universität
wien

DISSERTATION

Titel der Dissertation

"Condensation and Large Cardinals"

Verfasser

Peter Holy

angestrebter akademischer Grad

Doktor der Naturwissenschaften (Dr.rer.nat)

Wien, im Oktober 2010

Studienkennzahl lt. Studienblatt:	A 091 405
Dissertationsgebiet lt. Studienblatt:	Mathematik
Betreuer:	O.Univ.Prof. Sy-David Friedman

Abstract

We define Local Club Condensation, a principle which isolates and generalizes properties of Gödel's Condensation principle. We show that over any model of set theory we may perform a cofinality-preserving forcing to obtain a model of set theory which satisfies Local Club Condensation while at the same time preserving various very large cardinals; in particular, we show that Local Club Condensation is consistent with the existence of an ω-superstrong cardinal. We proceed similarly for Acceptability, another principle isolating and generalizing aspects of Gödel's Condensation principle. This continues the outer model program of Sy Friedman [3]. We also hint at a possible future application regarding the consistency strength of PFA of the above-described results at the end of this thesis.

Contents

0 A Guide through the thesis

The central result of this thesis is presented in section 8 as theorem 8.21. A much easier side-result is presented in section 6. The main concepts relevant to this thesis are defined in section 4. Some historical background is provided in sections 5 and 7. Sections 1 - 3 provide a small number of well-known lemmata and definitions which will be used later on; they may well be skipped by more experienced readers, who should start with section 4 and may then continue with either section 8 (Forcing Local Club Condensation) or section 6 (Forcing Acceptability) independently. Finally, section 9 presents a possible future application of the results of this thesis. Most of the material presented in this thesis is based on [3], which the less experienced reader might find helpful to consult at some points.

1 Canonical Functions

In this section we list some well-known and easy facts related to canonical functions, which will be important in section 8. Some proofs which might not be completely standard are given here for sake of completeness.

Fact 1.1 *Assume $f, f' \colon \operatorname{card}\beta \to \beta$ are both bijections from the cardinality of β, a regular uncountable cardinal, to β. Then*

$$\{\delta < \operatorname{card}\beta \colon f[\delta] = f'[\delta]\} \textit{ is club},$$

where $f[\delta]$ denotes the pointwise image of δ under f. $\square$

Definition 1.2 *Assume $f_\beta \colon \operatorname{card}\beta \to \beta$ is a bijection from the cardinality of β, a regular uncountable cardinal, to β. Let $g_\beta \colon \operatorname{card}\beta \to \operatorname{card}\beta$ be defined by $g_\beta(\delta) = \operatorname{ot} f_\beta[\delta]$, where for a set of ordinals X, $\operatorname{ot} X$ denotes the order-type of $(X, \in)$. We then say that g_β is a canonical function (for β) or also that g_β is the canonical function associated to f_β.*

Corollary 1.3 *Assume g_β and g'_β are canonical functions associated to f_β and f'_β (bijections from $\operatorname{card}\beta$ to β) respectively in the above sense. Then g_β and g'_β agree on a club, i.e. $\{\delta < \operatorname{card}\beta \colon g_\beta(\delta) = g'_\beta(\delta)\}$ contains a club.* $\square$

Lemma 1.4 *Assume g_β is a canonical function for β. Then*

$$\{g_\beta(\delta) \colon \delta < \operatorname{card}\beta\} \textit{ is unbounded in } \operatorname{card}\beta. \ \square$$

Lemma 1.5 *Assume $\mathfrak{M}$ is a structure with universe M for a language of size less than $\operatorname{card}\beta$ such that $\operatorname{Ord}(M) = \beta$. Then there is a club in $[M]^{<\operatorname{card}\beta}$ of universes N of elementary substructures $\mathfrak{N}$ of $\mathfrak{M}$ each satisfying $N \cap \operatorname{Ord} = f_\beta[\delta]$ for some $\delta < \operatorname{card}\beta$.*

Proof:

- $\{N \colon \mathfrak{N} \prec \mathfrak{M} \wedge \operatorname{card} N < \operatorname{card} \beta\}$ is club in $[M]^{<\operatorname{card} \beta}$, thus
- $\{N \cap \mathrm{Ord} \colon \mathfrak{N} \prec \mathfrak{M} \wedge \operatorname{card} N < \operatorname{card} \beta\}$ is club in $[\beta]^{<\operatorname{card} \beta}$.
- $\{f_\beta[\delta] \colon \delta < \operatorname{card} \beta\}$ is club in $[\beta]^{<\operatorname{card} \beta}$. □

Corollary 1.6 *Letting* $\mathfrak{M}$ *and* β *be as above,* $\{\delta < \operatorname{card} \beta \colon \exists \mathfrak{N} \prec \mathfrak{M} \operatorname{card} N < \operatorname{card} \beta \wedge N \cap \mathrm{Ord} = f_\beta[\delta]\}$ *is club.* □

Lemma 1.7 *There is a club of* $\delta < \operatorname{card} \beta$ *such that*

$$f_\alpha[\delta] = f_\beta[\delta] \cap \alpha \textit{ for all } \alpha \in f_\beta[\delta] \setminus \operatorname{card} \beta.$$

Proof: For every $\alpha \in [\operatorname{card} \beta, \beta)$, let $C_\alpha = \{\delta < \operatorname{card} \beta \colon f_\alpha[\delta] = f_\beta[\delta] \cap \alpha\}$. Each such C_α is club. Let $\langle i_\gamma \colon \gamma < \operatorname{card} \beta \rangle$ be any enumeration of $[\operatorname{card} \beta, \beta)$ and let C be the diagonal intersection of C_{i_γ} for $\gamma < \operatorname{card} \beta$. Then

$$C = \{\delta \colon \delta \in \bigcap_{\eta < \delta} C_{i_\eta}\} = \{\delta \colon \forall \eta < \delta \; f_{i_\eta}[\delta] = f_\beta[\delta] \cap i_\eta\}.$$

Now $\{\delta < \operatorname{card} \beta \colon \{i_\eta \colon \eta < \delta\} = f_\beta[\delta] \setminus \operatorname{card} \beta\}$ is club.
Intersecting this club with C yields the desired result. □

2 Large Cardinal Basics

Well-known notions of large cardinals which are relevant for this thesis are: superstrong, hyperstrong and n-superstrong for $n \leq \omega$, definitions of which can be found in [3]. For the definition of a supercompact cardinal, see [6]. Other, less common or previously unused large cardinal notions will be defined when they are first utilized.

3 Forcing Basics

Our notation concerning forcing is pretty standard (see e.g. [6]). We will only give our definition of η^+-strategic closure, since it is a central concept in this thesis:

Definition 3.1 *If* P *is a notion of forcing and* η *is a cardinal, we say that* P *is* η^+*-strategically closed iff Player I has a winning strategy in the following two player game of perfect information: Player I and Player II alternately make moves where in each move, each player plays a condition of* P*. Player I has to start and play* $\mathbf{1}_P$ *in the first move. Player II is allowed to play any condition stronger than the condition just played by Player I in each of his moves. Player I has to play a condition stronger than all previously played* *conditions in each move, Player I has to make a move at every limit step of the game. We say that Player I wins if he can find conditions to play in any such game of length η^+ (arriving at η^+, the game ends, no condition has to be played at stage η^+).*

4 Fragments of Condensation

In this section we will give definitions of Local Club Condensation and Acceptability; both of these definitions apply to models $\mathbf{M}$ of set theory with a hierarchy of levels of the form $\langle M_\alpha \colon \alpha \in \mathrm{Ord}\rangle$ with the properties that $\mathbf{M} = \bigcup_{\alpha\in\mathrm{Ord}} M_\alpha$, each M_α is transitive, $\mathrm{Ord}(M_\alpha) = \alpha$, if $\alpha < \beta$ then $M_\alpha \in M_\beta$ and if γ is a limit ordinal, $M_\gamma = \bigcup_{\alpha<\gamma} M_\alpha$. We will also let M_α denote the structure $(M_\alpha, \in, \langle M_\beta \colon \beta < \alpha\rangle)$, where context will usually clarify the intended meaning.

Full Condensation is the statement that if $X \prec (M_\alpha, \in, \langle M_\beta \colon \beta < \alpha\rangle)$, then X is isomorphic to some $(M_{\bar\alpha}, \in, \langle M_\beta \colon \beta < \bar\alpha\rangle)$.

Local Club Condensation is the statement that if α has uncountable cardinality κ and $\mathcal{A}_\alpha = (M_\alpha, \in, \langle M_\beta \colon \beta < \alpha\rangle, \ldots)$ is a structure for a countable language, then there exists a continuous chain $\langle \mathcal{B}_\gamma \colon \omega \le \gamma < \kappa\rangle$ of substructures of $\mathcal{A}_\alpha$ whose domains have union M_α, where each $\mathcal{B}_\gamma = (B_\gamma, \in, \langle M_\beta \colon \beta \in B_\gamma\rangle, \ldots)$ is s.t. B_γ has cardinality $|\gamma|$, contains γ as a subset and each $(B_\gamma, \in, \langle M_\beta \colon \beta \in B_\gamma\rangle)$ is isomorphic to some $(M_{\bar\alpha}, \in, \langle M_\beta \colon \beta < \bar\alpha\rangle)$.

Whenever we want to work with the above-defined notions, we will be in the situation that $\mathbf{M} = (\mathbf{L}[A], A)$ for some $A \subseteq \mathrm{Ord}$ and $\langle M_\alpha \colon \alpha \in \mathrm{Ord}\rangle = \langle L_\alpha[A] \colon \alpha \in \mathrm{Ord}\rangle$. We say that $\mathbf{M}$ is of the form $\mathbf{L}[A]$ in that case.

Acceptability is the statement that, assuming $\mathbf{M}$ is of the form $\mathbf{L}[A]$, for any ordinals $\gamma \ge \delta$, if there is a subset of δ in $M_{\gamma+1} \setminus M_\gamma$, then $H^{M_{\gamma+1}}(\delta) = M_{\gamma+1}$ (the Skolem hull also uses the predicate A).

Note: The above property might also be referred to as "Weak Acceptability" as in the literature, "Acceptability" is often used for the following, closely related notion (see also lemma 4.2): If there is a subset of δ in $M_{\gamma+1} \setminus M_\gamma$, then there is a surjection of δ onto M_γ in $M_{\gamma+1}$. We will stick to the term "Acceptability" for our above-defined notion though.

All of the above properties hold in $\mathbf{L}$, using the hierarchy of levels $\langle L_\alpha \colon \alpha \in \mathrm{Ord}\rangle$: Local Club Condensation easily follows from Full Condensation, which holds in $\mathbf{L}$ (see [1]). Acceptability also follows from Full Condensation:

Lemma 4.1
Full Condensation implies Acceptability.

Proof: Assume $\delta \leq \gamma$ and there is a subset x of δ in $M_{\gamma+1} \setminus M_\gamma$. Let $N := H^{M_{\gamma+1}}(\delta)$. By elementarity of N there is an x with the above property in N. Let $\bar{N} = \text{coll}(N)$; as $\delta \subseteq N$, $x \in \bar{N}$. But by Full Condensation, $\bar{N}$ is a level of $\mathbf{M}$, hence $\bar{N} = M_{\gamma+1}$ by minimality of γ. But $H^{\bar{N}}(\delta) = \bar{N}$. □

The following lemma will be used in section 6:

Lemma 4.2 *Assume* $\mathbf{M}$ *is of the form* $\mathbf{L}[A]$*. Assume there is a surjection from* β *onto* γ *in* $M_{\gamma+1}$*. If* $\beta \in H^{M_{\gamma+1}}(\beta)$*, then* $H^{M_{\gamma+1}}(\beta) = \mathbf{M}_{\gamma+1}$*.*

Proof: Let $H := H^{M_{\gamma+1}}(\beta)$. $\gamma \in H$ as γ is the largest ordinal of $M_{\gamma+1}$. Let f be a surjection from β onto γ in H, which exists by elementarity. Since $\beta \subseteq H$, it follows that $\gamma \subseteq H$. Thus $H = M_{\gamma+1}$. □

We conclude with the following, which will be tacitly used in section 8:

Lemma 4.3 *Local Club Condensation is equivalent to the following, seemingly weaker statement: If* α *has uncountable cardinality* κ*, then the structure* $\mathcal{A}_\alpha = (M_\alpha, \in, \langle M_\beta \colon \beta < \alpha \rangle, F)$ *has a continuous chain* $\langle \mathcal{B}_\gamma \colon \gamma \in C \rangle$ *of substructures* $\mathcal{B}_\gamma = (B_\gamma, \in, \langle M_\beta \colon \beta \in B_\gamma \rangle, F)$ *of* $\mathcal{A}_\alpha$ *with* $\bigcup_{\gamma \in C} B_\gamma = M_\alpha$*,* $C \subseteq \kappa$ *is club,* C *consists only of cardinals if* κ *is a limit cardinal, each* B_γ *has cardinality* $\text{card}\,\gamma$*, contains* γ *as a subset and each* $(B_\gamma, \in, \langle M_\beta \colon \beta \in B_\gamma \rangle)$ *is isomorphic to some* $(M_{\bar{\alpha}}, \in, \langle M_\beta \colon \beta < \bar{\alpha} \rangle)$*, where* F *denotes the function* $(f, x) \mapsto f(x)$ *whenever* $f \in M_\alpha$ *is a function with* $x \in \text{dom}(f)$*.*

Proof: Suppose $\langle M_\alpha \colon \alpha \in \text{Ord} \rangle$ witnesses the above-described, seemingly weaker property. First note that for any infinite cardinal κ, each subset x of κ belongs to M_α for some α less than κ^+: If not, let $\lambda > \kappa$ be the least cardinality of some α such that x belongs to M_α. By the assumed property, x also belongs to some B_γ of cardinality less than λ which contains κ as a subset s.t. $(B_\gamma, \in)$ is isomorphic to some $(M_{\bar{\alpha}}, \in)$. As x belongs to $M_{\bar{\alpha}}$ and $\bar{\alpha} < \lambda$, this contradicts leastness of λ.

Now we prove that Local Club Condensation holds by induction on κ: Assume α has uncountable cardinality κ and $\mathcal{E}_\alpha = (M_\alpha, \in, \langle M_\beta \colon \beta < \alpha \rangle, \ldots)$ is a structure for a countable language. By the previous paragraph, we may choose $\alpha' > \alpha$ of cardinality κ so that $\mathcal{E}_\alpha$ is an element of $M_{\alpha'}$. Applying the assumed property to $\mathcal{A}_{\alpha'} = (M_{\alpha'}, \in, \langle M_\beta \colon \beta < \alpha' \rangle, F)$, we obtain a continuous chain $\langle \mathcal{B}_\gamma \colon \gamma \in C \rangle$ of substructures $\mathcal{B}_\gamma = (B_\gamma, \in, \langle M_\beta \colon \beta \in B_\gamma \rangle, F)$ of $\mathcal{A}_{\alpha'}$ with the properties described in the statement of the lemma. We may assume that $\mathcal{E}_\alpha$ is an element of B_{γ_0}, where γ_0 is the least element of C. Then we obtain a continuous chain $\langle \mathcal{D}_\gamma \colon \gamma \in C \rangle$ of substructures

$\mathcal{D}_\gamma = (D_\gamma, \in, \langle M_\beta \colon \beta \in D_\gamma\rangle, \ldots)$ of $\mathcal{E}_\alpha$ such that $\bigcup_{\gamma\in C} D_\gamma = M_\alpha$ by setting $\mathcal{D}_\gamma = \mathcal{E}_\alpha \upharpoonright B_\gamma$, using the fact that F is part of the structure $\mathcal{A}_{\alpha'}$. Moreover each $(D_\gamma, \in, \langle M_\beta \colon \beta \in D_\gamma\rangle)$ is isomorphic to some $(M_{\bar\alpha}, \in, \langle M_\beta \colon \beta < \bar\alpha\rangle)$.

Now if $\kappa = \delta^+$, δ an uncountable cardinal, then by reindexing we can assume that $C = [\delta, \kappa)$, choose $\bar\alpha$ so that $(D_\delta, \in, \langle M_\beta \colon \beta \in D_\delta\rangle)$ is isomorphic to $(M_{\bar\alpha}, \in, \langle M_\beta \colon \beta < \bar\alpha\rangle)$ and can define $\mathcal{D}_\gamma$ for $\gamma < \delta$ by applying Local Club Condensation inductively to $\bar\alpha$. If κ is a limit cardinal, then we let $\langle \gamma_i \colon i < \operatorname{ot} C\rangle$ be the increasing enumeration of C and fill in $\langle \mathcal{D}_\gamma \colon \gamma \in C\rangle$ to $\langle \mathcal{D}_\gamma \colon \omega \le \gamma < \kappa\rangle$ by applying Local Club Condensation inductively to the $\bar\alpha_{i+1}$, where $\bar\alpha_{i+1}$ is such that $(D_{\gamma_{i+1}}, \in, \langle M_\beta \colon \beta \in D_{\gamma_{i+1}}\rangle)$ is isomorphic to $(M_{\bar\alpha_{i+1}}, \in, \langle M_\beta \colon \beta < \bar\alpha_{i+1}\rangle)$. $\Box$

5 History, Motivation

L, the constructible universe of sets as discovered by Gödel around 1937 (see for example [6] for more historical background), has a lot of very nice set theoretical properties, such as the GCH, a definable well-ordering, Jensen's global square principle $\Box$, the diamond principle $\diamondsuit$ and Gap 1 morasses. But **L** does not allow for very large cardinals, as $0^\sharp$ cannot exist within **L**. Since both **L**-like principles and large cardinals are often useful tools in set theoretic proofs, it seems very interesting to produce models of set theory which possess both: **L**-like properties and very large cardinals. There are two possible approaches to this goal: The first is the Inner Model Program, where one starts with a model of set theory containing large cardinals and tries to find an inner model with those large cardinals which also has **L**-like properties. At the moment, techniques for producing inner models allow for **L**-like inner models containing Woodin limits of Woodin cardinals, but there seems to be great difficulty to extend these methods to even larger large cardinals. The second approach is the Outer Model Program. Here the basic strategy is to start with a model of set theory containing very large cardinals and to then force over such a model to obtain **L**-like properties in the generic extension while at the same time preserving various very large cardinals. In [3], this is done for all the above-mentioned **L**-like properties.

The main theorems of this thesis are both contributions to the outer model program: we will show how to force Local Club Condensation and how to force Acceptability while at the same time preserving various very large cardinals. The forcings to achieve those results will both be shown to preserve GCH and to be cofinality-preserving, which implies that if we start with a ground model containing very large cardinals, satisfying GCH and some other **L**-like properties, which are preserved under cofinality-preserving forcing, as is $\Box$, we may obtain models of set theory with very large cardinals satisfying Local Club Condensation or Acceptability together with various other **L**-like principles.

It would be very interesting to produce models of set theory with very large cardinals which satisfy both Local Club Condensation and Acceptability, which seems to be surprisingly hard and which we unfortunately did not yet succeed to do. We will say more about this in section 9.

6 Forcing Acceptability

In this section we will show how to extend (by forcing) a ground model $\mathbf{V}$ satisfying GCH to a model of Acceptability of the form $\mathbf{L}[A]$ while preserving various large cardinals.

Notation: We use the appending operator $\frown$ repeatedly in this chapter in slightly varying contexts. For example, if $G \subseteq \kappa$ and $p\colon [\kappa, |p|) \to 2$, then $G^\frown p$ denotes the following subset of $|p|$: If $\alpha < \kappa$ then $\alpha \in G^\frown p$ iff $\alpha \in G$, if $\alpha \geq \kappa$ then $\alpha \in G^\frown p$ iff $p(\alpha) = 1$.

On the other hand, if $p\colon [\kappa, |p|) \to 2$ and $c \subseteq \kappa$, $p^\frown c$ denotes the following function with domain $[\kappa, |p| + \kappa)$: If $\alpha \leq |p|$ then $p^\frown c(\alpha) = p(\alpha)$, if $\alpha \geq |p|$ then $p^\frown c(\alpha) = 1$ iff $\alpha = |p| + \beta$ and $\beta \in c$.

Definition 6.1 *We refer to p as a κ^+ Cohen condition iff p is a function from $[\kappa, |p|)$ to 2, where $|p|$ is an ordinal of size κ. If $G \subseteq \kappa$ in some extension of $\mathbf{V}$, then a κ^+ Cohen condition $p \in \mathbf{V}[G]$ is acceptable with respect to $G \subseteq \kappa$ iff for every $\eta \in [\kappa, |p|)$, for every $\delta \in (\eta, |p|]$, if there is a new subset of η in $L_{\delta+1}[G^\frown p]$, then $H^{L_{\delta+1}[G^\frown p]}(\eta) = L_{\delta+1}[G^\frown p]$. The Skolem Hull inside $L_{\delta+1}[G^\frown p]$ also makes use of the predicate $G^\frown p$ here. We say that p is correct with respect to G iff $p = \emptyset$ or $L_{|p|}[G^\frown p] \models \kappa$ is the largest cardinal.*

Definition 6.2 *For an infinite cardinal κ and $G \subseteq \kappa$, we define the forcing*

$$\mathrm{AAdd}(\kappa^+, G) :=$$

$$\{p\colon p \textit{ is an acceptable, correct } \kappa^+\textit{-Cohen condition w.r.t. } G\},$$

where conditions are ordered by inclusion.

