DIPLOMARBEIT

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Differential Operators on Manifolds

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Abstract

In chapter 1 we give an overview of results from Differential Geometry, Distribution Theory, Functional Analyis and Measure Theory. Chapter 2 is dedicated to the algebraic definition of PDOs. We consider local operators, the principal symbol, elliptic operators and formal adjoints. In chapter 3 we introduce L^2- and Sobolev spaces of sections and analytical realizations. We prove the Sobolev embedding theorem, Young inequality, Interpolation inequalities for Sobolev norms, elliptic estimates, elliptic regularity theorem, Weyl's Lemma, Rellich's Theorem, Poincaré inequality and finally the Spectral Theorem for selfadjoint elliptic operators. We use the Spectral Theorem to define the Functional Calculus. Finally we consider two applications to partial differential equations in chapter 4.

Zusammenfassung

In Kapitel 1 geben wir einen Überblick über Ergebnisse aus Differentialgeometrie, Distributionentheorie, Funktionalanalyis und Maßtheorie. Kapitel 2 ist der algebraischen Definition der PDOs gewidmet. Wir betrachten lokale Operatoren, das Hauptsymbol, elliptische Operatoren und formale Adjungierte. In Kapitel 3 führen wir L^2- und Sobolev Schnitt-Räume und analytische Realisierungen ein. Wir beweisen das Sobolev Einbettungs Theorem, die Young Ungleichung, Interpolations Ungleichungen für Sobolevnormen, elliptische Abschätzungen, das elliptische Regularitätstheorem, Weyls Lemma, Rellichs Theorem, die Poincaré Ungleichung and schließlich das Spektraltheorem für selbstadjungierte elliptische Operatoren. Wir benutzen das Spektraltheorem um den Funktionalkalkül zu definieren. Abschließend betrachten wir zwei Anwendungen auf partielle Differentialgleichungen in Kapitel 4.

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Chapter 0

Introduction

The theory of differential equations and the corresponding differential operators is a huge field of science. Since *Isaac Newton* (1643-1727) and *Gottfried Wilhelm Leibniz* (1646-1716) introduced the theory of calculus in the 17^{th} century, legions of mathematicians aim to examine and solve differential equations. The first who introduced differential operators (but not in their full meaning) was *Joseph Fourier* (1768-1830) in his *Théorie de la Chaleur*.[1] The real advantage of the concept of differential operators became evident just around 1900 with the introduction of Functional Analysis.

Many differential equations and their associated initial/boundary value problems call for non-euclidean spaces (smooth manifolds). Nonetheless most of PDO Theory in the mathematical literature is done on $\mathbb{R}^n$ for the sake of simplicity. In many cases this approach is sufficient, since differentiation is a local concept. But there are questions which can not be answered directly from the Euclidean case. In addition a definition which has a clear global meaning is important in itself.

The aim of this thesis is to give a precise introduction to differential operators on manifolds. We cover the main concepts of the smooth theory and Spectral Theory and discuss some applications.

Since this topic is intertwined with different mathematical areas we summarize the necessary results from Differential Geometry, Distribution Theory, Functional Analysis and Measure Theory in chapter 1.

In chapter 2 we introduce a purely algebraic approach to differential operators on manifolds. This definition is chart independent. We observe the equivalence of local operators, differential operators and their local representations. Afterwards we define the principal symbol and elliptic operators, which will get very important in chapter 3. The last part of chapter 2 is dedicated to the concept of formal adjoints.

In chapter 3 we generalize PDOs by their analytical realizations and use the concepts from chapter 2 to prove the Spectral Theorem. We introduce the Functional Calculus, which provides an existence and uniqueness result for initial value problems on smooth manifolds with elliptic differential operators in the "space direction". This is a strong result since existence and uniqueness theorems for partial differential equations are rather sophisticated.

[1] see [Caj07], Differential and Integral Calculus

We conclude this work with applications to the generalized heat equation and the generalized wave equation in chapter 4.

Chapter 1

Prerequisites

In this chapter we will briefly summarize a few concepts from differential geometry, functional analysis, measure theory and the theory of distributions, which will be used throughout the whole thesis. Readers who are familiar with these topics can easily jump to chapter 2 for the main topic. The intention behind the summary is primarily to allow a wider branch of readers to follow the text and secondarily to make clear which versions of definitions and theorems are used (if there exist more than one).

1.1 Measure Theory

The basic notions σ-Algebra, measure, measure space and measureable function will be listed. Then we define the integral of measureable functions, L^p-spaces and state the Hölder inequality.

1.1.1 Definition: Let X be a set. A family of subsets $\mathcal{A} \subseteq \mathcal{P}(X)$ is called *σ-Algebra* on X, if

(A1) $X \in \mathcal{A}$

(A2) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

(A3) $A_n \in \mathcal{A}\ (n \in \mathbb{N}) \Rightarrow \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$

1.1.2 Definition: Let $(X, \mathcal{T})$ be a topological space. The *Borel-σ-Algebra* $\mathcal{B}_X$ is defined as the smallest σ-Algebra on X which contains the topology $\mathcal{T}$.

1.1.3 Definition: Let X be a set and $\mathcal{A}$ a σ-Algebra on X. A map $\mu\colon \mathcal{A} \to [0, \infty]$ is called *measure* on $(X, \mathcal{A})$, if

(M1) $\mu(\emptyset) = 0$

(M2) $A_n\ (n \in \mathbb{N})$ pairwise disjoint $\Rightarrow \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n=0}^{\infty} \mu(A_n)$

In this case $(X, \mathcal{A}, \mu)$ is called *measure space*.

1.1.4 Definition: Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces. $f\colon X \to Y$ is called *$\mathcal{A}$-$\mathcal{B}$-measureable* , if $f^{-1}\mathcal{B} \subseteq \mathcal{A}$. A map $f\colon X \to \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ is called *Borel-measureable* if f is A-$\mathcal{B}_{\mathbb{R}}$-measureble.

1.1.5 Definition: Let $(X, \mathcal{A}, \mu)$ be a measure space and $f\colon X \to \mathbb{R}$ Borel-measureable. Let f_n be a step function ($f_n = \sum_{i=1}^{k} \alpha_i \cdot 1_{A_i}$, $\alpha_i \in \mathbb{R}$), then the integral over f_n is defined as

$$\int_X f_n d\mu := \sum_{i=1}^{k} \alpha_i \cdot \mu(A_i)$$

The *(Lebesgue)-integral* of f is defined as

$$\int_X f d\mu := \lim_{n\to\infty} \int_X f_n d\mu$$

where (f_n) is a sequence of step functions which converges to f pointwise.[1]

1.1.6 Definition: Let $(X, \mathcal{A}, \mu)$ be a measure space. A property is true *almost everywhere* (*a.e.*), if the set $A \in \mathcal{A}$ on which the property is false has measure zero, i.e. $\mu(A) = 0$.

1.1.7 Definition: Let $(X, \mathcal{A}, \mu)$ be a measure space and $p \in [1, \infty[$. The space of *L^p-functions* on X is defined as

$$L^p(X) := \left\{ f : X \to \mathbb{R} \mid f \text{ is Borel-measureable and } \int_X |f|^p d\mu < \infty \right\} \Big/ \mathcal{N},$$

where $\mathcal{N} := \{f : X \to \mathbb{R} \mid f \text{ is Borel-measureable and } f = 0 \text{ a.e.}\}$.[2]

$$\|f\|_{L^p} := \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}}$$

is called the *p-norm* of f. The space of *L^∞-functions* on X is defined as

$$L^\infty(X) := \left\{ f : X \to \mathbb{R} \mid f \text{ is Borel-measureable and } |f| < \infty \text{ a.e.} \right\} \Big/ \mathcal{N},$$

with norm

$$\|f\|_{L^\infty} := \operatorname{essup}_X(|f|) := \inf_{A\in\mathcal{A}, \mu(A)=0} \big(\sup_{X\setminus A}(|f|) \big)$$

1.1.8 Remark (Hölder inequality): Let $p, q \in]1, \infty[$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then[3]:

$$\|f \cdot g\|_{L^1} \le \|f\|_{L^p} \cdot \|g\|_{L^q}$$

[1] this is well-defined, see [Zwe09], section 4.2

[2] $\mathcal{N}$ is a linear subspace, so the quotient is a vector space; see [Zwe09], text between Theorem 4.10 and Theorem 4.11

[3] see [Zwe09], Proposition 4.9

1.2 Differential Geometry

To define PDOs on manifolds we first need to introduce some basic terminology and results on manifolds and vector bundles. Thereafter we will consider tensors and Riemannian manifolds.

1.2.1 Definition: Let M be a set. A *chart* (ψ, U) of M is a bijective map $\psi\colon M \supseteq U \to V$, where V is an open subset of $\mathbb{R}^n$. Two Charts (ψ_1, U_1) and (ψ_2, U_2) are called *compatible* if $\psi_1 \circ \psi_2^{-1}\colon \psi_2(U_1 \cap U_2) \to \psi_1(U_1 \cap U_2)$ (and therefore $\psi_2 \circ \psi_1^{-1}\colon \psi_1(U_1 \cap U_2) \to \psi_2(U_1 \cap U_2)$) is a diffeomorphism of open subsets of $\mathbb{R}^n$. An C^∞-*atlas* of M is a family $\{(\psi_\alpha, U_\alpha) \mid \alpha \in A\}$ of pairwise compatible charts, such that $M = \bigcup_{\alpha \in A} U_\alpha$. Two atlasses $\mathcal{A}_1$ and $\mathcal{A}_2$ are called *equivalent* if one (and hence every) chart of $\mathcal{A}_1$ is compatible with one (and hence every) chart of $\mathcal{A}_2$. An equivalence class of atlasses of M is called a *smooth structure* on M. An atlas $\mathcal{A}$ is called *maximal* if every atlas $\mathcal{A}'$ compatible to $\mathcal{A}$ is a subset of $\mathcal{A}$.

1.2.2 Definition: A *(smooth) manifold (C^∞-mf)* is a second countable Hausdorff space M, together with a smooth structure on M.

1.2.3 Remark: Every open subset of a smooth manifold is a smooth manifold.

1.2.4 Remark: Let M be a manifold, $\psi\colon U \to V \subseteq \mathbb{R}^n$ a chart of M and $W \subseteq V$, such that $\psi(W)$ is open in $\mathbb{R}^n$. Then $(\psi|_W, W)$ is also a chart of M.[4]

1.2.5 Definition: Let M, N be smooth manifolds, $\psi_1\colon U_1 \to V_1 \subseteq \mathbb{R}^m$ and $\psi_2\colon U_2 \to V_2 \subseteq \mathbb{R}^n$ chart maps of M at p resp. N at $f(p)$. A map $f\colon M \to N$ is called *smooth* if $\psi_2 \circ f \circ \psi_1^{-1}\colon V_1 \to V_2$ is smooth.

1.2.6 Remark: Let M be a C^∞-mf and $\mathcal{A} = \{(\psi_\alpha, U_\alpha) \mid \alpha \in A\}$ a maximal atlas of M. Then $\mathcal{B} = \{U_\alpha \mid \alpha \in A\}$ is the basis of a topology on M.[5]

1.2.7 Definition: Let X be a topological space. A family of sets $\{V_\alpha \subseteq X \mid \alpha \in A\}$ is called *locally finite* if $\forall p \in M \quad \exists$ a neighborhood U of p, such that $\{\alpha \in A \mid V_\alpha \cap U \neq \emptyset\}$ is finite.

1.2.8 Definition: Let M be a C^∞-mf, $\mathcal{U}$ an open covering of M. A *partition of unity* subordinate to $\mathcal{U}$ is a family of smooth functions $\{\chi_\alpha\colon M \to [0,1] \mid \alpha \in A\}$ with the following properties:

(i) $\{\mathrm{supp}(\chi_\alpha) \mid \alpha \in A\}$ is locally finite

(ii) $\forall \alpha \in A \quad \exists U \in \mathcal{U}$ with $\mathrm{supp}(\chi_\alpha) \subseteq U$

[4] for a proof see [Kun08a], 2.2.4
[5] see [Kun08a], 2.2.5

(iii) $\forall p \in M$, $\sum_\alpha \chi_\alpha = 1$

1.2.9 Remark (Properties of C^∞-mfs[6]): Let M be a C^∞-mf. Then:

(i) For each open cover $\mathcal{U}$ of M there exists a partition of unity subordinate to $\mathcal{U}$. It is even possible to choose the same index set for cover and partition of unity, i.e. if $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ we can have a partition of unity $\{\chi_\alpha \mid \alpha \in A\}$ with $\mathrm{supp}(\chi_\alpha) \subset U_\alpha \ \forall \alpha \in A$.

(ii) M is locally compact

1.2.10 Definition: Let M be a manifold, $p \in M$ and $\psi\colon U \to V \subseteq \mathbb{R}^n$ a chart around p. The *dimension* of a manifold M *at* p is defined as $\dim(M)|_p := n$ (V is open in $\mathbb{R}^n$). If $\dim(M)|_p = \dim(M)|_q$ for all $p, q \in M$ we define the *dimension* of the manifold M as $\dim(M) = \dim(M)|_p$.

1.2.11 Remark: Let M be a manifold. $\dim(M)|_p$ is well defined, because $\psi_1 \circ \psi_2^{-1}$ is a diffeomorphism, for two charts $(\psi_1, U_1) \in \mathcal{A}_1$, $(\psi_2, U_2) \in \mathcal{A}_2$, where $\mathcal{A}_1$ and $\mathcal{A}_2$ are atlasses in the smooth structure of M. A chart around a point $p \in M$ is also a chart for nearby points, so $p \mapsto \dim(M)|_p$ is locally constant. Thus the dimension of a connected manifold is constant. Furthermore we will only consider manifolds with constant dimension.

1.2.12 Definition: Let M be a C^∞-mf, $p \in M$ and (ψ, U) a chart at p. The *tangent space* of M *at* p is defined as

$$T_pM := \{[c]_p \mid c\colon I \to M \quad C^\infty, \quad c(0) = p\}$$

where $[c_1]_p = [c_2]_p$ if $(\psi \circ c_1)' = (\psi \circ c_2)'$ and $c_1(0) = p = c_2(0)$. [7]

1.2.13 Remark: T_pM is an $\mathbb{R}$-vector space.

1.2.14 Definition: Let M, N be C^∞-mfs, $p \in M$, $f\colon M \to N \quad C^\infty$. The *tangent map* of f *at* p is defined as

$$T_pf\colon T_pM \to T_{f(p)}N, \quad T_pf = [c]_p \mapsto [f \circ c]_{f(p)}$$

1.2.15 Remark (chain rule [8]): Let M, N, P be C^∞-mfs, $p \in M$, $f\colon M \to N$, $g\colon N \to P \quad C^\infty$. Then

$$T_p(g \circ f) = T_{f(p)}g \circ T_pf \tag{1.1}$$

1.2.16 Definition: $TM := \bigcup_{p\in M} \big(\{p\} \times T_pM\big)$ is called *tangent space* of M. $Tf\colon TM \to TN, \quad Tf(p, v) := \big(f(p), T_pf(v)\big)$ is called *tangent map* of $f\colon M \to N$.

[6] see [Kun08a], 2.3.10, 2.3.14 & 2.3.6

[7] this definition is chart independent, see [Kun08a], 2.4.5

[8] for a proof see [Kun08a], 2.4.9

1.2.17 Definition: Let E and B be C^∞-mfs and $\pi\colon E \to B$ a smooth surjection. (E, B, π) is called $(\mathbb{K}-)$*vector bundle* if $\quad \forall b \in B$:

(i) $E_b := \pi^{-1}(b)$, the *fibre* over b, is a $(\mathbb{K}-)$vector space.

(ii) $\exists$ an open neighborhood $U^{(b)}$ of b in B and a fiberwise $(\mathbb{K}-)$linear diffeomorphism $\Phi\colon \pi^{-1}(U^{(b)}) \to U^{(b)} \times F$ (where F is a $(\mathbb{K}-)$vector space), such that $\pi = pr_1 \circ \Phi$

We may identify B with the subset $\phi^{-1}(U^{(b)})$ of E. A vector bundle is called *trivial* if $\quad \forall b \in B \quad U^{(b)} = B$ is possible.

1.2.18 Definition: Let (E, B, π) be a $\mathbb{K}$-vector bundle. A *vector bundle chart* is a fiberwise $\mathbb{K}$-linear diffeomorphism $\psi\colon E \supseteq U \to V \times \mathbb{K}^q$ $(q \in \mathbb{N})$ such that V is an open subset of $\mathbb{R}^n$. Clearly $\mathrm{pr}_1 \circ \psi|_{\pi(U)}\colon \pi(U) \to V$ is a chart map for B.

1.2.19 Definition: Let (E, B, π) be a vector bundle. A smooth map $X\colon B \to E$ is called *smooth section* of (E, B, π) if $\ \pi \circ X = id_B$. The space of smooth sections of (E, B, π) is denoted by $\Gamma^\infty(E)$.

1.2.20 Remark: Let M be a C^∞-mf and $\pi\colon TM \to M, \pi(p, v) := p$. Then (TM, M, π) is a vector bundle. $\mathfrak{X}(M) := \Gamma^\infty(TM)$

1.2.21 Definition: An $\mathbb{R}$-linear map $D\colon C^\infty(M) \to C^\infty(M)$ is called *derivation* of $C^\infty(M)$, if the following equation holds:

$$D(f \cdot g) = D(f) \cdot g + f \cdot D(g) \qquad \forall f, g \in C^\infty(M)$$

The space of derivations of $C^\infty(M)$ is called $\mathrm{Der}\big(C^\infty(M)\big)$.

1.2.22 Remark: $\mathrm{Der}\big(C^\infty(M)\big) \cong \mathfrak{X}(M)$ as $C^\infty(M)$-modules. [9]

1.2.23 Definition: Let V be a finite dimensional $\mathbb{K}$-vector space ($\mathbb{K} = \mathbb{R}$, $\mathbb{C}$). $T^r_s V := L^{r+s}(\overbrace{V^* \times \dots \times V^*}^{r\ times} \times \overbrace{V \times \dots \times V}^{s\ times}; \mathbb{K})$ is called the *space of* (r, s)-*tensors* over V. The *tensor product* of $t_1 \in T^{s_1}_{r_1} V$ and $t_2 \in T^{s_2}_{r_2} V$ is the tensor $t_1 \otimes t_2 \in T^{s_1+s_2}_{r_1+r_2} V$ defined by

$$\begin{aligned} t_1 \otimes t_2(\alpha^1, \dots, \alpha^{r_1}, \beta^1, \dots, \beta^{r_2}, v_1, \dots, v_{s_1}, w_1, \dots, w_{s_2}) &:= \\ = t_1(\alpha^1, \dots, \alpha^{r_1}, v_1, \dots, v_{s_1}) \cdot t_2(\beta^1, \dots, \beta^{r_2}, w_1, \dots, w_{s_2}) \end{aligned}$$

$$\forall \alpha^i, \beta^j \in V^* \quad \forall v_k, w_l \in V$$

[9] see [Kun08a], 2.5.13

1.2.24 Definition: Let (E, B, π) be a vector bundle. $T^r_s E := \bigcup_{b\in B}\big(\{b\} \times T^r_s E_b\big)$ is called the *(r,s)-tensor bundle* over E. Let M be a C^∞-mf. Then $T^r_s M := T^r_s(TM)$.

1.2.25 Definition: Let M be a C^∞-mf, $f \in C^\infty(M)$. $df\colon M \to TM^* := T^0_1 M$, $\quad df := p \mapsto T_p f$ is called *exterior derivative* of f.

1.2.26 Remark: Let M be a C^∞-mf, $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$. Then[10]:

$$df\big(X\big)(p) := df(p)\big(X(p)\big) = T_p f\big(X(p)\big) \cong X\big(f\big)(p) \quad \forall p \in M \tag{1.2}$$

1.2.27 Definition: Let M be a C^∞-mf. Elements of $\mathcal{T}^r_s M := \Gamma^\infty(T^r_s M)$ are called *(r,s)-tensor fields*.

1.2.28 Definition: Let M and N be smooth manifolds, $F : M \to N$ a smooth map and $\omega \in \mathcal{T}^0_s N$. The *pullback* $F^*\omega$ of ω under F is defined by

$$F^*\omega(p)\big(X_1, \dots, X_s\big) := \omega\big(F(p)\big)\Big(T_pF(X_1), \dots, T_pF(X_s)\Big),$$

where $X_1, \dots, X_s \in T_pM$. If F is a diffeomorphism we define the *push-forward* of $\mu \in \mathcal{T}^0_s M$ under F by

$$F_*\mu := (F^{-1})^*\mu$$

1.2.29 Definition: Let V be a finite dimensional $\mathbb{K}$-vector space ($\mathbb{K} = \mathbb{R}$, $\mathbb{C}$). $\Lambda^k V^* := L^k_{alt}(V^k; \mathbb{K})$ is the space of alternating multilinear maps from V^k to $\mathbb{K}$.

1.2.30 Remark: The *alternator* $Alt\colon T^0_k V \to T^0_k V$,

$$Alt(t)(v_1, \dots, v_k) := \frac{1}{k!} \cdot \sum_{\sigma\in S_k} sgn(\sigma) \cdot t(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

is a linear projection from $T^0_k V$ onto $\Lambda^k V^*$.[11]

1.2.31 Definition: Let $\alpha \in T^0_k V, \beta \in T^0_l V$. The *exterior product* (or *wedge product*) of α and β is defined as

$$\alpha \wedge \beta := \frac{(k+l)!}{k! \cdot l!} \cdot Alt(\alpha \otimes \beta) \tag{1.3}$$

1.2.32 Definition: Let $\omega \in \Lambda^k V^*$ and $v \in V$. We define the *interior product* by v, $i_v\colon \Lambda^k V^* \to \Lambda^{k-1} V^*$ on monomials by

$$i_v(\omega_1 \wedge \dots \wedge \omega_k) = \sum_{i=1}^{k} (-1)^{i+1} \omega_i(v) \cdot \omega_1 \wedge \dots \wedge \widehat{\omega_i} \wedge \dots \wedge \omega_k \tag{1.4}$$

[10] see [Kun08a], (2.4.3)
[11] see [Kun08a], 2.7.4

where $\widehat{\omega_i}$ means that ω_i is missing.

1.2.33 Definition: Let (E, B, π) be a vector bundle.

$$\Lambda^k E := \bigcup_{b \in B} \big(\{b\} \times \Lambda^k E_b\big)$$

Let M be a smooth manifold. We set $\Lambda^k M := \Lambda^k(TM^*)$ and call elements of $\Omega^k(M) := \Gamma^\infty(\Lambda^k M)$ *k-forms* on M.

1.2.34 Remark (The exterior derivative [12]): Let M be a smooth manifold. For every open subset U of M, there exists a unique family of maps $d := d^k(U) : \Omega^k(U) \to \Omega^{k+1}(U)$, the *exterior derivative* on U, such that:

(i) d is $\mathbb{R}$-linear and

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta \qquad \forall \alpha \in \Omega^k(U),\ \beta \in \Omega^l(U) \tag{1.5}$$

(ii) For $f \in \Omega^0(U) = C^\infty(U)$, df is the exterior derivative from 1.2.25

(iii) $d \circ d = 0$

(iv) For $U \subseteq V$, U, V open in M, $\quad d^k(U) = d^k(V)\big|_{\Omega^k(U)}$

1.2.35 Definition: Let (E, M, π) be a $\mathbb{K}$-vector bundle ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). A *(linear) connection* on E is a $\mathbb{K}$-linear map $\nabla\colon \Gamma^\infty(E) \to \Gamma^\infty(TM^* \otimes E)$, such that

$$\nabla(f \cdot u) = df \otimes u + f \cdot \nabla(u) \qquad \forall u \in \Gamma^\infty(E), \quad f \in C^\infty(M) \tag{1.6}$$

Let $\nabla u = \omega \otimes v \in \Gamma^\infty(TM^* \otimes E)$. The *covariant derivative* of u by X is defined as $\nabla_X u := \omega(X) \cdot v \in \Gamma^\infty(E)$.

1.2.36 Definition: Let M be a smooth manifold. A tensor field $g \in \mathcal{T}_2^0 M$ is called *metric (tensor)*, if $\forall p \in M\colon\ g_p\colon T_pM \times T_pM \to \mathbb{R}$ is a non degenerate, symmetric bilinearform. A smooth manifold equipped with a metric tensor is called *Semi-Riemannian manifold*. If in addition g_p is positive definite in every point p, we call (M, g) a *Riemannian manifold* and g a *Riemannian metric*.

1.2.37 Definition: Let $X, Y \in \mathfrak{X}(M)$. The *Lie bracket* $[X, Y]$ is defined as

$$[X, Y](f) := X\big(Y(f)\big) - Y\big(X(f)\big) \quad \forall f \in C^\infty(M)$$

1.2.38 Remark: Let (M, g) be a Semi-Riemannian manifold. There exists a unique connection[13] on TM, named *Levi-Civita connection* which satisfies

[12] for a proof see [Kun08a], 2.7.22
[13] see [Kun08b], 2.2.4

(i) $[X,Y] = \nabla_X Y - \nabla_Y X \quad \forall X,Y \in \mathfrak{X}(M)$

(ii) $Z\big(g(X,Y)\big) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \quad \forall X,Y,Z \in \mathfrak{X}(M)$

1.2.39 Definition: Let $\mathcal{A} = \{(\psi_\alpha, V_\alpha) \mid \alpha \in A\}$ be an atlas for the manifold M. $\mathcal{A}$ is called *oriented atlas* if $\det D(\psi_\alpha \circ \psi_\beta^{-1}) > 0 \quad \forall \alpha, \beta$ with $V_\alpha \cap V_\beta \neq 0$. A manifold together with an oriented atlas is called *oriented manifold.*

1.2.40 Definition: Let M be an n-dimensional oriented smooth manifold, $\mathcal{A} = \{(\psi_\alpha, V_\alpha) \mid \alpha \in A\}$ the corresponding oriented atlas and $\{\chi_\alpha \mid \alpha \in A\}$ a partition of unity subordinate to $\{V_\alpha \mid \alpha \in A\}$. The *integral*[14] of $\omega \in \Omega^n_c(M) = \{\omega \in \Omega^n(M) \mid \text{supp}(\omega) \text{ compact}\}$ is defined by

$$\int_M \omega := \sum_{\alpha \in A} \int_{\psi_\alpha(V_\alpha)} \psi_{\alpha *}(\chi_\alpha \cdot \omega)$$

1.2.41 Definition: A *manifold with boundary* is defined in the same way as an ordinary manifold, but demanding that charts are bijective maps onto (relatively) open subsets in $H^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 \leq 0\}$ instead of open subsets in $\mathbb{R}^n$ like in 1.2.1.

1.2.42 Remark (Stokes' theorem[15]): Let M be an oriented n-dimensional manifold with boundary, $\omega \in \Omega^{n-1}(M)$ and $\iota \colon \partial M \to M$ the canonical inclusion. Then:

$$\int_{\partial M} \iota^* \omega = \int_M d\omega \tag{1.7}$$

1.2.43 Definition: Let M be an n-dimensional manifold. An $\omega \in \Omega^n(M)$ with $\omega(p) \neq 0 \quad \forall p \in M$ is called *volume form.*

1.2.44 Remark: If (M,g) is an n-dimensional oriented Riemannian manifold there exists a canonical volume form, called the *metric volume* dV_g on M. In addition we have an isomorphism $*$ from $\Omega^k(M)$ to $\Omega^{n-k}(M)$, the *Hodge *-Operator*, uniquely defined by

$$\alpha \wedge *\beta := g_k(\alpha, \beta) \cdot dV_g \quad \forall \alpha, \beta \in \Omega^k(M) \tag{1.8}$$

where the metric $g_k \colon \Omega^k(M) \times \Omega^k(M) \to C^\infty(M)$ is defined on monomials by

$$\begin{aligned} g_k(\alpha_1 \wedge \dots \wedge \alpha_k, \beta_1 \wedge \dots \wedge \beta_k) : &= det\Big(\big(g(\alpha_i^*, \beta_j^*)\big)_{1 \leq i,j \leq k}\Big) = \\ &= det\Big(\big(\alpha_i(\beta_j^*)\big)_{1 \leq i,j \leq k}\Big) \end{aligned} \tag{1.9}$$

where α_i^* denotes the vector field equivalent to α_i, given by $\alpha_i = g(\alpha_i^*, .)$ and analogous for β_j. [16]

[14] Note that the right hand side is an integral in $\mathbb{R}^n$; This definition is independent of the chosen oriented atlas in the smooth structure of M, see [Kun08a], 2.8.6

[15] see [Kun08a], 2.8.12

[16] see [Nic07], page 149

1.2.45 Remark: Let (M, g) be an n-dimensional oriented Riemannian manifold, (ϕ, U) a chart and $f \in C^\infty(M)$ with $\mathrm{supp}(f) \subseteq U$. Then[17]

$$\int_U f dV_g = \int_{\phi(U)} \phi_* f \cdot \sqrt{\left|\det\left(g(X_i, X_j)\right)\right|} \circ \phi^{-1} dx$$

where $X_i := (T_p\phi)^{-1}(e_i)$ ($\{e_i \mid 1 \leq i \leq n$ is the standard basis of $\mathbb{R}^n$).

1.2.46 Definition: Let (M, g) be an n-dimensional Semi-Riemannian manifold and U an open subset of M. A selection of vectorfields $v_1, \ldots, v_n \in \mathfrak{X}(U)$ is called *frame* on U, if $\{v_1(p), \ldots, v_n(p)\}$ is an orthonormal basis of T_pM for every point $p \in U$.[18]

1.2.47 Remark: Let (M, g) be an oriented Riemannian manifold, $\{v_1, \ldots, v_n\}$ a local frame, $v_i^* := g(v_i, .)$ the dual 1-form to v_i $(1 \leq i \leq n)$ and $\sigma \in \Sigma_n$ a permutation. Then[19]:

$$*(v^*_{\sigma(1)} \wedge \cdots \wedge v^*_{\sigma(k)}) = \mathrm{sgn}(\sigma) \cdot v^*_{\sigma(k+1)} \wedge \cdots \wedge v^*_{\sigma(n)} \tag{1.10}$$

1.2.48 Corollary: *The inverse of* $*\colon \Omega^k(M) \to \Omega^{n-k}(M)$ *is given by*

$$*^{-1} = (-1)^{k(n-k)} * \tag{1.11}$$

Proof. Let $\{v_1, \ldots, v_n\}$ be a local frame. $*$ is linear, so it suffices to prove the assumption for the basis vectors $v^*_{\sigma(1)} \wedge \cdots \wedge v^*_{\sigma(k)}$ of $\Omega^k(M)$.

$$\begin{aligned} * * (v^*_{\sigma(1)} \wedge \cdots \wedge v^*_{\sigma(k)}) &= *(\mathrm{sgn}(\sigma) \cdot v^*_{\sigma(k+1)} \wedge \cdots \wedge v^*_{\sigma(n)}) = \\ &= \mathrm{sgn}(\sigma)\mathrm{sgn}(\mu) \cdot v^*_{\sigma(1)} \wedge \cdots \wedge v^*_{\sigma(k)} \end{aligned} \tag{1.12}$$

where

$$\mu = \begin{pmatrix} 1 & \ldots & n-k & n-k+1 & \ldots & n \\ \sigma(k+1) & \ldots & \sigma(n) & \sigma(1) & \ldots & \sigma(k) \end{pmatrix} \in \Sigma_n.$$

If we take the last k entries of μ and put them in the front, thereby pushing the first $n-k$ entries to the back we get σ. This can be realized by $n-k$ transpositions of the form $(i \quad i-1)$ for each of the k entries $\sigma(1)$ to $\sigma(k)$. Thus we have

$$\mathrm{sgn}(\mu) = (-1)^{k(n-k)}\mathrm{sgn}(\sigma)$$

We use this in equation (1.12) and conclude

$$** = (-1)^{k(n-k)}.$$

□

[17] see [Nic07], page 149

[18] For every $p \in M$ and every orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM there exist an open neighborhood U of p and a frame $\{v_1, \ldots, v_n\}$ on U, such that $v_i(p) = e_i \quad \forall 1 \leq i \leq n$. A proof can be found in [Kun08b], 2.9.3

[19] see [Kun07], 3.3.7(i)

1.3 Distribution Theory

We will define the notions of test function, distribution, the support of a distribution and see that distributions with support in one single point are given by an easy formula. We define Sobolev spaces and state the Sobolev embedding theorem. Then we consider distributional Fourier transform.

1.3.1 Definition: Let Ω be an open subset of $\mathbb{R}^n$. $\mathcal{D}(\Omega) := \{f \in C^\infty(\Omega) \mid \operatorname{supp}(f)$ is compact$\}$ is called the space of *test functions* on Ω. Let K be a compact subset of Ω ($K \subset\subset \Omega$), $\mathcal{D}(K) := \{f \in \mathcal{D}(\Omega) \mid \operatorname{supp}(f) \subseteq K\}$

1.3.2 Definition: A linear map $u \colon \mathcal{D}(\Omega) \to \mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$) is called a *distribution* if

$$\forall K \subset\subset \Omega \quad \exists C > 0 \quad \exists m \in \mathbb{N}_0 \colon\ |u(\phi)| \leq C \cdot \sum_{|\alpha| \leq m} \|\partial^\alpha \phi\|_{L^\infty(K)} \quad \forall \phi \in \mathcal{D}(K)$$

The space of distributions on Ω is called $\mathcal{D}'(\Omega)$.

1.3.3 Remark: The map $f \mapsto u_f$, where $u_f(\phi) := \int_\Omega f(x)\phi(x)dx$ for $\phi \in \mathcal{D}(\Omega)$, is a linear embedding from $L^1_{loc}(\Omega)$ to $\mathcal{D}'(\Omega)$.[20]

1.3.4 Definition: Let $u, u_k \in \mathcal{D}'(\Omega)$. We say that u_k *converges* to u *in* $\mathcal{D}'(\Omega)$, if $u_k(\phi)$ converges to $u(\phi)$ in $\mathbb{C}$ for every $\phi \in \mathcal{D}(\Omega)$.

1.3.5 Definition: Let $u \in \mathcal{D}'(\Omega)$, $X := \{x \in \Omega \mid \exists U$ open neighborhood of x, such that $u|_{\mathcal{D}(U)} = 0\}$. $\operatorname{supp}(u) := \Omega \setminus X$ is called the *support* of the distribution u.

