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A Jost-Schroer Theorem for String-Localized Quantum Fields

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Zusammenfassung

Diese Arbeit beschäftigt sich mit Quantenfeldern, deren Lokalisierungsgebiete raumartige Strahlen bzw. "Strings" im Minkowskiraum sind. Solche string-lokalisierten Felder können im Gegensatz zu punkt-lokalisierten Feldern mit den irreduziblen Darstellungen der Poincarégruppe zur Masse Null und Spin unendlich assoziiert werden. Unter anderem wird die Reeh-Schlieder Eigenschaft für diese Felder und ein Theorem zu trunkierten Vakuumserwartungswerten bewiesen. Letzteres Theorem sagt aus, dass trunkierte Vakuumserwartungswerte für mehr als zwei string-lokalisierte Feldoperatoren verschwinden, falls die Erwartungswerte in jeder Variable die Klein-Gordon Gleichung erfüllen. Jede der beiden erwähnten Eigenschaften liefert einen unabhängigen Beweis für ein Jost-Schroer Theorem für string-lokalisierte Felder. Dieses Theorem gibt ein einfaches Kriterium für ein Feld an, ein freies string-lokalisiertes Feld zu sein. Die Resultate beruhen auf einer Annahme über die analytischer Fortsetzbarkeit der betrachteten Objekte in den String-Variablen. Es wird gezeigt, dass unter Ausnützung von Kovarianzeigenschaften ein gewisser Teil dieser Analytizität für die n-Punkt-Funktionen bewiesen werden kann.

Contents

1 Introduction

Up until a few years ago, the Wigner representations [27] of mass zero and infinite spin were known to be problematic with regard to the construction of quantum fields as these representations are incompatible with point-like localized fields [29] which fulfil the general principles of quantum field theory [25]. In 2002, however, R. Brunetti, D. Guido and R. Longo [5] showed in their paper that for every irreducible positive energy representation of the Poincaré group there exists an associated free field algebra with localization in *space-like cones*[1] in Minkowski space having arbitrary small opening angles. This localization can be tightened up to double cone localization except for the massless infinite spin case. Here localization always means that two field operators commute if the corresponding localization regions are space-like separated.

Motivated by the results in [5] J. Mund, B. Schroer and J. Yngvason [20, 21] elaborated an explicit construction of free fields associated with the Wigner representations of mass zero and infinite spin, which solves the old problem [1, 13, 28, 29] of making these representations compatible with the principle of causality. These fields are localized in semi-infinite, *space-like strings*. Such a string can be pictured as the core of a space-like cone considered in [5] and so is the tightest localization region possible for this case. More precisely, a string is characterized by a point x in d-dimensional Minkowski space and a point e in the manifold of space-like directions,

$$H^{d-1} := \{e \in \mathbb{R}^d : e \cdot e = -1\}, \tag{1.1}$$

where the dot denotes the Minkowski scalar product, i.e.

$$x \cdot y = x^0 y^0 - \sum_{i=1}^{d-1} x^i y^i, \quad \forall x, y \in \mathbb{R}^d. \tag{1.2}$$

We then define the string $S_{x,e}$ extending from x to infinity in direction e by

$$S_{x,e} := x + \mathbb{R}_0^+ e. \tag{1.3}$$

The string-localized fields considered in [20, 21] are operator valued distributions $\varphi(x, e)$ on $\mathbb{R}^d \times H^{d-1}$ as shall be explained in detail in Chapter 2. These fields generalize the modified Wightman fields introduced by O. Steinmann [23]. The crucial difference is that in [23] all string directions e are fixed and coincide.

[1] A space-like cone is a set $\mathcal{C}$ of the form $\mathcal{C} := a + \bigcup_{\lambda \geq 0} \lambda D$, where the apex a is a point in Minkowski space and D is a double cone space-like separated from the origin.

The construction in [20, 21], however, also works for other irreducible representations of the Poincaré group. This is an interesting fact with regard to the construction of interacting string-localized objects. A "speculative outlook" on that topic can be found in [18].

We also want to mention that besides the Wigner infinite spin representations the irreducible massive positive energy representations of the Poincaré group in space-time dimension $d = 1 + 2$ with neither integer nor half-integer spin also lead to string localization. This case, however, gives rise to braid group statistics [15] and it has been shown that a mass-shell description of the corresponding fields is not possible [16].

Note that string localization as considered here has only little relation to the strings of String Theory. In particular, according to [7, 9] quantization of Nambu-Goto string Lagrangians does not lead to string-localized fields.

In this contribution, we present a generalization of the findings in [23], i. e. we prove a version of the Jost-Schroer theorem for string-localized fields using the concepts of [20, 21]. We study two different methods of proof: either by involving a Reeh-Schlieder property [25] for string-localized fields or by exploiting a theorem which states that the truncated vacuum expectation values of more than two string-localized fields vanish if they fulfil the Klein-Gordon equation in each variable. The latter theorem and the Reeh-Schlieder property, both depending on analyticity of the considered quantities in both arguments x and e, shall also be proved in this paper. We also show that to a certain extent analyticity in e can be proved for the vacuum expectation values by using covariance properties.

We also refer to a very recent publication by J. Mund [19] on an algebraic Jost-Schroer theorem. Although the results in this paper also apply to string-localized fields the methods leading to them differ from those used here. Mund's paper can therefore be understood as a complement to this contribution.

2 String-localization

This chapter is devoted to the definition of string-localization. The properties of the corresponding quantum fields and vacuum expectation values are pointed out. This shall be done in a modified Wightman framework following the basic concepts of [20, 21]. We restrict ourselves to one hermitian scalar field. A generalization to several fields with other transformation properties would require only minor modifications.

2.1 String-Localized Quantum Fields

In this section we want to specify the notion of a string-localized quantum field. We already mentioned in the introduction that a string-localized field $\varphi(x, e)$ is an operator valued distribution over $\mathbb{R}^d \times H^{d-1}$ with localization region being the space-like string $S_{x,e}$, cf. (1.3). We now want to state this description more precisely. To this end, we first have to define the properties of the underlying vector space on which we want our field to act as an operator. We make the following assumptions:

1. Let $\mathscr{H}$ denote a separable Hilbert space and $U(a, \Lambda)$ a continuous, unitary representation of the restricted Poincaré group $\mathcal{P}_+^\uparrow = \mathbb{R}^d \rtimes \mathcal{L}_+^\uparrow$ acting on $\mathscr{H}$, where $\mathcal{L}_+^\uparrow$ denotes the restricted Lorentz group.

   The joint spectrum of the generators P_μ of the translation group $U(a, 1)$, denoted by spec(P), lies in the closed forward light cone

   $$\overline{V}_+ := \{p \in \mathbb{R}^d : p \cdot p \geq 0, p^0 \geq 0\}, \tag{2.1}$$

   i.e. the translation group satisfies the spectral condition.

   There is an unit vector $\Omega \in \mathscr{H}$ satisfying

   $$U(a, \Lambda)\Omega = \Omega, \qquad \forall (a, \Lambda) \in \mathcal{P}_+^\uparrow. \tag{2.2}$$

   Ω is unique (up to a phase) and represents the vacuum.

Next, we want to discuss the properties defining a string-localized field. We start with the assumptions about the domain and continuity of the field.

2. For $f \in \mathscr{S}(\mathbb{R}^d)$[1] and $h \in \mathscr{S}(H^{d-1})$ there is an operator valued distribution $\varphi(f,h)$ which together with its adjoint $\varphi(f,h)^*$ is defined on a dense set $D \subset \mathscr{H}$. Furthermore,

$$\Omega \in D, \qquad U(a,\Lambda)D \subset D, \quad \varphi(f,h)D \subset D \quad \textit{and} \quad \varphi(f,h)^*D \subset D. \tag{2.3}$$

For $\Phi, \Psi \in D$, $\langle \Phi, \varphi(f,h)\Psi \rangle$ is a distribution with regard to f and h.

By the nuclear theorem of L. Schwartz [25] the separately continuous bilinear functional $\varphi(f,h)$ can be uniquely extended to a distribution $\varphi(g)$ with test functions $g \in \mathscr{S}(\mathbb{R}^d \times H^{d-1})$, such that for the special choice $g(x,e) = (f \otimes h)(x,e) = f(x)h(e)$ the functional $\varphi(g)$, symbolised by $\varphi(x,e)$, reduces to $\varphi(f,h)$. Formally, we have $\varphi(x,e)^*|_D = \varphi(x,e)$.

For what follows we have to point out that elements of the manifold H^{d-1} transform under the Poincaré group according to

$$g\,e := \Lambda e, \qquad \forall\, g = (a,\Lambda) \in \mathcal{P}_+^\uparrow, \quad e \in H^{d-1}. \tag{2.4}$$

That is to say, translations act trivially on H^{d-1}.
We then have the following transformation law of the field:

3. For all $(a,\Lambda) \in \mathcal{P}_+^\uparrow$, $f \in \mathscr{S}(\mathbb{R}^d)$ and $h \in \mathscr{S}(H^{d-1})$ holds

$$U(a,\Lambda)\varphi(f,h)U(a,\Lambda)^{-1} = \varphi(f_{(a,\Lambda)}, h_{(a,\Lambda)}), \tag{2.5}$$

where

$$f_{(a,\Lambda)}(x) := f(\Lambda^{-1}(x-a)) \quad \textit{and} \quad h_{(a,\Lambda)}(e) := h(\Lambda^{-1}e)^2. \tag{2.6}$$

We shall sometimes state this covariance property in the following formal way:
For all $(a,\Lambda) \in \mathcal{P}_+^\uparrow$ and $(x,e) \in \mathbb{R}^d \times H^{d-1}$ holds

$$U(a,\Lambda)\varphi(x,e)U(a,\Lambda)^{-1} = \varphi(\Lambda x + a, \Lambda e).$$

The next assumption defines the notion of string localization.

4. If the sets $\mathrm{supp}(f_1) + \bigcup_{\lambda \geq 0} \lambda \mathrm{supp}(h_1)$ and $\mathrm{supp}(f_2) + \bigcup_{\lambda \geq 0} \lambda \mathrm{supp}(h_2)$ are space-like separated, with $\mathrm{supp}(f_i) \in \mathbb{R}^d$ and $\mathrm{supp}(h_i) \in H^{d-1}$, $i = 1,2$, respectively, then

$$[\varphi(f_1,h_1), \varphi(f_2,h_2)] = 0. \tag{2.7}$$

[1] $\mathscr{S}$ is the set of C_∞ functions of fast decrease, the so-called Schwartz-space. Linear, continuous functionals over $\mathscr{S}$ are tempered distributions [14, 25] and are elements in $\mathscr{S}'$.
[2] This definition is in agreement with equation (2.4).

Alternatively, we may express this property in the following formal way:

$$[\varphi(x_1, e_1), \varphi(x_2, e_2)] = 0$$

if the strings $S_{x_1, e_1'}$ and S_{x_2, e_2} are space-like separated for all e_1' in some open neighbourhood of e_1[3].

We may sometimes also refer to the sets $\mathrm{supp}(f) + \mathbb{R}_0^+ \mathrm{supp}(h)$ as cones. Clearly, such a cone contains the string $S_{x,e}$.

Finally, let $\mathscr{P}$ denote the polynomial algebra of fields, i.e. the set containing all polynomials of the form

$$c + \sum_{i=1}^{N} \varphi(f_1^i, h_1^i) \cdots \varphi(f_i^i, h_i^i), \quad \text{with } \mathrm{supp}(f_k^i) \subset \mathbb{R}^d \text{ and } \mathrm{supp}(h_k^i) \subset H^{d-1}, \tag{2.8}$$

where c is any complex constant. We then require:

5. Ω is cyclic for the polynomial algebra of fields $\mathscr{P}$, that is, $\mathscr{P}\Omega$ is dense in $\mathscr{H}$.

2.2 Vacuum Expectation Values

The assumptions listed under point 2 in Section 2.1 allow us to define the quantities

$$W(f_1, h_1; \ldots; f_n, h_n) := \langle \Omega, \varphi(f_1, h_1) \cdots \varphi(f_n, h_n)\Omega \rangle \tag{2.9}$$

with $f_i \in \mathscr{S}(\mathbb{R}^d)$ and $h_i \in \mathscr{S}(H^{d-1})$, $i = 1, \ldots, n$, respectively. By application of the nuclear theorem this functional can be uniquely extended to a distribution $W(g)$ with test functions g over $\left((\mathbb{R}^d) \times (H^{d-1})\right)^n$. This distribution is formally given by

$$W(x_1, e_1; \ldots; x_n, e_n) = \langle \Omega, \varphi(x_1, e_1) \cdots \varphi(x_n, e_n)\Omega \rangle. \tag{2.10}$$

It is clear that the assumptions made in Section 2.1 lead to corresponding properties of the vacuum expectation values. These are as follows.

Hermiticity of the fields yields the property

1. $$W(g) = \overline{W}(g^*), \tag{2.11}$$

where

$$g^*(x_1, e_1; \ldots; x_n, e_n) := \overline{g}(x_n, e_n; \ldots; x_1, e_1) \tag{2.12}$$

and the bar denotes complex conjugation.

The fact that the norm of a vector in $\mathscr{H}$ is non-negative implies positivity of the distributions,

[3] That is, S_{x_1,e_1} and S_{x_2,e_2} are space-like separated as well as e_1 and e_2 are space-like separated. This follows from the distributional character of the fields.

2.
$$\sum_{i,j=0}^{N} W(g_i^* \otimes g_j) \geq 0 \tag{2.13}$$
for $N \in \mathbb{N}_0$ arbitrary and $g_i = g_i(x_1, e_1; \ldots; x_i, e_i)$.

