DIPLOMARBEIT

Titel der Diplomarbeit

Provability Logic
Completeness and incompleteness results

Verfasser

Georg Smejda

angestrebter akademischer Grad

Magister der Naturwissenschaften (Mag.rer.nat)

Wien, im Jänner 2011

Matrikelnummer:	9804232
Studienkennzahl lt. Studienblatt:	A 405
Studienrichtung lt. Studienblatt:	Mathematik
Betreuer:	O.Univ.-Prof. Sy-David Friedman

To my mother

Abstract

In this text we investigate the logic of the formal provability predicate. A definition of this predicate is given. The notions of the classes of always provable and always true sentences of PA - Peano Arithmetic - are defined. We introduce the modal system GL. In chapter 2 we show that in the propositional case that system completely axiomatizes the class of always provable sentences. From there we introduce the system GLS and show that this system does the same for the class of always true sentences.
In chapter 3 we investigate the same question, but this time for the quantificational case. Our results are negative. There are no systems that axiomatize the class of the always true or the always provable sentences. We show that these sets are $\Pi^0_{\omega+1}$- or Π^0_2-complete respectively.

Abstract in deutscher Sprache

In diesem Text untersuchen wir die Logik des formalisierten Beweisbarkeitsprädikates. Wir geben eine Definition dieses Prädikates. Wir definieren die Begriffe von immer beweisbaren und immer wahren Sätzen der Peano Arithmetik. Wir führen das modal-logische System GL ein.
In Kapitel 2 befassen wir uns mit den Klassen immer beweisbarer und immer wahrer Sätze, die ohne Prädikate gebildet werden können. Wir zeigen, dass GL erstere Klasse vollständig axiomatisiert. Aus GL gewinnen wir das System GLS und zeigen, dass dieses zweitere Klasse vollständig axiomatisiert.
In Kapitel 3 befassen wir uns mit den Klassen immer beweisbarer und immer wahrer Sätze, die mit Prädikaten gebildet werden können. Wir suchen Systeme, die diese Klassen axiomatisieren. Wir zeigen, dass es keine solchen Systeme gibt, da diese Klassen $\Pi^0_{\omega+1}$- bzw. Π^0_2-komplett sind.

Acknowledgements

I would like to thank Prof. Sy Friedman for his supervision and his support.
I also want to thank my many great teachers from whom to learn I had the pleasure over the last decade. Of course I can´t list all of them here, so three names exemplarily:
Prof. Harald Rindler, whose introductional lectures piqued my interest and curiosity for maths. Prof. Ralf Schindler, who introduced me to mathematical logic and who never grew tired of answering my questions. Prof. Joachim Schwermer who taught me that mathematics is an exact science and that details matter as much as the big picture.
Last but not least I want to thank my family - Melanie, Arthur and Leandra - for being so supportive during the time I wrote this paper.

Contents

1 Preliminaries

1.1 Definitions and Notation

In this chapter I will define all notions that will occur in this text as well as give a few remarks on some of the concepts. Let us start with the most basic definitions:

Definition 1.1 (Basic definitions and notation)**.** *$\mathcal{L}$ denotes - as usual - the language of arithmetic. That is: $\mathcal{L} = \{\bar{0}, s, +, \times\}$.*
The language $\mathcal{M}$ of propositional modal logic consists of: propositional variables: $p_0, p_1, p_2, \ldots$; boolean connectives: $\rightarrow, \top, \bot$ and one unary modality: $\Box$.
The language $\mathcal{QML}$ of quantified modal logic is $\mathcal{M}$ together with the quantifier symbols $\forall, \exists$ and predicate symbols: $P_0, P_1, \ldots$.
We assume that the variables $p_0, p_1, p_2, \ldots$ are common to the languages $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{QML}$.
`PA` *is the usual formalization of Peano arithmetic.*
Metavariables ranging over well-formed formulae in the language $\mathcal{L}$ are denoted by capital Greek letters: $\Phi, \Psi, \Xi, \ldots$.
Metavariables ranging over well-formed formulae in the languages $\mathcal{M}$ or $\mathcal{QML}$ are denoted by Greek letters: $\varphi, \psi, \chi, \ldots$. There will be no danger of confusion, since the two cases are handled in separate chapters.
Let $\bar{n}$ be the term representing the number n (the numeral for n). That is: $\bar{n} = s^n\bar{0}$.
For a formula Φ of `PA` *the Gödel number of Φ will be denoted by $\ulcorner\Phi\urcorner$.*
The Gödel number of a finite sequence of `PA` *formulae $\langle\Phi_0, \ldots, \Phi_n\rangle$ - like a* `PA` *proof - will be denoted by $\mathcal{G}$.*

We now come to the essential definition of the formalized proof predicate. If a strong enough formal system like `PA` was to reason about its own provability, it had to prove the formalized version of Gödel's second incompleteness theorem, ie.: "'if it is provable in `PA` that `PA` is consistent, then `PA` is inconsistent"'. Hilbert and Bernays formulated three derivability conditions shown to be sufficient enough for the proof of Gödel's second incompleteness theorem. These conditions were strengthened by Löb in 1950. But still, one had to construct a formalized proof predicate, that met these conditions. I will just sketch the construction, for an extensive overview see [4]

Definition 1.2. *First we specify an elementary (i.e.: all quantifiers occur bounded) formula* $\mathtt{Ax}\left(\ulcorner\Phi\urcorner\right)$ *which is true if and only if* Φ *is an axiom of* $\mathtt{PA}$*. From there one constructs in a standard way an elementary proof predicate* $\mathtt{Prf}\left(\mathcal{G},\ulcorner\Psi\urcorner\right)$*, which expresses "*$\mathcal{G}$ *codes a proof* $\langle\Phi_0,\ldots,\Phi_n\rangle$ *in* $\mathtt{PA}$ *for the formula* Ψ*". To that predicate a provability predicate and a consistency assertion are associated:*

$$\mathtt{Prov}\left(\ulcorner\Psi\urcorner\right) := \exists\mathcal{G} : \mathtt{Prf}\left(\mathcal{G},\ulcorner\Psi\urcorner\right) \quad \textit{and} \quad \mathtt{Con}\left(\mathtt{PA}\right) := \neg\mathtt{Prf}\left(\ulcorner\bot\urcorner\right).$$

The so constructed provability predicate $\mathtt{Prov}$ *satisfies the three derivability conditions of Bernays and Löb:*

1) $\mathtt{PA}\vdash\Phi \quad\Leftrightarrow\quad \mathtt{PA}\vdash\mathtt{Prov}\left(\ulcorner\Phi\urcorner\right)$
2) $\mathtt{PA}\vdash\mathtt{Prov}\left(\ulcorner\Phi\to\Psi\urcorner\right)\to\left(\mathtt{Prov}\left(\ulcorner\Phi\urcorner\right)\to\mathtt{Prov}\left(\ulcorner\Psi\urcorner\right)\right)$
3) $\mathtt{PA}\vdash\mathtt{Prov}\left(\ulcorner\Phi\urcorner\right)\to\mathtt{Prov}\left(\ulcorner\mathtt{Prov}\left(\ulcorner\Phi\urcorner\right)\urcorner\right)$

A corollary of the three derivability conditions is Löbs theorem, which I will state here:

Theorem 1.1.

$$\mathtt{PA}\vdash\mathtt{Prov}\left(\ulcorner\Phi\urcorner\right)\to\Phi \qquad\Leftrightarrow\qquad \mathtt{PA}\vdash\Phi.$$

Proof. see [6] □

Together with the Formalized Deduction Theorem due to Feferman this theorem can be viewed as a version of Gödel's second incompleteness

theorem for $\mathtt{PA} + \neg\Phi$. On the other hand Gödel's second incompleteness theorem can be obtained from Löbs theorem by setting $\Phi = \bot$.
The boxed version of Löbs theorem is taken as an axiom for the provability logic $\mathtt{GL}$, as we see in a moment:

Definition 1.3. $\mathtt{GL}$*, the Gödel - Löb provability logic, is the smallest collection of modal formulae containing the following axiom schemata and closed under the following rules of inference:*

Axiom schemata:

A0)	*Boolean tautologies*	
A1)	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	*(distribution)*
A2)	$\Box(\Box\varphi \to \varphi) \to \Box\varphi$	*(Löb´s axiom)*

Rules of inference:

R1)	$\dfrac{\varphi, \varphi \to \psi}{\psi}$	*(modus ponens)*
R2)	$\dfrac{\varphi}{\Box\varphi}$	*(necessitation)*

Distribution and necessitation are essentially just boxed versions of the derivability conditions. In the beginnings of provability logic $\mathtt{GL}$ was requested to have the transitivity axiom $\Box\varphi \to \Box\Box\varphi$ as well, which is now known to be redundant, as it is derivable from $\mathtt{GL}$:

Lemma 1.1. *1. If* $\mathtt{GL} \vdash \varphi \to \psi$*, then* $\mathtt{GL} \vdash \Box\varphi \to \Box\psi$.

2. $\mathtt{GL} \vdash \Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi$.

3. $\mathtt{GL} \vdash \Box\varphi \to \Box\Box\varphi$

Proof. 1. Say $\mathtt{GL} \vdash \varphi \to \psi$. Then by necessitation: $\mathtt{GL} \vdash \Box(\varphi \to \psi)$. By distribution: $\mathtt{GL} \vdash \Box(\varphi \to \psi) \to \Box\varphi \to \Box\psi$. Therefore, by modus ponens: $\mathtt{GL} \vdash \Box\varphi \to \Box\psi$.

2. Since $\varphi \wedge \psi \to \varphi$ and $\varphi \wedge \psi \to \psi$ are tautologies, we have by 1) of this lemma:

$$\mathtt{GL} \vdash \Box(\varphi \wedge \psi) \to \Box\varphi; \Box(\varphi \wedge \psi) \to \Box\psi$$

$$\mathtt{GL} \vdash \Box(\varphi \wedge \psi) \to \Box\varphi \wedge \Box\psi \qquad \text{by propositional logic}$$

On the other hand one has:

$$\mathtt{GL} \vdash \Box\varphi \to \Box(\psi \to (\varphi \wedge \psi))$$

by (1)

$$\mathtt{GL} \vdash \Box(\psi \to (\varphi \wedge \psi) \to \Box\psi \to \Box(\varphi \wedge \psi))$$

by distribution

$$\mathtt{GL} \vdash \Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$$

by the two above and propositional logic

3. I will make use of 1) and 2) without further mention.

$$\mathtt{GL} \vdash (\varphi \wedge \Box\varphi) \to \varphi$$

$$\mathtt{GL} \vdash \Box(\varphi \wedge \Box\varphi) \to \Box\varphi$$

$$\mathtt{GL} \vdash \varphi \to (\Box(\varphi \wedge \Box\varphi) \to (\varphi \wedge \Box\varphi))$$

$$\mathtt{GL} \vdash \Box\varphi \to \Box(\Box(\varphi \wedge \Box\varphi) \to (\varphi \wedge \Box\varphi))$$

$$\mathtt{GL} \vdash \Box\varphi \to \Box(\varphi \wedge \Box\varphi) \qquad \text{by Löb´s axiom} \quad (*)$$

One also has:

$$\mathtt{GL} \vdash (\varphi \wedge \Box\varphi) \rightarrow \Box\varphi$$
$$\mathtt{GL} \vdash \Box(\varphi \wedge \Box\varphi) \rightarrow \Box\Box\varphi$$
$$\mathtt{GL} \vdash \Box\varphi \rightarrow \Box\Box\varphi \qquad \text{by } *$$

$\Box$

The semantics for GL are of course given by Kripke models.

Definition 1.4. *A Kripke frame, or frame for short, is a pair $\langle K, \prec \rangle$, where K is a nonempty set of so-called worlds or nodes and $\prec$ is a binary Relation on K, the so-called accessibility relation. A forcing relation $\Vdash$ on a frame is a binary relation between nodes and propositional variables, which can uniquely be extended to a relation between nodes and all modal formulae via the conditions:*

1) $k \nVdash \bot, k \Vdash \top \qquad \forall k \in K$
2) $k \Vdash \varphi \rightarrow \psi$ *iff* $(k \nVdash \varphi$ *or* $k \Vdash \psi)$
3) $k \Vdash \Box\varphi$ *iff* $\forall k' \left(k \prec k' \rightarrow k' \Vdash \varphi\right)$

The triple $\mathcal{K}_{\Vdash} = \langle K, \prec, \Vdash \rangle$ is called a Kripke model. I will omit the subscript $\Vdash$ whenever it leads to no confusion.
I will be saying "'φ is true at k"' or "'φ holds at k"' if $k \Vdash \varphi$.
$\mathcal{K} \Vdash \varphi$ means that φ is forced at the root of K. Note that there is not necessarily a root.
$\mathcal{K} \vDash \varphi$ is short for: $\forall k \in K : k \Vdash \varphi$
I will say that φ is valid in a frame $\langle K, \prec \rangle$ if for every possible $\Vdash$ there is: $\mathcal{K}_{\Vdash} \vDash \varphi$

A very important concept is the one of an interpretation (some authors call it realization), which is a mapping from $\mathcal{M}$ or $\mathcal{QML}$ into $\mathcal{L}$. But here we have to be careful: The cases of $\mathcal{M}$ and $\mathcal{QML}$ are handled differently, so I will be giving the definitions in chapter 2 and chapter 3 respectively. We will also be needing a concept of the length of a propositional modal formula, which is a highly natural one:

Definition 1.5. *For each well-formed modal formula φ we define the length of the formula as a number $l(\varphi) \in \mathbb{N}$ as follows:*

1. *If p is a variable, $l(p) = 1$;*
2. *$l(\bot) = 1 = l(\top)$;*
3. *$l(\psi \to \chi) = l(\psi) + l(\chi) + 1$;*
4. *$l(\Box\varphi) = l(\varphi) + 1$.*

1.2 A short review of recursion theory

We will be needing some definitions and results of another branch of logic, namely of recursion theory. I will give a short review in this section. I will only be defining primitive recursive functions, since partial recursive functions don´t occur in our context. Note that the primitive recursive functions are fully contained in the class of the partial recursive functions.
Throughout this section ϕ will denote a (partial) recursive function.