1.3.6 Remark: Let $x_0 \in \Omega, u \in \mathcal{D}'(\Omega)$ with $\operatorname{supp}(u) = \{x_0\}$. Then[21]:

$$\exists m \in \mathbb{N}_0 \quad \exists a_\alpha \in \mathbb{K} \quad \forall \phi \in \mathcal{D}(\Omega)\colon \quad u(\phi) = \sum_{|\alpha| \leq m} a_\alpha \cdot \partial^\alpha \phi(x_0)$$

1.3.7 Definition (Differentiation of distributions): Let $u \in \mathcal{D}'(\Omega)$. We define the distribution[22] $\partial_i u$ for $1 \leq i \leq n$ by

$$\big(\partial_i u\big)(\phi) := -u(\partial_i \phi) \quad \forall \phi \in \mathcal{D}(\Omega)$$

1.3.8 Remark: Let $u, u_k \in \mathcal{D}'(\Omega)$. From $u = \sum_{k=0}^\infty u_k$ in $\mathcal{D}'(\Omega)$ it follows that $\partial^\alpha u = \sum_{k=0}^\infty \partial^\alpha u_k \quad \forall \alpha \in \mathbb{N}_0^n$ in $\mathcal{D}'(\Omega)$.[23]

[20] see [HS09], 1.27; Therefore we can consider $L^1_{loc}(\Omega)$ as a subspace of $\mathcal{D}'(\Omega)$ and often write the same symbol for f and u_f.

[21] for a proof see [HS09], 1.70

[22] see [HS09], 2.4; For (piecewise) differentiable functions this is the usual partial derivative

[23] see [HS09], 2.7

1.3.9 Definition (Multiplication by smooth functions): Let $u \in \mathcal{D}'(\Omega)$ and $f \in C^\infty(\Omega)$. The product[24] $f \cdot u$ is defined by

$$\big(f \cdot u\big)(\phi) := u(f \cdot \phi) \quad \forall \phi \in \mathcal{D}(\Omega)$$

1.3.10 Definition: Let $s \in \mathbb{N}_0$. We define the *Sobolev space* $H^s(\Omega)$ by

$$H^s(\Omega) := \{u \in \mathcal{D}'(\mathbb{R}^n) \mid \partial^\alpha u \in L^2(\Omega) \quad \forall |\alpha| \leq s\}$$

1.3.11 Remark (Sobolev embedding): For $s > k + \frac{n}{2}$ it follows that $H^s(\Omega)$ is a subset of $C^k(\Omega)$.

1.3.12 Definition: The *Fourier transform* $\mathcal{F}\colon L^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ is defined by

$$\mathcal{F}\big(f\big)(\xi) := \widehat{f}(\xi) := \int_{\mathbb{R}^n} \mathrm{e}^{-i\langle x,\xi\rangle} \cdot f(x) dx \qquad (\xi \in \mathbb{R}^n)$$

1.3.13 Definition: The space of *Schwartz functions* is defined as

$$\mathcal{S}(\mathbb{R}^n) := \Big\{\phi \in C^\infty(\mathbb{R}^n) \mid \forall \alpha \in \mathbb{N}_0^n \; \forall l \in \mathbb{N}_0 \; \exists C > 0 \colon \\ \forall x \in \mathbb{R}^n \colon |\partial^\alpha \phi(x)| \leq \frac{C}{(1+|x|)^l}\Big\}$$

1.3.14 Remark: Since $\mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ we can Fourier transform elements of $\mathcal{S}(\mathbb{R}^n)$. If $f \in \mathcal{S}(\mathbb{R}^n)$ it follows that $\widehat{f} \in \mathcal{S}(\mathbb{R}^n)$.

1.3.15 Definition: A linear map $u\colon \mathcal{S}(\mathbb{R}^n) \to \mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$) is called a *tempered distribution* if

$$\exists C > 0 \quad \exists m \in \mathbb{N}_0 \colon \; |u(\phi)| \leq C \cdot \sum_{|\alpha|,|\beta| \leq m} \|x^\beta \partial^\alpha \phi\|_{L^\infty(\mathbb{R}^n)} \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n)$$

The space of tempered distributions on $\mathbb{R}^n$ is called $\mathcal{S}'(\mathbb{R}^n)$.

1.3.16 Remark: $\mathcal{S}'(\mathbb{R}^n) \subseteq \mathcal{D}'(\mathbb{R}^n)$.

1.3.17 Definition: Let $u \in \mathcal{S}'(\mathbb{R}^n)$ We define the *Fourier transform of tempered distributions* $\mathcal{F}\colon \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ by

$$\widehat{u}(\phi) := u(\widehat{\phi}) \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n)$$

1.3.18 Remark (Exchange formulae): For $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$ the following equations hold[25]:

$$\mathcal{F}(\partial^\alpha u) = i^{|\alpha|} \xi^\alpha \cdot \mathcal{F}(u) \tag{1.13}$$

[24] this is again a distribution, see [HS09], 2.15
[25] see [HS09], 5.26

and

$$\mathcal{F}(i^{|\alpha|}x^{\alpha} \cdot u) = \partial^{\alpha}\mathcal{F}(u) \tag{1.14}$$

1.3.19 Remark (Plancherel Theorem[26]): For $f \in L^2(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ it follows that $\widehat{f} \in L^2(\mathbb{R}^n)$ and the following equations hold:

$$\int_{\mathbb{R}^n} f(x) \cdot \widehat{g}(x)dx = \int_{\mathbb{R}^n} \widehat{f}(\xi) \cdot g(\xi)d\xi \quad \forall f, g \in L^2(\mathbb{R}^n) \tag{1.15}$$

and

$$\|\widehat{f}\|_{L^2} = (2\pi)^{\frac{n}{2}}\|f\|_{L^2} \tag{1.16}$$

1.3.20 Remark: Let $u \in H^s(\mathbb{R}^n)$. The norms

$$\|u\|_s := \sum_{|\alpha|\leq s} \|\partial^{\alpha}u\|_{L^2}$$

and

$$\|u\|_{H^s} := (2\pi)^{-\frac{n}{2}}\|(1+|\xi|^2)^{\frac{s}{2}}\widehat{u}\|_{L^2}$$

are equivalent.[27]

1.3.21 Remark: Let $u \in \mathcal{S}'(\mathbb{R}^n)$. Then

$$\mathcal{F}^2(u) = (2\pi)^n u^-$$

where $u^-\big(\phi(.)\big) := u\big(\phi(-.)\big)$.

1.3.22 Definition: Let $u, v \in \mathcal{D}'(\mathbb{R}^n)$, at least one of them with compact support. The convolution of u and v is defined by

$$u * v(\phi) := u_{(x)}\Big(v_{(y)}\big(\phi(x+y)\big)\Big)$$

1.3.23 Remark: Let u, v as above and $\alpha \in \mathbb{N}_0^n$. Then[28]:

$$\partial^{\alpha}(u * v) = (\partial^{\alpha}u) * v = u * \partial^{\alpha}v \tag{1.17}$$

1.3.24 Remark: Let $\rho \in C^{\infty}(\mathbb{R}^n)$ and $u \in \mathcal{D}'(\mathbb{R}^n)$, one of them with compact support. Then $\rho * u$ is smooth.

1.3.25 Proposition: *We can also define the convolution on $\mathcal{S}' \times \mathcal{S}$ (see [HS09], 4.17). Let $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$\widehat{\phi \cdot u} = (2\pi)^{-n}(\widehat{\phi} * \widehat{u}) \tag{1.18}$$

[26] see [HS09], 5.36
[27] see [HS09], 6.17
[28] see [HS09], 4.5 (ii)

Proof. We have the convolution theorem[29]

$$\widehat{v * \psi} = \widehat{v} \cdot \widehat{\psi}$$

Set $v := \widehat{u}$ and $\psi := \widehat{\phi}$. Since $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$, v and ψ are also in $\mathcal{S}'$ and $\mathcal{S}$, respectively. So we get

$$\widehat{\widehat{u} * \widehat{\phi}} = \widehat{\widehat{u}} \cdot \widehat{\widehat{\phi}}$$

We Fourier transform this equation and use 1.3.21 to conclude

$$(2\pi)^n (\widehat{u} * \widehat{\phi})^- = \mathcal{F}((2\pi)^n u^- \cdot (2\pi)^n \phi^-) = (2\pi)^{2n} \widehat{u \cdot \phi}^{\,-}$$

and consequently

$$(2\pi)^{-n} (\widehat{u} * \widehat{\phi}) = \widehat{u \cdot \phi}$$ □

1.4 Functional Analysis

We will define the notions normed space, scalar product space, Banach and Hilbert space and state that every normed/scalar product space can be densely embedded into an almost unique Banach/Hilbert space. Thereafter we consider densely defined operators. We state the Spectral Theorem for selfadjoint, compact operators and the Arzelà-Ascoli theorem.

1.4.1 Definition: Let V be a vector space. If V is equipped with a norm $\|.\|$, we call $(V, \|.\|)$ a *normed space*. . If a *scalar product*[30] $\langle \cdot, \cdot \rangle$ on V is given we call $(V, \langle \cdot, \cdot \rangle)$ a *scalar product space* or *Pre-Hilbert space* .

1.4.2 Remark: Every scalar product space $(V, \langle \cdot, \cdot \rangle)$ is a normed space, where the norm is given by $\|.\| := \sqrt{\langle \cdot, \cdot \rangle}$.

1.4.3 Definition: A normed space $(V, \|.\|)$ is called *Banach space* if it is *complete*, i.e. if every Cauchy sequence in V with respect to $\|.\|$ has a limit in V. A complete Pre-Hilbert space is called *Hilbert space*.

1.4.4 Remark: Every normed space $(V, \|.\|)$ can be densely embedded into a Banach space $(V', \|.\|')$, such that $\|.\|'|_V = \|.\|$. Every Pre-Hilbert space $(V, \langle \cdot, \cdot \rangle)$ can be densely embedded into a Hilbert space $(V', \langle \cdot, \cdot \rangle')$, such that $\langle \cdot, \cdot \rangle'|_V = \langle \cdot, \cdot \rangle$. In both cases V' is called *completion* of V. Two completions of the same space are isometric.[31]

[29] see [HS09], 5.34
[30] a positive definite, symmetric (or Hermitian) (sesqui-)linear form
[31] see [Heu92], 41.3 & 41.5

1.4.5 Remark: Let C be a closed linear subspace of a Hilbert space $(V, \langle\cdot,\cdot\rangle)$ and $x_0 \in V \setminus C$. Then there exists a $y \in V$, such that $\langle y, x\rangle = 0$ for all $x \in C$ and $\langle y, x_0\rangle > 0$.[32]

1.4.6 Definition: A *semi-normed space* is a vector space V together with a system of *semi-norms*[33] $\mathcal{P}$. A subset B of a seminormed space is *bounded* if every $p \in \mathcal{P}$ is bounded on B.

1.4.7 Remark: Let Ω be an open subset of $\mathbb{R}^n$. $C^k(\Omega)$ $(k \in \mathbb{N}_0 \cup \{\infty\})$ together with the semi-norms $p_{\alpha,K}(f) := \|\partial^\alpha f\|_{L^\infty(K)}$ $(\alpha \in \mathbb{N}_0^n,\ K \subset \Omega$ compact) is a semi-normed space.

1.4.8 Definition: Let V and W be normed spaces and $T\colon V \to W$ a linear map. We call T *bounded* if the image of all bounded sets in V under T is bounded in W. Otherwise we call it *unbounded*.

1.4.9 Definition: Let V and W be Hilbert spaces over $\mathbb{K}$ $(= \mathbb{R}$ or $\mathbb{C})$. An operator $T\colon \mathrm{Dom}(T) \subseteq V \to W$ is called *densely defined* if $\mathrm{Dom}(T)$, the domain of T, is a dense linear subspace of V. T is called *closed* if its graph,

$$\mathrm{graph}(T) := \{(x, Tx) \in \mathrm{Dom}(T) \times W\}$$

is a closed subspace of $V \times W$.

1.4.10 Remark (Closed Graph Theorem[34]): Let V, W be Banach spaces and $T\colon V \to W$ linear $(\mathrm{Dom}(T) = V!)$. Then T is bounded if and only if T is closed.

1.4.11 Definition: Let $T\colon \mathrm{Dom}(T) \to W$ be a closed, densely defined linear operator. The *adjoint operator* of T is the operator $T^*\colon \mathrm{Dom}(T^*) \to V$ defined by the graph[35]

$$\mathrm{graph}(T^*) := \{(w, v) \in W \times V \mid \langle v, x\rangle_V = \langle w, Tx\rangle_W \quad \forall x \in \mathrm{Dom}(T)\}$$

1.4.12 Remark: T^* is a closed operator. Since T is closed we get that the domain of T^* is dense, so we can define T^{**}. Moreover $T^{**} = T$.[36]

1.4.13 Remark: $id^* = id$ and if T is invertible $(T^{-1})^* = (T^*)^{-1}$.[37]

[32] see [Gro08], 12.5 + 6.3
[33] maps $p\colon V \to \mathbb{R}$, which satisfy all norm axioms, except $p(v) = 0 \Rightarrow v = 0$
[34] see [RS80], III.12
[35] this is well defined since the scalar products are non-degenerated and $\mathrm{Dom}(T)$ is dense in V
[36] see [RS80], Theorem VIII.1
[37] see [Gro08], 6.5

1.4.14 Definition: A closed, densely defined operator $T\colon \mathrm{Dom}(T) \subseteq X \to X$ is called *selfadjoint* if $T = T^*$, more precisely:

1. $\langle Tx, y\rangle = \langle x, Ty\rangle \quad \forall x, y \in \mathrm{Dom}(T)$ and

2. $\mathrm{Dom}(T) = \mathrm{Dom}(T^*)$

if only the first condition holds we call T *symmetric*.

1.4.15 Definition: Let $T\colon V \supseteq \mathrm{Dom}(T) \to V$ be a densely defined linear operator. The *spectrum* of T is defined as

$$\begin{aligned}&\mathrm{spec}(T) := \\ &\quad = \{\lambda \in \mathbb{K} \mid \lambda \cdot id_{\mathrm{Dom}(T)} - T \text{ has no bounded inverse from V to } \mathrm{Dom}(T)\}\end{aligned}$$

1.4.16 Proposition: *Let T be as above and closed, then if an inverse to $\lambda \cdot id_{\mathrm{Dom}(T)} - T$ ($\lambda \in \mathbb{K}$) exists, it is bounded. Thus*

$$\mathrm{spec}(T) = \{\lambda \in \mathbb{K} \mid \lambda \cdot id_{\mathrm{Dom}(T)} - T \textit{ has no inverse from } V \textit{ to } \mathrm{Dom}(T)\}$$

Proof. T closed means that $\{(x, Tx) \mid x \in \mathrm{Dom}(T)\}$ is closed in $\mathrm{Dom}(T) \times V$. Thus

$$\{(x, \lambda \cdot x - Tx) \mid x \in \mathrm{Dom}(T)\}$$

is also closed in $\mathrm{Dom}(T)\times V$ for some fixed $\lambda \in \mathbb{K}$. We assume that $\lambda\cdot id_{\mathrm{Dom}(T)} - T$ has an inverse $Q\colon V \to \mathrm{Dom}(T)$. Then

$$\mathrm{graph}(Q) = \{(y, Qy) \mid y \in V\} = \{(\lambda \cdot x - Tx, x) \mid x \in \mathrm{Dom}(T)\}$$

is closed in $V \times \mathrm{Dom}(T)$ by the above. Thus Q is a linear closed operator from the Banach space V to $\mathrm{Dom}(T) \subseteq V$ and we use the closed graph theorem, 1.4.10 to conclude that Q is bounded. □

1.4.17 Definition: Let V and W be Banach spaces. An operator $T : V \to W$ is called *compact* if it maps bounded subsets of V to relatively compact subsets of W.

1.4.18 Remark (Spectral Theorem for compact operators[38]): Let V be an infinite-dimensional Hilbert space and $T\colon V \to V$ a compact and selfadjoint operator. The spectrum of T consists of 0 and countably many eigenvalues which accumulate only at 0. In addition

$$V = \bigoplus_{\lambda\in\mathrm{spec}(T)\setminus\{0\}} \ker(\lambda \cdot id - T)$$

[38] see [Gro08], 9.7

1.4.19 Definition: Let (X, d) and (X', d') be two metric spaces. A family of functions A, where $f\colon (X, d) \to (X', d')$ for all $f \in A$, is called *equicontinuous* if

$$\forall \epsilon > 0 \quad \forall x \in X \quad \exists \delta > 0 \quad \forall f \in A \quad \forall y \in X : \\ d\big(x, y\big) < \delta \;\Rightarrow\; d'\big(f(x), f(y)\big) < \epsilon$$

1.4.20 Remark (Arzelà-Ascoli[39]): Let (X, d) be a compact metric space and $A \subseteq C(X)$. A is relatively compact in $(C(X), \|.\|_\infty)$ if and only if A is pointwise bounded and equicontinuous.

[39] see [Gro08], 10.3.

Chapter 2

PDOs on Manifolds

We introduce a definiton of differential operators on manifolds which is chart-independent and show some basic properties, including local representation, the principal symbol and formal adjoint operators. In particular we will show that differential operators are exactly the local operators from $\Gamma^\infty(E) \to \Gamma^\infty(F)$.

This chapter will roughly follow the approach of [Nic07], section 10.1.

2.1 Differential/Local Operators

To get an idea how differential operators could be defined we investigate two examples of PDOs on $\mathbb{R}^n$:

- *The Laplacian*[1]: $\Delta\colon C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n), \quad \Delta u = -\sum_{i=1}^n \partial_i^2 u$
- *The exterior derivative*: $d\colon \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$

Remember that $\Omega^k(\mathbb{R}^n)$ is defined as the space of smooth sections of the vector bundle $\Lambda^k(T\mathbb{R}^n)$ over the manifold $\mathbb{R}^n$. $C^\infty(\mathbb{R}^n)$ can be interpreted as the space of smooth sections of the trivial vector bundle $\mathbb{R}^n \times \mathbb{R}$ over the manifold $\mathbb{R}^n$. We see that the above operators are both $\mathbb{R}$-linear and link smooth sections of some vector bundle to smooth sections of another vector bundle. So we concentrate on operators with these properties:

2.1.1 Definition: Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For $\mathbb{K}$-vector bundles E and F over the smooth manifold M we define

$$\mathbf{Op}(E,F) := \{P\colon \Gamma^\infty(E) \to \Gamma^\infty(F) \mid P \quad \mathbb{K}-\text{linear}\}$$

2.1.2 Remark: $\mathbf{Op}(E,E)$ is an associative $\mathbb{K}$-Algebra. $\Gamma^\infty(E)$ and $\Gamma^\infty(F)$ are modules over the ring $C^\infty(M)$.

[1] Please note the sign convention; The Laplacian with this sign is also often called *geometer's Laplacian*.

On $\mathbb{R}^n$, 0^{th}-order PDOs with smooth coefficients are just multiplication by some smooth function f, which are the $C^\infty(\mathbb{R}^n)$-linear maps. So, on an arbitrary C^∞-mf M, this should generalize to $C^\infty(M)$-linear maps $P\colon \Gamma^\infty(E) \to \Gamma^\infty(F)$. We can characterize the linearity by

$$[P,f]u := P(f\cdot u) - f\cdot P(u) = 0 \quad \forall f\in C^\infty(M) \quad \forall u \in \Gamma^\infty(E)$$

We see that each $f \in C^\infty(M)$ defines a map $ad(f) : \mathbf{Op}(E,F) \to \mathbf{Op}(E,F)$ by

$$P \mapsto ad(f)P := [P,f]$$

Thus we may define:

$$\mathbf{PDO}^0(E,F) := \{P\in \mathbf{Op}(E,F) \mid P \in ker\big(ad(f)\big) \quad \forall f\in C^\infty(M)\},$$

and for higher orders:

$$\begin{aligned}&\mathbf{PDO}^{(m)}(E,F) :=\\ &\qquad = \{P\in \mathbf{Op}(E,F)\mid P\in ker\big(ad(f_0)\cdots ad(f_m)\big) \quad \forall f_i \in C^\infty(M)\}\end{aligned}$$

The elements of $\mathbf{PDO}^{(m)}$ are called *partial differential operators (PDOs) of order $\leq m$.*

$$\mathbf{PDO}(E,F) := \bigcup_{m\geq 0} \mathbf{PDO}^{(m)}(E,F)$$

is the set of all PDOs from E to F.

Note that we have a recursive formula for $\mathbf{PDO}^{(m)}$:

$$\mathbf{PDO}^{(m)} = \{P\in\mathbf{Op}\mid [P,f]\in \mathbf{PDO}^{(m-1)} \quad \forall f\in C^\infty(M)\}$$

2.1.3 Remark:

$$\mathbf{PDO}^0(E,F) \underset{(1)}{=} \mathbf{Hom}_{C^\infty(M)}\big(\Gamma^\infty(E),\Gamma^\infty(F)\big) \underset{(2)}{\cong} \mathbf{Hom}(E,F)$$

where

$$\begin{aligned}&\mathbf{Hom}(E,F) :=\\ &= \{\phi\colon E\to F \mid \phi\ C^\infty, \pi_F\circ\phi = \pi_E \text{ and } \phi|_{E_p}: E_p \to F_p \text{ is linear } \forall p\in M\}\end{aligned}$$

(1) The property $[P,f] = 0$ means that P is a $C^\infty(M)$-linear map between the $C^\infty(M)$-modules $\Gamma^\infty(E)$ and $\Gamma^\infty(F)$, i.e. a $C^\infty(M)$-module homomorphism.

(2) See [GHV72], chapter II, §5, proposition XIII.

Now we will show that some known operators are PDOs in our sense, by using that each of these operators satisfies some kind of product rule:

2.1.4 Example: 1. The *partial derivative* $\partial_i \colon C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is a PDO of order ≤ 1:

Let $f, g \in C^\infty(\mathbb{R}^n), \quad u \in C^\infty(\mathbb{R}^n) = \Gamma^\infty(\mathbb{R}^n \times \mathbb{R})$

$$\begin{aligned}
[\partial_i, f](u) &= \partial_i(f \cdot u) - f \cdot \partial_i(u) = \\
&= \partial_i(f) \cdot u + f \cdot \partial_i(u) - f \cdot \partial_i(u) = \\
&= \partial_i(f) \cdot u
\end{aligned}$$

$$\big[[\partial_i, f], g\big](u) = \partial_i(f) \cdot (g \cdot u) - g \cdot \partial_i(f) \cdot u = 0$$

2. The *Euclidean Laplacian* $\Delta \colon C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n), \quad \Delta := -\sum_{i=1}^n \partial_i^2$ is a PDO of order ≤ 2:

We will just use the product rule for ∂_i. Let $f, g, h \in C^\infty(\mathbb{R}^n), \quad u \in C^\infty(\mathbb{R}^n) = \Gamma^\infty(\mathbb{R}^n \times \mathbb{R})$

$$\begin{aligned}
[\Delta, f](u) &= \Delta(f \cdot u) - f \cdot \Delta(u) = \\
&= -\sum_{i=1}^n \partial_i^2(f \cdot u) - f \cdot \Delta(u) = \\
&= -\sum_{i=1}^n \partial_i\big(\partial_i(f) \cdot u + f \cdot \partial_i(u)\big) - f \cdot \Delta(u) = \\
&= -\sum_{i=1}^n \big(\partial_i^2(f) \cdot u + 2 \cdot \partial_i(f) \cdot \partial_i(u) + f \cdot \partial_i^2(u)\big) - f \cdot \Delta(u) = \\
&= \Delta(f) \cdot u - 2 \cdot \sum_{i=1}^n \big(\partial_i(f) \cdot \partial_i(u)\big)
\end{aligned}$$

$$\begin{aligned}
\big[[\Delta, f], g\big](u) &= [\Delta, f](g \cdot u) - g \cdot [\Delta, f](u) = \\
&= \Delta(f) \cdot (g \cdot u) - 2 \cdot \sum_{i=1}^n \big(\partial_i(f) \cdot \partial_i(g \cdot u)\big) - \\
&\qquad - g \cdot \Big(\Delta(f) \cdot u - 2 \cdot \sum_{i=1}^n \big(\partial_i(f) \cdot \partial_i(u)\big)\Big) = \\
&= 2 \cdot \sum_{i=1}^n \big(- \partial_i(f) \cdot \partial_i(g) \cdot u - \partial_i(f) \cdot g \cdot \partial_i(u) + \\
&\qquad + g \cdot \partial_i(f) \cdot \partial_i(u)\big) = \\
&= -2 \cdot u \cdot \sum_{i=1}^n \big(\partial_i(f) \cdot \partial_i(g)\big)
\end{aligned}$$

$$
\begin{aligned}
\Big[\big[[\Delta, f], g\big], h\Big](u) &= \big[[\Delta, f], g\big](h \cdot u) - h \cdot \big[[\Delta, f], g\big](u) = \\
&= -2 \cdot (h \cdot u) \cdot \sum_{i=1}^{n} \big(\partial_i(f) \cdot \partial_i(g)\big) - \\
&\qquad\qquad - h \cdot \Big(-2 \cdot u \cdot \sum_{i=1}^{n} \big(\partial_i(f) \cdot \partial_i(g)\big)\Big) = \\
&= 0
\end{aligned}
$$

3. The *exterior derivative* $d \colon \Omega^k(M) \to \Omega^{k+1}(M)$ is a PDO of order ≤ 1:

   Let $f, g \in C^\infty(M), \quad \omega \in \Omega^k(M) = \Gamma^\infty(\Lambda^k(TM))$.
   By (1.5) we get

   $$[d, f](\omega) = d(f \cdot \omega) - f \cdot d(\omega) = df \wedge \omega + (-1)^0 f \wedge d\omega - f \cdot d\omega = df \wedge \omega,$$

   and by $C^\infty(M)$-linearity of the wedge-product (see (1.3))

   $$\big[[d, f], g\big](\omega) = [d, f](g \cdot \omega) - g \cdot [d, f](\omega) = df \wedge (g \cdot \omega) - g \cdot df \wedge \omega = 0$$

4. Let M be a smooth manifold, (E, M, π) a vector bundle and ∇ a connection on E. ∇ is an operator from $\Gamma^\infty(E)$ to $\Gamma^\infty(TM^* \otimes E)$ (see 1.2.35). ∇ is a PDO of order ≤ 1:
   Let $f, g \in C^\infty(M), \quad t \in \Gamma^\infty(E)$. By (1.6) we get

   $$[\nabla, f](t) = \nabla(f \cdot t) - f \cdot \nabla(t) = df \otimes t + f \cdot \nabla(t) - f \cdot \nabla(t) = df \otimes t$$

   and by $C^\infty(M)$-linearity of the tensor-product (see 1.2.23)

   $$\big[[\nabla, f], g\big](t) = [\nabla, f](g \cdot t) - g \cdot [\nabla, f](t) = df \otimes (g \cdot t) - g \cdot df \otimes \ t = 0$$

5. Let $X \in \mathfrak{X}(M)$. From the above and the definition of the *covariant derivative* in 1.2.35 we also get that ∇_X is a PDO of order ≤ 1 from $\Gamma^\infty(E)$ to $\Gamma^\infty(E)$. The first commutator (which will be of importance in the next section) is

   $$[\nabla_X, f](t) = df(X) \cdot t$$

2.1.5 Lemma: *Let $P \in \mathbf{PDO}(E, F)$, $u \in \Gamma^\infty(E)$, $f, g \in C^\infty(M)$. Then:*

$$[P, f \cdot g](u) = [P, f](g \cdot u) + f \cdot [P, g](u) \tag{2.1}$$

Proof.

$$
\begin{aligned}
[P, f \cdot g](u) &= P(f \cdot g \cdot u) - f \cdot g \cdot P(u) = \\
&= P(f \cdot g \cdot u) - f \cdot P(g \cdot u) + f \cdot P(g \cdot u) - f \cdot g \cdot P(u) = \\
&= [P, f](g \cdot u) + f \cdot [P, g](u)
\end{aligned}
$$

□

The following example will reveal what elements of $\mathbf{PDO}^{(1)}(M\times\mathbb{R}, M\times\mathbb{R})$ essentially are.

2.1.6 Example: Let M be a smooth manifold and $P \in \mathbf{PDO}^{(1)}(M\times\mathbb{R}, M\times\mathbb{R})$, $\quad f, g \in C^\infty(M)$. Then we have that

$$[P, f] \in \mathbf{PDO}^{(0)}(M\times\mathbb{R}, M\times\mathbb{R}) \underset{2.1.3}{\cong} \mathbf{Hom}(M\times\mathbb{R}, M\times\mathbb{R})$$

from which we conclude $\exists\sigma(f) \in C^\infty(M)$, such that

$$[P, f]u = \sigma(f)\cdot u \quad \forall u \in C^\infty(M)$$

By choosing $u = 1 \in C^\infty(M)$ in (2.1), we get

$$\underline{\underline{\sigma(f\cdot g)}} = [P, f\cdot g](1) = [P, f](g\cdot 1) + f\cdot[P, g](1) = \underline{\underline{\sigma(f)\cdot g + f\cdot\sigma(g)}},$$

which means that $f \mapsto \sigma(f)$ is a derivation of $C^\infty(M)$. So by 1.2.22 we conclude that $\exists X \in \mathfrak{X}(M)$, such that

$$\sigma(f) = X(f) \quad \forall f \in C^\infty(M)$$

Let $\mu := P(1) \in C^\infty(M)$. For all $u \in C^\infty(M)$, we have:

$$P(u) = P(u\cdot 1) = P(u\cdot 1) - u\cdot P(1) + u\cdot P(1) = [P, u](1) + u\cdot P(1) = X(u) + \mu\cdot u$$

Thus every $P \in \mathbf{PDO}^{(1)}(M\times\mathbb{R}, M\times\mathbb{R})$ can be written as a linear combination of a vector field and a homomorphism.

We even have $\mathbf{PDO}^{(1)}(M\times\mathbb{R}, M\times\mathbb{R}) = \mathfrak{X}(M) \oplus \mathbf{Hom}\big(C^\infty(M), C^\infty(M)\big)$:

Let $P \in \mathrm{Der}\big(C^\infty(M)\big) \cap \mathbf{Hom}\big(C^\infty(M), C^\infty(M)\big)$. Then

$$f\cdot g\cdot P(1) = P(f\cdot g) = P(f)\cdot g + f\cdot P(g) = 2\cdot f\cdot g\cdot P(1) \quad \forall f, g \in C^\infty(M)$$

Thus $P(1) = 0$ and since P is $C^\infty(M)$-linear $P = 0$.

2.1.7 Proposition: *Let E, F, G be $\mathbb{K}$-vector bundles over the smooth manifold M. From $P \in \mathbf{PDO}^{(m)}(F, G)$ and $Q \in \mathbf{PDO}^{(n)}(E, F)$ it follows that $P \circ Q \in$* $\mathbf{PDO}^{(m+n)}(E, G)$

Proof. Let $f \in C^\infty(M)$ be arbitrary. In the following calculations f will denote the operator $f\cdot$, multiplication by f.

$$\begin{aligned}[P\circ Q, f] &= (P\circ Q)\circ f - f\circ(P\circ Q) = \\ &= P\circ(Q\circ f) - P\circ(f\circ Q) + (P\circ f)\circ Q - (f\circ P)\circ Q = \\ &= P\circ[Q, f] + [P, f]\circ Q \end{aligned} \tag{2.2}$$

We argue by induction over $m+n$:

- $m+n=0$:

$$[P\circ Q,f] = P\circ\underbrace{[Q,f]}_{=0}+\underbrace{[P,f]}_{=0}\circ Q = 0 \quad \Rightarrow P\circ Q\in\mathbf{PDO}^0$$

- $m+n-1\mapsto m+n$:

$$[P\circ Q,f] = \underbrace{P}_{\in\mathbf{PDO}^{(m)}} \circ \underbrace{[Q,f]}_{\in\mathbf{PDO}^{(n-1)}\text{ or }=0\text{ (if }n=0)} + $$
$$+ \underbrace{[P,f]}_{\in\mathbf{PDO}^{(m-1)}\text{ or }=0\text{ (if }m=0)} \circ \underbrace{Q}_{\in\mathbf{PDO}^{(n)}} \underset{\text{induc.}}{\in} \mathbf{PDO}^{(m+n-1)}$$

$$\Rightarrow P\circ Q\in\mathbf{PDO}^{(m+n)}$$ □

2.1.8 Corollary: *Let $a_\alpha\in C^\infty(\mathbb{R}^n)$ for all $|\alpha|\leq m$. Then $P=\sum_{|\alpha|\leq m}a_\alpha\cdot\partial^\alpha\colon C^\infty(\mathbb{R}^n)\to C^\infty(\mathbb{R}^n)$ is a PDO of order $\leq m$.*

Proof. In 2.1.4, 1 we saw that ∂_i is a PDO of order ≤ 1. Multiplication with a smooth function is $C^\infty(\mathbb{R}^n)$-linear, hence a PDO of order 0. By 2.1.7 we obtain that P is a PDO of order $\leq m$. □

2.1.9 Corollary: *Let $A_\alpha\in\mathbf{Hom}(\mathbb{R}^n\times\mathbb{K}^p,\mathbb{R}^n\times\mathbb{K}^q)\quad\forall|\alpha|\leq m$. Then $P=\sum_{|\alpha|\leq m}A_\alpha\cdot\tilde{\partial}^\alpha\colon\Gamma^\infty(\mathbb{R}^n\times\mathbb{K}^p)\to\Gamma^\infty(\mathbb{R}^n\times\mathbb{K}^q)$ is a PDO of order $\leq m$, where $\tilde{\partial}^\alpha(v_1,\ldots,v_p):=(\partial^\alpha v_1,\ldots,\partial^\alpha v_p)$* [2] $\quad\forall(v_1,\ldots,v_p)\in\Gamma^\infty(\mathbb{R}^n\times\mathbb{K}^p)$.