Covariance (2.5) of the string-localized fields indicates the relativistic transformation law

3.
$$W(f_1, h_1; \ldots; f_n, h_n) = W(f_{1(a,\Lambda)}, h_{1(a,\Lambda)}; \ldots; f_{n(a,\Lambda)}, h_{n(a,\Lambda)}). \tag{2.14}$$

Proof
Using (2.2), we have

$$\begin{aligned} \langle \Omega, \varphi(f_1, h_1) \cdots \varphi(f_n, h_n)\Omega \rangle &= \langle \Omega, U(a,\Lambda)\varphi(f_1, h_1) \cdots \varphi(f_n, h_n) U(a,\Lambda)^{-1}\Omega \rangle \\ &= \langle \Omega, \varphi(f_{1(a,\Lambda)}, h_{1(a,\Lambda)}) \cdots \varphi(f_{n(a,\Lambda)}, h_{n(a,\Lambda)})\Omega \rangle. \quad \square \end{aligned}$$

By application of the nuclear theorem (2.14) leads to the formal expression

$$W(x_1, e_1; \ldots; x_n, e_n) = W(\Lambda x_1 + a, \Lambda e_1; \ldots; \Lambda x_n + a, \Lambda e_n). \tag{2.15}$$

Further, string locality (2.7) yields

4.
$$\begin{aligned} &W(f_1, h_1; \ldots; f_k, h_k; f_{k+1}, h_{k+1}; \ldots; f_n, h_n) \\ &\qquad = W(f_1, h_1; \ldots; f_{k+1}, h_{k+1}; f_k, h_k; \ldots; f_n, h_n) \end{aligned} \tag{2.16}$$

if $\mathrm{supp}(f_k) + \bigcup_{\lambda \geq 0} \lambda \mathrm{supp}(h_k)$ and $\mathrm{supp}(f_{k+1}) + \bigcup_{\lambda \geq 0} \lambda \mathrm{supp}(h_{k+1})$ are space-like separated.

Correspondingly,

$$\begin{aligned} &W(x_1, e_1; \ldots; x_k, e_k; x_{k+1}, e_{k+1}; \ldots; x_n, e_n) \\ &\qquad = W(x_1, e_1; \ldots; x_{k+1}, e_{k+1}; x_k, e_k; \ldots; x_n, e_n) \end{aligned} \tag{2.17}$$

if $S_{x_k, e_k'}$ and $S_{x_{k+1}, e_{k+1}}$ are space-like separated for all e_k' in some open neighbourhood of e_k.

In the interest of clarity we make the following definition:

$$W(x_1, \ldots, x_n; e_1, \ldots, e_n) := W(x_1, e_1; \ldots; x_n, e_n), \tag{2.18}$$

where the left-hand side is a distribution over $\mathbb{R}^{dn} \times H^{(d-1)n}$.
As a consequence of equation (2.15) concerning translations, there are distributions $\mathcal{W}(\xi_1, \dots, \xi_{n-1}; e_1, \dots, e_n)$ [25] satisfying

$$W(x_1, \dots, x_n; e_1, \dots, e_n) = \mathcal{W}(\xi_1, \dots, \xi_{n-1}; e_1, \dots, e_n), \tag{2.19}$$

with

$$\xi_i = x_i - x_{i+1}, \qquad i = 1, \dots, n-1. \tag{2.20}$$

$\mathcal{W}(\xi_1, \dots, \xi_{n-1}; e_1, \dots, e_n)$ is invariant under the restricted Lorentz group $\mathcal{L}_+^\uparrow$, that is

$$\mathcal{W}(\Lambda\xi_1, \dots, \Lambda\xi_{n-1}; \Lambda e_1, \dots, \Lambda e_n) = \mathcal{W}(\xi_1, \dots, \xi_{n-1}; e_1, \dots, e_n), \quad \Lambda \in \mathcal{L}_+^\uparrow. \tag{2.21}$$

The next properties of the vacuum expectation values we are going to discuss are best expressed in terms of the Fourier transforms. The Fourier transform of a distribution T is defined by $\widetilde{T}(\widetilde{f}) := T(f)$, where f is a test function in $\mathscr{S}$ and $\widetilde{f}$ is its Fourier transform. In our case the Fourier transforms of the vacuum expectation values are formally given by

$$\begin{aligned}\widetilde{W}&(p_1, \dots, p_n; e_1, \dots, e_n) \\ &= \int \exp\left(i \sum_{k=1}^{n} p_k \cdot x_k\right) W(x_1, \dots, x_n; e_1, \dots, e_n) d^d x_1 \cdots d^d x_n,\end{aligned} \tag{2.22}$$

and

$$\begin{aligned}\widetilde{\mathcal{W}}&(q_1, \dots, q_{n-1}; e_1, \dots, e_n) \\ &= \int \exp\left(i \sum_{k=1}^{n-1} q_k \cdot \xi_k\right) \mathcal{W}(\xi_1, \dots, \xi_{n-1}; e_1, \dots, e_n) d^d \xi_1 \cdots d^d \xi_{n-1}\end{aligned} \tag{2.23}$$

respectively, where the dot in the exponentials denotes the Minkowski scalar product (1.2).

Clearly

$$\begin{aligned}&\int d^d x_1 \cdots d^d x_n \exp\left(i \sum_{k=1}^{n} p_k \cdot x_k\right) W(x_1, \dots, x_n; e_1, \dots, e_n) \\ &\quad = \int d^d x_1 \cdots d^d x_n \exp\{i[p_1(x_1 - x_2) + (p_1 + p_2)(x_2 - x_3) + \cdots + (p_1 + \cdots \\ &\quad + p_{n-1})(x_{n-1} - x_n)]\} \exp\left\{\left(i \sum_{k=1}^{n} p_k\right) x_n\right\} \mathcal{W}(x_1 - x_2, \dots, x_{n-1} - x_n; e_1, \dots, e_n)\end{aligned}$$

and hence it follows the formal relation

$$\begin{aligned}\widetilde{W}&(p_1, \dots, p_n; e_1, \dots, e_n) \\ &= (2\pi)^d \delta\left(\sum_{i=1}^{n} p_i\right) \widetilde{\mathcal{W}}(p_1, p_1 + p_2, \dots, p_1 + p_2 + \cdots + p_{n-1}; e_1, \dots, e_n).\end{aligned} \tag{2.24}$$

Furthermore, we have the following theorem resulting from the spectral condition.

5.

Theorem 2.2.1.
$$\widetilde{\mathcal{W}}(q_1,\dots,q_{n-1};e_1,\dots,e_n)=0 \tag{2.25}$$
if any $q_i = p_1+\cdots+p_i \notin spec(P)$.

Proof
By the SNAG (Stone, Naimark, Ambrose, Godement) theorem [8] $U(a) := U(a,1)$ may be written as a Stieltjes integral over $\overline{V}_+$

$$U(a)=\int_{\overline{V}_+} e^{ip\cdot a}dE(p), \tag{2.26}$$

where E is a projection valued measure on $\overline{V}_+$. The joint spectrum of the generators P_μ of $U(a)$ is simultaneously the Lorentz invariant support of E. It is then evident that

$$\int \rho(a)U(a)d^d a = \int \widetilde{\rho}(p)dE(p) = 0$$

for all $\rho \in \mathscr{S}$ such that supp$(\widetilde{\rho})\cap$supp$(dE) = \emptyset$. Here $\widetilde{\rho}(p) := \int e^{ip\cdot a}\rho(a)d^d a$. This clearly implies for any $\phi,\psi \in D$

$$\int e^{-ip\cdot a}\langle\phi, U(a)\psi\rangle d^d a = 0, \tag{2.27}$$

unless p is contained in supp(dE) [25]. Thus, we have

$$\int e^{ip\cdot a}\langle\Omega, \varphi(f_1,h_1)\cdots U(-a)\varphi(f_{j+1},h_{j+1})\cdots\varphi(f_n,h_n)\Omega\rangle d^d a = 0$$

which because of

$$\begin{aligned}\langle\Omega, &\varphi(x_1,e_1)\cdots U(-a)\varphi(x_{j+1},e_{j+1})\cdots\varphi(x_n,e_n)\Omega\rangle\\ &= \langle\Omega, \varphi(x_1,e_1)\cdots\varphi(x_{j+1}-a,e_{j+1})\varphi(x_{j+2}-a,e_{j+2})\cdots\varphi(x_n-a,e_n)\Omega\rangle\end{aligned}$$

implies

$$\int e^{ip\cdot a}\mathcal{W}(\xi_1,\dots,\xi_j+a,\xi_{j+1},\dots,\xi_{n-1};e_1,\dots,e_n)d^d a = 0, \quad j=1,\dots,n$$

for $p \notin$ supp(dE). This yields $\widetilde{\mathcal{W}}(q_1,\dots,q_{n-1};e_1,\dots,e_n)=0$ unless each $q_i \in$ spec(P).□

The requirement of a unique vacuum gives a general result, namely

$$\lim_{\lambda\to\infty}\langle\phi, U(\lambda a)\psi\rangle = \langle\phi,\Omega\rangle\langle\Omega,\psi\rangle, \tag{2.28}$$

where $\phi,\psi \in D$. This equation holds $\forall a \neq 0$, $a \in \mathbb{R}^d$ and $d > 2$. The proof of this assertion is based on the Riemann-Lebesque-Lemma which states that the Fourier transform of a L^1-function tends to zero at infinity [14]. Equation (2.28) leads directly to the following cluster property.

6.

Theorem 2.2.2. *If a is a space-like vector, then*

$$\lim_{\lambda\to\infty} W(x_1,e_1;\ldots;x_i+\lambda a,e_i;\ldots;x_n+\lambda a,e_n)$$
$$= W(x_1,e_1;\ldots;x_{i-1},e_{i-1})W(x_i,e_i;\ldots;x_n,e_n). \quad (2.29)$$

We want to close this section by discussing that expressions of the form

$$\Psi(g)=\int \varphi(x_1,e_1)\cdots\varphi(x_n,e_n)g(x_1,e_1;\ldots;x_n,e_n)d^dx_1\cdots d^dx_n d\sigma(e_1)\cdots d\sigma(e_n)\Omega \quad (2.30)$$

make sense and exist, where σ denotes the Lorentz invariant measure on H^{d-1}.
Note first that $\left(\mathscr{S}(\mathbb{R}^d)\otimes\mathscr{S}(H^{d-1})\right)^{\otimes n}$ is dense in $\mathscr{S}\left((\mathbb{R}^d\times H^{d-1})^n\right)$. So there is a sequence of functions $\{g_K\}$ with

$$g_K(x_1,e_1;\ldots;x_n,e_n)=\sum_{i=1}^{K} f_1^i(x_1)h_1^i(e_1)\cdots f_n^i(x_n)h_n^i(e_n)$$

such that $g_K\to g$ for $K\to\infty$, where $g\in\mathscr{S}\left((\mathbb{R}^d\times H^{d-1})^n\right)$. Consequently, the sequence of vectors $\{\Psi(g_K)\}$ with

$$\Psi(g_K)=\sum_{i=1}^{K}\varphi(f_1^i,h_1^i)\cdots\varphi(f_n^i,h_n^i)\Omega$$

converges in norm. In order to prove this, consider

$$\|\Psi(g_K)-\Psi(g_L)\|^2=\|\Psi(g_K-g_L)\|^2=W\left((g_K-g_L)^*\otimes(g_K-g_L)\right)$$

which tends to zero for $K\to\infty$ and $L\to\infty$. Since $\mathscr{H}$ is complete the sequence $\{\Psi(g_K)\}$ has a limit vector $\Psi(g)\in\mathscr{H}$ which is given by equation (2.30) and which is a vector-valued distribution for test functions $g\in\mathscr{S}\left((\mathbb{R}^d\times H^{d-1})^n\right)$.

Finally, we have

$$\langle\Omega,\Psi(g)\rangle=W(g). \quad (2.31)$$

2.2.1 Truncated Vacuum Expectation Values

We also want to define the truncated vacuum expectation values (TVEV) recursively given by

$$W(x_1,e_1;\ldots;x_n,e_n)=\sum_{\text{partitions}} W^{\mathrm{T}}(x_{l_{1,1}},e_{l_{1,1}};\ldots;x_{l_{1,r_1}},e_{l_{1,r_1}})$$
$$\times W^{\mathrm{T}}(x_{l_{2,1}},e_{l_{2,1}};\ldots;x_{l_{2,r_2}},e_{l_{2,r_2}})\cdots W^{\mathrm{T}}(x_{l_{s,1}},e_{l_{s,1}};\ldots;x_{l_{s,r_s}},e_{l_{s,r_s}}) \quad (2.32)$$

where we sum over all partitions of the indices $1, \dots, n$ and take in each subset $l_{k,r_1}, \dots, l_{k,r_k}$ the indices in natural order. For example, for $n = 1$ and $n = 2$ we have

$$\begin{aligned} W^{\mathrm{T}}(x_1, e_1) &= W(x_1, e_1) \\ W^{\mathrm{T}}(x_1, e_1; x_2, e_2) &= W(x_1, e_1; x_2, e_2) - W(x_1, e_1)W(x_2, e_2). \end{aligned} \tag{2.33}$$

We shall, however, always assume that $W(x_1, e_1) = 0$.

As the TVEVs clearly satisfy the properties (1,3,4) listed in Section 2.2 we may define the quantities $\mathcal{W}^{\mathrm{T}}(\xi_1, \dots, \xi_{n-1}; e_1, \dots, e_n) := W^{\mathrm{T}}(x_1, \dots, x_n; e_1, \dots, e_n)$, $\xi_i = x_i - x_{i+1}$.

Further, the following theorem holds.

Theorem 2.2.3. $\widetilde{\mathcal{W}}^{\mathrm{T}}(q_1, \dots, q_{n-1}; e_1, \dots, e_n) = 0$ *unless each* $q_i = p_1 + \cdots + p_i \in \operatorname{supp}(dE) \setminus \{0\}$, $i = 1, \dots, n-1$.

The theorem may be proven just like in the point-like case [14].

3 Analytic Continuation of the Vacuum Expectation Values

In the theory of point-localized fields it is well known that the Wightman distributions are boundary values of analytic functions, the Wightman functions. Our aim is to find an analogous property for the string-localized case. The reason for this investigation is that the principle of analytic continuation shall be an important tool in the proofs discussed in the sequel.

