Diplomarbeit

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On the saturation of the nonstationary ideal

Verfasser

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Abstract

This work summarizes the many attempts to understand the saturation of the nonstationary ideal, starting with Solovays Splitting Theorem, proven by a generic ultrapower construction, which asserts that NS_κ cannot be κ-saturated. The natural arising question wheter NS_κ is κ^+ saturated is answered for $\kappa \neq \aleph_1$, even when the ideal is restricted to points of a certain cofinality, following the work of M. Gitik and S. Shelah. Further we prove the consistency of the claim that $NS_{\aleph_1}$ is $\aleph_2$ saturated, using a supercompact cardinal, following the joint work of M. Foreman, M. Magidor and S. Shelah. The last chapter, according to the work of M. Foreman, M. Magidor and D.R. Burke, Y. Matsubara respectively, investigates the saturation of the nonstationary ideal on $P_\kappa(\lambda)$. We prove that this ideal cannot be λ^+ saturated, unless $\kappa = \lambda = \omega_1$.

Contents

Chapter 1

Introduction

1.1 Clubs

Definition 1.1.1. *Let κ be a regular uncountable cardinal. A subset C of κ is a closed unbounded subset of κ (a club) if C is unbounded in κ and if for every sequence $\alpha_0, \alpha_1, ., .,$ of elements of C of length $\gamma < \kappa$ we have $lim_{\xi \to \gamma}\ \alpha_\xi \in C$.*

A subset S of κ is stationary if it has a nonempty intersection with every club of κ

Fact 1.1.2 (*[11], Theorem 8.3.*)**.** *The intersection of fewer than κ club subsets of κ is a club subset*

Definition 1.1.3. *Let $(X_\alpha : \alpha < \kappa)$ be a sequence of subsets of κ. The set*

$$\Delta X_\alpha := \{\xi < \kappa : \xi \in \cap_{\alpha<\xi} X_\alpha\}$$

is called the diagonal intersection of the X_α. Further we can define the dual notion, the diagonal sum of $(X_\alpha : \alpha < \kappa)$:

$$\sum_{\alpha<\kappa} X_\alpha := \{\xi < \kappa\ :\ \xi \in \bigcup_{\alpha<\xi} X_\alpha\}.$$

Fact 1.1.4 (*[11], Lemma 8.4.*)**.** *If C_α, $\alpha < \kappa$ is a sequence of clubs of length κ. Then the diagonal intersection of the C_α's is a club*

Definition 1.1.5. *An ordinal function f on a set $S \subset$ **On** is regressive if $f(\alpha) < \alpha$ for all $\alpha \in S$, $\alpha > 0$.*

Lemma 1.1.6 (**Fodor's Lemma**)**.** *If f is a regressive function on a stationary set $S \subset \kappa$ then there is a stationary set $T \subset S$ and some $\gamma < \kappa$ such that $f(\alpha) = \gamma$ for all $\alpha \in T$.*

Proof. Assume to the contrary that for each $\gamma < \kappa$ the set $\{\alpha \in S : f(\alpha) = \gamma\}$ is nonstationary. Therefore there is for each $\gamma < \kappa$ a club C_γ such that $C_\gamma \cap \{\alpha \in$

$S : f(\alpha) = \gamma\} = \emptyset$ i.e. $f(\alpha) \neq \gamma \forall \alpha \in C_\gamma$ Let $C := \Delta_{\gamma<\kappa} C_\gamma$. C is a club, hence $S \cap C$ is stationary hence contains a nonzero ordinal and for each $\alpha \in S \cap C$ we have $f(\alpha) \neq \gamma$ for every $\gamma < \alpha$, i.e. $f(\alpha) \geq \alpha$ which is a contradiciton □

For stationary sets we have this fundamental combinatorial property discovered by Robert Solovay in the 70's. Its proof uses a technique we will introduce later.

Theorem 1.1.7. *Let κ be a regular uncountable cardinal. Then every stationary subset of κ can be split into κ many disjoint stationary subsets of κ.*

Definition 1.1.8. *A filter $\mathcal{F}$ on a set $S \neq \emptyset$ is a collection of subsets of S such that*

1. $S \in \mathcal{F}$, $\emptyset \notin \mathcal{F}$
2. *If* $X, Y \in \mathcal{F}$ *then* $X \cap Y \in \mathcal{F}$
3. *If* $X \in \mathcal{F}$, $Y \subset S$ *and* $X \subset Y$ *then* $Y \in \mathcal{F}$

If λ is a regular cardinal then the filter is λ-complete if $\mathcal{F}$ is closed under the intersection of less than λ sets of $\mathcal{F}$, i.e. If $\gamma < \lambda$ and for all $\alpha < \gamma$ $X_\alpha \in \mathcal{F}$ then it follows that $\bigcap_{\alpha<\gamma} X_\alpha \in \mathcal{F}$

Definition 1.1.9. *An ideal on a nonempty set S is a collection I of subsets of S such that:*

1. $\emptyset \in I \wedge S \notin I$
2. $X \in I, Y \in I \Rightarrow X \cup Y \in I$
3. $X, Y \in S$, $X \in I \wedge Y \subset X \Rightarrow Y \in I$

If λ is a regular cardinal then the ideal I is λ-complete if it is closed under the union of less than λ sets of I, i.e. if $\gamma < \lambda$ and if for all $\alpha < \gamma$ $X_\alpha \in I$ then $\bigcup_{\alpha<\gamma} X_\alpha \in I$

Note that the concept of an ideal is closely related to the concept of a filter. If F is a filter on S then the set $I := \{S - X : X \in F\}$ is an ideal on S, the so called dual ideal. In a similar way we can define for each ideal I it's dual filter which will be denoted by I^*. Further we write I^+ to denote $P(S) - I$, i.e. $I^+ := \{X \subset S : X \in I\}$ and say that I^+ is the set of the I-positive subsets of S. Lastly if φ is an arbitrary property then we say that φ holds for almost all $x \in S$ if $\{x \in S : \neg\varphi(x)\} \in I$.

Definition 1.1.10. *The collection of all $X \subset \kappa$, such that there is a club $C \subset \kappa$ with $C \subset X$ is a filter F on κ, the closed unbounded filer on κ. Its dual ideal I is the ideal of all nonstationary subsets of κ, the nonstationary ideal NS_κ. Since F is κ complete NS_κ is κ complete as well.*

Definition 1.1.11. *A filter F on κ is normal if it is closed under diagonal intersections:*

$$\forall \alpha < \kappa \, X_\alpha \in F \Rightarrow \Delta_{\alpha<\kappa} X_\alpha \in F$$

An ideal I is normal if its dual filter is normal.

Fact 1.1.12 (*[11], Lemma 8.11.*). *If F is a normal filter on κ that contains all final segments $\{\alpha : \alpha_0 < \alpha < \kappa\}$ then F contains all clubs.*

For example the closed unbounded filter on κ is normal due to Fact 1.1.2. We can generalize the concept of a club on a regular κ:

Definition 1.1.13. *Let A be any set of cardinality at least κ*

- *A set $X \subset P_\kappa(A) = [A]^{<\kappa} := \{X \in P(A) : |X| < \kappa\}$ is unbounded if for every $x \in P_\kappa(A)$ there exists a $y \supset x$ such that $y \in X$*
- *A set $X \subset P_\kappa(A) = [A]^{<\kappa}$ is closed if for every chain $x_0 \subset x_1 \subset ..$ of length $\alpha < \kappa$ of sets in X, the union $\bigcup_{\xi<\alpha} x_\xi$ is in X.*
- *A set $C \subset P_\kappa(A) = [A]^{<\kappa}$ is closed unbounded (a club) if it is closed and unbounded*
- *A set $S \subset P_\kappa(A) = [A]^{<\kappa}$ is stationary if it has a nonempty intersection with each club of $P_\kappa(A)$*

The closed unbounded filter on $P_\kappa(A)$ is the filter generated by the clubs, i.e. the filter consisting of all sets that have a club as a subset.

The natural question now arises if we are allowed to transform important results obtained for clubs on κ to similar statements for the generalized concept of a club. This will later be the main theme of this work. At first we notice that the most basic results still hold in the more general frame:

Fact 1.1.14 (*[11], Theorem 8.22.*). *If $X_0, X_1, ..$ is a sequence of length $\gamma < \kappa$ of clubs of $P_\kappa(A)$ then $\bigcap_{\xi<\gamma}$ is a club in $P_\kappa(A)$*

Definition 1.1.15. *The diagonal intersection of $|A|$-many subsets X_a, $a \in A$ of $P_\kappa(A)$ is defined as:*

$$\Delta_{a\in A} X_a := \{x \in P_\kappa(A) : x \in \bigcap_{a\in x} X_a\}$$

Fact 1.1.16 (Jech). *([11], Theorem 8.24.) If f is a function on a stationary set $S \subset P_\kappa(A)$ and if f(x) $\in$ x for every nonempty set $x \in S$ then there exists a stationary set $T \subset S$ and some $a \in A$ such that f(x) = a fora all $x \in T$.*

Definition 1.1.17. *A set $D \subset P_\kappa(A)$ is directed if for all x and y in D there is a $z \in D$ such that $x \cup y \subset z$.*

Lemma 1.1.18. *If C is a closed subset of $P_\kappa(A)$ then for every directed set D $\subset$ C with $|D| < \kappa$, $\bigcup D \in C$.*

Proof. By induction on $|D|$. If D is finite the lemma holds, thus let $D = \{x_\alpha : \alpha < \gamma < \kappa\}$ be an enumeration of D of length γ. Assume the lemma holds for every directed set of cardinality less than γ. Let $\alpha < \gamma$ and D_α be the smallest directed subset of D such that $x_\alpha \in D_\alpha$ and $D_\alpha \supset \bigcup_{\beta<\alpha} D_\beta$.

We claim now that $|D_\alpha| < \gamma$ for all $\alpha < \gamma$ which we will prove by induction on α: Let $|D_\beta| < \gamma$ for all $\beta < \alpha$. Since D_α is the smallest directed set extending all the D_β's, $\beta < \alpha$, D_α consists of the following elements:

- all the x^i_β which denotes the i-th element of D_β
- all the $x^{i,j}_{\beta,\delta} \supset x^i_\beta \cup x^j_\delta$
- all the $x^{i,j,k,l}_{\beta,\delta,\xi,\theta} \supset x^{i,j}_{\beta,\delta} \cup x^{k,l}_{\xi,\theta}$

$\vdots$

- ...

$\vdots$

It follows that $|D_\alpha| < \gamma$ and the claim is proven. Now let $y_\alpha = \bigcup D_\alpha$. We have $y_\alpha \in C$ for all $\alpha < \gamma$, and $y_\beta \subset y_\alpha$ if $\beta < \alpha$. This implies $\bigcup D = \bigcup_{\alpha<\gamma} y_\alpha \in C$ which finishes the proof. □

Definition 1.1.19. *Let f: $[A]^{<\omega} \to P_\kappa(A)$ be a function. A set $x \in P_\kappa(A)$ is a closure point of f if the following holds for all finite subsets of A:*

$$e \subset x \Rightarrow f(e) \subset x$$

Note that the set C_f of all closure points of f form a closed unbounded subset of $P_\kappa(A)$. As a matter of fact the sets C_f generate the closed unbounded filter of $P_\kappa(A)$:

Lemma 1.1.20. *For every club C in $P_\kappa(A)$ there exists a function f: $[A]^{<\omega} \to P_\kappa(A)$ such that $C_f \subset C$.*

Proof. Since C is a club we can find for each finite subset e of A an infinite set $f(e) \in C$ such that $e \subset f(e)$ and that $f(e_1) \subset f(e_2)$ whenever $e_1 \subset e_2$. Let x be a closure point of f. As $x = \bigcup\{f(e) : e \in [x]^{<\omega}\}$ is the union of a directed subset of C of cardinality $< \kappa$ we have (by Lemma 1.1.18) that $x \in C$, hence $C_f \subset C$. □

Definition 1.1.21. *Let $A \subset B$ and $|A| \geq \kappa$.*

- *For $Y \subset P_\kappa(B)$ the projection of X to A is the set*

$$X \restriction A := \{x \cap A : x \in X\}$$

- *For $Y \subset P_\kappa(A)$ the lifting of Y to B is the set:*

$$Y := \{x \in P_\kappa(B) : x \cap A \in Y\}$$

Lemma 1.1.22 (Menas). *Let $A \subset B$*

- *If S is stationary in $P_\kappa(B)$ then $S\restriction A$ is stationary in $P_\kappa(A)$*
- *If S is stationary in $P_\kappa(A)$ then S^B is stationary in $P_\kappa(B)$*

Proof. The first assertion holds because if C is club in $P_\kappa(A)$ then C^B is club in $P_\kappa(B)$: C^B is closed since if $x_0 \subset x_1 \subset ...$ is a chain of length less than κ in C^B then $(x_\xi \cap A) \in C$ for all $\xi < \gamma$. It follows that $\bigcup_{\xi<\gamma} x_\xi \cap A \in C$ and hence $\bigcup_{\xi<\gamma} x_\xi \in C^B$. C^B is unbounded since for each $x \in P_\kappa(B), x \cap A \in P_\kappa(A)$ and there exists a $x_1 \in C$ such that $x_1 \supset x \cap A$ and it follows that $x_1 \cup (x - A) \supset x$ and $(x_1 \cup (x - A)) \cap A = x_1 \in C$, hence $x_1 \cup (x - A) \in C^B$ and C^B is in fact a club subset of $P_\kappa(B)$. This ends our proof of the first claim since if we would suppose that $S \restriction A$ is not stationary in $P_\kappa(A)$ then there would be a set C which is club in $P_\kappa(A)$ such that $C \cap S \restriction A = \emptyset$ which implies $C^B \cap S = \emptyset$, a contradiction to the fact that C^B is club.

For the second assertion we want to show that if $C \subset P_\kappa(B)$ is club then $C \restriction A$ contains a club which would implie the claim. If C is club in $P_\kappa(B)$ then by the lemma 1.1.20 there is a function $f : [B]^{<\omega} \to P_\kappa(B)$ such that $C_f \subset C$. Let $g : [A]^{<\omega} \to P_\kappa(A)$ be the following function: for each finite subset e of A let x be the smallest closure point of f such that $x \supset e$ and let $g(e) = x \cap A$. We claim that $C_f \restriction A = C_g$: Let $x \in C_f \restriction A, e \subset x$ then $g(e) = x_1 \cap A \subset x \cap A$ due to the definition of g, hence $x \in C_g$. Let $x \in C_g$ and $e_1 \subset x, e_1 \in [A]^{<\omega}$ and $g(e_1) = x_1 \cap A \subset x$. For $e_2 \subset x - (x_1 \cap A)$ there exists $x_2 \in C_f$ such that $g(e_2) = x_2 \cap A \subset x$ and we can continue this way until we obtain a chain $(x_\alpha)_{\alpha<\kappa}$ of elements of C_f such that $x = \bigcup x_\alpha \cap A$ holds. Hence $x \in C_f \restriction A$. This finishes our proof since $C_f \restriction A = C_g$ we have $C_g \subset C \restriction A$. □

Before ending the first section we state a result by Kueker, which will be important later in the text, and which improves Lemma 1.1.20 in the case where $\kappa = \omega_1$. Let A be an arbitrary set, let $[A]^\omega$ denote the set of subsets of A of size ω, then a function $F : [A]^{<\omega} \to A$ is called an operation on A. Moreover we say that $x \in [A]^\omega$ is closed under F if for every finite $e \subset x$ $F(e) \in x$.

Theorem 1.1.23 (Kueker). *Suppose that $C \subset [A]^\omega$ is a club, then there is an operation F on A such that*

$$\{x \in [A]^\omega \ : \ x \text{ is closed under } F\} =: C_F \subset C$$

i.e. the family of the C_F is a basis for the clubs in $P_\omega(A)$.

Proof. Suppose that C is a club. First we define a function $f : [A]^{<\omega} \to C$ with these two properties:

1. $\forall e \in [A]^{<\omega}$ $e \subset f(e)$
2. if $e_1 \subset e_2$ then $f(e_1) \subset f(e_2)$.

This can always be done since C is a club, and as in lemma 1.1.20 we have that $C_f \subset C$. Since $f(e)$ is countable for each e we may write $f(e) = \{f_k(e) : k \in \omega\}$. Let $n \mapsto (k_n, m_n)$ be a bijection.

Now we define an operation as follows: $F(\{\alpha\}) = \alpha + 1$ and if $\alpha_1 < \alpha_2 < ... < \alpha_n$ then let $F(\{\alpha_1, ..., \alpha_n\}) = f_{k_n}(\{\alpha_1, ..., \alpha_n\})$. It suffices to show that $C_F \subset Cf$.

Thus let x be closed under F, let $k \in \omega$ and let $e \in [x]^{<\omega}$. We want to show that $f_k(e) \in x$. If $e = \{\alpha_1, ..., \alpha_m\}$ and $\alpha_1 < ... < \alpha_m$, then let $n \geq m$ be so that $k = k_n$ and $m = m_n$. Since x is unbounded (as $F(\{\alpha\}) = \alpha + 1$), there are $\alpha_{m+1}, ..., \alpha_n \in x$ such that $f_k(\{\alpha_1, ..., \alpha_m\}) = F(\{\alpha_1, ..., \alpha_n\}) \in x$ which is all we need. □

Note that Kuekers theorem becomes a wrong statement for $\kappa > \omega_1$: Suppose that $\kappa > \omega_1$ then let $C := \{x \in P_\kappa(A) : |x| \geq \aleph_1\}$. It's hard not to show that C is a club in $P_\kappa(A)$. On the other hand for each operation F there exists a closed $x \in A$ such that $|x| = \omega$, so $C_F \subset C$ is impossible. Nevertheless we will later show that there is a reasonable generalization of Kuekers theorem for $\kappa > \omega_1$.

1.2 Measurability

The concept of measurability has its origin in the investigations of Henri Lebesgue in the beginning of the last century. The question wheter there exists a measure on an arbitrary set S led to the introduction of measurable cardinals, which turned out to be a too strong concept to be decided in ZFC alone. We start with the definition of the first large cardinals, the inaccessible ones:

Definition 1.2.1. *A cardinal $\kappa > \omega$ is weakly inaccessible if and only if κ is regular and a limit cardinal.*

A cardinal is (strongly) inaccessible if and only if κ is weakly inaccessible and a strong limit i.e. $\forall \lambda < \kappa\ 2^\lambda < \kappa$. We will suppress the adverb strongly from now on, thus an inaccessible cardinal will always be a strongly inaccessible one.

We will later derive the basic properties of inaccessible cardinals. Instead we now present the definition of measure on a set S, axiomatizing some elementary characteristics of capacity. A definition which looks very harmless at first sight is the following:

Definition 1.2.2. *Let S be an infinite set. A (nontrivial, σ-additive) measure on S is a real valued function μ on P(S) such that the following holds:*

(i) $\mu(\emptyset) = 0$

(ii) if $X \subseteq Y$ *then* $\mu(X) \leq \mu(Y)$

(iii) $\mu(\{a\}) = 0$ *for all* $a \in S$

(iv) if $X_n, n = 0, 1, 2\ldots$ are pairwise disjoint then:

$$\mu(\bigcup_{n=0}^{\infty} X_n) = \sum_{n=0}^{\infty} \mu(X_n)$$

The easiest case, and thus probably the first one that comes to mind, would be a two valued measure μ, i.e. a measure with the property that for every $X \subset S$ $\mu(X) = 0$ or $\mu(X) = 1$. If μ is so then

$$U = \{X \subset S : \mu(X) = 1\}$$

defines a nonprincial ultrafilter on S which is moreover $\aleph_1$-complete. Conversely such a filter defines a two-valued measure on S by:

$$\mu(X) = 1 \Leftrightarrow X \in U$$

$$\mu(X) = 0 \Leftrightarrow X \notin U.$$

So these two concepts are equivalent.

The next fact leads the way to the definition of a measurable cardinal:

Fact 1.2.3 (*[11], Lemma 10.2*)**.** *Suppose that κ is the least cardinal such that there is a nonprincipal, $\aleph_1$-complete ultrafilter, then U is even κ-complete.*

Thus the following is reasonable:

Definition 1.2.4. *An uncountable cardinal κ is measurable if there exists a κ-complete, nonprincipal ultrafilter U on κ.*

Fact 1.2.5 (*[11], Lemma 10.4*)**.** *Every measurable cardinal is inaccessible*

If we turn our attention to measures that are not necessarily two-valued, we discover an important property: Let μ be a measure on a set S and consider the ideal:

$$I_\mu = \{X \subset S : \mu(X) = 0\}$$

I_μ is a nonprincipal,$\aleph_1$-complete ideal (the so-called ideal of all null sets) and satisfies these two properties:

1. $\{x\} \in I$ *for every* $x \in S$
2. *every family of pairwise disjoint sets $X \subset S$ that are not in I is at most countable.*

A σ-complete,nonprincipal ideal I on S, for which those two properties hold is called σ-saturated.

Definition 1.2.6. *An uncountable cardinal κ is real valued measurable if there exists a nontrivial (i.e. $\mu(\{\gamma\}) = 0$ for all $\gamma < \kappa$) κ-additive measure μ on κ.*

Note that beeing real valued measurable implies beeing regular since every singular cardinal is the sum of less than κ many small sets, and κ additivity implies that small sets have measure zero.

Lemma 1.2.7. *A κ-complete ideal I on κ is normal iff for every $S_0 \notin I$ and every regressive function f on S_0 there is $S \subset S_0$, $S \notin I$ such that f is constant on S*

Proof. In the forward direction let I be normal and let $f : S_0 \to \kappa$ be a regressive function. Let us assume to the contrary that for each $\gamma < \kappa$, $f^{-1}(\gamma) \in I$. Since I is normal we claim that I is closed under diagonal unions: If $(A_\alpha)_{\alpha<\kappa}$ is a sequence in I then $(\kappa - A_\alpha)$ is in I^* and by normality we know that $\Delta_{\alpha<\kappa}(\kappa - A_\alpha \in I^*$. But $\Delta(\kappa - A_\alpha) = \{\xi < \kappa : \xi \in \bigcap_{\alpha<\xi}(\kappa - A_\alpha\} = \{\xi < \kappa : \xi \notin \bigcup_{\alpha<\xi} A_\alpha\} = \kappa - \sum A_\alpha$ hence $\sum A_\alpha \in I$ which shows that I is indeed closed under diagonal unions. Thus set $S := \sum_{\gamma<\kappa} f^{-1}(\gamma)$ then $S \in I$ and $\sum f^{-1}(\gamma) = \{\xi < \kappa : \xi \in \bigcup_{\gamma<\xi} f^{-1}(\gamma)\}$ which equals, as f is regressive $\bigcup_{\gamma<\kappa} f^{-1}(\gamma) = \kappa \in I$ which is a contradiction. Hence there must be a $\gamma < \kappa$ such that $f^{-1}(\gamma) \in I$.

To prove the backward direction we assume that I is an ideal which satisfies the latter assertion but is not normal. Thus let $(A_\alpha)_{\alpha<\kappa}$ be a sequence in I and assume that $\Delta_{\alpha<\kappa}(\kappa - A_\alpha) \notin I^*$. Consider the regressive function $f : (\kappa - \Delta(\kappa - A_\alpha)) \to \kappa$ defined by $f : \xi \mapsto \alpha < \xi$ where α is the least ordinal such that $\xi \notin \kappa - A_\alpha$. By our assumption f is defined on a set which is not in I hence there exists an $S \subset \kappa - (\Delta(\kappa - A_\alpha)$ such that f is constant on S. For each element $\xi \in S$ we have that $f(\xi) = \alpha$ for a fixed α, hence $\xi \notin \kappa - A_\alpha$, i.e. $S \cap (\kappa - A_\alpha) = \emptyset$ and hence $S \subset \kappa - (\kappa - A_\alpha) = A_\alpha$ which implies that $S \in I$ - a contradiction. □

Definition 1.2.8. *A normal measure on κ is a normal, κ-complete non principal ultrafilter on κ.*

Fact 1.2.9 (*[11], Lemma 10.19*)**.** *If D is a normal measure on κ then every set in D is stationary*

1.3 Ultrapowers

This section gives a brief introduction to ultrapowers, a useful tool to construct from a given model of ZFC a new one. We assume that the reader already feels familiar with model theoretic concepts. V will always denote a transitive model of ZFC.

Definition 1.3.1. *A formula of set theory is Δ_0 (Σ_0, Π_0) if*

1. *it has no quantifiers or*

2. *it is $\varphi \wedge \psi$, $\varphi \vee \psi$, $\neg\varphi$, $\varphi \to \psi$ or $\varphi \leftrightarrow \psi$ where φ and ψ are Δ_0 formulas or*

3. it is $(\exists x \in y)\varphi$ *or* $(\forall x \in y)\varphi$ *where* φ *is a* Δ_0 *formula*

Definition 1.3.2. *A formula is* Σ_{n+1} *if it is of the form* $\exists x_0 \exists x_1 ... \exists x_m \varphi$ *where* φ *is* Π_n *and* Π_{n+1} *if it is of the form* $\forall x_0 \forall x_1 ... \forall x_m \varphi$ *where* φ *is* Σ_n. φ *is* Σ_n^{ZF} *if there is a* ψ *which is* Σ_n *and* $ZF \vdash \varphi \leftrightarrow \psi$. Π_n^{ZF} *is defined in the same way;* Δ_n^{ZF} *means both* Π_n^{ZF} *and* Σ_n^{ZF}.

Fact 1.3.3 *([11], Lemma 12.9.)*. *If* $\varphi(v_0, v_1, \ldots, v_n)$ *is* Σ_0, *M a transitive class and* $x_0, x_1, \ldots, x_n \in M$ *then:*

$$M \models \varphi[x_0, x_1, \ldots, x_n] \leftrightarrow V \models \varphi[x_0, x_1, \ldots, x_n]$$

The same result holds for Δ_1^{ZF}- formulas:

Fact 1.3.4 *([13], Exc. 4.15.)*. *If* φ *is* Δ_1^{ZF}, *M a transitive model of ZF and* $x_0, x_1, \ldots, x_n \in$ *in M then:*

$$M \models \varphi[x_0, x_1, \ldots, x_n] \leftrightarrow V \models \varphi[x_0, x_1, \ldots, x_n]$$

Fact 1.3.5. *„*$rank(v_0) = v_1$*“ is* Δ_1^{ZF}; *„*$V_{v_0} = v_1$*“ is* Π_1^{ZF}

Proof. Let $\varphi(f)$ be the statement „ f is a function and $\forall x \in dom(f)(x \subset dom(f) \wedge f(x) = \bigcup\{f(y) + 1 \ : \ y \in x\}$ “. This can be written by a Δ_0 formula. „$rank(v_0) = v_1$“ can be written as $\exists f(\varphi(f) \wedge (v_0, v_1) \in f)$ or as $\forall f(\varphi(f) \wedge v_0 \in dom(f) \rightarrow (v_0, v_1) \in f)$. The first formula is Σ_1, the second one is Π_1.

„$V_{v_0} = v_1$“ iff $\forall v_2(v_2 \in v_1 \leftrightarrow \exists v_3 \in v_0(rank(v_2) = v_3))$ which is Π_1 □

Definition 1.3.6. *Let* $\#\varphi$ *denote the Gödel number of the formula* φ. *A formula T(x) is called a truth definition if it satisfies these two conditions:*

1. $\forall x(T(x) \rightarrow x \in \omega)$

2. *if* σ *is a sentence then* $\sigma \leftrightarrow T(\#\sigma)$

Fact 1.3.7 (Tarski). *([11], Theorem 12.7) A truth definition does not exist*

Proof. Assume to the contrary that there is a truth definition $T(x)$. Let $\psi(x)$ be the formula $x \in \omega \wedge \neg T(\#\varphi_x(x))$. Let k be such that $\varphi_k(x) = \psi(x)$. Let σ be the sentence $\psi(k)$. Then we have: $\sigma \leftrightarrow \psi(k) \leftrightarrow \varphi_k(k) \leftrightarrow \neg T(\#\varphi_k(k)) \leftrightarrow \neg T(\#\sigma)$ which is a contradiction. □

Although a truth definition does not exist we are able to formalize a satisfaction relation $\models_M^n$ for a transitive class M restricted to Σ_n formulas:

Definition 1.3.8. *We write* $\models_M^0 \varphi[x_1, .., x_k]$ *iff* $\varphi(v_1, .., v_k)$ *is* Σ_0 *and* $\exists y \in M$ *(y is transitive and* $x_1, .., x_k \in y$ *and* $(y, \in) \models \varphi[x_1, .., x_k]$*).*

And recursively $\models_M^{n+1}$ *iff* $\varphi(v_1, .., v_k)$ *is* Σ_{n+1} *say* $\exists v_{k+1} \ldots \exists v_{k+r} \neg\psi(v_1, \ldots, v_{k+r})$ *where* ψ *is* Σ_n, *and* $\exists y_1 \ldots \exists y_r \neg(\models_M^n \psi[x_1, \ldots, x_k, y_1, \ldots, y_r])$.

The definition 1.3.8. above is justified by the following fact:

Fact 1.3.9. *Let φ be a Σ_n formula with variables $v_1,.,.v_k$. Let$\#\varphi$ be its Gödel number. If M is a transitive class and $x_1,.,.,x_k \in M$ then the following holds:*

$$\varphi^M[x_1,.,.,x_k] \leftrightarrow \models^n_M \varphi(x_1,..,x_k).$$

Note that the relation on the right side is a relation in $\#\varphi$, $x_1,.,.,x_k$, n.

Definition 1.3.10. *Let $\mathcal{M}_0 = (M_0,.,.)$ and $\mathcal{M}_1 = (M_1,.,.)$ be structures (possibly of class-size) for a language $\mathcal{L}$. An injective function (which may be a class again) j: $M_0 \to M_1$ is an elementary embedding of $\mathcal{M}_0$ to $\mathcal{M}_1$ if it satisfies the elementarity schema: For any formula $\varphi(v_1,.,.,v_n)$ of $\mathcal{L}$ and $x_1,.,.,x_n \in M_0$:*

$$\mathcal{M}_0 \models \varphi[x_1,.,.,x_n] \leftrightarrow \mathcal{M}_1 \models \varphi[j(x_1),.,.,j(x_n)]$$

Note that the definition above is merely an informal one since the relation $\mathcal{M} \models \varphi[x_0,.,.,x_n]$ is not formalizable in general.

We can overcome this difficulty, using the following stratagem: We don't want the full elementarity schema to hold we just want it to hold for Σ_n formulas for which the satisfaction relation is formalizable.