Proof.

$$\partial_i\in\mathbf{PDO}^{(1)}(\mathbb{R}^n\times\mathbb{R},\mathbb{R}^n\times\mathbb{R})\quad\Rightarrow\quad\tilde{\partial}_i\in\mathbf{PDO}^{(1)}(\mathbb{R}^n\times\mathbb{K}^p,\mathbb{R}^n\times\mathbb{K}^p)$$

$A_\alpha\in\mathbf{PDO}^0(\mathbb{R}^n\times\mathbb{K}^p,\mathbb{R}^n\times\mathbb{K}^q)$ because of 2.1.3. Again by 2.1.7 we get that $P\in\mathbf{PDO}^{(m)}(\mathbb{R}^n\times\mathbb{K}^p,\mathbb{R}^n\times\mathbb{K}^q)$. □

Now we are going to prove the main result of this section, that locally PDOs and local operators are the same.

2.1.10 Definition: An operator $P\in\mathbf{Op}(E,F)$ is called *local operator* if

$$\forall u\in\Gamma^\infty(E)\colon\quad\mathrm{supp}(Pu)\subseteq\mathrm{supp}(u)$$

2.1.11 Proposition: *Let E and F be vector bundles over the manifold M, $P\in\mathbf{Op}(E,F)$ a local operator, U an open subset of M and $u,v\in\Gamma^\infty(E)$.*

$$u|_U=v|_U\quad\Rightarrow\quad(Pu)|_U=(Pv)|_U$$

[2] in the future we will also write ∂ instead of $\tilde{\partial}$ most of the time, the meaning of ∂ will be clear from the context

Proof.

$$\operatorname{supp}(u-v) \subseteq M \setminus U \quad \Rightarrow \quad \operatorname{supp}(Pu - Pv) = \operatorname{supp}\big(P(u-v)\big) \subseteq M \setminus U$$

□

2.1.12 Theorem: *Let E and F be vector bundles over the smooth manifold M. Every $P \in \mathbf{PDO}^{(m)}(E,F)$ is a local operator.*

Proof. We argue by induction over m:

- $m = 0$:

$$P \in \mathbf{Hom}(E,F) \Rightarrow \Big(u(x) = 0 \Rightarrow Pu(x) = 0 \quad \forall x \in M\Big)$$
$$\Rightarrow \operatorname{supp}(Pu) \subseteq \operatorname{supp}(u)$$

- $m \mapsto m+1$:

  Let $P \in \mathbf{PDO}^{(m+1)} \Rightarrow [P,f] \in \mathbf{PDO}^{(m)} \quad \forall f \in C^\infty(M)$

  $P(f\cdot u) = P(f\cdot u) - f \cdot P(u) + f \cdot P(u) = [P,f](u) + f\cdot P(u) \quad \forall f \in C^\infty(M)$

$$\Big[\text{we remember:} \quad (i)\ \operatorname{supp}(f\cdot g) \subseteq \operatorname{supp}(f) \cap \operatorname{supp}(g)$$
$$(ii)\ \operatorname{supp}(f+g) \subseteq \operatorname{supp}(f) \cup \operatorname{supp}(g)\Big]$$

  So we get

$$\operatorname{supp}(P(f\cdot u)) \underset{(ii)}{\subseteq} \underbrace{\operatorname{supp}\big([P,f](u)\big)}_{\substack{\subseteq \operatorname{supp}(u) \\ \text{by induction}}} \cup \underbrace{\operatorname{supp}\big(f \cdot P(u)\big)}_{\underset{(i)}{\subseteq} \operatorname{supp}(f) \cap \operatorname{supp}\left(P(u)\right)} \qquad \forall f \in C^\infty(M)$$

  For $g \in G := \{f \in C^\infty(M) \mid f\big|_{\operatorname{supp}(u)} \equiv 1\} \subseteq C^\infty(M)$ follows $g \cdot u \equiv u$ and we get

$$\operatorname{supp}\big(P(u)\big) \subseteq \operatorname{supp}(u) \cup \operatorname{supp}(g) \quad \forall g \in G \qquad (*)$$

  Let $p \in \operatorname{supp}(u)^c$. $\operatorname{supp}(u)^c$ is open, so it is a neighborhood of p. M is locally compact (see 1.2.9) so we find a compact neighborhood $K \subseteq \operatorname{supp}(u)^c$ of p. Because of 1.2.9 we get a partition of unity $\{\phi, \chi\}$ subordinate to $\{\operatorname{supp}(u)^c, K^c\}$.

$$\chi\big|_{\operatorname{supp}(u)} \equiv 1 \quad \Rightarrow \quad \chi \in G$$
$$\operatorname{supp}(\chi) \subseteq K^c \quad \Rightarrow \quad p \in K \subseteq \operatorname{supp}(\chi)^c$$

$$\Rightarrow p \in \operatorname{supp}(u)^c \cap \operatorname{supp}(\chi)^c = \big(\operatorname{supp}(u) \cup \operatorname{supp}(\chi)\big)^c \underset{(*)}{\subseteq} \operatorname{supp}\big(P(u)\big)^c$$

$$\Rightarrow \operatorname{supp}\big(P(u)\big) \subseteq \operatorname{supp}(u)$$

□

Note that the above theorem tells us that we can work in local coordinates. Now we will see that every local operator locally has the form of P in 2.1.9 and therefore is a PDO. First we need to prove the following preparatory statement:

2.1.13 Lemma: *Let $P \in \mathbf{Op}(\mathbb{R}^n \times \mathbb{K}, \mathbb{R}^n \times \mathbb{K})$ be a local operator, $x_0 \in \mathbb{R}^n$. There exist an open neighborhood Y of x_0, $m \in \mathbb{N}_0$ and $C > 0$, such that*

$$\|P\phi\|_{L^\infty} \leq C \cdot \underbrace{\max_{|\alpha|\leq m} \|\partial^\alpha \phi\|_{L^\infty}}_{=:\|\phi\|_{(m)}} \quad \forall \phi \in \mathcal{D}(Y \setminus \{x_0\}) \tag{2.3}$$

Proof. We prove the lemma by contradiction. Let Y_0 be an open neighborhood of x, such that $\overline{Y_0}$ is compact. We assume that (2.3) is false, that means

$$\forall Y \quad \forall m \in \mathbb{N}_0 \quad \forall C > 0 \quad \exists \phi \in \mathcal{D}(Y \setminus \{x_0\}): \quad \|P\phi\|_{L^\infty} > C \cdot \|\phi\|_{(m)} \tag{2.4}$$

So by choosing $Y = Y_0$, $m = 1$ and $C = 2^2$ there must exist a $\phi_1 \in \mathcal{D}(Y_0 \setminus \{x_0\})$, such that

$$\|P\phi_1\|_{L^\infty} > 2^2 \cdot \|\phi_1\|_{(1)}$$

Let Y_1 be open, such that $\operatorname{supp}(\phi_1) \subset Y_1 \subset \overline{Y_1} \subset Y_0$. We see that $Y_0 \setminus \overline{Y_1}$ is also an open neighborhood of x_0. So we can choose $Y = Y_0 \setminus \overline{Y_1}$, $m = 2$ and $C = 2^{2\cdot 2}$ in (2.4) and get a $\phi_2 \in \mathcal{D}\big(Y_0 \setminus (Y_1 \cup \{x_0\})\big)$, such that

$$\|P\phi_2\|_{L^\infty} > 2^{2\cdot 2} \cdot \|\phi_2\|_{(2)}$$

Inductively we get a $\phi_k \in \mathcal{D}\big(Y_0 \setminus (\bigcup_{i=1}^{k-1} \overline{Y_1} \cup \{x_0\})\big)$ and an open set Y_k, such that $\operatorname{supp}(\phi_k) \subset Y_k \subset \overline{Y_k} \subset Y_0$ and

$$\|P\phi_k\|_{L^\infty} > 2^{2k} \cdot \|\phi_k\|_{(k)} \tag{2.5}$$

Now we define

$$\phi := \sum_{k=1}^{\infty} \frac{1}{2^k \cdot \|\phi_k\|_{(k)}} \cdot \phi_k$$

The supports of the ϕ_k are disjoint, so at every point there is at most one $\phi_k \neq 0$ and therefore the sum is convergent. Obviously, the series defininig ϕ converges uniformly in all derivatives. Thus ϕ is smooth. By definition we have

$$\phi|_{Y_k} = \frac{1}{2^k \cdot \|\phi_k\|_{(k)}} \cdot \phi_k$$

P is local, so by 2.1.11 we get

$$P\phi|_{Y_k} = \frac{1}{2^k \cdot \|\phi_k\|_{(k)}} \cdot P\phi_k$$

Now we choose a $y_k \in Y_k \subset Y_0$ for every $k = 1, 2, 3, \ldots$, such that $|P\phi_k(y_k)| > 2^{2k} \cdot \|\phi_k\|_{(k)}$ (which is possible because of (2.5)) and obtain

$$|P\phi(y_k)| = \left| \frac{1}{2^k \cdot \|\phi_k\|_{(k)}} \cdot P\phi_k(y_k) \right| > \frac{1}{2^k \cdot \|\phi_k\|_{(k)}} \cdot 2^{2k} \cdot \|\phi_k\|_{(k)} = 2^k$$

But Y_0 is relatively compact and so the continuous (actually smooth) function $P\phi$ has to be bounded on Y_0, a contradiction. $\square$

2.1.14 Theorem: *Let E, F be $\mathbb{K}$-vector bundles over the smooth manifold M. Every local operator $P \in \mathbf{Op}(E, F)$ is locally given by*

$$P = \sum_{|\alpha| \leq m} A_\alpha \cdot \partial^\alpha \colon \Gamma^\infty(\mathbb{R}^n \times \mathbb{K}^p) \to \Gamma^\infty(\mathbb{R}^n \times \mathbb{K}^q)$$

For some $p, q, m \in \mathbb{N}_0,\ A_\alpha \in \mathbf{Hom}(\mathbb{R}^n \times \mathbb{K}^p, \mathbb{R}^n \times \mathbb{K}^q)$.

Proof. Let $x \in M$ and $U := U^{(x)}$ be a relatively compact trivializing neighborhood of x, like in 1.2.17, w.l.o.g. for both vector bundles E and F (if U_1 is such a neighborhood for E and U_2 for F, then the intersection $U_1 \cap U_2$ meets the requirements for both). So we have $E|_U \cong U \times \mathbb{K}^p$ and $F|_U \cong U \times \mathbb{K}^q$. Because of $\Gamma^\infty(U \times \mathbb{K}^t) \cong \Gamma^\infty(U \times \mathbb{K})^t$ and the $\mathbb{K}$-linearity of P it suffices to prove the theorem for p=q=1.

For $y \in U$ we define $P_y := \phi \mapsto P\phi(y)$ and prove that it is a distribution:
P_y is a linear map from $\mathcal{D}(U)$ to $\mathbb{K}$ by definition. Since U is relatively compact we can choose a finite cover $\{U_{x_i} \mid i = 1, \ldots, k\}$ of U such that each $U_{x_i} \subseteq U$ satisfies the assumptions of 2.1.13. Let $\{\chi_i \mid i = 1, \ldots, k\}$ be a partition of unity subordinate to $\{U_{x_i}\}$ and $\phi \in \mathcal{D}(\bigcup_{i=1}^k U_{x_i} \setminus \{x_i\})$ arbitrary. Then

$$\begin{aligned}
|P_y(\phi)| &= |P\phi(y)| \leq \|P\phi\|_{L^\infty} = \\
&= \left\| P\Big(\sum_{i=1}^k \chi_i \cdot \phi\Big) \right\|_{L^\infty} \leq \\
&\leq \sum_{i=1}^k \|P(\chi_i \cdot \phi)\|_{L^\infty} \leq \\
&\underset{2.1.13}{\leq} \sum_{i=1}^k C_i \cdot \max_{|\alpha| \leq m_i} \|\partial^\alpha(\chi_i \cdot \phi)\|_{L^\infty} = \\
&= \sum_{i=1}^k C_i \cdot \max_{|\alpha| \leq m_i} \left\| \sum_{|\beta| \leq |\alpha|} \binom{\alpha}{\beta} \partial^{\alpha-\beta}\chi_i \cdot \partial^\beta \phi \right\|_{L^\infty} \leq \\
&\leq \sum_{i=1}^k C_i \cdot \max_{|\alpha| \leq m_i} \left(\sum_{|\beta| \leq |\alpha|} \binom{\alpha}{\beta} \|\partial^{\alpha-\beta}\chi_i\|_{L^\infty} \cdot \|\partial^\beta \phi\|_{L^\infty} \right) \leq \\
&\leq \sum_{i=1}^k C_i \cdot \max_{|\alpha| \leq m_i} \left(\max_{|\beta| \leq |\alpha|} \left(\|\partial^\beta \phi\|_{L^\infty}\right) \cdot \sum_{|\beta| \leq |\alpha|} \binom{\alpha}{\beta} \|\partial^{\alpha-\beta}\chi_i\|_{L^\infty} \right) \leq \\
&\leq \sum_{i=1}^k C_i \cdot \max_{|\beta| \leq m_i} \left(\|\partial^\beta \phi\|_{L^\infty}\right) \cdot \max_{|\alpha| \leq m_i} \left(\sum_{|\beta| \leq |\alpha|} \binom{\alpha}{\beta} \|\partial^{\alpha-\beta}\chi_i\|_{L^\infty} \right) \leq
\end{aligned}$$

$$\leq \max_{|\beta|\leq m:=\max(m_i)} \big(\|\partial^\beta \phi\|_{L^\infty}\big) \cdot \underbrace{\sum_{i=1}^{k} C_i \cdot \max_{|\alpha|\leq m_i} \left(\sum_{|\beta|\leq|\alpha|} \binom{\alpha}{\beta} \|\partial^{\alpha-\beta}\chi_i\|_{L^\infty} \right)}_{=:C} \leq$$

$$\leq C \cdot \sum_{|\beta|\leq m} \|\partial^\beta \phi\|_{L^\infty}$$

This means that P_y is a distribution on $\bigcup_{i=1}^k U_{x_i} \setminus \{x_i\}$. The support of P_y is just the point y:

$$\begin{aligned} \phi \in \mathcal{D}(U \setminus \{y\}) \quad &\Rightarrow \quad \operatorname{supp}(\phi) \subseteq U \setminus \{y\} \\ &\Rightarrow \quad \operatorname{supp}(P\phi) \subseteq U \setminus \{y\} \\ &\Rightarrow \quad P_y(\phi) = P\phi(y) = 0 \end{aligned}$$

By 1.3.6 we now can conclude that

$$P\phi(y) = P_y(\phi) = \sum_{|\alpha|\leq m} a_\alpha(y) \cdot \partial^\alpha \phi(y) \quad \forall \phi \in \mathcal{D}\Big(\bigcup_{i=1}^k U_{x_i} \setminus \{x_i\}\Big) \tag{2.6}$$

where $a_\alpha(y) \in \mathbb{K}, m \in \mathbb{N}_0$. We check that $y \mapsto a_\alpha(y)$ is smooth: Let $\phi = y^\mu$ locally.

$$P\big(y^\mu\big)(y) = \sum_{|\alpha|\leq m} a_\alpha(y) \cdot \partial^\alpha \big(y^\mu\big)(y) = \sum_{\alpha\leq\mu, |\alpha|\leq m} a_\alpha(y) \cdot \frac{\mu!}{(\mu-\alpha)!} \cdot y^{\mu-\alpha}$$

For $\mu = 0$ we get $a_0 = P\big(1\big) \in C^\infty(U)$ ($P\colon C^\infty(U) \to C^\infty(U)$!). By induction over $|\mu|$ we get $a_\alpha \in C^\infty(U) \quad \forall |\alpha| \leq m$. Thereby we also see that both sides in (2.6) are smooth, from which we conclude that the formula has to hold for $\phi \in \mathcal{D}(\bigcup_{i=1}^k U_{x_i}) = \mathcal{D}(U)$ as well. Because of 2.1.11 and the fact that every smooth function is locally given by a smooth function with compact support we get

$$P\phi(y) = \sum_{|\alpha|\leq m} a_\alpha(y) \cdot \partial^\alpha \phi(y) \quad \forall \phi \in C^\infty(U)$$

□

Let us summarize what we have obtained:

2.1.15 Theorem: *Let E, F be $\mathbb{K}$-vector bundles over the smooth manifold M and $P \in \mathbf{Op}(E,F)$. TFAE:*

(i) P is a local operator

(ii) For every point $x \in M$ there exists a neighborhood U of x and a number $m \in \mathbb{N}_0$ such that $P|_U$ is given by

$$\sum_{|\alpha|\leq m} A_\alpha \cdot \partial^\alpha \colon \Gamma^\infty(\mathbb{R}^n \times \mathbb{K}^p) \to \Gamma^\infty(\mathbb{R}^n \times \mathbb{K}^q)$$

(iii) For every point $x \in M$ there exists a neighborhood U of x and a number $m \in \mathbb{N}_0$ such that $P|_U \in \mathbf{PDO}^{(m)}(E|_U, F|_U)$

Proof. $(i) \Rightarrow (ii)$: 2.1.14
$(ii) \Rightarrow (iii)$: 2.1.9
$(iii) \Rightarrow (i)$: We cover M by neighborhoods as in (iii) and choose a partition of unity $\{\chi_\alpha \mid \alpha \in A\}$ subordinate to this cover. By 2.1.12 $P|_{U_\alpha}$ is a local operator for every $\alpha \in A$. Since $P \in \mathbf{Op}(E, F)$ we get

$$\begin{aligned}
\operatorname{supp}(Pu) = \operatorname{supp}\left(P\left(\sum_{\alpha\in A}\chi_\alpha \cdot u\right)\right) &= \\
= \operatorname{supp}\left(\sum_{\alpha\in A} P(\chi_\alpha \cdot u)\right) &\subseteq \\
\subseteq \bigcup_{\alpha\in A} \underbrace{\operatorname{supp}\big(P(\chi_\alpha \cdot u)\big)}_{\subseteq \operatorname{supp}(\chi_\alpha \cdot u) \subseteq \operatorname{supp}(\mathrm{u})} &\subseteq \\
\subseteq \operatorname{supp}(u) &
\end{aligned}$$

□

2.2 The Principal Symbol

In this section we will give a definition for the principal symbol of a PDO, consider some examples and define ellipticity. For the main definition we need some preparatory statements:

2.2.1 Lemma: *Let E and F be vector bundles over the smooth manifold M.*

$$\forall P \in \mathbf{PDO}^{(m)}(E, F) \quad \forall f, g \in C^\infty(M): \quad ad(f)\big(ad(g)P\big) = ad(g)\big(ad(f)P\big)$$

Proof.

$$\begin{aligned}
ad(f)\big(ad(g)P\big) &= \big[[P, g], f\big] = \\
&= \big[P(g\cdot) - g \cdot P, f\big] = \\
&= \big(P(g \cdot f \cdot) - g \cdot P(f\cdot)\big) - \big(f \cdot P(g\cdot) - f \cdot g \cdot P\big) = \\
&= \big(P(f \cdot g \cdot) - f \cdot P(g\cdot)\big) - \big(g \cdot P(f\cdot) - g \cdot f \cdot P\big) = \\
&= \big[P(f\cdot) - f \cdot P, g\big] = \\
&= \big[[P, f], g\big] = \\
&= ad(g)\big(ad(f)P\big)
\end{aligned}$$

□

2.2.2 Lemma: *Let M be a smooth manifold of dimension n. For $p \in M$, U an open neighborhood of p, we define $\mathcal{I}_p := \{f \in C^\infty(U) \mid f(p) = 0\}$ and $\mathcal{J}_p := \{f \in C^\infty(U) \mid f(p) = 0$ and $df(p) = 0\}$.*

$$\forall p \in M \quad \exists U \subseteq M \text{ open neighborhood of } p \text{, such that:}$$

$$\forall f \in \mathcal{J}_p \quad \exists g_i, h_i \in \mathcal{I}_p: \quad f = \sum_{i=1}^{n} g_i \cdot h_i$$

Proof. Let (ψ, U) be a chart around p, with $\psi(p) = 0$. $\psi(U)$ is open in $\mathbb{R}^n$ and without loss of generality convex (choose an open and convex subset V of $\psi(U)$, such that $p \in \psi^{-1}(V)$, then $\big(\psi, \psi^{-1}(V)\big)$ is again a chart around p, see 1.2.4), so we can use Taylor's Theorem on $\tilde{f} := f \circ \psi^{-1} \colon \mathbb{R}^n \supseteq \psi(U) \to \mathbb{R}$:

$$\begin{aligned}
\tilde{f}(x) = & \underbrace{\tilde{f}(0)}_{=f(p)=0} + \int_0^1 \langle \operatorname{grad}\tilde{f}(t\cdot x), x\rangle dt = \\
= & \int_0^1 \sum_{i=1}^n \big(\partial_i \tilde{f}\big)(t\cdot x)\cdot x_i dt = \\
= & \sum_{i=1}^n \underbrace{x_i}_{=:\tilde{g}_i(x)} \cdot \underbrace{\int_0^1 \big(\partial_i \tilde{f}\big)(t\cdot x) dt}_{=:\tilde{h}_i(x)} = \sum_{i=1}^n \tilde{g}_i(x)\cdot \tilde{h}_i(x)
\end{aligned}$$

So by composition with ψ we get:

$$f = \tilde{f}\circ\psi = \sum_{i=1}^n \underbrace{\tilde{g}_i\circ\psi}_{=:g_i} \cdot \underbrace{\tilde{h}_i\circ\psi}_{=:h_i} = \sum_{i=1}^n g_i\cdot h_i$$

Finally we prove that $g_i, h_i \in \mathcal{I}_p$:

$$\begin{aligned}
g_i(p) &= \tilde{g}_i(0) = 0 \\
h_i(p) &= \tilde{h}_i(0) = \int_0^1 \underbrace{\big(\partial_i \tilde{f}\big)(0)}_{\underset{(*)}{=}0} dt = 0
\end{aligned}$$

$$\begin{aligned}
(*) \quad \big(\partial_i \tilde{f}\big)(0) &= T_0\tilde{f}(e_i) = T_0\big(f\circ\psi^{-1}\big)(e_i) = \\
&\underset{(1.1)}{=} T_{\underbrace{\psi^{-1}(0)}_{=p}} f \circ T_0\psi^{-1}(e_i) = \underbrace{df(p)}_{=0}\big(T_0\psi^{-1}(e_i)\big) = 0
\end{aligned}$$

$\tilde{g}$ and $\tilde{f}$ are smooth from $\psi(U)$ to $\mathbb{R}$, therefore f and g are smooth from U to $\mathbb{R}$. ☐

2.2.3 Proposition: *Let E, F be vector bundles over the manifold M. $P \in$ $\mathbf{PDO}^{(m)}(E,F)$, $p \in M$, $f_i, g_i \in C^\infty(M)$ $(i = 1, \dots, m)$, such that $df_i(p) = dg_i(p) \quad \forall i$. Then:*

$$\{ad(f_1)ad(f_2)\dots ad(f_m)P\}|_p = \{ad(g_1)ad(g_2)\dots ad(g_m)P\}|_p$$

Proof. Without loss of generality we can assume that $f_i(p) = g_i(p)$, because $ad(f) = ad(f) + \underbrace{ad(const)}_{=0} = ad(f + const)$. 2.2.1 tells us that it suffices to prove:

$$\{ad(f_1)ad(f_2)\dots ad(f_m)P\}|_p = \{ad(g_1)ad(f_2)\dots ad(f_m)P\}|_p$$

Let $Q := \underbrace{ad(f_2)\dots ad(f_m)}_{m-1} \underbrace{P}_{\in \mathbf{PDO}^{(m)}} \in \mathbf{PDO}^{(1)}$ and $\Phi := f_1 - g_1$. We see that $\Phi \in \mathcal{J}_p$, so by 2.2.2 there exist $\alpha_i, \beta_i \in \mathcal{I}_p$, such that $\Phi = \sum_{i=1}^n \alpha_i \cdot \beta_i$ in a neighborhood of p.

$$\begin{aligned}\{ad(\Phi)Q\}|_p &= \sum_{i=1}^n \{ad(\alpha_i \cdot \beta_i)Q\}|_p = \\ &\underset{(2.1)}{=} \sum_{i=1}^n \Big(\{ad(\alpha_i)Q\}|_p \circ \underbrace{\beta_i(p)}_{=0} \cdot + \underbrace{\alpha_i(p)}_{=0} \cdot \{ad(\beta_i)Q\}|_p \Big) = 0\end{aligned}$$

So we conclude $\{ad(f_1)Q\}|_p = \{ad(g_1)Q\}|_p$. $\square$

This proposition shows that the linear map $\{\frac{1}{m!} ad(f_1)\dots ad(f_m)P\}|_p \colon E_p \to F_p$ only depends on the $\xi_i := df_i(p) \in T_pM^*$, so we can factorize by $\sim$, where $f \sim g :\Leftrightarrow df(p) = dg(p)$ and get the map $\sigma\big(P\big)(\xi_1, \dots, \xi_m) \colon E_p \to F_p$, which is linear and symmetric in all slots (see 2.2.1). σ is uniquely defined by $\sigma_m\big(P\big)(\xi) := \sigma\big(P\big)(\xi, \dots, \xi)$ via polarization:

$$\begin{aligned}\frac{1}{m!}\frac{\partial^m}{\partial t_1 \dots \partial t_m}&\sigma_m\big(P\big)(t_1\xi_1 + \dots + t_m\xi_m) = \\ &= \frac{1}{m!}\frac{\partial^m}{\partial t_1 \dots \partial t_m}\sigma\big(P\big)(t_1\xi_1 + \dots + t_m\xi_m, \dots, t_1\xi_1 + \dots + t_m\xi_m) = \\ &= \frac{1}{m!}\frac{\partial^m}{\partial t_1 \dots \partial t_m} \sum_{i_1,\dots,i_m=1}^m t_{i_1} \cdots t_{i_m} \cdot \sigma\big(P\big)(\xi_{i_1}, \dots, \xi_{i_m}) = \\ &= \frac{1}{m!} \cdot m! \cdot \sigma\big(P\big)(\xi_1, \dots, \xi_m) = \\ &= \sigma\big(P\big)(\xi_1, \dots, \xi_m)\end{aligned}$$

Now we can define the order and symbol of a PDO, notions which are very important for our further study of PDOs. In particular we can define ellipticity of operators via the symbol.

2.2.4 Definition: Let $P \in \mathbf{PDO}^{(m)}$. If $\sigma_m(P) \neq 0$ we say P has *order* m and write $P \in \mathbf{PDO}^m$. In this case $\sigma_m(P)$ is called the *(principal) symbol* of P.

2.2.5 Definition: Let E, F be vector bundles over M. A partial differential operator $P \in \mathbf{PDO}^m(E, F)$ is called *elliptic* if for every point $p \in M$ and every $\xi \in T_pM^*, \xi \neq 0$ the principal symbol is an isomorphism from E_p to F_p.

2.2.6 Example: In 2.1.4 we showed for some operators that they are PDOs, by calculating multiple commutators. The principal symbol is per definition the highest nonvanishing multiple commutator, so we already calculated the principal symbols of the above operators and know which order they have:

1. The *partial derivative* $\partial_i \colon C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is a PDO of order 1 with principal symbol $\sigma_1\big(\partial_i\big)(\xi) = \xi_i \cdot \colon \mathbb{R} \to \mathbb{R}$.

2. The *Euclidean Laplacian* $\Delta\colon C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is a PDO of order 2 with principal symbol $\sigma_2\big(\Delta\big)(\xi) = -|\xi|^2\cdot\colon \mathbb{R} \to \mathbb{R}$. We see that $\sigma_2\big(\Delta\big)(\xi) \neq 0 \quad \forall \xi \neq 0$. Multiplication by a constant $\neq 0$ is an automorphism on $\mathbb{R}$, so Δ is elliptic.

3. The *exterior derivative* $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ is a PDO of order 1 with principal symbol $\sigma_1\big(d\big)(\xi) = \xi\wedge\colon \Lambda^k T_pM \to \Lambda^{k+1}T_pM$.

4. $\nabla\colon \Gamma^\infty(E) \to \Gamma^\infty(TM^* \otimes E)$ is a PDO of order 1 with principal symbol $\sigma_1\big(\nabla\big)(\xi) = \xi\otimes\colon E_p \to T_pM^* \otimes E_p$.

5. Let $X \in \mathfrak{X}(M)$. $\nabla_X\colon \Gamma^\infty(E) \to \Gamma^\infty(E)$ is a PDO of order 1 with principal symbol $\sigma_1\big(\nabla_X\big)(\xi) = \xi(X)\cdot\colon E_p \to E_p$.

For later purposes we want to prove the following proposition:

2.2.7 Proposition: *Let E, F, G be vector bundles over the smooth manifold M. $P \in \mathbf{PDO}^{(m)}(E,F)$ and $Q \in \mathbf{PDO}^{(n)}(F,G)$. Then:*

$$\sigma_{m+n}(Q \circ P) = \sigma_n(Q) \circ \sigma_m(P)$$

Proof. Let $f \in C^\infty(M)$, such that $df(p) = \xi$. Then

$$\begin{aligned}
\sigma_{m+n}\big(Q \circ P\big)(\xi) &= \left\{ \frac{1}{(m+n)!}\big(ad(f)\big)^{m+n}(Q \circ P) \right\}\Bigg|_p = \\
&\underset{(2.2)}{=} \left\{ \frac{1}{(m+n)!} \sum_{k=0}^{m+n} \binom{m+n}{k} \underbrace{\Big(\big(ad(f)\big)^k Q\Big)}_{=0 \text{ for } k>n} \circ \underbrace{\Big(\big(ad(f)\big)^{m+n-k} P\Big)}_{=0 \text{ for } k<n} \right\}\Bigg|_p = \\
&= \left\{ \frac{1}{(m+n)!} \cdot \frac{(m+n)!}{m! \cdot n!} \Big(\big(ad(f)\big)^n Q\Big) \circ \Big(\big(ad(f)\big)^m P\Big) \right\}\Bigg|_p = \\
&= \left\{ \frac{1}{n!}\Big(\big(ad(f)\big)^n Q\Big) \right\}\Bigg|_p \circ \left\{ \frac{1}{m!}\Big(\big(ad(f)\big)^m P\Big) \right\}\Bigg|_p = \\
&= \sigma_n\big(Q\big)(\xi) \circ \sigma_m\big(P\big)(\xi)
\end{aligned}$$

□

2.2.8 Example: Let $P = \sum_{|\alpha| \leq m} A_\alpha \cdot \partial^\alpha\colon \Gamma^\infty(\mathbb{R}^n \times \mathbb{K}^p) \to \Gamma^\infty(\mathbb{R}^n \times \mathbb{K}^q)$. We calculate the principal symbol of P:

$A_\alpha \in \mathbf{PDO}^0(\mathbb{R}^n \times \mathbb{K}^p, \mathbb{R}^n \times \mathbb{K}^q)$ and $\partial_i \in \mathbf{PDO}^1(\mathbb{R}^n \times \mathbb{K}^p, \mathbb{R}^n \times \mathbb{K}^p)$ with principal symbol $\xi_i\cdot$ (see 2.2.6), so by 2.2.7 we get:

$$\sigma_m\big(P\big)(\xi) = \sum_{|\alpha| \leq m} A_\alpha \cdot \sigma_m\big(\partial^\alpha\big)(\xi) = \sum_{|\alpha| = m} A_\alpha \cdot \xi^\alpha$$

In 2.1.9 we saw that $P \in \mathbf{PDO}^{(m)}(\mathbb{R}^n \times \mathbb{K}^p, \mathbb{R}^n \times \mathbb{K}^q)$. Together with the above we have $P \in \mathbf{PDO}^m(\mathbb{R}^n \times \mathbb{K}^p, \mathbb{R}^n \times \mathbb{K}^q)$ and σ_m is the principal symbol of P.

