_1 \in I^+(a) \subseteq I^+(A)$. Since $p \in I^+(a_1)$, we conclude that $p \in I^+(I^+(A))$. Now let $p \in I^+(I^+(A))$. Then there exists some $q \in I^+(A)$ and a piecewise smooth future-directed timelike curve c_1 that connects q with p . We find

some $a \in A$ and a piecewise smooth future-directed timelike curve c_2 that connects a with q . The concatenation $c_2 \cup c_1$ is a piecewise smooth future-directed timelike curve that connects a with p and hence $p \in I^+(A)$.

Next we show that $I^+(I^+(A)) = I^+(J^+(A))$. Since $I^+(A) \subseteq J^+(A)$ we obviously have that $I^+(I^+(A)) \subseteq I^+(J^+(A))$. Now let $p \in I^+(J^+(A))$. There exist $q \in J^+(A)$ and $a \in A$ such that $a \leq q \ll p$. By Proposition 3.1.7 we obtain that $a \ll p$, which means that $p \in I^+(a) \subseteq I^+(A)$. Part one of the proof shows that $p \in I^+(I^+(A))$.

To see that $I^+(I^+(A)) = J^+(I^+(A))$, first note that the inclusion $I^+(I^+(A)) \subseteq J^+(I^+(A))$ is obvious. For $p \in J^+(I^+(A))$ we find $q \in I^+(A)$ and $a \in A$ such that $a \ll q \leq p$. By Proposition 3.1.7 we obtain that $a \ll p$, hence $p \in I^+(a) \subseteq I^+(A)$. Again, by part one of the proof we conclude that $p \in I^+(I^+(A))$.

Since $I^+(A) \subseteq J^+(A)$ by definition we trivially obtain that $J^+(I^+(A)) \subseteq J^+(J^+(A))$.

We finally show that $J^+(J^+(A)) = J^+(A)$. To this end let $p \in J^+(A)$. We first assume that $p \in A \subseteq J^+(A)$. In this case we obviously obtain that $p \in J^+(J^+(A))$. For $p \in J^+(A) \setminus A$ there exists a piecewise smooth future-directed causal curve c that connects some $a \in A$ with p . We choose an arbitrary $a_1 \in \{c\} \setminus \{a, p\}$ and obtain that $a < a_1 < p$. Thus we have that $p \in J^+(a_1)$ and $a_1 \in J^+(a) \subseteq J^+(A)$. We conclude that $p \in J^+(J^+(A))$. Now let $p \in J^+(J^+(A))$. We assume that $p \notin J^+(A)$. By assumption there exist some $q \in J^+(A)$ and a piecewise smooth future-directed causal curve c_1 that connects q and p . If $q \in A$, we obtain a contradiction, hence there exists a piecewise smooth future-directed causal curve c_2 that connects some $a \in A$ with q . The concatenation $c_2 \cup c_1$ is a piecewise smooth future-directed causal curve that connects a and p and we conclude that $p \in J^+(A)$, which is a contradiction. Thus $p \in J^+(A)$.

An analogous proof establishes the opposite time direction. \square

We now introduce several causality conditions.

Definition 3.1.10. *Let $p \in M$. The chronology condition holds at p if $p \notin I^+(p)$, i.e. there are no closed piecewise smooth future-directed timelike curves through p .*

Definition 3.1.11. *Let $p \in M$. The causality condition holds at p if $J^+(p) \cap J^-(p) = \{p\}$. Hence the causality condition forbids the existence of closed future-directed causal curves.*

Definition 3.1.12. *A space-time (M, g) is called totally vicious if $I^+(p) \cap I^-(p) = M$ for some $p \in M$.*

Proposition 3.1.13. *Let (M, g) be totally vicious, then $I^\pm(A) = J^\pm(A) = M$ for all $A \subseteq M$ with $A \neq \emptyset$.*

Proof. Since the space-time (M, g) is totally vicious, there exists some $p \in M$ such that $I^+(p) \cap I^-(p) = M$. For an arbitrary $q \in M$ we obtain that $q \in I^-(p) = M$ and hence $p \in I^+(q)$. It follows that $M = I^+(p) \subseteq I^+(I^+(q)) = I^+(q)$. We furthermore obtain that $q \in I^+(p) = M$ and hence $p \in I^-(q)$. Thus we have that $M = I^-(p) \subseteq I^-(I^-(q)) = I^-(q)$. Since $I^+(q) \subseteq J^+(q)$ and $I^-(q) \subseteq J^-(q)$ for all $q \in M$ we conclude that

$$I^+(q) = J^+(q) = M \quad \text{and} \quad I^-(q) = J^-(q) = M$$

for all $q \in M$. The proposition now follows from Remark 3.1.4. \square

In general Lorentzian manifolds, it is possible for closed timelike curves to exist.

Definition 3.1.14. *Let $p \in M$. We say that causality (resp. chronology) is violated at p if there exists a closed, non trivial causal (resp. timelike) curve from p to p . The chronology violating set is given by*

$$\{p \in M : p \in I^+(p)\} = \{p \in M : p \in I^-(p)\}$$

and the causality violating set is given by

$$\{p \in M \text{ there is a non-trivial causal curve } \gamma \text{ from } p \text{ to } p\}$$

Remark 3.1.15. Now we can extend Definitions 3.1.10 and 3.1.11 and say that the chronology condition (resp. causality condition) holds on M if the chronology violating set (resp. the causality violating set) is empty. We also say that the space-time is chronological (resp. causal) if the chronological condition (resp. causal condition) holds.

Proposition 3.1.16. *The chronology (resp. causality) violating set of a space-time (M, g) consists of connected components of the form $I^+(p_i) \cap I^-(p_i)$ (resp. $J^+(p_i) \cap J^-(p_i)$), where $p_i \in M$ and $i \in \{1, 2, 3, \dots\}$.*

Proof. We only show the Proposition for the chronology violation. The proof for the causality violation is completely analogous. Let \mathfrak{C} be a connected component of the chronology violating set and let $p \in \mathfrak{C}$. Since \mathfrak{C} is connected we find for any point $q \in \mathfrak{C}$ a continuous curve $\{\gamma\} \subseteq \mathfrak{C}$ which connects p and q . Let $\gamma : [0, a] \rightarrow M$, $\gamma(0) = p$ and $\gamma(a) = q$. We claim that $\{\gamma\} \subseteq I^+(p)$. Since $I^+(p)$ is an open neighbourhood of $p = \gamma(0)$, there is some $t_0 > 0$ such that $\gamma([0, t_0]) \subseteq I^+(p)$. Therefore,

$$s := \sup\{t \in [0, a] : \gamma([0, t]) \subseteq I^+(p)\} > 0.$$

We set $A := \{t \in [0, a] : \gamma([0, t]) \subseteq I^+(p)\}$. Now suppose that $s < a$. Since $I^-(\gamma(s))$ is an open neighbourhood of $\gamma(s)$, there exists some s_0 such that $\gamma(s_0) \in I^-(\gamma(s))$ with $s_0 \in A$. Hence $\gamma(s) \in I^+(\gamma(s_0))$ and $\gamma(s_0) \in I^+(p)$, so $p \ll \gamma(s_0) \ll \gamma(s)$, i.e. $s \in A$. But then $I^+(\gamma(s))$ is an open neighbourhood of $\gamma(s)$, so there is some $t > s$ with $\gamma(t') \in I^+(\gamma(s)) \subseteq I^+(p)$ for all $t' \in [s, t]$, contradicting the definition of s . We conclude that $s = a$, so $q \in I^+(p)$. By the same argument there is a piecewise smooth past-directed timelike curve from q to p . We conclude that $\mathfrak{C} \subseteq I^+(p) \cap I^-(p)$ and since $I^+(p) \cap I^-(p)$ is connected, $\mathfrak{C} = I^+(p) \cap I^-(p)$. Since M is second countable, the claim follows. \square

Proposition 3.1.17. *Let (M, g) be a compact space-time. Then the chronology violating set of M is nonempty.*

Proof. The set $\mathcal{U} := \{I^+(p) : p \in M\}$ forms an open cover for (M, g) . Since (M, g) is a compact space-time by assumption, there exists a finite subcover, that is we find $\{p_1, \dots, p_k\} \in M$ such that $\{I^+(p_1), \dots, I^+(p_k)\}$ covers (M, g) . We may assume that this is the minimal number of such sets covering our space-time. If $p_1 \in I^+(p_i)$ for some $i \neq 1$, we would obtain that $I^+(p_1) \subseteq I^+(I^+(p_i)) = I^+(p_i)$, which is a contradiction. Hence $p_1 \in I^+(p_1)$ and we obtain a piecewise smooth future-directed timelike closed curve through p . \square

Definition 3.1.18. *A space-time (M, g) satisfies the future resp. past distinguishing condition at $p \in M$ if*

$$I^+(q) \neq I^+(p) \quad \text{resp.} \quad I^-(q) \neq I^-(p)$$

for all $q \neq p$.

Remark 3.1.19. A space-time (M, g) is future- (resp. past-) distinguishing if and only if for all points $p, q \in M$ $I^+(p) = I^+(q)$ (resp. $I^-(p) = I^-(q)$) implies $p = q$. Following [1], we furthermore define a space-time (M, g) to be distinguishing if for all points $p, q \in M$, either $I^+(p) = I^+(q)$ or $I^-(p) = I^-(q)$ implies $p = q$, hence a space-time is distinguishing if it is both future-distinguishing and past-distinguishing. Hawking and Ellis (cf. [5]) give an alternative definition of a future- (resp. past-) distinguishing space-time. A space-time should thus be future- (resp. past-) distinguishing at $p \in M$ if every neighbourhood of p contains a neighbourhood of p which no future- (resp. past-) directed piecewise smooth causal curve from p intersects more than once. They don't mention a pure "distinguishing-condition", but define that the strong causality condition holds at $p \in M$ if and only if every neighbourhood of p contains a neighbourhood of p which no piecewise smooth

causal curve intersects more than once. Anyway, the next Proposition will show that all these definitions are equivalent.

Proposition 3.1.20. *Let (M, g) be a space-time and $p \in M$. The following are equivalent.*

- (i) *The future- (or past-)distinguishing condition holds at p .*
- (ii) *For any neighbourhood \mathcal{U} of p there exists a neighbourhood $\mathcal{V} \subseteq \mathcal{U}$ of p such that no piecewise smooth future- (resp. past-)directed causal curve from p intersects \mathcal{V} more than once.*

Proof. (ii) \Rightarrow (i) Let $p \in M$ and let $q \in M \setminus \overline{I^+(p)}$ with $p \neq q$. We choose a neighbourhood \mathcal{U}_q of q that does not intersect $I^+(p)$. Obviously we obtain that $I^+(q) \cap I^+(p) \cap \mathcal{U}_q = \emptyset$. We conclude that $I^+(q) \neq I^+(p)$ and we are finished.

Now let $q \in \overline{I^+(p)}$ and $p \neq q$. Since M is Hausdorff we can find neighbourhoods \mathcal{U}_q and \mathcal{U}_p of q and p such that $\mathcal{U}_q \cap \mathcal{U}_p = \emptyset$. Suppose that, no matter how small we choose \mathcal{U}_p there exists some $s \in I^+(p) \cap I^+(q) \cap \mathcal{U}_p$. If we can derive a contradiction from this assumption, then $I^+(q) \neq I^+(p)$ and we are finished. Since $s \in I^+(q)$ we trivially have that $q \in I^-(s)$ and since $I^-(s)$ is open by Proposition 3.1.34 (i) we find a neighbourhood \mathcal{V}_q of q such that $\mathcal{V}_q \subseteq \mathcal{U}_q \cap I^-(s)$. By assumption $q \in \overline{I^+(p)}$, hence \mathcal{V}_q must intersect $I^+(p)$ and there exists an $r \in \mathcal{V}_q \cap I^+(p)$. Any piecewise smooth future-directed timelike curve γ that connects p and r will leave \mathcal{U}_p , since $\mathcal{V}_q \cap \mathcal{U}_p = \emptyset$. By construction we have that $r \in \mathcal{V}_q \subseteq I^-(s)$ and thus $s \in I^+(r)$, hence there is a piecewise smooth future-directed timelike curve $\tilde{\gamma}$ that connects r and s . The concatenation $\gamma \cup \tilde{\gamma}$ of γ and $\tilde{\gamma}$ defines a piecewise smooth future-directed timelike curve that leaves \mathcal{U}_p and returns to $s \in \mathcal{U}_p$. Summing up, if $s \in I^+(p) \cap I^+(q) \cap \mathcal{U}_p$, we find piecewise smooth future-directed timelike curves leaving and re-entering \mathcal{U}_p which is a contradiction to our assumption, since \mathcal{U}_p above can be chosen arbitrarily small. Therefore we conclude that $I^+(p) \neq I^+(q)$.

(i) \Rightarrow (ii) Let $p \in M$ and let $q \in \overline{I^+(p)}$ with $p \neq q$. We first show that there exists a neighbourhood $\mathcal{V}_{p,q}$ of p such that $\mathcal{V}_{p,q} \cap J^+(q) = \emptyset$. By Lemma 3.1.36 (iii) we obtain that $I^+(q) \subseteq I^+(p)$. By assumption we have that $I^+(p) \neq I^+(q)$, hence there exists some $r_q \in I^+(p) \setminus I^+(q)$. Since $p \in I^-(r_q)$ and $I^-(r_q)$ is open by Proposition 3.1.34 (i) we find a neighbourhood $\mathcal{V}_{p,q}$ of p such that $\mathcal{V}_{p,q} \subseteq I^-(r_q)$. Suppose there is a $p' \in \mathcal{V}_{p,q} \cap J^+(q)$. Then there is a piecewise smooth future-directed causal curve α that connects $q \in J^+(q)$ and $p' \in \mathcal{V}_{p,q}$. Since $\mathcal{V}_{p,q} \subseteq I^-(r_q)$ there exists a piecewise smooth future-directed timelike curve that connects p' and r_q , hence we obtain that

$q \leq p' \ll r_q$ and thus $q \ll r_q$ by Proposition 3.1.7. This implies that $r_q \in I^+(q)$ which is a contradiction to $r_q \in I^+(p) \setminus I^+(q)$ and we conclude that $\mathcal{V}_{p,q} \cap J^+(q) = \emptyset$. Now let \mathcal{N}_p be a normal neighbourhood of p and let \mathcal{U}_p be an open neighbourhood of p with compact closure in \mathcal{N}_p . Let γ be a piecewise smooth future-directed causal curve from p that is leaving \mathcal{U}_p . Let $r \in \overline{\mathcal{U}_p}$ be the first point of γ which is not in \mathcal{U}_p . Then $r \in J^+(p) \subseteq \overline{J^+(p)} = \overline{I^+(p)}$ by Lemma 3.1.36 (ii) and by the above there is a neighbourhood $\mathcal{V}_{p,r}$ of p with $\mathcal{V}_{p,r} \subseteq \mathcal{U}_p$ and $\mathcal{V}_{p,r} \cap J^+(r) = \emptyset$. γ from r to the future is itself in $J^+(r)$, hence if γ has left $\overline{\mathcal{U}_p}$ through r it cannot return to $\mathcal{V}_{p,r}$. By Lemma 3.1.30 (ii) causality in \mathcal{N}_p corresponds to causality in the Minkowski-space, hence γ cannot return to $\mathcal{V}_{p,r}$ within $\overline{\mathcal{U}_p}$ either. \square

Corollary 3.1.21. *The future- (or past-) distinguishing condition implies the causality condition.*

Proof. Immediate from Proposition 3.1.20. \square

Definition 3.1.22. *Let (M, g) be a space-time. The strong causality condition holds at $p \in M$ if for any neighbourhood V of p there is a neighbourhood $U \subseteq V$ of p such that any piecewise smooth causal curve intersects U at most once.*

Remark 3.1.23. It should be mentioned that Definition 3.1.22 implies the following weaker condition, which is often used as a definition of strong causality (cf. [15], section 14, Definition 11). Let (M, g) be a strongly causal space-time and $p \in M$. Then for each neighbourhood \mathcal{V} of p there exists a neighbourhood $\mathcal{U} \subseteq \mathcal{V}$ of p such that every piecewise smooth causal curve with endpoints in \mathcal{U} lies entirely in \mathcal{V} .

Remark 3.1.24. A strongly causal space-time (M, g) is both future- and past-distinguishing, since it is distinguishing by Proposition 3.1.20.

Definition 3.1.25. *A space-time (M, g) is said to be stably causal if there exists a global time function, i.e. a smooth function $t : M \rightarrow \mathbb{R}$ such that $\text{grad}(t)$ is timelike, i.e. $\langle \text{grad}(t), \text{grad}(t) \rangle_{T_p M}(p) < 0$ for all $p \in M$.*

Proposition 3.1.26. *Let (M, g) be a stably causal space-time. Then (M, g) satisfies the strong causality condition.*

Proof. Let $t : M \rightarrow \mathbb{R}$ be a global time function and let $p \in M$. Let \mathcal{N}_p be a normal neighbourhood of p , hence causality in \mathcal{N}_p corresponds to causality in the Minkowski space \mathbb{R}_1^n (cf. Lemma 3.1.30 below). Let

$$N := \{q \in M : t(q) = t(p)\}.$$

Since $\langle \text{grad}(t), \text{grad}(t) \rangle_{T_p M} < 0$ for all $p \in M$, we conclude that N is a hypersurface in M . Furthermore N is achronal (cf. Definition 3.4.1 below), since t is strictly monotonically increasing along every piecewise smooth future-directed timelike curve. We choose some $q_0 \in M$ such that $p \in I^-(q_0)$. Obviously $V := I^-(q_0) \cap N$ is open in N . For each $q \in M$ and for each $\epsilon > 0$ the sets $U_\epsilon(q)$ are defined to be the future causal cones with height ϵ and middle point q . Obviously the $U_\epsilon(q)$ form a fundamental system of neighbourhoods for q . Now let q_ϵ be the tip of $U_\epsilon(q)$, hence $U_\epsilon(q) = J^+(q_\epsilon) \cap \{q' : t(q') \leq t(q_\epsilon) + \epsilon\}$. Let $p' \in V$. Then there exists some $\epsilon > 0$ such that $U_\epsilon(p') \cap N \subseteq V$.

We claim that $U_\epsilon(p') \cap N = J^+(p_\epsilon) \cap N$. Obviously we have that $U_\epsilon(p') \cap N \subseteq J^+(p_\epsilon) \cap N$. We show the converse direction. Suppose there is some $z \in J^+(p_\epsilon) \cap N \setminus (U_\epsilon(p') \cap N)$ which implies that $z \in J^+(p_\epsilon) \setminus (U_\epsilon(p'))$. So $t(z) > t(p_\epsilon) + \epsilon > t(p)$, thus $z \notin N$. This is a contradiction.

For each $p' \in V$ we choose a suitable ϵ as above and set $p'' := p'_\epsilon$ and $W_p := U_\epsilon(p')$.

Finally we define

$$W := I^-(q_0) \cap \{q \in M : t(q) \geq t(p)\} \\ \cup \left(\bigcup_{p \in V} W_p \cap \{q \in M : t(q) \leq t(p)\} \right).$$

Each piecewise smooth future-directed causal curve γ can thus only enter W in $N_- := \{q \in M : t(q) \leq t(p)\}$ and leave it in $N_+ := \{q \in M : t(q) \geq t(p)\}$. Since t increases along every piecewise smooth future-directed causal curve, if any of these curves ever enters and leaves W , then it will not enter again. \square

Definition 3.1.27. *A space-time (M, g) is said to be causally simple if it is distinguishing and $J^\pm(p)$ is closed for every $p \in M$.*

Definition 3.1.28. *Let (M, g) be a space-time. Let U be an open subset of M and let A be an arbitrary subset of M . The future- resp. past-horismos $E^+(A, U)$ resp. $E^-(A, U)$ of A relative to U is defined as*

$$E^+(A, U) := J^+(A, U) \setminus I^+(A, U) \quad \text{resp.} \quad E^-(A, U) := J^-(A, U) \setminus I^-(A, U)$$

It is customary to write $E^+(A)$ resp. $E^-(A)$ for $E^+(A, M)$ resp. $E^-(A, M)$.

Lemma 3.1.29. *Let M be a Lorentzian manifold and $p \in M$. Suppose that $\beta : [0, b] \rightarrow T_p M$ is a piecewise smooth curve starting at $0 \in T_p M$ such that $\alpha := \exp_p \circ \beta$ is causal. Then β remains in a single timecone of $T_p M$.*

Proof. A proof can be found in [15], Chapter 5, Proposition 33. \square

Lemma 3.1.30. *Let $p \in M$ and \mathcal{U} be a normal neighbourhood of p such that $\exp_p : \tilde{\mathcal{U}} \rightarrow \mathcal{U}$ is a diffeomorphism. Then we have*

$$(i) \quad I^+(p, \mathcal{U}) = \exp_p(I^+(0) \cap \tilde{\mathcal{U}}) \text{ and}$$

$$(ii) \quad J^+(p, \mathcal{U}) = \exp_p(J^+(0) \cap \tilde{\mathcal{U}}).$$

Proof. (i) (\subseteq) Let $q \in I^+(p, \mathcal{U})$. Then there is a piecewise smooth future-directed timelike curve $\alpha : [0, 1] \rightarrow M$ in \mathcal{U} from $\alpha(0) = p$ to $\alpha(1) = q$. Since $\beta := \exp_p^{-1} \circ \alpha : [0, 1] \rightarrow T_p M \cap \tilde{\mathcal{U}}$ is a piecewise smooth curve and $\beta(0) = 0$ Lemma 3.1.29 shows that β remains in $I^+(0)$. It follows that $\exp_p^{-1}(q) \in I^+(0)$ and we obtain that $q \in \exp_p(I^+(0) \cap \tilde{\mathcal{U}})$. (\supseteq) Let $x \in I^+(0) \cap \tilde{\mathcal{U}}$ and let $\gamma : [0, 1] \rightarrow \tilde{\mathcal{U}}$, $\gamma(s) := sx$. γ is smooth and remains in $I^+(0) \cap \tilde{\mathcal{U}}$. We show that $c := \exp_p \circ \gamma$ is a future-directed timelike geodesic from p to $\exp_p(x)$ in \mathcal{U} . The map $t \mapsto \exp_p(tx)$ is a geodesic and by the Gauß - Lemma we obtain

$$\begin{aligned} \langle c'(s), c'(s) \rangle &= \langle T_{tx} \exp_p(x), T_{tx} \exp_p(x) \rangle \\ &= \langle x, x \rangle < 0 \end{aligned}$$

for all $s \in [0, 1]$, hence c is timelike. By Proposition 1.1.8 we find a future-directed timelike smooth vector field $X \in \mathfrak{X}(M)$. Let $f : [0, 1] \rightarrow \mathbb{R}$, $f(s) := \langle X(c(s)), c'(s) \rangle$. f is smooth and we obtain that $f(0) = \langle X(c(0)), c'(0) \rangle = \langle X(p), x \rangle < 0$. By continuity of f we conclude that f is strictly negative since otherwise there would be some $s_0 \in [0, 1]$ such that $\langle X(c(s_0)), c'(s_0) \rangle = 0$. Since $c'(s)$ is timelike for all $s \in [0, 1]$, Lemma 1.1.2 implies that $X(c(s_0))$ is spacelike, which is a contradiction. It follows by Lemma 1.1.3 that c is future-directed and we obtain $\exp_p(x) = c(1) \in I^+(p, \mathcal{U})$.

(ii) (\subseteq) Let $q \in J^+(p, \mathcal{U})$ and let $\{q_m\}_m \in \mathbb{N}$ be a convergent sequence in M with $q_m \rightarrow q$ and $q_m \gg q$ for all $m \in \mathbb{N}$. Without loss of generality let $q_m \in \mathcal{U}$ for all $m \in \mathbb{N}$. We obtain that $p \leq q \ll q_m$ and Lemma 3.1.7 implies that $p \ll q_m$ for all $m \in \mathbb{N}$, i.e. $q_m \in I^+(p, \mathcal{U})$ for all $m \in \mathbb{N}$. We have $\exp_p^{-1}(q_m) \in I^+(0) \cap \tilde{\mathcal{U}}$ by (i). Therefore,

$$\begin{aligned} \exp_p^{-1}(q) &= \lim_{m \rightarrow \infty} \exp_p^{-1}(q_m) \in \overline{I^+(0) \cap \tilde{\mathcal{U}}} \\ &\subseteq \overline{I^+(0)} \cap \tilde{\mathcal{U}} = J^+(0) \cap \tilde{\mathcal{U}}. \end{aligned}$$

Since $q \in \mathcal{U}$, we have $\exp_p^{-1}(q) \in \tilde{\mathcal{U}}$, hence $\exp_p^{-1}(q) \in J^+(0) \cap \tilde{\mathcal{U}}$.

(\supseteq) We proceed as in the proof of (\supseteq) in (i). The same arguments show that $c := \exp_p \circ \gamma$ is a causal geodesic. If c is timelike, we can use (i). If c is causal and not timelike, the condition $\langle X(c(s_0)), c'(s_0) \rangle = 0$ gives a contradiction by using Lemma 1.1.2, since $c'(s_0)$ is null and $X(c(s_0))^\perp$ is spacelike. \square

Remark 3.1.31. Let \mathfrak{C} be a convex open set in M . Then for any two points $p, q \in M$ there is a unique geodesic $\sigma_{pq} : [0, 1] \rightarrow \mathfrak{C}$ with $\sigma_{pq}(0) = p$ and $\sigma_{pq}(1) = q$. We set $\overrightarrow{pq} := \sigma'_{pq}(0)$. It turns out that the map $(p, q) \mapsto \overrightarrow{pq}$ is continuous (cf. [15], section 5, Lemma 9).

Lemma 3.1.32. *Let \mathfrak{C} be an open and convex subset of M .*

- (i) *For $p, q \in \mathfrak{C}$, $p \neq q$ we have $q \in I^+(p, \mathfrak{C})$ ($q \in J^+(p, \mathfrak{C})$) if and only if \overrightarrow{pq} is future-directed timelike (\overrightarrow{pq} is future-directed causal).*
- (ii) *$I^+(p, \mathfrak{C})$ is open in \mathfrak{C} and hence open in M .*
- (iii) *$J^+(p, \mathfrak{C})$ is the closure of $I^+(p, \mathfrak{C})$ in \mathfrak{C} .*
- (iv) *The relation \leq is closed on \mathfrak{C} , i.e. for $p_n, p, q_n, q \in \mathfrak{C}$ with $p_n \rightarrow p$ and $q_n \rightarrow q$ and $q_n \in J^+(p_n, \mathfrak{C})$ we have $q \in J^+(p, \mathfrak{C})$.*
- (v) *Let $K \Subset \mathfrak{C}$, i.e. K is compact and $K \subseteq M$ and let $\alpha : [0, b) \rightarrow K$ be a piecewise smooth causal curve. Then we can extend α continuously to $[0, b]$.*

Proof. (i) (\Rightarrow) Let $\alpha : [0, 1] \rightarrow M$ be a piecewise smooth future-directed timelike (causal) curve from $p = \alpha(0)$ to $q = \alpha(1)$. We define $\beta : [0, 1] \rightarrow T_p M$ by $\beta(s) := (\exp_p^{-1} \circ \alpha)(s)$. β is piecewise smooth and we have $\beta(0) = \exp_p^{-1}(\alpha(0)) = 0$. Since $\alpha = \exp_p \circ \beta$ is timelike we obtain that β remains in a single timecone (causal cone) by Lemma 3.1.29. We furthermore obtain that

$$\beta(1) = \exp_p^{-1}(\alpha(1)) = \exp_p^{-1}(q) = \overrightarrow{pq}$$

is timelike (causal) and future-directed. (\Leftarrow) σ_{pq} connects p and q within \mathfrak{C} and is timelike (causal), since $\overrightarrow{pq} = \sigma'_{pq}(0)$ is timelike (causal). The same arguments as in the proof of Lemma 3.1.30 (i) and (ii) show that σ_{pq} is future-directed.

(ii) and (iii) follow directly from Lemma 3.1.30 (i) and (ii), since \exp_p is a homeomorphism on \mathfrak{C} .

(iv) The case $p = q$ is trivial, hence let $p \neq q$ and without loss of generality let $p_n \neq q_n$ for all $n \in \mathbb{N}$. By assumption and (i) we obtain that $\overrightarrow{p_n q_n}$ is future-directed causal for all $n \in \mathbb{N}$. Since the map $\Phi : \mathfrak{C} \times \mathfrak{C} \rightarrow TM$, $(p, q) \mapsto \overrightarrow{pq}$ is continuous (cf. Remark 3.1.31), we find that

$$\begin{aligned} \lim_{n \rightarrow \infty} (\overrightarrow{p_n q_n}) &= \lim_{n \rightarrow \infty} (\Phi(p_n, q_n)) = \Phi(\lim_{n \rightarrow \infty} p_n, \lim_{n \rightarrow \infty} q_n) \\ &= \Phi(p, q) = \overrightarrow{pq}. \end{aligned}$$

We show that \vec{pq} is future-directed causal. To this end let X be some future-directed timelike smooth vector field. Obviously we obtain that $\langle \vec{p_n q_n}, X \rangle < 0$ for all $n \in \mathbb{N}$, hence $\langle \vec{pq}, X \rangle \leq 0$. If $\langle \vec{pq}, X \rangle = 0$, we would obtain that \vec{pq} is spacelike which is a contradiction. So $\langle \vec{pq}, X \rangle < 0$ and \vec{pq} is future-directed causal. By (i) it follows that $q \in J^+(p, \mathfrak{C})$.

(v) Let $\{s_i\}_{i \in \mathbb{N}}$ be a sequence with $s_i \rightarrow b$ and $0 \leq s_1 < s_2 < \dots$. Since K is a compact set we obtain that $\{\alpha(s_i)\}_{i \in \mathbb{N}}$ has at least one accumulation point. We show that there is only one accumulation point p . To this end let $q \neq p$ be another accumulation point. We choose a subsequence $\{s_{i_k}\}_{k \in \mathbb{N}}$ such that $s_{i_k} < s_{i_{k+1}}$ for all $k \in \mathbb{N}$ and $\alpha(s_{i_{2k}}) \rightarrow p$ and $\alpha(s_{i_{2k+1}}) \rightarrow q$. Then we have $\alpha(s_{i_{2k}}) \leq \alpha(s_{i_{2k+1}}) \leq \alpha(s_{i_{2k+2}})$ for all $k \in \mathbb{N}$. (iv) implies that $p \leq q \leq p$ and by (i) we see that \vec{pq} is future- and past-directed. Let $X \in \mathfrak{X}(M)$ be a future-directed timelike vector field. It follows that $\langle X, \vec{pq} \rangle \geq 0$ and $\langle X, \vec{pq} \rangle \leq 0$, hence $\langle X, \vec{pq} \rangle = 0$. This implies that \vec{pq} is spacelike, which is a contradiction to \vec{pq} causal. It follows that $p = q$. Hence $\alpha(s_i) \rightarrow p$ and we can extend α continuously to $[0, b]$. \square

Lemma 3.1.33. *The relation \ll is open, that is for $p \ll q$ there exist neighbourhoods \mathcal{U} of p and \mathcal{V} of q such that $p' \ll q'$ holds for all $p' \in \mathcal{U}$ and for all $q' \in \mathcal{V}$.*

Proof. By assumption there is a piecewise smooth future-directed timelike curve $\alpha : [0, 1] \rightarrow M$ that connects $p = \alpha(0)$ and $q = \alpha(1)$. Let \mathfrak{C} and \mathfrak{C}' be convex neighbourhoods of q and p . Let $\tilde{q} \in \mathfrak{C}$ be a point on α in \mathfrak{C} and let \tilde{p} be a point on α in \mathfrak{C}' . By Lemma 3.1.32 (ii) the sets $\mathcal{U} := I^-(\tilde{p}, \mathfrak{C}')$ and $\mathcal{U}' := I^+(\tilde{q}, \mathfrak{C})$ are open in M and \mathcal{U} and \mathcal{V} have the required property. \square

Proposition 3.1.34. *Let (M, g) be a space-time and $A \subseteq M$. Then*

(i) *the interior of $J^+(A)$ equals $I^+(A)$, in particular $I^+(A)$ is open.*

(ii) *$J^+(A)$ is a subset of the closure of $I^+(A)$.*

(iii) *$J^+(A)$ equals the closure of $I^+(A)$ if and only if $J^+(A)$ is closed.*

Analogous results hold for $I^-(A)$ and $J^-(A)$

Proof. (i) Let $A \neq \emptyset$. Obviously we have $I^+(A) \subseteq J^+(A)$ and since $I^+(A)$ is open by Lemma 3.1.33, we obtain that $I^+(A) \subseteq J^+(A)^\circ$. To prove the converse let $q \in J^+(A)^\circ$ and let $\mathfrak{C} \subseteq J^+(A)^\circ$ be a convex neighbourhood of q . By Lemma 3.1.30 (i) the set $I^-(q, \mathfrak{C})$ is not empty, thus we find that $I^-(q, \mathfrak{C}) \subseteq \mathfrak{C} \subseteq J^+(A)^\circ$. This shows that there is some $p \in J^+(A) \cap I^-(q, \mathfrak{C})$. It follows that $p \in J^+(A)$ and $q \in I^+(p)$, hence $q \in I^+(J^+(A)) = I^+(A)$, where we have used Proposition 3.1.9. We obtain that $J^+(A)^\circ \subseteq I^+(A)$.

(ii) It suffices to show that $J^+(p) \subseteq \overline{I^+(p)}$, since then we obtain $J^+(A) = \bigcup_{p \in A} J^+(p) \subseteq \bigcup_{p \in A} \overline{I^+(p)} \subseteq \overline{I^+(A)}$. To this end let $q \in J^+(p)$. For $p = q$ we take an arbitrary piecewise smooth future-directed timelike curve α through $p = \alpha(0)$ and see that $\alpha(\frac{1}{n}) \in I^+(p)$ for all $n \in \mathbb{N}$, hence $p \in \overline{I^+(p)}$. Now let $p < q$ and let α be a piecewise smooth future-directed causal curve that connects p and q and let \mathfrak{C} be a convex neighbourhood of q . Let q^- be a point of α in $J^-(q, \mathfrak{C})$. Then we obtain that $q \in J^+(q^-, \mathfrak{C}) \subseteq \overline{I^+(q^-, \mathfrak{C})}$, where we have used Lemma 3.1.32. Since we have $q^- \in J^+(p)$, we can use Proposition 3.1.9 to show that $I^+(q^-, \mathfrak{C}) \subseteq I^+(J^+(p)) = I^+(p)$, hence we have shown that $q \in \overline{I^+(p)}$.

(iii) Let $J^+(A) = \overline{I^+(A)}$. Then $J^+(A)$ is obviously closed. Now let $J^+(A)$ be a closed set. By (ii) we obtain that $J^+(A) \subseteq \overline{I^+(A)}$. $I^+(A) \subseteq J^+(A)$ and since $J^+(A)$ is closed we finally have that $\overline{I^+(A)} = J^+(A)$. \square

Proposition 3.1.35. *A distinguishing space-time (M, g) is causally simple if and only if*

$$\partial J^+(p) = E^+(p) \quad \text{and} \quad \partial J^-(p) = E^-(p)$$

for all $p \in M$.

Proof. We first assume that the space-time (M, g) is causally simple, that is in particular that $J^+(p)$ and $J^-(p)$ are closed subsets of M for every $p \in M$. Then

$$\overline{J^\pm(p)} = J^\pm(p)$$

and we obtain $\partial J^\pm(p) = \overline{J^\pm(p)} \setminus J^\pm(p)^\circ = J^\pm(p) \setminus I^\pm(p) = E^\pm(p)$ (where we have used Proposition 3.1.34(i)). Conversely let $\partial J^+(p) = E^+(p)$ and $\partial J^-(p) = E^-(p)$. We obtain $J^\pm(p) \setminus I^\pm(p) = E^\pm(p) = \partial J^\pm(p) = \overline{J^\pm(p)} \setminus J^\pm(p)^\circ = \overline{J^\pm(p)} \setminus I^\pm(p)$ (for the last equality see Proposition 3.1.34(i)), hence $\overline{J^\pm(p)} = J^\pm(p)$ and we are finished. \square

Lemma 3.1.36. *Let (M, g) be an arbitrary space-time and let A be a nonempty subset of M . Then*

(i) $I^\pm(\overline{A}) = I^\pm(A)$,

(ii) $\overline{J^\pm(A)} = \overline{I^\pm(A)}$,

(iii) $\overline{I^\pm(A)} = \{p \in M : I^\pm(p) \subseteq I^\pm(A)\}$ and

(iv) $\partial J^\pm(A) = \partial I^\pm(A)$.

Proof. (i) Obviously we have that $I^+(A) \subseteq I^+(\bar{A})$. To show that $I^+(\bar{A}) \subseteq I^+(A)$ let $q \in I^+(\bar{A})$. Then there exists some $p \in \bar{A}$ such that $p \ll q$. Since \ll is an open relation by Lemma 3.1.33, we find neighbourhoods \mathcal{U} and \mathcal{U}' of p and q such that $\tilde{p} \ll \tilde{q}$ for all $(\tilde{p}, \tilde{q}) \in \mathcal{U} \times \mathcal{U}'$. We only need to choose $\tilde{p} \in A$ and $\tilde{q} = q$.

(ii) By Proposition 3.1.34 (ii) $J^+(A)$ is a subset of the closure of $I^+(A)$, hence $\overline{J^+(A)} \subseteq \overline{I^+(A)} = \overline{I^+(A)}$. By definition of the chronological and causal future we have that $I^+(A) \subseteq J^+(A)$ and therefore $\overline{I^+(A)} \subseteq \overline{J^+(A)}$.

(iii) We set $\mathfrak{V} := \{p \in M : I^+(p) \subseteq I^+(A)\}$ and first show that $\mathfrak{V} \subseteq \overline{I^+(A)}$. To this end let $p \in \mathfrak{V}$. We choose an arbitrary neighbourhood \mathcal{U} of p and obtain that $\mathcal{U} \cap I^+(p) \neq \emptyset$, hence $\mathcal{U} \cap I^+(A) \neq \emptyset$. Since \mathcal{U} is an arbitrary neighbourhood of p , we find that $p \in \overline{I^+(A)}$, thus $\mathfrak{V} \subseteq \overline{I^+(A)}$. Now let $p \in \overline{I^+(A)}$. We show that $I^+(p) \subseteq I^+(A)$. To this end we assume that $I^+(p) \not\subseteq I^+(A)$. Then there is some $q \in I^+(p)$ with $q \notin I^+(A)$. Since $p \in I^-(q)$ and $I^-(q)$ is open by Proposition 3.1.34 (i) we obtain that $I^-(q) \cap I^+(A) \neq \emptyset$. Let $z \in I^-(q) \cap I^+(A)$. Obviously we have that $z \ll q$ and $a \ll z$ for some $a \in A$. We conclude that $a \ll z \ll q$ and thus $a \ll q$ implying that $q \in I^+(A)$. This is a contradiction, hence $I^+(p) \subseteq I^+(A)$.

