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The tree property

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Contents

Chapter 1

Introduction

For an infinite cardinal κ, a *κ-tree* is a tree T of height κ such that every level of T has size less than κ. A tree T is a *κ-Aronszajn tree* if T is a κ-tree which has no cofinal branches. We say that *the tree property holds at κ*, or $\mathrm{TP}(\kappa)$ *holds*, if every κ-tree has a cofinal branch, i.e. a branch of length κ through it. Thus, $\mathrm{TP}(\kappa)$ holds iff there is no κ-Aronszajn tree.

For example, $\mathrm{TP}(\aleph_0)$ holds in ZFC, and it is actually exactly the statement of the well-known König's lemma. Aronszajn showed also in ZFC that there is an $\aleph_1$-Aronszajn tree. Hence, $\mathrm{TP}(\aleph_1)$ fails in ZFC.

Large cardinals are needed once we consider trees of height greater than $\aleph_1$. For example, Silver proved that for $\kappa > \aleph_1$ $\mathrm{TP}(\kappa)$ implies κ is weakly compact in L, and Mitchell proved that given a weakly compact cardinal λ above a regular cardinal κ, one can make λ into κ^+ so that in the extension, κ^+ has the tree property. Moreover, if κ is the successor of a regular cardinal, then this can be done preserving cardinals up to and including κ. Thus, $\mathrm{TP}(\aleph_2)$ is equiconsistent with the existence of a weakly compact cardinal.

Natasha Dobrinen and Sy-D. Friedman [1] used a generalization of Sacks forcing to reduce the large cardinal strength required to obtain the tree property at the double successor of a measurable cardinal from a supercompact to a weakly compact hypermeasurable cardinal (see Definition 17).

In this thesis we extend the method of [1] to obtain improved upper bounds on the consistency strength of the tree property at the double successor of singular cardinals.

The thesis is organised as follows. Chapter 2 is reserved for preliminaries which roughly prepare even a nonlogician for understanding later chapters (or at least the basic statements). In Chapter 3 we give a systematic overview of the most significant theorems which have been proven about the tree property in the past, and thus prepare the contextual ground for the main theorem of this thesis. The main theorem is found in Chapter 4:

Theorem. *Assume that V is a model of ZFC and κ is a weakly compact hypermeasurable cardinal in V. Then there exists a forcing extension of V in which $\aleph_{\omega+2}$ has the tree property, $\aleph_\omega$ strong limit.*

Chapter 5 discusses some variations of the tree property. It also contains several remarks and open problems.

Chapter 2

Preliminaries

2.1 Some basic set theory for nonlogicians

2.1.1 Sets and numbers

The founder of set theory, Georg Cantor, defined sets to be collections of any objects (that can be thought of). However, the words *any* and *every* turned out to be relative. Russell's paradox* was a clear sign that a formal approach to set theory demands more precise definitions. One way to avoid troubles was to start with axioms and only consider 'worlds of objects' (also called models) in which these axioms are true. The most famous system of axioms for set theory is called $ZFC^{\dagger}$.

We fix a model of ZFC which becomes our universe. By a *set* we understand any object in that universe. (If an object is (possibly) not in the universe, we use the word *class* for it.) The set of all subsets of a set A is called the *powerset* of A and is denoted by $P(A)$. A set A is *countable* if there exists an injective function $f : A \to \mathbb{N}$, otherwise it is *uncountable.*

Relations on sets

Let A be a set. Any subset R of $P(A \times A) = \{(a, b) : a \in A,\ b \in A\}$ is called a (*binary*) *relation on* A. We usually write $a\ R\ b$ instead of $(a, b) \in R$.

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*Consider the collection of all objects which do not contain themselves. Is it contained in itself?

†Zermelo-Fraenkel axioms with the axiom of choice: there exists an empty set $\emptyset$ (can be thought of as a unit); there exists an infinite set; pairs, unions, powersets and certain subsets of sets exist (are sets); images of sets (under any function) are sets; two sets are same if and only if they have the same elements; every nonempty set has a $\in$-minimal element; every family of nonempty sets has a choice function. We refer to [8, Chapter 1] for a complete and formal description of ZFC axioms.

A relation R is said to be

reflexive if $a \; R \; a$ for every $a \in A$;

irreflexive if $a \not\!R \; a$ for every $a \in A$;

symmetric if $a \; R \; b$ implies $b \; R \; a$ for every $a, b \in R$;

transitive if $a \; R \; b \wedge b \; R \; c$ implies $a \; R \; c$; and

total if either $a \; R \; b$ or $b \; R \; a$ or $a = b$ for every $a, b \in A$.

Definition 1.

1. A binary relation $\leq_P$ on a set P is called a *quasi ordering* of P if it is reflexive and transitive.

2. A binary relation $<_P$ on a set P is called a *strict partial ordering* of P if it is irreflexive and transitive.

3. A total strict partial ordering on a set P is called a *linear ordering* of P.

4. A binary relation on a set P is called an *equivalence relation* on P if it is reflexive, symmetric and transitive.

There can be at the same time both a quasi ordering $\leq_P$ and a strict partial ordering $<_P$ on a set P; we identify P with $(P, \leq_P, <_P)$.

Definition 2. Fix a set P and let $\leq_P$ and $<_P$ be a quasi ordering and a strict partial ordering of P, respectively. For nonempty sets $X, Y \subseteq P$, and $p \in P$, we say that

p is a *maximal* element of X if $p \in X$ and $p \not<_P x$ for every $x \in X$;

p is a *minimal* element of X if $p \in X$ and there is no $q \in X$ such that $q \leq_P p$ and $p \not\leq_P q$;

p is a *least* element of X (in the relation $\leq_P$) if $p \in X$ and $p \leq_P x$ for every $x \in X$;

p is an *upper bound* of X (or p bounds X) if $x \leq_P p$ for every $x \in X$;

p is a $<_P$*-upper bound* of X (or p $<_P$-bounds X) if $x <_P p$ for every $x \in X$;

X is *cofinal in* Y in the relation $<_P$ (resp. $\leq_P$) if for every $b \in Y$ there is some $a \in X$ such that $b <_P a$ (resp. $b \leq_P a$) [we also say 'cofinal in $(Y, <_P)$' instead of cofinal in the relation $<_P$];

X is *bounded* in Y if there is an upper bound for X in Y;

p is an *exact upper bound* of X if p is a least upper bound of X and X is cofinal in $\{q \in P : q <_P p\}$ in the relation $\leq_P$.

So p is a *minimal upper bound* of X if p is an upper bound of X and there is no upper bound q of X such that $q \leq_P p$ and $p \not\leq_P q$; and p is a *least upper bound* of X if p is an upper bound of X and $p \leq_P q$ for every upper bound q of X (p is then also called a supremum of X (sup$X$)).

Suppose that R is an equivalence relation on a set P. For each $p \in P$, we define the *equivalence class* $[p] := \{q \in P : p \; R \; q\}$ of p. Every element of P is then in some equivalence class ($p \in [p]$), and no element is in two different classes. The *quotient* P/R *of* P *modulo* R is the collection of all equivalence classes.

Ordinal numbers

A linearly ordered set $(P, <_P)$ is *well-ordered* if every nonempty subset of it has a least element (in the linear ordering). By an *initial segment* of a well-ordered set P we mean a subset of the form $\{x \in P : x <_P r\}$ for some $r \in P$. It holds[‡] that any two well-ordered sets are comparible in the following sense; either they are isomorphic (with respect to the relation $<_P$) to each other, or one of them is isomorphic to an initial segment of the other one. If we define equivalence classes on the collection of all well-ordered sets by putting isomorphic well-ordered sets into the same class, we can think of ordinal numbers as the collection of the nicest representatives of these equivalence classes.

Definition 3. A set A is an *ordinal number* (an *ordinal*) if it is well-ordered by the relation $\in$ (is an element of), and if $a \subseteq A$ for every $a \in A$ (transitiveness).

Ordinals are usually denoted by lowercase greek letters α, β, etc., and the class (collection) of all ordinal numbers is denoted by *Ord*. A function f is called an *ordinal function* if range$(f) \subseteq Ord$. For ordinals α and β we also write $\alpha < \beta$ instead of $\alpha \in \beta$. We list some of the basic facts about ordinals without proving them. The proofs can be found in [8].

[‡]See [8] for a proof.

Proposition 1. *The following hold for any ordinal number α:*

1. *The empty set $\emptyset$ is an ordinal;*
2. *if $\beta \in \alpha$, then β is also an ordinal;*
3. $\alpha = \{\beta : \beta \in \alpha\}$*;*
4. $\alpha + 1 := \alpha \cup \{\alpha\}$ *is also an ordinal;*
5. *If X is a nonempty set of ordinals, then $\bigcup X$ is also an ordinal;*
6. *$<$ is a linear ordering of the class Ord;*
7. *each well-ordering P is isomorphic to exactly one ordinal, this ordinal is then called the order-type of P.*

Ordinals of the form $\alpha \cup \{\alpha\}$ are called *successor ordinals*. All other ordinals are called *limit ordinals*. Finite ordinals are also known as *natural numbers* and are written as follows

$$
\begin{aligned}
&0 = \emptyset,\\
&1 = 0 + 1 = \emptyset \cup \{\emptyset\} = \{\emptyset\},\\
&2 = 1 + 1 = \{\emptyset\} \cup \big\{ \{\emptyset\} \big\} = \big\{\emptyset, \{\emptyset\} \big\},\\
&\text{etc.}
\end{aligned}
$$

Cardinal numbers

Definition 4. An ordinal number α is a *cardinal number* (a *cardinal*) if there is no bijection between α and any $\beta < \alpha$.

We usually use κ, λ, μ... to denote cardinals. By the *cardinality* $|X|$ of a set X we mean the unique cardinal number κ for which there is a bijection $f : \kappa \to X$. (The existence of such a bijection is not trivial; it relies on the axiom of choice.) Note that each natural number is a cardinal number; the cardinality of a finite set is simply the natural number of its elements.

The infinite cardinals are called *alephs*. Since cardinals are linearly ordered by $<$, we can enumerate them by ordinal numbers; $\aleph_0$ (or ω) denotes the first infinite cardinal (the set of natural numbers), and $\aleph_\alpha$ denotes the α-th infinite cardinal. If α is a successor (limit) ordinal, then we say that $\aleph_\alpha$ is a *successor* (*limit*) cardinal. We also write $\aleph_\alpha^+$ for $\aleph_{\alpha+1}$.

The arithmetic operations on cardinals are defined as follows:

$$\kappa + \lambda := |A \cup B|, \quad \kappa \cdot \lambda := |A \times B|,$$

$$\kappa^\lambda := |A^B| = |\{f : f \text{ is a function from } B \text{ into } A\}|,$$

where A and B are any disjoint sets with cardinalities $|A| = \kappa$ and $|B| = \lambda$.

Proposition 2. *The following hold for any cardinals κ, λ:*

1. *If κ and λ are infinite cardinals, then $\kappa + \lambda = \kappa \cdot \lambda = max\,\{\kappa,\ \lambda\}$;*
2. *$+$ and $\cdot$ are associative, commutative and distributive;*
3. *$(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$, $\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$, $(\kappa^\lambda)^\mu = \kappa^{\lambda\cdot\mu}$;*
4. *$\kappa \leq \lambda$ implies $\kappa^\mu \leq \lambda^\mu$, and $0 < \lambda \leq \mu$ implies $\kappa^\lambda \leq \kappa^\mu$;*
5. *Cantor : $\kappa < 2^\kappa$. (If a set A has the cardinality $\kappa = |A|$, then $2^\kappa = |\{f : f$ is a function from A into $2\}| = |P(A)|$ is the cardinality of the powerset of A.)*

For a proof see [8].

We say that a set of ordinals A is *cofinal* in a set of ordinals B if for every $\beta \in B$ there is an $\alpha \in A$ such that $\beta < \alpha$. For any ordinal α define the *cofinality of* α, denoted as $cf(\alpha)$, to be the least cardinality of a subset of α which is cofinal in α. If α is a cardinal number and $cf(\alpha) = \alpha$, then α is called a *regular* cardinal. Otherwise, (that is, if $cf(\alpha) < \alpha$), α is called a *singular* cardinal. (We denote the class of regular cardinals by *Reg*.) One can show that for every α, $cf(cf(\alpha)) = cf(\alpha)$. Thus, $cf(\alpha)$ is always a regular cardinal.

The exponentiation of cardinal numbers, unlike addition and multiplication, which are trivial, is one of the main topics in set theory. In the following proposition we state some of the basic properties of the cardinal arithmetic.[§]

Proposition 3. *The following hold for any cardinals κ, λ:*

1. *If λ is infinite and $2 \leq \kappa \leq \lambda$, then $\kappa^\lambda = 2^\lambda$;*
2. *if $\lambda \geq cf(\kappa)$, then $\kappa < \kappa^\lambda$;*
3. *if I is any index set and $\kappa_i < \lambda_i$ for every $i \in I$, then $\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i$;*
4. *$(\kappa^+)^\lambda = \kappa^\lambda \cdot \kappa^+$ (Hausdorff formula).*

[§] For a proof of the proposition we refer the reader to [8, page 51]. In chapter 5 there are deeper results regarding cardinal arithmetic.

If κ is a cardinal and $2^\alpha < \kappa$ for every $\alpha < \kappa$, then κ is said to be an (*strongly*) *inaccessible cardinal.*

Closed unbounded sets

Let κ be a limit ordinal, and let $C \subseteq \kappa$. Any limit ordinal $\alpha < \kappa$ with sup $C \cap \alpha = \alpha$ is called a *limit point* of C. We say that C is *closed unbounded* (in κ) if it contains all its limit points and is cofinal in κ. For example, the set of all limit ordinals in κ is a closed unbounded set. The intersection of two closed unbounded sets is also closed unbounded.

Suppose that κ is a regular uncountable cardinal. A set $S \subseteq \kappa$ is said to be *stationary* (in κ) if $S \cap C \neq \emptyset$, for every closed unbounded set C in κ.

Filters and ultrafilters

Definition 5. A family $F \subseteq P(A)$ of subsets of a set A is called a *filter on* A if it satisfies the following conditions:

1. $\emptyset \notin F$ and $A \in F$;
2. if $X \in F$ and $Y \in F$, then $X \cap Y \in F$;
3. if $X, Y \subseteq A$, $X \in F$, and $X \subseteq Y$, then $Y \in F$.