Definition 1.6. *The class of primitive recursive functions is the the smallest class $\mathcal{C}$ of functions such that:*

1. *All constant functions: $\lambda x_1 x_2 \cdots x_k [m]$ are in $\mathcal{C}$ $\quad 1 \leq k, 0 \leq m$.*
2. *The successor function: $\lambda x [x+1]$ is in $\mathcal{C}$.*
3. *All identity functions: $\lambda x_1 \cdots x_k [x_i]$ are in $\mathcal{C}$ $\quad 1 \leq i \leq k$.*

4. *If f is a function of k variables in $\mathcal{C}$ and $g_1, \ldots, g_k$ are each functions of m variables in $\mathcal{C}$, then the function*
$\lambda x_1 \cdots x_m \left[f\left(g_1\left(x_1, \ldots, x_m\right), \ldots, g_k\left(x_1, \ldots, x_m\right)\right)\right]$ *is in* $\mathcal{C}$
$1 \leq k, m$.

5. *If h is a function of $k+1$ variables in $\mathcal{C}$ and g is a function of $k-1$ variables in $\mathcal{C}$, then the unique function f of k variables satisfying*

$$\begin{aligned} f\left(0, x_2, \ldots, x_k\right) &= g\left(x_2, \ldots, x_k\right) \\ f\left(y+1, x_2, \ldots, x_k\right) &= h\left(y, f\left(y, x_2, \ldots, x_k\right), x_2, \ldots, x_k\right) \end{aligned}$$

is in $\mathcal{C} \quad 1 \leq k$.

In 5) if $k = 1$, then "'function of zero variables in $\mathcal{C}$"' means a fixed integer.

It follows directly from this definition that a function f is primitive recursive iff there is a finite sequence $f_1, \ldots, f_n$ of functions with $f_n = f$ and for each $j \leq n$ f_j is either in $\mathcal{C}$ by 1), 2) or 3) or f_j is directly obtainable from some $f_i, i < j$ by 4) or 5). Such a sequence $f_1, \ldots, f_n$ is called a derivation for f as primitive recursive function. We use the Turing characterization for partial recursive functions.

Definition 1.7. *Consider a mechanical device which has associated with it a paper tape of infinite length in both directions, which we will refer to as L(eft) and R(ight). The tape is sectioned into spaces of equal length which we will refer to as cells. A cell can either be blank or it has a 1 written in it. The device is arranged, so that the tape runs through it and so that there is room for one cell to lie within it. At each calculation step the machine can perform exactly one of four basic operations: It can write "1" on the cell it is examining, if no "1" already appears in it; it can erase the cell it is examining - making it blank - if the cell is not blank already; it can shift its attention one cell to the right or it can*

shift its attention one cell to the left.
The device begins its calculation in a certain internal configuration and after each calculation step takes on one of a fixed finite set of possible configurations. So the action the machine takes is determined by its internal configuration or state and the content of the examined cell. In strict mathematical language:
Let $T = \{0, 1\}$ *and* $S = \{0, 1, L, R\}$*. Then a Turing machine is a mapping of a finite subset of* $\mathbb{N} \times T$ *into* $S \times \mathbb{N}$*.*
Here T *represents conditions of a tape cell,* $\mathbb{N}$ *represents possible labels for internal states and* S *represents operations to be performed; where operation "0" makes no change if the examined cell is already blank and operation "1" makes no change if the examined cell has already a "1" written in it.*

It is possible to associate a (partial) recursive function with each Turing machine, moreover it has been shown, that the characterization of Turing is equivalent to the characterization of Kleene, as well as to the characterizations of Church, Post, Markov and others. It has also been shown that these characterizations suffice to demonstrate a wide variety of partial functions to be partial recursive functions.
Note that it is possible to effectively list all Turing machines in an algorithmic way. Assume now that we have selected such a listing procedure and we keep it fixed.

Definition 1.8. T_x *is the Turing machine (i.e.: set of quadruples) associated with the integer* x *in the fixed list of all Turing machines. (It comes at the* $(x+1)$*st place in the list.)* x *is called index or Gödel number of* T_x*.*
ϕ_x *is the partial recursive function determined by* T_x*.* x *is also called the index or Gödel number of* ϕ_x*.*

We now give the shortest form of the recursion theorem and prove it.

Theorem 1.2. *Let f be any recursive function; then there exists an n such that:*

$$\phi_n = \phi_{f(n)}$$

n is called a fixed-point value for f.

Proof. Let any u be given. Define a recursive function ψ by the following instructions:
To compute $\psi(x)$, first use T_u with input u. If this terminates and gives output w, use T_w with input x. If this terminates, take its output as $\psi(x)$. This can be summarized:

$$\psi(x) = \begin{cases} \phi_{\phi_u(u)}(x), & \text{if } \phi_u(u) \text{ convergent;} \\ \text{divergent}, & \text{if } \phi_u(u) \text{ divergent.} \end{cases}$$

The instructions for ψ depend uniformly on u. Take $\tilde{g}$ to be the recursive function which yields, from u, the Gödel number for these instructions for ψ. Thus:

$$\phi_{\tilde{g}(u)}(x) = \begin{cases} \phi_{\phi_u(u)}(x), & \text{if } \phi_u(u) \text{ convergent;} \\ \text{divergent}, & \text{if } \phi_u(u) \text{ divergent.} \end{cases}$$

Now let any recursive function f be given. Then $f\tilde{g}$ is a recursive function. Let v be a Gödel number for $f\tilde{g}$. Since $\phi_v = f\tilde{g}$ is total, $\phi_v(v)$ is convergent. Therefore by putting v for u in the definition of $\tilde{g}$ we have

$$\phi_{\tilde{g}(v)} = \phi_{\phi_v(v)} = \phi_{f\tilde{g}(v)}.$$

Therefore $n = \tilde{g}(v)$ is a fixed-point value. $\square$

We will also be needing some basic facts about the arithmetical hierarchy, which I will state now.

Definition 1.9. *A set A (therefore also a relation R) is recursive if it possesses a recursive characteristic function, i.e.: there exists a recursive function f such that for all x:*

$$x \in a \Rightarrow f(x) = 1 \quad \textit{and} \quad x \in \overline{A} \Rightarrow f(x) = 0.$$

Informally, A is recursive if there is an effective procedure which decides for any given x if $x \in A$ or $x \notin A$.
A is called recursively enumerable (r.e.) if either $A = \emptyset$ or there exists a recursive f such that $A =$ range f. Informally, A is recursively enumerable, if there exists an effective procedure to list the members of A.

Note that if A is recursive, so is $\overline{A}$. There is a connection between these two concepts

Theorem 1.3.

$$A \textit{ is recursive} \quad \Leftrightarrow \quad A \textit{ and } \overline{A} \textit{ are both recursively enumerable.}$$

Proof. Can be found in [9]. □

Definition 1.10. *A relation R is Σ^0_0 in A, or Π^0_0 in A if and only if R is recursive in A.*
A n-place relation R is Σ^0_{m+1} in A if for some (n+1)-place Relation S that is Π^0_m in A one has: $R = \{\mathtt{i} | \exists j S(\mathtt{i}, j)\}$.
A n-place relation R is Π^0_{m+1} in A if for some (n+1)-place Relation S that is Σ^0_m in A one has: $R = \{\mathtt{i} | \forall j S(\mathtt{i}, j)\}$.

Note that R is Σ^0_m in A iff $\bar{R}$ is Π^0_m in A. Also the Σ^0_1 relations in A, are the recursively enumerable relations in A.

Theorem 1.4.

$$\Sigma^0_m \cup \Pi^0_m \subset \Sigma^0_{m+1} \cap \Pi^0_{m+1}$$

Proof. Can be found in [9]. □

A set is called arithmetical if it can be defined by a formula in the language $\mathcal{L}$ of arithmetic. One can show that a set is arithmetical iff it is Σ^0_m or Π^0_m for some m.

Definition 1.11. *A set S is called Π^0_m-complete in A ($m > 0$), if it is Π^0_m in A and for every other set S' that is Π^0_m in A there is a recursive function f, so that $S' = \{i|f(i) \in S\}$.*
"'Σ^0_m-complete"' is defined analogously.

Kleene´s enumeration theorem, which I will not state here, allows one to show, that if a set S is Π^0_m-complete in A, then it is not Σ^0_m in A. Therefore a Π^0_2-complete set is not Σ^0_2, thus not Π^0_1, Σ^0_1 or recursive, and a Π^0_1-complete set is not Σ^0_1, thus not recursive.

2 Propositional provability logic: The Solovay theorems

In this chapter I will discuss the Solovay´s first and second arithmetic completeness theorem, which basically state that propositional reasoning about the formalized provability predicate can sufficiently be described in arithmetic itself and even more, that it is decidable. I will stay closely to Solovay´s paper [1], so all results are due to him, if not otherwise noted. However I have changed his notation into a more contemporary one, following the notation of [5]. As mentioned above we will need the concept of an interpretation.

Definition 2.1. *An interpretation of $\mathcal{M}$ in $\mathcal{L}$ is a mapping $(.)^*$ from the set of the propositional variables to the set of arithmetical sentences, additionally satisfying following requirements:*

- $(\bot)^* = \bot$; $(\top)^* = \top$
- $(\varphi \to \psi)^* = \varphi^* \to \psi^*$
- $(\Box\varphi)^* = \mathtt{Prov}(\ulcorner\varphi^*\urcorner)$

We will say that a modal formula φ is PA*-valid or always provable if for every interpretation $(.)^*$; φ^* is a theorem of* PA*. (i.e.: $\forall * \ \mathtt{PA} \vdash \varphi^*$). We call a modal formula φ always true or $\mathbb{N}$-valid if it is true in the standard model for every interpretation. (i.e.: $\forall * \ \mathbb{N} \vDash \varphi^*$).*

At first I prove that GL is sound with respect to the arithmetical interpretation and I do so, because in the proof we will see, that Löb´s Axiom is essentially a reformulation of Gödel´s second incompleteness theorem.

Lemma 2.1. *Every theorem of* GL *is* PA*-valid, i.e.: for all interpretations $*$*

$$\text{If } \mathtt{GL} \vdash \varphi, \text{ then } \mathtt{PA} \vdash \varphi^*.$$

Proof. The proofs works by induction on the number of axioms and rules of inference used in a GL-proof of φ. So we need to check different cases,

according to which rule or axiom scheme was the last to be cited in the proof of φ.
The case of Boolean tautologies and of the modus ponens is evident.
It can be shown in $\mathtt{PA}$ that the theorems of $\mathtt{PA}$ are closed under the modus ponens, which handles A1.
Note that $\mathtt{Prov}(\ulcorner\varphi\urcorner)$ is a Σ_1^0 formula. That means it is provably equivalent in $\mathtt{PA}$ to a formula of the form $\exists y : R\,(x, y)$, where R is a primitive recursive predicate. It is a known fact about Σ_1^0 sentences, that if they are true in a model of $\mathtt{PA}$, they are provable in $\mathtt{PA}$. So the lemma is true, if the last step in a $\mathtt{GL}$ proof is the necessitation rule.
It remains to show, what happens if the last instance of a $\mathtt{GL}$ proof is Löb´s axiom. I argue in $\mathtt{PA}$. Given a sentence φ such that $\mathtt{PA} \vdash \mathtt{Prov}\,(\ulcorner\varphi\urcorner) \to \varphi$ we must show $\mathtt{PA} \vdash \varphi$. One has:

$$\begin{aligned}
&\mathtt{PA} \vdash \mathtt{Prov}\,(\ulcorner\varphi\urcorner) \to \varphi && \Rightarrow \\
&\mathtt{PA} \vdash \neg\varphi \to \neg\mathtt{Prov}\,(\ulcorner\varphi\urcorner) && \Rightarrow \\
&\mathtt{PA} \vdash \neg\varphi \to \mathtt{Con}\,(\mathtt{PA} + \ulcorner\neg\varphi\urcorner)
\end{aligned}$$

But then $\mathtt{PA}+\neg\varphi$ proves its own consistency, so by Gödel´s second incompleteness theorem the system $\mathtt{PA}+\neg\varphi$ is inconsistent. So

$\mathtt{PA} \vdash \varphi$ □

The next goal is to identify those Kripke frames in which the theorems of $\mathtt{GL}$ are valid. Recall that this means to characterize those Kripke frames $\langle K, \prec\rangle$ so that for every possible corresponding Kripke model $\mathcal{K} = \langle K, \prec, \Vdash\rangle$ and for every $\varphi \in \mathcal{T}$, $\mathcal{K} \vDash \varphi$.

