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Universal properties and categories of modules

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1 Category theory

1.1 Foundations and elementary definitions

1.1.1 Remark. There are several set theories on which category theory can be built. For us it is sufficient to adapt a system such as Morse-Kelley set theory, which allows us to talk about two different kinds of entities, namely small classes (sets) and proper classes. Each class has only small classes as elements. Anything that can be done in ZFC set theory we may also do with sets in Morse-Kelley set theory, but beyond that we may also speak of arbitrary collections of sets, which is essential in category theory. Proper classes, for example the class of all sets, are not elements of any class.

There is another way to deal with the collection of all sets. To the axioms of ZFC set theory we may add an axiom due to Alexander Grothendieck which states that any set is an element of a set U that has the properties that (1) if $x \in U$, then x is transitive (that is, $\bigcup\limits_{y\in x} y \subseteq x$), (2) $U \times U \subseteq U$, (3) if $x \in U$, then the powerset of x is an element of U and (4) if $(x_i)_{i\in I}$ is a family in U and $I \in U$, then $\bigcup\limits_{i\in I} x_i \in U$. Such a set is called a *Grothendieck universe.* All of "ordinary mathematics" can be done within any particular uncountable Grothendieck universe U. Call the elements of U *small sets.* In contrast to Morse-Kelley set theory however, we do not simply have small sets and sets having small sets as elements. Instead, the sets are arranged in layers in such a way that the lowest layer consists of all small sets and such that there is no topmost layer.

In all what follows, we do not place special emphasis on the choice of set theoretic foundations. The only important point is that we have at least two kinds of objects at our disposal. We use the terms "small class" and "set" exchangeably to denote objects of the smaller kind and "proper class" to denote objects of the larger kind. Everything is a class.

1.1.2 Definition. A function $\circ$ is called a *partial binary operation on a class M* if there is a subclass D of $M \times M$ such that $\circ : D \to M$. In this context, the function $\circ$ is called *composition.* We say that the composition of two elements a and b of M is *defined* if $(a, b) \in D$. An element e of M is called an *identity* if $e \circ a = a$ whenever $e \circ a$ is defined and $a \circ e = a$ whenever $a \circ e$ is defined.

A class $\mathscr{C}$ is called a *category* if the following conditions are satisfied.

(1) $\mathscr{C}$ is a pair $(M, \circ)$, where $\circ$ is a partial binary operation on M.

(2) For all $a, b, c \in M$, the following statements are equivalent:

   (a) The compositions $a \circ b$ and $b \circ c$ are defined.

   (b) The compositions $a \circ b$ and $(a \circ b) \circ c$ are defined.

   (c) The compositions $b \circ c$ and $a \circ (b \circ c)$ are defined.

   If so, we have $(a \circ b) \circ c = a \circ (b \circ c)$.

(3) For all $a \in M$, there exist identities c and d such that $a \circ d$ and $c \circ a$ are defined.

(4) For all identities c and d, the class $\{a \in M : a \circ d \text{ and } c \circ a \text{ are defined}\}$ is a set.

Elements of M are called *morphisms* of the category $\mathscr{C}$. For the class M we write $\mathrm{Mor}(\mathscr{C})$. Identities $e \in M$ are also called *objects* of the category $\mathscr{C}$. The class of objects of $\mathscr{C}$ is written as $\mathrm{Ob}(\mathscr{C})$. A category $\mathscr{C}$ is called *small* if it is a set.

1.1.3 Lemma. *Let $\mathscr{C}$ be a category.*

(1) *For each morphism a of $\mathscr{C}$ there exist uniquely determined identities c and d such that $c \circ a$ and $a \circ d$ are defined. The identity d is called the* domain *of a and is written as $\operatorname{dom} a$. The identity c is called the* codomain *of a and is written as $\operatorname{cod} a$.*

(2) *Let e be an object of $\mathscr{C}$. Then $\operatorname{cod} e = e = \operatorname{dom} e$.*

(3) *For each object e of $\mathscr{C}$, the composition $e \circ e$ is defined and $e = e \circ e$.*

(4) *For all objects e and f of $\mathscr{C}$, the composition $e \circ f$ is defined if and only if $e = f$.*

(5) *If a and b are morphisms of $\mathscr{C}$, then the composition $a \circ b$ is defined if and only if $\operatorname{dom} a = \operatorname{cod} b$.*

Proof. (1) If c' is another identity such that $c' \circ a$ is defined, then $c' \circ (c \circ a) = c' \circ a$ is defined, therefore $c' \circ c$ is defined; since c and c' are identities, $c = c \circ c' = c'$. The proof of the other statement is similar. (2) All of e, $\operatorname{dom} e$ and $\operatorname{cod} e$ are identities, therefore $\operatorname{cod} e = (\operatorname{cod} e) \circ e = e = e \circ \operatorname{dom} e = \operatorname{dom} e$. (3) $e = e \circ \operatorname{dom} e = e \circ e$. (4) If $e \circ f$ is defined, we have $e = \operatorname{cod} f = f$, since e is an identity; if $e = f$, then $f = \operatorname{dom} e$, therefore $e \circ f$ is defined. (5) First, $a \circ b = (a \circ \operatorname{dom} a) \circ ((\operatorname{cod} b) \circ b)$. Therefore, $a \circ b$ is defined if and only if $\operatorname{dom} a \circ \operatorname{cod} b$ is defined. Since $\operatorname{dom} a$ and $\operatorname{cod} b$ are identities, the claim follows from (4). $\square$

For objects d and c of the category $\mathscr{C}$, we write $\hom_{\mathscr{C}}(d, c)$ for the set $\{a \in M : \operatorname{dom} a = d \text{ and } \operatorname{cod} a = c\}$ of morphisms from d to c. The class M is the disjoint union of the sets $\hom_{\mathscr{C}}(d, c)$, where $c, d \in \mathrm{Ob}(\mathscr{C})$.

1.1.4 Remark. Very often categories are defined by specifying a class $\mathscr{O}$, for given elements a and b of $\mathscr{O}$ some nonempty set $A(b, a)$ and for elements a, b and c of $\mathscr{O}$ a function $\circ_{c,b,a} : A(c, b) \times A(b, a) \to A(c, a)$. The class of morphisms of the category to be defined is then the class $\mathscr{M} = \{(b, f, a) : b, a \in \mathscr{O}, f \in A(b, a)\}$. Composition of morphisms (c, g, b') and (b, f, a) is defined whenever $b = b'$. If so, $(c, g, b) \circ (b, f, a) = (c, \circ_{c,b,a}(g, f), a)$. Assume that the pair $\mathscr{C} = (\mathscr{M}, \circ)$ constructed this way is a category. Then for each $a \in \mathscr{O}$ there exists a unique element $e_a \in A(a, a)$ such that (a, e_a, a) is an identity of $\mathscr{C}$. Each identity is of this form. Via the assignment $a \mapsto (a, e_a, a)$ we may identify $\mathscr{O}$ with the class of objects of $\mathscr{C}$. Using this identification, for each morphism (b, f, a) of $\mathscr{C}$ we have $\operatorname{cod}(b, f, a) = b$ and $\operatorname{dom}(b, f, a) = a$ and for given objects a and b of $\mathscr{C}$ we have $\hom_{\mathscr{C}}(a, b) = A(b, a)$.

1.1.5 Examples.

(1) The *empty category* is the category $(\emptyset, \emptyset)$.

(2) Let $\mathscr{O}$ be the class of all (small) sets. For sets a and b, let $A(b, a)$ be the set of all functions from a to b. If a, b and c are sets, let the function $\circ_{c,b,a} : A(c, b) \times A(b, a) \to A(c, a)$ be given by the usual composition of functions. The construction above gives us a category, the *category of sets*, which is denoted **Set**.

(3) The category **Top** has as objects the topological spaces; morphisms from $(X, \mathscr{T})$ to $(X', \mathscr{T}')$ are given by the functions f from X to X' that are continuous with respect to $\mathscr{T}$ and $\mathscr{T}'$. Composition in **Top** is the usual composition of functions.

(4) Other examples of this kind include the category of groups (with group homomorphisms as morphisms), the category of (R, S)-bimodules (with (R, S)-linear functions as morphisms), the category of R-algebras and so on.

(5) To each preordered set $(I, \leq)$ we construct an associated small category $\mathscr{I}$: The set of morphisms of $\mathscr{I}$ is the set $\{(i,j) \in I \times I : j \leq i\}$. Composition of (i,j) and (j',k) is defined if and only if $j = j'$. Then $(i,j) \circ (j,k) = (i,k)$. Objects of $\mathscr{I}$ are the pairs (i,i) for $i \in I$. The function $i \mapsto (i,i)$ from I to $\mathrm{Ob}(\mathscr{I})$ is a bijection. Using this identification, we have $\mathrm{dom}\,(i,j) = j$ and $\mathrm{cod}\,(i,j) = i$. For given objects i and j of $\mathscr{I}$, we have $\hom_{\mathscr{I}}(i,j) = \{(j,i)\}$ if $i \leq j$ and $\emptyset$ in the other case.