We say that that a set $H \subseteq P(A)$ generates a filter F, if F is the closure of H under supersets and finite intersections.

A filter F is an *ultrafilter* if for every $X \subseteq A$, either $X \in F$, or $A \backslash X \in F$, where $A \backslash X = \{a \in A : a \notin X\}$ denotes the complement of X in A. It holds that every filter can be extended to an ultrafilter.

Measurable cardinals

Let κ be a cardinal number. An ultrafilter U on κ is said to be κ-complete iff it is closed under intersections of fewer than κ elements. A principal ultrafilter on κ is a filter of the form $\{X \subseteq \kappa | \alpha \in X\}$, $\alpha \in \kappa$.

Definition 6. An uncountable cardinal κ is *measurable* if there exists a κ-complete nonprincipal ultrafilter U on κ.

The word *measurable* comes from the fact that an ultrafilter on κ induces a function called *measure* on the power set of κ:

Definition 7 ([8])**.** Let S be an infinite set. A (*nontrivial* σ-*additive*) *measure* on S is a real-valued function μ on $P(S)$ such that

1. $\mu(\emptyset) = 0$ and $\mu(S) = 1$,

2. if $X \subset Y$, then $\mu(X) \leq \mu(Y)$,

3. $\mu(\{a\}) = 0$ for all $a \in S$ (nontriviality), and

4. if $X_n, n \in \omega$, are pairwise disjoint, then

$$\mu(\bigcup_{n\in\omega} X_n) = \sum_{n\in\omega} \mu(X_n) \text{ (additivity).}$$

If U is an ultrafilter on κ then the function $\mu : P(\kappa) \to \{0,1\}$ defined by $\mu(X) = 1$ if $X \in U$ and $\mu(X) = 0$ if $X \notin U$ is actually a *two-valued measure* on κ.

A cardinal number on which there is a nontrivial κ-additive (not neccessarily two-valued) measure is called *real-valued measurable*.

2.1.2 Structures and embeddings

At the beginning we mentioned 'worlds of objects' in which certain axioms hold. We say that these 'worlds of objects' are *models* for the given axioms. There we were referring to possible frameworks of set theory (mathematics), and this remains our course, but the notion of a model is much more general. An algebraic group and a linear ordering are also examples of models for a bunch of axioms. A more general name for models is *structures*.

The models of set theory are precisely defined as follows: The language of set theory, beside the logical symbols $\forall$ (universal quantifier *for all*), $\exists$ (existential quantifier *exists*), $=$ (equality), and v_i (variables), consists of a single non-logical symbol $\in$. A structure $\mathbb{A}$ (for the language of set theory) is a function which assigns a set V to the symbol $\forall$ and a binary relation $\in^{\mathbb{A}} \subseteq V \times V$ to the symbol $\in$. In other words, $\mathbb{A} = (V, \in^{\mathbb{A}})$ consists of a *universe* V over which $\forall$ ranges, and a binary relation $\in$ on V. We identify $\in^{\mathbb{A}}$ with $\in$, and instead of $\mathbb{A}$ we simply write V.

The formulas of set theory are built up from the *atomic formulas* $x \in y$ and $x = y$ by means of connectives $\wedge$ (and), $\vee$ (or), $\urcorner$ (not) and $\to$ (implies), and quantifiers $\forall$ and $\exists$. We already referred the reader to [8, Chapter 1] for a precise definition of logical formulas and a complete and formal description of ZFC axioms. We also assume that the reader understands what it means for a formula ϕ to be true in a structure V, it should namely express a correct statement about the universe V and the relation $\in$ (this is denoted by $V \models \phi$).

Our goal is to relate models of set theory to each other by elementary embeddings. These embeddings are one of the most powerful tools in set theory.

Let V and M be two structures in the language of set theory. A function $j : V \to M$ is said to be an *elementary embedding of* V *into* M if for every formula ϕ and $x_1, ..., x_n \in V$:

$$V \models \phi(x_1, ..., x_n) \text{ if and only if } M \models \phi(j(x_1), ..., j(x_n)).$$

2.1.3 Forcing and consistency results

A set of axioms (formulas or statements) is said to be consistent if no contradiction can be produced from them. Equivalently, a set of axioms is consistent if there exists a model in which these statements hold simultaneously.

Since many statements, like the tree property or the inaccessibility of a cardinal number, are neither provable nor refutable from ZFC (that is, ZFC is incomplete), investigating which axioms or statements are consistent with ZFC or with each other became a central topic in the modern set theory.

The consistency results are of the form $\mathrm{Con}(ZFC+\phi) \to \mathrm{Con}(ZFC+\psi)$, where ϕ and ψ are two statements and $\mathrm{Con}(ZFC+\phi)$ means that ϕ is consistent with ZFC. In other words, it is assumed that there is a model in which ZFC and ϕ hold, and then another model is constructed in which ZFC and ψ hold. The main method for obtaining the second model from the first one is called *forcing* and the constructed model is then called *forcing extention.* If the implication above holds we say that ϕ has a higher consistency strength than ψ.

In most consistency results the statements ϕ and ψ are statements about cardinal numbers known as *large cardinal properties* (such as the existence of a measurable cardinal) because they are the perfect scale for consistency strength - any property ψ can be forced (to hold together with ZFC in some other model) with their help, and they can be forced from some properties.

Forcing was discovered by Paul Cohen who was also the first one to use it for a consistency result. Namely, he proved $\mathrm{Con}(ZFC) \to \mathrm{Con}(ZFC+\neg CH)$, where CH is the statement $2^{\aleph_0} = \aleph_1$, i.e. any infinite subset of reals is either countable or as big as the reals.

Forcing is a rather complicated technical method which we will not describe here. We refer the reader to [17] for a detailed introduction to forcing.

2.2 Trees and branches

Definition 8. A *tree* is a strict partial ordering $(T, <)$ with the property that for each $x \in T$, the set $\{y : y < x\}$ is well-ordered by $<$.

Trees belong to the most fundamental objects in combinatorial set theory and are frequently used in many different contexts.

The αth *level* of a tree T consists of all x such that $\{y : y < x\}$ has order-type α. The *height* of T is the least α such that the αth level of T is empty. A *branch* in T is a maximal linearly ordered subset of T. We say that a branch is *cofinal* in T if it hits every level of T.

2.2.1 Definition of the tree property

The tree property is a combinatorial principle which was introduced by Erdős and Tarski in 1961. It is the main subject of this thesis.

Definition 9. An infinite cardinal κ has the *tree property* if every tree of height κ whose levels have size $< \kappa$ has a cofinal branch.

A tree of height κ whose levels have size $< \kappa$ is called a *κ-Aronszajn tree* if it has no cofinal branches. Therefore, an equivalent formulation of the tree property is nonexistence of Aronszajn trees.

2.3 Forcing notions

We define some forcing notions which will be used in the later chapters, and state their most important properties.

2.3.1 Lévy collapse

Let κ be a regular cardinal and let $\alpha > \kappa$ be a cardinal. Let P_α be the set of all functions p such that

1. $\mathrm{dom}(p) \subset \kappa$ and $|\mathrm{dom}(p)| < \kappa$, and
2. $\mathrm{ran}(p) \subset \alpha$,

and let $p < q$ if and only if $p \supset q$.

If G is a generic filter over P_α, then $f := \bigcup G$ is a function from κ onto α, i.e. forcing with P_α collapses α to κ. P_α is $< \kappa$-closed and therefore all cardinals $\leq \kappa$ are preserved. If $\alpha^{<\kappa} = \alpha$, then $|P_\alpha| = \alpha$ and hence also all cardinals $\geq \alpha^+$ are preserved.

The Lévy collapse is a forcing for collapsing cardinals below an inaccessible cardinal λ and it is basically the product of P_α's for $\alpha < \lambda$:

Definition 10 ([8])**.** Let κ be a regular cardinal and let $\lambda > \kappa$ be an inaccessible cardinal. *Lévy collapse* is the set of all functions p on subsets of $\lambda \times \kappa$ such that

1. $|\mathrm{dom}(p)| < \kappa$, and

2. $p(\alpha, \xi) < \alpha$ for each $(\alpha, \xi) \in \mathrm{dom}(p)$,

ordered by $p < q$ if and only if $p \supset q$.

Let G be generic over P, and for each $\alpha < \lambda$, let G_α be the projection of G on P_α. Then G_α is generic over P_α. It follows that P collapses every $\alpha < \lambda$ to κ.

The Lévy collapse P is $< \kappa$-closed and therefore all cardinals $\leq \kappa$ are preserved. For an inaccessible cardinal λ, the product with supports of size $< \kappa$ of λ many forcings of size $< \lambda$ satisfies the λ-chain condition. Thus, P preserves cardinals $\geq \lambda$ as well, in particular $\lambda = \kappa^+$ in the extension.

2.3.2 Sacks forcing

Definition 11. Let ρ be a strongly inaccessible cardinal. Then Sacks(ρ) denotes the following forcing notion. A condition p is a subset of $2^{<\rho}$ such that:

1. $s \in p$, $t \subseteq s \rightarrow t \in p$.

2. Each $s \in p$ has a proper extension in p.

3. For any $\alpha < \rho$, if $\langle s_\beta : \beta < \alpha \rangle$ is a sequence of elements of p such that $\beta < \beta' < \alpha \rightarrow s_\beta \subseteq s_{\beta'}$, then $\bigcup\{s_\beta : \beta < \alpha\} \in p$.

4. Let Split(p) denote the set of $s \in p$ such that both $s^\frown 0$ and $s^\frown 1$ are in p. Then for some club denoted $C(p) \subseteq \rho$, $\mathrm{Split}(p) = \{s \in p : \mathrm{length}(s) \in C(p)\}$.

The conditions are ordered as follows: $q \leq p$ iff $q \subseteq p$, where $q \leq p$ means that q is stronger than p.

Given $p \in \mathrm{Sacks}(\rho)$, let $\langle \gamma_\alpha : \alpha < \rho \rangle$ be the increasing enumeration of $C(p)$. For $\alpha < \rho$, the α-th splitting level of p, $\mathrm{Split}_\alpha(p)$, is the set of $s \in p$ of length γ_α. For $\alpha < \rho$ we write $q \leq_\alpha p$ iff $q \leq p$ and $\mathrm{Split}_\beta(q) = \mathrm{Split}_\beta(p)$ for all $\beta < \alpha$.

Sacks(ρ) satisfies the following *ρ-fusion* property: Every decraesing sequence $\langle p_\alpha : \alpha < \rho \rangle$ of elements in Sacks(ρ) such that for each $\alpha < \rho$, $p_{\alpha+1} \leq_\alpha p_\alpha$, has a lower bound, namely $\bigcap_{\alpha<\rho} p_\alpha \in \mathrm{Sacks}(\rho)$.

This closure under certain sequences of length κ, namely fusion sequences, is a big advantage of Sacks(κ) forcing over, say, κ-Cohen forcing, and it will play a crucial role in the proofs of the fourth chapter.

The forcing notion Sacks(ρ) is also $< \rho$ -closed, satisfies the ρ^{++}-c.c., and preserves ρ^+. For a proof see [3] or [1].

Definition 12. Let ρ be a strongly inaccessible cardinal and let $\lambda > \rho$ be a regular cardinal. Sacks(ρ, λ) denotes the λ-length iteration of Sacks(ρ) with supports of size $\leq \rho$.

Sacks(ρ, λ) satisfies the *generalized ρ-fusion* property which we describe next: For $\alpha < \rho$, $X \subseteq \rho$ of size less than ρ, and $p, q \in$ Sacks(ρ, λ), we write $q \leq_{\alpha,X} p$ iff $q \leq p$ (i.e. $q \restriction i \Vdash q(i) \leq p(i)$ for each $i < \lambda$) and in addition, for each $i \in X$, $q \restriction i \Vdash q(i) \leq_\alpha p(i)$. Every decreasing sequence $\langle p_\alpha : \alpha < \rho \rangle$ of elements in Sacks(ρ, λ) such that for each $\alpha < \rho$, $p_{\alpha+1} \leq_{\alpha, X_\alpha} p_\alpha$, where the X_α's form an increasing sequence of subsets of λ each of size less than ρ whose union is the union of the supports of the p_α's, has a lower bound. [The lower bound is q where $q(0) = \bigcap_{\alpha<\rho} p_\alpha(0)$, $q(1)$ is a name s.t. $q(0) \Vdash q(1) = \bigcap_{\alpha<\rho} p_\alpha(1)$, etc.]

Assuming $2^\rho = \rho^+$, Sacks(ρ, λ) is $< \rho$-closed, satisfies the λ-c.c., preserves ρ^+, collapses λ to ρ^{++} and blows up 2^ρ to ρ^{++}. For a proof see [3] or [1].

Chapter 3

The tree property

This chapter is a survey of ZFC and consistency results on the tree property.

3.1 ZFC results on the tree property

3.1.1 The tree property at inaccessible cardinals

Equivalent formulations

Before we start talking about deeper results on the tree property, we would like to give the reader some idea about the importance of this cardinal property by stating all of its equivalent formulations for inaccessible cardinals.

We first define some notions which are needed for these formulations and which will be used later on. The most fundamental one among them, including the tree property, relates to the subject of compactness which is a milestone of logic. Thus, we want to start by introducing compactness.

Compactness

Let α and β be infinite cardinal numbers. The language $\mathcal{L}_{\alpha,\beta}$ is the language obtained by closing the usual first order language under infinitary conjuctions, disjunctions and quantifications, more precisely:

1. if $\delta < \alpha$ and $(\phi_i)_{i<\delta}$ is a sequence of formulas in $\mathcal{L}_{\alpha,\beta}$, then $\bigvee_{i<\delta} \phi_i$ and $\bigwedge_{i<\delta} \phi_i$ are also formulas in $\mathcal{L}_{\alpha,\beta}$; and
2. if $\sigma < \beta$, $(x_j)_{j<\sigma}$ is a sequence of variables, and ϕ is a formula in $\mathcal{L}_{\alpha,\beta}$, then $\forall (x_j)_{j<\sigma}\phi$ and $\exists (x_j)_{j<\sigma}\phi$ are also formulas in $\mathcal{L}_{\alpha,\beta}$.

The usual finitary language $\mathcal{L}_{\omega,\omega}$ satisfies the following *Compactness The-*

orem: If Σ is a set of sentences such that every finite subset $S \subseteq \Sigma$ has a model, then Σ has a model.