Lemma 6.3 $\mathrm{AAdd}(\kappa^+, G)$ *is κ^+-closed.*

Proof: Assume $\langle p_i\colon i < \alpha\rangle$ is a strictly descending sequence of conditions in $\mathrm{AAdd}(\kappa^+, G)$ of limit length $\alpha < \kappa^+$. Let $q := \bigcup_{i<\alpha} p_\alpha$. We have to show that q is acceptable and correct w.r.t. G. Correctness of q w.r.t. G follows immediately from the correctness of the p_i w.r.t. G. Now assume there is a new subset of κ in $L_{|q|+1}[G^\frown q]$ and let x be the $<_{L_{|q|+1}[G^\frown q]}$-least such.

Let $H = H^{L_{|q|+1}[G^\frown q]}(\kappa)$. If $\kappa \leq \eta < |q|$, $\eta \in H$, then $L_{|q|}[G^\frown q] \models \eta \cong \kappa$, hence the same is true in H, but since $\kappa \subseteq H$, it follows that $\eta \subseteq H$, i.e. H is transitive below $|q|$. But this means that $\text{coll}(H) = L_{\gamma+1}[G^\frown q]$ for some limit ordinal $\gamma \leq |q|$. But as $x \subseteq \kappa$, it follows that $x \in L_{\gamma+1}[G^\frown q]$ and hence $\gamma = |q|$. Finally if $\kappa < \eta < |q|$ and there is a new subset of η in $L_{|q|+1}[G^\frown q]$, then since there is a bijection from η to κ in $L_{|q|}[G^\frown q]$, it follows that there is a new subset of κ in $L_{|q|+1}[G^\frown q]$.□

Lemma 6.4 *If $L_\kappa[G] = H_\kappa$, then for every $\alpha < \kappa^+$, any condition in* $\text{AAdd}(\kappa^+, G)$ *can be extended to a condition of length at least α.*

Proof: Assume $p \in \text{AAdd}(\kappa^+, G)$ and $\alpha < \kappa^+$ are given. We may assume without loss of generality that α is a limit ordinal and that $\alpha \geq |p| + \kappa$.

Claim 6.5 *We can find a κ^+-Cohen condition q extending p of length α such that for every $\delta \in [|p|+\kappa, |q|]$, δ is collapsed to κ in $L_{\delta+1}[G^\frown q \cap (|p|+\kappa)]$ and hence in $L_{\delta+1}[G^\frown q]$.*

Proof: Let f be a bijection from κ to α. Let $\bar{c} \subseteq \kappa^2$ be such that $(x_1, x_2) \in \bar{c}$ iff $f(x_1) \in f(x_2)$. Let s_κ be the **L**-least bijection from κ to κ^2 and define $c \subseteq \kappa$ by setting $\gamma \in c \iff s_\kappa(\gamma) \in \bar{c}$. Note that $s_\kappa \in L_{\kappa+1}$. Let $q = p^\frown c^\frown \vec{0}$ such that $|q| = \alpha$, where $\vec{0}$ denotes a string of zeroes of appropriate length, which is appended to $p^\frown c$ in the obvious way. Then $c, \bar{c} \in L_{|p|+\kappa+1}[G^\frown q]$. Fix $\delta \in [|p| + \kappa, |q|]$. First assume δ is a limit ordinal. Let $h\colon \delta \to \kappa$ be s.t.

$$\forall\theta < \delta\, \forall\Omega < \theta\ (h(\Omega), h(\theta)) \in \bar{c} \ \wedge\ \forall\nu \in \kappa\ (\exists\theta < \delta\, (\nu, h(\theta)) \in \bar{c} \ \rightarrow\ \nu \in \text{range}\, h).$$

If h satisfies the above definition, $h = f^{-1} \restriction \delta$, hence h is an injection from δ to κ. $h \subseteq L_\delta$ and is definable over $L_\delta[G^\frown q \cap (|p|+\kappa)]$ by the above, absolute for $L_\delta[G^\frown q \cap (|p| + \kappa)]$, formula.

If δ is a successor ordinal, let γ be the largest limit ordinal below δ and use the injection from γ to κ (obtained as above) to build an injection from δ to κ and code it by techniques described in [1] (a paper on how to handle successor levels of the constructible hierarchy by using a reasonable definition of ordered pairs which does not increase rank) to obtain (a code for) that injection within $L_{\delta+1}[G^\frown q \cap (|p| + \kappa)]$. □

Claim 6.6 *Assume q is a κ^+-Cohen condition extending p with $|q| \geq |p|+\kappa$. For every $\delta \in (|p|, |p| + \kappa)$, if there is a new subset of κ in $L_{\delta+1}[G^\frown q]$, then every element of $L_{\delta+1}[G^\frown q]$ is definable in $L_{\delta+1}[G^\frown q]$ from parameters in κ.*

Proof: Assume there is a subset of κ in $L_{\delta+1}[G^\frown q] \setminus L_\delta[G^\frown q]$ and let x be the $<_{L_{\delta+1}[G^\frown q]}$-least such. Let $H := H^{L_{\delta+1}[G^\frown q]}(\kappa)$. As $L_{|p|}[G^\frown q] \models \kappa$

is the largest cardinal, it follows that H is transitive below $|p|$. Now as $L_\kappa[G] = H_\kappa$ and $q \cap [|p|, \delta) \in H_\kappa$, it follows that $L_{\delta+1}[G^\frown q] = L_{\delta+1}[G^\frown p]$, hence the transitive collapse of H is of the form $L_{\gamma+1}[G^\frown q]$ for some $\gamma \leq \delta$. As $\kappa \subset H$, it follows that $x \in \text{coll}(H)$, hence $\gamma = \delta$ by minimality of δ. As every element of $\text{coll}(H)$ is definable from parameters in κ it follows that the collapsing map of H is the identity map, i.e. $H = L_{\delta+1}[G^\frown q]$. $\square$

The above claims together imply that we can find an extension q of p, as in the proof of claim 6.5, in $\text{AAdd}(\kappa^+, G)$ of length at least α: Assume $\delta < \gamma \leq \alpha$ and there is a new subset of δ in $L_{\gamma+1}[G^\frown p]$. We have to show that $H^{L_{\gamma+1}}(\delta) = L_{\gamma+1}$. If $\gamma \leq |p|$, this follows as p is acceptable. We may thus assume that $\gamma > |p|$. As $L_\kappa[G] = H_\kappa$, we may also assume $\delta \geq \kappa$. If $\gamma \geq |p| + \kappa$, our desired property follows immediately from claim 6.6. We may thus assume that $\gamma < |p| + \kappa$.

Case 1 - $\delta = \kappa$: Follows directly from claim 6.5.

Case 2 - $\delta \in (\kappa, |p|)$: Follows by correctness of p.

Case 3 - $\delta = |p|$: Note that γ is collapsed to $|p|$ in $L_{\gamma+1}$.

Case 4 - $\delta \in (|p|, |p| + \kappa)$: Note that δ is collapsed to $|p|$ in $L_{\delta+1}$.

$\square_{\text{Lemma 6.4}}$

Theorem 6.7 *Assume* GCH. *There is a cofinality-preserving forcing extension of the universe of the form* $\mathbf{L}[A]$ *satisfying Acceptability and* GCH. *Moreover we may preserve a given large cardinal of any of the following kinds: superstrong, hyperstrong, n-superstrong for any $n \leq \omega$.*

Proof: Let $\mathbf{P}$ be the class-sized reverse Easton iteration with Easton support of $\text{AAdd}(\kappa^+, G_\kappa)$ over all infinite cardinals κ, where for each κ, G_κ denotes the generic predicate obtained by P_κ, the iteration below κ. Let A be the generic subset of Ord obtained by the full iteration P. By a density argument using the proof of Lemma 6.4, $\mathbf{V}^P = L[A]$. The following is a standard claim, using that we work with a reverse Easton iteration where at stage κ, we apply a κ^+-closed forcing of size κ^+ (see for example [2]):

Claim 6.8 *If κ is regular, then P_κ has a dense subset of size κ. Moreover, each P_κ preserves cofinalities and hence* $\mathbf{P}$ *preserves cofinalities. After forcing with P_κ, $\forall \lambda \leq \kappa\ L_\lambda[G_\kappa] = H_\lambda$. Therefore after forcing with* $\mathbf{P}$, GCH *holds.* $\square$

Claim 6.9 $(\mathbf{L}[A], A)$ *satisfies Acceptability.*

Proof: Assume $\gamma > \kappa$ are ordinals and there is a new subset of κ in $\mathbf{L}_{\gamma+1}[A]$. Let γ^- denote the largest cardinal below or equal to γ. Since $\mathbf{L}_\lambda[A] = H_\lambda{}^{\mathbf{L}[A]}$ for all cardinals λ, it follows that $\kappa \geq \gamma^-$. Now Acceptability follows directly from the fact that A is built up from acceptable Cohen conditions. $\square$

Large Cardinal Preservation: By exactly the same arguments as given for the GCH forcing in [3], we may force Acceptability preserving a given cardinal of one of the following kinds: superstrong, hyperstrong, n-superstrong for any $n \leq \omega$. $\square_{\text{Theorem 6.7}}$

Note: Moreover all kinds of large cardinals which may be preserved by the forcing to achieve Local Club Condensation (see below) may also be preserved (by much easier arguments) by the Acceptability forcing. It is also possible to preserve various of those large cardinals simultaneously while forcing Acceptability as above, as well as many other kinds of large cardinals.

7 A small history of fragments of Condensation

In [9], Hugh Woodin defines the principle of Strong Condensation, which may be reformulated in the context of models with a hierarchy of levels as in section 4 as follows:

Definition 7.1 *Strong Condensation is the principle that for each α, there is a structure $(M_\alpha, \in, \langle M_\beta \colon \beta < \alpha \rangle, \ldots)$ for a countable language s.t. for every substructure $(B, \in, \langle M_\beta \colon \beta \in B \rangle, \ldots)$, $(B, \in, \langle M_\beta \colon \beta \in B \rangle)$ is isomorphic to some $(M_{\bar\alpha}, \in, \langle M_\beta \colon \beta < \bar\alpha \rangle)$.*

In [4], it is shown that Strong Condensation is inconsistent with the existence of an ω_1-Erdös cardinal. It is unknown whether it is possible to force Strong Condensation up to ω_3 with a small forcing. Also, Sy Friedman defines the principle of Stationary Condensation in [4] (again using a model $\mathbf{M}$ with a hierarchy of levels as in section 4):

Definition 7.2 *Stationary Condensation is the principle that for each α and infinite cardinal $\kappa \leq \alpha$, any structure $(M_\alpha, \in, \langle M_\beta \colon \beta < \alpha \rangle, \ldots)$ for a countable language has a substructure $(B, \in, \langle M_\beta \colon \beta \in B \rangle, \ldots)$ with B of size κ, containing κ as a subset such that $(B, \in, \langle M_\beta \colon \beta \in B \rangle)$ is isomorphic to some $(M_{\bar\alpha}, \in, \langle M_\beta \colon \beta < \bar\alpha \rangle)$.*

Sy Friedman shows ([4]) that starting with a model of GCH, the reverse Easton iteration of adding a Cohen subset to every successor cardinal κ with Easton support produces a model of Stationary Condensation. This forcing is easily seen to preserve many large cardinals, as in the case of the Acceptability forcing above.

The principle of Local Club Condensation lies between those above-mentioned principles in the sense that it is implied by Strong Condensation and that it implies Stationary Condensation, both easily seen. We will show that Local Club Condensation is, in contrast to Strong Condensation, consistent with the existence of very large cardinals in the following.

8 Forcing Local Club Condensation

In this section we will show how to extend (by forcing) a given model of set theory to a model of Local Club Condensation while preserving large cardinals. This is the central result of the thesis. We assume that the universe $\mathbf{V}$ we start with satisfies GCH, that R is a predicate well-ordering $\mathbf{V}$ and work in the model $(\mathbf{V}, R)$. Note that the definition of the forcing iteration given below depends on the predicate R and we will see in the proof of theorem 8.21 that a careful choice of R will be important for large cardinal preservation.

Definition of basic objects:
For each ordinal α, fix f_α as the R-least bijection from the cardinality of α (denoted as $\operatorname{card}\alpha$) to α. Let S denote the forcing poset consisting of the conditions $\{\mathbf{1}, 0, 1\}$ where $0 \leq_S \mathbf{1}$, $1 \leq_S \mathbf{1}$, $0 \perp_S 1$. An S-generic filter simply decides for either 0 or 1. For two compatible conditions s_0 and s_1 in S, let $s_0 \cup s_1$ denote the stronger of both. Whenever $\operatorname{card}\alpha$ is regular and $g \subseteq (\alpha + 1)$, let $C_\alpha(g)$ denote the following forcing poset:

If $\operatorname{card}\alpha$ is a successor cardinal, $\operatorname{card}\alpha = \theta^+$, q^{**} is a condition in $C_\alpha(g)$ iff

- q^{**} is a closed, bounded subset of $[\theta, \theta^+)$ and
- $\forall \eta \in q^{**}\ g(\operatorname{ot} f_\alpha[\eta]) = g(\alpha)$.

If $\operatorname{card}\alpha$ is inaccessible, q^{**} is a condition in $C_\alpha(g)$ iff

- q^{**} is a closed, bounded set of cardinals below $\operatorname{card}\alpha$ and
- $\forall \eta \in q^{**}\ g(\operatorname{ot} f_\alpha[\eta]) = g(\alpha)$.

Conditions in $C_\alpha(g)$ are ordered by end-extension (in both cases).

We identify sets with their characteristic functions and vice versa in the above (i.e. $g(\beta) = 1 \leftrightarrow \beta \in g$); we will continue to do so in the following.

Definition of the Forcing: We will force with P, a reverse Easton-like iteration of $Q(\alpha)$, $\alpha \in \mathrm{Ord}$. If $\alpha < \omega$, $Q(\alpha)$ denotes the trivial forcing. If $\operatorname{card}\alpha = \omega$ or $\operatorname{card}\alpha$ is singular, $Q(\alpha) = Q(\alpha)(0) = S$. If $\operatorname{card}\alpha$ is regular, $Q(\alpha) = Q(\alpha)(0) * Q(\alpha)(1)$ with $Q(\alpha)(0) = S$ and $Q(\alpha)(1) = C_\alpha(G_{\alpha+1})$, where $G_{\alpha+1}$ denotes the generic predicate obtained from the generic for $P^{\oplus}_{\alpha}$ (where $P^{\oplus}_{\alpha} = P_\alpha * Q(\alpha)(0)$, with P_α denoting the iteration P below α) as follows: $G_{\alpha+1} \restriction \omega = 0$. For any ordinal $\beta \in [\omega, \alpha]$, $G_{\alpha+1}(\beta)$ is either 0 or 1, depending on whether the $P^{\oplus}_{\alpha}$-generic decided for either 0 or 1 at $Q(\beta)(0)$. To complete the definition of P, we need to specify the supports used; before doing so, we introduce some further notation:

If $\operatorname{card} \alpha$ is regular, we write p_α instead of $p(\alpha)(0)$ and we write p^{**}_{α} instead of $p(\alpha)(1)$. If $\operatorname{card} \alpha$ is singular, we write p_α instead of $p(\alpha)$ and say that $p^{**}_{\alpha} = \emptyset$ for notational simplicity. For $\alpha < \beta$, $p[\alpha, \beta)$ denotes $p \restriction [\alpha, \beta)$ and $P[\alpha, \beta)$ denotes the iteration P restricted to the interval $[\alpha, \beta)$. Whenever we use such notation, we will tacitly assume that α is a cardinal (which will not necessarily be the case for β) and whenever we talk about properties of $P[\alpha, \beta)$, we will tacitly assume that we are in some generic extension after forcing with P_α (with generic predicate G_α). We will later show that forcing with P_α preserves cardinals, cofinalities and the GCH.

If $\operatorname{card} \beta$ is regular, we will write $p[\alpha, \beta)^{\oplus}$ to denote $p[\alpha, \beta)^\frown p(\beta)(0)$. If $p \in P[\alpha, \beta)$, we also write $p \restriction \gamma$ for $p[\alpha, \gamma)$ and $p \restriction \gamma^{\oplus}$ for $p[\alpha, \gamma)^{\oplus}$. We use respective notation for $p \in P_\alpha$. $P[\alpha, \beta)^{\oplus}$ denotes $P[\alpha, \beta) * Q(\beta)(0)$.

Let $\check{\mathbf{1}}$ denote the standard name for the weakest condition $\mathbf{1}$ of a forcing. For a condition $p \in P$ (or $P[\alpha, \beta)$), we call $\{\gamma \colon p_\gamma \neq \check{\mathbf{1}}\}$ the string support of p and denote it by S-supp(p), we call $\{\gamma \colon p^{**}_{\gamma} \neq \check{\mathbf{1}}\}$ the club support of p and denote it by C-supp(p).

Now $p \in P$ iff:

1. S-supp(p) is bounded below every regular α,
2. C-supp$(p) \subseteq$ S-supp(p) and
3. $\operatorname{card}(\text{C-supp}(p) \cap [\alpha, \alpha^+)) < \alpha$ for every regular cardinal α.

This completes the definition of P. Note that by 2, S-supp(p) = supp(p), the support of p. We will usually assume our conditions to satisfy the following properties (possible as a dense subset of conditions does):

A1. $\forall \gamma \; \mathbf{1}_{P_\gamma} \Vdash p_\gamma \in S$.

A2. $\forall \gamma \; \mathbf{1}_{P_\gamma{}^{\oplus}} \Vdash p^{**}_{\gamma} \in C_\gamma(G_{\gamma+1})$.

A3. $\forall \gamma \; \big((p_\gamma = \check{\mathbf{1}}) \vee (\mathbf{1}_{P_\gamma} \Vdash p_\gamma \neq \mathbf{1})\big)$.

We will at some points have to temporarily cease from assumption A2 above. We will explicitly mention whenever we do so.

Fact 8.1 *If $p \parallel q$ in P, then they have a greatest lower bound $p \sqcap q$ in P. Moreover the same holds in $P[\alpha, \beta)$ for every pair of ordinals $\alpha < \beta$.*

Proof: Each of the iterands of P has canonical greatest lower bounds for any two of its conditions (their componentwise union) and this property is preserved by the iteration. □

Claim 8.2 (String Extendibility) *Work in any P_α-generic extension. Assume $\beta > \alpha$. Assume f is a function with domain $d \subseteq [\alpha, \beta)$ such that for every $\gamma \in d$ $f(\gamma)$ is a $P[\alpha, \gamma)$-name which is forced by the trivial condition to equal either 0 or 1. Assume d is bounded below every regular cardinal. Then any given $p \in P[\alpha, \beta)$ with* S-supp$(p) \cap d = \emptyset$ *can be extended to $q \leq p$ such that $\Vdash_{P[\alpha,\gamma)} q_\gamma = f(\gamma)$ whenever $\gamma \in d$.* □

Definition 8.3 (strategically closed part of a condition) *Given a cardinal $\eta \in [\alpha, \beta)$ and $p \in P[\alpha, \beta)$, we define $u_\eta(p) \in P[\alpha, \beta)$ as follows:*

- $(u_\eta(p))_\gamma = \begin{cases} \mathbf{1} & \text{if } \alpha \leq \gamma < \eta \\ p_\gamma & \text{otherwise} \end{cases}$

- $(u_\eta(p))^{**}_\gamma = \begin{cases} \mathbf{1} & \text{if } \alpha \leq \gamma < \eta^+ \\ p^{**}_\gamma & \text{otherwise} \end{cases}$

and call $u_\eta(p)$ the η^+-strategically closed part of p. We let $u_\eta(P[\alpha, \beta)) :=$ $\{u_\eta(p) \colon p \in P[\alpha, \beta)\}$ and call it the η^+-strategically closed part of $P[\alpha, \beta)$.

Note:

- The fact that $u_\eta(p) \in P[\alpha, \beta)$ heavily uses assumptions A1 and A2.
- If $\eta = \omega$ or η is a singular cardinal, then $u_\eta(P[\eta, \beta)) = P[\eta, \beta)$.
- We may think of $u_\eta(p)$ as the condition extracting from p it's string of bits in the interval $[\eta, \eta^+)$ and everything at and above η^+.

Definition 8.4 (small part of a condition)
If $\eta \in [\alpha, \beta)$ is a cardinal and $p \in P[\alpha, \beta)$, we define $l_\eta(p)$ as follows:

- $(l_\eta(p))_\gamma = \begin{cases} \mathbf{1} & \text{if } \beta > \gamma \geq \eta \\ p_\gamma & \text{otherwise} \end{cases}$

- $(l_\eta(p))^{**}_\gamma = \begin{cases} \mathbf{1} & \text{if } \beta > \gamma \geq \eta^+ \\ p^{**}_\gamma & \text{otherwise} \end{cases}$

where γ ranges over the interval $[\alpha, \beta)$ and call $l_\eta(p)$ the η-sized part of p. Note that $l_\eta(p)$ is in general not a condition in $P[\alpha, \beta)$. Note also that $l_\eta(p)$ complements $u_\eta(p)$ in the sense that it carries exactly all information about p not contained in $u_\eta(p)$.