Definition 2.2. *A partially ordered set P is said to satisfy the ascending chain condition (acc) if every ascending chain of elements eventually* *terminates. Equivalently, given any sequence of elements of P*

$$a_1 \le a_2 \le a_3 \le \cdots ,$$

there exists a positive integer n such that

$$a_n = a_{n+1} = a_{n+2} = \cdots .$$

Theorem 2.1. *Let $\langle K, \prec \rangle$ be a Kripke frame. Then the following are equivalent:*

1) Every theorem of `GL` *is valid in $\langle K, \prec \rangle$*
2) $\prec$ is transitive and has the ascending chain condition (acc)

Proof. Let Δ be the set of modal formulas valid in a frame $\langle K, \prec \rangle$. It is easy to check, that Δ is closed under the two rules of inference and contains all instances of the axiom schemata A0 and A1.
($\Rightarrow$): Suppose A2 is valid in $\langle K, \prec \rangle$ and that $x \prec y$, $y \prec z$.
Let p be some propositional variable. Choose $\Vdash$ so, that $w \Vdash p$ iff $x \prec w$. Then $x \Vdash \Box p$. By hypothesis and Lemma 1.1(3) $x \Vdash \Box\Box p$. That means $y \Vdash \Box p$ and therefore $z \Vdash p$. Hence $x \prec z$. So $\prec$ is transitive.
Suppose now that $\prec$ has not got the acc.
Then there is an infinitively ascending chain $a_0 \prec a_1 \prec a_2 \prec \ldots$. Let A be the set containing precisely all the a_i. Let p be a propositional variable. Choose $\Vdash$ so that $x \nVdash p$ iff $x \in A$.
Since A is an infinite chain $x \nVdash \Box p$ for all $x \in A$. That means $\Box p \to p$ holds at every point in K. (The hypothesis is false at points in A, the conclusion is true at points not in A.) So $\mathcal{K}_{\Vdash} \vDash \Box(\Box p \to p)$. Finally $\Box(\Box p \to p) \to \Box p$ is false at each point in A, which contradicts Löb´s axiom.
($\Leftarrow$): Now let $\prec$ be transitive and have the acc.
We need to show, that Löb´s axiom is valid in $\langle K, \prec \rangle$. Suppose not.
Let $x \in K$, $\varphi \in \mathcal{F}$ and $\Vdash$ be so, that: $x \nVdash \Box(\Box\varphi \to \varphi) \to \Box\varphi$, which

means that $x \Vdash \Box(\Box\varphi \to \varphi)$ and $x \nVdash \Box\varphi$.
Let $A := \{a \in K : x \prec a \text{ and } a \nVdash \varphi\}$. A is not empty and since $\prec$ has the acc, A has a maximal element $\bar{a}$. Now let $y \in K$ be arbitrary with: $\bar{a} \prec y$. Since $\prec$ is transitive and $\bar{a}$ is maximal in A: $x \prec y$ and $y \notin A$. Hence $y \Vdash \varphi$. Since y was arbitrary $\bar{a} \Vdash \Box\varphi$. Because $x \Vdash \Box(\Box\varphi \to \varphi)$ we have that $\bar{a} \Vdash \Box\varphi \to \varphi$ and therefore $\bar{a} \Vdash \varphi$ which contradicts $\bar{a} \in A$. $\Box$

Theorem 2.2. *Let φ be a modal formula which is not a theorem of* GL. *Then there is a model $\mathcal{K} = \langle K, \prec, \Vdash \rangle$ and a $k_0 \in K$ such that:*

1. *$k_0 \nVdash \varphi$*
2. *$\prec$ is transitive and has the acc. (So by the previous theorem* GL *is valid in $\langle K, \prec \rangle$)*
3. *If $k \in K$ then $k_0 \preceq k$*
4. *K is finite*

Proof. Fix a modal formula φ which is not a theorem of GL. Let E be the set of subformulae of φ and let $A \subseteq E$ consist of propositional variables and formulae with principal connective $\Box$, say $A = \{\chi_1, \ldots, \chi_n\}$. Think of A as the set of atomic subformulae of φ. Each truth assignment β from the set A to the truth set $2 = \{0, 1\}$ has a canonical prolongation, which I will again denote by β to a map $\beta : E \longrightarrow 2$. This follows from the fact that each formula in E is a boolean combination of formulae in A. Let $B = \{\beta | \beta : A \longrightarrow 2\}$. We construct a modal formula $\Phi(\beta)$ associated to each truth assignment.

$$\Phi(\beta) = \chi_1' \wedge \ldots \wedge \chi_n', \quad \text{where: } \chi_i' = \begin{cases} \chi_i, & \text{if } \beta(\chi_i) = 1 \\ \neg\chi_i, & \text{if } \beta(\chi_i) = 0 \end{cases}$$

Let $K_1 \subseteq B$ be the subset

$$K_1 := \{\beta | \beta : A \longrightarrow 2 \text{ and } \neg\Phi(\beta) \text{ is not a theorem of } \mathtt{GL}\}.$$

Note that for some $\beta \in K_1$ one must have: $\beta(\varphi) = 0$, otherwise one would have $\beta(\Phi(\beta) \to \varphi) = 1$ for each $\beta \in K_1$. But that would mean, that φ is a theorem of GL.
Pick a $\beta_0 \in K_1$ so that $\beta_0(\varphi) = 0$.

For $\beta \in K_1$ let $rank(\beta)$ be the number of formulae in A of the form $\Box\psi$ such that $\beta(\Box\psi) = 1$.
Now for the definition of a binary relation $\lhd$ on K_1:
Put $\beta \lhd \beta'$ iff

1. $rank(\beta) < rank(\beta')$
2. If $\beta(\Box\psi) = 1, \qquad$ then $\beta'(\Box\psi) = 1$ and $\beta'(\psi) = 1$

It is easily checked, that $\lhd$ is transitive and has the ACC. Put $K := \{\beta \in K_1 | \beta_0 \unlhd \beta\}$. Define a binary relation $\prec$ on K by putting $\beta' \prec \beta$ iff $\beta' \lhd \beta$ for $\beta, \beta' \in K$. Since A is a finite set so is K. Even more: A has at most $l(\varphi)$ elements, so K has at most $2^{l(\varphi)}$ elements.
So we constructed the frame of our model. It remains to construct the forcing relation, so that the condition 1) is met. To do so it is necessary to prove an auxiliary lemmata and a helpful claim first.

Lemma 2.2. *Let $\beta \in K$. Let $\Box\psi \in A$ such that: $\beta(\Box\psi) = 0$. Then there is a $\beta' \in K$ with $\beta \prec \beta'$ and $\beta'(\psi) = 0$.*

Proof. Let $\psi_1, \ldots, \psi_n$ be those formulae in E such that $\Box\psi_i \in A$ and $\beta(\Box\psi_i) = 1$. We show that there is a truth assignment $\beta' \in K_1$ such that:

1. $\beta'(\psi_i) = 1$ and $\beta'(\Box\psi_i) = 1$

2. $\beta'(\psi) = 0$ and $\beta'(\Box\psi) = 1$

It then readily follows that $rank(\beta) < rank(\beta')$ and therefore $\beta \lhd \beta'$, so $\beta_0 \lhd \beta'$ and finally $\beta' \in K$.
So suppose now for every β' which satisfies 1) and 2) we have $\beta' \notin K_1$. That means that $\Phi(\beta')$ is refutable in GL. Let $\vartheta = \psi_1 \wedge \ldots \wedge \psi_n$. Lemma 1.1 (2) shows that $\texttt{GL} \vdash \Box\vartheta \leftrightarrow (\Box\psi_1 \wedge \ldots \wedge \Box\psi_n)$. Now if for every β' satisfying 1) and 2) $\Phi(\beta')$ is refutable, that means $\texttt{GL} \vdash \neg\Box\vartheta \vee \neg\Box\psi$. Therefore:

$$\texttt{GL} \vdash \Box\vartheta \wedge \vartheta \rightarrow (\Box\psi \rightarrow \psi) \tag{i}$$

$$\texttt{GL} \vdash \Box(\Box\vartheta \wedge \vartheta) \rightarrow \Box(\Box\psi \rightarrow \psi) \qquad \text{by Lemma 1.1 (1)} \tag{ii}$$

$$\texttt{GL} \vdash \Box(\Box\psi \rightarrow \psi) \rightarrow \Box\psi \qquad \text{by Löb´s axiom} \tag{iii}$$

$$\texttt{GL} \vdash \Box\vartheta \rightarrow \Box\Box\vartheta \qquad \text{by Lemma 1.1 (3)} \tag{iv}$$

$$\texttt{GL} \vdash \Box\vartheta \rightarrow \Box\Box\vartheta \wedge \Box\vartheta \qquad . \tag{v}$$

$$\texttt{GL} \vdash \Box\vartheta \rightarrow \Box(\Box\vartheta \wedge \vartheta) \qquad \text{by Lemma 1.1(2).} \tag{vi}$$

$$\texttt{GL} \vdash \Box\vartheta \rightarrow \Box\psi \qquad \text{by (ii), (iii) and (vi).}$$

But since $\Phi(\beta)$ has among others the conjuncts $\Box\vartheta$ and $\neg\Box\psi$ this entails that $\Phi(\beta)$ is refutable in GL, contradicting $\beta \in K_1$. So the lemma is proven. $\Box$

We now define a forcing relation $\Vdash \subseteq K \times P$, where K is the set we earlier constructed and P is the set of propositional variables. If $p \in E$ we put $\beta \Vdash p$ if $\beta(p) = 1$ and $\beta \nVdash p$ if $\beta(p) = 0$. If $p \notin E$ we put $\beta \Vdash p$.

Claim 2.1. *Let $\psi \in E$ and $\beta \in K$. Then:*

$$\beta \Vdash \psi \quad \Leftrightarrow \quad \beta(\psi) = 1$$
$$\beta \nVdash \psi \quad \Leftrightarrow \quad \beta(\psi) = 0$$

The proof is by induction on the length of ψ. The cases were ψ is a variable or $\perp$ or has principal connective $\rightarrow$ are evident from the definitions. Suppose now that $\psi = \Box\vartheta$ and consider two sub cases:
Case 1: $\beta(\Box\vartheta) = 1$
By induction it suffices to show that for all β' with $\beta \prec \beta'$ we have $\beta'(\vartheta) = 1$. But this follows from the definition of $\prec$ and the hypothesis.
Case 2: $\beta(\Box\vartheta) = 0$
We have to show that there exists a β' with $\beta \prec \beta'$ such that $\beta'(\vartheta) = 0$. We apply Lemma 2.2 to get such an β'. This proves the Claim.
Now since $\beta_0(\varphi) = 0$, this also finally proves assertion 1) of the theorem. □

From the theorem two corollaries follow readily:

Corollary 2.1. GL $\vdash \varphi$ *iff* φ *is valid in every finite frame* $\langle K, \prec \rangle$ *in which* $\prec$ *is transitive and has ACC.*

This is straightforward, so I will omit the proof.

Corollary 2.2. *The set of theorems of* GL *is recursive.*

Proof. We will assume the formulae of $\mathcal{M}$ encoded as binary strings. Now when given a modal formula φ let n be the length of the encoding of this formula.
By the above theorem it suffices to check if $\mathcal{K} \vDash \varphi$ for frames of size at most 2^n. There obviously is only a finite number of frames of size at most 2^n. All that matters about the forcing relation $\Vdash$ is what happens with pairs (k, p), where the variable p occurs in φ. So again, each frame underlies a finite number of essentially different models. Given a certain finite model it only takes a finite number of steps to check whether φ holds at each node. Therefore we can efficiently test for validity in each model in a finite number of steps. □

So from now on, if we talk about a binary relation $\prec$ it will be clear, that $\prec$ is transitive and has ACC.
It is remarkable, that (3) of the theorem entails, that every constructed Model has a root k_0.