The mentioned boundary values have to be understood in the sense of distribution theory. This means for a function $g(x+iy)$ holomorphic in e. g. $\mathbb{R} + i\Gamma$, with Γ some convex open set of $\mathbb{R}$, and satisfying a certain boundedness condition, namely it has at most polynomial growth in x for each fixed $y \in \Gamma$, that the limit

$$\lim_{y \to 0} \int g(x+iy) f(x) dx = G(f)$$

exists for all $f \in \mathscr{S}$ and $G(f)$ is a tempered distribution. Then $G(f)$ is said to be the boundary value of g in the sense of $\mathscr{S}'$.

3.1 Analytic Continuation to complex "ξ"

Since by Theorem 2.2.1 $\widetilde{\mathcal{W}}(q_1, \dots, q_{n-1}; e_1, \dots, e_n) = 0$ unless each $q_i \in \overline{V}_+$ we have in analogy to the usual Wightman theory the following result.

Theorem 3.1.1. *The distributions $\mathcal{W}(\xi_1, \dots, \xi_{n-1}; e_1, \dots, e_n)$ introduced in Section 2.2 are boundary values in the sense of $\mathscr{S}'$ of quantities $\mathcal{W}(\zeta_1, \dots, \zeta_{n-1}; e_1, \dots, e_n)$ which are analytic functions in $(\zeta_1, \dots, \zeta_{n-1}) \in \mathscr{T}_-^{n-1}$, where*

$$\mathscr{T}_- := \{\zeta = \xi - i\eta \in \mathbb{C}^d : \eta \in V_+\} \tag{3.1}$$

is referred to as the backward tube.

V_+ denotes the open forward light cone defined by $V_+ := \{x \in \mathbb{R}^d : x \cdot x > 0, x^0 > 0\}$. Holomorphy of $\mathcal{W}(\zeta_1, \dots, \zeta_{n-1}; e_1, \dots, e_n)$ in the variables $\zeta_1, \dots, \zeta_{n-1}$ follows from Theorem 2.2.1 and general results on the Laplace transform of a distribution [25]. In particular,

$$\begin{aligned}&\mathcal{W}(\zeta_1,\ldots,\zeta_{n-1};e_1,\ldots,e_n)\\&=(2\pi)^{-(n-1)d}\int\exp\left(-i\sum_{k=1}^{n-1}q_k\cdot\zeta_k\right)\widetilde{\mathcal{W}}(q_1,\ldots,q_{n-1};e_1,\ldots,e_n)d^dq_1\cdots d^dq_{n-1}\end{aligned}\tag{3.2}$$

exists and is holomorphic in $(\zeta_1,\ldots,\zeta_{n-1})\in\mathscr{T}_-^{n-1}$.

Note that $\mathcal{W}(\zeta_1,\ldots,\zeta_{n-1};e_1,\ldots,e_n)$ is still a distribution with respect to $(e_1,\ldots,e_n)$.

Corollary 3.1.2. *$\mathcal{W}(\zeta_1,\ldots,\zeta_{n-1};e_1,\ldots,e_n)$ is $\mathcal{L}_+^\uparrow$-invariant.*

Proof

For $\zeta_k=\xi_k+i\eta_k$, $k=1,\ldots,n-1$, rewrite

$$\mathcal{W}(\zeta_1,\ldots,\zeta_{n-1};e_1,\ldots,e_n)=\mathcal{W}(\xi_1,\ldots,\xi_{n-1};e_1,\ldots,e_n;\eta_1,\ldots,\eta_{n-1}),$$

where the right-hand side can be interpreted as a distribution in the variables $\xi_1,\ldots,$ $\xi_{n-1},e_1,\ldots,e_n$, which depends on the parameters $\eta_1,\ldots,\eta_{n-1}$. Thus, in terms of test functions this distribution is given by $\mathcal{W}(g,\eta_1,\ldots,\eta_{n-1})$ and the relation to its boundary value $\mathcal{W}(g)$ is given by

$$\mathcal{W}(g,\eta_1,\ldots,\eta_{n-1})=\widetilde{\mathcal{W}}\left(\widetilde{g}\cdot\exp\left(-\sum_k\eta_kq_k\right)\right).$$

We know from equation (2.21) that $\widetilde{\mathcal{W}}(\widetilde{G})=\widetilde{\mathcal{W}}(\widetilde{G_\Lambda})$, where $G_\Lambda(\xi_1,\ldots,\xi_{n-1};e_1,\ldots,e_n):=G(\Lambda^{-1}\xi_1,\ldots,\Lambda^{-1}\xi_{n-1};\Lambda^{-1}e_1,\ldots,\Lambda^{-1}e_n)$.
Define $\widetilde{G}(q_1,\ldots,q_{n-1};e_1,\ldots,e_n):=\widetilde{g}(q_1,\ldots,q_{n-1};e_1,\ldots,e_n)\cdot\exp\left(-\sum_k\eta_kq_k\right)$, then we have

$$\begin{aligned}\mathcal{W}(g,\eta_1,\ldots,\eta_{n-1})&=\widetilde{\mathcal{W}}(\widetilde{G})=\widetilde{\mathcal{W}}(\widetilde{G_\Lambda})=\\&\widetilde{\mathcal{W}}\left(\widetilde{g_\Lambda}\cdot\exp\left(-\sum_k\eta_k\Lambda^Tq_k\right)\right)=\mathcal{W}(g_\Lambda,\Lambda\eta_1,\ldots,\Lambda\eta_{n-1}),\end{aligned}$$

where we used $\widetilde{G_\Lambda}(q,e)=\widetilde{G}(\Lambda^Tq,\Lambda^{-1}e)$. □

3.2 Analytic Continuation to complex "*e*"

We now want to exploit the results of the previous section in order to obtain an analytic continuation of the distributions $\mathcal{W}(\xi_1,\ldots,\xi_{n-1};e_1,\ldots,e_n)$ with regard to the manifold $H^{(d-1)n}$. To this end, some preparation is necessary.

First of all, we define by

$$H_c := \{\hat{e} \in \mathbb{C}^d : \hat{e} \cdot \hat{e} = -1\} \tag{3.3}$$

the complexification of H^{d-1}, where the dot indicates bilinear extension of the Minkowski metric to $\mathbb{C}^d$,

$$\hat{e} \cdot \hat{e} := e' \cdot e' - e'' \cdot e'' + 2i\, e' \cdot e'' \quad \text{if} \quad \hat{e} = e' + ie''. \tag{3.4}$$

Moreover, we denote by $\mathcal{T}_+ \subset H_c$ the tuboid [3] consisting of all $\hat{e} \in H_c$ with $\mathrm{Im}\, \hat{e} \in V_+$, i. e.

$$\mathcal{T}_+ := H_c \cap (\mathbb{R}^d + iV_+). \tag{3.5}$$

We shall also consider

$$\mathcal{T}_n := H_c^n \cap T_n, \tag{3.6}$$

where

$$T_n := \left\{ (\hat{e}_1, \dots, \hat{e}_n) \in \mathbb{C}^{dn} : \mathrm{Im}(\hat{e}_i - \hat{e}_{i-1}) \in V_+, i = 2, \dots, n \right\}. \tag{3.7}$$

According to [3, 4] the set $\mathcal{T}_n$ is a domain, i. e. an open connected set, and a tuboid. It is pointed out in [3, 4] that from this domain it is possible to take the distributional boundary value of an analytic function in a well-defined way.

Next, we consider certain subregions in Minkowski space, namely so-called wedge regions. Each wedge W arises from a fixed reference wedge, e. g.

$$W_0 = \left\{ x \in \mathbb{R}^d : x^1 > |x^0| \right\}, \tag{3.8}$$

by a Poincaré transformation, i. e. $W = (a, \Lambda) W_0$ for $(a, \Lambda) \in \mathscr{P}_+^\uparrow$. Further, to every wedge W one can assign a one-parameter group $\Lambda_W(\theta)$ of Lorentz boosts and the reflection j_W across the edge of W satisfying

1. $j_W\, W = W'$ and $j_{g\,W} = g\, j_W\, g^{-1}$, $g = (a, \Lambda) \in \mathscr{P}_+^\uparrow$, and
2. $\Lambda_W(\theta)\, W = W$ and $\Lambda_{g\,W}(\theta) = g \Lambda_W(\theta) g^{-1}$, $\theta \in \mathbb{R}$, $g = (a, \Lambda) \in \mathscr{P}_+^\uparrow$,

where W' denotes the causal complement of W. Indeed, these properties are consistent because every Poincaré transformation leaving the wedge W invariant commutes with $\Lambda_W(\theta)$ and j_W.
In particular,

$$\Lambda_{W_0}(\theta) = \begin{pmatrix} \cosh(\theta) & \sinh(\theta) & 0 & \dots & 0 \\ \sinh(\theta) & \cosh(\theta) & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \tag{3.9}$$

leave the reference wedge W_0 invariant and j_{W_0} inverts the signs of the coordinates x^0 and x^1 leaving the other ones unchanged.
We may also define wedge regions W_H in H^{d-1} arising from Minkowski wedges $W = (a, \Lambda)W_0$ by

$$W_H := \Lambda W_0 \cap H^{d-1}, \tag{3.10}$$

where the wedge ΛW_0 arises from W by translation and contains the origin in its edge.

In the following we shall also deal with complex Lorentz transformations, in particular we intend to consider transformations $\Lambda_W(i\theta)$ with real θ. Thus, we want to say a few words on the complex Lorentz group $\mathcal{L}(\mathbb{C})$. It is composed of all complex matrices A satisfying $A^T G A = G$, where G =diag$(1, -1, \ldots, -1)$. In contrast to the real case, only the distinction $\det(A) = \pm 1$ makes sense and the corresponding subgroups $\mathcal{L}_\pm(\mathbb{C})$ are connected. We are, however, only interested in the group $\mathcal{L}_+(\mathbb{C})$. We choose the Cayley parametrization [6, 14] for a regular parametrization of the group $\mathcal{L}_+(\mathbb{C})$ as we shall later need an analytic parametrization of a neighbourhood of the unit element in $\mathcal{L}_+(\mathbb{C})$ and the fulfilment of the condition that real parameters correspond to real Lorentz transformations in $\mathcal{L}_+^\uparrow$. The Cayley parameters fulfil these requirements.

As two complex Lorentz transformations may be related to each other via real transformations we define

Definition 3.2.1. *The complex Lorentz transformations A and B are equivalent with respect to $\mathcal{L}_+^\uparrow$, if $\Lambda_1, \Lambda_2 \in \mathcal{L}_+^\uparrow$ exist such that $A = \Lambda_1 B \Lambda_2$.*

In this context, there is the following theorem [14].

Theorem 3.2.2. *Any $A \in \mathcal{L}_+(\mathbb{C})$ is equivalent to one of the following normal forms N:*

1. *For even dimension:*
   a) *Normal case:*

$$N = \begin{pmatrix} L(i\alpha) & & & 0 \\ & K(i\beta_1) & & \\ & & \ddots & \\ 0 & & & K(i\beta_k) \end{pmatrix}, \quad |\alpha| \le \pi;\ \alpha, \beta_1, \ldots, \beta_k\ \textit{real}, \tag{3.11}$$

   b) *Exceptional case:*

$$N = \sigma \begin{pmatrix} M(i\tau) & & & & 0 \\ & K(i\beta_1) & & & \\ & & \ddots & & \\ & & & K(i\beta_{k-1}) & \\ 0 & & & & 1 \end{pmatrix}, \quad \tau \neq 0;\ \tau, \beta_1, \ldots, \beta_{k-1}\ \textit{real}, \quad \sigma = \pm 1, \tag{3.12}$$

where the index k is determined by $k = \frac{d-2}{2}$;

2. *For odd dimension:*

   a) *Normal case:*

      i.
$$N = \begin{pmatrix} L(i\alpha) & & & & 0 \\ & K(i\beta_1) & & & \\ & & \ddots & & \\ & & & K(i\beta_k) & \\ 0 & & & & 1 \end{pmatrix}, \tag{3.13}$$

      ii.
$$N = \begin{pmatrix} 1 & & & 0 \\ & K(i\beta_1) & & \\ & & \ddots & \\ 0 & & & K(i\beta_{k+1}) \end{pmatrix}, \tag{3.14}$$

      iii.
$$N = \begin{pmatrix} 1 & & & 0 \\ & K(i\beta_1) & & \\ & & \ddots & \\ 0 & & & K(i\beta_{k+1}) \end{pmatrix} R, \tag{3.15}$$

   b) *Exceptional case:*

      i.
$$N = \begin{pmatrix} M(i\tau) & & & 0 \\ & K(i\beta_1) & & \\ & & \ddots & \\ 0 & & & K(i\beta_{k-1}) \end{pmatrix}, \tag{3.16}$$

      ii.
$$N = \begin{pmatrix} M(i\tau) & & & 0 \\ & K(i\beta_1) & & \\ & & \ddots & \\ 0 & & & K(i\beta_{k-1}) \end{pmatrix} R, \tag{3.17}$$

   with $k = \frac{d-3}{2}$,

where

$$L(\alpha) = \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix}, \quad K(\beta) = \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix} \tag{3.18}$$

and

$$M(\tau) = \begin{pmatrix} 1+\frac{\tau^2}{2} & -\frac{\tau^2}{2} & -\tau \\ \frac{\tau^2}{2} & 1-\frac{\tau^2}{2} & -\tau \\ -\tau & \tau & 1 \end{pmatrix}, \quad R = \begin{pmatrix} -1 & & & & 0 \\ & -1 & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix}. \tag{3.19}$$

Finally, the following statements hold true:

Lemma 3.2.3. *The complex Lorentz transformation $\Lambda_W(i\theta)$ maps every element $x \in W$ to an element $z = \Lambda_W(i\theta)x$ in the forward tube $\mathscr{T}_+ = \mathbb{R}^d + iV_+$, where W is some wedge containing the origin in its edge and $0 < \theta < \pi$.*

Proof

Consider first $z = \Lambda_{W_0}(i\theta)x$, $x \in W_0$. Then $\mathrm{Im}(z) = \sin(\theta)\,\big(x^1, x^0, 0, \dots, 0\big)$, which is in V_+ for $0 < \theta < \pi$. Clearly, $z = \Lambda_{W_0}(i\theta)x$ if and only if $\Lambda z = \Lambda_W(i\theta)x'$, where $\Lambda \in \mathcal{L}_+^\uparrow$ such that $\Lambda W_0 = W$ and $\Lambda x = x' \in W$. Therefore, $\mathrm{Im}(\Lambda z) \in V_+$ for all $\Lambda \in \mathcal{L}_+^\uparrow$. □

It is obvious that lemma 3.2.3 also holds if $\mathscr{T}_+$ is replaced by $\mathscr{T}_-$ and θ varies between $(-\pi, 0)$.