Definition 1.3.11. *The injective function j is a Σ_n-elementary embedding, denoted j: $\mathcal{M}_0 \prec_n \mathcal{M}_1$ if the elementary schema holds for all the Σ_n-formulas.*

We will often denote a model $\mathcal{M} = (M, \in, ..)$ simply with M, and hope the reader doesn't lose his faith in the overwhelming accuracy of mathematical notation. For inner models we obtain the following basic result:

Fact 1.3.12 (*[12], Proposition 5.1.*)**.** *Let M_0 and M_1 be inner models and let j: $M_0 \prec_1 M_1$. then the following holds:*

- *For any ordinal α $j(\alpha)$ is an ordinal and $j(\alpha) \geq \alpha$*
- *If j is not the identity and if either $M_1 \subset M_0$ or $M_0 \models AC$ then $j(\delta) > \delta$ for some ordinal δ.*
- *For any $n \in \omega$ j: $M_0 \prec_n M_1$.*

Motivated by the third statemant of 1.3.12. an elementary embedding j will always be a Σ_1-elementary embedding, and we will treat them informally.

Definition 1.3.13 (**Ultrapower of the universe**)**.** *Let S be a set, U be an ultrafilter on S and consider the class of the functions with domain S. We define:*

- *$f =^* g \leftrightarrow \{x \in S | f(x) = g(x)\} \in U$*
- *$f \in^* g \leftrightarrow \{x \in S | f(x) \in g(x)\} \in U$*

For each f: $S \to V$ define $[f] := \{g \mid f =^* g \wedge \forall h(h =^* f \to rank(g) \leq rank(h))\}$ and let $Ult_U(V)$ be the class of all $[f]$. The model $(Ult_U(V), \in^*)$ is the so called ultrapower of V, and we will sometimes just write Ult to denote it.

Our next theorem answers the question why ultapowers have been studied, and moreover states the elementary equivalence of V and $Ult_U(V)$:

Fact 1.3.14 (Łoś). *([11], Theorem 12.3.) For every formula $\varphi(v_1, v_2, ..v_n)$ of $\mathcal{L}_\in$ (i.e. the standard language of set theory containing the relation symbol $\in$ and nothing more) and $f_1, .., f_n$ functions : $S \to V$ the following holds:*

$$(Ult_U(V), \in^*) \models \varphi[[f_1], [f_2], ..., [f_n]] \leftrightarrow \{i \in S | \varphi[(f_1(i), ..., f_n(i)]\} \in U$$

Proof. Informal induction over the complexity of φ. □

Note that the fact above is in fact a schema of theorems, one for each φ. The function $j = j_U$ defined by $j_U(a) = [c_a]$ (where $c_a : S \to \{a\}$ is the constant function) is an elementary embedding of V in Ult. Of particular interest is the case where Ult is well founded, i.e.

- Every nonempty set $X \subset Ult$ has an $\in^*$-minimal element
- $\{[g] : g \in^* f\}$ is a set for each f

The second condition is always satisfied and using AC the first one is equivalent to the non-existence of an infinite descending $\in^*$-sequence. We have the following characterization:

Lemma 1.3.15. *U is a σ-complete ultrafilter if and only if (Ult, $\in^*$) is a well founded model.*

Proof. $'\Rightarrow'$: Assume to the contrary that there is such a descending sequence of infinite length $f_1, f_2, ...$ Let $X_n := \{x \in S : f_{n+1}(x) \in f_n(x)\}$. Each $X_n \in U$ and so $\bigcap X_n$ is not empty (since it is an element of U). This would lead to an infinite descending sequence $f_1(x) \ni f_2(x) \ni ....$ for each x $\in \bigcap X_n$ which is a contradiction.

For the reverse direction we assume that there would be a sequence $(X_i)_{i<\omega}$ of elements of U such that $\bigcap X_i \notin U$. For each $k < \omega$ let $g_k : S \to V$ be defined as follows:

$$g_k(i) := \begin{cases} n - k & \text{if } i \in \bigcap_{m<n} X_m - X_n \\ 0 & \text{otherwise} \end{cases}$$

Then $\{i \in S : g_{k+1}(i) < g_k(i)\} \supset \bigcap_{m\leq k} X_m - \bigcap_{n<\omega} X_n \in U$ for $k \in \omega$ and so $([g_n])_{n<\omega}$ is an infinite descending sequence in Ult which is again a contradiction. □

Now if the ultrapower Ult is well founded we can apply the Mostowski Collapsing Theorem to obtain an isomorphic transitive model $M = \pi(Ult)$. Its elements $\pi([f])$ will be, in an abusive way, denoted again with $[f]$ to simplify the notation.

Lemma 1.3.16. *If U is a κ-complete ultrafilter over a measurable cardinal κ and $j : V \prec M$ its canonical embedding then crit(j):= the smallest ordinal such that $j(\alpha) > \alpha$ equals κ*

Proof. At first we show that for each $\alpha < \kappa$ $j(\alpha) = \alpha$. If not then let $\alpha < \kappa$ be the least ordinal such that $j(\alpha) > \alpha$. If $[f] = \alpha$ we have $\{\xi < \kappa : f(\xi) < \alpha\} \in U$ due to our assumption. Let $X_\beta := \{\xi < \kappa : f(\xi) = \beta(< \alpha)\}$ for each $\beta < \alpha$. That's a partition of an element of U into α-many sets and κ-completeness implies that there is a $\beta < \alpha$ such that $X_\beta \in U$ which would implie that $[f] = j(\beta) = \beta = \alpha$ a contradiction.

Next if we consider the diagonal function $d : \kappa \to \kappa$ $d(\alpha) := \alpha$ we observe first that $\alpha < [d]$ for each $\alpha < \kappa$ due to the κ-completeness, further $[d] < j(\kappa)$, which combined gives us $\kappa \leq [d] < \jmath(\kappa)$ and so κ is the critical point of j. □

The converse is also true:

Lemma 1.3.17. *Let j be a nontrivial elementary embedding j: $V \prec M$ then there exists a measurable cardinal*

Proof. If j is such an embedding then there exists an ordinal α such that j(α) $> \alpha$ due to Fact 1.3.12. Set κ:= crit(j) and we have $\kappa > \omega$. What remains to show is that κ is measurable: Let D be the collection of subsets of κ defined as follows:

$$X \in D \leftrightarrow \kappa \in j(X) \quad (X \subset \kappa)$$

We have that $\kappa < j(\kappa)$ and so $\kappa \in D$, also $\emptyset \notin D$ because $j(\emptyset) = \emptyset$. If $X, Y \in D$ then $\kappa \in j(X) \cap j(Y) \leftrightarrow \kappa \in j(X \cap Y) \leftrightarrow X \cap Y \in D$. If $X \in D$ and $Y \supset X$ then $\kappa \in j(X) \subset j(Y) \to Y \in D$. And finally if $X \notin D$ then because of $j(\kappa - X) = j(\kappa) - j(X) = j(\kappa - X)$ and so $\kappa - X \in D$. Also $j(\{\alpha\}) = \{j(\alpha)\} = \{\alpha\} \not\ni \kappa$ for all $\alpha < \kappa$ and so D is a nonprincipal ultrafilter.

It remains to show that D is κ-complete. Thus let $\gamma < \kappa$ and let $(X_\alpha)_{\alpha<\gamma}$ be a sequence of elements of D. We have: $j((X_\alpha)_{\alpha<\gamma}) = (j(X_\alpha))_{\alpha<j(\gamma)} = (j(X_\alpha))_{\alpha<\gamma}$ and so if $X = \bigcap_{\alpha<\gamma} X_\alpha$ we get $j(X) = j(\bigcap_{\alpha<\gamma} X_\alpha) = \bigcap_{\alpha<\gamma} j(X_\alpha)$ and so $\kappa \in j(X)$ hence $X \in D$. □

Remark : The measure $D = \{X \subset \kappa : \kappa \in j(X)\}$ is even a normal one. This follows from the following observation, together with Fact 1.2.10. : Let f be a regressive function on some $X \in D$. Then $(jf)(\kappa) < \kappa$. Let $\gamma = (jf)(\kappa)$. We have: $\gamma = (jf)(\kappa) < \kappa \leftrightarrow \kappa \in \{\alpha : (jf)(\alpha) = \gamma\} \leftrightarrow \kappa \in j(\{\alpha : f(\alpha) = \gamma\}) \leftrightarrow \{\alpha : f(\alpha) = \gamma\} \in D \leftrightarrow f(\alpha) = \gamma$ for almost all $\alpha < \kappa$

Lemma 1.3.18. *Let D be a nonprincipal κ-complete ultrafilter on κ. Then the following are equivalent:*

(i) *D is normal*

(ii) *In the ultrapower $Ult_D(V)$: $\kappa = [d]$ where d is the diagonal function :$\kappa \to \kappa$, $d(\alpha) = \alpha$*

(iii) *For every $X \subset \kappa$, $X \in D$ if and only if $\kappa \in j_D(X)$.*

Proof. (i) $\to$ (ii): If $[f] \in^* [d]$ then there exists a $g : \kappa \to \kappa$ such that $g =^* f$ and g is regressive. Due to Fact 1.3.12.(1) g is constant on a set in D hence

$[f] = [g] = \alpha < \kappa$ and we have $[d] \subset \kappa$. To show that $\kappa \subset [d]$ let $\alpha < \kappa$. Since $\alpha = j(\alpha) = [\alpha] \in^* [d]$ we are done.

(ii) $\to$ (iii): If $X \subset \kappa$ then $X \in D \leftrightarrow d(\alpha) \in X$ for almost all $\alpha \leftrightarrow [d] \in j(X)$. If $[d] = \kappa$ we get $X \in D \leftrightarrow \kappa \in j_D(X)$.

(iii) $\to$ (i): by the remark preceding the lemma □

Lemma 1.3.19. *Let U be a nonprincipal κ-complete ultrafilter on κ, let $M = Ult_U(V)$ and let $j = j_U$ be the canonical elementary embedding of V in M.*

(i) $j(x) = x$ *for every* $x \in V_\kappa$ *and so* $V_\kappa^M = V_\kappa$; $j(X) \cap V_\kappa = X$ *for every* $X \subset V_\kappa$ *and so* $V_{\kappa+1}^M = V_{\kappa+1}$ *and* $(\kappa^+)^M = \kappa^+$.

(ii) $M^\kappa \subset M$, *i.e. every κ-sequence $(a_\alpha)_{\alpha<\kappa}$ of elements of M is itself a member of M*

(iii) $U \notin M$

(iv) $2^\kappa \leq (2^\kappa)^M < j(\kappa) < (2^\kappa)^+$

(v) *If λ is a limit ordinal and if $cf(\lambda) = \kappa$ then $j(\lambda) > lim_{\alpha\to\lambda} j(\alpha)$, if $cf(\lambda) \neq \kappa$ then $j(\lambda) = lim_{\alpha\to\lambda} j(\alpha)$*

(vi) *If $\lambda > \kappa$ is a strong limit cardinal and $cf(\lambda) \neq \kappa$ then $j(\lambda) = \lambda$.*

1.4 Some Large Cardinals

We have already introduced measurable cardinals and derived some easy consequences. In this section we will define some other large cardinals that will later in this text become important. We start with a property which all large cardinals have in common and which is crucial for its theory. The existence of an inaccessible cardinal is not provable in ZFC, further the relative consistency of the statement there exists an inaccessible cardinal is not provable:

Fact 1.4.1 (*[12], Theorem 1.2.*)**.** *Suppose that κ is inaccessible. Then:*

(i) *If $x \subset V_\kappa$ then: $x \in V_\kappa \leftrightarrow |x| < \kappa$*

(ii) $(V_\kappa, \in) \models$ *ZFC*

Fact 1.4.2 (*[11], Theorem 12.12.*)**.** *(i) ZFC $\nvdash$ (Con(ZFC) $\to$ There exists an inaccessible cardinal)*

(ii) ZFC $\nvdash$ (Con(ZFC) $\to$ Con(ZFC + There exists an inaccessible cardinal))

The next large cardinal notion is due to Paul Mahlo

Definition 1.4.3. *Let κ be a limit cardinal*

(i) *κ is weakly Mahlo if and only if $\{\alpha < \kappa : \alpha$ is regular$\}$ is stationary in κ.*

(ii) κ *is (strongly) Mahlo if and only if* $\{\alpha < \kappa : \alpha \text{ is inaccessible}\}$ *is stationary in* κ*. Again the strongly will be supressed*

A weakly Mahlo cardinal is regular (to see this assume to the contrary that there is an unbounded $X \subset \kappa$ such that $|X| < \kappa$. The limit points of $X - |X|$ other than κ form a club which doesn't contain any regular cardinal). Hence a weakly Mahlo cardinal is weakly inaccessible. Moreover a Mahlo cardinal is regular (since it is weakly Mahlo) and obviously a strong limit. Thus a Mahlo cardinal is inaccessible.

Fact 1.4.4 (*[12], Proposition 6.2.*)**.** *The following holds:*

(i) κ *is inaccessible if and only if* $\forall R \subset V_\kappa \exists \alpha < \kappa \ (V_\alpha, \in, R \cap V_\alpha) \prec (V_\kappa, \in, R)$

(ii) κ *is Mahlo if and only if* $\forall R \subset V_\kappa \exists \alpha < \kappa$ *such that* α *is inaccessible and* $(V_\alpha, \in, R \cap V_\alpha) \prec (V_\kappa, \in, R)$

The next large cardinal we want to introduce is the weakly compact: To motivate our definition we need some definitions first.

Definition 1.4.5 (**Arrow notation**)**.** *Let* κ*,* λ *be infinite cardinals, let* n *be a natural number and let* m *be a (finite or infinite) cardinal. We write*

$$\kappa \to (\lambda)^n_m$$

if every partition of $[\kappa]^n$ *into* m *pieces has a homogeneous set of size* λ*, i.e. every* $F : [\kappa]^n \to m$ *is constant on* $[H]^n$ *for some* $H \subset \kappa$ *such that* $|H| = \lambda$

The arrow notation was invented to faciliate the study of possible generalizations of the quite famous theorem of Ramsey, who could prove that $\aleph_0 \to (\aleph_0)^n_k$ holds for all $n, k \in \omega$. An obvious question is if there are uncountable cardinals κ that satisfy $\kappa \to (\kappa)^2_2$, which is the weakest case of $\kappa \to (\kappa)^n_m$ (as $\kappa \to (\lambda)^n_m$ remains true if n, m are made smaller).

Definition 1.4.6. *An uncountable cardinal* κ *is weakly compact if it satisfies* $\kappa \to (\kappa)^2_2$

This is a stronger principle than inaccessibility:

Fact 1.4.7 (*[11], Lemma 9.9.*)**.** *Every weakly compact cardinal is inaccessible*

It is also stronger than Mahloness

Fact 1.4.8 (*[11], Theorem 17.19.*)**.** *Every weakly compact cardinal* κ *is a Mahlo-cardinal. Moreover the set of Mahlo cardinals below* κ *is stationary*

Its name refers to another equivalent definition: It can be shown that for weakly inaccessible cardinals the infinitary language $\mathcal{L}_{\kappa,\kappa}$ satisfies a variation of the Compactness Theorem in Model Theory, which explains its name. We will later use another characterization of weak compactness using some tree terminology:

Definition 1.4.9. *Let κ be a regular uncountable cardinal. κ has the tree property if every tree of height κ whose levels have cardinality $< \kappa$ has a branch of cardinality κ.*

This concept is closely related to an old result by Aronszajn who showed that ω_1 doesn't have the tree property.

Fact 1.4.10 (*[11], Theorem 9.26.*). *(i) If κ is weakly compact then κ has the tree property*

(ii) If κ is inaccessible and has the tree property then κ is weakly compact.

The next lemma determines the position of measurability in the hierachy of the large cardinals that we have already defined:

Lemma 1.4.11. *Every measurable cardinal is weakly compact*

Proof. We already know that κ is inaccessible, thus 1.4.10 tells us that we only have to show that κ has the tree property. Let $(T, <)$ be a tree of height κ with levels of size $< \kappa$, and let U be a κ-complete ultrafilter on T.

We consider the first level $T_1 := \{x_\xi : \xi < \gamma\}\gamma < \kappa$ of the tree T. For each $x_\xi \in T_1$ let U_ξ be the set of all successors of x_ξ in$T$. The $(U_\xi)_{\xi<\gamma}$ form a partition of T of size less than κ, hence due to κ-completeness there is a $\xi \in \gamma$ such that $U_\xi \in U$. We apply the same argument to the set $T_2 \cap U_\xi$, where T_2 is the set of elements of the tree of level 2 and get an $x'_{\xi'} \in U_\xi \cap T_2$ such that the set of the successors of $x'_{\xi'}$ is in U. Continuing this way we obtain a branch of T whose length is κ. This shows that κ has the tree property. □

The last of the large cardinals we will need is the supercompact cardinal.

Definition 1.4.12. *A κ-complete filter F on $P_\kappa(A)$ is normal if for every $a \in A$, $\{x \in P_\kappa(A) : a \in x\} \in F$ and if F is closed under diagonal intersections.*

Lemma 1.4.13. *Let U be a κ-complete ultrafilter on $P_\kappa(A)$ and assume that for every $a \in A$, $\{x \in P_\kappa(A) : a \in x\} \in F$. Then U is normal if and only if for every function $f : P_\kappa(A) \to A$ such that f is regressive, i.e. $f(x) \in x$ holds for each x, f is constant on a set in U.*

Definition 1.4.14. *A cardinal κ is supercompact if for every A such that $|A| \geq \kappa$ there exists a normal ultrafilter on $P_\kappa(A)$.*

We will see soon that a supercompact cardinal is always measurable, in fact it is the κ-th measurable cardinal.

Fact 1.4.15 (*[11], Lemma 20.13.*). *Let U be a normal ultrafilter on $P_\kappa(A)$, let d:$P_\kappa(A) \to P_\kappa(A)$ be the diagonal function i.e. d(x)=x for each x in $P_\kappa(A)$. Then if we consider $[d]$ in $Ult_U(V)$ we have $[d] = \{j(\gamma) : \gamma < \lambda\}= j``\lambda$. Moreover for every $X \subset P_\kappa(A)$:*

$$X \in U \text{ if and only if } j``\lambda \in j(X)$$

It follows that if f and g are functions in $P_\kappa(\lambda)$ then: $[f] = [g] \leftrightarrow X := \{x : f(x) = g(x)\} \in U \leftrightarrow j``\lambda \in j(X) \leftrightarrow j``\lambda \in \{y : jf(y) = jg(y)\} \leftrightarrow jf(j``\lambda) = jg(j``\lambda)$. So

(1) $[f] = [g]$ if and only if $jf(j``\lambda) = jg(j``\lambda)$

Moreover $[f] = [g] \leftrightarrow X := \{x : f(x) \in g(x)\} \in U \leftrightarrow j``\lambda \in j(X) \leftrightarrow j``\lambda \in \{y : jf(y) \in jg(y)\} \leftrightarrow jf(j``\lambda) \in jg(j``\lambda)$ So

(2) $[f] \in [g] \leftrightarrow jf(j``\lambda) \in jg(j``\lambda)$

(1) and (2) implies that

$(*)$ $[f] = jf(j``\lambda)$.

Lemma 1.4.16. *Let U be a normal ultrafilter on $P_\kappa(\lambda)$. Consider the ultrapower $Ult_U(V)$. Then:*

(i) λ is represented in $Ult_U(V)$ by the function $f : x \mapsto ot(x)$, where $ot(x)$ denotes the ordertype of x.

(ii) κ is represented by the function $g : x \mapsto x \cap \kappa$.

Proof. (i): Let $f : x \mapsto ot(x)$. Then $[f] = jf(j``\lambda)$. Due to elementarity of $j$ the function $jf$ still maps each$x$ to $ot(x)$, thus $[f] = jf(j``\lambda) = ot(j``\lambda) = \lambda$. Note that this implies, as $ot(x) < \kappa$ for each x, $j(\kappa) > \lambda$.

(ii): Let $g : x \mapsto \kappa \cap x$. Then $jg : x \mapsto j(\kappa) \cap x$ and hence $[g] = jg(j``\lambda) = j(\kappa) \cap j``\lambda$. Now, by the usual argument, the κ-completeness of U implies that $j(\gamma) = \gamma$ for each $\gamma < \kappa$, whereas $j(\kappa) > \lambda$ by (i). Hence $[g] = j(\kappa) \cap j``\lambda = \kappa$. □

Fact 1.4.17 (*[12], Lemma 22.4.*)**.** *Let U be an ω_1-complete ultrafilter over a set S, and let $j : V \prec M \cong Ult_U(V)$ be the canonical elementary embedding. Then*

(i) If $j``X \in M$ for some set X and $Y \subset M$ is such that $|Y| \leq |X|$, then $Y \in M$.

(ii) For any γ, $j``\gamma \in M \leftrightarrow M^\gamma \subset M$.

(iii) $j``(|S|^+) \notin M$

(iv) $U \notin M$

The last lemma together with $(*)$ imply the following characterization of supercompact cardinals:

Fact 1.4.18 (**Solovay,Reinhardt**)**.** *([11], Lemma 20.14.) Let $\lambda \geq \kappa$ be two cardinals, the latter should be regular. A normal ultrafilter on $P_\kappa(\lambda)$ exists if and only if there exists an elementary embedding $j : V \to M$ such that:*

(i) $j(\gamma) = \gamma$ for all $\gamma < \kappa$;

(ii) $j(\kappa) > \kappa$;

(iii) $M^\lambda \subset M$; *i.e. every sequence* $(a_\alpha : \alpha < \lambda)$ *of elements of* M *is a member of* M

Note that the last theorem is not a result of ZFC since the expression „there exists an elementary embedding j“ is not formalizable (j could be a class). Nevertheless this doesn't make any problem in the $\Rightarrow$ -direction since the elementary embedding j is a definable class. Hence in this direction the Theorem should read: „if there is a normal ultrafilter then the class j defined by $\varphi(x)$ (where $\varphi(x)$ is so chosen that it defines the class j) satisfies the elemetarity schema“.

This doesn't work for the other direction though. To mention a frame for which the theorem remains exprssible consider the usual language $\mathcal{L}_\in$, together with a function symbol j and a predicat for M, and augment ZFC with the formula schema, verifying that j is an elementary embedding satisfying conditions (i)-(ii) in the last theorem, and M is a model of ZFC satisfying (iii). Then this new axiom system proves that there exists a normal measure on $P_\kappa(\lambda)$.

Definition 1.4.19. *Let* κ *be a cardinal that satisfies the conditions (i)-(iii) in the theorem of Solovay and Reinhardt. Then* κ *is called* λ*-supercompact.*

Lemma 1.4.20 (*[11], Lemma 20.16.*)**.** *If* κ *is supercompact then there exists a normal measure* D *on* κ *such that almost every* $\alpha < \kappa$ *(modD) is measurable. In particular,* κ *is the* κ*-th measurable cardinal.*

1.5 Forcing

I will only give a short review of the basic facts of forcing. The reader will find more in Bells [1], Jechs [11] and Kunens [13] books.

Definition 1.5.1. *A forcing notion* P *satisfies the* κ*-chain condition (*κ*-c.c.) if every antichain in* P *is of size less than* κ*.* P *has the countable chain condition (c.c.c.) if* P *satifies the* ω_1*-chain condition.*

Fact 1.5.2 (*[11], Theorem 15.3.*)**.** *If* κ *is a regular cardinal in* V *(where* V *denotes as always the ground model) and if* P *has the* κ*-c.c. then* κ *remains a regular cardinal in the generic extesion* $V[G]$*.*

Fact 1.5.3 (*[12], Proposition 10.5.*)**.** *Let* P *be a notion of forcing,* κ *be a regular cardinal,* P *has the* κ*-c.c., and let* λ *be an arbitrary cardinal. If we let* $\theta = (|P|^{<\kappa})^\lambda$ *and* G *be a generic filter then*

$$V[G] \models 2^\lambda \leq \theta$$

Definition 1.5.4. *A partial order* P *is* κ*-distributive if the intersection of less than* κ *open dense sets is open dense.*

Definition 1.5.5. *Let* κ *be a cardinal. A forcing notion* P *is* κ*-closed if for every* $\lambda < \kappa$*, every descending sequence* $(p_\alpha : \alpha < \lambda)$ *has a lower bound. We say that* P *is* σ*-closed if* P *is* ω_1*-closed.*

Fact 1.5.6 (*[11], Lemma 15.8.*)**.** *If P is κ-closed then it is κ-distributive.*

Fact 1.5.7 (*[11], Theorem 15.6.*)**.** *Let V be the ground model. If the forcing notion P is κ-distributive, if G is a V-generic filter for P, and if $f \in V[G]$ is a function from λ to V (where $\lambda < \kappa$), then $f \in V$. As a consequence such λ have no new subsets in $V[G]$.*

Definition 1.5.8. *Suppose that P and Q are two notions of forcing. The partial order $P \times Q$ is defined as the set of all pairs (p, q) such that $p \in P \wedge q \in Q$ and such that*

$$(p_1, q_1) \leq (p_2, q_2) \text{ if and only if } p_1 \leq p_2 \wedge q_1 \leq q_2$$

If G is a generic filter on $P \times Q$ let

$$G_1 := \{p \in P \,:\, \exists q\,(p, q) \in G\}, \qquad G_2 := \{q \in Q \,:\, \exists p\,(p, q) \in G\}$$

Fact 1.5.9 (**The Product Lemma**)**.** *([11], Lemma 15.9.) Let P and Q be two notions of forcing. Then the following are equivalent:*

- *$G \subset P \times Q$ is V-generic.*
- *$G = G_1 \times G_2$ and $G_1 \subset P$ is V-generic filter and $G_2 \subset Q$ is $V[G_1]$-generic.*

As a consequence $V[G] = V[G_1][G_2]$. Further if G_1 is generic over V and G_2 is generic over $V[G_1]$ then G_1 is generic over $V[G_2]$, and $V[G_1][G_2] = V[G_2][G_1]$

Fact 1.5.10 (*[11], Lemma 15.12.*)**.** *If P and Q are λ-closed partial orders then $P \times Q$ is λ-closed.*

A generalization of product forcing is the iterated forcing:

Definition 1.5.11. *Let P be a notion of forcing and let $\dot{Q} \in V^P$ be a name for a partial ordering in V^P. Then*

(i) $P * \dot{Q} := \{(p, \dot{q}) \,:\, p \in P \wedge \Vdash_P \dot{q} \in \dot{Q}\}$

(ii) $(p_1, \dot{q}_1) \leq (p_2, \dot{q}_2)$ *if and only if* $p_1 \leq p_2 \wedge\, p_1 \Vdash \dot{q}_1 \leq \dot{q}_2$.

Fact 1.5.12 (*[11], Theorem 16.2.*)**.** *Let P and $\dot{Q}$ be as above in the definition*

(i) Let G be a V-generic filter on P, let $Q = \dot{Q}^G$, and let H be $V[G]$-generic filter on Q. Then

$$G * \dot{H} := \{(p, \dot{q}) \in P * \dot{Q} \,:\, p \in G \wedge \dot{q}^G \in H\}$$

is a V-genric filter on $P * \dot{Q}$ and $V[G * H] = V[G][H]$.

(ii) Let K be a V-genric filter on $P * \dot{Q}$. Then

$$G := \{p \in P \,:\, \exists \dot{q}\,(p, \dot{q}) \in K\} \text{ and } H := \{\dot{q}^G \,:\, \exists p (p, \dot{q}) \in K\}$$

are, respectively, a V-generic filter on P and a $V[G]$-generic filter on $Q = \dot{Q}^G$, and $K = G * H$.

Definition 1.5.13. *Let $\{P_i, : i \in I\}$ be a family of partial orders, each having a greatest element 1. Let κ be a regular cardinal. Then the κ-product P of the P_i is the set of all functions p on I with $p(i) \in I$ such that the support s for each p, i.e. the set $\{i \in I : p(i) \neq 1\}$, has size less than κ. The ordering of the κ-product is coordinatewise:*

$$\forall p, q \in P : \ p \leq q \leftrightarrow p(i) \leq q(i) \text{ for each } i \in I$$

Definition 1.5.14. *Let I be a class (or a set) of ordinals. Let $\{P_i : i \in I\}$ be a collection of notions of forcing. The Easton product P of the P_i is the set of all functions $p \in \prod_{i \in I} P_i$, ordered as usual coordinatewise, which satisfy this additional condition:*

For all regular cardinals γ : $|s(p) \cap \gamma| < \gamma$ where $s(p)$ denotes the support

Lemma 1.5.15 (*[11], Lemma 15.19*). *Let $G \times H$ be a generic filter on $P \times Q$, where P is λ^+-closed and Q satisfies the λ^+-chain condition. Then every function $f : \lambda \to M$ in $M[G \times H]$ is in M[H]. In particular:*

$$P^{M[G \times H]}(\lambda) = P^{M[H]}(\lambda)$$

Definition 1.5.16 (**Iteration of arbitrary length**). *Let $\alpha \geq 1$. A forcing notion P_α is an iteration of length α if it is a set of sequences of length α satisfying the following properties:*

1. *If $\alpha = 1$ then there exists a forcing notion Q_0 such that:*
   (a) *P_1 is the set of all sequences of length 1 (p(0)) where $p(0) \in Q_0$.*
   (b) *(p(0)) $\leq_1$ (q(0)) if and only if p(0) $\leq_{Q_0}$ q(0)*
2. *If $\alpha = \beta + 1$ then $P_\beta = P_\alpha \restriction \beta = \{p \restriction \beta : p \in P_\alpha\}$ is an iteration of length β, and there exists a notion of forcing $\dot{Q}_\beta \in V^{P_\beta}$ such that*
   (a) *$p \in P_\alpha$ if and only if $p \restriction \beta \in P_\beta$ and $\Vdash_\beta p(\beta) \in \dot{Q}_\beta$*
   (b) *$p \leq_\alpha q$ if and only if $p \restriction \beta \leq_\beta q \restriction \beta$ and $p \restriction \beta \Vdash_{beta} p(\beta) \leq q(\beta)$.*
3. *If α is a limit ordinal, then for every $\beta < \alpha$, $P_\beta = P_\alpha \restriction \beta = \{p \restriction \beta : p \in P_\alpha\}$ is an iteration of length β and:*
   (a) *the α-sequence constant 1, (1,1,...,1) is in P_α*
   (b) *if $p \in P_\alpha, \beta < \alpha$ and if $q \in P_\beta$ is such that $q \leq_\beta p \restriction \beta$ then $r \in P_\alpha$ where for all $\xi < \alpha, r(\xi) = q(\xi)$ if $\xi < \beta$ and $r(\xi) = p(\xi)$ if $\beta \leq \xi < \alpha$;*
   (c) *$p \leq_\alpha q$ if and only if $\forall \beta < \alpha p \restriction \beta \leq_\beta q \restriction \beta$.*

A general iteration depends not only on the $\dot{Q}_\beta$ but also on the limit stages of the iteration.