We will now work our way to the proof of a central result in provability logic, known as Solovay´s theorem:

Theorem 2.3.

$$\texttt{GL} \vdash \varphi \qquad \textit{iff} \qquad \texttt{PA} \vdash \varphi^* \quad \textit{for all } *$$

For the proof of this theorem Solovay invented the method of "embedding" Kripke models into Peano arithmetic, a construction which I will now describe.
Let $\mathcal{K} = \langle K, \prec, \Vdash \rangle$ be a finite Kripke model. Let´s assume without loss of generality that $K = \{0, \ldots, n\}$ and that 0 is the root of $\mathcal{K}$. With aid of the recursion theorem a recursive function $h : \omega \longrightarrow K$ can be defined to satisfy the following equations provably in PA:

$$h(0) = 0$$
$$h(m+1) = \begin{cases} z, & \text{if } z \in K, h(m) \prec z \text{ and } \texttt{Prf}\left(m, \ulcorner l \neq \mathbf{z} \urcorner\right); \\ h(m), & \text{otherwise.} \end{cases}$$

Here $l = z$ is short for the arithmetical formula $\exists m \forall n > m : h(n) = z$ and $l \neq z$ denotes $\neg (l = z)$. Artemov gives an informal illustration of the behavior of the function$h$.
Think of a refugee who is admitted to enter a country only if he/she provides a proof not to stay in that country forever. But refugee also

is not allowed to enter a country he/she has previously been to. Since there are only a finite number of countries the refugee eventually must stop somewhere. So, an honest refugee would never be able to leave his/her country of origin. $h(m) = z$ is the statement, that the refugee is in the country z at the moment m.
The following lemma provides some basic facts about the behavior of the function:

Lemma 2.3. *The following statements are provable in* $\mathtt{PA}$*:*

(i) $\bigvee_{z \in K} l = \boldsymbol{z}$*;*

(ii) $\forall u, v\,(l = u \wedge l = v \to u = v)$*;*

(iii) $l = \boldsymbol{z} \to \mathtt{Prov}\left(\ulcorner \bigvee_{z \prec w} l = \boldsymbol{w} \urcorner\right)$*, if* $z \in K$ *and* $0 \prec z$*;*

(iv) $l = \boldsymbol{z} \to \neg\mathtt{Prov}\,(\ulcorner l \neq \boldsymbol{u} \urcorner)$*, if* $z, u \in K$ *and* $z \prec u$*.*

Proof. Statements (i) and (ii) readily follow from the facts that provably in $\mathtt{PA}$ values of h belong to K and that h is weakly increasing in the sense of the ordering $\preceq$.
To prove (iii) we argue in $\mathtt{PA}$ as follows:
If $l = z$ then for some m, $h(m) = z$. Then by Σ_1-completeness (i.e.: the fact that any true Σ_1 sentence is provable in $\mathtt{PA}$), $\mathtt{PA} \vdash \exists m : h(m) = \mathbf{z}$. Since h is provably monotone $\mathtt{PA} \vdash \exists m \forall n > m : h(n) \succeq \mathbf{z}$. Therefore $\mathtt{PA} \vdash \bigvee_{z \preceq w} l = \mathbf{w}$.
If on the other hand $l = z$ and $0 \prec z$, then there is a least m such that $h(m+1) = z$, which by definition of h means that $\mathtt{PA} \vdash l \neq \mathbf{z}$. So we obtain $\mathtt{PA} \vdash \bigvee_{z \prec w} l = \mathbf{w}$.

To show (iv) we formalize the following argument in $\mathtt{PA}$:

If $l = z$ where $z \prec u$, then for some m $\forall k \geq m : h(k) = z$ and $\mathtt{Prf}(m, \ulcorner l \neq \mathbf{u} \urcorner)$. (This is due to the fact, that the usual Gödel numbering of proofs of $\mathtt{PA}$ has the property that each theorem has infinitely many proofs.) But then, by definition of h, there is $h(m+1) = u$. So, since h is weakly increasing that implies $l \neq z$, a contradiction. $\square$

The function

$$f(p) = \bigvee_{z \in K, z \Vdash p} l = \mathbf{z}$$

is called *Solovay realization.* Here the right hand side is a finite disjunction. Should the right hand side be the empty disjunction, put $f(p) = \bot$

We now prove an important technical lemma, concerning properties of the Solovay realization.

Lemma 2.4. *For all well-formed modal formulas φ and all $z \in K; 0 \prec z$:*
(i) If $z \Vdash \varphi$, then $\mathtt{PA} \vdash l = \boldsymbol{z} \to f(\varphi)$;
(ii) If $z \nVdash \varphi$, then $\mathtt{PA} \vdash l = \boldsymbol{z} \to \neg f(\varphi)$.

Proof. The proof of (i) and (ii) is simultaneously by induction on $l(\varphi)$. The case when φ is a variable p is clear from the definition of $f(p)$. The cases $\varphi = \bot, \top$ and if φ has principal connective $\to$ are straightforward. We consider the most important case when φ has the form $\Box\psi$.

(i) If $z \Vdash \Box\psi$, then $\forall u \succ z : u \Vdash \psi$. Hence by induction hypothesis:

$$\mathtt{PA} \vdash \bigvee_{z \prec u} l = \mathbf{u} \to f(\psi).$$

Using the previous Lemma (iii) we obtain:

$$\begin{aligned} \mathtt{PA} \vdash l = \mathbf{z} \quad &\to \quad \mathtt{Prov}\left(\ulcorner \textstyle\bigvee_{z \prec u} l = \mathbf{u} \urcorner\right) \\ &\to \quad \mathtt{Prov}\left(\ulcorner f(\psi) \urcorner\right) \\ &\to \quad f(\Box\psi). \end{aligned}$$

(ii) If $z \nVdash \Box\psi$, then $\exists u \succ z : u \nVdash \psi$. By induction hypothesis

$$\mathtt{PA} \vdash l = \mathbf{u} \to \neg f(\psi),$$

therefore

$$\mathtt{PA} \vdash \neg\mathtt{Prov}\left(\ulcorner l \neq \mathbf{u} \urcorner\right) \to \neg\mathtt{Prov}\left(\ulcorner f(\psi) \urcorner\right).$$

Using the previous Lemma (iv) we obtain:

$$\begin{aligned} \mathtt{PA} \vdash l = \mathbf{z} \quad &\rightarrow \quad \neg\mathtt{Prov}\,(\ulcorner l = \mathbf{u} \urcorner) \\ &\rightarrow \quad \neg\mathtt{Prov}\,(\ulcorner f\,(\psi) \urcorner) \\ &\rightarrow \quad \neg f\,(\Box\psi). \end{aligned}$$

□

Claim 2.2. *If* $0 \leq z \leq n$ *then* $\mathtt{Con}(\mathtt{PA} + "l = \boldsymbol{z}")$

Proof. Arguments in this proof cannot be completely formalized in PA For $z = 0$ the standard model is a model of PA+"$l = \mathbf{z}$". For $z \geq 1$ Lemma 2.3 (iv) provides the proof □

Proof of theorem 2.3. One direction of the proof is already given by Lemma 2.1. It remains to show that:

$$\mathtt{PA} \vdash \varphi^* \quad \Rightarrow \quad \mathtt{GL} \vdash \varphi$$

We will do so by modus tollens.
If $\mathtt{GL} \nvdash \varphi$, then by theorem 2 there is a finite Kripke model $\mathcal{K}_\prime$ such that $\mathcal{K}_\prime \nVdash \varphi$. We may assume that $K_0 = \{1, \ldots, n\}$ and that 1 is the root of $\mathcal{K}_0$. We extend $\mathcal{K}_0$ to a new model $\mathcal{K}$ by a new node 0 stipulating $K = K_0 \cup \{0\}$ and $0 \prec z$, for all $z \in K_0$. The forcing of propositional variables is defined at 0 arbitrarily. As arithmetical interpretation we take the Solovay interpretation f applied to $\mathcal{K}$. Since $1 \nVdash \varphi$ this yields

$$\mathtt{PA} \vdash l = 1 \rightarrow \neg f\,(\varphi)\,.$$

By claim 2 $\mathtt{Con}(\mathtt{PA} + "l = \mathbf{1}")$. Therefore $\mathtt{Con}(\mathtt{PA} + \neg f\,(\varphi))$, i.e.: $f\,(\varphi)$ is not a theorem of PA. □

We have now characterized the modal schemata provable in PA.
What about the class of always true sentences? We begin by introducing a new system of modal logic.

Definition 2.3. *The system* GLS *has two axiom schemata and one rule* *of inference.* ***Axiom schemata:***

$$\begin{array}{ll} \textit{A3)} & \textit{All theorems of } \texttt{GL}. \\ \textit{A4)} & \Box\varphi \to \varphi \end{array}$$

Rule of inference:

$$\textit{R1)} \quad \frac{\varphi, \varphi \to \psi}{\psi} \qquad\qquad \textit{(modus ponens)}$$

Then Solovay´s second arithmetic completeness theorem states:

Theorem 2.4.

$$\texttt{GLS} \vdash \varphi \qquad \textit{iff} \qquad \mathbb{N} \vDash \varphi^* \quad \textit{for all } *$$

Before we commence the proof two lemmata:

Lemma 2.5. *Let φ be a formula. Suppose that for every subformula of φ of the form $\Box\psi$ $1 \Vdash \Box\psi \to \psi$. Then if χ is a subformula of φ:*

1. *If* $1 \Vdash \chi$,
$$\texttt{PA} \vdash l = \boldsymbol{0} \to \chi^*$$
2. *If* $1 \nVdash \chi$,
$$\texttt{PA} \vdash l = \boldsymbol{0} \to \neg\chi^*$$

Proof. The only problematic case is the one, where χ is of the form $\Box\theta$. As usual we have to look at two possible cases:
case 1 $1 \Vdash \Box\theta$
If $1 < i \leq n$ then $i \Vdash \theta$. Also by hypothesis of the lemma $1 \Vdash \theta$. It follows from Lemma 2.4. and the induction hypotheses that if $0 \leq i \leq n$ then $\texttt{PA} \vdash l = \mathbf{i} \to \theta^*$. By Lemma 2.3 (i) $\texttt{PA} \vdash \mathbf{0} \leq l \leq \mathbf{n}$. Therefore $\texttt{PA} \vdash \theta^*$. So $(\Box\theta)^* = \texttt{Prov}(\ulcorner\theta^*\urcorner)$ is a true Σ^0_1 sentence. Therefore $\texttt{PA} \vdash (\Box\theta)^*$. So even more $\texttt{PA} \vdash l = \mathbf{0} \to (\Box\theta)^*$.
case 2 $1 \nVdash \Box\theta$
Then for some j with $1 < j \leq n$, $j \nVdash \theta$. So by Lemma 2.4 $\texttt{PA} \vdash l = \mathbf{j} \to \neg\theta^*$ and by Lemma 2.3(iv) $\texttt{PA} \vdash l = \mathbf{0} \to \texttt{Con}\,(\texttt{PA} + l = \mathbf{j})$. So $\texttt{PA} \vdash l = \mathbf{0} \to \texttt{Con}\,(\texttt{PA} + \neg\theta^*)$. That is $\texttt{PA} \vdash l = \mathbf{0} \to \neg(\Box\theta)^*$. $\Box$

Now let φ be a modal formula and let $\Box\psi_1, \ldots, \Box\psi_n$ enumerate all the subformulae of φ with principal connective $\Box$. Then let:

$$S(\varphi) := \bigwedge_{i=1}^{n} (\Box\psi_i \to \psi_i).$$

Lemma 2.6. *Let φ be a modal formula. Then if*

$$S(\varphi) \to \varphi$$

is not a theorem of $\mathtt{GL}$*, then there is an interpretation of* $\mathcal{M}$ *in* $\mathtt{PA}$ *such that* φ^* *is false in the standard model, i.e.:* $\mathbb{N}$*.*

Proof. Apply theorem 2.3 to $S(\varphi) \to \varphi$. We get a model $\mathcal{K}$ as in the theorem and additionally:

$$1 \vDash \Box\psi_i \to \psi_i \qquad (1 \leq i \leq n).$$

By the previous lemma:

$$\mathtt{PA} \vdash l = \mathbf{0} \to \neg\varphi^*$$

It is not hard to prove that $l = \mathbf{0}$ is true (the honest refugee must stay in his or her country). So φ^* is false as desired. □

proof of the theorem. $\Leftarrow$:
Obviously the set of $\mathbb{N}$-valid formulae is closed the modus ponens. By Lemma 2.1 it contains all instances of A3. Since the theorems of $\mathtt{PA}$ all hold in $\mathbb{N}$, all instances of A4 are $\mathbb{N}$-valid.
$\Rightarrow$:
If $S(\varphi) \to \varphi$ is a theorem of $\mathtt{GL}$, then φ is a theorem of $\mathtt{GLS}$. So if φ is a not theorem of $\mathtt{GLS}$, then φ is not $\mathbb{N}$-valid. □

3 Quantified provability logic

After chapter 2, where we showed that propositional reasoning about the formalized provability predicate is decidable and can moreover be appropriately specified in arithmetic itself, it is only natural to ask, whether there are formal systems that axiomatize the classes of always true and always provable sentences of predicate modal logic, just as `GL` and `GLS` do for the classes of always true and always provable sentences of propositional modal logic.
After Solovay published his first and second arithmetic completeness theorems in 1976, it was long believed that one could axiomatize the class of always provable sentences by simply adding ordinary quantificational logic to the system `GL`. But Vardanyan´s Π^0_2-completeness theorem, published in 1985, ended the search for possible axiomatizations for quantified provability logic.
In fact we will show, that not only in the case of the set of Gödel numbers of always provable sentences but also in the case of set of Gödel numbers of always true sentences, the worst possible case takes effect.
Informally: The set of Gödel numbers of always provable sentences is Π^0_2-complete at worst ($\forall$ interpretations* $\exists$ a proof ...) and the set of Gödel numbers of always true sentences is Π^0_1-complete relative to the truth set V for arithmetic at worst ($\forall$ interpretations*: φ^* is true). And this really is the case with both the former and the latter.
We now give the definition of an interpretation in the case of $\mathcal{QML}$. We need to be more careful than in the case of $\mathcal{L}$, since a predicate modal formula φ may contain a predicate P_0 and we have to forbid P_0^* to contain quantifiers which bind variables that appear in φ.