(iv) By (ii) we have that $J^+(A)^\circ \cup \partial J^+(A) = \overline{J^+(A)} = \overline{I^+(A)} = I^+(A)^\circ \cup \partial I^+(A)$. Since $I^+(A)$ is open by Proposition 3.1.34 (i) we see that $I^+(A) = I^+(A)^\circ$. We furthermore have $J^+(A)^\circ = I^+(A)$ (again by Proposition 3.1.34 (i)) and therefore we obtain that $\partial J^+(A) = \partial I^+(A)$. \square

Definition 3.1.37. A space-time (M, g) is said to be reflecting if $I^+(q) \subseteq I^+(p)$ implies $I^-(p) \subseteq I^-(q)$ and $I^-(p) \subseteq I^-(q)$ implies $I^+(q) \subseteq I^+(p)$ for all $p, q \in M$. (M, g) is said to be strictly reflecting, if $I^+(q) \subsetneq I^+(p)$ implies $I^-(p) \subsetneq I^-(q)$ and $I^-(p) \subsetneq I^-(q)$ implies $I^+(q) \subsetneq I^+(p)$ for all $p \neq q \in M$.

Proposition 3.1.38. Let (M, g) be a space-time that is causally simple. Then it is stably causal.

Proof. See [21], Proposition 2.25. \square

Definition 3.1.39. A space-time (M, g) is said to be globally hyperbolic if it satisfies the strong causality condition and if

$$J(p, q) := J^+(p) \cap J^-(q)$$

is compact for all $p, q \in M$.

Remark 3.1.40. We can extend Definition 3.1.39 for arbitrary subsets H of M by additionally requiring that $J^+(p, q)$ is a subset of H for all $p, q \in H$. Thus a subset H of M is called globally hyperbolic if it is strongly causal and $J(p, q)$ is a compact subset of H for all $p, q \in H$.

Remark 3.1.41. The next Proposition will complete the causal chain by showing that globally hyperbolic space-times are automatically causally simple. Hence global hyperbolicity is the 'strongest' causality condition.

Proposition 3.1.42. *Let (M, g) be a globally hyperbolic space-time. Then (M, g) is causally simple.*

Proof. By Definition 3.1.39 a globally hyperbolic space-time satisfies the strong causality condition and hence is distinguishing by Remark 3.1.24. We have to show that $J^+(p)$ and $J^-(p)$ are closed for every $p \in M$. We only show this for the former case. Suppose there exists a $q \in \overline{J^+(p)} \setminus J^+(p)$ for some $p \in M$. We show that this leads to a contradiction. We choose an $r \in I^+(q)$. Then obviously we have $q \in I^-(r)$. We show that $r \in I^+(p)$. To this end we choose a sequence $\{q_n\}_{n \in \mathbb{N}}$ with $q_n \in J^+(p)$ for all $n \in \mathbb{N}$ that converges to q . By Proposition 3.1.34 (i) $I^-(r)$ is an open neighbourhood of q and thus we find $p \leq q_n$ and $q_n \ll r$ for sufficiently large $n \in \mathbb{N}$. By Proposition 3.1.7 we obtain $p \ll r$. Consequently we have $q \in \overline{J^+(p) \cap J^-(r)} \setminus J^+(p) \cap J^-(r)$. But this is impossible since $J^+(p) \cap J^-(r)$ is compact and hence closed. \square

Theorem 3.1.43. *All causality conditions introduced in this chapter are given in increasing order of restriction. This means we have the following implications for an arbitrary space-time (M, g) . "M is globally hyperbolic" \Rightarrow "M is causally simple" \Rightarrow "M is stably causal" \Rightarrow "M is strongly causal" \Rightarrow "M is distinguishing" \Rightarrow "M is causal" \Rightarrow "M is chronological".*

Proof. The statements follow directly from Proposition 3.1.42, Proposition 3.1.38, Proposition 3.1.26, Remark 3.1.24, Corollary 3.1.21 and Definitions 3.1.11 and 3.1.10. \square

3.2 Energy conditions

Definition 3.2.1. *Let (M, g) be a space-time and let $I := [a, b] \subseteq \mathbb{R}$, where $a, b \in \mathbb{R}$. A timelike geodesic $c : I \rightarrow M$ satisfies the generic condition if there is an $s_0 \in I$ such that the curvature endomorphism*

$$R : N(c(s_0)) \rightarrow N(c(s_0)) \quad N(c(s_0)) \ni v \mapsto R(v, c'(s_0))c'(s_0)$$

is not identically zero.

Definition 3.2.2. *Let (M, g) be an arbitrary space-time and let $[a, b] =: I \subseteq \mathbb{R}$ for $a, b \in \mathbb{R}$. Let $\gamma : I \rightarrow M$ be a piecewise smooth curve in M . We say that a point $p \in M$ is the endpoint of γ corresponding to $s = b$ if*

$$\lim_{s \rightarrow b} \gamma(s) = p$$

If γ is a piecewise smooth future- (resp. past-)directed causal curve with endpoint p corresponding to $s = b$, the point p is called a future- (resp. past-)endpoint of γ . A piecewise smooth causal curve is said to be future (resp. past) inextendible if it has no future- (resp. past-)endpoint. We say that a piecewise smooth causal curve is inextendible if it is both future and past-inextendible.

Definition 3.2.3. A space-time (M, g) satisfies the generic condition if each inextendible timelike geodesic satisfies the generic condition.

Definition 3.2.4. A space-time (M, g) satisfies the strong energy condition if $\text{Ric}(v, v) \geq 0$ for all causal tangent vectors $v \in TM$.

Lemma 3.2.5. Let $s \mapsto A(s)$ be a differential map taking values in $n \times n$ -matrices such that $A(s)^{-1}$ exists for all $s \in I \subseteq \mathbb{R}$. Then

$$\det'(A(s))(A'(s)) = \det(A(s))\text{tr}(A(s)^{-1}A'(s))$$

Proof. We have the following chain of equations:

$$\begin{aligned} & \det'(A(s))(A'(s)) \\ &= \frac{d}{dt}\Big|_{t=0} \det(A(s) + tA'(s)) \\ &= \frac{d}{dt}\Big|_{t=0} \det(A(s)(E(s) + tA(s)^{-1}A'(s))) \\ &= \frac{d}{dt}\Big|_{t=0} \det(tA(s)\left(\frac{E(s)}{t} + A(s)^{-1}A'(s)\right)) \\ &= \frac{d}{dt}\Big|_{t=0} \det(tA(s))\det\left(\frac{E(s)}{t} + A(s)^{-1}A'(s)\right) \\ &= \frac{d}{dt}\Big|_{t=0} t^n \det(A(s))\left(\frac{1}{t^n} + \frac{1}{t^{n-1}}\text{tr}(A(s)^{-1}A'(s)) + \dots + \det(A(s)^{-1}A'(s))\right) \\ &= \frac{d}{dt}\Big|_{t=0} \det(A(s))(1 + t\text{tr}(A(s)^{-1}A'(s)) + \dots + t^n \det(A(s)^{-1}A'(s))) \\ &= \det A(s)\text{tr}(A(s)^{-1}A'(s)) \end{aligned}$$

□

Lemma 3.2.6. Let A be an arbitrary $\binom{1}{1}$ -tensor field such that $A(s)^{-1}$ exists for all $s \in I \subseteq \mathbb{R}$. Then

$$\frac{d}{ds}(\det A(s))(\det A(s))^{-1} = \text{tr}\left(\frac{\nabla A}{ds}(s) \circ A(s)^{-1}\right)$$

Proof. By Remark A.3.7 we obtain a frame field for A such that

$$A(s) = A_j^i(s)E_i(s) \otimes E^{*j}(s).$$

By Proposition 2.3.3 all coefficient functions A_j^i are smooth. We differentiate the local representation and find

$$\begin{aligned} \frac{\nabla A}{ds}(s) &= \frac{dA_j^i}{ds}(s)E_i(s) \otimes E^{*j}(s) + A_j^i(s)\frac{\nabla}{ds}(E_i(s) \otimes E^{*j}(s)) \\ &= \frac{dA_j^i}{ds}(s)E_i(s) \otimes E^{*j}(s), \end{aligned}$$

where $A_j^i(s)\frac{\nabla}{ds}(E_i(s) \otimes E^{*j}(s)) = 0$ since $E_i \in \mathfrak{X}(M)$ and $E^{*j} \in \Omega^1(M)$ are parallel. We see that

$$\begin{aligned} \frac{d}{ds}(\det(A(s))) &= \det'(A(s))\left(\left(\frac{dA_j^i}{ds}(s)\right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}\right) \\ &= \det'(A(s))\frac{\nabla A}{ds}(s), \end{aligned}$$

hence the induced covariant derivation and the 'usual' derivation coincide in a frame field. Now the statement follows from Lemma 3.2.5. \square

Proposition 3.2.7. *Let $c : I \rightarrow M$ be an inextendible timelike geodesic as in Definition 3.2.2 that satisfies $\text{Ric}(c'(s), c'(s)) \geq 0$ for all $s \in I$. Let A be a Lagrangian tensor field along c as in Definition 2.4.2. Suppose that the expansion $\theta(s) := \text{tr}\left(\left(\frac{\nabla A}{ds}\right)(s) \circ A^{-1}(s)\right)$ (cf. Definition 2.5.1) has a negative (resp. positive) value $\theta_1 := \theta(s_1)$ at $s_1 \in I$. Then $\det(A(s)) = 0$ for some s in the interval $[s_1, s_1 - \frac{(n-1)}{\theta_1}]$ (resp. some s in the interval $[s_1 - \frac{(n-1)}{\theta_1}, s_1]$), provided that $s \in I$. Here, n denotes the dimension of the space-time, i.e. $n := \dim(M)$.*

Proof. By Lemma 3.2.6 θ remains finite on any compact interval where $\det(A(s)) \neq 0$. Thus if we can show that $|\theta| \rightarrow \infty$ as $s \rightarrow s_0$, then necessarily $\det(A(s_0)) = 0$.

For our further calculations we set

$$t_1 := \frac{n-1}{\theta_1}.$$

Since $\text{Ric}(c'(s), c'(s)) \geq 0$ by assumption and $\text{tr}((\sigma(s))^2) \geq 0$ by Remark 2.5.4 for all $s \in I$, the vorticity-free Raychaudhuri-equation (cf. Proposition 2.5.6) yields the inequality

$$\frac{d\theta(s)}{ds} \leq -\frac{(\theta(s))^2}{n-1}$$

and hence

$$\frac{d\theta(s)}{ds} < 0.$$

This means that θ is strictly monotonically decreasing.

We first assume that $\theta_1 < 0$, thus $t_1 < 0$. Let $s \in [s_1, s_1 - t_1)$. We obtain that $s_1 \leq s < s_1 - t_1$ and hence $s + t_1 - s_1 < 0$. Since θ is strictly monotonically decreasing and since $s_1 \leq s$ we obtain that $\theta(s) \leq \theta_1 < 0$ for all $s \in [s_1, s_1 - t_1)$.

We integrate the above equation and obtain

$$\int_{\theta(s_1)}^{\theta(s)} \frac{1}{\tilde{\theta}^2} d\tilde{\theta} \leq -\frac{(s - s_1)}{n - 1} = \frac{s_1 - s}{n - 1}$$

and hence

$$-\left(\frac{1}{\theta(s)} - \frac{1}{\theta_1}\right) \leq \frac{s_1 - s}{n - 1}.$$

This means that

$$t_1 - \frac{(n - 1)}{\theta(s)} \leq s_1 - s$$

implying that $\frac{s+t_1-s_1}{n-1} \leq \frac{1}{\theta(s)}$ and finally

$$\theta(s) \leq \frac{n - 1}{s + t_1 - s_1} \rightarrow -\infty$$

as $s \rightarrow s_1 - t_1$. Since θ is strictly monotonically decreasing, the divergence $\lim_{s \rightarrow s_0} \theta(s) = -\infty$ also implies $\lim_{s \rightarrow s'} \theta(s) = -\infty$ for all $s' \geq s$. Hence there is some $s' \in [s_1, s_1 - t_1]$ such that $\lim_{s \rightarrow s'} \theta(s) = -\infty$.

Now suppose that $\theta_1 > 0$ and let $s \in [s_1 - t_1, s_1]$. Since θ is strictly monotonically decreasing, we obtain that $\theta_1 \leq \theta(s)$, so $\theta(s) > 0$. We integrate the above equation and obtain that

$$\int_{\theta(s)}^{\theta(s_1)} \frac{1}{\tilde{\theta}^2} d\tilde{\theta} \leq -\frac{(s_1 - s)}{n - 1} = \frac{s - s_1}{n - 1}.$$

Hence we conclude that $-\frac{1}{\theta_1} + \frac{1}{\theta} \leq \frac{s-s_1}{n-1}$ and thus $\frac{1}{\theta} \leq \frac{s+t_1-s_1}{n-1}$. We finally have that $\theta \geq \frac{n-1}{s+t_1-s_1} \rightarrow +\infty$ as $s \rightarrow s_1 - t_1$. Again, there is some $s'' \in [s_1 - t_1, s_1]$ with $\lim_{s \rightarrow s''} \theta(s) = \infty$. \square

Remark 3.2.8. The proof of the previous Proposition also holds for $I = \mathbb{R}$, i.e. if the geodesic becomes complete.

Remark 3.2.9. Let (M, g) be an arbitrary space-time and let $c : [a, b] \rightarrow M$ be a geodesic. Recall that two points $c(s_1)$ and $c(s_2)$ with $s_1 \neq s_2$ are conjugated along c if there is a non-vanishing Jacobian vector field J along c with $J(a) = 0$ and $J(b) = 0$. Furthermore we have the following

Proposition 3.2.10. *Let (M, g) be a space-time and let $c : [0, b] \rightarrow M$ be a geodesic starting at $p \in M$. Then the following statements are equivalent.*

- (i) $c(b)$ is a conjugate point of $p = c(0)$ along c .
- (ii) There is a nontrivial variation x of c through geodesics starting at p such that $x_v(b, 0) = 0$.
- (iii) The exponential map $\exp_p : T_p M \rightarrow M$ is singular at $bc'(0)$, that is, there is a nonzero tangent vector $z \in T_{bc'(0)} T_p M$ such that $T_{bc'(0)} \exp_p(z) = 0$.

Proof. See [15], section 10, Proposition 10. □

Our next aim is to show that every timelike geodesic in a space-time (M, g) that satisfies the generic condition and the strong energy condition has to be incomplete or else has to have a pair of conjugate points. We will follow [1].

Lemma 3.2.11. *Let (M, g) be a space-time and let $c : [a, b] \rightarrow M$ be a timelike geodesic with $c(a) = p$ and let $\varphi, \psi \in L(N(p))$ such that the condition*

$$\ker(\varphi) \cap \ker(\psi) = \{0\}$$

holds. Then there is a unique Jacobian tensor field A on $V^\perp(c)$ with $A(a) = \varphi$ and $\frac{\nabla A}{ds}(a) = \psi$. In particular if we set $\varphi := 0$, we obtain a unique Lagrangian tensor field A on $V^\perp(c)$ with $A(a) = 0$ and $\frac{\nabla A}{ds}(a) = \psi$ by Proposition 2.4.5.

Proof. We can choose a parallel $\binom{1}{0}$ -frame field $\{E_1, \dots, E_{n-1}\}$ along c such that $\{E_1(t), \dots, E_{n-1}(t)\}$ is a basis for $N(c(t))$. In particular all E_i are spacelike for $i = 1, \dots, n-1$. Let J_1, \dots, J_{n-1} denote the unique Jacobian vector fields along c with $J_i(a) = \varphi(E_i(a))$ and $\frac{\nabla}{ds} J_i(a) = \psi(E_i(a))$. Now let A be the unique $\binom{1}{1}$ -tensor field along c with $A(t) : N(c(t)) \rightarrow N(c(t))$ and $A(t)E_i(t) = J_i(t)$. Then we obtain that $(A'' + RA)(t)E_i(t) = (AE_i)''(t) + RA(t)E_i(t) = J_i'' + RJ_i(t) = 0$ for all $i = 1, \dots, n-1$ and all $t \in [a, b]$. So $A'' + RA = 0$ and A satisfies the first condition of a Jacobian tensor field. In particular A has the required initial values, since $A(a)E_i(a) = J_i(a) = \varphi(a)E_i(a)$, hence $A(a) = \varphi$ and $(\frac{\nabla}{ds} A(a))E_i(a) = \frac{\nabla}{ds} J_i(a) = \psi(a)E_i(a)$, hence $\frac{\nabla}{ds} A(a) = \psi$.

It remains to show that $\ker(A(t)) \cap \ker(\frac{\nabla}{ds}A(t)) = \{0\}$ for all $t \in [a, b]$. To this end we suppose that there is some $t \in [a, b]$ and some $0 \neq v \in N(c(t))$ such that $A(t)v = 0$ and $(\frac{\nabla}{ds}A(t))v = 0$. We can use the parallel $\binom{1}{0}$ -frame field and obtain that $v = \sum_{i=1}^{n-1} \lambda_i E_i(t)$. Let V be the parallel vector field along c with $V(a) = v$, hence $V(s) = \sum_{i=1}^{n-1} \lambda_i E_i(s)$. We show that $J := AV$ is a Jacobian vector field. We have that $J'' = A''V = -RAV = -R_{AV}\gamma', \gamma' = -R_{J\gamma'}\gamma'$. By assumption we obtain that $J(t) = (AV)(t) = 0$ and $\frac{\nabla}{ds}J(t) = \frac{\nabla}{ds}(AV)(t) = 0$, hence $AV = 0$. We finally arrive at

$$0 = (AV)(a) = \sum_{i=1}^{n-1} \lambda_i A(a)E_i(a) = \sum_{i=1}^{n-1} \lambda_i \varphi(a)E_i(a) = \varphi(V(a))$$

and

$$0 = \frac{\nabla}{ds}(AV)(a) = \sum_{i=1}^{n-1} \lambda_i (\frac{\nabla}{ds}A(a))E_i(a) = \sum_{i=1}^{n-1} \lambda_i \psi(a)E_i(a) = \psi(V(a)).$$

We conclude that $0 \neq v = V(a) \in \ker(\varphi) \cap \ker(\psi)$, which is a contradiction to our assumption.

We finally have to prove uniqueness. It suffices to show that $A'' + RA = 0$ and $A(0) = 0$ and $\frac{\nabla}{ds}A(0) = 0$ implies that $A = 0$. Now let $J_i := AE_i$ for $i = 1, \dots, n-1$. Obviously J_i is a Jacobian vector field that satisfies $J_i(0) = 0$ and $\frac{\nabla}{ds}J_i(0) = 0$. This means that $J_i = 0$ for all $i = 1, \dots, n-1$ and hence $A = 0$. \square

Remark 3.2.12. The proof of Lemma 3.2.11 shows that for arbitrary $\varphi, \psi \in L(N(c(s)))$ there exists a unique $\binom{1}{1}$ -tensor field along a geodesic c with $A'' + RA = 0$ and $A(0) = \varphi$ and $\frac{\nabla}{ds}A(0) = \psi$. Obviously A satisfies the first condition of a Jacobian tensor field along a geodesic c . Therefore we will call such tensor fields pseudo Jacobian tensor fields .

Lemma 3.2.13. *Let (M, g) be a space-time and let $c : [a, b] \rightarrow M$ be a timelike geodesic without conjugate points. Then there is a unique $\binom{1}{1}$ -tensor field A on $V^\perp(c)$ (cf. Definition 2.2.6) which satisfies the differential equation*

$$A'' + RA = 0$$

with given boundary conditions $A(a)$ and $A(b)$.

Proof. Let \mathfrak{T} denote the vector space of all $\binom{1}{1}$ -tensor fields A on $V^\perp(c)$ with $A'' + RA = 0$ and let $L(N(c(s)))$ be the set of all linear endomorphisms $\varphi : N(c(s)) \rightarrow N(c(s))$, where $s \in [a, b]$. We define

$$\phi : \mathfrak{T} \rightarrow L(N(c(a))) \times L(N(c(b)))$$

by

$$\phi : A \mapsto (A(a), A(b)).$$

Obviously ϕ is a linear transformation. We find that

$$\dim(\mathfrak{T}) = 2(n-1)^2 = \dim(L(N(c(a)))) + \dim(L(N(c(b)))),$$

where we have used Remark 3.2.12. Hence it suffices to show that ϕ is injective. Then we obtain that ϕ is an isomorphism and we have established the existence of a unique solution A . To this end we assume that $\phi(A) = (A(a), A(b)) = (0, 0)$. Let Y be an arbitrary parallel vector field along c . We define $J(s) := A(s)Y(s)$ and find that J is a Jacobian vector field, since $J'' = A''Y = -RAY = -R_{AY}\gamma', \gamma' = -R_{J\gamma'}\gamma'$. We furthermore obtain that $J(a) = 0$ and $J(b) = 0$. Since there are no conjugate points along c we obtain that $J = 0$. Y was an arbitrary parallel vector field along c , therefore we conclude that $A = 0$. This shows that ϕ is one-to-one. \square

Remark 3.2.14. Let (M, g) be a space-time and let $c : [s_1, \infty) \rightarrow M$ be a timelike geodesic without conjugate points. We choose an arbitrary $t \in (s_1, \infty)$. By Lemma 3.2.13 we find a unique $\binom{1}{1}$ -tensor field D_t on $V^\perp(c)$ that satisfies

$$D_t'' + RD_t = 0$$

with boundary conditions $D_t(s_1) = E$ and $D_t(t) = 0$, where E denotes the $(n-1) \times (n-1)$ unit-matrix and 0 is the $(n-1) \times (n-1)$ null-matrix. Since $D_t(s_1) = E$, we have $\ker(D_t(s_1)) \cap \ker(\frac{\nabla D_t}{ds}(s_1)) = \{0\}$ and we show that D_t is a Jacobian tensor field. We have to show that $\ker(D_t(s_0)) \cap \ker(\frac{\nabla D_t}{ds}(s_0)) = \{0\}$ for all s_0 . We assume that there is some $0 \neq v \in \ker(D_t(s_0)) \cap \ker(\frac{\nabla D_t}{ds}(s_0))$. Let V be the parallel vector field along c with $V(s_0) = v$ and define $J(s) := D_t(s)V(s)$. J is a Jacobian vector field, since $J'' = A''V = -RAV = -R_{AV}\gamma', \gamma' = -R_{J\gamma'}\gamma'$ holds. By assumption we obtain that $J(s_0) = D_t(s_0)V(s_0) = 0$ and $\frac{\nabla J}{ds}(s_0) = (\frac{\nabla D_t}{ds}(s_0))V(s_0) = 0$, hence $J = 0$ and we finally conclude that $0 = D_t(s_1)V(s_1) = EV(s_1) = V(s_1) \neq 0$, which is a contradiction. By Proposition 2.4.5 we obtain, that D_t is a Lagrangian tensor field, since $D_t(t) = 0$.

Lemma 3.2.15. Let (M, g) be a space-time and let $c : [s_1, \infty) \rightarrow M$ be a timelike geodesic without conjugate points. Let A be the unique Lagrangian tensor field on $V^\perp(c)$ with initial conditions $A(s_1) = 0$ and $\frac{\nabla A}{ds}(s_1) = E$. Then for each $t \in (s_1, \infty)$ the unique Lagrangian tensor field D_t on $V^\perp(c)$ with $D_t(s_1) = E$ and $D_t(t) = 0$ satisfies the equation

$$D_t(s) = A(s) \int_s^t P_{(1,1;\tau,s)}(c)((A^*A)^{-1}(\tau))d\tau$$

for all $s \in (s_1, t]$, where $P_{(1,1;\tau,s)}(c)$ is the $\binom{1}{1}$ -parallel transport along c (cf. Definition A.1.4). Moreover, $D_t(s)$ is nonsingular for all $s \in (s_1, t)$, i.e. $D_t(s)(v) = 0$ for some $s \in (s_1, t]$ implies that $v = 0$.

Proof. We set $X(s) := A(s) \int_s^t P_{(1,1;\tau,s)}(c)((A^*A)^{-1}(\tau))d\tau$ and by Lemma 3.2.11 we have to show that $X'' + RX = 0$, $X(t) = D_t(t) = 0$ and $\frac{\nabla X}{ds}(t) = \frac{\nabla D_t}{ds}(t)$. We first show that $A(s)^{-1}$ exists for all $t \neq s_1$. We have to show that $\ker(A(t)) = 0$ for all $t \neq s_1$. This implies that $A(t)$ is one-to-one and hence bijective. We assume that there is some $t \neq s_1$ and some $v \neq 0$ with $A(t)v = 0$. Let V be the parallel vector field along c with $V(t) = v$. We define $J := AV$. J is a Jacobian vector field, since $J'' = A''V = -RAV = -R_{AV}\gamma' = -R_{J\gamma'}\gamma'$ holds. Furthermore we have that $J(t) = 0$ and $A(s_1) = 0$, hence $J(s_1) = 0$. Since c does not have any conjugate points by assumption, we obtain that $J = 0$. Finally $\frac{\nabla}{ds}J(s_1) = (\frac{\nabla}{ds}A(s_1))V(s_1) = EV(s_1) = V(s_1) \neq 0$ gives a contradiction.

We differentiate $X(s)$. To this end let $E_i \otimes E^{*j}$ be a $\binom{1}{1}$ -frame field along c (cf. Remark A.3.7) such that $(A^*A)^{-1}(\tau) = A_j^i(\tau)(E_i \otimes E^{*j})(\tau)$, where $A_j^i : (s, t) \rightarrow \mathbb{R}$ are smooth functions. We obtain that $P_{(1,1;\tau,s)}(c)((A^*A)^{-1}(\tau)) = A_j^i(\tau)(E_i \otimes E^{*j})(s)$ and hence

$$\begin{aligned} A(s) \int_s^t P_{(1,1;\tau,s)}(c)((A^*A)^{-1}(\tau))d\tau &= A(s) \int_s^t A_j^i(\tau)(E_i \otimes E^{*j})(s)d\tau \\ &= A(s) \int_s^t A_j^i(\tau)d\tau(E_i \otimes E^{*j})(s). \end{aligned}$$

Now we have

$$\begin{aligned} \frac{\nabla X}{ds}(s) &= \frac{\nabla A}{ds}(s) \left(\int_s^t A_j^i(\tau)d\tau \right) (E_i \otimes E^{*j})(s) - A(s)A_j^i(s)(E_i \otimes E^{*j})(s) \\ &\quad + A(s) \left(\int_s^t A_j^i(\tau)d\tau \right) \left(\frac{\nabla}{ds}(E_i \otimes E^{*j})(s) \right) \\ &= \frac{\nabla A}{ds}(s) \left(\int_s^t A_j^i(\tau)d\tau \right) (E_i \otimes E^{*j})(s) - A(s)A_j^i(s)(E_i \otimes E^{*j})(s) \\ &= \frac{\nabla A}{ds}(s) \left(\int_s^t A_j^i(\tau)d\tau \right) (E_i \otimes E^{*j})(s) - A(s)(A^*A)^{-1}(s) \end{aligned}$$

and

$$\begin{aligned}
X'' &= A''(s) \left(\int_s^t A_j^i(\tau) d\tau \right) (E_i \otimes E^{*j})(s) \\
&\quad - \frac{\nabla A}{ds}(s) A_j^i(s) (E_i \otimes E^{*j})(s) - \frac{\nabla}{ds}((A^*)^{-1})(s) \\
&= A''(s) \left(\int_s^t A_j^i(\tau) d\tau \right) (E_i \otimes E^{*j})(s) \\
&\quad - \frac{\nabla A}{ds}(s) (A^* A)^{-1}(s) - \frac{\nabla}{ds}((A^*)^{-1})(s) \\
&= A''(s) \left(\int_s^t A_j^i(\tau) d\tau \right) (E_i \otimes E^{*j})(s) \\
&\quad - \frac{\nabla A}{ds}(s) A^{-1}(s) (A^*)^{-1}(s) + (A^*)^{-1}(s) \frac{\nabla A^*}{ds}(s) (A^*)^{-1}(s),
\end{aligned}$$

where we have used Proposition 2.4.4 (iv). Since A is a Lagrangian tensor field, we obtain that $\frac{\nabla A^*}{ds} = A^* \frac{\nabla A}{ds} A^{-1}$ and thus $(A^*)^{-1} \frac{\nabla A^*}{ds} (A^*)^{-1} = \frac{\nabla A}{ds} A^{-1} (A^*)^{-1}$. We have shown that

$$\begin{aligned}
X''(s) &= A''(s) \left(\int_s^t A_j^i(\tau) d\tau \right) (E_i \otimes E^{*j})(s) \\
&= A''(s) \int_s^t P_{(1,1;\tau,s)}(c) ((A^* A)^{-1}(\tau)) d\tau.
\end{aligned}$$

Since A is a Lagrangian tensor field, and in particular a Jacobian tensor field, we can use $A''(s) + R(s)A(s) = 0$ to show that

$$\begin{aligned}
X''(s) + R(s)X(s) &= A''(s) \int_s^t P_{(1,1;\tau,s)}(c) ((A^* A)^{-1}(\tau)) d\tau \\
&\quad + R(s)A(s) \int_s^t P_{(1,1;\tau,s)}(c) ((A^* A)^{-1}(\tau)) d\tau \\
&= (A''(s) + R(s)A(s)) \int_s^t P_{(1,1;\tau,s)}(c) ((A^* A)^{-1}(\tau)) d\tau = 0.
\end{aligned}$$

By setting $s = t$ we obtain that

$$X(t) = A(t) \int_t^t P_{(1,1;\tau,s)}(c) ((A^* A)^{-1}(\tau)) d\tau = 0$$

and

$$\begin{aligned}
\frac{\nabla X}{ds}(t) &= \frac{\nabla A}{ds}(t) \left(\int_t^t A_j^i(\tau) d\tau \right) (E_i \otimes E^{*j})(t) \\
&\quad - A(t) (A^* A)^{-1}(t) \\
&= -A(t) (A^* A)^{-1}(t) = -(A^*)^{-1}(t).
\end{aligned}$$

It remains to show that $\frac{\nabla D_t}{ds}(t) = -(A^*)^{-1}(t)$. Since $R^* = R$ we find

$$\begin{aligned} \frac{\nabla(\frac{\nabla A^*}{ds}D_t - A^*\frac{\nabla D_t}{ds})}{ds} &= (A^*)''D_t + \frac{\nabla A^*}{ds}\frac{\nabla D_t}{ds} - \frac{\nabla}{ds}(A^*)\frac{\nabla D_t}{ds} - A^*D_t'' \\ &= (A^*)''D_t - A^*D_t'' \\ &= -A^*R^*D_t + A^*RD_t = 0. \end{aligned}$$

Therefore, $\frac{\nabla A^*}{ds}D_t - A^*\frac{\nabla D_t}{ds}$ is parallel along c . At $s = s_1$, the initial conditions $A(s_1) = 0$ and $\frac{\nabla A}{ds}(s_1) = E$ for A imply that $A^*(s_1) = 0$ and $\frac{\nabla A^*}{ds}(s_1) = (\frac{\nabla A}{ds})^*(s_1) = E$. Since $D_t(s_1) = E$, we obtain

$$\left(\frac{\nabla A^*}{ds}D_t - A^*\frac{\nabla D_t}{ds}\right)(s_1) = E.$$

Since $\frac{\nabla A^*}{ds}D_t - A^*\frac{\nabla D_t}{ds}$ is parallel along c we furthermore find $(\frac{\nabla A^*}{ds}D_t - A^*\frac{\nabla D_t}{ds})(s) = E$ for all s , where we have used that $E = \delta_i^j E_i \otimes E^{*j}$ is parallel along c and has the same initial value. By setting $s = t$ we have

$$E = \left(\frac{\nabla A^*}{ds}D_t - A^*\frac{\nabla D_t}{ds}\right)(t) = -\left(A^*\frac{\nabla D_t}{ds}\right)(t),$$

which implies that

$$\frac{\nabla D_t}{ds}(t) = -(A^*)^{-1}(t) = \frac{\nabla X}{ds}(t).$$

It remains to show that $D_t(s)$ is nonsingular for all $s \in (s_1, t)$. To this end let $D_t(s)v = 0$ for some $s \in (s_1, t)$. We obtain that

$$0 = D_t(s)v = A(s) \int_s^t P_{(1,1;\tau,s)}(c)((A^*A)^{-1}(\tau))d\tau v$$

which is equivalent to

$$\begin{aligned} 0 &= \int_s^t P_{(1,1;\tau,s)}(c)((A^*A)^{-1}(\tau))d\tau v \\ &= \int_s^t P_{(1,1;\tau,s)}(c)((A^*A)^{-1}(\tau))vd\tau, \end{aligned}$$

since A is a regular tensor field by what we have shown in the beginning of the proof. Hence

$$\int_s^t \langle P_{(1,1;\tau,s)}(c)((A^*A)^{-1}(\tau))v, v \rangle_{T_{c(s)}M} d\tau = 0.$$

We set $B(\tau) := P_{(1,1;\tau,s)}(c)((A^*A)^{-1}(\tau))$, so that

$$\int_s^t \langle B(\tau)v, v \rangle_{T_{c(s)}M} d\tau = 0.$$

Since $B(\tau)$ is positive definite and self-adjoint we obtain that

$$\langle B(\tau)v, v \rangle_{T_{c(s)}M} \geq 0$$

for all $\tau \in (s, t)$ and therefore $\langle B(\tau)v, v \rangle_{T_{c(s)}M} = 0$ for all $\tau \in [s, t]$. In particular we have $\langle B(s)v, v \rangle_{T_{c(s)}M} = 0$, implying that

$$\langle P_{(1,s;1,s)}((A^*A)^{-1}(s))v, v \rangle_{T_{c(s)}M} = \langle ((A^*A)^{-1}(s))v, v \rangle_{T_{c(s)}M} = 0.$$

Since A is self-adjoint, we have $\langle ((A)^{-1}(s))^*v, ((A)^{-1}(s))^*v \rangle_{T_{c(s)}M} = 0$ and hence $((A)^{-1}(s))^*v = 0$. This means that $v = 0$. \square

Lemma 3.2.16. *Let (M, g) be a space-time and let $c : [a, \infty) \rightarrow M$ be a timelike geodesic without conjugate points. For $s_1 > a$ and $t \in [a, \infty) \setminus \{s_1\}$, let D_t be the unique Lagrangian tensor field along c with initial conditions $D_t(s_1) = E$ and $D_t(t) = 0$ (cf. Remark 3.2.14). Then*

$$D(s) := \lim_{t \rightarrow \infty} D_t(s)$$

is a Lagrangian tensor field. Furthermore, $D(s)$ is nonsingular for all $s_1 < s < \infty$.