Definition 8.5 (Strategic Belowness)
For $p, q \in P[\alpha, \beta)$ and a cardinal $\eta \in [\alpha, \beta)$, we say that q is η^+-strategically below p with respect to $\langle C_\theta : \theta \in [\eta^+, \beta) \text{ regular}\rangle$ iff:

(1a) $u_\eta(q) \leq u_\eta(p)$,

(1b) for every regular $\theta \in [\eta^+, \beta)$, $C_\theta \subseteq \theta$ is club and

(2) whenever $\theta \in [\eta^+, \beta)$ is regular and $\text{C-supp}(p) \cap [\theta, \theta^+) \neq \emptyset$,

(2a) for all $\gamma \in \text{C-supp}(p) \cap [\theta, \theta^+)$, $u_\eta(q) \restriction \gamma$ forces that p_γ has a $P[\alpha, \sup(\text{S-supp}(q) \cap \theta))$-name,

(2b) for all $\gamma \in \text{C-supp}(p) \cap [\theta, \theta^+)$, $u_\eta(q) \restriction \gamma^\oplus$ forces that $\max q^{**}_\gamma > \sup(\text{S-supp}(p) \cap \theta)$ *and* $\sup(\text{S-supp}(q) \cap \theta) > \max p^{**}_\gamma$,

(2c) $\sup(\text{S-supp}(q) \cap \theta)$ *is larger than or equal to some element of C_θ above* $\sup(\text{S-supp}(p) \cap \theta)$ *and*

(2d) if θ is inaccessible, $\sup(\text{S-supp}(q) \cap \theta) > \text{card}(\text{C-supp}(p) \cap [\theta, \theta^+))$.

The same definition applies for $p, q \in P[\alpha, \beta)^\oplus$.

Claim 8.6 (Persistence of Strategic Belowness)

- *For $p, q, r \in P[\alpha, \beta)$ and a cardinal $\eta \in [\alpha, \beta)$, if q is η^+-strategically below p with respect to C and $u_\eta(r) \leq u_\eta(q)$, then r is η^+-strategically below p with respect to C. Analogously for $P[\alpha, \beta)^\oplus$.*

- *For $p, q, r \in P[\alpha, \beta)$ and a cardinal $\eta \in [\alpha, \beta)$, if $u_\eta(q) \leq u_\eta(p)$ and r is η^+-strategically below q with respect to C, then r is η^+-strategically below p with respect to C. Analogously for $P[\alpha, \beta)^\oplus$.*

Proof: Follows straightforwardly from definition 8.5. □

Notation: Assume $\langle s^i : i < \delta \rangle$ is a decreasing sequence of conditions in S. Then $\langle s^i : i < \delta \rangle$ is eventually constant and we denote it's limit by $\bigcup_{i<\delta} s^i$.

Definition 8.7 (Strategic lower bound) *Given a cardinal $\eta \in [\alpha, \beta)$ and a sequence $\langle p^i : i < \delta \rangle$ of conditions in $P[\alpha, \beta)$ of limit length $\delta < \eta^+$ s.t. $\langle u_\eta(p^i) : i < \delta \rangle$ is a decreasing sequence of conditions, form r s.t.*

- *$\forall \gamma \geq \eta$, $r_\gamma = \bigcup_{i<\delta} p^i_\gamma$ and*
- *$\forall \gamma \geq \eta^+$, $r^{**}_\gamma = \bigcup_{i<\delta} (p^i)^{**}_\gamma$,*

For γ other than the above, set $r_\gamma = \mathbf{1}$ and $r^{**}_\gamma = \emptyset$. r is not necessarily a condition in $P[\alpha, \beta)$ (some r^{**}_γ may not be closed), but note that $\text{S-supp}(r)$ *and* $\text{C-supp}(r)$ *can be calculated as if r were a condition and* $\text{S-supp}(r)$ *is bounded below every regular cardinal,* $\text{C-supp}(r) \cap [\theta, \theta^+)$ *has size less than θ*

for every regular θ*. More explicitly, we let* S-supp$(r) := \bigcup_{i<\delta}$ S-supp$(u_\eta(p^i))$ *and* C-supp$(r) := \bigcup_{i<\delta}$ C-supp$(u_\eta(p^i))$.

We would like to form q *by setting, for every* $\gamma \in$ C-supp(r)*:*

(1) $q_{\mathrm{ot}\, f_\gamma[\sup r^{**}_\gamma]} := r_\gamma$.

(2) $q^{**}_\gamma := r^{**}_\gamma \cup \sup r^{**}_\gamma$.

If such q *exists, we call* q *the* η^+*-strategic lower bound for* $\langle p^i \colon i < \delta\rangle$*. Whenever we want to apply the above, we will be in a situation where each* $\sup r^{**}_\gamma$ *will have been decided to equal an actual ordinal value (and is not just a name for an ordinal).*

Fact 8.8 *Given a cardinal* $\eta \in [\alpha, \beta)$ *and a sequence* $\langle p^i \colon i < \delta\rangle$ *of conditions in* $P[\alpha, \beta)$ *of limit length* $\delta < \eta^+$ *such that* $\langle u_\eta(p^i) \colon i < \delta\rangle$ *is a decreasing sequence of conditions, if we can form the* η^+*-strategic lower bound* q *as above such that* $q \in u_\eta(P[\alpha, \beta))$*, then* $u_\eta(q) \leq u_\eta(p^i)$ *for each* $i < \delta$*. Analogously for* $P[\alpha, \beta)^\oplus$*.* □

Claim 8.9 *If* $\alpha < \beta < \beta'$*,* $\eta \in [\alpha, \beta)$ *is a cardinal,* $p, q \in P[\alpha, \beta')$ *and* q *is* η^+*-strategically below* p *with respect to* C*, then* $q \restriction \beta$ *is* η^+*-strategically below* $p \restriction \beta$ *with respect to* C*. Analogously for* $P[\alpha, \beta')^\oplus$ *and/or* $q \restriction \beta^\oplus$*.*

Proof: Follows straightforwardly from definition 8.5. □

Note: In the above claim, if $C = \langle C_\theta \colon \theta \in [\eta^+, \beta') \text{ regular}\rangle$, then - strictly speaking - the conclusion should be that q is η^+-strategically below p with respect to $\langle C_\theta \colon \theta \in [\eta^+, \beta) \text{ regular}\rangle$. But since the conclusion - as we originally wrote it - should have an obvious meaning (as explained), we will stick - here and in the following - to such (and similar) notation for sake of simplicity and clarity.

Definition 8.10 *If* θ *is a regular cardinal,* $v \subseteq [\theta, \theta^+)$ *of size less than* θ *and* $C \subseteq \theta$ *is a club set, we say that* C *is a separating club for* v *iff:*

- $\forall \gamma \in v \, \forall \eta \in C \;\; f_\gamma[\eta] \supseteq \eta$
- $\forall \gamma_0 < \gamma_1 \in v \, \forall \eta \in C \;\; f_{\gamma_0}[\eta]$ *is a proper initial subset of* $f_{\gamma_1}[\eta]$*.*

Fact 8.11 *For every* v *as above, there exists a separating club for* v*.*

Proof: Assume $v \subseteq [\theta, \theta^+)$ of size less than θ, θ regular. For any $\gamma \in v$, $\{\eta < \theta \colon f_\gamma[\eta] \supseteq \eta\}$ is obviously club in θ. So as v has size less than θ, $D_0 := \{\eta < \theta \colon \forall \gamma \in v \; f_\gamma[\eta] \supseteq \eta\}$ is club in θ.

Similarly, for any $\gamma_0 < \gamma_1 \in v$, $\{\eta < \theta \colon f_{\gamma_0}[\eta]$ is a proper initial subset of $f_{\gamma_1}[\eta]\}$ is club in θ. Using again that v has size less than θ, there are less than θ-many possible choices of γ_0, γ_1 from v, hence we may intersect all correpsonding clubs to obtain that $D_1 := \{\eta < \theta \colon \forall \gamma_0 < \gamma_1 \in v \; f_{\gamma_0}[\eta]$ is a proper initial subset of $f_{\gamma_1}[\eta]\}$ is club in θ.

Finally, $D_0 \cap D_1$ is a separating club for v. $\square$

Observations:

- If $C' \subseteq C$ and C is a separating club for v then C' is a separating club for v.
- If $v' \supseteq v$ and C is a separating club for v' then C is a separating club for v.

Notation: For a sequence of clubs of the form $C = \langle C_{\theta,i} \colon \theta \in [\eta^+, \beta)$ regular, $i < \delta\rangle$, we write $C(i)$ for $\langle C_{\theta,i} \colon \theta \in [\eta^+, \beta)$ regular$\rangle$ and we write $\bigcap_{j<i} C(j)$ for $\langle \bigcap_{j<i} C_{\theta,j} \colon \theta \in [\eta^+, \beta)$ regular$\rangle$.

Claim 8.12 (Existence of strategic lower bounds)
Assume $\eta \in [\alpha, \beta)$ is a cardinal, $\langle p^i \colon i < \delta\rangle$ is a sequence of conditions of limit length $\delta < \eta^+$ in $P[\alpha, \beta)$ such that $\langle u_\eta(p^i) \colon i < \delta\rangle$ is a decreasing sequence of conditions, $C_{\theta,i}$ is a separating club for C-supp$(p^i) \cap [\theta, \theta^+)$ *for every $C_{\theta,i}$ in $C = \langle C_{\theta,i} \colon \theta \in [\eta^+, \beta)$ regular, $i < \delta\rangle$. Assume that for all $i < \delta$, p^{i+1} is η^+-strategically below p^i with respect to $\bigcap_{j \leq i} C(j)$. Then the η^+-strategic lower bound q for $\langle p^i \colon i < \delta\rangle$ exists. Analogously within $P[\alpha, \beta)^\oplus$.*

Proof: By induction on $\beta > \eta$. For any $\gamma < \beta$, given that the claim holds within $P[\alpha, \gamma)$, it immediately follows that it holds within $P[\alpha, \gamma)^\oplus$. Assume we want to show the claim holds for β. Inductively, for $\gamma < \beta$, let q^γ be the η^+-strategic lower bound for $\langle p^i {\restriction} \gamma \colon i < \delta\rangle$, let $q^{\gamma^\oplus}$ be the η^+-strategic lower bound for $\langle p^i {\restriction} \gamma^\oplus \colon i < \delta\rangle$. This is possible using claim 8.9.

Now form r (from $\langle p^i \colon i < \delta\rangle$) as in definition 8.7. We first show that the sequence $\langle p^i \colon i < \delta\rangle$ has the property that for every regular $\theta \in [\eta^+, \beta)$, either C-supp$(p^i) \cap [\theta, \theta^+) = \emptyset$ for all $i < \delta$ or the following hold:

i. sup(S-supp$(r) \cap \theta) >$ sup(S-supp$(p^i) \cap \theta)$ for all $i < \delta$,

ii. for $\gamma \in$ C-supp$(r) \cap [\theta, \theta^+)$, $q^{\gamma^\oplus} \Vdash \sup r^{**}_\gamma =$ sup(S-supp$(r) \cap \theta)$,

iii. for $\gamma \in$ C-supp$(r) \cap [\theta, \theta^+)$, $f_\gamma[$sup(S-supp$(r) \cap \theta)] \supseteq$ sup(S-supp$(r) \cap \theta)$,

iv. for $\gamma_0 < \gamma_1$ both in C-supp$(r) \cap [\theta, \theta^+)$, $f_{\gamma_0}[$sup(S-supp$(r) \cap \theta)]$ is a proper initial segment of $f_{\gamma_1}[$sup(S-supp$(r) \cap \theta)]$

v. for $\gamma \in$ C-supp$(r) \cap [\theta, \theta^+)$, $q^{\gamma^\oplus}$ forces that r_γ has a $P_{\text{sup(S-supp}(r)\cap\theta)}$-name.

vi. if θ is inaccessible, $\sup(\text{S-supp}(r) \cap \theta) \geq \text{card}(\text{C-supp}(r) \cap [\theta, \theta^+))$.

Property (i) immediately follows from property (2b) in definition 8.5. Property (ii) also follows from property (2b) in definition 8.5, using that $q^{\gamma^\oplus}$ is stronger than $u_\eta(p^i \upharpoonright \gamma^\oplus)$ for every $i < \delta$. Properties (iii) and (iv) easily follow, as property (2c) in definition 8.5 implies that for every regular $\theta \in [\eta^+, \beta)$, $\sup(\text{S-supp}(r) \cap \theta)$ belongs to $\bigcap_{i<\delta} C_{\theta,i}$, a separating club for $\text{C-supp}(r) \cap [\theta, \theta^+)$. Property (v) follows from property (2a) in definition 8.5, property (vi) follows from property (2d) in definition 8.5.

Now we show, using (i)-(vi), that we can form the η^+-strategic lower bound q for $\langle p^i : i < \delta \rangle$ as in definition 8.7: Note that if we can form q out of r as in definition 8.7, then $q \upharpoonright \gamma \leq q^\gamma$ and $q \upharpoonright \gamma^\oplus \leq q^{\gamma^\oplus}$ for every $\gamma < \beta$. Similarly whenever $\gamma_0 < \gamma_1 < \beta$, $q^{\gamma_1} \upharpoonright \gamma_0 \leq q^{\gamma_0}$ and $q^{\gamma_1} \upharpoonright \gamma_0^\oplus \leq q^{\gamma_0 \oplus}$.

Assume $\theta \in [\eta^+, \beta)$ is regular, $\text{card}\, \gamma = \theta$. Given (i)-(iv), $q^{\gamma^\oplus}$ decides $\sup r^{**}_\gamma$ and forces that $\text{ot} f_\gamma[\sup r^{**}_\gamma] \geq \sup(\text{S-supp}(r) \cap \theta)$ is distinct from $\text{ot} f_\xi[\sup r^{**}_\xi]$ for every $\xi < \gamma$. By (v), $q^{\gamma^\oplus}$ forces that r_γ has a $P_{\sup r^{**}_\gamma}$-name, allowing us to satisfy (1) as in definition 8.7. (2) in definition 8.7 can obviously be satisfied. Finally (vi) implies that $\text{S-supp}(q) \setminus \text{S-supp}(r)$ (and hence $\text{S-supp}(q)$) is bounded below every regular cardinal and hence q actually is a condition in $u_\eta(P[\alpha, \beta))$. $\square$

Note: To be exact, note that we assumed our conditions p to satisfy property A2: $\forall \gamma\ \mathbf{1}_{P[\alpha,\gamma)^\oplus} \Vdash p^{**}_\gamma \in C_\gamma(G_{\gamma+1})$. This will usually not be the case for q as obtained above. But, as can be seen from the construction, it will be the case that

$$\forall \gamma\ u_\eta(q) \upharpoonright \gamma^\oplus \Vdash q^{**}_\gamma \in C_\gamma(G_{\gamma+1}).$$

Thus we may replace q by an η^+-strategically equivalent q' satisfying A2, where we say that q and q' are η^+-strategically equivalent iff $u_\eta(q') \leq u_\eta(q)$ and $u_\eta(q) \leq u_\eta(q')$.

Claim 8.13 (Existence of strategic lower bounds for singulars) *Assume $\langle \eta_i : i < \text{cof}\, \eta \rangle$ is an increasing sequence of regular cardinals with singular limit $\eta \leq \beta$, $\langle p^i : i < \text{cof}\, \eta \rangle$ is a decreasing sequence of conditions in $P[\alpha, \beta)$, $\eta_0 \geq \max\{\text{cof}\, \eta, \alpha\}$. Assume further that for every $i < \text{cof}\, \eta$, p^{i+1} is η_i^+-strategically below p^i with respect to $C_i = \langle \bigcap_{j \leq i} C_{\theta,j} : \theta \in [\eta_i{}^+, \alpha)$ regular$\rangle$, $l_{\eta_i}(p^{i+1}) = l_{\eta_i}(p^i)$ and at limit ordinals $i < \text{cof}\, \eta$, p^i is a lower bound for $\langle p^j : j < i \rangle$, where $C_{\theta,j}$ is a separating club for* $\text{C-supp}(p^j) \cap [\theta, \theta^+)$ *for every regular $\theta \in [\eta_j{}^+, \beta)$ and $j < \text{cof}\, \eta$. Then a greatest lower bound q for $\langle p^i : i < \text{cof}\, \eta \rangle$ exists. Analogously within $P[\alpha, \beta)^\oplus$.*

Proof: By induction on β. First assume $\beta = \eta$ and form q as the componentwise union of the p^i, $i < \text{cof}\, \eta$. Note that for every $\gamma < \eta$, $\langle p^i(\gamma) : i < \text{cof}\, \eta \rangle$

is eventually constant and $\eta_0 \geq \operatorname{cof}\eta$. Thus q is indeed a condition in $P[\alpha, \eta)$ and is the greatest lower bound for the p^i, $i < \operatorname{cof}\eta$. For $\beta > \eta$, the proof is very similar to the proof of claim 8.12, except that we will take r as the componentwise union of the p^i (for all components, note that the p^i here form a decreasing sequence of conditions), but in the end, we will again calculate q from r as in definition 8.7. □

Note: in the above claim, q again will not satisfy property A2. It will just be the case that

$$\forall\gamma\ q{\restriction}\gamma^{\oplus} \Vdash q^{**}_{\gamma} \in C_\gamma(G_{\gamma+1}).$$

Still we may replace q by an equivalent q' satisfying A2, where we say that q and q' are equivalent iff $q' \leq q$ and $q \leq q'$. This is sufficient in this case as we are requiring the p^i to form a decreasing sequence of conditions (in contrast to claim 8.12).

Claim 8.14 (Induced Strategic Belowness)
Assume $\eta \in [\alpha, \beta)$ is a cardinal, β is a limit ordinal, $p, q \in P[\alpha, \beta)$, $\langle \beta_j : j < \operatorname{cof}\beta\rangle$ is cofinal in β and increasing with $\beta_0 > \eta$ such that for every $j < \operatorname{cof}\beta$, $q{\restriction}\beta_j$ is η^+-strategically below $p{\restriction}\beta_j$ with respect to C. Then q is η^+-strategically below p with respect to C. Analogously within $P[\alpha, \beta)^{\oplus}$.

Proof: Immediate from definition 8.5. □

Claim 8.15 (Existence of induced strategic lower bounds)
Assume $\eta \in [\alpha, \beta)$ is a cardinal, β is a limit ordinal, $\langle p^i : i < \delta\rangle$ is a sequence of conditions of limit length $\delta < \eta^+$ in $P[\alpha, \beta)$ such that $\langle u_\eta(p^i) : i < \delta\rangle$ is a decreasing sequence of conditions, $C_{\theta,i}$ is a separating club for C-supp$(p^i) \cap [\theta, \theta^+)$ *for every $C_{\theta,i}$ in $C = \langle C_{\theta,i} : \theta \in [\eta^+, \beta)$ regular, $i < \delta\rangle$, $\langle \beta_j : j < \operatorname{cof}\beta\rangle$ is cofinal in β and increasing such that $\beta_0 > \eta$ and:*

- *$\forall i < \delta$ there exists $n < \operatorname{cof}\beta$ such that $p^{i+1}{\restriction}\beta_n$ is η^+-strategically below $p^i{\restriction}\beta_n$ with respect to $\bigcap_{j \leq i} C(j)$ and $p^{i+1}[\beta_n, \beta) = p^i[\beta_n, \beta)$.*
- *$\forall j < \operatorname{cof}\beta$ there are unboundedly many $i < \delta$ for which there exists $n \geq j$ s.t. $p^{i+1}{\restriction}\beta_n$ is η^+-strategically below $p^i{\restriction}\beta_n$ w.r.t. $\bigcap_{j \leq i} C(j)$.*

Then the η^+-strategic lower bound for $\langle p^i : i < \delta\rangle$ exists and is η^+-strategically below p^0 with respect to $C(0)$. Analogously within $P[\alpha, \beta)^{\oplus}$.

Proof: By claims 8.9, 8.6 and 8.12, we know that for every $j < \operatorname{cof}\beta$, the η^+-strategic lower bound for $\langle p^i{\restriction}\beta_j : i < \delta\rangle$ exists and denote it by q^j. Let p^δ be the componentwise union of the q^j, $j < \operatorname{cof}\beta$, and note that whenever $j < k < \operatorname{cof}\beta$, $q^k \leq q^j$ and for every γ of regular cardinality, $\langle (q^j)^{**}_\gamma : j < \operatorname{cof}\beta\rangle$ is eventually constant. It is thus easily seen that p^δ is a condition in $P[\alpha, \beta)$ extending each p^i. The final statement of the claim follows by claims 8.6 and 8.14. □

Theorem 8.16 *Suppose $\omega \leq \bar{\alpha} < \alpha$, $\bar{\alpha} \in$ **Card**. Then the following hold:*

1. *[Early names on the club support]*
   If α^-, the largest cardinal below α exists, is regular and $\bar{\alpha} < \alpha^-$, then every $p \in P[\bar{\alpha}, \alpha)$ can be strengthened to $q \leq p$ such that for some $\gamma < \alpha^-$, $\langle p_i : i \in \text{C-supp}(p) \cap [\alpha^-, \alpha)\rangle$ and $\langle \sup p_i^{**} : i \in \text{C-supp}(p) \cap [\alpha^-, \alpha)\rangle$ are both forced by q to have a $P[\bar{\alpha}, \gamma)$-name. Analogously for $p \in P[\bar{\alpha}, \alpha)^{\oplus}$.

   Moreover, if $\exists \eta\ \eta^+ = \alpha^-$, this can be done s.t. $l_\eta(q) = l_\eta(p)$ and if α^- is inaccessible, then for any $\eta < \alpha^-$, this can be done s.t. $l_\eta(q) = l_\eta(p)$.