Definition 3.1. *An interpretation $(.)^*$ for predicate modal formulae φ is a function $(.)^* : \mathcal{QML} \to \mathcal{L}$ that assigns to each predicate symbol P_i that occurs in φ an arithmetic formula $P_i^* (p_0, \ldots, p_{n-1})$, whose bound variables do not occur in φ and whose free variables are just the first n variables of the alphabetical list of the variables of $\mathcal{L}$ if n is the arity of P_i. For any interpretation $*$ for φ we define φ^* by the following induction on the complexity of φ:*

- *In the atomic cases:*
  $(P_i (x_1, \ldots, x_n))^* = P_i^* (x_1, \ldots, x_n)$, $(\bot)^* = \bot$; $(\top)^* = \top$
- $(\varphi \to \psi)^* = \varphi^* \to \psi^*$

- $(\exists x \varphi)^* \; = \exists x \, (\varphi^*)$
- $(\Box \varphi)^* \; = \mathtt{Prov}(\ulcorner \varphi^* \urcorner)$

Note that φ^* always contains the same free variables as φ.
We will call a sentence φ of $\mathcal{QML}$ always provable or PA*-valid if for all interpretations $*$, φ^* is a theorem of* PA*.*
We will call a sentence φ of $\mathcal{QML}$ always true or $\mathbb{N}$-valid if for all interpretations $*$, φ^* is true in the standard model $\mathbb{N}$.

Our first goal in this chapter is to prove that the set of always true formulae of $\mathcal{QML}$ is not arithmetical. So there is no formula in the language $\mathcal{L}$ that is true of exactly the Gödel numbers of the always true sentences. We now develop the tools for doing so:

Let G be a one-place predicate letter. Then $\mathcal{L}^+ = \mathcal{L} \cup \{G\}$.
For each atomic formula Φ of $\mathcal{L}^+$, let $\widehat{\Phi}$ be some standard logical equivalent of Φ with the same free variables, built up by conjunction and existential quantification from atomic formulae of one of the six forms:

$$u = v; \quad \mathbf{0} = u; \quad \mathbf{s}u = v; \quad u + v = w; \quad u \times v = w; \quad Gu$$

For example: If Φ is the formula $ss\mathbf{0} + s\mathbf{0} = x$ then $\widehat{\Phi}$ could be $\exists y_1 \exists y_2 \exists y_3 \, (\mathbf{0} = y_1 \wedge sy_1 = y_2 \wedge sy_2 = y_3 \wedge y_2 + y_3 = x)$.
If Φ is a non-atomic formula, then $\widehat{\Phi}$ is defined by letting $\widehat{(.)}$ commute with quantifiers and boolean connectives.
Now let P_0 be a one-place predicate letter other than G, $P_=$ and P_s two two-place predicate letters and P_+ and $P_\times$ two three-place predicate letters.
For each formula Φ of $\mathcal{L}^+$ let $\widetilde{\Phi}$ be the formula obtained from $\widehat{\Phi}$ by replacing each occurrence of $u = v$ with $P_= uv$; $\mathbf{0} = u$ with $P_0 u$; $su = v$ with $P_s uv$; $u + v = w$ with $P_+ uvw$; $u \times v = w$ with $P_\times uvw$ respectively. Formulae Gu are left alone. $\widetilde{\Phi}$ is a formula of pure predicate calculus that contains exactly the same free variables as Φ.
So $\widetilde{(.)}$ translates a formula of the language $\mathcal{L}^+$ into a formula of pure predicate calculus. That means that $\mathcal{L}^+$ can be seen as fragment of $\mathcal{QML}$ and therefore an interpretation * also determines interpretations of the predicate symbols $P_=, P_s, P_+$ and $P_\times$, which then form "the model

determined by *".

We now introduce a certain sentence Θ in the language $\mathcal{L}$. We basically plug "'truths"' about arithmetic into it, so it is a conjunction of axioms of arithmetic, equality axioms, valid sentences that express that $s, +$ and $\times$ define functions, recursion axioms for zero, successor, plus and times and certain theorems of arithmetic. So $\widetilde{\Theta}$ is a sentence of predicate calculus and if we choose an interpretation $*$, $\left(\widetilde{\Theta}\right)^*$, or $\widetilde{\Theta}^*$ for short, makes sure the axioms of arithmetic and all the other conjuncts in Θ are interpreted to be true in the model determined by $*$.

For any interpretation let $R^*(x, y)$ be the following formula of $\mathcal{L}$:

$$\exists s(\mathrm{FinS}(s) \wedge \mathrm{lh}(s) = x + 1 \wedge \mathrm{end}(s) = y \wedge P_0^*(s_0) \wedge \\ \wedge \forall z < x \; P_s^*(s_z, s_{z+1})).$$

Here $FinS(s)$ is short for "'*s is the code of a finite sequence*"', $lh(s)$ denotes the length of the sequence s and $end(s)$ denotes the last element of the sequence s. Informally $R^*(x, y)$ expresses that the number x is represented by the term s^y0 in the model determined by $*$. Most of the time x will be represented by many s^y0, but they turn out to be an equivalence class with respect to $P^*_=$, because the axioms phrasing reflexivity, transitivity and symmetry of the identity relation are conjuncts in Θ.

Now let * be an arbitrary interpretation.

Lemma 3.1. $\mathtt{PA} \vdash \widetilde{\Theta}^* \to \forall x \exists y \; R^*(x, y)$.

Proof. We work in $\mathtt{PA}$ and assume that $\widetilde{\Theta}^*$ holds. We prove the lemma by induction on x.
First set $x = 0$.
We may assume that one of the conjuncts of Θ is the sentence $\exists x \; \mathbf{0} = x$. Therefore one of the conjuncts of $\widetilde{\Theta}$ is $\exists x \;\; P_0 x$ and thus one of the conjuncts of $\widetilde{\Theta}^*$ is $\exists x \;\; P_0^* x$. So for some $y : P_0^* y$. Let s be the finite sequence of length 1 such that $s_0 = y$. Then $R^*(0, y)$
For the induction step suppose that for some y; $R^*(x, y)$.
Let s be a finite sequence as in the definition of R such that s witnesses

the truth of $R^*(x,y)$. Since we may assume that one of the conjuncts of Θ is $\forall x \exists x'\ \mathbf{s}x = x'$ one of the conjuncts of $\widetilde{\Theta}^*$ is $\forall x \exists x'\ P^*_s(x,x')$. Thus for some y'; $P^*_s(y,y')$. Let s' be the finite sequence of length $x+2$ extending s such that $s'_{x+1} = x'$. Then s' witnesses the truth of $R^*(x+1,y')$. $\square$

Lemma 3.2. $\texttt{PA} \vdash \widetilde{\Theta}^* \wedge R^*(x,y) \wedge P^*_=(y,y') \to R^*(x,y')$

Proof. This proof is an induction on x just like the one before.
For the basis step we assume that Θ contains
$\forall x \forall x'\ (\mathbf{0} = x \wedge x = x' \to \mathbf{0} = x')$.
For the induction step we assume that Θ contains
$\forall x \forall x' \forall x''\ (\mathbf{s}x = x' \wedge x' = x'' \to \mathbf{s}x = x'')$ $\square$

Lemma 3.3.

$$1)\ \texttt{PA} \vdash \widetilde{\Theta}^* \wedge R^*(x,y) \wedge R^*\left(x',y'\right) \to \left(x = x' \leftrightarrow P^*_=\left(y,y'\right)\right)$$
$$2)\ \texttt{PA} \vdash \widetilde{\Theta}^* \wedge R^*(x,y) \to (\boldsymbol{0} = x \leftrightarrow P^*_0(y))$$
$$3)\ \texttt{PA} \vdash \widetilde{\Theta}^* \wedge R^*(x,y) \wedge R^*\left(x',y'\right) \to \left(\boldsymbol{s}x = x' \leftrightarrow P^*_s\left(y,y'\right)\right)$$
$$4)\ \texttt{PA} \vdash \widetilde{\Theta}^* \wedge R^*(x,y) \wedge R^*\left(x',y'\right) \wedge R^*\left(x'',y''\right) \to$$
$$\to \left(x + x' = x'' \leftrightarrow P^*_+\left(y,y',y''\right)\right)$$
$$5)\ \texttt{PA} \vdash \widetilde{\Theta}^* \wedge R^*(x,y) \wedge R^*\left(x',y'\right) \wedge R^*\left(x'',y''\right) \to$$
$$\to \left(x \times x' = x'' \leftrightarrow P^*_\times\left(y,y',y''\right)\right)$$

Proof. The proof of each of these is very similar to the proof of lemma 3.1. First we observe, that a certain finite number of simple theorems about the natural numbers can be proved in $\texttt{PA}$. These theorems can be assumed to be conjuncts of Θ, so their tilde and star versions may be assumed to follow from $\widetilde{\Theta}^*$. One then uses the facts stated in these versions to prove the lemma by induction $\square$

Now let $\mathtt{x}$ abbreviate $(x_1,\ldots,x_n)$, $\mathtt{y}$ abbreviate $(y_1,\ldots,y_n)$ and $\mathtt{R}(\mathtt{x},\mathtt{y})$ abbreviate $R^*(x_1,y_1) \wedge \ldots \wedge R^*(x_n,y_n)$.

Lemma 3.4. *Let* $\Phi(x)$ *be any formula of* $\mathcal{L}$*. Then*
$\texttt{PA} \vdash \widetilde{\Theta}^* \wedge \forall y \exists x\ R^*(x,y) \wedge \mathtt{R}(\mathtt{x},\mathtt{y}) \to \Big(\Phi(\mathtt{x}) \leftrightarrow \widetilde{\Phi}^*(\mathtt{y})\Big)$.

Proof. Again, a proof by induction, this time on the construction of Φ. We can assume that each atomic formula of Φ is of one of the forms $u = v; \mathbf{0} = u; su = v; u + v = w$ or $u \times v = w$.
To these cases the previous lemma applies; truth-functional cases are handled as ever.
For the quantifier case we have $\forall x \exists y\ R^*(x, y)$ following from $\widetilde{\Theta}^*$ by Lemma 3.1 and we have $\forall y \exists x\ R^*(x, y)$, which is a conjunct of the antecedent. These two formulae suffice for the deduction of the lemma for $\exists\Phi$ from the lemma for Φ. $\square$

We now want to find a formula ϑ of $\mathcal{QML}$, so that its interpreted version can replace the conjunct $\forall y \exists x\ R^*(x, y)$ of the antecedent in lemma 3.4.
A bounded formula of $\mathcal{L}$ is one that is built up from atomic formulae and their negations by truth-functional operations and bounded quantification.

Lemma 3.5. *Let* $\Phi(\mathtt{x})$ *be a bounded formula of* $\mathcal{L}$. *Then*
$\mathtt{PA} \vdash \widetilde{\Theta}^* \wedge \mathtt{R}^*(\mathtt{x}, \mathtt{y}) \to \Big(\Phi(\mathtt{x}) \leftrightarrow \widetilde{\Phi}^*(\mathtt{y})\Big)$.

Proof. Induction on the construction of $\Phi(\mathtt{x})$.
Lemma 3.3 handles atomic cases, truth-functional cases are handled as usual.
In the bounded quantifier case suppose that $\Phi(\mathtt{x}, x)$ is $\forall z < x\ \Psi(\mathtt{x}, z)$ and that the Lemma holds for $\Psi(\mathtt{x}, z)$. Suppose that the formulae $\widetilde{\Theta}^*$, $\mathtt{R}^*(\mathtt{x}, \mathtt{y})$ and $R^*(x, y)$ hold. Now proceed by induction on x in $\mathtt{PA}$.
Suppose $x = 0$.
Then surely $\Phi(\mathtt{x}, x)$. Since $R^*(0, y)$ there is $P_0^* y$ (Lemma 3.3(2)). We assume that $\forall z\ \neg z < 0$ is a conjunct of Θ. So we have $\forall z\ \neg z <^* y$. But then there is $\forall z <^* y\ \Psi^*(\mathtt{y}, y)$ which is exactly $\widetilde{\Phi}^*(\mathtt{y}, y)$.
Now suppose $x = sx'$
Then for some y': $R^*(x', y')$ and $P_s^*(y', y)$. And then

$$\begin{aligned}
\Phi(\mathtt{x}, x) &\Leftrightarrow \Phi(\mathtt{x}, x') \text{ and } \Psi(\mathtt{x}, x') \\
&\Leftrightarrow \text{(by induction hypothesis and the Lemma for } \Psi(\mathtt{x}, x)) \\
&\widetilde{\Phi}^*(\mathtt{y}, y') \text{ and } \widetilde{\Psi}^*(\mathtt{y}, y') \\
&\Leftrightarrow \quad \forall z <^* y'\ \widetilde{\Psi}^*(\mathtt{y}, z) \text{ and } \widetilde{\Psi}^*(\mathtt{y}, y').
\end{aligned}$$

Since $P_s^*(y, y')$ and since we may take

$$\forall x' \forall z \ \left(z < sx \leftrightarrow z < x' \vee z = x'\right)$$

as a conjunct of Θ we have: $z <^* y$ iff $z <^* y'$ or $z =^* y'$. Obviously if $z =^* y'$ then $\widetilde{\Psi}^*(\mathtt{y}, y') \ \Leftrightarrow \ \widetilde{\Psi}^*(\mathtt{y}, z)$.
Thus $\forall z <^* y' \ \widetilde{\Psi}^*(\mathtt{y}, z)$ and $\widetilde{\Psi}^*(\mathtt{y}, y')$ iff $\forall z <^* y \ \widetilde{\Psi}^*(\mathtt{y}, z)$
which is exactly $\widetilde{\Phi}^*(\mathtt{y}, y)$. $\square$

Lemma 3.6. *Let $\Phi(\mathtt{x})$ be any Σ formula of $\mathcal{L}$. Then*
$\mathtt{PA} \vdash \widetilde{\Theta}^* \wedge \mathtt{R}^*(\mathtt{x}, \mathtt{y}) \to (\Phi(\mathtt{x}) \to \widetilde{\Phi}^*(\mathtt{y}))$.