There are the following special cases. If $\leq$ is the discrete order on I, the associated category $\mathscr{I}$ has no nontrivial morphisms. Such a category is also called a *discrete category.* If I is a two-element set $\{a,b\}$ and $\leq$ is the preorder $I \times I$, we may illustrate the category $\mathscr{I}$ as $\bullet \rightleftarrows \bullet$. Likewise, the category originating from a three-element set I together with the order $I \times I$ may be visualized as

$$\begin{tikzcd} \bullet \arrow[r, shift left] \arrow[dr, shift left] & \bullet \arrow[l, shift left] \arrow[dl, shift left] \\ & \bullet \arrow[ul, shift left] \arrow[u, phantom] \arrow[ur, shift left] \end{tikzcd} .$$

(6) Let (V,A) be a directed multigraph, that is, V is a set (of *vertices*) and A is a function from a set I (of *edges*) to $V \times V$. Let X be the image of A and assume that the multigraph has the properties that (a) for all $v \in V$, we have $(v,v) \notin X$ and (b) whenever $(c,b) \in X$ and $(b,a) \in X$ there is a unique $i \in I$ such that $(c,a) = A(i)$.

Then a category $\mathscr{C}$ is defined as follows. The set of morphisms is the disjoint union M of I and V. Extend the function A to M by setting $A(v) = (v,v)$ for $v \in V$. Composition of morphisms j and i is defined if there exist a,b,c such that $A(j) = (c,b)$ and $A(i) = (b,a)$, and in that case,

$$j \circ i = \begin{cases} \text{the } k \in I \text{ such that } A(k) = (c,a) & \text{if} \quad i,j \in I \\ j & \text{if} \quad i \in V \\ i & \text{if} \quad j \in V. \end{cases}$$

For example, let $I = \{1,2\}$ and $V = \{1,2\}$. The function $1 \mapsto (1,2)$, $2 \mapsto (1,2)$ from I to V defines a category which we may depict as $\bullet \rightrightarrows \bullet$.

The function $1 \mapsto (1,2)$, $2 \mapsto (1,2)$, $3 \mapsto (2,3)$, $4 \mapsto (1,3)$ from $I = \{1,2,3,4\}$ to $V = \{1,2,3\}$ defines the category $\begin{tikzcd} \bullet \arrow[r, shift left] \arrow[r, shift right] \arrow[rr, bend left] & \bullet \arrow[r] & \bullet \end{tikzcd}$.

1.1.6 Definition. (1) Let $\mathscr{C} = (M, \circ)$ be a category. The *dual* or *opposite category of* $\mathscr{C}$ is the pair $(M, \circ^{\mathrm{op}})$, where $\circ^{\mathrm{op}}$ is defined as follows: If $D \subseteq M \times M$ is chosen in such a way that $\circ : D \to M$, then $\circ^{\mathrm{op}}$ is the function from $\{(a,b) : (b,a) \in D\}$ to M defined by $a \circ^{\mathrm{op}} b = b \circ a$.

(2) Let $\mathscr{C} = (M, \circ)$ and $\mathscr{D} = (N, \circ')$ be categories. The category $\mathscr{D}$ is called a *subcategory of* $\mathscr{C}$ if $N \subseteq M$, $\mathrm{Ob}(\mathscr{D}) \subseteq \mathrm{Ob}(\mathscr{C})$ and $\circ'$ is the restriction of $\circ$ that has the property that whenever $a \circ b$ is defined for $a, b \in N$, then $a \circ' b$ is defined.

(3) Let $\mathscr{C} = (M, \circ)$ and $\mathscr{D} = (N, \circ')$ be categories. The *product category* $\mathscr{C} \times \mathscr{D}$ is the category $(M \times N, (\circ, \circ'))$. That is, morphisms (f_1, f_2) and (g_1, g_2) of $\mathscr{C} \times \mathscr{D}$ are composable if and only if f_1 and g_1 are composable in $\mathscr{C}$ and f_2 and g_2 are composable in $\mathscr{D}$. Then $(f_1, f_2) \circ (g_1, g_2) = (f_1 \circ g_1, f_2 \circ g_2)$.

More general, let $\mathbf{P}$ denote a formula and let $\mathbf{i}$, $\mathbf{I}$, $\mathbf{C}$, $\mathbf{f}$ and $\mathbf{g}$ denote distinct variables of which $\mathbf{I}$, $\mathbf{f}$ and $\mathbf{g}$ do not occur freely in $\mathbf{P}$. Let $\mathbf{P}$ have the following property:

$$\begin{gathered} \text{There is a set } \mathbf{I} \text{ whose elements are all } \mathbf{i} \text{ for which there exists a unique } \mathbf{C} \text{ such that } \mathbf{P} \\ \text{and for all } \mathbf{i} \in \mathbf{I}, \text{ there is a category } \mathbf{C} \text{ such that } \mathbf{P}. \end{gathered} \tag{1.1.1}$$

Then the class $\{\mathbf{f} : \mathbf{f}$ is a family having the index set $\mathbf{I}$ and $\forall \mathbf{i} \in \mathbf{I}\, \exists \mathbf{C} (\mathbf{P}$ and $\mathbf{f}(\mathbf{i}) \in \mathrm{Mor}(\mathbf{C}))\}$ together with the composition defined by $\mathbf{f} \circ \mathbf{g} = (\mathbf{f_i} \circ \mathbf{g_i})_{\mathbf{i} \in \mathbf{I}}$ is a category, the *product of the categories* $\mathbf{C}$, *where* $\mathbf{i} \in \mathbf{I}$ *and* $\mathbf{P}$.

1.1.7 Example. For rings with unity R and S (not necessarily commutative), we denote by ${}_R\mathbf{M}_S$ the category of (R,S)-bimodules together with (R,S)-linear maps. Let $R_0,\dots,R_n$ be rings and let P be the formula "$C = {}_{R_{i-1}}\mathbf{M}_{R_i}$ and $1 \le i \le n$". The product of the categories ${}_{R_{i-1}}\mathbf{M}_{R_i}$ where $1 \le i \le n$ has as morphisms all tuples $(f_1,\dots,f_n)$ such that $f_i \in \mathrm{Mor}\left({}_{R_{i-1}}\mathbf{M}_{R_i}\right)$ for $1 \le i \le n$. Composition of $(f_1,\dots,f_n)$ and $(g_1,\dots,g_n)$ is defined if and only if for all $1 \le i \le n$ the composition $f_i \circ g_i$ is defined in ${}_{R_{i-1}}\mathbf{M}_{R_i}$. If so, we have $(f_1,\dots,f_n)\circ(g_1,\dots,g_n) = (f_1\circ g_1,\dots,f_n\circ g_n)$, where the composition in the i-th coordinate is taken in the category ${}_{R_{i-1}}\mathbf{M}_{R_i}$.

1.1.8 Remark. The construction of product categories takes a simpler form if we accept the axiom of Grothendieck universes, since in this theory it is possible to talk about families of categories. In Morse-Kelley set theory this is not possible since categories that are not small cannot occur as elements, and therefore they can not be members of a family. Nevertheless, we use phrases such as "Let C_i be a category for $i \in I$" if we want to suppose that a formula $\mathbf{P}$ having the property (1.1.1) is given.

1.2 Functors and natural transformations

1.2.1 Definition. A class F is called a *functor from $\mathscr{A}$ to $\mathscr{B}$* if the following conditions are satisfied.

(1) $\mathscr{A}$ and $\mathscr{B}$ are categories.

(2) F is a function from $\mathrm{Mor}(\mathscr{A})$ to $\mathrm{Mor}(\mathscr{B})$.

(3) If $a \in \mathrm{Mor}(\mathscr{A})$ is an identity, then $F(a)$ is an identity.

(4) For all $\mathscr{A}$-morphisms f and g, $\operatorname{cod} f = \operatorname{dom} g$ implies $\operatorname{cod} F(f) = \operatorname{dom} F(g)$ and $F(g\circ f) = F(g)\circ F(f)$.

To indicate this situation, we also write $F : \mathscr{A} \to \mathscr{B}$. We also use phrases such as "Let $F : \mathscr{A} \to \mathscr{B}$ be a functor". Note however that the categories $\mathscr{A}$ and $\mathscr{B}$ are not determined by F alone: It is easy to see that a functor from $\mathscr{A}$ to $\mathscr{B}$ is also a functor from $\mathscr{A}^{\mathrm{op}}$ to $\mathscr{B}^{\mathrm{op}}$. This fact simplifies the process of "reversing all the arrows in a category", that is, passing to the dual of a "category theoretic statement". Alternatively, we could also define a functor as a triple $(\mathscr{A}, F, \mathscr{B})$ such that the properties (1) to (4) are satisfied. We do not use this definition.