Lemma 3.2.4. *Every element in H_c is of the form $\hat{e} = \Lambda_W(i\theta)e$, where W is some wedge containing the origin in its edge, $e \in H^{d-1}$ and $0 \le \theta < \pi$.*

The proof of this lemma can be found in [21, Lemma A.2].

Having collected all the equipment we need to make an investigation on the analytic continuation of the distributions $\mathcal{W}(\xi_1, \dots, \xi_{n-1}; e_1, \dots, e_n)$ with regard to the manifold $H^{(d-1)n}$ we start by noting that equation (2.21) gives meaning to the following relation

$$\mathcal{W}(\xi_1, \dots, \xi_{n-1}; \Lambda e_1, \dots, \Lambda e_n) = \mathcal{W}(\Lambda^{-1}\xi_1, \dots, \Lambda^{-1}\xi_{n-1}; e_1, \dots, e_n), \quad \Lambda \in \mathcal{L}_+^\uparrow. \tag{3.20}$$

Passing on to complex transformations and using Lemma 3.2.3 and 3.2.4 we may formally define the quantities

$$\begin{aligned} &\mathcal{W}(\xi_1, \dots, \xi_{n-1}; \Lambda_W(i\theta)e_1, \dots, \Lambda_W(i\theta)e_n) \\ &\qquad := \mathcal{W}(\Lambda_W(-i\theta)\xi_1, \dots, \Lambda_W(-i\theta)\xi_{n-1}; e_1, \dots, e_n), \\ &\qquad\qquad \Lambda_W(i\theta) \in \mathcal{L}_+(\mathbb{C}), \quad 0 < \theta < \pi, \end{aligned} \tag{3.21}$$

with $(\Lambda_W(-i\theta)\xi_1, \dots, \Lambda_W(-i\theta)\xi_{n-1}) \in \mathscr{T}_-^{n-1}$, provided $\xi_1, \dots, \xi_{n-1} \in W$. To begin with, the left-hand side of equation (3.21) depends on e_i, $i = 1, \dots, n$, θ and W and is in general a distribution in e_i, but we want to show that it depends only on $\hat{e}_i = \Lambda_W(i\theta)e_i$ for $(\hat{e}_1, \dots, \hat{e}_n)$ in a certain domain which we shall call Ξ. Taking all wedges W into account, then the right-hand side of (3.21) implies due to Theorem 3.1.1 holomorphy of the left-hand side in $(\hat{e}_1, \dots, \hat{e}_n) \in \Xi \subset H_c^n$, where Ξ is determined as follows. In order to take distributional boundary values of regular functions from the set Ξ in a well-defined way it must hold $\Xi \subset \mathcal{T}_n$ according to [3], where $\mathcal{T}_n$ is given by equation (3.6). So the vector $(\Lambda_W(i\theta)e_1, \dots, \Lambda_W(i\theta)e_n)$ is in $\mathcal{T}_n$ if and only if $(e_1, \dots, e_n) \in \Delta_W$, with

$$\Delta_W := \left\{ (e_1, \dots, e_n) \in H^{n(d-1)} : (e_i - e_{i-1}) \in W, i = 2, \dots, n \right\}. \tag{3.22}$$

Therefore, Ξ is given by

$$\Xi := \bigcup_{\substack{W \\ \theta \in (0,\pi)}} \Lambda_W(i\theta)\Delta_W, \tag{3.23}$$

which is clearly a subset of $\mathcal{T}_n$.

In the following we may abbreviate $(\xi_1, \dots, \xi_{n-1})$ by ξ, $(e_1, \dots, e_n)$ by e, etc. and $\mathscr{T}_-^{n-1}$ by $\mathscr{T}_-$, $H^{n(d-1)}$ by H, etc.. Equation (3.21) then reads $\mathcal{W}(\xi; \Lambda_W(i\theta)e) = \mathcal{W}(\Lambda_W(-i\theta)\xi; e)$. We now have to show that the analytic continuation to the quantity $\mathcal{W}(\xi; \hat{e})$ is single-valued. That is, if a point $\hat{e} \in \Xi$ can be reached from two different points e_1 and e_2 in H, i. e. $\hat{e} = \Lambda_{W_1}(i\theta_1)e_1 = \Lambda_{W_2}(i\theta_2)e_2$, $e_1 \in \Delta_{W_1}$, $e_2 \in \Delta_{W_2}$, then both definitions of $\mathcal{W}(\xi; \hat{e})$ via (3.21) lead to the same value. This would guarantee the single-valuedness of the continuation.

That is, if

$$\Lambda_{W_1}(i\theta_1)e_1 = \Lambda_{W_2}(i\theta_2)e_2$$

we have to show that

$$\mathcal{W}(\Lambda_{W_1}(-i\theta_1)\xi; e_1) = \mathcal{W}(\Lambda_{W_2}(-i\theta_2)\xi; e_2).$$

This is, however, equivalent to

$$\mathcal{W}(A\zeta_2; Ae_2) = \mathcal{W}(\zeta_2; e_2), \tag{3.24}$$

where $A := \Lambda_{W_1}(-i\theta_1)\Lambda_{W_2}(i\theta_2)$, $e_1 = Ae_2$, $\zeta_i := \Lambda_{W_i}(-i\theta_i)\xi$ and $\zeta_1 = A\zeta_2$, consequently $(A\zeta_2, Ae_2) \in \mathscr{T}_- \times H$. Thus, for the single-valuedness we have to prove that equation (3.24) holds for those $A \in \mathcal{L}_+(\mathbb{C})$ of the above for which $(A\mathscr{T}_- \times AH) \cap (\mathscr{T}_- \times H) \neq \emptyset$. Note that both sides of equation (3.24) have to be understood in the sense of distributions with regard to e_2 and $Ae_2 = e_1$ respectively. Now, we take a closer look at the set $(A\mathscr{T}_- \times AH) \cap (\mathscr{T}_- \times H) \neq \emptyset$:

$$(A\mathscr{T}_- \times AH) \cap (\mathscr{T}_- \times H) = (A\mathscr{T}_- \cap \mathscr{T}_-) \times (AH \cap H) \neq \emptyset. \tag{3.25}$$

Let

$$\mathscr{A} := \{A \in \mathcal{L}_+(\mathbb{C}) : A\mathscr{T}_- \cap \mathscr{T}_- \neq \emptyset\},$$

further let

$$\mathscr{B} := \{A \in \mathcal{L}_+(\mathbb{C}) : AH \cap H \neq \emptyset\}.$$

From the $\mathcal{L}_+^\uparrow$-invariance of $\mathscr{T}_-$ and H respectively these definitions imply that $\Lambda_1 \mathscr{A} \Lambda_2 = \mathscr{A}$ and $\Lambda_1 \mathscr{B} \Lambda_2 = \mathscr{B}$ for all $\Lambda_1, \Lambda_2 \in \mathcal{L}_+^\uparrow$. These sets are therefore sets of equivalence classes of $\mathcal{L}_+(\mathbb{C})$ relative to $\mathcal{L}_+^\uparrow$. Thus, as a consequence of Theorem 3.2.2 it suffices to determine the normal forms contained in $\mathscr{A}$ and $\mathscr{B}$. According to the notations in Theorem 3.2.2 the normal forms contained in the set $\mathscr{A}$ are the following ones [14]:

1. For even dimension:
   a) Normal case: $|\alpha| < \pi$, rest of parameters arbitrary,
   b) Exceptional case: $\sigma = +1$, rest of parameters arbitrary.
2. For odd dimension:
   a) Normal case:
      i. $|\alpha| < \pi$, rest of parameters arbitrary,
      ii. all cases,
   b) Exceptional case:
      i. all cases.

The normal forms contained in set $\mathscr{B}$ are as follows:

1. For even dimension:
   a) Normal case: α, β_i, $i = 1, \ldots, \hat{j}, \ldots, k$, arbitrary, $\beta_j = 0$, or β_i, $i = 1, \ldots, k$, arbitrary and $\alpha = 0$,
   b) Exceptional case: all cases.
2. For odd dimension:
   a) Normal case:
      i. all cases,
      ii. β_i, $i = 1, \ldots, \hat{j}, \ldots, k$, arbitrary, $\beta_j = 0$,
      iii. β_i, $i = 1, \ldots, \hat{j}, \ldots, k$, arbitrary, $\beta_j = 0$,
   b) Exceptional case:
      i. τ, β_i, $i = 1, \ldots, \hat{j}, \ldots, k-1$, arbitrary, $\beta_j = 0$,
      ii. τ, β_i, $i = 1, \ldots, \hat{j}, \ldots, k-1$, arbitrary, $\beta_j = 0$.

Further, let

$$\mathscr{D} := \{A \in \mathcal{L}_+(\mathbb{C}) : (A\mathscr{T}_- \cap \mathscr{T}_-) \times (AH \cap H) \neq \emptyset\}. \tag{3.26}$$

It follows

$$\mathscr{D} = \mathscr{A} \cap \mathscr{B}. \tag{3.27}$$

Thus, $\mathscr{D}$ is determined by the joint normal forms of $\mathscr{A}$ and $\mathscr{B}$.

The sets $\mathscr{A}, \mathscr{B}$ and $\mathscr{D}$ are connected as all the normal forms contained in them can continuously be connected with the identity by decreasing the parameters in absolute value to zero. Moreover, $A\mathscr{T}_- \cap \mathscr{T}_- \neq \emptyset$ is a connected set, as it is the intersection of two convex sets. The set $AH \cap H \neq \emptyset$, however, is initially not connected. In order to obtain a connected set out of $AH \cap H \neq \emptyset$ we have to restrict firstly the dimension d to $d \geq 4$ and secondly the normal forms in $\mathscr{B}$ to the following ones:

1. For even dimension:
   a) Normal case: $\alpha, \beta_i, i = 1, \ldots, \hat{j}, \ldots, k$, arbitrary, $\beta_j = 0$,
   b) Exceptional case: $\tau, \beta_i, i = 1, \ldots, \hat{j}, \ldots, k-1$, arbitrary, $\beta_j = 0$.
2. For odd dimension:
   a) Normal case:
      i. $\alpha, \beta_i, i = 1, \ldots, \hat{j}, \ldots, k$, arbitrary, $\beta_j = 0$, or $\beta_i, i = 1, \ldots, k$, arbitrary and $\alpha = 0$,
      ii. $\beta_i, i = 1, \ldots, \hat{j}, \ldots, k$, arbitrary, $\beta_j = 0$,
      iii. $\beta_i, i = 1, \ldots, \hat{j}, \ldots, k$, arbitrary, $\beta_j = 0$,
   b) Exceptional case:
      i. $\tau, \beta_i, i = 1, \ldots, \hat{j}, \ldots, k-1$, arbitrary, $\beta_j = 0$,
      ii. $\tau, \beta_i, i = 1, \ldots, \hat{j}, \ldots, k-1$, arbitrary, $\beta_j = 0$.

Denote the set containing these normal forms by $\mathscr{B}'$. Indeed, every $A \in \mathscr{B}'$ yields a connected intersection $AH \cap H \neq \emptyset$. This is easily seen by applying a normal form $N \in \mathscr{B}'$ to an arbitrary vector in H. For convenience consider $d = 4$. The corresponding normal form N is then given by

$$N = \Lambda_{W_0}(i\alpha) = \begin{pmatrix} \cosh(i\alpha) & \sinh(i\alpha) & 0 & 0 \\ \sinh(i\alpha) & \cosh(i\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Except for the trivial case $\alpha = 0$ the set $NH \cap H = \{e \in H : e^0 = e^1 = 0\} \neq \emptyset$ is topologically S^1, hence a circle and that is connected. In the case of n four-vectors

$e_1, \ldots, e_n$ the intersection is simply the n-dimensional torus $T^n = S^1 \times \cdots \times S^1$ which again is connected. This implies that the set $AH \cap H \neq \emptyset$ is also connected as the map $N \mapsto \Lambda_1 N \Lambda_2 = A$ is continuous for all Λ_1 and Λ_2 in $\mathcal{L}_+^\uparrow$. The assertion is analogously proven for $d > 4$.

This implies, however, that the set $(A\mathscr{T}_- \cap \mathscr{T}_-) \times (AH \cap H) \neq \emptyset$ is also a connected set for all $A \in \mathscr{D}' = \mathscr{A} \cap \mathscr{B}'$ as it is the Cartesian product of connected sets. It is therefore sufficient to prove that (3.24) holds in the neighbourhood of one point. Moreover, if $\mathcal{W}(A\zeta, Ae) = \mathcal{W}(\zeta, e)$ applies for some complex Lorentz transformation A_0 (e. g. the identity) then this will still be the case for some A varying in a suitable neighbourhood of A_0.