Proof. We choose an arbitrary sequence $\{t_n\}_{n \in \mathbb{N}} \in [a, \infty) \setminus \{s_1\}$ with $t_n \rightarrow \infty$ and without restriction we assume that $t_n \geq 1$. We show that $\{\frac{\nabla D_{t_n}}{ds}(s_1)\}_{n \in \mathbb{N}}$ has a self-adjoint limit as $t_n \rightarrow \infty$. Since D_{t_n} is a Lagrangian tensor field, we have $(\frac{\nabla(D_{t_n})^*}{ds}D_{t_n})(s_1) = (D_{t_n}^* \frac{\nabla D_{t_n}}{ds})(s_1)$ and hence $\frac{\nabla(D_{t_n})^*}{ds}(s_1) = \frac{\nabla D_{t_n}}{ds}(s_1)$ by assumption. Thus the limit of $\{\frac{\nabla D_{t_n}}{ds}(s_1)\}_{n \in \mathbb{N}}$ must be a self-adjoint map which we will denote by $\frac{\nabla D}{ds}(s_1) : N(c(s_1)) \rightarrow N(c(s_1))$ if it exists. Consequently, we only have to show that for each $y \in N(c(s_1))$ the value of $\left\langle \frac{\nabla D_{t_n}}{ds}(s_1)y, y \right\rangle_{T_{c(s_1)}M}$ converges to some value $\left\langle \frac{\nabla D}{ds}(s_1)y, y \right\rangle_{T_{c(s_1)}M}$. We show that the sequence $n \mapsto \left\langle \frac{\nabla D_{t_n}}{ds}(s_1)y, y \right\rangle_{T_{c(s_1)}M}$ is monotonically increasing for all t_n with $s_1 < t_n < \infty$ and is bounded from above by $\left\langle \frac{\nabla D_a}{ds}(s_1)y, y \right\rangle_{T_{c(s_1)}M}$ to establish the existence of the limit. To this end let $r \in (s_1, t)$. By the proof of Lemma 3.2.15 we have

$$\frac{\nabla D_t}{ds}(s) = \frac{\nabla A}{ds}(s) \int_s^t P_{(1,1;\tau,s)}(c)((A^*A)^{-1}(\tau))d\tau - (A^*)^{-1}(s),$$

where A is the unique Lagrangian tensor field that satisfies $A(s_1) = 0$ and $\frac{\nabla A}{ds}(s_1) = E$. Thus for $s \in (s_1, t)$ we obtain that

$$\begin{aligned} & \left\langle \frac{\nabla D_t}{ds}(s)Y(s), Y(s) \right\rangle_{T_{c(s)}M} \\ &= \left\langle \left(\frac{\nabla A}{ds}(s) \int_s^t P_{(1,1;\tau,s)}(c)((A^*A)^{-1}(\tau))d\tau \right) Y(s), Y(s) \right\rangle_{T_{c(s)}M} \\ & - \left\langle (A^*)^{-1}(s)Y(s), Y(s) \right\rangle_{T_{c(s)}M}, \end{aligned}$$

where Y is the unique parallel vector field along c with $Y(s_1) = y$. Thus for s with $s_1 < s < r$ it follows that

$$\left\langle \frac{\nabla D_t}{ds}(s)Y(s), Y(s) \right\rangle_{T_{c(s)}M} - \left\langle \frac{\nabla D_r}{ds}(s)Y(s), Y(s) \right\rangle_{T_{c(s)}M}$$

is given by

$$\left\langle \left(\frac{\nabla A}{ds}(s) \int_r^t P_{(1,1;\tau,s)}(c)((A^*A)^{-1}(\tau))d\tau \right) Y(s), Y(s) \right\rangle_{T_{c(s)}M}.$$

We let $s \rightarrow s_1$ and use that $Y(s_1) = y$ and $\frac{\nabla A}{ds}(s_1) = E$ to obtain that

$$\begin{aligned} & \left\langle \frac{\nabla D_t}{ds}(s_1)y, y \right\rangle_{T_{c(s_1)}M} - \left\langle \frac{\nabla D_r}{ds}(s_1)y, y \right\rangle_{T_{c(s_1)}M} \\ &= \left\langle \left(\int_r^t P_{(1,1;\tau,s_1)}(c)((A^*A)^{-1}(\tau))d\tau \right) Y(s_1), Y(s_1) \right\rangle_{T_{c(s_1)}M}. \end{aligned}$$

Since Y is parallel along c , we obtain that $\frac{\nabla}{d\tau}Y(\tau) = 0$. Thus we have that

$$\begin{aligned} & \left\langle \left(\int_r^t P_{(1,1;\tau,s_1)}(c)((A^*A)^{-1}(\tau))d\tau \right) Y(s_1), Y(s_1) \right\rangle_{T_{c(s_1)}M} \\ &= \left\langle \left(\int_r^t A_j^i(\tau)d\tau(E_i \otimes E^{*j})(s_1) \right) Y(s_1), Y(s_1) \right\rangle_{T_{c(s_1)}M} \\ &= \int_r^t A_j^i(\tau) \langle (E_i \otimes E^{*j})(s_1)Y(s_1), Y(s_1) \rangle_{T_{c(s_1)}M} d\tau \\ &= \int_r^t A_j^i(\tau) \langle (E_i \otimes E^{*j})(\tau)Y(\tau), Y(\tau) \rangle_{T_{c(\tau)}M} d\tau \\ &= \int_r^t \langle P_{(1,1;\tau,\tau)}(c)((A^*A)^{-1}(\tau)Y(\tau), Y(\tau) \rangle_{T_{c(\tau)}M} d\tau \\ &= \int_r^t \langle (A^*A)^{-1}(\tau)Y(\tau), Y(\tau) \rangle_{T_{c(\tau)}M} d\tau. \end{aligned}$$

Moreover,

$$\begin{aligned}
& \int_r^t \langle (A^*A)^{-1}(\tau)Y(\tau), Y(\tau) \rangle_{T_{c(\tau)}M} d\tau \\
&= \int_r^t \langle A^{-1}(\tau)(A^*)^{-1}(\tau)Y(\tau), Y(\tau) \rangle_{T_{c(\tau)}M} d\tau \\
&= \int_r^t \langle (A^*)^{-1}(\tau)Y(\tau), (A^*)^{-1}(\tau)Y(\tau) \rangle_{T_{c(\tau)}M} d\tau,
\end{aligned}$$

which must be positive because $(A^*)^{-1}(\tau)Y(\tau)$ is an element of $N(c(t))$, hence spacelike for each $\tau \in [r, t]$. We finally obtain that

$$\left\langle \frac{\nabla D_t}{ds}(s_1)y, y \right\rangle_{T_{c(s_1)}M} - \left\langle \frac{\nabla D_r}{ds}(s_1)y, y \right\rangle_{T_{c(s_1)}M} > 0$$

and the sequence $n \mapsto \left\langle \frac{\nabla D_{t_n}}{ds}(s_1)y, y \right\rangle_{T_{c(s_1)}M}$ is monotonically increasing for all $t_n > s_1$ as required. We now show that

$$\left\langle \frac{\nabla D_t}{ds}(s_1)y, y \right\rangle_{T_{c(s_1)}M} < \left\langle \frac{\nabla D_a}{ds}(s_1)y, y \right\rangle_{T_{c(s_1)}M}$$

for all $t > s_1$ and any $y \in N(c(s_1))$. To this end let Y be (again) the unique parallel vector field along c with $Y(s_1) = y$. Let J be the piecewise smooth Jacobian vector field along $c|_{[a, t]}$ given by

$$J(s) := \begin{cases} D_a(s)Y(s), & \text{for } a \leq s \leq s_1 \\ D_t(s)Y(s), & \text{for } s_1 \leq s \leq t \end{cases}$$

J is well defined at $s = s_1$ since $D_a(s_1) = E$ and $D_t(s_1) = E$. Recall that for a smooth curve $\alpha : [a, b] \rightarrow M$ the Index-form I on $V^\perp(\alpha)$ is defined to be the symmetric bilinear form $I : V^\perp(\alpha) \times V^\perp(\alpha) \rightarrow \mathbb{R}$ given by $I(X, Y) := -\int_a^b \langle \frac{\nabla X}{ds}(s), \frac{\nabla Y}{ds}(s) \rangle_{(T_{\alpha(s)}M)} - \langle R(X(s), \alpha'(s))\alpha'(s), Y(s) \rangle_{(T_{\alpha(s)}M)} ds$ (cf. [15], section 10 for details). $I(X, Y)$ can be written as

$$\begin{aligned}
I(X, Y) &= - \left\langle \frac{\nabla X}{ds}(s), Y(s) \right\rangle_{(T_{\alpha(s)}M)} \Big|_a^b \\
&\quad + \int_a^b \langle X''(s) + R(X(s), \alpha'(s))\alpha'(s), Y(s) \rangle_{(T_{\alpha(s)}M)} ds
\end{aligned}$$

and one can show that for a future-directed timelike geodesic $\alpha : [a, b] \rightarrow M$ the nonexistence of a conjugate point in $(a, b]$ implies negative definiteness

of the Index-form (cf. [15], section 10, Theorem 17). We set $J_a := J_{|[a, s_1]}$ and $J_t := J_{|[s_1, t]}$. With these preparations we now obtain that

$$\begin{aligned}
I(J, J)_a^t &= I(J, J)_a^{s_1} + I(J, J)_{s_1}^t \\
&= - \left\langle \frac{\nabla J_a}{ds}(s), J_a(s) \right\rangle_{T_{c(s)}M} \Big|_a^{s_1} - \left\langle \frac{\nabla J_t}{ds}(s), J_t(s) \right\rangle_{T_{c(s)}M} \Big|_{s_1}^t \\
&= - \left\langle \frac{\nabla J_a}{ds}(s_1), J_a(s_1) \right\rangle_{T_{c(s_1)}M} + \left\langle \frac{\nabla J_a}{ds}(a), J_a(a) \right\rangle_{T_{c(a)}M} \\
&\quad - \left\langle \frac{\nabla J_t}{ds}(t), J_t(t) \right\rangle_{T_{c(t)}M} + \left\langle \frac{\nabla J_t}{ds}(s_1), J_t(s_1) \right\rangle_{T_{c(s_1)}M} \\
&= - \left\langle \frac{\nabla J_a}{ds}(s_1), J_a(s_1) \right\rangle_{T_{c(s_1)}M} + \left\langle \frac{\nabla J_t}{ds}(s_1), J_t(s_1) \right\rangle_{T_{c(s_1)}M},
\end{aligned}$$

where we have used that $J_a(a) = 0$ and $J_t(t) = 0$. Furthermore,

$$\begin{aligned}
I(J, J)_a^t &= - \left\langle \frac{\nabla J_a}{ds}(s_1), J_a(s_1) \right\rangle_{T_{c(s_1)}M} + \left\langle \frac{\nabla J_t}{ds}(s_1), J_t(s_1) \right\rangle_{T_{c(s_1)}M} \\
&= - \left\langle \frac{\nabla D_a}{ds}(s_1)Y(s_1), Y(s_1) \right\rangle_{T_{c(s_1)}M} + \left\langle \frac{\nabla D_t}{ds}(s_1)Y(s_1), Y(s_1) \right\rangle_{T_{c(s_1)}M} \\
&= - \left\langle \frac{\nabla D_a}{ds}(s_1)y, y \right\rangle_{T_{c(s_1)}M} + \left\langle \frac{\nabla D_t}{ds}(s_1)y, y \right\rangle_{T_{c(s_1)}M},
\end{aligned}$$

where we have used that $Y(s_1) = y$. Since c has no conjugate points in $[a, \infty)$ by assumption, we have $I(J, J) < 0$ and hence

$$\left\langle \frac{\nabla D_t}{ds}(s_1)y, y \right\rangle_{T_{c(s_1)}M} < \left\langle \frac{\nabla D_a}{ds}(s_1)y, y \right\rangle_{T_{c(s_1)}M}$$

for all $t > s_1$. Hence the tensor $\frac{\nabla D}{ds}(s_1) := \lim_{t_n \rightarrow \infty} \frac{\nabla D_{t_n}}{ds}(s_1)$ exists. Now we define $D(s)$ by setting $D(s)$ equal to the unique Jacobian tensor field along c which satisfies $D(s_1) = E$ and $\frac{\nabla D}{ds}(s_1) = \lim_{t_n \rightarrow \infty} \frac{\nabla D_{t_n}}{ds}(s_1)$. Since $D(s)$ and $D_{t_n}(s)$ both satisfy the differential equation $A'' + RA = 0$ and the initial conditions of D_{t_n} approach the initial conditions of D as $t_n \rightarrow \infty$, it follows that $D(s) = \lim_{t_n \rightarrow \infty} D_{t_n}(s)$ and $\frac{\nabla D}{ds}(s) = \lim_{t_n \rightarrow \infty} \frac{\nabla D_{t_n}}{ds}(s)$. This implies that the limit $D(s)$ of the Lagrangian tensor fields $D_t(s)$ must be a Lagrangian tensor field. The last statement follows as in the proof of Lemma 3.2.15 from the representation formula

$$D(s) = A(s) \int_s^\infty P_{(1,1;\tau,s)}((A^*A)^{-1}(\tau))d\tau.$$

□

Remark and Definition 3.2.17. For our further considerations, it turns out to be useful to divide the Lagrangian tensor fields along a complete timelike geodesic $c : (-\infty, +\infty) \rightarrow (M, g)$ that satisfy the generic and strong energy conditions into two classes \mathfrak{L}_+ and \mathfrak{L}_- . Let \mathfrak{L} denote the set of all Lagrangian tensor fields along c with $A(s_1) = E$ for some $s_1 \in \mathbb{R}$ that satisfy $\text{Ric}(c', c') \geq 0$ and $R(-, c'(s_1))c'(s_1) \neq 0$. Then we define

$$\mathfrak{L}_+ := \{A : A \in \mathfrak{L} \text{ and } \text{tr}\left(\frac{\nabla A}{ds}(s_1)\right) \geq 0\}$$

and

$$\mathfrak{L}_- := \{A : A \in \mathfrak{L} \text{ and } \text{tr}\left(\frac{\nabla A}{ds}(s_1)\right) \leq 0\}.$$

Obviously we have

$$\mathfrak{L} = \mathfrak{L}_+ \cup \mathfrak{L}_- \quad \text{and in general} \quad \mathfrak{L}_+ \cap \mathfrak{L}_- \neq \emptyset$$

Lemma 3.2.18. *Let (M, g) be a space-time. Then every $A \in \mathfrak{L}_-$ satisfies $\det(A(s)) = 0$ for some $\mathbb{R} \ni s > s_1$ and each $A \in \mathfrak{L}_+$ satisfies $\det(A(s)) = 0$ for some $\mathbb{R} \ni s < s_1$.*

Proof. It suffices to prove the statement for $A \in \mathfrak{L}_-$. Since $A(s_1) = E$ we obtain $A^{-1}(s_1) = E$ and find that $\theta(s_1) = \text{tr}\left(\frac{\nabla A}{ds}(s_1)A^{-1}(s_1)\right) = \text{tr}\left(\frac{\nabla A}{ds}(s_1)\right) \leq 0$. Since c satisfies the strong energy condition, i.e. $\text{Ric}(c', c') \geq 0$ and $\text{tr}(\sigma^2) \geq 0$ by Remark 2.5.4, the vorticity-free Raychaudhuri equation (cf. Proposition 2.5.6) reduces to

$$\frac{d\theta}{ds}(s) \leq 0$$

for all s and we find $\theta(s) \leq 0$ for all $s \geq s_1$. Now assume $\theta(s_0) < 0$ for some $s_0 > s_1$. Then the Lemma is established by Proposition 3.2.7 resp. by Remark 3.2.8. Therefore we consider the case where $\theta(s) = 0$ for all $s \geq s_1$. Then

$$\frac{d\theta}{ds}(s) = 0$$

for all $s \geq s_1$ and Proposition 2.5.6 implies that $\text{tr}(\sigma^2) = 0$ and $\text{Ric}(c', c') = 0$. We find that $\sigma = 0$ for $s \geq s_1$ by Remark 2.5.4. Since B is self-adjoint by Proposition 2.5.3 and since σ and θ vanish for all $s \geq s_1$, $B = 0$ by Definition 2.5.1 (iii). The proof of Proposition 2.5.2 now shows that $R = -B^2 - \frac{\nabla B}{ds} = 0$ for $s \geq s_1$, in contradiction to $R(s_1) \neq 0$ by our assumption in Remark 3.2.17. □

We are now ready to prove the following

Theorem 3.2.19. *Let (M, g) be a space-time and let $c : \mathbb{R} \rightarrow M$ a complete timelike geodesic that satisfies the generic and the strong energy condition, then c has a pair of conjugate points.*

Proof. By assumption $c : \mathbb{R} \rightarrow M$ is a complete timelike geodesic that satisfies $\text{Ric}(c'(s), c'(s)) \geq 0$ for all $s \in \mathbb{R}$ and $R(-, c'(s_1))c'(s_1) \neq 0$ for some $s_1 \in \mathbb{R}$. We now assume that c has no conjugate points. Let $D(s) := \lim_{t \rightarrow \infty} D_t(s)$ be the Lagrangian tensor field with $D(s_1) = E$ from Lemma 3.2.16. Since $c|_{[s_1, \infty)}$ has no conjugate points, $D(s)$ is non-singular for all $s \geq s_1$ (cf. Lemma 3.2.16). Lemma 3.2.18 now shows that $D \notin \mathfrak{L}_-$. By Remark 3.2.17 we find that $D \in \mathfrak{L}_+$ and since $D \notin \mathfrak{L}_-$, we obtain $\text{tr}(\frac{\nabla D}{ds}(s_1)) > 0$. The proof of Lemma 3.2.16 shows that $\frac{\nabla D}{ds}(s_1) = \lim_{t \rightarrow \infty} \frac{\nabla D_t}{ds}(s_1)$, hence there is a $t > s_1$ such that $\text{tr}(\frac{\nabla D_t}{ds}(s_1)) > 0$. Lemma 3.2.18 guarantees the existence of an $s_2 < s_1$ and a nonzero tangent vector $v \in N(c(s_2))$ with $D_t(s_2)(v) = 0$. Recall from the proof of Lemma 3.2.15 that $D_t(t) = 0$, but $\frac{\nabla D_t}{ds}(t) = (A^*)^{-1}(t)$ is nonsingular. Now let $Y \in V^\perp(c)$ be the unique parallel vector field along c with $Y(s_2) = v$ and define $J := D_t Y$. It turns out that J is a nontrivial Jacobian vector field along c , since $J'' = D_t'' Y = -R D_t Y = -R_{D_t(s)Y} \gamma', \gamma' = -R_{J\gamma'} \gamma'$. Furthermore we obtain $J(s_2) = 0$ and $J(t) = 0$. This is a contradiction. \square

Theorem 3.2.20. *Let (M, g) be a space-time that satisfies the generic and the strong energy condition (cf. Definitions 3.2.3 and 3.2.4). Then each timelike geodesic in (M, g) is either incomplete or else has a pair of conjugate points.*

Proof. This follows immediately from Theorem 3.2.19. \square

One can formulate the same results for null geodesics. A way how to do this can be found in [1], section 9 and section 11.

Theorem 3.2.21. *Let (M, g) be a space-time of dimension at least three which satisfies the generic and the strong energy condition. Then each causal geodesic in (M, g) is either incomplete or else has a pair of conjugate points. Thus every causal geodesic in M without conjugate points is incomplete.*

3.3 Cluster and limit curves

In this section we will use the concept of continuous causal curves to study sequences of causal curves and their limits. Nonetheless, to reach more general conclusions many results will be stated for piecewise smooth causal curves.

We will follow [6]. (M, g) denotes an arbitrary space-time. We start with the following

Definition 3.3.1. A continuous curve γ is called causal (timelike) if every point p on γ has a convex neighbourhood \mathfrak{C} such that any point $q \neq p$ on $\{\gamma\} \cap \mathfrak{C}$ can be connected to p by a causal (timelike) \mathcal{C}^1 curve which is contained in \mathfrak{C} .

Remark 3.3.2. Definition 3.3.1 generalizes the concept of piecewise smooth causal curves treated in the past sections. By a timelike (or causal) curve we will mean a continuous timelike (or causal) curve. Differentiability will be additionally emphasized.

Lemma 3.3.3. Let $q \in M$. Then there is a convex coordinate neighbourhood \mathfrak{C} of q , a constant $k > 0$ and coordinates (x^0, \dots, x^{n-1}) such that all continuous causal curves γ in \mathfrak{C} can be parametrized by $t = x^0$ and the coordinate inequality

$$\left(\sum_{a=0}^{n-1} (\gamma^a(t_1) - \gamma^a(t_2))^2 \right)^{\frac{1}{2}} \leq k |t_1 - t_2|$$

holds for all t_1, t_2 .

Proof. By Lemma 3.1.30 causality in \mathfrak{C} corresponds to causality in \mathbb{R}_1^n and we choose the coordinate system such that $e_0 := (1, 0, \dots, 0)$ is timelike and $e_i := (0, \dots, 1, \dots, 0)$ is spacelike, where $i = 1, \dots, n-1$.

We show that there exists some $k_0 > 0$ such that every vector $v \in T_p M$ that is causal relative to g_p (the standard metric) for some $p \in \overline{\mathfrak{C}}$ is also causal relative to the metric

$$h_{k_0} := -k_0 dt \otimes dt + \sum_{i=1}^{n-1} (dx^i \otimes dx^i).$$

We show this for future-directed causal vectors. The proof for past-directed causal vectors is analogous. Let

$$A := \{(p, v) \in \overline{\mathfrak{C}} \times S^{n-1} : g_p(v, v) \leq 0\}.$$

Since $\overline{\mathfrak{C}} \times S^{n-1}$ is compact and A is closed, A is compact. Let $(p, v) \in A$ and let v be future-directed. Then we obtain that $g_p(v, e_0) < 0$, since e_0 is timelike. Let $f : A \rightarrow \mathbb{R}$, $f(p, v) := g_p(v, e_0)$. Obviously f is continuous and negative. Since A is compact and f is continuous, we obtain that f assumes a maximum which we will denote by $c \in \mathbb{R}$, hence $c := \max_{(p,v) \in A} \{f((p,v))\}$.

Thus we have $g_p(v, e_0) \leq c < 0$.

Let

$$W := \{v \in S^{n-1} : \text{there is some } p \in \overline{\mathfrak{C}} : g_p(v, e_0) \leq c\}.$$

Together with the above we obtain that $S^{n-1} \cap J_g^+(\bar{\mathcal{C}}) \subseteq W$. We furthermore have that $W \cap (\{0\} \times \mathbb{R}^{n-1}) = \emptyset$ and since W is closed and hence compact there is some $k_0 \in \mathbb{R}$ such that $k_0 > 0$ and $W \subseteq J_{h_{k_0}} \cap S^{n-1}$. Hence every future-directed causal vector relative to g is future-directed causal relative to h_{k_0} .

Now let γ be a continuous causal curve from $\gamma(t_1)$ to $\gamma(t_2)$. By definition there is some causal \mathcal{C}^1 -curve μ with $\mu(t_1) = \gamma(t_1)$ and $\mu(t_2) = \gamma(t_2)$. We assume that μ is causal future-directed with $\mu(t) = (x^0(t), \dots, x^{n-1}(t))$. Since e_0 is timelike and $\mu'(t)$ is a causal vector, we obtain that $-k_0 x^{0'}(t) = \langle \mu'(t), e_0 \rangle < 0$ for all t . Hence we can parameterize μ by x^0 . The same procedure holds for causal past-directed \mathcal{C}^1 -curves. Thus without restriction we can assume that $\mu^0 = t$, hence $\dot{\mu}^0 = 1$.

We find that $k_0 = k_0(\dot{\mu}^0)^2 \geq \sum_{i=1}^{n-1} (\dot{\mu}^i)^2$. By definition of the euclidean norm we have $\|v^a e_a\|_{\text{eucl}} = (\sum_{i=0}^{n-1} (\dot{\mu}^i)^2)^{\frac{1}{2}}$ and we obtain

$$\begin{aligned} \|\gamma^a(t_1)e_a - \gamma^a(t_2)e_a\|_{\text{eucl}} &= \|\mu^a(t_1)e_a - \mu^a(t_2)e_a\|_{\text{eucl}} \\ &= \left\| \int_{t_1}^{t_2} \dot{\mu}^a(\tau) d\tau e_a \right\|_{\text{eucl}} \leq \int_{t_1}^{t_2} \|\dot{\mu}^a(\tau)e_a\|_{\text{eucl}} d\tau \\ &= \int_{t_1}^{t_2} \left(\sum_{a=0}^{n-1} (\dot{\mu}^a)^2(\tau) \right)^{\frac{1}{2}} d\tau = \int_{t_1}^{t_2} \left((\dot{\mu}^0)^2 + \sum_{a=1}^{n-1} (\dot{\mu}^a)^2(\tau) \right)^{\frac{1}{2}} d\tau \\ &\leq \int_{t_1}^{t_2} (1 + k_0)^{\frac{1}{2}} d\tau \leq (1 + k_0)^{\frac{1}{2}} (t_2 - t_1). \end{aligned}$$

Now we set $k := (1 + k_0)^{\frac{1}{2}}$ and the lemma is proved. \square

Definition 3.3.4. Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \rightarrow Y$ is called *Lipschitz continuous* if there exists a real constant $k \geq 0$ such that $d_Y(f(x_1), f(x_2)) \leq k d_X(x_1, x_2)$ holds for all $x_1, x_2 \in X$. k is called the (a) *Lipschitz constant* of the map f . $f : X \rightarrow Y$ is called *locally Lipschitz continuous* if for each $x \in X$ there is some neighbourhood \mathcal{U} on which f is Lipschitz continuous.

Corollary 3.3.5. Causal curves are locally Lipschitz continuous and therefore differentiable almost everywhere.

Proof. This is Rademacher's Theorem. See [2], section 5, Theorem 6 for a proof. \square

The concept of limit curves is closely related to the Hausdorff closed limit.

Definition 3.3.6. Let $\{A_n\}_{n \in \mathbb{N}}$ be an arbitrary sequence of subsets (not necessarily curves) of M . The Hausdorff upper and lower limits of $\{A_n\}_{n \in \mathbb{N}}$ are defined by

$$\text{limsup}\{A_n\} := \{p \in M : \text{each neighbourhood of } p \\ \text{intersects infinitely many of the sets } A_n\}$$

and

$$\text{liminf}\{A_n\} := \{p \in M : \text{each neighbourhood of } p \\ \text{intersects all but a finite number of the sets } A_n\}.$$

Remark 3.3.7. The Hausdorff upper and lower limits always exist, although they may be empty. Obviously we have

$$\text{liminf}\{A_n\}_{n \in \mathbb{N}} \subseteq \text{limsup}\{A_n\}_{n \in \mathbb{N}}.$$

If these limits are equal, then the Hausdorff closed limit of $\{A_n\}_{n \in \mathbb{N}}$, denoted by $\lim\{A_n\}_{n \in \mathbb{N}}$, is defined to be

$$\lim\{A_n\}_{n \in \mathbb{N}} := \text{liminf}\{A_n\}_{n \in \mathbb{N}} = \text{limsup}\{A_n\}_{n \in \mathbb{N}}.$$

Definition 3.3.8. Let $x, y \in M$. We denote the space of all continuous causal curves from x to y by $\mathcal{C}_{\text{causal}}^0(x, y)$. We define $\mathcal{C}_{\text{time}}^1(x, y)$ to be the space of all timelike curves from x to y that are \mathcal{C}^1 (cf. Remark 3.3.2).

The following definition is based on Definition 3.3.6.

Definition 3.3.9. Let $\gamma : [a, b] \rightarrow M$ and $\gamma_i : [a, b] \rightarrow M$ ($i \in \mathbb{N}$) be arbitrary curves in the space-time M . The sequence $\{\gamma_i\}_{i \in \mathbb{N}}$ converges to γ in the \mathcal{C}^0 -topology if for every neighbourhood \mathcal{U} of γ in M there exists an $i_0 \in \mathbb{N}$ such that $\gamma_i \subseteq \mathcal{U}$ for all $i > i_0$. The curve γ is called the limit curve of the sequence $\{\gamma_i\}_{i \in \mathbb{N}}$.

Definition 3.3.10. Let $\gamma : [a, b] \rightarrow M$ and $\gamma_i : [a, b] \rightarrow M$ ($i \in \mathbb{N}$) be arbitrary curves in M . γ is said to be a cluster curve of the sequence $\{\gamma_i\}_{i \in \mathbb{N}}$ if there exists a subsequence $\{\gamma_{i_j}\}_{j \in \mathbb{N}}$ such that for all $x \in \{\gamma\}$ each neighbourhood of x intersects all but finitely many of the curves γ_{i_j} . We will say that the sequence $\{\gamma_{i_j}\}_{j \in \mathbb{N}}$ distinguishes the cluster curve γ .

Proposition 3.3.11. Let (M, g) be a strongly causal space-time (cf. Definition 3.1.22) and γ a cluster curve of a sequence $\{\gamma_i\}_{i \in \mathbb{N}}$ of causal curves, then γ is causal.

Proof. Since $\{\gamma\}$ is compact and (M, g) is strongly causal, we can cover $\{\gamma\}$ by finitely many convex neighbourhoods $\{\mathfrak{C}_1, \dots, \mathfrak{C}_n\}$ such that no causal curve can enter any of these neighbourhoods twice. We choose an arbitrary convex set \mathfrak{C}_i ($i \in \{1, \dots, n\}$), and denote it by \mathfrak{C} . Let $p, q \in \{\gamma\} \cap \mathfrak{C}$ and denote by $\{\gamma_{i_j}\}_{j \in \mathbb{N}}$ a subsequence which distinguishes γ . We find sequences $\{x_j\}_{j \in \mathbb{N}}, \{y_j\}_{j \in \mathbb{N}} \in \{\gamma_{i_j}\}_{j \in \mathbb{N}} \cap \mathfrak{C}$ with $x_j \rightarrow x$ and $y_j \rightarrow y$. Since $y_j \in J^+(x_j, \mathfrak{C})$ Lemma 3.1.30 (ii) implies the existence of a causal vector v_j with $\exp_{x_j}(v_j) = y_j$. These vectors have an accumulation point v with $\exp_p(v) = y$. The vector v must be causal since the set of causal vectors is closed by Lemma 3.1.32 (iii). This implies $y \in J^+(x, \mathfrak{C})$. If x and y are arbitrary points on $\{\gamma\}$, there are finitely many neighbourhoods $\mathfrak{C}_1, \dots, \mathfrak{C}_k$ such that the segment from x to y is covered by $\bigcup_{i=1}^k \mathfrak{C}_i$. We can now apply the preceding argument finitely often to conclude that x and y are causally related. \square

Definition 3.3.12. Let $A \subseteq \mathbb{R}^k$ be a compact set, i.e. $A \in \mathbb{R}^k$. We define

$$\mathcal{C}^0(A, \mathbb{R}^l) := \{f : A \rightarrow \mathbb{R}^l : f \text{ is continuous}\}$$

to be the space of all continuous functions defined on A .

Remark 3.3.13. $(\mathcal{C}^0(A, \mathbb{R}^l), \|\cdot\|_\infty)$, where $\|f\|_\infty := \sup_{x \in A} \{|f(x)|\}$, is a Banach space.

Lemma 3.3.14. Let $B \subseteq \mathcal{C}^0(A, \mathbb{R}^l)$ be a closed set and assume that for every $\epsilon > 0$ there are finitely many balls $\{B_\epsilon^1(x_1), \dots, B_\epsilon^{j(\epsilon)}(x_{j(\epsilon)})\}$ with radius ϵ and $B \subseteq \bigcup_{i=1}^{j(\epsilon)} B_\epsilon^i(x_i)$. Then B is compact in $\mathcal{C}^0(A, \mathbb{R}^l)$.

Proof. If the lemma is not true, then there exists an open cover $(\mathcal{U}_\lambda)_{\lambda \in \Lambda}$ of B such that there is no finite subset $\{\mathcal{U}_{\lambda_1}, \dots, \mathcal{U}_{\lambda_k}\}$ that covers B . Let $\{B_1^1(x_1), \dots, B_1^{j(1)}(x_{j(1)})\}$ be a finite set of balls of radius 1 which cover B . By our assumption one of these balls cannot be covered by finitely many \mathcal{U}_λ (otherwise we would obtain a finite cover of B by finitely many sets which are in turn finitely covered by sets \mathcal{U}_λ). We denote this ball by B_0 and assume that we have constructed balls $\{B_i\}_{i=0, \dots, k-1}$ such that the following three conditions hold.

- (i) Any two consecutive balls intersect.
- (ii) Each ball B_i has radius 2^{-i} .
- (iii) None of these balls can be covered by finitely many \mathcal{U}_λ .

Obviously B_0 satisfies the conditions (ii) and (iii). We can find balls

$$\{B_{2^{-k}}^1(x_1), \dots, B_{2^{-k}}^{j(2^{-k})}(x_{j(2^{-k})})\}$$

which cover B and therefore also B_{k-1} . Since B_{k-1} cannot be covered by finitely many \mathcal{U}_l there exists at least one $B_{2^{-k}}^m(x_m)$ which intersects B_{k-1} and cannot be covered by finitely many \mathcal{U}_l . Denoting $B_{2^{-k}}^m(x_m)$ by B_k we have inductively defined a sequence $\{B_i\}_{i \in \mathbb{N} \cup \{0\}}$ of balls which satisfy conditions (i), (ii) and (iii). We denote the centers of these balls by y_i ($i \in \{0, \dots, k\}$). For any natural number $m < n$ we obtain

$$\begin{aligned} \|y_n - y_m\| &\leq \sum_{i=m+1}^n \|y_i - y_{i-1}\| \\ &\leq \sum_{i=m+1}^n (2^{-i} + 2^{-i+1}) \leq 3 \cdot 2^{-m}. \end{aligned}$$

This shows that $\{y_i\}_{i \in \mathbb{N} \cup \{0\}}$ is a Cauchy sequence. Since $(\mathcal{C}^0(A, \mathbb{R}^l), \|\cdot\|_\infty)$ is a Banach space (cf. Remark 3.3.13) $\{y_i\}_{i \in \mathbb{N} \cup \{0\}}$ converges. We denote the limit of $\{y_i\}_{i \in \mathbb{N} \cup \{0\}}$ by y and find an \mathcal{U}_l which contains y and a number $r \in \mathbb{N}$ such that the ball $B_{4(2^{-(r+1)})}(y)$ is contained in \mathcal{U}_l . But this implies $B_{2^{-r-1}}(y_{r+1}) \subseteq \mathcal{U}_l$ in contradiction to (iii). \square

Definition 3.3.15. Let (X, d_X) and (Y, d_Y) be metric spaces. A set \mathfrak{F} of functions $f : X \rightarrow Y$ is called *equi-continuous* in $x_0 \in X$, if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\sup_{f \in \mathfrak{F}} d_Y(f(x), f(x_0)) \leq \epsilon$$

for all $x \in X$ with $d_X(x, x_0) \leq \delta$. The set \mathfrak{F} is said to be *equi-continuous* if it is equi-continuous in each point $x_0 \in X$.

With these preparations we can now prove the theorem of Ascoli.

Theorem 3.3.16. Let $A \subseteq \mathbb{R}^k$ be a compact set and $f_i : A \rightarrow \mathbb{R}^l$ ($i \in \mathbb{N}$) be an equi-continuous sequence of continuous functions such that for all $a \in A$ the set $\overline{\bigcup_{i \in \mathbb{N}} f_i(a)}$ is compact. Then there is a continuous function $f : A \rightarrow \mathbb{R}^l$ and a subsequence $\{f_{i_j}\}_{j \in \mathbb{N}}$ of $\{f_i\}_{i \in \mathbb{N}}$ which converges uniformly to f .

Proof. We first show that the subset $\overline{\bigcup_{i=1}^\infty \{f_i\}}$ is compact in the Banach space $(\mathcal{C}^0(A, \mathbb{R}^l), \|\cdot\|_\infty)$. By Lemma 3.3.14 we only have to show that for any $\epsilon > 0$ there is a finite number of balls with diameter less than ϵ which cover $\overline{\bigcup_{i=1}^\infty \{f_i\}}$. Let $\epsilon > 0$ and $a \in A$. By assumption $\{f_i\}_{i \in \mathbb{N}}$ is equi-continuous.

Therefore, for each $a \in A$ there exists a neighbourhood \mathcal{U}_a of a such that for all f_j ($j \in \mathbb{N}$) and all $y \in \mathcal{U}_a$ the inequality

$$\|f_j(a) - f_j(y)\| < \frac{\epsilon}{4}$$

holds, where $\|\cdot\|$ denotes the standard norm on \mathbb{R}^l . The set $\{\mathcal{U}_a : a \in A\}$ forms an open cover for A . Since A is compact we can cover A with finitely many such neighbourhoods U_{a_l} ($l \in \{1, \dots, k\}$). The union

$$K := \bigcup_{l=1}^k \overline{\bigcup_{i=1}^{\infty} \{f_i(a_l)\}}$$

is compact since $\overline{\bigcup_{i=1}^{\infty} \{f_i(a)\}}$ is compact for each $a \in A$. We cover K by finitely many balls of radius $\frac{\epsilon}{4}$ and denote their centers by $x_s \in K$, where $s \in \{1, \dots, r\}$ for some $r \in \mathbb{N}$. We consider the finite set of all maps

$$\sigma : \{1, \dots, k\} \rightarrow \{1, \dots, r\}$$

and define

$$\mathcal{V}_\sigma := \left\{ h \in \overline{\bigcup_{i=1}^{\infty} \{f_i\}} : \|h(a_l) - x_{\sigma(l)}\| < \frac{\epsilon}{4} \text{ for all } l \in \{1, \dots, k\} \right\}.$$

Let $\hat{f} \in \overline{\bigcup_{i=1}^{\infty} \{f_i\}}$. Then we find a subsequence of functions $\{f_{i_j}\}_{j \in \mathbb{N}}$ with $f_{i_j} \in \overline{\bigcup_{i=1}^{\infty} \{f_i\}}$ and $f_{i_j} \rightarrow \hat{f}$. Since K is covered by finitely many balls of radius $\frac{\epsilon}{4}$ and with centers x_1, \dots, x_r and since $\hat{f}(a_l) \in K$ for all $l \in \{1, \dots, k\}$ there is for each $l \in \{1, \dots, k\}$ an x_{s_l} such that

$$\|\hat{f}(a_l) - x_{s_l}\| < \frac{\epsilon}{4}.$$

Furthermore \hat{f} is continuous.

By defining $\hat{\sigma}(l) := s_l$ we see that $\hat{f} \in \mathcal{V}_{\hat{\sigma}}$. Hence the sets \mathcal{V}_σ cover all of $\overline{\bigcup_{i=1}^{\infty} \{f_i\}}$. Now let $h \in \mathcal{V}_\sigma \subseteq \overline{\bigcup_{i=1}^{\infty} \{f_i\}}$. Then $\|h - f_{i_r}\|_\infty \rightarrow 0$ and with $a \in \mathcal{U}_{a_l}$ we obtain that

$$\|f_{i_r}(a) - f_{i_r}(a_l)\| < \frac{\epsilon}{4},$$

hence

$$\|h(a) - h(a_l)\| < \frac{\epsilon}{4}.$$

Likewise, for \hat{h} , we obtain that

$$\left\| \hat{h}(a) - \hat{h}(a_l) \right\| < \frac{\epsilon}{4}.$$

This implies

$$\begin{aligned} \left\| h(a) - \hat{h}(a) \right\| &\leq \left\| h(a) - h(a_l) \right\| + \left\| h(a_l) - x_{\sigma(l)} \right\| \\ &\quad + \left\| x_{\sigma(l)} - \hat{h}(a_l) \right\| + \left\| \hat{h}(a_l) - \hat{h}(a) \right\| \\ &\leq 4 \cdot \frac{\epsilon}{4} = \epsilon. \end{aligned}$$

Hence each set \mathcal{V}_σ is contained in a ball of radius $\epsilon > 0$. This implies that $\overline{\bigcup_{i=1}^{\infty} \{f_i\}}$ is covered by finitely many balls of radius $\epsilon > 0$ and therefore compact. The theorem follows since in a compact subset of a normed space every sequence has a convergent subsequence. \square

Proposition 3.3.17. *Let $\mathfrak{C} \subseteq M$ be a convex neighbourhood of some point $q \in M$ with compact closure and let $\{\gamma_i\}_{i \in \mathbb{N}}$ be a sequence of causal curves in $\overline{\mathfrak{C}}$ which are inextendible in $\overline{\mathfrak{C}}$. If $p \in M$ is an accumulation point of this sequence, i.e. each neighbourhood of p intersects infinitely many curves of $\{\gamma_i\}_{i \in \mathbb{N}}$, then there is a causal cluster curve γ through p which is inextendible in \mathfrak{C} .*

Proof. By considering a subsequence we can assume without loss of generality that for each $i \in \mathbb{N}$ there is an s_i such that $\gamma_i(s_i) \rightarrow p$. We choose the same coordinates as in Lemma 3.3.3 and view the curves γ_i ($i \in \mathbb{N}$) as continuous maps from finite intervals $[a_i, b_i]$ to \mathbb{R}_1^n ($i \in \mathbb{N}$). In order to apply Theorem 3.3.16 all curves γ_i ($i \in \mathbb{N}$) have to be defined on a common interval $[a, b]$. To this end we trivially enlarge the domain $[a_i, b_i]$ of the curve γ_i ($i \in \mathbb{N}$) to $[a, b] := [\inf_{i \in \mathbb{N}} \{a_i\}, \sup_{i \in \mathbb{N}} \{b_i\}]$ by setting $\gamma_i(s) := \gamma_i(a_i)$ for $s \in [a, a_i]$ and $\gamma_i(s) := \gamma_i(b_i)$ for $s \in [b_i, b]$. These curves are equi-continuous by Lemma 3.3.3. For all $s \in [a, b]$ the sets $\overline{\bigcup_{i \in \mathbb{N}} \gamma_i(s)}$ are compact since they are closed subsets of the compact set $\overline{\mathfrak{C}}$. Theorem 3.3.16 now implies that a subsequence of these curves converges uniformly to a continuous curve γ , hence γ is a cluster curve. Furthermore $p \in \{\gamma\}$, since the so obtained subsequence, which we denote by $\{\gamma_{i_j}\}_{j \in \mathbb{N}}$, converges uniformly to γ : suppose that $p \notin \{\gamma\}$. Then we find neighbourhoods \mathcal{U} of γ and \mathcal{V} of p with $\mathcal{U} \cap \mathcal{V} = \emptyset$ such that all but finitely many curves of the subsequence are contained in \mathcal{U} and $p \in \mathcal{V}$. But then we have that $\gamma_{i_j}(s_{i_j}) \in \mathcal{U}$ does not converge to $p \in \mathcal{V}$ as $j \rightarrow \infty$, which is a contradiction to our assumption.