Proof. By Lemma 3.5 we only need to deduce this for $\exists x \ \Phi$ from the hypotheses that the lemma holds for $\Phi = \Phi(\mathtt{x}, x)$.
Work in $\mathtt{PA}$. Suppose $\widetilde{\Theta}^*$ and $\mathtt{R}^*(\mathtt{x}, \mathtt{y})$.
By Lemma 3.1 we have for some y: $R^*(x, y)$. Thus if $\Phi(\mathtt{x}, x)$, then by this lemma for $\Phi(\mathtt{x}, x)$: $\widetilde{\Phi}^*(\mathtt{y}, y)$ and then also $\exists x \ \widetilde{\Phi}^*(\mathtt{y}, x)$, i.e.: $(\widetilde{\exists x \ \Phi})^*$. $\square$

Let ϑ be the following formula of $\mathcal{QML}$:

$$\neg\Box\neg\top\wedge$$
$$\forall x \ (P_0 x \to \Box P_0 x) \wedge \forall x \ (\neg P_0 x \to \Box\neg P_0 x)\wedge$$
$$\forall x \forall y \ (P_= xy \to \Box P_= xy) \wedge \forall x \forall y \ (\neg P_= xy \to \Box\neg P_= xy)\wedge$$
$$\forall x \forall y \ (P_s xy \to \Box P_s xy) \wedge \forall x \forall y \ (\neg P_s xy \to \Box\neg P_s xy)\wedge$$
$$\forall x \forall y \forall z \ (P_+ xyz \to \Box P_+ xyz) \wedge \forall x \forall y \forall z \ (\neg P_+ xyz \to \Box\neg P_+ xyz)\wedge$$
$$\forall x \forall y \forall z \ (P_\times xyz \to \Box P_\times xyz) \wedge \forall x \forall y \forall z \ (\neg P_\times xyz \to \Box\neg P_\times xyz).$$

Let $\Phi(\mathtt{x})$ be any arithmetic formula. Then there is a formula of arithmetic stating that $\Phi(\mathtt{x})$ *defines a recursive relation.* The formula states that there is a Turing machine T that, when given a n-tuple of natural numbers $\mathtt{i}$ as input, outputs 1 (yes) if $\Phi(\mathtt{i})$ holds and 0 (no) if $\Phi(\mathtt{i})$ does not hold.

Lemma 3.7. $\mathtt{PA} \vdash \vartheta^* \to P_0^*, P_=^*, P_s^*, P_+^*$ *and* $P_\times^*$ *define recursive relations.*

Proof. Work in $\mathtt{PA}$.
Suppose ϑ^* holds. Then $(\neg\Box\neg\top)^*$ holds, i.e.: arithmetic is consistent.

Now consider for example P_+^*.
Then we can effectively construct an algorithm that decides P_+^*:
Given any numbers x, y, z the algorithm runs through all the proofs in `PA` until a proof of $P_+^*(\mathbf{x}, \mathbf{y}, \mathbf{z})$ or a proof of $\neg P_+^*(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is found. If a proof of the former is found first output 1; if a proof of the latter is found first output 0. Since arithmetic is consistent not both of them have a proof. Either one or the other holds by the law of the excluded middle and by the conjuncts of ϑ^* dealing with P_+^* whichever holds has a proof. □

Lemma 3.8. `PA` $\vdash \widetilde{\Theta}^* \wedge \vartheta^* \to \forall y \exists x\ R^*(x, y)$.

Proof. Let $\Xi(e, i, m)$ be a Σ formula stating: "e is the Gödel number of a Turing machine that halts on input i with output m".
Let $\Xi_0(x)$ and $\Xi_1(x)$ abbreviate $\Xi(x, x, \mathbf{0})$ and $\Xi(x, x, \mathbf{1})$ respectively.
We may assume that Θ implies $\forall x\ \neg(\Xi_0(x) \wedge \Xi_1(x))$.
Let $B_0(a, b, i)$ and $B_1(a, b, i)$ be the formulae

$$\exists q\ (q \times (\mathbf{1} + (i + \mathbf{1}) \times b) = a)$$

and

$$\exists q\ ((q \times (\mathbf{1} + (i + \mathbf{1}) \times b)) + \mathbf{1} = a).$$

β-function methods for coding sequences (see section 3.8 of [10]) show that for any formula $\Phi(x)$ of arithmetic, the following sentence is a theorem of `PA`:

$$\forall k \exists a \exists b \forall j < k\ ((\Phi(j) \leftrightarrow B_0(a, b, j)) \wedge (\neg\Phi(j) \leftrightarrow B_1(a, b, j))).$$

In particular the sentence Γ:

$$\forall k \exists a \exists b \forall j < k\ ((\Xi_0(j) \leftrightarrow B_0(a, b, j)) \wedge (\neg\Xi_0(j) \leftrightarrow B_1(a, b, j)))$$

is a theorem of `PA`. We assume that Θ implies Γ.
Now work in `PA` and assume that $\widetilde{\Theta}^*$ and ϑ^* hold.
The proof is now by contradiction and the common method of constructing a Turing machine that yields a contradiction, when given its own Gödel number as input. So suppose that $\forall y \exists x\ R^*(x, y)$ is false. Let k be a witness for that, so there is no r such that $R^*(r, k)$.

Since Θ implies Γ, $\widetilde{\Theta}^*$ implies $\widetilde{\Gamma}^*$ and $\widetilde{\Gamma}^*$ yields numbers a, b such that for every $j <^* k$:

$$\begin{aligned} \widetilde{\Xi_0}^*(j) &\Leftrightarrow \widetilde{B_0}^*(a,b,j) \text{ and} \\ \text{not}\widetilde{\Xi_0}^*(j) &\Leftrightarrow \widetilde{B_1}^*(a,b,j) \end{aligned}$$

By lemma 3.7 and since we assumed ϑ^*; $P_0^*, P_=^*, P_s^*, P_+^*$ and $P_\times^*$ define recursive relations. Now since $\widetilde{B_0}$ and $\widetilde{B_1}$ are equivalent to formulae built up from $P_0, P_=, P_s, P_+, P_\times$ by existential quantification, conjunction and disjunction, $\widetilde{B_0}^*$ and $\widetilde{B_1}^*$ define recursive enumerable relations. $R^*(x,y)$ apparently also defines a recursive enumerable relation. Now we look at action of a certain Turing machine T.
When given any number $i$as input, T first finds a number j such that $R^*(i,j)$ holds. Because of Lemma 3.1 such a j will always exist and because $R^*(x,y)$ defines a recursively enumerable relation T can find such a j effectively.
Afterwards T searches witnesses for the truth of either $\widetilde{B_0}^*(a,b,j)$ or $\widetilde{B_1}^*(a,b,j)$. If it finds a witness to truth of the former first it outputs 1; if it finds a witness to the truth of the latter first it outputs 0.
T gives an output for every input; i. e. T is totally defined, as we will see in a moment:
To show this it will suffice to show that $R^*(i,j) \Rightarrow j <^* k$ for if $j <^* k$ then - since $\Xi_0(j)$ either holds or does not hold - a witness to the truth of either $\widetilde{B_0}^*(a,b,j)$ or $\widetilde{B_1}^*(a,b,j)$ will exist and so T will give an output for every input.

We may assume that Θ implies

$$\forall x\ \mathbf{0} \le x, \forall x \forall x' \ \big(x < x' \to sx \le x'\big)$$

and

$$\forall x \forall x' \ \big(x \le x' \wedge x \ne x' \to x < x'\big).$$

If $R^*(0,j)$ then by $\widetilde{\Theta}^*$ (Lemma 3.3(2)) there is $j \le^* k$. But by assumption $\neg R^*(0,k)$ and by Lemma 3.2 $\neg P_=^*(j,k)$. Thus $j <^* k$.
For the induction step suppose that $R^*(i+1,j)$.

Then for some j' we have that $P_s^*(j',j)$, $R^*(i,j')$ and by induction hypothesis $j' <^* k$ whence $j' \widetilde{\leq}^* k$. But since $\neg R^*(i+1,k)$ there is $\neg P_=^*(j,k)$ and therefore $j \widetilde{<}^* k$.
Thus for every i: If $R^*(i,j)$ then $j \widetilde{<}^* k$. So T is totally defined.
Now let i be an arbitrary number. Let j be the number that is found by T when given input i. So $R^*(i,j)$ and by the above $j \widetilde{<}^* k$. If $\Xi_0(i)$ holds, then by Lemma 3.6 $\widetilde{\Xi_0}^*$ and therefore $\widetilde{B_0}^*(a,b,j)$ but not $\widetilde{B_1}^*(a,b,j)$ - thus T outputs 1.
If $\Xi_1(i)$ holds then by Lemma 3.6 $\widetilde{\Xi_1}^*$ and then - since Θ implies $\forall x \neg (\Xi_0(x) \wedge \Xi_1(x))$ - not $\widetilde{\Xi_0}^*$ therefore $\widetilde{B_1}^*(a,b,j)$ and not $\widetilde{B_0}^*(a,b,j)$ - thus T outputs 0.
So for every natural number i:

$$\begin{aligned} T(i) = 1 \quad &\Leftrightarrow \quad \widetilde{\Xi_0}^*(i) \\ T(i) = 0 \quad &\Leftrightarrow \quad \widetilde{\Xi_1}^*(i) \end{aligned}$$

Now let e be the Gödel number of the Turing machine T.
Then if

$$\Xi_0(e) \;\Leftrightarrow\; T(e) = 1 \;\Leftrightarrow \Xi(e,e,1) \;\Leftrightarrow\; \Xi_1(e) \;\Leftrightarrow\; \neg\Xi_0(e)$$

and if

$$\neg\Xi_0(e) \;\Leftrightarrow\; T(e) = 0 \;\Leftrightarrow\; \Xi(e,e,0) \;\Leftrightarrow\; \Xi_0(e)$$

This contradiction finally proves the Lemma. □

Lemma 3.9 (Artemov´s lemma). *Let* $\Phi(\mathtt{x})$ *be any formula of* $\mathcal{L}$. *Then* $\mathtt{PA} \vdash \widetilde{\Theta}^* \wedge \vartheta^* \wedge \mathtt{R}^*(\mathtt{x},\mathtt{y}) \to \Big(\Phi(\mathtt{x}) \leftrightarrow \widetilde{\Phi}^*(\mathtt{y})\Big)$.

Proof. Artemov´s lemma follows directly from lemmata 3.4 and 3.8. □

Theorem 3.1. *The class of always true sentences is not arithmetical.*

Proof. We will show that there is a one-one effective function f, that reduces the truth set V to the class of always true sentences of $\mathcal{QML}$. Since by Tarski´s theorem V is not arithmetical, neither the class of always true sentences of $\mathcal{QML}$ will be.