2.2.9 Example: Let (M, g) be an oriented n-dimensional Riemannian manifold. We calculate the principal symbol for $\delta := *d*\colon \Omega^k(M) \to \Omega^{k-1}(M)$:

Let $f \in C^\infty(M)$, such that $df(p) = \xi$. $*$ is $C^\infty(M)$-linear (see (1.8)) so we have

$$\sigma_1\big(\delta\big)(\xi) = \sigma_1\big(* \, d \, * \big)(\xi) = \big\{[*d*, f]\big\}\Big|_p = *\big\{[d, f]\big\}\Big|_p * = *\sigma_1\big(d\big)(\xi)* \underset{2.2.6}{=} *\xi \wedge *$$

and analogously $\sigma_k = 0 \quad \forall k > 1$ (see 2.2.6). So δ is a 1^{st}-order PDO. In what follows we will obtain an explicit formula for the principal symbol:
For notational simplicity we assume without loss of generality that $|\xi| = 1$ and include it in a positively oriented (that means $\xi_1 \wedge \dots \wedge \xi_n = +dV_g(p)$) orthonormal basis $\{\xi = \xi_1, \xi_2, \dots, \xi_n\}$ of T_pM^*. First we calculate the symbol for basis vectors $\omega = \xi_{i_1} \wedge \dots \wedge \xi_{i_k} \quad (i_1 < i_2 < \dots < i_k)$ of $\Lambda^k T_pM^*$. The formula for arbitrary elements of $\Lambda^k T_pM^*$ is determined by this special case because of the linearity of σ_1. By (1.10) we get

$$*(\xi_{i_1} \wedge \dots \wedge \xi_{i_k}) = \operatorname{sgn}(\sigma) \cdot \xi_{j_1} \wedge \dots \wedge \xi_{j_{n-k}} \tag{2.7}$$

where $\{i_1, \dots, i_k\} \cup \{j_1, \dots, j_{n-k}\} = \{1, \dots, n\}$ and

$$\sigma = \begin{pmatrix} 1 & \dots & k & k+1 & \dots & n \\ i_1 & \dots & i_k & j_1 & \dots & j_{n-k} \end{pmatrix} \in \Sigma_n.$$

If $i_1 \neq 1$ we have:

$$\begin{aligned} \sigma_1\big(\delta\big)(\xi)\omega &= * \underbrace{\xi}_{=\xi_1} \wedge * \, \omega = \\ &= *(\pm \underbrace{\xi_1 \wedge \xi_1}_{=0} \wedge \dots \wedge \widehat{\xi_{i_1}} \wedge \dots \wedge \widehat{\xi_{i_k}} \wedge \dots \wedge \xi_n) = \\ &= 0 = (-1)^{(n-k)(k-1)} \underbrace{i_{\xi_1^*}\omega}_{\underset{(1.4)}{=} 0} \end{aligned}$$

And if $i_1 = 1$ we get

$$\begin{aligned} \sigma_1\big(\delta\big)(\xi)\omega &= * \underbrace{\xi}_{=\xi_1} \wedge * \, \omega = \\ &\underset{(2.7)}{=} *\Big(\xi_1 \wedge \big(\operatorname{sgn}(\sigma) \cdot \xi_{j_1} \wedge \dots \wedge \xi_{j_{n-k}}\big)\Big) = \\ &\underset{(1.10)}{=} \operatorname{sgn}(\mu) \cdot \operatorname{sgn}(\sigma) \cdot \xi_{i_2} \wedge \dots \wedge \xi_{i_k} \\ &\underset{(1.4)}{=} \operatorname{sgn}(\mu) \cdot \operatorname{sgn}(\sigma) \cdot i_{\xi_1^*}\omega \end{aligned}$$

where

$$\mu = \begin{pmatrix} 1 & 2 & \dots & n-k+1 & n-k+2 & \dots & n \\ \underbrace{1}_{=i_1} & j_1 & \dots & j_{n-k} & i_2 & \dots & i_k \end{pmatrix} \in \Sigma_n.$$

We compare the two permutations σ and μ: In μ the $k-1$ indices $i_2, \dots, i_k$ are in the back, we can put them to the front if we make $n-k$ transpositions of the form $\big(\mu(l)\ \mu(l-1)\big)$, thereby constructing σ out of μ and $(n-k)(k-1)$ transpositions. For the signum we get

$$\operatorname{sgn}(\mu) = (-1)^{(n-k)(k-1)} \cdot \operatorname{sgn}(\sigma).$$

Altogether (by using the linearity of σ_1 as mentioned above) we now conclude that

$$\sigma_1\big(\delta\big)(\xi) = (-1)^{(n-k)(k-1)} i_{\xi^*}.$$

2.2.10 Example (Hodge-deRham operator and Hodge Laplacian): Let M be as above. We calculate the symbol of the *Hodge-deRham operator* $d+d^*$, where $d^* := (-1)^{(n-k)(k-1)+1}\delta\colon \Omega^k(M) \to \Omega^{k-1}(M)$ $(k \geq 1)$. d and δ are of order 1 (see 2.2.6 & 2.2.9), therefore $d+d^*$ is of order ≤ 1.

$$\sigma_1\big(d+d^*\big)(\xi) = \sigma_1\big(d\big)(\xi) + \sigma_1\big(d^*\big)(\xi) \underset{2.2.6\ \&\ 2.2.9}{=} \underbrace{\xi\wedge}_{=:e_\xi} - i_{\xi^*}$$

Hence $d+d^*$ is of order 1. By 2.1.7 the *Hodge Laplacian* $(d+d^*)^2$ is a PDO of order ≤ 2. We calculate σ_2:

$$\begin{aligned}
\sigma_2\Big(\big(d+d^*\big)^2\Big)(\xi) &\underset{2.2.7}{=} \Big(\sigma\big(d+d^*\big)(\xi)\Big)^2 = \\
&= (e_\xi - i_{\xi^*})^2 = \\
&= \underbrace{e_\xi \circ e_\xi}_{=0} - e_\xi \circ i_{\xi^*} - i_{\xi^*} \circ e_\xi + \underbrace{i_{\xi^*} \circ i_{\xi^*}}_{=0} = \\
&\underset{(*)}{=} -|\xi|_g^2 \cdot .
\end{aligned}$$

$$\begin{aligned}
(*)\quad & \big(e_\xi \circ i_{\xi^*} + i_{\xi^*} \circ e_\xi\big)(\omega_1 \wedge \cdots \wedge \omega_t) = \\
&= e_\xi\Big(\sum_{i=1}^k (-1)^{i+1}\omega_i(\xi^*) \cdot \omega_1 \wedge \cdots \wedge \widehat{\omega_i} \wedge \cdots \wedge \omega_t\Big) + i_{\xi^*}(\underbrace{\xi}_{=:\omega_0} \wedge \omega_1 \wedge \cdots \wedge \omega_t) = \\
&= \sum_{i=1}^k (-1)^{i+1}\omega_i(\xi^*) \cdot \underbrace{\xi}_{=:\omega_0} \wedge \omega_1 \wedge \cdots \wedge \widehat{\omega_i} \wedge \cdots \wedge \omega_t + \\
&\qquad + \sum_{i=0}^k (-1)^{i}\omega_i(\xi^*) \cdot \omega_0 \wedge \cdots \wedge \widehat{\omega_i} \wedge \cdots \wedge \omega_t = \\
&= g(\xi^*, \xi^*) \cdot \omega_1 \wedge \cdots \wedge \omega_t + \\
&\qquad + \sum_{i=1}^k \big(\underbrace{(-1)^{i+1} + (-1)^i}_{=0}\big)\omega_i(\xi^*) \cdot \omega_0 \wedge \cdots \wedge \widehat{\omega_i} \wedge \cdots \wedge \omega_t = \\
&= |\xi|_g^2 \cdot \omega_1 \wedge \cdots \wedge \omega_t
\end{aligned}$$

We conclude that $(d+d^*)^2$ is a PDO of order 2. Since $|\xi| \neq 0$ for all $\xi \neq 0$, $-|\xi|_g^2\cdot$ is an isomorphism, i.e. $(d+d^*)^2$ is elliptic.

If we compare the symbol of the euclidean Laplacian (2.2.6) with the symbol of the Hodge Laplacian from the last example, we see that the first is a special case of the second, $-|\xi|^2\cdot$ is $-|\xi|_g^2\cdot$, where g is the standard metric on $\mathbb{R}^n$. Thus we define:

2.2.11 Definition: Let E be a smooth vector bundle over the Riemannian manifold (M,g). A second order PDO $P\colon \Gamma^\infty(E)\to\Gamma^\infty(E)$ is called *generalized Laplacian* if $\sigma_2\big(P\big)(\xi) = -|\xi|_g^2\cdot$.

We observe that generalized Laplacians are elliptic operators.

2.2.12 Example: Let (M,g) be an oriented Riemannian manifold. Every $X \in \mathfrak{X}(M) = \Gamma^\infty(TM)$ defines an $\omega_X \in \Omega^1(M) = \Gamma^\infty(TM^*)$ by

$$\omega_X(Y) := g(X,Y) \quad \forall Y \in \mathfrak{X}(M)$$

The map $\phi := X \mapsto \omega_X \colon \Gamma^\infty(TM) \to \Gamma^\infty(TM^*)$ is $C^\infty(M)$-linear and therefore a PDO of order 0. By 2.1.7 and 2.2.10 we see that the *divergence* with respect to g, defined by

$$div_g(X) := -d^*(\omega_X)$$

is a PDO from $\mathfrak{X}(M)$ to $C^\infty(M)$. The map ϕ is bijective[3], so we may also assign a vector field $X_\omega \in \mathfrak{X}(M)$ to every $\omega \in \Omega^1(M)$ by

$$X_\omega := \phi^{-1}(\omega)$$

ϕ^{-1} is also $C^\infty(M)$-linear:

$$f\cdot\phi^{-1}\circ\phi(X) = f\cdot X = \phi^{-1}\circ\phi(f\cdot X) = \phi^{-1}\big(f\cdot\phi(X)\big) \quad \forall X \in \mathfrak{X}(M)$$

ϕ is bijective, so we get $f\cdot\phi^{-1}(\omega) = \phi^{-1}(f\cdot\omega)$ for all $\omega \in \Omega^1(M)$. As above we conclude that the *gradient* with respect to g, defined by

$$grad_g(f) := X_{df}$$

is a PDO from $C^\infty(M)$ to $\mathfrak{X}(M)$. The *Laplace-Beltrami operator* is defined by

$$\Delta_g := -div_g \circ grad_g = (d^*|_{\Omega^1(M)}\circ\phi)\circ(\phi^{-1}\circ d|_{\Omega^0(M)}) = d^*d|_{\Omega^0(M)}$$

Δ_g is a generalized Laplacian:

$$\begin{aligned}\sigma_2\big(\Delta_g\big)(\xi) &= \sigma_1\big(d^*|_{\Omega^1(M)}\big)(\xi)\circ\sigma_1\big(d|_{\Omega^1(M)}\big)(\xi) = \\ &= -i_{\xi^*}|_{\Lambda^1(T_pM)}\circ e_\xi|_{\Lambda^0(T_pM)} = \\ &= -|\xi|_g^2\cdot\end{aligned}$$

2.3 Formal Adjoints

We will define formal adjoints, see that every PDO has a unique formal adjoint, define the notion of formally selfadjoint operators and look at some examples. In this section (M,g) is a smooth oriented Riemannian manifold and dV_g

[3] see [Kun08b], 2.2.3

denotes the corresponding metric volume as introduced in 1.2.44. (E_i, M, π_i) $(i = 0, 1, 2, \ldots)$ are real or complex vector bundles and $\langle\cdot,\cdot\rangle_i$ are *symmetric* or *Hermitian metrics*[4] on them. Such vector bundles are called *Riemannian* or *Hermitian* vector bundles. We deal with both cases simultaneously and write complex conjugates whenever they are needed for the complex case, which is no problem for the real valued case since $\overline{a} = a$ for $a \in \mathbb{R}$.

2.3.1 Definition: Let $P \in \mathbf{PDO}(E_1, E_2)$. The operator $Q \in \mathbf{PDO}(E_2, E_1)$ is called a *formal adjoint* of P if

$$\int_M \langle Pu, v\rangle_2 dV_g = \int_M \langle u, Qv\rangle_1 dV_g \qquad \forall u \in \Gamma_c^\infty(E_1), v \in \Gamma_c^\infty(E_2)$$

where Γ_c^∞ are the smooth sections with compact support.

2.3.2 Lemma: *Any $P \in \mathbf{PDO}(E_1, E_2)$ has at most one formal adjoint, which we will denote by P^* hereafter (if it exists).*

Proof. Let $Q_1, Q_2 \in \mathbf{PDO}(E_2, E_1)$ be two formal adjoints of P, $Q := Q_1 - Q_2 \in \mathbf{PDO}(E_2, E_1)$. By definition and (sesqui-)linearity of metric and integral we get

$$\begin{aligned}\int_M \langle u, Qv\rangle_1 dV_g &= \int_M \langle u, Q_1 v\rangle_1 dV_g - \int_M \langle u, Q_2 v\rangle_1 dV_g = \\ &= \int_M \langle Pu, v\rangle_2 dV_g - \int_M \langle Pu, v\rangle_2 dV_g = 0 \\ &\qquad\qquad \forall u \in \Gamma_c^\infty(E_1), v \in \Gamma_c^\infty(E_2)\end{aligned}$$

$\langle\cdot,\cdot\rangle_1$ is positive definite, which implies $Qv = 0 \quad \forall v \in \Gamma_c^\infty(E_2)$. If $v \in \Gamma^\infty(E_2)$ does not necessarily have a compact support we can choose a partition of unity $(\chi_\alpha)_{\alpha \in A}$, such that every χ_α has compact support. Let U be an arbitrary relatively compact subset of M. $\{\chi_\alpha\}$ is locally finite, so we get

$$v|_U = \sum_{\alpha \in B} (\chi_\alpha \cdot v)|_U$$

where B is a finite subset of A. We use the locality of Q to conclude

$$(Qv)|_U = Q(v|_U) = \sum_{\alpha \in B} Q\big((\chi_\alpha \cdot v)|_U\big) = \sum_{\alpha \in B} \underbrace{Q(\chi_\alpha \cdot v)}_{=0}|_U = 0$$

U was arbitrary, so we get

$$Qv = 0 \quad \forall v \in \Gamma^\infty(E_2)$$

Which means that $Q = 0$, i.e. $Q_1 = Q_2$. $\square$

[4] $\langle\cdot,\cdot\rangle$ is called symmetric resp. Hermitian metric on the vector bundle (E, B, π) if $\langle\cdot,\cdot\rangle|_p \colon E_p \times E_p \to \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ resp. $\mathbb{C}$; $E_p := \pi^{-1}(p)$ is the fibre over p) is a positive definite symmetric bilinear- resp. Hermitian sesquilinear form and $p \mapsto \langle\cdot,\cdot\rangle|_p$ is smooth.

2.3.3 Proposition: *Let $P \in \mathbf{PDO}(E_0, E_1)$, $Q \in \mathbf{PDO}(E_1, E_2)$ with formal adjoints P^* and Q^*, respectively. Then $Q \circ P$ has a formal adjoint given by*

$$(Q \circ P)^* = P^* \circ Q^*$$

Proof. Let $u \in \Gamma_c^\infty(E_0)$ and $v \in \Gamma_c^\infty(E_2)$ be arbitrary. Then

$$\int_M \langle QPu, v\rangle dV_g = \int_M \langle Pu, Q^* v\rangle dV_g = \int_M \langle u, P^* Q^* v\rangle dV_g \qquad \square$$

Now we are going to prove the existence of formal adjoints. First we want to look into the following special case:

2.3.4 Lemma: *Let $X = X^i \partial_i \in \mathfrak{X}(\mathbb{R}^n)$, $\langle \cdot, \cdot \rangle$ the standard symmetric resp. Hermitian metric on $\mathbb{K}^s$ defined by $\langle u(p), v(p)\rangle|_p = \sum_{1 \leq i \leq s} u_i(p) \cdot \overline{v_i(p)}$ ($u, v \in \Gamma^\infty(\mathbb{R}^n \times \mathbb{K}^s)$). The PDO $\nabla_X \colon \Gamma^\infty(\mathbb{R}^n \times \mathbb{K}^s) \to \Gamma^\infty(\mathbb{R}^n \times \mathbb{K}^s)$ given by $\nabla_X(u) = X^i \tilde{\partial}_i(u)$ ($\tilde{\partial}_i$ as in 2.1.9) has the formal adjoint*

$$\nabla_X^* = -\nabla_X - div_g(X) \cdot,$$

where $div_g(X)$ is the divergence of X with respect to the Riemannian metric g on $\mathbb{R}^n$ (see 2.2.12).

Proof. ∇_X is a PDO by 2.1.9. Let $u, v \in \Gamma_c^\infty(\mathbb{R}^n \times \mathbb{K}^s)$.

$$\langle u, v\rangle(p) = \sum_{1 \leq i,j \leq s} u_i(p) \cdot \overline{v_j(p)}.$$

By the product rule for ∂_i we get

$$X(\langle u, v\rangle) = \langle \nabla_X u, v\rangle + \langle u, \nabla_X v\rangle.$$

u and v have compact support, so there exists a constant $R \in \mathbb{R}^n$ such that $\operatorname{supp}(u), \operatorname{supp}(v) \in B_R := \{x \in \mathbb{R}^n \colon |x| < R\}$. We deduce

$$\begin{aligned} \int_{\mathbb{R}^n} \langle \nabla_X u, v\rangle dV_g &= \int_{\overline{B}_R} \langle \nabla_X u, v\rangle dV_g = \\ &= \int_{\overline{B}_R} X(\langle u, v\rangle) dV_g - \int_{\overline{B}_R} \langle u, \nabla_X v\rangle dV_g \end{aligned} \tag{2.8}$$

The second term on the right hand side is already in the form $\int \langle u, Pv\rangle dV_g$, as we would like it. We try to recast the first term into such a form as well ($f := \langle u, v\rangle$, ω_X as in 2.2.12):

$$\begin{aligned} \int_{\overline{B}_R} X(f) dV_g &\underset{(1.2)}{=} \int_{\overline{B}_R} df(X) dV_g = \\ &\underset{(1.9)}{=} \int_{\overline{B}_R} g_1(df, \omega_X) dV_g = \\ &\underset{(1.8)}{=} \int_{\overline{B}_R} df \wedge * \omega_X = \end{aligned}$$

$$\underset{(1.5)}{=} \int_{\overline{B}_R} d(f \wedge *\omega_X) - (-1)^0 \int_{\overline{B}_R} f \wedge d(*\omega_X) =$$

$$\underset{(1.7)\&(1.8)}{=} \underbrace{\int_{\partial B_R} \iota^*(f \wedge *\omega_X)}_{=0,\text{ since } f|_{\partial B_R}=0} - \int_{\overline{B}_R} g_0\big(f, \underbrace{\underbrace{*^{-1}}_{\underset{(1.11)}{=}(-1)^{n(n-n)}*} d*}_{=\delta=(-1)^{(n-1)(1-1)+1}d^*} (\omega_X)\big) dV_g =$$

$$= -\int_{\overline{B}_R} f \cdot (-d^*\omega_X) dV_g =$$

$$\underset{2.2.12}{=} \int_{\overline{B}_R} \langle u, v\rangle\big(- div_g(X)\big) dV_g =$$

$$\underset{div_g(X)\in C^\infty(\mathbb{R})}{=} \int_{\overline{B}_R} \langle u, -div_g(X) \cdot v\rangle dV_g \tag{2.9}$$

We insert (2.9) into (2.8) and get

$$\int_{\mathbb{R}^n} \langle \nabla_X u, v\rangle dV_g = \int_{\overline{B}_R} \langle u, (-\nabla_X - div_g(X)\cdot)v\rangle dV_g =$$
$$= \int_{\mathbb{R}^n} \langle u, (-\nabla_X - div_g(X)\cdot)v\rangle dV_g$$

Where we used that supp(u) and supp(v) $\subset \overline{B}_R$ and ∇_X and $div_g(X)\cdot$ are local operators, to substitute $\overline{B}_R$ by $\mathbb{R}^n$. So by uniqueness of the formal adjoint (see 2.3.2) we have proved that $-\nabla_X - div_g(X)\cdot$ is *the* formal adjoint of ∇_X. □

For the general case we need the following Definition and Lemma:

2.3.5 Definition: Let $(E, \langle\cdot,\cdot\rangle)$ be a Riemannian or Hermitian vector bundle over the manifold M and U an open subset of M. A selection of smooth sections $\{e_1, \ldots, e_t\}$, $e_i \in \Gamma^\infty(E|_U)$ is called *E-frame*[5], on U if $\{e_1(p), \ldots, e_t(p)\}$ is an orthonormal basis of E_p with respect to the scalar product $\langle\cdot,\cdot\rangle|_p$ on E_p for every $p \in U$.

2.3.6 Lemma: *Let $(E, \langle\cdot,\cdot\rangle)$ be a Riemannian or Hermitian vector bundle over the manifold M. For every point $p \in M$ there exists a neighborhood U which admits an E-frame.*

Proof. Let p be an arbitrary point in M. By definition of a $\mathbb{K}$-vector bundle there exist an open neighborhood $U^{(p)}$ of p and a diffeomorphism $\Phi\colon \pi^{-1}(U^{(p)}) \to U^{(p)} \times \mathbb{K}^t$, such that $\pi = pr_1 \circ \Phi$. The plan is to pick an orthonormal basis in p, push it to the trivialization, copy the vectors from $\{p\}\times\mathbb{K}^t$ to $\{q\}\times\mathbb{K}^t$ ($q \in U^{(p)}$) and pull them back to the vector bundle $E|_{U^{(p)}}$. After a little manipulation using the Gram-Schmidt process we will create an E-frame on $U^{(p)}$ out of them:
We choose an orthonormal basis $\{e_{1,p}, \ldots, e_{t,p}\}$ of E_p and define the sections $\tilde{e}_1, \ldots, \tilde{e}_t$ by

$$\tilde{e}_i(q) := \Phi^{-1}\big(q, pr_2 \circ \Phi(e_{i,p})\big) \quad \forall 1 \le i \le t$$

[5]this definition is a generalization of the usual frame in 1.2.46

The map $q \mapsto \tilde{e}_i(q)$ is a composition of smooth functions and thereby smooth. Thus $\tilde{e}_1, \ldots, \tilde{e}_t \in \Gamma^\infty(E|_{U^{(p)}})$. Moreover

$$\tilde{e}_i(p) = \Phi^{-1}\big(\underbrace{p}_{=\pi(e_{i,p})=pr_1\circ\Phi(e_{i,p})}, pr_2 \circ \Phi(e_{i,p})\big) = \Phi^{-1} \circ \Phi(e_{i,p}) = e_{i,p}$$

This implies the linear independence of $\tilde{e}_1(p), \ldots, \tilde{e}_t(p)$ and by continuity of $q \mapsto \tilde{e}_i(q)$ we get that $\tilde{e}_1(q), \ldots, \tilde{e}_t(q)$ are linearly independent for $q \in U$, where U is a sufficiently small open subset of $U^{(p)}$ containing p. We restrict the $\tilde{e}_i$'s to U, use the Gram-Schmidt process for $\tilde{e}_i(q)$ (separately in each q) to get orthonormal vectors $e_i(q)$ and observe that the Gram-Schmidt process maps $\tilde{e}_i(q)$ smoothly to $e_i(q)$, so the composition $q \mapsto \tilde{e}_i(q) \mapsto e_i(q)$ is smooth as well. Thus $\{e_1, \ldots, e_t\}$ is an E-frame on U. □

By now we have everything prepared to prove the existence of a formal adjoint in the general case:

2.3.7 Theorem: *Let $(E_i, \langle\cdot,\cdot\rangle_i)$ $(i = 1, 2)$ be two arbitrary Riemannian or Hermitian vector bundles over the n-dimensional oriented Riemannian manifold (M, g). For Every $P \in \mathbf{PDO}(E_1, E_2)$ exists a unique formal adjoint $P^* \in \mathbf{PDO}(E_2, E_1)$.*

Proof. Uniqueness holds by 2.3.2. We choose an open covering $\mathcal{U} := \{U_\alpha \mid \alpha \in A\}$ of M such that every U_α is trivializing (cf. the proof of 2.1.14) and admits an E-Frame. Thereby $E_i|_{U_\alpha}$ $(i = 1, 2)$ are trivial vector bundles and P is given by a local formula as in 2.1.14 inside of every U_α. Now we use a partition of unity $\{\chi_\alpha \mid \alpha \in A\}$, subordinate to $\mathcal{U}$ to localize P:

$$P_\alpha := \chi_\alpha \cdot P$$

If we find formal adjoints to all P_α, we get a formal adjoint of P by

$$P^* := \sum_{\alpha\in A} P^*_\alpha \tag{2.10}$$

We choose an E-frame $\{e_1, \ldots, e_s\}$ on U_α and get

$$\langle u(p), v(p)\rangle|_p = \sum_{1\le i,j\le s} u_i(p) \cdot \overline{v_j(p)}\, \underbrace{\langle e_i(p), e_j(p)\rangle|_p}_{=\delta_{ij}} = \sum_{1\le i\le s} u_i(p) \cdot \overline{v_i(p)}$$

for all $p \in U_\alpha$. By 2.3.3 we now have

$$P^*_\alpha = \sum_\beta (\partial^\beta)^* \circ A^*_\beta \cdot : \Gamma^\infty(\mathbb{R}^n \times \mathbb{K}^r) \to \Gamma^\infty(\mathbb{R}^n \times \mathbb{K}^s) \tag{2.11}$$

where $A_\beta \in C^\infty\big(\mathbb{R}^n, M_{r,s}(\mathbb{K})\big)$. The formal adjoint of A_β is clearly $\overline{A}^t_\beta$, the pointwise conjugate transpose. Lemma 2.3.4 shows that ∂^β has a formal adjoint. Thus by using (2.10) and (2.11) the theorem is proved. □

2.3.8 Proposition: *Let E, F be two Riemannian or Hermitian vector bundles over the oriented Riemannian manifold M, $P \in \mathbf{PDO}^m(E,F)$. Then:*

(i) $P^* \in \mathbf{PDO}^m(F,E)$

(ii) $\sigma_m(P^*) = (-1)^m \sigma_m(P)^*$, *where* $\sigma_m(P)^*$ *denotes the conjugate transpose of the linear map* $\sigma_m(P)\colon E_p \to F_p$

Proof. Let $f_i \in C^\infty(M)$ $(i = 0, \dots, m)$. Then

$$\big(ad(f_0)P\big)^* = \big(P \circ f_0 - f_0 \circ P\big)^* \underset{2.3.3}{=} f_0 \circ P^* - P^* \circ f_0 = -ad(f_0)P^*$$

Recursively we get

$$ad(f_0)\cdots ad(f_{m-1})P^* = (-1)^m \big(ad(f_0)\cdots ad(f_{m-1})P\big)^* \tag{2.12}$$

and

$$ad(f_0)\cdots ad(f_m)P^* = (-1)^{m+1} \big(ad(f_0)\cdots ad(f_m)P\big)^* \tag{2.13}$$

If the right hand side of (2.12) or (2.13) is zero, so is the left hand side. From this, (i) follows. (2.12) at a point $p \in M$ multiplied by $\frac{1}{m!}$ is exactly (ii). P is of order m, so $\big(ad(f_0)\cdots ad(f_{m-1})P\big)\colon \Gamma^\infty(E) \to \Gamma^\infty(F)$ is a $C^\infty(M)$-linear map. In the proof of 2.3.4 we saw that the formal adjoint of such a map is the pointwise conjugate transpose. We restrict this map to p and multiply it by $\frac{1}{m!}$ to get the symbol, which proves the last claim. $\square$

2.3.9 Definition: Let E be a Riemannian or Hermitian vector bundle over the oriented Riemannian manifold M. An operator $P \in \mathbf{PDO}(E,E)$ is called *formally selfadjoint* if $P = P^*$.

We will see that formally selfadjoint operators have many nice properties, but first we want to give a few examples of formally selfadjoint operators.

2.3.10 Example: Let (M,g) be an oriented Riemannian manifold. Similar to the calculations in the proof of 2.3.4 we show that the formal adjoint of d (with respect to the metric g_k induced on $\Lambda^k(TM^*)$ by g, see 1.2.44) is d^* from 2.2.10 (hence the notation):
Let $\alpha \in \Omega_c^k(M)$ and $\beta \in \Omega_c^{k+1}(M)$.

$$\int_M g_{k+1}(d\alpha, \beta) dV_g =$$

$$\underset{(1.8)}{=} \int_M d\alpha \wedge *\beta =$$

$$\underset{(1.5)}{=} \int_M d(\alpha \wedge *\beta) - (-1)^k \int_M \alpha \wedge d(*\beta) =$$

$$\underset{(1.7)\&(1.8)}{=} \underbrace{\int_M \iota^*(\alpha \wedge *\beta)}_{=0,\text{ since supp}(\alpha)\text{ and supp}(\beta)\text{ are compact}} +$$

$$+ (-1)^{k-1} \int_M g_k(\alpha, \underbrace{*^{-1}}_{\underset{(1.11)}{=}(-1)^{(n-k)(n-(n-k))}*} d * \beta) dV_g =$$

$$= (-1)^{k-1+(n-k)k} \int_M g_k(\alpha, \underbrace{*d*}_{=\delta=(-1)^{(n-(k+1))((k+1)-1)+1}d^*} \beta) dV_g =$$

$$= (-1)^{k-1+(n-k)k+(n-k-1)k+1} \int_M g_k(\alpha, d^*\beta) dV_g$$

Because of

$$\begin{aligned} k-1+(n-k)k+(n-k-1)k+1 &= k(1+n-k+n-k-1) = \\ &= 2k(n-k) \equiv 0 \pmod 2 \end{aligned}$$

we may conclude

$$\int_M g_{k+1}(d\alpha, \beta) dV_g = \int_M g_k(\alpha, d^*\beta) dV_g$$

Therefore the Hodge-deRham operator $d+d^*$ and the Laplace-Beltrami operator $\Delta_g = d^*d|_{\Omega^0(M)}$ are formally selfadjoint:

$$(d+d^*)^* = d^* + d = d + d^* \quad \text{and} \quad (d^*d)^* = d^*d^{**} = d^*d$$

and so is the Hodge Laplacian $(d+d^*)^2$ by 2.3.3.

2.3.11 Example: Let E be a Riemannian or Hermitian vector bundle over the oriented Riemannian manifold M. For an arbitrary connection ∇ we define the *Bochner Laplacian* by

$$\Delta_\nabla := \nabla^*\nabla$$

Δ_∇ is a generalized Laplacian (and therefore elliptic):
∇ is an order 1 PDO (see 2.1.4) and by 2.3.8 so is ∇^*, therefore Δ_∇ is of order ≤ 2 (see 2.1.7). The symbol of ∇ is $\xi\otimes$ (see 2.2.6). By 2.3.8 we have

$$\sigma_1(\nabla^*) = -\sigma_1(\nabla)^*, \tag{2.14}$$

where $\sigma_1(\nabla)^*$ is the usual (not formally!) adjoint operator of $\sigma_1(\nabla)$ with respect to the metrics on E_p and $T_pM^* \otimes E_p$. We calculate $\sigma_1(\nabla)^*$:
Let $\omega \in T_pM^*$, $s, t \in E_p$ be arbitrary.

$$\begin{aligned} \left\langle \Big(\sigma_1(\nabla)(\xi)\Big)^*(\omega \otimes s), t \right\rangle_{E_p} &= \langle \omega \otimes s, \sigma_1(\nabla)(\xi) t \rangle_{T_pM^* \otimes E_p} = \\ &= \langle \omega \otimes s, \xi \otimes t \rangle_{T_pM^* \otimes E_p} = \\ &= g_1(\omega, \xi) \cdot \langle s, t \rangle_{E_p} = \\ &= \omega(\xi^*) \cdot \langle s, t \rangle_{E_p} \end{aligned}$$

We use (2.14) and get

$$\sigma_1\big(\nabla^*\big)(\xi)\big(\omega \otimes s\big) = -\omega(\xi^*) \cdot s$$

By 2.2.7 we conclude

$$\begin{aligned}
\sigma_2\big(\Delta_\nabla\big)(\xi)\big(s\big) &= \sigma_1\big(\nabla^*\big)(\xi) \circ \sigma_1\big(\nabla\big)(\xi)\big(s\big) \\
&= \sigma_1\big(\nabla^*\big)(\xi)\big(\xi \otimes s\big) \\
&= -\xi(\xi^*) \cdot s \\
&= -|\xi|_g^2 \cdot s \qquad \forall s \in E_p
\end{aligned}$$

Δ_∇ is formally selfadjoint:

$$\Delta_\nabla^* = (\nabla^* \nabla)^* \underset{2.3.3}{=} \nabla^* \underbrace{\nabla^{**}}_{=\nabla} = \Delta_\nabla$$

In chapter 3 we will see that formally selfadjoint, elliptic operators (like the Hodge Laplacian or the Bochner Laplacian) have very nice spectral properties.

Chapter 3

Spectral Theory for elliptic Operators

This chapter is dedicated to the proof of the spectral theorem for formally self-adjoint elliptic operators. Along the way we define many important notions, like L^2 and the Sobolev spaces on manifolds and analytical realizations of PDOs. In addition we prove the essentiell tools of elliptic regularity, in particular Weyl's Lemma and the Poincaré inequality. Applications of the Spectral Theory can be found in chapter 4.
Throughout the whole chapter (M, g) is an oriented, Riemannian manifold, dV_g will denote the metric volume defined by g and E and F are complex vector bundles over M with Hermitian metrics $\langle\cdot,\cdot\rangle_E$ and $\langle\cdot,\cdot\rangle_F$, respectively on them (the real-valued case is analogous with symmetric metrics instead of Hermitian).

Many concepts in this chapter are from [Nic07], section 10.2-10.4. For the proofs we use ideas from [AF03], [BC09], [Fol76] and [GT83] as well.

3.1 Analytical Realizations

The PDOs from chapter 2 are defined only on smooth sections. In this section we extend them to L^2-sections (which form a Hilbert space) by their analytical realizations. First we have to define what L^2-sections are, for which we need the following proposition:

3.1.1 Proposition: $(\Gamma_c^\infty(E), \langle\cdot,\cdot\rangle)$ *where* $\langle\psi,\phi\rangle := \int_M \langle\psi,\phi\rangle_E dV_g$ *is a Pre-Hilbert space.*

Proof. $\Gamma_c^\infty(E)$ is a $\mathbb{C}$-vector space with respect to the pointwise addition of sections $\big(\phi+\psi := x \mapsto \big(x, \phi(x)+\psi(x)\big)\big)$. We have to prove that $\langle\cdot,\cdot\rangle$ is a scalar product:
Sesquilinearity and hermiticity of $\langle\cdot,\cdot\rangle$ follow immediatly from the respective

properties of $\langle\cdot,\cdot\rangle_E$. $\langle\cdot,\cdot\rangle$ is also positive definite:

$$\langle\psi,\psi\rangle = \int_M \underbrace{\langle\psi,\psi\rangle_E}_{\geq 0} dV_g \geq 0$$

and from

$$0 = \langle\psi,\psi\rangle = \int_M \underbrace{\langle\psi,\psi\rangle_E}_{\geq 0} dV_g$$

it follows that $\langle\psi,\psi\rangle_E = 0$ almost everywhere. The positive definitness of $\langle\cdot,\cdot\rangle_E$ gives $\psi = 0$ almost everywhere, thus everywhere by smoothness of ψ. $\square$

3.1.2 Definition: $L^2(E)$ is defined as the Hilbert space which is the completion of $\Gamma_c^\infty(E)$ with respect to $\langle\cdot,\cdot\rangle$.[1]

In the next example we see that in general it is not possible to find an extension of a PDO $P\colon \Gamma^\infty(E) \to \Gamma^\infty(F)$ that maps the whole of $L^2(E)$ to $L^2(F)$:

3.1.3 Example: Let $M =]0,1[$ with the Riemannian metric $g\colon (v_1,v_2) \mapsto v_1 \cdot v_2$ ($v_1, v_2 \in \mathfrak{X}(M) = C^\infty(]0,1[)$; thereby we get $dV_g = dx$), $E =]0,1[\times\mathbb{R}$ a vector bundle with the same metric $\langle\cdot,\cdot\rangle = g$ and $P = \frac{d}{dx} \in \mathbf{PDO}(E,E)$. Let $u = (x, x^{-\frac{1}{3}}) \in \Gamma^\infty(E)$. u is in $L^2(E)$:

$$\int_{]0,1[} \langle u,u\rangle\, dx = \int_{]0,1[} \left(x^{-\frac{1}{3}}\right)^2 dx = \int_{]0,1[} x^{-\frac{2}{3}}\, dx = 3x^{\frac{1}{3}}\Big|_0^1 = 3$$

But $u' = \frac{d}{dx}u = (x, -\frac{1}{3}x^{-\frac{4}{3}})$ is not in $L^2(E)$:

$$\int_{]0,1[} \langle u',u'\rangle\, dx = \int_{]0,1[} \left(-\frac{1}{3}x^{-\frac{4}{3}}\right)^2 dx = \int_{]0,1[} \frac{1}{9}x^{-\frac{8}{3}}\, dx =$$
$$= -\frac{1}{15}x^{-\frac{5}{3}}\Big|_0^1 = \infty$$

Thus we are led to define the Sobolev spaces. We will see that these spaces are exactly the right domain to define the extensions of PDOs on. First we observe:

1. A connection ∇^E on the vector bundle E is a PDO of order 1 from $\Gamma^\infty(E)$ to $\Gamma^\infty(TM^*\otimes E)$ (see 2.2.6). ∇^E induces a connection $\nabla^{TM^*\otimes E}$ from $\Gamma^\infty(TM^*\otimes E)$ to $\Gamma^\infty(TM^*\otimes TM^*\otimes E)$ by definition on splitting tensors:

$$\nabla^{TM^*\otimes E}(\omega\otimes s) := (\nabla^{TM^*}\omega)\otimes s + \omega\otimes(\nabla^E s)$$

for all $\omega\in\Omega^1(M)$ and $s\in\Gamma^\infty(E)$, where ∇^{TM^*} is the dual connection of the Levi-Civita connection on M, i.e.

$$\big(\nabla_X^{TM^*}\omega\big)(Y) := X\big(\omega(Y)\big) - \omega\big(\nabla_X^{TM}Y\big) \quad \forall Y\in\mathfrak{X}(M),$$

[1] By 1.4.4 this completion exists and is unique (if we identify isometric spaces)

where $X \in \mathfrak{X}(M)$, $\omega \in \Omega^1(M)$ and ∇^{TM} is the Levi-Civita connection on M. $\nabla^{TM^*\otimes E}$ also induces a connection $\nabla^{TM^*\otimes TM^*\otimes E}$ from $\Gamma^\infty(TM^* \otimes TM^* \otimes E)$ to $\Gamma^\infty(TM^* \otimes TM^* \otimes TM^* \otimes E)$ by an analogous formula. If we continue recursively we obtain connections $\nabla^{TM^{*\otimes k}\otimes E}$ from $\Gamma^\infty(TM^{*\otimes k} \otimes E)$ to $\Gamma^\infty(TM^{*\otimes k+1} \otimes E)$ $(k = 0, 1, 2, \ldots)$. We denote all these connections by ∇. By 2.1.7 the composition

$$\nabla^m := \nabla^{TM^{*\otimes m-1}\otimes E} \circ \cdots \circ \nabla^E$$

is a PDO of order $\leq m$ from $\Gamma^\infty(E)$ to $\Gamma^\infty(TM^{*\otimes m} \otimes E)$.