In order to see this keep (ζ, e) fixed in $\mathscr{U}_{A_0} := (A_0\mathscr{T}_- \times A_0 H) \cap (\mathscr{T}_- \times H)$[1] and consider $\mathcal{W}(A\zeta, Ae)$ as a function of A. We introduce local coordinates in the neighbourhood of A_0, setting $A = BA_0$, where B is expressed in Cayley parameters. The vector (ζ, e) will also be in $\mathscr{U}_A$ if B varies over a suitable, sufficiently small connected neighbourhood of the identity. The same will hold for all points in a neighbourhood of (ζ, e) if we keep the neighbourhood B is varying over small enough. $\mathcal{W}(A\zeta, Ae)$, however, is an analytic function of the Cayley parameters of B because of the analyticity in ζ and because the assumption $Ae \in AH \cap H \neq \emptyset$ yields Ae is real and therefore constant in the complex Cayley parameters. Obviously, this function is constant for real parameters, i. e. for $B \in \mathcal{L}_+^\uparrow$ we have $\mathcal{W}(A\zeta, Ae) = \mathcal{W}(A_0\zeta, A_0 e)$ by corollary 3.1.2. Hence it is also constant for complex parameters. This arises from the fact that the coefficients of the power series of a holomorphic function can be obtained by differentiation in the direction of the real axis. Therefore, it follows $\mathcal{W}(A\zeta, Ae) = \mathcal{W}(A_0\zeta, A_0 e)$ for all B in the neighbourhood of the identity and all elements in a neighbourhood of (ζ, e). This implies $\mathcal{W}(A\zeta, Ae) = \mathcal{W}(\zeta, e)$ in $\mathscr{U}_A = (A\mathscr{T}_- \times AH) \cap (\mathscr{T}_- \times H)$.

Finally, because $\mathscr{D}'$ is connected, as all normal forms in this set can continuously be connected with the identity (see discussion above), any $A \in \mathscr{D}'$ can be joined to the identity by a finite chain of overlapping neighbourhoods, to each of which the above reasoning applies. Thus, we proved that equation (3.24) holds throughout $(A\mathscr{T}_- \times AH) \cap (\mathscr{T}_- \times H)$ for all complex Lorentz transformations $A \in \mathscr{D}'$.

Now it remains to discuss whether this already proves the single-valuedness of the continuation to $\mathcal{W}(\xi, \hat{e})$, because so far we ignored those A which are in $\mathscr{D}$ but not in $\mathscr{D}'$. Indeed, the proof for the single-valuedness is complete at this point. This is so because, obviously, our discussion was more general than necessary: Firstly, we considered far more complex Lorentz transformations A than needed as we were only considering transformations of the form $A = \Lambda_{W_1}(i\alpha)\Lambda_{W_2}(i\beta)$ with W_i arbitrary wedges. Secondly, we were only considering real vectors $e = (e_1, \ldots, e_n)$ in $\bigcup_W \Delta_W$, cf. equation (3.22). Therefore, taking these two factors into account the transition from the set $\mathscr{D}$ to $\mathscr{D}'$ does

[1] With regard to the variable e this is to be understood in the sense of distributions, namely that the support of the corresponding test function is contained in a fixed region in $A_0 H \cap H$.

not exclude any $A = \Lambda_{W_1}(i\alpha)\Lambda_{W_2}(i\beta)$ for the following reason: Consider $d = 4$, then the normal forms in $\mathscr{D}$ to which such an A is equivalent are either $N = \Lambda_{W_0}(i\theta)$ or $N = \begin{pmatrix} M(i\tau) & 0 \\ 0 & 1 \end{pmatrix}$. The latter normal form, however, can be excluded for $n > 2$ as for $n > 2$ this normal form will only leave such $e \in H$ real which are not in Δ_W for any wedge W, namely these are the vectors $e = (e_1, \dots, e_n)$ with e_i equals e_k up to a sign. Therefore, the only eligible normal form that is left in this case is $N = \Lambda_{W_0}(i\theta)$ which, however, is also contained in $\mathscr{D}'$. The same arguments also apply to higher dimensions.

Thus, the following theorem holds.

Theorem 3.2.5. *For $n > 2$ and $d > 3$ the distribution $\mathcal{W}(\xi_1, \dots, \xi_{n-1}; e_1, \dots, e_n)$ over $\mathbb{R}^{d(n-1)} \times \bigcup_W \Delta_W$ introduced in Section 2.2 is the boundary value in the sense of $\mathscr{S}'$ of the quantity $\mathcal{W}(\xi_1, \dots, \xi_{n-1}; \hat{e}_1, \dots, \hat{e}_n)$ which is an analytic function in $(\hat{e}_1, \dots, \hat{e}_n) \in \Xi$.*

As already stated in this theorem, the real boundary of Ξ does not contain the whole manifold $H^{(d-1)n}$, but only the subset $\bigcup_W \Delta_W$. However, we desire analyticity in a domain which is just big enough so that its boundary contains all vectors in $H^{(d-1)n}$. To this end, we make the following *assumption*:

Assumption 3.2.6. *The distribution $\mathcal{W}(\xi_1, \dots, \xi_{n-1}; e_1, \dots, e_n)$ introduced in Section 2.2 is the boundary value in the sense of $\mathscr{S}'$ of the quantity $\mathcal{W}(\xi_1, \dots, \xi_{n-1}; \hat{e}_1, \dots, \hat{e}_n)$ which is an analytic function in $(\hat{e}_1, \dots, \hat{e}_n) \in \mathcal{T}_n$.*

According to [3], this assumption is meaningful, i.e. $\mathcal{W}(\xi_1, \dots, \xi_{n-1}; \hat{e}_1, \dots, \hat{e}_n)$ admits $\mathcal{W}(\xi_1, \dots, \xi_{n-1}; e_1, \dots, e_n)$ as its boundary value in the sense of distributions. Clearly, the real boundary of $\mathcal{T}_n$ fulfils the above required property.

It is shown in [4] that as a consequence of Assumption 3.2.6 the vector-valued distribution $\Psi(x_1, \dots, x_n; e_1, \dots, e_n)$ given by equation (2.30) is the boundary value in the sense of $\mathscr{S}'$ of the quantity $\Psi(x_1, \dots, x_n; \hat{e}_1, \dots, \hat{e}_n)$ which is an analytic function in $\mathscr{G}_n$, where

$$\mathscr{G}_n := H_c^n \cap \mathcal{G}_n \tag{3.28}$$

with $\mathcal{G}_n = \{(\hat{e}_1, \dots, \hat{e}_n) \in \mathbb{C}^{dn} : \mathrm{Im}(\hat{e}_1) \in V_+, \mathrm{Im}(\hat{e}_i - \hat{e}_{i-1}) \in V_+, i = 2, \dots, n\}$. Therefore, it is not an extra assumption that for any $\Phi \in \mathscr{H}$ the distribution $\langle \Phi, \Psi(x_1, \dots, x_n; e_1, \dots, e_n)\rangle$ is the boundary value of the quantity $\langle \Phi, \Psi(x_1, \dots, x_n; \hat{e}_1, \dots, \hat{e}_n)\rangle$ which is a holomorphic function in $(\hat{e}_1, \dots, \hat{e}_n) \in \mathscr{G}_n$.

The appearance of difference variables $\hat{e}_i - \hat{e}_j$ in the analyticity domains considered above may seem to come out of the blue in the absence of translational invariance in the string directions. However it is claimed in [4], a Lorentz invariant function $f(\hat{e}_1, \dots, \hat{e}_n)$ that is analytic in a subdomain of H_c^n is a function of the invariants $\hat{e}_i \cdot \hat{e}_j$. This in turn implies that such a function depends only on $(\hat{e}_i - \hat{e}_j)^2$ as $\hat{e}_i \cdot \hat{e}_i = -1$. We did not investigate in the scope of this thesis whether or not this statement fully applies to the quantities considered here as these do not depend exclusively on the variables $\hat{e}_i$. In particular, there may occur other invariants besides the scalar products according to invariant theory [14, p. 89 et seq.].

Note that by Theorem 3.2.5 some of the assumed analyticity in Assumption 3.2.6 is proven. To this end, analyticity of $\mathcal{W}(\zeta_1, \dots, \zeta_{n-1}; e_1, \dots, e_n)$ in $\mathscr{T}_-^{n-1}$ resulting from the spectral condition was used. Then by exploiting that the complex Lorentz transformations $\Lambda_W(i\theta)$, $-\pi < \theta < 0$, map the wedge W into $\mathscr{T}_-$ and by using the relativistic transformation law (2.21) we obtained Theorem 3.2.5. It may be possible, however, to extend the "primitive" analyticity domain $\mathscr{T}_-^{n-1}$ by making use of a version of the Glaser-Streater theorem [14] for the string-localized case. Then we may be able to consider additional complex Lorentz transformations besides the complex rotations $\Lambda_W(i\theta)$. As a result some more of the assumed analyticity in Assumption 3.2.6 could be proven. In the scope of this thesis we did not study a generalized version of the Glaser-Streater theorem for the string-localized case.

We want to conclude this section by noting that the analyticity assumption 3.2.6 is in accordance with the results for the free string-localized fields of [20, 21], see also Section 5.2. In particular, the two-point functions for the free string-localized fields $\mathcal{W}(\xi; e_1, e_2)$ can be analytically continued into $\mathcal{T}_2$.

4 General Theorems for String-Localized Fields

In Section 2.1 we introduced the polynomial algebra of fields $\mathscr{P}$, cf. equation (2.8). We are now interested in subalgebras $\mathscr{P}_{\mathcal{O},\mathcal{U}}$ of $\mathscr{P}$, where $\mathcal{O}$ and $\mathcal{U}$ denote arbitrary open sets in $\mathbb{R}^d$ and H^{d-1} respectively. Such an algebra is the set of all polynomials of the form

$$c + \sum_{i=1}^{N} \varphi(f_1^i, h_1^i) \cdots \varphi(f_i^i, h_i^i), \quad \text{with } \operatorname{supp}(f_k^i) \subset \mathcal{O} \text{ and } \operatorname{supp}(h_k^i) \subset \mathcal{U}, \tag{4.1}$$

where c is any complex constant.

The following theorem yields the Reeh-Schlieder property [25] for string-localized fields.

Theorem 4.0.7. *Let $\mathcal{O}$ and $\mathcal{U}$ be arbitrary open sets in $\mathbb{R}^d$ and H^{d-1} respectively, and let the analyticity assumption 3.2.6 hold. Then Ω is cyclic for the polynomial algebra of fields $\mathscr{P}_{\mathcal{O},\mathcal{U}}$ given by equation (4.1), if it is cyclic for $\mathscr{P}$ given by equation (2.8). That is, $\mathscr{P}_{\mathcal{O},\mathcal{U}}\Omega$ is dense in $\mathscr{H}$.*

Proof

We have to show that $(\mathscr{P}_{\mathcal{O},\mathcal{U}}\Omega)^{\perp} = \{0\}$. To this end, firstly consider a vector $\Psi \in (\mathscr{P}_{\mathcal{O},\mathcal{U}}\Omega)^{\perp}$. Secondly, it can be shown that

$$\langle \Psi, \Psi(g) \rangle = \int \langle \Psi, \varphi(x_1, e_1) \cdots \varphi(x_n, e_n)\Omega \rangle g(x_1, \ldots, x_n; e_1, \ldots, e_n) d^d x_1 \cdots d\sigma(e_n)$$

makes sense for all $g \in \mathscr{S}$ and any $\Psi \in \mathscr{H}$ using the same arguments that justify equation (2.30). This expression, however, vanishes for all $g \in \mathscr{S}(\mathcal{O}^n \times \mathcal{U}^n)$. Likewise, the distribution F formally defined by

$$\begin{aligned} F(-x_1, x_1 - x_2, \ldots, x_{n-1} - x_n; e_1, \ldots, e_n) &:= \langle \Psi, \Psi(x_1, \ldots, x_n; e_1, \ldots, e_n) \rangle \\ &= \langle \Psi, \varphi(x_1, e_1) \cdots \varphi(x_n, e_n)\Omega \rangle \end{aligned}$$

gives zero for $-x_1, x_1 - x_2, \ldots, x_{n-1} - x_n$ in an open set determined by $x_1, \ldots, x_n \in \mathcal{O}$ and $e_1, \ldots, e_n \in \mathcal{U}$.

The Fourier transform of F vanishes unless each of the variables conjugate to $-x_1$, $x_i - x_{i+1}$, $i = 1, \dots, n-1$, lies in $\mathrm{spec}(P)$. This may be proved just like Theorem 2.2.1. It therefore follows that there exists an analytic continuation of F to a quantity which is an analytic function in $((-x_1) - i\eta_1,\ (x_1 - x_2) - i\eta_2, \dots, (x_{n-1} - x_n) - i\eta_n) \in \mathscr{T}_-^n$. Moreover, it follows from the analyticity assumption 3.2.6 that analytic continuation of F into $(\hat{e}_1, \dots, \hat{e}_n) \in \mathscr{G}_n$ is also provided. Therefore, the holomorphic function defined on $\mathscr{T}_-^n \times \mathscr{G}_n$ has F as its distributional boundary value. Now as a consequence of the edge of the wedge theorem [25] this function vanishes throughout its domain of definition as its boundary value gives zero in an open set determined by $x_1, \dots, x_n \in \mathcal{O}$ and $e_1, \dots, e_n \in \mathcal{U}$. Therefore, $F(-x_1, x_1 - x_2, \dots, x_{n-1} - x_n; e_1, \dots, e_n)$ vanishes for all $x_1, \dots, x_n \in \mathbb{R}^d$ and for all $e_1, \dots, e_n \in H^{d-1}$. That is, $\Psi \in (\mathscr{P}\Omega)^\perp$. This, however, implies $\Psi = 0$ as by assumption $\mathscr{P}\Omega$ spans the Hilbert space $\mathscr{H}$. □

The next theorem reveals Ω as a separating vector for $\mathscr{P}_{\mathcal{O},\mathcal{U}}$. It combines the Reeh-Schlieder property with string-locality.

Theorem 4.0.8. *Let $T \in \mathscr{P}_{\mathcal{O},\mathcal{U}}$, then*

$$T\Omega = 0 \tag{4.2}$$

implies $T = 0$.