Definition 1.5.17. *Suppose that P_α is an iteration of length α, and α is a limit ordinal. We say that P_α is a direct limit if for every α-sequence p:*

$$p \in P_\alpha \Leftrightarrow \exists \beta < \alpha p \restriction \beta \in P_\beta \text{ and } \forall \xi \geq \beta p(\xi) = 1$$

Further P_α is an inverse limit if for every α-sequence p the following holds:

$$p \in P_\alpha \Leftrightarrow \forall \beta < \alpha p \restriction \beta \in P_\beta$$

Fact 1.5.18 (*[11], Theorem 16.30.*)**.** *Suppose that $\kappa \geq \aleph_0$ is regular, and α is a limit ordinal. If P_α is an iteration with the property that for each $\beta < \alpha$ $P_\beta = P_\alpha \restriction \beta$ satisfies the κ-c.c. and if P_α is a direct limit, and either $cf(\alpha) \neq \kappa$ or (if $cf(\alpha) = \kappa$)) there is a stationary set of $\beta < \alpha$ with P_β a direct limit, then P_α has the κ-c.c. too.*

Definition 1.5.19 (**Countable support iteration**)**.** *Let α be an ordinal and let I be the ideal on α, consisting of the at most countable sets. Then we say that the iteration P_α has countable support if and only if the following holds for each limit ordinal $\gamma \leq \alpha$*

$$p \in P_\gamma \text{ if and only if } \forall \beta < \gamma \; p \restriction \beta \in P_\beta \wedge \; s(p) \in I.$$

Where $s(p)$ denotes the support of p, i.e. the set $\{\beta < \gamma : \nVdash_\beta p(\beta) = 1\}$.

We have seen that a c.c.c. notion of forcing doesn't collaps $\aleph_1$. However the c.c.c. property is a quite strong one, therefore set theorists were looking for a weaker characteristic for a notion of forcing to preserve $\aleph_1$. Shelah eventually arrived at this definition:

Definition 1.5.20. *We call a notion of forcing $(P, <)$ proper if for every uncountable cardinal λ, every stationary subset of $[\lambda]^\omega$ remains stationary in the generic extension $V[G]$ via P.*

Lemma 1.5.21 (*[11], Lemma 31.2.*)**.** *Let P be a c.c.c. notion of forcing. Then P is proper*

Fact 1.5.22 (*[11], Lemma 31.3.*)**.** *Let P be an ω-closed notion of forcing then P is proper.*

Lemma 1.5.23 (*[11], Lemma 31.4.*)**.** *Let P be a proper notion of forcing and G be a V-generic filter over P. If A is a set of countable ordinals in V[G] then there is a set B, that is countable in V such that $A \subset B$. As a consequence $\aleph_1$ is preserved in V[G].*

There are some other definitions of proper forcing which we will use later in this text. First we introduce a notion which is often used in the context of proper forcing. Let P be any fixed poset and let λ be a cardinal such that $\lambda > 2^{|P|}$. then we say that λ is *sufficiently large.*

For any cardinal λ we may define the set H_λ which is the collection of sets whose transitive collapse has size less than λ, i.e.

$$H_\lambda := \{x : |tc(x)| < \lambda\}.$$

Each H_λ is transitive and if λ is regular then H_λ is a model of ZFC with the power set axiom deleted.

Definition 1.5.24. *Let P be a notion of forcing, λ a sufficiently large cardinal. Let H_λ denote (in an abusive way) the structure $(H_\lambda, \in, <, P, ..)$ where $<$ is a wellordering of H_λ and let $M \prec H_\lambda$. Then a condition q is (M,P)-generic if and only if for every maximal antichain $A \in M$, the set $A \cap M$ is predense below q, i.e. every $p \leq q$ is compatible with an element of $A \cap M$.*

This gives us a new characterization of properness:

Lemma 1.5.25 (*[14], Theorem 3.2.8.*)**.** *A notion of forcing P is proper if and only if for all sufficiently large cardinals λ there is a club C of countable elementary submodels $M \prec (H_\lambda, \in, <, P, ...)$ such that*

$$\forall p \in M\, \exists q \leq p\, (q \text{ is } (M,P)\text{-generic})$$

Lemma 1.5.26 (*[11], Theorem 31.16.*)**.** *A notion of forcing P is proper if and only if for every $p \in P$, every sufficiently large cardinal λ and every countable $M \prec (H_\lambda, \in, <, P, ...)$ with $p \in M$, there exists a $q \leq p$ that is (M,P)-generic.*

Definition 1.5.27 (**The proper game**)**.** *Let P be a notion of forcing and let $p \in P$. The proper game for P below p is defined as follows: Player I plays P-names $\dot\alpha_n$ for ordinals and player II plays ordinals β_n. II wins if and only if there is some $q \leq p$ such that for each n:*

$$q \Vdash \forall n \exists k\, \dot\alpha_n = \beta_k.$$

Theorem 1.5.28 (*[10], Theorem 2.10.*)**.** *A notion of forcing is proper if and only if for every $p \in P$ player II has a winning strategy for the proper game.*

Definition 1.5.29 (**The semiproper game**)**.** *Let P be a poset $p \in P$. The semiproper game for P below p has the following rules: Player I plays P-names for countable ordinals and player II plays countable ordinals. Again player II wins if and only if there exists a $q \leq p$ such that:*

$$q \Vdash \forall n \exists k \dot\alpha_n = \beta_k$$

A notion of forcing is semiproper if and only if for each $p \in P$ player II has a winning strategy in the semiproper game for P below p.

The notion of semiproperness has again equivalent definitions:

Definition 1.5.30. *Let P be an arbitrary poset, λ be sufficiently large and let $M \prec (H_\lambda, \in, <, P, ...)$ be a countable elementary submodel. A condition $q \in P$ is (M, P)-semigeneric if and only if for every name $\dot{\alpha} \in M$ such that $\Vdash \dot{\alpha}$ is a countable ordinal the following holds:*

$$q \Vdash \exists \beta \in M \dot{\alpha} = \beta$$

Lemma 1.5.31 (*[11], Exercise 37.6.*)**.** *P is a semiproper notion of forcing if and only if the following holds: Let λ be a sufficiently large cardinal, $M \prec (H_\lambda, \in, <)$ countable, and let $p \in P$ be arbitrary such that $p \in M$ and $P \in M$. Then there exists a $q \leq p$ such that q is (M,P)-semigeneric.*

Definition 1.5.32 (**PFA, SPFA**)**.** *The proper forcing axiom (PFA) is the assertion that for any proper notion of forcing P and for any collection D of $\aleph_1$ dense subsets of P there exists a D-generic filer on P i.e. a filter which meets every element of D. The semiproper forcing axiom (SPFA) is defined the same way but with the different assumption that P is semiproper instead of proper.*

A very useful fact is the following:

Theorem 1.5.33 (**Shelah**)**.** *([14], Theorem 3.3.2.] If P_α is a countable support iteration of $\{\dot{Q}_\beta : \beta < \alpha\}$ such that every $\dot{Q}_\beta$ is a proper forcing notion in $V^{P_\alpha \restriction \beta}$, then P_α is proper.*

Chapter 2

The saturation of NS_κ

2.1 The saturation of an ideal

We now turn to the central notion of this work, introduced by Tarski:

Definition 2.1.1. *Let $\kappa > \omega$ and λ be cardinals. A κ-complete ideal I on κ is λ-saturated if and only if for any set $\{X_\alpha : \alpha < \lambda\}$ with $X_\alpha \notin I$ (or equivalently $X_\alpha \in I^+$) for each α, there are $\beta < \gamma < \lambda$ such that $X_\beta \cap X_\gamma \in I^+$. Moreover set*

$$sat(I) := \text{ the least } \lambda \text{ such that } I \text{ is } \lambda-\text{saturated.}$$

We can think of $sat(I)$ as a measure of how close I is to being a maximal ideal. The lower $sat(I)$ is, the more maximal is I and $sat(I) = 2$ if and only if I is maximal. The natural question arising now is the following: Are there ideals I_n, $n \in \omega$ such that $sat(I_n) = n$? The answer is yes assuming the existence of a measurable cardinal. From now on κ will always denote a regular, uncountable cardinal.

Lemma 2.1.2. *Let κ be a measurable cardinal then there exists an ideal I such that I is κ-complete and $sat(I)$=3*

Proof. Let I be a maximal ideal on κ. Let $f : \kappa \times \{0, 1\} \to \kappa$ be a bijection which induces a function, also denoted by $f : P(\kappa \times \{0, 1\}) \to P(\kappa)$. Then $f(I \times \{0\} \cup I \times \{1\})$ generates a κ complete ideal J on κ for which the following holds: $\forall X \subset \kappa(X \in J \Leftrightarrow \exists Y_1 \in I \times \{0\} \exists Y_2 \in I \times \{1\} X \subset f(Y_1) \cup f(Y_2))$. Thus if $X \in J^+$ and $X \subset f(Y_1) \cup f(Y_2)$ then at least one $Y_i \in I^+$ for $i \in \{0, 1\}$. On the other hand if without loss of generality $Y_1 \in I^+$ and $X = f(Y_1) \cup f(Y_2)$, then $X \notin J$.

Now we claim that $sat(J) = 3$. If we set $X_1 = f(\kappa \times \{0\})$ and $X_2 = f(\kappa \times \{1\})$ then $X_1 \notin J$, $X_2 \notin J$ and $X_1 \cap X_2 = \emptyset$. Hence $sat(J) > 2$. But if $X_1, X_2, X_3 \in J^+$ then $X_1 = f(Y_0^1) \cup f(Y_1^1)$ and without loss of generality $Y_0^1 \in I^+$, $X_2 = f(Y_0^2) \cup f(Y_1^2)$ and without loss of generality $Y_1^2 \in I^+$, last $X_3 = f(Y_0^3) \cup f(Y_1^3)$ and we may assume that $Y_0^3 \in I^+$. We conclude that

$X_1 \cap X_3 = f(Y_0^1 \cap Y_0^3) \cup f(Y_1^1 \cap Y_1^3)$ and as $Y_0^1, Y_0^3 \in I^+$ and $sat(I) = 2$ we conclude that $Y_0^1 \cap Y_0^3 \in I^+$ hence $X_1 \cap X_3 \in J^+$ and so $sat(J) = 3$. □

If we generalize this construction in an obvious way we see that ideals of any finite saturation are possible. The next step would be an ideal such that $sat(I) = \omega$. This is impossible though. To faciliate our next proof we briefly introduce some useful notions. Let I be an ideal on a set, which should be without loss of generality a cardinal κ. Then an *I-partition* of κ is a maximal family W of I-positive subsets of κ (i.e. the sets belong to I^+), that are pairwise almost disjoint (i.e. if $X, Y \in W$ and $X \neq Y$ then $X \cap Y \in I$). If $X \in I^+$ then X is an *atom* if there are no $Y, Z \in I^+$, with $Y \cap Z \in I$ such that $X = Y \cup Z$. If X is not an atom then we say that X *splits*. Moreover if $X \in I^+$ then we define $sat(X) := sat(I \restriction X)$ (see 2.2.12 for the definition of $I \restriction X$), and say that $X \in I^+$ is *stable* if $sat(X) = sat(Y)$ for all $Y \subset X$ such that $Y \in I^+$.

Lemma 2.1.3. *Let I be a κ-complete ideal over κ. If sat(I) is infinite then sat(I) is a regular, uncountable cardinal.*

Proof. Assume first that $sat(I) = \omega$. We will construct an antichain in I^+ of size ω, contradicting our assumption. Let T be a tree whose elements are coded by sequences $s \in 2^{<\omega}$, defined inductively like this: The first level of T consists of two almost disjoint elements of I^+, T_0 and T_1, coded by the sequence 0 and 1 respectively. Now if $X \in I^+$ is in T and is coded by an $s \in 2^n$, $n \in \omega$, and if X splits, i.e. $X = Y \cup Z$ then Y and Z are the successors of X in T, coded by $s^\frown 0$ and $s^\frown 1$. If X does not split then X has no successors in T. Now as $sat(I) = \omega$ each so defined tree T has heigth ω and each level of T is finite, hence has an infinite branch b. If we pick for each $X \in b$ the $Y \in T$ such that Y is the offshoot of X (i.e. the Y in T whose code differs from the code of X only in the last digit) then this defines an antichain of length ω.

Now we turn to the regularity of $sat(I) := \mu$. We assume that μ is singular and build again an antichain of length $sat(I)$. First we observe that μ equals the smallest cardinal such that there is no antichain of its length in the Boolean algebra $P(\kappa)/I$. As $P(\kappa)/I$ is dense in its completion B this still holds for B, thus we may say that μ is the least cardinal such that there is no partition of B of its length, and assume its singularity to obtain a contradiction.

Let S be the set of stable elements of B, then it is dense in B as can be seen as follows: Assume not then there would be a descending sequence $u_0 > u_1 > u_2 ...$ with a corresponding decreasing sequence $sat(u_0) > sat(u_1) ...$ which is a contradiction. Let T be a maximal subset of pairwise disjoint elements of S. From its maximality we deduce that $sup\{sat(u) : u \in T\} = \mu$, as for every regular $\lambda < \mu$ there is a partition W of B of size λ, which yields a partition of an element $u \in T$ of size λ.

Now we split into cases

1. If there is an $u \in T$ such that $sat(u) = \mu$ then since $cf(\mu) < \mu$ there is a partition W of u: $W = \{u_\alpha : \alpha < cf(\mu)\}$. Let $(\mu_\alpha)_{\alpha < cf\mu}$ be an increasing sequence with limit μ, then we let for each α W_α be a partition of u_α of size μ_α. $\bigcup_{\alpha < cf\mu} W_\alpha$ is a partition of u of size μ.

2. If for all $u \in T$ $sat(u) < \mu$ holds then again let $(\mu_\alpha)_{\alpha<cf\mu}$ converge to μ. As $sup\{sat(u) : u \in T\} = \mu$ we can define by induction a sequence $(u_\alpha)_{\alpha<cf\mu}$ in T such that each u_α admits a partition W_α of size μ_α. Then $\bigcup_{\alpha<cf\mu} W_\alpha$ is an antichain of length μ in B.

□

The question whether there are any λ-saturated ideals on λ needs at least the assumption of the existence of weakly inaccessible cardinals by the next lemma:

Lemma 2.1.4. *For any λ, there is no λ^+-saturated, λ^+-complete ideal over λ^+*

Proof. Assume that there exists a λ^+-saturated, λ^+-complete ideal I on λ^+. For each $\xi < \lambda^+$ let f_ξ be a function $f_\xi : \lambda \to \lambda^+$ such that $ran(f_\xi) \supset \xi$. Further let $A_{\alpha,\eta}$, $\alpha < \lambda^+ \eta < \lambda$ be a family of subsets of λ^+ defined by:

$$\xi \in A_{\alpha,\eta} \text{ if and only if } f_\xi(\eta) = \alpha$$

The family of the $A_{\alpha,\eta}$'s form a so-called *Ulam-matrix*, that is it satisfies:

(i) $A_{\alpha,\eta} \cap A_{\beta,\eta} = \emptyset$ whenever α and β are nonequal ordinals below λ^+ and $\eta < \lambda$.

(ii) $|\lambda^+ - \bigcup_{\eta<\lambda} A_{\alpha,\eta}| \leq \lambda$ for each $\alpha < \lambda^+$

Now if we fix an $\alpha < \lambda^+$ and consider the $(A_{\alpha,\eta})_{\eta<\lambda}$, then by the λ^+-completeness of I and (ii) there must be an η_α such that $A_{\alpha,\eta} \in I^+$. Hence there is an η and a $W \subset \lambda^+$ of size λ^+ such that $\eta_\alpha = \eta$ for each $\alpha \in W$. Then $(A_{\alpha,\eta})_{\alpha\in W}$ is a pairwise disjoint sequence of elements of I^+ of length λ^+, a contradiction.

□

Lemma 2.1.5. *Let I be a κ-complete ideal over κ then the following holds:*

(i) If I is λ-saturated where $2^{<\lambda} < \kappa$ then κ is measurable.

(ii) If I is κ-saturated and κ is weakly compact, then κ is measurable.

Proof. Assume first that I has an atom A. Then it is easy to check that the set $\{X \subset \kappa : X \cap A \in I^+\}$ is a κ-complete ultrafilter, hence witnessing the measurability of κ. So we can assume that in both cases (i) and (ii), I has no atoms. We shall derive a contradiction.

We build a tree T ordered by $\supset$ inductively: We consider the set of the sequences $s \in 2^{<\kappa}$ and the corresponding $X_s \subset \kappa$ defined inductively like this: Set $X_\emptyset = \kappa$ and if X_s is already defined then let $X_{s^\frown(0)}$ and $X_{s^\frown(1)}$ be two I-positive sets witnessing that X_s splits. Finally if δ is a limit and $s \in 2^\delta$ then let $X_s = \bigcap_{\alpha<\delta} X_{s\restriction\alpha}$. This definition makes sense as I is assumed to have no atoms.

The tree T has the following property:

(*) If $\gamma \leq \kappa$ and $s \in 2^\gamma$, then the set $\{X_{s\restriction\alpha^\frown(i)} : \alpha < \gamma \wedge X_{s\restriction(\alpha+1)}$ is defined $\wedge$ $s(\alpha) \neq i\}$ is pairwise disjoint.

For (i) (*) and λ-saturation imply that T has heigth at most λ. But then κ is the union of the $2^{<\lambda} < \kappa$ many X_s's without tree successor, contradicting the κ-completeness.

For (ii) note that if T had heigth less than κ, then by the inaccessibility of κ and the argument above, the κ-completeness of I would again be violated. Thus we may assume that T has heigth κ and by the weak compactness of κ there must be a branch of length κ. Again with (*) this yields a sequence in I^+ of length κ, contradicting the κ-saturation of I.

□

The next method we want to present is the Kunen-Paris-Forcing. Let U be a normal ultrafilter over a measurable cardinal κ. Let $j : V \prec Ult_U(V) \cong M$ be the canonical elementary embedding. Let P be a forcing poset in V and let $j(P)$, which is again a partial order in V, satisfy two certain conditions, that will enable us to extend the elementary embedding j. Let us assume that:

(i) $j(P) \cong P * \dot{Q}$ under an identification through which

(ii) $j(p) = (p, \dot{1}_Q)$ for every $p \in P$ ($\dot{1}_Q$ denotes the maximal element of $\dot{Q}$).

Now let G be a $j(P)$-generic filter over V. Then by Fact 1.5.10. there exist two filters G_0, G_1, such that $G = G_0 * G_1$, where G_0 is P-generic over V and G_1 is $\dot{Q}^{G_0}$-generic over $V[G_0]$, and $V[G] = V[G_0][G_1]$. Further by (ii)

(*) If $p \in G_0$ then $j(p) \in G$

As $Ult_U(V) \cong M \subset V$ we observe that G is also a $j(P)$-generic filter over M, and now we can construct the generic extension $M[G]$. The crucial point is that with (*) the canonical elementary embedding $j : V \prec M$ can be extended to an elementary embedding $\overline{j} : V[G_0] \prec M[G]$, if we apply j to the names:

Let $x \in V[G_0]$ and let $\dot{x}$ be a name for it. Then due to elementarity $j(\dot{x})$ is a $j(P)$ name and we can therefore set:

$$\overline{j}(x) := j(\dot{x})^{M[G]}$$

The next easy lemma tells us that our concept does make sense:

Lemma 2.1.6. *Let $\overline{j}$ be defined as above. Then $\overline{j}$ is a function, moreover satisfies the elementarity schema, and extends j, i.e. $j(x) = \overline{j}(x)$ for all $x \in V$.*

Proof. At first we notice that $\overline{j}$ is indeed a well-defined function: Suppose that $\dot{x}$, $\dot{y}$ are two different names for $x \in V[G_0]$. Then there is a $p \in G_0$ such that $p \Vdash_P \dot{x} = \dot{y}$ and by (*) $j(p) \in G$ and $j(p) \Vdash_{j(P)} j(\dot{x}) = j(\dot{y})$, hence $(j(\dot{x}))^{M[G]} = (j(\dot{y}))^{M[G]}$ and $\overline{j}$ is well-defined.

Next we show that $\overline{j}$ is elementary: Let $\varphi(v_1, ..., v_n)$ be a formula, let $x_1, ..., x_n \in V[G_0]$, and let $p \in G_0$ be such that $p \Vdash_P \varphi[\dot{x_1}, ..., \dot{x_n}]$. Then $j(p) \in G$ and $j(p) \Vdash_{j(P)} \varphi[j(\dot{x_1}), ..., j(\dot{x_n})]$, hence $M[G] \models \varphi[\overline{j}(x_0), ..., \overline{j}(x_n)]$.

The last thing remaining is to show that $\overline{j}$ extends j: If $x \in V$ then $x = \check{x}^{G_0}$ and $\overline{j}(x) = (j(\check{x}))^{M[G]} = (\check{j(x)})^{M[G]} = j(x)$ and the proof is done. □

Lemma 2.1.7. *Let G, G_1, G_0, j, $\overline{j}$ be as above and let $\overline{U}$ be defined by:*

$$X \in \overline{U} \leftrightarrow \ X \in P(\kappa) \cap V[G_0] \wedge \kappa \in \overline{j}(X)$$

Then $\overline{U}$ is a $V[G_0]$-normal ultrafilter over κ which extends U. With $V[G_0]$-normal we mean that $\overline{U}$ is an ultrafilter on $P(\kappa) \cap V[G_0]$ and further that for any function f in $V[G_0]$ with domain κ, which is almost everywhere regressive (in the sense of the ultrafilter $\overline{U}$), we may conclude that f is almost everywhere constant.

Note now that if the forcing with $\dot{Q}$ in $j(P) \cong P * \dot{Q}$ adds no new subsets of κ then the $\overline{U}$ of the lemma above witnesses the measurability of κ in $V[G]$.

Definition 2.1.8. *Suppose that N is a model of ZFC. We say that J is a N-λ-saturated ideal over κ if and only if:*

(i) J is an ideal on $P(\kappa) \cap N$ such that if $\gamma < \kappa$ and $f \in J^\gamma \cap N$, then $\bigcup_{\alpha<\gamma} f(\alpha) \in J$

(ii) For any function $g : \lambda \to J^+$ which is in N there are $\alpha < \beta < \lambda$ such that $g(\alpha) \cap g(\beta) \in J^+$.

Lemma 2.1.9. *Let κ, λ be regular cardinals, and let P be a notion of forcing which has the λ-c.c. then: If*

$$\Vdash_P \dot{J} \textit{ is a } \check{V}\textit{-}\lambda\textit{-saturated ideal over } \kappa$$

then

$$I := \{X \subset \kappa : \Vdash_P \check{X} \in \dot{J}\}$$

is a λ-saturated ideal over κ.

Proof. Assume that $\{X_\alpha \subset \kappa : \alpha < \lambda\} \subset I^+$. Then there exists a $p \in P$ such that

$$p \Vdash |\{\alpha < \lambda : \check{X}_\alpha \in \dot{J}^+\}| = \lambda.$$

Because: Suppose not then by the regularity of λ and the λ-c.c. of P there would be a $\gamma < \lambda$ such that

$$\Vdash \{\alpha < \lambda : \check{X}_\alpha \in \dot{J}^+\} \subset \gamma.$$

But then $X_\gamma \in I$ by the definition of I, which is a contradiction.

Now $p \Vdash |\{\alpha < \lambda : \check{X}_\alpha \in \dot{J}^+\}| = \lambda$ together with the $\check{V}$-λ-saturation of $\dot{J}$ implies that there is a $q \leq p$ and $\alpha < \beta < \lambda$ such that $q \Vdash \check{X}_\alpha \cap \check{X}_\beta \in \dot{J}^+$, i.e. $X_\alpha \cap X_\beta \in I^+$. □

The next theorem gives us the consistency of $sat(I) = \kappa$:

Theorem 2.1.10 (Kunen-Paris). *Let κ be a measurable cardinal. Then there is a partial order E such that every E-generic extension of V satisfies the following three claims:*

(i) $2^{\aleph_0} = \kappa$

(ii) There is a κ-saturated ideal I over κ

(iii) For no $\lambda < \kappa$ is there a λ-saturated ideal I over κ.

Therefore: $V[G] \models sat(I) = \kappa$.

Proof. Let P be Easton product of the forcing notion which adds κ new subsets to ω and to each successor cardinal below κ. That is P is the set of all functions p with $dom(p) = \kappa \times \kappa \times \kappa$ and $ran(p) \subset 2$, such that the folllowing two conditions are satisfied:

(i) For any $(\xi, \zeta, \eta) \in dom(p)$ we have that $\xi = \omega$ or a successor cardinal below κ , and $\zeta < \xi$.

(ii) For any regular ν, $|\{(\xi, \zeta, \eta) \in dom(p) : \xi \in \nu\}| < \nu$.

P is ordered by reverse inclusion: $p \leq q$ if and only if $p \supset q$. Note that this definition implies that $|p| < \kappa$ for every $p \in P$.

We note also that P has the κ-c.c.: Suppose that $A \subset P$ is a maximal antichain. We have to show that $|A| < \kappa$. Let $\delta < \kappa$ be inaccessible such that

$$(V_\delta, \in, P \cap V_\delta, A \cap V_\delta) \prec (V_\kappa, \in, P, A)$$

Such a δ always exists due to Lemma 1.4.5.(ii). Then for any $p \in P$ $|p \cap V_\delta| < \delta$ (stipulation (ii)) and therefore, due to Lemma 1.4.2.(i), $p \cap V_\delta \in V_\delta$. So by the maximality of $A \cap V_\delta$ in V_δ, $p \cap V_\delta$ is compatible with some element of $A \cap V_\delta$. But then p itself is compatible with that element. Hence $A \cap V_\delta$ is a maximal antichain of P, and so $A = A \cap V_\delta$ and $|A| < \kappa$.

Suppose now that U is a normal ultrafilter over κ and $j : V \prec M \cong Ult_U(V)$ the canonical elementary embedding. Let

$$P_1 := \{p \in j(P) : p \subset (j(\kappa) - \kappa \times j(\kappa) \times j(\kappa) \times 2\}$$

$$P_2 := \{p \in j(P) : p \subset \kappa \times \kappa \times (j(\kappa) - \kappa) \times 2\}$$

Then:

$$j(P) \cong P \times P_1 \times P_2$$

through a natural identification and if $p \in P$ then $j(p)$ may be identified with $(p, 1_{P_1}, 1_{P_2})$, since $|p| < \kappa$ and hence $j(p) = p$. So the Kunen-Paris conditions are satisfied.

We now proceed to show that $P \times P_1$ is our desired notion of forcing E. Let G_0 be P-generic over V and let G_1 be P_1-generic over $V[G_0]$. To invoke the Kunen- Paris scheme let G_2 be P_2-generic over $V[G_0][G_1]$ and set $G := G_0 \times G_1 \times G_2$. Since P_1 is κ^+-closed (as κ is not a successor cardinal), it follows from fact 1.5.15. that forcing with P_1 after P adds no further subsets of κ. Therfore:

(∗) The ultrafilter $\overline{U}$ from lemma 2.1.7. is even a $V[G_0][G_1]$-normal ultrafilter

We continue the proof by noting that $\Vdash_{P\times P_1} \check{P}_2$ has the κ-c.c.. To see this let $\{p_\alpha : \alpha < \kappa\} \subset P_2$, and let

$$X := \{\eta : \exists\alpha\exists\xi\exists\zeta\,(\xi, \zeta, \eta) \in dom(p_\alpha)\} \subset j(\kappa) - \kappa.$$

Then by stipulation (ii) $\forall p \in P_2\,|p| < \kappa$, hence $|X| \leq \kappa$, and so an injection $g : X \to \kappa$ associates to each p_α a $q_\alpha \in P$ by taking the $\eta \in j(\kappa) - \kappa$ to the corresponding $g(\eta) \in \kappa$. Note that this preserves incompatibility i.e. $p_\alpha \perp p_\beta$ if and only if $q_\alpha \perp q_\beta$. Thus the κ-c.c. of P implies the κ-c.c. of P_2. Now because of the κ-c.c. of P κ is preserved in V^P hence $\Vdash_P \check{P}_2$ has the κ-c.c.. Moreover P_1 is κ^+-closed and so we apply fact 1.5.15. to obtain our desired $\Vdash_{P\times P_1} \check{P}_2$ has the κ-c.c..

Now we can conclude: By (∗) there exists a $V[G_0][G_1]$-2-saturated ideal over κ, and by the κ-c.c in $V^{P\times P_1}$ we may apply 2.1.9. to see that $V[G_0][G1]$ contains a κ-saturated ideal over κ.

Moreover note that $\Vdash_{P\times P_1} 2^{\aleph_0} = \kappa$ holds due to fact 1.5.3. and the κ-c.c. which can be seen as follows: We have already noticed earlier that forcing with $P \times P_1$ doesn't raise the size of $P(\kappa)$, compared to forcing only with P. In particular $(2^{\aleph_0})^{V[G_0][G_1]} = (2^{\aleph_0})^{V[G_0]}$ and invoking fact 1.5.3. gives us $\Vdash_P 2^{\aleph_0} \leq \theta$ where $\theta = (|P|^{<\kappa})^{\aleph_0}$; as P consists of functions p with $|p| < \kappa$ we may conclude with the measurability of κ that $|P| = \kappa$, moreover $\kappa^{<\kappa} = \kappa$ which gives us $\theta = \kappa$. Hence $(2^{\aleph_0})^{V[G_0][G_1]} = \kappa$ which is what we wanted to show.

Thus it is left to show that there are no λ-saturated ideals over κ in $V^{P\times P_1}$: We show this by contradiction: Suppose that there is an ideal which is λ-saturated for a $\lambda < \kappa$. Due to the already shown fact that each measurable cardinal is Mahlo, we may assume without loss of generality that $\lambda^{<\lambda} = \lambda$. We split the forcing notion P into two parts:

- $(P)_\lambda := \{p \in P : p \subset \lambda \times \kappa \times \kappa \times 2\}$
- $(P)^\lambda := \{p \in P : p \cap (\lambda \times \kappa \times \kappa \times 2) = \emptyset\}$.

It is possible to show that $P \cong (P)_\lambda \times (P)^\lambda$ and further that $(P)_\lambda$ has the λ^+-c.c. and $(P)^\lambda$ is λ^+-closed. G_0 can thus be considered as $H_0 \times H_1$ where $H_0 \subset (P)_\lambda$ and $H_1 \subset P^\lambda$. If $V[G_0][G_1]$ is regarded as a generic extension of $V[H_1][G_1]$ using $(P)_\lambda$, then by 2.1.9. there is a λ^+-saturated ideal in $V[H_1][G_1]$. As $(P)^\lambda$ is λ^+-closed, P_1 is κ^+-closed, we know by fact 1.5.10. that $(P)^\lambda \times P_1$ is λ^+-closed. As I is λ^+ saturated in $V[H_1][G_1]$ and $(2^{<\lambda^+})^{V[H_1][G_1]} = (2^\lambda)^{V[H_1][G_1]} = (2^\lambda)^V$ (the last equality holds due to the λ^+-closedness of $(P)^\lambda \times P_1$) and the latter term is less than κ; hence we may use 2.1.5(i) to conclude that κ is measurable in $V[H_1][G_1]$. However $2^{\lambda^+} = \kappa$ also holds in the model because $(P)^\lambda$ adds to $P(\lambda^+)$ κ-many elements. This is our desired contradiction. □

Theorem 2.1.11 (**Kunen-Paris**). *Suppose that κ is a measurable cardinal. Then there is a notion of forcing E such that the following three properties hold in V^E:*

(i) κ is weakly compact

(ii) There is a κ^+-saturated ideal over κ

(iii) There is no κ-saturated ideal over κ.