Let us say that the language $\mathcal{L}_{\kappa,\kappa}$ satisfies *Weak Compactness Theorem* if whenever Σ is a set of sentences of $\mathcal{L}_{\kappa,\kappa}$ of cardinality κ such that every $S \subset \Sigma$ with $|S| < \kappa$ has a model, then Σ has a model.

Definition 13. An inaccessible cardinal κ is called *weakly compact* iff $\mathcal{L}_{\kappa,\kappa}$ satisfies Weak Compactness Theorem.

If κ is an inaccessible cardinal then $\mathcal{L}_{\kappa,\kappa}$ satisfies the Weak Compactness Theorem iff $\mathcal{L}_{\kappa,\omega}$ satisfies the Weak Compactness Theorem. The left to right direction is trivial because $\mathcal{L}_{\kappa,\omega} \subset \mathcal{L}_{\kappa,\kappa}$. For a proof of the converse see [8], Thm. 17.13.

Unfoldability

Unfoldability is a large cardinal property expressed in terms of elementary embeddings:

Definition 14. [11] A cardinal number κ is λ-*unfoldable* if and only if for every transitive model N_0 of ZF^- (ZF without Power Set Axiom) with $|N_0| = \kappa$, $\kappa \in N_0$ and ${}^{<\kappa}N_0 \subseteq N_0$ there exists a non-trivial elementary embedding $k : N_0 \to N_1$, where N_1 is also a transitive model of ZF^-, ${}^{<\kappa}N_1 \subseteq N_1$, $\mathrm{crit}(k) = \kappa$ and $k(\kappa) \geq \lambda$.

If κ is κ-*unfoldable* and N_0, k are as above, then one can assume in addition that N_0, k are elements of N_1. For a proof see [10].

We remark here that some authors use the name *unfoldability* for *the extension property* from Theorem 1.

Indescribability

Recall that in the usual first order logic (first order calculus) all variables range only over the universe. We define higher orders as follows: In the second order logic there are also variables which range over the power set of the universe. Correspondingly, the $\in$-relation extends to the power set of the universe. The third order logic has even variables which range over the power set of the power set of the universe. Etc.

A Π^n_m formula is a formula of order $n+1$ of the form $\forall x \exists y ... \phi$ (m quantifiers in front of ϕ), where $x, y, ...$ are $(n+1)$th order variables and ϕ is such that all quantified variables in it are of order at most n.

A cardinal number is said to be indescribable in the sense that it can not be distinguished from smaller cardinals in some given language. Namely, the reflection in the definition of indescribability makes many smaller cardinals have similar properties.

Definition 15. A cardinal κ is Π^n_m-*indescribable* iff whenever U is a subset of V_κ and σ is a Π^n_m sentence such that $(V_\kappa, \in, U) \models \sigma$, then for some $\alpha < \kappa$, $(V_\alpha, \in, U \cap V_\alpha) \models \sigma$.

Partition properties

Unlike the notions above, partition properties are purely combinatorial principles. We denote by

$$\kappa \longrightarrow (\lambda)^n_m$$

the following partition property: Every function $F : [\kappa]^n \longrightarrow m$ is constant on $[H]^n$ for some $H \subset \kappa$ with $|H| = \lambda$.

Theorem 1. *The following are equivalent for an inaccessible cardinal* κ*:*

1. κ *is weakly compact;*
2. κ *has the tree property;*
3. $\kappa \longrightarrow (\kappa)^2_2$*;*
4. κ *is* κ*-unfoldable;*
5. κ *is* Π^1_1*-indescribable;*
6. κ *has the extension property: for any* $R \subseteq V_\kappa$ *there is a transitive set* $X \neq V_\kappa$ *and an* $S \subseteq X$ *such that* $(V_\kappa, \in, R) \preccurlyeq (X, \in, S)$*;*
7. *every linear order of cardinality* κ *has an ascending or a descending sequence of order type* κ*;*
8. *for every set* $S \subseteq P(\kappa)$ *of cardinality* κ *there is a nontrivial* κ*-complete filter that decides* S*.*

The proof of the equivalence of 1., 2., 3., 5. and 6. can be found in [12], Theorems 4.5, 6.4, 7.8; the proof of the equivalence of 4. and 6. can be found in [11], Theorem 4.1; and the proof of the equivalence of 1., 7. and 8. (actually 1., 2., 3., 5., 6., 7. and 8.) can be found in [9], Ch. 10, Theorem 2.1.

Cardinal strength

The existence of an inaccessible cardinal which has the tree property (i.e. weakly compact) is not provable from ZFC - it is a large cardinal property. These equivalent formulations of the weak compactness actually say a lot about its cardinal strength. For example, from 3. and 5. in Theorem 1 one can easily define stronger or weaker partition and indescribablity properties. Let us mention some of the stronger properties. A cardinal κ is called *Ramsey* iff $\kappa \longrightarrow (\kappa)_2^{<\omega}$; *totally indescribable* iff it is Π_m^n-indescribable for every $m, n \in \omega$; *unfoldable* iff it is λ-unfoldable for each ordinal λ; etc.

Here is a little broader context of the cardinal strength of weak compactness in terms of the famous large cardinal properties which we have already mentioned:

Definition 16. A cardinal κ is called (*strongly*) *Mahlo* iff $\{\alpha < \kappa \mid \alpha$ is inaccessible$\}$ is stationary in κ.

The Mahlo cardinals are obviously inaccessible, however, the weakly compact cardinals are stronger:

Proposition 4. *If κ is weakly compact, then κ is Mahlo.*

For a proof see [12], 4.7, or [8], 17.19. On the other side, measurable cardinals are stronger than the weakly compacts:

Proposition 5. *If κ is measurable, then κ is weakly compact.*

For a proof see [8], 10.18. Without going further into the cardinal hierarchy, we refer the reader to [12] for a complete survey of large cardinal properties.

3.1.2 The tree property at small cardinals

It is very popular in set theory to investigate which (combinatorial) properties of large cardinals small cardinals have. Especially if these properties characterise the large cardinals. For example, in our case, the tree property at an inaccessible cardinal is equivalent to the weak compactness of that cardinal. So it is very interesting to ask whether, say, the smallest uncountable cardinals, have the tree property, and hence are 'weak compact' in some sense.

The tree property at $\aleph_0$, known as *König's Lemma*, holds in every model of *ZFC*. It is namely very easy in *ZFC* to construct an infinite branch through a given tree of hight ω whose levels have finite size.

If we look at the first uncountable cardinal $\aleph_1$ and ask whether it has the tree property, we also get an answer in *ZFC*, but this time a negative one:

Theorem 2 (Aronszajn)**.** *There is an $\aleph_1$-Aronszajn tree.*

For a proof see [12], 7.10.

Whether the tree property holds at the other small uncountable cardinals can not be decided in ZFC. One can use large cardinals to build models of ZFC in which other small uncountable cardinals have the tree property. We will make an overview of these consistency results in section 3.2.

3.1.3 Other ZFC implications

We finally state some general ZFC results on the tree property which don't apply only for inaccessible cardinals. Here we want to remark that there have not been discovered many ZFC implications about the tree property until the late 1980's, and in the last two decades neither, because the consistency results have been attracting all the interests. However, here are first the few old results:

For a singular cardinal κ there is always a κ-Aronszajn tree: Let $\kappa - \{0\} = \bigcup_{\alpha<\delta} X_\alpha$ be a disjoint union with $\delta < \kappa$ and $|X_\alpha| < \kappa$ for each $\alpha < \delta$, and consider $(\kappa, <_T)$ where $\xi <_T \zeta$ iff $\xi < \zeta$ and $\xi, \zeta \in \{0\} \cup X_\alpha$ for some $\alpha < \delta$. This tree obviously has no cofinal branches. Therefore, a singular cardinal can never have the tree property.

Kurepa proved in 1935 that for regular cardinals κ uniformly thin trees have a cofinal branch, i.e if $(T, <_T)$ is a κ-tree such that for some $\delta < \kappa$ all levels of T have size less than δ, then $(T, <_T)$ has a cofinal branch. For a proof see [12], 7.9.

Kurepa's result has the following two consequences:

Corollary 1 (Silver)**.** If κ is real-valued measurable, then κ has the tree property.

For a proof see [12], 7.12.

An ideal I on κ is said to be *δ-saturated* iff for any $\{X_\alpha | \alpha < \delta\} \subseteq P(\kappa) \backslash I$ there are $\beta < \gamma < \delta$ such that $X_\beta \cap X_\gamma \notin I$.

Corollary 2. If there exists a *δ-saturated* ideal on κ for some $\delta < \kappa$, then κ has the tree property.

For a proof see [12], 16.4.

In 1949 Specker generalised the result from Theorem 2 as follows:

Theorem 3. *If $\kappa^{<\kappa} = \kappa$, then there is a κ^+-Aronszajn tree.*

For a proof see [12], 7.10. A straightforward consequence of this theorem is that if tree property holds at some successor cardinal κ^{++} then 2^κ is at least κ^{++}.

Beside these older results, we mention here the following theorem of Magidor and Shelah from 1996.

Theorem 4. *If a singular cardinal λ is a limit of strongly compact cardinals, then there are no Aronszajn trees of height λ^+.*

3.2 Consistency results on the tree property

The questions whether small cardinals and consecutive cardinals can have the tree property led to many nice consistency results. We state the most important ones.

Theorem 5 (Mitchell [13]). *If there exists a weakly compact cardinal, then for any regular cardinal $\kappa > \omega$, there is an extension in which κ^+ has the tree property.*

Baumgartner and Laver simplified Mitchell's forcing notion and got a little weaker result. They obtained the tree property at $\aleph_2$ by using a weakly compact length countable support iteration of Sacks forcing. This generalizes to the following ([3]): If κ is strongly inaccessible and $\lambda > \kappa$ is weakly compact, then the iteration of Sacks(κ) of length λ, with supports of size κ, yields the tree property at κ^{++} in the extension (for a proof see [1]). This was improved by Kanamori who, assuming $\diamondsuit_\kappa$, showed that the same holds for any regular cardinal κ ([3]).

In fact, Baumgartner showed that the countable support iteration of many other forcings (including ω-Cohen forcing) of weakly compact length produces models in which $\aleph_2$ has the tree property.

The assumption in the theorem above turns out to be optimal, having the same consistency strength as the tree property:

Theorem 6 (Silver [14]). *If a cardinal $\kappa > \aleph_1$ has the tree property, then it is weakly compact in L.*

Mitchell's question whether two consecutive cardinals can simultaneously have the tree property, and whether two weakly compacts would suffice, was answered in 1983:

Theorem 7 (Abraham [5]). *If there exist cardinals $\delta < \kappa < \lambda$ which are respectively regular, supercompact and weakly compact, then there is an extension in which $2^\delta = \delta^{++} = \kappa, 2^{\delta^+} = \delta^{+++} = \lambda$, and κ, λ still have the tree property.*

The lower bound of the consistency strength of two successive cardinals having the tree property was first explored by M. Magidor who proved the following:

Theorem 8 (Magidor [5]). *If there exists a model with two successive cardinals having the tree property, then there is an inner model with a measurable cardinal.*

There have been efforts to improve this lower bound. A partial success in this direction is:

Theorem 9 (Foreman, Magidor, Schindler [15]). *If there exists a pair of successive cardinals with the tree property, and either $2^{\aleph_0} \leq \kappa$ or there exists a measurable cardinal, then Π^1_2 Determinacy holds.*

For another formulation of this result in terms of Woodin cardinals see [6]. The definition of Determinacy can be found in [8].

Theorem 10 (Cummings, Foreman [4]). *If there exist ω supercompact cardinals, then there is an extension in which there are no $\aleph_n$-Aronszajn trees for $2 \leq n < \omega$.*

A lower bound on the consistency strength of ω consecutive cardinals having the tree property (with an additional condition) was also proven in [15]:

Theorem 11 (Foreman, Magidor, Schindler [15]). *If there are ω pairs of consecutive cardinals with the tree property, and their supremum is a strong limit, then Projective Determinacy holds.*

For another formulation of this result see [6]. The definition of Projective Determinacy can be found in [8].

Schindler has shown in an unpublished work that if κ is an inaccessible limit of cardinals $\delta < \kappa$ such that both δ and δ^+ have the tree property, then the Axiom of Determinacy holds in $L(\mathbb{R}^*)$, where $\mathbb{R}^*$ denotes the reals of the Levy collapse $V^{Coll(<\kappa,\omega)}$.

Theorem 12 (Magidor, Shelah [7]). *If there exist a (roughly) huge cardinal and ω supercompact cardinals above it, then in some extension $\aleph_{\omega+1}$ has the tree property.*

This result generalizes to successors of singular cardinals. Note that it did not give an answer to Woodins question from 1980s whether the tree property at $\aleph_{\omega+1}$ implies SCH at $\aleph_\omega$ (SCH is the following *Singular cardinal hypothesis*: if $2^{cf(\kappa)} < \kappa$, then $\kappa^{cf(\kappa)} = \kappa^+$). However, this question was recently negatively answered by Itay Neeman:

Theorem 13 (Neeman [16]). *Relative to the existence of ω supercompact cardinals, there is a model with a strong limit cardinal κ of cofinality ω such that $2^\kappa = \kappa^{++}$ and κ^+ has the tree property.*

The consistency strength of the tree property at $\aleph_{\omega+1}$ is quite high. By a result of Jensen from 1972 the existence of a special κ^+-Aronszajn tree is equivalent to *Weak Square* principle, and the consistency strength of the failure of Weak Square principle at a singular cardinal is proven to be at least one Woodin cardinal.

There are more consistency theorems on the tree property but we decide to complete this list of most significant results by discussing the tree property at the double successors in the next chapter which is the real beginning of this thesis.

Chapter 4

The tree property at the double successors

We start this chapter with an easy observation that the tree property can hold at κ^{++} for a supercompact cardinal κ. The strategy for showing this is as follows. Start with a model in which there exist a supercompact κ and a weakly compact $\lambda > \kappa$. First do the Laver preparation for preservation of supercompactness by $<\kappa$-directed-closed forcings, and then force with $\mathrm{Sacks}(\kappa, \lambda)$. This produces the desired model.

It is also possible to make the tree property hold at the double successor of a measurable cardinal, assuming the existence of something called *weakly compact hypermeasurable* which is the optimal requirement (equiconsistency holds, see [1]), but this requires more work as will be indicated in the following section.