We show that γ is causal. To this end let $s, t \in [a, b]$ with $s < t$. Since γ_i ($i \in \mathbb{N}$) is causal we have $\gamma_i(t) \in J^+(\gamma_i(s), \mathfrak{C})$ ($i \in \mathbb{N}$). Furthermore we have

$\gamma_i(t) \rightarrow \gamma(t)$ and $\gamma_i(s) \rightarrow \gamma(s)$. The relation \leq is closed by Lemma 3.1.32 (iv) and we obtain $\gamma(t) \in J^+(\gamma(s), \mathfrak{C})$. Since s and t were arbitrary in $[a, b]$ γ must be a causal curve. Since $\gamma(a)$ is an accumulation point of $\gamma_i(a) \in \overline{\mathfrak{C}} \setminus \mathfrak{C}$ this point must also lie in $\overline{\mathfrak{C}} \setminus \mathfrak{C}$. Analogously for $\gamma(b)$. This implies that γ is inextendible in \mathfrak{C} . \square

Note that all previous results that have been established in this section were proved for continuous causal curves. We now have to switch to piecewise smooth causal curves to obtain more global results. We will follow [1].

To this end we recall the following

Definition 3.3.18. *Let (M, g) be a semi-Riemannian manifold with $\dim(M) = n$ and let $\alpha : [a, b] \rightarrow M$ be a piecewise smooth curve in M . The arc-length of α , $L(\alpha)$, is defined by*

$$L(\alpha) := \int_a^b \|\alpha'(s)\| ds,$$

where $\|\alpha'(s)\| := |\langle \alpha'(s), \alpha'(s) \rangle|^{\frac{1}{2}}$. For a chart $(\mathcal{U}, \varphi := (x^1, \dots, x^n))$ we obtain that

$$\|\alpha'(s)\| = \left| g_{ij}(\alpha(s)) \frac{d(x^i \circ \alpha)}{ds}(s) \frac{d(x^j \circ \alpha)}{ds}(s) \right|^{\frac{1}{2}}.$$

Definition 3.3.19. *Let (M, h) be a Riemannian manifold and let $p, q \in M$. We define $\Omega(p, q)$ to be the set*

$$\Omega(p, q) := \{\alpha : \alpha \text{ is a piecewise smooth curve that connects } p \text{ and } q\}.$$

We define the Riemannian distance $d_0(p, q)$ of p and q to be

$$d_0(p, q) := \inf_{\alpha \in \Omega(p, q)} \{L(\alpha)\}.$$

Now we recall the theorem of Hopf and Rinow

Theorem 3.3.20. *Let (N, h) be a connected Riemannian manifold. Then the following are equivalent.*

- (i) *The metric space (N, d_0) is complete.*
- (ii) *For any $v \in TN$, the geodesic c in N with $c'(0) = v$ is defined for all real numbers $s \in \mathbb{R}$.*
- (iii) *For some $p \in N$, the exponential map \exp_p is defined on the entire tangent space $T_p N$ to N at p .*

(iv) Every subset K of N that is d_0 -bounded, i.e. $\sup\{d_0(p, q) : p, q \in K\} < \infty$ has compact closure.

Furthermore, if any of (i) to (iv) holds, then there is for each $p, q \in M$ a geodesic segment c from p to q with $L_0(c) = d_0(p, q)$, where L_0 is the Riemannian arc length.

Proof. For a proof see [15], section 5, Theorem 21 and Proposition 22. \square

A Riemannian manifold (N, h) is said to be complete provided any one (and hence all) of conditions (i) through (iv) in the theorem of Hopf and Rinow is satisfied. By the theorem of Nomizu and Ozeki (cf. [14], Theorem 1) it is even possible to obtain that (N, h) is a complete Riemannian manifold.

Remark 3.3.21. Let (M, g) be a space-time and let h be a complete Riemannian metric on M with distance function d_0 . By the theorem of Hopf and Rinow we obtain that the closed balls

$$\{q \in M : d_0(p, q) \leq r\}$$

are compact for all fixed $p \in M$ and $0 \leq r < \infty$. Let \mathcal{U} be a relatively compact convex neighbourhood of p . By Lemma 3.3.3 we obtain that all piecewise smooth causal curves $\gamma : [a, b] \rightarrow \mathcal{U}$ can be parametrised by $t = x^0$ such that the coordinate inequality

$$\left(\sum_{a=0}^{n-1} (x^a(t) - x^a(s))^2\right)^{\frac{1}{2}} \leq k |t - s|$$

holds for all $t, s \in [a, b]$. Thus γ is even Lipschitz continuous with Lipschitz constant $k \geq 0$. We furthermore obtain that $|x'_i| \leq k$ and we conclude that

$$L_0(\gamma) \leq nH^{\frac{1}{2}}k |a - b|,$$

where L_0 is the Riemannian arc length (cf. Definition 3.3.18) and H is the maximum of $|h_{ij}|$ on the compact set $\overline{\mathcal{U}}$ for $0 \leq i, j \leq n - 1$. Thus, we can parametrize γ such that

$$d_0(\gamma(s_1), \gamma(s_2)) \leq |s_1 - s_2|$$

for all $s_1, s_2 \in [a, b]$

The next proposition is based on a more general version of the theorem of Ascoli.

Theorem 3.3.22. *Let X be a locally compact Hausdorff space with a countable basis and let (M, h) be a complete Riemannian manifold with distance function d_0 . Assume that the sequence $\{f_n\}_{n \in \mathbb{N}}$ of functions $f_n : X \rightarrow M$ is equi-continuous and that for each $x_0 \in X$ the set $\bigcup_{n \in \mathbb{N}} \{f_n(x_0)\}$ is bounded with respect to d_0 . Then there exists a continuous function $f : X \rightarrow M$ and a subsequence $\{f_m\}_{m \in \mathbb{N}}$ of $\{f_n\}_{n \in \mathbb{N}}$ which converges to f uniformly on each compact subset of X .*

Proof. See [12], section 6, Theorem 6.1. □

Lemma 3.3.23. *Let (M, g) be a space-time and let h be a complete Riemannian metric on M with distance function d_0 . Let $\gamma : \mathbb{R} \rightarrow M$ be a piecewise smooth causal curve. Then there exists a parametrization of γ such that*

$$d_0(\gamma(s), \gamma(t)) \leq |s - t|$$

for all $s, t \in \mathbb{R}$.

Proof. By Remark 3.3.21 we can cover γ by countably many relatively compact convex sets $\{\mathfrak{C}_i\}_{i \in \mathbb{N}}$ with arbitrarily chosen Lipschitz constants. In particular we can choose the Lipschitz constants less than 2^{-i-1} on \mathfrak{C}_i and by the triangle inequality we obtain that

$$d_0(\gamma(s), \gamma(t)) \leq \sum_{i=0}^{\infty} 2^{-i-1} |s - t| = |s - t|.$$

□

Lemma 3.3.24. *Let (M, g) be a space-time and let h be a complete Riemannian metric on M . Let $\gamma : \mathbb{R} \rightarrow M$ be a continuous causal and hence locally Lipschitz continuous curve. Then γ possesses a parametrization by arc length relative to h .*

Proof. Without loss of generality, we can assume that $M = \mathbb{R}_1^n$. Since γ is locally Lipschitz, its derivation γ' exists almost everywhere and $\gamma' \in L_{\text{loc}}^\infty$. Now let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$,

$$\varphi(t) := \int_0^t \|\gamma'(s)\|_h ds.$$

Obviously φ is differentiable almost everywhere and $\varphi' = \|\gamma'\|_h$. We show that $\gamma'(s) \neq 0$ almost everywhere. Suppose that $\gamma' = 0$ on an open set. Then we obtain that $\gamma(s) = p$ on a whole interval $I = [s_0, s_1]$ for suitable s_0, s_1 . Since γ is continuous causal, there exists a causal \mathcal{C}^1 -curve μ that connects $\gamma(s_0)$ and $\gamma(s_1)$. For sufficiently small intervals $|s_0 - s_1|$, the curve segment $\gamma|_{[s_0, s_1]}$ is contained in a convex neighbourhood and the corresponding

causality is Minkowskian (cf. Lemma 3.1.30). But then μ obviously cannot be causal. Hence $\gamma'(s) \neq 0$ almost everywhere implying that φ is strictly monotonically increasing. Thus φ is invertible. We now define $\tilde{\gamma}(\tau) := \gamma(\varphi^{-1}(\tau))$. We obtain that $\tilde{\gamma}(\tau) = \gamma'(\varphi^{-1}(\tau))(\varphi^{-1})'(\tau)$ and since $(\varphi^{-1})'(\tau) = \frac{1}{\varphi'(\varphi^{-1}(\tau))} = \frac{1}{\|\gamma'(\varphi^{-1}(\tau))\|_h}$, we finally conclude that $\|\tilde{\gamma}(\tau)\|_h = 1$ for all τ . \square

Proposition 3.3.25. *Let $\{\gamma_n\}_{n \in \mathbb{N}}$ be a sequence of piecewise smooth (future-) inextendible causal curves in (M, g) . If $p \in M$ is an accumulation point of the sequence $\{\gamma_n\}_{n \in \mathbb{N}}$, then there is a continuous causal cluster curve γ of the sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ such that $p \in \{\gamma\}$. Furthermore γ is a (future-) inextendible causal curve.*

Proof. We only show the result for piecewise smooth inextendible curves since the proof for piecewise smooth future-inextendible curves is very similar.

Let h be a complete Riemannian metric for M with distance function d_0 as above. Without loss of generality we can assume that the domain of each γ_n with $n \in \mathbb{N}$ is \mathbb{R} since they are inextendible by assumption. By shifting parameterizations if necessary, we may then choose a subsequence $\{\gamma_{n_m}\}_{m \in \mathbb{N}}$ of $\{\gamma_n\}_{n \in \mathbb{N}}$ such that $\gamma_{n_m}(0) \rightarrow p$ for $m \rightarrow \infty$ since p is an accumulation point of the sequence $\{\gamma_n\}_{n \in \mathbb{N}}$. Now for each $m \in \mathbb{N}$ and each $s_1, s_2 \in \mathbb{R}$ we obtain that

$$d_0(\gamma_{n_m}(s_1), \gamma_{n_m}(s_2)) \leq |s_1 - s_2|$$

by Lemma 3.3.23. Thus the piecewise smooth curves $\{\gamma_{n_m}\}_{m \in \mathbb{N}}$ form an equi-continuous family.

Furthermore, since $\gamma_{n_m}(0) \rightarrow p$ for $m \rightarrow \infty$ there exists some $N \in \mathbb{N}$ such that $d_0(\gamma_{n_m}(0), p) < 1$ whenever $m \geq N$. This implies that for each fixed $s_0 \in \mathbb{R}$ and each $m \in \mathbb{N}$ the curve $\gamma_{n_m} : [-s_0, s_0] \rightarrow M$ of the subsequence $\{\gamma_{n_m}\}_{m \in \mathbb{N}}$ lies in the compact set $A := \{q \in M : d_0(p, q) \leq s_0 + 1\}$ whenever $m \geq N$. This follows since we have that $d_0(p, q) \leq d_0(p, \gamma_{n_m}(0)) + d_0(\gamma_{n_m}(0), q = \gamma_{n_m}(s_0)) < 1 + s_0$. A is compact by the theorem of Hopf and Rinow (cf. Remark 3.3.21). Hence the family $\{\gamma_{n_m}\}_{m \in \mathbb{N}}$ satisfies all assumptions in Theorem 3.3.22 and we thus obtain a continuous curve $\gamma : \mathbb{R} \rightarrow M$ and a subsequence $\{\gamma_{n_{m_k}}\}_{k \in \mathbb{N}}$ of $\{\gamma_{n_m}\}_{m \in \mathbb{N}}$ such that $\{\gamma_{n_{m_k}}\}_{k \in \mathbb{N}}$ converges to γ uniformly on each compact subset of \mathbb{R} . Obviously we have that $\gamma_{n_{m_k}}(0) \rightarrow p = \gamma(0)$. The convergence of $\{\gamma_{n_{m_k}}\}_{k \in \mathbb{N}}$ to γ also yields the inequality

$$d_0(\gamma(s_1), \gamma(s_2)) \leq |s_1 - s_2|$$

for all $s_1, s_2 \in \mathbb{R}$.

Hence it remains to show that γ is causal and inextendible. We show that γ is causal. To this end we fix $s_1 \in \mathbb{R}$. Let \mathcal{U} be a convex neighbourhood

of (M, g) that contains $\gamma(s_1)$. We choose some $\delta > 0$ such that the set $\{q \in M : d_0(\gamma(s_1), q) < \delta\}$ is contained in \mathcal{U} . If $s_1 < s_2 < s_1 + \delta$ holds, we can use $d_0(\gamma_{n_{m_k}}(s_1), \gamma_{n_{m_k}}(s_2)) \leq |s_1 - s_2|$ and the uniform convergence on compact subsets to obtain that $\gamma_{n_{m_k}}(s_1, s_2)$ lies in \mathcal{U} for large $k \in \mathbb{N}$. Since $\gamma_{n_{m_k}}(s_1) \rightarrow \gamma(s_1)$, $\gamma_{n_{m_k}}(s_2) \rightarrow \gamma(s_2)$ and $\gamma_{n_{m_k}}(s_1) \leq \gamma_{n_{m_k}}(s_2)$ in \mathcal{U} for all large $k \in \mathbb{N}$ and since \mathcal{U} is a convex neighbourhood, we conclude that $\gamma(s_1) \leq \gamma(s_2)$ in \mathcal{U} by Proposition 3.3.11. Thus $\gamma : [s_1, s_2] \rightarrow M$ is a continuous future-directed causal curve in \mathcal{U} . It follows that γ is a continuous future-directed causal curve in (M, g) .

It remains to show that γ is inextendible. We only give the proof of the future-inextendibility since the past-inextendibility is proved similarly. To this end we assume that γ is not future-inextendible. Then $\gamma(s) \rightarrow q_0$ for some $q_0 \in M$ as $s \rightarrow \infty$. Let \mathcal{U} be a convex neighbourhood of q_0 such that $\bar{\mathcal{U}}$ is a compact set contained in a chart (V, x) of M with local coordinates x_0, \dots, x_{n-1} such that $f = x_0 : \mathcal{U} \rightarrow \mathbb{R}$ is a time function for \mathcal{U} . Let $\gamma : [s_1, \infty) \rightarrow \mathcal{U}$. By Remark 3.3.21, there exists some $\delta > 0$ such that no piecewise smooth causal curve in \mathcal{U} from the level set $f^{-1}(f(\gamma(s_1)))$ to the level set $f^{-1}(f(q_0))$ can have arc length with respect to h greater than δ . On the other hand we have that $f(\gamma_{n_{m_k}}(s_1)) \rightarrow f(\gamma(s_1))$ and $f(\gamma_{n_{m_k}}(s_1 + 1)) \rightarrow f(\gamma(s_1 + 1))$. Since $f(\gamma(s_1)) < f(\gamma(s_1 + 1))$, we find some $k \in \mathbb{N}$ such that $f(\gamma(s_1)) \leq f(\gamma_{n_{m_k}}(s_1 + 1))$. Likewise we obtain that $f(\gamma_{n_{m_k}}(s_1 + \delta + 2)) \leq f(q_0)$ and we clearly have $f(\gamma_{n_{m_k}}(s_1 + 1)) \leq f(\gamma_{n_{m_k}}(s_1 + \delta + 2))$. Summing up, we obtain that

$$f(\gamma(s_1)) \leq f(\gamma_{n_{m_k}}(s_1 + 1)) \leq f(\gamma_{n_{m_k}}(s_1 + \delta + 2)) \leq f(q_0)$$

and we conclude that $\gamma_{n_{m_k}}(s_1 + 1, s_1 + \delta + 2) \subseteq f^{-1}(f(\gamma(s_1)), f(q_0))$. By Lemma 3.3.24 we find reparameterizations of $\{\gamma_{n_{m_k}}\}_{k \in \mathbb{N}}$ such that

$$L_0(\gamma_{n_{m_k}}(s_1 + 1, s_1 + \delta + 2)) = \delta + 1$$

for all $k \in \mathbb{N}$. This is a contradiction. \square

Remark 3.3.26. Proposition 3.3.25 now ensures the existence of a causal cluster limit γ in the sense of Definition 3.3.1. In particular γ is a priori not even piecewise smooth. This will lead us to obvious problems in the further sections (cf. the proof of Theorem 3.5.5). The point is that our literature uses piecewise smooth cluster curves by quoting results like Proposition 3.3.25 (cf. for instance [1], section 11, Proposition 11.39). At least there is no trivial reason why one could always obtain differentiability on cluster curves as in Proposition 3.3.25. A carefully thought out alternative approach to limit curves is given in [15], which leads us away from the previous ideas of studying

continuous cluster curves. The following quite technical and finicky results do save our proofs of the singularity theorems in the final chapter. We start with a

Definition 3.3.27. *Let (M, g) be a semi-Riemannian manifold. A convex covering \mathfrak{U} of (M, g) is a covering of (M, g) by convex open sets such that if elements \mathcal{U} and \mathcal{V} of \mathfrak{U} meet then $\mathcal{U} \cap \mathcal{V}$ is convex.*

Lemma 3.3.28. *Let (M, g) be a semi-Riemannian manifold and let \mathfrak{C} be an open covering of M . Then there exists a convex covering \mathfrak{U} of M such that each element in \mathfrak{U} is contained in some element of \mathfrak{C} .*

Proof. See [15], section 5, Lemma 10. □

Definition 3.3.29. *Let $\{\alpha_n\}_{n \in \mathbb{N}}$ be an infinite sequence of piecewise smooth future-directed causal curves in M and let \mathfrak{U} be a convex covering of M . A limit sequence for $\{\alpha_n\}_{n \in \mathbb{N}}$ relative to \mathfrak{U} is a finite or infinite sequence*

$$p = p_0 < p_1 < \dots$$

in M such that there are a subsequence $\{\alpha_{n_m}\}_{m \in \mathbb{N}}$ and, for each $m \in \mathbb{N}$, numbers $t_{m,0} < t_{m,1} < \dots$ such that

- (L1) (a) $\lim_{m \rightarrow \infty} \alpha_{n_m}(t_{m,j}) = p_j$ for all $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and
 (b) for each $j \in \mathbb{N}_0$ there is a $\mathcal{U}_j \in \mathfrak{U}$ and some $m(j)$ such that $p_j, p_{j+1} \in \mathcal{U}_j$ and $\alpha_{n_m}([t_{m,j}, t_{m,j+1}]) \subseteq \mathcal{U}_j$ for all $m \geq m(j)$.

- (L2) *If the sequence $p = p_0 < p_1 < \dots$ is infinite, it is nonconvergent. If $p = p_0 < p_1 < \dots < p_k$ (for some $k \in \mathbb{N}$) is finite, it contains more than one point and no strictly longer sequence satisfies (L1).*

To show the existence of limit curves, we have to establish a

Lemma 3.3.30. *Let $\alpha : [0, b) \rightarrow M$ be a piecewise smooth future-directed causal curve with $\alpha(0) = p$. Let $\tilde{\alpha} : [0, b] \rightarrow M$ be a continuous extension of α with $\tilde{\alpha}(b) = q$. Then we have*

- (i) $p \leq q$.
 (ii) *If \mathcal{U} is a neighbourhood of $\tilde{\alpha}([0, b])$, there exists a piecewise smooth future-directed causal curve β in \mathcal{U} that connects p with q .*

Proof. (i) Let \mathfrak{C} be a convex neighbourhood of q and let $b_i \nearrow b$ as $i \rightarrow \infty$ such that $\alpha(b_i) \in \mathfrak{C}$ for all $i \in \mathbb{N}$. We obviously have that $p \leq \alpha(b_i)$ for all $i \in \mathbb{N}$. Now let $i_0 \in \mathbb{N}$ be arbitrary. We obtain that $p \leq \alpha(b_{i_0}) \leq \alpha(b_i) \rightarrow q$ for all $i \geq i_0$ and hence $p \leq q$ by Lemma 3.1.32 (iv).

(ii) Let \mathfrak{C} be a convex neighbourhood of q such that $\mathfrak{C} \subseteq \mathcal{U}$. We choose some $t_0 \in [0, b]$ such that $\alpha(t_0) \in \mathfrak{C}$ for all $t \geq t_0$. As in the proof of (i) we obtain that $\alpha(t_0) \leq q$. By Lemma 3.1.32 (i) we conclude that the radial geodesic γ that connects $\alpha(t_0)$ and q is future-directed and causal, since $\overrightarrow{\alpha(t_0)q}$ is future-directed and causal by assumption that $\alpha(t_0) \leq q$ and since $\overrightarrow{\alpha(t_0)q}$ is the initial speed of γ . Now we can set $\beta := \alpha|_{[0, b]} \cup \gamma$ and the lemma is proved. \square

The following result shows the existence of limit curves as in Definition 3.3.29 under mild prerequisites.

Proposition 3.3.31. *Let $\{\alpha_n\}_{n \in \mathbb{N}}$ be a sequence of piecewise smooth future-directed causal curves such that the following conditions hold.*

- (i) $\alpha_n(0)$ converges to some $p \in M$.
- (ii) There is some neighbourhood of p that contains only finitely many of the curves α_n ($n \in \mathbb{N}$).

Let \mathfrak{U} be a convex covering of M . Then $\{\alpha_n\}_{n \in \mathbb{N}}$ has a limit sequence relative to \mathfrak{U} starting at p .

Proof. Since M is paracompact, there is a locally finite refinement \mathfrak{V} of \mathfrak{U} , such that for each $\mathcal{V} \in \mathfrak{V}$ the closure $\overline{\mathcal{V}}$ of \mathcal{V} is compact and contained in some $\mathcal{U} \in \mathfrak{U}$. We furthermore set $t_{m,0} := 0$ for all $m \in \mathbb{N}$.

By (i) and (ii) we can assume that there exists a neighbourhood $\mathcal{V}_0 \in \mathfrak{V}$ of $p \in M$ such that infinitely many (even all but finitely many) α_n ($n \in \mathbb{N}$) start in \mathcal{V}_0 and leave $\overline{\mathcal{V}_0}$. We denote this subsequence of $\{\alpha_n\}_{n \in \mathbb{N}}$ by $\{\alpha_{f_1(m)}\}_{m \in \mathbb{N}}$, where $f_1 : \mathbb{N} \rightarrow \mathbb{N}$ is strictly monotonically increasing.

We set

$$\tilde{t}_{m,1} := \inf\{t > 0 : \alpha_{f_1(m)}(t) \notin \mathcal{V}_0\}$$

for $m \in \mathbb{N}$, hence $\alpha_{f_1(m)}(\tilde{t}_{m,1})$ is the first point of $\alpha_{f_1(m)}$ in the boundary of \mathcal{V}_0 . Since $\partial\mathcal{V}_0$ is compact, there is a subsequence of $\{\alpha_{f_1(m)}(\tilde{t}_{m,1})\}_{m \in \mathbb{N}}$, without restriction of generality the sequence itself, that converges to some point p_1 in the boundary $\partial\mathcal{V}_0$.

We now choose some $\mathcal{V}_1 \in \mathfrak{V}$ such that $p_1 \in \mathcal{V}_1$. In the case that there are infinitely many $\alpha_{f_1(m)}$ that leave \mathcal{V}_1 , the same procedure as above gives us a subsequence $\{\alpha_{f_1(f_2(m))}\}_{m \in \mathbb{N}}$ of $\{\alpha_{f_1(m)}\}_{m \in \mathbb{N}}$. We again define

$$\tilde{t}_{m,2} := \inf\{t > 0 : \alpha_{f_1(f_2(m))}(t) \notin \mathcal{V}_1\}$$

and since $\partial\mathcal{V}_1$ is compact, a subsequence of $\{\alpha_{f_1(f_2(m))}\}_{m \in \mathbb{N}}$, without loss of generality $\{\alpha_{f_1(f_2(m))}\}_{m \in \mathbb{N}}$ itself, converges to some point $p_2 \in \partial\mathcal{V}_1$.

We now proceed by induction and repeat this procedure as long as possible, i.e. as long as there are infinitely many curves $\alpha_{f_1 \circ \dots \circ f_k(m)}$ that leave \mathcal{V}_k for some $k \in \mathbb{N}$. During this process we follow the following selection procedure of the \mathcal{V}_k ($k \in \mathbb{N}$). If there is more than one $\mathcal{V} \in \mathfrak{V}$ such that $p_k \in \mathcal{V}$, then let \mathcal{V}_k be the element of \mathfrak{V} that has been used the least often (if there are more than one with that property then choose any of these).

We can now construct a certain subsequence of $\{\alpha_n\}_{n \in \mathbb{N}}$ and define $t_{m,j}$ for $m \in \mathbb{N}$ and $j \in \mathbb{N}_0$ such that all conditions of Definition 3.3.29 are satisfied. To this end we have to distinguish between two cases.

In the first case the above procedure does not end. In this case we define

$$n_m := f_1 \circ \dots \circ f_m(m)$$

for all $m \in \mathbb{N}$ and set

$$t_{m,j} := \begin{cases} \text{arbitrary, but so that } t_{m,j} < t_{m,j+1} < \dots, & \text{for } m \leq j \\ \tilde{t}_{f_{j+1} \circ \dots \circ f_m(m),j}, & \text{for } m > j \end{cases}$$

for all $m \in \mathbb{N}$ and for all $j \in \mathbb{N}_0$. We show that $\{\alpha_{n_m}\}_{m \in \mathbb{N}}$ is a subsequence of $\{\alpha_n\}_{n \in \mathbb{N}}$ that satisfies (L1) in Definition 3.3.29. To this end let $j \in \mathbb{N}_0$ be arbitrary. For $m > j$ we obtain that

$$\alpha_{n_m}(t_{m,j}) = \alpha_{f_1 \circ \dots \circ f_j(f_{j+1} \circ \dots \circ f_m(m))}(\tilde{t}_{f_{j+1} \circ \dots \circ f_m(m),j})$$

is a subsequence of

$$\alpha_{f_1 \circ \dots \circ f_j(m)}(\tilde{t}_{m,j}).$$

Hence it converges to p_j . This shows (L1)(a). (L1)(b) is obvious by construction.

In the second case the above procedure does end after k steps. In this case we define

$$n_m := f_1 \circ \dots \circ f_k(m)$$

for all $m \in \mathbb{N}$ and set

$$t_{m,j} := \begin{cases} \text{arbitrary, but so that } t_{m,j} < t_{m,j+1} < \dots, & \text{for } m \leq k \\ \tilde{t}_{f_{j+1} \circ \dots \circ f_k(m),j}, & \text{for } j < k < m \\ \tilde{t}_{m,k}, & \text{for } j = k < m \end{cases} .$$

We show that $\{\alpha_{n_m}\}_{m \in \mathbb{N}}$ is a subsequence of $\{\alpha_n\}_{n \in \mathbb{N}}$ that satisfies (L1) in Definition 3.3.29. In fact, let $m > k$. Then we obtain that

$$\alpha_{n_m}(t_{m,j}) = \begin{cases} \alpha_{f_1 \circ \dots \circ f_j(f_{j+1} \circ \dots \circ f_k(m))}(\tilde{t}_{f_{j+1} \circ \dots \circ f_k(m),j}), & \text{for } j < k \\ \alpha_{f_1 \circ \dots \circ f_k(m)}(\tilde{t}_{m,k}), & \text{for } j = k \end{cases}$$

is a subsequence of $\alpha_{f_1 \circ \dots \circ f_j(m)}(\tilde{t}_{m,j})$, hence converges to p_j . This shows (L1)(a). (L2)(b) is again obvious by construction.

Furthermore, by construction we obtain that $\alpha_{n_m}(t_{m,j}) < \alpha_{n_m}(t_{m,j+1})$ for all $j < m$. Since $\alpha_{n_m}(t_{m,j}) \rightarrow p_j$ and $\alpha_{n_m}(t_{m,j+1}) \rightarrow p_{j+1}$, we can use Lemma 3.1.32 (iv) and conclude that $p_j \leq p_{j+1}$ for all $j \in \mathbb{N}_0$. Since $p_j \neq p_{j+1}$ for all $j \in \mathbb{N}_0$, we obtain that $p_j < p_{j+1}$.

It remains to show (L2). We distinguish two cases. In the first case we assume that the sequence $\{p_i\}_{i \in \mathbb{N}_0}$ is infinite, hence the above procedure does not terminate. We have to show that $\{p_i\}_{i \in \mathbb{N}_0}$ does not converge. We assume that $\{p_i\}_{i \in \mathbb{N}_0}$ converges to some $q \in M$. We find some $\mathcal{V} \in \mathfrak{V}$ such that $q \in \mathcal{V}$, hence all but finitely many p_i are contained in \mathcal{V} . Since $\overline{\mathcal{V}}$ is compact and since \mathfrak{V} is locally finite, there are only finitely many elements of \mathfrak{V} that have nonempty intersection with \mathcal{V} . Hence there is some $\mathcal{W} \in \mathfrak{V}$ that has been chosen infinitely often to be \mathcal{V}_i . We show that \mathcal{V} has only been chosen finitely many times to be \mathcal{V}_i . At each time when $\mathcal{V} = \mathcal{V}_i$ we obtain that $p_{i+1} \in \partial\mathcal{V}$. But all but finitely many of the p_i are contained in \mathcal{V} , hence not in $\partial\mathcal{V}$. This leads to a contradiction, since \mathcal{V} was infinitely many often a candidate for \mathcal{V}_i , but \mathcal{W} was always chosen instead of \mathcal{V} .

In the second case we assume that the sequence $\{p_i\}_{i \in \mathbb{N}_0}$ is finite, hence the above procedure terminates. Obviously only finitely many of the curves $\alpha_{f_1 \circ \dots \circ f_k(m)}$ can leave the set \mathcal{V}_k ($k \in \mathbb{N}$). We define $\{\alpha_{f_1 \circ \dots \circ f_{k+1}(m)}\}_{m \in \mathbb{N}}$ to be the subsequence of piecewise smooth future-directed causal curve that remain in \mathcal{V}_k . By a reparametrization we can obtain that each $\alpha_{f_1 \circ \dots \circ f_{k+1}(m)}$ ($m \in \mathbb{N}$) is defined on a finite interval $[0, b_m)$ with $b_m < \infty$. By Lemma 3.1.32 (v) we can extend each $\alpha_{f_1 \circ \dots \circ f_{k+1}(m)}$ ($m \in \mathbb{N}$) continuously to $[0, b_m]$. By Lemma 3.3.30 (ii) we can even assume that $\alpha_{f_1 \circ \dots \circ f_{k+1}(m)}$ ($m \in \mathbb{N}$) is defined on $[0, b_m]$. Since $\overline{\mathcal{V}_k}$ is compact, there exists a subsequence of $\{\alpha_{f_1 \circ \dots \circ f_{k+1}(m)}\}_{m \in \mathbb{N}}$, without restriction the sequence itself, that converges to some $q \in \overline{\mathcal{V}_k}$. We first assume that $q = p_k$ and show that in this case we can not extend $p_0 < \dots < p_k$ without violating (L1). Let us suppose that we can extend this sequence. Then there is a $p_{k+1} > p_k$ such that $p_0 < \dots < p_k < p_{k+1}$. On the other hand,

$$\alpha_{f_1 \circ \dots \circ f_{k+1}(m)}(\tilde{t}_{m,k+1}) \leq \alpha_{f_1 \circ \dots \circ f_{k+1}(m)}(b_m),$$

and $\alpha_{f_1 \circ \dots \circ f_{k+1}(m)}(\tilde{t}_{m,k+1}) \rightarrow p_{k+1}$ and $\alpha_{f_1 \circ \dots \circ f_{k+1}(m)}(b_m) \rightarrow p_k$, hence $p_{k+1} \leq p_k$ by Lemma 3.1.32 (iv). This is a contradiction since it would result (by Lemma 3.1.32 (iv)) in the existence of a closed causal curve within a convex set from \mathcal{U} (containing \mathcal{V}_k). On the other hand, if $q \neq p_k$, then (L1) and (L2) are satisfied for $p_0 < \dots < p_k < p_{k+1} := q$. As in the first case it follows that the sequence cannot be extended without violating (L1).

Finally, the sequence $\{p_i\}_{i \in \mathbb{N}_0}$ must contain more than one single point

by assumptions (i) and (ii). □

Definition 3.3.32. *If $\{p_i\}_{i \in \mathbb{N}_0}$ is a limit sequence for $\{\alpha_n\}_{n \in \mathbb{N}}$ as in Proposition 3.3.31, let λ_i with $i \in \mathbb{N}_0$ be the (future-directed causal) geodesic from p_i to p_{i+1} in a convex set \mathcal{U}_i as in (L1). Assembling these segments for all $i \in \mathbb{N}_0$ gives a broken geodesic $\lambda := \sum_{i=0}^{\infty} \lambda_i$. λ is called the quasi-limit of $\{\alpha_n\}_{n \in \mathbb{N}}$ with vertices p_i ($i \in \mathbb{N}_0$). Thus λ is a future-directed causal broken geodesic that starts in $p \in M$.*

Remark 3.3.33. *If $\{p_i\}_{i \in \mathbb{N}_0}$ is infinite, then by (L2), λ is future-inextendible. In the finite case $p_0 < \dots < p_k$ (with $k \in \mathbb{N}_0$), the quasi-limit λ connects p_0 with p_k .*

3.4 Achronal and acausal sets

Definition 3.4.1. *A set $A \subseteq M$ is said to be achronal if there are no points $p, q \in A$ with $p \ll q$, i.e. if no two points in A are chronologically related. This means that no piecewise smooth timelike curve meets A in two different points. A subset A' of M is called acausal if there are no points $p, q \in A'$ with $p < q$, i.e. if no piecewise smooth causal curve meets A' in two different points. Every acausal subset of M is achronal, but not conversely.*

Remark 3.4.2. *Let $A \subseteq M$ be an achronal set and let $B \subseteq A$. Then obviously B is achronal. The closure \overline{A} of an achronal set A is achronal. To see this assume that there are $p, q \in \overline{A}$ with $p \ll q$. We choose sequences $\{p_n\}_{n \in \mathbb{N}}$ and $\{q_n\}_{n \in \mathbb{N}}$ in A with $p_n \rightarrow p$ and $q_n \rightarrow q$. For sufficiently large $n \in \mathbb{N}$ we have $p_n \ll q_n$ by Lemma 3.1.33, which is a contradiction to the achronality of A . We also mention that a subset A of M is achronal if and only if $A \cap I^+(A) = \emptyset$ if and only if $A \cap I^-(A) = \emptyset$. The latter two conditions are used as a definition of achronality in several books.*

Definition 3.4.3. *Let A be an achronal set. Then the edge of A , $\text{edge}(A)$, is the set of all points $p \in \overline{A}$ such that for any neighbourhood \mathcal{U} of p there exists a pair of points p^\pm in $I^\pm(p, \mathcal{U})$ that can be joined by a piecewise smooth timelike curve which is contained in \mathcal{U} and does not intersect A .*

Our next aim is to prove that the boundary of so called future (or past) sets are achronal topological hypersurfaces. To make this precise, we need some preparations. We start with the following

Lemma 3.4.4. *Let $A \subseteq M$ be an achronal set. Then we have*

- (i) $\overline{A} \setminus A$ is a subset of the edge of A and

(ii) the edge of A is closed.

Proof. (i) Let $p \in \overline{A} \setminus A$ and let \mathcal{U} be a neighbourhood of p . By Lemma 3.1.30 (i) there exists a piecewise smooth future-directed timelike curve α through p that intersects $I^-(p, \mathcal{U})$ and $I^+(p, \mathcal{U})$. By Remark 3.4.2 we obtain that \overline{A} is achronal, hence α intersects \overline{A} only in p . Since $p \in \overline{A} \setminus A$, α does not intersect A and it follows that $p \in \text{edge}(A)$.

(ii) Let $p \in \text{edge}(A)$. We show that $p \in \text{edge}(A)$. Let \mathcal{U} be an open neighbourhood of p in M . By construction and Proposition 3.1.34 (i) the intersection $\mathfrak{J} := I^+(I^-(p, \mathcal{U}), \mathcal{U}) \cap I^-(I^+(p, \mathcal{U}), \mathcal{U})$ is an open neighbourhood of p and we can find an open neighbourhood \mathcal{V} such that $\mathcal{V} \subseteq \mathfrak{J}$. Since $p \in \text{edge}(A)$, the set $\mathcal{V} \cap \text{edge}(A)$ is nonempty. Let $p' \in \mathcal{V} \cap \text{edge}(A)$. By Definition 3.4.3 we find a piecewise smooth future-directed timelike curve $\alpha : [-1, 1] \rightarrow \mathcal{V}$ with $p^- := \alpha(-1) \in I^-(p', \mathcal{V})$ and $p^+ := \alpha(1) \in I^+(p', \mathcal{V})$ that does not intersect A . Since $p^- \in \mathcal{V} \subseteq I^+(I^-(p, \mathcal{U}), \mathcal{U})$, we can extend α to a piecewise smooth timelike curve $\alpha^- : [-2, 1] \rightarrow \mathcal{U}$ with $\alpha^-(-2) \in I^-(p, \mathcal{U})$. Analogously we can extend α^- to a piecewise smooth timelike curve $\tilde{\alpha} : [-2, 2] \rightarrow \mathcal{U}$ with $\tilde{\alpha}(2) \in I^+(p, \mathcal{U})$. We show that $\tilde{\alpha}|_{[-2, -1]}$ and $\tilde{\alpha}|_{[1, 2]}$ are not meeting A . We only show this for the former restriction and assume that $\tilde{\alpha}|_{[-2, -1]}$ meets A . We have $p^- \in I^-(p', \mathcal{V})$, hence $p' \in I^+(p^-, \mathcal{V})$. Since $I^+(p^-, \mathcal{V})$ is open by Proposition 3.1.34 (i) and $p' \in \text{edge}(A) \subseteq \overline{A}$ we find a $p'' \in A \cap I^+(p^-, \mathcal{V})$. Let β be a piecewise smooth future-directed timelike curve from p^- to p'' . Then the concatenation $\tilde{\alpha}|_{[-2, -1]} \cup \beta$ is a piecewise smooth timelike curve that meets A twice. This is a contradiction to the achronality of A . Hence $\tilde{\alpha} : [-2, 2] \rightarrow \mathcal{U}$ starts in $I^-(p, \mathcal{U})$ and ends in $I^+(p, \mathcal{U})$ without intersecting A . We obtain that $p \in \text{edge}(A)$. \square

Definition 3.4.5. A subset S of a smooth manifold M with $n := \dim(M)$ is said to be a topological hypersurface if for each point $p \in S$ there is an open neighbourhood \mathcal{U} of p in M , an open $\mathcal{V} \subseteq \mathbb{R}^n$ and a homeomorphism $\varphi : \mathcal{U} \rightarrow \mathcal{V}$ such that $\varphi(\mathcal{U} \cap S) = \mathcal{V} \cap (\{0\} \times \mathbb{R}^{n-1})$.