Proof. Let P be the Easton product of the notions of forcing for adding a generic subset to each regular cardinal less than κ, i.e. P consists of the functions p with $dom(p) \subset \kappa \times \kappa$ and $ran(p) \subset 2$ that additionally satisfy these two conditions:

1. if $(\xi, \zeta) \in dom(p)$ then ξ is a regular cardinal less than κ and $\zeta < \xi$
2. for any regular μ, $|\{(\xi, \zeta) \in dom(p) : \xi \leq \mu\}| < \mu$.

Again P has the κ-c.c..

Suppose that U is a normal ultrafilter over κ and $j : V \prec M \cong Ult_U(V)$. We split:

- $P_1 := \{p \in j(P) : p \subset (j(\kappa) - \kappa + 1) \times j(\kappa) \times 2\}$
- $P_2 := \{p \in j(P) : p \subset \{\kappa\} \times \kappa \times 2\}$

Then:

$$j(P) \cong P \times P_1 \times P_2.$$

and the Kunen-Paris conditions are satisfied if we identify $j(p)$ with $(p, 1, 1)$. Moreover by the usual sum-argument P_1 is κ^+-closed, and P_2 is κ-closed. Also P_2 has the κ^+-c.c.

We now proceed very similar to our last proof, and show that $P \times P_1$ is our desired notion of forcing. Let G_0 be P-generic over V and let G_1 be P_1-generic over $V[G_0]$. Again, to invoke the Kunen-Paris scheme, we let G_2 be a P_2-generic filter over $V[G_0][G_1]$ and $G := G_0 \times G_1 \times G_2$. As in the previous proof, the κ^+ -closure of P_1 implies that the normal $V[G_0]$-ultrafilter is in fact a normal $V[G_0][G_1]$ ultrafilter over κ. And with lemma 2.1.9 we may derive from the $\kappa^+ - c.c.$ of P_2 that there exists a κ^+-saturated ideal over κ in $V[G_0][G_1]$.

The next thing is to show that κ is forced to be weakly compact. We already know by theorem 1.4.11. that κ is weakly compact if and only if κ is inaccessible and has the tree property, thus we continue the proof by showing these two characteristics. Due to fact 1.5.15. it suffices to show them in $V[G_0]$. First κ is inaccessible in $V[G_0]$ because: Let γ be a cardinal $< \kappa$ such that $\gamma^{<\gamma} = \gamma$ (for example if γ is inaccessible) then again set

$$(P)_\gamma := \{p \in P : p \leq \gamma \times \kappa \times 2\}$$

$$(P)^\gamma := \{p \in P : p \cap (\gamma \times \kappa \times 2) = \emptyset\}$$

.

We know that $(P)_\gamma \times (P)^\gamma \cong P$ and $(P)_\gamma$ has the γ^+-c.c. and $(P)^\gamma$ is γ^+-closed. Hence by fact 1.5.15. forcing with $(P)^\gamma$ after $(P)_\gamma$ adds no new subsets to γ, γ^+ is preserved and since the inaccessible cardinals below κ form an unbounded set we know that κ remains inaccessible after forcing with P and thus in $V[G_0]$.

Next suppose that $(T, <)$ is a κ-tree in $V[G_0]$, $T \subset \kappa$. Our goal is to produce a branch in $V[G_0]$ of length κ. Let $\overline{j} : V[G_0] \prec M[G]$ be the canonical extension of j. Then $(\overline{j}(T), \overline{j}(<))$ is a $\overline{j}(\kappa)$-tree in $M[G]$. We know that $j(T) \cap \kappa = T$ and therefore γ which is an element of the κ-th level of $j(T)$ determines a branch of length κ in T. Unfortunately this branch sits in $M[G] \subset V[G]$. Thus we have to show that this branch actually lies in $V[G_0]$.

We work in $V[G_0]$ and use the fact that $V[G]$ is a generic extension by $P_1 \times P_2$ which is a κ-closed notion of forcing. We already know that for each generic $G_1 \times G_2$ there is a branch τ in $V[G]$. Therefore

$$\Vdash_{P_1 \times P_2} \exists \tau (\tau \text{ is a } \kappa\text{-branch in } \check{\mathrm{T}})$$

and by fullness there is a $P_1 \times P_2$-name τ such that

$$\Vdash_{P1 \times P_2} (\tau \text{ is a } \kappa\text{-branch in } \check{\mathrm{T}})$$

. Now we can define recursively conditions $p_\alpha \in P_1 \times P_2$ and sets $b_\alpha \subset \kappa$ for $\alpha < \kappa$ (the κ-closedness of $P_1 \times P_2$ is crucial at the limit steps) such that

(i) $p_\alpha \Vdash \tau \cap \alpha = \check{b}_\alpha$

(ii) $p_\beta \leq p_\alpha$

If we set $b := \bigcup_{\alpha < \kappa} b_\alpha$ then b is a κ branch in $V[G_0]$ (because $p_\alpha \Vdash \tau \cap \alpha = \check{b}_\alpha \wedge (\check{b}_\alpha$ is an α-branch) and the latter formula written inside the brackets is Δ_0 thus $||\ \check{b}_\alpha$ is an α-branch$|| = 1$ hence $\check{b}_\alpha$ is an α-branch and $\bigcup b_\alpha$ is a κ branch).

To finish the proof we have to show that there are no κ-saturated ideals over κ in $V[G_0][G_1]$. Using lemma 2.1.5.(ii) it suffices to show that in $V[G_0][G_1]$ κ is not measurable: Let $\alpha < \kappa$ be regular then P added a generic set $g(\alpha) \subset \alpha$ such that $g(\alpha) \cap \xi \in V$ for each $\xi < \alpha$. Now assume that κ were measurable. Let W be a normal ultrafilter over κ, let $x = [g]_W$. Then due to normality x is a subset of κ in the ultrapower hence by 1.3.19.(i) $[g]_W = x = j_W(x) \cap \kappa$ which means that $\{\alpha : g(\alpha) = x \cap \alpha\} \in W$. But if $\alpha_1 < \alpha_2$ are elements of this set then $g(\alpha_2) = x \cap \alpha_2 \supset x \cap \alpha_1 = g(\alpha_1) = x \cap \alpha_1 = (x \cap \alpha_2) \cap \alpha_1 = g(\alpha_2) \cap \alpha_1 \in V$. □

2.2 The saturation of the nonstationary ideal

In this section we focus our interest on the saturation of the nonstationary ideal on κ, denoted with NS_κ. For this purpose we introduce the so called generic ultrapower construction, a useful method which combines forcing and ultrapowers.

Let κ be a regular cardinal and let I be an ideal on κ. Consider this notion of forcing: The set of conditions P is the set of all I-positive sets and $p \in P$ is stronger than $q \in P$ if and only if $p \subset q$. Let V denote the ground model and let G be a V-generic filter on P. Certain properties of the ideal are conveyed to G:

Lemma 2.2.1 (*[11], Lemma 22.13*). *(i) G is a V-ultrafilter on κ extending the filter dual to I*

(ii) If I is a κ-complete then so is G

(iii) The normality of I implies that G is normal

Having constructed a V-ultrafilter on κ we are now able to construct in $V[G]$ the ultrapower $Ult_G(V)$. This ultrapower is the so called generic ultrapower, which is a model of ZFC though not necessarily well-founded. Of course Łoś theorem still holds here and we have:

$$Ult_G(V) \models \varphi([f_1], ..., [f_n]) \Leftrightarrow \{\alpha : V \models \varphi(f_1(\alpha), ..., f_n(\alpha))\} \in G$$

where $f_1, ...f_n \in V$ are functions from κ to V. Again $j : V \to Ult_G(V)$ denotes the canonical elementary embedding.

Note that the generic ultrapower may be constructed with an arbitrary set A instead of κ, and an ideal I on A instead on κ. Thus the last lemma, and the generic ultrapower construction apply also in the case where $A = P_\kappa(\lambda)$.

Lemma 2.2.2 (*[11], Lemma 22.14*). *Let κ be a regular cardinal and let I be a κ-complete ideal on κ, containing all singletons. Let $M = Ult_G(V)$ be the generic ultrapower. Then the following holds:*

(i) $\forall \gamma < \kappa\ j(\gamma) = \gamma$, hence Ord^M (though maybe not well-ordered) has an initial segment of order type κ.

(ii) $j(\kappa) \neq \kappa$

(iii) If I is normal then $[d] = \kappa$ where $d(\alpha) = \alpha$ is the diagonal function.

As already mentioned the ultrapower $Ult_G(V)$ needs not to be well-founded, even if G is countably complete, as G is only a V-ultrafilter. Nevertheless the well foundedness of the generic ultrapower is a very useful thing, thus it is reasonable to consider ideals for which the generic ultrapower is well founded:

Definition 2.2.3. *Let κ be a regular cardinal, and let I be a κ-complete ideal, containing all singletons. Then I is precipitous if and only if $Ult_G(V)$ is well-founded for every generic filter G.*

We will characterize precipitous ideals soon. First of all we need some new notions. Suppose that I is an ideal as in the definition above, and $S \subset \kappa$ with $S \notin I$. Then we say that $W \subset P(S)$ is an I-partition of S if W is a mximal antichain in S with respect to I, i.e. W is maximal with the property that each X in W has positive measure and if $X, Y \in W$ then $X \cap Y \in I$. Further an I-partition W_1 is a refinement of an I-partition W_2, $W_1 \leq W_2$, if every $X \in W_1$ is a subset of some $Y \in W_2$.

A functional F on S is a family of functions such that $W_F = \{dom(f) : f \in F\}$ is an I-partition of S. Moreover we say that for two functionals F, G $F < G$ holds if

1. $\forall f \in F \cup G$ f is a function into the ordinals
2. $W_F \leq W_G$
3. if $f \in F$ and $g \in G$ are so chosen that $dom(f) \subset dom(g)$ the $f(\alpha) < g(\alpha)$ for each α in the domain of f.

With these notions we assert:

Lemma 2.2.4 ([11], Lemma 22.19)**.** *The following are equivalent:*

(i) I is precipitous

(ii) Whenever S is a set such that $S \notin I$ and $\{W_n : n < \omega\}$ are I-partitions of S such that $W_0 \geq W_1 \geq ...W_n \geq ...$, then there exists a sequence of sets $X_0 \supset X_1 \supset X_2...$ such that $X_n \in W_n$ for each n, and $\bigcap X_n \neq \emptyset$.

(iii) There is no set S of postive measure such that there is a sequence of functionals on S with $F_0 > F_1 > F_2...$

Lemma 2.2.5 (**Solovay**)**.** *If I is a κ-complete, κ^+-saturated ideal where κ is a regular cardinal then I is precipitous*

Proof. Let S be a set of positive measure and let $W_0 \supset W_1, \supset ...$ be I-partitions of S. We want to show that there are $X_0 \supset X_1 \supset X_2 \supset ...$, $X_i \in W_i$ such that $\bigcap_{n=0}^{\infty} X_n$ is nonempty.

We start to modify by induction on n the partitions W_i to obtain more suitable $W_i^{'}$. Let $W_0 = \{X_\alpha : \alpha < \theta\}$, $\theta < \kappa$ be an enumeration of W_0. For each $\alpha < \theta$ let $X_\alpha^{'} := X_\alpha - \bigcup_{\beta<\alpha} X_\beta$ and set $W_0^{'} := \{X_\alpha^{'} : \alpha < \theta\}$. $W_0^{'}$ is clearly a pairwise disjoint family of sets, moreover it is an I-partition of S since for each $X_\alpha^{'}$ we have $X_\alpha - X_\alpha^{'} = (X_\alpha \cap \bigcup_{\beta<\alpha} X_\beta) = \bigcup_{\beta<\alpha}(X_\alpha \cap X_\beta) \in I$ due to the κ-completeness of I; and hence $X_\alpha^{'} \notin I$.

Having constructed $W_n^{'}$ we enumerate $W_{n+1} = \{X_\alpha : \alpha < \theta\}$, $\theta < \kappa$ and let $X_\alpha^{'} = (X_\alpha - \bigcup_{\beta<\alpha} X_\beta) \cap Z$ where Z is the unique $Z \in W_n^{'}$ that is almost all of the unique $Y \in W_n$ such that $X_\alpha \subset Y$. We set $W_{n+1}^{'} = \{X^{'} : X \in W_{n+1}\}$. Again $W_{n+1}^{'}$ is a partition of $S_{n+1} := \bigcup W_{n+1}^{'}$, $S - S_{n+1} \in I$ and $X_\alpha - X_\alpha^{'} \in I$ for all $\alpha < \theta$ (because $X_\alpha - X_\alpha^{'} = (X_\alpha \cap \bigcup_{\beta<\alpha} X_\beta) - Z$ and $\bigcup_{\beta<\alpha} X_\beta \in I$ due to κ-completeness).

To finish the proof we observe that $\bigcap_{n<\omega} S_n \neq \emptyset$ (otherwise $S - \bigcap S_n = S$ and $S - \bigcap S_n = \bigcup_i (S - S_i) \in I$ - a contradiction). Let $z \in \bigcap S_n$. For each n there is a unique $Y_n \in W_n^{'}$ such that $z \in Y_n$ and let X_n be the unique $X_n \in W_n$ such that $X_n \supset Y_n$ (If there would be a second $X_n^{'} \in W_n$ such that $X_n^{'} \supset Y_n$ then $X_n \cap X_n^{'} \supset Y_n \notin I$ - a contadiction to the fact that $W_n^{'}$ is an I- partition). Now we have that $X_0 \supset X_1 \supset ...$ because if $X_{n+1} \not\subset X_n$ then, since W_{n+1} is a refinement of W_n, there would be a $X^{'} \in W_n$ such that $X^{'} \supset X_{n+1}$. But now $Y_{n+1} \subset X' \cap X_n \in I$ would follow which is a contradiction) and $\bigcap X_n \neq \emptyset$. $\square$

Note that our last lemma still applies if I is an ideal over $P_\kappa(\lambda)$.

The next few lemmata will be used later to prove the splitting theorem of Solovay.

Lemma 2.2.6. *(i) If κ carries a κ-saturated, κ-complete ideal then κ is weakly inaccessible.*

(ii) If there exists a κ-saturated, κ-complete ideal on an uncountable cardinal κ then there exists a normal κ-saturated, κ-complete ideal on κ.

(iii) Let I be a normal κ-saturated, κ-cpmplete ideal on κ. If $S \notin I$ and if $f : S \to \kappa$ is regressive, then there is a $\gamma < \kappa$ such that $f(\alpha) < \gamma$ for almost all $\alpha \in S$.

Proof. (i) Follows immediatly from 2.1.3. and 2.1.4.
For the proof of part (ii) take a look at [11] 22.3 (i) pp 410.
(iii) Due to lemma 1.2.7 for every $X \subset S$ of positive measure there is a $Y \subset X$ of positive measure where f is constant. Thus let W be a maximal disjoint family of sets $X \subset S$ of positive measure such that f is constant on X. W has cardinality less than κ and hence $f(\bigcup W) < \gamma < \kappa$ and $\bigcup W$ has measure one due to the maximality of W. □

Lemma 2.2.7. *Let $\kappa > \omega$ be a regular cardinal, P be a notion of forcing, which has the κ-c.c., V[G] be its generic extension. Then every club $C \subset \kappa$ in V[G] has a closed and unbounded subset $D \in V$. It follows that each $S \subset V$ which is stationary in V remains stationary in the generic extension.*

Proof. For $\alpha < \kappa$ set

$$X_\alpha := \{\xi < \kappa \,:\, \exists q \leq p\, (q \Vdash \xi \text{ is least member of } \dot{C} - (\alpha + 1))$$

By the κ-c.c. $|X_\alpha| < \kappa$, so set $f(\alpha) := sup(X_\alpha) < \kappa$. Finally, let

$$D := \{\beta < \kappa \,:\, \forall \alpha < \beta\, (f(\alpha) < \beta)\}$$

which is a club as can be seen easily.

To show that $p \Vdash \check{\mathrm{D}} \subset \dot{C}$, assume to the contrary that there is a $q \leq p$ and a $\beta \in D$ such that $q \Vdash\notin \dot{C}$. Then for some $r \leq q$ and $\alpha < \beta$, $r \Vdash \alpha = sup((\dot{C}) \cap \beta)$. For some $s \leq r$ and $\xi \in X_\alpha$, ξ is the least element of $\dot{C} - (\alpha + 1)$. But $\xi < \beta$ as $\beta \in D$, contradicting the choice of α.

□

Theorem 2.2.8. *Let κ be a regular cardinal $> \omega$ and let I be a κ-complete, κ-saturated ideal on κ. Then the following holds:*

(i) κ is weakly Mahlo

(ii) The set of the weakly Mahlo cardinals below κ is an stationary subset of κ

(iii) *If I is additionally normal and if $X \subset \kappa$ has measure one then $X \cap M(X)$ has measure one, where M(X), the Mahlo operation is defined as*

$$M(X) := \{\alpha < \kappa : cf(\alpha) > \omega \wedge X \cap \alpha \text{ is stationary in} \alpha\}$$

Proof. The existence of a κ-saturated ideal implies that κ is weakly inaccessible and also implies the existence of a normal κ-saturated ideal by lemma 2.2.6.(i) and (ii). Thus we assume that I is normal. We start with the following

Claim: If $S \subset \kappa$ is stationary then for I-almost all $\alpha < \kappa$ $S \cap \alpha$ is stationary.

We want to prove the claim first: Assume to the contrary that there is an $X \notin I$ such that $(S \cap \alpha)$ is not stationary in α for all $\alpha \in X$. Choose a generic filter G such that $X \in G$ and build $Ult_G(M)$. We have $Ult_G(M) \models$ "$[c_S] \cap [d]$ is nonstationary" where d is the diagonal function $d : \kappa \to \kappa$, $d(\alpha) = \alpha$. Since I has the κ-c.c. $Ult_G(M)$ is wellfounded, thus we can identify it with its transitive collaps $N \overset{\pi}{\cong} Ult_G(M)$ and also the canonical elementary embedding $j : M \overset{\sim}{\to} N$ exists. Since I is normal, κ is represented by d, thus $N \models$ "$(\pi([c_S]) \cap \kappa)$ is not stationary". And since $S = j(S) \cap \kappa = \pi([c_S]) \cap \kappa$ we obtain $N \models$ "S is not stationary". However, the notion of forcing is κ-saturated and hence κ is a regular cardinal in $M[G]$ and by lemma 2.2.7. $M[G] \models$ "S is stationary", but $N \subset M[G]$, hence $N \models$ "S is stationary" which is a contradiction and the proof of our Claim is finished.

Now we can tackle (iii): At first we notice that for I-almost all $\alpha < \kappa$ cf(α) $> \omega$: If there would be a set X of positive measure such that for all $\alpha \in X$ cf$(\alpha) = \omega$ we could choose again a generic G and construct $Ult_G(M)$ to get $Ult_G(M) \models cf[d] = \omega$ which would lead to $N \models cf(\kappa) = \omega$. But κ is a regular cardinal in M and I is κ-saturated, thus κ is regular in $M[G]$, but $N \subset M[G]$ - a contradiciton.

Now if X has measure one, then since Lemma 2.2.6.(iii) X is stationary and the set $\{\alpha < \kappa : cf(\alpha) > \omega\}$ has measure one (due to our claim). Also $\{\alpha < \kappa : cf(\alpha) > \omega\}$ has measure one and so $X \cap M(X)$ has measure one which ends our proof of (iii).

To prove (i) it suffices to show that I-almost all $\alpha < \kappa$ are regular cardinals. Otherwise let $X \notin I$ be such that all $\alpha \in X$ are singular. Let G be generic with $X \in G$. Then $Ult_G(M) \models$ "$[d]$ is singular" hence $N \models$ "κ is singular" - again a contradiction.

And for (ii) let $\{\alpha < \kappa : \alpha \text{ is regular}\}$ then $M(X) = \{\alpha < \kappa : cf(\alpha) > \omega \wedge X \cap \alpha \text{ is stationary in } \alpha\}$ has measure one and so: $X \cap M(X) = \{\alpha < \kappa : \alpha \text{ is weakly Mahlo}\}$ has measure one, hence stationary. □

To prove eventually Solovays Splitting Theorem we need one last Lemma:

Lemma 2.2.9. *Let S be a stationary subset of κ and assume that every $\alpha \in S$ is a regular uncountable cardinal, Then $T = S \cap (\kappa - M(S)) = \{\alpha \in S : S \cap \alpha \text{ is not stationary in } \alpha\}$ is stationary in κ.*

Proof. Let C be any club subset of κ and let C' be the set of the limit points of C which is also a club. Hence $S \cap C' \neq \emptyset$ and let α be its minimal element. Since α is regular, and a limit point of C we have that $C \cap \alpha$ is a club in α, yet $(C \cap \alpha)' = C' \cap \alpha$. As α is the least element of $S \cap C'$, $(C' \cap \alpha) \cap (S \cap \alpha) = \emptyset$ and so $S \cap \alpha$ is nonstationary in α, hence $\alpha \in T \cap C$ $\square$

Theorem 2.2.10 (Solovay). *If $S \subset \kappa$ is stationary then S can be split into κ-many sationary, pairwise disjoint subsets $(S_\xi)_{\xi<\kappa}$.*

Proof. If not then let S be a stationary set for which such a decomposition is impossible. Consider the set $I := \{X \subset \kappa : X \cap S \text{ is stationary }\}$. It is straightforward to show that I is an ideal which is κ-complete, normal, and due to our assumption κ-saturated. The Claim in the proof of Theorem 2.2.8. tells us that $M(S)$ has measure one, thus $S - M(S)$ has measure zero, i.e. is nonstationary ($S - M(S) \in I$ implies $S - M(S) \cap S = S - M(S)$ is nonstationary) which is a contradiction to Lemma 2.2.9. $\square$

Remark: The proof above is Solovays original proof and the theorem is one of the most prominent basic results of set theory which were discovered first in the context of large cardinals. An elementary proof which avoids large cardinals can be found in [11] (pp. 94-95).

Definition 2.2.11. *Let κ be a regular uncountable cardinal. Then*

$$NS_\kappa := \{X \subset \kappa : X \text{ is nonstationary } \}$$

is an ideal, the nonstationary ideal on κ

Remark: NS_κ is κ-complete and normal, since its dual filter is the closed and unbounded filter. The theorem of Solovay shows that NS_κ is not κ-saturated. The question arises if NS_κ is at least κ^+-saturated and that is what we want to start to investigate now. Our goal will be a result of Gitik and Shelah who showed that the answer is no if $\kappa > \aleph_1$. We will prove this assertion in a series of lemmas:

Definition 2.2.12. *If I is an ideal on κ, $S \in I^+$ then we define*

$$I \restriction S := \{X \subset \kappa : X \cap S \in I\}$$

We say that $I \restriction S$ concentrates on S

Lemma 2.2.13. *If I is a normal κ-complete ideal then so is $I \restriction S$*

Proof. Straightforward: Let $(X_\xi)_{\xi<\gamma}$, $\gamma < \kappa$ be a sequence in $I \restriction S$, i.e. $X_\xi \cap S \in I \quad \forall \xi < \gamma$ then $\bigcup X_\xi \cap S \in I \Rightarrow \bigcup X_\xi \in I \restriction S$. Moreover by definition $I \restriction S$ is normal if and only if its dual filter $(I \restriction S)^*$ is so. Thus assume that $\forall \xi < \kappa X_\xi \in (I \restriction S)^*$. We want that $\Delta_{\xi<\kappa} X_\xi \in (I \restriction S)^*$. Since $X_\xi \in (I \restriction S)^*$ it follows that $(\kappa - X_\xi) \cap S \in I$ and hence $\kappa - ((\kappa - S) \cup X_\xi) \in I$, thus $(\kappa - S) \cup X_\xi \in I^*$ which implies that $\Delta((\kappa - S) \cup X_\xi) \in I^*$ due to the normality of I. But $\Delta((\kappa - S) \cup X_\xi) = (\kappa - S) \cap \Delta X_\xi \in I^*$. Hence $\kappa - ((\kappa - S) \cap \Delta X_\xi) \in I \Leftrightarrow S \cup (\kappa - \Delta X_\xi) \in I \Rightarrow (\kappa - \Delta X_\xi) \cap S \in I \Leftrightarrow \Delta X_\xi \in (I \restriction S)^*$ which we wanted to show. $\square$

Lemma 2.2.14 (Smith-Tarski). *([12], Lemma 16.5) If I is a κ^+-saturated ideal over κ then $P(\kappa)/I$, i.e. the Boolean Algebra $P(\kappa)$ modulo the ideal I is a complete one.*

Lemma 2.2.15. *Let I be a normal, κ-complete, κ^+-saturated ideal on κ. Let $(P,<)$ be the forcing with I-positive sets, G be the generic ultrafilter and let $M = Ult_G(V)$. Then $P^M(\kappa) = P^{V[G]}(\kappa)$ and all cardinals and cofinalities $< \kappa$ are preserved.*

Proof. We first state again that the Boolean algebra $B = P(\kappa)/I$ is complete. Hence if $\dot{A}$ is a name for a subset $A \subset \kappa$, $A \in V[G]$, there are sets $S_\alpha \notin I$ such that $\|\alpha \in \dot{A}\| = [S_\alpha]$. If $j : V \to M$ is the canonical embedding we get for each α: $\alpha \in A \Leftrightarrow S_\alpha \in G \Leftrightarrow \kappa \in j(S_\alpha)$ (the last equivalence is due to lemma 1.3.18). Moreover, since for each α $j(S_\alpha) \in M$ we have that the sequence $(j(S_\alpha)_{\alpha<\kappa}$ is in M (which is a consequence of Lemma 1.3.19 (ii)) hence $A = \{\alpha \in \kappa : \kappa \in j(S_\alpha)\}$ is a set defineable in M, thus $A \in M$ and we have $P^{V[G]}(\kappa) = P^M(\kappa)$.

If $\lambda < \kappa$ is a cardinal then since κ is the critical point of j (Lemma 2.2.2) we have that $j(\lambda) = \lambda$ and λ remains a cardinal in M due to elementarity of j. Since $P^{V[G]}(\kappa) = P^M(\kappa)$, λ is a cardinal in $V[G]$. □

Definition 2.2.16. *Let λ be a cardinal and let $\alpha < \lambda^+$ be a limit ordinal. A family $\{X_\xi : \xi < \lambda^+\}$ of subsets of α is called strongly almost disjoint if every $X_\xi \subset \alpha$ is unbounded and if for every $\nu < \lambda^+$ there exist ordinals $\delta_\xi < \alpha$ for $\xi < \nu$ such that the sets $X_\xi - \delta_\xi$, $\xi < \nu$ are pairwise disjoint.*

Lemma 2.2.17. *Let κ be a regular cardinal then there exists a strongly almost disjoint family of κ^+-many subsets of κ.*

Proof. We first observe that there are κ^+-many almost disjoint functions from κ to κ: It suffices to show that given a list f_ν of κ almost disjoint functions, there is an f which isn't included in the list and which is almost disjoint from each f_ν, $\nu < \kappa$. We define $f(\alpha) \neq f_\nu(\alpha)\ \forall \nu < \alpha$, which is well defined because of the regularity of κ. Thus we have at least κ^+-many almost disjoint functions $f : \kappa \to \kappa$.

Next we consider a bijection $h : \kappa \times \kappa \to \kappa$. If $(f_\alpha)_{\alpha<\kappa^+}$ is an almost disjoint family of functions from κ to κ then $(h'' f_\alpha)_{\alpha<\kappa^+})$ is an almost disjoint family of subsets of κ.

And now it can be seen that each almost disjoint family of subsets of κ is in fact strongly almost disjoint: Let $(X_\alpha)_{\alpha<\kappa}$ be an almost disjoint family. We define inductively $X'_0 := X_0$, $(X_\alpha)' := X_\alpha - \delta_\alpha$ where δ_α is an ordinal such that $(X_\alpha - \delta_\alpha) \cap X_\beta = \emptyset\ \forall \beta < \alpha$. Such a δ_α always exists due to the regularity of κ. The $(X_\alpha)'$ are strongly almost disjoint. □

Lemma 2.2.18. *If $\alpha < \lambda^+ \wedge cf(\alpha) \neq cf(\lambda)$ then there exists no strongly almost disjoint family of subsets of α of size λ^+.*

Proof. If not then there would be a family $(X_\xi)_{\xi<\lambda^+}$ which is strongly almost disjoint. Since by the definition each X_ξ is cofinal in α, we may assume that each X_ξ has ordertype cf(α). Let f be a function that maps λ onto α.

Claim: For each $\xi < \lambda^+$ there exists some $\gamma_\xi < \lambda$ such that $X_\xi \cap f''\gamma_\xi$ is cofinal in α

We proof now the claim: Since $cf(\alpha) \neq cf(\lambda)$ we can distinct two cases:

(a) $cf(\alpha) > cf(\lambda)$: Assume to the contrary that there is a $\beta < \lambda^+$ such that for all $\gamma < \lambda X_\beta \cap f''\gamma$ is bounded inα. Let $(\gamma_\xi)_{\xi < cf(\lambda)}$ be a sequence cofinal in λ. Let δ' be any element of α then there is a $\delta > \delta'$ such that $\delta \in X_\beta$. If $\nu \in \lambda$ denotes an element such that $f(\nu) = \delta$ then there is a $\xi < cf(\lambda)$ such that $\gamma_\xi > \nu$, hence $\delta \in f''\gamma_\xi \cap X_\beta$ and $\delta < sup(f''\gamma_\xi \cap X_\beta) \neq \alpha$ (due to our assupmtion). Thus the non-decreasing sequence $(sup(f''\gamma_\xi \cap X_\beta))_{\xi < cf\lambda}$ is unbounded inα which implies $cf\alpha <= cf\lambda$ which is a contradiction.

(b) $cf\alpha < cf\lambda$: Let $(x_\xi)_{\xi < cf\alpha}$ be an enumeration of any X_β, $\beta < \lambda^+$, then for each x_i, $i < cf\alpha$ there is a $y_i < \lambda$ such that $x_i \in f(y_i)$. We obtain a sequence $(y_i)_{i < cf\alpha}$ in λ, hence $\gamma_\beta := lim(y_i) \in \lambda$ and $X_\beta \cap f''\gamma_\beta \supset (x_\xi)_{\xi < cf\alpha} = X_\beta$ which ends our proof of the claim.