Our aim later in the chapter is to build up on this result singularizing this measurable cardinal such that it remains a strong limit and the tree property at its double successor is preserved. This is an improvement of a result of Matthew Foreman from [4] who uses supercompact cardinals to build a model in which the double successor of a singular strong limit cardinal has the tree property.

4.1 The tree property at the double successor of a measurable

Definition 17. We say that κ is *weakly compact hypermeasurable* if there is weakly compact cardinal $\lambda > \kappa$ and an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$ such that $H(\lambda)^V = H(\lambda)^M$.

Let κ be a weakly compact hypermeasurable cardinal. Define a forcing notion P as follows. Let ρ_0 be the first inaccessible cardinal and let λ_0 be the least weakly compact cardinal above ρ_0. For $k < \kappa$, given λ_k, let ρ_{k+1} be the least inaccessible cardinal above λ_k and let λ_{k+1} be the least weakly compact cardinal above ρ_{k+1}. For limit ordinals $k < \kappa$, let ρ_k be the least inaccessible cardinal greater than or equal to $\sup_{l<k}\lambda_l$ and let λ_k be the least weakly compact cardinal above ρ_k. Note that $\rho_\kappa = \kappa$ and λ_κ is the least weakly compact cardinal above κ.

Let $P_0 = \{1_0\}$. For $i < \kappa$, if $i = \rho_k$ for some $k < \kappa$, let $\dot{Q}_i$ be a P_i-name for the direct sum $\bigoplus_{\eta\leq\lambda_k}\text{Sacks}(\rho_k,\eta)$:=$\{\langle\text{Sacks}(\rho_k,\eta),p\rangle$: η is an inaccessible $\leq \lambda_k$ and $p \in \text{Sacks}(\rho_k,\eta)\}$, where $\langle\text{Sacks}(\rho_k,\eta),p\rangle \leq \langle\text{Sacks}(\rho_k,\eta'),p'\rangle$ iff $\eta = \eta'$ and $p \leq_{\text{Sacks}(\rho_k,\eta)} p'$. Otherwise let $\dot{Q}_i$ be a P_i-name for the trivial forcing. Let $P_{i+1} = P_i * \dot{Q}_i$. Let P_κ be the iteration $\langle\langle P_i,\dot{Q}_i\rangle : i < \kappa\rangle$ with reverse Easton support.

Theorem 14 (N. Dobrinen, S. Friedman)**.** *Assume that V is a model of ZFC in which GCH holds and κ is a weakly compact hypermeasurable cardinal in V. Let $\lambda > \kappa$ be a weakly compact cardinal and let $j : V \to M$ be an elementary embedding with crit(j) $= \kappa$, $j(\kappa) > \lambda$ and $H(\lambda)^V = H(\lambda)^M$, witnessing the weakly compact hypermeasurability of κ. Let $G*g$ be a generic subset of $P = P_\kappa * \dot{Sacks}(\kappa,\lambda)$ over V. Then in $V[G][g]$, $2^\kappa = \kappa^{++}$, κ^{++} has the tree property, and κ is still measurable, i.e. the embedding $j : V \to M$ can be lifted to an elementary embedding $j : V[G][g] \to M[G][g][H][h]$, where $G * g * H * h$ is a generic subset of $j(P)$ over M.*

For a proof see [1].

4.2 The tree property at the double successor of a singular

Theorem 15. *Assume that V is a model of ZFC and κ is a weakly compact hypermeasurable cardinal in V. Then there exists a forcing extension of V in which cof(κ) $= \omega$ and κ^{++} has the tree property.*

Proof. Let $\lambda > \kappa$ be a weakly compact cardinal and let $j : V \to M$ be an elementary embedding with $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $H(\lambda)^V = H(\lambda)^M$. We may assume that M is of the form $M = \{j(f)(\alpha) : \alpha < \lambda, f : \kappa \to V, f \in V\}$. First force as in Theorem 14 with $P = P_\kappa * \dot{\text{Sacks}}(\kappa,\lambda)$ over V to get a model $V[G][g]$ in which $2^\kappa = \kappa^{++}$, κ^{++} has the tree property, and κ is still measurable, i.e. there is an elementary embedding $j : V[G][g] \to M[G][g][H][h]$, where $G * g * H * h$ is a generic subset of $j(P)$ over M.

Now force with the usual Prikry forcing which we will denote by $R := \{(s, A) : s \in [\kappa]^{<\omega}, A \in U\}$, where U is the normal measure on κ derived from j. We say that s is *the lower part* of (s, A). A condition (t, B) is stronger than a condition (s, A) iff s is an initial segment of t, $B \subseteq A$, and $t - s \subset A$. The Prikry forcing preserves cardinals and introduces an ω-sequence of ordinals which is cofinal in κ. It remains to show that it also preserves the tree property on $\kappa^{++} = \lambda$.

In order to get a contradiction suppose that there is a κ^{++}-Aronszajn tree in some R-extension of $V[G][g]$. Then in $V[G]$ there is a Sacks$(\kappa, \lambda) * \dot{R}$ - name $\dot{T}$ of size λ (because Sacks$(\kappa, \lambda) * \dot{R}$ satisfies λ-c.c.) and a condition $(p, \dot{r}) \in$ Sacks$(\kappa, \lambda) * \dot{R}$ which forces $\dot{T}$ to be a κ^{++}-Aronszajn tree. Recall that λ is a weakly compact cardinal in $V[G]$. Therefore, by Theorem 1, there exist in $V[G]$ transitive ZF^--models N_0, N_1 of size λ and an elementary embedding $k : N_0 \to N_1$ with critical point λ, such that $N_0 \supseteq H(\lambda)^{V[G]}$ and $G, \dot{T} \in N_0$.

Since g is also Sacks(κ, λ)-generic over N_0 and the critical point of k is λ, k can be lifted to $k^* : N_0[g] \to N_1[g][K]$, where K is any $N_1[g]$-generic subset of Sacks$(\kappa, [\lambda, k(\lambda)))$ in some larger universe (and where Sacks$(\kappa, [\lambda, k(\lambda)))$ is the quotient Sacks$(\kappa, k(\lambda))/$Sacks(κ, λ), i.e. the iteration of Sacks(κ) indexed by ordinals between λ and $k(\lambda)$). Consider the forcing $R^* := k^*(\dot{R}^g)$ in $N_1[g][K]$ and choose any generic C^* for it such that $k^*(r) \in C^*$, where $r = \dot{r}^g$. Let $C := (k^*)^{-1}[C^*]$ be the pullback of C^* under k^*. Then C is an $N_0[g]$-generic subset of R, because if $\Delta \in N_0[g]$ is a maximal antichain of R then $k^*(\Delta) = k^*[\Delta]$ (since crit(k)=λ and R has the κ^+-c.c.) and by elementarity $k^*(\Delta)$ is maximal in $k^*(R) = R^*$, so $k^*[\Delta]$ meets C^* and hence Δ meets C. It follows that there is an elementary embedding $k^{**} : N_0[g][C] \to N_1[g][K][C^*]$ extending k^*.

We have $r \in C$. So it follows that the evaluation T of $\dot{T}$ in $N_0[g][C]$ is a λ-Aronszajn tree. By elementarity $k^{**}(T)$ is a $k^{**}(\lambda)$-Aronszajn tree in $N_1[g][K][C^*]$ which coincides with T up to level λ. Hence T has a cofinal branch b in $N_1[g][K][C^*]$. We will show that b has to belong to $N_1[g][C]$ (i.e. the quotient Q of the natural projection π : Sacks$(\kappa, k(\lambda)) * \dot{R}^* \to RO($Sacks$(\kappa, \lambda) * \dot{R})$ can not add a new branch), and thereby reach the desired contradiction!

Let us first analyse the quotient Q of the projection above. In $N_1[g][C]$ we have $Q = \{(p^*, (s^*, \dot{A}^*)) \in$ Sacks$(\kappa, k(\lambda)) * \dot{R}^* \mid$ for all $(p, (s, \dot{A})) \in g * C$, $(p, (s, \dot{A}))$ does not force that $(p^*, (s^*, \dot{A}^*))$ is not a condition in the quotient$\}$. Observe that $(p, (s, \dot{A}))$ forces that $(p^*, (s^*, \dot{A}^*))$ is not a condition in Q iff the two conditions are incompatible, which is the case iff one of the following holds:

1. $p^* \restriction \lambda$ is incompatible with p.

2. $s^* \subsetneq s$ and $s \subsetneq s^*$.

3. $p^* \restriction \lambda$ is compatible with p, $s^* \subseteq s$, and $p^* \cup p$ forces that $s - s^* \subsetneq \dot{A}^*$.

4. $p^* \restriction \lambda$ is compatible with p, $s \subseteq s^*$, and $p^* \restriction \lambda \cup p$ forces that $s^* - s \subsetneq \dot{A}$.

It follows that $Q = \{(p^*, (s^*, \dot{A}^*)) \in \mathrm{Sacks}(\kappa, k(\lambda)) * \dot{R}^* \mid (p^*, (s^*, \dot{A}^*))$ is compatible with all $(p, (s, \dot{A})) \in g * C\}$, i.e. Q is the set of all $(p^*, (s^*, \dot{A}^*)) \in \mathrm{Sacks}(\kappa, k(\lambda)) * \dot{R}^*$ such that for all $(p, (s, \dot{A})) \in g * C$ either

1. $p^* \restriction \lambda$ is compatible with p, $s^* \subseteq s$, and $p^* \cup p$ does not force that $s - s^* \subsetneq \dot{A}^*$, or

2. $p^* \restriction \lambda$ is compatible with p, $s \subseteq s^*$, and $p^* \restriction \lambda \cup p$ does not force that $s^* - s \subsetneq \dot{A}\}$.

Equivalently, Q is the set of all $(p^*, (s^*, \dot{A}^*)) \in \mathrm{Sacks}(\kappa, [\lambda, k(\lambda))) * \dot{R}^*$ such that

1. $p^* \in \mathrm{Sacks}(\kappa, [\lambda, k(\lambda)))$,

2. s^* is an initial segment of $S(C)$ (the Prikry ω-sequence arising from C)

3. p^* forces that $\dot{A}^*$ is in $\dot{U}^*$, and

4. for any finite subset x of $S(C)$, some extension q of p^* forces x to be a subset of $s^* \cup \dot{A}^*$.

We now again argue indirectly. Assume that b is not in $N_1[g][C]$, and let $\dot{b}$ in $N_1[g]$ be an $R * \dot{Q}$ - name for b. Identify $k(\dot{T})$ with the $R * \dot{Q}$ - name defined by interpreting the $\mathrm{Sacks}(\kappa, k(\lambda)) * \dot{R}^*$ - name $k(\dot{T})$ in N_1 as an $R * \dot{Q}$ - name in $N_1[g]$. Let $((s_0, A_0), (p_0, (t_0, \dot{A}_0)))$ be an $R * \dot{Q}$ - condition forcing that the Prikry-name $\dot{T}$ is a λ-tree and that $\dot{b}$ is a branch through $\dot{T}$ not belonging to $N_1[g][\dot{C}]$.

Let us take a closer look at the condition $((s_0, A_0), (p_0, (t_0, \dot{A}_0)))$. Note that the forcing Q lives in $N_1[g][C]$, but its elements are in $N_1[g]$, so we can assume that $(p_0, (t_0, \dot{A}_0))$ is a real object and not just a Prikry-name. The Prikry condition (s_0, A_0) forces that p_0 is an element of $\mathrm{Sacks}(\kappa, [\lambda, k(\lambda)))$, that t_0 is an initial segment of $S(\dot{C})$, and that for all finite subsets x of $S(\dot{C})$, some extension of p_0 forces x to be a subset of $t_0 \cup \dot{A}_0$. This simply means that t_0 is an initial segment of s_0 and for every finite subset x of $s_0 \cup A_0$, some extension of p_0 forces x to be a subset of $t_0 \cup \dot{A}_0$.

Moreover, we can assume that s_0 equals t_0. Namely, from the next claim follows that the set of conditions of the form $((s, A), (p, (s, \dot{A})))$ is dense in $R * \dot{Q}$.

Claim. Suppose that p is an element of Sacks$(\kappa, [\lambda, k(\lambda)))$ which forces that $\dot{A}$ is in $\dot{U}^*$. Then there is $A(p) \in U$ such that whenever x is a finite subset of $A(p)$, there is $q \leq p$ forcing x to be contained in $\dot{A}$.

Proof of the claim. Define the function $f : [\kappa]^{<\omega} \to 2$ by

$$f(x) = \begin{cases} 1 & \text{if } \exists q \leq p \; q \Vdash x \subseteq \dot{A} \\ 0 & \text{otherwise.} \end{cases}$$

By normality f has a homogeneous set $A(p) \in U$. It follows that for each $n \in \omega$, $f \restriction [A(p)]^n$ has the constant value 1: Assume on the contrary that there is some $n \in \omega$ such that $f \restriction [A(p)]^n$ has the constant value 0. Then $p \Vdash x \not\subseteq \dot{A}$ for every $x \in [A(p)]^n$, but this is in contradiction with the facts that the measure U^* extends U, $p \Vdash \dot{A} \in U^*$, and $A(p) \in U$.

It is now easy to show that the set of conditions of the form $((s, A), (p, (s, \dot{A})))$ is dense in $R * \dot{Q}$. Assume that $((s, A), (p, (t, \dot{A})))$ is an arbitrary condition in $R * \dot{Q}$. We have $t \subseteq s$. There is some $q \leq p$ which forces that $x := s - t$ is contained in $\dot{A}$. Now by shrinking A to $A(q)$ we get that $((s, A(q)), (q, (s, \dot{A})))$ is a condition which is below $((s, A), (p, (t, \dot{A})))$. We will from now on work with this dense subset of $R * \dot{Q}$.