2. *[Strategic Successors, Strategic Closure]*
   If $\eta \in [\bar{\alpha}, \alpha)$ is a cardinal, $p \in P[\bar{\alpha}, \alpha)$ and $C = \langle C_\theta : \theta \in [\eta^+, \alpha)$ regular$\rangle$ is such that each $C_\theta \subseteq \theta$ is club, then for any $q \leq p$ there exists $r \leq q$ which is η^+-strategically below p with respect to C. Analogously for $p \in P[\bar{\alpha}, \alpha)^{\oplus}$.

   Consequently, $u_\eta(P[\bar{\alpha}, \alpha))$ and $u_\eta(P[\bar{\alpha}, \alpha)^{\oplus})$ are both η^+-strategically closed.

3. *[Early Club Information]*
   $P[\bar{\alpha}, \alpha)$ has a dense subset of conditions p for which $p \restriction i^{\oplus}$ forces that p_i^{**} has a $P[\bar{\alpha}, \operatorname{card} i)$-name for each $i \in [\bar{\alpha}, \alpha)$ of regular cardinality. Similar for $P[\bar{\alpha}, \alpha)^{\oplus}$.

4. *[Chain Condition]*
   Assume $\eta \in [\bar{\alpha}, \alpha)$ is a regular cardinal. If J is an antichain of $P[\bar{\alpha}, \alpha)$ such that whenever p and q are in J, $u_\eta(p) \parallel u_\eta(q)$, then $|J| \leq \eta$. Similar for $P[\bar{\alpha}, \alpha)^{\oplus}$.

5. *[Early names]*

   - *Let $\eta \in [\bar{\alpha}, \alpha)$ be regular. Let $\dot{f}$ be a $P[\bar{\alpha}, \alpha)$-name for an ordinal-valued function with domain η. Then any condition in $P[\bar{\alpha}, \alpha)$ can be strengthened to a condition q with the same η-sized part forcing that for every $i < \eta$, there is a maximal antichain of size at most η below q deciding $\dot{f}(i)$, where for every element a of that antichain, $u_\eta(a) = u_\eta(q)$. In particular, q forces that $\dot{f}$ has a $P[\bar{\alpha}, \gamma)$-name for some $\gamma < \eta^+$. Similar for a $P[\bar{\alpha}, \alpha)^{\oplus}$-name $\dot{f}$.*
   - *Let $\eta \in [\bar{\alpha}, \alpha]$ be a singular cardinal. Let $\dot{f}$ be a $P[\bar{\alpha}, \alpha)$-name for an ordinal-valued function with domain η. Then for any $\zeta < \eta$, any condition in $P[\bar{\alpha}, \alpha)$ can be strengthened to a condition q with the same ζ-sized part, forcing that for every $i < \eta$, there is a maximal antichain of size less than η below q deciding $\dot{f}(i)$, where for every element a of that antichain, $u_\eta(a) = u_\eta(q)$. In particular, q forces that $\dot{f}$ has a P_η-name. Similar for a $P[\bar{\alpha}, \alpha)^{\oplus}$-name $\dot{f}$.*

6. *[Distributivity]*
   For any θ, $P[\theta,\alpha)$ *is* θ*-distributive.*

7. *[Smallness of the iteration]*
   Let $\eta \in [\bar\alpha,\alpha)$ *be a cardinal. If* α *is regular,* $u_\eta(P[\bar\alpha,\alpha))$ *has a dense subset of size* α*. Otherwise* $u_\eta(P[\bar\alpha,\alpha))$ *has a dense subset of size* α^+*.*

8. *[Preservation of the* GCH*]*
   After forcing with P_α*,* GCH *holds.*

9. *[Covering, Preservation of Cofinalities]*
   For every cardinal θ*, for every* $p \in P_\alpha$ *and every* P_α*-name* $\dot x$ *for a set of ordinals of size* θ *there is a set* X *in* $\mathbf{V}$ *of size* θ *and an extension* q *of* p *such that* $q \Vdash \dot x \subseteq X$*.*

   Therefore forcing with P_α *preserves all cofinalities.*

10. *[Factorization]*
    Whenever $\alpha^* > \alpha$*, there exists a canonical dense embedding from* $P[\bar\alpha,\alpha^*)$ *into* $P[\bar\alpha,\alpha) * \dot P[\alpha,\alpha^*)$*.*

11. *[Club Extendibility]*
    Assume $\gamma < \alpha$ *has regular cardinality,* $\bar\delta < \operatorname{card}\gamma$*. For any condition* $p \in P[\bar\alpha,\alpha)$*, there is* $q \le p$ *such that* $q{\restriction}\gamma^\oplus$ *forces that* $\max q^{**}_\gamma \ge \bar\delta$*.*

    Moreover, if $I \subseteq [\bar\alpha,\alpha)$ *is such that* $\operatorname{card}(I \cap [\theta,\theta^+)) < \theta$ *for all* θ*,* I *is bounded below every inaccessible,* $I \subseteq \bigcup_{\theta\ regular}[\theta,\theta^+)$ *and* $\langle \bar\delta^i : i \in I\rangle$ *is such that* $\bar\delta_i < \operatorname{card} i$ *for every* $i \in I$*, then for every* $p \in P[\bar\alpha,\alpha)$*, there is* $q \le p$ *such that* $\forall i \in I\ q{\restriction}i^\oplus \Vdash \max q^{**}_i \ge \bar\delta_i$*.*

Proof: By induction on α. But before proving (inductively) any of the above statements to actually hold at α, we will prove various implications between those statements at α, (inductively) assuming that 1-11 hold below α.

1→2 Let $\eta \in [\bar\alpha,\alpha)$ be a cardinal, let $p \in P[\bar\alpha,\alpha)$, let $C = \langle C_\theta : \theta \in [\eta^+,\alpha)$ regular$\rangle$ be such that each $C_\theta \subseteq \theta$ is club. Assume $q \le p$ and set $q^0 := q$. We may assume that $u_\eta(q^0) \le u_\eta(p)$ (if not, let q^0 be the (componentwise) union of p and q).

Let θ_0 be least above η such that C-supp$(p) \cap [\theta_0,{\theta_0}^+) \neq \emptyset$.

Case 1: $\theta_0 = \kappa^+$ is a successor: Strengthen $q^0{\restriction}{\theta_0}^+$ using 1 to a condition forcing that for some $\gamma < \theta_0$, both $\langle p_i : i \in$ C-supp$(p) \cap [\theta_0,{\theta_0}^+)\rangle$ and $\langle \sup p^{**}_i : i \in$ C-supp$(p) \cap [\theta_0,{\theta_0}^+\rangle$ have a $P[\bar\alpha,\gamma)$-name, keeping the κ-sized part of q^0 unchanged.

Case 2: θ_0 is inaccessible: Choose $\kappa < \theta_0$ above $\sup(\text{S-supp}(q^0) \cap \theta_0)$ and strengthen $q^0 \restriction {\theta_0}^+$ using 1 to a condition forcing that for some $\gamma < \theta_0$, both $\langle p_i \colon i \in \text{C-supp}(p) \cap [\theta_0, {\theta_0}^+)\rangle$ and $\langle \sup p_i^{**} \colon i \in \text{C-supp}(p) \cap [\theta_0, {\theta_0}^+)\rangle$ have a $P[\bar{\alpha}, \gamma)$-name, keeping the κ-sized part of q^0 unchanged.

Let $q^1 \leq q^0$ be the result of strengthening q^0 as above. Let θ_1 be the least cardinal above θ_0 such that $\text{C-supp}(p) \cap [\theta_1, {\theta_1}^+) \neq \emptyset$ and perform the above construction for θ_1 instead of θ_0 starting with q^1 instead of q^0 to obtain $q^2 \leq q^1$. Continue to perform the above successively for increasingly large cardinals $\theta > \eta$ with $\text{C-supp}(p) \cap [\theta, \theta^+) \neq \emptyset$, taking (componentwise) unions at limit steps. Those limits exist as for every γ and every limit ordinal δ, $\langle (q^i)^{**}_\gamma \colon i < \delta\rangle$ is eventually constant.

Let r denote the final condition obtained by the above after considering every cardinal θ with $\text{C-supp}(p) \cap [\theta, \theta^+) \neq \emptyset$. We may easily ensure that r satisfies condition (2a) in definition 8.5 by sufficiently increasing it's string support below every θ considered above. Now using 11 inductively, it is easy to make sure the first clause of condition (2b) in definition 8.5 holds for r. For the second clause of (2b) to hold, we need the following:

"Whenever $\text{C-supp}(p) \cap [\theta, \theta^+) \neq \emptyset$ with $\theta > \eta$, $u_\eta(r) \restriction \theta^+$ forces that $\sup(\text{S-supp}(r) \cap \theta)$ is greater than $\sup\{\max p_i^{**} \colon i \in \text{C-supp}(p) \cap [\theta, \theta^+)\}$."

To ensure this property, fix some regular cardinal $\theta > \eta$. Note that r forces that $\langle \max p_i^{**} \colon i \in \text{C-supp}(p) \cap [\theta, \theta^+)\rangle$ has - by the above construction - a $P[\bar{\alpha}, \xi)$-name for some $\xi < \theta$. But using 7 inductively, $P[\bar{\alpha}, \xi)$ has a dense subset of size θ, thus there is some ordinal $\zeta < \theta^+$ such that r forces that $\sup\{\max p_\gamma^{**} \colon \gamma \in \text{C-supp}(p) \cap [\theta, \theta^+)\}$ is less than ζ. So we may also ensure that r satisfies this part of condition (2b) by sufficiently increasing it's string support below θ for every regular θ with $\text{C-supp}(p) \cap [\theta, \theta^+) \neq \emptyset$.

Finally, we may once more increase the string support of r sufficiently below every regular θ with $\text{C-supp}(p) \cap [\theta, \theta^+) \neq \emptyset$ to ensure that r satisfies conditions (2c) and (2d) in definition 8.5. As (2a), (2b), (2c) and (2d) are each indestructible by extension, it follows that we finally obtained a condition $r \leq q$ which is η^+-strategically below p with respect to C.

The final statement about strategic closure follows immediately from claim 8.12 (the strategy is to play some η^+-strategic successor in each step, so we can always play η^+-strategic lower bounds at limit steps).

2→3: We first need the following:

Claim 8.17 *If $\beta < \alpha$, $\eta \in [\bar{\alpha}, \beta)$ is a cardinal and $p \in P[\bar{\alpha}, \beta)$, then there is $q \leq p$ with $l_\eta(q) = l_\eta(p)$ such that for every $\gamma \geq \eta^+$ in $\text{C-supp}(p)$, $q \restriction \gamma^\oplus$ forces that p_γ^{**} has a $P[\bar{\alpha}, \text{card}\,\gamma)$-name. We say that "$q$ gives early club information about p above η" in this case.*

Proof: This is similar to the proof of 1→2: Inductively apply 5 successively at every regular $\theta \in [\eta^+, \beta)$ with C-supp$(p) \cap [\theta, \theta^+) \neq \emptyset$ to obtain q as desired, making sure at each step that for every γ of cardinality θ, $q \restriction \gamma^\oplus$ forces that p^{**}_γ has a $P[\bar\alpha, \theta)$-name, preserving $l_\eta(q) = l_\eta(p)$. $\square_{\text{of claim}}$

For an ordinal $\xi > \bar\alpha$, we say that $p \in P[\bar\alpha, \xi)$ satisfies (*) iff whenever $\gamma \in [\bar\alpha, \xi)$ has regular cardinality, then $p \restriction \gamma^\oplus$ forces that p^{**}_γ has a $P[\bar\alpha, \operatorname{card}\gamma)$-name. We have to show that given any $p \in P[\bar\alpha, \alpha)$, we can find $q \leq p$ satisfying (*). If $\alpha > \operatorname{card}\alpha$ is of singular cardinality, this is immediate using 3 inductively, as C-supp(p) is then bounded in α. Otherwise:

Case 1: α is a successor ordinal:
Given a condition $p \in P[\bar\alpha, \alpha)$ with $\alpha = i + 1$, apply 5 inductively to strengthen $p \restriction i^\oplus$ to $q \leq p \restriction i^\oplus$ forcing that p^{**}_i has a $P[\bar\alpha, \operatorname{card} i)$-name, then apply 3 inductively to strengthen $q \restriction i$ to r such that r satisfies (*) and set $s = r^\frown(i, (q_i, p^{**}_i))$. Then $s \leq p$ satisfies (*).

Case 2: α is a limit ordinal, $\operatorname{cof}\alpha = \operatorname{card}\alpha$**:**
Given any $p \in P[\bar\alpha, \alpha)$, C-supp$(p)$ is bounded in α, thus the claim follows using 3 inductively.

Case 3: $\alpha > \bar\alpha^+$ is a limit ordinal, $\operatorname{cof}\alpha < \operatorname{card}\alpha$**:**
Let $\eta = \max\{\bar\alpha, \operatorname{cof}\alpha\}$. Let $\langle \alpha_i \colon i < \operatorname{cof}\alpha \rangle$ be increasing and cofinal in α such that $\alpha_0 \geq \eta^+$. Given any condition $p \in P[\bar\alpha, \alpha)$, we may inductively assume that $p \restriction \alpha_0$ satisfies (*). We extend $p^0 := p$ to $p^{\operatorname{cof}\alpha}$ in $\operatorname{cof}\alpha$-many steps, where at stage $j + 1 < \operatorname{cof}\alpha$, we extend p^j to p^{j+1} such that

- p^{j+1} gives early club information about p^j above η (use claim 8.17),
- p^{j+1} is chosen below $\langle p^k \colon k \leq j \rangle$ according to the strategy for η^+-strategic closure of $u_\eta(P[\bar\alpha, \alpha))$

and let $u_\eta(p^j)$ be the η^+-strategic lower bound of $\langle p^k \colon k < j \rangle$ at limit steps $j \leq \operatorname{cof}\alpha$ (keeping $l_\eta(p^j) = l_\eta(p)$). Note that for every γ, $(p^{\operatorname{cof}\alpha})^{**}_\gamma = \bigcup_{j < \operatorname{cof}\alpha} (p^j)^{**}_\gamma \cup \sup(\bigcup_{j < \operatorname{cof}\alpha} (p^j)^{**}_\gamma)$ and for every $j < \operatorname{cof}\alpha$, $p^{\operatorname{cof}\alpha}$ forces that $(p^j)^{**}_\gamma$ has a $P[\bar\alpha, \operatorname{card}\gamma)$-name. Thus $p^{\operatorname{cof}\alpha}$ forces that $(p^{\operatorname{cof}\alpha})^{**}_\gamma$ has a $P[\bar\alpha, \operatorname{card}\gamma)$-name for every γ, i.e. $p^{\operatorname{cof}\alpha}$ satisfies (*).

Case 4: $\alpha < \bar\alpha^+$ is a limit ordinal, $\operatorname{cof}\alpha < \bar\alpha$**, $\bar\alpha$ successor:** Let $\langle \alpha_i \colon i < \operatorname{cof}\alpha \rangle$ be cofinal and increasing in α such that $\alpha_0 > \bar\alpha$. We show that for any $\bar p \in P_{\bar\alpha}$ and $p \in P[\bar\alpha, \alpha)$, we can extend $(\bar p, p)$ to $(\bar q, q)$ such that $\bar q$ forces that q satisfies (*), which is sufficient. We may inductively assume that $\bar p$ forces that $p \restriction \alpha_0$ satisfies (*). Choose $p^* \in P_\alpha$ such that $p^* \restriction \bar\alpha = \bar p$ and that $p^* \restriction \bar\alpha \Vdash p^*[\bar\alpha, \alpha) = p$. Let $\eta := \bar\alpha^-$. Now extend $p^0 := p^*$ to $p^{\operatorname{cof}\alpha}$ in cof α-many steps, where at stage $j+1 < \operatorname{cof}\alpha$, we choose p^{j+1} such that, using 5 inductively,

- $p^{j+1}\restriction\alpha_{j+1}$ gives early club information about $p^j\restriction\alpha_{j+1}$ above η and
- p^{j+1} is chosen below $\langle p^k\colon k \le j\rangle$ according to the strategy for η^+-strategic closure of $u_\eta(P[\eta,\alpha))$.

Let $u_\eta(p^j)$ be the η^+-strategic lower bound of $\langle p^k\colon k < j\rangle$ at limit steps $j \le \operatorname{cof}\alpha$, keeping $l_\eta(p^j) = l_\eta(p^*)$; note that $p^j\restriction\bar\alpha$ forces that $p^j[\bar\alpha,\alpha_j)$ satisfies (*), as for every $\gamma \in$ C-supp(p^j), $(p^j)^{**}_\gamma = \bigcup_{k<j}(p^k)^{**}_\gamma \cup \sup(\bigcup_{k<j}(p^k)^{**}_\gamma)$, which is forced by p^j to have a $P_{\bar\alpha}$-name. Thus in the end, $p^{\operatorname{cof}\alpha}\restriction\bar\alpha$ forces that $p^{\operatorname{cof}\alpha}[\bar\alpha,\alpha) \le p$ satisfies (*).

Case 5: $\alpha < \bar\alpha^+$ is a limit ordinal, $\operatorname{cof}\alpha < \bar\alpha$, $\bar\alpha$ inaccessible: Let $\langle\alpha_i\colon i < \operatorname{cof}\alpha\rangle$ be cofinal and increasing in α such that $\alpha_0 > \bar\alpha$. Choose some cardinal $\eta > \operatorname{cof}\alpha$ below $\bar\alpha$. Now proceed as in case 4.

3→4: Apply 7 inductively to obtain a dense subset $P[\bar\alpha,\eta)^*$ of $P[\bar\alpha,\eta)$ of size η and apply 3 to obtain a dense subset $P[\eta,\alpha)^*$ of $P[\eta,\alpha)$ of conditions satisfying (*) (as described in the proof of 2→3 above). Assume for a contradiction that $|J| > \eta$ for some antichain J of $P[\bar\alpha,\eta)^* * \dot P[\eta,\alpha)^*$ (we use 10 inductively here).

As $P[\bar\alpha,\eta)^*$ has size η, $p[\bar\alpha,\eta)$ is the same for η^+-many conditions $p \in J$, hence there is $\bar p \in P[\bar\alpha,\eta)$ and $J' \subseteq P[\eta,\alpha)^*$ such that $\bar p \Vdash J'$ is an antichain of $\dot P[\eta,\alpha)$ of size η^+. Work in any P_η-generic extension with $\bar p$ contained in the P_η-generic. As GCH holds by 8 inductively, by a Δ-system argument, there is $W \subseteq J'$ of size η^+ and a size less than η subset A of η^+ s.t. C-supp$(p) \cap$ C-supp$(q) \cap [\eta,\eta^+) = A$ whenever $p \ne q$ are both in W. Again using GCH, there are only η-many possibilities for $\langle p_i^{**}\colon i \in A\rangle$ for $p \in P[\eta,\alpha)^*$. Hence for η^+-many conditions p in W, $\langle p_i^{**}\colon i \in A\rangle$ is the same (modulo equivalence). But - using the assumption that $u_\eta(p) \parallel u_\eta(q)$ - any two such conditions are compatible, thus W (and hence also J) is not an antichain.

2→5: First assume $\dot f$ is a $P[\bar\alpha,\alpha)$-name for an ordinal-valued function with domain some regular cardinal $\eta \in [\bar\alpha,\alpha)$. Start with $i = 0$. Fix an arbitrary condition $p^0 \in P[\bar\alpha,\alpha)$ and choose $q^0 \le p^0$ such that q^0 decides $\dot f(i)$. At stage $j+1$, let $p^{j+1} \le p^0$ be any condition in $P[\bar\alpha,\alpha)$ incompatible to all q^k, $k \le j$ s.t. $u_\eta(p^{j+1}) = u_\eta(q^j)$ (if such exists) and choose q^{j+1} as follows:

- $q^{j+1} \le p^{j+1}$,
- q^{j+1} decides $\dot f(i)$ and

- $u_\eta(q^{j+1})$ is chosen with respect to the strategy for η^+-strategic closure below $\langle u_\eta(q^k)\colon k \leq j\rangle$.

At limit stages j, let $p^j \leq p^0$ be any condition in $P[\bar{\alpha}, \alpha)$ incompatible to all q^k, $k < j$ s.t. for all $k < j$, $u_\eta(p^j) \leq u_\eta(q^k)$ if such exists. Note that a p^j satisfying the latter condition can always be found by the strategic choice of the $u_\eta(q^k)$. Choose $q^j \leq p^j$ deciding $\dot{f}(i)$ with $u_\eta(q^j) \leq u_\eta(p^j)$.

Proceed with this until arriving at some stage j where no condition p^j as above can be chosen. By 4, this j will have cardinality at most η. We can then find $r \in u_\eta(P[\bar{\alpha}, \alpha))$ s.t. $r \leq u_\eta(q^k)$ for every $k < j$. Hence we may strengthen every q^k to $\bar{q}^k$ such that $u_\eta(\bar{q}^k) = u_\eta(r)$ and $l_\eta(\bar{q}^k) = l_\eta(q^k)$. Let $s \in P[\bar{\alpha}, \alpha)$ be such that $l_\eta(s) = l_\eta(p^0)$ and $u_\eta(s) = u_\eta(r)$. Then $\{\bar{q}^k\colon k < j\}$ is a maximal antichain of $P[\bar{\alpha}, \alpha)$ below s deciding $\dot{f}(i)$.