We continue with the proof of the lemma: There exixts some γ and a set $W \subset \lambda^+$ of size λ such that $\gamma_\xi = \gamma$ for all $\xi \in W$. Let $\eta > sup(W)$. By our assumption on the X_ξ there exist ordinals $\delta_\xi < \alpha$, $\xi < \eta$ such that the $X_\xi - \delta_\xi$ are pairwise disjoint. Thus $f^{-1}(X_\xi - \delta_\xi)$, $\xi \in W$ are λ pairwise disjoint nonempty subsets of $\gamma < \lambda$ which is a contradiction. □

Corollary 2.2.19. *If κ is a regular cardinal and if a notion of forcing makes $cf\kappa \neq cf|\kappa|$, then P collapses κ^+*

Proof. Assume to the contrary that κ^+ is preserved. Then $V[G]$ thinks that $(\kappa^+)^V = \lambda^+$ where $\lambda = |\kappa|$. By Lemma 2.2.17. there exists a strongly almost disjoint family $(X_\xi)_{\xi < \kappa^+}$ of subsets of κ and it remains strongly almost disjoint in $V[G]$, but its length changes to λ^+. Since $cf\kappa \neq cf|\kappa| = cf\lambda$ (in $V[G]$). This is a contradiction to Lemma 2.2.17. □

Corollary 2.2.20. *If $\kappa = \lambda^+$, if $\nu < \lambda$, $\nu \neq cf\lambda$ is regular and if I is a normal, κ-complete, κ^+-saturated ideal on κ. Then $E^\kappa_\nu := \{\alpha < \kappa : cf\alpha = \nu\} \in I$.*

Proof. Assume that $E^\kappa_\nu \notin I$, let G be generic on $P(\kappa)/I$ such that $E^\kappa_\nu \in G$. By Lemma 2.2.14. all cardinals $< \kappa$ i.e. $\leq \lambda$ are preserved as well as κ^+ (I is κ^+-saturated, hence the notion of forcing $P(\kappa)/I$ has the κ^+.c.c.). We construct the generic ultrapower $M = Ult_G(V)$ in $V[G]$, which is well founded due to lemma 2.2.5. We have $E^\kappa_\nu \in G \Leftrightarrow \{\alpha < \kappa : cf\alpha = \nu\} \in G \Leftrightarrow M \models cf[d] = \nu \Leftrightarrow M \models cf\kappa = \nu$ (The last equality holds because I is normal hence G) Hence $V[G] \models cf\kappa = \nu$. We also have that $\{\alpha \in \kappa : |\alpha| = \lambda\} \in G$ which implies $M \models |\kappa| = \lambda$ and $V[G] \models |\kappa| = \lambda$ follows. Thus $V[G] \models cf\kappa = \nu \wedge cf|\kappa| = cf\lambda \wedge \nu \neq cf\lambda$ which contradicts corollary 2.2.19. □

Now we can easily proof the Theorem of Gitik and Shelah, at least for successor cardinals:

Theorem 2.2.21 (Gitik-Shelah). *Let κ be a successor-cardinal such that $\kappa > \aleph_1$. Then the nonstationary ideal on κ is not κ^+-saturated. Moreover the ideal $NS_\kappa \restriction E^\kappa_\nu$ is not κ^+-saturated for all regular $\nu \neq cf\lambda$ where λ is the predecessor of κ.*

Proof. The sets E^κ_ν are stationary subsets of κ. Thus if $NS_\kappa \restriction E^\kappa_\nu$ would be κ^+-saturated then because of Corollary 2.2.20. $E^\kappa_\nu \in NS_\kappa \restriction E^\kappa_\nu$ since $NS_\kappa \restriction E^\kappa_\nu$ is a κ-complete normal ideal on κ which is impossible. □

What is now still opened is the question wheter NS_κ is κ^+-saturated when κ is a limit cardinal. Its answer is again a negative one and its proof goes in a diffrent direction: Again we will obtain our desired result in a series of lemmas. We start with:

Definition 2.2.22. *Let κ be a regular uncountable cardinal and let E be a stationary subset of κ. Let C be a club on κ and let for each $\alpha \in E$ c_α be a cofinal subset of α. Then we say that the sequence $(c_\alpha)_{\alpha\in E}$ guesses C if for all $\alpha \in E$ $\exists\beta < \alpha\, C \supset c_\alpha - \beta$ i.e. C contains for each $\alpha \in E$ an end segment of c_α.*

Lemma 2.2.23. *Let κ and λ be regular, $\lambda \geq \aleph_1$ and $\lambda^+ < \kappa$. Then there exists no sequence $(c_\alpha : \alpha \in E^\kappa_\lambda)$ with each $c_\alpha \subset \alpha$ closed and unbounded that guesses every club $C \subset \kappa$ almost everywhere [i.e. the set*

$$G(C) := \{\alpha \in E^\kappa_\lambda : \exists\beta < \alpha\, C \supset c_\alpha - \beta\}$$

has measure one in the ideal NS_κ] and additionaly satisfies this property:

- *(∗) if $\alpha \in E$ is a limit of ordinals of cofinality greater than λ, then all nonlimit elements of c_α have cofinality greater than λ*

Proof. Assume to the contrary that $(c_\alpha : \alpha \in E^\kappa_\lambda)$ is such a sequence. Set $E_0 := \{\alpha \in \kappa : \alpha \text{ is a limit of ordinals of cofinality } > \lambda\}$. E_0 is clearly a club in κ. Now we define inductively: If the club E_n is already defined then we consider the club E'_n, the set of the limit points of E_n. Due to our assumption the set $G(E'_n) = \{\alpha \in E^\kappa_\lambda : \exists\beta\, C \supset c_\alpha - \beta\}$ has measure one, i.e. it is an element of the dual filter of NS_κ, the closed unbounded filter, hence $G(E'_n)$ contains a club C. Notice that if $\alpha \in C$ then E'_n contains a final segment of c_α. Set $E_{n+1} := E'_n \cap C$.

Finally we let $E := \bigcap E_n$, which is a club. Due to the stationarity of E^κ_λ, $E \cap E^\kappa_\lambda$ is nonempty and we set $\delta := \min(E \cap E^\kappa_\lambda)$. For every $n < \omega$ E'_n contains a final segment of c_δ since $\delta \in E_{n+1}$. But $cf\delta = \lambda > \aleph_0$ and so a final segment of c_δ is contained in E. Pick some $\beta \in E \cap c_\delta$. By (∗) $cf(\beta) > \lambda$. Since $E'_n \supset E_{n+1}$ $\beta \in E'_n$ for every $n \in \omega$. So it is a limit point of E_n. Hence $E_n \cap \beta$ is a club in β for every $n \in \omega$. Since $cf\beta > \aleph_0$ also $E \cap \beta$ is a club in β. But $cf\beta > \lambda$, hence there is some $\gamma \in E \cap \beta$ of cofinality λ which contradicts the minimality of δ_0. □

A only slightly different concept is this one, invented again by Shelah:

Definition 2.2.24. *Let κ, θ be regular cardinals, $\kappa > \aleph_2 \wedge \kappa > \theta^+$. If $S \subset E^\kappa_\theta$ is stationary in κ then $\diamondsuit'_{club}(S)$ denotes the following: There exists a sequence $(S_\alpha : \alpha \in E^\kappa_\theta)$ which satisfies:*

(i) $S_\alpha \subset \alpha$

(ii) *sup* $S_\alpha = \alpha$

(iii) $|S_\alpha| = \theta$

(iv) *if α is a limit of ordinals of cofinality $> \theta$ (and $> \aleph_1$ if $\theta = \aleph_0$) then for every $\beta \in S_\alpha$ $cf\beta > \theta$ (and if $\theta = \aleph_0$ then $cf\beta > \aleph_1$)*

(v) *For every club $C \subset \kappa$ the set $\{\alpha \in S : \exists \beta < \alpha\ C \supset S_\alpha - \beta\}$ is stationary*

The next very small lemma emphasizes the similarites between the two notions, $\diamondsuit'_{club}(S)$ and the sequence defined in Lemma 2.2.23.

Lemma 2.2.25. *If $\diamondsuit'_{club}(E)$ holds then there exists a squence $(c_\alpha)_{\alpha \in E}$ such that:*

(i) *$c_\alpha \subset \alpha$ is a club for each $\alpha \in E$*

(ii) *if $\alpha \in E$ is a limit of ordinals of cofinality greater than θ then all nonlimit elements of c_α have cofinality greater than θ*

(iii) *if C is a club then$\{\alpha \in E : \exists \beta \in \alpha\, C \supset c_\alpha - \beta\}$ is stationary.*

Proof. Just define $c_\alpha := \overline{S}_\alpha$ where $\overline{S}_\alpha$ denotes the completion under the limit of sequences in S_α, and S_α is the $\diamondsuit'_{club}(E)$- sequence. Then the c_α have the desired properties. □

We see that the only difference to the concept of lemma 2.2.23. is point (*iii*), where we have postive measure instead of measure one. Surprisingly this turns the existence upside down since:

Theorem 2.2.26. *Let $\kappa > \theta$ are regular cardinals, $\kappa > \theta^+ \wedge \kappa > \aleph_2$. If $S \subset E^\kappa_\theta$ is stationary then $\diamondsuit'_{club}(S)$ holds in ZFC.*

Proof. Suppose otherwise that $\theta^+ \geq \aleph_2$ (otherwise we do the same proof only with θ^+ replaced by θ^{++}). We will show that there is even a sequence $(S_\alpha)_{\alpha \in S}$ witnessing $\diamondsuit'_{club}(S)$ such that for every club $C \subset \kappa$ $\{\alpha \in S : C \supset S_\alpha\}$ is stationary. Suppose that for some stationary $S \subset E^\theta_\kappa$ $\diamondsuit'_{club}(S)$ fails. We shall define sequences $(C_i : i < \theta^+)$ of clubs on κ, $(T^i_\alpha : i < \theta^+, \alpha \in S)$ of trees which nodes are elements of α and subsets $(S^i_\alpha : i < \theta^+, \alpha \in S)$ of α. We start with the definition of the trees T^i_α: Let C be a club with nonlimit points of cofinality $> \theta$ and let $\alpha \in E^\kappa_\theta$ be an ordinal which is the limit of ordinals of cofinality$> \theta$.We define in this case a canonical tree $T_\alpha(C)$. The first level of $T_\alpha(C)$ will consist of a closed cofinal in α sequence F of order type θ, whose nonlimit points have cofinality $> \theta$. We pick such a sequence to be the least one in some fixed well ordering. Now let η be a point from the first level. Let η^* be the predeccessor of η in the sequence F if there exists one and let $\eta' := sup(C \cap (\eta+1))$. Now we want to define the set of the immediate successors of η in the tree $T_\alpha(C)$, i.e. the set $suc_{T_\alpha(C)}(\eta)$. We distinguish these three cases:

(a) if η is a limit point of F then $suc_{T_\alpha(C)}(\eta) = \emptyset$

(b) if η is not a limit point and $\eta' < \eta^*$ then again $suc_{T_\alpha(C)}(\eta) = \emptyset$

(c) $\eta^* < \eta' < \eta$ and

  (i) if $cf\eta' > \theta$ then set $suc(\eta) = \{\eta'\}$ and $suc(\eta') = \emptyset$

  (ii) if $cf\eta' \leq \theta$ then as above we pick the least closed cofinal in η' sequence F' of length $cf\eta'$ with nonlimit points of cofinality $> \theta$ and the first element above η^* if it exists. Otherwise set $suc(\eta) = \emptyset$.

Using the points $(a) - (c)$ we continue to define $T_\alpha(C)$ above η. $T_\alpha(C)$ is a tree of cardinality $\leq \theta$ and is wellfounded.

Now let us define the sequences of clubs: Let C_0 be a club with nonlimit points of cofinality $> \theta$. For $\alpha \in S$ which is a limit of ordinals of cofinality $> \theta$ we set $T^0_\alpha := T_\alpha(C_0)$ and $S^0_\alpha := T^0_\alpha \cap C_0 \cap E^\kappa_{>\theta}$ i.e. the set of all points of all the levels of T^0_α which are in C_0 and have cofinality bigger than θ. We have that for all but nonstationary α's in S, S^0_α is unbounded in α (if $C_0 \cap \alpha$ is a club in α then $S^0_\alpha = T^0_\alpha \cap C_0 \cap E^\kappa_{>\theta}$ is unbounded in α, and the set $\{\alpha \in \kappa : \alpha \cap C_0 \text{ is a club in } \alpha\}$ is a club in κ, hence the set $\{\alpha \in S : \alpha \cap C_0 \text{ is bounded }\}$ has measure zero). We will ignore this nonstationary set where S^0_α is not defined or bounded and let S^0_α be an unbounded, in α cofinal sequence of length θ in these cases.

Now due to our assumption, there exists a club $C_1 \supset C_0$ such that every nonlimit point has cofinality $> \theta$, and C_1 witnesses the failure of $(S_\alpha : \alpha \in S)$ being a $\diamondsuit'_{club}(S)$- sequence. We define T^1_α and S^1_α as above with C_1 replacing C_0 and continue by induction. At limit stages we take C_i to be a club subset of $\bigcap_{j<i} C_j$ with nonlimit points of cofinality $> \theta$.

Finally let $D = \bigcap_{i<\theta^+} C_i$, and D is a club since $\kappa > \theta^+$. Let $\alpha \in D \cap S$ be such that the elements of D of cofinality $> \theta$ are unbounded in it (such an α always exists: Let $\tilde{D}$ be a club subset of D such that every nonlimit element has a cofinality $> \theta$, and consider the club $\tilde{D}'$ (i.e. the set of all limit points of $\tilde{D}$). Then $\tilde{D}' \cap S \neq \emptyset$ and if $\alpha \in \tilde{D}' \cap S$ then it has the desired property). Now let us show that the trees $T_\alpha(C_i)$, $i < \theta^+$ must stabilize, i.e. there is an $i_\alpha < \theta^+$ such that $T_\alpha(C_i) = T_\alpha(C_j)$ for all i, $j > i_\alpha$: If not then there would be a $\eta_0 \in \mathrm{Lev}_1(T^0_\alpha)$ such that there is no stabilisator above it (this is due to the fact that there would be θ^+-many diffrent trees but only θ-many elements at level 1). The first level of all the trees is the same by definition. η_0 can not be a limit, because it would have no successors in the tree in this case (by (a) in the definition of the tree), so let η^*_0 be again the predeccessor of η_0 in the sequence or 0 if there is none. Since there is no stabilization above η_0 in the trees and C_i's are decreasing, the sequence $(\eta_{0,i} : i < \theta^+)$ where $\eta_{0,i} := sup(C_i \cap (\eta_0 + 1))$ will be a nonincreasing sequence of ordinals inside the interval $(\eta^0, \eta]$. Hence it is eventually constant, so there is some $\eta_1 \in (\eta_0^*, \eta_0, cf\eta_1 \leq \theta$ such that $\eta_{0,i} = \eta_1$ for all i bigger than some i_1. By the definition of the trees the set of the immediate successors of η_1 will be the same in every $T_\alpha(C_i)$ $(i \geq i_1)$. Pick η_2 to be one of them with no stabilization above it. Deal with it as it was done with η_0. We will obtain $\eta_2, \eta_3, \ldots$ which form an infinite decreasing sequence of ordinals which is impossible.

Hence there is an $i_\alpha < \theta^+$ such that $T_\alpha^i = T_\alpha^{i_\alpha}$ for every $i \geq i_\alpha$. Then there are $i^* < \theta^+$ and a stationary $S^* \subset S$ such that for every $\alpha \in S^* : i_\alpha = i^* \wedge supS_\alpha^{i^*} = \alpha$. But this contradicts the choice of C_{i^*+1}.

□

Now we can prove the Theorem of Gitik-Shelah for every cardinal $> \aleph_2$:

Theorem 2.2.27 (Gitik-Shelah). *Let κ, λ be regular uncountable cardinals, $\kappa \geq \aleph_3$. Then the ideal $NS_\kappa \restriction E_\lambda^\kappa$ can not be κ^+-saturated.*

Proof. Let $E \subset E_\lambda^\kappa$ be stationary. Then by Lemma 2.2.25. there exists a sequence $(c_\alpha)_{\alpha \in E}$ that witnesses $\diamondsuit'_{club}(E)$. We have the following

Claim: If $NS_\kappa \restriction E_\lambda^\kappa$ is κ^+-saturated then there exists a stationary set $\tilde{E} \subset E$ such that for every club C, $\tilde{E} - G(C) := \{\alpha \in E : \exists \beta < \alpha\, C \supset c_\alpha - \beta$ is nonstationary. (With Definition 2.2.20. in mind we can say that C is guessed by (c_α) at almost every $\alpha \in \tilde{E}$).

We prove now the claim: If not then for every stationary $S \subset E$ there exists a club C such that $S - G(C)$ is stationary. Thus there exists a family of pairs (S_i, C_i) which has cardinality at most κ such that $W = \{S_i - G(C_i : i < \kappa\}$ is a maximal antichain in $P(\kappa)/NS_\kappa$ below E. If we set $C := \Delta_{i<\kappa}(C_i$ then for every $i < \kappa$ C_i contains an end segment of C thus $G(C_i)$ contains an end segment of $G(C)$ and hence $G(C)$ is stationary. But we have that $G(C) - (S_i - G(C_i) \in NS_\kappa$ because $G(C) \cap (S_i - G(C_i)) \subset G(C) \cap (\kappa - G(C_i))$ is bounded hence nonstationary. This contradicts the maximality of W, which ends the proof of the claim.

We continue with the proof of the theorem: Assume now to the contrary that $NS_\kappa \restriction E_\lambda^\kappa$ is κ^+-saturated. By the saturation there exists a maximal antichain $\{S_i : i < \gamma\}$ of pairwise disjoint stationary subsets of E_λ^κ with $\gamma \leq \kappa$ and for each i there exists a sequence of clubs $(c_\alpha)_{\alpha \in S_i}$ (by lemma 2.2.25.) witnessing $\diamondsuit'_{club}(S_i)$. The sets S_i are so chosen that every club $C \subset \kappa$ is guessed at almost every $\alpha \in S_i$. Then $(c_\alpha : \alpha \in \bigcup_{i<\kappa} S_i)$ guesses every C almost everywhere which contradicts lemma 2.2.23.

□

The theorem above doesn't tell us anything about the case where $\lambda = \omega$. Indeed its proof doesn't work if λ is countable as the proof of lemma 2.2.23 fails. However we can define a sequence in the sense of 2.2.23 which works for ω too, quickly leading to the assertion that the theorem of Gitik-Shelah still holds for $\lambda = \omega$. We follow the lines of [9]:

Definition 2.2.28. *Let κ and λ be two regular cardinals with $\kappa > \aleph_2$ and $\kappa > \lambda^+$. Then $\diamondsuit^*_{club}(\kappa, \lambda)$ denotes the following: There exists a sequence $(S_\alpha : \alpha \in E^\lambda_\kappa)$ with the following properties:*

1. *$\forall \alpha \in E^\lambda_\kappa \; S_\alpha \subset \alpha$*
2. *$\forall \alpha \in E^\lambda_\kappa \; sup(S_\alpha) = \alpha$*
3. *$\forall \alpha \in E^\lambda_\kappa \; |S_\alpha| = \lambda$*
4. *if α is a limit of ordinals of cofinality $> \lambda$ (and $> \aleph_1$ if $\lambda = \omega$) then for every $\beta \in S_\alpha \; cf(\beta) > \lambda$ (and $cf(\beta) > \aleph_1$ if $\lambda = \omega$).*
5. *for every club $C \subset \kappa$ the set*

$$G(C) := \{\alpha \in E^\lambda_\kappa : \exists \beta < \alpha C \supset S_\alpha - \beta$$

has measure one in NS_κ, i.e. contains a club intersected with E^λ_κ.

Note that if $\lambda > \omega$ then a $\diamondsuit^*_{club}(\kappa, \lambda)$ sequence is exactly the sequence of lemma 2.2.23, hence such a sequence can't exist. This remains true in the countable case:

Lemma 2.2.29. *$\diamondsuit^*_{club}(\kappa, \omega)$ doesn't hold in ZFC for each regular κ.*

Proof. Similar to the proof of 2.2.23: Let $E_0 := \{\beta < \kappa : \beta$ is a limit of ordinals of cofinality $> \aleph_1\}$. E_0 is a club, hence so is E_0', the set of its limits, thus there exists a club $C \subset \kappa$ such that for each $\alpha \in C \cap E^\omega_\kappa$, E_0' contains a final segement of S_α. Let $E_1 := E_0' \cap C$ and continue this way. We obtain a sequence $(E_i : i < \omega_1)$ and set $E := \bigcap_{i<\omega_1} E_i$. As in the proof of 2.2.23 let δ denote the minimum of $E \cap E^\omega_\kappa$. We know by (3) of the definition that $|S_\delta| = \omega$, thus let $(s_n : n \in \omega)$ be a sequence in S_δ, cofinal in δ. Since E_{i+1} contains a final segment of S_δ for each i, there must be an $n_0 \in \omega$ such that for $\aleph_1$ many i's $E_{i+1} \supset \{s_n : n \geq n_0\}$. But the E_i's are decreasing hence for each $i \in \aleph_1 \; E_i \supset \{s_n : n \geq n_0\}$. Hence there is a $\beta \in E \cap S_\delta$, which satisfies $cf(\beta) > \aleph_1$ ($\delta \in E$ and point 4 in the definition). β is a limit point of each E_n as $E_n' \supset E_{n+1}$, which implies that $E_n \cap \beta$ is a club for each n. Since $cf(\beta) > \aleph_1$ also $E \cap \beta$ is a club in β, thus verifying the exsistence of a γ in $E \cap \beta$ of cofinality ω which contradicts the minimality of δ. $\square$

Lemma 2.2.30. *Let κ, λ be regular cardinals with $\kappa > \lambda^+$ and $\kappa > \aleph_2$. Assume that $NS_\kappa \restriction E^\lambda_\kappa$ is κ^+-saturated. Then for any stationary $S \subset E^\lambda_\kappa$ and for any $\diamondsuit'_{club}(S)$ sequence $(S_\alpha : \alpha \in S)$ there exists a stationary $S^* \subset S$ such that for every club $C \subset \kappa$ the set*

$$G_{S^*}(C) := \{\alpha \in S^* : \exists \beta < \alpha \, C \supset S_\alpha - \beta\}$$

contains a club intersected with S^*.

Proof. Assume that the lemma is false. Let $S \subset E^\lambda_\kappa$ witness this. We define inductively a modulo NS_κ decreasing sequence of clubs $(C_\alpha : \alpha < \kappa^+)$ and an almost disjoint sequence $(A_\alpha : \alpha < \kappa^+)$ of stationary subsets of S:

If C is a club then $G(C)$ denotes as always the set $\{\alpha \in S : \exists \beta < \alpha\, C \supset S_\alpha - \beta\}$. Let C_0 be a club such that $S - G(C_0)$ is stationary. Set $A_0 := S - G(C_0)$. Now assume that $(C_\beta : \beta < \alpha)$ and $(A_\beta : \beta < \alpha)$ are already defined. First take a club C which is almost contained in every C_β for $\beta < \alpha$ (such a C always exists, take for example the diagonal intersection of the previous C_β). Then consider the stationary $G(C)$ which is almost contained in every $G(C_\beta)$ for $\beta < \alpha$. As $G(C)$ is stationary there exists a club $C_\alpha \subset \kappa$ such that $\{\gamma \in G(C) : \exists \beta < \gamma\, C_\alpha \supset S_\gamma - \beta\}$ does not contain a club intersected with $G(C)$, i.e. there exists a club C_α such that $G(C) - G(C_\alpha)$ remains stationary. Finally let $A_\alpha := G(C) - G(C_\alpha)$. This completes the inductive definition.

Now we obtain a sequence of length κ^+ of stationary subsets A_α, whose intersections are nonsationary, contradicting the κ^+-saturatedness of $NS_\kappa \restriction E^\lambda_\kappa$. □

Now the proof of the full theorem is easy:

Theorem 2.2.31 (**Gitik-Shelah**). *As always let κ and λ be two regular cardinals, $\kappa > \aleph_2$ and $\lambda^+ < \kappa$. Then $NS_\kappa \restriction E^\lambda_\kappa$ is not κ^+-saturated.*

Proof. Assume that $NS_\kappa \restriction E^\lambda_\kappa$ is κ^+-saturated. Then we pick a maximal sequence of almost disjoint stationary subsets S^*, given by our last lemma, which must be of size κ. Then we make them completely disjoint, and glue the $\diamondsuit'_{club}$-sequences together to finally obtain $\diamondsuit^*_{club}(\kappa, \lambda)$ which is a contradiction. □

2.3 The saturation of $NS_{\aleph_1}$

The theorem of Gitik-Shelah leaves the case $\kappa = \aleph_1$ unanswered, thus our next attempt will be to present a solution to the question: May $NS_{\aleph_1}$ be $\aleph_2$-saturated? We show in this section that relative to the existence of a supercompact cardinal, it is consistent that $NS_{\aleph_1}$ is $\aleph_2$-saturated. Hence the result of Gitik and Shelah is best possible. We start with a technical lemma:

Lemma 2.3.1 (**Laver function**). *Suppose that κ is a supercompact cardinal. There exists a function $f : \kappa \to V_\kappa$ such that for every set x and every $\lambda \geq \kappa$ with $x \in H_{\lambda^+}$, there exists a normal measure U on $P_\kappa(\lambda)$ such that $j_U(f)(\kappa) = x$. Such an f is called a Laver function.*

Proof. By contradiction. Thus assume the lemma is false. For each $f : \kappa \to V_\kappa$ let λ_f be the least cardinal such that there exists an $x \in H_{\lambda^+}$ and x witnesses the Non-Laverness of f, i.e. $j_U(f)(\kappa) \neq x$ for every normal measure U on $P_\kappa(\lambda_f)$. Choose a ν, greater than all of the λ_f's and let $j : V \to M$ witness the ν-supercompactness of κ.

Let $\varphi(g, \delta)$ be the following statement: There exists a cardinal α, $g : \alpha \to V_\alpha$ and δ is the least cardinal $\delta \geq \alpha$ for which there exists an x with $x \in H_{\delta^+}$ such that there is no normal measure U on $P_\alpha(\delta)$ with $j_U(g)(\alpha) = x$. Again let λ_g denote this δ. Since κ is the critical point of j we know that each $f : \kappa \to V_\kappa$ in V is also a function $f : \kappa \to (V_\kappa)^M$, hence $f : \kappa \to M$ and since $M^\kappa \subset M^\nu \subset M$, we know by elementarity that $M \models \varphi(f, \lambda_f)$ for all $f : \kappa \to V_\kappa$. Let A be the set of all $\alpha < \kappa$ such that $\varphi(g, \lambda_g)$ holds for all $g : \alpha \to V_\alpha$. By our assumption we know that $\kappa \in j(A)$, hence A cannot be empty.

Now inductively define $f : \kappa \to V_\kappa$ as follows: If $\alpha \in A$ we let $f(\alpha) = x_\alpha$ where x_α witnesses $\varphi(f \restriction \alpha, \lambda_{f\restriction\alpha})$, otherwise $f(\alpha) = \emptyset$.

From now on the proof will be more sketchy. Let $x = jf(\kappa)$. It follows from the definition of f that x witnesses $\varphi(f, \lambda_f)$ in M and hence in V. Let $U := \{x \in P_\kappa(\lambda) : j``\lambda \in j(x)\}$, which is a normal ultrafilter and induces an elementary embedding j_U due to the Solovay-Reinhardt characterization of supercompact cardinals. Thus we can write $j = k \circ j_U$ where $k : Ult_U(V) \to M$ is an elementary embedding. It's not hard to verify that $k(x) = x$ and therfore $j_U(f)(\kappa) = k^{-1}(jf)(\kappa) = k^{-1}(x) = x$ which is a contradiction. □

Next the mother of all the forcing axiom ever studied - Martin's Axiom

Definition 2.3.2 (**Martin's Axiom**). *Suppose that $(P, <)$ is a partial order with the c.c.c and let D be a collection of fewer than $2^{\aleph_0}$ dense subsets of P, then there exists a D-generic filter on P.*

Further let κ be an infinite cardinal, then MA_κ is the following assertion: Let $(P, <)$ be a partial order, satisfying the c.c.c. and let D be a collection of at most κ many dense subsets of P, then there exists a D-generic filter on P.

Note that PFA implies $MA_{\aleph_1}$. We will need only one (of the many interesting consequences) of MA_κ:

Fact 2.3.3 (*[11], Exercise 16.10*). *MA_κ implies that $\kappa < 2^{\aleph_0}$*

Theorem 2.3.4. *If κ is a supercompact cardinal then there is a proper forcing extension in which κ equals $\aleph_2$ and PFA holds.*

Proof. Let $f : \kappa \to V_\kappa$ be a Laver function. We construct a countable support iteration P_κ of $\{\dot{Q}_\alpha : \alpha < \kappa\}$ as follows: At stage α, if $f(\alpha)$ is a pair $((\dot{P}), (\dot{D}))$ of P_α names such that $\dot{P}$ is proper and $\dot{D}$ is a γ-sequence of dense subsets of $\dot{P}$ for some $\gamma < \kappa$ then set $\dot{Q}_\alpha = \dot{P}$. Otherwise let $\dot{Q}_\alpha$ be the trivial forcing. Note that each notion of forcing used in the iteration is proper by definition, hence P_κ is proper and $\aleph_1$ is preserved. Further each P_α (i.e. the iteration of $\{\dot{Q}_\beta : \beta < \alpha\}$) has size less than κ, as $f(\alpha) \in V_\kappa$, hence (due to fact 1.5.18.) P_κ has the κ-c.c. and all cardinals $\geq \kappa$ are preserved. Let G be a generic filter on P_κ. Now we

Claim: In $V[G]$ holds that if P is proper and $D := \{D_\alpha : \alpha < \kappa\}$ with $\gamma < \kappa$ is a family of dense subsets of P, then there is a D-generic filter on P.

First we prove this claim: Let $\dot{P}, \dot{D}$ be P_κ names for P and D. Let $\lambda > 2^{2^{|P|}}$ and w.l.o.g. we may assume that $P \leq \lambda$. Remember that f is a Laver function, therefore there is an elementary embedding $j : V \to M$ with critical point

κ such that $j(\kappa) > \lambda, M^\lambda \subset M$ and $jf(\kappa) = (\dot{P}, \dot{D})$. By our assumption P is proper in $V[G]$ and by lemma 1.5.25. this is witnessed by a club C of elementary submodels $M \prec (H_\eta, ...)$ where $2^{|P|} < \eta < \lambda$. Since $M^\lambda \subset M$ and P_κ has the κ-c.c., $V[G]$ thinks that $M[G]^\lambda \subset M[G]$ (let $f \in M[G]^\lambda$ then f has a name $\dot{f}$, hence each $f \in M[G]^\lambda$ can be seen as a $\overline{f} \in M^\lambda$ which is itself in M and so $f \in M[G]$). Thus C is a club in$M[G]$, witnessing that P is proper in $M[G]$.