Proof

Let Φ be in D, the domain of definition of the fields $\varphi(f, h)$, and $\Psi = P'\Omega$, where $P' \in \mathscr{P}_{\mathcal{O}',\mathcal{U}'}$ with $\mathcal{O}'$ and $\mathcal{U}'$ such that the cones $\mathcal{O}' + \mathbb{R}_0^+\mathcal{U}'$ and $\mathcal{O} + \mathbb{R}_0^+\mathcal{U}$ are space-like separated. Consider any $T \in \mathscr{P}_{\mathcal{O},\mathcal{U}}$ satisfying (4.2), then

$$\langle \Psi, T^*\Phi \rangle = \langle TP'\Omega, \Phi \rangle = \langle P'T\Omega, \Phi \rangle = 0.$$

By Theorem 4.0.7, however, vectors of the form Ψ span the Hilbert space $\mathscr{H}$, which yields $T^*\Phi = 0$. On the other hand $T\Psi = 0$, for $\Psi \in D$ since $\langle \Psi, T^*\Phi \rangle = \langle T\Psi, \Phi \rangle$ and D is dense, which implies $T = 0$. □

The preceding theorems are basically sufficient in order to prove the Jost-Schroer theorem for string-localized fields, which we shall also present in the next chapter. However, as we already mentioned in the introduction we want to provide an alternative proof that depends on the following theorem which is a generalisation of a theorem by O. Steinmann [23, Theorem 1] to the case where the string directions are not fixed.

Theorem 4.0.9. *Consider $W^{\mathrm{T}}(x_1, e_1; \dots; x_n, e_n)$ with the properties stated in Section 2.2.1 including the analyticity assumption 3.2.6. If*

$$(\Box_i + m_i^2) W^{\mathrm{T}}(x_1, e_1; \dots; x_n, e_n) = 0, \quad \forall i = 1, \dots, n, \quad m_i \geq 0, \tag{4.3}$$

then

$$W^{\mathrm{T}}(x_1, e_1; \ldots; x_n, e_n) = 0, \quad \text{for} \quad n \geq 3. \tag{4.4}$$

Proof

Equation (4.3) formally implies

$$(p_i^2 - m_i^2)\widetilde{W}^{\mathrm{T}}(p_1, \ldots, p_n; e_1, \ldots, e_n) = 0. \tag{4.5}$$

Thus the p_i-support of $\widetilde{W}^{\mathrm{T}}(p_1, \ldots, p_n; e_1, \ldots, e_n)$ is contained in $p_i \in \{p_i^2 = m_i^2\}$, $i = 1, \ldots, n$. Moreover, it follows from the spectral condition that the p_1-support lies in $\overline{V}_+$, i. e. $p_1^2 = m_1^2$, $(p_1)_0 \geq 0$, cf. Theorem 2.2.3.

We proceed by making the following definitions

Definition 4.0.10.

$$X := x_1 + x_2, \qquad \xi := x_1 - x_2, \qquad P := \frac{1}{2}(p_1 + p_2), \qquad Q := \frac{1}{2}(p_1 - p_2). \tag{4.6}$$

Definition 4.0.11.

$$\begin{aligned} V_{12}(X, \xi, x_3, \ldots, x_n; e_1, \ldots, e_n) \;&:=\; W^{\mathrm{T}}(x_1, e_1; \ldots; x_n, e_n) \\ V_{21}(X, \xi, x_3, \ldots, x_n; e_1, \ldots, e_n) \;&:=\; W^{\mathrm{T}}(x_2, e_2; x_1, e_1; x_3, e_3; \ldots; x_n, e_n). \end{aligned} \tag{4.7}$$

The Fourier transforms $\widetilde{V}_{ij}(P, Q, p_3, \ldots, p_n; e_1, \ldots, e_n)$, $i, j = 1, 2$, $i \neq j$, of the formal expressions defined in equation (4.7) have both their P-support in $\overline{V}_+$ which results again from the spectral condition. However, $P \neq 0$ as the contribution of the vacuum state was eliminated by considering truncated vacuum expectation values.[1]

The determination of the Q-support requires some discussion. First of all look at the Q-support of $\widetilde{V}_{12}$ for $P, p_3, \ldots, p_n$ being held fix and P as stated above. We recall $Q := \frac{1}{2}(p_1 - p_2)$ and $p_1 \in \overline{V}_+$. Then

Case 1: $(p_2)_0 \geq 0$ yields compact Q-support;

Case 2: $(p_2)_0 \leq 0$ yields, by considering the conditions $(P + Q)^2 = m_1^2$, $P_0 + Q_0 \geq 0$, $(P - Q)^2 = m_2^2$, $P_0 - Q_0 \leq 0$ for this case,

a. compact Q-support if $P^2 > 0$, or if $P^2 = 0$ and at least one of the masses m_1, m_2 does not vanish;

b. Q-support which is contained in the half-line $Q = \lambda P$, $\lambda \geq 0$, if $P^2 = 0$ and $m_1 = m_2 = 0$.

[1] These statements on the P-support can only be made for $n > 2$.

By a similar discussion we find that the Q-support of $\widetilde{V}_{21}$ is also compact except for the case $P^2 = 0$ and $m_1 = m_2 = 0$. It is then contained in $Q = \lambda P$, $\lambda \leq 0$.

Consider now adequate test functions $\widetilde{f}(P, p_3, \dots, p_n) \in \mathscr{D}$[2], that is functions $\widetilde{f}$ whose P-support is either in the set $\{P^2 > 0\}$ or in a sufficiently small neighbourhood of a null point not containing the origin. Then the Q-support of

$$\widetilde{F}_+(Q, e_1, \dots, e_n) = \int d^d P \prod_{i=3}^{n} d^d p_i \widetilde{V}_{12}(P, Q, p_3, \dots, p_n; e_1, \dots, e_n) \widetilde{f}(P, p_3, \dots, p_n) \quad (4.8)$$

is contained in the union of a compact set and a narrow cone C_+ around a positive null direction with apex at the origin. Analogous the Q-support of

$$\widetilde{F}_-(Q, e_1, \dots, e_n) = \int d^d P \prod_{i=3}^{n} d^d p_i \widetilde{V}_{21}(P, Q, p_3, \dots, p_n; e_1, \dots, e_n) \widetilde{f}(P, p_3, \dots, p_n) \quad (4.9)$$

is in the union of a compact set and a narrow cone $C_- = -C_+$ around a negative null direction. These support properties have the consequence that the Fourier transforms

$$F_\pm(\xi, e_1, \dots, e_n) = \int d^d X \prod_{i=3}^{n} d^d x_i V_{ij}(X, \xi, x_3, \dots, x_n; e_1, \dots, e_n) f(X, x_3, \dots, x_n), \quad (4.10)$$

$i, j = 1, 2$, $i \neq j$, are boundary values of quantities $F_\pm(\zeta, e_1, \dots, e_n)$ that are analytic functions in the tubes

$$\begin{aligned} \mathscr{T}(F_\pm) &= \{\zeta = \xi + i\eta : \eta \in \Gamma_\pm\}, \\ \Gamma_\pm &= \{\eta : \eta \cdot Q \lessgtr 0, \forall Q \in C_+ \backslash \{0\}\}. \end{aligned} \quad (4.11)$$

Note that $\Gamma_\pm$ also contains space-like points.

Due to string-locality the boundary values $F_\pm(\xi, e_1, \dots, e_n)$ coincide if the strings S_{x_1,e_1} and S_{x_2,e_2} are causally disjoint. In order to determine the open set of all real ξ, however, for which F_+ and F_- coincide by string-locality we have to control the directions e_1 and e_2. Consider therefore suitable neighbourhoods U_1 and U_2 of e_1 and e_2 respectively such that, for all $e_1' \in U_1$ and $e_2' \in U_2$, $F_\pm(\xi, e_1', e_2', e_3, \dots, e_n)$ coincide by string-locality in the real open set

$$\mathscr{R} := \{\xi = x_1 - x_2 : S_{x_1,e_1'} \text{ space-like separated from } S_{x_2,e_2'}, \forall e_1' \in U_1, \forall e_2' \in U_2\}. \quad (4.12)$$

$\mathscr{R}$ contains only space-like ξ by lemma $A.1$ in [21], but in general not all space-like ξ. However, for certain e_1 and e_2 and therefore for their (small) neighbourhoods U_1 and

[2]The space of C^∞-functions with compact support.

U_2 respectively $\mathscr{R}$ contains all ξ with $(\xi_0)^2 - \cdots - (\xi_{d-2})^2 < 0$.[3] Further as the choice of U_1 and U_2 is arbitrary we can assume this scenario.

It follows by the edge of the wedge theorem that $F_\pm(\zeta, e_1', e_2', e_3, \ldots, e_n)$ are analytic continuations of each other in the variable ζ and that the function $F(\zeta, e_1', e_2', e_3, \ldots, e_n)$ they define is holomorphic for all $\zeta \in \mathscr{A}$, where

$$\mathscr{A} := \mathscr{T}(F_+) \cup \mathscr{T}(F_-) \cup \mathscr{N}(\mathscr{R}). \tag{4.13}$$

$\mathscr{N}(\mathscr{R})$ is an open complex neighbourhood of $\mathscr{R}$ and e_1', e_2' are in U_1 and U_2 respectively.

Consider now some vector ξ whose components are all real and fixed except for one ξ_i, $i = 2, \ldots, d-1$, such that $\mathrm{Im}(\xi_i) \neq 0$ is contained either in Γ_+ or in Γ_-. Then the vector ξ is either in $\mathscr{T}(F_+)$ or in $\mathscr{T}(F_-)$, therefore in $\mathscr{A}$. Indeed, such a vector exists for some i. If $\mathrm{Im}(\xi_i) = 0$ consider the straight line in the i-direction through the point $(\xi_0, \ldots, \xi_{i-1}, 0, \xi_{i+1}, \ldots, \xi_{d-1})$. Then this straight line intersects the complement of $\mathscr{R}$, $\mathscr{R}^c$, if $(\xi_0)^2 - \cdots - (\xi_{i-1})^2 - (\xi_{i+1})^2 - \cdots - (\xi_{d-2})^2 \geq 0$.[4] This inequality, however, is only fulfilled for ξ_i in a compact set. On the other hand, if $(\xi_0)^2 - \cdots - (\xi_{i-1})^2 - (\xi_{i+1})^2 - \cdots - (\xi_{d-2})^2 < 0$ our straight line never intersects $\mathscr{R}^c$ for any real ξ_i. In the latter case the vector ξ is in $\mathscr{A}$ for all ξ_i, i.e. also for $\mathrm{Im}(\xi_i) \neq 0$ as chosen above. Starting from this region we may change $\xi_0, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_{d-2}$ until the straight line intersects the boundary of $\mathscr{R}$ and hence of $\mathscr{A}$ in a compact real set in the complex ξ_i-plane. But this is exactly the setting needed to apply the continuity theorem [26]. By this theorem $F(\zeta, e_1', e_2', e_3, \ldots, e_n)$ can be continued analytically into an open neighbourhood of the mentioned compact set. In this manner it can be shown that the holomorphy envelope of $\mathscr{A}$ contains all real $\xi \in \mathbb{R}^d$.[5] As a consequence $F_+(\xi, e_1', e_2', e_3, \ldots, e_n) = F_-(\xi, e_1', e_2', e_3, \ldots, e_n)$, $\forall \xi \in \mathbb{R}^d$, $e_1' \in U_1$, $e_2' \in U_2$. Therefore $W^{\mathrm{T}}(x_1, e_1'; x_2, e_2'; \ldots; x_n, e_n) = W^{\mathrm{T}}(x_2, e_2'; x_1, e_1'; x_3, e_3; \ldots; x_n, e_n)$, $\forall x_i \in \mathbb{R}^d$, $e_1' \in U_1$, $e_2' \in U_2$.

Next, we formally define the distribution

$$\begin{aligned} &T(x_1, \ldots, x_n; e_1', e_2', e_3, \ldots, e_n) \\ &\qquad := W^{\mathrm{T}}(x_1, e_1'; x_2, e_2'; \ldots; x_n, e_n) - W^{\mathrm{T}}(x_2, e_2'; x_1, e_1'; x_3, e_3; \ldots; x_n, e_n). \end{aligned}$$

Now T vanishes in an open set constituted by $e_1' \in U_1$, $e_2' \in U_2$. The analyticity assumption 3.2.6 provides analytic continuation of T into a set of complex string directions, namely $\mathcal{T}_n$. Thus, by analytic continuation and application of the edge of the wedge theorem $T(x_1, \ldots, x_n; e_1, e_2, e_3, \ldots, e_n)$ vanishes for all $e_i \in H^{d-1}$. Therefore

[3] This includes the case discussed in [23].

[4] For the moment we consider $(d-1)$-dimensions.

[5] This statement is made in [23] for $d = 4$. Here it is a generalisation of this statement to $d \geq 4$.

$$W^{\mathrm{T}}(x_1, e_1; x_2, e_2; \dots; x_n, e_n) = W^{\mathrm{T}}(x_2, e_2; x_1, e_1; x_3, e_3; \dots; x_n, e_n), \tag{4.14}$$

$\forall x_i \in \mathbb{R}^d$ and $\forall e_i \in H^{d-1}$, $i = 1, \dots, n$.