Define f as follows: for any sentence Ψ of $\mathcal{L}$ set $f(\Psi) := \widetilde{\Theta} \wedge \vartheta \rightarrow \widetilde{\Psi}$, which is a sentence of $\mathcal{QML}$. Ψ is true if and only if $f(\Psi)$ is always true:
Applying Artemov´s lemma to the case where $\Phi(\mathtt{x})$ is a sentence of $\mathcal{L}$, we see that for every interpretation $*$ the sentence $\widetilde{\Theta}^* \wedge \vartheta^* \rightarrow \left(\Psi \leftrightarrow \widetilde{\Psi}^*\right)$ is a theorem of $\mathtt{PA}$ and therefore true.
Suppose now that Ψ is true. Then $\widetilde{\Theta}^* \wedge \vartheta^* \rightarrow \widetilde{\Psi}^*$ is true for every $*$ and therefore $f(\Psi)$ is always true.
Conversely suppose that $f(\Psi)$ is always true. Take $*$ as the special interpretation that assigns $\mathbf{0} = p_0, p_0 = p_1, sp_0 = p_1, p_0 + p_1 = p_2$ and $p_0 \times p_1 = p_2$ to $P_0, P_=, P_s, P_+, P_\times$ respectively. Then $\widetilde{\Theta}^* \wedge \vartheta^* \rightarrow \widetilde{\Psi}^*$ is true. But $\widetilde{\Theta}^*$ is equivalent to Θ, hence true and by consistency of arithmetic and provable Σ-completeness ϑ^* is also true. Thus $\widetilde{\Psi}^*$ is true. But $\widetilde{\Psi}^*$ is equivalent to Ψ. Therefore Ψ is true. $\square$

Before we strengthen this theorem, we turn our attention to the class of the always provable sentences of $\mathcal{QML}$. As we informally stated above the set of always provable sentences is Π^0_2. In more detail: Let $R(i, j, k)$ hold if and only if i is the Gödel number of a sentence φ of $\mathcal{QML}$ and j is the Gödel number of an interpretation of the predicate symbols in φ and k is the Gödel number of a proof in $\mathtt{PA}$ for φ^*. R is a recursive relation and the set of always provable sentences can be written as $\{\varphi | \forall j \exists k \, R(\ulcorner \varphi \urcorner, j, k)\}$. We are now going to show that this set is in fact Π^0_2-complete, so there is no simpler characterization for this set, in particular there is no way of axiomatizing it.

To do so we will need an alternative characterization of Π^0_2 sets.

Lemma 3.10. *S is a Π^0_2 set if and only if for some recursive relation P: $S = \{n \,|\, \forall i \exists j \; (i < j \wedge P(n, j))\}$.*

Proof. First suppose that $S = \{n \,|\, \forall e \exists m \; R(n, e, m)\}$, with R a recursive relation. Define $P(n, j)$ so that it holds iff j is the Gödel number of a finite sequence such that for all $e < |j|$ - where $|j|$ denotes the length of j - j_e is the least m such that $R(n, e, m)$. Then P is a recursive relation. If $n \in S$, then for every natural number e there will be such a finite sequence of length $e + 1$ and thus there will be infinitely many such sequences. And if there are infinitely many such sequences, then - since any two of them have the same values for arguments less than their length - there will be at least one such sequence y of length $e+1$. And then $R(n,e,j_e)$. Thus $S = \{n|\forall i \exists j \ (j > i \wedge P(n,j))\}$. Conversely, if P is recursive and $S = \{n|\forall i \exists j \ (j > i \wedge P(n,j))\}$, then S is visibly Π^0_2. $\square$

Theorem 3.2. *The class of always provable sentences is Π^0_2-complete.*

Proof. Since we know that the class of always provable sentences is Π^0_2, we need to show that for any other set S that is Π^0_2, there exists a recursive function f such that

$$S = \{n|f(n) \text{ is an always provable sentence of } \mathcal{QML}\}.$$

We will show how to effectively affiliate a sentence ϕ_n of $\mathcal{QML}$ with every natural number n, such that $S = \{n| \text{ for all } * : \mathtt{PA} \vdash \phi^*_n\}$.
We will write $S = \{n|\forall i \exists j \ (j > i \wedge P(n,j))\}$.
Since P is recursive by Lemma 3.10 it can be defined by some Σ formula.
Let $Q_n(y)$ be the formula holding iff $P(n,y)$.
Let E be the following of $\mathcal{QML}$ sentence $\forall z \forall z' \ (P_= zz' \to (\Box Gz \leftrightarrow \Box Gz'))$.
Let $H(v,z)$ be the Σ formula formalizing "v is the Gödel number of a Turing machine that halts on input z". $H(v,z)$ can be written as $\exists y \ \Xi(v,z,y)$ with Ξ as in the proof of Lemma 8.
Finally, define ϕ_n for each $n \in S$ to be the following sentence of $\mathcal{QML}$:

$$\widetilde{\Theta} \wedge \vartheta \wedge E \to \exists v \exists w \ \left(v \widetilde{<} w \wedge \widetilde{Q_n}(w) \wedge \forall z \ \left(\Box Gz \leftrightarrow \widetilde{H}(v,z)\right)\right).$$

We need to show that $n \in S$ iff for every $*$: $\mathtt{PA} \vdash \phi^*_n$.
Suppose $n \in S$ and $*$ arbitrary. Then

Claim 3.1. *For some natural number x:*

$$\mathtt{PA} \vdash \vartheta^* \to ((\exists z \ (R^*(z_0,z) \wedge \mathtt{Prov}(\ulcorner G^*(z) \urcorner))) \leftrightarrow H(\boldsymbol{x}, z_0)).$$

Proof of the Claim. Work in PA. Suppose ϑ^* holds. Then by arguments we used to prove Lemma 3.7, $P^*_0, P^*_=, P^*_s, P^*_+$ and $P^*_\times$ define recursive relations and are therefore equivalent to Σ formulae. For example: $P^*_s(\mathbf{i},\mathbf{j})$ iff there is a proof of $P^*_s(\mathbf{i},\mathbf{j})$ with a smaller Gödel number than any proof of $\neg P^*_s(\mathbf{i},\mathbf{j})$. This property for a pair of numbers can also be defined by a Σ formula. Therefore $R^*(z_0, z)$ is a Σ formula and since $\mathtt{Prov}\,(\ulcorner G^*\,(z)\urcorner)$ is also Σ, so is the left-hand side of the consequent. One can show by the usual induction on the construction of strict Σ formulae, that for every strict Σ formula $\Phi(\mathtt{z})$, there exists a Turing machine T with Gödel number e with the property that it is provable in $\mathtt{PA}$ that T halts on just those numbers that satisfy $\Phi(\mathtt{z})$. But the left side of the consequent is a Σ formula and hence equivalent to some strict Σ formula. $\Box$

Now fix the number x as in Claim 3.1.

Claim 3.2.

$$\mathtt{PA} \vdash \widetilde{\Theta}^* \wedge \vartheta^* \wedge E^* \wedge R^*(z_0, z) \to (\mathtt{Prov}(\ulcorner G^*(z)\urcorner) \leftrightarrow H(\boldsymbol{x}, z_0)).$$

Proof of the claim. Work in $\mathtt{PA}$and assume $\widetilde{\Theta}^* \wedge \vartheta^* \wedge E^* \wedge R^*(z_0, z)$.
By Claim 3.1: if $\mathtt{Prov}(\ulcorner G^*(z)\urcorner)$ then $H(\mathbf{x}, z_0)$.
On the other hand: if $H(\mathbf{x}, z_0)$, then by Lemma 3.6 there is $R^*(z_0, z')$ and $\mathtt{Prov}(\ulcorner G^*(z')\urcorner)$ for some z'. By Lemma 3.8 since $R^*(z_0, z)$ and $R^*(z_0, z')$ there is $P^*_=(z, z')$. But then $\mathtt{Prov}(\ulcorner G^*(z)\urcorner)$. $\Box$

Claim 3.3.

$$\mathtt{PA} \vdash \widetilde{\Theta}^* \wedge \vartheta^* \wedge R^*(\boldsymbol{x}, z) \wedge R^*(z_0, z) \to \Big(H(\boldsymbol{x}, z_0) \leftrightarrow \widetilde{H}^*(v, z)\Big).$$

Proof of the claim. This is an instance of Artemovs Lemma (Lemma 3.9). $\Box$

Now by Claim 3.2 and Claim 3.3 we have:

$$\begin{aligned} \mathtt{PA} \vdash\ & \widetilde{\Theta}^* \wedge \vartheta^* \wedge E^* \wedge R^*(\mathbf{x}, z) \wedge R^*(z_0, z) \\ & \to \Big(\mathtt{Prov}(\ulcorner G^*(z)\urcorner) \leftrightarrow \widetilde{H}^*(v, z)\Big) \end{aligned} \tag{1}$$

By Lemma 3.8 and (1):

$$\begin{aligned} \mathtt{PA} \vdash\ & \widetilde{\Theta}^* \wedge \vartheta^* \wedge E^* \wedge R^*(\mathbf{x}, z) \\ & \to \forall z\ \Big(\mathtt{Prov}(\ulcorner G^*(z)\urcorner) \leftrightarrow \widetilde{H}^*(v, z)\Big) \end{aligned} \tag{2}$$

Since $n \in S$ as we have supposed, there is a number y, such that $x < y$ and $Q_n(y)$ holds. We apply Artemov´s Lemma again:

$$\begin{aligned} \mathtt{PA} \vdash\ & \widetilde{\Theta}^* \wedge \vartheta^* \wedge R^*(\mathbf{x}, v) \wedge R^*(\mathbf{y}, w) \\ & \to \Big((\mathbf{x} < \mathbf{y} \wedge Q_n(\mathbf{y})) \leftrightarrow \Big(v \widetilde{<}^* w \wedge \widetilde{Q_n}^*(w) \Big) \Big) \end{aligned} \tag{3}$$

Since Q and $x < y$ are Σ-formulae:

$$\mathtt{PA} \vdash \mathbf{x} < \mathbf{y} \wedge Q_n(\mathbf{y}) \tag{4}$$

From (3) and (4) we obtain:

$$\mathtt{PA} \vdash \widetilde{\Theta}^* \wedge \vartheta^* \wedge R^*(\mathbf{x}, v) \wedge R^*(\mathbf{y}, w) \to \Big(v \widetilde{<}^* w \wedge \widetilde{Q_n}^*(w) \Big) \tag{5}$$

Which together with (2) yields:

$$\begin{aligned} \mathtt{PA} \vdash\ & \widetilde{\Theta}^* \wedge \vartheta^* \wedge R^*(\mathbf{x}, v) \wedge R^*(\mathbf{y}, w) \\ & \to \Big(v \widetilde{<}^* w \wedge \widetilde{Q_n}^*(w) \wedge \forall z\ \Big(\mathtt{Prov}\,(\ulcorner G^*\,(z) \urcorner) \leftrightarrow \widetilde{H}^*\,(v, z) \Big) \Big) \end{aligned} \tag{6}$$

By predicate calculus:

$$\begin{aligned} \mathtt{PA} \vdash\ & \widetilde{\Theta}^* \wedge \vartheta^* \wedge E^* \wedge \exists v\ R^*\,(\mathbf{x}, v) \wedge \exists w\ R^*\,(\mathbf{y}, w) \\ & \to \exists v \exists w\ (v \widetilde{<}^* w \wedge \widetilde{Q_n}^*(w) \wedge \\ & \wedge \forall z\ (\mathtt{Prov}(\ulcorner G^*(z) \urcorner) \leftrightarrow \widetilde{H}^*(v, z))) \end{aligned} \tag{7}$$

Since by Lemma 6 there is $\mathtt{PA} \vdash \widetilde{\Theta}^* \to \exists v\ R^*(\mathbf{x}, v) \wedge \exists w\ R^*(\mathbf{y}, w)$ it follows from (7) that:

$$\begin{aligned} \mathtt{PA} \vdash\ & \widetilde{\Theta}^* \wedge \vartheta^* \wedge E^* \\ & \to \exists v \exists w\ \Big(v \widetilde{<}^* w \wedge \widetilde{Q_n}^*(w) \wedge \wedge \forall z\ \Big(\mathtt{Prov}\,(\ulcorner G^*\,(z) \urcorner) \leftrightarrow \widetilde{H}^*\,(v, z) \Big) \Big) \end{aligned} \tag{8}$$

which is exactly $\mathtt{PA} \vdash \phi_n^*$.

On the other hand assume that for all $*$, $\mathtt{PA} \vdash \phi_n^*$. We need to show that

$n \in S$.
Consider a family of interpretations $*_i$, which all interpret $*_i$ P_0, $P_=$, P_s, P_+, $P_\times$ in the standard way. (i.e.: $P_+^{*_i}$ is the formula $v_0 + v_1 = v_2$, etc.). The only difference between these interpretations is what they assign to G. $G^{*_i}(v_0)$ is the formula $v_0 = \mathbf{i}$.
First observe that every theorem of PA is true and that for every interpretation $*$ in which $P_0, P_=, P_s, P_+, P_\times$ are standardly interpreted $\widetilde{\Theta}^* \wedge \vartheta^* \wedge E^*$ is true. Thus for each i:

$$\left(\exists v \exists w \ \left(v \widetilde{<} w \wedge \widetilde{Q_n}(w) \wedge \forall z \ \left(\Box Gz \leftrightarrow \widetilde{H}(v,z)\right)\right)\right)^{*_i}$$

is true. In English language that means that for every i, there exist natural numbers v, w such that $v < w$, $Q_n(w)$ holds and for all z, $\mathbf{z} = \mathbf{i}$ is provable if and only if the Turing machine with Gödel number v halts on z. By consistency of arithmetic, $\mathbf{z} = \mathbf{i}$ is provable if and only if $z = i$ and therefore for each i there exist natural numbers v, w such that $v < w$, $Q_n(w)$ holds and the Turing machine with Gödel number v halts on i and i alone. Of course if the Turing machine with Gödel number v halts on i and i alone, the Turing machine with Gödel number v' halts on i' and i' alone and $i \neq i'$, then $v \neq v'$. Thus for each i there exist numbers v, w - with different v for different i - such that $v < w$ and $Q_n(w)$ holds. Thus there are infinitely many numbers v such that for some w: $v < w$ and $Q_n(w)$. Thus for every x, for some w: $x < w$ and $Q_n(w)$, which is exactly $n \in S$. This proves the theorem. $\Box$

We now strengthen the result of Artemov and show, that the class of always true sentences is Π_1^0-complete in V, where V denotes the truth set of arithmetic. To do so we will have to show some auxiliary lemmata first. Let $\Phi(\mathtt{x})$ be any formula of $\mathcal{L}^+$. We say that $\Phi(a_1, \ldots, a_n)$ holds at the set $A \subseteq \mathbb{N}$ if $\Phi(\mathtt{x})$ is satisfied by numbers $a_1, \ldots, a_n$ when A is assigned to the predicate letter G.
Let $\zeta = \widetilde{\Theta} \wedge \theta \wedge \forall x \forall x' (Gx' \wedge P_= xx' \rightarrow Gx)$.