Now in $N_1[g]$ build a κ-tree E of conditions in Sacks$(\kappa, [\lambda, k(\lambda)))$, whose branches will be fusion sequences, together with a sequence of ordinals $\langle \lambda_\beta : \beta < \kappa \rangle$, each $\lambda_\beta < \lambda$, as follows:

Consider an enumeration $\langle s_\beta : \beta < \kappa \rangle$ of all possible lower parts of conditions in R, i.e. all finite increasing sequences of ordinals less than κ, in which every lower part appears cofinally often. Start building the tree E below the condition p_0 (p_0 was chosen such that $((s_0, A_0), (p_0, (s_0, \dot{A}_0)))$ forces $\dot{b}$ to be a bad branch). Assume that the tree E is built up to level β. Then, at stage β of the construction of the tree, at each node v (a condition in Sacks$(\kappa, [\lambda, k(\lambda))))$, is associated an $X_v \subset [\lambda, k(\lambda)), |X_v| < \kappa$; we will find stronger (incompatible) conditions v_0 and v_1 which on all indices in X_v equal v below level β (for purposes of fusion), i.e. $v_0, v_1 \leq_{\beta, X_v} v$. (The sets X_v can be chosen in different ways, the only condition they have to satisfy is that at the end of the construction of the tree E for every branch through the tree the union of the supports of the conditions (nodes) on the branch is equal to the union of the corresponding X's.) Before we start the construction of the level $\beta + 1$ of the tree E we need to set some notation. Given $i \in [\lambda, k(\lambda))$, let S_i denote Sacks$(\kappa, [\lambda, i))$. For a node v on level β, let $\delta_v = o.t.(X_v)$ and $d_v = |^{\delta_v}(^{\beta+1}2)|$. Let $\langle i^v_\epsilon : \epsilon < \delta_v \rangle$ be the strictly increasing enumeration of X_v and let $i_{\delta_v} = \sup\{i^v_\epsilon : \epsilon < \delta_v\}$. For each $\epsilon < \delta_v$ there are $S_{i^v_\epsilon}$- names $\dot{s}^v_{\epsilon,\zeta}$ ($\zeta \in {}^{\beta+1}2$) such that $S_{i^v_\epsilon} \Vdash$ ($\dot{s}^v_{\epsilon,\zeta}$ is the ζ-th node of Split$_{\beta+1}(v(i^v_\epsilon)))$, where the nodes of $\mathrm{Split}_{\beta+1}(v(i^v_\epsilon))$) are ordered canonically lexicographically (by choosing an $S_{i^v_\epsilon}$- name for an isomorphism between $v(i^v_\epsilon)$ and ${}^{<\kappa}2$). Let $\langle u^v_l : l < d_v\rangle$ enumerate ${}^{\delta_v}({}^{\beta+1}2)$ (the δ_v-length sequences whose entries are elements of ${}^{\beta+1}2$) so that $u^v_l = \langle u^v_l(\epsilon) : \epsilon < \delta_v\rangle$, where each $u^v_l(\epsilon) \in {}^{\beta+1}2$. We now need the following two facts:

Fact 1. Suppose that v is a node and $l < d_v$. We can construct a condition $r \leq v$ called v *thinned through* u_l, denoted by $(v)^{u_l}$, in the following manner: $r \restriction i^v_0 = v \restriction i^v_0$, for each $\epsilon < \delta_v$, $r(i^v_\epsilon) = v(i^v_\epsilon) \restriction \dot{s}^v_{\epsilon,u^v_l(\epsilon)}$, $r \restriction (i^v_\epsilon, i^v_{\epsilon+1}) = v \restriction (i^v_\epsilon, i^v_{\epsilon+1})$ and $r \restriction (i_{\delta_v}, k(\lambda)) = v \restriction (i_{\delta_v}, k(\lambda))$, where $v(i^v_\epsilon) \restriction \dot{s}^v_{\epsilon,u^v_l(\epsilon)}$ is the subtree of $v(i^v_\epsilon)$ whose branches go through $\dot{s}^v_{\epsilon,u^v_l(\epsilon)}$.

Fact 2. Suppose that v and r are conditions in $\mathrm{Sacks}(\kappa, [\lambda, k(\lambda)))$ with $r \leq (v)^{u_l}$. Then there is a condition v' such that $v' \leq_{\beta,X_v} v$ and $(v')^{u_l} \sim r$ (i.e. $(v')^{u_l} \leq r$ and $r \leq (v')^{u_l}$). We say that v' is v *refined through* u_l to r.

Let $\langle v_j : j < 2^{\beta+1}\rangle$ be an enumeration of level β of the tree E and let $\langle u_m\rangle_{m<\sum_{j<2^{\beta+1}} d_{v_j}}$ be an enumeration of $Y := \bigcup_{j<2^{\beta+1}}\{u^{v_j}_l : l < d_{v_j}\}$. In order to construct the next level of the tree we will first thin out all the nodes on level β (by considering all the pairs in Y) and then split each of them into two incompatible nodes. The thinning out is done as follows: Consider u_0 and u_1. If they belong to the same node, i.e. if there is $j < 2^{\beta+1}$ and $l_0, l_1 < d_{v_j}$ s.t. $u_0 = u^{v_j}_{l_0}$ and $u_1 = u^{v_j}_{l_1}$, then no thinning takes place. So assume that u_0 and u_1 belong to different nodes, say v_{j_0} and v_{j_1}, respectively. Use Fact 1 to construct conditions $r_{01} = (v_{j_0})^{u_0}$ and $r_{10} = (v_{j_1})^{u_1}$, i.e. thin v_{j_0} and v_{j_1} through u_0 and u_1 to r_{01} and r_{10}, respectively. Now ask whether there exist extensions r'_{01} and r'_{10} of r_{01} and r_{10}, respectively, such that for some $\gamma_{01} < \lambda$ and some A_{01}, A_{10}, $\dot{A}_{01}$, $\dot{A}_{10}$, $((s_\beta, A_{01}), (r'_{01}, (s_\beta, \dot{A}_{01})))$ and $((s_\beta, A_{10}), (r'_{10}, (s_\beta, \dot{A}_{10})))$ force different nodes on level γ_{01} of $\dot{T}$ to lie on $\dot{b}$. If the answer is 'yes', use Fact 2 to refine v_{j_0} and v_{j_1} through r'_{01} and r'_{10}, respectively, and continue with the next pair: u_0, u_2. And if the answer is 'no', go to the pair u_0, u_2 without refining v_{j_0} and v_{j_1}. The next pairs are u_1, u_2; u_0, u_3 and so on, i.e. all pairs of the form u_δ, u_η, for $\eta < \sum_{j<2^{\beta+1}} d_{v_j}$ and $\delta < \eta$. At the limit stages take lower bounds, they exist since the forcing is κ-closed. Let λ_β be the supremum of (the increasing sequence of) $\gamma_{\delta\eta}$'s. Now extend each node v on level β (after thinning out the whole level) to two incompatible conditions v_o and v_1, such that $v_0, v_1 \leq_{\beta,X_v} v$.

Let α be the supremum of λ_β's. Note that $\alpha < \lambda$, because $\lambda = (\kappa^{++})^{N_1[g]}$. Let p be the result of a fusion along a branch through E. By the claim we can choose $A_0(p) \subseteq A_0$ in U such that $((s_0, A_0(p)), (p, (s_0, \dot{A}_0)))$ is a condition. Extend this condition to some $((s_1(p), A_1(p)), (p^*, (s_1(p), \dot{A}_1(p))))$ which decides $\dot{b}(\alpha)$, say it forces $\dot{b}(\alpha) = x_p$.

As level α of $\dot{T}$ has size $< \lambda$, there exist limits p, q of κ-fusion sequences arising from distinct κ-branches through E for which x_p equals x_q and $s_1(p)$ equals $s_1(q)$. Moreover, we can intersect $A_1(p)$ and $A_1(q)$ to get a common A_1. Say, $((s_1, A_1), (p^*, (s_1, \dot{A}_1(p))))$ and $((s_1, A_1), (q^*, (s_1, \dot{A}_1(q))))$ force $\dot{b}(\alpha) = x$.

Now choose a Prikry generic C containing (s_1, A_1) (and therefore containing (s_0, A_0)). As $\dot{b}$ is forced by $((s_0, A_0), (p_0, (s_0, \dot{A}_0)))$ to not belong to $N_1[g][\dot{C}]$ and $((s_1, A_1), (p^*, (s_1, \dot{A}_1(p))))$ extends $((s_0, A_0), (p_0, (s_0, \dot{A}_0)))$, we can extend $((s_1, A_1), (p^*, (s_1, \dot{A}_1(p))))$ to incompatible conditions $((s_{2_0}, A_{2_0}), (p_0^{**}, (s_{2_0}, \dot{A}_{2_0})))$, $((s_{2_1}, A_{2_1}), (p_1^{**}, (s_{2_1}, \dot{A}_{2_1})))$, with $(s_{2_0}, A_{2_0}), (s_{2_1}, A_{2_1}) \in C$ and $p_0^{**}, p_1^{**} \leq p^*$, which force a disagreement about $\dot{b}$ at some level γ above α.

Now extend $((s_1, A_1), (q^*, (s_1, \dot{A}_1(q))))$ to some $((s_3, A_3), (q^{**}, (s_3, \dot{A}_3)))$ deciding $\dot{b}(\gamma)$ with (s_3, A_3) in C. We can assume without loss of generality that $((s_3, A_3), (q^{**}, (s_3, \dot{A}_3)))$ and $((s_{2_0}, A_{2_0}), (p_0^{**}, (s_{2_0}, \dot{A}_{2_0})))$ disagree about $\dot{b}(\gamma)$. Also w.l.o.g. we can assume that $s_3 \supseteq s_{2_0}$.

Using the claim extend $((s_{2_0}, A_{2_0}), (p_0^{**}, (s_{2_0}, \dot{A}_{2_0})))$ to a stronger condition $((s_3, A_3'), (p^{***}, (s_3, \dot{A}_{2_0})))$ with $A_3' \in U$ and $p^{***} \leq p_0^{**}$.

Now, for some $\beta < \kappa$ we have $s_3 = s_\beta$ where s_β is the βth element of the enumeration of the lower parts (s_3 is not the third element!). Since s_β appears cofinally often in the construction of the tree E, we can assume that the branches which fuse to p and q split in E at some node below level β and go through some nodes v_{j_0} and v_{j_1} at level β. It follows that for some $l < d_{v_{j_0}}$ and $k < d_{v_{j_1}}$,

$$r_1 := ((s_3, A_3'((p^{***})^{u_l^{v_{j_0}}})), ((p^{***})^{u_l^{v_{j_0}}}, (s_3, \dot{A}_{2_0})))$$

and

$$r_2 := ((s_3, A_3((q^{**})^{u_k^{v_{j_1}}})), ((q^{**})^{u_k^{v_{j_1}}}, (s_3, \dot{A}_3)))$$

force different nodes to lie on $\dot{b}$ at level $\gamma > \alpha$. By construction, this means that for some $\eta < \sum_{j<2^{\beta+1}} d_{v_j}$ and $\delta < \eta$,

$$r_3 := ((s_\beta, A_{\delta\eta}), (r'_{\delta\eta}, (s_\beta, \dot{A}_{\delta\eta})))$$

and

$$r_4 := ((s_\beta, A_{\eta\delta}), (r'_{\eta\delta}, (s_\beta, \dot{A}_{\eta\delta})))$$

force different nodes on level $\gamma_{\delta\eta}(< \alpha)$ of $\dot{T}$ to lie on $\dot{b}$. Say, $\dot{b}(\gamma_{\delta\eta}) = y_0$ and $\dot{b}(\gamma_{\delta\eta}) = y_1$, respectively.

On the other side, conditions r_1 and r_2 extend $((s_1, A_1), (p^*, (s_1, \dot{A}_1(p))))$ and $((s_1, A_1), (q^*, (s_1, \dot{A}_1(q))))$, respectively. Therefore we have that r_1 and r_2 also force $\dot{b}(\alpha) = x$.

Note that $(p^{***})^{v_{j0}}_{u_l} \leq r'_{\delta\eta}$ and $(q^{**})^{v_{j1}}_{u_k} \leq r'_{\eta\delta}$. Since any two $R * \dot{Q}$ conditions with the same lower part and compatible Sacks conditions are compatible, we have that $r_1 \parallel r_3$ and $r_2 \parallel r_4$. Let $((s_3, B'), (\bar{p}, (s_3, \dot{B}')))$ be a common lower bound of r_1 and r_3, and let $((s_3, B''), (\bar{q}, (s_3, \dot{B}'')))$ be a common lower bound of r_2 and r_4. The first condition forces $\dot{b}(\gamma_{\delta\eta}) = y_0$ and $\dot{b}(\alpha) = x$, and the second condition forces $\dot{b}(\gamma_{\delta\eta}) = y_1$ and $\dot{b}(\alpha) = x$.

Finally, let $\bar{B} := B' \cap B''$. Then $(s_3, \bar{B})$ forces that $y_0, y_1 <_{\dot{T}} x$ in the ordering of the tree $\dot{T}$, because $\dot{T}$ is a Prikry-name, i.e. all the relations between the nodes of $\dot{T}$ are determined by the Prikry parts of the conditions above. Contradiction. □

4.3 The tree property at $\aleph_{\omega+2}$

Using a forcing notion which makes κ into $\aleph_\omega$ instead of Prikry forcing in the proof of Theorem 15 one can get from the same assumptions the tree property at $\aleph_{\omega+2}$, $\aleph_\omega$ strong limit.

Theorem 16. *Assume that V is a model of ZFC and κ is a weakly compact hypermeasurable cardinal in V. Then there exists a forcing extension of V in which $\aleph_{\omega+2}$ has the tree property.*

Proof. Let $\lambda > \kappa$ be a weakly compact cardinal and let $j : V \to M$ be an elementary embedding with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $H(\lambda)^V = H(\lambda)^M$. We may assume that M is of the form $M = \{j(f)(\alpha) : \alpha < \lambda, f : \kappa \to V, f \in V\}$. First force as in Theorem 14 with $P = P_\kappa * \dot{\mathrm{Sacks}}(\kappa, \lambda)$ over V to get a model $V[G][g]$ in which $2^\kappa = \kappa^{++}$, κ^{++} has the tree property, and κ is still measurable, i.e. there is an elementary embedding $j : V[G][g] \to M[G][g][H][h]$, where $G * g * H * h$ is a generic subset of $j(P)$ over M. Let $M^* := M[G][g][H][h]$.