Composition of functors is the usual composition of functions – the arising function turns out to be a functor as well. Associativity of this kind of composition is evident, and so is the fact that the functor $\mathrm{id}_{\mathrm{Mor}(\mathscr{A})} : \mathscr{A} \to \mathscr{A}$, denoted by $\mathbb{1}_{\mathscr{A}}$, acts as an identity for composition.

1.2.2 Definition. A class η is called *natural transformation* from F to G (*with respect to $\mathscr{A}$ and $\mathscr{B}$*) if the following conditions are satisfied.

(1) F and G are functors from $\mathscr{A}$ to $\mathscr{B}$.

(2) η is a function from the class of identities of $\mathscr{A}$ to the class of morphisms of $\mathscr{B}$.

(3) For each identity a of $\mathscr{A}$ we have $\eta(a) : F(a) \to G(a)$.

(4) If $f : a \to b$ is a $\mathscr{C}$-morphism, then $\eta_b \circ F(f) = G(f)\circ \eta_a$.

If confusion is unlikely to occur, we will omit the part "with respect to $\mathscr{A}$ and $\mathscr{B}$". The notation $\eta : F \to G$ is synonymous with "η is a natural transformation from F to G".

There are two kinds of composition of natural transformations.

(1) The *vertical composition* or simply *composition*. Let categories, functors and natural transformations be given as in the following diagram.

$$\begin{tikzcd}[column sep=large] \mathscr{A} \arrow[rr, bend left=60, "F"{name=F}] \arrow[rr, "G"{name=G, description}] \arrow[rr, bend right=60, "H"'{name=H}] && \mathscr{B} \arrow[from=F, to=G, "\varepsilon"] \arrow[from=G, to=H, "\eta"] \end{tikzcd}$$

We define $(\eta \circ \varepsilon)(a) = \eta(a) \circ \varepsilon(a)$ for objects a of $\mathscr{A}$. Then $\eta \circ \varepsilon$ is a natural transformation from F to H with respect to $\mathscr{A}$ and $\mathscr{B}$. Vertical composition of natural transformations is associative, as can be checked easily by gluing together two commutative squares. Let $\mathscr{A}$ and $\mathscr{B}$ are categories and let $F : \mathscr{A} \to \mathscr{B}$ be a functor. The natural transformation $\mathbb{1}_F$ from F to F, defined by $\mathbb{1}_F(a) = F(a)$ for $a \in \mathrm{Ob}(\mathscr{A})$, acts as an identity for the vertical composition, since $F(a)$ is an identity of $\mathscr{B}$ for all $\mathscr{A}$-objects a.

A natural transformation $\eta : F \to G$ is called a *natural isomorphism* if $\eta(i)$ is an isomorphism for all i. This is the case if and only if there is a natural transformation $\varepsilon : G \to F$ such that $\varepsilon \circ \eta = \mathbb{1}_F$ and $\eta \circ \varepsilon = \mathbb{1}_G$.

(2) The *horizontal composition* or *star product*. First, let categories, functors and a natural transformation be given as in the following diagram.

$$\begin{tikzcd} \mathscr{A} \arrow[r, "M"] & \mathscr{B} \arrow[r, bend left=50, "F"{name=F}] \arrow[r, bend right=50, "G"'{name=G}] & \mathscr{C} \arrow[r, "K"] & \mathscr{D} \arrow[from=F, to=G, "\varepsilon"] \end{tikzcd}$$

A short calculation shows that the function $a \mapsto \varepsilon_{M(a)}$ is a natural transformation from $F \circ M$ to $G \circ M$ and $b \mapsto K(\varepsilon_b)$ is a natural transformation from $K \circ F$ to $K \circ G$. These functions will turn out to be special cases of the star product yet to be defined. Now consider the following diagram.

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left=50, "F"{name=F}] \arrow[r, bend right=50, "G"'{name=G}] & \mathscr{B} \arrow[r, bend left=50, "K"{name=K}] \arrow[r, bend right=50, "L"'{name=L}] & \mathscr{C} \arrow[from=F, to=G, "\varepsilon"] \arrow[from=K, to=L, "\mu"] \end{tikzcd}$$

We have two natural transformations from $K \circ F$ to $L \circ G$, specifically $a \mapsto L(\varepsilon_a) \circ \mu_{F(a)}$ and $a \mapsto \mu_{G(a)} \circ K(\varepsilon_a)$. Since μ is a natural transformation from K to L, for all $a \in \mathrm{Ob}(\mathscr{A})$ the following diagram is commutative and hence the two natural transformations just defined are equal.

$$\begin{tikzcd} KFa \arrow[r, "\mu_{F(a)}"] \arrow[d, "K(\varepsilon_a)"'] & LFa \arrow[d, "L(\varepsilon_a)"] \\ KGa \arrow[r, "\mu_{G(a)}"'] & LGa \end{tikzcd}$$

The natural transformation thus defined is denoted by $\mu{\star}\varepsilon$. Horizontal composition is associative; identities are the natural transformations $\mathbb{1}_{\mathbb{1}_{\mathscr{A}}} : a \mapsto a$, where $\mathscr{A}$ is a category.

There is the following special case. Let functors $\mathscr{A} \xrightarrow{M} \mathscr{B} \overset{F}{\underset{G}{\rightrightarrows}} \mathscr{C} \xrightarrow{K} \mathscr{D}$ be given and let $\varepsilon : F \to G$ be a natural transformation. By abuse of notation, we write $K \star \varepsilon$ instead of $\mathbb{1}_K \star \varepsilon$ and $\eta \star F$ instead of $\eta \star \mathbb{1}_F$. Then $(K \star \varepsilon)(b) = K(\varepsilon_b)$ for objects b of $\mathscr{B}$ and $(\eta \star F)(a) = \eta_{F(a)}$ for objects a of $\mathscr{A}$. Also, the equation $K \star F = \mathbb{1}_{K \circ F}$ holds whenever F and K are composable functors. Given functors $\mathscr{A} \overset{F}{\underset{G}{\rightrightarrows}} \mathscr{B} \overset{K}{\underset{L}{\rightrightarrows}} \mathscr{C}$ and natural transformations $\varepsilon : F \to G$ and $\eta : K \to L$, the following diagram is commutative.

$$\begin{tikzcd}
K \circ F \arrow[r, "\mu \star F"] \arrow[d, "K \star \varepsilon"'] \arrow[dr, "\mu \star \varepsilon" description] & L \circ F \arrow[d, "L \star \varepsilon"] \\
K \circ G \arrow[r, "\mu \star G"'] & L \circ G
\end{tikzcd}$$

There are the following distributive laws for natural transformations.

Given functors $\mathscr{A} \xrightarrow{M} \mathscr{B} \overset{F}{\underset{H}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xrightarrow{G}}}}} \mathscr{C} \xrightarrow{K} \mathscr{D}$ and natural transformations $\varepsilon : F \to G$ and $\eta : G \to H$, we have $K \star (\eta \circ \varepsilon) = (K \star \eta) \circ (K \star \varepsilon)$ and $(\eta \circ \varepsilon) \star M = (\eta \star M) \circ (\varepsilon \star M)$.

There is also the following (more general) theorem on the relationship between horizontal and vertical composition. Given the following diagram, we have $(\nu \circ \mu) \star (\eta \circ \varepsilon) = (\nu \star \eta) \circ (\mu \star \varepsilon)$, the so–called *interchange law*.

$$\begin{tikzcd}
& F \arrow[d, "\varepsilon"] \arrow[dr] & & K \arrow[d, "\mu"] \arrow[dr] & \\
\mathscr{A} \arrow[ur, no head] \arrow[r, no head] \arrow[dr, no head] & G \arrow[r] \arrow[d, "\eta"] & \mathscr{B} \arrow[ur, no head] \arrow[r, no head] \arrow[dr, no head] & L \arrow[r] \arrow[d, "\nu"] & \mathscr{C} \\
& G \arrow[ur] & & M \arrow[ur] &
\end{tikzcd}$$

Now we wish to define the "category of natural transformations between functors from J to $\mathscr{C}$". It is, however, not enough to take the class of natural transformations as the class of morphisms, since in general a pair of functors (F, G) such that η is a natural transformation from F to G is not unique, as the following example shows.