Now consider the forcing notion $j(P_\kappa)$ in M. Due to elementarity it is a countable support iteration of length $j(\kappa)$, generated by the Laver function $j(f)$. We note that $j(P_\kappa) \upharpoonright \kappa = P_\kappa$ (as $j \upharpoonright V_\kappa$ is the identity), and as $jf(\kappa) = (\dot{P}, \dot{D})$, P proper in $M[G]$ (which implies that if $\dot{P}$ is a name of P in $M[G]$, then $\dot{P}$ is proper in $j(P_\kappa) \upharpoonright \kappa = P_\kappa$ in M) it follows that $j(\dot{Q})_\kappa = \dot{P}$ and $j(P_\kappa) = P_\kappa * \dot{P} * \dot{R}$ for a forcing notion $\dot{R}$. Thus $j(P_\kappa)$ satisfies the Kunen-Paris schema and if $H * K$ denotes a $V[G]$-generic filter on $\dot{P} * \dot{R}$, then in $V[G * H * K]$ we may extend j to the elementary embedding $\overline{j} : V[G] \to M[G*H*K]$. The filter H on P is $V[G]$-generic, hence has nonempty intersection with every $D_\alpha \in D$. Let $E = \{\overline{j}(p) : p \in H\}$ then E belongs to $M[G * H * K]$ and generates a filter on $\overline{j}(P)$ that is $\overline{j}(D)$-generic. Hence $M[G * H * K] \models$ "there exists a $\overline{j}(D)$ generic filter on $\overline{j}(P)$" and due to elementarity we have $V[G] \models$ "there exists a D-generic filter on P". This is the end of the proof of the claim.

Now the rest is easy: For every $\gamma < \kappa$ let P be the forcing that collapses γ onto ω_1, with countable conditions and for $\alpha < \gamma$ let $D_\alpha := \{p \in P : \alpha \in ran(p)\}$. Due to our claim there exists a bijection between γ and ω_1 and as $V[G]$ preserves κ (P_κ has the κ-c.c.) we conclude that $V[G] \models \kappa = \aleph_2$. The already proven claim thus shows that in $V[G]$ PFA holds. The last thing remaining is to show that $2^{\aleph_0} = \aleph_2$ in $V[G]$. On the one hand we have that $(2^{\aleph_0})^{V[G]} \leq (|P_\kappa|^{\aleph_0})^V = (\kappa^{\aleph_0})^V = \kappa$. On the other hand PFA implies $MA_{\aleph_1}$ and so $2^{\aleph_0} > \aleph_1$ and so $2^{\aleph_0} \geq \aleph_2 = \kappa$, thus $2^{\aleph_0} = \kappa = \aleph_2$ in $V[G]$. □

There exists a notion of iteration, the so called revised support iteration (RCS) for which semiproperness of the $\{\dot{Q}_\beta : \beta < \alpha\}$ implies semiproperness of P_α. We can use the RCS to obtain a result, similar to our last theorem:

Theorem 2.3.5. *If κ is a supercompact cardinal than there is a semiproper forcing extension in which SPFA and $\kappa = \aleph_2$ holds.*

Thus SPFA and PFA are consistent relative to the existence of a supercompact cardinal.

Definition 2.3.6 (Martin's Maximum (MM)). *If $(P, <)$ is a stationary set preserving notion of forcing, and if D is a collection of $\aleph_1$ dense subsets of P then there exists a D-generic filter on P.*

Definition 2.3.7. *A sequence $(M_\alpha : \alpha < \omega_1)$ of countable elementary submodels of $H_\lambda = (H_\lambda, \in, <)$ is called an elementary chain if it satisfies additionally $M_\alpha \subset M_\beta$ for $\alpha \leq \beta$, $M_\alpha \in M_\beta$ for $\alpha < \beta$ and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for each limit α.*

MM is consistent relative to the existence of a large cardinal. This follows from the next theorem:

Theorem 2.3.8 (Shelah). *SPFA implies that every stationary set preserving notion of forcing is also semiproper. As a consequence SPFA implies MM.*

Proof. First we define a new notion: If X is a set of countable elementary submodels of $H_\lambda = (H_\lambda, \in, <)$ then:

$X^\perp := \{M \in [H_\lambda]^\omega : M \prec H_\lambda$ and$N \notin X$ for every countable N that satisfies $M \prec N \prec H_\lambda$ and $N \cap \omega_1 = M \cap \omega_1\}$

Lemma 2.3.9. *Assume SPFA and let $\omega_1 \leq \kappa < \lambda$ with λ regular and sufficiently large. Let $Y \subset [H_\kappa]^\omega$ be stationary and let $X = \{M \in [H_\lambda]^\omega : M \cap H_\kappa \in Y\}$ be the lifting of Y to H_λ. Then there is an elementary chain $(M_\alpha : \alpha < \omega_1)$ of submodels of $(H_\lambda, \in<)$ such that $M_\alpha \in X \cup X^\perp$ for all $\alpha < \omega_1$.*

Moreover the assertion remains true if we consider any $X' \supset X$

Proof. Let P be the forcing that shoots on elementary chain through $X \cup X^\perp$, i.e. the conditions of P are elementary chains $(M_\alpha : \alpha < \gamma < \omega_1)$ in $X \cup X^\perp$ and stronger conditions are extensions of the weaker ones. In order to show that this definition of P does make sense (and is not the trivial forcing) we prove first that the sets $\{(M_\alpha \in X \cup X^\perp : \alpha \leq \gamma) : \gamma \geq \xi\} = D_\xi$ form dense sets in P.

Thus let $(M_\alpha \in X \cup X^\perp : \alpha < \gamma)$ be a sequence of countable length in V. Let $\overline{P}$ be the forcing notion that collapses $|H_\lambda|$ to ω_1 with countable conditions. Let $\overline{G}$ be V-generic over $\overline{P}$. In $V[\overline{G}]$ there exists an elementary chain C of length ω_1 with limit $(H_\lambda)^V$. Since $X \cup X^\perp$ is stationary, and the forcing is proper, and the elementary chain is a club, hence we know that $X \cup X^\perp \cap C$ is itself a stationary subset of $[(H_\lambda)^V]^\omega$ of size $\aleph_1$. Due to the theorem of H. Friedman [7] we know that $(X \cup X^\perp) \cap C$ contains a closed subset of length $\gamma < \omega_1$ for any $\gamma < \omega_1$. This subset is an elementary chain. Thus in $V[\overline{G}]$ holds that for every $\gamma < \omega_1$ there is an elementary chain C of length γ such that $C \subset X \cup X^\perp$. Since $\overline{P}$ is $< \omega_1$-closed and therefore all functions $f : \gamma \to V$ in $V[\overline{G}]$ with γ countable are in V. Hence V contains elementary chains through $X \cup X^\perp$ of any countable length.

The next thing we want to show is that the forcing notion P is semiproper: Let $\mu > \lambda$, let $M \prec (H_\mu, \in, <)$ be countable with $P \in M$ and let $p \in P \cap M$. Due to fact 1.5.31. we only need to find a $q \leq p$ that is (M, P)-semigeneric.

Claim: There exists a countable N with $N \prec M \prec H_\mu$ such that $N \cap \omega_1 = M \cap \omega_1$ and $N \cap H_\lambda \in X \cup X^\perp$

We proof this claim: If $M \cap H_\lambda \in X^\perp$ then set $N = M$. If not then there exists a countable $N' \prec H_\lambda$ such that $M \cap H_\lambda \subset N'$, $N' \cap \omega_1 = M \cap \omega_1$ and $N' \in X$. Let N be the Skolem hull of $M \cup (N' \cap H_\kappa)$ in $(H_\mu, \in, <)$. Then $N \cap H_\kappa \supset (M \cup (N' \cap H_\kappa)) \cap H_\kappa = (M \cap H_\kappa) \cup (N' \cap H_\kappa)$ and as $M \cap H_\lambda \subset N'$, $N' \cap H_\kappa \supset M \cap H_\kappa$, hence $N \cap H_\kappa \supset N' \cap H_\kappa$. On the other hand

$N \cap H_\kappa \subset N'$ as for every Skolem function h for H_μ, $h \cap H_\kappa \in M \cap H_\lambda \subset N'$. Thus $N \cap H_\kappa = N' \cap H_\kappa$, which gives us $N \cap \omega_1 = N' \cap \omega_1$ and $N \cap H_\kappa \in Y$ (as $N' \prec H_\lambda$, $N' \in X \Rightarrow N' \cap H_\kappa \in Y \Rightarrow N \cap H_\kappa \in Y$) hence $N \cap H_\lambda \in Y$ and this ends the proof of our claim.

Using this claim we can finish the proof of lemma 2.3.9. We can find a decreasing sequence of conditions $p_n \in N$ with $p = p_0$ such that $p_n = (M_\alpha : \alpha < \gamma_n$ and such that every name for a countable ordinal in N is decided by some p_n (as an ordinal in N). We may also assume that $\bigcup_{n\in\omega} \bigcup_{\alpha \le \gamma_n} M_\alpha = N \cap H_\lambda$. Set $\gamma = \bigcup \gamma_n$ and $M_\gamma = N \cap H_\lambda$, then as already shown in the proof of our claim $M_\gamma \in X \cup X^\perp$, hence $q = (M_\alpha : \alpha \le \gamma)$ is a condition in P, stronger than p and by construction (N, P)-semigeneric. Since $M \subset N$ and $M \cap \omega_1 = N \cap \omega_1$, q is also (M, P)-semigeneric which is all we wanted to prove our lemma $\square$

Now we are finally ready to finish the proof of the theorem. We go on indirectly, assume that SPFA holds, further that Q is a stationary set preserving notion of forcing, that is not semiproper. Let κ be sufficiently large so that all Q names for countable ordinals are in H_κ.

By our assumption Q is not semiproper, hence there is a condition $p \in Q$ for which the set $Y = \{M \in [H_\kappa]^\omega : M \prec H_\kappa \wedge \text{ there is no } (M,Q)\text{-semigeneric } q \le p\}$ is stationary. Let $\lambda > \kappa$ be regular and lift the set Y to H_λ, i.e. consider $X = \{M \in [H_\lambda]^\omega : M \cap H_\kappa \in Y\}$ which remains stationary. Due to our choice of κ we may rewrite: $X = \{M \in [H_\lambda]^\omega : M \cap H_\kappa \prec H_\kappa \wedge \text{ there is no } (M \cap H_\kappa, Q)\text{-semigeneric } q \le p\ \} = \{M \in [H_\lambda]^\omega : M \cap H_\kappa \prec H_\kappa \wedge \text{ there is no } (M,Q)\text{-semigeneric } q \le p\ \}$. By our last lemma there exists an elementary chain $(M_\alpha : \alpha < \omega_1)$ in $X \cup X^\perp$. We

Claim The set $\{\alpha < \omega_1 : M_\alpha \in X\}$ is nonstationary in $V[G]$ hence in V.

We proceed by proving the claim: Towards a contradiction we assume that S is stationary. Let G be a generic filter on Q, $p \in G$. Remember that Q is assumed to be stationary set preserving, hence S remains stationary in $V[G]$. Let $\dot\delta_\xi$, $\xi \in \omega_1$ be an enumeration of all the names in $\bigcup_{\alpha<\omega_1} M_\alpha$ for countable ordinals. We define in $V[G]$ the set $C := \{\alpha < \omega_1 : M_\alpha \cap \omega_1 = \alpha\} \cap \{\alpha < \omega_1 : \forall \xi < \alpha (\dot\delta_\xi \in M_\alpha)\} \cap \{\alpha < \omega_1 : \forall \xi < \alpha\, \dot\delta_\xi^G < \alpha\} \cap \{\alpha < \omega_1 : \text{ each name of a countable ordinal in } M_\alpha \text{ is in } \{\dot\delta_\xi : \xi < \alpha\}\}$. It is straightforward to check that each of the sets is a club, hence C is a club itself. Moreover if $\alpha \in C$ then there is a $q \in G$ below p such that for every $\dot\delta_\xi \in M_\alpha\, q \Vdash \exists \beta \in \alpha \dot\delta_\xi = \beta$ and as $\alpha \subset M_\alpha$ we have $q \Vdash \exists \beta \in M_\alpha\, \dot\delta_\xi = \beta$. Thus q is (M_α, Q)-semigeneric. Hence if S is indeed stationary then there is an $\alpha \in C \cap S$ which leads to an $M_\alpha \in X$ which includes a (M_α, Q)-semigeneric condition $\le p$, which is a contradiction to the way we defined X. Hence S has to be nonstationary and our claim is proven.

With the last claim in mind we derive that there exists an elementary chain $(M_\alpha : \alpha < \omega_1)$ in $X^\perp$. Let $\mu > \lambda$ be sufficiently large, pick a countable $M \prec (H_\mu, \in, <, Q, (M_\alpha : \alpha < \omega_1))$ with $p \in M$ and set $\delta = M \cap \omega_1$, which implies $M_\delta \subset M \cap H_\lambda$ and $\delta = M_\delta \cap \omega_1$. Since $M_\delta \in X^\perp$ we know that

$M \cap H_\lambda \notin X$ and by the definition of X there exists an (M, Q)-semigeneric q below p. So for each $p \in Q$ there is a club of countable $M \prec (H_\mu, \in, <)$ and an (M, Q)-semigeneric $q \leq p$. If we take the diagonal intersection then we obtain a club of countable $M \prec (H_\lambda, \in, <)$ such that for every $p \in M$ there is an (M, Q)-semigeneric condition below p, establishing the semiproperness of Q.

□

Theorem 2.3.10. *MM implies that $NS_{\aleph_1}$ is $\aleph_2$-saturated.*

Proof. Let $\{A_i : i < \delta\}$ be a maximal, almost disjoint collection of stationary subsets of $\omega 1$. Assume that MM holds. Our goal is to find a subset $Z \subset \delta$ such that $|Z| \leq \aleph_1$, and $\sum_{i \in Z} A_i$ contains a club ($\sum$ denotes the diagonal union). This suffices because: Suppose that S is a stationary subset, that doesn't equal any A_i. Suppose further that $S \cap A_i \in NS$ for each i then $\sum_{i \in \delta}(S \cap A_i) \in NS$, and this contradicts the fact that $\sum A_i$ contains a club.

Let P be the set of all pairs (p, q) with the following properties:

1. $q : \gamma + 1 \to \delta$ for some $\gamma < \omega_1$
2. $p \subset \omega_1$ is a closed subset of ω_1 of countable size such that $\alpha \in p \Rightarrow \alpha \in \bigcup_{\xi < \alpha} A_{q(\xi)}$

On P we consider the following ordering: $(q', p') < (q, p)$ if and only if $q' \supset q$ and there is an α such that $p' \cap \alpha = p$, i.e. p' is an end extension of p.

Thus P is a 2-step iteration $P = Q * R$ where Q collapses $|\delta|$ to ω_1 and R shoots a club through $\dot{S} := \sum_{i \in \delta} A_i$, the diagonal union of the A_i. P is stationary set preserving: Let $A \subset \omega_1$ be stationary then, by maximality there is some $i < \delta$ such that $A \cap A_i$ is stationary. As Q is ω-closed, it preserves stationarity, hence $A \cap A_i \cap \dot{S}$ is stationary inV^Q. We further

Claim: The forcing for shooting a club through the stationary set $\dot{S}$ preserves stationarity of any stationary subset of $\dot{S}$.

We prove the claim: To faciliate our notation we write S instead of $\dot{S}$ (our ground model is V^Q but we will not denote its elements with a dot) Our forcing notion consists of closed subsets $p \subset S$ of countable length (they exist due to H. Friedmans theorem [7]), and p is stronger than q if and only if there is an $\alpha < \omega_1$ such that $p \cap \alpha = q$. We shall show that if $T \subset S$ is stationary and if p is a condition such that $p \Vdash \dot{C}$ is a club in ω_1 then there is a $q \leq p$ and a $\lambda \in T$ such that $q \Vdash \lambda \in \dot{C}$.

We construct a sequence of sets of conditions like this: Set $A_0 = \{p\}$ and $A_\alpha := \bigcup_{\beta < \alpha} A_\beta$ if α is a limit. If A_α is already defined we let $\gamma_\alpha := sup\{max(q) : q \in A_\alpha\}$ and further we let $r(q)$ be a condition stronger than q such that $max(r) > \gamma_\alpha$ and there exists a $\beta(q) > \gamma_\alpha$ with $r(q) \Vdash \beta(q) \in \dot{C}$. We may additionally assume that $max(r(q)) > \beta(q)$. Then we let $A_{\alpha+1} := A_\alpha \cup \{r(q) : q \in A_\alpha\}$.

Consider the set $C := \{\lambda < \omega_1 : \alpha < \lambda \Rightarrow \gamma_\alpha < \lambda\}$. C is a club hence $C \cup T$ contains limit ordinals, let λ be one of these, then there is a sequence of α_n that converges to λ. As $\lambda \in C$ we also have that $\gamma_{\alpha_n} \to \lambda$. Let $p_{\alpha_n} \in A_{\alpha_n}$ be a corresponding, decreasing sequence of conditions, then $sup\{max(p_{\alpha_n}) : \alpha_n < \lambda\} = \lambda$, hence $q := \bigcup p_{\alpha_n} \cup \{\lambda\}$ is a condition and we have: $q \Vdash \forall n < \omega \beta(p_{\alpha_n}) \in \dot{C}$ which leads to $q \Vdash \exists \beta = lim \beta(p_{\alpha_n}) \in \dot{C}$ but $lim \beta(p_{\alpha_n}) = lim \gamma_{\alpha_n} = \lambda$ and therefore $q \Vdash \lambda \in \dot{C} \cap S$ which gives us the claim.

Therefore $A \cap A_i \cap \dot{S}$ remains stationary in V^P, and so does A.

Now define for each $\alpha < \omega_1$: $D_\alpha := \{(q,p) \in P : \alpha \leq max(p)\}$. D_α is dense for each $\alpha < \omega_1$, and since MM holds, there is a generic filter G on P that meets every D_α. Set

$$F := \bigcup \{q : (q,p) \in G \text{ for some } p\} \quad C := \bigcup \{p : (q,p) \in G \text{ for some } q\}$$

Then C is a club and F a function $\omega_1 \to |\delta|$ and moreover $C = \{\alpha : \exists \xi < \alpha (\alpha \in A_{F(\xi)})\}$. Now it follows that if A is an arbitray stationary set then $A \cap C$ is a stationary subset of C, and due to the closedness of $NS_{\aleph_1}$ under diagonal unions we know that there must be $\beta < \omega_1$ such that $C \cap A \cap A_{F(\beta)}$ is stationary which means that $\{A_{F(\alpha)} : \alpha < \omega_1\}$ is already a maximal pairwise almost disjoint collection. Thus $NS_{\aleph_1}$ is $\aleph_2$-saturated. $\square$

Chapter 3

The saturation of the generalized NS

3.1 The generalized Splitting Theorem

The first result we obtained, concerning the question of the saturation of the nonstationary ideal on κ, was Solovays Splitting Theorem, asserting that NS_κ cannot be κ-saturated. It is reasonable to think about an analogue in the more general notion of stationarity, defined in 1.1.13.

Our first expectation would be something like „every stationary subset of $P_\kappa(\lambda)$ can be partitioned into $(\lambda)^{<\kappa}$ pairwise disjoint stationary subsets“. However this is impossible as we will see after some lemmas.

Lemma 3.1.1. *There exists a stationary set $S \subset [\omega_2]^\omega$ of size $\aleph_2$.*

Proof. Due to the Theorem of Kueker the clubs and the strong clubs coincide on $[\omega_2]^\omega$. Thus there is for every club C a functional $F : [\omega_2]^{<\omega} \to \omega_2$ such that $C = C_F$.

For each uncountable $\alpha < \omega_2$ let $f_\alpha : \alpha \to \omega_1$ be a bijection and set

$$X_{\alpha,\xi} := \{\beta < \alpha \, : \, f_\alpha(\beta) < \xi\}.$$

Now we claim that the set

$$S := \{X_{\alpha,\xi} \, : \, \alpha < \omega_2, \xi < \omega_1\}$$

is stationary.

Let $f : [\omega_2]^{<\omega} \to [\omega_2]^\omega$ be an arbitrary function. It suffices to show that there are $\alpha < \omega_2$ and $\xi < \omega_1$ such that $X_{\alpha,\xi}$ is closed under f. We first claim that there is an $\alpha < \omega_2$ such that α is closed under f: Let $\alpha_0 := \omega_1$ and consider the union $\bigcup_{e \in [\omega_1]^{<\omega}} g(e)$. As ω_2 is regular there exists an $\alpha_1 < \omega_2$ such that $\bigcup g(e) \subset \alpha_1$. We continue like this: Again build $\bigcup_{e \in [\alpha_1]^{<\omega}} g(e)$, for which an

$\alpha_2 < \omega_2$ exists such that $\bigcup g(e) \subset \alpha_2$ and so on. If we let $\alpha := sup_{i\in\omega}\alpha_i$ then this is our desired ordinal less than ω_2, which is closed under f.

Next let $\xi_0 := \omega$. Consider X_{α,ξ_0} and take the union $\bigcup_{e\in[X_{\alpha,\xi}]^{<\omega}} g(e)$. As X_{α,ξ_0} is, by our assumption countable, $\bigcup g(e)$ is countable as well, moreover $\bigcup g(e) \subset \alpha$, hence there exists an $\alpha_1 < \omega_1$ such that $\bigcup g(e) \subset X_{\alpha,\xi_1}$. We loop this construction to obtain a sequence $X_{\alpha,\xi_0}, X_{\alpha,\xi_1}, ...$ and if we let $\xi := sup_{i\in\omega}\xi_i$ then $X_{\alpha,\xi}$ is closed under f, which ends the proof of the lemma.

□

The lemma above shows that a partition of a stationary subset of $P_\kappa(\lambda)$ into $\lambda^{<\kappa}$ many sets might be impossible. For instance if κ and λ are regular cardinals such that $\lambda^{<\kappa} > \kappa$ and a stationary subset S of $P_\kappa(\lambda)$ of size λ exists then a partition of S into $\lambda^{<\kappa}$ many parts is impossible.

But there is even another reason why our first attempt of the generalization of Solovays Theorem fails; Large cardinals affect the size of $sat(NS)$ too.

Theorem 3.1.2 (Gitik). *([8]) If there exists a model of ZFC in which a supercompact cardinal exists then there exists a model of ZFC and a stationary set $S \subset P_\kappa(\kappa^+)$ in it, such that S cannot be split into κ^+ many stationary sets.*

Although we know now these delimiting results a similar assertion to Solovays theorem is still provable.

Lemma 3.1.3. *If κ is a regular cardinal and λ a cardinal $\geq \kappa$. Then let $E := \{x \in P_\kappa(\lambda) : |x \cap \kappa| = |x|\}$. We have*

(i) E is stationary

(ii) If κ is a successor cardinal then E contains a club.

Proof. For (i) let C be a club, $x_0 \in C$. Assume that $|x_0 > |x_0 \cap \kappa|$. Let $|x_0| = \xi_0 < \kappa$, set $y_1 := x_0 \cup \xi_0$ and notice that $|y_1 \cap \kappa| = |y_1|$. Let $x_1 \in C$ be such that $x_1 \supset y_1$ and check again if $|x_1| = |x_1 \cap \kappa|$. If not then let $y_2 := x_1 \cup \xi_1$ where $\xi_1 = |x_1|$. If we loop this contruction then it yields an increasing sequence $(x_i)_{i\in\omega}$ and $x := \bigcup_{i\in\omega} x_i$ is in $C \cap E$.

For (ii) let $E' := \{x \in P_\kappa(\lambda) : |x| = |x \cap \kappa| = |\kappa^-|\}$ where κ^- denotes the predecessor of κ. It is easy to see that E' is a club. □

Lemma 3.1.4. *Assume the same sitution as in the lemma above and define E in the same way. Then every stationary subset $S \subset E$ can be split into λ-many disjoint stationary sets.*

Proof. If $\kappa = \lambda$ then for each club $C \subset P_\kappa(\lambda)$, the set $\tilde{C} := \{x \in C : x \in On\}$ is a club subset of C. Hence if S is stationary then $\tilde{S} := \{x \in S : x \in On\}$ remains stationary in $P_\kappa(\kappa)$, and can be considered as a stationary set in κ. Therefore there exists a partition of S $\{S_i : i < \kappa\}$ and each S_i is still stationary in $P_\kappa(\kappa)$ which is all we wanted.

Thus assume $\kappa > \lambda$ and assume first that λ is a regular cardinal. Let $S \subset E$ be an arbitrary stationary subset. For each $x \in S$ we may choose a bijection $f_x : x \to x \cap \kappa$. Now an elegant trick: for arbitrary $x \in S$ and $\alpha \in x$ we set

$$g_\alpha(x) := f_x(\alpha).$$

It is immediate that g_α is a function, defined on the stationary set $S \cap \{x \in P_\kappa(\lambda) : \alpha \in x\}$, and $g_\alpha(x) = f_x(\alpha) \in x \cap \kappa \subset x$, hence g_α is a regressive function on $S \cap \{x \in P_\kappa(\lambda) : \alpha \in x\}$. By Fodors theorem there exists for each $\alpha < \lambda$ a stationary set $S_\alpha \subset S$ such that g_α is constant on it with value $\gamma_\alpha < \kappa$.

Now there must be a set $X \subset \lambda$ of size λ such that $\gamma_\alpha = \gamma$ for each $\alpha \in X$, and we claim that the corresponding subsets $\{S_\alpha : \alpha \in X\}$ are pairwise disjoint. If not then $x \in S_\alpha \cap S_\beta$ and $g_\alpha(x) = g_\beta(x) = \gamma$ but $g_\alpha(x) = f_x(\alpha)$ and $g_\beta(x) = f_x(\beta)$ and f_x is one to one, which is a contradiction. Thus every stationary subset $S \subset E$ can be split into λ-many pairwise disjoint stationary sets.

If λ is singular then the argument just described shows that E can be partitioned into γ many subsets, where γ is an arbitrary cardinal less than λ. If E is not scissible into λ many parts then $sat(NS \upharpoonright E) = \lambda$, but this is a contradiction as $sat(I)$, if infinite, is always a regular cardinal. Hence E can be devided into λ many parts as well. □

Lemma 3.1.5. *Let κ be as always a regular cardinal and let $\lambda \geq \kappa$. Assume that GCH holds. If $cf(\lambda) < \kappa$ then every stationary set in $P_\kappa(\lambda)$ can be partitioned into λ^+ disjoint stationary sets.*

Proof. Note first that the regularity of κ and $cf(\lambda) < \kappa$ implies that $\lambda \neq \kappa$ hence $\lambda > \kappa$. We observe that $|P_\kappa(\lambda)| = \lambda^{<\kappa} = sup_{\mu<\kappa}\lambda^\mu$. And as $cf(\lambda) < \kappa < \lambda$ we may conclude with GCH that $sup_{\mu<\kappa}\lambda^\mu = \lambda^+$. Moreover every unbounded $Y \subset P_\kappa(\lambda)$ has size λ^+ as $P_\kappa(\lambda) = \bigcup_{x\in Y} P(x)$.

Let $(f_\alpha)_{\alpha<\lambda^+}$ be an enumeration of the set of all functions $f_\alpha : [\lambda]^{<\omega} \to P_\kappa(\lambda)$ such that each function appears cofinally often. We remember lemma 1.1.20. which states that for each club $C \subset P_\kappa(\lambda)$ there is a function $f : [\lambda]^{<\omega} \to P_\kappa(\lambda)$ such that the club of the closure points $C_f \subset C$. Thus for every club $C \subset P_\kappa(\lambda)$ and every $\gamma < \lambda^+$ there is an $\alpha > \gamma$ such that $C \supset C_{f_\alpha} = \{x : f(e) \subset x \text{ for all finite } e \subset x\}$.

Now let $S \subset P_\kappa(\lambda)$ be stationary. We define inductively sequences $(x_\xi^\alpha : \xi < \alpha) \subset S \cap C_{f_\alpha}$ for each $\alpha < \lambda^+$, such that $(x_\xi^\alpha : \xi < \alpha)$ and $(x_\xi^\beta : \xi < \beta)$ are pairwise disjoint if $\alpha \neq \beta$ (note here that $S \cap C_{f_\alpha}$ is an unbounded set of size λ^+, hence the pairwise disjointness causes no problem). For each $\xi < \lambda^+$ we set

$$S_\xi := \{x_\xi^\alpha : \xi < \alpha < \lambda^+\}.$$

The S_ξ form a set of size λ^+ of pairwise disjoint subsets of S. What remains to show is that each S_ξ is stationary. Fix an S_ξ, if C is an arbitrary club then there is an $\alpha > \xi$ such that $C \supset C_{f_\alpha}$, and $x_\xi^\alpha \in S_\xi \cap C_{f_\alpha}$, hence each S_ξ is stationary. □

In the light of 3.1.1 and 3.1.2 the next theorem is our best possible result-not as good as expected though:

Theorem 3.1.6. *Let κ be a regular uncountable cardinal and let $\lambda \geq \kappa$. Then the following holds:*

(i) $P_\kappa(\lambda)$ can be partitioned into λ pairwise disjoint stationary subsets.

(ii) If κ is a successor cardinal then every stationary subset of $P_\kappa(\lambda)$ can be split into λ disjoint stationary subsets.

(iii) If GCH holds then $P_\kappa(\lambda)$ can be partitioned into $\lambda^{<\kappa}$ stationary subsets.

Proof. (i) Follows directley from 3.1.4

(ii) Let $S \subset P_\kappa(\lambda)$ be stationary. Then as κ is a successor cardinal, E contains a club, hence $S \cap E$ is stationary and by 3.1.4 can be split into λ disjoint stationary sets.

(iii) If $cf(\lambda) < \kappa$ then by the last lemma $P_\kappa(\lambda)$ can be split into λ^+ many disjoint stationary subsets. In this case also $\lambda^{<\kappa} = \lambda^+$ holds by the GCH. Hence (iii) is true. If $cf(\lambda) \geq \kappa$ then aagain by the GCH for all $\mu < \kappa$ $\lambda^\mu = \lambda$ and part (i) of the theorem applies. □

3.2 The saturation of the generalized NS

We enventually arrived at the highlight of this work. When considering the result of Gitik-Shelah, it is natural to ask wheter their result still holds, when exchanging the nonstationary ideal on κ with the the nonstationary ideal on $P_\kappa(\lambda)$. This chapter is devoted to the answer of this question. Since it's proof is long we will give a short preview of the things to come.