We now have that M^* is the ultrapower of $V[G][g]$ (by the normal measure U induced by j), i.e. every element in M^* is of the form $j(f)(\kappa)$ for some function $f : \kappa \to V[G][g], f \in V[G][g]$. This is because of the following two facts: every element in M^* is of the form $j(f)(\alpha)$ for some $\alpha < \lambda$ and some $f : \kappa \to V[G][g], f \in V[G][g]$; and every $\alpha < \lambda$ is of the form $j(g)(\kappa)$ for some $g : \kappa \to V[G][g], g \in V[G][g]$. To see the first fact recall that every element in M^* is of the form $j(f)(\alpha)^{G*g*H*h}$ for some $\alpha < \lambda$ and some $f : \kappa \to V, f \in V$. Define $f' : \kappa \to V[G][g], f' \in V[G][g]$ by setting $f'(\alpha)$ to be $f(\alpha)^{G*g}$ whenever $f(\alpha)$ is a name, and 0 otherwise. Then, by elementarity, $j(f')(\alpha) = j(f)(\alpha)^{G*g*H*h}$. To see the second fact note that $2^\kappa = \kappa^{++}$ in $V[G][g]$. Let $\langle X_\alpha : \alpha < \kappa^{++}\rangle$ be an enumeration of the subsets of κ. Identify X_α with α and let $g(\beta) := X_\alpha \upharpoonright \beta$ for $\beta < \kappa$. Then again by elementarity we have $j(g)(\kappa) = j(X_\alpha) \upharpoonright \kappa = X_\alpha$.

Claim. Define $Q' := \mathrm{Coll}((\kappa^{+++})^{M^*}, j(\kappa))^{M^*}$, the forcing that collapses each ordinal less than $j(\kappa)$ to $(\kappa^{+++})^{M^*}$ using conditions of size $\leq (\kappa^{++})^{M^*}$. There exists G' in $V[G][g]$, a generic subset of Q' over M^*.

Proof of the claim. Every maximal antichain $\Delta \subset Q'$ in M^* is actually in $M[G][g][H]$, and thus of the form σ^{G*g*H} for some $j(P_\kappa)$-name σ in M. It follows that Δ is of the form $j(f)(\alpha)^{G*g*H}$ for some $\alpha < \lambda = (\kappa^{++})^{M^*}$, and some $f : \kappa \to V, f \in V$. Since we can assume that $\sigma = j(f)(\alpha)$ is in $V_{j(\kappa)}$ (because $|j(P_\kappa)| = j(\kappa)$ and $j(P_\kappa)$ has $j(\kappa)$-c.c.), it follows that we can assume that $f : \kappa \to V_\kappa$, by modifying it.

For a fixed $f : \kappa \to V_\kappa$ we have that $F_f := \{\Delta \subset Q' \mid \Delta$ maximal antichain, $\Delta \in M[G][g][H]$, and $j(f)(\alpha)^{G*g*H} = \Delta$ for some $\alpha < (\kappa^{++})^{M^*}\}$ is an element of $M[G][g][H]$. Therefore, since Q' is $(\kappa^{+++})^{M^*}$-distributive in $M[G][g][H]$, there exists a single condition $p_f \in Q'$ which lies below every antichain in F_f.

Now, there are $2^\kappa = \kappa^+$ functions $f : \kappa \to V_\kappa$ in V. Enumerate them as f_1, f_2, f_3... We can find conditions $q_\gamma \in Q'$ for $\gamma < \kappa^+$ such that q_γ is a lower bound of $(p_{f_\beta})_{\beta<\gamma}$, because $M[G][g][H]^\kappa \cap V[G][g] \subseteq M[G][g][H]$ and Q' is $(\kappa^+)^V$-closed in $M[G][g][H]$. The sequence $\{q_\gamma \mid \gamma < \kappa^+\}$ generates a filter G' for Q' in $V[G][g]$, which is generic over $M[G][g][H]$. Here ends the proof of the claim.

We now define in $V[G][g]$ a κ^+-c.c. forcing notion $R(G', U)$, or just R, called *Collapse Prikry*, which makes κ into $\aleph_\omega$ and preserves the tree property on κ^{++}: An element $p \in R$ is of the form $(\aleph_0, f_0, \alpha_1, f_1, ..., \alpha_{n-1}, f_{n-1}, A, F)$ where

1. $\aleph_0 < \alpha_1 < \cdots < \alpha_{n-1} < \kappa$ are inaccessibles
2. $f_i \in \mathrm{Coll}(\alpha_i^{+++}, \alpha_{i+1})$ for $i < n-1$ and $f_{n-1} \in \mathrm{Coll}(\alpha_{n-1}^{+++}, \kappa)$
3. $A \in U$, $\min A > \alpha_{n-1}$
4. F is a function on A such that $F(\alpha) \in \mathrm{Coll}(\alpha^{+++}, \kappa)$
5. $[F]_U$, which is an element of $\mathrm{Coll}((\kappa^{+++})^{M^*}, j(\kappa))^{M^*}$, belongs to G'.

The conditions in R are ordered as follows:
$(\aleph_0, g_0, \beta_1, g_1, .., \beta_{m-1}, g_{m-1}, B, H) \leq (\aleph_0, f_0, \alpha_1, f_1, .., \alpha_{n-1}, f_{n-1}, A, F)$ iff

1. $m \geq n$

2. $\forall i < n\ \beta_i = \alpha_i,\ g_i \supseteq f_i$
3. $B \subseteq A$
4. $\forall i \geq n\ \beta_i \in A,\ g_i \supseteq F(\beta_i)$
5. $\forall \alpha \in B\ H(\alpha) \supseteq F(\alpha)$.

We often abbreviate the lower part of a condition by a single letter and write (s, A, F) instead of $(\aleph_0, f_0, \alpha_1, f_1, ..., \alpha_{n-1}, f_{n-1}, A, F)$ where $|s| = n$ denotes the length of the lower part. Let S denote the 'generic sequence', i.e. the Prikry sequence together with the generic collapsing functions.

Claim. R satisfies κ^+-c.c.

Proof of the claim. There are only κ lower parts and any two conditions with the same lower part are compatible, so no antichain has size bigger than κ.

Claim. Let $(s, A, F) \in R$ and let σ be a statement of the forcing language. There exists a stronger condition (s', A^*, F^*) with $|s| = |s'|$ which decides σ.

For a proof see [2].

Claim. Let C be a $V[G][g]$-generic subset of R and let $\langle \aleph_0, \alpha_1, ..., \alpha_n, ...\rangle$ be the Prikry sequence in κ introduced by R. For $j \in \omega$, define $R \upharpoonright j := \mathrm{Coll}(\aleph_0^{+++}, \alpha_1)\times$ $\mathrm{Coll}(\alpha_1^{+++}, \alpha_2) \times ... \times$ $\mathrm{Coll}(\alpha_{j-1}^{+++}, \alpha_j)$. Then $V[G][g][C]$ and $V[G][g][C \upharpoonright j]$ have the same cardinal structure below $\alpha_j + 1$, namely $\aleph_1, \aleph_2, \aleph_3,\ \alpha_1, \alpha_1^+, \alpha_1^{++}, \alpha_1^{+++}, ..., \alpha_{j-1}, \alpha_{j-1}^+, \alpha_{j-1}^{++}, \alpha_{j-1}^{+++}, \alpha_j$, where $C \upharpoonright j$ is the restriction of C to $R \upharpoonright j$.

Proof of the claim. Write R as $R \upharpoonright j * R/(\dot{R} \upharpoonright j)$, where the quotient $R/(\dot{R} \upharpoonright j)$ is defined in the same way as R (using only inaccessibles between α_j and κ). We need to show that $R/(R \upharpoonright j)$ does not add bounded subsets of α_j, but this follows immediately from the last claim.

So we proved that R makes κ into $\aleph_\omega$. It remains to show that it also preserves the tree property on $\kappa^{++} = \lambda$.

In order to get a contradiction suppose that there is a κ^{++}-Aronszajn tree in some R-extension of $V[G][g]$. Then in $V[G]$ there is a Sacks$(\kappa, \lambda) * \dot{R}$ - name $\dot{T}$ of size λ (because Sacks$(\kappa, \lambda) * \dot{R}$ satisfies λ-c.c.) and a condition $(p, \dot{r}) \in$ Sacks$(\kappa, \lambda) * \dot{R}$ which forces $\dot{T}$ to be a κ^{++}-Aronszajn tree. Let $\dot{G}'$ be a Sacks(κ, λ)-name in $V[G]$ for G' of size λ (there is such a name because Sacks(κ, λ) has the λ-c.c. and $|Q'| = \lambda$). We can assume w.l.o.g. that p forces $\dot{G}'$ to be generic over Q'. Recall that λ is a weakly compact cardinal in $V[G]$. Therefore, there exist in $V[G]$ transitive ZF^--models N_0, N_1 of size

λ and an elementary embedding $k : N_0 \to N_1$ with critical point λ, such that $N_0 \supseteq H(\lambda)^{V[G]}$ and $G, \dot{T}, \dot{G}' \in N_0$.

Since g is also Sacks(κ, λ)-generic over N_0 and the critical point of k is λ, k can be lifted to $k^* : N_0[g] \to N_1[g][K]$, where K is any $N_1[g]$-generic subset of Sacks$(\kappa, [\lambda, k(\lambda)))$ in some larger universe (and where Sacks$(\kappa, [\lambda, k(\lambda)))$ is the quotient Sacks$(\kappa, k(\lambda))$/Sacks(κ, λ), i.e. the iteration of Sacks(κ) indexed by ordinals between λ and $k(\lambda)$). Consider the forcing $R^* := k^*(R) = R(k(G'), k(U))$ in $N_1[g][K]$ and choose any generic C^* for it such that $k^*(r) \in C^*$, where $r = \dot{r}^g, R = \dot{R}^g, G' = \dot{G'}^g$. Let $C := (k^*)^{-1}[C^*]$ be the pullback of C^* under k^*. Then C is an $N_0[g]$-generic subset of R because crit(k)=λ and R has the κ^+-c.c. It follows that there is an elementary embedding $k^{**} : N_0[g][C] \to N_1[g][K][C^*]$ extending k^*.

We have $r \in C$. So it follows that the evaluation T of $\dot{T}$ in $N_0[g][C]$ is a λ-Aronszajn tree. By elementarity $k^{**}(T)$ is a $k^{**}(\lambda)$-Aronszajn tree in $N_1[g][K][C^*]$ which coincides with T up to level λ. Hence T has a cofinal branch b in $N_1[g][K][C^*]$. We will show that b has to belong to $N_1[g][C]$ and thereby reach the desired contradiction!

Let us first analyse the quotient Q arising from the natural projection π : Sacks$(\kappa, k(\lambda)) * \dot{R}^* \to RO($Sacks$(\kappa, \lambda) * \dot{R})$. As in the previous section, Q is the set of all $(p^*, (\aleph_0, f_0, \alpha_1, f_1, ..., \alpha_{n-1}, f_{n-1}, \dot{A}^*, \dot{F}^*)) \in$ Sacks$(\kappa, k(\lambda)) * \dot{R}^*$ which are compatible with each $(p, (\aleph_0, g_0, \beta_1, g_1, ..., \beta_{m-1}, g_{m-1}, \dot{A}, \dot{F})) \in g * C$, that is, either

1. $p^* \restriction \lambda$ is compatible with p,

2. $n < m$,

3. for all $i < n$ $\alpha_i = \beta_i \wedge f_i \parallel g_i$,

4. there is $q \leq p \cup p^*$ such that $q \Vdash$ "$\beta_n, ..., \beta_{m-1} \subset \dot{A}^*$ and $\dot{F}^*(\beta_i) \parallel g_i$ for $n \leq i < m$",

or

1. $p^* \restriction \lambda$ is compatible with p,

2. $n \geq m$,

3. for all $i < m$ $\alpha_i = \beta_i \wedge f_i \parallel g_i$,

4. there is $q \leq p \cup p^*$ such that $q \Vdash$ "$\alpha_m, ..., \alpha_{n-1} \subset \dot{A}$ and $\dot{F}(\alpha_i) \parallel f_i$ for $m \leq i < n$".

[Note that in both cases the condition q also forces $\dot{F}$ and $\dot{F}^*$ to be compatible on a measure one set. This is because the weaker condition p (by definition) forces $j(\dot{F})(\kappa)$ to be in $\dot{G}'$, and therefore, by elementarity, also forces $k(j)(k(\dot{F}))(\kappa)$ to be in $k(\dot{G}')$, but $k(j)(k(\dot{F}))(\kappa)$ is the same as $k(j)(\dot{F})(\kappa) = [\dot{F}]_{U^*}$, since the trivial condition forces $k(\dot{F}) = \dot{F}$.]

Equivalently, Q is the set of conditions $(p^*, (\aleph_0, f_0, ..., \alpha_{n-1}, f_{n-1}, \dot{A}^*, \dot{F}^*))$ in $\text{Sacks}(\kappa, [\lambda, k(\lambda))) * \dot{R}^*$ such that

1. $p^* \in \text{Sacks}(\kappa, [\lambda, k(\lambda)))$,
2. $\langle \aleph_0, \alpha_1 ..., \alpha_{n-1} \rangle$ is an initial segment of $S(C)$ (the Prikry sequence arising from C),
3. the collapsing function $\bar{g}_i : \alpha_i^{+++} \to \alpha_{i+1}$ arising from C extends f_i, $i < n$,
4. p^* forces that $\dot{A}^*$ is in $\dot{U}^*$, and that $\dot{F}^*$ is a function on $\dot{A}^*$ such that $\dot{F}^*(\alpha) \in \text{Coll}(\alpha^{+++}, \kappa)$ for each $\alpha \in \dot{A}^*$,
5. for every finite subset $x = \langle \beta_n, ..., \beta_{m-1} \rangle$ of $S(C)$ and every sequence of functions $\langle g_n, ..., g_{m-1} \rangle$ with $g_i \subseteq \bar{g}_i$, $n \leq i < m$, there is some extension q of p^* which forces that x is a subset of $\{\aleph_0, \alpha_1, ..., \alpha_{n-1}\} \cup \dot{A}^*$ and that $\dot{F}^*(\beta_i) \parallel g_i$ for $n \leq i < m$.

We now again argue indirectly. Assume that b is not in $N_1[g][C]$, and let $\dot{b}$ in $N_1[g]$ be an $R * \dot{Q}$ - name for b. Identify $k(\dot{T})$ with the $R * \dot{Q}$ - name defined by interpreting the $\text{Sacks}(\kappa, k(\lambda)) * \dot{R}^*$ - name $k(\dot{T})$ in N_1 as an $R * \dot{Q}$ - name in $N_1[g]$. Let $((s_0, A_0, F_0), (p_0, (t_0, \dot{A}_0, \dot{F}_0)))$ be an $R * \dot{Q}$ - condition forcing that the Prikry-name $\dot{T}$ is a λ-tree and that $\dot{b}$ is a branch through $\dot{T}$ not belonging to $N_1[g][\dot{C}]$.