Lemma 3.11. *Let $\Phi(\mathtt{x})$ be any formula of $\mathcal{L}^+$ and let $*$ be any interpretation of ζ. Suppose that ζ^* is true and that $R^*(a_1, b_1), \ldots, R^*(a_n, b_n)$ hold.*
Let $A = \{a | \text{for some } b,\ R^*(a,b) \text{ and } G^*(b) \text{ hold}\}$.
Then $\Phi(a_1, \ldots, a_n)$ holds at A if and only if $\widetilde{\Phi}^*(b_1, \ldots, b_n)$ holds.

Proof. As usual the proof is by induction just like the proof of Lemma 3.4. The atomic cases - with exception of the one, where the formula is of the form Gu - are handled by Lemma 3.3.
So we look at that case.
Suppose $R^*(a,b)$ holds. Assume that $G(a)$ holds at A. Then we have for some b', $R^*(a,b')$ and $G^*(b')$. By Lemma 3.3(1) $P^*_= bb'$ holds. $\forall x \forall y (Gx \wedge P_= xy \rightarrow Gy)$ is certainly true and therefore G^*b holds. The other direction is straightforward from the definition of A.
Therefore if $R^*(a,b)$ holds, then $G(a)$ holds at A iff G^*b holds.
Truth-functional cases are handled as always and since $\widetilde{\Theta}^*$ and θ^* are true, so are by Lemmata 3.1 and 3.8 $\forall x \exists y\, R^*(x,y)$ and $\forall y \exists x\, R^*(x,y)$. This is enough to prove the quantifier cases. □

Lemma 3.12. *Let Φ be any sentence of $\mathcal{L}^+$ and let $*$ be any interpretation of ζ. Suppose that ζ^* is true and let*

$$A = \{a | \textit{for some } b,\ R^*(a,b) \ \textit{ and } G^*(b) \ \textit{ hold}\}.$$

Then Φ holds at A if and only if $\widetilde{\Phi}^*$ is true.

Proof. This is just the special case of the previous lemma. □

We will say that two sets $A, B \subseteq \mathbb{N}$ are k-equivalent if for every $m \leq k, m \in A$ iff $m \in B$.
We will also be needing the relativized Kleene T-predicate. I will define this concept for an oracle machine, which is a kind of Turing machine.

Definition 3.2. *An oracle machine is a Turing machine, that is able to stop its computation to gain information from an external source, an oracle. To be more precise: In the table of an oracle machine there are instructions $\langle i, s, j_1, j_2 \rangle$. Given such an instruction the machine acts in the following way. When in state i scanning a symbol of type s, the machine stops and asks the oracle whether the number of 1s on its tape belong to a certain set or not. If it receives a positive answer it will enter the state j_1 and continue its computation; if it receives a negative answer it will enter the state j_2 and continue its computation.*
A halting computation by an oracle machine is correct for the set A, if A is the set the oracle machine asks about.
We may assume, that we have chosen a formulation of the concept of an

oracle machine that satisfies the following condition: if k is the Gödel number of a computation by the oracle machine and n is a number about which the machine asks the oracle, then $n \leq k$.
The relativized Kleene T-predicate T is a relation among numbers e, i, k and a set A. This relation holds if and only if k is the Gödel number of a halting computation by the oracle machine with Gödel number e that is correct for the set A when given input i. We will write $T^A(e, i, k)$, when this relation holds.
(In this paper T also denotes a Turing machine, but there will be no danger of confusion)

Lemma 3.13. *If $T^A(e, i, k)$ and A and B are k-equivalent, then $T^B(e, i, k)$.*

Proof. Since A and B are k-equivalent, any number about which the oracle is questioned during a computation is less than the Gödel number of that computation. Thus if k is correct for A, k is also correct for B. $\square$

We say that A m-approximates V if the following condition is met:

$[(m$ is not (the Gödel number of) a sentence of $\mathcal{L} \rightarrow m \notin A) \wedge$
$(m$ is a sentence of $\mathcal{L} \rightarrow$
$\forall n(n$ is a subsentence of the sentence $m \rightarrow$
$[n$ is an atomic sentence $\rightarrow (n \in A \leftrightarrow n \in V_0)] \wedge$
$[n$ is a conditional $\Phi \rightarrow \Phi' \rightarrow \big(n \in A \leftrightarrow \big(\Phi \in A \rightarrow \Phi' \in A\big)\big)] \wedge$
$[n$ is a universal quantification $\forall x \Phi \rightarrow (n \in A \leftrightarrow$ for all $i, \Phi_x(i) \in A)]))]$

Here V_0 is the recursive set of the Gödel numbers of true atomic sentences of $\mathcal{L}$. Note that $\Phi_x(\mathbf{i})$ counts as a subsentence of $\forall x \Phi$ for each i.

Now let $F(x, y, G)$ be the formula of $\mathcal{L}^+$ expressing:

$$\forall m \left(\forall j < m \, \neg T^G(x, y, j) \rightarrow G \text{ } m\text{-approximates } V\right).$$

Lemma 3.14. *Suppose that A is arithmetical and that $F(\boldsymbol{e}, \boldsymbol{i}, A)$ holds. Then for some k there is $T^A(e, i, k)$.*

Proof. If for all k, $\neg T^A(e, i, k)$, then A would m-approximate V for all m, hence be identical with V. But V is not arithmetical. $\square$

For each e, i let $\psi_{e,i}$ be the sentence $\zeta \wedge \widetilde{F(\mathbf{e}, \mathbf{i})}$.

Lemma 3.15.

$$\exists k\, T^V(e,i,k) \quad \textit{iff} \quad \textit{for some } *, \psi^*_{e,i} \textit{ is true.}$$

Proof. Suppose $T^V(e,i,k)$.
Let r be a number greater than the number of occurrences of logical operators in any sentence of $\mathcal{L}$ with Gödel number $\leq k$. Let A be the set of Gödel numbers of true sentences of $\mathcal{L}$ that contain $< r$ occurrences of the logical operators. Now if $m \leq k$ and m is the Gödel number of a sentence of $\mathcal{L}$, then the number of occurrences of logical symbols in that sentence is smaller than r and therefore $m \in A$ iff $m \in V$.
On the other hand if m is not the Gödel number of a sentence then neither $m \in A$ nor $m \in V$.
So A is an arithmetical set that is k-equivalent to V.
Therefore by Lemma 3.13 $T^A(e,i,k)$. That means, that $F(\mathbf{e}, \mathbf{i}, A)$ holds. For if $\forall j < m\ \neg T^A(e,i,j)$, then there must be $m \leq k$, and since A is the set of Gödel numbers of true sentences of $\mathcal{L}$ that contain $< r$ occurrences of the logical operators, A m-approximates V.
We define $*$ as follows. Let $P_0, P_=, P_s, P_+, P_\times$ receive their standard interpretations. ($P^*_+ p_0 p_1 p_2$ is $p_0 + p_1 = p_2$, etc.) and let G^* be $B(p_0)$, where $B(p_0)$ is a formula of $\mathcal{L}$ that defines the set A. Under this conditions ϑ^*, $\widetilde{\Theta}^*$ and $\forall x \forall x'\ (Gx' \wedge P_= xx' \to Gx)^*$ are true, therefore ζ^* is true.
Since $P_=$ received the standard interpretation $R^*(a,b)$ holds iff $a = b$ and therefore $A = \{a|$ for some $b, R^*(a,b)$ and $G^*(b)$ hold$\}$. Hence by Lemma 3.12 $\widetilde{F(\mathbf{e}, \mathbf{i}, A)}^*$ is true and therefore also $\psi^*_{e,i}$.
Conversely suppose that $\psi^*_{e,i}$ is true.
Let $A = \{a|$ for some $b, R^*(a,b)$ and $G^*(b)$ hold$\}$. Since P^*_0 and $P^*_=$ define arithmetical relations $R^*(x,y)$ also does. Since G^* also defines an arithmetical set, A is arithmetical. Since I^* and $\widetilde{F(\mathbf{e}, \mathbf{i}, A)}^*$ are true $F(\mathbf{e}, \mathbf{i}, A)$ by Lemma 3.12. By Lemma 3.14 for some k: $T^A(e,i,k)$. Now suppose $m \leq k$. Then $\forall j < m\ \neg T^A(e,i,j)$ and since $F(\mathbf{e}, \mathbf{i}, A)$ holds, A m-approximates V. Thus if m is not the Gödel number of a sentence, m is not in A or V, but if m is the Gödel number of a sentence Φ, then by induction on sub sentences Φ' of Φ, if n is the Gödel number of a

subsentence Φ' of Φ then $n \in A$ iff Φ' is true iff $n \in V$. Therefore $m \in A$ iff $m \in V$, A is k-equivalent to V and by Lemma 3.13 $T^V(e,i,k)$. $\square$

We can now prove:

Theorem 3.3. *The class of always true sentences is Π^0_1-complete in V.*

Proof. Let $U(i,j)$ hold if and only if:
i is the Gödel number of a sentence φ of $\mathcal{QML}$ and if j is the Gödel number of an interpretation $*$ that assigns formulas of $\mathcal{L}$ to all and only predicate letters of S, then the result of substituting in φ for those predicate letters the formulae assigned to them by $*$ is true.
U is recursive in V and a sentence is always true iff its Gödel number is in $\{i|\forall j\ U(i,j)\}$.
Now let A be an arbitrary set that is Π^0_1 in V. Then $\mathbb{N}\setminus A$ is Σ^0_1 in V and thus for some e: $\mathbb{N} \setminus A = \big\{i|\exists k\ T^V(e,i,k)\big\}$. By Lemma 3.15: $\mathbb{N} \setminus A = \Big\{i|$ for some $*, \psi^*_{e,i}$ is true$\Big\}$. Therefore $A = \{i|\neg\psi_{e,i}$ is always true$\}$. So we showed that we can construct a sentence φ_i of $\mathcal{QML}$ for an arbitrary i such that $A = \{i|\varphi_i$ is always true$\}$. Take $\varphi_i = \neg\psi_{e,i}$. $\square$

Bibliography

[1] Robert M. Solovay. *Provability interpretations of modal logic.* Israel Journal of Mathematics **25** (1976) 287-304.

[2] Per Lindström. *Provability logic - a short introduction.* Theoria **62**(1-2)(1996) 19-61.

[3] George S. Boolos. *The Logic of Provability.* Cambridge University Press (1996).

[4] Solomon Feferman. *Artihmetization of methamathematics in a general setting.* Fundamenta Mathematicae **49** (1960) 35-92.

[5] Sergei N. Artemov, Lev D. Beklemishev. *Provability Logic.* Handbook of Philosophical Logic **13** (2005) 189-360.

[6] Martin Hugo Löb. *Solution of a Problem of Leon Henkin* The Journal of Symbolic Logic **20/2** (1955) 115-118.

[7] Samuel R. Buss (ed.). *Handbook of Proof Theory* Elsevier-Verlag (1998).

[8] C. Smorynski. *Self-Reference and Modal Logic.* Springer-Verlag (1985).

[9] Hartley Rogers Jr. *Theory of Recursive Functions and Effective Computability* MIT Press (1992).

[10] Herbert Enderton. *A Mathematical Introduction to Logic, 2nd Edition* Academic Press (2011)

Curriculum vitae

Personal Data

Name: Georg Smejda

Date and place of Birth: 5^{th} of May 1980, Vienna

Martial status: Living in partnership, two children

email: georgsmejda@yahoo.de

Education

September 1998: Inscription at the University of Vienna for undergraduate studies of mathematics

June 1998: Final Examination with honours at BG8

Some related and unrelated work activities

September 2009 - June 2010: Teacher at the Montessorischulen Pragerstrasse

July 2009 : Trainee at Baxters Chain Supply Management

Summers 2006 - 2008: seasonal worker for the harvest

Spring 2004: Production assistant for the documentary "'Momentaufnahme"' in India (awarded a silver dolphin at the Cannes Corporate Media & Tv Awards 2010)

January 9, 2011