Question: If $\kappa \leq \lambda$ is a regular cardinal, NS the nonstationary ideal on $P_\kappa(\lambda)$. May NS be λ^+-saturated?

Answer: No, it is impossible, unless $\kappa = \lambda = \omega_1$ (where 2.3.10. applies).
We show this answer in this way:

- First we will prove a result of Burke-Matsubara stating that if

  (i) κ limit and $cf(\lambda) > \kappa$ or

  (ii) κ a successor, $\kappa \geq \omega_2$ and $cf(\lambda) \geq \kappa$

  then NS cannot be λ^+-saturated. In the next section we consider the case where

- $\kappa = \omega_1$ and $\lambda > \aleph_1$ is arbitrary. We show again that NS cannot be λ^+-saturated.

- $cf(\lambda) < \kappa$, where κ is arbitrary but regular. We even show that NS cannot be λ^{++}-saturated.

- The only remainig case is: κ a limit, and $cf(\lambda) = \kappa$. Then NS cannot be λ^+-saturated.

3.2.1 The Result of Burke and Matsubara

If we look back to the proof of Corollary 2.2.20. we can detect the following strategy: Define a subset S of κ which is so chosen that it changes the cofinality of κ in $Ult_G(V)$ and hence in $V[G]$ (where G is a generic filter for the forcing below S), in order to obtain a contradiction to Corollary 2.2.19. This argument does also work (with the right assumptions) in the more general frame: the nonstationary ideal NS on $P_\kappa(\lambda)$.

For practical reasons we state Cor 2.2.19. again:

Theorem 3.2.1 (**Shelah**). *Let λ be a regular cardinal and let P be a notion of forcing that preserves λ^+. Then for all generic $G \subset P$: $V[G]$ satisfies $cf(|\lambda|) = cf(\lambda)$.*

There exists a variation of the theorem above, which works for singular λ under different circumstances:

Theorem 3.2.2 (**Cummings**). *([3]) Suppose that λ is a singular cardinal and P is a notion of forcing that preserves stationary subsets of λ^+. Then:*

$$\Vdash_P cf(|\lambda|) = cf(\lambda).$$

Note that any notion of forcing P that satisfies the λ^+-chain condition satisfies the hypothesis of the both theorems above.

Theorem 3.2.3. *([4], Theorem 2.25. pp 903) Let I be a normal ideal on $P_\kappa(\lambda)$ and assume that I is λ^+-saturated. Then I is precipitous and further the generic ultrapower M=$Ult_G(V)$ is closed under sequences of length λ, i.e. $M^\lambda \cap M = M^\lambda \cap V[G]$.*

Lemma 3.2.4. *Let κ and λ be cardinals, and let κ be regular.*

(i) Suppose that $cf(\lambda) > \kappa$ and that μ, ν are regular cardinals less than κ. Let

$$S = \{x \in P_\kappa(\lambda) : |x| = |x \cap \kappa|,\ cf(x \cap \kappa) = \mu,\ \text{and } cf(sup(x) = \nu\}$$

Then S is stationary in $P_\kappa(\lambda)$

(ii) Suppose that $\kappa = \theta^+$ and $cf(\lambda) \geq \kappa$. If we let

$$S = \{x \in P_\kappa(\lambda) : cf(x \cap \kappa) = cf(sup(x)) \neq cf(\theta)\}$$

then S is stationary.

Theorem 3.2.5 (**Burke-Matsubara**). *([2]) Let NS denote the nonstationary ideal on $P_\kappa(\lambda)$, where κ is a regular cardinal and λ a cardinal $\geq \kappa$. Assume further that one of these two assumptions holds:*

(i) $cf(\lambda) > \kappa$ and κ is a limit

(ii) κ is a successor cardinal, $\kappa \geq \omega_2$ and $cf(\lambda) \geq \kappa$

Then the nonstationary ideal NS is not λ^+-saturated.

Proof. (i):Thus let $cf(\lambda) > \kappa$ and let κ be a limit. Assume to the contrary that NS is λ^+-saturated. Then by lemma 3.2.4 the set $S := \{x \in P_\kappa(\lambda) : |x| = |x \cap \kappa|, cf(x \cap \kappa) = \mu$ and $cf(sup\, x) = \nu\}$ is stationary in $P_\kappa(\lambda)$. Let P be the forcing with NS-positive sets below S and let G be a generic filter.

The next thing we observe is that in the ultrapower $Ult_G(V)$ $cf(\kappa)$ is represented by the function $f : x \mapsto cf(ot(\kappa \cap x))$ because then $jf : x \mapsto cf(ot(j(\kappa) \cap x))$ and $[f] = jf(f``\lambda) = cf(ot(j(\kappa) \cap j``\lambda)) = cf(ot(\kappa)) = \kappa$.

Further $cf(\lambda)$ is represented by the function $g : x \mapsto cf(sup\, x)$ as $jg : x \mapsto cf(sup\, x)$ and $[g] = jg(j``\lambda) = cf(sup(j``\lambda)) = cf(\lambda)$. This together with the representation of κ we obtained in lemma 1.4.16 tells us that

$$Ult_G(V) \cong M \models |\lambda| = |\kappa| \wedge cf(\kappa) = \mu \wedge cf(\lambda) = \nu$$

As M is closed under $V[G]$ sequences of length λ we know that:

$$V[G] \models |\lambda| = |\kappa| \wedge cf(\kappa) = \mu \wedge cf(\lambda) = \nu$$

And as κ is a limit cardinal it remains a cardinal in M hence in $V[G]$. Thus

$$V[G] \models |\lambda| = \kappa \wedge cf(\kappa) = \mu \wedge cf(\lambda) = \nu$$

which leads to

$$V[G] \models cf(|\lambda|) = cf(\kappa) \neq cf(\lambda)$$

and this is a contradiction to Shelah's theorem if λ is regular. If λ is singular then this is a contradiction to Cumming's theorem.

(ii): Assume again that NS is λ^+-saturated. If $\kappa = \mu^+$ and $\mu \geq \omega_1$ and $cf(\lambda) \geq \kappa$ then $S = \{x \in P_\kappa(\lambda) : cf(x \cap \kappa) = cf(sup(x)) \neq cf(\mu)\}$ is stationary due to lemma 3.2.4. If we force again with NS-positive sets below S then $S \in G$ (where G is the generic filter) hence

$$Ult_G(V) \cong M \models |\lambda| = \mu \wedge cf(\lambda) \neq cf(\mu)$$

as the set $\{x : |ot(x)| = \mu\} \in G$ and as $\{x : cf(sup(x)) \neq cf(\mu)\} \supset S \in G$. Because M is closed under $V[G]$-sequences of length λ we know that

$$V[G] \models |\lambda| = \nu \wedge cf(\lambda) \neq cf(\mu)$$

. Again this contradicts either the theorem of Shelah or the theorem of Cummings. □

3.2.2 Mutual stationarity

Definition 3.2.6. *Let A be a set. An Algebra $\mathcal{A}$ on A is a structure $(A, f_{i<\omega})$, where $f_{i<\omega} : A^i \to A$. A is the so called universe of $\mathcal{A}$. Equivalently an algebra is a structure (A, F) where F is a operation on A. A subalgebra $\mathcal{B} \prec \mathcal{A}$ is a subset of the universe of $\mathcal{A}$ which is closed under the operation F.*

Definition 3.2.7. *Let K be a set of regular uncountable cardinals, let $sup(K) := \delta$, and let $S_\kappa \subset \kappa$ for each $\kappa \in K$. Then the collection $(S_\kappa : \kappa \in K)$ is mutually stationary if for every algebra $\mathcal{A}$ on δ there is an $\mathcal{N} \prec \mathcal{A}$ such that*

$$\forall \kappa \in N \cap K \quad sup(N \cap K) \in S_\kappa$$

where N denotes the universe from $\mathcal{N}$.

We will derive some easy consequences from the notion of mutual stationarity:

Lemma 3.2.8. *Let K be again a set of regular cardinals and S_κ be subsets of κ. Then the following holds:*

(i) If $(S_\kappa : \kappa \in K)$ is a sequence such that for every κ: S_κ is a club in κ then the sequnce of the S_κ is mutually stationary.

(ii) If $(S_\kappa : \kappa \in K)$ is mutually stationary then S_κ is a stationary subset of κ for each $\kappa \in K$.

Proof. (i) Suppose $\mathcal{A}$ is an algebra on $\delta := sup(K)$. Choose a countable subalgebra $\mathcal{N}_0 \prec \mathcal{A}$.Let N_0 denote the universe of $\mathcal{N}_0$. If $K \cap N_0 = \emptyset$ we are done, thus let $\kappa \in K \cap N_0$. If $sup(\kappa \cap N_0) \in S_\kappa$ we are finished again, thus assume that $sup(\kappa \cap N_0) \notin S_\kappa$. Since the S_κ is a club, there exists an $x_0 \in S_\kappa$ such that $x_0 > sup(\kappa \cap N_0)$. Build now $N_1 := \langle N_0 \cup \{x_0\} \rangle$ (i.e. the subalgebra generated by the elements of N_0 and x_0) and continue with checking wheter $sup(N_1 \cap \kappa) \in S_\kappa$ or not. In the latter case there is an $x_1 \in S_\kappa$ such that $x_1 > sup(N_1 \cap \kappa)$ and we set $N_2 := \langle N_1 \cup \{x_1\} \rangle$. We obtain this way an increasing chain of subalgebras $\mathcal{N}_i$ of $\mathcal{A}$: $\mathcal{N}_0 \prec \mathcal{N}_1 \prec \ldots$. Let $\tilde{N} := \bigcup N_i$ then $\tilde{N}$ is closed under the operations of (A), hence universe of a subalgebra $\tilde{\mathcal{N}}$ and we have that $sup(\tilde{N} \cap \kappa) = sup(\kappa \cap \bigcup N_i) = sup(\kappa \cap N_i) \in S_\kappa$.

This construction can easily be generalized to handle the case where $N \cap \kappa$ has more than one element and we are finished.

(ii) We start with a

Claim: Let C be a club on $\kappa \in K$. Then there exists an Algebra $\mathcal{A}$ on $\delta := sup(K)$ such that for all nonempty $\mathcal{N} \prec \mathcal{A}$: $\kappa \in N$ and if $sup(N \cap \kappa) \in \kappa$ then $sup(N \cap \kappa) \in C$.

Using this claim we can continue as follows: Let C be a club on κ. We want to show that $C \cap S_\kappa \neq \emptyset$ which would justify the stationarity of S_κ. We choose for C an algebra $\mathcal{A}$ which satisfies the properties of our claim. Due to the mutual stationarity of the S_κ there is an $\mathcal{N} \prec \mathcal{A}$ such that $sup(N \cap \kappa) \in S_\kappa$ which implies that $sup(N \cap \kappa) \in \kappa$, hence $sup(N \cap \kappa) \in C$.

What remains to prove is our claim: Consider $(E^\kappa_{\aleph_0} \cap C)$ and for each $x_i \in E^\kappa_{\aleph_0} \cap C)$ a cofinal sequence $(x_i^j)_{j<\omega} \to x_i$. For each $\alpha < \kappa$ there is an $x_i \in E^\kappa_{\aleph_0} \cap C)$ such that x_i is the least element of $E^\kappa_{\aleph_0} \cap C) \geq \alpha$. Define now

$$f_1(\alpha) := \begin{cases} x_i^j & \text{if } \alpha \neq x_i^j \ \forall j \text{ where } x_i^j \text{ is the least element such that } x_i^j > \alpha \\ x_i^{j+1} & \text{if } \alpha = x_i^j \\ \beta & \text{if } \alpha = x_i \text{ or } \alpha > \kappa \text{ and } \beta \text{ is any ordinal bigger than } \kappa \end{cases}$$

and $f_2(\alpha, \beta) = \kappa \ \forall \alpha, \beta \in \delta$, $f_3(\alpha, \beta, \gamma) = \kappa \ \forall \alpha, \beta, \gamma \in \delta$... and so on. Then $(\delta, f_1, f_2, \dots)$ has the desired attributes. $\square$

The last lemma states that mutual stationarity is a property stronger than the usual stationarity but weaker than closed unboundedness. The concept of mutual stationarity can be reformulated by the means of the notion of a strongly closed unbounded set (a strong club). A subset $X \subset P_\kappa(\lambda)$ is strongly closed and unbounded if there is an algebra $\mathcal{A}$ on λ such that $X = \{N \in P_\kappa(\lambda) : N \prec \mathcal{A}\}$. A set X is strongly (or generally) stationary if and only if it has, as one would expect, nonempty intersection with every strong club, i.e. for each algebra $\mathcal{A}$ there is an $N \prec \mathcal{A}$, $N \in P_\kappa(\lambda)$ such that $N \in S$. It is immediate that $\{S_\kappa : \kappa \in K\}$ is mutually stationary if and only if the set $\{X \subset sup(K) : \forall \kappa \in X \cap K (sup(X \cap \kappa) \in S_\kappa)\}$ is a strongly stationary subset of $P(sup(K))$. Using the following lemma we will obtain another despriction of mutual stationarity:

Lemma 3.2.9 *([6], Lemma 0)***.** *Let $\mu \leq \kappa < \lambda$ be regular cardinals, $\mathcal{F}_s(\lambda, \mu)$, $\mathcal{F}_s(\kappa, \mu)$ be the filters generated by the strong clubs on $[\lambda]^{<\mu}$ and on $[\kappa]^{<\mu}$ respectively. $\mathcal{F}(\kappa, \mu)$, $\mathcal{F}(\lambda, \mu)$ denote the corresponding club filter. Then the following holds:*

(i) $\mathcal{F}(\lambda, \mu)$ is the filter generated by $\mathcal{F}_s(\lambda, \mu) \cup \{\{z \in [\lambda]^{<\mu} : z \cap \mu \in \mu\}\}$

(ii) If $C \subset [\lambda]^{<\mu}$ is a strong club then $\{y \cap \kappa : y \in C\}$ is a strong club in $[\kappa]^{<\mu}$.

(iii) If $C \subset [\kappa]^{<\mu}$ is a strong club then $\{z \in [\lambda]^{<\mu} : z \cap \kappa \in C\}$ is a strong club in $[\lambda]^{<\mu}$.

Lemma 3.2.10. *Let K be as always a set of regular cardinals, and let Y be a set with $sup(K) \subset Y$. Then $\{S_\kappa : \kappa \in K\}$ is mutually stationary if and only if $\{X \subset Y : \forall \kappa \in X \cap \kappa \, (sup(X \cap \kappa) \in S_\kappa)\}$ is a strongly stationary subset of $P(Y)$.*

Thus if Y is a set with $sup(K) \subset Y$ then $(S_\kappa : \kappa \in K)$ is mutually stationary if and only if the set $\{X \subset Y : \forall \kappa \in K \cap X (sup(X \cap \kappa) \in S_\kappa)\}$ is stationary in $P(Y)$.

We will use the notion of mutual stationarity to investigate the saturation of the nonstationary ideal on $P_\kappa(\lambda)$ in the case where $\kappa = \omega_1$

3.2.3 The generalized saturation of NS when $\kappa = \omega_1$

Next we consider the case where $\kappa = \omega_1$ and λ is an arbitrary cardinal. We shall show that the nonstationary ideal in $P_{\omega_1}(\lambda)$ cannot be λ^+ saturated. At first let $\lambda > \omega_1$ be a regular cardinal. then we already know by 2.2.31 that the nonstationary ideal on λ, restricted to the elements whose cofinality is ω is not λ^+-saturated. Thus the next lemma suffices to show that the nonstationary ideal on $P_{\omega_1}(\lambda)$ is not λ^+-saturated whenever λ is regular:

Lemma 3.2.11. *Let $\kappa < \lambda$ be two regular cardinals and let $A \subset \lambda$ be such that $cf(\alpha) = \gamma < \kappa$ for every element α of A. Then*

$$A \textit{ is stationary iff } \tilde{A} := \{x \in P_\kappa(\lambda) : sup(x) \in A\} \textit{ is stationary in } P_\kappa(\lambda)$$

Proof. We start with the direction from the left to the right: Suppose that $C \subset P_\kappa(\lambda)$ is a club. Let $f : [\lambda]^{<\omega} \to P_\kappa(\lambda)$ be such that the set of the closure points of f, C_f is a subset of C. Let D be the club in λ, consisting of the ordinals which are closed under f. Then $D \cap A \neq \emptyset$, and if $\beta \in A \cap D$, then there exists a sequence $(\beta_\alpha)_{\alpha<\gamma}$ which converges to β. Build the closure B of $(\beta_\alpha)_{\alpha<\gamma}$ under f, then $B \in C$ and $sup(B) = \beta \in A$, hence $B \in C \cap \tilde{A}$ and $\tilde{A}$ is stationary in $P_\kappa(\lambda)$.

For the other direction assume that C is a club in λ, then $\tilde{C} := \{X \in P_\kappa(\lambda) : sup(X) \in C\}$ is a club in $P_\kappa(\lambda)$, hence there is an $X \in \tilde{C} \cap \tilde{A}$ and $sup(X) \in C \cap A$, witnessing the stationarity of A. □

Corollary 3.2.12. *If $\lambda > \omega_1$ is regular then the nonstationary ideal on $P_{\omega_1}(\lambda)$ is not λ^+-saturated.*

Note that the just described strategy isn't limited to the case $\kappa = \omega_1$. In fact it works for all regular κ and all regular $\lambda > \kappa$.

The remainig case is where λ is a singular cardinal. Unfortunately non-saturation is way harder to show here. It turns out that the notion of the mutually stationarity is crucial in that case. We start with a small technical result:

Lemma 3.2.13. *Suppose that $\kappa < \theta$ are regular cardinals. Then*

$$S \subset \kappa \textit{ is stationary } \Leftrightarrow \forall \mathcal{A} = (H(\theta), \in, < ..)\, \exists M \prec \mathcal{A},\, |M| < \kappa \textit{ with } sup(M \cap \kappa) \in S$$

Proof. We start with the $\Rightarrow$ direction. Consider for a fixed $\mathcal{A}$ the set

$$C_{\mathcal{A}} := \{sup(M \cap \kappa) : M \prec \mathcal{A}, |M| < \kappa\}.$$

Then $C_{\mathcal{A}}$ contains a club as we can define a sequence of length κ of elementary submodels $(M_\alpha)_{\alpha<\kappa}$ of $\mathcal{A}$ for which $M_\xi \prec M_\eta$ holds whenever $\xi < \eta$, and $|M_\alpha| < \kappa$ for each $\alpha < \kappa$ and moreover $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ holds for every limit α. Now if we consider $sup(M_\alpha \cap \kappa)$ then this is a club and a subset of $C_{\mathcal{A}}$, thus $C_{\mathcal{A}} \cap S \neq \emptyset$ as S is stationary and this is what we wanted.

For the other direction let C be an arbitrary club in κ. We claim that there exists a structure $\mathcal{A} = (H(\theta), ..)$ such that for each $M \prec \mathcal{A}$ $sup(M \cap \kappa) \in C$. Indeed let $f : \kappa \to \kappa$ be an increasing function with $f : \alpha \mapsto \beta \in C$ where β is the least such ordinal that is larger than α. Now set $\mathcal{A} := (H(\theta), \in, <, f)$ and whenever $M \prec \mathcal{A}$ with $|M| < \kappa$ then $sup(M \cap \kappa)$ can be approached by a sequence of the form $(f(\beta))_{\beta<\gamma}$ which is a sequence in C hence $sup(M \cap \kappa) \in C$. Now by our assumption there is an $M \prec \mathcal{A}$ and $sup(M \cap \kappa) \in S$, thus S is stationary. □

Our main result is this one:

Theorem 3.2.14 (Foreman-Magidor). *Let* $(\kappa_\alpha : \alpha \in \gamma)$ *be an increasing sequence of regular cardinals, let* $(S_\alpha : \alpha \in \gamma)$ *be a sequence of stationary subsets of* κ_α*, such that each* S_α *consists of points of countable cofinality. If* $\lambda := sup_{\alpha<\gamma}(\kappa_\alpha)$ *and* $\mathcal{A}$ *is an algebra on* λ*, then there is a countable* $N \prec \mathcal{A}$ *such that for all* $\kappa_\alpha \in K \cap N$ $sup(N \cap \kappa_\alpha) \in S_\alpha$*, i.e. if the* S_α *are stationary for each* $\alpha < \gamma$ *and each* $x \in S_\alpha$ *has countable cofinality then* $(S_\alpha : \alpha \in \gamma)$ *is mutually stationary.*

Proof. Let $\mathcal{A}$ be any algebra on λ and let $\tau \subset \lambda^{<\omega}$ be a tree constructed in a way that there is a function $l : \tau \to \{\kappa_\alpha : \alpha < \gamma\}$ which satisfies these two conditions:

1. If $\sigma \in \tau$ and $l(\sigma) = \kappa_\alpha$ then $\{\gamma : \sigma^\frown\gamma \in \tau\} \subset \kappa_\alpha$ and has cardinality κ_α.
2. If $\sigma \in \tau$ and $\kappa_\alpha \in sk^{\mathcal{A}}(\sigma)$ then there are infinitely many $n \in \omega$ such that if $\sigma' \supset \sigma$ and σ' has length n then $l(\sigma') = \kappa_\alpha$.

Let τ' be a subtree of τ with stem σ_0 and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be a finite collection of the κ_α's such that each $\lambda_i \in sk^{\mathcal{A}}(\sigma_0)$. We say that τ' is an acceptable subtree (for $\{\lambda_1, \ldots, \lambda_n)$ if and only if for all nodes $\sigma \in \tau'$:

- if $l(\sigma) \notin \{\lambda_1, \lambda_2, .., \lambda_n\}$ then $\{\gamma : \sigma \frown \gamma \in \tau'\}$ is a subset of $l(\sigma)$ with cardinality $l(\sigma)$.
- and if $l(\sigma) \in \{\lambda_1, ..., \lambda_n\}$ then there is a unique γ such that $\sigma \frown \gamma \in \tau'$.

We say that the tree τ' is fixed for κ_α if τ' is acceptable for a set $\{\lambda_1, ..., \lambda_n\}$ and $\kappa_\alpha \in \{\lambda_1, ..., \lambda_n\}$. Our goal is to produce a decreasing sequence of subtrees τ_n, and a non-decreasing sequence of finite subsets A_n of $\{\kappa_\alpha : \alpha \in \gamma\}$ such that the following three conditions hold:

(i) τ_n is an acceptable subtree for A_n and the length of the stem of σ_n is at least n.

(ii) If κ_α is in the skolem hull of the stem of one of the τ_n then there is an $m > n$ such that τ_m is fixed for κ_α.

(iii) If τ_n is fixed for κ_α then there is a $\beta_\alpha \in S_\alpha$ such that for all branches b through τ_n: $sup(sk^{\mathcal{A}}(b) \cap \kappa_\alpha) = \beta_\alpha$.

If we could prove the existence of such a sequence then we could end the proof of the theorem: Due to (i) the intersection of the τ_n is nonempty and is a branch b through each τ_n. Now if $N := sk^{\mathcal{A}}(b)$ then $N \prec \mathcal{A}$, $|N| = \omega$ and if $\kappa_\alpha \in N$ then there is a stem σ_n of a tree τ_n such that $\kappa_\alpha \in sk^{\mathcal{A}}(\sigma_n)$. Due to (ii) there is an $m > n$ such that τ_m is fixed for κ_α and (iii) tells us that there is a $\beta_\alpha \in S_\alpha$ such that $sup(N \cap \kappa_\alpha) = \beta_\alpha \in S_\alpha$ which is exactly what we wanted to prove. Thus we need to show that there is such a decreasing sequence of the τ_n, which is done if we could prove:

Lemma 3.2.15. *Suppose that τ is an acceptable tree for $\lambda_1, ..., \lambda_n$ and $\kappa_\alpha \in sk^{\mathcal{A}}(\sigma)$, where σ is the stem of τ. Then there is a $\beta_\alpha \in S_\alpha$ and a subtree $\tau' \subset \tau$ such that τ' is acceptable for $\{\lambda_1, ..., \lambda_n, \kappa_\alpha\}$ and such that for all branches b through τ' : $sup(sk^{\mathcal{A}}(b) \cap \kappa_\alpha) = \beta_\alpha$.*

Proof. We proof this lemma by defining a game G_δ for each ordinal $\delta \in \kappa$, played on the tree τ with two players G (good) and B (bad), who play nodes of the tree determining a branch b of the tree. At a stage of the game where a node $\sigma \in \tau$ has been determined:

1. If $l(\sigma) \in \{\lambda_1, ..., \lambda_n\}$ then there is only one γ such that $\sigma \frown \gamma \in \tau$. B must play this γ.

2. If $l(\sigma) < \kappa_\alpha$ then B has to play a γ such that $\sigma \frown \gamma \in \tau$.

3. If $l(\sigma) > \kappa_\alpha$ then B chooses a subset D of $l(\sigma)$ with $|D| < l(\sigma)$ and G chooses an element of $\{\gamma : \sigma^\frown\gamma \in \tau\} - D$.

4. If $l(\sigma) = \kappa_\alpha$ then B chooses an ordinal $\beta < \delta < \kappa_\alpha$ and G chooses a $\gamma > \beta$ such that $\sigma^\frown\gamma \in \tau$.

If the game described above determines a branch of τ such that $sk^{\mathcal{A}}(b) \cap \kappa_\alpha \leq \delta$ then G wins, otherwise B. Note that if B wins then $sk^{\mathcal{A}}(b) \cap \kappa_\alpha > \delta$ after a finite number of steps, hence the game G_δ is determined for every $\delta \in \kappa_\alpha$.

Claim: There is a closed unbounded set $\delta < \kappa_\alpha$ such that G has a winning strategy in the game G_δ

Assume to the contrary that $S \subset \kappa_\alpha$ is stationary and G doesn't have a winning strategy S_δ for each $\delta \in S$. Let θ be a cardinal $> \lambda^{+5}$ and $N \prec (H(\theta), \in, <, (S_\delta : \delta \in S), ..)$ be such that $|N| < \kappa_\alpha$ and $N \cap \kappa_\alpha = \delta_0 \in S$. Such an N always exists as the previous lemma justifies the existence of an $N' \prec (H(\theta), \in, ..)$ such that $sup(N' \cap \kappa_\alpha \in S$. Now let $N \prec N'$ be such that $N' \cap \kappa_\alpha \subset N$ and $|N| < \kappa_\alpha$. This N has now the desired properties.

We will derive a contradiction by exhibiting a game according to S_{δ_0} that produces a branch b with $sk^{\mathcal{A}}(b) \cap \kappa_\alpha \leq \delta_0$. As B plays with strategy S_{δ_0} we only need to describe what G does. We will show that each player in each step plays ordinals that are elements of N and as N is closed under finite sequences by elementarity we can say that for all n $b \restriction n \in N$.

Assume inductively that the play has constructed σ of length n and $l(\sigma) \in N$.

1. If $l(\sigma) \in \{\lambda_1, ..., \lambda_n$ then there is exactly one γ such that $\sigma^\frown\gamma \in \tau$ and this γ also lies in N.

2. At a stage where $l(\sigma) < \kappa_\alpha$, B has to play a γ so that $\sigma^\frown\gamma \in \tau$. As $N \cap \kappa_\alpha \in \kappa_\alpha$ any $\gamma < l(\sigma)$ is in N.

3. If the play has arrived at a σ of length n so that $l(\sigma) > \kappa_\alpha$ then as $l(\sigma) \in N$, and as the cardinality of $U = \bigcup\{S_\delta(\sigma) : \delta \in S\}$ is less than $l(\sigma)$ we know that $N \models l(\sigma) - U \neq \emptyset$, and so G plays an element $\gamma \in N \cap (l(\sigma) - U)$. Since $\gamma \in N$ $\sigma^\frown\gamma \in N$, and by construction $\gamma \notin S_\delta(\sigma)$.

4. Suppose now the game has constructed a σ of length n and $l(\sigma) = \kappa_\alpha$, and S_{δ_0} forces B to play an ordinal $\beta < \delta_0$. Then G plays an ordinal γ so that $\sigma^\frown\gamma \in \tau$ and $\beta < \gamma < \delta_0$, and such a $\gamma \in N$ always exists as by elementarity $N \models$ " the successors of σ are unbounded in κ_α ".

Now let b be the resulting branch through τ and let $M = (sk^{\mathcal{A}}(b))^N$. Then $M \prec N$ and hence $supM \cap \kappa_\alpha \leq \delta_0$. This contradicts the fact that b is the result of a game according to the winning strategy for B S_{δ_0}. Thus our claim is proven.

□

We go back to the proof of the theorem. Since each S_α is stationary there is a $\beta_\alpha \in S_\alpha$ such that G has a winnig strategy for G_{β_α}. Let S denote this strategy. Due to our assumption β_α has countable cofinality thus let $(\delta_m)_{m\in\omega}$ be a sequence cofinal in β_α. Let us now define the subtree τ', which should have the desired attributes: We take a look at each $\sigma \in \tau$ and decide wheter σ is in τ' or not. Assume inductively that each $\sigma \in \tau'$ is the result of a partial play by G according to S and that if $l(\sigma) \notin \{\lambda_1, .., \lambda_n, \kappa_\alpha\}$ then the set $\{\gamma : \alpha^\frown\gamma\}$ has cardinality $l(\sigma)$.

Suppose we have put $\sigma \in \tau'$. Then:

1. If $l(\sigma) \in \{\lambda_1, ..., \lambda_n\}$ then σ has a unique successor in τ. We put this successor into τ'. Notice that this is done by B, thus it doesn't contradict our strategy.

2. If $l(\sigma) < \kappa_\alpha$ then we let the successor of σ in τ be the successors of τ'. Note again that this is a legal play of the game according to S.

3. If $l(\sigma) > \kappa_\alpha$ then in the game B chooses a subset D of $l(\sigma)$ of cardinality $< l(\sigma)$ and then G an element of $\{\gamma : \sigma^\frown\gamma\}$. To ensure that the set of successors of σ has cardinality $l(\sigma)$, we define by induction on $\nu < l(\sigma)$ ordinals $\gamma_\nu \in l(\sigma)$ such that γ_ν is the response by S to B playing $\{\gamma_{\nu'} : \nu' < \nu\}$. Then we let the set of successors of σ in τ' be $\{\sigma^\frown\gamma_\nu : \nu \in l(\sigma)\}$

4. If $l(\sigma) = \kappa_\alpha$ then we want τ' be fixed for κ_α i.e. σ has only one successor in τ'. The rules of the game require B to play an ordinal $\beta < \beta_\alpha$ and G to play an ordinal γ such that $\gamma > \beta$ and $(\sigma^\frown\gamma) \in \tau$. If the length of σ is m then we let $\{\gamma : \sigma^\frown\gamma \in \tau'\}$ be the well defined γ such that γ is G's response to B playing δ_m.