Definition 3.4.6. An n -dimensional topological manifold T is a Hausdorff space such that each point has a neighbourhood homeomorphic to an open set in \mathbb{R}^n .

Remark 3.4.7. For the next proof we now state a theorem of Brouwer: let $\mathcal{U} \subseteq \mathbb{R}^n$ be open and let $f : \mathcal{U} \rightarrow \mathbb{R}^n$ be a one-to-one and continuous map. Then $f(\mathcal{U})$ is open and f is a homeomorphism $f : \mathcal{U} \rightarrow f(\mathcal{U})$. A proof can be found in [11], section 7, Theorem 7.12.

Proposition 3.4.8. Let A be an achronal subset of M . Then the set $A \cap \text{edge}(A)$ is empty if and only if A is a topological hypersurface.

Proof. Let $A \cap \text{edge}(A) = \emptyset$ and let $p \in A$. Since $p \notin \text{edge}(A)$ there exists an open neighbourhood \mathcal{U} of p such that every piecewise smooth timelike curve from $I^-(p, \mathcal{U})$ to $I^+(p, \mathcal{U})$ that remains in \mathcal{U} intersects A . Without restriction let \mathcal{U} be a chart neighbourhood of the chart (φ, \mathcal{U}) with $\varphi : \mathcal{U} \rightarrow \varphi(\mathcal{U}) \subseteq \mathbb{R}^n$ and $\varphi = (x^0, \dots, x^{n-1})$. By choosing $\varphi := \exp_p$ we can reach by Lemma 3.1.30 (i) that ∂_{x^0} is future-directed timelike. By Lemma 3.1.30 (i) we choose an open subset \mathcal{V} of \mathcal{U} such that the following holds for some $a, b \in \mathbb{R}$ and for some $\delta > 0$.

- (i) $\varphi(\mathcal{V}) = (a - \delta, b + \delta) \times N \subset \mathbb{R} \times \mathbb{R}^{n-1}$ for an open subset N of \mathbb{R}^{n-1} .
- (ii) $\{x \in \mathcal{V} : x_0 = a\} \subseteq I^-(p, \mathcal{U})$ and $\{x \in \mathcal{V} : x_0 = b\} \subseteq I^+(p, \mathcal{U})$.

Let $y \in N \subseteq \mathbb{R}^{n-1}$. Then the piecewise smooth curve $\alpha : [a, b] \rightarrow \mathcal{V}$, defined by $s \mapsto \varphi^{-1}(s, y)$ is timelike, starts in $I^-(p, \mathcal{U})$, ends in $I^+(p, \mathcal{U})$ and hence intersects the set A by definition of \mathcal{U} . Since A is achronal there is a unique $h(y) \in (a, b)$ such that $\varphi^{-1}(h(y), y) \in A$. We show that $h : N \rightarrow (a, b)$ is continuous. To this end let $\{y_m\}_{m \in \mathbb{N}}$ be a sequence in N that converges to $y \in N$. We assume that $\{h(y_m)\}_{m \in \mathbb{N}}$ does not converge to $h(y)$. Since $[a, b]$ is compact and $h(N) \subseteq [a, b]$ there exists a subsequence, without restriction $\{y_m\}_{m \in \mathbb{N}}$ itself such that $\{h(y_m)\}_{m \in \mathbb{N}}$ converges to $r \in [a, b]$ with $r \neq h(y)$. Let $q := \varphi^{-1}(h(y), y) \in A$. Since the curve $s \mapsto \varphi^{-1}(s, y)$ is timelike and contains q and $\varphi^{-1}(r, y) \neq q$, we obtain

$$\varphi^{-1}(r, y) \in I^-(q, \mathcal{V}) \cup I^+(q, \mathcal{V}).$$

By Lemma 3.1.34 (i) this set is open and since \exp_p^{-1} is continuous, the sequence $\{\varphi^{-1}(h(y_m), y_m)\}_{m \in \mathbb{N}}$ converges to $\varphi^{-1}(r, y)$. Hence there exists an $m_0 \in \mathbb{N}$ with $A \ni \varphi^{-1}(h(y_{m_0}), y_{m_0}) \in I^-(q, \mathcal{V}) \cup I^+(q, \mathcal{V})$, but this is a contradiction to the achronality of A . This shows that h is continuous. We obtain that

$$\mathcal{V} \cap A = \varphi^{-1}(\{(h(y), y) : y \in N\}),$$

that is with respect to φ $\mathcal{V} \cap A$ is the graph of the continuous map h . Let $(\varphi^0, \varphi^1, \dots, \varphi^{n-1}) = (\varphi^0, \varphi')$ be the component functions of φ and let $\psi : V \rightarrow \mathbb{R}^n$, $\psi(p) := (\varphi^0(p) - h(\varphi'(p)), \varphi'(p))$. Obviously ψ is continuous. Then $\psi^{-1}(x^0, x') = \varphi^{-1}(x^0 + h(x'), x')$. Indeed,

$$\begin{aligned} \psi(\psi^{-1}(x^0, x')) &= \psi(\varphi^{-1}(x^0 + h(x'), x')) = (\varphi^0(\varphi^{-1}(x^0 + h(x'), x')) \\ &\quad - h(\varphi'(\varphi^{-1}(x^0 + h(x'), x'))), \varphi'(\varphi^{-1}(x^0 + h(x'), x'))) \\ &= (x^0 + h(x') - h(x'), x') = (x^0, x'). \end{aligned}$$

Similarly we verify that $\psi^{-1}(\psi(p)) = p$, hence ψ is bijective. $\psi(\mathcal{V})$ is open since $\varphi(\mathcal{V})$ is open and $\psi \circ \varphi^{-1}(x) = (x^0 - h(x'), x')$ is continuous and injective

on $\psi(\mathcal{V})$. By the theorem of Brouwer we obtain that $\psi : \mathcal{V} \rightarrow \psi(\mathcal{V})$ is a homeomorphism and $\psi(\mathcal{V})$ is open. Now we have

$$\begin{aligned}\psi(\mathcal{V} \cap A) &= \psi \circ \varphi^{-1}(\{(h(y), y) : y \in N\}) \\ &= \{(0, y) : y \in N\} \\ &= \psi(\mathcal{V}) \cap (\{0\} \times \mathbb{R}^{n-1}).\end{aligned}$$

To prove the converse statement let $p \in A$. Since we show a local statement we can assume without restriction that $M = \mathbb{R}_1^n$ (cf. Lemma 3.1.30 (i)). Let (φ, \mathcal{U}) be a chart as in Definition 3.4.5. We can assume that \mathcal{U} is connected. Since A is a topological hypersurface, we have

$$\varphi(\mathcal{U} \cap A) = \mathcal{V} \cap (\{0\} \times \mathbb{R}^{n-1}) =: \mathcal{V}_1.$$

In particular there is a homeomorphism

$$\varphi_1 := \varphi|_{\mathcal{U} \cap A} : \mathcal{U} \cap A \rightarrow \mathcal{V}_1.$$

Let $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ be the canonical projection, $\pi : \mathbb{R} \times \mathbb{R}^{n-1} \ni (x^0, x') \mapsto x' \in \mathbb{R}^{n-1}$. Since A is achronal and every vertical straight line $g : \mathbb{R} \ni t \mapsto (t, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$ is timelike, g intersects A at most once, hence $\pi|_A : A \rightarrow \mathbb{R}^{n-1}$ is one-to-one, implying that $\pi|_{\mathcal{U} \cap A} : \mathcal{U} \cap A \rightarrow \mathbb{R}^{n-1}$ is one-to-one. We furthermore obtain that $\pi \circ \varphi_1^{-1} : \mathcal{V}_1 \rightarrow \pi(\mathcal{U} \cap A)$ is bijective and continuous. By the theorem of Brouwer (cf. Remark 3.4.7) $\pi \circ \varphi_1^{-1}$ is a homeomorphism and $\pi(\mathcal{U} \cap A)$ is an open subset of \mathbb{R}^{n-1} , hence $\pi : \mathcal{U} \cap A \rightarrow \pi(\mathcal{U} \cap A)$ is a homeomorphism. Let $pr_0 : \mathbb{R}^n \rightarrow \mathbb{R}$, $pr_0 : (x^0, x') \mapsto x^0$ be the projection onto the first factor. We define the map $f : \pi(\mathcal{U} \cap A) \rightarrow \mathbb{R}$ by $f(x') := pr_0 \circ \pi^{-1}(x')$. Since pr_0 and π^{-1} are continuous, so is f . We obtain that

$$\mathcal{U} \cap A = \text{graph}(f) = \{(f(x'), x') : x' \in \pi(A \cap \mathcal{U})\},$$

hence $\mathcal{U} \setminus A$ splits in the following two connected components

- (i) $\mathcal{U}^+ := \{(x^0, x') \in \mathcal{U} : x^0 > f(x')\}$ and
- (ii) $\mathcal{U}^- := \{(x^0, x') \in \mathcal{U} : x^0 < f(x')\}$.

By Proposition 3.1.34 (i) the sets $I^+(p, \mathcal{U})$ and $I^-(p, \mathcal{U})$ are open and connected (cf. also Lemma 3.1.30 (i)) and since A is achronal they are subsets of $\mathcal{U} \setminus A$, hence they lie in \mathcal{U}^+ or \mathcal{U}^- . The vertical straight line through $p \in A$ intersects $I^+(p, \mathcal{U})$, $I^-(p, \mathcal{U})$, \mathcal{U}^+ and \mathcal{U}^- . It follows that $I^-(p, \mathcal{U}) \cap \mathcal{U}^- \neq \emptyset$ and hence we have $I^-(p, \mathcal{U}) \subseteq \mathcal{U}^-$. Analogously we obtain that $I^+(p, \mathcal{U}) \subseteq \mathcal{U}^+$. Hence every piecewise smooth curve α that remains in \mathcal{U} and connects $I^+(p, \mathcal{U})$ with $I^-(p, \mathcal{U})$ has to intersect $A \cap \mathcal{U}$, but that means that $p \notin \text{edge}(A)$. \square

Corollary 3.4.9. *Let A be an achronal subset of M . Then the edge of A , $\text{edge}(A)$ is empty if and only if A is a closed topological hypersurface.*

Proof. Let $\text{edge}(A) = \emptyset$. Obviously $A \cap \text{edge}(A) = \emptyset$. By Proposition 3.4.8 A is a topological hypersurface. With Lemma 3.4.4 (i) we find that $\overline{A} \setminus A \subseteq \text{edge}(A) = \emptyset$, hence $A = \overline{A}$ and A is closed. Now let A be a closed topological hypersurface. By Proposition 3.4.8 we obtain that $A \cap \text{edge}(A) = \emptyset$. By Definition 3.4.3 we have $\text{edge}(A) \subseteq \overline{A}$. Since $A = \overline{A}$ we finally have $\text{edge}(A) = \emptyset$. \square

Definition 3.4.10. *Let B be a subset of M . B is called future-set if $I^+(B) \subseteq B$. B is called past-set if $I^-(B) \subseteq B$.*

Remark 3.4.11. Let B be a future-set. We show that $M \setminus B$ is a past-set. We have to show that $I^-(M \setminus B) \subseteq M \setminus B$. To this end we assume there was some $q \in M \setminus B$ such that $I^-(q)$ is not a subset of $M \setminus B$. We find a $b \in B$ such that $b \in I^-(q)$ and hence $q \in I^+(b)$. It follows that $q \in I^+(B) \subseteq B$, since B is a future-set. This gives a contradiction. Of course the dual statement holds as well, hence if B is a past-set, then $M \setminus B$ is a future-set.

With these preparations we can now prove that the boundary of a future-set (past-set) is a hypersurface.

Proposition 3.4.12. *Let B be a nonempty future-set of M with $B \neq M$. Then its boundary ∂B is an achronal closed topological hypersurface.*

Proof. By Corollary 3.4.9 we have to show that the boundary of B is achronal and its edge is empty. To this end let $p \in \partial B$ and $q \in I^+(p)$. Obviously $p \in I^-(q)$ and $I^-(q)$ is an open neighbourhood of $p \in \partial B$ by Proposition 3.1.34 (i). It follows that $I^-(q) \cap B \neq \emptyset$ and since B is a future-set we obtain that $q \in I^+(B) \subseteq B$, hence $I^+(p) \subseteq I^+(B) \subseteq B$. Since $I^+(p)$ is open, we even have $I^+(p) \subseteq \text{int}(B) = B^\circ$. On the other hand let $p \in \partial B$ and $q \in I^-(p)$. It follows that $I^+(q)$ is an open neighbourhood of $p \in \partial B$ and we obtain that $I^+(q) \cap M \setminus B \neq \emptyset$. Hence $q \in I^-(M \setminus B)$ and by Remark 3.4.11 we have $q \in M \setminus B$. Since $I^-(M \setminus B)$ is open by Proposition 3.1.34 (i), we obtain that $q \in (M \setminus B)^\circ$ and it follows that $I^-(p) \subseteq (M \setminus B)^\circ$. Therefore $I^+(p) \cap \partial B = \emptyset$ and $I^-(p) \cap \partial B = \emptyset$ for all $p \in \partial B$ and hence ∂B is achronal. We show that the edge of the boundary of B is empty. To this end let $p \in \partial B$ and let α be a piecewise smooth timelike curve from $I^-(p)$ to $I^+(p)$. α starts in $(M \setminus B)^\circ$ and ends in B° , hence it intersects the boundary of B . It follows that $p \notin \text{edge}(\partial B)$, hence $\text{edge}(\partial B) = \emptyset$. \square

Corollary 3.4.13. *Let A be an arbitrary subset of M such that $J^\pm(A)$ is not empty and $J^\pm(A) \neq M$. Then $\partial J^\pm(A)$ is a closed achronal topological hypersurface.*

Proof. By Proposition 3.1.9 we obtain that $I^\pm(J^\pm(A)) = I^\pm(A) \subseteq J^\pm(A)$, hence $J^\pm(A)$ is a future-set. By Proposition 3.4.12 $\partial J^\pm(A)$ is an achronal closed topological hypersurface. \square

Proposition 3.4.14. *Let A be a spacelike topological hypersurface. Then the edge of A , $\text{edge}(A)$, is a subset of the boundary of A .*

Proof. Let $p \in A \setminus \partial A$. Since A is spacelike, by Lemma 3.1.30 there exists a neighbourhood \mathcal{U} of p such that

$$(I^+(p, \mathcal{U}) \cup I^-(p, \mathcal{U})) \cap A = \emptyset.$$

The same argument as in the second part of the proof of Proposition 3.4.8 shows that $\mathcal{U} \setminus A$ splits into two connected components \mathcal{U}^+ and \mathcal{U}^- which contain $I^+(p, \mathcal{U})$ and $I^-(p, \mathcal{U})$. But this implies that every piecewise smooth causal curve from $I^-(p, \mathcal{U})$ to $I^+(p, \mathcal{U})$ intersects A . \square

Proposition 3.4.15. *Let $A \subseteq M$ be a spacelike submanifold and let $p \in M$. Then $p \in E^+(A, \mathcal{U})$ (cf. Definition 3.1.28) holds if and only if there is a null geodesic from A to p which is completely contained in \mathcal{U} and does not have focal points before p .*

Proof. A proof can be found in [6], section 8, Lemma 8.3.4. \square

3.5 Cauchy developments

Definition 3.5.1. *Let $A \subseteq M$. The future Cauchy development (or the future domain of dependence) $D^+(A)$ is the set of all points $p \in M$ such that all past-inextendible piecewise smooth causal curves through p intersect A . The past Cauchy development (or the past domain of dependence) $D^-(A)$ is the set of all points $p \in M$ such that all future-inextendible piecewise smooth causal curves through p intersect A . The union*

$$D(A) := D^+(A) \cup D^-(A)$$

is called the Cauchy development of A (or the domain of dependence). It should be mentioned that in some books Cauchy developments are only defined for achronal sets.

Proposition 3.5.2. *Let A be an achronal set. Then we have*

$$I^-(\overline{D^+(A)}) \cap \overline{I^+(A)} \subseteq D^+(A).$$

Proof. Let $p \in I^-(\overline{D^+(A)}) \cap \overline{I^+(A)}$. Since $p \in I^-(\overline{D^+(A)})$, there exists a $\tilde{q} \in \overline{D^+(A)}$ such that $p \in I^-(\tilde{q})$, hence $\tilde{q} \in I^+(p)$. Since $I^+(p)$ is open by Proposition 3.1.34 (i) it follows that it intersects $D^+(A)$. Therefore we find a $\tilde{p} \in D^+(A) \cap I^+(p)$. Let γ be a piecewise smooth future-directed timelike curve from p to \tilde{p} . We assume that $p \notin D^+(A)$. Then there is a past-inextendible piecewise smooth causal curve μ through p which does not intersect A . The concatenation $\tilde{\gamma}$ of μ and γ is a past-inextendible piecewise smooth causal curve through \tilde{p} and hence intersects A at some point $z \in A$. We obtain $z \in I^+(p)$ and since $p \in \overline{I^+(A)}$ we have $z \in I^+(A)$ by Lemma 3.1.36 (iii). This implies that there is a piecewise smooth timelike curve which intersects A twice, but this is a contradiction to the achronality of A . \square

Our next aim is to prove that the interior of the Cauchy development of an achronal set is either globally hyperbolic (cf. Definition 3.1.39) or empty. This will require some preparations. We start with the following

Lemma 3.5.3. *Let $A \subseteq M$ be a closed set and let $\alpha : [0, b) \rightarrow M \setminus A$ be a piecewise smooth past-directed causal curve with $\alpha(0) = p$ that is past-inextendible in M . Then for each $q \in I^+(p, M \setminus A)$ there exists a piecewise smooth past-directed timelike curve $\tilde{\alpha} : [0, b) \rightarrow M \setminus A$ with $\tilde{\alpha}(0) = q$ that is past-inextendible in M .*

Proof. Since α is past-inextendible, we can assume without loss of generality that $b = \infty$ and $\{\alpha(n)\}_{n \in \mathbb{N}}$ is not convergent. Let d be a metric that generates the topology of M . Since A is closed, $M \setminus A$ is open and therefore a submanifold of M . By \ll we denote the relation on $M \setminus A$ (that implies the relation \ll in M). Let $M \setminus A \ni p_0 := q \gg p = \alpha(0)$. By definition of α we have $\alpha(1) \leq \alpha(0) = p \ll q = p_0$ and by Lemma 3.1.7 we obtain that $\alpha(1) \ll p_0$, hence there exists a piecewise smooth past-directed timelike curve α_1 that starts at p_0 and ends at $\alpha(1)$. We choose some $p_1 \in M \setminus A$ on α_1 with $0 < d(p_1, \alpha(1)) < 1$. Again, by definition of α we have $\alpha(2) \leq \alpha(1) \ll p_1$, hence $\alpha(2) \ll p_1$ and we can choose some $p_2 \in M \setminus A$ on a piecewise smooth past-directed timelike curve α_2 that connects $\alpha(2)$ with p_1 and satisfies $d(p_2, \alpha(2)) < \frac{1}{2}$. By induction we find $p_k \in M \setminus A$ ($k \geq 3$) with $\alpha(k) \ll p_k \ll p_{k-1}$ and $d(\alpha(k), p_k) < \frac{1}{k}$. Now we can connect all p_i ($i \geq 0$) by a piecewise smooth past-directed timelike curve $\tilde{\alpha}$ in $M \setminus A$, that contains all p_i ($i \geq 0$) and satisfies $\tilde{\alpha}(k) = p_k$ for all $k \geq 0$.

We show that $\tilde{\alpha}$ is past-inextendible. Assume that $\tilde{\alpha}$ was past extendible. Then we could find some $p_\infty \in M$ such that $\tilde{\alpha}(k) \rightarrow p_\infty$. But then we have $d(\alpha(k), p_\infty) \leq d(\alpha(k), p_k) + d(p_k, p_\infty) \rightarrow 0$ for $k \rightarrow \infty$, hence $\alpha(k) \rightarrow p_\infty$ which contradicts our assumption. \square

Lemma 3.5.4. *Let $A \subseteq M$ be an achronal set. Then every past- (future-)inextendible piecewise smooth causal curve which passes through $p \in \text{int}(D(A))$ intersects $I^-(A)$ ($I^+(A)$).*

Proof. We only prove this for the past-inextendible case. Let $\alpha : [0, b) \rightarrow M$ be a piecewise smooth past-inextendible causal curve that starts at $\alpha(0) = p \in \text{int}D(A)$. Since α is past-inextendible we can assume that it is defined on $[0, \infty)$. We first show that $D(A) \subseteq I^-(A) \cup I^+(A) \cup A$. To this end let $\tilde{p} \in D^+(A)$. Let $\tilde{\alpha}$ be a piecewise smooth past-inextendible timelike curve through \tilde{p} . By definition of the future Cauchy development $\tilde{\alpha}$ intersects A , hence $\tilde{p} \in A \cup I^+(A)$ and we obtain $D^+(A) \subseteq A \cup I^+(A)$. An analogous argument holds for $\tilde{p} \in D^-(A)$ and we have $D^-(A) \subseteq A \cup I^-(A)$. Since $D(A) = D^+(A) \cup D^-(A)$ we obtain $D(A) \subseteq I^-(A) \cup I^+(A) \cup A$.

If $p \in I^-(A)$, then we are finished. Assume that $p \in I^+(A) \cup A$. Recall that every neighbourhood of p intersects $I^+(p)$ by Lemma 3.1.30. Since $p \in \text{int}D(A)$, $D(A)$ is a neighbourhood of p and therefore there exists some $q \in I^+(p) \cap D(A)$. We show that $I^+(p) \cap D(A) = I^+(p) \cap D^+(A)$. To this end we show that $I^+(p) \cap D^-(A) = \emptyset$. Let $\tilde{p} \in I^+(p) \cap D^-(A)$ and first assume that $p \in A$. Then we have $\tilde{p} \in D^-(A) \cap I^+(A)$ and in particular $\tilde{p} \in I^+(A)$, hence there exists a piecewise smooth future-directed timelike curve $\tilde{\alpha}$ that connects $q \in A$ and \tilde{p} . We find a piecewise smooth future-inextendible timelike curve β with $\beta(0) = \tilde{p}$. Since $\tilde{p} \in D^-(A)$, β intersects A and hence the concatenation of $\tilde{\alpha}$ and β intersects A twice. But this gives a contradiction to the achronality of A . Now assume that $p \in I^+(A)$. Then we have $\tilde{p} \in D^-(A) \cap I^+(I^+(A)) = D^-(A) \cap I^+(A)$ and we can use the same argument to obtain a contradiction. Hence we have proved that $I^+(p) \cap D(A) = I^+(p) \cap D^+(A)$. By Lemma 3.5.3 there exists a piecewise smooth past-inextendible timelike curve $\gamma : [0, \infty) \rightarrow M$ that starts in $q \in I^+(p) \cap D(A)$. In addition, the proof of Lemma 3.5.3 shows that for every $s \in [0, \infty)$ there exists a $k \in \mathbb{N}$ with $\alpha(k) \in I^-(\gamma(s))$ since by construction for $k > s$ we have that $\alpha(k) \ll p_k = \gamma(k) \ll \gamma(s)$. Since $q \in D^+(A)$, γ intersects the set A in some point $\gamma(s)$. The corresponding $\alpha(k)$ therefore lies in $I^-(A)$. \square

Theorem 3.5.5. *Let A be an achronal set. Then the interior of the Cauchy development, $\text{int}D(A) = D(A)^\circ$, is globally hyperbolic or empty.*

Proof. We will divide the proof into four parts.

(i) We show that $\text{int}D(A)$ satisfies the causality condition. To this end we assume that there is a closed piecewise smooth causal curve α through some $p \in \text{int}D(A)$. By Lemma 3.5.4 α contains points q_+ and q_- such that $q_+ \in I^+(A)$ and $q_- \in I^-(A)$. We choose $q'_+, q'_- \in A$ such that $q_+ \in I^+(q'_+)$ and $q_- \in I^-(q'_-)$. Since $q_+, q_- \in \{\alpha\}$ and since α is closed and causal, we obtain that $q_+ \leq q_-$. We conclude that $q'_+ \ll q_+ \leq q_- \ll q'_-$ and hence $q'_+ \ll q'_-$. This is a contradiction to the achronality of A and we have proved the claim.

(ii) We show that $\text{int}D(A)$ satisfies the strong causality condition. To this end we assume that the strong causality condition is violated. Then there exist some $p \in D(A)^\circ$, a neighbourhood \mathcal{U} of p and a sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ of piecewise smooth future-directed causal curves $\alpha_n : [0, 1] \rightarrow M$ with $\alpha_n(0) \rightarrow p$, $\alpha_n(1) \rightarrow p$ and $\alpha_n([0, 1]) \not\subseteq \mathcal{U}$ for all $n \in \mathbb{N}$. Thus $\{\alpha_n\}_{n \in \mathbb{N}}$ has a limit sequence $p = p_0 < p_1 < \dots$ by Proposition 3.3.31.

We first assume that the limit sequence is finite, hence $p =: p_0 < p_1 < \dots < p_k$ for some $k \in \mathbb{N}$. We set $q := \lim_{n \rightarrow \infty} \alpha_n(1) = p$ and obtain that $p_k = \lim_{n \rightarrow \infty} \alpha_n(1) = p$ (cf. the proof of Proposition 3.3.31). We find that $p = p_0 < p_k = p$ which is a contradiction to (i).

Thus the limit sequence is infinite and we denote the corresponding quasi-limit by λ (cf. Definition 3.3.32). By Remark 3.3.33 λ is future-inextendible, hence by Lemma 3.5.4 it intersects the open set $I^+(A)$. Therefore there exists some s_0 in the domain of λ such that $\lambda(s_0) \in I^+(A)$. We also have that $\lambda(s) \geq \lambda(s_0)$ for all $s \geq s_0$ and since $\lambda(s_0) \in I^+(A)$, we obtain that $\lambda(s) \in J^+(\lambda(s_0)) \subseteq J^+(I^+(A)) = I^+(A)$, where we have used Proposition 3.1.9. This means that λ remains in $I^+(A)$ after it has passed $\lambda(s_0)$. In particular there is some $i_0 > 0$ such that $p_{i_0} \in I^+(A)$.

By Definition 3.3.29 (L1)(a) resp. Proposition 3.3.31 there are a subsequence of $\{\alpha_n\}_{n \in \mathbb{N}}$, without restriction $\{\alpha_n\}_{n \in \mathbb{N}}$ itself, and numbers $t_{n,i_0} \in [0, 1]$ such that $\alpha_n(t_{n,i_0}) \rightarrow p_{i_0}$, where $i_0 \in \mathbb{N}_0$ (even $i_0 \in \mathbb{N}$) is fixed. By reparameterizations we can achieve that $t_{n,i_0} = t_{0,i_0} \in [0, 1]$ for all $n \in \mathbb{N}$, hence there is some $t_0 := t_{0,i_0} \in [0, 1]$ such that $\alpha_n(t_0) \rightarrow p_{i_0}$. In particular we obtain that $\alpha_n(t_0) \in I^+(A)$ for suitable $n \in \mathbb{N}$.

Since $p < p_{i_0}$, $p \neq p_{i_0}$. Now let $\beta_n : [t_0, 1] \rightarrow M$ be defined by $\beta_n(t) := \alpha_n(t_0 + 1 - t)$. Obviously β_n is piecewise smooth and past-directed causal for all $n \in \mathbb{N}$, since α_n is piecewise smooth and future-directed causal for all $n \in \mathbb{N}$. We furthermore obtain that $\beta_n(t_0) = \alpha_n(1) \rightarrow p$ and since $\beta_n(1) = \alpha_n(t_0) \rightarrow p_{i_0} \neq p$, we conclude that there is a neighbourhood of p that contains only finitely many of the curves β_n . By Proposition

3.3.31 (resp. by its analogue for piecewise smooth past-directed causal curves) the sequence $\{\beta_n\}_{n \in \mathbb{N}}$ has a limit sequence $p =: q_0 > q_1 > \dots$

We first assume that the limit sequence is finite, hence there exists some $l \in \mathbb{N}_0$ such that $q_0 > q_1 > \dots > q_l$. By the same argument as above, we would obtain that $q_l = p_{i_0}$, hence $p < p_{i_0} = q_l < \dots < q_0 = p$. This is a contradiction to (i).

Hence we conclude that the limit sequence is infinite and the corresponding quasi-limit, we denote it by $\tilde{\lambda}$, is a past-inextendible causal piecewise geodesic that starts in $D(A)^\circ$. By Lemma 3.5.4 $\tilde{\lambda}$ intersects $I^-(A)$. By (L1)(a) in Definition 3.3.31 there exist an $n \in \mathbb{N}$ and some $t \in [0, 1]$ with $\alpha_n(t_0 + 1 - t) = \beta_n(t) \in I^-(A)$

We have shown that there is some α_n ($n \in \mathbb{N}$ suitable) that intersects $I^-(A)$ and $I^+(A)$. We set $t_1 := t_0 + 1 - t \in [t_0, 1]$ and obtain that $\alpha_n(t_0) \leq \alpha_n(t_1)$. Now we choose some $a_0 \in A$ such that $a_0 \ll \alpha_n(t_0)$ and some $a_1 \in A$ such that $\alpha_n(t_1) \ll a_1$. But this means that $a_0 \ll \alpha_n(t_0) \leq \alpha_n(t_1) \ll a_1$ and hence $a_0 \ll a_1$. This is a contradiction to the achronality of A . Hence we have proved claim (ii).

- (iii) We show that $J(p, q) = J^+(p) \cap J^-(q)$ is a compact set for arbitrary $p, q \in D(A)^\circ$. We first assume that $p \not\leq q$. In this case we obtain that $J(p, q) = \emptyset$, hence $J(p, q)$ is compact. Now let $p = q$. We show that $J(p, q) = J(p, p) = \{p\}$, hence $J(p, q)$ is compact. To this end we assume that there is some $r \in J(p, q)$ with $r \neq p$. This means that $p < r < p$ which is a contradiction to (i).

Hence it suffices to show the case where $p \neq q$ and $p < q$. Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in $J(p, q)$. We show that there is a subsequence of $\{x_n\}_{n \in \mathbb{N}}$ that converges to some element in $J(p, q)$. For each $x_n \in J(p, q)$ with $n \in \mathbb{N}$ there exists a piecewise smooth future-directed causal curve $\alpha_n : [0, 1] \rightarrow M$ that connects p and q and intersects x_n . Now let \mathfrak{U} be a convex covering of M such that each $\mathcal{U} \in \mathfrak{U}$ is convex, open and has a compact closure that is contained in some convex set. By Proposition 3.3.31 $\{\alpha_n\}_{n \in \mathbb{N}}$ has a limit sequence $p =: p_0 < p_1 < \dots$ relative to \mathfrak{U} . We distinguish two cases.

In the first case we assume that at least one limit sequence is finite, hence $p_0 < p_1 < \dots < p_N$ for some $N \in \mathbb{N}$. The same procedure as above shows that $p_N = q$. Now let $0 = t_{m,0} < t_{m,1} < \dots < t_{m,N} = 1$ as in Definition 3.3.29 (note that by the proof of Proposition 3.3.31, $t_{m,N} = 1$ is really achievable). For each $m \in \mathbb{N}$ the point x_m is contained in $\alpha_m([0, 1])$ and in particular contained in one of the intervals

$\alpha_m([t_{m,j}, t_{m,j+1}])$ for a suitable $0 \leq j \leq N-1$. By the pigeonhole principle there is one $j \in \{0, \dots, N-1\}$ such that infinitely many of the x_m are contained in $\alpha_m([t_{m,j}, t_{m,j+1}])$. Possibly restricting to a subsequence we can assume without loss of generality that $x_n \in \alpha_m([t_{m,j}, t_{m,j+1}])$ for this certain $j \in \{0, \dots, N-1\}$ and all $m \in \mathbb{N}$. By (L1)(b) we obtain that all x_m are contained in only one $\mathcal{U} \in \mathfrak{U}$. Since $\overline{\mathcal{U}}$ is compact, there is a subsequence of $\{x_m\}_{m \in \mathbb{N}}$, without restriction the sequence $\{x_m\}_{m \in \mathbb{N}}$ itself, such that $\{x_m\}_{m \in \mathbb{N}}$ converges to some $x \in \overline{\mathcal{U}} \subseteq \mathcal{V} \in \mathfrak{V}$, where \mathfrak{V} was chosen as a convex covering such that \mathfrak{U} is a refinement of \mathfrak{V} . Since \mathcal{V} is convex and since $\alpha_m(t_{m,j}) \leq x_m \leq \alpha_m(t_{m,j+1})$, Lemma 3.1.32 (iv) implies that $p_j \leq x \leq p_{j+1}$ and hence $p \leq x \leq q$. We obtain that $x \in J(p, q)$

In the second case we assume that all limit sequences are infinite and construct a contradiction, hence we show that this case cannot obtain. Now let $p = p_0 < p_1 < \dots$ be such a limit sequence. As in the proof of statement (ii) we obtain a future-inextendible causal quasi-limit λ and (by a reparameterization and choosing a subsequence) some $t_0 \in [0, 1]$ such that $\alpha_n(t_0) \rightarrow p_{i_0} \in I^+(A)$. As q is the endpoint of all α_n and since $p = p_0 < p_1 < \dots$ does not end, we obtain that $q \neq p_{i_0}$. Again, as in the proof of statement (ii) we define $\beta_n := -(\alpha_n|_{[t_0, 1]})$, $\beta_n : [t_0, 1] \rightarrow M$. Then we have that $\beta_n(t_0) = \alpha_n(1) = q$ for all $n \in \mathbb{N}$ and $\beta_n(1) = \alpha_n(t_0) \rightarrow p_{i_0} \neq q$, hence there is a neighbourhood of q that contains only finitely many of the curves β_n . By Proposition 3.3.31 $\{\beta_n\}_{n \in \mathbb{N}}$ has a limit sequence $q =: q_0 > q_1 > \dots$. We first assume that this sequence is finite, hence $q_0 > \dots > q_N$ for some $N \in \mathbb{N}$. As in the proof of (ii) we would obtain that $q_N = p_{i_0}$. But this means that $p = p_0 < p_1 < \dots < p_{i_0} = q_N > \dots > q_0 = q$, which is a finite limit sequence of $\{\alpha_n\}_{n \in \mathbb{N}}$ in contradiction to our assumption. Hence $q_0 > q_1 > \dots$ is infinite. The corresponding quasi-limit will be denoted by $\tilde{\lambda}$. $\tilde{\lambda}$ is past-inextendible and intersects (with the same proof as in (ii)) $I^-(A)$. As in (ii) we obtain a contradiction to the achronality of A .

- (iv) We finally show that $J(p, q) \subseteq D(A)^\circ$ for arbitrary $p, q \in D(A)^\circ$. For $p = q$ as well as for $p \not\leq q$ there is nothing to show (cf. the proof of (iii)). Now let $p < q$ and $p \neq q$. By the proof of Lemma 3.5.4 we have that $D^+(A) \subseteq A \cup I^+(A)$ and $D^-(A) \subseteq A \cup I^-(A)$. Hence p and q are contained in $I^+(A)$ or p and q are contained in $I^-(A)$ or p is contained in $J^-(A)$ and q is contained in $J^+(A)$. By symmetry reasons the case that $p \in J^+(A)$ and $q \in J^-(A)$ does not have to be treated. Therefore we distinguish the following cases.

In the first case we assume that $p, q \in I^+(A)$. The same proof holds for $p, q \in I^-(A)$ by interchanging p and q and $+$ and $-$. By Proposition 3.1.30 (i) each neighbourhood of p (in particular $D(A)$) intersects $I^+(q)$, hence there exists some $q^+ \in I^+(q) \cap D(A)$ and since $I^+(q) \subseteq I^+(A)$ by assumption, we obtain that $q^+ \in D^+(A)$. We now define

$$\mathcal{U} := I^+(A) \cap I^-(q^+).$$

Then $J(p, q) = J^+(p) \cap J^-(q) \subseteq J^+(I^+(A)) \cap J^-(I^-(q^+)) = I^+(A) \cap I^-(q^+) = \mathcal{U}$.

Hence it suffices to show that $\mathcal{U} \subseteq D(A)$. To this end let $x \in \mathcal{U}$. Since $\mathcal{U} \subseteq I^-(q^+)$ there is a piecewise smooth future-directed timelike curve α that connects x with q^+ and since $x \in I^+(A)$ there is a piecewise smooth future-directed timelike curve β that connects some $a \in A$ with x . We assume that α intersects A . Then we define the concatenation of α and β , $\alpha \cup \beta$, which is a piecewise smooth future-directed timelike curve that intersects A twice. This is a contradiction to the achronality of A . Hence α does not intersect A .

Now let γ be a piecewise smooth past-inextendible causal curve that starts in x . Then the concatenation $(-\alpha) \cup \gamma$ is a piecewise smooth past-inextendible causal curve through $q^+ \in D^+(A)$, hence it intersects A . This means that γ intersects A and hence $x \in D^+(A)$.

We now consider the second case, i.e. we let $p \in J^-(A)$ and $q \in J^+(A)$. As in the first case we can choose a $p_- \in I^-(p) \cap D(A)$ and a $q_+ \in I^+(q) \cap D(A)$. We define

$$\mathcal{U} := I^+(p_-) \cap I^-(q_+).$$

Obviously \mathcal{U} is open and clearly $p \in I^+(p_-)$ and $q \in I^-(q_+)$. Hence we conclude that $J(p, q) = J^+(p) \cap J^-(q) \subseteq J^+(I^+(p_-)) \cap J^-(I^-(q_+)) = I^+(p_-) \cap I^-(q_+) = \mathcal{U}$.