1.2.3 Example. Consider the following category $\mathscr{A}$.

$$\begin{tikzcd}
a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[dr, "k"'] & b \arrow[d, "h"] \\
& c
\end{tikzcd}$$

Let three functors from $\mathscr{A}$ to $\mathscr{A}$ be defined as follows: F is the identity functor, G is the functor that exchanges f and g and H is defined by $H(f) = H(g) = k$, $H(k) = k$ and $H(h) = c$. This gives $H(a) = a$, $H(b) = H(c) = c$. Let also a function η be defined by $\eta(a) = a$, $\eta(b) = h$ and $\eta(c) = c$. Then η is a natural transformation from F to H, as the following diagrams, corresponding respectively to the morphisms f, g, k and h, show.

$$\begin{tikzcd}
a \arrow[r, "a"] \arrow[d, "f"'] & a \arrow[d, "k"] \\
b \arrow[r, "h"'] & c
\end{tikzcd}
\quad
\begin{tikzcd}
a \arrow[r, "a"] \arrow[d, "g"'] & a \arrow[d, "k"] \\
b \arrow[r, "h"'] & c
\end{tikzcd}
\quad
\begin{tikzcd}
a \arrow[r, "a"] \arrow[d, "k"'] & a \arrow[d, "k"] \\
c \arrow[r, "c"'] & c
\end{tikzcd}
\quad
\begin{tikzcd}
b \arrow[r, "h"] \arrow[d, "h"'] & c \arrow[d, "c"] \\
c \arrow[r, "c"'] & c
\end{tikzcd}$$

Analogously, η is a natural transformation from G to H.

1.2.4 Definition. Let $\mathscr{C}$ be a category and J be a small category. According to Remark 1.1.4 we define a category, written as $[J, \mathscr{C}]$ or $\mathscr{C}^J$ as follows: The objects are the functors from J to $\mathscr{C}$, the morphisms are given by natural transformations between such functors and the composition of morphisms is the vertical composition of natural transformations. The category $\mathscr{C}^J$ is called the *category of functors from J to $\mathscr{C}$*.

1.2.5 Remark. **(1)** The class of morphisms of the category $\mathscr{C}^J$ is the class of triples (G, η, F), where $\eta : F \to G$ with respect to J and $\mathscr{C}$. Composition of (H, μ, G') and (G, η, F) is defined if and only if $G = G'$, and given that, we have $(H, \mu, G) \circ (G, \eta, F) = (H, \mu \circ \eta, F)$.

(2) When working in Morse-Kelley set theory, the claim that J be a small category is essential: If J is a proper class and if $\mathscr{C}$ is not the empty category, then a natural transformation η between functors from J to $\mathscr{C}$ is itself a proper class and can therefore not be considered as a morphism of a category.

1.3 The dual of a statement

1.3.1 Remark and Definition. For each statement involving one or more categories, we want to be able to speak of the statement that results from "reversing all morphisms" in the appearing categories. To avoid entering into a discussion about what a "category-theoretic statement" could be, we give an elementary definition. This definition requires us to speak of "the dual $\mathscr{C}^{\mathrm{op}}$ of the category $\mathscr{C}$" without any restriction on $\mathscr{C}$ – that is, even if $\mathscr{C}$ turns out not to be a category at all. In the remaining part of the paper however we will use the notation $\mathscr{C}^{\mathrm{op}}$ only in the case that we know that $\mathscr{C}$ is a category, since we do not have to say very much about $\mathscr{C}^{\mathrm{op}}$ in the other case.

Let $\mathbf{C}_1, \ldots, \mathbf{C}_n$ denote pairwise distinct variables and let $\mathbf{P}$ denote a formula. Let $\mathbf{Q}$ be the formula that arises from $\mathbf{P}$ by replacing each free occurrence of $\mathbf{C}_1, \ldots, \mathbf{C}_n$ by $\mathbf{C}_1^{\mathrm{op}}, \ldots, \mathbf{C}_n^{\mathrm{op}}$ respectively. We call any formula equivalent to "$\mathbf{C}_1, \ldots, \mathbf{C}_n$ are categories and $\mathbf{Q}$" *the dual of* $\mathbf{P}$ *with respect to* $\mathbf{C}_1, \ldots, \mathbf{C}_n$.

1.3.2 Lemma. *Let* $\mathbf{P}$ *be a formula and let* $\mathbf{C}_1, \ldots, \mathbf{C}_n$ *be pairwise distinct variables.*

(1) The dual of the dual of $\mathbf{P}$ *with respect to* $\mathbf{C}_1, \ldots, \mathbf{C}_n$ *is the formula "*$\mathbf{C}_1, \ldots, \mathbf{C}_n$ *are categories and* $\mathbf{P}$*".*

(2) If $\mathbf{P}$ *is a theorem and* $\mathbf{P}^{\mathrm{op}}$ *is the dual of* $\mathbf{P}$ *with respect to* $\mathbf{C}_1, \ldots, \mathbf{C}_n$*, then the formula "*$\mathbf{C}_1, \ldots, \mathbf{C}_n$ *are categories* $\Rightarrow \mathbf{P}^{\mathrm{op}}$*" is also a theorem.*

□

1.3.3 Example. The statement that x is an initial object of $\mathscr{C}$ is the following: "$\mathscr{C}$ is a category and for all $z \in \mathrm{Ob}(\mathscr{C})$ there is a unique $\mathscr{C}$-morphism $f : x \to z$". The dual statement with respect to $\mathscr{C}$ is "$\mathscr{C}$ is a category and for all $z \in \mathrm{Ob}(\mathscr{C}^{\mathrm{op}})$ there is a unique $\mathscr{C}^{\mathrm{op}}$-morphism $f : x \to z$". The categories $\mathscr{C}$ and $\mathscr{C}^{\mathrm{op}}$ have the same morphisms and objects. Furthermore, when passing to the dual category, domain and codomain of a morphism exchange. Therefore the dual statement is "$\mathscr{C}$ is a category and for all $z \in \mathrm{Ob}(\mathscr{C})$ there is a unique $\mathscr{C}$-morphism $f : z \to x$", which means that x is a final object of $\mathscr{C}$.
Also, consider the notion of an isomorphism of objects: "$\mathscr{C}$ is a category, x and y are objects of $\mathscr{C}$ and there exist $\mathscr{C}$-morphisms $f : x \to y$ and $g : y \to x$ such that $f \circ g = \mathbb{1}_y$ and $g \circ f = \mathbb{1}_x$". The dual statement is "$\mathscr{C}$ is a category, x and y are objects of $\mathscr{C}^{\mathrm{op}}$ and there exist $\mathscr{C}^{\mathrm{op}}$-morphisms $f : x \to y$ and $g : y \to x$ such that $f \circ^{\mathrm{op}} g = \mathbb{1}_y$ and $g \circ^{\mathrm{op}} f = \mathbb{1}_x$". This is equivalent to "$\mathscr{C}$ is a category, x and y are objects of $\mathscr{C}$ and there exist $\mathscr{C}$-morphisms $f : y \to x$ and $g : x \to y$ such that $g \circ f = \mathbb{1}_y$ and $f \circ g = \mathbb{1}_x$", which is the original statement up to renaming of bound variables. Isomorphism of objects is *self-dual*.

Now consider the following theorem:
"If x and y are initial objects of a category $\mathscr{C}$, then x and y are isomorphic in $\mathscr{C}$". Dualizing this proposition, we get the following theorem: "If $\mathscr{C}$ is a category and x and y are initial objects of $\mathscr{C}^{\mathrm{op}}$, then x and y are isomorphic in $\mathscr{C}^{\mathrm{op}}$". We arrive at the following corollary: "If $\mathscr{C}$ is a category and x and y are final objects of $\mathscr{C}$, then x and y are isomorphic in $\mathscr{C}$".

This is the right place to examine what happens to functors, natural transformations and functor categories when we pass to the dual categories. From now on, unless otherwise noted, duals are always to be taken with respect to all of the involved categories. Identifying these categories will always be an easy task.

1.3.4 Remark. **(1)** The class F is a functor from $\mathscr{A}$ to $\mathscr{B}$ if and only if F is a functor from $\mathscr{A}^{\mathrm{op}}$ to $\mathscr{B}^{\mathrm{op}}$. Since $(\mathscr{C}^{\mathrm{op}})^{\mathrm{op}} = \mathscr{C}$ for any category $\mathscr{C}$, we have to work out only one direction of this claim. Let $F : \mathscr{A} \to \mathscr{B}$ be a functor. It is a trivial fact that the function F maps objects of $\mathscr{A}^{\mathrm{op}}$ to objects of $\mathscr{B}^{\mathrm{op}}$. If f and g are morphisms of $\mathscr{A}^{\mathrm{op}}$ such that $\mathrm{cod}^{\mathrm{op}} f = \mathrm{dom}^{\mathrm{op}} g$, then $F(g \circ^{\mathrm{op}} f) = F(f \circ g) = F(f) \circ F(g) = F(g) \circ^{\mathrm{op}} F(f)$, and the statement is proved. It follows that the statement "F is a functor from $\mathscr{A}$ to $\mathscr{B}$" is self-dual.