Now do the same for $i = 1$, starting with s instead of p_0, and successively handle every $i < \eta$, taking lower bounds (as given by the strategy for η^+-strategic closure) at limit steps. In the end, this gives some $q \leq p^0$ with $l_\eta(q) = l_\eta(p^0)$ such that for each $i < \eta$, there is a maximal antichain of size at most η below q deciding $\dot{f}(i)$ with the property that for every condition t in that antichain, $u_\eta(t) = u_\eta(q)$. But as q has bounded support in η^+, this means that q forces that $\dot{f}$ has a $P[\bar{\alpha}, \gamma)$-name for some $\gamma < \eta^+$. The proof for a $P[\bar{\alpha}, \alpha)^{\oplus}$-name $\dot{f}$ is identical.

Now assume $\dot{f}$ is a $P[\bar{\alpha}, \alpha)$-name for an ordinal-valued function with domain some singular cardinal $\eta \in [\bar{\alpha}, \alpha]$ (note the possibility of the case $\eta = \alpha$ here) and $\zeta < \eta$. Let $\eta = \bigcup_{i < \operatorname{cof} \eta} \eta_i$ with each η_i regular and η_0 greater than both $\operatorname{cof} \eta$ and ζ. Let $p^0 \in P[\bar{\alpha}, \alpha)$ be arbitrary. Using the "regular case" of 5, we find some condition $p^1 \in P[\bar{\alpha}, \alpha)$, $p^1 \leq p^0$ with $l_{\eta_0}(p^1) = l_{\eta_0}(p^0)$ and $u_{\eta_0}(p^1) \leq u_{\eta_0}(p^0)$ such that for each $i < \eta_0$, there is a maximal antichain of size at most η_0 below p^1 deciding $\dot{f}(i)$, where for every element a of that antichain, $u_{\eta_0}(a) = u_{\eta_0}(p^1)$. By 2, we may also assume that p^1 is ${\eta_0}^+$-strategically below p^0 with respect to some sequence of clubs $C_0 = \langle C_{\theta,0}\colon \theta \in [{\eta_0}^+, \alpha) \text{ regular}\rangle$, where each $C_{\theta,0}$ is a separating club for C-supp$(p^0) \cap [\theta, \theta^+)$. Now do the same starting with p^1 instead of p^0 and use η_1 instead of η_0 to obtain $p^2 \leq p^1$ with $l_{\eta_1}(p^2) = l_{\eta_1}(p^1)$ and $u_{\eta_1}(p^2) \leq u_{\eta_1}(p^1)$ such that for each $i < \eta_1$, there is a maximal antichain of size at most η_1 below p^2 deciding $\dot{f}(i)$, where for every element a of that antichain, $u_{\eta_1}(a) = u_{\eta_1}(p^2)$. Again by 2, we may also assume that p^2 is ${\eta_1}^+$-strategically below p^1 with respect to $C_1 = \langle C_{\theta,0} \cap C_{\theta,1}\colon \theta \in [{\eta_1}^+, \alpha) \text{ regular}\rangle$ where each $C_{\theta,1}$ is a separating club for C-supp$(p^1) \cap [\theta, \theta^+)$. Continue like this for $j < \operatorname{cof} \eta$, taking lower bounds at limit steps, which is possible by claim 8.13. In the end this gives some condition $q \leq p^0$ with $l_\zeta(q) = l_\zeta(p^0)$ such that for each $i < \eta$, there is a maximal antichain of size less than η below q deciding $\dot{f}(i)$ with the property that for every condition t in that antichain, $u_\eta(t) = u_\eta(q)$. In particular, this means that q forces that $\dot{f}$ has a $P[\bar{\alpha}, \eta)$-name.

5→1: Immediate.

Proof of 1-5: As we already know that at each stage of the induction, 1-5 are equivalent, we will, at each stage α only prove either 1 or 2, again assuming (inductively) that 1-11 hold below α. By the above implications, we then know that 1-5 hold at stage α.

Case 1: α is a limit cardinal
We prove 1, which is trivial as α^- does not exist.

Case 2: $\operatorname{card}\alpha < \alpha$ is singular or $\operatorname{cof}\alpha = \operatorname{card}\alpha$
We prove 1. Let $p \in P[\bar{\alpha}, \alpha)$ be given. Then C-supp(p) is bounded in α and thus the desired statement of the claim follows inductively using 1.

Case 3: α is a successor ordinal, $\operatorname{card}\alpha$ is regular
We prove 1. Assume $\alpha = \beta + 1$ and let $p \in P[\bar{\alpha}, \alpha)$ be given. We may assume that $\beta \in$ C-supp(p) as otherwise the claim is immediate using 1 inductively for β. Apply 5 inductively to obtain that $p \restriction \beta^{\oplus}$ can be strengthened to $q \leq p \restriction \beta^{\oplus}$ forcing that both p_β and $\sup p^{**}_\beta$ (this is a set of ordinals of size 2) have a $P[\bar{\alpha}, \gamma_0)$-name for some $\gamma_0 < \operatorname{card}\alpha$. Now apply 1 inductively for β to obtain that $q[\bar{\alpha}, \beta)$ can be strengthened to $r \leq q[\bar{\alpha}, \beta)$ forcing that for some $\gamma_1 < \operatorname{card}\alpha$, $\langle p_i \colon i \in \text{C-supp}(p) \cap [\operatorname{card}\alpha, \beta)\rangle$ and $\langle \sup p^{**}_i \colon i \in \text{C-supp}(p) \cap [\operatorname{card}\alpha, \beta)\rangle$ are both forced by r to have a $P[\bar{\alpha}, \gamma_1)$-name.

Then $r^\frown(\beta, (q_\beta, p^{**}_\beta))$ forces the desired statement of the claim.

Case 4: $\operatorname{cof}\alpha < \operatorname{card}\alpha$, $\operatorname{card}\alpha$ is regular, α is a limit ordinal
We prove 2. We want to show that for every $p \in P[\bar{\alpha}, \alpha)$, any cardinal $\eta \in [\bar{\alpha}, \alpha)$ and any sequence of clubs $C = \langle C_\theta \colon \theta \in [\eta^+, \alpha) \text{ regular}\rangle$, there exists q which is η^+-strategically below p with respect to C. This is trivial if $\eta = \operatorname{card}\alpha$, so assume $\eta < \operatorname{card}\alpha$. Let $\langle \alpha_n \colon n < \operatorname{cof}\alpha\rangle$ be a cofinal, increasing sequence in α such that $\alpha_0 > \operatorname{card}\alpha$.

First assume that $\operatorname{cof}\alpha \leq \eta$: Let $q_0 := p$ and $C_0 = \langle C_{\theta,0} \colon \theta \in [\eta^+, \alpha) \text{ regular}\rangle$ such that $C_{\theta,0} = C_\theta$ for all regular $\theta \in [\eta^+, \alpha)$. Using 2 inductively, let $q^1 \leq q^0$ be such that $q^1 \restriction \alpha_1$ is η^+-strategically below $q^0 \restriction \alpha_1$ with respect to C_0 and such that $q^1[\alpha_1, \alpha) = q^0[\alpha_1, \alpha)$. Set $C_1 := C_0 \cup \langle C_{\theta,1} \colon \theta \in [\eta^+, \alpha) \text{ regular}\rangle := \langle C_{\theta,0}, C_{\theta,1} \colon \theta \in [\eta^+, \alpha) \text{ regular}\rangle$ such that each $C_{\theta,1}$ is a separating club for C-supp$(q^1) \cap [\theta, \theta^+)$. Go on like this, letting (in general) q^{l+1} be such that $q^{l+1} \restriction \alpha_{l+1}$ is η^+-strategically below $q^l \restriction \alpha_{l+1}$ with respect to $\bigcap_{i \leq l} C_l(i)$ and set $C_{l+1} := C_l \cup \langle C_{\theta,l+1} \colon \theta \in [\eta^+, \alpha) \text{ regular}\rangle$ such that each $C_{\theta,l+1}$ is a separating club for C-supp$(q^{l+1}) \cap [\theta, \theta^+)$; let

$q^{l+1}[\alpha_{l+1}, \alpha) = q^l[\alpha_{l+1}, \alpha)$. At limit steps $l \leq \operatorname{cof} \alpha$, we may take q^l as the canonical lower bound of $\langle q^k \colon k < l \rangle$ by claim 8.15 and let $C_l := \bigcup_{k<l} C_k$. In the end, we let $q = q^{\operatorname{cof} \alpha}$ and by the final statement of claim 8.15, q is η^+-strategically below p with respect to C.

Finally, assume $\operatorname{cof} \alpha > \eta$ and we want to find q as above. First extend p to q^0 such that $q^0 \restriction (\operatorname{cof} \alpha)^+$ is η^+-strategically below $p \restriction (\operatorname{cof} \alpha)^+$ with respect to C using 2 inductively (note that $(\operatorname{cof} \alpha)^+ \leq \operatorname{card} \alpha$) and $q^0 \restriction [(\operatorname{cof} \alpha)^+, \alpha) = p \restriction [(\operatorname{cof} \alpha)^+, \alpha)$. Then we may proceed as above (using $\operatorname{cof} \alpha$ instead of η) to extend the $(\operatorname{cof} \alpha)^+$-strategically closed part of q^0 and obtain $q = q^{\operatorname{cof} \alpha}$ in $\operatorname{cof} \alpha$-many steps, preserving the $\operatorname{cof} \alpha$-sized part of q^0 (i.e. $l_{\operatorname{cof} \alpha}(q) = l_{\operatorname{cof} \alpha}(q^0)$) such that q is $(\operatorname{cof} \alpha)^+$-strategically below q^0 with respect to C.

We claim that q is η^+-strategically below p with respect to C: As $q^0 \restriction (\operatorname{cof} \alpha)^+$ is η^+-strategically below $p \restriction (\operatorname{cof} \alpha)^+$ with respect to C, we know by claim 8.6 that $q \restriction (\operatorname{cof} \alpha)^+$ is η^+-strategically below $p \restriction (\operatorname{cof} \alpha)^+$ with respect to C. We also know by the above construction that q is $(\operatorname{cof} \alpha)^+$-strategically below q^0 and hence below p (see claim 8.6) with respect to C. Now a detailed, but completely straightforward analysis of definition 8.5 yields that those two properties together imply that q is η^+-strategically below p with respect to C as desired.

Proof of 6: Assume $\dot{x}$ is a P_α-name for a sequence of ordinals of size less than θ. Then by 5, below any $p \in P_\alpha$ there is $q \leq p$ forcing that $\dot{x}$ has a P_θ-name.

Proof of 7: Assume for simplicity of notation that $\eta = \bar{\alpha}$ is singular and hence $u_\eta(P[\bar{\alpha}, \alpha)) = P[\bar{\alpha}, \alpha)$. Other cases are similar. We prove that $D_\alpha := \{p \in P[\bar{\alpha}, \alpha) \colon \forall \theta \, \exists \gamma \;\; \text{S-supp}(p) \cap [\theta, \theta^+) = [\theta, \gamma)\}$ has an equivalent dense subset E_α of size α if α is regular and of size α^+ if α is singular, in the sense that for every $p \in D_\alpha$, there is $p' \in E_\alpha$ such that $p \leq p' \leq p$. Note that D_α itself is dense in $P[\bar{\alpha}, \alpha)$.

Regular Cardinals: If α is regular, conditions in $P[\bar{\alpha}, \alpha)$ have bounded support below α, thus the claim follows by 7 inductively.

Successor Ordinals: Assume $p \in D_\alpha$, $\alpha = \beta + 1$ and D_β has an equivalent dense subset E_β of size α^+ inductively. p_β can be identified with an antichain of E_β below $p \restriction \beta$. Since for any two elements a_0, a_1 of such an antichain, $u_{\operatorname{card} \alpha}(a_0) \parallel u_{\operatorname{card} \alpha}(a_1)$, such an antichain will have size at most $\operatorname{card} \alpha$ using 4 inductively, thus there are α^+-many possible choices for p_β. p^{**}_β can be identified with a collection of less than $\operatorname{card} \alpha$-many antichains of E_β below $p \restriction \beta$, each elementwise paired with ordinals below $\operatorname{card} \alpha$, thus using similar arguments as before, there are α^+-many possible choices for p^{**}_β. Thus D_α has an equivalent dense subset of size α^+.

Singular Ordinals: If α is singular and $p \in D_\alpha$, we can modify p to an equivalent p' such that for every $\gamma < \alpha$, $p' \restriction \gamma \in E_\gamma$. Hence D_α has an equivalent dense subset of size $\prod_{\gamma<\alpha} \gamma^+ \leq \alpha^+$.

Proof of 8: P_α has a dense subset of size $\leq \alpha^+$ by 7. Thus $\Vdash_{P_\alpha} 2^\theta = \theta^+$ for $\theta \geq \alpha$. For $\theta < \alpha$, note that $P_\alpha \cong P_{\theta^+} * P[\theta^+, \alpha)$, where P_{θ^+} has a dense subset of size θ^+ and hence preserves $2^\theta = \theta^+$. If $\theta^+ = \alpha$, we are done. Otherwise, the result follows by 6.

Proof of 9: As P_α has a dense subset of size $\leq \alpha^+$ by 7, this is immediate for $\theta \geq \alpha^+$. If α is regular, P_α has a dense subset of size α and hence this is immediate for $\theta = \alpha$ in that case. If $\theta < \alpha$ and $\theta^+ < \alpha$, this follows inducively, using that $P[\theta^+, \alpha)$ does not add new sets of size θ. If $\alpha = \theta^+$, we use 5 to obtain $q \leq p$ forcing that for every $i < \theta$, there exists an antichain of size at most θ below q deciding the i^{th} element of $\dot{x}$, thus q forces that we can cover $\dot{x}$ by some $X \in \mathbf{V}$ of size θ. If $\theta = \alpha$ is singular, note that the "singular case" of 5 also holds in the case that $\eta = \alpha$ and thus we may apply 5 as above to obtain $q \leq p$ forcing that for every $i < \theta$, there exists an antichain of size less than θ below q deciding the i^{th} element of $\dot{x}$. □

Proof of 10: We show that the mapping $p \mapsto (p \restriction \alpha, p[\alpha, \alpha^*))$ is a dense embedding from $P[\bar{\alpha}, \alpha^*)$ to $P[\bar{\alpha}, \alpha) * P[\alpha, \alpha^*)$: Given any $(p, \sigma) \in P[\bar{\alpha}, \alpha) * \dot{P}[\alpha, \alpha^*)$, the only problem is that the supports (string and club support) of σ are P_α-names, which we need to force to be covered by ground model objects of the same size, where ground model here refers to the model after forcing with $P_{\bar{\alpha}}$. Use claim 9 three times to successively extend p to q forcing that by ground model objects of the same size, C-supp$(\sigma) \cap [\alpha, \alpha^+)$, C-supp$(\sigma) \cap [\alpha^+, \alpha^{++})$ and S-supp$(\sigma) \cap [\alpha, \alpha^+)$ are covered. For supports at larger cardinals, this is immediate (without extending q) as P_α has a dense subset of size at most α^+ by 7. Now extend (q, σ) to (q, σ') such that the supports of σ' are equal to those covering sets from the ground model obtained above (also those obtained without extending p). Then (q, σ') is in the range of our above-defined dense embedding.

Proof of 11: Let $p \in P[\bar{\alpha}, \alpha)$ and $\gamma < \alpha$ of regular cardinality. We want to extend p to q such that $q \restriction \gamma^\oplus$ forces that $\max q^{**}_\gamma \geq \bar{\delta}$. We may assume that $p_\gamma \neq \mathbf{1}$, as we may just set $p_\gamma = 0$ otherwise. We may also assume that $p \restriction \gamma$ decides p_γ as we may strengthen $p \restriction \gamma$ to do so otherwise. First assume $\gamma \geq \bar{\alpha}^+$. Choose $\delta \geq \bar{\delta}$ such that $\operatorname{ot} f_\gamma[\delta] > \sup \text{S-supp}(p) \cap \operatorname{card} \gamma$ and strengthen p to q by setting $q_{\operatorname{ot} f_\gamma[\delta]} = q_\gamma$ and $q^{**}_\gamma = p^{**}_\gamma \cup \{\delta\}$. Let the other components of q be identical to the respective ones of p. Now $q\restriction\gamma^{\oplus}$ forces that q^{**}_γ is a condition in $C_\gamma(G_{\gamma+1})$, hence we may replace q with an equivalent q' (i.e. $q \leq q'$ and $q' \leq q$) such that $\Vdash_{P[\bar\alpha,\gamma)^\oplus} q^{**}_\gamma \in C_\gamma(G_{\gamma+1})$.

Now we consider the case that $\gamma < \bar\alpha^+$. Note that by an easy density argument, for ϵ either 0 or 1, $S_\epsilon := \{\xi < \bar\alpha \colon G_{\bar\alpha}(\xi) = \epsilon\}$ intersects every unbounded ground model subset of $\bar\alpha$ unboundedly often below $\bar\alpha$. Let $\epsilon \in \{0,1\}$ be s.t. $p\restriction\gamma \Vdash p_\gamma = \epsilon$, choose $\delta \geq \bar\delta$ such that $\operatorname{ot} f_\gamma[\delta] \in S_\epsilon$ and set $q^{**}_\gamma = p^{**}_\gamma \cup \{\delta\}$.

Finally assume we have to handle a whole set I as in the statement of the claim, i.e. for any given sequence of ordinals $\langle \bar\delta_\gamma \colon \gamma \in I\rangle$, we want to find $q \leq p$ such that for every $\gamma \in I$, $q\restriction\gamma^\oplus \Vdash q^{**}_\gamma \geq \bar\delta_\gamma$. We may assume that $p(\gamma) \neq \mathbf{1}$ for every $\gamma \in I$ as above. We may also assume that $p\restriction\gamma$ forces that p_γ has a $P[\bar\alpha, \xi_\gamma)$-name for some $\xi_\gamma < \operatorname{card}\gamma$ for every $\gamma \geq \bar\alpha^+$ in I as this is a dense property; this is shown as in the proof of 1$\to$2, using 5 instead of 1. For $\gamma < \bar\alpha^+$, note that (by the same argument) below every $\bar p \in P_{\bar\alpha}$ there exists $\bar q \leq \bar p$ and $q \leq p$ such that $\bar q$ forces that $q\restriction\gamma$ decides q_γ for every $\gamma < \bar\alpha^+$ in I, i.e. we may also assume that $p\restriction\gamma$ decides p_γ for every $\gamma < \bar\alpha^+$ in I. This allows us to handle all $\gamma \in [\bar\alpha, \bar\alpha^+)$ as above. To handle all $\gamma \geq \bar\alpha^+$, we simply have to choose a separating club C_θ for $I \cap [\theta, \theta^+)$ for every $\theta \geq \bar\alpha^+$ and choose δ_γ above both $\sup\{\bar\delta_\gamma \colon \gamma \in I \cap [\theta,\theta^+)\}$ and $\sup\{\min\{\alpha \colon \operatorname{ot} f_\gamma[\alpha] \geq \xi_\gamma\} \colon \gamma \in I \cap [\theta, \theta^+)\}$ inside $C_{\operatorname{card}\gamma}$ such that $\operatorname{ot} f_\gamma[\delta_\gamma] > \sup \text{S-supp}(p) \cap \operatorname{card}\gamma$ for every $\gamma \in I \cap [\theta,\theta^+)$ and choose the same δ_γ for all γ of cardinality θ. If $\operatorname{card}\gamma$ is inaccessible, we have to additionally make sure to choose δ_γ at least as big as $\operatorname{card}(I \cap [\theta,\theta^+))$. Then we may extend p to q as above, i.e. we let $q^{**}_\gamma = p^{**}_\gamma \cup \{\delta_\gamma\}$ for every $\gamma \in I$ and set $q_{\operatorname{ot} f_\gamma[\delta_\gamma]} = q_\gamma$ for every $\gamma \geq \bar\alpha^+$ in I. $\square_{\text{of theorem 8.16}}$

Corollary 8.18 *P preserves* ZFC*, cofinalities, cardinals and the* GCH.

Proof: Preservation of cofinalities, cardinals and the GCH is immediate. Note that whenever κ is singular, $P[\kappa,\infty)$, the iteration starting from κ, is κ^+-strategically closed and thus κ^+-distributive for definable sequences of dense classes. Now it can be seen easily from [2], section 2.2 that this suffices to show that P is tame and thus preserves ZFC. $\square$

Note: For every i of regular cardinality, $\bigcup_{p\in G} p^{**}_i$ is club in $\operatorname{card} i$ for any P-generic G. This is immediate from theorem 8.16, 11 above.