Using this result we are now going to prove by induction that W^{T} is invariant under all permutations of $(x_1, e_1; \dots, ; x_n, e_n)$. Let therefore W^{T} be invariant under permutations of $(x_1, e_1; \dots, ; x_\alpha, e_\alpha)$, $2 \le \alpha < n$. Then make the following definitions

Definition 4.0.12.

$$Y := \frac{1}{2}(x_\alpha + x_{\alpha+1}), \qquad y := x_{\alpha+1} - x_\alpha, \tag{4.15}$$

Definition 4.0.13.

$$K_+(x_1, \dots, x_{\alpha-1}, Y, y, x_{\alpha+2}, \dots, x_n; e_1, \dots, e_n) := W^{\mathrm{T}}(x_1, e_1; \dots; x_n, e_n), \tag{4.16}$$

$$K_-(x_1, \dots, x_{\alpha-1}, Y, y, x_{\alpha+2}, \dots, x_n; e_1, \dots, e_n)$$
$$:= W^{\mathrm{T}}(x_1, e_1; \dots; x_{\alpha+1}, e_{\alpha+1}; x_\alpha, e_\alpha; \dots; x_n, e_n).$$

Integration over a suitable test function $g(y) \in \mathscr{D}$ leads to

$$K'_\pm(x_1, \dots, x_{\alpha-1}, Y, x_{\alpha+2}, \dots, x_n; e_1, \dots, e_n) = \int d^d y K_\pm(\dots, y, \dots) g(y). \tag{4.17}$$

This implies, considering e.g. K_+, expressions of the form

$$\int \varphi(Y - \frac{y}{2}, e_\alpha) \varphi(Y + \frac{y}{2}, e_{\alpha+1}) g(y) d^d y.$$

Such an expression can, however, be treated like a single field that is localized in a space-like cone $\mathcal{C}$ which contains the strings S_{x_α, e'_α} and $S_{x_{\alpha+1}, e'_{\alpha+1}}$ for all e'_α and $e'_{\alpha+1}$ in some small open neighbourhoods U_α of e_α and $U_{\alpha+1}$ of $e_{\alpha+1}$ respectively. We want to assume this case. Thus, by the spectral condition $K'_\pm$ are boundary values of analytic functions, regular in

$$\{\mathrm{Im}(x_1 - x_2) \in V_+, \dots,$$
$$\mathrm{Im}(x_{\alpha-1} - Y) \in V_+, \mathrm{Im}(Y - x_{\alpha+2}) \in V_+, \dots, \mathrm{Im}(x_{n-1} - x_n) \in V_+\}.$$

Furthermore, this statement holds also for the difference

$$\Delta' := K'_+ - K'_-.$$

Now, there is a non-empty open set constituted by real $x_{\alpha-1}$ and Y such that $S_{x_{\alpha-1}, e'_{\alpha-1}}$ is causally disjoint from $S_{x_\alpha, e'_\alpha} \cup S_{x_{\alpha+1}, e'_{\alpha+1}}$ for all $e'_{\alpha-1}$ in some small open neighbourhood

$U_{\alpha-1}$ of $e_{\alpha-1}$, $e'_\alpha \in U_\alpha$ and $e'_{\alpha+1} \in U_{\alpha+1}$. Moreover, this string-locality holds for all $y \in$supp(g). Therefore, in these sets it follows for $\Delta := K_+ - K_-$ that

$$\begin{aligned}\Delta(x_1, \ldots, x_{\alpha-1}, Y, y, x_{\alpha+2}, \ldots, x_n; e_1, \ldots, e'_{\alpha-1}, e'_\alpha, e'_{\alpha+1}, \ldots, e_n)&\\ = \Delta(x_1, \ldots, Y, y, x_{\alpha-1}, x_{\alpha+2}, \ldots, x_n; e_1, \ldots, e'_\alpha, e'_{\alpha+1}, e'_{\alpha-1}, \ldots, e_n)&\\ = W^{\mathrm{T}}(x_1, e_1; \ldots; x_\alpha, e_\alpha; x_{\alpha+1}, e_{\alpha+1}; x_{\alpha-1}, e_{\alpha-1}; \ldots; x_n, e_n) -&\\ W^{\mathrm{T}}(x_1, e_1; \ldots; x_{\alpha+1}, e_{\alpha+1}; x_\alpha, e_\alpha; x_{\alpha-1}, e_{\alpha-1}; \ldots; x_n, e_n) = 0&\end{aligned}$$

as by the inductive assumption W^{T} is invariant under permutations of its first α arguments. Hence, also Δ' vanishes for $x_{\alpha-1}, Y, e'_{\alpha-1}, e'_\alpha, e'_{\alpha+1}$ in the open sets named above. By analytic continuation Δ' gives zero for all $x_1, \ldots, x_{\alpha-1}, Y, x_{\alpha+2}, \ldots, x_n \in \mathbb{R}^d$ and $e'_{\alpha-1}, e'_\alpha, e'_{\alpha+1}$ in $U_{\alpha-1}$, U_α and $U_{\alpha+1}$ respectively. This implies, however, that

$$W^{\mathrm{T}}(x_1, \ldots, x_\alpha, x_{\alpha+1}, \ldots, x_n; e \ldots) = W^{\mathrm{T}}(x_1, \ldots, x_{\alpha+1}, x_\alpha, \ldots, x_n; e \ldots)$$

for all $x_i \in \mathbb{R}^d$, $i = 1, \ldots, n$ and $e'_{\alpha-1}, e'_\alpha, e'_{\alpha+1}$ in $U_{\alpha-1}$, U_α and $U_{\alpha+1}$ respectively. By the analyticity assumption 3.2.6 it follows using the same arguments leading to (4.14) that

$$W^{\mathrm{T}}(x_1, e_1; \ldots; x_\alpha, e_\alpha; x_{\alpha+1}, e_{\alpha+1}; \ldots; x_n, e_n) = W^{\mathrm{T}}(x_1, e_1; \ldots; x_{\alpha+1}, e_{\alpha+1}; x_\alpha, e_\alpha; \ldots; x_n, e_n) \tag{4.18}$$

for all $x_i \in \mathbb{R}^d$ and $e_i \in H^{d-1}$, $i = 1, \ldots, n$. Thus, W^{T} is invariant under commutation of (x_α, e_α) and $(x_{\alpha+1}, e_{\alpha+1})$ and therefore under all permutations of $(x_1, e_1; \ldots, ; x_{\alpha+1}, e_{\alpha+1})$. Hence, we showed that W^{T} is invariant under all permutations of $(x_1, e_1; \ldots, ; x_n, e_n)$.

This result, however, yields

$$\widetilde{W}^{\mathrm{T}}(p_1, \ldots, p_n; e_1, \ldots, e_n) = \widetilde{W}^{\mathrm{T}}(p_2, \ldots, p_n, p_1; e_2, \ldots, e_n, e_1). \tag{4.19}$$

As the left-hand side of (4.19) has its p_1-support contained in $\overline{V}_+$ and the right-hand side in $p_1 \in \overline{V}_-$ it follows that

$$\widetilde{W}^{\mathrm{T}}(p_1, \ldots, p_n; e_1, \ldots, e_n) = 0, \quad \text{for} \quad n \geq 3, \tag{4.20}$$

because support at $p_1 = 0$ was excluded by considering truncated vacuum expectation values. This finishes the proof. □

At this point we want to make the following remark. If we omit the analyticity assumption 3.2.6 in Theorem 4.0.9 then it is possible to prove the theorem only for a certain configuration of the string directions $e_1, \ldots, e_n$. This configuration must be such that there are causally disjoint strings S_{x_i, e_i} and S_{x_j, e_j} for all $i \neq j$. Then the above reasoning still applies even without analytic continuation in the string directions, with the difference that the results hold only for the chosen configuration of the space-like directions $e_1, \ldots, e_n$. This is so because in the proof presented above string-locality was used amongst others to obtain the results. Due to the special configuration of the string directions, however, string-locality still applies. The principle of analytic continuation in the space-like directions was then used to obtain general results without any restrictions on the supports of the test functions, which arises from string-locality. So by omitting the analyticity assumption 3.2.6 Theorem 4.0.9 only holds for this special choice of the space-like directions.

5 Free String-Localized Fields and the Jost-Schroer Theorem

This chapter is devoted to free string-localized fields. We shall call a hermitian scalar string-localized field $\varphi(f,h)$ a free field if it obeys the equation

$$\left(\Box_x + m^2\right) \varphi(f,h) = 0 \tag{5.1}$$

with $\Box = \left(\frac{\partial}{\partial x^0}\right)^2 - (\nabla_{\vec{x}})^2$ and $m \geq 0$. The commutator

$$[\varphi(f_1,h_1), \varphi(f_2,h_2)] = \varphi(f_1,h_1)\varphi(f_2,h_2) - \varphi(f_2,h_2)\varphi(f_1,h_1) \tag{5.2}$$

of free fields is a c-number, i. e. a multiple of the identity operator, with $f_i \in \mathscr{S}(\mathbb{R}^d)$ and $h_i \in \mathscr{S}(H^{d-1})$, $i = 1,2$.
On the other hand, the vacuum expectation values $\langle \Omega, \varphi(f_1,h_1) \cdots \varphi(f_n,h_n)\Omega \rangle$ for the theory of a free field are only composed of two-point functions $\langle \Omega, \varphi(f_1,h_1)\varphi(f_2,h_2)\Omega \rangle$.

However, it can be useful to have a criterion for a field to be a free string-localized field. For the case of point-localized fields there is the well known theorem by R. Jost and B. Schroer [25]. Thus, in the first section of this chapter we want to prove a generalisation of this theorem for string-localized fields[1]. In the second section we give an overview of the construction of free string-localized fields as achieved in [21].

5.1 A Jost-Schroer Theorem for String-Localized Fields

Theorem 5.1.1. *Let $\varphi(x,e)$ be a hermitian scalar string-localized field as defined in Section 2.1 for which the vacuum Ω is cyclic and consider $W(x_1,e_1;x_2,e_2) = \langle \Omega, \varphi(x_1,e_1)\varphi(x_2,e_2)\Omega \rangle$ with the properties stated in Section 2.2 including the analyticity assumption 3.2.6. If*

$$\left(\Box_{x_1} + m^2\right) \langle \Omega, \varphi(x_1,e_1)\varphi(x_2,e_2)\Omega \rangle = 0, \tag{5.3}$$

then $\varphi(x,e)$ is a free field of mass m.

Analogous to the point-like case it therefore follows from this theorem that free string-localized fields have the property that they create only single particle states from the

[1] A version of the Jost-Schroer theorem for string-localized fields has been proved by O. Steinmann [23], considering only the case where all space-like directions e coincide and are fixed.

vacuum and that they are of the form $a^*(\varphi(x,e)\Omega) + a(\varphi(x,e)^*\Omega)$ up to unitary equivalence. Here a^* and a are the creation and annihilation operators in Fock space over the single particle space.

Note, however, that in two spacetime dimensions there are concrete examples of string-localized fields [11] which have the just mentioned properties, but are not free fields. In particular, the corresponding n-point functions are not only composed of two-point functions. Theorem 5.1.1, however, applies to $d > 2$.

We present two different ways of proving Theorem 5.1.1. The first of which shall be referred to as "proof 1" and involves Theorems 4.0.7 and 4.0.8. It is only valid in the case of $m > 0$. The alternative "proof 2" based on Theorem 4.0.9, however, includes the case of $m = 0$. "Proof 1" goes as follows.

Proof 1

Define

$$j(f,h) := \left(\Box_x + m^2\right)\varphi(f,h). \tag{5.4}$$

It follows from (5.3) that

$$\begin{aligned}&||j(f,h)\Omega||^2\\ &= \int \bar{f}(x)\bar{h}(e)\left(\Box_x + m^2\right)\left(\Box_y + m^2\right)\langle\Omega, \varphi(x,e)\varphi(y,e')\Omega\rangle f(y)h(e')d^dx d^dy d\sigma(e)d\sigma(e')\\ &= 0.\end{aligned}$$

Thus, $j(f,h)\Omega = 0$ which, by Theorem 4.0.8 implies $j(f,h) = 0$. Hence, the field φ satisfies the equation of a free field, i. e.

$$\left(\Box_x + m^2\right)\varphi(f,h) = 0. \tag{5.5}$$

It remains to show, however, that the commutator is a c-number.
Equation (5.5) formally yields $(p^2 - m^2)\widetilde{\varphi}(p,e) = 0$ in momentum space. The support of $\widetilde{\varphi}(p,e)$ is contained in $(H_m^+ \cup H_m^-) \times H^{d-1}$, where $H_m^\pm := \{p \in \mathbb{R}^4 : p\cdot p = m^2, p^0 \gtrless 0\}$. Accordingly, we have a decomposition of φ into a positive and a negative frequency part

$$\varphi(f,h) = \varphi_+(f,h) + \varphi_-(f,h). \tag{5.6}$$

This can be understood as follows. $\varphi_+(g)$ e. g. is defined for a test function $g(x,e)$ by

$$\varphi_+(g) := \widetilde{\varphi}(\widetilde{\theta}\widetilde{g}),$$

where $\widetilde{\theta}$ is an infinitely differentiable function with the property

$$\widetilde{\theta} = \begin{cases} 0 & \text{if} \quad p^2 = m^2, \quad p_0 < 0 \\ 1 & \text{if} \quad p^2 = m^2, \quad p_0 > 0. \end{cases}$$

Negative energies (= positive frequencies), however, do not exist, we therefore have

$$\varphi_+(f,h)\Omega = 0. \tag{5.7}$$

For the commutator being a c-number it suffices to proof the following

$$[\varphi(f_1,h_1),\varphi(f_2,h_2)]\Omega = \langle\Omega,[\varphi(f_1,h_1),\varphi(f_2,h_2)]\Omega\rangle\Omega. \tag{5.8}$$

Equations (5.6) and (5.7) yield

$$\begin{aligned}&[\varphi(f_1,h_1),\varphi(f_2,h_2)]\Omega\\ &= \{\varphi_+(f_1,h_1)\varphi_-(f_2,h_2) - \varphi_+(f_2,h_2)\varphi_-(f_1,h_1)\}\,\Omega + [\varphi_-(f_1,h_1),\varphi_-(f_2,h_2)]\Omega.\end{aligned} \tag{5.9}$$

Now, take a closer look at the vector

$$\varphi_+(f_1,h_1)\varphi_-(f_2,h_2)\Omega, \tag{5.10}$$

whose properties may be best investigated by considering a distribution G defined by

$$G(f_1,h_1,f_2,h_2) := \langle\Psi,\varphi_+(f_1,h_1)\varphi_-(f_2,h_2)\Omega\rangle,$$

with $\Psi \in D$. The formal Fourier transform of G, $\widetilde{G}(p_1,p_2;e_1,e_2)$, vanishes unless $p_1 \in H_m^+$ and $p_2 \in H_m^-$. Therefore the vector $p_1 + p_2$ is either space-like or zero. By the spectral condition the case of space-like $p_1 + p_2$ can be excluded. Thus, it follows that the vector in (5.10) is translation invariant and due to the uniqueness of the vacuum a multiple of Ω. We obtain

$$\begin{aligned}\varphi_+(f_1,h_1)\varphi_-(f_2,h_2)\Omega &= \langle\Omega,\varphi_+(f_1,h_1)\varphi_-(f_2,h_2)\Omega\rangle\Omega\\ &= \langle\Omega,\varphi(f_1,h_1)\varphi(f_2,h_2)\Omega\rangle\Omega = W(f_1,h_1;f_2,h_2)\Omega.\end{aligned}$$

Hence, (5.9) can be expressed by

$$\begin{aligned}&[\varphi(f_1,h_1),\varphi(f_2,h_2)]\Omega\\ &\qquad= \langle\Omega,[\varphi(f_1,h_1),\varphi(f_2,h_2)]\Omega\rangle\Omega + [\varphi_-(f_1,h_1),\varphi_-(f_2,h_2)]\Omega.\end{aligned} \tag{5.11}$$

It is obvious that this result yields $[\varphi_-(f_1,h_1),\varphi_-(f_2,h_2)]\Omega = 0$ in case of string-localization (2.7). We now want to proof that the latter is true not only for string-localization but also without any conditions on the supports of the test functions. We therefore define for any $\Psi \in D$ a distribution

$$F(f_1,f_2,h_1,h_2) := \langle\Psi,[\varphi_-(f_1,h_1),\varphi_-(f_2,h_2)]\Omega\rangle. \tag{5.12}$$

Its Fourier transform, formally denoted by $F(p_1,p_2;e_1,e_2)$ vanishes unless p_1 and p_2 lie in $V_- = -V_+$. Hence $F(x_1,x_2;e_1,e_2)$ is the boundary value of a function analytic in the tube $\mathrm{Im}(x), \mathrm{Im}(y) \in V_+$. Furthermore, it follows from the analyticity assumption 3.2.6 that analytic continuation of F to complex string-directions $(\hat{e}_1, \hat{e}_2) \in \mathscr{G}_2$ is also provided. Thus, as a consequence of the edge of the wedge theorem [25] it follows that F vanishes not only in case of sting-locality in an open set but also for all $x_1, x_2 \in \mathbb{R}^d$ and all $e_1, e_2 \in H^{d-1}$.