Again, it suffices to show that $\mathcal{U} \subseteq D(A)$. To this end let $x \in \mathcal{U}$. Since $A \subseteq D(A)$, we can assume that $x \notin A$. Since $x \in I^+(p_-)$ there is a piecewise smooth future-directed timelike curve α_- that connects p_- with x and since $x \in I^-(q_+)$ there is a piecewise smooth future-directed timelike curve α_+ that connects x with q_+ . Since A is achronal, the concatenation $\alpha := \alpha_- \cup \alpha_+$ intersects A at most once. Hence, since $x \notin A$, α_- or α_+ does not intersect A . Without loss of generality we assume that the curve α_+ does not intersect A . Let γ be a piecewise smooth past-inextendible causal curve that starts in x . Then the concatenation

$(-\alpha_+) \cup \gamma$ is a piecewise smooth past-inextendible causal curve through $q_+ \in D^+(A)$, hence it intersects A . This means that γ intersects A and it follows that $x \in D^+(A)$.

□

3.6 Causal Disconnectedness

Definition 3.6.1. Let (X, τ) be a noncompact topological space. An infinite sequence of points in X is said to diverge to infinity if given any compact subset \mathfrak{C} , only finitely many elements of this sequence are contained in \mathfrak{C} .

Definition 3.6.2. A noncompact space-time (M, g) is said to be causally disconnected by a compact set K if there exist two infinite sequences $\{p_n\}_{n \in \mathbb{N}}$ and $\{q_n\}_{n \in \mathbb{N}}$ diverging to infinity such that for each $n \in \mathbb{N}$, $p_n \leq q_n$, $p_n \neq q_n$ and all future-directed piecewise smooth causal curves from p_n to q_n meet K . A space-time (M, g) that is causally disconnected by some compact K is said to be causally disconnected.

Definition 3.6.3. Let (M, g) be an arbitrary space-time. We denote by $\Omega(p, q)$ the set of all piecewise smooth future-directed causal curves from p to q . The Lorentzian distance function (or time-separation) $d(p, q)$ is defined by

$$d(p, q) = \begin{cases} \sup\{L(c) : c \in \Omega(p, q)\} & \text{for } p < q \\ 0 & \text{else} \end{cases}$$

Definition 3.6.4. Let (M, g) be an arbitrary space-time. A past- and future-inextendible future-directed causal geodesic $\gamma : (a, b) \rightarrow M$ is said to be a causal geodesic line, if

$$L(\gamma|_{[s, t]}) = d(\gamma(s), \gamma(t))$$

holds for all s, t with $a < s \leq t < b$. d denotes the Lorentzian distance function.

For completeness we will quote an important statement. The proof can be found [1], section 7, Theorem 7.13.

Theorem 3.6.5. Let (M, g) be a strongly causal space-time (cf. Definition 3.1.22) which is causally disconnected by a compact set K . Then, for numbers $a, b \in \mathbb{R}$, M contains a piecewise smooth inextendible causal geodesic line $\gamma : (a, b) \rightarrow M$ which intersects K .

Remark 3.6.6. The space-time (M, g) in Theorem 3.6.5 is not completely arbitrary. We need noncompactness for causal disconnectedness. But since (M, g) satisfies the strong causality condition, it automatically satisfies the chronology condition by Theorem 3.1.43, thus there are no closed piecewise smooth timelike curves in (M, g) , hence (M, g) is not compact by Proposition 3.1.17.

Definition 3.6.7. Let (M, g) be an arbitrary space-time and let S be an achronal and closed subset of M . S is called *future-trapped* (*past-trapped*), if its future-horismos $E^+(S)$ (*past-horismos* $E^-(S)$) (cf. Definition 3.1.28) is compact.

Theorem 3.6.8. Let (M, g) be a strongly causal space-time. If S is future-trapped in (M, g) , then there is some piecewise smooth future-inextendible timelike curve γ contained in $D^+(E^+(S))$ (cf. Definitions 3.1.28 and 3.5.1).

Proof. We will prove a more general statement later (cf. Corollary 4.2.18). □

Remark 3.6.9. This section follows the book of Beem and Ehrlich. It should be mentioned that their definition of the Cauchy development is a bit weaker than ours, since they only define $D^+(S)$, $D^-(S)$ and hence $D(S)$ for closed subsets $S \subseteq M$. Since S is future-trapped in Theorem 3.6.8, $D^+(E^+(S))$ is nonetheless well defined.

In this section we finally study singularity theorems. By the existence of a singularity we mean timelike or causal geodesically incompleteness, which is caused by natural assumptions on the space-times. Singularity theorems are thus existence theorems for singularities. We will formulate and prove the most important singularity theorems in general relativity.

In the entire chapter we follow [1], [4] [5], [6], [15] and [21]. We start with

4.1 A first singularity theorem

The following rudimentary result already yields a first existence theorem of singularities. A (sketch of a) proof was found in [4], Lemma 2.10.

Proposition 4.1.1. *If (M, g) is a chronological (cf. Definition 3.1.10) space-time such that each inextendible null geodesic has a pair of conjugate points, then (M, g) is strongly causal.*

Proof. We assume that (M, g) fails to be strongly causal at some point $p \in M$ and derive a contradiction. We divide the proof into five parts.

- (i) Since strong causality is violated (cf. Remark 3.1.23), there exists an open neighbourhood \mathcal{U} of p such that $\bar{\mathcal{U}}$ is compact and the following holds. There are a sequence of neighbourhoods $\mathcal{V}_k \subseteq \mathcal{U}$ ($k \in \mathbb{N}$) which converges to p and piecewise smooth causal curves c_k ($k \in \mathbb{N}$) connecting some $x_k \in \mathcal{V}_k$ with some $y_k \in \mathcal{V}_k$ which leave \mathcal{U} and return to \mathcal{V}_k ($k \in \mathbb{N}$). Furthermore we can shrink \mathcal{U} such that there exists some convex neighbourhood \mathcal{U}' of p with $\bar{\mathcal{U}} \not\subseteq \mathcal{U}'$. Hence these curves intersect the boundary of \mathcal{U} , $\partial\mathcal{U}$. Let q_k ($k \in \mathbb{N}$) be the first intersection

point of the curve c_k with $\partial\mathcal{U}$ ($k \in \mathbb{N}$). Since $\partial\mathcal{U}$ is compact, there is a subsequence of the sequence $\{q_k\}_{k \in \mathbb{N}}$, without loss of generality the sequence $\{q_k\}_{k \in \mathbb{N}}$ itself, such that $q_k \rightarrow q \in \partial\mathcal{U}$. Since \mathcal{U}' is convex by assumption, there is a geodesic γ_{pq} that connects p and q . We show that γ_{pq} is a null geodesic. For each $k \in \mathbb{N}$ we obtain that $x_k \leq q_k$, since c_k is causal, hence $q_k \in J^+(x_k, \mathcal{U}')$. Since $q_k \rightarrow q$ and $x_k \rightarrow p$, we obtain that $q \in J^+(p, \mathcal{U}')$ and the vector \vec{pq} is causal. The latter two statements follow from Lemma 3.1.32 (i) and (iv). We show that \vec{pq} is null. We assume that \vec{pq} is timelike. By Lemma 3.1.33 there exist some $k \in \mathbb{N}$ and a neighbourhood \mathcal{W} of q such that $\mathcal{V}_k \ll \mathcal{W}_k$. By Proposition 3.1.7 we conclude for $y_k \in \mathcal{V}_k$ and $q_k \in \mathcal{W}_k$ that $y_k \ll q_k \leq y_k$ and thus $y_k \ll y_k$. This gives a contradiction to the assumption that (M, g) is chronological.

- (ii) We show that strong causality is violated in each point of γ_{pq} . We first show that strong causality fails in q itself. To see this, choose a sequence $\{\tilde{\mathcal{V}}_k\}_{k \in \mathbb{N}}$ of neighbourhoods of q that converge to q , whose closure is contained in some fixed convex neighbourhood $\tilde{\mathcal{U}}$ of q and such that $y_k \notin \tilde{\mathcal{U}}$ for all $k \in \mathbb{N}$.

Choose $\tilde{q}_k \in \tilde{\mathcal{V}}_k \cap I^+(q)$. Then $q \in I^-(\tilde{q}_k)$ and (by choosing a subsequence of $\{q_k\}_{k \in \mathbb{N}}$ if necessary) we may assume without loss of generality that $q_k \in I^-(\tilde{q}_k) \cap \mathcal{V}_k$ for all $k \in \mathbb{N}$ (since $q_k \rightarrow q$). Moreover, $I^-(\tilde{q}_k)$ is an open neighbourhood of p (since $p \leq q \ll \tilde{q}_k$) and $y_k \rightarrow p$, so again without loss of generality we may assume that $y_k \in I^-(\tilde{q}_k)$ for all $k \in \mathbb{N}$. Let \tilde{c}_k be a piecewise smooth future-directed timelike curve from y_k to \tilde{q}_k . Then $\hat{c}_k := c_k \cup \tilde{c}_k$ (where we let c_k start in q_k) is a piecewise smooth future-directed causal curve from q_k to \tilde{q}_k which contains $y_k \notin \tilde{\mathcal{V}}_k$. This sequence of curves shows that strong causality fails at q .

Next we show that strong causality in fact fails at each q'' on γ_{pq} . To show this we first note that each such q'' lies on the boundary of a suitable neighbourhood \mathcal{U}'' of \mathcal{U} with the same properties as \mathcal{U} above. In addition, we may assume that $\{\gamma_{pq}\} \cap \partial\mathcal{U}'' = \{q''\}$. Further, without restriction $\mathcal{V}_k \subseteq \mathcal{U}''$ for all $k \in \mathbb{N}$. Finally, we may replace the initial part of c_k from x_k to q_k by the unique geodesic $\gamma_{x_k q_k}$ connecting x_k and q_k in \mathcal{U}' . Denote by q_k'' the first point where $\gamma_{x_k q_k}$ intersects $\partial\mathcal{U}''$.

Then the initial velocity of $\gamma_{x_k q_k}$ is the causal vector $\overrightarrow{x_k q_k}$ and by construction there exists some $\lambda_k \in]0, 1]$ such that $\overrightarrow{x_k q_k''} = \lambda_k \overrightarrow{x_k q_k}$. Since $\partial\mathcal{U}''$ is compact, we may without loss of generality suppose that $\lambda_k \rightarrow \lambda \in [0, 1]$ and $q_k'' \rightarrow \tilde{q}'' \in \partial\mathcal{U}''$. Thus $\overrightarrow{p \tilde{q}''} = \lambda \overrightarrow{pq}$, so $\tilde{q}'' \in \{\gamma_{pq}\}$.

Since also $\tilde{q}'' \in \partial\mathcal{U}''$ it follows that $\tilde{q}'' \in \partial\mathcal{U}'' \cap \{\gamma_{pq}\} = \{q''\}$, i.e. $\tilde{q}'' = q''$. By what we have shown above, strong causality is violated in $\tilde{q}'' = q''$.

(iii) As we have seen in (ii), there exists a convex neighbourhood $\tilde{\mathcal{V}}_1$ of q such that the curves \hat{c}_k , starting from q_k leave $\tilde{\mathcal{V}}_1$ and re-enter $\tilde{\mathcal{V}}_k$. Therefore we may repeat the construction from (i) with q instead of p and $\tilde{\mathcal{V}}_1$ instead of \mathcal{U} . Note that \hat{c}_k coincides with c_k within $\tilde{\mathcal{V}}_1$ (in fact, up to y_k), so the curves used to obtain the new sequence of boundary points $\{r_k\}_{k \in \mathbb{N}}$ are in fact the c_k themselves. In this way we obtain a point $r = \lim_{k \rightarrow \infty} r_k \in M$ and a null geodesic γ_{qr} such that strong causality fails along γ_{qr} . We show that γ_{qr} extends γ_{pq} as an unbroken null geodesic. We assume that this is not the case. By Proposition 4.2.14 below we obtain that $p \ll r$, hence there exists a neighbourhood $\tilde{\mathcal{W}}$ of r and some $k \in \mathbb{N}$ such that $\mathcal{V}_k \ll \tilde{\mathcal{W}}$ by Lemma 3.1.33. For some $r_k \in \tilde{\mathcal{W}} \cap \{c_k\}$ we obtain that $r_k \leq y_k \ll r_k$. This means that $r_k \ll r_k$ for some $k \in \mathbb{N}$ by Proposition 3.1.7 which is a contradiction.

(iv) Now we can iterate this procedure both into the future and into the past to obtain an inextendible null geodesic γ along which strong causality fails.

(v) By assumption γ has conjugate points, hence there exist points $\tilde{p}, \tilde{r} \in \{\gamma\}$ with $\tilde{p} \ll \tilde{r}$ (cf. Proposition 4.2.14 below). We first assume that \tilde{p} is contained in the future-inextendible part of γ and \tilde{r} is contained in the past-inextendible part of γ . By construction we have that $p \leq \tilde{p}$. Since $\tilde{p} \ll \tilde{r}$, we obtain that $p \ll \tilde{r}$, hence by Lemma 3.1.33 we find some $k \in \mathbb{N}$ and neighbourhoods \mathcal{V}_k of p and $\tilde{\mathcal{V}}$ of \tilde{r} such that $\mathcal{V}_k \ll \tilde{\mathcal{V}}$. By construction of \mathcal{V}_k and since $\mathcal{V}_k \ll \tilde{\mathcal{V}}$ there is some $\tilde{r}_k \in \tilde{\mathcal{V}} \cap \{c_k\}$ such that $y_k \ll \tilde{r}_k$. Since $\tilde{r}_k \leq y_k$ by construction of the curve c_k , we conclude that $y_k \ll y_k$ by Proposition 3.1.7, which is a contradiction.

Now we assume that both points \tilde{p} and \tilde{r} are contained in the future-inextendible part of γ . Then $\tilde{p} \ll \tilde{r}$ and by construction $p \leq \tilde{p}$, hence $p \ll \tilde{r}$ by Proposition 3.1.7. Again, we can find some $k \in \mathbb{N}$ and neighbourhoods \mathcal{V}_k of p and $\tilde{\mathcal{V}}$ of \tilde{r} such that $\mathcal{V}_k \ll \tilde{\mathcal{V}}$. Obviously there is some $k \in \mathbb{N}$ and some $\tilde{r}_k \in \{c_k\}$ such that $y_k \ll \tilde{r}_k$. By construction of c_k we have that $\tilde{r}_k \leq y_k$. By Proposition 3.1.7 we obtain that $y_k \ll y_k$, which is a contradiction.

We finally assume that both points \tilde{p} and \tilde{r} are contained in the past-inextendible part of γ . Then $\tilde{p} \ll \tilde{r}$ and by construction we obtain that $\tilde{r} \leq p$, hence $\tilde{p} \ll p$ by Proposition 3.1.7. We find some neighbourhoods $\tilde{\mathcal{U}}$ of \tilde{p} and \mathcal{V}_k (for some $k \in \mathbb{N}$ suitable) such that $\tilde{\mathcal{U}} \ll \mathcal{V}_k$.

Choose k and \tilde{p}_k such that $\tilde{p}_k \in \tilde{\mathcal{U}} \cap \{c_k\}$. Then by construction, $x_k \leq \tilde{p}_k \ll x_k$, so $x_k \ll x_k$, contradiction.

□

The proof of the previous Proposition 4.1.1 establishes the following

Corollary 4.1.2. *Assume that (M, g) is chronological. If it fails to be strongly causal at some $p \in M$, then there is an achronal, inextendible null geodesic γ through p along which strong causality is violated. In particular we have that for chronological space-times the set where strong causality is violated is always generated by null geodesics.*

Theorem 4.1.3. *Let (M, g) be a chronological space-time with $\dim(M) \geq 3$ which satisfies the generic condition and the strong energy condition (cf. Definitions 3.2.1 and 3.2.4). Then (M, g) is either strongly causal or null incomplete.*

Proof. This follows directly from Proposition 4.1.1 and Theorem 3.2.21. □

With Theorem 3.6.5 one can prove a very similar result found in [1] Theorem 11.41.

Theorem 4.1.4. *Let (M, g) be a chronological space-time with $\dim(M) \geq 3$ which is causally disconnected (cf. Definition 3.6.2). If (M, g) satisfies the generic condition and the strong energy condition, then (M, g) is causally incomplete.*

4.2 The singularity theorem of Hawking and Penrose, Version 1

Definition 4.2.1. *Let F be a future-set of M . The null-boundary $\partial^{\text{null}}F$ is defined to be the set*

$$\partial^{\text{null}}F := \{p \in \partial F : \text{there is a neighbourhood } \mathcal{U} \text{ of } p \text{ with } I^+(F \setminus \mathcal{U}) = I^+(F)\}.$$

The acausal boundary of F is defined to be

$$\partial^{\text{ac}}F := \partial F \setminus \partial^{\text{null}}F.$$

Definition 4.2.2. *Let B be a nonempty future-set of M with $B \neq M$. Then its boundary ∂B is called a proper achronal boundary .*

Remark 4.2.3. Recall the definition of a future-endpoint (cf. Definition 3.2.2). We give a slightly different but equivalent definition. Let (M, g) be an arbitrary space-time. A point $p \in M$ is a future-endpoint of a piecewise smooth future-directed causal curve $\gamma : [a, b] \rightarrow M$ if for every neighbourhood \mathcal{U}_p of p there exists some $u_0 \in [a, b]$ such that $\gamma(u) \in \mathcal{U}_p$ for every $u \in [a, b]$ with $u \geq u_0$. Similarly we define a past-endpoint.

Remark 4.2.4. In Definition 4.2.2 we could have assumed that the future-set is open by simply taking its interior, since $I^+(B^\circ) \subseteq I^+(B) \subseteq B^\circ$ (cf. the proof of Proposition 3.4.12). Thus, every nonempty proper achronal boundary ∂B is the boundary of an open future-set. We denote this open future-set by B^+ . Any proper achronal boundary can be divided into four disjoint subsets B_A, B_N, B_F and B_P according to the following classification. B_A is acausal. B_N is the set of all points through which there passes a null geodesic segment contained in ∂B . B_F is the set of all future-endpoints of null geodesic segments in ∂B that are not in B_N . B_P is the set of all past-endpoints of null geodesic segments in ∂B that are not in B_N . For more details see [21], section 2. The following Proposition characterizes these subsets.

Proposition 4.2.5. *Let ∂B be the boundary of the open future set B^+ . If there is a neighbourhood \mathcal{U}_p of $p \in \partial B$ such that $B^+ = I^+(B^+ \setminus \mathcal{U}_p)$ (i.e. $p \in \partial^{\text{null}} B^+$) then $p \in B_N \cup B_F$.*

Proof. A proof can be found in [21], section 2, Proposition 2.17. □

Corollary 4.2.6. *Let $A \subseteq M$ and let $B := \partial J^+(A) = \partial I^+(A)$ (cf. Lemma 3.1.36 (iv)). Then, $B \setminus \bar{A} \subseteq B_N \cup B_F$.*

Proof. For a proof see [21], section 2, Corollary 2.3. □

Definition 4.2.7. *Let A be an achronal subset of M . The future Cauchy horizon $H^+(A)$ is defined to be the set*

$$H^+(A) := \overline{D^+(A)} \setminus I^-(D^+(A)).$$

The past Cauchy horizon $H^-(A)$ is defined to be the set

$$H^-(A) := \overline{D^-(A)} \setminus I^+(D^-(A)).$$

The Cauchy horizon $H(A)$ is the union $H(A) = H^+(A) \cup H^-(A)$ of the future and past Cauchy horizon.

Remark 4.2.8. Let $p \in \overline{D^+(A)}$. Obviously we have $p \in H^+(A) = \overline{D^+(A)} \setminus I^-(D^+(A))$ if and only if $p \notin I^-(D^+(A))$ if and only if there is no $q \in D^+(A)$ such that $q \gg p$ if and only if $I^+(p) \cap D^+(A) = \emptyset$. Hence we obtain that $H^+(A) = \overline{\{p \in \overline{D^+(A)} : I^+(p) \cap D^+(A) = \emptyset\}}$. Similarly we have that $H^-(A) = \overline{\{p \in \overline{D^-(A)} : I^-(p) \cap D^-(A) = \emptyset\}}$.

Lemma 4.2.9. *Let A be a closed achronal subset of M . Then the following hold.*

(i) *The closure of the future Cauchy development is the set of all points $p \in M$ such that every piecewise smooth past-inextendible timelike curve from p meets A and*

(ii) $I^+(H^+(A)) = I^+(A) \setminus \overline{D^+(A)}$.

Proof. (i) We define X to be the set

$$X := \{p \in M : \text{every piecewise smooth past-inextendible timelike curve through } p \text{ intersects } A\}$$

and show that $\overline{D^+(A)} = X$.

We show that $\overline{D^+(A)} \subseteq X$. To this end assume that $p \in \overline{D^+(A)} \setminus X$. Then there exists a piecewise smooth past-inextendible timelike curve $\alpha : [0, b) \rightarrow M$ with $\alpha(0) = p$, that does not intersect A . In particular we obtain that $p \notin A$. Since A is closed there is a convex neighbourhood \mathcal{U} of p with $\mathcal{U} \cap A = \emptyset$. Obviously there is some $s \in [0, b)$ such that $p \in I^+(q, \mathcal{U})$ with $q := \alpha(s)$. Since $I^+(q, \mathcal{U})$ is an open neighbourhood of $p \in \overline{D^+(A)}$ by Proposition 3.1.34 (i), we find some $r \in I^+(q, \mathcal{U}) \cap D^+(A)$. Let γ be the unique past-directed timelike geodesic that connects r and q in \mathcal{U} (cf. Proposition 3.1.30). Since $\mathcal{U} \cap A = \emptyset$, γ does not intersect A . The concatenation $\gamma \cup \alpha|_{[s, b)}$ is a piecewise smooth past-inextendible timelike curve that starts in $r \in D^+(A)$, hence it must intersect A by definition of $D^+(A)$. Since γ does not intersect A , $\alpha|_{[s, b)}$ has to intersect A , but this is a contradiction.

We show that $X \subseteq \overline{D^+(A)}$. Let $p \notin \overline{D^+(A)}$. We show that $p \notin X$. To this end we choose some $q \in I^-(p, M \setminus \overline{D^+(A)})$. In particular we have that $q \in M \setminus \overline{D^+(A)}$. Since $q \notin \overline{D^+(A)}$ we find a piecewise smooth past-inextendible causal curve α in M that starts in q and does not intersect A . Since $A \subseteq \overline{D^+(A)}$ and $q \in I^-(p, M \setminus \overline{D^+(A)})$, we obviously have that $p \in I^+(q, M \setminus \overline{D^+(A)}) \subseteq I^+(q, M \setminus A)$. By Lemma 3.5.3 there is some piecewise smooth past-inextendible timelike curve through p that does not intersect A . But this means that $p \notin X$.

(ii) We first show that $I^+(H^+(A)) \subseteq I^+(A) \setminus \overline{D^+(A)}$. By Remark 4.2.8 we have that $I^+(H^+(A)) \cap D^+(A) = \emptyset$. Since $I^+(H^+(A))$ is open by Proposition

3.1.34 (i) we obtain that $I^+(H^+(A)) \cap \overline{D^+(A)} = \emptyset$, hence it suffices to show that $I^+(H^+(A)) \subseteq I^+(A)$. To this end let $p \in I^+(H^+(A))$, thus we find some $q \in \overline{H^+(A)}$ such that $q \ll p$. Since $q \in H^+(A)$ we in particular have that $q \in \overline{D^+(A)}$. By part (i) of the Lemma there exists a piecewise smooth past-inextendible timelike curve through q that intersects A , hence there is some $a \in A$ with $a \ll q$. We conclude that $a \ll q \ll p$ and hence $a \ll p$. This means that $p \in I^+(a) \subseteq I^+(A)$.

We show that $I^+(H^+(A)) \supseteq I^+(A) \setminus \overline{D^+(A)}$. To this end let $p \in I^+(A) \setminus \overline{D^+(A)}$. Then there exists some $a \in A$ with $a \ll p$, i.e. $p \in I^+(a)$. We show that there exists some $q \in H^+(A)$ such that $q \ll p$. Let α be a piecewise smooth future-directed timelike curve that connects a with p . Since $A \subseteq \overline{D^+(A)} \subseteq \overline{D^+(A)}$ and $p \notin \overline{D^+(A)}$, α intersects $\partial \overline{D^+(A)}$ in some point $q \in \partial \overline{D^+(A)}$. We show that $q \in H^+(A)$. To this end we assume that $q \in \overline{D^+(A)} \setminus H^+(A)$. Then there exists some $r \in I^+(q) \cap D^+(A)$. Obviously $q \in I^-(r)$ and $I^+(A) \cap I^-(r)$ is an open neighbourhood of q . We show that $I^+(A) \cap I^-(r) \subseteq D^+(A)$ implying that $q \in D^+(A)^\circ$, which is a contradiction to $q \in \partial \overline{D^+(A)}$. Hence let $r' \in I^+(A) \cap I^-(r)$ and let α be a piecewise smooth past-inextendible causal curve that starts in r' . Since $r' \in I^-(r)$, there exists a piecewise smooth past-directed timelike curve γ that connects r and r' . Since γ is timelike and $r' \in I^+(A)$, γ is contained in $I^+(A)$. Since A is achronal we have that $A \cap I^+(A) = \emptyset$, hence γ does not intersect A . Since $r \in D^+(A)$, the concatenation $\gamma \cup \alpha$ intersects A , hence α intersects A . But this means that $r' \in D^+(A)$. Since $q \ll p$ we obtain that $p \in I^+(H^+(A))$. \square

Lemma 4.2.10. *Let A be a closed and achronal subset of M . Then we obtain that*

$$(i) \ I^+(\text{edge}(A)) \cap \overline{D^+(A)} = \emptyset \text{ and}$$

$$(ii) \ \text{edge}(H^+(A)) = \text{edge}(A).$$

Proof. (i) Let $s \in I^+(\text{edge}(A))$. Since A is closed there exists some $p \in \text{edge}(A) \subseteq \overline{A} = A$ such that $s \in I^+(p)$. By Lemma 3.1.33 the relation \ll is open, hence we find a neighbourhood \mathcal{U}_p of p such that $\mathcal{U}_p \ll s$. Since $p \in \text{edge}(A)$, there exist $r \in I^+(p, \mathcal{U}_p)$ and $q \in I^-(p, \mathcal{U}_p)$ and a piecewise smooth past-directed timelike curve β that connects r and q such that $\{\beta\} \cap A = \emptyset$. By the above we have that $\mathcal{U}_p \ll s$ and thus $r \ll s$, hence there exists some piecewise smooth past-directed timelike curve α that connects s and r . Now let γ be a piecewise smooth past-inextendible timelike curve that starts in q . Since $\{\alpha\} \subseteq I^+(A)$ and since A is achronal, we obtain that $\{\alpha\} \cap A = \emptyset$. Similarly we have that $\{\gamma\} \cap A = \emptyset$. We conclude that the concatenation

$\alpha \cup \beta \cup \gamma$ is a piecewise smooth past-inextendible timelike curve that does not intersect A , hence $s \notin \overline{D^+(A)}$ by Lemma 4.2.9 (i). This proves (i).

(ii) We first show that $\text{edge}(A) \subseteq \overline{D^+(A)}$. Obviously $A \subseteq D^+(A)$, since for $p \in A$ every piecewise smooth past-inextendible causal curve through p intersects A , hence we obtain that $\overline{\text{edge}(A)} \subseteq \overline{A} \subseteq \overline{D^+(A)}$. Let $p \in \text{edge}(A)$. By (i) we obtain that $I^+(p) \cap \overline{D^+(A)} = \emptyset$ so that $p \notin I^-(\overline{D^+(A)}) = I^-(D^+(A))$ (cf. Lemma 3.1.36 (i)). We then have $p \in \overline{D^+(A)} \setminus I^-(D^+(A)) = H^+(A)$. We show that in fact $p \in \text{edge}(H^+(A))$. Since $p \in \text{edge}(A)$, there exists a piecewise smooth past-directed timelike curve γ that connects some $r \in I^+(p, \mathcal{U}_p)$ with $q \in I^-(p, \mathcal{U}_p)$ such that $\{\gamma\} \cap A = \emptyset$. Let γ_1 be a piecewise smooth past-inextendible timelike curve that starts in q . We assume that γ intersects $H^+(A)$ in some point s . Then the concatenation $\tilde{\gamma} := \gamma|_{[s,q]} \cup \gamma_1$ is a piecewise smooth past-inextendible timelike curve and since $s \in H^+(A) \subseteq \overline{D^+(A)}$, $\tilde{\gamma}$ intersects A by Lemma 4.2.9 (i). Since $\{\gamma_1\} \cap A = \emptyset$, γ already intersects A . This is a contradiction, hence $p \in \text{edge}(H^+(A))$. The converse direction is very similar. \square

Now we can show a fundamental result

Proposition 4.2.11. *Let A be a closed achronal subset of M . Then $H^+(A)$ is generated by null geodesic segments which are either past-inextendible or have past-endpoint at $\text{edge}(A)$*

Proof. We only give a sketch of the proof (cf. also [21], Proposition 2.27). The set $B^+ := I^+(H^+(A)) = I^+(A) \setminus \overline{D^+(A)}$ is open by Lemma 4.2.9 and Proposition 3.1.34 (i). Furthermore B^+ is an open future-set by Proposition 3.1.9. Its boundary ∂B^+ is a proper achronal boundary in the sense of Definition 4.2.2. We show that $H^+(A)$ is closed. Obviously $\overline{D^+(A)}$ is closed and by Proposition 3.1.34 (i) the set $I^-(D^+(A))$ is open. Hence $H^+(A) = \overline{D^+(A)} \setminus I^-(D^+(A))$ is closed. We show that $H^+(A)$ is achronal. By Remark 4.2.8 we have that $I^+(H^+(A)) \cap D^+(A) = \emptyset$. Since $I^+(H^+(A))$ is open by Proposition 3.1.34 (i) we obtain that $I^+(H^+(A)) \cap \overline{D^+(A)} = \emptyset$. By definition of $H^+(A)$ we have $H^+(A) \subseteq \overline{D^+(A)}$ and hence $I^+(H^+(A)) \cap H^+(A) = \emptyset$. Thus $H^+(A)$ is achronal by Remark 3.4.2. Obviously we have the following chain of inclusions. $H^+(A) \subseteq J^+(H^+(A)) \subseteq \overline{I^+(H^+(A))} = \overline{B^+}$ and since $I^+(H^+(A)) \cap H^+(A) = \emptyset$ we obtain that $H^+(A) \subseteq \overline{B^+} \setminus B^+ = \partial B^+$. Hence $H^+(A)$ is a closed subset of ∂B^+ . Let $p \in H^+(A) \setminus A$. Since $p \in \overline{D^+(A)}$ by definition of the future Cauchy horizon, we obtain a piecewise smooth future-directed timelike curve γ from A to p by Lemma 4.2.9. Let $q \in \gamma \cap I^-(p)$. We find a neighbourhood \mathcal{U}_p of p such that $\overline{\mathcal{U}_p} \subseteq I^+(q)$. From every $r \in \mathcal{U}_p \cap B^+$, since $B^+ = I^+(A) \setminus \overline{D^+(A)}$, there exists a piecewise smooth past-inextendible timelike curve that does not intersect A (by Lemma 4.2.9 (i)) and therefore

does not meet $H^+(A)$. This curve has to intersect $\overline{\mathcal{U}_p}$ in some point $s \in \overline{\mathcal{U}_p}$. Since $s \in I^+(q)$, we find a piecewise smooth future-directed timelike curve $\tilde{\gamma}$ that connects q and s and does not intersect \mathcal{U}_p . $\tilde{\gamma}$ is necessarily crossing $H^+(A)$, since $p \notin \text{edge}(H^+(A))$. By joining $\tilde{\gamma}$ and γ we obtain a piecewise smooth future-directed timelike curve from $H^+(A) \setminus \mathcal{U}_p$ to any $r \in \mathcal{U}_p \cap B^+$ and we have that $\mathcal{U}_p \cap B^+ \subseteq I^+(H^+(A) \setminus \mathcal{U}_p)$. This means that the condition of Corollary 4.2.6 holds and we obtain that $p \in B_N \cup B_F$. Analogously, if $p \in (H^+(A) \cap A) \setminus \text{edge}(A)$, we choose the neighbourhood \mathcal{U}_p within $I^+(q) \cap I^-(r)$ for some points $q \in I^-(A)$ and $r \in I^+(A)$, and such that every timelike curve in $I^+(q) \cap I^-(r)$ meets $H^+(A)$ and A . Then, the same reasoning as before proves that $p \in B_N \cup B_F$. \square

Lemma 4.2.12. *Let A be a closed and achronal subset of M . Let $p \in \overline{D^-(A)} \setminus H^-(A)$. Then every piecewise smooth future-inextendible causal curve from p intersects $A \setminus H^-(A)$ and $I^+(A)$.*

Proof. Let γ be a piecewise smooth future-inextendible causal curve that starts at $p \in \overline{D^-(A)} \setminus H^-(A)$. Since $p \notin H^-(A)$, the set $I^-(p) \cap D^-(A)$ is nonempty by Remark 4.2.8. Let $q \in I^-(p) \cap D^-(A)$. Let λ be a piecewise smooth future-inextendible timelike curve from q . Then for each $r \in \lambda$ there is an $r_\lambda \in \gamma \cap I^+(r)$. Since $q \in D^-(A)$, λ intersects A at some point $s \in A$. We show that s cannot be in $H^-(A)$. To this end we assume that $s \in H^-(A)$. We obviously have that $q \in I^-(s) \subseteq \overline{I^-(H^+(A))}$ and since $q \in D^-(A) \subseteq \overline{D^-(A)}$, we obtain that $q \in I^-(H^-(A)) \cap \overline{D^-(A)} = \emptyset$. This is a contradiction. Furthermore there is a point $s_\lambda \in \gamma \cap I^+(s)$ by construction, thus γ enters into $I^+(A)$. \square

Lemma 4.2.13. *Let K be a strongly causal compact subset of an arbitrary space-time (M, g) . Then there is no piecewise smooth future-inextendible causal curve remaining in K or which enters and re-enters infinitely many times in K .*

Proof. We cover K with normal neighbourhoods and extract a finite cover \mathcal{U}_i with $i = \{1, \dots, n\}$. Since strong causality holds every piecewise smooth future-directed causal curve cannot re-enter into any of the \mathcal{U}_i ($i \in \{1, \dots, n\}$) once it has left it. Consequently, any piecewise smooth future-inextendible causal curve γ cannot remain in K . Similarly, γ can leave K and then re-enter into K through some \mathcal{U}_i ($i \in \{1, \dots, n\}$) not yet crossed by γ , but this can be done only a finite number of times. \square

For the next lemma we need the following result from the calculus of variations.

Proposition 4.2.14. *Let M be a Lorentzian manifold and let $p \in M$. Let α be a piecewise smooth causal curve that connects p with some $q \in M$. Then there exists a piecewise smooth timelike curve from p to q arbitrarily close to α unless α is a null pregeodesic along which there are no conjugate points from p to q .*

Proof. See [15], section 10, Theorem 51. □

Lemma 4.2.15. *Let (M, g) be a space-time and let $A \subseteq M$. Let α be a piecewise smooth future-directed causal curve that connects $p \in A$ with some $q \in E^+(A) = J^+(A) \setminus I^+(A)$. Then α is a null geodesic without conjugate points before q that does not intersect $I^+(A)$.*

Proof. We assume that α is not a null geodesic without conjugate points before q . By Proposition 4.2.14 there exists a piecewise smooth timelike curve β that connects p and q which is arbitrarily close to α . β is a variation of α , hence it is future-directed as well. But this means that $q \in I^+(A)$ which is a contradiction.

Now we assume that α intersects $I^+(A)$ in some point $\alpha(s_0) \in I^+(A)$. Obviously there is a piecewise smooth future-directed timelike curve β that connects some $p_1 \in A$ with $\alpha(s_0)$. The concatenation $\gamma := \beta \cup \alpha|_{[s_0, 1]}$ defines a piecewise smooth causal curve that connects p_1 and q . Since γ is not a null pregeodesic, we obtain that $q \in I^+(A)$ by Lemma 3.1.6. This is a contradiction and we have proved the lemma. □

The next result will be fundamental to what follows.

Proposition 4.2.16. *Let A be a closed and achronal subset of M . If the strong causality condition holds on $\overline{J^+(A)}$, then $H^+(\overline{E^+(A)})$ is non-compact or empty.*

Proof. We only give a sketch of the proof (cf. [21], Lemma 2.7 (iv)). By Corollary 4.2.6 we obtain that $\partial J^+(A) \setminus A = \partial J^+(A) \setminus \overline{A} \subseteq B_N \cup B_F$, where $\partial J^+(A)$ is the proper achronal boundary that can be divided into four disjoint subsets B_A, B_N, B_F and B_P (cf. Remark 4.2.4).

Now let $p \in \partial J^+(A) \setminus A$ and let $q \in E^+(A) = J^+(A) \setminus I^+(A)$. Obviously there is some piecewise smooth future-directed causal curve α and some $q \in A$ such that α connects q and p . Since $p \in E^+(A)$ we conclude that α is a null geodesic that does not intersect $I^+(A)$ by Lemma 4.2.15 and since $J^+(A)^\circ = I^+(A)$ by Proposition 3.1.34 we have that α remains in $\partial J^+(A)$. Hence $p \in B_N$ and there is a past-directed null geodesic through p with past-endpoint q that remains in $\partial J^+(A)$.

Now assume that $p \in (\partial J^+(A) \setminus A) \setminus E^+(A)$. We obtain that

$$\begin{aligned} \partial J^+(A) \setminus E^+(A) &= \partial J^+(A) \setminus (J^+(A) \setminus I^+(A)) \\ &= \partial J^+(A) \cap (J^+(A) \cap I^+(A)^c) \\ &= \partial J^+(A) \cap (J^+(A)^c \cup I^+(A)) \\ &= \partial J^+(A) \cap J^+(A)^c, \end{aligned}$$

hence $p \in \partial J^+(A) \cap J^+(A)^c$. Hence there is no geodesic segment that connects some $a \in A$ with p . We obtain that $p \notin B_N$, thus $p \in B_F$. The past-directed null geodesic segment through p lying in $\partial J^+(A)$ can be maximally extended to the past, hence there is a piecewise smooth past-inextendible null geodesic through $p \in \partial J^+(A)$.