Claim 8.19 *P forces Local Club Condensation.*

Proof: Let G be P-generic. Let A be the generic predicate obtained from G, i.e. $\alpha \in A \leftrightarrow \exists p \in G\ p\restriction\alpha \Vdash p_\alpha = 1$. Note that $\mathbf{V}[G] = \mathbf{L}[A]$ as any set of ordinals in $\mathbf{V}$ is coded into A. We claim that $\langle M_\alpha \colon \alpha \in \mathrm{Ord}\rangle$ witnesses Local Club Condensation in $\mathbf{V}[G]$ with $M_\alpha = L_\alpha[A]$. If α has regular uncountable cardinality κ then Local Club Condensation is guaranteed by the forcing P: Note that for each $\beta \in \alpha \setminus \kappa$ we have $A(\beta) = A(\text{ot}\, f_\beta[\delta])$ for all δ in the club $\bigcup_{p\in G} p^{**}_\beta \subseteq \kappa$. It follows that for a club C of $\delta < \kappa$, $A(\beta) = A(\text{ot}\, f_\beta[\delta])$ and moreover $f_\beta[\delta] = f_\alpha[\delta] \cap \beta$ for all $\beta \in f_\alpha[\delta] \setminus \kappa$; this is seen using lemma 1.7. Let, as in lemma 4.3, F denote the function $(f, x) \mapsto f(x)$ whenever $f \in M_\alpha$ is a function with $x \in \text{dom}(f)$. Now let $M^*_\alpha = (M_\alpha, \in, \langle M_\beta \colon \beta < \alpha\rangle, F, \ldots)$ be a Skolemized structure for a countable language and for any $X \subseteq \alpha$ let $M^*_\alpha(X)$ be the least substructure of M^*_α containing X as a subset. Consider the continuous chain $\langle M^*_\alpha(f_\alpha[\delta]) \colon \delta \in D\rangle$, where D consists of all elements δ of C s.t. $\delta \subseteq f_\alpha[\delta] = M^*_\alpha(f_\alpha[\delta]) \cap \text{Ord}$ and $f_\alpha[\delta] \cap \kappa \in \text{Ord}$. Then $(M^*_\alpha(f_\alpha[\delta]), \in, \langle M_\beta \colon \beta \in M^*_\alpha(f_\alpha[\delta])\rangle)$ is isomorphic to some $(M_{\bar\alpha}, \in, \langle M_\beta \colon \beta < \bar\alpha\rangle)$ for each δ in D.

Finally we must verify Local Club Condensation for α when α has singular cardinality κ. Suppose that $\dot S \in \mathbf{V}$ is a name for a Skolemized structure $(M_\alpha, \in, \langle M_\beta \colon \beta < \alpha\rangle, R, F, \ldots)$ for a countable language in $\mathbf{L}[A]$ such that the $\dot S$-closure of κ is all of M_α, with F as above, R the given well-ordering of $\mathbf{V}$ (from which in particular the canonical functions $\langle f_i \colon i \in \text{Ord}\rangle$ were chosen). We show that any condition p has an extension q which forces that there is a continuous chain $\langle Y_\gamma \colon \gamma \in C\rangle$ of substructures of $\dot S$ whose domains $\langle y_\gamma \colon \gamma \in C\rangle$ have union M_α such that $\langle y_\gamma \cap \text{Ord} \colon \gamma \in C\rangle$ belongs to the ground model, where C is a closed unbounded subset of $\mathbf{Card} \cap \kappa$, each y_γ has cardinality γ, contains γ as a subset and each $(y_\gamma, \in, \langle M_\beta \colon \beta \in y_\gamma\rangle)$ is isomorphic to some $(M_{\bar\alpha}, \in, \langle M_\beta \colon \beta < \bar\alpha\rangle)$. Choose C to be any club subset of $\mathbf{Card} \cap \kappa$ of ordertype $\text{cof}\,\kappa$ whose minimum is either ω or a singular cardinal and is at least $\text{cof}\,\kappa$. Write C in increasing order as $\langle \gamma_i \colon i < \text{cof}\,\kappa\rangle$.

Claim 8.20 *For any condition $r \in P$, $\gamma \in C$ and any set of ordinals s of size γ, there is x in $\mathbf{V}$ of size γ such that some extension r' of r agreeing with r below* $\min C$ *forces that the set of ordinals in the $\dot S$-closure of s is covered by x and that $A \cap x$ has a P_ξ-name for some $\xi < \gamma^+$.*

Proof: An immediate application of theorem 8.16, 9 and 5. □

Proof of claim 8.19 continued: We want to find a sequence of conditions $\langle p^i \colon i < \text{cof}\,\kappa \cdot \omega\rangle$ with componentwise union r, following the strategy for ${\gamma_0}^+$-strategic closure of $P[\gamma_0, \alpha)$ with greatest lower bound q, letting $p^0 = p$, and a sequence of models $\langle Y_\gamma \colon \gamma \in C\rangle$ with domains $\langle y_\gamma \colon \gamma \in C\rangle$ with the following properties for every $\gamma \in C$:

(1) $\sup(\text{S-supp}(p^1) \cap \theta) > \sup(C \cap \theta)^+$ for every inaccessible $\theta \in (\gamma_0, \kappa)$,

(2) y_γ is transitive below γ^+,

(3) $y_\gamma \cap [\gamma, \gamma^+) = \text{S-supp}(r) \cap [\gamma, \gamma^+)$,

(4) $y_\gamma \cap [\gamma^+, \gamma^{++}) = \text{C-supp}(r) \cap [\gamma^+, \gamma^{++})$,

(5) q forces that the $\dot{S}$-closure of y_γ intersected with Ord equals y_γ and

(6) q forces that $A \cap y_\gamma$ has a $P_{y_\gamma \cap \gamma^+}$-name.

For if we are in such a situation, let π_γ be the collapsing map of y_γ. If $\xi \in y_\gamma \cap [\gamma^+, \gamma^{++})$, f_ξ is a bijection from γ^+ to ξ, hence $f_\xi \restriction (y_\gamma \cap \gamma^+)$ is a bijection from $y_\gamma \cap \gamma^+$ to $y_\gamma \cap \xi$ by elementarity, i.e. $\pi_\gamma(\xi) = \text{ot}(f_\xi[y_\gamma \cap \gamma^+])$, therefore $q(\pi_\gamma(\xi)) = r(\xi)$. Now extend q such that for every $\xi \in y_\gamma$, $\xi \geq \gamma^{++}$, we have $q(\pi_\gamma(\xi)) = r(\xi)$; this is possible since as the chain of p^i follows the strategy for strategic closure, it will be the case that whenever $\text{C-supp}(r) \cap [\theta, \theta^+) \neq \emptyset$ and θ is inaccessible, $\sup(r^{**}_\zeta) = \sup(\text{S-supp}(r) \cap \theta) > \sup(C \cap \theta)^+$ for every $\zeta \in \text{C-supp}(r) \cap [\theta, \theta^+)$, hence when we form q out of r and have to set $q(\text{ot}\, f_\zeta[\sup(r^{**}_\zeta)])$ to be equal to $q(\zeta)$ for $\zeta \in \text{C-supp}(r) \cap [\theta, \theta^+)$, we do not make any new requirements in the interval $[\gamma, \gamma^+)$ - note that $\text{ot}\, f_\zeta[\sup(r^{**}_\zeta)] \geq \sup(r^{**}_\zeta)$ as the p^i follow the strategy for strategic closure. We thus made sure that q forces that $(y_\gamma, \in, \langle M_\beta \colon \beta \in y_\gamma \rangle)$ is isomorphic to some $(M_{\bar{\alpha}}, \in, \langle M_\beta \colon \beta < \bar{\alpha} \rangle)$ for every $\gamma \in C$.

It remains to construct sequences $\langle p^i \colon i < \text{cof}\, \kappa \cdot \omega \rangle$ and $\langle Y_\gamma \colon \gamma \in C \rangle$ with the above-described properties: (1) is obviously easy to satisfy, noting that $\gamma_0 \geq \text{cof}\, \kappa$. We will concentrate on (2)-(6): Successively extend $p^0 = p$ without changing it below γ_0, following the strategy for ${\gamma_0}^+$-strategic closure of $u_{\gamma_0}(P_\alpha)$, to $p = p^0 \geq p^1 \geq \cdots$ in a sequence of length $\text{cof}\, \kappa + 1$ so that for each i we obtain $x_i \in \mathbf{V}$ of size γ_i, transitive below ${\gamma_i}^+$, such that p^{i+1} forces x_i to cover the set of ordinals in the $\dot{S}$-closure of $\sup(\text{S-supp}(p^i) \cap {\gamma_i}^+) \cup (\text{C-supp}(p^i) \cap [{\gamma_i}^+, {\gamma_i}^{++}))$. We can also arrange that $\langle x_i \colon i < \text{cof}\, \kappa \rangle$ forms a continuous, increasing sequence. Let $p' = p^{\text{cof}\, \kappa}$. Now perform a similar construction starting with p' instead of p, successively extending p' without changing it below γ_0 to $p' = {p'}^0 \geq {p'}^1 \geq \cdots$ in a sequence of length $\text{cof}\, \kappa + 1$ so that for each i and some x'_i in the ground model of size γ_i, ${p'}^{i+1}$ forces x'_i to cover the set of ordinals in the $\dot{S}$-closure of $\sup(\text{S-supp}({p'}^i) \cap {\gamma_i}^+) \cup (\text{C-supp}({p'}^i) \cap [{\gamma_i}^+, {\gamma_i}^{++}))$, but additionally at each step $i < \text{cof}\, \kappa$, make sure that ${p'}^{i+1}$ forces that $A \cap x_i$ has a P_{ξ_i}-name for some $\xi_i < {\gamma_i}^+$ and make sure that this ξ_i is contained (as an element) in $\text{S-supp}\, {p'}^{i+1}$; also make sure that $\text{S-supp}\, {p'}^{i+1} \supseteq x_i \cap [\gamma_i, {\gamma_i}^+)$ and that $\text{C-supp}\, {p'}^{i+1} \supseteq x_i \cap [{\gamma_i}^+, {\gamma_i}^{++})$. Let $p'' = {p'}^{\text{cof}\, \kappa}$. Continuing in this way, following the strategy for strategic closure, we construct an ω-sequence $\langle p', p'', \ldots, p^{(n)}, \ldots \rangle$ of decreasing conditions. Let r be the componentwise union of the $p^{(n)}$, $n < \omega$, let q be the greatest lower bound for the $p^{(n)}$, $n < \omega$. Let, for each $i < \text{cof}\, \kappa$, y_i :=the union of the ω-sequence $x_i \subseteq x'_i \subseteq \cdots$. Then $y_i \cap {\gamma_i}^{++} = \sup(\text{S-supp}(r) \cap {\gamma_i}^+) \cup (\text{C-supp}(r) \cap [{\gamma_i}^+, {\gamma_i}^{++}))$ and the $\dot{S}$-closure of $y_i \cap {\gamma_i}^{++}$ equals the $\dot{S}$-closure of y_i equals y_i. Moreover $\langle y_i \colon i < \text{cof}\, \kappa \rangle$ forms a continuous, increasing sequence and q forces that $A \cap y_i$ has a $P_{\sup(\text{S-supp}(r) \cap {\gamma_i}^+)}$-name for each $i < \text{cof}\, \kappa$. □

Theorem 8.21 *Local Club Condensation is consistent with the existence of an ω-superstrong cardinal.*

Proof: Assume κ is ω-superstrong, witnessed by the embedding $j\colon \mathbf{V} \to \mathbf{M}$. Let A be a well-ordering of V_κ (viewed as a function $A\colon \kappa \to V_\kappa$). We use A to build a well-ordering of $V_{j^\omega(\kappa)}$ as follows: By elementarity of j, $j(A)$ is a well-ordering of $M_{j(\kappa)}$ extending A. But $V_{j(\kappa)} = M_{j(\kappa)}$, hence $j(A)$ is in fact a well-ordering of $V_{j(\kappa)}$. Similarly, $j(j(A))$ is a well-ordering of $V_{j^2(\kappa)}$ extending $j(A)$. Going on like this for ω steps, using that $V_{j^\omega(\kappa)} = M_{j^\omega(\kappa)}$, we obtain a well-ordering $B := \bigcup_{n\in\omega} j^n(A)$ of $V_{j^\omega(\kappa)}$ such that $j(B) = B$. Now we perform a class forcing T to add a predicate R extending B which well-orders $\mathbf{V}$: A condition in T is a function f from an ordinal into $\mathbf{V}$ extending B; f is stronger than g in T iff f extends g. Forcing with T does not add new sets and adds a predicate R which well-orders $\mathbf{V}$ with the property that $j(R{\restriction}j^\omega(\kappa)) = R{\restriction}j^\omega(\kappa)$. Since no new sets are added, j is an elementary embedding from $(\mathbf{V}, R)$ to $(\mathbf{M}, j(R))$ with $j(R) := \bigcup_{\alpha\in\mathrm{Ord}} j(R{\restriction}\alpha)$.

Let P be the Local Club Condensation forcing relative to R as defined at the beginning of this section, letting, for each ordinal γ, f_γ be the R-least bijection from the cardinality of γ to γ. Let $\langle f^*_\gamma\colon \gamma \in \mathrm{Ord}\rangle$ denote the $\mathbf{M}$-version of $\langle f_\gamma\colon \gamma \in \mathrm{Ord}\rangle$ - letting each f^*_γ be the $j(R)$-least bijection from the cardinality of γ in $\mathbf{M}$ to γ. Let P^* denote the $\mathbf{M}$-version of P (using the definition of P in $\mathbf{M}$ relative to $\langle f^*_\gamma\colon \gamma \in \mathrm{Ord}\rangle$). Note that by our choice of R, $f_\gamma = f^*_\gamma$ for $\gamma < j^\omega(\kappa)$ and hence we made sure that for every $n < \omega$, $P_{j^n(\kappa)} = P^*_{j^n(\kappa)}$. We want to find a $\mathbf{V}$-generic $G \subseteq P$ and an $\mathbf{M}$-generic $G^* \subseteq P^*$ such that $j''G \subseteq G^*$ and $V[G]_{j^\omega(\kappa)} \subseteq M[G^*]$. Let $G_{j(\kappa)}$ be generic for $P_{j(\kappa)}$, let $G^*_{j(\kappa)} = G_{j(\kappa)}$. Trivially, $j''G_\kappa = G_\kappa \subseteq G_{j(\kappa)}$ and thus we may lift j to $j^*\colon \mathbf{V}[G_\kappa] \to \mathbf{M}[G_{j(\kappa)}]$. For simplicity of notation, we will denote j^* (and any further liftings of j^*) by j again. We want to show that we can arrange that for every $n \in \omega$, $j''G[j^n(\kappa), j^{n+1}(\kappa))$ has a lower bound in $P[j^{n+1}(\kappa), j^{n+2}(\kappa))$ which is contained in $G[j^{n+1}(\kappa), j^{n+2}(\kappa))$. We will then set $G^*_{j^n(\kappa)} = G_{j^n(\kappa)}$ for every $n \in \omega$. We start with $j''G[\kappa, j(\kappa))$. Let r be such that for every $\gamma \in [j(\kappa), j^2(\kappa))$,

- $r_\gamma = \bigcup_{p\in G[\kappa,j(\kappa))} j(p)_\gamma$,
- $r^{**}_\gamma = \bigcup_{p\in G[\kappa,j(\kappa))} j(p)^{**}_\gamma$.

To simplify notation, we will abbreviate this in the following as

$$r = \bigcup_{p\in G[\kappa,j(\kappa))} j(p),$$

an obvious abuse of notation, thinking of $\bigcup$ as the componentwise union here. We will use similar abbreviations in similar cases. As we did earlier, we write S-supp(r) for $\{\gamma\colon r_\gamma \neq \check{\mathbf{1}}\}$ and C-supp(r) for $\{\gamma\colon r^{**}_\gamma \neq \check{\mathbf{1}}\}$. We first want to show that S-supp(r) is bounded below every regular cardinal and that card(C-supp$(r) \cap [\theta, \theta^+)) < \theta$ for every regular cardinal θ.

Assume $\theta \in [j(\kappa)^+, j^2(\kappa)]$ is regular.

$$\text{S-supp}(r) \cap \theta = \bigcup_{p \in G[\kappa, j(\kappa))} \text{S-supp}(j(p)) \cap \theta.$$

But for every $p \in G[\kappa, j(\kappa))$, $j(p) \in P[j(\kappa), j^2(\kappa))$, so S-supp$(j(p)) \cap \theta$ is bounded below θ, hence using that $P[\kappa, j(\kappa))$ has a dense subset of size $j(\kappa)$ and $\theta > j(\kappa)$ is regular, it follows that S-supp$(r) \cap \theta$ is bounded in θ.

Claim 8.22 C-supp$(r) \cap [j(\kappa), j(\kappa)^+) = j''[\kappa, \kappa^+)$.

Proof: Assume $\gamma \in$ C-supp$(r) \cap [j(\kappa), j(\kappa)^+)$. Then $\gamma \in$ C-supp$(j(p)) \cap [j(\kappa), j(\kappa)^+) = j($C-supp$(p) \cap [\kappa, \kappa^+))$ for some $p \in G[\kappa, j(\kappa))$.

But C-supp$(p) \cap [\kappa, \kappa^+)$ has order-type less than κ, thus $j($C-supp$(p) \cap [\kappa, \kappa^+)) = j''($C-supp$(p) \cap [\kappa, \kappa^+))$. □

We have thus shown that C-supp$(r) \cap j(\kappa)^+$ has size $\kappa^+ < j(\kappa)$. Assume now that $\theta \in [j(\kappa)^{++}, j^2(\kappa))$ is a successor of a regular cardinal:

$$\text{C-supp}(r) \cap \theta = \bigcup_{p \in G[\kappa, j(\kappa))} \text{C-supp}(j(p)) \cap \theta.$$

It follows as for the string support above that card(C-supp$(r) \cap \theta) < \theta^-$.

Having shown that r has appropriate supports, we want to form q^ξ out of r for every $\xi \in [j(\kappa), j^2(\kappa)]$ by setting, for every $\gamma \in$ C-supp(r) below ξ:

- $(q^\xi)^{**}_\gamma = r^{**}_\gamma \cup \sup r^{**}_\gamma$ and
- $(q^\xi)_{\operatorname{ot} f_\gamma[\sup r^{**}_\gamma]} = r_\gamma$ if $\operatorname{card} \gamma > j(\kappa)$.

Of course we want to set $(q^\xi)_\gamma = r_\gamma$ for $\gamma < \xi$, γ in S-supp(r). We want to show, by induction on ξ, that q^ξ is a condition in $P[j(\kappa), \xi)$ for every $\xi \in [j(\kappa), j^2(\kappa)]$. In that case, each q^ξ is a lower bound for $\{j(p)\restriction\xi\colon p \in G[\kappa, j(\kappa))\}$ and $q := q^{j^2(\kappa)}$ is then the desired lower bound for $j''G[\kappa, j(\kappa))$. For each ξ as above, let $(q^\xi)^\oplus$ be such that $(q^\xi)^\oplus_\xi = r_\xi$ and $(q^\xi)^\oplus\restriction\xi = q^\xi$. If q^ξ is a condition in $P[j(\kappa), \xi)$, then $(q^\xi)^\oplus$ is a condition in $P[j(\kappa), \xi)^\oplus$.

Claim 8.23 $\forall \gamma \in [\kappa, \kappa^+)$ $\operatorname{ot} j(f_\gamma)[\kappa] = \gamma$.

Proof: If $\alpha < \kappa$, then $j(f_\gamma)(\alpha) = j(f_\gamma(\alpha))$, thus $j(f_\gamma)[\kappa] = j''f_\gamma[\kappa]$, which has order-type γ as j is order-preserving. □

Claim 8.24 *If* $\xi \in \text{C-supp}(r) \cap [j(\kappa), j(\kappa)^+)$, *then*

$$q^\xi \Vdash G(ot f_\xi[\sup r^{**}_\xi]) = r_\xi.$$

Proof: Note that $\sup r^{**}_\xi = \kappa$. Let γ be such that $j(\gamma) = \xi$, $\gamma \in [\kappa, \kappa^+)$. Then $\text{ot}\, f_\xi[\kappa] = \gamma$. Let $p \in G[\kappa, j(\kappa))$ such that $p{\restriction}\gamma$ decides p_γ. Then $q^\xi \leq j(p){\restriction}\xi \Vdash r_\xi = j(p_\gamma)$ and $G(\gamma) = p_\gamma$. □

We have thus shown that $q^{j(\kappa)^+}$ is a condition in $P[j(\kappa), j(\kappa)^+)$. Now assume ξ has regular cardinality $\theta \in [j(\kappa)^+, j^2(\kappa))$, $\xi \in \text{C-supp}(r)$.

Claim 8.25 $(q^\xi)^\oplus \Vdash \sup r^{**}_\xi \geq \sup(\text{range}\, j \cap \theta)$.

Proof: $\exists p \in G[\kappa, j(\kappa))\ \xi \in \text{C-supp}(j(p))$. For every δ,

$$D_\delta := \{t \in P[\kappa, j(\kappa)) \colon \forall i \geq \delta^+\ i \in \text{C-supp}(t) \to t \Vdash \max t^{**}_i \geq \delta\}$$

is dense in $P[\kappa, j(\kappa))$. Assume $\beta < \theta$, $\beta \in \text{range}(j)$ and choose $t \leq p$ in $D_{j^{-1}(\beta)} \cap G[\kappa, j(\kappa))$. Then $\forall i \geq \theta\ i \in \text{C-supp}(j(t)) \to j(t){\restriction}i^\oplus \Vdash \max j(t)^{**}_i \geq \beta$. Thus $(q^\xi)^\oplus \Vdash \sup r^{**}_\xi \geq \sup(\text{range}\, j \cap \theta)$. □

Claim 8.26
If $\gamma \in \text{C-supp}(r)$ *has cardinality* θ, $\gamma < \xi$, *then* $(q^\xi)^\oplus \Vdash \sup r^{**}_\gamma = \sup r^{**}_\xi$.