Let us take a closer look at the condition $((s_0, A_0, F_0), (p_0, (t_0, \dot{A}_0, \dot{F}_0)))$. Say, $s_0 = \langle \aleph_0, f_0, \alpha_1, f_1, ..., \alpha_{n-1}, f_{n-1} \rangle$ and $t_0 = \langle \aleph_0, g_0, \beta_1, g_1, ..., \beta_{m-1}, g_{m-1} \rangle$. Note that the forcing Q lives in $N_1[g][C]$, but its elements are in $N_1[g]$, so we can assume that $(p_0, (t_0, \dot{A}_0, \dot{F}_0))$ is a real object and not just an R-name. The condition (s_0, A_0, F_0) forces $(p_0, (t_0, \dot{A}_0, \dot{F}_0))$ to be an element of $\dot{Q}$. But this simply means that:

1. p_0 is an element of $\text{Sacks}(\kappa, [\lambda, k(\lambda)))$,
2. $\langle \aleph_0, \beta_1, ..., \beta_{m-1} \rangle$ is an initial segment of $\langle \aleph_0, \alpha_1, ..., \alpha_{n-1} \rangle$,
3. $g_i \subseteq f_i$ for $i < m$, and

4. for every finite subset $x = \langle \delta_1, ..., \delta_l \rangle$ of $\{\aleph_0, \alpha_1, ..., \alpha_{n-1}\} \cup A_0$ and every sequence of functions $\langle g_{\delta_1}, ..., g_{\delta_l} \rangle$ with $g_{\delta_i} \supseteq F_0(\delta_i)$ if $\delta_i > \alpha_{n-1}$, and $g_{\delta_i} \supseteq f_i$ if $\delta_i = \alpha_i$ (for some $i < n$), some extension of p_0 forces that x is a subset of $\{\aleph_0, \beta_1, ..., \beta_{m-1}\} \cup \dot{A}_0$ and that $\dot{F}_0(\delta_i) \parallel g_{\delta_i}$ for $i < l$.

Moreover, we can assume that $s_0 = t_0$. Namely, the following claim gives us a nice dense subset of $R * \dot{Q}$ on which we will work from now on.

Claim. Let $((s, A, F), (p, (t, \dot{A}, \dot{F})))$ be an arbitrary condition in $R*\dot{Q}$. There is a stronger condition $((s', A', F'), (p', (s', \dot{A}, \dot{F})))$ with the property that for each $\alpha \in A'$ $p' \Vdash F'(\alpha) \leq \dot{F}(\alpha)$.

Proof of the claim. Say, s is of the form $\langle \aleph_0, f_0, \alpha_1, f_1, ..., \alpha_{n-1}, f_{n-1} \rangle$ and t is of the form $\langle \aleph_0, g_0, \beta_1, g_1, ..., \beta_{m-1}, g_{m-1} \rangle$. Let q be an extension of p which forces that $\{\alpha_m, ..., \alpha_{n-1}\}$ is a subset of $\dot{A}$ and that $f_i \parallel \dot{F}(\alpha_i)$ for $m \leq i < n$. Extend q further to q' to decide $\dot{F}(\alpha_i)$ and let $f'_i := f_i \cup \dot{F}(\alpha_i)$. Define s' to be $\langle \aleph_0, f_0, \alpha_1, f_1, ..., \alpha_{m-1}, f_{m-1}, \alpha_m, f'_m, ..., \alpha_{n-1}, f'_{n-1} \rangle$.

Using the fusion property of $\mathrm{Sacks}(\kappa, [\lambda, k(\lambda)))$ we can find a condition $q'' \leq q'$ and a ground model function F^* on A with $|F^*(\alpha)| \leq \alpha^{++}$ for each α such that $q'' \Vdash \dot{F}(\alpha) \in \mathrm{Coll}(\alpha^{+++}, \kappa) \cap F^*(\alpha)$. It follows that q'' forces that in $\mathrm{Ult}(N_1[g], U)$, the ultrapower of $N_1[g]$ by U, $j_U(\dot{F})(\kappa) \in \mathrm{Coll}(\kappa^{+++}, j_U(\kappa)) \cap j_U(F^*)(\kappa)$, where $|j_U(F^*)(\kappa)| \leq \kappa^{++}$, that is, q'' forces that there are fewer than κ^{+++} possibilities for $j_U(\dot{F})(\kappa)$. Note that the forcing $\mathrm{Coll}(\kappa^{+++}, j_U(\kappa))$ of $\mathrm{Ult}(N_1[g], U)$ is the same as $\mathrm{Coll}(\kappa^{+++}, j_U(\kappa))$ of $\mathrm{Ult}(N_0[g], U)$, because these two ultrapowers agree below $j_U(\kappa)$.

Since $\mathrm{Coll}(\kappa^{+++}, j_U(\kappa))$ is κ^{+++}-closed we can densely often find conditions in $\mathrm{Coll}(\kappa^{+++}, j_U(\kappa))$ which are either stronger than or incompatible with all elements in $j_U(F^*)(\kappa)$. Therefore we can find in G' some $j_U(F')(\kappa) \leq j_U(F)(\kappa)$ with this property, i.e. $q'' \Vdash j_U(F')(\kappa) \leq j_U(\dot{F})(\kappa) \ \vee \ j_U(F')(\kappa) \perp j_U(\dot{F})(\kappa)$. But actually we have $q'' \Vdash j_U(F')(\kappa) \leq j_U(\dot{F})(\kappa)$, because for any generic K below q'', $j_U(F')(\kappa)$ and $j_U(\dot{F}^K)(\kappa)$ can not be incompatible as $k(j_U(F')(\kappa))$ and $k(j_U(\dot{F}^K)(\kappa)) = j_{k(U)}(\dot{F}^K)(\kappa)$ both belong to the guiding generic $k(G')$.

It follows that q'' forces that for some $B \in U, B \subseteq A$, for each $\alpha \in B$, $q'' \Vdash F'(\alpha) \leq \dot{F}(\alpha)$. Extend q'' to some p' deciding B.

Finally, using the claim from the previous section, shrink B to some A' such that every finite subset of A' is forced by some extension of p' to belong to $\dot{A}$. Then we have $((s', A', F'), (p', (s', \dot{A}, \dot{F}))) \leq ((s, A, F), (p, (t, \dot{A}, \dot{F})))$ such that for each $\alpha \in A'$ $p' \Vdash F'(\alpha) \leq \dot{F}(\alpha)$. This proves the claim.

Now in $N_1[g]$ build a κ-tree E of conditions in $\mathrm{Sacks}(\kappa, [\lambda, k(\lambda)))$, whose branches will be fusion sequences, together with a sequence of ordinals $\langle \lambda_\beta :$

$\beta < \kappa\rangle$, each $\lambda_\beta < \lambda$, in the same way as in the last section (using the same notation, Fact 1 and Fact 2):

Let $\langle v_j : j < 2^{\beta+1}\rangle$ be an enumeration of level β of the tree E and let $\langle u_m\rangle_{m<\sum_{j<2^{\beta+1}} d_{v_j}}$ be an enumeration of $Y := \bigcup_{j<2^{\beta+1}}\{u_l^{v_j} : l < d_{v_j}\}$. In order to construct the next level of the tree we will first thin out all the nodes on level β (by considering all the pairs in Y) and then split each of them into two incompatible nodes. The thinning out is done as follows: Consider u_0 and u_1. If they belong to the same node, i.e. if there is $j < 2^{\beta+1}$ and $l_0, l_1 < d_{v_j}$ s.t. $u_0 = u_{l_0}^{v_j}$ and $u_1 = u_{l_1}^{v_j}$, then no thinning takes place. So assume that u_0 and u_1 belong to different nodes, say v_{j_0} and v_{j_1}, respectively. Use Fact 1 to construct conditions $r_{01} = (v_{j_0})^{u_0}$ and $r_{10} = (v_{j_1})^{u_1}$, i.e. thin v_{j_0} and v_{j_1} through u_0 and u_1 to r_{01} and r_{10}, respectively. Now ask whether there exist extensions r'_{01} and r'_{10} of r_{01} and r_{10}, respectively, such that for some $\gamma_{01} < \lambda$ and some A_{01}, A_{10}, F_{01}, F_{10}, $\dot{A}_{01}$, $\dot{A}_{10}$, $\dot{F}_{01}$, $\dot{F}_{10}$, $((s_\beta, A_{01}, F_{01}), (r'_{01}, (s_\beta, \dot{A}_{01}, \dot{F}_{01})))$ and $((s_\beta, A_{10}, F_{10}), (r'_{10}, (s_\beta, \dot{A}_{10}, \dot{F}_{10})))$ force different nodes on level γ_{01} of $\dot{T}$ to lie on $\dot{b}$. If the answer is 'yes', use Fact 2 to refine v_{j_0} and v_{j_1} through r'_{01} and r'_{10}, respectively, and continue with the next pair: u_0, u_2. And if the answer is 'no', go to the pair u_0, u_2 without refining v_{j_0} and v_{j_1}. The next pairs are u_1, u_2; u_0, u_3 and so on, i.e. all pairs of the form u_δ, u_η, for $\eta < \sum_{j<2^{\beta+1}} d_{v_j}$ and $\delta < \eta$. At the limit stages take lower bounds, they exist since the forcing is κ-closed. Let λ_β be the supremum of (the increasing sequence of) $\gamma_{\delta\eta}$'s. Now extend each node v on level β (after thinning out the whole level) to two incompatible conditions v_o and v_1, such that $v_0, v_1 \leq_{\beta, X_v} v$.

Let α be the supremum of λ_β's. Note that $\alpha < \lambda$, because $\lambda = (\kappa^{++})^{N_1[g]}$. Let p be the result of a fusion along a branch through E. As before we can find $A_0(p) \subseteq A_0$ such that $((s_0, A_0(p), F_0), (p, (s_0, \dot{A}_0, \dot{F}_0)))$ is a condition. Extend this condition to some $((s_1(p), A_1(p), F_1(p)), (p^*, (s_1(p), \dot{A}_1(p), \dot{F}_1(p))))$ which decides $\dot{b}(\alpha)$, say it forces $\dot{b}(\alpha) = x_p$.

As level α of $\dot{T}$ has size less than λ, there exist limits p, q of κ-fusion sequences arising from distinct κ-branches of the tree E for which x_p equals x_q and $s_1(p)$ equals $s_1(q)$. Moreover, we can extend $(s_1(p), A_1(p), F_1(p))$ and $(s_1(q), A_1(q), F_1(q))$ to get a common (s_1, A_1, F_1). Say, $((s_1, A_1, F_1)$, $(p^*, (s_1, \dot{A}_1(p), \dot{F}_1(p))))$ and $((s_1, A_1, F_1), (q^*, (s_1, \dot{A}_1(q), \dot{F}_1(q))))$ force $\dot{b}(\alpha) = x$.

Now choose a Collapse Prikry generic C containing (s_1, A_1, F_1) (and hence containing (s_0, A_0, F_0)). As $((s_0, A_0, F_0), (p_0, (s_0, \dot{A}_0, \dot{F}_0))) \Vdash \dot{b} \notin N_1[g][\dot{C}]$ and $((s_1, A_1, F_1), (p^*, (s_1, \dot{A}_1(p), \dot{F}_1(p)))) \leq ((s_0, A_0, F_0), (p_0, (s_0, \dot{A}_0, \dot{F}_0)))$, we can extend $((s_1, A_1, F_1), (p^*, (s_1, \dot{A}_1(p), \dot{F}_1(p))))$ to two incompat. conditions, $((s_{2_0}, A_{2_0}, F_{2_0}), (p_0^{**}, (s_{2_0}, \dot{A}_{2_0}, \dot{F}_{2_0})))$ and $((s_{2_1}, A_{2_1}, F_{2_1}), (p_1^{**}, (s_{2_1}, \dot{A}_{2_1}, \dot{F}_{2_1})))$, with $(s_{2_0}, A_{2_0}, F_{2_0})$, $(s_{2_1}, A_{2_1}, F_{2_1}) \in C$ and $p_0^{**}, p_1^{**} \leq p^*$, which force a disagreement about $\dot{b}$ at some level γ above α.

Now extend $((s_1, A_1, F_1), (q^*, (s_1, \dot{A}_1(q), \dot{F}_1(q))))$ to some stronger condition $((s_3, A_3, F_3), (q^{**}, (s_3, \dot{A}_3, \dot{F}_3)))$ which decides $\dot{b}(\gamma)$ with (s_3, A_3, F_3) in C. Say, $((s_3, A_3, F_3), (q^{**}, (s_3, \dot{A}_3, \dot{F}_3)))$ and $((s_{2_0}, A_{2_0}, F_{2_0}), (p_0^{**}, (s_{2_0}, \dot{A}_{2_0}, \dot{F}_{2_0})))$ don't agree about $\dot{b}(\gamma)$, and say, s_3 is of the form $\langle \aleph_0, f_0, \alpha_1, f_1, ..., \alpha_{n-1}, f_{n-1}\rangle$, and s_{2_0} is of the form $\langle \aleph_0, g_0, \beta_1, g_1, ..., \beta_{m-1}, g_{m-1}\rangle$.

Assume w.l.o.g. that $m < n$. As both (s_3, A_3, F_3) and $(s_{2_0}, A_{2_0}, F_{2_0})$ are in C, we have $\langle \aleph_0, \beta_1, ..., \beta_{m-1}\rangle$ is an initial segment of $\langle \aleph_0, \alpha_1, ..., \alpha_{n-1}\rangle$, $g_i \parallel f_i$ for $i < m$, $\{\alpha_m, ..., \alpha_{n-1}\} \subset A_{2_0}$, and $F_{2_0}(\alpha_i) \parallel f_i$ for $m \le i < n$. Let $f_i' := f_i \cup g_i$ for $i < m$, and $f_i' := f_i \cup F_{2_0}(\alpha_i)$ for $m \le i < n$. Define s_3' to be $\langle \aleph_0, f_0', \alpha_1, f_1', ..., \alpha_{n-1}, f_{n-1}'\rangle$.