We show that $p \notin \overline{D^+(\partial J^+(A))} \setminus H^+(\partial J^+(A))$ in the case that $p \in \partial J^+(A) \setminus A$ and $p \in \partial J^+(A) \setminus E^+(A)$. The piecewise smooth past-inextendible null segment remains in $\partial J^+(A)$ and thus it cannot enter in $I^-(\partial J^+(A))$. By Lemma 4.2.12 we obtain that $p \notin \overline{D^+(\partial J^+(A))} \setminus H^+(\partial J^+(A))$. Furthermore we have proved that

$$\partial J^+(A) \setminus \overline{E^+(A)} \subseteq H^+(\partial J^+(A)).$$

From this equation it follows that

$$\overline{D^+(\partial J^+(A))} \setminus \overline{D^+(\overline{E^+(A)})} = \partial J^+(A) \setminus \overline{E^+(A)} \subseteq H^+(\partial J^+(A)),$$

because this is the set of points from which there is a piecewise smooth past-inextendible timelike curve γ that intersects $\partial J^+(A) \setminus \overline{E^+(A)}$. γ must in fact start in $\partial J^+(A) \setminus \overline{E^+(A)}$ due to the achronality of $H^+(\partial J^+(A))$ and Lemma 4.2.9 (i).

We also have that

$$\text{int}D^+(\overline{E^+(A)}) = \text{int}D^+(\partial J^+(A))$$

and

$$H^+(\partial J^+(A)) \setminus H^+(\overline{E^+(A)}) = \partial J^+(A) \setminus \overline{E^+(A)}.$$

We now suppose that $H^+(\overline{E^+(A)})$ is compact. We can cover $H^+(\overline{E^+(A)})$ with a finite number of convex neighbourhoods $\{\mathcal{U}_i\}_{i=1, \dots, n}$ with compact closure such that the closure K of their union would still be compact and $K \supseteq H^+(\overline{E^+(A)})$.

We show that $I^-(\partial J^+(A)) \cap A = \emptyset$. To this end we assume that $p \in A$ and $p \in I^-(\partial J^+(A))$. Then there is some piecewise smooth past-directed timelike

curve that connects p with $q \in \partial J^+(A)$, i.e. $p \in I^-(q)$. Obviously we have that $q \in I^+(p)$ and we finally obtain that $q \in I^+(p) \subseteq I^+(A) = J^+(A)^\circ$. This is a contradiction since $q \in \partial J^+(A)$.

Every piecewise smooth past-directed causal curve from any $q \in J^+(A)$ to A must intersect $D^+(\overline{E^+(A)})$ because of the above properties and the fact that $I^-(\partial J^+(A)) \cap A = \emptyset$.

Now suppose that $H^+(\overline{E^+(A)})$ is nonempty. We will show a contradiction. We first obtain that $J^+(A) \setminus D^+(\partial J^+(A)) \cap K \neq \emptyset$. Hence there would be some piecewise smooth past-inextendible causal curve α_1 from this set that would not intersect $\partial J^+(A)$ and thus remaining in $J^+(A)$. In addition α_1 would not intersect $D^+(\overline{E^+(A)})$.

If α_1 remained in K , it would contradict strong causality by Lemma 4.2.13, hence α_1 leaves K . Obviously there is some point $q_1 \in \{\alpha_1\} \setminus K$ and since $q_1 \in J^+(A)$, there is some piecewise smooth past-directed causal curve $\tilde{\alpha}_1$ that connects q_1 with some point $a \in A$. But this means that $\tilde{\alpha}_1$ intersects $D^+(\overline{E^+(A)})$ which in turns means that $\tilde{\alpha}_1$ would have previously entered into $J^+(A) \setminus D^+(\partial J^+(A)) \cap K$ again. We choose another piecewise smooth past-directed causal curve α_2 with the same properties as α_1 and repeat the procedure. This yields to $\tilde{\alpha}_2$. We proceed by induction. The combination of all these curves would produce a piecewise smooth past-inextendible causal curve that enters, re-enters or remains in K , which is a contradiction to Lemma 4.2.13. \square

Definition 4.2.17. *Let (M, g) be a space-time and let \mathfrak{D} be an open subset of M . A congruence in \mathfrak{D} is a family of piecewise smooth curves such that through each $p \in \mathfrak{D}$ there passes precisely one such curve in this family. Thus, the tangent vectors to a congruence yield a vector field in \mathfrak{D} and, conversely every smooth vector field generates a congruence of curves. The congruence is said to be smooth if the corresponding vector field is smooth. A future-(past-)directed timelike congruence in \mathfrak{D} is a congruence in \mathfrak{D} such that all curves are future- (past-)directed timelike.*

Corollary 4.2.18. *Let (M, g) be a space-time such that there exists some future-trapped set A . If the strong causality condition holds on $J^+(A)$, then there is a piecewise smooth future-inextendible timelike curve γ contained in $D^+(\overline{E^+(A)})$.*

Proof. We only give a sketch of the proof (cf. [21], Lemma 4.1). We show that $E^+(A) \neq \emptyset$. Suppose that $E^+(A) = \emptyset$. Then $J^+(A) = I^+(A)$ and we obtain that $A \subseteq J^+(A) = I^+(A)$, hence $A \cap I^+(A) \neq \emptyset$, which is a contradiction to the achronality of A . By Proposition 4.2.16 the set $H^+(\overline{E^+(A)})$ is non-compact or empty. If $H^+(\overline{E^+(A)})$ is empty, then the result is trivial. Thus

we assume that $H^+(\overline{E^+(A)})$ is nonempty and non-compact. Since (M, g) is time-oriented, we can choose a future-directed timelike congruence in M . Since $H^+(E^+(A))$ is achronal, each piecewise smooth curve of the congruence that passes through $E^+(A)$ can intersect $H^+(E^+(A))$ at most once. If all of them intersected $H^+(E^+(A))$, there would be a one-to-one continuous map from $E^+(A)$ to $H^+(E^+(A))$, which is impossible because $E^+(A)$ is compact (since it is future-trapped) and $H^+(E^+(A))$ is not by assumption. Hence there must be some curves of the congruence not intersecting $H^+(E^+(A))$ and thus remaining in $D^+(E^+(A))$. \square

Definition 4.2.19. *Let (M, g) be a spacetime. A piecewise smooth inextendible causal curve which has compact closure and hence is contained in a compact set is said to be imprisoned. Let $K \subseteq M$ be a compact subset of M . A piecewise smooth future-directed causal curve $\gamma : [a, b) \rightarrow M$ is said to be future imprisoned in K if there is some $a \leq s_0 < b$ such that $\gamma(s) \in K$ for all $s_0 < s < b$. γ is said to be partially future imprisoned in K if there exists an infinite sequence $\{s_n\}_{n \in \mathbb{N}}$ that converges to b with $\gamma(s_n) \in K$ for each $n \in \mathbb{N}$. A similar definition is given for past imprisoned.*

Remark 4.2.20. Let (M, g) be a strongly causal space-time and let K be a compact subset of M . Then K may be covered with a finite number of normal neighbourhoods \mathcal{U}_i ($i = 1, \dots, n$) such that no piecewise smooth causal curve which leaves some \mathcal{U}_i ($i \in \{1, \dots, n\}$) ever returns to that \mathcal{U}_i ($i \in \{1, \dots, n\}$). This implies the following

Proposition 4.2.21. *Let (M, g) be a strongly causal space-time. Then no piecewise smooth inextendible causal curve can be partially future (or past) imprisoned in any compact set.*

Proposition 4.2.22. *Let (M, g) be a chronological space-time with $\dim(M) \geq 3$ such that each inextendible null geodesic has a pair of conjugate points. If (M, g) contains a future- (past-)trapped set S , then (M, g) is causally disconnected by $E^+(S)$ ($E^-(S)$).*

Proof. Let S be a future-trapped subset of M . By Theorem 3.6.8 we find a piecewise smooth future-inextendible timelike curve γ in the Cauchy development $D^+(E^+(S))$. We can extend γ to a piecewise smooth future and past-inextendible timelike curve in (M, g) . This extension will again be denoted by γ . We show that $E^+(S)$ is an achronal set. To this end let $p \in I^+(E^+(S)) = I^+(J^+(S) \setminus I^+(S)) \subseteq I^+(J^+(S)) = I^+(S)$, where we have used Proposition 3.1.9, hence $p \notin J^+(S) \setminus I^+(S)$ and therefore we obtain that

$$E^+(S) \cap I^+(E^+(S)) = (J^+(S) \setminus I^+(S)) \cap (I^+(J^+(S) \setminus I^+(S))) = \emptyset,$$

(cf. Remark 3.4.2). Since γ is a piecewise smooth inextendible and causal curve, it intersects by definition of the future Cauchy development the future-horismos and since $E^+(S)$ is achronal, γ intersects it in a unique point $r \in E^+(S) \cap \{\gamma\}$. Since (M, g) is strongly causal, we can use Proposition 4.2.21 and choose two sequences $\{p_n\}_{n \in \mathbb{N}}$ and $\{q_n\}_{n \in \mathbb{N}}$ on γ which diverge to infinity and satisfy $p_n \ll r \ll q_n$ for each $n \in \mathbb{N}$. We show that $\{p_n\}_{n \in \mathbb{N}}$, $\{q_n\}_{n \in \mathbb{N}}$ and $E^+(S)$ causally disconnect (M, g) . To this end we show that for each $n \in \mathbb{N}$ every piecewise smooth causal curve $\lambda : [0, 1] \rightarrow M$ with $\lambda(0) = p_n$ and $\lambda(1) = q_n$ meets $E^+(S)$. We can extend such a given λ to a piecewise smooth past-inextendible curve $\tilde{\lambda}$ by traversing γ up to p_n ($n \in \mathbb{N}$) and then traversing λ from p_n to q_n ($n \in \mathbb{N}$). Since $q_n \in D^+(E^+(S))$, the piecewise smooth curve $\tilde{\lambda}$ must intersect $E^+(S)$. Since γ meets $E^+(S)$ only at r , it follows that λ intersects $E^+(S)$. If S is past-trapped, we can use an analogous argument. \square

Definition 4.2.23. *A geodesic σ in M is cospacelike provided the subspace $\sigma'(s)^\perp$ of $T_{\sigma(s)}M$ is spacelike for one (hence every) s . In particular every timelike geodesic is cospacelike.*

Proposition 4.2.24. *Let $\sigma : [0, b] \rightarrow M$ be a cospacelike geodesic that connects p and q . If there is a conjugate point $\sigma(r)$ of p along σ with $0 < r < b$, then σ does not maximize the distance after it has passed its first conjugate point.*

Proof. For a proof see [15], section 10, Theorem 17 and Remark 18. \square

Now we can prove the first singularity theorem of Hawking and Penrose.

Theorem 4.2.25. *Let (M, g) be a space-time with $\dim(M) \geq 3$ that satisfies the generic condition and the strong energy condition. Then (M, g) cannot satisfy all of the following three requirements together.*

- (i) (M, g) contains no closed timelike curves.
- (ii) Every inextendible causal geodesic in (M, g) contains a pair of conjugate points.
- (iii) There exists a future- or past-trapped set S in (M, g) .

Proof. We assume that all three conditions are satisfied and show a contradiction. We first notice that (M, g) is strongly causal by Proposition 4.1.1. Without restriction we can assume that there exists a future-trapped set S in (M, g) , hence by Proposition 4.2.22 we obtain that (M, g) is causally disconnected by $E^+(S)$. By Theorem 3.6.5 there exists an inextendible causal

geodesic line γ and by assumption (ii) γ possesses a pair of conjugate points. We first assume that γ is timelike. By Proposition 4.2.24 γ does not maximize the distance which is a contradiction. Now let γ be a null geodesic. By 4.2.14 there exists a timelike curve arbitrarily close to γ . Again, γ does not maximize the distance, which gives a contradiction. Hence all three conditions cannot hold simultaneously. \square

4.3 The singularity theorem of Hawking and Penrose, Version 2

A slight variation of Theorem 4.2.25 gives a statement similar to Theorem 4.1.4

Theorem 4.3.1. *Let (M, g) be a chronological space-time with $\dim(M) \geq 3$ which satisfies the generic condition and the strong energy condition. If (M, g) contains a trapped set, then (M, g) is causally incomplete.*

Proof. Since (M, g) is chronological and contains a trapped set, there exists an inextendible causal geodesic γ that contains a pair of conjugate points (cf. Theorem 4.2.25). Hence γ is incomplete by Theorem 3.2.21. \square

4.4 The singularity theorem of Hawking and Penrose, Version 3

Recall the first version of the singularity theorem of Hawking and Penrose (cf. Theorem 4.2.25). In parts of the literature this Theorem is called the Lemma of Hawking and Penrose. Since it will be the crucial statement in the proof of the third version of the singularity theorem of Hawking and Penrose, we state it again.

Theorem 4.4.1. *(Lemma of Hawking and Penrose) The following three statements cannot hold simultaneously in any space-time (M, g) with $\dim(M) \geq 3$ that satisfies the generic condition and the strong energy condition.*

- (i) *Every piecewise smooth inextendible causal geodesic has a pair of conjugate points,*
- (ii) *the chronology condition is satisfied and*
- (iii) *there is a trapped set $A \subseteq M$.*

Definition 4.4.2. Let $A : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ be $\mathcal{C}^\infty(M)$ -linear, hence A is a $\binom{1}{1}$ -tensor field. Then the trace of A $tr(A)$ is defined as the $\binom{1}{1}$ -contraction of A , hence

$$tr(A) := C_1^1(A).$$

Remark 4.4.3. Let $n := \dim(M)$ and let E_1, \dots, E_n be a local $\binom{1}{0}$ -frame field. Let A be a $\binom{1}{1}$ -tensor field. Then one easily shows that

$$tr(A) = \sum_{i=1}^n \epsilon_i \langle E_i, A(E_i) \rangle.$$

Now let P be a semi-Riemannian submanifold of M and let

$$II : \mathfrak{X}(P) \times \mathfrak{X}(P) \rightarrow \mathfrak{X}(P)^\perp,$$

$$II(V, W) := \text{nor} \bar{\nabla}_V W$$

be its second fundamental form. We can contract the second fundamental form II to obtain the mean curvature vector field H . For $p \in P$ H_p is defined as

$$H_p := \frac{1}{l} \sum_{i=1}^l \epsilon_i II(e_i, e_i),$$

where $l := \dim(P)$ and e_1, \dots, e_l is a $\binom{1}{0}$ -frame at p . With these preparations we can give the next

Definition 4.4.4. Let P be a semi-Riemannian submanifold of M with mean curvature vector field H . The convergence of P is the real valued function k on the normal bundle NP such that

$$k(z) := \langle z, H(p) \rangle,$$

where $z \in T_p P^\perp$

Remark 4.4.5. Recall that for $V \in \mathfrak{X}(P)$ and $Z \in \mathfrak{X}(P)^\perp$ we can define

$$\tilde{II} : \mathfrak{X}(P) \times \mathfrak{X}(P)^\perp \rightarrow \mathfrak{X}(P),$$

$$\tilde{II}(V, Z) := \tan \bar{\nabla}_V Z$$

and one easily shows that this is \mathcal{C}^∞ -bilinear. By setting $S_Z V := -\tilde{II}(V, Z)$, we obtain that

$$\langle S_Z V, W \rangle = \langle V, S_Z W \rangle,$$

where $W \in \mathfrak{X}(P)$. $S_Z : \mathfrak{X}(P) \rightarrow \mathfrak{X}(P)$ is called the form operator in direction Z and we conclude that

$$k(z) = \frac{1}{l} tr(S_z)$$

for $l := \dim(P)$ and $z \in T_p P^\perp$. Details on this remark can be found in [15], section 10. We also recall the following

Proposition 4.4.6. *Let (M, g) be a space-time with $\dim(M) = n$ and let P be a spacelike $n - 2$ -dimensional submanifold of M with mean curvature vector field H of P . Let σ be a null geodesic normal to P at $p = \sigma(0)$ such that the following conditions hold.*

- (i) $k(\sigma'(0)) = \langle \sigma'(0), H_p \rangle_{TM} > 0$ and
- (ii) $\text{Ric}(\sigma', \sigma') \geq 0$.

Then there is a first focal point $\sigma(r)$ of P along σ with $0 < r \leq \frac{1}{k}$, where $k = k(\sigma'(0))$, provided σ is defined on this interval.

Proof. See [15], section 10, Proposition 43. □

Definition 4.4.7. *Let S be a compact spacelike submanifold of (M, g) without boundary and with $\dim(S) = n - 2$. Let E_{n-1} and E_n be future-directed null vector fields on S such that $(E_{n-1}, E_n) \in \mathfrak{X}(S)^\perp \times \mathfrak{X}(S)^\perp$ and such that $\{E_{n-1}(p), E_n(p)\}$ form a basis of $T_p M^\perp$ for all $p \in M$. Let S_1 and S_2 be the form operators in direction E_n and E_{n-1} . S is said to be a closed trapped surface if $\text{tr}(S_1)$ and $\text{tr}(S_2)$ are both either always positive or always negative on S .*

Remark 4.4.8. Note that in general, a closed trapped surface need not be a trapped set and vice versa (cf. Definition 3.6.7).

Definition 4.4.9. *Let (M, g) be a space-time. (M, g) satisfies the null convergence condition if $\text{Ric}(v, v) \geq 0$ for all null vectors $v \in TM$.*

Proposition 4.4.10. *Let (M, g) be a space-time with $\dim(M) \geq 3$ that satisfies the null convergence condition. If (M, g) contains a closed trapped surface S , then one of the following two conditions holds.*

- (i) *At least one of the sets $E^+(S)$ or $E^-(S)$ is compact.*
- (ii) *(M, g) is null incomplete.*

Proof. Let $\text{tr}(S_1) > 0$ and $\text{tr}(S_2) > 0$. We assume that (M, g) is null complete and show that $E^+(S)$ is compact. By Proposition 4.4.6 each null geodesic σ with $\sigma'(0) \perp S$ contains a geodesic segment which goes from a point $q \in S$ to a first focal point p . Furthermore, the derivation of each null geodesic orthogonal to S is proportional to E_{n-1} or E_n . Again, using Proposition 4.4.6 and the compactness of S , it follows that the union of all such null geodesic segments from S to a focal point is contained in a compact set K consisting of null geodesic segments starting in S .

Now let $r \in E^+(S)$. Then r can be joined to S by a past-directed null geodesic but not by a piecewise smooth past-directed timelike curve (cf. Lemma 4.2.15). Thus $r \in K$ and hence $E^+(S) \subseteq K$. To show that $E^+(S)$ is closed let $\{p_n\}_{n \in \mathbb{N}}$ be a sequence of points of $E^+(S)$ with limit p . Since K is closed, $p \in K$. From the definition of K we have $p \in J^+(S)$. We assume that $p \in I^+(S)$. Then the open set $I^+(S)$ must contain some elements of the sequence $\{p_n\}_{n \in \mathbb{N}}$, contradicting $p_n \in E^+(S)$ for all $n \in \mathbb{N}$. Thus $p \notin I^+(S)$ which yields $p \in E^+(S)$. This shows that $E^+(S)$ is a closed subset of the compact set K and hence is compact.

If we assume that (M, g) is null complete and that $tr(S_1) < 0$ and $tr(S_2) < 0$, then the same arguments show that $E^-(S)$ is compact. Hence we have proved the Proposition. \square

Proposition 4.4.11. *Let (M, g) be a strongly causal space-time with $\dim(M) \geq 3$ that satisfies the null convergence condition. If (M, g) contains a closed trapped surface S , then at least one of the following conditions hold.*

- (i) $H := E^+(S) \cap S$ (or $H := E^-(S) \cap S$) is a trapped set.
- (ii) (M, g) is null incomplete.

Proof. We assume that (M, g) is null complete and show that H is a trapped set. Then the proposition is proved. By Proposition 4.4.10 we obtain that either $E^+(S)$ or $E^-(S)$ is compact. We consider the case that $E^+(S)$ is compact. Obviously H is compact as a closed subset of the compact set $E^+(S)$. By Remark 3.4.2 H is achronal, since it is a subset of the achronal set $E^+(S)$ (cf. the proof of Proposition 4.2.22).

We show that H is nonempty. Since $E^+(S) = J^+(S) \setminus I^+(S)$, the set $H = S \setminus I^+(S)$ will be nonempty if and only if S contains some points which are not in $I^+(S)$. But if S was contained in $I^+(S)$, there would be a finite cover of the compact set S by open sets $I^+(p_1), \dots, I^+(p_n)$ with all $p_i \in S$. By the proof of Proposition 3.1.17 this would imply the existence of a piecewise smooth closed timelike curve in (M, g) which would contradict the strong causality of (M, g) . Hence $H \neq \emptyset$.

It remains to show that $E^+(H)$ is compact. We show that $E^+(H) = E^+(S)$. We will demonstrate this by showing that $I^+(H) = I^+(S)$ and $J^+(H) = J^+(S)$. To this end we cover the compact set S by a finite number of open sets $\mathcal{U}_1, \dots, \mathcal{U}_k$ of (M, g) such that each \mathcal{U}_i with $i \in \{1, \dots, k\}$ is a normal neighbourhood and no piecewise smooth causal curve which leaves \mathcal{U}_i with $i \in \{1, \dots, k\}$ ever returns.

Since S is spacelike by assumption, we may in addition assume that each $\mathcal{U}_i \cap S$ with $i \in \{1, \dots, k\}$ is achronal by choosing the \mathcal{U}_i with $i \in \{1, \dots, k\}$

sufficiently small. Since $H \subseteq S$, we obviously obtain that $I^+(H) \subseteq I^+(S)$. We show that $I^+(S) \subseteq I^+(H)$. To this end suppose there exists some $q \in I^+(S) \setminus I^+(H)$. Then there exists some $p_1 \in S$ with $p_1 \ll q$ and we conclude that $p_1 \in \mathcal{U}_{i(1)} \cap S$ for some index $i(1)$. Since $q \notin I^+(H)$, we obtain that $p_1 \notin H$ and hence $p_1 \notin E^+(S)$. Thus there exists $p_2 \in S$ with $p_2 \ll p_1$. Since $\mathcal{U}_{i(1)} \cap S$ is achronal, $p_2 \notin \mathcal{U}_{i(1)}$. Now $p_2 \in \mathcal{U}_{i(2)} \cap S$ for some $i(2) \neq i(1)$. Again $q \notin I^+(H)$ yields $p_2 \notin E^+(S)$. Thus there exists $p_3 \in S$ with $p_3 \ll p_2$. Furthermore, by construction of the sets \mathcal{U}_i (with $i \in \{1, \dots, k\}$) we have $p_3 \notin \mathcal{U}_{i(1)} \cup \mathcal{U}_{i(2)}$. Thus $p_3 \in \mathcal{U}_{i(3)} \cap S$ for some $i(3)$ different from $i(1)$ and $i(2)$. If we proceed in this manner we obtain an infinite sequence p_1, p_2, \dots in S with corresponding sets $\mathcal{U}_{i(1)}, \mathcal{U}_{i(2)}, \dots$ such that $i(j_1) \neq i(j_2)$ if $j_1 \neq j_2$. This contradicts the finiteness of the number of sets \mathcal{U}_i (with $i \in \{1, \dots, k\}$) of the given cover. Hence $I^+(H) \supseteq I^+(S)$.

It remains to show that $J^+(H) = J^+(S)$. Since $H \subseteq S$ we obviously obtain that $J^+(H) \subseteq J^+(S)$. We show that $J^+(S) \subseteq J^+(H)$. To this end suppose there exists some $q \in J^+(S) \setminus J^+(H)$. By the above we conclude that $q \notin I^+(H) = I^+(S)$ and hence there is a piecewise smooth future-directed null curve from some point $p \in S$ to the point q by Lemma 4.2.15. Since $p \in S$ and $p \leq q$, $p \notin I^+(S)$ (otherwise it would follow that $q \in I^+(H)$) and we have that $p \in E^+(S)$. Thus $p \in E^+(S) \cap S = H$ which yields $q \in J^+(H)$. But this is a contradiction.

Now we conclude that

$$E^+(H) = E^+(S)$$

which is compact by assumption. □

Lemma 4.4.12. *Let (M, g) be a space-time and let $\Sigma \subseteq M$ be a closed achronal set without edge. Then*

$$E^+(\Sigma) = E^-(\Sigma) = \Sigma.$$

In particular, if Σ is compact, then Σ is both future- and past-trapped.

Proof. We first show that $\Sigma \subseteq E^+(\Sigma)$. Since Σ is achronal, we obtain that $\Sigma \cap I^+(\Sigma) = \emptyset$ by Remark 3.4.2, hence $\Sigma \subseteq J^+(\Sigma) \setminus I^+(\Sigma) = E^+(\Sigma)$.

Next we show that $E^+(\Sigma) \subseteq \Sigma$. Suppose there exists $p \in \partial J^+(\Sigma) \setminus \Sigma$. Then there is a past-directed null geodesic segment γ through p that remains in $\partial J^+(\Sigma)$ and has a past-endpoint in Σ (cf. the proof of Proposition 4.2.16). Let $y \in \Sigma$ be a past-endpoint of γ . We show that $y \in \text{edge}(\Sigma)$. Since $\text{edge}(\Sigma) = \emptyset$ this will show that $\partial J^+(\Sigma) \setminus \Sigma = \emptyset$.

We first note that we can find a neighbourhood \mathcal{W} of $\{\gamma\} \setminus \{y\}$ which does not intersect Σ . Assume that $y \in \Sigma \setminus \text{edge}(\Sigma)$. We will show a contradiction.

Since $y \notin \text{edge}(\Sigma)$, we can find a convex neighbourhood \mathcal{U} of y such that for every pair of points \tilde{z}^\pm in $I^\pm(y, \mathcal{U})$ every piecewise smooth timelike curve λ which connects \tilde{z}^- and \tilde{z}^+ intersects Σ . Note that causality in \mathcal{U} corresponds to causality in the Minkowski-space.

Now let $x \in \mathcal{U} \cap \partial J^+(\Sigma) \setminus \Sigma$ and let $z^- \in I^-(y, \mathcal{U}) \subseteq I^-(x, \mathcal{U})$. We can choose some $z^+ \in I^+(y, \mathcal{U})$ such that the set

$$I^+(z^-, \mathcal{U}) \cap I^-(z^+, \mathcal{U}) \cap \{\gamma\} \cap \mathcal{W} \neq \emptyset.$$

Let $z_0 \in I^+(z^-, \mathcal{U}) \cap I^-(z^+, \mathcal{U}) \cap \{\gamma\} \cap \mathcal{W}$. Obviously there is a piecewise smooth future-directed timelike curve λ_1 that connects z^- with z_0 . Furthermore there exists some point $\hat{z}^+ \in I^+(y, \mathcal{U}) \cap \mathcal{W}$ and a piecewise smooth future-directed timelike curve $\lambda_2 \subseteq \mathcal{W}$ from z_0 to \hat{z}^+ . We define λ to be the concatenation of λ_1 and λ_2 , $\lambda := \lambda_1 \cup \lambda_2$. Since $y \in \Sigma \setminus \text{edge}(\Sigma)$, λ intersects Σ . By construction of \mathcal{W} we obtain that λ_2 cannot intersect Σ . Hence there is some point $z \in \lambda_1 \cap \Sigma$. We finally conclude that

$$x \in J^+(z_0) \subseteq J^+(I^+(z)) = I^+(z) \subseteq I^+(\Sigma)$$

in contradiction to $x \in \partial J^+(\Sigma)$.

Since $\text{edge}(\Sigma) = \emptyset$ by assumption, there cannot be any past-directed null geodesic segments that have past-endpoints in $\text{edge}(\Sigma)$, hence $\partial J^+(\Sigma) \setminus \Sigma = \emptyset$ and we conclude that

$$E^+(\Sigma) = J^+(\Sigma) \setminus I^+(\Sigma) = J^+(\Sigma) \setminus J^+(\Sigma)^\circ \subseteq \Sigma.$$

Very similar arguments show that $E^-(\Sigma) = \Sigma$. This finishes the proof. \square

Let us briefly summarize our results about the set $\partial J^+(A) \setminus A$ for a closed set $A \subseteq M$.

Corollary 4.4.13. *Let $A \subseteq M$ be a closed set. Then $\partial J^+(A)$ is an achronal topological hypersurface. Through every $p \in \partial J^+(A) \setminus A$ there passes a null geodesic segment that is either past-inextendible or has past-endpoint in $\text{edge}(A)$.*

Proof. By Corollary 3.4.13 we have that A is an achronal topological hypersurface. By the proof of Proposition 4.2.16 we obtain that through every $p \in \partial J^+(A) \setminus A$ there passes some null geodesic segment that is either past-inextendible or has past-endpoint in A . By the proof of Lemma 4.4.12 we finally conclude that these past-endpoints must be contained in $\text{edge}(A)$. \square

Now we can show the fundamental result of this section.

Theorem 4.4.14. (*Singularity theorem of Hawking and Penrose*) *Let (M, g) be a space-time with $\dim(M) \geq 3$ that satisfies the chronology condition, the generic condition and the strong energy condition. If one of the following conditions holds, then (M, g) is causally geodesically incomplete.*

- (i) *There exists a compact achronal set Σ without edge.*
- (ii) *There exists a closed trapped surface S .*

Proof. We first assume that there exists a compact achronal set Σ without edge. Suppose that (M, g) is causally geodesically complete. By Theorem 3.2.21 we obtain that each inextendible causal geodesic has a pair of conjugate points. By Lemma 4.4.12 Σ is both past- and future-trapped. Since (M, g) is chronological by assumption, we arrive at a contradiction to Theorem 4.4.1.

We now assume that there exists a closed trapped surface S . Again, suppose that (M, g) is causally geodesically complete. By Theorem 4.1.3 (M, g) is strongly causal. Since the strong energy condition implies the null convergence condition (cf. Definitions 3.2.4 and 4.4.9) Proposition 4.4.11 proves the existence of a trapped set H which is given by $H = E^+(S) \cap S$. By Theorem 3.2.21 each inextendible causal geodesic has a pair of conjugate points. This is again a contradiction to Theorem 4.4.1. \square

4.5 Singularities in globally hyperbolic space-times

Definition 4.5.1. *Let (M, g) be a space-time. A subset Σ of M is said to be a Cauchy hypersurface, if every piecewise smooth inextendible timelike curve intersects Σ precisely once.*

Proposition 4.5.2. *Let $\Sigma \subseteq M$ be a Cauchy hypersurface. Then*

- (i) Σ is achronal,
- (ii) Σ is a closed topological hypersurface and
- (iii) every piecewise smooth inextendible causal curve intersects Σ .

Proof. (i) We assume that there exists some piecewise smooth timelike curve α that intersects Σ twice. We can extend α to a piecewise smooth inextendible timelike curve $\tilde{\alpha}$. Obviously $\tilde{\alpha}$ intersects Σ at least twice. This is a contradiction to the definition of a Cauchy hypersurface, hence Σ is achronal.

(ii) We first show that

$$M = I^-(\Sigma) \sqcup \Sigma \sqcup I^+(\Sigma).$$

To this end let $p \in M$ and let α be a piecewise smooth inextendible timelike curve through p . Let $q \in M$ be the unique intersection point of α with Σ , hence $q \in \Sigma$. We obtain that $p \in I^-(q) \cup \{q\} \cup I^+(q) \subseteq I^-(\Sigma) \cup \Sigma \cup I^+(\Sigma)$ and therefore $M \subseteq I^-(\Sigma) \cup \Sigma \cup I^+(\Sigma)$. Obviously the converse direction holds, i.e. $I^-(\Sigma) \cup \Sigma \cup I^+(\Sigma) \subseteq M$. Thus we obtain that $M = I^-(\Sigma) \cup \Sigma \cup I^+(\Sigma)$. We show that this union is disjoint. Assume there was some $q \in I^\pm(\Sigma) \cap \Sigma$ or there was some $q \in I^-(\Sigma) \cap I^+(\Sigma)$. Then there exists some piecewise smooth timelike curve that intersects Σ twice which is a contradiction to the achronality of Σ . It follows that

$$\Sigma = M \setminus (I^-(\Sigma) \cup I^+(\Sigma))$$

is closed. We show that

$$\Sigma = \partial I^+(\Sigma) = \partial I^-(\Sigma).$$

By the previous part we have that $I^\pm(\Sigma) \sqcup \Sigma = M \setminus I^\mp(\Sigma)$ is closed, hence $I^\pm(\Sigma) \sqcup \Sigma = \overline{I^\pm(\Sigma) \sqcup \Sigma}$ and we conclude that $\partial I^\pm(\Sigma) = \overline{I^\pm(\Sigma)} \cap \overline{M \setminus I^\pm(\Sigma)} \subseteq (I^\pm(\Sigma) \sqcup \Sigma) \cap (I^\mp(\Sigma) \sqcup \Sigma) = \Sigma$. On the other hand we obviously have that $\Sigma \subseteq \partial I^\pm(\Sigma)$. We show that

$$\text{edge}(\Sigma) = \emptyset.$$

To this end we show that even every piecewise smooth timelike curve α from $I^-(\Sigma)$ to $I^+(\Sigma)$ intersects Σ . Assume that α did not intersect Σ , then we would obtain that $\alpha([a, b]) = (\alpha([a, b]) \cap I^-(\Sigma)) \sqcup (\alpha([a, b]) \cap I^+(\Sigma))$ which is a contradiction since $\alpha([a, b])$ is connected. Hence (ii) follows from Corollary 3.4.9.

(iii) We suppose that there is some piecewise smooth inextendible causal curve α that does not intersect Σ . Since $M = I^-(\Sigma) \sqcup \Sigma \sqcup I^+(\Sigma)$, we can assume without loss of generality that α is contained in $I^+(\Sigma)$. We choose a point $p \in \{\alpha\}$ and some $q \in I^+(p, M \setminus \Sigma)$. Since Σ is a closed set, we can use Lemma 3.5.3 to obtain a piecewise smooth past-directed timelike curve $\tilde{\alpha} : [0, b) \rightarrow M \setminus \Sigma$ such that $\tilde{\alpha}$ is past-inextendible in M . $\tilde{\alpha}$ does not intersect Σ , hence it is contained in $I^+(\Sigma)$. Now we can extend $\tilde{\alpha}$ to the future such that it becomes a piecewise smooth inextendible timelike curve that remains in $I^+(\Sigma)$, thus it does not intersect Σ . This is a contradiction to the definition of a Cauchy hypersurface. This proves (iii). \square

Proposition 4.5.3. *Let Σ be an achronal subset of M . Σ is a Cauchy hypersurface if and only if its domain of dependence equals M , i.e. $D(\Sigma) = M$.*

Proof. Let Σ be a Cauchy hypersurface of M . The proof of Proposition 4.5.2 states that $M = I^-(\Sigma) \sqcup \Sigma \sqcup I^+(\Sigma)$. Furthermore we show that

$$D^\pm(\Sigma) = \Sigma \cup I^\pm(\Sigma).$$

We only show this for the future-case. By the proof of Lemma 3.5.4 we obtain that $D^+(\Sigma) \subseteq \Sigma \cup I^+(\Sigma)$. We also have $\Sigma \subseteq D^+(\Sigma)$ (cf. the proof of Lemma 4.2.10). It remains to show that $I^+(\Sigma) \subseteq D^+(\Sigma)$. To this end let $q \in I^+(\Sigma)$ and let α be a piecewise smooth past-inextendible causal curve through q . Let $\tilde{\alpha}$ be a piecewise smooth future-inextendible extension of α such that $\tilde{\alpha} = \alpha \cup \beta$. By Proposition 4.5.2 (iii) $\tilde{\alpha}$ intersects the set Σ . If β intersected Σ in some point $q_1 \in \Sigma$, we would obtain that $q_1 \in J^+(q) \subseteq J^+(I^+(\Sigma)) = I^+(\Sigma)$, where we have used Proposition 3.1.9, implying that $q_1 \in I^+(\Sigma)$ and $q_1 \in \Sigma$. This is a contradiction to $\Sigma \cap I^+(\Sigma) = \emptyset$. Hence α intersects Σ , that is $q \in D^+(\Sigma)$. A very similar proof shows that $\Sigma \subseteq D^-(\Sigma)$ and $I^-(\Sigma) \subseteq D^-(\Sigma)$. Hence we obtain that $D(\Sigma) = D^+(\Sigma) \cup D^-(\Sigma) = M$.

Conversely, let $D(\Sigma) = M$. By definition of the Cauchy development $D(\Sigma)$ every piecewise smooth inextendible timelike curve intersects Σ . Since Σ is achronal by Proposition 4.5.2 (i), this curve intersects Σ at most once, hence precisely once. This shows that Σ is a Cauchy hypersurface. \square

Corollary 4.5.4. *Let (M, g) be a space-time that contains a Cauchy hypersurface Σ . Then (M, g) is globally hyperbolic.*

Proof. By Theorem 3.5.5 and Proposition 4.5.2 (i) we obtain that $\text{int}D(\Sigma) = D(\Sigma)^\circ$ is globally hyperbolic. By Proposition 4.5.3 we have that $D(\Sigma) = M$, hence $D(\Sigma)^\circ = M$. \square

Proposition 4.5.5. *Let $\Sigma \subseteq M$ be a Cauchy hypersurface and let X be a smooth timelike vector field on M . For each $p \in M$ let $\rho(p)$ be the unique intersection point of the maximal integral curve of X through p with Σ . Then ρ is well defined, continuous and open. Furthermore we obtain that $\rho|_\Sigma = \text{id}_\Sigma$ and that Σ is connected.*

Proof. Let $p \in M$ and let c be the unique maximal integral curve of X through p . If the domain of c is finite, then c cannot be continuously extended. Hence we assume that the domain of c is not bounded (for instance to the right). We have to show that the limit $\lim_{t \rightarrow \infty} c(t)$ does not exist. To this end we assume that the limit exists, hence there is some $p \in M$ such

that $p = \lim_{t \rightarrow \infty} c(t)$. We show that in this case $X(p) = 0$, which contradicts the fact that X is timelike. We assume that $X(p) \neq 0$. By the straightening out theorem there exists a chart $(\tilde{\mathcal{V}}, \psi = (y^1, \dots, y^n))$ such that $X = \partial_{y^1}$. In these coordinates the geodesic c can be written as $t \mapsto x_0 + te_1$, hence it does not converge as $t \rightarrow \infty$. This is a contradiction. Hence c has a unique intersection point $\rho(p)$ with Σ . Furthermore ρ is well-defined.