Proof: Assume $\exists u \leq (q^\xi)^\oplus\ u \Vdash \sup r^{**}_\gamma < \sup r^{**}_\xi$. Then there is $p \in G[\kappa, j(\kappa))$ with $u \Vdash \max j(p)^{**}_\xi > \sup r^{**}_\gamma$. We may assume $\gamma \in \text{C-supp}(j(p))$. $D := \{t \leq p \colon \forall \eta \forall \delta \in \text{C-supp}(p) \cap [\eta, \eta^+)\ t \Vdash \max t^{**}_\delta > \sup\{\max p^{**}_i \colon i \in \text{C-supp}(p) \cap [\eta, \eta^+)\}\}$ is dense below p. Choose $t \in D \cap G[\kappa, j(\kappa))$. Then $(q^\xi)^\oplus \leq j(t){\restriction}\xi^\oplus \Vdash \max j(t)^{**}_\gamma > j(p)^{**}_\xi$, hence $u \Vdash \max j(t)^{**}_\gamma > \sup r^{**}_\gamma$, a contradiction. Assuming that $\exists u \leq (q^\xi)^\oplus\ u \Vdash \sup r^{**}_\gamma > \sup r^{**}_\xi$ analogously leads to a contradiction. □

Claim 8.27 *If* $\gamma \in \text{C-supp}(r)$ *has cardinality* θ, $\gamma < \xi$, *then*

$$(q^\xi)^\oplus \Vdash \text{ot}\, f_\gamma[\sup r^{**}_\gamma] < \text{ot}\, f_\xi[\sup r^{**}_\xi].$$

Proof: Choose $p \in G[\kappa, j(\kappa))$ with γ, ξ both in $\text{C-supp}(j(p))$. We already know that $(q^\xi)^\oplus \Vdash \sup r^{**}_\gamma = \sup r^{**}_\xi$. Let $u \leq (q^\xi)^\oplus$ decide $\sup r^{**}_\xi$ and denote that value by s. We want to show that u forces that there exists a separating club for $\text{C-supp}(r) \cap [\theta, \theta^+)$ which contains s. Choose $C = \langle C_\eta \colon \text{C-supp}(p) \cap [\eta, \eta^+) \neq \emptyset \rangle$ such that for every cardinal η, C_η is a separating club for $\text{C-supp}(p) \cap [\eta, \eta^+)$. Then $j(C) = \langle E_\eta \colon \text{C-supp}(j(p)) \cap [\eta, \eta^+) \neq \emptyset \rangle$ is such that for every cardinal η, E_η is a separating club for $\text{C-supp}(j(p)) \cap [\eta, \eta^+)$.

Assume for a contradiction that E_θ is bounded in s by some $\alpha < s$. Choose $t \leq p$ in $G[\kappa, j(\kappa))$ such that $t \Vdash \alpha \leq \max j(t)^{**}_\gamma = \max j(t)^{**}_\xi \in E_\theta$.

This is possible since $\exists p' \leq p$ in $G[\kappa, j(\kappa))$ such that $j(p')^{**}_\gamma \geq \alpha$ and $D := \{t\colon \forall \eta\, \forall \delta_0, \delta_1 \in \text{C-supp}(t) \cap [\eta, \eta^+)\ t \Vdash \max t^{**}_{\delta_0} = \max t^{**}_{\delta_1} \in C_\eta\}$ is dense in $P[\kappa, j(\kappa))$, so we may choose $t \in D \cap G[\kappa, j(\kappa))$ below p'. t is then as desired. But $u \Vdash \max j(t)^{**}_\gamma \leq \sup r^{**}_\gamma = s$, thus u forces that E_θ is not bounded by α below s, a contradiction as desired. Therefore $u \Vdash s \in E_\theta$, a separating club for $\text{C-supp}(j(p)) \cap [\theta, \theta^+) \supseteq \{\gamma, \xi\}$. $\Box$

Claim 8.28 $(q^\xi)^\oplus \Vdash \text{ot}\, f_\xi[\sup r^{**}_\xi] \geq \sup \text{S-supp}(r) \cap \theta$.

Proof: Note that $\sup \text{S-supp}(r) \cap \theta$ is a limit ordinal and assume for a contradiction that $\exists u \leq (q^\xi)^\oplus\ u \Vdash \text{ot}\, f_\xi[\sup r^{**}_\xi] < \alpha < \sup \text{S-supp}(r) \cap \theta$ for some α. Choose $p \in G[\kappa, j(\kappa))$ such that $\sup(\text{S-supp}(j(p)) \cap \theta) \geq \alpha$ and $\xi \in \text{C-supp}(j(p))$. Now note that $D := \{t\colon t \Vdash \forall \eta\, \forall \delta \in \text{C-supp}(p) \cap [\eta, \eta^+)\ \max t^{**}_\delta \geq \sup(\text{S-supp}(p) \cap \eta)$ and $f_\delta[\max t^{**}_\delta] \supseteq \max t^{**}_\delta\}$ is dense in $P[\kappa, j(\kappa))$ below p. Choose $t \in D \cap G[\kappa, j(\kappa))$. Then $j(t) \Vdash \max(j(t)^{**}_\xi) \geq \sup(\text{S-supp}(j(p)) \cap \theta) \geq \alpha$ and $f_\xi[\max j(t)^{**}_\xi] \supseteq \max j(t)^{**}_\xi$. Thus $(q^\xi)^\oplus \leq j(t) \restriction \xi^\oplus \Vdash \text{ot}\, f_\xi[\sup r^{**}_\xi] \geq \text{ot}\, f_\xi[\max(j(t)^{**}_\xi)] \geq \alpha$, a contradiction. $\Box$

Claim 8.29
If θ is inaccessible, then $\sup(\text{S-supp}(r) \cap \theta) \geq \text{card}(\text{C-supp}(r) \cap [\theta, \theta^+))$.

Proof: $D := \{p\colon \forall \eta$ inaccessible $\sup(\text{S-supp}(p) \cap \eta) \geq \text{card}(\text{C-supp}(p) \cap [\eta, \eta^+))\}$ is dense in $P[\kappa, j(\kappa))$. Hence

$$\sup(\text{S-supp}(r) \cap \theta) = \bigcup_{p \in G[\kappa, j(\kappa))} \sup(\text{S-supp}(j(p)) \cap \theta)$$

is greater or equal than

$$\bigcup_{p \in G[\kappa, j(\kappa))} \text{card}(\text{C-supp}(j(p)) \cap [\theta, \theta^+)).$$

So for every $p \in G[\kappa, j(\kappa))$, $\sup(\text{S-supp}(r) \cap \theta) \geq \text{card}(\text{C-supp}(j(p)) \cap [\theta, \theta^+))$. As $P[\kappa, j(\kappa))$ has a dense subset of size $j(\kappa)$, it suffices to show that $\sup(\text{S-supp}(r) \cap \theta) \geq j(\kappa)$, which is true as $j(\kappa) \in \text{S-supp}(r)$. $\Box$

Claim 8.30 *q^ξ forces that r_ξ has a $P[j(\kappa), \sup(\text{S-supp}(r) \cap \theta))$-name.*

Proof: Choose $p \in G[\kappa, j(\kappa))$ such that $\xi \in \text{C-supp}(j(p))$.
Note that $D := \{t \leq p\colon \forall \eta\, \forall \delta \in \text{C-supp}(p) \cap [\eta, \eta^+)\ t \restriction \delta$ forces that t_δ has a $P[\kappa, \sup(\text{S-supp}(t) \cap \eta))$-name$\}$ is dense in $P[\kappa, j(\kappa))$ below p.

Choose $t \in D \cap G[\kappa, j(\kappa))$. Then $j(t) \restriction \xi$ forces that $j(t)_\xi = r_\xi$ has a $P[j(\kappa), \sup(\text{S-supp}(j(t)) \cap \theta))$-name. The claim follows as $\sup(\text{S-supp}(j(t)) \cap \theta) \leq \sup(\text{S-supp}(r) \cap \theta)$. $\Box$

Now by the above claims, we may set $q_{\mathrm{ot}\, f_\xi[\sup r^{**}_\xi]} = r_\xi$ and $q^{**}_\xi = r^{**}_\xi \cup \{\sup r^{**}_\xi\}$, i.e. given that q^ξ is a condition in $P[j(\kappa), \xi)$, we get that $q^{\xi+1}$ is a condition in $P[j(\kappa), \xi+1)$. If ξ is a limit ordinal, q^ξ is a condition in $P[j(\kappa), \xi)$, as for each $\zeta < \xi$, $q^\xi \restriction \zeta$ is a condition in $P[j(\kappa), \zeta)$ inductively and q^ξ has appropriate supports. So we finally obtain $q \in P[j(\kappa), j^2(\kappa))$ which is below $j''G[\kappa, j(\kappa))$, our desired master condition. If we choose our $P[j(\kappa), j^2(\kappa))$-generic $G[j(\kappa), j^2(\kappa))$ to contain q we have ensured that $j''G[\kappa, j(\kappa)) \subseteq G[j(\kappa), j^2(\kappa))$ and we may thus lift the embedding $j\colon \mathbf{V}[G_\kappa] \to \mathbf{M}[G_{j(\kappa)}]$ to $j\colon \mathbf{V}[G_{j(\kappa)}] \to \mathbf{M}[G_{j^2(\kappa)}]$. But in order to be able to further lift the embedding j, we have to demand a little more from $G[j(\kappa), j^2(\kappa))$: We will define a condition $t \in P[j(\kappa), j^2(\kappa))$, show that t and q are compatible, demand that $G[j(\kappa), j^2(\kappa))$ contains both t and q and show how this helps us to obtain that $j''G[j(\kappa), j^2(\kappa))$ has a lower bound in $P[j^2(\kappa), j^3(\kappa))$. This will finally enable us to lift $j\colon \mathbf{V}[G_{j(\kappa)}] \to \mathbf{M}[G_{j^2(\kappa)}]$ to $j\colon \mathbf{V}[G_{j^2(\kappa)}] \to \mathbf{M}[G_{j^3(\kappa)}]$. The further liftings of j up to $j\colon \mathbf{V}[G_{j^\omega(\kappa)}] \to \mathbf{M}[G_{j^\omega(\kappa)}]$ then work the same way (more strictly speaking, it will be immediate to find $q \in P_{j^\omega(\kappa)}$ such that if we demand that $q \in G_{j^\omega(\kappa)}$, then $j''G_{j^\omega(\kappa)} \subseteq G_{j^\omega(\kappa)}$).

- Let $c := \bigcup\{j(A)\colon A \subseteq [j(\kappa), j(\kappa)^+), |A| < j(\kappa)\}$,
- let $d := \sup(\mathrm{range}(j) \cap j^2(\kappa))$.

Note: Whichever $G[j(\kappa), j^2(\kappa))$ we choose, if we then let

$$r := \bigcup_{p \in G[j(\kappa), j^2(\kappa))} j(p),$$

it will be the case that

$$\text{C-supp}(r) \cap [j^2(\kappa), j^2(\kappa)^+) = c$$

and for $\gamma \in \text{C-supp}(r) \cap [j^2(\kappa), j^2(\kappa)^+)$, $\sup r^{**}_\gamma = d$.

Definition of t: For every $\gamma \in c$, let A_γ be a maximal antichain in $P[j(\kappa), j(\kappa)^+)$ which j-decides the bit at γ, in the sense that for every $a \in A_\gamma$, $j(a) \restriction \gamma$ decides $j(a)_\gamma$: this is possible as the set D of conditions p in $P[j(\kappa), j(\kappa)^+)$ such that p decides $\{p_\delta\colon \delta \in \text{C-supp}(p)\}$ and such that $\gamma \in \text{C-supp}(j(p))$ is dense in $P[j(\kappa), j(\kappa)^+)$. But for any such p, $j(p)$ decides $j(p)_\gamma$ by elementarity. Now we let, for every $\gamma \in c$,

$$t_{\mathrm{ot}\, f_\gamma[d]} := \{(a, \epsilon)\colon a \in A_\gamma \wedge j(a) \Vdash j(a)_\gamma = \epsilon\}.$$

Similar to claim 8.27, one may show that $\mathrm{ot}\, f_\gamma[d]$ is different for different $\gamma \in c$. We let $t_\delta = \mathbf{1}$ for all δ which are not as above and let $t^{**}_\delta = \emptyset$ for all δ. Note that each t_δ is a $P[j(\kappa),\delta)$-name, since $d > j(\kappa)^+$. We need to show that t has sufficiently small supports in order to be a condition in $P[j(\kappa), j^2(\kappa))$. The following is clearly sufficient:

Claim 8.31 card$(c) \leq d$.

Proof: For each $A \subseteq [j(\kappa), j(\kappa)^+)$ of size less than $j(\kappa)$, $\mathrm{card}(j(A)) \in \mathrm{range}(j) \cap j^2(\kappa)$. There are only $j(\kappa)^+$-many possibilities for A and thus the claim follows as $d > j(\kappa)^+$. □

$t \parallel q$**:** For $\gamma \in c$, $\mathrm{ot}\, f_\gamma[d] \geq d$. It suffices to note that whenever $\delta \in$ S-supp(q), then $\delta < d$. This allows us to demand that $G[j(\kappa), j^2(\kappa))$ contains both q and t.

lifting:
We want to lift $j\colon \mathbf{V}[G_{j(\kappa)}] \to \mathbf{M}[G_{j^2(\kappa)}]$ to $j\colon \mathbf{V}[G_{j^2(\kappa)}] \to \mathbf{M}[G_{j^3(\kappa)}]$. Let $r = \bigcup_{p \in G[j(\kappa), j^2(\kappa))} j(p)$. As before, one shows that r has appropriate supports. We want to form $\tilde{q}$ out of r by setting, for every $\gamma \in$ C-supp(r):

- $\tilde{q}^{**}_\gamma = r^{**}_\gamma \cup \sup r^{**}_\gamma$ and
- $\tilde{q}_{\mathrm{ot}\, f_\gamma[\sup r^{**}_\gamma]} = r_\gamma$ if $\mathrm{card}\, \gamma > j(\kappa)$.

Of course we want to set $\tilde{q}_\gamma = r_\gamma$ for γ in S-supp(r). We want to show that $\tilde{q}$ is a condition in $P[j^2(\kappa), j^3(\kappa))$. In that case, $\tilde{q}$ is obviously a lower bound for $j''G[j(\kappa), j^2(\kappa))$. Note that since $t \in G[j(\kappa), j^2(\kappa))$, we have that $G(\mathrm{ot}\, f_\gamma[\sup r^{**}_\gamma]) = r_\gamma$ for every $\gamma \in [j^2(\kappa), j^2(\kappa)^+)$ (to be exact, there exists $p \in G[j(\kappa), j^2(\kappa))$ such that $j(p) \restriction \gamma$ decides r_γ and thus forces the above), which shows that $\tilde{q} \restriction j^2(\kappa)^+$ is a condition in $P[j^2(\kappa), j^2(\kappa)^+)$. The rest of the proof that $\tilde{q}$ is a condition in $P[j^2(\kappa), j^3(\kappa))$ works as the proof for q above.

master condition: Continue as above for ω-many steps, in this way defining a master condition $u \in P_{j^\omega(\kappa)}$ with the property that $u \leq j''G_{j^\omega(\kappa)}$ and choose a $P_{j^\omega(\kappa)}$-generic $G_{j^\omega(\kappa)}$ containing u. Let $G^*_{j^\omega(\kappa)} := G_{j^\omega(\kappa)} \cap P^*_{j^\omega(\kappa)}$.

Claim 8.32 *$G^*_{j^\omega(\kappa)}$ is $P^*_{j^\omega(\kappa)}$-generic over* $\mathbf{M}$.

Proof: Suppose $D \in \mathbf{M}$ is open dense on $P^*_{j^\omega(\kappa)}$ and write D as $j(f)(a)$ where $\mathrm{dom}(f) = V_{j^\omega(\kappa)}$ and $a \in V_{j^{n+1}(\kappa)}$ for some $n \in \omega$. We may assume that every element of $\mathbf{M}$ is of this form. Choose $p \in G_{j^\omega(\kappa)}$ such that p reduces $f(\bar{a})$ below $j^n(\kappa)$ whenever $\bar{a}$ belongs to $V_{j^n(\kappa)}$ and $f(\bar{a})$ is open dense on $P_{j^\omega(\kappa)}$, in the sense that if q extends p then q can be further extended into $f(\bar{a})$ without changing $u_{j^n(\kappa)}(q)$. The existence of p as above is shown similar to the proof of theorem 8.16, 5, using that $V_{j^n(\kappa)}$ has size $j^n(\kappa)$. Then $j(p)$

belongs to $j''G_{j^\omega(\kappa)} \subseteq G^*_{j^\omega(\kappa)}$ and reduces D below $j^{n+1}(\kappa)$, i.e. if $q \leq j(p)$ then $\exists r \leq q\ r \in D \wedge u_{j^{n+1}(\kappa)}(r) = u_{j^{n+1}(\kappa)}(q)$.

Hence $E := \{q \in P_{j^{n+2}(\kappa)} \colon q^\frown j(p)[j^{n+2}(\kappa), j^\omega(\kappa)) \in D\}$ is dense below $j(p)\restriction j^{n+2}(\kappa)$ in $P_{j^{n+2}(\kappa)}$. Since $G_{j^{n+2}(\kappa)}$ contains $j(p)\restriction j^{n+2}(\kappa)$ and is $P_{j^{n+2}(\kappa)}$-generic over $\mathbf{M}$, $G_{j^{n+2}(\kappa)} \cap E \neq \emptyset$. Choose a condition q in that intersection. Then $q^\frown j(p)[j^{n+2}(\kappa), j^\omega(\kappa)) \in D \cap G^*_{j^\omega(\kappa)}$. $\square$

By the above, we obtain a lifted embedding $j\colon \mathbf{V}[G_{j^\omega(\kappa)}] \to \mathbf{M}[G^*_{j^\omega(\kappa)}]$. As $P[j^\omega(\kappa), \infty)$ is $j^\omega(\kappa)^+$-distributive by theorem 8.16, we may choose an arbitrary $P[j^\omega(\kappa), \infty)$-generic $G[j^\omega(\kappa), \infty)$, assume that j is given by an ultrapower and apply lemma 3 of [3] to find a P^*-generic G^* extending $G^*_{j^\omega(\kappa)}$ and an elementary embedding $j\colon \mathbf{V}[G] \to \mathbf{M}[G^*]$ extending $j\colon \mathbf{V} \to \mathbf{M}$. As $\mathbf{V}[G]_{j^\omega(\kappa)} = \mathbf{V}_{j^\omega(\kappa)}[G^*_{j^\omega(\kappa)}] \subseteq \mathbf{M}[G^*]$, j witnesses ω-superstrength of κ in $\mathbf{V}[G]$. $\square_{\text{theorem 8.21}}$

For possible applications (see also section 9), it may also be interesting that we can preserve a variety of smaller large cardinals while forcing Local Club Condensation. There are many kinds of cardinals which may be preserved, we give some examples in the following:

Definition 8.33 *A cardinal κ is called ω-hyperstrong iff $\exists j\colon \mathbf{V} \to \mathbf{M}$ with $\operatorname{crit}(j) = \kappa$ and $H_{j(\kappa)^{+\omega}} \subseteq \mathbf{M}$, where $j(\kappa)^{+\omega}$ denotes the ω^{th} cardinal successor of $j(\kappa)$. We may equivalently demand that $V_{j(\kappa)+\omega} \subseteq \mathbf{M}$.*

Theorem 8.34 *Assume* GCH *holds and κ is ω-hyperstrong. Then we may force Local Club Condensation and preserve the ω-hyperstrength of κ.*

Proof: The proof of this theorem is very similar to and easier than the proof of the preservation of an ω-superstrong cardinal above: Assume κ is ω-hyperstrong, witnessed by the embedding $j\colon \mathbf{V} \to \mathbf{M}$. Let A be a well-ordering of V_κ (viewed as a bijection $A\colon \kappa \to V_\kappa$). We use A to obtain a well-ordering of $\mathbf{V}$ as follows: By elementarity of j, $j(A)$ is a well-ordering of $M_{j(\kappa)}$ extending A. But $V_{j(\kappa)} = M_{j(\kappa)}$, hence $j(A)$ is in fact a well-ordering of $V_{j(\kappa)}$. $B := j(j(A))$ is thus a well-ordering of $H_{j^2(\kappa)}{}^{\mathbf{M}}$ extending $j(A)$. Now we perform a class forcing T to add a predicate R extending B which well-orders $\mathbf{V}$: A condition in T is a function f from an ordinal into $\mathbf{V}$ extending B; f is stronger than g in T iff f extends g. Forcing with T does not add new sets and adds a predicate R which well-orders $\mathbf{V}$. Since no new sets are added, j is an elementary embedding from $(\mathbf{V}, R)$ to $(\mathbf{M}, j(R))$ with $j(R) := \bigcup_{\alpha \in \mathrm{Ord}} j(R\restriction\alpha)$; we ensured that $j(R)$ and R agree on how to well-order $H_{j(\kappa)^{+\omega}}$.