Then $((s_3', A_3, F_3), (q^{**}, (s_3', \dot{A}_3, \dot{F}_3))) \le ((s_3, A_3, F_3), (q^{**}, (s_3, \dot{A}_3, \dot{F}_3)))$ is also a condition.

Since $\{\alpha_m, ..., \alpha_{n-1}\} \subset A_{2_0}$, there exists some $p^{***} \le p_0^{**}$ which forces that $\{\alpha_m, ..., \alpha_{n-1}\} \subset \dot{A}_{2_0}$. It follows that there is also some $A_3' \in U$ such that $((s_3', A_3', F_{2_0}), (p^{***}, (s_3', \dot{A}_{2_0}, \dot{F}_{2_0}))) \le ((s_{2_0}, A_{2_0}, F_{2_0}), (p_0^{**}, (s_{2_0}, \dot{A}_{2_0}, \dot{F}_{2_0})))$.

Now, for some $\beta < \kappa$ we have $s_3' = s_\beta$ where s_β is the βth element of the enumeration of the lower parts. Since s_β appears cofinally often in the construction of the tree E, we can assume that the branches which fuse to p and q split in E at some node below level β and go through some nodes v_{j_0} and v_{j_1} at level β. It follows that for some $l < d_{v_{j_0}}$ and $k < d_{v_{j_1}}$,

$$r_1 := ((s_3', A_3'((p^{***})^{u_l^{v_{j_0}}}), F_{2_0}), ((p^{***})^{u_l^{v_{j_0}}}, (s_3', \dot{A}_{2_0}, \dot{F}_{2_0})))$$

and

$$r_2 := ((s_3', A_3((q^{**})^{u_k^{v_{j_1}}}), F_3), ((q^{**})^{u_k^{v_{j_1}}}, (s_3', \dot{A}_3, \dot{F}_3)))$$

force different nodes to lie on $\dot{b}$ at level $\gamma > \alpha$. By construction, this means that for some $\eta < \sum_{j<2^{\beta+1}} d_{v_j}$ and $\delta < \eta$,

$$r_3 := ((s_\beta, A_{\delta\eta}, F_{\delta\eta}), (r_{\delta\eta}', (s_\beta, \dot{A}_{\delta\eta}, \dot{F}_{\delta\eta})))$$

and

$$r_4 := ((s_\beta, A_{\eta\delta}, F_{\eta\delta}), (r_{\eta\delta}', (s_\beta, \dot{A}_{\eta\delta}, \dot{F}_{\eta\delta})))$$

force different nodes on level $\gamma_{\delta\eta}(< \alpha)$ of $\dot{T}$ to lie on $\dot{b}$. Say, $\dot{b}(\gamma_{\delta\eta}) = y_0$ and $\dot{b}(\gamma_{\delta\eta}) = y_1$, respectively.

On the other side, r_1 and r_2 extend $((s_1, A_1, F_1), (p^*, (s_1, \dot{A}_1(p), \dot{F}_1(p))))$ and $((s_1, A_1, F_1), (q^*, (s_1, \dot{A}_1(q), \dot{F}_1(q))))$, respectively. Hence we have that r_1 and r_2 also force $\dot{b}(\alpha) = x$.

Note that $(p^{***})^{u_l^{v_{j_0}}} \le r_{\delta\eta}'$ and $(q^{**})^{u_k^{v_{j_1}}} \le r_{\eta\delta}'$. Since any two $R * \dot{Q}$ conditions with the same lower part and compatible Sacks conditions are compatible (this follows by the same arguments used in the proof of the last claim), we have that $r_1 \parallel r_3$ and $r_2 \parallel r_4$. Let $((s_3', B', H'), (\bar{p}, (s_3', \dot{B}', \dot{H}')))$ be a common lower bound of r_1 and r_3, and let $((s_3', B'', H''), (\bar{q}, (s_3', \dot{B}'', \dot{H}'')))$ be a common lower bound of r_2 and r_4. The first condition forces $\dot{b}(\gamma_{\delta\eta}) = y_0$ and $\dot{b}(\alpha) = x$, and the second condition forces $\dot{b}(\gamma_{\delta\eta}) = y_1$ and $\dot{b}(\alpha) = x$.

Finally, let $\bar{B} := B' \cap B''$ and $\bar{H} := H' \cap H''$. Then $(s_3', \bar{B}, \bar{H})$ forces that $y_0, y_1 <_{\dot{T}} x$ in the ordering of the tree $\dot{T}$, because $\dot{T}$ is a Collapse Prikry-name, i.e. all the relations between the nodes of $\dot{T}$ are determined by the Collapse Prikry parts of the conditions above. Contradiction. $\square$

Chapter 5

Appendix

5.1 Variations of the tree property

Recall that an inaccessible cardinal is weakly compact if and only if it has the tree property. There are also other constructions like trees and principles like the tree property which yield nice characterizations of large cardinal properties.

Lists

Let κ be a regular uncountable cardinal. A set $D = \{d_\alpha : \alpha < \kappa\}$ is called a *κ-list* if $d_\alpha \subseteq \alpha$ for all $\alpha < \kappa$. We say that $d \subseteq \kappa$ is

- a *branch* for D if for all $\alpha < \kappa$ there is $\beta < \kappa$, $\beta \geq \alpha$, such that $d \cap \alpha = d_\beta \cap \alpha$.
- an *ineffable branch* for D if there is a stationary set $S \subseteq \kappa$ such that $d \cap \alpha = d_\alpha$ for all $\alpha \in S$.

Fact 3. A cardinal κ is weakly compact iff every κ-list has a branch.

Fact 4. A cardinal κ is ineffable iff every κ-list has an ineffable branch.

There is now a straightforward generalization of the concept of a κ-list which yields a nice characterization of strong compactness and supercompactness:

Let $\lambda \geq \kappa$. A set $D = \{d_a : a \in [\lambda]^{<\kappa}\}$ is called a *$P_\kappa\lambda$-list* if $d_a \subseteq a$ for all $a \in [\lambda]^{<\kappa}$. We say that $d \subseteq \lambda$ is

- a *branch* for D if for all $a \in [\lambda]^{<\kappa}$ there is $b \in [\lambda]^{<\kappa}$, $b \supseteq a$, such that $d \cap a = d_b \cap a$.

- an *ineffable branch* for D if there is a stationary set $S \subseteq [\lambda]^{<\kappa}$ such that $d \cap a = d_a$ for all $a \in S$.

Theorem 17 (Jech). *A cardinal κ is strongly compact iff for every $\lambda \geq \kappa$, every list on $[\lambda]^{<\kappa}$ has a branch.*

Theorem 18 (Magidor). *A cardinal κ is supercompact iff for every $\lambda \geq \kappa$, every list on $[\lambda]^{<\kappa}$ has an ineffable branch.*

Note that the above characterizations 'contain' inaccessibility, unlike the tree property. However, inaccessibility can be subtracted from them and corresponding weakenings of the above principles in terms of the so-called *thin* and *slender* lists can be defined. For more on this subject we refer the reader to [22].

Other trees

Beside Aronszajn trees there are also other beautiful trees in set theory. We will just mention some of them:

A *special Aronszajn tree* is an Aronszajn tree from which there exists an orderpreserving function into the rational numbers. It is consistent that all Aronszajn trees are special.

A κ^+-*Suslin tree* is a tree of height κ^+ such that every branch and every antichain have cardinality at most κ. Jensen showed that if $V = L$, then for every infinite cardinal κ there exists a κ^+-Suslin tree. Shelah proved that adding a single Cohen real adds a Suslin tree as well.

A *Kurepa* tree is a tree of height ω_1 whose each level is countable and which has at least $\aleph_2$ uncountable branches. Solovay showed that if $V = L$, then there exists a Kurepa tree. The nonexistance of Kurepa trees is actually equiconsistent with an inaccessible cardinal.

Other tree properties

R. Hinnion and O. Esser recently developed the notion of tree property for both directed and ordered sets. This notion is another generalization of the usual tree property. The tree property for directed sets is connected to compactification problems, fixed-point problems in class-theory and large cardinals. See [23] for details and other relevant references.

A. Leshem in [24] considers only $\aleph_1$-trees which are first order definable over $(H_{\omega_1}, \in)$ and shows that the corresponding tree property at $\aleph_1$ is equiconsistent with the existence of a Π^1_1-reflecting cardinal.

5.2 SCH

We have seen from Theorem 13 and Woodin's question which was mentioned there, as well as from Specker's result, that SCH and tree property interact.

Precisely, SCH at $\aleph_\omega$ and the tree property at $\aleph_{\omega+1}$ like each other, it took Neeman ω supercompacts to separate them. But on the other side SCH at $\aleph_\omega$ strong limit kills the tree property at $\aleph_{\omega+2}$. This gives us a flavour of the big open question whether it is consistent to have the tree property at $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$ simultaneously.

In our model SCH fails at $\aleph_\omega$, because $\aleph_\omega$ remains a strong limit in the extension. The best known lower bound for the consistency strength of $\aleph_\omega$ being strong limit with the tree property at $\aleph_{\omega+2}$ is a weakly compact λ such that for each $n < \omega$ there exists $\kappa < \lambda$ with $o(\kappa) = \kappa^{+n}$. This lower bound is necessary and sufficient for GCH (and SCH) to fail at $\aleph_\omega$ strong limit with $2^{\aleph_\omega}$ weakly compact in the core model K. See [18] and [19] for proofs.

It would be interesting to investigate the consistency strength of the tree property at $\aleph_{\omega+2}$ without requiring that $\aleph_\omega$ is a strong limit, or particularly the consistency strength of the tree property at $\aleph_{\omega+2}$ together with $2^{\aleph_1} = \aleph_{\omega+2}$ or with $2^\omega \geq \aleph_\omega$, so that SCH vacuously holds.

Note that Mitchell's forcing can not be used for getting the tree property at $\aleph_{\omega+2}$ from a weakly compact cardinal: An alternative definition of Mitchell's forcing is alternating a κ-Cohen and a κ^+-Cohen for a weakly compact length. This gives the tree property at κ^{++}. But since this works only for regular κ, it is of no use if one wants to get the tree property at $\aleph_{\omega+2}$.

5.3 Aronszajn trees

The tree property on small or on successor cardinals is obtained by starting with a large cardinal which has the tree property and showing by means of the preservation theorems that the forcing which is being used to make this large cardinal into a small cardinal or into a successor preserves the tree property. So there are no Aronszajn trees neither at the beginning nor at the end. However, Aronszajn trees themselves are also very interesting (not only their absence) and have been studied a lot since they were invented by Aronszajn. Abraham, Todorčević, Shelah, Moore and others have many papers on Aronszajn trees.

We refer the reader to Abraham's work for detailed analysis of Aronszajn trees, see for example his recent paper [20] on constructing (special) Aronszajn trees (this paper contains many other good references regarding Aronszajn trees).

Jensen showed that if κ is a strong limit, then $\square_\kappa$ implies that there exists a special κ^+-Aronszajn tree.

The deep interest in Aronszajn trees is also evident from the fact that even nonspecial Aronszajn trees are being studied separately. There is also some work done on specializing Aronszajn trees, etc.

5.4 Open problems

1. What is the consistency strength of $\aleph_\omega$ strong limit with the tree property at $\aleph_{\omega+2}$? [The best known lower bound is a weakly compact λ such that for each $n < \omega$ there exists $\kappa < \lambda$ with $o(\kappa) = \kappa^{+n}$.]

2. What is the consistency strength of $\mathrm{TP}(\aleph_{\omega+2})$ without requiring that $\aleph_\omega$ is a strong limit?

3. What is the consistency strength of $\mathrm{TP}(\aleph_{\omega+2}) + \mathrm{TP}(\aleph_{\omega+3})$?

4. What is the consistency strength of the tree property at every even successor cardinal?

5. Is it consistent with ZFC to have the tree property at each $\aleph_n$, $1 < n < \omega$, and $\aleph_{\omega+2}$?

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Abstract

We say that κ is weakly compact hypermeasurable if there is weakly compact cardinal $\lambda > \kappa$ and an elementary embedding $j : V \to M$ with crit(j) $= \kappa$ such that $H(\lambda)^V = H(\lambda)^M$. Assuming the existence of a weakly compact hypermeasurable cardinal we prove that in some forcing extension $\aleph_\omega$ is a strong limit cardinal and $\aleph_{\omega+2}$ has the tree property. This improves a work of Matthew Foreman who got the same result using stronger assumption, namely he assumed the existence of a supercompact cardinal with a weakly compact above it.

The thesis builds on a paper by Natasha Dobrinen and Sy-D. Friedman who used a generalization of Sacks forcing to reduce the large cardinal strength required to obtain the tree property at the double successor of a measurable cardinal from a supercompact to a weakly compact hypermeasurable cardinal. In the thesis we extend the method of Dobrinen and Friedman to obtain improved upper bounds on the consistency strength of the tree property at the double successor of singular cardinals and at $\aleph_{\omega+2}$ by showing that forcing over Dobrinen's and Friedman's model with Prikry forcing and Collapse Prikry forcing preserves the tree property at the double successor.

Zusammenfassung

Eine Kardinalzahl κ ist schwach kompakt hypermessbar falls es eine schwach kompakte Kardinalzahl $\lambda > \kappa$ und eine elementare Einbettung $j : V \to M$ mit kritischem Punkt κ gibt, so dass $H(\lambda)^V = H(\lambda)^M$. Aus der Annahme dass es eine schwach kompakt hypermessbare Kardinalzahl κ gibt, konstruieren wir ein Modell in dem $\aleph_\omega$ eine unerreichbare Kardinalzahl ist und $\aleph_{\omega+2}$ die Baumeigenschaft hat. Das ist eine Verbesserung des Resultats von Matthew Foreman in dem er Superkompaktheit verwendet um die gleiche Konsistenz zu beweisen.

Diese Dissertation baut auf einem Werk von Natasha Dobrinen und Sy-D. Friedman in dem eine Generalisierung der Sacks-Erzwingungsmethode benutzt wird um eine bessere obere Schranke der Konsistenzstärke der Baumeigenschaft an dem zweiten Nachfolger einer messbaren Kardinalzahl zu finden. In dieser Dissertation erweitern wir die Methode von Dobrinen und Friedman um eine bessere obere Schranke der Konsistenzstärke der Baumeigenschaft an dem zweiten Nachfolger einer singulären Kardinalzahl, bzw. $\aleph_{\omega+2}$, zu finden.

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