(2) A family η is a natural transformation from F to G with respect to $\mathscr{A}$ and $\mathscr{B}$ if and only if η is a natural transformation from G to F with respect to $\mathscr{A}^{\mathrm{op}}$ and $\mathscr{B}^{\mathrm{op}}$. To prove this, assume that the former is true. For each $\mathscr{A}^{\mathrm{op}}$-morphism m we have $F(m) \circ^{\mathrm{op}} \eta(\mathrm{dom}^{\mathrm{op}} m) = \eta(\mathrm{cod}\, m) \circ F(m) = G(m) \circ \eta(\mathrm{dom}\, m) = \eta(\mathrm{cod}^{\mathrm{op}} m) \circ^{\mathrm{op}} G(m)$. The converse follows from the identity $\left(\mathscr{C}^{\mathrm{op}}\right)^{\mathrm{op}} = \mathscr{C}$.

The dual of the statement "η is a natural transformation from $F : \mathscr{A} \to \mathscr{B}$ to $G : \mathscr{A} \to \mathscr{B}$ (with respect to $\mathscr{A}$ and $\mathscr{B}$)" is therefore "η is a natural transformation from G to F (with respect to $\mathscr{A}$ and $\mathscr{B}$)". Dualizing reverts natural transformations.

(3) The dual of a functor category. Let $\mathscr{C}$ be a category and let J be a small one. The category $[J, \mathscr{C}]^{\mathrm{op}}$ has the same morphisms as $[J, \mathscr{C}]$; the category $[J^{\mathrm{op}}, \mathscr{C}^{\mathrm{op}}]$ has as morphisms the triples (F, η, G) such that η is a natural transformation from G to F with respect to $\mathscr{A}^{\mathrm{op}}$ and $\mathscr{B}^{\mathrm{op}}$, that is, such that η is a natural transformation from F to G with respect to $\mathscr{A}$ and $\mathscr{B}$. Hence (G, η, F) is a morphism of $[\mathscr{A}, \mathscr{B}]^{\mathrm{op}}$ if and only if (F, η, G) is a morphism of $[\mathscr{A}^{\mathrm{op}}, \mathscr{B}^{\mathrm{op}}]$. In fact $A(G, \eta, F) = (F, \eta, G)$ defines an isomorphism from $[\mathscr{A}, \mathscr{B}]^{\mathrm{op}}$ onto $[\mathscr{A}^{\mathrm{op}}, \mathscr{B}^{\mathrm{op}}]$: Since objects of these categories are of the form $(F, \mathbb{1}_F, F)$, objects are mapped to objects; if (G, η, F) and (H, ε, G) are morphisms of $[\mathscr{A}, \mathscr{B}]^{\mathrm{op}}$, then $A((G, \eta, F) \underset{[\mathscr{A}, \mathscr{B}]^{\mathrm{op}}}{\circ} (H, \varepsilon, G)) = A((H, \varepsilon, G) \underset{[\mathscr{A}, \mathscr{B}]}{\circ} (G, \eta, F)) = A(H, \varepsilon \circ \eta, F) = (F, \varepsilon \circ \eta, H) = (F, \eta \circ^{\mathrm{op}} \varepsilon, H) = (F, \eta, G) \underset{[\mathscr{A}^{\mathrm{op}}, \mathscr{B}^{\mathrm{op}}]}{\circ} (G, \varepsilon, H) = A(G, \eta, F) \underset{[\mathscr{A}^{\mathrm{op}}, \mathscr{B}^{\mathrm{op}}]}{\circ} A(H, \varepsilon, G)$. Here $\circ$ denotes composition of natural transformations with respect to $\mathscr{A}$ and $\mathscr{B}$, and $\circ^{\mathrm{op}}$ denotes composition of natural transformations with respect to $\mathscr{A}^{\mathrm{op}}$ and $\mathscr{B}^{\mathrm{op}}$. Bijectivity of A is obvious, and the claim is proved.

(4) Summarizing: Dualization reverts the orientation of morphisms and natural transformations, but not the orientation of functors. Furthermore, the categories $[\mathscr{A}, \mathscr{B}]^{\mathrm{op}}$ and $[\mathscr{A}^{\mathrm{op}}, \mathscr{B}^{\mathrm{op}}]$ are isomorphic.

1.4 Adjunctions

1.4.1 Definition. Let $R : \mathscr{B} \to \mathscr{A}$ be a functor and let a be an object of $\mathscr{A}$. We call a pair (b, ζ) an *initial morphism for a with respect to R* or an *R-initial morphism for a*, if the following, so-called *universal property* is satisfied:

> $b \in \mathrm{Ob}(\mathscr{B})$ and $\zeta : a \to R(b)$ and if $b' \in \mathrm{Ob}(\mathscr{B})$ and $g : a \to R(b')$, there is a unique $\mathscr{B}$-morphism $\overline{g} : b \to b'$ such that $R(\overline{g}) \circ \zeta = g$.

$$\begin{tikzcd} R(b) \arrow[rr, "R(\overline{g})"] & & R(b') \\ & a \arrow[ul, "\zeta"] \arrow[ur, "g"'] & \end{tikzcd}$$

Let $L : \mathscr{A} \to \mathscr{B}$ be a functor and let b be an object of $\mathscr{B}$. We call a pair (a, ζ) a *final morphism for b with respect to L* or an *L-final morphism for b*, if the following *universal property* is satisfied:

> $a \in \mathrm{Ob}(\mathscr{A})$ and $\zeta : L(a) \to b$ and if $a' \in \mathrm{Ob}(\mathscr{A})$ and $f : L(a') \to b$, there is a unique $\mathscr{A}$-morphism $\overline{f} : a' \to a$ such that $\zeta \circ L(\overline{f}) = f$.

$$\begin{tikzcd} L(a) \arrow[dr, "\zeta"'] & & L(a') \arrow[ll, "L(\overline{f})"'] \arrow[dl, "f"] \\ & b & \end{tikzcd}$$

Apart from the variable names the two notions are dual to each other. A *universal morphism* is either an initial morphism or a final morphism: A universal morphism from a to R is an R-initial morphism for a and a universal morphism from L to b is an L-final morphism for b.

1.4.2 Remark. Initial morphisms for a with respect to $R : \mathscr{B} \to \mathscr{A}$ are precisely the initial objects of a so-called *comma category.* The objects of this category are pairs (b, π), where $b \in \mathrm{Ob}(\mathscr{B})$ and $\pi : a \to R(b)$ is an $\mathscr{A}$-morphism. The set of morphisms from (b, π) to (b', π') consists of those $\mathscr{B}$-morphisms $\rho : b \to b'$ satisfying $R(\rho) \circ \pi = \pi'$. Likewise, final morphisms are the final objects of a similarly defined category.

1.4.3 Examples.

(1) The free group. Let **Grp** be the category of groups and group homomorphisms and let $U : \mathbf{Grp} \to \mathbf{Set}$ be the forgetful functor. This functor transforms each group into its underlying set and each group homomorphism into its underlying function. A U-initial morphism for a set A is a pair (G, η), where G is a group and η is a function from A to the underlying set of G with the following property:

> For each group H and each function g from A to the underlying set of H there exists one and only one group homomorphism $\overline{g} : G \to H$ such that $\overline{g} \circ f = g$.

This property characterizes the free group generated by A up to isomorphism. The map η is also called *insertion of generators.* This example carries over, for example, to free R-modules and free R-algebras, by making only the obvious changes.

(2) Define a functor $\Delta : \mathbf{Grp} \to \mathbf{Grp} \times \mathbf{Grp}$ by $\Delta(a) = (a, a)$ for groups and group homomorphisms a. Let G and H be groups. Then a Δ-final morphism for (G, H) is a pair (A, ε), where A is a group and ε is a pair (g, h) of group homomorphisms, $g : A \to G$ and $h : A \to H$, having the following property:

> For each group B and for all group homomorphisms $g' : B \to G$ and $h' : B \to H$ there exists a unique group homomorphism $f : B \to A$ such that $g \circ f = g'$ and $h \circ f = h'$.

This is the universal property of the product of groups. The example generalizes easily to arbitrary index sets and also to many other categories than **Grp**.

(3) Let $\Delta : \mathbf{Grp} \to \mathbf{Grp} \times \mathbf{Grp}$ be the functor introduced in the previous example. A Δ-initial morphism for a pair (G, H) of groups is a group A together with pair (g, h) of group homomorphisms, $g : G \to A$ and $h : H \to A$ such that the following property is satisfied.

> For each group B and for all group homomorphisms $g' : G \to B$ and $h' : H \to B$ there exists a unique group homomorphism $f : A \to B$ such that $f \circ g = g'$ and $f \circ h = h'$.