Let P be the Local Club Condensation forcing relative to R as defined at the beginning of this section, letting, for each ordinal γ, f_γ be the R-least bijection from the cardinality of γ to γ. Let $\langle f^*_\gamma \colon \gamma \in \mathrm{Ord}\rangle$ denote the $\mathbf{M}$-version of $\langle f_\gamma \colon \gamma \in \mathrm{Ord}\rangle$ - letting each f^*_γ be the $j(R)$-least bijection from the cardinality of γ in $\mathbf{M}$ to γ. Let P^* denote the $\mathbf{M}$-version of P (using the definition of P in $\mathbf{M}$ relative to $\langle f^*_\gamma \colon \gamma \in \mathrm{Ord}\rangle$). Note that by our choice of R, $f_\gamma = f^*_\gamma$ for $\gamma < j(\kappa)^{+\omega}$ and hence we made sure that for every $n < \omega$, $P_{j(\kappa)^{+n}} = P^*_{j(\kappa)^{+n}}$. We want to find a $\mathbf{V}$-generic $G \subseteq P$ and an $\mathbf{M}$-generic $G^* \subseteq P^*$ such that $j''G \subseteq G^*$ and $(H_{j(\kappa)^{+\omega}})^{\mathbf{V}[G]} \subseteq M[G^*]$. Let $G_{j(\kappa)}$ be generic for $P_{j(\kappa)}$, let $G^*_{j(\kappa)} = G_{j(\kappa)}$. Trivially, $j''G_\kappa \subseteq G_{j(\kappa)}$ and thus we may lift j to $j^* \colon \mathbf{V}[G_\kappa] \to \mathbf{M}[G_{j(\kappa)}]$. For simplicity of notation, we will denote j^* (and any further liftings of j^*) by j again. We want to show that $j''G[\kappa, \kappa^{+\omega})$ has a lower bound in $P^*[j(\kappa), j(\kappa)^{+\omega})$. Note first that $(j(\kappa)^{+n})^{\mathbf{M}} = j(\kappa)^{+n}$ and thus $j''\kappa^{+n}$ is bounded in $j(\kappa)^{+n}$ for every $n \in \omega$. Let $r := \bigcup_{p \in G[\kappa, j(\kappa))} j(p)$.

Claim 8.35 S-supp(r) *is bounded below* $j(\kappa)^{+n}$ *for every* $n \in \omega$.

Proof:

$$\text{S-supp}(r) \cap [j(\kappa)^{+n}, j(\kappa)^{+n+1}) =$$
$$= \bigcup_{p \in G[\kappa^{+n}, \kappa^{+n+1})} j(\text{S-supp}(p)) \subseteq \bigcup_{\alpha < \kappa^{+n+1}} j(\alpha) < j(\kappa)^{+n+1}. \ \square$$

Claim 8.36 C-supp$(r) \cap [j(\kappa), j(\kappa)^+) = j''[\kappa, \kappa^+)$.

Proof: As in the proof of theorem 8.21. □

Claim 8.37 card(C-supp$(r) \cap j(\kappa)^{+n+1}) < j(\kappa)^{+n}$ *for every* $n \in \omega$.

Proof:

$$\text{C-supp}(r) \cap [j(\kappa)^{+n}, j(\kappa)^{+n+1}) = \bigcup_{p \in G[\kappa^{+n}, \kappa^{+n+1})} j(\text{C-supp}(p)).$$

But for every $p \in G[\kappa^{+n}, \kappa^{+n+1})$, $j(\text{C-supp}(p))$ has size $< j(\kappa)^{+n}$ and $P_{\kappa^{+n+1}}$ has a dense subset of size κ^{+n+1}. □

Having shown that r has appropriate supports, we want to form q^ξ out of r for every $\xi \in [j(\kappa), j(\kappa)^{+\omega}]$ by setting, for every $\gamma \in$ C-supp(r) below ξ:

- $(q^\xi)^{**}_\gamma = r^{**}_\gamma \cup \sup r^{**}_\gamma$ and
- $(q^\xi)_{\mathrm{ot}\, f_\gamma[\sup r^{**}_\gamma]} = r_\gamma$ if $\mathrm{card}\, \gamma > j(\kappa)$.

Of course we want to set $(q^\xi)_\gamma = r_\gamma$ for $\gamma < \xi$, γ in S-supp(r). We want to show, by induction on ξ, that q^ξ is a condition in $P[j(\kappa), \xi)$ for every $\xi \in [j(\kappa), j(\kappa)^{+\omega}]$. In that case, each q^ξ is a lower bound for $\{j(p)\restriction\xi \colon p \in G[\kappa, \kappa^{+\omega})\}$ and $q := q^{j(\kappa)^{+\omega}}$ is then the desired lower bound for $j''G[\kappa, \kappa^{+\omega})$. For each ξ as above, let $(q^\xi)^\oplus$ be such that $(q^\xi)^\oplus_\xi = r_\xi$ and $(q^\xi)^\oplus\restriction\xi = q^\xi$. If q^ξ is a condition in $P[j(\kappa), \xi)$, then $(q^\xi)^\oplus$ is a condition in $P[j(\kappa), \xi)^\oplus$.

Claim 8.38 *If* $\xi \in$ C-supp$(r) \cap [j(\kappa), j(\kappa)^+)$, *then*

$$q^\xi \Vdash G(otf_\xi[\sup r^{**}_\xi]) = r_\xi.$$

Proof: As in the proof of theorem 8.21. □

We have thus shown that $q^{j(\kappa)^+}$ is a condition in $P[j(\kappa), j(\kappa)^+)$. Now assume ξ has regular cardinality $\theta \in [j(\kappa)^+, j(\kappa)^{+\omega})$, $\xi \in$ C-supp(r).

Claim 8.39

- $(q^\xi)^\oplus \Vdash \sup r^{**}_\xi \geq \sup(\operatorname{range} j \cap \theta)$.
- *If* $\gamma \in$ C-supp(r) *has cardinality* θ, $\gamma < \xi$, *then*

$$(q^\xi)^\oplus \Vdash \sup r^{**}_\gamma = \sup r^{**}_\xi.$$

- *If* $\gamma \in$ C-supp(r) *has cardinality* θ, $\gamma < \xi$, *then*

$$(q^\xi)^\oplus \Vdash \operatorname{ot} f_\gamma[\sup r^{**}_\gamma] < \operatorname{ot} f_\xi[\sup r^{**}_\xi].$$

- $(q^\xi)^\oplus \Vdash \operatorname{ot} f_\xi[\sup r^{**}_\xi] \geq \sup \text{S-supp}(r) \cap \theta$.
- q^ξ *forces that* r_ξ *has a* $P[j(\kappa), \sup(\text{S-supp}(r) \cap \theta))$*-name.*

Proof: Exactly as in the proof of theorem 8.21. □

Note: Since $\theta \in$ range(j), we get that $(q^\xi)^\oplus \Vdash \sup r^{**}_\xi = \sup(\operatorname{range} j \cap \theta)$, which immediately implies the second of the above properties. Many of the proofs of the above facts can be simplified using those easy observations.

Now by the above claims, we may set $q_{\operatorname{ot} f_\xi[\sup r^{**}_\xi]} = r_\xi$ and $q^{**}_\xi = r^{**}_\xi \cup \{\sup r^{**}_\xi\}$, i.e. given that q^ξ is a condition in $P[j(\kappa), \xi)$, we obtain that $q^{\xi+1}$ is a condition in $P[j(\kappa), \xi + 1)$. If ξ is a limit ordinal, q^ξ is a condition in $P[j(\kappa), \xi)$, as for each $\zeta < \xi$, $q^\xi\restriction\zeta$ is a condition in $P[j(\kappa), \zeta)$ inductively and q^ξ has appropriate supports. So we finally obtain a condition $q \in P[j(\kappa), j(\kappa)^{+\omega})$ which is below $j''G[\kappa, \kappa^{+\omega})$, our desired master condition. Choose a $P[j(\kappa), j(\kappa)^{+\omega})$-generic $G[j(\kappa), j(\kappa)^{+\omega})$ containing q and set $G^*[j(\kappa), j(\kappa)^{+\omega}) := G[j(\kappa), j(\kappa)^{+\omega}) \cap P^*[j(\kappa), j(\kappa)^{+\omega})$. We have now ensured that $j''G[\kappa, \kappa^{+\omega}) \subseteq G^*[j(\kappa), j(\kappa)^{+\omega})$. In order to be able to lift $j\colon \mathbf{V}[G_\kappa] \to \mathbf{M}[G_{j(\kappa)}]$ to $j\colon \mathbf{V}[G_{\kappa^{+\omega}}] \to \mathbf{M}[G^*_{j(\kappa)^{+\omega}}]$, it remains to show the following:

Claim 8.40 *$G^*_{j(\kappa)^{+\omega}}$ is $P^*_{j(\kappa)^{+\omega}}$-generic over $\mathbf{M}$.*

Proof: Suppose $D \in \mathbf{M}$ is open dense on $P^*_{j(\kappa)^{+\omega}}$ and write D as $j(f)(a)$ where $\mathrm{dom}(f) = H_{\kappa^{+\omega}}$ and $a \in H_{j(\kappa)^{+n}}$ for some $n \in \omega$. We may assume that every element of $\mathbf{M}$ is of this form. Choose $p \in G_{\kappa^{+\omega}}$ such that p reduces $f(\bar{a})$ below κ^{+n} whenever $\bar{a}$ belongs to $H_{\kappa^{+n}}$ and $f(\bar{a})$ is open dense on $P_{\kappa^{+\omega}}$, in the sense that if q extends p then q can be further extended into $f(\bar{a})$ without changing $u_{\kappa^{+n}}(q)$. The existence of p as above is shown similar to the proof of theorem 8.16, 5, using that $H_{\kappa^{+n}}$ has size κ^{+n}. Then $j(p)$ belongs to $j''G_{\kappa^{+\omega}} \subseteq G^*_{j(\kappa)^{+\omega}}$ and reduces D below $j(\kappa)^{+n}$, i.e. if $q \leq j(p)$ then $\exists r \leq q\ r \in D \wedge u_{j(\kappa)^{+n}}(r) = u_{j(\kappa)^{+n}}(q)$.

Hence $E := \{q \in P_{j(\kappa)^{+n+1}} : q^\frown j(p)[j(\kappa)^{+n+1}, j(\kappa)^{+\omega}) \in D\}$ is dense below $j(p)\restriction j(\kappa)^{+n+1}$ in $P_{j(\kappa)^{+n+1}}$. Since $G_{j(\kappa)^{+n+1}}$ contains $j(p)\restriction j(\kappa)^{+n+1}$ and is $P_{j(\kappa)^{+n+1}}$-generic over $\mathbf{M}$, $G_{j(\kappa)^{+n+1}} \cap E \neq \emptyset$. Choose a condition q in that intersection. Then $q^\frown j(p)[j(\kappa)^{+n+1}, j(\kappa)^{+\omega}) \in D \cap G^*_{j(\kappa)^{+\omega}}$. □

By the above, we obtain a lifted embedding $j\colon \mathbf{V}[G_{\kappa^{+\omega}}] \to \mathbf{M}[G^*_{j(\kappa)^{+\omega}}]$. As $P[\kappa^{+\omega}, \infty)$ is $\kappa^{+\omega+1}$-distributive by theorem 8.16, we may choose an arbitrary $P[j(\kappa)^{+\omega}, \infty)$-generic $G[j(\kappa)^{+\omega}, \infty)$, assume that j is given by an ultrapower and apply lemma 3 of [3] to find a P^*-generic G^* extending $G^*_{j(\kappa)^{+\omega}}$ and an elementary embedding $j\colon \mathbf{V}[G] \to \mathbf{M}[G^*]$ extending $j\colon \mathbf{V} \to \mathbf{M}$. As $(H_{j(\kappa)^{+\omega}})^{\mathbf{V}[G]} = H_{j(\kappa)^{+\omega}}[G^*_{j(\kappa)^{+\omega}}] \subseteq \mathbf{M}[G^*]$, j witnesses ω-hyperstrength of κ in $\mathbf{V}[G]$. $\Box_{\text{theorem 8.34}}$

Definition 8.41 *A cardinal κ is called α-hyperstrong iff $\exists j\colon \mathbf{V} \to \mathbf{M}$ with* $\mathrm{crit}(j) = \kappa$, *$j(\kappa) > \alpha$ and $H_{j(\kappa)^{+\alpha}} \subseteq \mathbf{M}$, where $j(\kappa)^{+\alpha}$ denotes the α^{th} cardinal successor of $j(\kappa)$.*

Theorem 8.42 *Assume* GCH *holds, $\alpha < \kappa$ is a limit ordinal and κ is α-hyperstrong. Then we may force Local Club Condensation and preserve the α-hyperstrength of κ.*

Proof of theorem: Very similar to the proof of theorem 8.34, replacing ω by α and using that $\alpha < \mathrm{crit}(j) = \kappa$ and hence $j(\kappa)^{+\alpha} = j(\kappa^{+\alpha})$. □

Definition 8.43 *A cardinal κ is called $j(\kappa)+\omega$-hyperstrong iff $\exists j\colon \mathbf{V} \to \mathbf{M}$ with* $\mathrm{crit}(j) = \kappa$ *and $H_{j(\kappa)^{+j(\kappa)+\omega}} \subseteq \mathbf{M}$.*

Theorem 8.44 *Assume* GCH *holds and κ is $j(\kappa)+\omega$-hyperstrong. Then we may force Local Club Condensation and preserve the $j(\kappa)+\omega$-hyperstrength of κ.*

Proof: Note that $j(\kappa)^{+j(\kappa)+\omega} < j^2(\kappa)$, since $j(\kappa)^{+j(\kappa)+\omega} = (j(\kappa)^{+j(\kappa)+\omega})^{\mathbf{M}}$ and $j^2(\kappa)$ is inaccessible in $\mathbf{M}$. Also $j(\kappa)^{+j(\kappa)+\omega} = j(\kappa^{+\kappa+\omega})$ and $\kappa^{+\kappa+\omega}$ is singular. Moreover, as is needed for the proof of claim 8.40, j is cofinal in $j(\kappa)^{+j(\kappa)+\omega}$. Using those facts, the proof is very similar to the proofs of theorem 8.42 and theorem 8.21. □

Definition 8.45 *We say that a cardinal κ is n-α-superstrong iff $\exists j\colon \mathbf{V} \to \mathbf{M}$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \alpha$ and $H_{j^n(\kappa)^{+\alpha}} \subseteq \mathbf{M}$.*

Theorem 8.46 *Assume GCH holds, $\alpha < \kappa$ is a limit ordinal and κ is n-α-superstrong. Then we may force Local Club Condensation and preserve the n-α-superstrength of κ.*

Proof: Very similar to the above theorems. □

Note: It is also easily possible to preserve many large cardinals simultaneously while forcing Strong Condensation, given that they are sufficiently spaced.

9 A possible future application

In this section we want to present a possible future application of the technique of forcing fragments of condensation. The presented application is hypothetical, since it would require models of Local Club Condensation and Acceptability (simultaneously) which have very large cardinals, the existence of such models is yet an open question. Therefore we only hint at the ideas and give no details of proof. This application is closely related to [8].

Definition 9.1 *The Proper Forcing Axiom (PFA) is the statement that whenever P is a proper notion of forcing and $\mathcal{D}$ is a collection of $\aleph_1$ dense subsets of P, then there exists a $\mathcal{D}$-generic filter on P.*

An old and well-known result of James Baumgartner is that given a model with a supercompact cardinal, one can do a proper iteration of proper forcings to obtain a model of PFA, yielding an upper bound for the consistency strength of PFA. It is unknown whether the consistency strength of PFA actually is that of a supercompact cardinal.

The result we would like to obtain using models of Local Club Condensation, Acceptability and other **L**-like properties with very large cardinals is the following:

Conjecture 9.2 *Let $\varphi(\kappa)$ describe a large cardinal property of κ consistency-wise weaker than supercompactness. It is then consistent that there is κ which satisfies $\varphi(\kappa)$ but no proper forcing extension satisfies PFA.*

Since any method to force PFA over a model with large cardinals is believed to be an iteration of proper forcing notions which is itself proper, we want to abbreviate the above conjecture with "A supercompact cardinal is a quasi-lower bound for PFA". The model to verify the above consistency result will be an **L**-like model (which in particular satisfies Local Club Condensation and Acceptability) with large cardinals, the proof is a generalization of the proof of the second part of [8], noting that the proof in [8] can be transferred to the context of **L**-like models (from the context of extender models), noting that Local Club Condensation (together with Acceptability) is a sufficient replacement for the condensation principle used in the proof of [8] and noting that the proof in [8], which actually works with a hierarchy of fragments of PFA, can be extended along that hierarchy to actually reach up to a supercompact cardinal. The basis case (which is described in [8] in different context) is the following:

Theorem 9.3 *Assume* $\mathbf{M}$ *is of the form* $\mathbf{L}[A]$, $M_\kappa = H_\kappa$ *for all cardinals* κ *and* $\mathbf{M}$ *satisfies Acceptability, Local Club Condensation and* $\square$ *at small cofinalities. If there is a proper forcing extension* $\mathbf{V}$ *of* $\mathbf{M}$ *in which* PFA($\mathbf{c}^+$-*linked) holds and* $\tau = \omega_2^{\mathbf{V}}$, *then* $[\tau, (\tau^+)^{\mathbf{M}}]$ *is* Σ^2_1*-indescribable in* $\mathbf{M}$.

We need the following definitions:

Definition 9.4 *A notion of forcing* P *is* τ*-linked if it can be written as a union of sets* P_ξ, $\xi < \tau$ *so that for each* ξ, *the conditions in* P_ξ *are pairwise compatible.*

Definition 9.5 PFA(τ*-linked) is the statement that whenever* P *is a proper,* τ*-linked notion of forcing and* $\mathcal{D}$ *is a collection of* $\aleph_1$ *dense subsets of* P, *then there exists a* $\mathcal{D}$*-generic filter on* P.

For the definition of a Σ^2_1-indescribable interval of cardinals, we refer the reader to [8]. For the definition of a subcompact cardinal, we refer the reader to [3] or [6], for the definition of $\square$ at small cofinalities, we refer the reader to [3]. We note that it is shown in [3] how to force $\square$ at small cofinalities and preserve various large cardinals. Now a forcing iteration to obtain Local Club Condensation and Acceptability which is cofinality-preserving will preserve $\square$ at small cofinalities. Thus the following is the main question left open by this thesis:

Question 9.6 *Given a model of Set Theory which satisfies* GCH *and has (very) large cardinals, can we do a cofinality-preserving forcing to obtain a model of Local Club Condensation and Acceptability while preserving certain (very) large cardinals?*

We think (and hope) that the answer to this question is a positive one, yet finding a solution seems surprisingly hard.

References

[1] George Boolos. *On the Semantics of the Constructible Levels*. Zeitschr. f. math. Logik und Grundlagen d. Math. 16, pp 139-148, 1970.

[2] Sy D. Friedman. *Fine Structure and Class Forcing*. De Gruyter Series in Logic and its applications 3, 2000.

[3] Sy D. Friedman. *Large cardinals and L-like universes*. Set theory: recent trends and applications, Quaderni di Matematica, vol. 17 pp. 93-110, 2007.

[4] Sy D. Friedman. *Unpublished notes*. 2008.

[5] Thomas Jech. *Set Theory. The Third Millennium Edition, Revised and Expanded*. Springer, 2003.

[6] Akihiro Kanamori. *The Higher Infinite. Second Edition*. Springer, 2005.

[7] Itay Neeman, Ernest Schimmerling. *Hierarchies of Forcing Axioms I*. Journal of Symbolic Logic 73, pp 343-362, 2008.

[8] Itay Neeman. *Hierarchies of Forcing Axioms II*. Journal of Symbolic Logic 73, pp 522-542, 2008.

[9] Hugh Woodin. *The Axiom of Determinacy, Forcing Axioms and the Nonstationary Ideal*. De Gruyter Series in Logic and its applications 1, 1999.

Short Summary / Kurzzusammenfassung

We define Local Club Condensation, a principle which isolates and generalizes properties of Gödel's Condensation principle. We show that we can force over any model of set theory to obtain a model which satisfies this principle while at the same time preserving various very large cardinals; in particular we show that Local Club Condensation is consistent with the existence of an ω-superstrong cardinal. We proceed similarly for Acceptability, another principle isolating and generalizing aspects of Gödel's Condensation principle. This continues the outer model program of Sy Friedman [3]. We also hint at a possible future application of the above-described results at the end of this thesis regarding the consistency strength of PFA.

Wir definieren lokale Clubmengenkondensation (Local Club Condensation), ein Prinzip, welches Eigenschaften von Gödels Kondensationsprinzip isoliert und verallgemeinert. Wir zeigen, dass wir über einem beliebigen Modell der Mengenlehre durch die Erzwingungsmethode zu einem Modell der Mengenlehre gelangen können, welches lokale Clubmengenkondensation erfüllt und zugleich verschiedene grosse Kardinalzahlen erhalten werden können; insbesondere zeigen wir, dass lokale Clubmengenkondensation mit der Existenz einer ω-superstarken Kardinalzahl konsistent ist. Wir gehen ähnlich für Acceptability vor, ein weiteres Prinzip welches Aspekte von Gödels Kondensationsprinzip isoliert und verallgemeinert. Dies setzt das Outer Model Program (zu deutsch Programm der äusseren Modelle) von Sy Friedman ([3]) fort. Wir führen auch eine mögliche Zukunftsanwendung in Bezug auf das Proper Forcing Axiom (PFA) oben beschriebener Ergebnisse am Ende der Arbeit auf.

Lebenslauf

Peter HOLY Geboren am 23. September 1982 in Mödling, Österreich.

2000-2004

Studium der Technischen Mathematik im Zweig Mathematik in den Computerwissenschaften an der Technischen Universität Wien

2004-2007

Studium der Mathematik im Zweig Mathematische Logik an der Universität Wien

Diplomarbeit: Absoluteness Results in Set Theory

Betreuer: Sy-David Friedman

seit 2007

Doktoratsstudent am Kurt Gödel Research Center, Universität Wien

Betreuer: Sy-David Friedman