2. A metric $\langle \cdot, \cdot \rangle_E$ on E induces a metric on $TM^{*\otimes k} \otimes E$ by defining on splitting tensors:

$$\begin{aligned}\langle \omega_1 \otimes \cdots \otimes \omega_k \otimes u, \mu_1 \otimes \cdots \otimes \mu_k \otimes v \rangle_{TM^{*\otimes k}\otimes E} :=& \\ = g_1(\omega_1, \mu_1) \cdots g_1(\omega_k, \mu_k) \cdot \langle w, v \rangle_E&\end{aligned}$$

where g_1 is the metric dual to g as introduced in (1.9).

3.1.4 Definition: Let $u \in L^2(E)$, $v \in L^2(F)$ and $P \in \mathbf{PDO}(E, F)$. We say that $Pu = v$ *weakly* (v is called the *weak derivative* of u with respect to P) if

$$\int_M \langle v, \phi \rangle_F dV_g = \int_M \langle u, P^*\phi \rangle_E dV_g \quad \forall \phi \in \Gamma_c^\infty(F)$$

Let ∇ be an arbitrary connection on E. We define the *Sobolev space* of order $m \in \mathbb{N}_0$ as

$$\begin{aligned}&H^m(E) := \\ &\quad = \{u \in L^2(E) \mid \forall j = 1, \ldots, m \; \exists v_j \in L^2(TM^{*\otimes j} \otimes E) : \; \nabla^j u = v_j \text{ weakly}\}\end{aligned}$$

and set

$$\langle u, v \rangle_m := \langle u, v \rangle_m^\nabla := \sum_{j=0}^m \langle \nabla^j u, \nabla^j v \rangle_{L^2}.$$

and

$$\|u\|_m := \|u\|_m^\nabla := \sqrt{\langle u, u \rangle_m}$$

In addition we define the *local Sobolev space* of order m as

$$H^m_{\text{loc}}(E) := \{u \in L^2(E) \mid \forall \phi \in C_c^\infty(M) \colon \phi \cdot u \in H^m(E)\}$$

3.1.5 Remark: 1. $H^m(E)$ depends on the choice of the metrics on M and E and the connection ∇. If M is non-compact this dependence actually matters.

2. For $t \leq s$ it follows that $H^t(E)$ embeds into $H^s(E)$.

3. $H^m(E) \subseteq H^m_{\text{loc}}(E)$.

3.1.6 Proposition: *There exists at most one weak derivative with respect to a given PDO P.*

Proof. Let $v, w \in L^2(F)$ such that $Pu = v$ weakly and $Pu = w$ weakly. Then

$$\begin{aligned}\langle v-w,\phi\rangle &= \int_M \langle v-w,\phi\rangle_F dV_g = \\ &= \int_M \langle v,\phi\rangle_F dV_g - \int_M \langle w,\phi\rangle_F dV_g = \\ &= \int_M \langle u,P^*\phi\rangle_E dV_g - \int_M \langle u,P^*\phi\rangle_E dV_g = 0 \quad \forall \phi \in \Gamma_c^\infty(F)\end{aligned}$$

$\langle\cdot,\cdot\rangle$ is continuous and $\Gamma_c^\infty(F)$ is dense in $L^2(F)$, so $\langle v-w,\phi\rangle = 0$ for all $\phi \in L^2(F)$. $\langle\cdot,\cdot\rangle$ is positive definite, from which we conclude that $v = w$ in $L^2(F)$. □

3.1.7 Remark: For $u \in \Gamma^\infty(E)$ there exists the usual derivative, which is a weak derivative by definition of the formal adjoint. By 3.1.6 we have $Pu = v$ weakly if and only if $Pu = v$ in the usual sense.

For later purposes we need:

3.1.8 Proposition: $\big(H^m(E), \langle\cdot,\cdot\rangle_m\big)$ *are Hilbert spaces for all* $m \in \mathbb{N}_0$.

Proof. Sesquilinearity, hermiticity and positive semidefiniteness of $\langle\cdot,\cdot\rangle_m$ follow easily from the corresponding properties of $\langle\cdot,\cdot\rangle_{L^2}$. Moreover, we have

$$\langle u,u\rangle_m = \sum_{j=0}^m \langle\nabla^j u,\nabla^j u\rangle_{L^2} = \langle u,u\rangle_{L^2} + \underbrace{\sum_{j=1}^m \langle\nabla^j u,\nabla^j u\rangle_{L^2}}_{\geq 0} \geq \langle u,u\rangle_{L^2},$$

from which positive definiteness of $\langle\cdot,\cdot\rangle_m$ follows by positive definiteness of $\langle\cdot,\cdot\rangle_{L^2}$. Hence $\langle\cdot,\cdot\rangle_m$ is a scalar product. $\big(H^m(E), \langle\cdot,\cdot\rangle\big)$ is complete:
Let (u_n) be a Cauchy sequence in $H^m(E)$. Thus $(\nabla^j u_n)$ is a Cauchy sequence in $L^2(TM^{*\otimes j} \otimes E)$ for all $j = 0,\dots,m$. $L^2(TM^{*\otimes j}\otimes E)$ is complete, so there exist $u^{(j)} \in L^2(TM^{*\otimes j}\otimes E)$, such that $\nabla^j u_n$ converges to $u^{(j)}$ in $L^2(TM^{*\otimes j}\otimes E)$.
$u^{(j)} = \nabla^j u^{(0)}$ weakly:
Let $\phi \in \Gamma_c^\infty(TM^{\otimes j}\otimes E)$. Then

$$\begin{aligned}\int_M \langle u^{(j)},\phi\rangle_{TM^{*\otimes j}\otimes E} dV_g &= \int_M \langle \lim_{n\to\infty} \nabla^j u_n,\phi\rangle_{TM^{*\otimes j}\otimes E} dV_g = \\ &= \lim_{n\to\infty}\int_M \langle \nabla^j u_n,\phi\rangle_{TM^{*\otimes j}\otimes E} dV_g = \\ &= \lim_{n\to\infty}\int_M \langle u_n,(\nabla^j)^*\phi\rangle_E dV_g = \\ &= \int_M \langle \lim_{n\to\infty} u_n,(\nabla^j)^*\phi\rangle_E dV_g = \\ &= \int_M \langle u^{(0)},(\nabla^j)^*\phi\rangle_E dV_g\end{aligned}$$

Hence (u_n) converges to $u := u^{(0)}$ in $H^m(E)$. □

So far we have seen that the Sobolev spaces, which depend on the chosen connection are suitable spaces for PDOs which are multiple compositions of that connection. The next Lemmata and Proposition demonstrate that Sobolev spaces in fact are the right spaces for arbitrary PDOs on compact manifolds.

3.1.9 Lemma: *Let M be compact and ∇ an arbitrary connection on (E, M, π). The Space $\mathbf{PDO}(E, F)$ is generated by successive covariant derivatives and $\mathbf{Hom}(E, F)$.*

Proof. Let $P \in \mathbf{PDO}^m(E, F)$. Claim: P is a combination of covariant derivatives and elements of $\mathbf{Hom}(E, F)$. We prove this by induction over m.

- $m = 0$: $\mathbf{PDO}^0(E, F) = \mathbf{Hom}(E, F)$ by 2.1.3.
- $m - 1 \mapsto m$:
  The symbol of P is given by $\sigma_m\big(P\big)(\xi) = \sigma\big(P\big)(\xi, \dots, \xi)$, where

  $$\sigma\big(P\big)|_p(\xi_1, \dots, \xi_m)\colon T_pM^* \times \cdots \times T_pM^* \to \mathbf{Hom}(E_p, F_p)$$

  is a multilinear map (see section 2.2). Thus

  $$\sigma\big(P\big)|_p \in T_pM \otimes \cdots \otimes T_pM \otimes \mathbf{Hom}(E_p, F_p)$$

  Since $\sigma\big(P\big)|_p$ depends smoothly on p we can write

  $$\sigma\big(P\big)|_U = \sum_{i_1, \dots, i_m} A_{i_1, \dots, i_m} \otimes X_{i_1} \otimes \cdots \otimes X_{i_m}$$

  where U is a sufficiently small neighborhood of p, $X_{i_j} \in \mathfrak{X}(U)$ $(j = 1, \dots, m)$ and $A_{i_1, \dots, i_m} \in \mathbf{Hom}(E|_U, F|_U)$. We define

  $$Q_U := \sum_{i_1, \dots, i_m} A_{i_1, \dots, i_m} \cdot \nabla_{X_{i_1}} \circ \cdots \circ \nabla_{X_{i_m}}.$$

  By 2.1.4 and 2.1.7 we get $Q_U \in \mathbf{PDO}^{(m)}(E|_U, F|_U)$. Hence $P|_U - Q_U \in \mathbf{PDO}^{(m)}(E|_U, F|_U)$. By 2.2.7 and 2.2.6 we get

  $$\begin{aligned}
  &\sigma_m\big(P|_U - Q_U\big)(\xi) = \\
  &\quad = \sigma_m\big(P|_U\big)(\xi) - \sigma_m\big(Q_U\big)(\xi) = \\
  &\quad = \sigma\big(P|_U\big)(\xi, \dots, \xi) - \\
  &\qquad\qquad - \sum_{i_1, \dots, i_m} A_{i_1, \dots, i_m} \cdot \sigma_1\big(\nabla_{X_{i_1}}\big)(\xi) \circ \cdots \circ \sigma_1\big(\nabla_{X_{i_m}}\big)(\xi) = \\
  &\quad = \sum_{i_1, \dots, i_m} A_{i_1, \dots, i_m} \cdot \xi(X_{i_1}) \cdots \xi(X_{i_m}) \cdot - \\
  &\qquad\qquad - \sum_{i_1, \dots, i_m} A_{i_1, \dots, i_m} \cdot \xi(X_{i_1}) \cdots \xi(X_{i_m}) \cdot = \\
  &\quad = 0
  \end{aligned}$$

  Thus $P|_U - Q_U \in \mathbf{PDO}^{(m-1)}(E|_U, F|_U)$. Since M is compact there exists a finite cover $\{U_k \mid 1 \leq k \leq N\}$ of such neighborhoods. We choose a partition of unity $\{\chi_k \mid 1 \leq k \leq N\}$ subordinate to $\{U_k \mid 1 \leq k \leq N\}$ and define

$$Q := \sum_{k=1}^{N} \chi_k \cdot Q_{U_k}$$

We conclude

$$\begin{aligned} P - Q &= \sum_{k=1}^{N} \chi_k \cdot P|_{U_k} - \sum_{k=1}^{N} \chi_k \cdot Q_{U_k} = \\ &= \sum_{k=1}^{N} \chi_k \cdot (\underbrace{P|_{U_k} - Q_{U_k}}_{\in \mathbf{PDO}^{(m-1)}(E|_U, F|_U)}) \in \mathbf{PDO}^{(m-1)}(E,F) \end{aligned}$$

By induction $P-Q$ is a combination of covariant derivatives and elements of $\mathbf{Hom}(E,F)$. Q is a combination of the Q_{U_k}'s, which are combinations of covariant derivatives $\nabla_{X_{i_j}}$ and the maps $A_{i_1,\dots,i_m} \in \mathbf{Hom}(E,F)$. Thus $P = P - Q + Q$ is also such a combination.

We conclude that $\mathbf{PDO}(E,F)$ is generated by multiple covariant derivatives and $\mathbf{Hom}(E,F)$. □

3.1.10 Lemma: *Let E be a Hermitian vector bundle over the manifold M, $u \in T_pM^{*\otimes k} \otimes E_p = L^k(\underbrace{T_pM \times \cdots \times T_pM}_{k\ times}; E)$ and $v \in E_p$. Then:*

$$\langle u, X_1^* \otimes \cdots \otimes X_k^* \otimes v \rangle_{T_pM^{*\otimes k} \otimes E_p} = \langle u(X_1, \dots, X_k), v \rangle_{E_p} \quad \forall X_1, \dots, X_k \in T_pM$$

where $X_i^* \in T_pM^*$ is the metrically equivalent co-vector to X_i for $i = 1, \dots, k$.

Proof. u is a finite linear combination of elements of the form $\omega_1 \otimes \cdots \otimes \omega_k \otimes w$, where $\omega_i \in T_pM^*$, $w \in E_p$. It suffices to prove the formula for $u = \omega_1 \otimes \cdots \otimes \omega_k \otimes w$ because of the sesqui-linearity of the metrics involved.

$$\begin{aligned} &\langle u, X_1^* \otimes \cdots \otimes X_k^* \otimes v \rangle_{T_pM^{*\otimes k} \otimes E_p} = \\ &\qquad = \langle \omega_1 \otimes \cdots \otimes \omega_k \otimes w, X_1^* \otimes \cdots \otimes X_k^* \otimes v \rangle_{T_pM^{*\otimes k} \otimes E_p} = \\ &\qquad = \langle \omega_1, X_1^* \rangle_{T_pM^*} \cdots \langle \omega_k, X_k^* \rangle_{T_pM^*} \cdot \langle w, v \rangle_{E_p} = \\ &\qquad = \omega_1(X_1) \cdots \omega_k(X_k) \cdot \langle w, v \rangle_{E_p} = \\ &\qquad = \langle \omega_1(X_1) \cdots \omega_k(X_k) \cdot w, v \rangle_{E_p} = \\ &\qquad = \langle \omega_1 \otimes \cdots \otimes \omega_k \otimes w(X_1, \dots, X_k), v \rangle_{E_p} = \\ &\qquad = \langle u(X_1, \dots, X_k), v \rangle_{E_p} \end{aligned}$$

□

3.1.11 Lemma: *Let ∇ be a connection on the Hermitian vector bundle E. Then:*

$$(\nabla^k)^*(X_1^* \otimes \cdots \otimes X_k^* \otimes v) = \nabla_{X_k}^* \circ \cdots \circ \nabla_{X_1}^* v$$

for all $X_1, \dots, X_k \in \mathfrak{X}(M)$, $v \in \Gamma^\infty(E)$.

Proof. Let $w \in \Gamma^\infty(E)$ arbitrary. Then

$$\begin{aligned}\langle w, (\nabla^k)^*(X_1^* \otimes \cdots \otimes X_k^* \otimes v)\rangle_E &= \langle \nabla^k w, X_1^* \otimes \cdots \otimes X_k^* \otimes v\rangle_{TM^{*\otimes k}\otimes E} = \\ &\underset{3.1.10}{=} \langle \nabla^k w(X_1, \dots, X_k), v\rangle_E = \\ &= \langle \nabla_{X_1} \circ \cdots \circ \nabla_{X_k} w, v\rangle_E = \\ &= \langle w, \nabla_{X_k}^* \circ \cdots \circ \nabla_{X_1}^* v\rangle_E \qquad \square\end{aligned}$$

3.1.12 Lemma: *Let $u \in L^2(E)$. Then $\nabla^k u = v$ weakly in $L^2(TM^{*\otimes k} \otimes E)$ if and only if for any $X_1, \dots, X_k \in \mathfrak{X}(M)$, $\nabla_{X_1} \circ \cdots \circ \nabla_{X_k} u = v(X_1, \dots, X_k)$ weakly in $L^2(E)$.*

Proof. $\nabla^k u = v$ weakly means that

$$\int \langle v, \phi\rangle dV_g = \int \langle u, (\nabla^k)^*\phi\rangle dV_g \quad \forall \phi \in \Gamma_c^\infty(TM^{*\otimes k} \otimes E) \tag{3.1}$$

Since ϕ has compact support, it is a finite $C^\infty(M)$-linear combination of sections of the form $\omega_1 \otimes \cdots \otimes \omega_k \otimes w$ with $\omega_i \in \Omega^1(M)$, $w \in \Gamma^\infty(E)$. Because of the sesqui-linearity of the metrics on $L^2(TM^{*\otimes k} \otimes E)$ and $L^2(E)$ it suffices to consider the case $\phi = \omega_1 \otimes \cdots \otimes \omega_k \otimes w$. Without loss of generality we may assume that w has compact support.

$\nabla_{X_1} \circ \cdots \circ \nabla_{X_k} u = v(X_1, \dots, X_k)$ weakly means that

$$\int \langle v(X_1, \dots, X_k), w\rangle dV_g = \int \langle u, \nabla_{X_k}^* \circ \cdots \circ \nabla_{X_1}^* w\rangle dV_g \quad \forall w \in \Gamma_c^\infty(E) \tag{3.2}$$

- Let $\nabla^k u = v$ weakly and $w \in \Gamma_c^\infty(E)$ be arbitrary. We use 3.1.10 and 3.1.11 to conclude

  $$\begin{aligned}\int \langle v(X_1, \dots, X_k), w\rangle dV_g &\underset{3.1.10}{=} \int \langle v, X_1^* \otimes \cdots \otimes X_k^* \otimes w\rangle dV_g = \\ &\underset{(3.1)}{=} \int \langle u, (\nabla^k)^*(X_1^* \otimes \cdots \otimes X_k^* \otimes w)\rangle dV_g = \\ &\underset{3.1.11}{=} \int \langle u, \nabla_{X_k}^* \circ \cdots \circ \nabla_{X_1}^* w\rangle dV_g\end{aligned}$$

  for all $X_1, \dots, X_k \in \mathfrak{X}(M)$. This means that

  $$\nabla_{X_1} \circ \cdots \circ \nabla_{X_k} u = v(X_1, \dots, X_k)$$

  weakly for all $X_1, \dots, X_k \in \mathfrak{X}(M)$.

- Conversely, let $\nabla_{X_1} \circ \cdots \circ \nabla_{X_k} u = v(X_1, \dots, X_k)$ weakly for all vector fields $X_1, \dots, X_k \in \mathfrak{X}(M)$. By the above it suffices to prove that (3.1) holds for $\phi = \omega_1 \otimes \cdots \otimes \omega_k \otimes w$. Let X_i such that $X_i^* = \omega_i$. Then

$$\begin{aligned}\int \langle u, (\nabla^k)^* \phi \rangle dV_g &= \int \langle u, (\nabla^k)^* (\omega_1 \otimes \cdots \otimes \omega_k \otimes w) \rangle dV_g = \\ &\underset{3.1.11}{=} \int \langle u, \nabla^*_{X_k} \circ \cdots \circ \nabla^*_{X_1} w \rangle dV_g = \\ &\underset{(3.2)}{=} \int \langle v(X_1, \ldots, X_k), w \rangle dV_g = \\ &\underset{3.1.10}{=} \int \langle v, \omega_1 \otimes \cdots \otimes \omega_k \otimes w \rangle dV_g = \\ &= \int \langle v, \phi \rangle dV_g \qquad \square\end{aligned}$$

3.1.13 Proposition: *Let M be a compact manifold. If $u \in H^m(E)$ and $P \in$ $\mathbf{PDO}^{(m)}(E, F)$ then there exists a (unique by 3.1.6) weak derivative $Pu \in$ $L^2(E)$.*

Proof. If $u \in H^m(E)$, there exist $v_j \in L^2(TM^{*\otimes j} \otimes E)$ $(j = 1, \ldots, m)$ such that $\nabla^j u = v_j$ weakly. Let $X_i \in \mathfrak{X}(M)$ $(i = 1, \ldots, j)$ be arbitrary. By 3.1.12 $\nabla_{X_1} \circ \cdots \circ \nabla_{X_j} u = v_j(X_1, \ldots, X_j)$ weakly. Since M is compact we have $v_j(X_1, \ldots, X_j) \in L^2(E)$. By 3.1.9 every PDO of order m is a combination of covariant derivatives (up to m compositions) and vector bundle homomorphisms. By the above all multiple covariant derivatives exist in L^2. Hence the weak derivative Pu, which is the sum over the above derivatives multiplied with the bundle homomorphisms $A_{i_1, \ldots, i_j}$, is in $L^2(E)$, since M is compact. $\square$

Now we can define:

3.1.14 Definition: Let M be compact and $P \in \mathbf{PDO}^{(m)}(E, F)$. The operator

$$\tilde{P} := u \mapsto Pu \text{ (in the weak sense)}: H^m(E) \to L^2(E)$$

is called *analytical realization* of P.

We often use the same symbol for a PDO and its analytical realization if there is no risk of confusion. Now we prove a few technical properties of Sobolev spaces, which will be very important to us in the next section.

3.1.15 Lemma: *Let $v \in H^l(E)$ and χ be a smooth function with compact support. Then there exists a constant $C > 0$ such that*

$$\|\chi \cdot v\|_l \leq C \|v\|_l$$

Proof. If $v \in \Gamma^\infty(E)$, then

$$\begin{aligned}
\|\chi \cdot v\|_l &= \sum_{j=0}^{l} \|\nabla^j(\chi \cdot v)\|_{L^2} = \\
&= \sum_{j=0}^{l} \left\| \sum_{h=0}^{j} \frac{j!}{h!(j-h)!} \nabla^{j-h}(\chi) \otimes \nabla^h(v) \right\|_{L^2} \leq \\
&\leq \sum_{j=0}^{l} \sum_{h=0}^{j} \left\| \frac{j!}{h!(j-h)!} \nabla^{j-h}(\chi) \otimes \nabla^h(v) \right\|_{L^2} \leq \\
&\leq \sum_{j=0}^{l} \sum_{h=0}^{j} \underbrace{\sup_{x \in \operatorname{supp}(\chi)} \left(\left\| \frac{j!}{h!(j-h)!} \nabla^{j-h} \chi(x) \right\|_{TM^{*\otimes j-h}} \right)}_{=:C_{hj}} \|\nabla^h v\|_{L^2} \leq \\
&\leq \max_{1 \leq h,j \leq l} (C_{hj}) \sum_{j=0}^{l} \|v\|_j \leq \\
&\leq \underbrace{\max_{1 \leq h,j \leq l} (C_{hj}) \cdot l}_{=:C} \|v\|_l
\end{aligned}$$

By density the claim follows for arbitrary $v \in H^l(E)$. ☐

3.1.16 Definition: Let (E, M, π) be a vector bundle, $m = \dim(M)$ and $\phi\colon E \supseteq U \to W \times \mathbb{K}^n$ a vector bundle chart. We define a norm on $H^s(E|_{\pi(U)})$ by

$$\|u\|_s^\phi := \left(\sum_{i=1}^{n} \sum_{|\alpha| \leq s} \int_W |\partial^\alpha (\phi_* u)_i|^2 dx \right)^{\frac{1}{2}}$$

3.1.17 Lemma: *Let (E, M, π) a Hermitian vector bundle over the manifold M, (ϕ, U) a vector bundle chart for E, where U is sufficiently small (specified in the proof) and ∇ a connection on E. Then $\|.\|_m$ (which depends on ∇) and $\|.\|_m^\phi$ (which depends on ϕ) are equivalent on $H^s(E|_V)$ ($V := \pi(U)$), i.e. there exist $C_1, C_2 > 0$, such that*

$$C_1 \|u\|_s^\phi \leq \|u\|_s \leq C_2 \|u\|_s^\phi \quad \forall u \in H^s(E|_V) \tag{3.3}$$

Proof. We prove the statement for $s = 1$, the general case is completely analogous. In addition it suffices to prove the claim for $u \in \Gamma_c^\infty(E)$, because $\Gamma_c^\infty(E)$ is dense in $H^s(E)$ and the norms involved are continuous on $H^s(E)$. By definition of a vector bundle chart $\psi := \mathrm{pr}_1 \circ \phi|_M$ is a chart map for M. Let $m := \dim(M)$ and $\{e_1, \ldots, e_m\}$ be the standard basis of $\mathbb{R}^m$. We define $\widetilde{X}_j := p \mapsto (T_p\psi)^{-1}(e_j) \in \mathfrak{X}(M)$ for $1 \leq j \leq m$. Thus $\{\widetilde{X}_j(p) \mid j = 1, \ldots, m\}$ is a basis of T_pM for all $p \in V$ and we may choose a frame $\{X_j(p) \mid j = 1, \ldots, m\}$ on V such that $X_i = A \cdot \widetilde{X}_i$ for all $1 \leq i \leq m$ where A is invertible and bounded on V.

Let $n := \dim(E_p)$ and $\{f_1, \dots, f_n\}$ be the standard basis of $\mathbb{R}^n$. We define $\widetilde{E}_i := p \mapsto \big(\phi^{-1}(\psi(p), f_i)\big) \in \Gamma_c^\infty(E)$ for $1 \le i \le n$. Thus $\{\widetilde{E}_i(p) \mid i = 1, \dots, n\}$ is a basis of E_p for all $p \in V$ and we may choose an E-frame $\{E_j(p) \mid j = 1, \dots, n\}$ on V such that $\|E_i\|$ and $\|\nabla_{X_j} E_i\|$ are bounded on V for all $1 \le i \le n$ and $E_i = B \cdot \widetilde{E}_i$ for all $1 \le i \le n$ where B is invertible and bounded on V.
Therefore we may write every $u \in \Gamma_c^\infty(E|_V)$ as $u = \sum_{i=1}^n u_i E_i = \sum_{l=1}^n \widetilde{u}_l \widetilde{E}_l$, where $u_i, \widetilde{u}_l \in C^\infty(V)$. We have

$$\nabla_{X_j} u = \nabla_{X_j} \left(\sum_{i=1}^n u_i E_i \right) = \sum_{i=1}^n \big(X_j(u_i) E_i + u_i \nabla_{X_j} E_i \big) \tag{3.4}$$

Thus

$$\begin{aligned}
&\langle \nabla_{X_j} u, \nabla_{X_l} u \rangle_{L^2} = \\
&\quad = \int_V \langle \nabla_{X_j} u, \nabla_{X_l} u \rangle dV_g = \\
&\quad = \int_V \sum_{i,k=1}^n \langle X_j(u_i) E_i + u_i \nabla_{X_j} E_i, X_l(u_k) E_k + u_k \nabla_{X_l} E_k \rangle dV_g = \\
&\quad = \int_V \Bigg(\sum_{i,k=1}^n \langle X_j(u_i) E_i, X_l(u_k) E_k \rangle + \sum_{i,k=1}^n \langle X_j(u_i) E_i, u_k \nabla_{X_l} E_k \rangle + \\
&\qquad + \sum_{i,k=1}^n \langle u_i \nabla_{X_j} E_i, X_l(u_k) E_k \rangle + \sum_{i,k=1}^n \langle u_i \nabla_{X_j} E_i, u_k \nabla_{X_l} E_k \rangle \Bigg) dV_g
\end{aligned}$$

For each term we use the following implication of the Cauchy Schwarz inequality:
Let $a_i, b_k \in \mathbb{K}$ and $v_i, w_k \in E_p$ $(1 \le i, k \le n)$, then

$$\begin{aligned}
\sum_{i,k=1}^n \langle a_i \cdot v_i, b_k \cdot w_k \rangle \underset{CSI}{\le} & \sum_{i,k=1}^n \|a_i \cdot v_i\| \cdot \|b_k \cdot w_k\| = \\
&= \sum_{i,k=1}^n |a_i| \cdot |b_k| \cdot \|v_i\| \cdot \|w_k\| \le \\
&\le \max_{i,k}(\|v_i\| \cdot \|w_k\|) \sum_{i,k=1}^n |a_i|^2 + |b_k|^2 = \\
&= \underbrace{n \cdot \max_{i,k}(\|v_i\| \cdot \|w_k\|)}_{=:C} \sum_{i=1}^n |a_i|^2 + |b_i|^2
\end{aligned}$$

Hence

$$\begin{aligned}
&\langle \nabla_{X_j} u, \nabla_{X_l} u \rangle_{L^2} \le \\
&\quad \le \int_V \Bigg(C_a \sum_{i=1}^n (|X_j(u_i)|^2 + |X_l(u_i)|^2) + C_b \sum_{i=1}^n (|X_j(u_i)|^2 + |u_i|^2) + \\
&\qquad + C_c \sum_{i=1}^n (|u_i|^2 + |X_l(u_i)|^2) + C_d \sum_{i=1}^n (|u_i|^2 + |u_i|^2) \Bigg) dV_g \le
\end{aligned}$$

$$\leq \underbrace{\sup_{p\in V}\big(C_a(p)+C_b(p)+C_c(p)+2C_d(p)\big)}_{=:C'}\cdot$$
$$\cdot\sum_{i=1}^{n}\int_V |X_j(u_i)|^2+|X_l(u_i)|^2+|u_i|^2 dV_g$$

$C' < \infty$ since C_a, C_b, C_c and C_d are bounded in V, because $\|E_i(p)\|$ and $\|\nabla_{X_j}E_i(p)\|$ are bounded for all $1 \leq i \leq n$ by definition. We use [Kun08a], (2.4.3) and 2.4.14 and the definitions of $\widetilde{X}_j$ and $\widetilde{E}_i$ to conclude

$$\begin{aligned}\widetilde{X}_j(\widetilde{u}_i) &= \partial_j(\mathrm{id}_{\mathbb{R}}\circ\widetilde{u}_i\circ\psi^{-1})\circ\psi = \\ &= \partial_j(\mathrm{pr}_i\circ\phi\circ u\circ\psi^{-1})\circ\psi = \\ &= \partial_j\big((\underbrace{\phi\circ u\circ\psi^{-1}}_{=\phi_* u})_i\big)\circ\psi\end{aligned}\tag{3.5}$$

Thus by 1.2.45

$$\begin{aligned}&\langle\nabla_{X_j}u,\nabla_{X_l}u\rangle_{L^2}\leq\\ &\quad\leq C'\sum_{i=1}^{n}\int_V |X_j(u_i)|^2+|X_l(u_i)|^2+|u_i|^2dV_g\leq\\ &\quad\leq \underbrace{C'\sup\big((\|A\|_{op}^2+1)\cdot\|B\|_{op}\big)}_{=:C''}\sum_{i=1}^{n}\int_V|\widetilde{X}_j(\widetilde{u}_i)|^2+|\widetilde{X}_l(\widetilde{u}_i)|^2+|\widetilde{u}_i|^2dV_g=\\ &\quad= C''\sum_{i=1}^{n}\int_V|\partial_j(\phi_*u)_i\big(\psi(p)\big)|^2+|\partial_l(\phi_*u)_i\big(\psi(p)\big)|^2+|(\phi_*u)_i\big(\psi(p)\big)|^2dV_g=\\ &\quad= C''\sum_{i=1}^{n}\int_W\Big(|\partial_j(\phi_*u)_i(x)|^2+|\partial_l(\phi_*u)_i(x)|^2+\\ &\qquad\qquad\qquad+|(\phi_*u)_i(x)|^2\Big)\sqrt{|\det(g_{ab}(x))|}dx\leq\\ &\quad\leq \underbrace{2C''\cdot\sup_{x\in W}\big(\sqrt{\det(g_{ab}(x))}\big)}_{=:C'''}\sum_{i=1}^{n}\sum_{|\alpha|\leq 1}\int_W|\partial^\alpha(\phi_*u)_i(x)|^2dx=\\ &\quad= C'''\big(\|u\|_1^\phi\big)^2\end{aligned}$$

$C''' < \infty$ since $|\det(g_{ab})|$ is bounded on $\psi(V)$. Finally

$$\begin{aligned}\|u\|_1^2 &= \langle\nabla u,\nabla u\rangle_{L^2}+\langle u,u\rangle_{L^2}=\\ &= \langle\sum_{j=1}^{m}X_j^*\otimes\nabla_{X_j}u,\sum_{l=1}^{m}X_l^*\otimes\nabla_{X_l}u\rangle_{L^2}+\int_V\|u\|_E^2dV_g=\\ &= \sum_{j=1}^{m}\sum_{l=1}^{m}\delta_{jl}\cdot\langle\nabla_{X_j}u,\nabla_{X_l}u\rangle+\int_{\psi(V)}\sum_{i=1}^{n}|\partial^\alpha(\phi_*u)_i|^2\sqrt{|\det(g_{ab})|}dx\leq\end{aligned}$$

$$\leq \sum_{j=1}^{m} \langle \nabla_{X_j} u, \nabla_{X_j} u \rangle + \underbrace{\sup(\sqrt{|\det(g_{ab})|})}_{=:C_g} \big(\underbrace{\|u\|_0^\phi}_{\leq \|u\|_1^\phi} \big)^2 \leq$$

$$\leq \sum_{j=1}^{m} C''' \big(\|u\|_1^\phi\big)^2 + C_g \big(\|u\|_1^\phi\big)^2 =$$

$$= \underbrace{(\max(C''') \cdot m + C_g)}_{=:C_2^2} \big(\|u\|_1^\phi\big)^2$$

which proves the second inequality in (3.3) for $s = 1$. For the first inequality we use (3.5) and 3.4 to conclude

$$\langle \nabla_{X_j} u, E_k \rangle = \sum_{i=1}^{n} \Big(\partial_j \big((\phi_* u)_i\big) \circ \psi \langle E_i, E_k \rangle + u_i \langle \nabla_{X_j} E_i, E_k \rangle \Big)$$

Since $\langle E_i, E_k \rangle = \delta_{ik}$ we may write

$$\partial_j \big((\phi_* u)_i\big) = \left(\langle \nabla_{X_j} u, E_i \rangle - \sum_{l=1}^{n} u_l \langle \nabla_{X_j} E_l, E_i \rangle \right) \circ \psi^{-1}$$

Through calculation as above we get

$$\|u\|_1^\phi = \sqrt{\sum_{i=1}^{n} \sum_{|\alpha| \leq 1} \int_W |\partial^\alpha (\phi_* u)_i|^2 dx} \leq$$

$$\leq C_1 \sqrt{\sum_{j=0}^{1} \int_V \langle \nabla^j u, \nabla^j u \rangle dV_g} = C_1 \|u\|_1$$

where C_1 depends on $\|X_j\|_g$ $(j = 1, \ldots, m)$ and n, which are positive finite numbers on V. $\square$

3.1.18 Corollary: *All norms $\|.\|_m^\nabla, \|.\|_m^{\nabla'}, \ldots$ on a vector bundle E over a compact manifold M are equivalent.*

Proof.

$$\|.\|_m^\nabla \sim \|.\|_m^\phi \text{ on } W \quad \wedge \quad \|.\|_m^\phi \sim \|.\|_m^{\nabla'} \text{ on } W' \quad \Rightarrow \quad \|.\|_m^\nabla \sim \|.\|_m^{\nabla'} \text{ on } W \cap W'$$

Since M is compact we may cover the manifold by finitely many open sets of the form $W \cap W'$, choose a partition of unity $\{\chi_j \mid j = 1, \ldots, N\}$ subordinate to this cover and conclude

$$\|u\|_m^\nabla = \|\sum_{j=1}^{N} \chi_j \cdot u\|_m^\nabla \leq \sum_{j=1}^{N} \underbrace{\|\chi_j \cdot u\|_m^\nabla}_{\leq C_{2,j} \|\chi_j \cdot u\|_m^{\nabla'}} \underset{3.1.15}{\leq} \underbrace{\sum_{j=1}^{N} C_{2,j} \cdot C}_{=:C_2} \|u\|_m^{\nabla'} = C_2 \|u\|_m^{\nabla'}$$

for all $u \in H^m(E)$. The formula $C_1 \|u\|_m^{\nabla'} \leq \|u\|_m^\nabla$ follows by symmetry. $\square$

To conclude this chapter, we consider the Sobolev embedding theorem, which describes the regularity of Sobolev spaces.