Therefore, equation (5.11) reduces to

$$[\varphi(f_1, h_1), \varphi(f_2, h_2)]\Omega = \langle \Omega, [\varphi(f_1, h_1), \varphi(f_2, h_2)]\Omega \rangle \Omega,$$

which is the desired result. □

We now want to present an alternative proof of Theorem 5.1.1 based on Theorem 4.0.9. In contrast to the first one, this proof includes not only the case of $m > 0$ but also $m = 0$.

Proof 2

Define

$$j(f, h) := \left(\Box_x + m^2\right) \varphi(f, h). \tag{5.13}$$

It then follows from (5.3) that

$$||j(f, h)\Omega||^2 = 0,$$

therefore

$$\left(\Box_x + m^2\right) \varphi(f, h)\Omega = 0. \tag{5.14}$$

Now $W^{\mathrm{T}}(x_1, \ldots, x_n; e_1, \ldots, e_n)$ possess due to Theorem 2.2.3 an analytic continuation into $\mathrm{Im}(z_i - z_{i+1}) \in V_+$ for $i = 1, \ldots, n-1$. Moreover, analytic continuation to complex string directions in $\mathcal{T}_n$ is also provided by Assumption 3.2.6. Then the distribution defined by

$$F_i(x_1, \ldots, x_n; e_1, \ldots, e_n) := \left(\Box_i + m^2\right) W^{\mathrm{T}}(x_1, \ldots, x_n; e_1, \ldots, e_n) \tag{5.15}$$

has the same analyticity properties.

Further, if the string $S_{x_i, e_i'}$ is space-like separated from all the strings S_{x_j, e_j} with $j > i$ for all e_i' in an open neighbourhood of e_i then string-locality yields

$$F_i(x_1, \ldots, x_n; e_1, \ldots, e_n) = \left(\Box_i + m^2\right) W^{\mathrm{T}}(x_1, e_1; \ldots; x_{i-1}, e_{i-1}; x_{i+1}, e_{i+1}; \ldots; x_n, e_n; x_i, e_i).$$

This in turn vanishes due to equation (5.14). That is, $F_i = 0$ in an open set determined by $x_i, \ldots, x_n, e_i, \ldots, e_n$. Thus, by analytic continuation F_i vanishes identically, which is a consequence of the edge of the wedge theorem [25]. By Theorem 4.0.9 this leads to the result that $W^{\mathrm{T}}(x_1, \ldots, x_n; e_1, \ldots, e_n) = 0$ for $n > 2$. Therefore, $W(x_1, \ldots, x_n; e_1, \ldots, e_n)$ is only composed of two-point functions $W(x_1, x_2; e_1, e_2)$, see Section 2.2.1. By assumption these are all free, cf. (5.3). Hence, $\varphi(x, e)$ is a free string-localized field. □

5.2 Modular Localization and the Construction of Free String-Localized Fields

In this section we summarise the basic results of [21] on the construction of free bosonic string-localized quantum fields. This construction uses the concept of modular localization introduced by Brunetti, Guido and Longo [5] and Schroer [10] which we also want to sketch here, referring to [5, 10] for details. This concept is based on the Tomita-Takesaki modular theory of operator algebras and is not associated with any position operators, which would lead to inconsistencies with relativistic covariance and causality. In particular, the main input is the Bisognano-Wichmann theorem: Given a local algebra $\mathscr{A}(W)$ on a Hilbert space $\mathscr{H}$ containing a vector Ω that is cyclic and separating for $\mathscr{A}(W)$ then according to the Bisognano-Wichmann theorem the Tomita involution $S_W : A\Omega \mapsto A^*\Omega, \forall A \in \mathscr{A}(W)$, associated with a wedge region W is determined by the representations of the boosts $\Lambda_W(\theta)$ and the reflection j_W, cf. Section 3.2. Now, modular localization inverts the argument and defines S_W via representations of the Poincaré group. This gives rise to a natural localization structure on the representation space for any positive energy representation of the Poincaré group. Via second quantisation one obtains then a local net of operator algebras on the Fock space over the representation space.

More precisely, let U be an (anti-)unitary representation of the proper Poincaré group $\mathcal{P}_+$ acting on a Hilbert space $\mathscr{H}$, that is $U(g)$ is unitary if $g \in \mathcal{P}_+^\uparrow$ and antiunitary if $g \in \mathcal{P}_+^\downarrow$, respectively. Furthermore, it shall also be assumed that the energy spectrum is nonnegative.

Then define for some wedge W

$$\Delta_W^{it} := U(\Lambda_W(-2\pi t)), \qquad J_W := U(j_W) \tag{5.16}$$
$$S_W := J_W \Delta_W^{1/2}. \tag{5.17}$$

$\Delta_W^{1/2}$ is a densely defined, closed, positive linear operator on $\mathscr{H}$, J_W is an anti-linear involution and S_W is anti-linear, closed, defined on the dense set $D(W)$ and satisfies $S_W^2 \subset id$. In particular, $D(W)$ consists of vectors ψ such that $U(\Lambda_W(t))\psi$ is analytic in the strip $0 \leq \mathrm{Im}(t) \leq \pi$. We may then define the real linear subspace of $\mathscr{H}$

$$K(W) := \{\phi \in D(W) : S_W\phi = \phi\} \tag{5.18}$$

which satisfies $K(W) \cap iK(W) = \{0\}$ and $K(W) + iK(W)$ is dense in $\mathscr{H}$. Any $\phi \in K(W)$ is said to be localized in W in the modular sense. Applying the operator J_W to K_W yields the symplectic complement K'_W of K_W which is defined by

$$K'_W := \{\psi \in \mathscr{H} : \mathrm{Im}(\psi, \phi) = 0, \forall \phi \in K_W\}. \tag{5.19}$$

Moreover, the following holds

$$K'_W = K_{W'}. \tag{5.20}$$

Second quantisation allows the definition of field operators $\Phi(\psi)$ on the Fock space over $\mathscr{H}$. Then the commutator given by $[\Phi(\psi), \Phi(\phi)] = i\,\mathrm{Im}(\psi, \phi)$ will vanish if ψ and ϕ are localized in causally disjoint wedges, cf. equations (5.19) and (5.20).

By intersecting the spaces K_W it is possible to sharpen the localization to any region $\mathcal{O}$ in Minkowski space. The corresponding space $K(\mathcal{O})$ is then defined by $K(\mathcal{O}) := \bigcap_{W \supset \mathcal{O}} K_W$, i. e. intersection over all wedges containing $\mathcal{O}$. If $\mathcal{O}$ is a space-like cone $\mathcal{C}$, that is a set of the form $\mathcal{C} := a + \bigcup_{\lambda \geq 0} \lambda D$, where the apex a is a point in Minkowski space and D is a double cone space-like separated from the origin, then $K(\mathcal{C}) + iK(\mathcal{C})$ is also dense in $\mathscr{H}$ [5]. This result, however, cannot be expected for general regions $\mathcal{O}$.

We may now turn to the construction of string-localized quantum fields, referring to [21] for details. By Theorem 5.1.1 free fields create only single particle states from the vacuum, i. e. $\varphi(f,h)\Omega \in \mathscr{H}_1$. Therefore, the construction is reduced to a single particle problem. Namely, using modular localization we may construct single particle vectors $\psi(f,h) := \varphi(f,h)\Omega$ for given representations U_1 on $\mathscr{H}_1$ and then obtain the field via second quantisation. This goes as follows.

Firstly, recall that an irreducible unitary positive-energy representation U_1 of the Poincaré group $\mathcal{P}_+^\uparrow$ in 4-dimensional Minkowski space is determined by an irreducible unitary representation V of the so-called little group G_p. It is a subgroup in $\mathcal{P}_+^\uparrow$ with the property that for a fixed vector p in the mass shell

$$H_m^+ := \{p \in \mathbb{R}^4 : p \cdot p = m^2, p^0 > 0, m \geq 0\} \tag{5.21}$$

it holds $G_p = \{\Lambda \in \mathcal{L}_+^\uparrow : \Lambda p = p\}$. G_p is therefore also called the stabiliser group of p. The case $m > 0$ leads to $G_p = SU(2)$ whereas for $m = 0$ G_p is isomorphic to the Euclidean group $E(2)$.

U_1 acts then on the representation space $\mathscr{H}_1 := L^2(H_m^+, d\mu; h)$, with $d\mu$ a Lorentz invariant measure on H_m^+ and h the representation space of V, according to

$$\left(U_1(a,\Lambda)\psi\right)(p) = e^{ip\cdot a} V(R(\Lambda,p))\psi(\Lambda^{-1}p). \tag{5.22}$$

The so-called Wigner rotation $R(\Lambda, p) \in G_{\overline{p}}$ is defined by

$$R(\Lambda, p) := L_p^{-1} \Lambda L_{\Lambda^{-1}p} \tag{5.23}$$

with L_p a Lorentz transformation which maps $\bar{p}$ to p, for almost all $p \in H_m^+$.

In the usual setting of point-localized fields one would then introduce intertwiner functions which help to overcome the difficulties that arise from the p-dependence of the Wigner-Rotation $R(\Lambda, p)$. The authors in [21], however, proceeded in a way that in contrast to the one just mentioned includes also the case of massless infinite spin representations. They introduced intertwiner functions $u(e,.)$ which also depend on the space-like directions $e \in H^{d-1}$. In particular, u satisfies the intertwiner property

$$V(R(\Lambda,p))u(\Lambda^{-1}e, \Lambda^{-1}p) = u(e,p) \tag{5.24}$$

for $(e, p) \in \mathcal{T}_+ \times H_m^+$ and $\Lambda \in \mathcal{L}_+^\uparrow$. Here $\mathcal{T}_+$ is defined as in equation (3.5).

Moreover, for almost all p, $u(e, p)$ is analytic in the tuboid $\mathcal{T}_+$. For fixed p, $u(e, p)$ is of moderate growth near the real boundary H^{d-1}, which leads to a distributional boundary value $u(h, p)$ with $\text{supp}(h) \subset H^{d-1}$.

As a result, the smeared $u(h, p)$ generate single particle vectors $\psi(f, h)$ which transform covariantly under U_1 and are modular-localized in $\mathcal{O} + \mathbb{R}_0^+\mathcal{U}$, where $\mathcal{O}$ and $\mathcal{U}$ are bounded subsets of $\mathbb{R}^d$ and H^{d-1} respectively, and $\text{supp}(f) \subset \mathcal{O}$, $\text{supp}(h) \subset \mathcal{U}$. For $f \in \mathscr{S}(\mathbb{R}^d)$ and $h \in \mathscr{D}(H^{d-1})$ these vectors are given by

$$\psi(f, h)(p) := Ef(p)u(h, p), \tag{5.25}$$

where $p \in H_m^+$ and Ef is the restriction to the mass shell of the Fourier transform of f.

If it is assumed that the intertwiners satisfy a certain self-conjugacy property then $\psi(f, h)$ is in $K_1(\mathcal{O} + \mathbb{R}_0^+\mathcal{U})$ for real valued f and h.

Now the last step towards a string-localized free quantum field is second quantisation. Hence, let $a^*(\psi)$ and $a(\psi)$, $\psi \in \mathscr{H}_1$ denote the creation and annihilation operators acting on the symmetrized Fock space over $\mathscr{H}_1$. Using the symbolic notation

$$a^*(\psi) =: \int_{H_m^+} d\mu(p)\psi(p) \circ a^*(p), \qquad a(\psi) =: \int_{H_m^+} d\mu(p)\overline{\psi(p)} \circ a(p),$$

where $\psi(p) \in h$ and the circle "$\circ$" stands for contraction over indices of a basis in h, we arrive at a hermitian field

$$\varphi(x, e) := \int_{H_m^+} d\mu(p) \left\{ e^{ip\cdot x}u(e, p) \circ a^*(p) + e^{-ip\cdot x}\overline{u(e, p)} \circ a(p) \right\}. \tag{5.26}$$

This field is in fact a string-localized field in sense of Section 2.1. Hermiticity of the field results from the assumption that the intertwiners are self-conjugate. Moreover, it is shown in [21] that every string-localized free field is of the above from and that the intertwiner function u is unique up to multiplication with a function $F(e \cdot p)$ that is meromorphic in the upper half plane.

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