Let $\mathcal{U} \subseteq \mathbb{R} \times M$ be the maximal open domain of the flow Fl^X of the vector field X . By Proposition 4.5.2 (ii) Σ is a topological hypersurface, so $\mathbb{R} \times \Sigma$ is a topological hypersurface in $\mathbb{R} \times M$ and thus $\mathcal{U}(\Sigma) := (\mathbb{R} \times \Sigma) \cap \mathcal{U}$ is a topological hypersurface in \mathcal{U} .

We show that the map $\Psi := Fl^X|_{\mathcal{U}(\Sigma)} : \mathcal{U}(\Sigma) \rightarrow M$ is bijective. To this end let $p \in M$. The map $t \rightarrow Fl_t^X(p)$ intersects Σ precisely once, hence there exists a t_0 such that $Fl_{t_0}^X(p) = q \in \Sigma$ implying that $p = Fl^X(-t_0, q)$ with $(-t_0, q) \in \mathcal{U}(\Sigma)$. This means that Ψ is onto. We assume that $Fl^X(t_1, q_1) = Fl^X(t_2, q_2)$ with $(t_i, q_i) \in \mathcal{U}(\Sigma)$ ($i = 1, 2$), hence we obtain that $q_2 = Fl^X(t_1 - t_2, q_1)$. This means that the flow line through $q_1 \in \Sigma$ intersects Σ in q_2 implying that $q_1 = q_2$ and $t_1 = t_2$. This shows that Ψ is one-to-one. Furthermore we obtain that Ψ is continuous.

We show that Ψ is an open mapping. To this end let (\mathcal{W}, φ) be a chart of the topological hypersurface $\mathcal{U}(\Sigma)$ and let (\mathcal{V}, χ) be a chart of M . Without restriction we can assume that $\Psi(\mathcal{W}) \subseteq \mathcal{V}$. Obviously the map $\chi \circ \Psi \circ \varphi^{-1} : \varphi(\mathcal{W}) \rightarrow \mathbb{R}^n$ is continuous and one-to-one, hence a homeomorphism by a theorem of Brouwer (cf. Remark 3.4.7). This shows that $\Psi(\mathcal{W})$ is open. Summing up, the map $\Psi : \mathcal{U}(\Sigma) \rightarrow M$ is a homeomorphism.

Let $pr_2 : \mathbb{R} \times M \rightarrow M$ the projection. pr_2 is continuous and an open mapping. Since $\rho = pr_2 \circ \Psi^{-1}$, ρ is continuous and an open mapping. If $p \in \Sigma$, then p itself is the unique intersection point of the unique integral curve through p with Σ , i.e. $\rho(p) = p$ and we obtain that $\rho|_{\Sigma} = id_{\Sigma}$. Since M is connected as a space-time and since ρ is continuous with $\rho(M) = \Sigma$, Σ is connected. \square

Theorem 4.5.6. (*Singularity theorem of Penrose*) *Let (M, g) be a space-time that satisfies the null convergence condition. If (M, g) contains a non-compact Cauchy hypersurface Σ and a closed trapped surface S , then the space-time is null geodesically incomplete.*

Proof. We assume that S is a future-trapped submanifold of M . If the space-time (M, g) was null geodesically complete, its future-horismos $E^+(S)$ would be a compact set by Remark 4.4.10. Since (M, g) contains a Cauchy hypersurface Σ , it is globally hyperbolic by Corollary 4.5.4 and therefore it is causally simple by Proposition 3.1.42. By Proposition 3.1.35 we find that

$E^+(S) = \partial J^+(S)$, hence the future-horismos of S equals the boundary of the causal future of S . By Definition 4.2.2 $E^+(S)$ is a compact proper achronal boundary.

Let X be a smooth timelike vector field on M . For $p \in M$ let $\rho(p)$ be the unique intersection point of the unique maximal integral curve of X through p with Σ (cf. Proposition 4.5.5). Now we consider $\rho|_{E^+(S)} : E^+(S) \rightarrow \Sigma$. Since $E^+(S)$ is achronal, ρ is one-to-one: we suppose that $Fl^X(t_1, q_1) = Fl^X(t_2, q_2)$ with $q_i \in E^+(S)$. As in the proof of Proposition 4.5.5 we obtain that $q_2 = Fl^X(t_1 - t_2, q_1)$, hence the timelike flow line of X through $q_1 \in E^+(S)$ intersects $E^+(S)$ again in q_2 , which is a contradiction in the case that $q_1 \neq q_2$. Hence $\rho|_{E^+(S)} : E^+(S) \rightarrow \Sigma$ is one-to-one and continuous (again, cf. the proof of Proposition 4.5.5). Since $E^+(S)$ and Σ are topological hypersurfaces, we can use the same proof as in Proposition 4.5.5 to obtain that $\rho|_{E^+(S)}$ is an open mapping. Hence $\rho(E^+(S))$ is open in Σ and since $E^+(S)$ is compact we obtain that $\rho(E^+(S))$ is compact, hence closed in Σ . By Proposition 4.5.5 Σ is connected, hence $\Sigma = \rho(E^+(S))$ is compact, which is a contradiction. \square

For our further considerations it will be necessary to extend the definition of global hyperbolicity (cf. Definition 3.1.39). To this end we give the following

Definition 4.5.7. *Let (M, g) be a space-time. M is said to be globally hyperbolic if it is stably causal (cf. Definition 3.1.25) and if there exists a time function $t : M \rightarrow \mathbb{R}$ such that the time slices $S_a := t^{-1}(a)$ (for $a \in \mathbb{R}$) satisfy $D(S_a) = M$, hence the Cauchy development of S_a is the whole space-time (cf. Definition 3.5.1).*

Remark 4.5.8. Since the time function $t : M \rightarrow \mathbb{R}$ is smooth and satisfies $\langle \text{grad}t, \text{grad}t \rangle < 0$ everywhere, we obtain that for arbitrary $a \in \mathbb{R}$ the set S_a is a semi Riemannian hypersurface with signum -1 . In particular it possesses a global unit normal vector field which we will denote by $n := \frac{\text{grad}t}{\|\text{grad}t\|}$.

Remark 4.5.9. Let (M, g) be a globally hyperbolic space-time and let S be an arbitrary time slice. One can show that for $p \in D^+(S)$ the set $D^+(S) \cap J^-(p)$ is compact. As a Corollary of this we obtain that $J(p, q) = J^+(p) \cap J^-(q)$ is compact and $J^+(p)$ is closed, hence Definition 4.5.7 indeed generalizes Definition 3.1.39. For more details on this remark see [13], Proposition 3.11 and Corollary 3.12.

Remark 4.5.10. In what follows we denote by θ the trace of the second fundamental form or the extrinsic curvature, which is also called the expansion. We furthermore recall the following three Propositions from the

calculus of variations. For the proofs see [13], Proposition 3.8, Proposition 3.9 and Theorem 3.13.

Proposition 4.5.11. *Let (M, g) be a globally hyperbolic space-time satisfying the strong energy condition (cf. Definition 3.2.4) and let S be an arbitrary time slice and $p \in S$ a point where $\theta := \theta_0 < 0$. Denote by c_p the unique timelike geodesic with $c_p(0) = p$ and $c_p'(0) = n_p = n(p)$, where n is the global unit normal vector field from Remark 4.5.8. If the geodesic c_p can be extended to a distance $t_0 := -\frac{n}{\theta_0}$ to the future of S , then it contains at least one point conjugate to S .*

Proposition 4.5.12. *Let (M, g) be a globally hyperbolic space-time, S a time slice, $p \in M$ and c a piecewise smooth timelike geodesic through p orthogonal to S . If there exists some conjugate point between S and p then c does not maximize length (among the timelike curves connecting S to p).*

Proposition 4.5.13. *Let (M, g) be a globally hyperbolic space-time with time slice S and $p \in D^+(S)$. Then among all timelike curves connecting p to S there exists a timelike curve with maximal length. This curve is a timelike geodesic that is orthogonal to S .*

With these preparations we can now formulate a fundamental result in singularity theory that is based on global hyperbolicity.

Theorem 4.5.14. *Let (M, g) be a globally hyperbolic space-time satisfying the strong energy condition (cf. Definition 3.2.4) and suppose that the expansion satisfies $\theta \leq \theta_0 < 0$ on a time slice S . Then (M, g) is not timelike geodesically complete.*

Proof. Let $c : [a, b] \rightarrow M$ be a future-directed timelike geodesic orthogonal to S . Recall that the length of c is given by $\tau(c) := \int_a^b \|\dot{c}(t)\| dt$. $\tau(c)$ is interpreted as the proper time measured by the particle between events $c(a)$ and $c(b)$. We will show that no future-directed timelike geodesic orthogonal to S can be extended to proper time greater than $\tau_0 := -\frac{n}{\theta_0}$ to the future of S , where n is the dimension of M . We assume that this was wrong. Then there would exist some future-directed timelike geodesic c orthogonal to S defined on the interval $[0, \tau_0 + \epsilon]$ for some $\epsilon > 0$. We set $p := c(\tau_0 + \epsilon)$. Since $M = D^+(S)$ we can use Proposition 4.5.13 to obtain some piecewise smooth timelike geodesic γ with maximal length connecting S and $p \in M$ that is orthogonal to S . Since $\tau(c) = \tau_0 + \epsilon$, we would necessarily have $\tau(\gamma) \geq \tau_0 + \epsilon$. Proposition 4.5.11 guarantees that γ would develop a conjugate point at a distance of at most τ_0 to the future of S . Proposition 4.5.12 states that γ would cease to be maximizing beyond this point. This leads to a contradiction. \square

4.6 The basic structure of singularity theorems

We finally investigate the general structure of a singularity theorem. As we have seen, all singularity theorems have the same basic structure: given some space-time (M, g) that satisfies

- (i) an energy condition,
- (ii) a causality condition and
- (iii) a boundary or initial condition,

then (M, g) contains at least one incomplete causal geodesic.

APPENDIX A

PARALLEL TRANSPORTS AND FRAMES

A.1 General parallel transports

Definition A.1.1. An $\binom{r}{s}$ -tensor field T along a smooth curve $\alpha : I \rightarrow M$ is said to be parallel along α if its first induced covariant derivation vanishes, i.e. if

$$\frac{\nabla T}{ds} = 0.$$

Now let $\alpha : I \rightarrow M$ (with $n := \dim(M)$) be a smooth curve and let $T \in \Gamma(I, T_s^r M, \alpha)$. We derive a local representation for $\frac{\nabla T}{ds}$ in terms of Christoffel-symbols. We first recall that by definition of the Christoffel-symbols we have that

$$\nabla_{\partial_{x^i}} \partial_{x^j} = \Gamma_{ij}^k \partial_{x^k}.$$

The next lemma gives us a local representation of the dual expression.

Lemma A.1.2. Let α be a smooth curve and let $\omega \in \Gamma(I, T_1^0 M, \alpha)$. Then we obtain that

$$\nabla_{\partial_{x^i}} dx^j = -\Gamma_{ik}^j dx^k.$$

Proof. We first observe that for $m \in \mathbb{N}$ we have $\nabla_{\partial_{x^j}}(m) = \partial_{x^j}(m) = 0$, hence $0 = \nabla_{\partial_{x^j}}(\delta_k^i) = \nabla_{\partial_{x^j}}(dx^i(\partial_{x^k})) = (\nabla_{\partial_{x^j}}(dx^i))(\partial_{x^k}) + dx^i(\nabla_{\partial_{x^j}}(\partial_{x^k}))$. From this we conclude that

$$(\nabla_{\partial_{x^j}}(dx^i))(\partial_{x^k}) = -dx^i(\nabla_{\partial_{x^j}}(\partial_{x^k})) = -dx^i(\Gamma_{jk}^l \partial_{x^l}) = -\Gamma_{jk}^i.$$

□

By using that $\nabla_{\partial_{x^j}}$ is a tensor derivation we now conclude that

$$\begin{aligned}
& (\nabla_{\partial_{x^j}}(\partial_{x^{i_1}} \otimes \dots \otimes dx^{j_s})) \\
&= \Gamma_{j i_1}^{k_1}(\partial_{x^{k_1}} \otimes \dots \otimes dx^{j_s}) \\
&+ \dots + \\
&+ \Gamma_{j i_r}^{k_r}(\partial_{x^{i_1}} \otimes \dots \otimes \partial_{x^{k_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s}) \\
&- \Gamma_{j k_1}^{j_1}(\partial_{x^{i_1}} \otimes \dots \otimes \partial_{x^{i_r}} \otimes dx^{k_1} \otimes dx^{j_2} \otimes \dots \otimes dx^{j_s}) \\
&- \dots - \\
&- \Gamma_{j k_s}^{j_s}(\partial_{x^{i_1}} \otimes \dots \otimes dx^{k_s}).
\end{aligned}$$

For $T = T_{j_1, \dots, j_s}^{i_1, \dots, i_r} \partial_{x^{i_1}} \otimes \dots \otimes dx^{j_s}$ we obtain

$$\begin{aligned}
& \nabla_{\partial_{x^j}}(T_{j_1, \dots, j_s}^{i_1, \dots, i_r} \partial_{x^{i_1}} \otimes \dots \otimes dx^{j_s}) \\
&= (\partial_{x^j}(T_{j_1, \dots, j_s}^{i_1, \dots, i_r})) \partial_{x^{i_1}} \otimes \dots \otimes dx^{j_s} + T_{j_1, \dots, j_s}^{i_1, \dots, i_r} \nabla_{\partial_{x^j}}(\partial_{x^{i_1}} \otimes \dots \otimes dx^{j_s}) \\
&= (\partial_{x^j}(T_{j_1, \dots, j_s}^{i_1, \dots, i_r})) \partial_{x^{i_1}} \otimes \dots \otimes dx^{j_s} \\
&+ T_{j_1, \dots, j_s}^{i_1, \dots, i_r} (\Gamma_{j i_1}^{k_1}(\partial_{x^{k_1}} \otimes \dots \otimes dx^{j_s}) \\
&+ \dots + \\
&+ \Gamma_{j i_r}^{k_r}(\partial_{x^{i_1}} \otimes \dots \otimes \partial_{x^{k_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s}) \\
&- \Gamma_{j k_1}^{j_1}(\partial_{x^{i_1}} \otimes \dots \otimes \partial_{x^{i_r}} \otimes dx^{k_1} \otimes dx^{j_2} \otimes \dots \otimes dx^{j_s}) \\
&- \dots - \\
&- \Gamma_{j k_s}^{j_s}(\partial_{x^{i_1}} \otimes \dots \otimes dx^{k_s})).
\end{aligned}$$

Locally we have that (cf. the uniqueness proof of Proposition 2.3.4)

$$\begin{aligned}
\frac{\nabla T}{ds}(s) &= \frac{dT_{j_1, \dots, j_s}^{i_1, \dots, i_r}}{ds}(s) (\partial_{x^{i_1}} \circ \alpha)(s) \otimes \dots \otimes (dx^{j_s} \circ \alpha)(s) \\
&+ T_{j_1, \dots, j_s}^{i_1, \dots, i_r}(s) \nabla_{\alpha'(s)}(\partial_{x^{i_1}} \otimes \dots \otimes dx^{j_s}).
\end{aligned}$$

Furthermore we obtain that

$$\begin{aligned}
& \nabla_{\alpha'(s)}(\partial_{x^{i_1}} \otimes \dots \otimes dx^{j_s}) \\
&= \nabla_{\frac{d(x^j \circ \alpha)}{ds} \partial_{x^j}}(\partial_{x^{i_1}} \otimes \dots \otimes dx^{j_s}) \\
&= \frac{d(x^j \circ \alpha)}{ds} \nabla_{\partial_{x^j}}(\partial_{x^{i_1}} \otimes \dots \otimes dx^{j_s}),
\end{aligned}$$

thus we find that

$$\begin{aligned}
\frac{\nabla T}{ds}(s) &= \frac{dT_{j_1, \dots, j_s}^{i_1, \dots, i_r}}{ds}(s)(\partial_{x^{i_1}} \circ \alpha)(s) \otimes \dots \otimes (dx^{j_s} \circ \alpha)(s) \\
&\quad + T_{j_1, \dots, j_s}^{i_1, \dots, i_r}(s) \frac{d(x^j \circ \alpha)}{ds}(s) \nabla_{\partial_{x^j}} (\partial_{x^{i_1}} \otimes \dots \otimes dx^{j_s}) \\
&= \frac{dT_{j_1, \dots, j_s}^{i_1, \dots, i_r}}{ds}(s)(\partial_{x^{i_1}} \circ \alpha)(s) \otimes \dots \otimes (dx^{j_s} \circ \alpha)(s) \\
&\quad + T_{j_1, \dots, j_s}^{i_1, \dots, i_r}(s) \frac{d(x^j \circ \alpha)}{ds}(s) (\Gamma_{j_1}^{k_1} (\partial_{x^{k_1}} \otimes \dots \otimes dx^{j_s})) \\
&\quad + \dots + \\
&\quad + \Gamma_{j_r}^{k_r} (\partial_{x^{i_1}} \otimes \dots \otimes \partial_{x^{k_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s}) \\
&\quad - \Gamma_{j_{k_1}}^{j_1} (\partial_{x^{i_1}} \otimes \dots \otimes \partial_{x^{i_r}} \otimes dx^{k_1} \otimes dx^{j_2} \otimes \dots \otimes dx^{j_s}) \\
&\quad - \dots - \\
&\quad - \Gamma_{j_{k_s}}^{j_s} (\partial_{x^{i_1}} \otimes \dots \otimes dx^{k_s})
\end{aligned}$$

Hence locally $\frac{\nabla T}{ds} = 0$ is a system of linear ordinary differential equations and we obtain the following

Proposition A.1.3. *Let $\alpha : I \rightarrow M$ be a smooth curve in a semi-Riemannian manifold (M, g) . For $a \in I$ and $t \in T_{\alpha(a)}M \otimes \dots \otimes T_{\alpha(a)}M \otimes T_{\alpha(a)}M^* \otimes \dots \otimes T_{\alpha(a)}M^*$ there is a unique parallel tensor field $T \in \Gamma(I, T_s^r M, \alpha)$ such that $T(a) = t$.*

Now we can define the $\binom{r}{s}$ - parallel transport.

Definition A.1.4. *Let $\alpha : I \rightarrow M$ be a smooth curve in a semi-Riemannian manifold (M, g) and let $a, b \in I$. We set $p := \alpha(a)$ and $q := \alpha(b)$. For $t \in T_p M \otimes \dots \otimes T_p M \otimes T_p M^* \otimes \dots \otimes T_p M^*$ let T_t be the unique parallel $\binom{r}{s}$ -tensor field along α with $T_t(a) = t$. The map*

$$P_{(r,s)} := P_{(r,s;a,b)}(\alpha) : T_p M \otimes \dots \otimes T_p M^* \rightarrow T_q M \otimes \dots \otimes T_q M^*,$$

$$P_{(r,s)}(t) := T_t(b)$$

is called the $\binom{r}{s}$ - parallel transport of t along α from p to q .

A.2 Examples

In the previous section we have derived the $\binom{r}{s}$ - parallel transport. For our further considerations it is necessary to treat $\binom{1}{1}$ - and $\binom{0}{1}$ - parallel transports.

We proceed in the same manner as in the previous section and show that the parallel transport equations take an easy form.

Let $A \in \Gamma(I, T_0^1 M, \alpha)$ be a smooth vector field along a smooth curve $\alpha : I \rightarrow M$. Locally, A is given by

$$\frac{\nabla A}{ds}(s) = \left(\frac{dA^k}{ds}(s) + \Gamma_{ij}^k(\alpha(s)) \frac{d(x^j \circ \alpha)}{ds}(s) A^i(s) \right) \partial_{x^k}(\alpha(s)),$$

which follows immediately from the previous section.

Remark A.2.1. It should be noted that we obtained the local representation for A directly from the previous section. An alternative approach is the following. For $A \in \Gamma(I, T_0^1 M, \alpha)$, the proof of Proposition 2.3.4 shows that A is locally given by

$$\frac{\nabla A}{ds}(s) = \frac{dA^i}{ds}(s) (\partial_{x^i} \circ \alpha)(s) + A^i(s) \nabla_{\alpha'(s)} \partial_{x^i}.$$

By using that

$$\begin{aligned} \nabla_{\alpha'(s)} \partial_{x^i} &= \nabla_{\frac{d(x^j \circ \alpha)}{ds} \partial_{x^j}} \partial_{x^i} = \frac{d(x^j \circ \alpha)}{ds} \nabla_{\partial_{x^j}} \partial_{x^i} \\ &= \frac{d(x^j \circ \alpha)}{ds} \Gamma_{ij}^k \partial_{x^k}, \end{aligned}$$

we obtain that

$$\frac{\nabla A}{ds}(s) = \left(\frac{dA^k}{ds}(s) + \Gamma_{ij}^k(\alpha(s)) \frac{d(x^j \circ \alpha)}{ds}(s) A^i(s) \right) \partial_{x^k}(\alpha(s)),$$

which coincides with the above.

Now let ω be a smooth covector field along a smooth curve α . Again, by the previous section we obtain that

$$\frac{\nabla \omega}{ds}(s) = \frac{d\omega_l}{ds}(s) (dx^l \circ \alpha)(s) - \omega_l(s) \frac{d(x^j \circ \alpha)}{ds}(s) \Gamma_{jk}^l(\alpha(s)) dx^k(\alpha(s)).$$

Our next aim is to show that $\binom{r}{s}$ -parallel transport is a linear isometry. To this end we will develop general frames along smooth curves. This will be done in the next section.

A.3 Frames

We first recall the following lemma from Linear Algebra.

Lemma A.3.1. *Let E and F be finite dimensional vector spaces over the field \mathbb{R} . Let $a := \{a_i\}_{i \in I}$ be a basis of E , $b := \{b_j\}_{j \in J}$ a basis of F and let $\{c_{i,j}\}_{(i,j) \in I \times J}$ be an arbitrary matrix over \mathbb{R} . Then there is a unique bilinear map $\phi : E \times F \rightarrow \mathbb{R}$ such that $\phi(a_i, b_j) = c_{i,j}$ holds for all $(i, j) \in I \times J$.*

Remark A.3.2. Let M be a semi-Riemannian manifold and TM its tangent bundle, i.e.

$$TM := \bigsqcup_{p \in M} T_p M = \bigcup_{p \in M} \{p\} \times T_p M.$$

Recall that the triple (TM, M, π_M) is a vector bundle. The map $\pi_M : TM \rightarrow M$, $(p, v) \mapsto p$ is called the canonical projection. For all $p \in M$ we have $T_p M = (\pi_M)^{-1}(p)$. We set $(T_p M)_s^r := T_s^r(T_p M)$ (cf. Remark 2.1.3) and define the $\binom{r}{s}$ -tensor bundle over TM by

$$\begin{aligned} T_s^r(TM) &:= \bigsqcup_{p \in M} T_s^r((\pi_M)^{-1}(p)) = \bigsqcup_{p \in M} T_s^r(T_p M) \\ &= \bigcup_{p \in M} \{p\} \times T_s^r(T_p M) = \bigcup_{p \in M} \{p\} \times (T_p M)_s^r. \end{aligned}$$

Let $\pi_s^r : T_s^r(TM) \rightarrow M$, $\pi_s^r(t) := p \in M$ for $t \in (T_p M)_s^r$ denote the canonical projection. Then $(T_s^r(TM), M, \pi_s^r)$ is a vector bundle. Let $P : T_p M \rightarrow (T_p M)^*$, $P(v)(v') := \langle v, v' \rangle_{T_p M}$ be the canonical isomorphism. We define the (unique) scalar product $g_1^0(p)$ on $(T_p M)^*$, the cotangent space by $g_1^0(p) := \langle P(v), P(v') \rangle_{(T_p M)^*} := \langle v, v' \rangle_{T_p M}$, where $v, v' \in T_p M$. We also denote the scalar product on $T_p M$ by $g_0^1(p) := \langle v, v' \rangle_{T_p M}$. Uniqueness of $g_1^0(p)$ follows directly from Lemma A.3.1. Thus we constructed a unique scalar product on $(T_p M)^*$ by declaring the canonical isomorphism P to be an isometry. Now we can extend the scalar product $g_1^0(p)$ on $(T_p M)^*$ pointwise to $(TM)^*$, the cotangent bundle. We will denote the scalar product on $(TM)^*$ by g_1^0 . Our next aim is to define a scalar product g_s^r on $(T_s^r(TM), M, \pi_s^r)$, so that $((T_s^r(TM), M, \pi_s^r), g_s^r)$ is semi-Riemannian. We first define a scalar product on $T_s^r(T_p M) = T_p M \otimes \dots \otimes T_p M \otimes (T_p M)^* \otimes \dots \otimes (T_p M)^*$. To this end we define $g_s^r(p) \equiv \langle \cdot, \cdot \rangle_{T_s^r(T_p M)}$ by

$$\begin{aligned} &\langle \alpha^1 \otimes \dots \otimes \alpha^r \otimes f_1 \otimes \dots \otimes f_s, \beta^1 \otimes \dots \otimes \beta^r \otimes g_1 \otimes \dots \otimes g_s \rangle_{T_s^r(T_p M)} \\ &:= \langle \alpha^1, \beta^1 \rangle_{T_p M} \cdots \langle \alpha^r, \beta^r \rangle_{T_p M} \langle f_1, g_1 \rangle_{(T_p M)^*} \cdots \langle f_s, g_s \rangle_{(T_p M)^*}, \end{aligned}$$

where $\alpha^1 \otimes \dots \otimes \alpha^r \otimes f_1 \otimes \dots \otimes f_s \in T_s^r(T_p M)$ and $\beta^1 \otimes \dots \otimes \beta^r \otimes g_1 \otimes \dots \otimes g_s \in T_s^r(T_p M)$. Clearly $g_s^r(p) : T_s^r(T_p M) \times T_s^r(T_p M) \rightarrow \mathbb{R}$ is bilinear, symmetric and nondegenerate. Lemma A.3.1 shows that $g_s^r(p)$ is unique such that $\{e^{i_1} \otimes \dots \otimes e^{i_r} \otimes e_{j_1}^* \otimes \dots \otimes e_{j_s}^* : 1 \leq i_k \leq n, 1 \leq j_l \leq n\}$ is an orthonormal basis of $T_s^r(T_p M)$. Now we can pointwise extend our scalar product construction to $T_s^r(TM)$ and we have proved the following Proposition.

Proposition A.3.3. *Let (M, g) be a semi-Riemannian manifold. Then the manifold $(T_s^r(TM), g_s^r)$ is semi-Riemannian.*

Definition A.3.4. *Let (M, g) be a semi-Riemannian manifold with $n := \dim(M)$.*

- (i) *An $\binom{r}{s}$ -frame on M at $p \in M$ is an orthonormal basis for the tensor product $T_p M \otimes \dots \otimes T_p M \otimes (T_p M)^* \otimes \dots \otimes (T_p M)^*$.*
- (ii) *A set of n^{r+s} orthonormal tensor fields on some $U \subseteq M$ open is called an $\binom{r}{s}$ -frame field on U . An $\binom{r}{s}$ -frame field assigns an $\binom{r}{s}$ -frame to each point $p \in U$.*
- (iii) *Let $\alpha : I \rightarrow M$ be a smooth curve in M . A set of n^{r+s} orthonormal tensor fields along α is called an $\binom{r}{s}$ -frame field along α .*

Our next aim is to prove that there always exists a parallel $\binom{r}{s}$ -frame field along a given smooth curve α . A standard procedure (which can be found in [15] Chapter 3) shows that $\binom{1}{0}$ -parallel transports are always linear isometries. Existence and uniqueness of (even parallel) $\binom{1}{0}$ -frame fields along α is ensured by the following

Proposition A.3.5. *Let (M, g) be a semi-Riemannian manifold with $n := \dim(M)$ and $\alpha : I \rightarrow M$ a smooth curve in M . Let $\{e_1, \dots, e_n\}$ be a $\binom{1}{0}$ -frame at $\alpha(0)$. Then there exists a unique parallel $\binom{1}{0}$ -frame field $\{E_1, \dots, E_n\}$ along α with $E_i(0) = e_i$ for all $i \in \{1, \dots, n\}$.*

Proof. A proof can be found in [15], Chapter 3, Corollary 46. □

We now show that there is always a $\binom{0}{1}$ -frame field along a smooth curve.

Remark A.3.6. Let $\{E_1, \dots, E_n\}$ be a parallel $\binom{1}{0}$ -frame field along α . Let $\{E^{*1}, \dots, E^{*n}\} \in \Gamma(I, T_1^0 M, \alpha)$ be the metrically equivalent covector fields along α (cf. Remark 1.1.10). By definition of $g_s^r(\alpha(s))$ it follows that

$$\langle E^{*j}(s), E^{*i}(s) \rangle_{(T_p M)^*} = \langle E_j(s), E_i(s) \rangle_{T_p M} = \delta_{ij}$$

for all $s \in I$, hence $\{E^{*1}, \dots, E^{*n}\}$ is a $\binom{0}{1}$ -frame field. Furthermore

$$\begin{aligned} 0 &= \frac{\nabla}{ds}(E^{*j}(E_i)) = \frac{\nabla}{ds}(\mathcal{C}(E_i \otimes E^{*j})) \\ &= \mathcal{C}\left(\frac{\nabla}{ds}(E_i \otimes E^{*j})\right) = \mathcal{C}\left(\frac{\nabla}{ds}(E_i) \otimes E^{*j} + E_i \otimes \frac{\nabla}{ds}(E^{*j})\right) \\ &= E^{*j}\left(\frac{\nabla}{ds}(E_i)\right) + \frac{\nabla}{ds}(E^{*j})(E_i) \end{aligned}$$

by Proposition 2.3.7 (i) and (ii). By assumption all E_i are parallel for $i \in \{1, \dots, n\}$, i.e. $\frac{\nabla}{ds}(E_i) = 0$. We conclude that $\frac{\nabla}{ds}(E^{*j}) = 0$ for an arbitrary $j \in \{1, \dots, n\}$, hence $\{E^{*1}, \dots, E^{*n}\}$ is a parallel $\binom{0}{1}$ -frame field.

Remark A.3.7. By induction we can now construct a general frame field for $\binom{r}{s}$ -tensor fields along smooth curves by tensorizing frame fields and dual frame fields. The scalar product on the spaces $T_s^r(T_p M)$ and $T_s^r(T_p M)^*$ are then well defined by Remark A.3.2. This concept will be needed below.

Before we can prove the main result of this section, we have to state the following

Lemma A.3.8. *Let $A, B \in \Gamma(I, T_s^r M, \alpha)$. Then*

$$\frac{d}{ds} \langle A, B \rangle_{T_s^r(T_{\alpha(s)} M)} = \left\langle \frac{\nabla}{ds} A, B \right\rangle_{T_s^r(T_{\alpha(s)} M)} + \left\langle A, \frac{\nabla}{ds} B \right\rangle_{T_s^r(T_{\alpha(s)} M)}.$$

Proof. By Remark A.3.7 we can choose a parallel $\binom{r}{s}$ -frame field

$$\{E_{i_1} \otimes \dots \otimes E_{i_r} \otimes E^{*j_1} \otimes \dots \otimes E^{*j_s} : 1 \leq i_k, j_k \leq n\}$$

along α such that

$$A = A_{j_1, \dots, j_s}^{i_1, \dots, i_r} E_{i_1} \otimes \dots \otimes E_{i_r} \otimes E^{*j_1} \otimes \dots \otimes E^{*j_s}$$

and

$$B = B_{j_1, \dots, j_s}^{i_1, \dots, i_r} E_{i_1} \otimes \dots \otimes E_{i_r} \otimes E^{*j_1} \otimes \dots \otimes E^{*j_s}.$$

We obtain that

$$\begin{aligned} \frac{d}{ds} (\langle A, B \rangle) &= \frac{d}{ds} (\langle A_{j_1, \dots, j_s}^{i_1, \dots, i_r} E_{i_1} \otimes \dots \otimes E^{*j_s}, B_{j_1, \dots, j_s}^{i_1, \dots, i_r} E_{i_1} \otimes \dots \otimes E^{*j_s} \rangle) \\ &= \frac{d}{ds} \left(\sum_{\substack{i_1, \dots, i_r, \\ j_1, \dots, j_s}} A_{j_1, \dots, j_s}^{i_1, \dots, i_r} B_{j_1, \dots, j_s}^{i_1, \dots, i_r} \right) \\ &= \left\langle \frac{\nabla}{ds} A, B \right\rangle_{T_s^r(T_{\alpha(s)} M)} + \left\langle A, \frac{\nabla}{ds} B \right\rangle_{T_s^r(T_{\alpha(s)} M)}. \end{aligned}$$

□

Proposition A.3.9. $\binom{r}{s}$ -parallel transport is a linear isometry.

Proof. Let $\alpha : I \rightarrow M$ be a smooth curve in a semi-Riemannian manifold (M, g) and let $a, b \in I$. We set $p := \alpha(a)$ and $q := \alpha(b)$. For $t_1, t_2 \in T_p M \otimes \dots \otimes T_p M \otimes (T_p M)^* \otimes \dots \otimes (T_p M)^*$ let $T_1, T_2 \in \Gamma(I, T_s^r M, \alpha)$ be the unique parallel $\binom{r}{s}$ -tensor fields along α that satisfy $T_1(a) = t_1$ and $T_2(a) = t_2$. Then we have $\frac{\nabla}{ds}(T_1 + T_2) = \frac{\nabla}{ds}(T_1) + \frac{\nabla}{ds}(T_2) = 0$ and $(T_1 + T_2)(a) = T_1(a) + T_2(a) = t_1 + t_2$ and we obtain that $P_{(r,s)}(t_1 + t_2) = (T_1 + T_2)(b) = T_1(b) + T_2(b) = P_{(r,s)}(t_1) + P_{(r,s)}(t_2)$. Let $t \in T_p M \otimes \dots \otimes T_p M \otimes (T_p M)^* \otimes \dots \otimes (T_p M)^*$ and let $T \in \Gamma(I, T_s^r M, \alpha)$ be the unique parallel $\binom{r}{s}$ -tensor field along α that satisfies $T(a) = t$. For $\lambda \in \mathbb{R}$ we obtain that $\frac{\nabla}{ds}(\lambda T) = \lambda \frac{\nabla}{ds} T = 0$ and $(\lambda T)(a) = \lambda T(a) = \lambda t$. Hence we have that $P_{(r,s)}(\lambda t) = (\lambda T)(b) = \lambda T(b) = \lambda P_{(r,s)}(t)$. Thus $P_{(r,s)} : T_p M \otimes \dots \otimes (T_p M)^* \rightarrow T_q M \otimes \dots \otimes (T_q M)^*$ is linear. Now let $P_{(r,s)}(t) = 0$ for some $t \in T_p M \otimes \dots \otimes T_p M \otimes (T_p M)^* \otimes \dots \otimes (T_p M)^*$. By uniqueness of the corresponding $\binom{r}{s}$ -tensor field T along α we obtain that $T = 0$, hence $t = 0$ and $P_{(r,s)} : T_p M \otimes \dots \otimes (T_p M)^* \rightarrow T_q M \otimes \dots \otimes (T_q M)^*$ is a linear isomorphism, since $\dim(T_p M \otimes \dots \otimes T_p M \otimes (T_p M)^* \otimes \dots \otimes (T_p M)^*) = \dim(T_q M \otimes \dots \otimes T_q M \otimes (T_q M)^* \otimes \dots \otimes (T_q M)^*) = n^{r+s}$, where $n := \dim(M)$. By Lemma A.3.8 we have that $\frac{d}{ds} \langle T_1, T_2 \rangle_{T_s^r(T_{\alpha(s)} M)} = \langle \frac{\nabla}{ds} T_1, T_2 \rangle_{T_s^r(T_{\alpha(s)} M)} + \langle T_1, \frac{\nabla}{ds} T_2 \rangle_{T_s^r(T_{\alpha(s)} M)} = 0$, hence $\langle T_1, T_2 \rangle_{T_s^r(T_{\alpha(s)} M)}$ is constant for all $s \in I$. Therefore we find that

$$\begin{aligned} \langle t_1, t_2 \rangle_{T_s^r(T_p M)} &= \langle T_1(a), T_2(a) \rangle_{T_s^r(T_p M)} \\ &= \langle T_1(b), T_2(b) \rangle_{T_s^r(T_q M)} = \langle P_{(r,s)}(t_1), P_{(r,s)}(t_2) \rangle_{T_s^r(T_q M)} \end{aligned}$$

which shows that $P_{(r,s)} : T_p M \otimes \dots \otimes (T_p M)^* \rightarrow T_q M \otimes \dots \otimes (T_q M)^*$ is a linear isometry. \square

The following proposition shows that the constructed $\binom{r}{s}$ -frame field (cf. Remark A.3.7) are in fact unique.

Proposition A.3.10. Let $\alpha : I \rightarrow M$ be a smooth curve in some semi-Riemannian manifold (M, g) and let $\{e_{i_1} \otimes \dots \otimes e_{i_r} \otimes \alpha^{j_1} \otimes \dots \otimes \alpha^{j_s} : 1 \leq i_k, j_l \leq n\}$ be a chosen $\binom{r}{s}$ -frame on M at $\alpha(0) \in M$. Let E_{i_k} be the parallel vector fields along α with $E_{i_k}(0) = e_{i_k}$. By Remark A.3.6 E^{*j_l} are the parallel covector fields along α . Then $\{E_{i_1} \otimes \dots \otimes E_{i_r} \otimes E^{*j_1} \otimes \dots \otimes E^{*j_s} : 1 \leq i_k, j_l \leq n\}$ is the unique $\binom{r}{s}$ -frame field along α with $(E_{i_1} \otimes \dots \otimes E_{i_r} \otimes E^{*j_1} \otimes \dots \otimes E^{*j_s})(0) = e_{i_1} \otimes \dots \otimes e_{i_r} \otimes \alpha^{j_1} \otimes \dots \otimes \alpha^{j_s}$ for all $1 \leq i_k, j_l \leq n$.

Proof. We obviously have that $(E_{i_1} \otimes \dots \otimes E_{i_r} \otimes E^{*j_1} \otimes \dots \otimes E^{*j_s})(0) = e_{i_1} \otimes \dots \otimes e_{i_r} \otimes \alpha^{j_1} \otimes \dots \otimes \alpha^{j_s}$ for all $1 \leq i_k, j_l \leq n$. Since E_{i_k} and E^{*j_l} are parallel we obtain that $\{E_{i_1} \otimes \dots \otimes E_{i_r} \otimes E^{*j_1} \otimes \dots \otimes E^{*j_s} : 1 \leq i_k, j_l \leq n\}$ are parallel. \square

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