3.1.19 Theorem (Sobolev embedding): *Let E be a Hermitian vector bundle over the n-dimensional manifold M. If $s > k + \frac{n}{2}$ it follows that $H^s_{\mathrm{loc}}(E)$ embeds into $\Gamma^k(E)$ (the k-times differentiable sections of E).*

Proof. Let $u \in H^s_{\mathrm{loc}}(E)$. Now choose a partition of unity $\{\chi_\alpha \mid \alpha \in A\}$ subordinate to a covering by relatively compact, trivializing chart neighborhoods. Then $\chi_\alpha \cdot u \in H^s(E|_{V_\alpha})$. We choose an E-frame on V_α and may write $\chi_\alpha \cdot u = (u_1^\alpha, \ldots, u_p^\alpha) \circ \phi$, where $u_i^\alpha \in H^s_c(V_\alpha) \subset H^s(\mathbb{R}^n)$ for all $1 \le i \le p$ (p is the dimension of the fibres of E) by 3.1.17. We use 1.3.11 to conclude that $u_i^\alpha \in C^k(\mathbb{R}^n)$. Thus $\chi_\alpha \cdot u \in \Gamma^k(E)$. A partition of unity is locally finite, thus u locally is a sum over finitely many Γ^k-sections and therefore $u \in \Gamma^k(E)$. $\square$

3.2 Elliptic Operators

In what follows we concentrate on elliptic operators. Ellipticity guarantees a kind of invertibility, which will be a key ingredient for our further study of (elliptic) PDOs. In particular the proof of the spectral theorem (see section 3.3) relies heavily on the following implications of ellipticity.

3.2.1 Lemma (Young inequality): *Let $a, b \in \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) be arbitrary. For every $\epsilon > 0$ we have*

$$-\frac{1}{2}(a\overline{b} + \overline{a}b) \le \epsilon|a|^2 + \frac{1}{4\epsilon}|b|^2$$

Proof. Let $\epsilon > 0$ be arbitrary. Then

$$\begin{aligned} 0 \le \left|\sqrt{\epsilon}a + \frac{b}{2\sqrt{\epsilon}}\right|^2 &= \left(\sqrt{\epsilon}a + \frac{b}{2\sqrt{\epsilon}}\right)\overline{\left(\sqrt{\epsilon}a + \frac{b}{2\sqrt{\epsilon}}\right)} = \\ &= \epsilon|a|^2 + \frac{a\overline{b}}{2} + \frac{\overline{a}b}{2} + \frac{1}{4\epsilon}|b|^2 \end{aligned}$$

Hence

$$-\frac{1}{2}(a\overline{b} + \overline{a}b) \le \epsilon|a|^2 + \frac{1}{4\epsilon}|b|^2$$

$\square$

3.2.2 Lemma (Interpolation inequalities[2]): *Let M be a manifold of dimension n, $U \subseteq M$ be open and relatively compact. Then:*

$$\forall k \in \mathbb{N} \setminus \{0\} \quad \forall \epsilon > 0 \quad \exists C_\epsilon \quad \forall u \in H^k(E|_U): \quad \|u\|_{k-1} \le \epsilon\|u\|_k + C_\epsilon\|u\|_{L^2}$$

[2] The idea for part 1 of the proof is from [GT83], Problem 2.15, which is a prototype of the general Interpolation inequalities: the L^2-norm of the gradient of a C^2-function u on $\mathbb{R}^n$ is interpolated by the L^2-norm of u and the L^2-norm of Δu.

Proof. For $k=1$ this is clear (choose $C_\epsilon = 1$). We prove the case $k \geq 2$ in three steps.

1. We prove that for all $\epsilon > 0$ there exists a C_ϵ such that

$$\|u\|_{k-1} \leq \epsilon \|u\|_k + C_\epsilon \|u\|_{k-2} \tag{3.6}$$

for u compactly supported in a chart neighborhood V with the properties of 3.1.17. Since $\|.\|_k$ and $\|.\|_k^\phi$ are equivalent on V by 3.1.17 ist suffices to prove (3.6) for the chart norm $\|.\|_k^\phi$. Let $\epsilon > 0$ be arbitrary and set $\epsilon' := \frac{\epsilon^2}{\binom{n+k-2}{k-1}}$. Clearly (set $\widetilde{u} := \phi_* u$)

$$\left(\|u\|_{k-1}^\phi\right)^2 = \sum_{|\alpha|=k-1} \sum_{i=1}^p \int_W \underbrace{|\partial^\alpha \widetilde{u}_i|^2}_{=(\partial^\alpha \widetilde{u}_i)(\partial^\alpha \overline{\widetilde{u}_i})} dx + \left(\|u\|_{k-2}^\phi\right)^2$$

Let α be an arbitrary multiindex of order $k-1$. There exists at least one index (which belongs to some coordinate direction j), which is not zero. Hence we may write

$$\partial^\alpha = \partial_j \partial^{\alpha - e_j}$$

We use integration by parts to get

$$\begin{aligned}
&\sum_{i=1}^p \int_W (\partial^\alpha \widetilde{u}_i)(\partial^\alpha \overline{\widetilde{u}_i}) dx = \\
&= \sum_{i=1}^p \left(\frac{1}{2} \int_W (\partial^\alpha \widetilde{u}_i)(\partial^\alpha \overline{\widetilde{u}_i}) dx + \frac{1}{2} \int_W (\partial^\alpha \widetilde{u}_i)(\partial^\alpha \overline{\widetilde{u}_i}) dx \right) = \\
&= \sum_{i=1}^p \frac{1}{2} \left(\underbrace{(\partial^{\alpha - e_j} \widetilde{u}_i)(\partial^\alpha \overline{\widetilde{u}_i}) \Big|_{\partial W}}_{=0} - \int_W (\partial^{\alpha - e_j} \widetilde{u}_i)(\partial^{\alpha + e_j} \overline{\widetilde{u}_i}) dx + \right. \\
&\left. + \underbrace{(\partial^\alpha \widetilde{u}_i)(\partial^{\alpha - e_j} \overline{\widetilde{u}_i}) \Big|_{\partial W}}_{=0} - \int_W (\partial^{\alpha + e_j} \widetilde{u}_i)(\partial^{\alpha - e_j} \overline{\widetilde{u}_i}) dx \right) = \\
&= \sum_{i=1}^p \int_W \underbrace{-\frac{1}{2}\left((\partial^{\alpha - e_j} \widetilde{u}_i)(\partial^{\alpha + e_j} \overline{\widetilde{u}_i}) + (\partial^{\alpha + e_j} \widetilde{u}_i)(\partial^{\alpha - e_j} \overline{\widetilde{u}_i})\right)}_{\underset{3.2.1}{\leq} \epsilon' |\partial^{\alpha + e_j} \widetilde{u}_i|^2 + \frac{1}{4\epsilon'} |\partial^{\alpha - e_j} \widetilde{u}_i|^2} dx \leq \\
&\leq \epsilon' \sum_{i=1}^p \int_W |\partial^{\alpha + j} \widetilde{u}_i|^2 dx + \frac{1}{4\epsilon'} \sum_{i=1}^p \int_W |\partial^{\alpha - j} \widetilde{u}_i|^2 dx \leq \\
&\leq \epsilon' \sum_{|\beta|=k} \sum_{i=1}^p \int_W |\partial^\beta \widetilde{u}_i|^2 dx + \frac{1}{4\epsilon'} \sum_{|\beta|=k-2} \sum_{i=1}^p \int_W |\partial^\beta \widetilde{u}_i|^2 dx \leq \\
&\leq \epsilon' \left(\|u\|_k^\phi\right)^2 + \frac{1}{4\epsilon'} \left(\|u\|_{k-2}^\phi\right)^2
\end{aligned}$$

Hence

$$\begin{aligned}
\left(\|u\|^{\phi}_{k-1}\right)^2 &= \sum_{|\alpha|=k-1} \sum_{i=1}^{p} \int_W |\partial^\alpha \widetilde{u}_i|^2 dx + \left(\|u\|^{\phi}_{k-2}\right)^2 \leq \\
&\leq \sum_{|\alpha|=k-1} \left(\epsilon' \left(\|u\|^{\phi}_{k}\right)^2 + \frac{1}{4\epsilon'} \left(\|u\|^{\phi}_{k-2}\right)^2 \right) + \left(\|u\|^{\phi}_{k-2}\right)^2 = \\
&= \underbrace{\binom{n+k-2}{k-1}\epsilon'}_{=\epsilon^2} \left(\|u\|^{\phi}_{k}\right)^2 + \underbrace{\left(\binom{n+k-2}{k-1}\frac{1}{4\epsilon'} + 1\right)}_{=:(C^{\phi}_{\epsilon})^2} \left(\|u\|^{\phi}_{k-2}\right)^2
\end{aligned}$$

And therefore

$$\|u\|^{\phi}_{k-1} \leq \epsilon \|u\|^{\phi}_{k} + C_\epsilon \|u\|^{\phi}_{k-2}$$

2. We prove that (3.6) holds on U. U is relatively compact, so there exists a finite cover $\{V_i \mid 1 \leq i \leq a\}$ of $\overline{U}$ (and hence of U itself), where the V_i are chart neighborhoods as in step 1. Let $\{\chi_i \mid 1 \leq i \leq a\}$ be a partition of unity subordinate to $\{V_i \mid 1 \leq i \leq a\}$. Let $\epsilon > 0$ be arbitrary. 3.1.15 tells us that $\|\chi_i \cdot u\|_t \leq C_{i,t}\|u\|_t$. We define $\epsilon' := \frac{\epsilon}{\sum_{i=1}^{a} C_{i,k}}$ and use step 1 for ϵ' to get

$$\begin{aligned}
\|u\|_{k-1} = \|\sum_{i=1}^{a} \chi_i \cdot u\|_{k-1} &\leq \\
&\leq \sum_{i=1}^{a} \|\chi_i \cdot u\|_{k-1} \leq \\
&\leq \sum_{i=1}^{a} (\epsilon' \|\chi_i \cdot u\|_k + C_{\epsilon'} \|\chi_i \cdot u\|_{k-2}) \leq \\
&\leq \sum_{i=1}^{a} (\epsilon' C_{i,k} \|u\|_k + C_{\epsilon'} C_{i,k-2} \|u\|_{k-2}) \leq \\
&\leq \epsilon \|u\|_k + \underbrace{\sum_{i=1}^{a} C_{\epsilon'} C_{i,k-2}}_{=:C} \|u\|_{k-2}
\end{aligned}$$

3. Let $\epsilon > 0$ be arbitrary. Define $\delta_1 := \frac{\epsilon}{2}$. Using step 2 for the $k-1$-norm and δ_1 we get

$$\|u\|_{k-1} \leq \delta_1 \|u\|_k + C_{\delta_1} \|u\|_{k-2}$$

We set $\delta_2 := \frac{1}{2(k-2) \cdot C_{\delta_1}}$ and use step 2 for the $k-2$-norm and δ_2 to get

$$\|u\|_{k-1} \leq \delta_1 \|u\|_k + \frac{1}{2(k-2)} \|u\|_{k-1} + C_{\delta_1} \cdot C_{\delta_2} \|u\|_{k-3}$$

Now setting $\delta_3 := \frac{1}{2(k-2)\cdot C_{\delta_1}\cdot C_{\delta_2}}$ and using step 3 again, we obtain

$$\begin{aligned}\|u\|_{k-1} &\leq \delta_1\|u\|_k + \frac{1}{2(k-2)}\|u\|_{k-1} + \frac{1}{2(k-2)}\|u\|_{k-2} + \\ &\quad + C_{\delta_1}\cdot C_{\delta_2}\cdot C_{\delta_3}\|u\|_{k-4} \leq \\ &\leq \delta_1\|u\|_k + \frac{1}{2(k-2)}\|u\|_{k-1} + \frac{1}{2(k-2)}\|u\|_{k-1} + \\ &\quad + C_{\delta_1}\cdot C_{\delta_2}\cdot C_{\delta_3}\|u\|_{k-4}\end{aligned}$$

Iterating this procedure we conclude

$$\|u\|_{k-1} \leq \delta_1\|u\|_k + (k-2)\frac{1}{2(k-2)}\|u\|_{k-1} + C_{\delta_1}\cdots C_{\delta_{k-1}}\|u\|_{L^2}$$

and therefore

$$\|u\|_{k-1} \leq \underbrace{2\delta_1}_{=\epsilon}\|u\|_k + \underbrace{2C_{\delta_1}\cdots C_{\delta_{k-1}}}_{=:C_\epsilon}\|u\|_{L^2}$$

□

3.2.3 Theorem (Elliptic estimates[3]): *Let $P \in \mathbf{PDO}^m(E,F)$ be elliptic and $k \in \mathbb{N}_0$ arbitrary. For every open, relatively compact set $U \subseteq M$ there exists a constant C such that*

$$\|u\|_{k+m} \leq C\big(\|Pu\|_k + \|u\|_{L^2}\big) \quad \forall u \in H^{k+m}(E|_U) \tag{3.7}$$

Proof. We prove the theorem in six steps. First we prove the weaker inequality

$$\|u\|_m \leq C\big(\|Pu\|_{L^2} + \|u\|_{m-1}\big) \tag{3.8}$$

for PDOs (and sections) of a very special form. Then in every step we get rid of some assumption or improve (3.8) towards (3.7).

1. Let $u \in H^m(E)$ be compactly supported in a trivializing chart neighborhood V with corresponding vector bundle map ϕ as in 3.1.17. There the local representation P_r of P is given by

$$P_r = \sum_{|\alpha|\leq m} A_\alpha\partial^\alpha\colon \Gamma^\infty(\mathbb{R}^n\times\mathbb{C}^p) \to \Gamma^\infty(\mathbb{R}^n\times\mathbb{C}^q)$$

(see 2.1.15) and $\widetilde{u} := \phi_* u$ where $\widetilde{u}_j \in H^m_c(\mathbb{R}^n\times\mathbb{C})$ for $1 \leq j \leq p$. In addition we will assume that $A_\alpha = 0$ if $|\alpha| < m$ and that A_α is constant on V. Let $\beta \in \mathbb{N}_0^n$ such that $|\beta| = m$. P has constant coefficients, so we get

$$\partial^\beta(P_r\widetilde{u}) = P_r(\partial^\beta\widetilde{u}) = \sum_{|\alpha|=m} A_\alpha\partial^\alpha\partial^\beta\widetilde{u}$$

[3] A proof for $M = \mathbb{R}^n$ is given in [Nic07], section 10.3.1 and [Fol76], 6.25. We use ideas from both of these books and give a proof for arbitrary M.

We Fourier transform this equation componentwise $\big(\widehat{\widetilde{u}} := (\widehat{\widetilde{u}}_1, \dots, \widehat{\widetilde{u}}_p)\big)$ and use the exchange formula (1.13) to get

$$i^m \xi^\beta \widehat{P_r \widetilde{u}} = i^m \sum_{|\alpha|=m} A_\alpha \xi^\alpha \widehat{\partial^\beta \widetilde{u}}$$

P is elliptic, so the principal symbol $\sigma_m\big(P\big)(\xi) = \sum_{|\alpha|=m} A_\alpha \xi^\alpha$ is invertible for all $\xi \neq 0$. Thus we may write

$$\widehat{\partial^\beta \widetilde{u}} = \underbrace{\Big(\sigma_m\big(P\big)(\xi)\Big)^{-1} \xi^\beta}_{=:B(P,\xi,\beta)} \widehat{P_r \widetilde{u}} \quad \forall \xi \neq 0$$

Since σ_m is homogeneous of degree m, σ_m^{-1} is homogeneous of degree $-m$ and B is homogeneous of degree 0. Hence for

$$C_\beta := \max_{|\xi|=1} \big(1, B(P,\xi,\beta)\big) < \infty$$

we get

$$\|\widehat{\partial^\beta \widetilde{u}}\|_{L^2} \leq C_\beta \|\widehat{P_r \widetilde{u}}\|_{L^2}$$

Using (1.16) in Plancherel's theorem we have

$$\|\partial^\beta \widetilde{u}\|_{L^2} \leq C_\beta \|P_r \widetilde{u}\|_{L^2} \tag{3.9}$$

Thus we conclude

$$\begin{aligned}
\big(\|u\|_m^\phi\big)^2 &= \sum_{|\beta|=m} \|\partial^\beta \widetilde{u}\|^2_{L^2(E)} + \big(\|u\|^\phi_{m-1}\big)^2 \leq \\
&\leq \underbrace{\binom{n+m-1}{m} \cdot \max_{|\beta|=m} (C_\beta^2)}_{=:C^2>1} \cdot \|P_r \widetilde{u}\|^2_{L^2} + \big(\|u\|^\phi_{m-1}\big)^2
\end{aligned}$$

and therefore

$$\|u\|^\phi_m \leq C\big(\|Pu\|^\phi_{L^2} + \|u\|^\phi_{m-1}\big)$$

This shows (3.8) for $u \in H^m(E)$ compactly supported in a chart neighborhood V and P "homogeneous" and with constant coefficients.

2. Let $u \in H^{m+k}(E)$ and P as above. We generalize (3.8):

$$\begin{aligned}
\big(\|u\|^\phi_{m+k}\big)^2 &= \sum_{|\alpha|\leq m+k} \|\partial^\alpha \widetilde{u}\|^2_{L^2} = \\
&= \sum_{k\leq|\alpha|\leq m+k} \|\partial^\alpha \widetilde{u}\|^2_{L^2} + \sum_{|\alpha|\leq k-1} \|\partial^\alpha \widetilde{u}\|^2_{L^2} = \\
&= \sum_{k\leq|\alpha|\leq m+k} \|\partial^\alpha \widetilde{u}\|^2_{L^2} + \big(\|u\|^\phi_{k-1}\big)^2 = \\
&= \sum_{j=0}^{m} \sum_{|\gamma|=k} \sum_{|\beta|=j} \|\partial^\beta \partial^\gamma \widetilde{u}\|^2_{L^2} + \big(\|u\|^\phi_{k-1}\big)^2 = \\
&= \sum_{|\gamma|=k} \big(\|\phi^*(\partial^\gamma \widetilde{u})\|^\phi_m\big)^2 + \big(\|u\|^\phi_{k-1}\big)^2
\end{aligned}$$

we use the statement of step 1 and get

$$
\begin{aligned}
&\|u\|^{\phi}_{m+k} \leq \\
&\leq \sum_{|\gamma|=k} \|\phi^*(\partial^\gamma \widetilde{u})\|^{\phi}_{m} + \|u\|^{\phi}_{k-1} \leq \\
&\leq \sum_{|\gamma|=k} C'\big(\|P_r(\partial^\gamma \widetilde{u})\|_{L^2} + \|\phi^*(\partial^\gamma \widetilde{u})\|^{\phi}_{m-1}\big) + \|u\|^{\phi}_{k-1} \leq \\
&\leq C' \left(\sum_{|\gamma|=k} \|\partial^\gamma (P_r \widetilde{u})\|_{L^2} + \sum_{|\gamma|=k} \|\phi^*(\partial^\gamma \widetilde{u})\|^{\phi}_{m-1} \right) + \|u\|^{\phi}_{k-1} \leq \\
&\leq C'\big(\|Pu\|^{\phi}_{k} + \|u\|^{\phi}_{m-1+k}\big) + \|u\|^{\phi}_{k-1} \leq \\
&\leq \underbrace{\max(1, C')}_{=:C} \Big(\|Pu\|^{\phi}_{k} + \|u\|^{\phi}_{m+k-1}\Big) \qquad (3.10)
\end{aligned}
$$

3. Let u be as in step 2 and P represented by $\sum_{|\alpha|=m} A_\alpha \partial^\alpha$ where A_α is smooth, but not necessary constant. For every $x_i \in V$ we may define P_{x_i} such that its local representation is

$$\sum_{|\alpha|=m} A_\alpha(x_i)\partial^\alpha$$

From step 2 we know that there exists a $C_i > 0$ such that

$$\|u\|^{\phi}_{m+k} \leq C_i\big(\|P_{x_i} u\|^{\phi}_{k} + \|u\|^{\phi}_{m+k-1}\big)$$

We estimate the difference of Pu and $P_{x_i}u$:

$$
\begin{aligned}
\|Pu - P_{x_i}u\|^{\phi}_{k} &= \left\| \sum_{|\alpha|=m} \phi^*\Big(\big(A_\alpha(.) - A_\alpha(x_i)\big)\partial^\alpha \widetilde{u}\Big) \right\|^{\phi}_{k} \leq \\
&\leq \sum_{|\alpha|=m} \sup_{x \in \operatorname{supp}(\widetilde{u})} \big(\|A_\alpha(x) - A_\alpha(x_i)\|_{op}\big) \cdot \|\phi^*(\partial^\alpha \widetilde{u})\|^{\phi}_{k}
\end{aligned}
$$

where $\|.\|_{op}$ is the operator norm of operators from E_x to F_x. Since A_α is smooth we have that for every $\epsilon_i > 0$ there exists a $\delta_{x_i} > 0$, such that

$$\|A_\alpha(x) - A_\alpha(x_i)\|_{op} < \epsilon_i \quad \forall x \in B_{\delta_{x_i}}(x_i)$$

Letting $\epsilon_i = \frac{1}{2C_i}$, we choose for every x_i an appropriate δ_{x_i} and get for every compactly supported $u \in H^{m+k}(E|_{B_{\delta_{x_i}}(x_i)})$:

$$
\begin{aligned}
\|u\|^{\phi}_{m+k} &\leq C_i\big(\|P_{x_i}u\|^{\phi}_{k} + \|u\|^{\phi}_{m+k-1}\big) = \\
&= C_i\big(\|Pu - Pu + P_{x_i}u\|^{\phi}_{k} + \|u\|^{\phi}_{m+k-1}\big) \leq \\
&\leq C_i\big(\|Pu\|^{\phi}_{k} + \|Pu - P_{x_i}u\|^{\phi}_{k} + \|u\|^{\phi}_{m+k-1}\big) \leq
\end{aligned}
$$

$$\leq C_i\big(\|Pu\|_k^\phi + \|u\|_{m+k-1}^\phi\big) + \\ + C_i \cdot \sum_{|\alpha|=m} \sup_{x\in\mathrm{supp}(\widetilde{u})} \big(\|A_\alpha(x) - A_\alpha(x_i)\|_{op}\big) \cdot \|\phi^*(\partial^\alpha \widetilde{u})\|_k^\phi \leq$$

$$\leq C_i\big(\|Pu\|_k^\phi + \|u\|_{m+k-1}^\phi\big) + \\ + C_i \cdot \sum_{|\alpha|=m} \frac{1}{2C_i} \|\phi^*(\partial^\alpha \widetilde{u})\|_k^\phi \leq$$

$$\leq C_i\big(\|Pu\|_k^\phi + \|u\|_{m+k-1}^\phi\big) + \frac{1}{2}\|u\|_{m+k}^\phi$$

We conclude

$$\|u\|_{m+k}^\phi \leq \underbrace{2C_i}_{=:C_i'} \big(\|Pu\|_k^\phi + \|u\|_{m+k-1}^\phi\big) \tag{3.11}$$

for $u \in H^{m+k}(E|_{B_{\delta_{x_i}}(x_i)})$ compactly supported.

4. Let $u \in H^{m+k}(E|_{B_{\delta_{x_i}}(x_i)})$ be compactly supported ($B_{\delta_{x_i}}(x_i)$ such that (3.11) holds) and $P \in PDO^m(E,F)$ an arbitrary elliptic operator. Since we are working on a trivializing chart V, P is represented by

$$\sum_{|\alpha|\leq m} A_\alpha \partial^\alpha = \underbrace{\sum_{|\alpha|=m} A_\alpha \partial^\alpha}_{=:P_0} + \underbrace{\sum_{|\alpha|\leq m-1} A_\alpha \partial^\alpha}_{=:P_1}$$

For P_0 we have the estimate

$$\|u\|_{m+k}^\phi \leq C'\big(\|P_0 u\|_k^\phi + \|u\|_{m+k-1}^\phi\big)$$

by step 3. We calculate the norm of $P_1 u$:

$$\|P_1 u\|_k^\phi = \left\| \sum_{|\alpha|\leq m-1} \phi^*(A_\alpha \partial^\alpha \widetilde{u}) \right\|_k^\phi \leq \\ \leq \sum_{|\alpha|\leq m-1} \sup_{x\in\mathrm{supp}(\widetilde{u})} \big(\|A_\alpha\|_{op}\big) \cdot \|\phi^*(\partial^\alpha \widetilde{u})\|_k^\phi \leq \\ \leq \underbrace{\sup_{|\alpha|\leq m-1} \Big(\sup_{x\in\mathrm{supp}(\widetilde{u})} \big(\|A_\alpha\|_{op}\big)\Big)}_{=:C''} \cdot \sum_{|\alpha|\leq m-1} \|\phi^*(\partial^\alpha \widetilde{u})\|_k^\phi \leq \\ \leq C'' \cdot \|u\|_{m+k-1}^\phi$$

Together we conclude

$$\|u\|_{m+k}^\phi \leq C'\big(\|P_0 u\|_k^\phi + \|u\|_{m+k-1}^\phi\big) = \\ = C'\big(\|P_0 u + P_1 u - P_1 u\|_k^\phi + \|u\|_{m+k-1}^\phi\big) \leq \\ \leq C'\big(\|Pu\|_k^\phi + \|P_1 u\|_k^\phi + \|u\|_{m+k-1}^\phi\big) \leq \\ \leq C'\big(\|Pu\|_k^\phi + C'' \cdot \|u\|_{m+k-1}^\phi + \|u\|_{m+k-1}^\phi\big) \leq \\ \leq \underbrace{\max(C',1)\cdot(C''+1)}_{=:C} \big(\|Pu\|_k^\phi + \|u\|_{m+k-1}^\phi\big)$$

By 3.1.17 there exists a $C > 0$, such that

$$\|u\|_{m+k} \leq C\big(\|Pu\|_k + \|u\|_{m+k-1}\big) \tag{3.12}$$

for $u \in H^{m+k}(E|_{B_{\delta_{x_i}}(x_i)})$.

5. For every chart (ψ_V, V),

$$\{B_{\delta_{x_i}}(x_i) \mid x_i \in V, (3.12) \text{ holds on } B_{\delta_{x_i}}(x_i)\}$$

is an open covering of V. So we can cover the whole manifold M by

$$\{\psi_V^{-1}\big(B_{\delta_{x_i}}(x_i)\big) \mid (\psi, V) \text{ chart around } x_i, (3.12) \text{ holds on } B_{\delta_{x_i}}(x_i)\}$$

For a relatively compact $U \subseteq M$ there exists a finite subcover

$$\mathcal{V} := \{\psi_V^{-1}\big(B_{\delta_{x_i}}(x_i)\big) \mid 1 \leq i \leq N\}$$

We choose a partition of unity $\{\chi_i \mid 1 \leq i \leq N\}$ subordinate to $\mathcal{V}$. Let $u \in H^{k+m}(E|_U)$ be arbitrary. It follows that $\chi_i \cdot u \in H_c^{k+m}(E|_{\psi_V^{-1}\left(B_{\delta_{x_i}}(x_i)\right)})$ for $1 \leq i \leq N$. By (3.12) and 3.1.15 we get

$$\begin{aligned}
&\|u\|_{m+k} = \\
&\quad = \|\sum_{i=1}^N \chi_i \cdot u\|_{m+k} \leq \\
&\quad \leq \sum_{i=1}^N \|\chi_i \cdot u\|_{m+k} \leq \\
&\quad \leq \sum_{i=1}^N C_i\big(\|P(\chi_i \cdot u)\|_k + \|\chi_i \cdot u\|_{m+k-1}\big) \leq \\
&\quad \leq \underbrace{\max_{i=1,\dots,N}(C_i)}_{=:C^{(1)}} \sum_{i=1}^N \big(\|P(\chi_i \cdot u)\|_k + \|\chi_i \cdot u\|_{m+k-1}\big) \leq \\
&\quad \leq C^{(1)} \sum_{i=1}^N \big(\underbrace{\|\chi_i \cdot Pu\|_k}_{\leq C^{(2)}\|Pu\|_k} + \|\underbrace{[P, \chi_i]}_{\in \mathbf{PDO}^{m-1}} u\|_k + \underbrace{\|\chi_i \cdot u\|_{m+k-1}}_{\leq C^{(3)}\|u\|_{m+k-1}}\big)
\end{aligned}$$

In addition for every compactly supported $Q \in \mathbf{PDO}^r$ there exists a $C^{(4)}$ such that

$$\|Qu\|_{L^2} \leq C^{(4)}\|u\|_r$$

because of 3.1.9. Altogether we conclude

$$\begin{aligned}
&\|u\|_{m+k} \leq \\
&\quad \leq C^{(1)} \sum_{i=1}^N \big(C^{(2)}\|Pu\|_k + C^{(4)}\|u\|_{m+k-1} + C^{(3)}\|u\|_{m+k-1}\big) \leq \\
&\quad \leq \underbrace{C^{(1)} \max(C^{(2)}, 1) \max(C^{(3)} + C^{(4)}, 1)}_{=:C'} \cdot \sum_{i=1}^N \big(\|Pu\|_k + \|u\|_{m+k-1}\big) = \\
&\quad = \underbrace{C'N}_{=:C} \big(\|Pu\|_k + \|u\|_{m+k-1}\big)
\end{aligned}$$

6. In step 5 we proved

$$\|u\|_{m+k} \leq C'\big(\|Pu\|_k + \|u\|_{m+k-1}\big)$$

for an arbitrary elliptic operator $P \in \mathbf{PDO}^m(E,F)$ and $u \in H^{k+m}(E|_U)$, where $U \subseteq M$ is relatively compact. The last step is to exchange the $m+k-1$-norm by the L^2-norm of u, therefore we need the interpolation inequalities from 3.2.2. We set $\epsilon := \frac{1}{2C'}$ and get

$$\begin{aligned}
&\|u\|_{m+k} \leq \\
&\quad \leq C'\big(\|Pu\|_k + \|u\|_{m+k-1}\big) \leq \\
&\quad \leq C'\left(\|Pu\|_k + \frac{1}{2C'}\|u\|_{m+k} + C_\epsilon\|u\|_{L^2}\right) = \\
&\quad = C'\|Pu\|_k + \frac{1}{2}\|u\|_{m+k} + C'C_\epsilon\|u\|_{L^2}
\end{aligned}$$

So we conclude

$$\|u\|_{m+k} \leq \underbrace{2C'\max(1,C_\epsilon)}_{=:C}\big(\|Pu\|_k + \|u\|_{L^2}\big)$$

□

3.2.4 Remark: The requirement in 3.2.2 and 3.2.3 that the sections on $E|_U$, where U is already relatively compact, must have compact support is absolutely necessary, since the chart image $\psi(U)$ of U is not necessarily relatively compact (Take for example $U =]0,1[$ and $\psi = \ln$). Thereby the coordinate functions of the chart representation $(u_1,\ldots,u_l)$ of $u \in H^k(E|_U)$ are not necesarily in $H^k(\mathbb{R}^n)$. This feature can only be guaranteed if we require $u \in H^k_c(E|_U)$.