The group A together with g and h is a free product of the groups G and H.

(4) Consider the diagonal functor $\Delta : \mathbf{Set} \to \mathbf{Set} \times \mathbf{Set}$, where $\Delta(x) = (x, x)$ for sets and functions x. A Δ-initial morphism for a pair (A, B) of sets is a set C together with two functions, $a : A \to C$ and $b : B \to C$, such that the universal property of the disjoint union of two sets is satisfied: Each pair of functions $a' : A \to D$, $b' : B \to D$ can be continued uniquely to a function from C to D.

(5) Let **CRing** be the category of commutative rings with unity, where ring homomorphisms are assumed to be unitary. For a given ring R, we denote by $R^\times$ the group of invertible elements of R. We define a category $\mathscr{A}$ as follows: Objects of $\mathscr{A}$ are pairs (R, S), where R is a ring and S is a submonoid of $(R, \cdot)$. Morphisms $(R, S) \to (R', S')$ are given by ring homomorphisms $f : R \to R'$ with the property that $f(S) \subseteq S'$. We further define a functor $G : \mathbf{CRing} \to \mathscr{A}$ by $G(R) = (R, R^\times)$ for rings R and $G(f) = f$ for morphisms $f : R \to R'$.

Let R be a ring and S a submonoid of $(R, \cdot)$. An G-initial morphism for (R, S) is a pair (T, e), where T is a ring and $e : R \to T$ is a ring homomorphism satisfying $e(S) \subseteq T^\times$ and the following property:

> For each ring T' and each ring homomorphism $g : R \to T'$ satisfying $g(S) \subseteq T'^\times$ there is a unique ring homomorphism $\overline{g} : T \to T'$ such that $\overline{g} \circ e = g$.

This is the universal property of the ring of fractions: T is the ring of fractions of R with denominators in S.

(6) Let R be a commutative ring and ${}_R\mathbf{M}$ be the category of (left) R-modules and R-linear maps. We then have the *internal hom-functor* ${}_R\mathrm{Hom}$ which assigns to each pair (E, F) of R-modules the R-module of R-linear maps $f : E \to F$ and to each pair (f, g) of R-module homomorphisms $f : E' \to E$, $g : F \to F'$ the R-linear map ${}_R\mathrm{Hom}(f, g) : {}_R\mathrm{Hom}(E, F) \to {}_R\mathrm{Hom}(E', F')$, $x \mapsto g \circ x \circ f$. Let F be an R-module. In what follows we are concerned with the (covariant) functor ${}_R\mathrm{Hom}(F, \_\,) : {}_R\mathbf{M} \to {}_R\mathbf{M}$.

Let E be an R-module. A pair (G, η), where G is an R-module and $\eta : E \to {}_R\mathrm{Hom}(F, G)$ is an R-linear map, is a ${}_R\mathrm{Hom}(F, \_\,)$-initial morphism for E if and only if the following holds:

> For each R-module H and each R-linear map $h : E \to {}_R\mathrm{Hom}(F, H)$ there is one and only one R-linear map $\overline{h} : G \to H$ such that ${}_R\mathrm{Hom}(F, \overline{h}) \circ \eta = h$.

The formula ${}_R\mathrm{Hom}(F, \overline{h}) \circ \eta = h$ is equivalent to saying that $\overline{h}(\eta(e)(f)) = h(e)(f)$ for all $(e, f) \in E \times F$. Using the canonical correspondence between bilinear maps $E \times F \to L$ and linear maps $E \to {}_R\mathrm{Hom}(F, L)$, we translate the statement above and arrive at the universal property of the tensor product of the R-modules E and F:

> If H is an R-module and $h : E \times F \to H$ is R-bilinear, then there exists a unique R-linear map $\overline{h} : G \to H$ such that $\overline{h}(e \otimes f) = h(e, f)$ for all $(e, f) \in E \times F$.

Here $\otimes$ denotes the bilinear map $(e, f) \mapsto \eta(e)(f)$.

Analogously, using the canonical correspondence between bilinear maps from $E \times F$ to L and linear maps from F to ${}_R\mathrm{Hom}(E, L)$, the tensor product $E \underset{R}{\otimes} F$ can be seen as a ${}_R\mathrm{Hom}(E, \_\,)$-initial morphism for F.

(7) The initial topology on a set. Let I be a small category and let $\mathscr{C}$ be another category. For each functor $F : I \to \mathscr{C}$ we have a category $\underset{I,\mathscr{C}}{\mathrm{Cone}}(F)$, the *category of cones over* F: Its objects are families of $\mathscr{C}$-morphisms $(f_i)_{i \in I}$ such that there exists an $\mathscr{C}$-object X with the properties that $f_i : X \to F(i)$ for all i and if $a : i \to j$ is an I-morphism, then $F(a) \circ f_i = f_j$. A morphism from $(f_i)_i$ to $(g_i)_i$ is given by a $\mathscr{C}$-morphism $h : X = \mathrm{dom}\, f_i \to Y = \mathrm{dom}\, g_i$ with the property that $g_i \circ h = f_i$ for all i.

$$\begin{tikzcd}
Y \arrow[dr, "g_i"] \arrow[ddr, "g_j"'] & & X \arrow[ll, "h"'] \arrow[dl, "f_i"'] \arrow[ddl, "f_j"] \\
& F(i) \arrow[d, "F(a)"] & \\
& F(j) &
\end{tikzcd}$$

In the following, whenever a letter is used both in boldface and normal font, we speak of a topological space resp. a **Top**-morphism (in boldface) and the underlying set resp. the underlying function (normal font). Now let I be a set (which we identify with the associated small discrete category), let $(\mathbf{Y}_i)_{i\in I} = (Y_i, \mathscr{T}_i)_{i\in I}$ be a family of topological spaces (which we identify with a functor from I to **Top**) and let X be a set. Furthermore, let $(f_i)_{i\in I}$ be a family of functions, where $f_i : X \to Y_i$. If $U : \mathbf{Top} \to \mathbf{Set}$ is the forgetful functor, this family is an object of the category $\underset{I,\mathbf{Set}}{\mathrm{Cone}}(U \circ \mathbf{Y}) = \underset{I,\mathbf{Set}}{\mathrm{Cone}}(Y)$. We have the *forgetful functor* $W : \underset{I,\mathbf{Top}}{\mathrm{Cone}}(\mathbf{Y}) \to \underset{I,\mathbf{Set}}{\mathrm{Cone}}(Y)$ which assigns to a family of **Top**-morphisms the family of corresponding **Set**-morphisms and also to a **Top**-morphism the corresponding **Set**-morphism. We intend to prove the following: If $\mathbf{X}$ denotes the set X provided with the initial topology with respect to the functions f_i, and if $\mathbf{f}_i$ denotes the now continuous function $\mathbf{X} \to \mathbf{Y}_i$, the family $(\mathbf{f}_i)_{i\in I}$ together with the identity function on X forms a final morphism for $(f_i)_{i\in I}$ with respect to the functor W.

For a pair $(\mathbf{G}, k)$ to be a W-final morphism it is necessary and sufficient that the following properties be fulfilled.

(1) $\mathbf{G}$ is a family $(\mathbf{g}_i)_{i\in I}$ and there exists a topological space $\mathbf{X}'$ such that for all i, the element $\mathbf{g}_i$ is a **Top**-morphism from $\mathbf{X}'$ to $\mathbf{Y}_i$. The space $\mathbf{X}'$ is then uniquely determined.

(2) k is a function from X' to X with the property that $f_i \circ k = g_i$ for all i.

(3) If $\mathbf{X}''$ is a topological space, $\mathbf{h}_i$ is a continuous function from $\mathbf{X}''$ to $\mathbf{Y}_i$ for all i and if $l : X'' \to X$ is a function with the property that $f_i \circ l = h_i$ for all i, then there exists a unique continuous function $\mathbf{m} : \mathbf{X}'' \to \mathbf{X}$ such that $k \circ m = l$.

$$\begin{tikzcd}
U(\mathbf{X}') \arrow[dr, "k"] \arrow[ddr, "U(\mathbf{g}_i)"'] & & U(\mathbf{X}'') \arrow[ll, "U(\mathbf{m})"'] \arrow[dl, "l"'] \arrow[ddl, "U(\mathbf{h}_i)"] \\
& X \arrow[d, "f_i"] & \\
& U(\mathbf{Y}_i) &
\end{tikzcd}$$

The pair $((\mathbf{f}_i)_i, \mathrm{id}_X)$ satisfies these properties: Choose $\mathbf{X}' = \mathbf{X}$. Then (1) clearly holds, and (2) is trivial. Let $\mathbf{h}_i : \mathbf{X}'' \to \mathbf{Y}_i$ be a **Top**-morphism for all i and let $l : X'' \to X$ be a function satisfying $f_i \circ l = h_i$ for all i. Since h_i is a continuous function, all of the functions $f_i \circ l$ are continuous. The characteristic property of the initial topology now implies continuity of l. Letting $\mathbf{m}$ be such that $U(\mathbf{m}) = l$, the diagram above is commutative. Since U is a faithful functor (that is, each restriction of U to a hom-set is injective) and k is the identity on X, the choice is clearly unique.