This defines a subtree $\tau' \subset \tau$, and we checked in each step that τ' is acceptable for $\{\lambda_1, ..., \lambda_n, \kappa_\alpha\}$. Thus it remains to show that for all branches b in τ' $sup(sk^{\mathcal{A}}(b) \cap \kappa_\alpha) = \beta_\alpha$. Since such a branch is the result of a game with G following his winning strategy S we have that $sup(sk^{\mathcal{A}}(b) \cap \kappa_\alpha) \leq \beta_\alpha$. On the other hand, since κ_α is in the skolem hull of the stem of σ of τ we have that there are infinetly many m such that $l(b \restriction m) = \kappa_\alpha$ and for each such m we have that the unique γ with $b \restriction m^\frown\gamma \in \tau'$ is bigger than δ_m cofinal in β_α, hence the equality is verified.

□

Theorem 3.2.16 (**Foreman-Magidor**). *Let λ be a singular cardinal with cofinality μ. Then the nonstationary ideal on $P_{\omega_1}(\lambda)$ is not λ^μ-saturated.*

Proof. We pick an increasing sequence $(\kappa_\alpha : \alpha < \mu)$ of regular cardinals, cofinal in λ. For each α we divide the stationary set of the elements of κ_α with countable cofinality into κ_α pairwise disjoint stationary sets $(S^\alpha_\beta : \beta < \kappa_\alpha)$. Then we define for each function $f \in \prod_{\alpha\in\mu} \kappa_\alpha$:

$$S_f := \{N \in P_{\omega_1}(\lambda) : \forall \kappa_\alpha \in N \, (sup(N \cap \kappa_\alpha) \in S^\alpha_{f(\alpha)}\}$$

Our last theorem witnesses the strong stationarity of each S_f in $P_{\omega_1}(\lambda)$. Due to Kueker's theorem strongly stationary and stationary are the same in $P_{\omega_1}(\lambda)$, hence each S_f is even stationary in $P_{\omega_1}(\lambda)$. Further if $f \neq g$ then there is an α such that $f(\alpha) \neq g(\alpha)$, hence $S_f \cap S_g$ is nonstationary as it has empty intersection with the club $\{N \in P_{\omega_1}(\lambda) : \kappa_\alpha \in N\}$. So the set $\{S_f : f \in \prod_{\alpha<\mu} \kappa_\alpha\}$ forms an antichain in $P_{\omega_1}(\lambda)$ of cardinality λ^μ. □

3.2.4 The cofinality of λ is less than κ

Now we consider the case where, as already mentioned, the cofinality of λ is less than κ. We will show an even better result than the one we already got by Burke-Matsubara: namely that the nonstationary ideal cannot be λ^{++}-saturated. Our proof relies on a fact by Shelah which will be stated without proof, and makes heavy use of the PCF theory. Therefore we need to introduce first some definitions and properties of this theory.

Definition 3.2.17. *Suppose that λ is a singular cardinal, $(\lambda_i)_{i\in cf(\lambda)}$ is a cofinal sequence of regular cardinals and I is an ideal on $cf(\lambda)$. Then a scale in $\prod_{i\in cf(\lambda)} \lambda_i/I$ is a sequence of functions $(f_\alpha : \alpha < \eta)$ such that each $f_\alpha \in \prod_{i\in cf(\lambda)} \lambda_i$ and the following two properties hold:*

(i) $\forall \alpha < \alpha' \, \{i : f_\alpha(i) \geq f_{\alpha'}(i)\} \in I$, i.e. the sequence is increasing.

(ii) $\forall g \in \prod_{i\in cf(\lambda)} \lambda_i \, \exists \alpha < \eta \, \{i : g(i) \geq f_\alpha(i)\} \in I$, i.e. the sequence is cofinal

Fact 3.2.18. *Let λ be a singular cardinal, let I be the ideal of the bounded subsets of $cf(\lambda)$, and let $(\lambda_i)_{i\in cf(\lambda)}$ be a cofinal sequence in λ. Then there is a scale in $\prod \lambda_i/I$ of length λ^+. We will say that $\prod \lambda_i/I$ has true cofinality λ^+.*

Definition 3.2.19. *Let $(f_\alpha : \alpha < \eta)$ be a sequence of functions $f_\alpha : cf(\lambda) \to \lambda$, and moreover let $g \in \lambda^{cf(\lambda)}$. Then g is called an exact upper bound for the sequence if and only if g is greater than each f_α almost everywhere, i.e. $\{i : f_\alpha(i) \geq g(i)\} \in I$, and if $h \in \prod_{i\in cf(\lambda)} \lambda_i$ is such that $\{i : h(i) \geq g(i)\} \in I$, then there exists an α such that $\{i : h(i) \geq f_\alpha(i)\} \in I$.*

Moreover we say that a scale is continous if and only if for all β, whenever there is an exact upper bound for $(f_\alpha : \alpha < \beta)$ then f_β is the exact upper bound.

Definition 3.2.20. *Let $(f_\alpha : \alpha < \eta)$ be a scale and let β be an ordinal. Then β is called good if and only if there exists a set $B := \{h_\xi : \xi < cf(\beta)\} \subset \prod_{i\in cf(\lambda)} \lambda_i$ and a set $S \in I$ such that:*

1. *for all* $\xi < \eta < cf(\beta)$, $i \in cf(\beta)$, $i \in cf(\lambda) - S$ $h_\xi(i) < h_\eta(i)$

2. *for all* $\alpha < \beta$ *there is an* $h \in B$ *such that* $\{i : f_\alpha(i) \geq h(i)\} \in I$.

Fact 3.2.21. *There exists always a stationary set of good points*

Fact 3.2.22 (Shelah). *([15]) Suppose that* λ *is a singular cardinal, and* $\mu < \lambda$ *is a regular uncountable cardinal. then there exists a set* $R \subset \lambda^+$ *and a stationary set* $A \subset \lambda^+$ *consisting of ordinals of cofinality* μ*, such that whenever* $N \prec (H(\theta), \in, <, R, ..)$*, if* $\alpha = N \cap \lambda^+ \in A$*, then there is a cofinal sequence* $C \subset \alpha$ *of order type* μ *such that for all* $\beta < \alpha$, $C \cap \beta \in N$.

Fact 3.2.23 *([5]). Let* λ *be a singular cardinal, and let* μ *be a regular cardinal less than* κ*. Then there exists a stationary set* $A \subset \lambda^+$ *such that for all stationary* $B \subset A$ *and all expansions of* $H(\theta) := (H(\theta), \in, <, R, ..)$ *-where* θ *is as always a sufficiently large cardinal, and* R *is as in our previous fact- there is an elementary submodel* $N \prec H(\theta)$ *such that:*

(i) $|N| < \kappa$ *and* $N \cap \kappa \in \kappa$

(ii) $sup(N \cap \lambda^+) \in B$

(iii) N *is internally approachable of length* μ

Definition 3.2.24 (Internally approachable). *Let* N *be an arbitrary set and* μ *an ordinal, then* N *is internally approachable of length* μ *if and only if there is a sequence* $(N_\alpha : \alpha < \mu)$ *such that* $N = \bigcup_{\alpha<\mu} N_\alpha$ *and for all* $\beta < \mu$, $(N_\alpha : \alpha < \beta) \in N$.

Note that if $N \prec (H(\theta), \in)$ and N is internally approachable of length μ, then due to the definability of μ as the length of the sequence $(N_i : i < \mu)$ and by the elementarity of N, $\mu \subset N$ and moreover $N_i \in N$ for each $i < \mu$.

Lemma 3.2.25. *Let* λ *be a singular cardinal,* $cf(\lambda) < \kappa$*, let* $(\lambda_i)_{i<cf\lambda}$ *be a sequence of regular cardinals cofinal in* λ *and let* $(f_\beta : \beta < \lambda^+)$ *be a continous scale. Further let* I *be the ideal of the bounded subsets of* $cf(\lambda)$*, and* $N \prec (H(\theta), \in, <, (f_\beta : \beta < \lambda^+), ...)$ *be an internally approachable structure of length* $\mu \neq cf(\lambda)$ *(where* μ *is a regular cardinal), with* $|N| < \kappa, cf(\lambda) \in N \cap \kappa \in \kappa$ *and* $sup(N \cap \lambda^+) = \alpha$ *then*

$$\chi_N = f_\alpha \text{ almost everywhere modulo } I$$

Proof. Pick a sequence $(N_i : i < \mu)$ that witnesses that N is internally approachable. We already know that $N_i \in N$ for all $i < \mu$, as well as $\mu \subset N$. As $N \prec H(\theta)$ we may assume that the N_i's are increasing, i.e. $N_i \subset N_j$ if $i \leq j$. Additionally we may assume that each $\lambda_j \in N$, which gives us that $\chi_{N_i} \in N$ for every $i < \mu$. Now we

Claim: There are cofinal subsets $X \subset \mu$ and $Y \subset \alpha \cap N$ and a $j_0 < cf(\lambda)$ such that if i, i' are successive in X then there is a unique $\beta \in Y$ such that for all $j > j_0$:

$$\chi_{N_i}(j) < f_\beta(j) < \chi_{N_{i'}}(j)$$

We prove the claim. Let $\chi_{N_i} \in N$ be the characteristic function for an arbitrary $i < \mu$. As the $(f_\beta : \beta < \lambda^+)$ form a scale in N there exists a $\beta \in N \cap \lambda^+$ such that $\chi_{N_i} <_I f_\beta$. Moreover as $\mu \neq cf(\lambda)$ and as the cofinality of the approaching sequence is uniquely determined, there exists an i' such that $f_\beta \subset N'_i$. For the N'_i there exists again a $\beta' \in N \cap \lambda^+$ such that $f_\beta \geq_I N'_i$ and so on. We obtain a cofinal $X \subset \mu$ and $Y \subset \alpha \cap N$ and for each successive $i, i' \in X$ there is exactly one $\beta \in Y \cap \alpha$ such that $\chi_{N_i} <_I f_\beta < \chi_{N'_i}$, i.e. there is a $j_i < cf(\lambda)$ such that for all $j > j_i$ $\chi_{N_i}(j) < f_\beta(j) < \chi_{N'_i}(j)$. Thus it is only left to show that there is even a $j_0 < cf(\lambda)$ such that the inequality above holds for all $j > j_0$. We break into cases:

1. $\mu < cf(\lambda)$. Then $sup_{i<\mu} j_i < cf(\lambda)$, thus there is a j_0 such that $sup j_i < j_0 < cf(\lambda)$ which works.
2. $\mu > cf(\lambda)$. Then since the function $h : i \mapsto j_i$ goes from μ to $cf(\lambda)$, we know that there is a $j_0 \in cf(\lambda)$ such that j_0 is hit by μ-many i's. If we let $X' := h^{-1}(j_0)$ then $X' \subset X$ and if we build Y' just in the same way as our Y, then we obtain a cofinal $X' \subset \mu$ and a cofinal $Y' \subset \alpha \cap N$. For these X' and Y' now the claim holds.

This ends the proof of our claim.

Now we continue with the proof of the lemma. Again we break into cases

1. $cf(\lambda) > \mu$. Then as $\alpha = sup(N \cap \lambda^+)$ which implies that $cf(\alpha) \leq \mu$ and as $|Y| = \mu$ we know that $\forall j < j_0 : f_\alpha(j) = sup_{\beta \in Y} f_\beta(j) = sup_{i \in X} \chi_{N_i}(j) = \chi_N(j)$.
2. $cf(\lambda) < \mu$. Then Y and j_0 witness that α is a good point, hence again for sufficiently large $j > j_0$: $f_\alpha(j) = sup_{\beta \in Y} f_\beta(j) = sup_{i \in X} \chi_{N_i}(j) = \chi_N(j)$ and we are done

□

Theorem 3.2.26. *If $\kappa \geq \omega_2$ is a regular cardinal and $\lambda > \kappa$ is a cardinal wirh $cf(\lambda) < \kappa$. Then the nonstationary ideal on $P_\kappa(\lambda)$ is not λ^{++}-saturated.*

Proof. As $\kappa \geq \aleph_2$ we may pick a regular cardinal $\mu < \kappa$ such that $\mu \neq cf(\lambda)$. Let $A \subset \lambda^+$ be a set as in fact 3.2.23. As λ^+ is regular we already know that there is an antichain D of cardnality λ^{++} in $P(A)/NS$. Thus let $\{B_\alpha : \alpha < \lambda^{++}\}$ be an enumeration of this antichain. For each $\xi, \eta < \lambda^{++}$ let $C_{\xi,\eta}$ be a club such that $B_\xi \cap B_\eta \cap C_{\xi,\eta} = \emptyset$. By taking a bijection between the $\lambda^{++} \times \lambda^{++}$ and λ^{++} we define C_ξ for all $\xi < \lambda^{++}$ and build $\Delta_{\xi<\lambda^{++}} C_\xi$. We set $B'_\alpha := B_\alpha \cap \Delta_{\xi<\lambda^{++}} C_\xi$ for every $\alpha < \lambda^{++}$. Then each B'_α is stationary and if $\alpha \neq \beta$ then $B'_\alpha \cap B'_\beta$ is bounded. Thus we may assume without loss of generality that for all $B_\xi, B_\eta \in D$ $B_\xi \cap B_\eta$ is bounded.

Each stationary $B \subset A$ induces a stationary set

$$S_B := \{N \in P_\kappa(H(\theta)) : sup(N \cap \lambda^+) \in B \text{ and } N \text{ is I.A. of length } \mu\}.$$

Because if C is a club in $P_\kappa H(\theta)$ then by 3.2.9 there is a function $F : [H(\theta)]^{<\omega} \to H(\theta)$ such that $\{x \in P_\kappa(H(\theta)) : x \cap \kappa \in \kappa \wedge F``[x]^{<\omega} \subset x\}$ is a subset of C. Now fact 3.2.23 witnesses that there exists such an $N \prec (H(\theta), F, ...)$ which additionally has $sup(N \cap \lambda^+) \in B$ and is I.A. of length μ, hence S_B is stationary.

If $(f_\alpha : \alpha < \lambda^+$ is a continous scale then the set $\{N \in P_\kappa(H(\theta)) : (f_\alpha : \alpha < \lambda^+) \in N\}$ is a club and we may assume that each element of S_B additionally contains the scale.

Now we set for each $B \in D$ $T_B := \{N \cap \lambda : N \in S_B\}$ which is a projection of a stationary set and therefore remains stationary. We shall show that

if B, C are distinct elements of D then $T_B \cap T_S$ is nonstationary.

Let $\gamma < \lambda^+$ be large enough that $B \cap C - \gamma = \emptyset$. Let $N \in S_B$ and assume that $f_\gamma(j) \in N \cap \lambda \in T_B$ for each $j < cf(\lambda)$ (this is always possible since the set $\{A \in P_\kappa(\lambda) : \forall j\, f_\gamma(j) \in A\}$ is a club). As $N \in S_B$ we know by lemma 3.2.25 that there is an $\alpha \in B - \gamma$ such that $\chi_N =_I f_\alpha$. Assume now that there is an $M \in S_C$ such that $M \cap \lambda = N \cap \lambda$, then $\chi_N(j) = sup(N \cap \lambda_j) = sup(M \cap \lambda_j) = \chi_M(j)$, but $\chi_N =_I f_\alpha$, $\alpha \in B - \gamma$ and $\chi_M =_I f_\beta$, $\beta \in C - \gamma$, which is a contradiction.

Thus we have shown that if E is the club in $P_\kappa(\lambda)$ whose elements contain all the ordinals $f_\gamma(j)$ for $j < cf(\lambda)$, then $E \cap T_B \cap T_C = \emptyset$, which shows that T_B and T_C have indeed nonstationary intersection, hence $P_\kappa(\lambda)$ is not λ^{++} saturated.

□

3.2.5 κ is weakly inaccessible

Our last remaining case is where κ is weakly inaccessible and λ has cofinality κ. We shall show:

Theorem 3.2.27. *Let κ be weakly inaccessible and let $\lambda > \kappa$ be such that $cf(\lambda) = \kappa$. Then the nonstationary ideal on $P_\kappa(\lambda)$ is not λ^+-saturated.*

The theorem will be proven after a series of lemmas are settled. Our proof is indirect, thus we assume within the whole section that the nonstationary ideal *is* λ^+-saturated. Define a map $\pi : P_\kappa(\lambda^+) \to P_\kappa(\lambda)$ by $\pi(x) = x \cap \lambda$. Moreover this function induces a map, which we will denote confusingly again with $\pi : P(P_\kappa(\lambda^+)) \to P(P_\kappa(\lambda))$. The second π preserves stationarity, as well as it's inverse function π^{-1}.

Lemma 3.2.28. *If $S \subset P_\kappa(\lambda^+)$ is a stationary set, and if the nonstationary ideal restricted to $\pi(S)$ is λ^+-saturated, then there is a club $C \subset P_\kappa(\lambda^+)$ such that for each stationary $S' \subset \pi(S \cap C)$, the set $S \cap \pi^{-1}(S')$ is stationary.*

Proof. Let $B \subset \pi(S)$ be stationary. We say that B is bad if and only if the set $\{N \in S : N \cap \lambda \in B\} = S \cap \pi^{-1}(B)$ is not stationary. Suppose that $A \subset P(P_\kappa(\lambda)$ is a maximal antichain of bad sets. By the λ^+-saturation we may assume that $A = \{B_\alpha : \alpha < \lambda\}$. Since for each $\alpha < \lambda$ $D_\alpha := \{x \in P_\kappa(\lambda) : \alpha \in x\}$ is a club, we may also assume that if $N \in B_\alpha$ then $\alpha \in N$.

Now since each B_α is bad, for every α there exists a club $C_\alpha \subset P_\kappa(\lambda^+)$ such that $C_\alpha \cap S \subset \{N \in S : N \cap \lambda \notin B_\alpha\}$. Let $C := \Delta_{\alpha<\lambda} C_\alpha$ and let $T \subset \pi(C \cap S)$ be stationary.

If $S \cap \pi^{-1}(T)$ is not stationary, i.e. $\{N \in S : N \cap \lambda \in T\} \in NS$, then T is bad and by the maximality of B_α, there must be an α such that $B_\alpha \cap T$ is stationary. Hence we may assume without loss of generality that $T \subset B_\alpha$ for an $\alpha < \lambda$. Let $N \in S \cap C$ with $N \cap \lambda \in T$. Then since $T \subset B_\alpha$, $N \cap \lambda \in B_\alpha$ and thus $\alpha \in N$. As $\Delta C_\alpha = \{x \in P_\kappa(\lambda) : x \in \bigcap_{\alpha \in x} C_\alpha\}$, $\alpha \in N$ and $N \in C \cap S = \Delta C_\alpha \cap S$, $N \in C_\alpha \cap S$ and so $N \cap \lambda \notin B_\alpha$, thus $N \cap \lambda \notin T$ which is a contradiction. So $\{N \in S : N \cap \lambda \in T\}$ must be stationary. $\square$

Now we pick a cofinal sequence of regular cardinals $(\lambda_i : i < \kappa)$ such that the true cofinality of the reduced product of the λ_i's modulo the bounded sets on κ is λ^+. We choose a continous scale $(f_\alpha : \alpha < \lambda^+)$ in this reduced product.

Lemma 3.2.29. *Let T be the subset of $P_\kappa(\lambda^+)$ such that its elements M satisfy*

(i) $cf(M \cap \kappa) = \omega_1$

(ii) there is a sequence $(\delta_n : n \in \omega) \subset M \cap \lambda^+$ and an $i_0 \in M \cap \kappa$ so that $\chi_M(i) := sup(M \cap \lambda_i) = sup(f_{\delta_n})(i) : n \in \omega)$ for all $i \in \kappa \cap M$, $i > i_0$.

Then T is stationary

Proof. Let θ be a sufficiently large regular cardinal such that all the neccessary things (the scale, the cardinals mentioned) lie in $H(\theta)$. We shall show that there exists an elementary submodel $M \prec H(\theta)$ with $M \in T$, such that $M \cap \kappa \in \kappa$. This suffices as we know by lemma 3.2.9 that $\{N \in P_\kappa(\lambda^+ : N \prec (H(\theta), \in, <, ...) \wedge N \cap \kappa \in \kappa\}$ forms a basis for the clubs in $P_\kappa(\lambda^+)$.

Pick an $M \prec H(\theta)$ such that $\kappa \in M$, M has cardinality κ. We may demand that M is internally approachable by an increasing sequence $(N_k : k \in \omega)$ of length ω as can be seen as follows: Let $M_0 \prec H(\theta)$ such that $|M_0| = \kappa$, $\kappa \subset M_0$. Pick an $M_1 \prec H(\theta)$ with $M_0 \subset M_1$, $M_0 \in M_1$ and $|M_1| = kappa$. Moreover let $M_2 \supset M_1$ be such that $M_2 \prec H(\theta)$, $(M_0, M_1) \in M_2$, $|M_2| = \kappa$ and so on. Then $M := \bigcup_{i \in \omega} M_i$ has the desired properties.

Then (maybe after thinning the internally approaching sequence out) for all k there is a $\delta_k \in N_{k+1}$ such that $\chi_{N_k} <^* f_{\delta_k} <^* \chi_{N_{k+1}}$. Next we choose an increasing sequence $(M_\alpha : \alpha < \kappa)$ such that $(M_\alpha \cap \kappa) \in \kappa$, $|M_\alpha| < \kappa$, each $N_k \in M_0$ and $M_\alpha \subset M$. Then for some α with $cf(\alpha) = \omega_1$, $(M_\alpha, (f_{\delta_k})) \prec (M, (f_{\delta_k}))$.

Now we claim that for this M_α, $M_\alpha \cap \lambda \in T$. Note that since each $f_{\delta_k} \in M_\alpha$ and $cf(\alpha) = \omega_1$ there is an $i_0 \in M_\alpha \cap \kappa$ such that for all i with $i_0 < i < \kappa$, and all $k \in \omega$ $\chi_{N_k}(i) < f_{\delta_k}(i) < \chi_{N_{k+1}}(i)$. Thus for all i between i_0 and κ $\chi_{M_\alpha}(i) = sup\{\chi_{N_k}(i) : k \in \omega\} = sup\{f_{\delta_k}(i) : k \in \omega\}$. $\square$

Now for every $N \in T$ we fix an increasing sequence $(\alpha_i^N : i \in \omega_1)$ of ordinals cofinal in $N \cap \kappa$. Moreover we pick a $G \subset P(P_\kappa(\lambda))/NS$ that is generic with $\pi(T)$ in G and build the generic ultrapower of V by G. This gives us an embedding $j : V \to M \subset V[G]$, M transitive, and $j``\lambda \in M$ and the sequence

$(\alpha_i^{j``\lambda} : i \in \omega_1)$ is a cofinal increasing sequence in κ. This sequence determines a subsequence of the cardinals $(\lambda_i : i \in \kappa)$, which we will denote by $(\lambda_i^* : i \in \omega_1)$. Thus it is reasonable to view each f_α as an element of $\prod_{i\in\omega_1} \lambda_i^*$. Further for $N \in T$, the sequence $(\alpha_i^N : i \in \omega_1)$ determines a version of the sequence λ_i^* relative to $N \cap \lambda$, and for $i > i_0$ each $N \cap \lambda_i^*$ has cofinality ω. By our penultimate lemma by intersecting T with a club if necessary we may assume that for all stationary $S \subset \pi(T)$, $\pi^{-1}(S) \cap T$ is stationary.

Lemma 3.2.30. *For every generic $G \subset P(P_\kappa(\lambda))/NS$ with $\pi(T) \in G$,*

$$V[G] \models ((f_\alpha^* : \alpha \in \lambda^+) \textit{ is unbounded in } \prod_{i<\omega_1} \lambda_i^* \textit{ (mod the filter of countable sets))}$$

Proof. Assume to the contrary that the lemma is false, take G generic with $\pi(T) \in G$ as a counterexample and let $j : V \to M \subset V[G]$ be the generic elementary embedding, where M is transitive. We may assume that $\pi(T) \Vdash ((f_\alpha^* : \alpha < \lambda^+)$ is bounded in $\prod_{i\in\omega_1} \lambda_i^*)$. Hence there is an $h \in \prod_{i\in\omega_1} j``\lambda_i^*$ such that for all $\alpha \in \lambda^+$ and all large enough $i < \omega_1$, $h(i) > f_{j(\alpha)}^*(i)$. Note that $h \in M$. Hence by the saturation there is a $g : \pi(T) \to V$ such that for almost every $x \in \pi(T)$, $g(x) \in \prod_{i\in\omega_1}(\lambda_i^* \cap x)$, and such that for all $\alpha < \lambda^+$, $\pi(T) \Vdash$ (for all sufficiently large $i \in \omega_1$, $[g]_M(i) > f_{j(\alpha)}^*(i)$.

Let $N \in T$ then $g(N \cap \lambda) \in \prod_{i\in\omega_1}(\lambda_i^* \cap N)$. In particular for all $i \in \omega_1$, $g(N \cap \lambda)(i) < \chi_N^*(i)$ (where $\chi_N^*(i) = sup(N \cap \lambda_i^*)$). Since $N \in T$, there is a sequence $\delta_n : n \in \omega)$ such that for all large enough $i \in \omega_1$, $\chi_N^*(i) = sup\{f_{\delta_n}^*(i) : n \in \omega\}$. Hence for all large $i \in \omega_1$, there is an n such that $g(N \cap \lambda)(i) < f_{\delta_n}^*(i)$. Hence there is an unbounded set of $i < \omega_1$, and an n such that $f_{\delta_n}^*(i) > g(N \cap \lambda)(i)$.

Now by Fodor there is a δ and a stationary $T' \subset T$ such that for all $N \in T'$, $g(N \cap \lambda)(i) < f_\delta^*(i)$ for cofinally many $i \in omega_1$.

Assume that $\pi(T') \in G$, then $j(g)(j``\lambda)(i) < f_{j(\delta)}^*(i)$ holds in M for unboundedly many $i < \omega_1$, a contradiction. □

proof of theorem 3.2.27. We are finally ready to prove the theorem 3.2.27: With the help of our last lemmas we can find a generic $G \subset P(P_\kappa(\lambda))$ such that in $V[G]$ the following holds:

(a) $(f_\alpha^* : \alpha < \lambda^+)$ is unbounded in $\prod_{i\in\omega_1} \lambda_i^*$

(b) $cf(\kappa) = \omega_1$ and for all i, $cf(\lambda_i^*) = \omega$.

Working now in $V[G]$ we may apply Shelahs trichotomy theorem ([15], *Claim 2.1.2*) to see that in $\mathbf{On}^{\omega_1}/$ bounded sets the sequence $(f_\alpha^* : \alpha < \lambda^+)$ either

1. has an exact upper bound g in $\mathbf{On}^{\omega_1}/$ bounded sets such that for almost all $i, j \in \omega_1$, $cf(g(j)) = cf(g(i))$, or
2. there are sets $A_i \subset \lambda_i^*$, with $|A_i| \le \omega_1$, and an ultrafilter $D \subset P(\omega)$ such that for all $\alpha < \lambda^+$ there is a $\beta < \lambda^+$ and a $g \in \prod_{i<\omega_1} A_i$ such that $f_\alpha^* <_D g <_D f_\beta^*$ or

3. there is a $g \in \prod_{i<\omega_1} \lambda_i^*$ such that the sequence of equivalence classes of $\{i : f_\alpha^*(i) \leq g(i)\}$ modulo the ideal of bounded sets in ω_1 is not eventually constant.

Note that by our last lemma, if the sequence $(f_\alpha^* : \alpha < \lambda^+)$ has an exact upper bound then it must be given by the function $g(i) = \lambda_i^*$. As each λ_i^* has cofinality ω, we can choose cofinal countable sets $A_i \subset \lambda_i^*$. Since g is an exact upper bound, for every function $h \in \prod_{i\in\omega_1} A_i$, there is a β such that $h <^* f_\beta^*$. Further since the A_i are cofinal, for all $\beta < \lambda^+$, there is an $h \in \prod_{i<\omega_1} A_i$ such that $f_\beta^* <^* h$. Hence if there is an exact upper bound the sets$\{A_i\}$ are a witness to being in case 2 for any ultrafilter D.

We shall show that both cases 2 and 3 lead to a contradiction. In either case there is an ordinal $\delta \in \lambda^+$ of cofinality ω_2 (in V and $V[G]$) such that either for all $\alpha < \delta$ there is a $\beta < \delta$ and a $g \in \prod_{i<\omega_1} A_i$ such that $f_\alpha^* <_D g <_D f_\beta^*$ (in case 2). Or the sequence of equivalence classes of $\{i : f_\alpha^* \leq g_\alpha^*\}$ (modulo the ideal of bounded sets in ω_1) for $\alpha < \delta$ is not eventually constant (in case 3).

We return to V now and choose there a cofinal $X \subset \delta$ of order type ω_2. Then since κ is regular in V there is a $j < \kappa$ such that for all $i > j$ and all $\alpha < \beta$ in X, $f_\alpha(i) < f_\beta(i)$. Thus in $V[G]$ there is an $i_0 \in \omega_1$ such that for all $\alpha < \beta$, $\alpha, \beta \in X$ and all $i > i_0$, $f_\alpha^*(i) < f_\beta^*(i)$.

We work in $V[G]$ again: If 2 holds then we construct a cofinal set $X' \subset X$ such that for all $\alpha < \beta$ in X' there is a $g \in \prod_{i<\omega_1} A_i$ such that $f_\alpha^* <_D g <_D f_\beta^*$. If $\alpha < \alpha'$ are successive elements of X', pick $i_\alpha > i_0$ and a $g_\alpha \in \prod_{i\in\omega_1} A_i$ such that $f_\alpha(i_\alpha) < g_\alpha(i_\alpha) < f_{\alpha'}(i_\alpha)$. Then there is a $j > i_0$ such that for cofinally many $\alpha \in X'$ we have that $i_\alpha = j$, and hence we can assume that for all $\alpha \in X'$, $i_\alpha = j$. But then, if $\alpha < \beta$ are arbitrary elements of X'

$$f_\alpha^*(j) < g_\alpha(j) < f_{\alpha'}^*(i) < f_\beta^*(j) < g_\beta(j)$$

which is a contradiction, since it implies that A_j has cardinality at least ω_2.

If 3 holds then choose $X' \subset X$ cofinal in δ so that if $\alpha, \beta \in X'$ are distinct then modulo the bounded subsets of ω_1, $[\{i : f_\alpha(i) \leq g(i)\}] \neq [\{i : f_\beta(i) \leq g(i)\}]$. Since the f_α's are increasing with $\alpha \in X'$ at every $i > i_0$, the sets $\{i > i_0 : f_\alpha(i) \leq g(i)\}$ are strictly decreasing with $\alpha \in X'$, however it is impossible to have a stricitly decreasing sequence of subsets of ω_1 of length ω_2. So our theorem is finally proven. $\square$

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Lebenslauf

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