The above estimates imply the following Lemma, which provides an easy proof of the elliptic regularity theorem.

3.2.5 Theorem (Elliptic Regularity[4]): *Let $P \in \mathbf{PDO}^m(E,F)$ be elliptic, $u \in H^m_{\text{loc}}(E)$ and $s \in \mathbb{N}_0$ arbitrary. From $Pu \in H^s_{\text{loc}}(F)$ it follows that $u \in H^{m+s}_{\text{loc}}(E)$.*

Proof. Let $u \in H^m_{\text{loc}}(E)$ and $Pu \in H^s_{\text{loc}}(F)$. We have to show that $\chi \cdot u \in H^{m+s}(E)$ for every $\chi \in C^\infty_c(M)$. We cover $\text{supp}(\chi)$ by coordinate charts (finitely many suffice) and consider the problem locally. Since

$$v \in H^k_{loc}(E|_U) \Leftrightarrow \psi(v)_i \in H^k_{loc}(\psi(U)) \quad \forall 1 \leq i \leq p$$

where (ψ,U) is a chart and p is the dimension of the fibres of E (see 3.1.17) it suffices to prove the theorem on $\mathbb{R}^n$. This requires some analytical work with difference quotients which will not be included in this book. We refer to [Fol76], (6.21)-(6.30) for a very intelligible proof. □

[4] A proof for two special cases can be found in [Nic07], Theorem 10.3.6. An explicit proof for $M = \mathbb{R}^n$ is given in [Fol76], (6.21)-(6.30)

3.2.6 Corollary (Weyl's Lemma): *Let $P \in \mathbf{PDO}^m(E,F)$ be elliptic and $u \in H^m_{\text{loc}}(E)$. From $Pu \in \Gamma^\infty(F)$ it follows that $u \in \Gamma^\infty(E)$.*

Proof.

$$\begin{aligned} Pu \in \Gamma^\infty(F) \quad &\Rightarrow \quad Pu \in H^s_{\text{loc}}(F) \quad \forall s \in \mathbb{N}_0 \\ &\underset{3.2.5}{\Rightarrow} \quad u \in H^{m+s}_{\text{loc}}(E) \quad \forall s \in \mathbb{N}_0 \\ &\underset{3.1.19}{\Rightarrow} \quad u \in \Gamma^{m+s-\lfloor \frac{n}{2} \rfloor -1} \quad \forall s \in \mathbb{N}_0 \\ &\Rightarrow \quad u \in \Gamma^\infty(E) \end{aligned}$$ □

We use these results about elliptic operators to prove some properties of their analytical realizations:

3.2.7 Proposition: *Let M be a compact manifold. The analytical realization $\widetilde{P}$ of an elliptic PDO $P \in \mathbf{PDO}^m(E,F)$ is a closed, densely defined linear operator.*

Proof. $\widetilde{P}$ is linear by definition. We have $\Gamma^\infty_c(E) \subseteq H^m(E) \subseteq L^2(E)$ and $\Gamma^\infty_c(E)$ is dense in $L^2(E)$. Thus $H^m(E)$ is dense in $L^2(E)$. We have to prove that $\operatorname{graph}(P) = \{(u, Pu) \mid u \in H^m(E)\}$ is closed in $L^2(E) \times L^2(F)$.
Let $u_n \in H^m(E)$ for all $n \in \mathbb{N}$ and (u_n, Pu_n) converge to (u,v) in $L^2(E)\times L^2(F)$. Hence (u_n) is a Cauchy sequence in $L^2(E)$ and (Pu_n) is a Cauchy sequence in $L^2(F)$. We use 3.2.3 (M is compact!) and get

$$\|u_n - u_l\|_m \leq C\big(\underbrace{\|P(u_n-u_k)\|_{L^2}}_{\to 0 \text{ for } n,k\to\infty} + \underbrace{\|u_k - u_l\|_{L^2}}_{\to 0 \text{ for } k,l\to\infty} \big) \to \ 0 \text{ for } n,l \to \infty$$

Thus (u_n) is a Cauchy sequence in $H^m(E)$. In 3.1.8 we saw that $H^m(E)$ is complete, so (u_n) converges to some w in $H^m(E) \subset L^2(E)$. But (u_n) converges to u in $L^2(E)$, so $w = u$. Thus (Pu_n) converges to Pu in $L^2(F)$, which implies $v = Pu$. Hence $\operatorname{graph}(P)$ is closed in $L^2(E) \times L^2(F)$. □

Thus we can define the adjoint operator of $\tilde{P}$ (see 1.4.11). This is not the same as the formal adjoint of P from section 2.3 (since it is not defined on the same spaces), but:

3.2.8 Proposition: *Let M be compact and $P \in \mathbf{PDO}^m(E,F)$ an elliptic operator. The adjoint of the analytical realization is the analytical realization of the formal adjoint:*

$$(\widetilde{P})^* = \widetilde{P^*}$$

Proof. 3.2.7 shows that $\widetilde{P}$ is closed, densely defined and linear, so we can define the adjoint operator $(\widetilde{P})^*$. By definition of the adjoint operator, 1.4.11, we have

$$\langle \widetilde{P}u, v\rangle = \langle u, (\widetilde{P})^* v\rangle \quad \forall u \in H^m(E), v \in H^m(F)$$

Thus $\mathrm{Dom}((\widetilde{P})^*) \supseteq H^m(F) = \mathrm{Dom}(\widetilde{P^*})$ and

$$(\widetilde{P})^* = \widetilde{P^*} \quad \text{on } \mathrm{Dom}(\widetilde{P^*})$$

$\mathrm{Dom}((\widetilde{P})^*) \subseteq \mathrm{Dom}(\widetilde{P^*})$: Let $u \in \mathrm{Dom}\big((\widetilde{P})^*\big)$. Then there exists a $v \in L^2(E)$ such that

$$\langle v, \phi\rangle = \langle u, \widetilde{P}\phi\rangle = \langle u, P\phi\rangle \quad \forall \phi \in \Gamma_c^\infty(E)$$

This means that $v = P^*u$ weakly. We use the elliptic regularity theorem 3.2.5 to conclude $u \in H^m(F)$. So $\mathrm{Dom}\big((\widetilde{P})^*)\big) \subseteq H^m(F) = \mathrm{Dom}(\widetilde{P^*})$. $\square$

3.2.9 Corollary: *The analytical realization of a formally selfadjoint elliptic operator on a compact manifold is selfadjoint.*

Proof. $(\widetilde{P})^* = \widetilde{P^*} = \widetilde{P}$ $\square$

The last aim of this section is to prove the Poincaré inequality. On the way we need to prove the Theorem of Rellich. For $M = \mathbb{R}^n$ proofs can be found in [AF03], 6.3, [BC09], section 4.5, [Fol76], 6.19 and [Nic07], Theorem 10.2.24. We use the approach of [BC09] and generalize it to arbitrary manifolds.

3.2.10 Lemma: *Let $B \subseteq C^1(\mathbb{R}^n)$ be bounded. Then B is relatively compact in $C(\mathbb{R}^n)$.*

Proof. B bounded means that for all $K \in \mathbb{R}^n$ compact there exist C_K and C'_K, such that $\|f\|_{L^\infty(K)} \le C_K$ and $\|df\|_{L^\infty(K)} \le C'_K$ for all $f \in B$. The first part implies that B is bounded in $C(\mathbb{R}^n)$. Now let K be some compact ball. Since

$$f(x) - f(y) = \int_0^1 df\big(y + t(x-y)\big)\big(x-y\big)dt$$

we get

$$|f(x) - f(y)| \le C'_K|x-y| \quad \forall f \in B \quad \forall x, y \in K$$

which means that B is equicontinuous. Using the theorem of Arzelà and Ascoli, 1.4.20, we deduce that B is relatively compact in $C(\mathbb{R}^n)$. $\square$

3.2.11 Theorem (Rellich): *Let M be compact, $s, t \in \mathbb{N}_0$ and $t > s$. The embedding $\iota\colon H^t(E) \hookrightarrow H^s(E)$ is compact*[5].

Proof. We prove the theorem in three steps.

1. First we prove the theorem locally and for $t = 1$ and $s = 0$.
   Let U be a trivializing chart neighborhood. We use the chart map to identify U with an open subset of $\mathbb{R}^n$. Let $B \subset H^1(E|_U) \subset H^1(\mathbb{R}^n \times \mathbb{K}^q)$ be bounded and all $u \in B$ supported in the same compact set $K \subset U$.

[5] bounded sets are mapped to relatively compact sets

Let (u_n) be a sequence in B. We show that there exists an L^2-convergent subsequence of u_n. Since $L^2(E|_U)$ is a Hilbert space it suffices to find a Cauchy-subsequence of (u_n).
Let $\epsilon > 0$ be arbitrary. We want to use 3.2.10 for this proof, so we first smooth out the u_n's. For this purpose we use componentwise convolution: Let $\rho \in \mathcal{D}\big(B_1(0)\big)$, such that $\int_{B_1(0)} \rho = 1$. We define

$$\rho_\delta(x) := \frac{1}{\delta^n}\rho(\frac{x}{\delta})$$

for all $\delta > 0$. Clearly $\operatorname{supp}(\rho_\delta) \subseteq B_\delta(0)$. The mollified sections

$$u_n^\delta := \rho_\delta * u_n$$

(componentwise convolution with ρ) are smooth by 1.3.24 for every $\delta > 0$. We prove that u_n^δ converges to u_n in L^2 as $\delta \to 0$:
In what follows every integral over a section is meant componentwise.

$$\begin{aligned}
\|u_n - u_n^\delta\|_{L^2} &= \int_K \|\big(u_n(x) - \rho_\delta * u_n(x)\big)\|^2 dx = \\
&= \int_K \left\| u_n(x) \underbrace{\int_K \rho(y)dy}_{=1} - \int_{B_\delta(0)} \rho_\delta(z)u_n(x-z)dz \right\|^2 dx = \\
&= \int_K \left\| \int_{B_1(0)} \rho(y)u_n(x) - \rho_\delta(\delta\cdot y)u_n(x-\delta\cdot y)\delta^n dy \right\|^2 dx = \\
&= \int_K \left\| \int_{B_1(0)} \rho(y)\big(u_n(x) - u_n(x-\delta\cdot y)\big)dy \right\|^2 dx \leq \\
&\leq \int_K \int_{B_1(0)} |\rho(y)| \left\| \int_0^\delta \partial_t u_n(x - t\cdot y)\cdot y dt \right\|^2 dydx \leq \\
&\leq \int_K \int_{B_1(0)} |\rho(y)\cdot y|\delta \cdot \operatorname{essup}_{B_\delta(x)}\big(\|\partial_t u_n\|^2\big)dydx = \\
&= \delta \underbrace{\int_K \int_{B_1(0)} |\rho(y)\cdot y| \cdot \operatorname{essup}_{K_\delta}\big(\|\partial_t u_n\|^2\big)dydx}_{=:C(K,u_n,\rho,\delta)}
\end{aligned}$$

where $K_\delta := \{y \mid d(y,K) \leq \delta\}$. Since $K_\delta \subseteq K_{\delta_0}$ for $\delta < \delta_0$ we have $C(K,u_n,\rho,\delta) \leq C(K,u_n,\rho,\delta_0)$, thus u_n^δ converges to u_n in L^2 as $\delta \to 0$. Hence we find a $\delta_n > 0$ such that

$$\|u_n - u_n^{\delta_n}\|_{L^2} \leq \frac{\epsilon}{3}$$

The sequence $(u_n^{\delta_n})_{n\in\mathbb{N}}$ is bounded in $\Gamma^1(\mathbb{R}^n \times \mathbb{K}^q)$: Let α be a multiindex of length ≤ 1

$$\begin{aligned}
\|\partial^\alpha u_n^{\delta_n}(x)\| &= \left\|\partial^\alpha \int_{B_{\delta_n}(0)} \rho_{\delta_n}(z) u_n(x-z)dz\right\| = \\
&= \left\|\partial^\alpha \int_{B_1(0)} \rho(y) u_n(x-\delta_n y)dy\right\| = \\
&= \left\|\int_{B_1(0)} \rho(y)\big(\partial^\alpha u_n\big)(x-\delta_n y)dy\right\| \leq \\
&\leq \int_{B_1(0)} |\rho(y)| \cdot \|\big(\partial^\alpha u_n\big)(x-\delta_n y)\| dy \leq \\
&\underset{1.1.8}{\leq} \|\rho\|_{L^2} \cdot \Big(\int_{B_1(0)} \|\big(\partial^\alpha u_n\big)(x-\delta_n y)\|^2\Big)^{\frac{1}{2}} dy = \\
&= \|\rho\|_{L^2} \cdot \|\partial^\alpha u_n\|_{L^2}
\end{aligned}$$

Hence

$$\begin{aligned}
\|u_n^{\delta_n}\|_{\Gamma^1} &= \sum_{|\alpha|\leq 1} \sup_{x\in K} \|u_n^{\delta_n}(x)\| \leq \\
&\leq \sum_{|\alpha|\leq 1} \|\rho\|_{L^2} \cdot \|\partial^\alpha u_n\|_{L^2} \leq \\
&\underset{3.1.17}{\leq} C\|\rho\|_{L^2}\|u_n\|_1
\end{aligned}$$

Since $\{u_n \mid n \in \mathbb{N}\}$ is bounded in H^1, $u_n^{\delta_n}$ is bounded in Γ^1. Now we choose an E-frame and take a look at the component functions, $u_n^{\delta_n} =: (f_n^1, \ldots, f_n^q)$. From the above we get that (f_n^1) is a bounded sequence in $C^1(\mathbb{R}^n)$, so by 3.2.10 there exists a $C(\mathbb{R}^n)$-convergent subsequence $(f_{n_k}^1)$. The subsequence $u_{n_k}^{\delta_{n_k}}$ of $u_n^{\delta_n}$ is again bounded and we can consider the component sequence $f_{n_k}^2$, which is bounded in $C^1(\mathbb{R}^n)$. We use 3.2.10 again and get a $C(\mathbb{R}^n)$-convergent subsequence $(f_{n_{k_l}}^2)$. The corresponding subsequence of $u_n^{\delta_n}$ now converges in the first two components in $C(\mathbb{R}^n)$. We continue with this procedure until we have constructed a subsequence for which all component sequences converge, thus the whole subsequence converges in $\Gamma(\mathbb{R}^n \times \mathbb{K}^q)$ and since $u_n^{\delta_n}$ is supported in the compact set K for all $n \in \mathbb{N}$, it converges in $L^2(E)$. We call this convergent subsequence $(v_n^{\delta_n})$ and (v_n) the corresponding subsequence of (u_n). For $m, n \in \mathbb{N}$ big enough we have

$$\|v_n^{\delta_n} - v_m^{\delta_m}\|_{L^2} < \frac{\epsilon}{3}.$$

Finally,

$$\|v_n - v_m\|_{L^2} \leq \underbrace{\|v_n - v_n^{\delta_n}\|_{L^2}}_{\leq \frac{\epsilon}{3}} + \underbrace{\|v_n^{\delta_n} - v_m^{\delta_m}\|_{L^2}}_{< \frac{\epsilon}{3}} + \underbrace{\|v_m^{\delta_m} - v_m\|_{L^2}}_{\leq \frac{\epsilon}{3}} < \epsilon$$

2. We show that the inclusion $H^1(E) \hookrightarrow L^2(E)$ is compact.
   Let (u_n) be a bounded sequence in $H^1(E)$. We cover the manifold M by trivializing chart neighborhoods (finitely many suffice, since M is compact)

and choose a partition of unity $\{\chi_i \mid 1 \leq i \leq N\}$ subordinate to this covering. Since $\mathrm{supp}(\chi_1)$ is compact we can use step 1 to construct a convergent subsequence of $(\chi_1 \cdot u_n)$, which defines a subsequence of (u_n). We take this subsequence (u_{n_k}) and consider $(\chi_2 \cdot u_{n_k})$. Again we get a convergent subsequence and continue with χ_3. In finitely many steps we construct a subsequence of (u_n), such that $(\chi_i \cdot u_n)$ converges in the L^2 sense for every i. Therefore $u_n = \sum_{i=1}^n \chi_i \cdot u_n$ has an $L^2(E)$-convergent subsequence.

3. Let $s, t \in \mathbb{N}_0$ such that $t > s$. The inclusion $H^t(E) \hookrightarrow H^s(E)$ is compact: Let (u_n) be a bounded sequence in $H^t(E)$. Because of $t \geq s+1$ and 3.1.5 the sequence (u_n) is also bounded in $H^{s+1}(E)$. Thus $(\nabla^j u_n)$ $(1 \leq j \leq s)$ is a bounded sequence in $H^1(E)$, so we can use step 2 to conclude that $(\nabla^j u_n)$ has an $L^2(E)$-convergent subsequence for all $1 \leq j \leq s$. Hence (u_n) has a subsequence which converges in $H^s(E)$. □

3.2.12 Lemma: *Let M be compact, $m > 0$, $P \in \mathbf{PDO}^m(E, F)$ and $(u_n)_{n\in\mathbb{N}}$ a sequence in $H^m(E)$, such that $\{\|u_n\|_{L^2} + \|Pu_n\|_{L^2} \mid n \in \mathbb{N}\}$ is bounded. Then there exists a subsequence of (u_n) which converges in $L^2(E)$.*

Proof. From the elliptic estimates in 3.2.3 we deduce

$$\|u_n\|_m \leq C\big(\underbrace{\|Pu_n\|_{L^2} + \|u_n\|_{L^2}}_{\text{bounded}}\big) \leq const.$$

Thus (u_n) is bounded in $H^m(E)$. Since $H^m(E)$ embeds compactly into $L^2(E)$ by 3.2.11, (u_n) contains a convergent subsequence. □

3.2.13 Proposition (Poincaré inequality): *Let M be a compact manifold, $m > 0$ and $P \in \mathbf{PDO}^m(E, F)$ elliptic. There exists a constant $C > 0$, such that*

$$\|u\|_{L^2} \leq C\|Pu\|_{L^2}$$

for all $u \in H^m(E)$ which are orthogonal to $\ker(P)$, *i.e*

$$\int_M \langle u, v\rangle_E dV_g = 0 \quad \forall v \in \ker(P)$$

Proof. Denote by $\big(\ker(P)\big)^\perp$ the subspace of sections which are orthogonal to $\ker(P)$.

We argue by contradiction. We assume that for all $C \in \mathbb{R}$ there exists a section $u \in \big(\ker(P)\big)^\perp$, such that

$$\|u\|_{L^2} > C\|Pu\|_{L^2}$$

In particular for every $n \in \mathbb{N}$, there exists a $u'_n \in \big(\ker(P)\big)^\perp$ such that

$$\|u'_n\|_{L^2} > n\|Pu'_n\|_{L^2}$$

We define $u_n := \frac{u'_n}{\|u'_n\|} \in \big(\ker(P)\big)^{\perp}$. P is linear, therefore we get

$$1 = \|u_n\|_{L^2} > n\|Pu_n\|_{L^2}$$

Now we have a sequence $(u_n)_{n\in\mathbb{N}}$ in $\big(\ker(P)\big)^{\perp}$ such that

$$\|u_n\|_{L^2} = 1 \text{ and } \|Pu_n\|_{L^2} < \frac{1}{n}$$

Hence $\{\|u_n\|_{L^2} + \|Pu_n\|_{L^2}\}$ is bounded and we can apply 3.2.12 to get a convergent subsequence (u_{n_k}). We denote the limit by u. Clearly $\|u\|_{L^2} = 1$. $\big(\ker(P)\big)^{\perp}$ is a closed[6] subspace, so $u \in \big(\ker(P)\big)^{\perp}$. Since P is a closed operator we have $u \in \mathrm{Dom}(P)$ and $Pu = \lim_{k\to\infty} Pu_{n_k} = 0$. Thus $u \in \ker(P) \cap \big(\ker(P)\big)^{\perp} = \{0\}$, which contradicts $\|u\|_{L^2} = 1$. $\square$

3.3 The Spectral Theorem

In this section M will always be compact. We prove the spectral theorem and give some short remarks. One very useful feature of the spectral theorem is the functional calculus, which we consider in section 3.4.

3.3.1 Lemma: *Let $k > 0$ and $P \in \mathbf{PDO}^k(E, E)$ be elliptic. Then* $\ker(P)$ *is finite dimensional.*

Proof. Let $\emptyset \neq B \subseteq \ker(P)$ be open and bounded. Since B is open and not empty the dimension of B and the dimension of $\ker(P)$ are the same. We show that B is relatively compact:
Let (u_n) be a sequence in B. Since B is bounded in $L^2(E)$ the set

$$\{\|u_n\|_{L^2} + \|\underbrace{Pu_n}_{=0}\|_{L^2}\} = \{\|u_n\|_{L^2}\}$$

is bounded in $\mathbb{R}$. By 3.2.12 there exists a convergent subsequence.
Thus $C := \overline{B}$ is compact and the cover $\{c + \frac{1}{2}C \mid c \in B\}$ of C contains a finite subcover $\{c_i + \frac{1}{2}C \mid 1 \leq i \leq N\}$. We define $Y := \mathrm{span}(\{c_i \mid 1 \leq i \leq N\})$. Clearly

$$C \subseteq Y + \frac{1}{2}C \subseteq \underbrace{Y + \frac{1}{2}Y}_{=Y} + \frac{1}{2^2}C \subseteq \cdots \subseteq Y + \frac{1}{2^m}C \quad \forall m \in \mathbb{N}_0$$

Since C is bounded we conclude that $C \subseteq \overline{Y} = Y$. Hence

$$\dim\big(\ker(P)\big) = \dim(B) \leq \dim(C) \leq \dim(Y) = N < \infty$$ $\square$

[6] the orthogonal space $S^{\perp}$ of any set S is closed: Let $u_n \in S^{\perp}$ converge to u and $v \in S$ be arbitrary.

$$\langle u, v\rangle = \langle \lim_{n\to\infty} u_n, v\rangle = \lim_{n\to\infty} \underbrace{\langle u_n, v\rangle}_{0} = 0$$

3.3.2 Lemma: *Let $P \in \mathbf{PDO}^k(E, E)$ be elliptic. Then* $\mathrm{im}(P)$ *is closed and*

$$L^2(E) = \mathrm{im}(P) \oplus \ker(P^*)$$

Proof.

- $\mathrm{im}(P) \subseteq \big(\ker(P^*)\big)^\perp$: Let $v \in \mathrm{im}(P)$, i.e. there exists a $u \in H^k(E)$ such that $v = Pu$. Let $w \in \ker(P^*)$ be arbitrary. Then

$$\langle v, w\rangle_{L^2} = \langle Pu, w\rangle_{L^2} = \langle u, \underbrace{P^*w}_{=0}\rangle_{L^2} = 0.$$

  Hence $v \perp w$ for all $w \in \ker(P^*)$.

- $\mathrm{im}(P)$ is closed: Let (v_n) be a sequence in $\mathrm{im}(P)$ which converges to a section v in $L^2(E)$. For each v_n there exists a unique[7] $u_n \in \big(\ker(P)\big)^\perp$ such that $Pu_n = v_n$. We use the Poincaré inequality, 3.2.13 and deduce

$$\|u_n - u_m\|_{L^2} \le C\|P(u_n - u_m)\|_{L^2} = C\|v_n - v_m\|_{L^2}$$

  Together with the elliptic estimates, 3.2.3 we get

$$\|u_n - u_m\|_k \le C'(\|P(u_n - u_m)\|_{L^2} + \|u_n - u_m\|_{L^2}) \le (C' + C)\|v_n - v_m\|_{L^2}$$

  Thus (u_n) is a Cauchy sequence in $H^k(E)$. By 3.1.8 there exists a $u \in H^k(E)$ such that (u_n) converges to u in $H^k(E)$. Hence $(v_n) = (Pu_n)$ converges to Pu. Thus $v = Pu$, i.e. $v \in \mathrm{im}(P)$.

- $\big(\ker(P^*)\big)^\perp \subseteq \mathrm{im}(P)$: We argue by contradiction: Let $u \in \big(\ker(P^*)\big)^\perp$, but $u \notin \mathrm{im}(P)$. By 1.4.5 there exists a $v \in H^k(E)$ such that

$$\langle v, Pw\rangle_{L^2} = 0 \quad \forall w \in H^k(E)$$

  and

$$\langle v, u\rangle_{L^2} = c > 0$$

  From the first equation we conclude that

$$\langle P^*v, w\rangle_{L^2} = \langle v, Pw\rangle_{L^2} = 0 \quad \forall w \in H^k(E)$$

  and since $H^k(E)$ is dense in $L^2(E)$, $P^*v = 0$, i.e. $v \in \ker(P^*)$. Since $u \in \big(\ker(P^*)\big)^\perp$ the second equation shows that $v \notin \big(\ker(P^*)\big)^{\perp\perp} = \overline{\ker(P^*)} = \ker(P^*)$, a contradiction. □

3.3.3 Theorem (Spectral Theorem for selfadjoint elliptic Operators): *Let $k > 0$ and $P \in \mathbf{PDO}^k(E, E)$ be a formally selfadjoint, elliptic PDO. Then:*

(i) The spectrum of P consists only of real eigenvalues of P.

(ii) Every eigenspace is finite dimensional and consists of smooth sections.

[7] this follows from 3.2.13

(iii) $\operatorname{spec}(P)$ *is a closed, countable, discrete and unbounded (and thereby non empty) subset of* $\mathbb{R}$.

(iv) There exists an orthogonal decomposition

$$L^2(E) = \bigoplus_{\lambda \in \operatorname{spec}(P)} \ker(\lambda \cdot id - P)$$

with projections $pr_\lambda \colon L^2(E) \to \ker(\lambda \cdot id - P)$.

(v) $H^k(E) = \{u \in L^2(E) \mid \sum_\lambda \lambda^2 \|pr_\lambda u\|^2_{L^2} < \infty\}$

Proof. (i) All eigenvalues of P are real: Let μ be an eigenvalue with eigenvector v, $\|v\| = 1$. Then by 3.2.9,

$$\mu = \mu\langle v, v\rangle = \langle \mu v, v\rangle = \langle Pv, v\rangle = \langle v, \underbrace{P^*}_{=P} v\rangle = \langle v, \mu v\rangle = \overline{\mu}\langle v, v\rangle = \overline{\mu}$$

$\lambda \in \operatorname{spec}(P)$ means that $(\lambda \cdot id - P)\colon H^k(E) \to L^2(E)$ has no two-sided inverse map from $L^2(E)$ to $H^k(E)$ (see 1.4.16). So $\lambda \cdot id - P$ is not injective or not surjective. If it is not injective, there exists a $0 \neq v \in H^k(E)$ such that $\lambda \cdot v - Pv = 0$, i.e. λ is an eigenvalue of P. If $\lambda \cdot id - P$ is not surjective we have $\operatorname{im}(\lambda \cdot id - P) \neq L^2(E)$. Since $k > 0$ and P is elliptic, we have that $\lambda \cdot id - P$ is elliptic. We use 3.3.2 to conclude that $\ker(\overline{\lambda} \cdot id - P^*) = \ker(\lambda \cdot id - P)^* \neq \{0\}$, which means that $\overline{\lambda}$ is an eigenvalue of $P^* = P$. But all eigenvalues of P are real, so $\lambda = \overline{\lambda}$ is again an eigenvalue of P. Altogether $\operatorname{spec}(P)$ consists only of real eigenvalues.

(ii) Let λ be an eigenvalue of P. Let $v \in \ker(\lambda \cdot id - P)$, i.e. $(\lambda \cdot id - P)v = 0$. Since $k > 0$ we have

$$\sigma_k(\lambda \cdot id - P) = \sigma_k(P)$$

We use Weyl's Lemma, 3.2.6 and smoothness of 0 to conclude that v is smooth. The eigenspace $\ker(\lambda \cdot id - P)$ is finite dimensional because of 3.3.1 and the ellipticity of $\lambda \cdot id - P$.

(iii) Let $t \in \mathbb{R} \setminus \operatorname{spec}(P)$. Thus $t \cdot id - P$ has a bounded inverse, which we call T. Clearly T is invertible. T is selfadjoint:

$$\begin{aligned} T^* &= \big((t \cdot id - P)^{-1}\big)^* \underset{1.4.13}{=} \big((t \cdot id - P)^*\big)^{-1} = \\ &= (\overline{t} \cdot \underbrace{id^*}_{\underset{1.4.13}{=} id} - P^*)^{-1} = (t \cdot id - P)^{-1} = T \end{aligned}$$

T is compact: Let v_n be a bounded sequence in $L^2(E)$. We have to show that $(u_n) := (Tv_n)$ admits an L^2-convergent subsequence. Since T is bounded, (u_n) is a bounded sequence in $L^2(E)$. We have

$$v_n = T^{-1}u_n = (t \cdot id - P)u_n$$

As in part (ii) from $k > 0$ it follows that $(t \cdot id - P)$ is elliptic, so we can use the elliptic estimates, 3.2.3 and the L^2-boundedness of u_n and v_n to conclude that u_n is bounded in $H^k(E)$:

$$\|u_n\|_k \leq C\big(\|(t \cdot id - P)u_n\|_{L^2} + \|u_n\|_{L^2}\big) = C\big(\underbrace{\|v_n\|_{L^2}}_{\text{bounded}} + \underbrace{\|u_n\|_{L^2}}_{\text{bounded}}\big)$$

Finally we use 3.2.11 to conclude that (u_n) has an L^2-convergent subsequence.
The Spectral Theorem for compact selfadjoint operators, 1.4.18 shows that $\mathrm{spec}(T)$ is bounded, countable, closed and has its only accumulation point at 0, all $x \in \mathrm{spec}(T) \setminus \{0\}$ are eigenvalues. The eigenvalues of P and T correspond in the following way:

$$\begin{aligned}
\lambda \text{ is an eigenvalue of } P \quad &\Leftrightarrow \quad \ker(\lambda \cdot id - P) \neq \{0\} \\
&\Leftrightarrow \quad \ker\big(\lambda \cdot id - (t \cdot id - T^{-1})\big) \neq \{0\} \\
&\Leftrightarrow \quad \exists v \neq 0 : \ (\lambda - t)v + T^{-1}v = 0 \\
&\Leftrightarrow \quad \exists v \neq 0 : \ v = T\big((t-\lambda)v\big) = (t-\lambda)Tv \\
&\Leftrightarrow \quad \exists v \neq 0 : \ \frac{1}{t-\lambda}v = Tv \\
&\Leftrightarrow \frac{1}{t-\lambda} \text{ is an eigenvalue of } T
\end{aligned}$$

Or equivalently

$$\mathrm{spec}(P) = \{t - \frac{1}{\mu} \mid \mu \in \mathrm{spec}(T) \setminus \{0\}\}$$

From this we conclude that $\mathrm{spec}(P)$ is closed, countable and unbounded, since $\mathrm{spec}(T)$ is closed, countable and has accumulation point 0. In $\mathrm{spec}(T)$ there does not exist any other accumulation point, thus $\mathrm{spec}(P)$ is discrete.

(iv) 1.4.18 also gives that

$$L^2(E) = \bigoplus_{\mu \in \mathrm{spec}(T) \setminus \{0\}} \ker(\mu \cdot id - T)$$

In part (iii) we saw that the eigenvectors of T to the eigenvalue μ are precisely the eigenvectors of P to the eigenvalue $\lambda := t - \frac{1}{\mu}$. Hence

$$L^2(E) = \bigoplus_{\mu \in \mathrm{spec}(T) \setminus \{0\}} \ker\big((t - \frac{1}{\mu}) \cdot id - P\big)$$

and further (we use the bijection between eigenvalues of T and P)

$$L^2(E) = \bigoplus_{\lambda \in \mathrm{spec}(P)} \ker(\lambda \cdot id - P)$$

(v) Let $u \in H^k(E)$, then $Pu \in L^2(E)$. By part (iv) we may write every L^2-section v as sum over its projections on the eigenspaces of P. We use the orthogonality of the different eigenspaces to conclude

$$\begin{aligned}\sum_\lambda \lambda^2 \|pr_\lambda u\|_{L^2}^2 = \sum_\lambda \|\lambda \cdot pr_\lambda u\|_{L^2}^2 &= \\ &= \left\| \sum_\lambda P(pr_\lambda u) \right\|_{L^2}^2 = \\ &= \left\| \sum_\lambda pr_\lambda P u \right\|_{L^2}^2 = \\ &= \|Pu\|_{L^2}^2 < \infty\end{aligned}$$

Conversely: Let $u \in L^2(E)$ such that $\sum_\lambda \lambda^2 \|pr_\lambda u\|_{L^2}^2 < \infty$. This condition guarantees that $v := \sum_\lambda \lambda \cdot pr_\lambda u$ is in $L^2(E)$ and $v_n := \sum_{|\lambda| \leq n} \lambda \cdot pr_\lambda u$ converge to v in $L^2(E)$. We define $u_n := \sum_{|\lambda| \leq n} pr_\lambda u$ and observe that

$$Pu_n = P\left(\sum_{|\lambda|\leq n} pr_\lambda u\right) = \sum_{|\lambda|\leq n} P(pr_\lambda u) = \sum_{|\lambda|\leq n} \lambda \cdot pr_\lambda u = v_n$$

(u_n) clearly converges to u in $L^2(E)$. We use the elliptic estimates, 3.2.3 to show that (u_n) is a Cauchy sequence in $H^k(E)$:

$$\begin{aligned}\|u_n - u_m\|_k &\leq C\big(\|P(u_n - u_m)\|_{L^2} + \|u_n - u_m\|_{L^2}\big) = \\ &= C\big(\underbrace{\|v_n - v_m\|_{L^2}}_{\to 0} + \underbrace{\|u_n - u_m\|_{L^2}}_{\to 0}\big) \to 0\end{aligned}$$

By completeness of H^k, (u_n) converges in H^k. Since (u_n) already converges to u in L^2, u has to be in H^k. $\square$

3.3.4 Remark: 3.3.3, (iv)+(v) show that we may write

$$id_{L^2(E)} = \sum_\lambda pr_\lambda$$

on $L^2(E)$ and

$$P = \sum_\lambda \lambda \cdot pr_\lambda$$

on $\mathrm{Dom}(P) = H^k(E)$.

In the next example we see that the Fourier decomposition of periodic functions is just a spectral decomposition of an elementary PDO on the one dimensional sphere.