1.4.4 PROPOSITION. *Let L and M be functors from $\mathscr{A}$ to $\mathscr{B}$ and let $b \in \mathrm{Ob}(\mathscr{B})$. Assume that (a, ζ) is an L-final morphism for b.*

(1) Assume that $(\overline{a}, \xi)$ is another final morphisms for b with respect to L. There is a unique morphism $g : \overline{a} \to a$ such that $\zeta \circ L(g) = \xi$. It is an isomorphism.

(2) Let $g : \overline{a} \to a$ be an isomorphism. Then $(\overline{a}, \zeta \circ L(g))$ is an L-final morphism for b.

(3) If $\tau : M \to L$ is a natural isomorphism, then $(a, \zeta \circ \tau_a)$ is an M-final morphism for b. In particular, the object parts of final morphisms with respect to naturally isomorphic functors are isomorphic.

(4) If $f : b \to \overline{b}$ is an isomorphism, then $(a, f \circ \zeta)$ is an L-final morphism for $\overline{b}$.

(5) Assume that the following diagram is commutative, where $g : a \to \overline{a}$ and $f : b \to \overline{b}$ are isomorphisms.

$$\begin{tikzcd} L(a) \arrow[r, "L(g)"] \arrow[d, "\zeta"'] & L(\overline{a}) \arrow[d, "\xi"] \\ b \arrow[r, "f"'] & \overline{b} \end{tikzcd}$$

Then $(\overline{a}, \xi)$ is an L-final morphism for $\overline{b}$.

Proof.

(1) There is exactly one $\mathscr{A}$-morphism $h : a \to \overline{a}$ satisfying $\xi \circ L(h) = \zeta$ and exactly one $\mathscr{A}$-morphism $g : \overline{a} \to a$ satisfying $\zeta \circ L(g) = \xi$. Then $g \circ h$ is the unique morphism $f : a \to a$ satisfying $\zeta \circ F(f) = \zeta$, therefore $g \circ h = \mathbb{1}_a$.

$$\begin{tikzcd} L(a) \arrow[dr, "\zeta"'] & L(\overline{a}) \arrow[l, "L(g)"] \arrow[d, "\xi"] & L(a) \arrow[l, "L(h)"] \arrow[dl, "\zeta"] \arrow[ll, bend right, "L(g \circ h)"'] \\ & b & \end{tikzcd}$$

Also, $h \circ g$ is the unique morphism $f : \overline{a} \to \overline{a}$ such that $\xi \circ F(f) = \xi$.

(2) Set $\xi = \zeta \circ L(g)$. Assume that a' is an $\mathscr{A}$-object and that $f : L(a') \to b$ is a $\mathscr{B}$-morphism. There exists an $\mathscr{A}$-morphism $\overline{f} : a' \to a$ such that $\zeta \circ L(\overline{f}) = f$; setting $h = g^{-1} \circ \overline{f}$ it follows that $\xi \circ L(h) = \zeta \circ L(g) \circ L(g^{-1}) \circ L(\overline{f}) = f$. If h' is another $\mathscr{A}$-morphism satisfying $\xi \circ L(h') = f$, then $g \circ h'$ is a morphism a such that $\zeta \circ L(a) = \zeta \circ L(g) \circ L(h') = \xi \circ L(h') = f$, that is, $g \circ h' = \overline{f}$. It follows that $h = h'$.

(3) Suppose that $\overline{a}$ is an $\mathscr{A}$-object and that $\xi : M(\overline{a}) \to b$. Since $\xi \circ \tau_{\overline{a}}^{-1} : L(\overline{a}) \to b$, there is a unique $\mathscr{A}$-morphism $k : \overline{a} \to a$ such that $\zeta \circ L(k) = \xi \circ \tau_{\overline{a}}^{-1}$. Using the naturality of τ this means that there is a unique $\mathscr{A}$-morphism $k : \overline{a} \to a$ such that $\xi = \zeta \circ L(k) \circ \tau_{\overline{a}} = (\zeta \circ \tau_a) \circ M(k)$.

(4) If a' is an $\mathscr{A}$-object and $\psi : L(a') \to \overline{b}$, there is a unique morphism $h : a' \to a$ such that $\zeta \circ L(h) = f^{-1} \circ \psi$, that is, such that $(f \circ \zeta) \circ L(h) = \psi$.

(5) By (4), $(a, f \circ \zeta)$ is an L-final morphism for $\overline{b}$. From (2) it follows that $\big(\overline{a}, f \circ \zeta \circ L(g^{-1})\big) = (\overline{a}, \xi)$ is an L-final morphism for $\overline{b}$.

$\square$

1.4.5 Corollary (Dual of Proposition 1.4.4). *Let $R, S : \mathscr{B} \to \mathscr{A}$ be functors and let $a \in \mathrm{Ob}(\mathscr{A})$.*

(1) Assume that (b, ζ) and $(\overline{b}, \xi)$ are initial morphisms for a with respect to R. There is a unique morphism $f : b \to \overline{b}$ such that $R(f) \circ \zeta = \xi$. It is an isomorphism.

(2) Let (b, ζ) be an R-initial morphism for a and let $f : b \to \overline{b}$ be an isomorphism. Then $(\overline{b}, R(f) \circ \zeta)$ is an R-initial morphism for a.

(3) If (b, ζ) is an R-initial morphism for a and $\tau : R \to S$ is a natural isomorphism, then $(b, \tau_b \circ \zeta)$ is an S-initial morphism for a. In particular, the object parts of initial morphisms with respect to naturally isomorphic functors are isomorphic.

(4) If (b, ζ) is an R-initial morphism for a and if $g : \overline{a} \to a$ is an isomorphism, then $(b, \zeta \circ g)$ is an R-initial morphism for $\overline{a}$.

(5) Let (b, ζ) be be an R-initial morphism for a. Assume that $f : \overline{b} \to b$ is a $\mathscr{B}$-isomorphism and that $g : \overline{a} \to a$ is an $\mathscr{A}$-isomorphism. If $\zeta \circ g = R(f) \circ \xi$, then $(\overline{b}, \xi)$ is an R-initial morphism for $\overline{a}$. $\square$

1.4.6 Proposition. *Let functors $\mathscr{A} \xrightarrow{L} \mathscr{B} \xrightarrow{S} \mathscr{C}$ be given and let C be an object of $\mathscr{C}$.*

(1) If (B, s) is a final morphism for C with respect to S and (A, l) is a final morphism for B with respect to L, then $(A, s \circ S(l))$ is a final morphism for C with respect to $S \circ L$.

(2) Let (B, s) be a final morphism for C with respect to S and let (A, t) be any such for C with respect to $S \circ L$. There exists a final morphism for B with respect to L.

Proof.

(1) First, we have $s : S(B) \to C$, $l : L(A) \to B$, hence $s \circ S(l) : (S \circ L)(A) \to C$. Let X be an $\mathscr{A}$-object and $f : (S \circ L)(X) \to C$.

$$\begin{tikzcd}
S(L(A)) \arrow[dr, "S(l)"'] & & S(L(X)) \arrow[ll, "S(L(a))"'] \arrow[dl, "S(b)"] \arrow[dr, "f"] & \\
& S(B) \arrow[rr, "s"'] & & C
\end{tikzcd}$$

Because (B, s) is a final morphism, there is a uniquely determined $\mathscr{B}$-morphism $b : L(X) \to B$ such that $s \circ S(b) = f$. As (A, l) is one, there is a unique $\mathscr{A}$-morphism $a : X \to A$ such that $l \circ L(a) = b$. This proves the existence of an $\mathscr{A}$-morphism $g : X \to A$ satisfying $f = s \circ S(l \circ L(g)) = (s \circ S(l)) \circ (S \circ L)(g)$. If h is another morphism with this property, then $l \circ L(h)$ is a $\mathscr{B}$-morphism $k : L(X) \to B$ with the property that $s \circ S(k) = f$. It follows that $l \circ L(h) = b$ due to the first uniqueness, and then $h = a$ because of the second.