3.3.5 Example (Fourier decomposition): Let $M = S^1$ with the induced metric from $\mathbb{R}^2$ (thus $dV_g = d\theta$, where θ is the angle in polar coordinates), $E = S^1 \times \mathbb{C}$ with the (standard) hermitian metric

$$\langle (p, z_1), (p, z_2) \rangle := z_1 \overline{z_2} \in \mathbb{C}$$

and

$$P := -i\frac{\partial}{\partial\theta} \colon \Gamma^\infty(S^1 \times \mathbb{C}) \to \Gamma^\infty(S^1 \times \mathbb{C})$$

P is elliptic: Since P is clearly of first order we calculate $\sigma_1(P)$. Let $u \in \Gamma^\infty(S^1 \times \mathbb{C}) \cong C^\infty(S^1, \mathbb{C})$, $f \in C^\infty(S^1, \mathbb{R})$ and $\xi := df$. Then

$$\begin{aligned}
[P, f]u &= -i\frac{\partial}{\partial\theta}(f \cdot u) - f \cdot \left(-i\frac{\partial}{\partial\theta}\right) = \\
&= -i\frac{\partial f}{\partial\theta} \cdot u - f \cdot i\frac{\partial u}{\partial\theta} + f \cdot i\frac{\partial u}{\partial\theta} = \\
&= -i\frac{\partial f}{\partial\theta} \cdot u
\end{aligned}$$

Hence

$$\sigma_1\big(P\big)(\xi_p) = [P, f]\big|_p = -i\frac{\partial f}{\partial\theta}\bigg|_p \cdot = -i \cdot df\left(\frac{\partial}{\partial\theta}\right)\bigg|_p \cdot = -i\xi_p\left(\frac{\partial}{\partial\theta}\bigg|_p\right)\cdot$$

Since $\{d\theta|_p\}$ is an orthonormal basis of the one dimensional vector space $(T_pS^1)^*$ and $\xi_p \in (T_pS^1)^*$ we may write $\xi_p = a \cdot d\theta|_p$, where $a \in \mathbb{R} \setminus \{0\}$ if $\xi_p \neq 0 \in (T_pS^1)^*$. Finally

$$\sigma_1\big(P\big)(\xi_p) = -i \cdot a \cdot \underbrace{d\theta_p\left(\frac{\partial}{\partial\theta}\bigg|_p\right)}_{=1} \cdot \neq 0 \qquad \forall \xi_p \neq 0$$

P is formally selfadjoint: Let $u, v \in \Gamma_c^\infty(S^1 \times \mathbb{C}) \cong C^\infty(S^1, \mathbb{C})$ be arbitrary.

$$\begin{aligned}
\int_{S^1} \langle u, Pv\rangle d\theta &= \int_{S^1} u \cdot \overline{Pv} d\theta = \\
&= \int_{S^1} u \cdot \overline{-i\frac{\partial v}{\partial\theta}} d\theta = \\
&= \int_{S^1} u \cdot i\frac{\partial \overline{v}}{\partial\theta} d\theta = \\
&= \int_{S^1} i\frac{\partial}{\partial\theta}(u \cdot \overline{v}) d\theta - \int_{S^1} i\frac{\partial u}{\partial\theta} \cdot \overline{v} d\theta = \\
&= \underbrace{i \cdot u \cdot \overline{v}\big|_0^{2\pi}}_{=0} + \int_{S^1} Pu \cdot \overline{v} d\theta = \\
&= \int_{S^1} \langle Pu, v\rangle d\theta
\end{aligned}$$

Hence $P^* = P$.
Eigenvalues and eigenfunctions of P: From

$$-i\frac{\partial}{\partial\theta}u_\lambda = \lambda \cdot u_\lambda$$

it follows that

$$u_\lambda(\theta) = C \cdot \exp(i\lambda\theta)$$

The u_λ's are functions on the circle, i.e. $u_\lambda(0) = u_\lambda(2\pi)$, thus $\lambda \in \mathbb{Z}$. By the spectral theorem, 3.3.3 we get that

$$L^2(S^1 \times \mathbb{C}) = \bigoplus_{k\in\mathbb{Z}} \ker(k \cdot id - P) = \bigoplus_{k\in\mathbb{Z}} \operatorname{span}_{\mathbb{C}}\{\underbrace{\exp(ik\theta)}_{=:e_k}\}$$

and

$$f = \sum_{k\in\mathbb{Z}} pr_k f$$

where

$$\big(pr_k f\big)(\theta) = f_k \cdot \exp(ik\theta) \quad \text{with} \quad f_k := \langle f, e_k\rangle = \int_{S^1} f(\theta) \cdot \exp(-ik\theta) d\theta$$

which is the usual complex Fourier decomposition for periodic functions.

3.4 The Functional Calculus

The functional calculus is a useful tool to define functions of an operator, which we will use to find solutions for initial value problems where the corresponding differential operator is elliptic in the space-directions. We use the functional calculus in chapter 4 to calculate solutions for heat and wave equations.
One may define a functional calculus in many different ways (and in a more general setting), for which we refer to [Con85], [KR83] and [RS80]. In this section M is compact and $P \in \mathbf{PDO}^k(E, E)$ is an elliptic PDO.

3.4.1 Definition: Let $f\colon \operatorname{spec}(P) \to C^\infty(I)$ (where $I \subseteq \mathbb{R}$ is an interval) and $\{\phi_n \mid n \in \mathbb{N}\}$ a basis of $L^2(E)$ of eigenfunctions of P (which exists because of 3.3.3). We define the linear operator $f(P)\colon \operatorname{Dom}(P) \to L^2(E)$ by

$$f(P, t)\phi_n := f\big(\lambda_n\big)(t) \cdot \phi_n$$

where λ_n is the eigenvalue of P for the eigenfunction ϕ_n. The map $f \mapsto f(P, t)$ is called *functional calculus* for P.

3.4.2 Remark: If the $f(\lambda_n)$ $(n \in \mathbb{N})$ are uniformly bounded functions, the operators $f(P, t)$ are bounded for all $t \in I$ and thus defined on all of $L^2(E)$ by the formula

$$f(P, t)u = \sum_{n\in\mathbb{N}} a_n \cdot f\big(\lambda_n\big)(t) \cdot \phi_n$$

where $u = \sum_{n\in\mathbb{N}} a_n \cdot \phi_n$.

The following theorem will provide a solution $u\colon M \times [0, \infty[\to E$ to many initial value problems of the form

$$\begin{aligned} D_t u(x, t) + P u(x, t) &= 0 \\ u(x, 0) &= v(x) \in H^k(E) \end{aligned} \tag{3.13}$$

where D_t is a linear PDO in the t-variable.

3.4.3 Theorem: *Let $f(\lambda_n) \in C^\infty([0,\infty[)$ solve the equation*

$$\begin{aligned} D_t f(\lambda_n) + \lambda_n \cdot f(\lambda_n) &= 0 \\ f(\lambda_n)(0) &= 1 \end{aligned} \tag{3.14}$$

for all $n \in \mathbb{N}$ and let $f(\lambda_n)$ be uniformly bounded. Then $f(P,t)v \in H^k(E)$ is the unique solution of (3.13).

Proof. Let $\{\phi_n \mid n \in \mathbb{N}\}$ be an orthonormal basis of $L^2(E)$ of eigenfunctions of P (see 3.3.3). Thus we may write v as

$$v = \sum_{n\in\mathbb{N}} a_n \cdot \phi_n$$

We claim that $f(P,t)v \in H^k(E)$ for all $t \in [0,\infty[$: Indeed, since $v \in H^k(E)$ by 3.3.3, (v) we have

$$C := \sum_n \lambda_n^2 \|a_n \cdot \phi_n\|_{L^2}^2 = \sum_n \lambda_n^2 \|pr_{\lambda_n} v\|_{L^2}^2 < \infty$$

and hence

$$\begin{aligned} \sum_n \lambda_n^2 \|pr_{\lambda_n} f(P)v\|_{L^2}^2 &= \sum_n \lambda_n^2 \|a_n f(\lambda_n)\phi_n\|_{L^2}^2 \leq \\ &\leq \sup_{n\in\mathbb{N}} \big(|f(\lambda_n)(t)|\big) \sum_n \lambda_n^2 \|a_n \cdot \phi_n\|_{L^2}^2 = \\ &= \sup_{n\in\mathbb{N}} \big(|f(\lambda_n)(t)|\big) \cdot C < \infty \end{aligned}$$

which means, again by 3.3.3, (v), that $f(P,t)v \in H^k(E)$, for all $t \in [0,\infty[$ since the $f(\lambda_n)$ are uniformly bounded.
$(t,x) \mapsto f(P,t)v(x)$ solves (3.13): Since $f(P,t)v \in H^k(E)$ for each $t \in [0,\infty[$ we may calculate

$$\begin{aligned} P\big(f(P,t)v\big) &= P\left(\sum_{n=0}^{\infty} a_n \cdot f\big(\lambda_n\big)(t) \cdot \phi_n\right) = \\ &= \sum_{n=0}^{\infty} a_n \cdot f\big(\lambda_n\big)(t) \cdot P\phi_n = \\ &= \sum_{n=0}^{\infty} a_n \cdot \underbrace{f\big(\lambda_n\big)(t) \cdot \lambda_n}_{=-D_t f\left(\lambda_n\right)(t)} \cdot \phi_n \end{aligned}$$

Since the partial sums of the right hand side converge uniformly to the infinite sum we may switch differentiation and summation and get

$$\begin{aligned} P\big(f(P,t)v\big) &= -D_t\left(\sum_{n=0}^{\infty} a_n \cdot f\big(\lambda_n\big)(t) \cdot \phi_n\right) = \\ &= -D_t\big(f(P,t)v\big) \end{aligned}$$

And for the initial value:

$$f(P,0)v(x) = \sum_{n=0}^{\infty} a_n \cdot \underbrace{f(\lambda_n)(0)}_{=1} \cdot \phi_n(x) = v(x)$$

Since $f(\lambda_n)$ is the (unique!) solution of the linear ODE (3.14), $(t,x) \mapsto f(P,t)v(x)$ is the unique solution of (3.13), since it is uniquely determined by f, v and the eigenvalues of P. $\square$

Chapter 4

Applications

We give solutions for the heat- and wave-equation for arbitrary elliptic PDOs and provide explicit examples for applications. As usual M denotes a compact oriented Riemannian manifold, E is a Hermitian vector bundle over M and $P \in \mathbf{PDO}^k(E,E)$ is a formally self-adjoint, elliptic operator.

4.1 The Heat Equation

From the last chapter, Theorem 3.4.3 we know that we can find a solution $u\colon M \times [0,\infty[\to E$ for the homogeneous heat equation

$$\begin{aligned} \big(\partial_t + P\big)u(t,x) &= 0 \\ u(0,x) &= v(x) \end{aligned} \tag{4.1}$$

by spectral decomposition of $L^2(E)$ in eigenspaces of P and the solution of the linear ODE

$$\begin{aligned} \big(\partial_t + \lambda_n\big)f_n(t) &= 0 \\ f_n(0) &= 1 \end{aligned} \tag{4.2}$$

which is $f_n(t) := \exp(-\lambda_n t)$. The f_n are uniformly bounded by 1 on $[0,\infty[$, if $\lambda_n \geq 0$ for all $n \in \mathbb{N}$. In this case $\exp(-t\cdot P)v$ is the unique solution of (4.1).

4.1.1 Example (Heating the Stanford Torus[1]): Let $R > r > 0$ and M be a two dimensional torus[2] with external diameter $2(R+r)$ and internal diameter $2(R-r)$, i.e.

$$M = \left\{ \begin{pmatrix} (R + r\cos\alpha)\cos\beta \\ (R + r\cos\alpha)\sin\beta \\ r\sin\alpha \end{pmatrix} \in \mathbb{R}^3 \colon \alpha, \beta \in]-\pi,\pi] \right\}$$

[1] The Stanford Torus is a suggested design for a space habitat, proposed during the 1975 NASA Summer Study. It has the shape of a torus and artificial gravity of about $1g$ is created through rotation. For the purpose of heating the space habitat it would be interesting to know how thermal energy flows from one point to another. For simplicity we will assume that it flows only along the hull of the torus and no energy is radiated into the universe. For more information on the Stanford Torus see [HJ77].

[2] For the Stanford Torus: $R = 830m$, $r = 65m$

in appropriate coordinates. We endow M with the induced metric g from $\mathbb{R}^3$. We consider the Laplace-Beltrami operator $\Delta_g = d^*d\colon C^\infty(M) \to C^\infty(M)$, which is elliptic (see 2.2.12) and formally selfadjoint (see 2.3.10). The temperature flow on M is given by the heat equation for Δ_g (see [Dem06], (10.40)):

$$\partial_t T + \lambda_T \cdot \Delta_g T = 0 \tag{4.3}$$

Here $T \in C^\infty(M \times [0,\infty[)$ is the temperature function and λ_T is a constant which describes the temperature conductivity of the material of M.
By the above it suffices to calculate a spectral decomposition of $L^2(M \times \mathbb{R})$ into eigenspaces of Δ_g. For notational simplicity we consider the even bigger section space $L^2(M \times \mathbb{C})$ and find a spectral decomposition of it. Since the eigenvalues of a self-adjoint operator are real, a real valued initial temperature distribution causes real valued temperature distributions for all $t \in]0,\infty[$. The functions

$$u_{nm}(\alpha,\beta) := e^{i(n\alpha+m\beta)}$$

with $(n,m) \in \mathbb{Z}^2$ are an orthonormal basis of $L^2(M \times \mathbb{C})$ by Fourier series theory (see for example [Heu92], §23). Fortunately these functions are eigenfunctions of Δ_g (this is an extension of the above operator to complex functions by $\Delta_g(u+iv) := \Delta_g u + i\Delta_g v$): By definition of d^* (see 2.2.10 and 2.2.9) and d (see 1.2.34 and 1.2.25) and the chain rule we get

$$\begin{aligned}
\Delta_g u_{nm} &= d^*d(u_{nm}) = \\
&= (-1)^{(2-1)(1-1)+1} * d * d(u_{nm}) = \\
&= - * d * d\left(e^{i(n\alpha+m\beta)}\right) = \\
&= - * d * \left(in \cdot e^{i(n\alpha+m\beta)} d\alpha + im \cdot e^{i(n\alpha+m\beta)} d\beta\right)
\end{aligned}$$

Since $\{\partial_\alpha := (d\alpha)^*, \partial_\beta := (d\beta)^*\}$ is a frame for M, we use (1.10) to conclude

$$\begin{aligned}
&*\big(in \cdot e^{i(n\alpha+m\beta)} d\alpha + im \cdot e^{i(n\alpha+m\beta)} d\beta\big) = \\
&\qquad = in \cdot e^{i(n\alpha+m\beta)} d\beta - im \cdot e^{i(n\alpha+m\beta)} d\alpha
\end{aligned}$$

Thus again by 1.2.34 and (1.10)

$$\begin{aligned}
\Delta_g u_{nm} &= - * d\big(in \cdot e^{i(n\alpha+m\beta)} d\beta - im \cdot e^{i(n\alpha+m\beta)} d\alpha\big) = \\
&= - * \big(- n^2 \cdot e^{i(n\alpha+m\beta)} d\alpha \wedge d\beta + m^2 \cdot e^{i(n\alpha+m\beta)} d\beta \wedge d\alpha\big) = \\
&= *\big((n^2+m^2) \cdot e^{i(n\alpha+m\beta)} d\alpha \wedge d\beta\big) = \\
&= (n^2+m^2) \cdot e^{i(n\alpha+m\beta)} = \\
&= (n^2+m^2) \cdot u_{nm}
\end{aligned}$$

Hence u_{nm} is an eigenfunction of Δ_g to the eigenvalue n^2+m^2 for all $n,m \in \mathbb{Z}$. Since $n^2+m^2 \geq 0$ we get a unique solution of (4.3) for every initial temperature distribution $T_0 \in L^2(M \times \mathbb{R})$ by the considerations in the beginnig of this section. Let for example $T_0 = 1_A$ with

$$A = \left\{ p = \begin{pmatrix} (R + r\cos\alpha)\cos\beta \\ (R + r\cos\alpha)\sin\beta \\ r\sin\alpha \end{pmatrix} : \alpha, \in \left]-\frac{\pi}{8r}, \frac{\pi}{8r}\right[, \beta \in \left]-\frac{\pi}{8R}, \frac{\pi}{8R}\right[\right\}$$

where we assume that $R, r > \frac{1}{8}$ (otherwise we take a number greater than 8 in the definition of A). By this definition A is a rectangle[3] in M with edgelength $\frac{\pi}{4}$ (independent of R and r, the condition $R, r > \frac{1}{8}$ guarantees that there is enough space for A on M). We calculate the coefficients of 1_A:

$$\langle 1_A, u_{nm} \rangle_{L^2} = \int_M 1_A \cdot \overline{u_{nm}} dV_g = \int_A e^{-i(n\alpha + m\beta)} d\alpha \wedge d\beta$$

If $n \neq 0$ and $m \neq 0$ we have

$$e^{-i(n\alpha+m\beta)} d\alpha \wedge d\beta = d\left(\frac{1}{-2in} e^{-i(n\alpha+m\beta)} d\beta - \frac{1}{-2im} e^{-i(n\alpha+m\beta)} d\alpha\right)$$

We may use the Theorem of Stokes, (1.7) to conclude

$$\begin{aligned}
\langle 1_A, u_{nm} \rangle_{L^2} &= \frac{i}{2} \int_{\partial A} \frac{1}{n} e^{-i(n\alpha+m\beta)} d\beta - \frac{1}{m} e^{-i(n\alpha+m\beta)} d\alpha = \\
&= \frac{i}{2} \left(\int_{\frac{\pi}{8R}}^{-\frac{\pi}{8R}} \frac{1}{n} e^{-i\left(n\frac{-\pi}{8r} + m\beta\right)} d\beta - \int_{\frac{\pi}{8r}}^{-\frac{\pi}{8r}} \frac{1}{m} e^{-i\left(n\alpha + m\frac{\pi}{8R}\right)} d\alpha + \right. \\
&\qquad \left. + \int_{-\frac{\pi}{8R}}^{\frac{\pi}{8R}} \frac{1}{n} e^{-i\left(n\frac{\pi}{8r} + m\beta\right)} d\beta - \int_{-\frac{\pi}{8r}}^{\frac{\pi}{8r}} \frac{1}{m} e^{-i\left(n\alpha + m\frac{-\pi}{8R}\right)} d\alpha \right) = \\
&= \frac{i}{2} \left(\frac{1}{n} e^{i\frac{n\pi}{8r}} \frac{1}{-im} e^{-im\beta} \Big|_{\frac{\pi}{8R}}^{-\frac{\pi}{8R}} - \frac{1}{m} e^{-i\frac{m\pi}{8R}} \frac{1}{-in} e^{-in\alpha} \Big|_{\frac{\pi}{8r}}^{-\frac{\pi}{8r}} + \right. \\
&\qquad \left. + \frac{1}{n} e^{-i\frac{n\pi}{8r}} \frac{1}{-im} e^{-im\beta} \Big|_{-\frac{\pi}{8R}}^{+\frac{\pi}{8R}} - \frac{1}{m} e^{i\frac{m\pi}{8R}} \frac{1}{-in} e^{-in\alpha} \Big|_{-\frac{\pi}{8r}}^{+\frac{\pi}{8r}} \right) = \\
&= \frac{1}{2mn} \Big(- e^{i\frac{n\pi}{8r}} \left(e^{i\frac{m\pi}{8R}} - e^{-i\frac{m\pi}{8R}} \right) + e^{-i\frac{m\pi}{8R}} \left(e^{i\frac{n\pi}{8r}} - e^{-i\frac{n\pi}{8r}} \right) - \\
&\qquad - e^{-i\frac{n\pi}{8r}} \left(e^{-i\frac{m\pi}{8R}} - e^{i\frac{m\pi}{8R}} \right) + e^{i\frac{m\pi}{8R}} \left(e^{-i\frac{n\pi}{8r}} - e^{i\frac{n\pi}{8r}} \right) \Big) = \\
&= \frac{-2}{2mn} \left(e^{i\frac{n\pi}{8r}} - e^{-i\frac{n\pi}{8r}} \right) \left(e^{i\frac{m\pi}{8R}} - e^{-i\frac{m\pi}{8R}} \right) = \\
&= \frac{4}{mn} \sin\left(\frac{n\pi}{8r}\right) \sin\left(\frac{m\pi}{8R}\right)
\end{aligned}$$

for $m, n \neq 0$. If $n \neq 0$ and $m = 0$ we have

$$e^{-in\alpha} d\alpha \wedge d\beta = d\left(\frac{1}{-in} e^{-in\alpha} d\beta\right)$$

Again by (1.7) we have

$$\begin{aligned}
\langle 1_A, u_{n0} \rangle_{L^2} &= \frac{i}{n} \int_{\partial A} e^{-in\alpha} d\beta = \\
&= \frac{i}{n} \left(\int_{\frac{\pi}{8R}}^{-\frac{\pi}{8R}} e^{i\frac{n\pi}{8r}} d\beta + \int_{-\frac{\pi}{8R}}^{\frac{\pi}{8R}} e^{-i\frac{n\pi}{8r}} d\beta + \right) = \\
&= \frac{-i}{n} \left(e^{i\frac{n\pi}{8r}} - e^{-i\frac{n\pi}{8r}} \right) \int_{-\frac{\pi}{8R}}^{\frac{\pi}{8R}} 1 d\beta =
\end{aligned}$$

[3]This could be the shape of a heating installation inside the Stanford Torus.

$$
\begin{aligned}
&= \frac{2}{n} \sin\left(\frac{n\pi}{8r}\right) \frac{\pi}{4R} = \\
&= \frac{\pi}{2nR} \sin\left(\frac{n\pi}{8r}\right)
\end{aligned}
$$

for $n \neq 0$. Because of the symmetry of u_{nm} we directly get

$$
\langle 1_A, u_{0m} \rangle_{L^2} = \frac{\pi}{2mr} \sin\left(\frac{m\pi}{8R}\right)
$$

And finally for $n = 0$ and $m = 0$ we have

$$
1 d\alpha \wedge d\beta = d\left(\alpha \cdot d\beta\right)
$$

and thus

$$
\begin{aligned}
\langle 1_A, u_{00} \rangle_{L^2} &= \int_{\partial A} \alpha d\beta = \\
&= \int_{\frac{\pi}{8R}}^{-\frac{\pi}{8R}} \frac{-\pi}{8r} d\beta + \int_{-\frac{\pi}{8R}}^{\frac{\pi}{8R}} \frac{\pi}{8r} d\beta = \\
&= \frac{\pi}{4r} \int_{-\frac{\pi}{8R}}^{\frac{\pi}{8R}} 1 d\beta = \\
&= \frac{\pi^2}{16rR}
\end{aligned}
$$

We may use the function

$$
\operatorname{si}(x_0) := \lim_{x \to x_0} \frac{\sin x}{x}
$$

to get a simple formula for the coefficients in the series expansion:

$$
\langle 1_A, u_{nm} \rangle_{L^2} = \frac{\pi^2}{16rR} \operatorname{si}\left(\frac{n\pi}{8r}\right) \operatorname{si}\left(\frac{m\pi}{8R}\right)
$$

Thus

$$
1_A(\alpha, \beta) = \frac{\pi^2}{16rR} \sum_{n,m \in \mathbb{Z}} \operatorname{si}\left(\frac{n\pi}{8r}\right) \operatorname{si}\left(\frac{m\pi}{8R}\right) \cdot e^{i(n\alpha + m\beta)}
$$

and

$$
\begin{aligned}
T(\alpha, \beta, t) &= e^{-\Delta_g} 1_A(\alpha, \beta, t) = \\
&= \frac{\pi^2}{16rR} \sum_{n,m \in \mathbb{Z}} \operatorname{si}\left(\frac{n\pi}{8r}\right) \operatorname{si}\left(\frac{m\pi}{8R}\right) \cdot e^{-\lambda_T (n^2+m^2)t} e^{i(n\alpha + m\beta)}
\end{aligned}
$$

is the unique L^2-solution of the heat equation. One can see that the mean temperature on M (which is given by $\langle T, 1 \rangle_{L^2} = \frac{\pi^2}{16rR}$) is constant, hence the thermal energy of the system is conserved.

4.2 The Wave Equation

Similarly as in section 4.1 we get a solution $u \colon M \times [0, \infty[\to E$ for the homogeneous wave equation

$$
\begin{aligned}
\left(\partial_t^2 + P\right) u(t, x) &= 0 \\
u(0, x) &= v(x)
\end{aligned} \tag{4.4}
$$

by spectral decomposition of $L^2(E)$ in eigenspaces of P and the solution of the linear ODE

$$\begin{aligned}\big(\partial_t^2 + \lambda_n\big) f_n(t) &= 0 \\ f_n(0) &= 1\end{aligned} \tag{4.5}$$

which is $f_n(t) := \cos(\sqrt{\lambda_n} t)$. The f_n are uniformly bounded by 1 on $[0, \infty[$ and real if $\lambda_n \geq 0$ for all $n \in \mathbb{N}$. In this case $\cos(t\sqrt{P})v$ is the unique solution of (4.4).

4.2.1 Example: Let M be a two dimensional sphere with radius $r > 0$. In appropriate coordinates M is the following subset of $\mathbb{R}^3$:

$$M = \left\{ \begin{pmatrix} r\cos\phi\sin\psi \\ r\sin\phi\sin\psi \\ r\cos\psi \end{pmatrix} \in \mathbb{R}^3 \colon \phi \in]-\pi, \pi], \psi \in [0, \pi] \right\}$$

We consider the Euclidean Laplacian on $\mathbb{R}^3$, $\Delta = -\sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2}$, which is elliptic (see 2.2.6) and formally selfadjoint (use integration by parts). Its restriction to M in the above coordinates is given by[4]

$$\Delta_M = -\frac{1}{r^2 \sin\psi} \frac{\partial}{\partial\psi} \left(\sin\psi \frac{\partial}{\partial\psi} \right) - \frac{1}{r^2 \sin^2\psi} \frac{\partial^2}{\partial\phi^2}$$

Thus Δ_M is elliptic and formally selfadjoint on M with the induced metric[5] from $\mathbb{R}^3$. The *spherical harmonics*

$$Y_{lm}(\psi, \phi) := \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} \cdot P_l^m(\cos\psi) \exp(im\phi) \qquad (l \in \mathbb{N}, -l \leq m \leq l)$$

where P_l^m are the *associated Legendre polynomials* defined by

$$P_l^m(x) := \frac{(-1)^m}{2^l l!} (1 - x^2)^{\frac{m}{2}} \frac{d^{l+m}}{dx^{l+m}} (x^2 + 1)^l$$

are an orthonormal basis of $L^2(M \times \mathbb{R})$ (see [Fli05], p.101-103). In addition the spherical harmonics are eigenfunctions of Δ_M with eigenvalues $\frac{l(l+1)}{r^2}$ (see [Fli05], (11.8) where we use that we have a minus in the definition of the Laplacian), i.e.

$$\Delta_M Y_{lm} = \frac{l(l+1)}{r^2} Y_{lm}$$

Thus the unique solution of

$$\begin{aligned}\big(\partial_t^2 + \Delta_M\big) u(t, \psi, \phi) &= 0 \\ u(0, \psi, \phi) &= v(\phi, \psi)\end{aligned}$$

with

$$v = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}$$

[4] see [Fli05], (9.11) where we used our sign convention for the Laplacian

[5] note that Δ_M is indeed the Laplace-Beltrami operator for this induced metric

is given by

$$\big(\cos(\sqrt{\Delta_M})v\big)(t,\psi,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}Y_{lm}(\psi,\phi)\cos\left(\frac{\sqrt{l(l+1)}}{r}t\right)$$

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Notation

This list includes short descriptions for commonly used symbols in the text. The page number refers to the definition of the symbol, if it is defined in the text.

$*$	Hodge-$*$-operator, p. 18
$[.,.]$	Lie bracket, p. 17
$\cong$	isomorph
Δ	Laplace operator
Δ_∇	Bochner Laplace operator, p. 49
Δ_g	Laplace-Beltrami operator, p. 43
$\Gamma^\infty(E)$	space of smooth sections over the vector bundle E, p. 15
$\Gamma_c^\infty(E)$	space of smoth sections over E with compact support
$\Gamma^k(E)$	space of k-times differentiable sections over the vector bundle E
ι	inclusion map
$\Lambda^k V^*$	space of alternating multilinear maps from V^k to $\mathbb{K}$, p. 16
$\langle \cdot, \cdot \rangle_s$	Sobolev metric of order s, p. 53
$\mathbb{K}$	a field, usually $\mathbb{R}$ or $\mathbb{C}$
$\mathbb{N} = \mathbb{N}_0$	natural numbers including zero
$\mathbb{N}^*$	natural numbers without zero
$\mathbb{N}_0^n$	multi-index
$\mathbf{Hom}(E, F)$	space of vector bundle homomorphism from E to F, p. 28
$\mathbf{Op}(E, F)$	space of $\mathbb{K}$-linear operators from $\Gamma^\infty(E)$ to $\Gamma^\infty(F)$, p. 27
$\mathbf{PDO}^{(m)}$	space of partial differential operators of order $\leq m$, p. 28
$\mathbf{PDO}^m$	space of partial differential operators of order m, p. 39
$\mathcal{B}_X$	Borel σ-Algebra on the set X, p. 11
$\mathcal{D}'(\Omega)$	space of distributions on $\Omega \subseteq \mathbb{R}^n$, p. 20
$\mathcal{D}(\Omega)$	space of test functions on $\Omega \subseteq \mathbb{R}^n$, p. 20

$\mathcal{F}$ or $\widehat{\ }$	Fourier transform, p. 21
$\mathcal{P}(X)$	power set of the set X
$\mathcal{S}'(\Omega)$	space of tempered distributions on $\Omega \subseteq \mathbb{R}^n$, p. 21
$\mathcal{S}(\Omega)$	space of Schwartz functions on $\Omega \subseteq \mathbb{R}^n$, p. 21
$\mathcal{T}^r_s M$	space of (r, s)-tensor fields over the manifold M, p. 16
$\mathfrak{X}(M)$	space of vector fields over the manifold M, p. 15
$\mathrm{Der}(R)$	space of derivations of the ring R, p. 15
$\dim(M)$	dimension of the smooth manifold M, p. 14
$\mathrm{dom}(T)$	domain of the operator T
$\mathrm{essup}(f)$	essential supremum of f
$\mathrm{graph}(T)$	graph of the operator T
$\mathrm{im}(P)$	image of the operator P
$\ker(P)$	kernel of the operator P
$\mathrm{span}(C)$	linear span of the set C
$\mathrm{spec}(T)$	spectrum of the operator T, p. 25
$\mathrm{supp}(u)$	support of the function, section or distribution u, p. 20
∇	linear connection, p. 17
∇_X	covariant derivative by X, p. 17
$\Omega^k(M)$	space of k-forms on the manifold M, p. 17
$\otimes$	tensor product, p. 15
$\Vert\,.\,\Vert_s$	Sobolev norm of order s, p. 53
$\Vert\,.\,\Vert_s^\phi$	chart norm of order s with respect to the chart ϕ, p. 59
$\Vert\,.\,\Vert_{L^p}$	L^p-norm, p. 12
$\Vert\,.\,\Vert_{op}$	operator norm
$\sigma_m(P)$	symbol of $P \in \mathbf{PDO}^m$, p. 39
Σ_n	space of permutations of n numbers
$\widetilde{P}$	analytical realization of the PDO P, p. 58
$\xi\wedge = e_\xi$	exterior product by $\xi \in \Omega^1(M)$, p. 16
A^c	complement of the set A
$ad(f)$	p. 28
Alt	alternator, p. 16
$B_r(x)$	metric ball around x with radius r

$C^\infty(M)$	space of smooth functions from the manifold M to $\mathbb{R}$
d	exterior derivative, p. 17
$div_g(X)$	divergence of $X \in \mathfrak{X}(M)$, p. 43
dV_g	metric volume for the metric g, p. 18
E_b	fibre of the vector bundle (E, B, π) over $b \in B$, p. 15
$F^*\omega$	pullback of the differential form ω under the function F, p. 16
$F_*\omega$	push-forward of the differential form ω under the function F, p. 16
g_k	metric for k-forms, induced by the Riemannian metric g, p. 18
$grad_g(f)$	gradient of $f \in C^\infty(M)$, p. 43
$H^s(\Omega)$	Sobolev space of order s on $\Omega \subseteq \mathbb{R}^n$, p. 21
$H^s(E)$	Sobolev (section) space of order s on the vector bundle E, p. 53
$H^s_c(E)$	space of Sobolev sections of order s on E with compact support
$H^s_{\mathrm{loc}}(E)$	local Sobolev space of order s on the vector bundle E, p. 53
i_v	interior product by a vector v, p. 16
id_X	identity map on the set X
$K \subset\subset U$	K is a compact subset of U
$L^2(E)$	square-integrable sections on the manifold E, p. 52
$L^\infty(X)$	bounded measureable functions from the set X to $\mathbb{R}$, p. 12
$L^p(X)$	p-integrable functions from the set X to $\mathbb{R}$, p. 12
$L^p_{\mathrm{loc}}(X)$	locally p-integrable functions from the topological space X to $\mathbb{R}$
pr_i	projection on the i-th factor of a Cartesian product
T^*	(formal) adjoint operator of T, p. 44 and 24, respectively
$T^r_s E$	(r, s)-tensor bundle over the vector bundle E, p. 16
$T^r_s M$	(r, s)-tensor bundle over the vector bundle TM, p. 16
$T^r_s V$	space of (r, s)-tensors over the vector space V, p. 15
$T_p f$	tangent map of the function f at the point p, p. 14
$T_p M$	tangent space of the manifold M at the point p, p. 14
Tf	tangent map of the function f, p. 14
TM	tangent space of the manifold M, p. 14
V^*	algebraic dual space of the vector space V
$V^\perp$	orthogonal complement of V

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Curriculum Vitae

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Name	Simon Ernst Rößler
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Education

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06/2005	Matura (school leaving examination) passed with honors
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1996–2000	Bundesrealgymnasium Rahlgasse, grammar school
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Employment

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10/2009–02/2010	Tutor at the Faculty of Mathematics, University of Vienna
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