(2) There is one and only one $\mathscr{B}$-morphism $l : L(A) \to B$ with the property that $s \circ S(l) = t$. We show that (A, l) is a final morphism for B with respect to L. Assume that X is an $\mathscr{A}$-object and $f : L(X) \to B$ is a $\mathscr{B}$-morphism.

$$\begin{tikzcd}
& S(L(A)) \arrow[dl, "t"'] \arrow[dr, "S(l)"] & & S(L(X)) \arrow[ll, "S(L(a))"'] \arrow[dl, "S(f)"'] \\
C & & S(B) \arrow[ll, "s"] &
\end{tikzcd}$$

Due to the universality of (A, t) there is a unique $\mathscr{A}$-morphism $a : X \to A$ such that $t \circ (S \circ L)(a) = s \circ S(f)$. We then have $s \circ S(l \circ L(a)) = s \circ S(l) \circ (S \circ L)(a) = t \circ (S \circ L)(a) = s \circ S(f)$. Since (B, s) is universal, there is a unique $\mathscr{B}$-morphism $z : L(X) \to B$ such that $s \circ S(z) = s \circ S(f)$. This fact gives us $f = l \circ L(a)$. Uniqueness: Assume that $b : X \to A$ is another morphism with the property that $l \circ L(b) = f$. Then $t \circ (S \circ L)(b) = s \circ S(l) \circ (S \circ L)(b) = s \circ S(f)$, which implies $a = b$.

$\square$

We state the dual version of this proposition.

1.4.7 Corollary. *Let functors* $\mathscr{C} \xrightarrow{T} \mathscr{B} \xrightarrow{R} \mathscr{A}$ *be given and let* $A \in \mathrm{Ob}(\mathscr{A})$.

(1) If (B, r) *is an initial morphism for* A *with respect to* R *and* (C, t) *is an initial morphism for* B *with respect to* T, *then* $(C, R(t) \circ r)$ *is an initial morphism for* A *with respect to* $R \circ T$.

(2) Let (B, r) *be an initial morphism for* A *with respect to* R *and* (C, u) *be an initial morphism for* A *with respect to* $R \circ T$. *Then there exists an initial morphism for* B *with respect to* T.

1.4.8 Examples.

(1) Let $\Delta_1 : \mathbf{Grp} \to \mathbf{Grp} \times \mathbf{Grp}$ and $\Delta_2 : \mathbf{Set} \to \mathbf{Set} \times \mathbf{Set}$ be the diagonal functors (defined in 1.4.3(2), (4)) and let $V_1 : \mathbf{Grp} \to \mathbf{Set}$ and $V_2 : \mathbf{Grp} \times \mathbf{Grp} \to \mathbf{Set} \times \mathbf{Set}$ be the forgetful functors which assign to each group (resp. pair of groups) the underlying set (resp. pair of sets) and to each group homomorphism (resp. pair of group homomorphisms) the underlying function (resp. pair of functions). Then the following diagram is commutative.

$$\begin{tikzcd}
\mathbf{Grp} \arrow[r, "\Delta_1"] \arrow[d, "V_1"'] & \mathbf{Grp} \times \mathbf{Grp} \arrow[d, "V_2"] \\
\mathbf{Set} \arrow[r, "\Delta_2"'] & \mathbf{Set} \times \mathbf{Set}
\end{tikzcd}$$

Suppose that A and B are sets. Let F_A and F_B be the free groups generated by A and B respectively. Denote by $F_A * F_B$ their free product and by $A \sqcup B$ the disjoint union of A and B. Using Example 1.4.3(1,3,4), we infer the following facts:

1. The pair (F_A, F_B) is the object part of a V_2-initial morphism for (A, B).
2. The group $F_A * F_B$ is the object part of a Δ_1-initial morphism for (F_A, F_B).
3. The set $A \sqcup B$ is the object part of a Δ_2-initial morphism for (A, B).

Hence $F_A * F_B$ is the object part of an initial morphism for (A, B) with respect to $V_2 \circ \Delta_1 = \Delta_2 \circ V_1$ and is therefore isomorphic to a free group with generating set $A \sqcup B$. In other words, the free product of free groups is free.

(2) Suppose that R is a commutative ring. Let ${}_R\mathrm{Hom} : {}_R\mathbf{M}^{\mathrm{op}} \times {}_R\mathbf{M} \to {}_R\mathbf{M}$ be the internal hom-functor as in example 1.4.3(6). Let E, F and G be R-modules. The R-module $E \underset{R}{\otimes} F$ is the object part of a ${}_R\mathrm{Hom}(E, \_)$-initial morphism for F and $(E \underset{R}{\otimes} F) \underset{R}{\otimes} G$ is the object part of a ${}_R\mathrm{Hom}(G, \_)$-initial morphism for $E \underset{R}{\otimes} F$. Hence $(E \underset{R}{\otimes} F) \underset{R}{\otimes} G$ is the object part of a ${}_R\mathrm{Hom}(E, {}_R\mathrm{Hom}(G, \_))$-initial morphism for F.

Analogously, $F \underset{R}{\otimes} G$ is the object part of a ${}_R\mathrm{Hom}(G, \_)$-initial morphism for F and $E \underset{R}{\otimes} (F \underset{R}{\otimes} G)$ is the object part of a ${}_R\mathrm{Hom}(E, \_)$-initial morphism for $F \underset{R}{\otimes} G$. Therefore $E \underset{R}{\otimes} (F \underset{R}{\otimes} G)$ is the object part of a ${}_R\mathrm{Hom}(G, {}_R\mathrm{Hom}(E, \_))$-initial morphism for F.

By a straightforward evaluation of the expressions $\big(\varphi(B) \circ {}_R\mathrm{Hom}(E, {}_R\mathrm{Hom}(G, f))\big)(\zeta)(e)(g)$ and $\big({}_R\mathrm{Hom}(G, {}_R\mathrm{Hom}(E, f)) \circ \varphi(A)\big)(\zeta)(e)(g)$, where $\varphi(A) : {}_R\mathrm{Hom}(E, {}_R\mathrm{Hom}(G, A)) \to {}_R\mathrm{Hom}(G, {}_R\mathrm{Hom}(E, A))$ is the isomorphism defined by $\varphi(A)(\zeta)(g)(e) = \zeta(e)(g)$ for each R-module A, one can see that φ is a natural isomorphism from ${}_R\mathrm{Hom}(E, {}_R\mathrm{Hom}(G, \_))$ to ${}_R\mathrm{Hom}(G, {}_R\mathrm{Hom}(E, \_))$. Proposition 1.4.4(3) yields the existence of an isomorphism

$$\psi : E \underset{R}{\otimes} (F \underset{R}{\otimes} G) \to (E \underset{R}{\otimes} F) \underset{R}{\otimes} G.$$

Later we will see that the isomorphism ψ can be chosen in such a way that $\psi(e \otimes (f \otimes g)) = (e \otimes f) \otimes g$ for all $(e, f, g) \in E \times F \times G$.

1.4.9 Lemma. *Let $T : I \to \mathscr{B}$ and $L : \mathscr{A} \to \mathscr{B}$ be functors.*

(1) Assume that $f : x \to y$ is an I-morphism. Let there exist L-final morphisms $(\overline{x}, \varepsilon_x)$ for $T(x)$ and $(\overline{y}, \varepsilon_y)$ for $T(y)$. There is a unique $\mathscr{A}$-morphism $\overline{f} : \overline{x} \to \overline{y}$ such that $\varepsilon_y \circ L(\overline{f}) = T(f) \circ \varepsilon_x$.

$$\begin{tikzcd}
L(\overline{y}) \arrow[d, "\varepsilon_y"'] & L(\overline{x}) \arrow[l, "L(\overline{f})"'] \arrow[d, "\varepsilon_x"] \\
T(y) & T(x) \arrow[l, "T(f)"]
\end{tikzcd}$$

(2) For all $i \in \mathrm{Ob}(I)$ let (A_i, ε_i) be a final morphism for $T(i)$ with respect to L. There exists a uniquely determined functor $R : I \to \mathscr{A}$ satisfying the following properties.

(a) For all $i \in \mathrm{Ob}(I)$, we have $R(i) = A_i$.

(b) The family $(\varepsilon_i)_{i \in \mathrm{Ob}(I)}$ is a natural transformation from $L \circ R$ to T.

(3) For all $i \in \mathrm{Ob}(I)$ let (A_i, ε_i) and $(\hat{A}_i, \hat{\varepsilon}_i)$ be final morphisms for $T(i)$ with respect to L. Define functors R and $\hat{R}$ according to (2), corresponding to the families $(A_i, \varepsilon_i)_{i \in \mathrm{Ob}(I)}$ and $(\hat{A}_i, \hat{\varepsilon}_i)_{i \in \mathrm{Ob}(I)}$ respectively. For $i \in \mathrm{Ob}(I)$ let $\varphi_i : \hat{A}_i \to A_i$ be the unique $\mathscr{A}$–morphism having the property that $\varepsilon_i \circ L(\varphi_i) = \hat{\varepsilon}_i$. Then $(\varphi_i)_{i \in \mathrm{Ob}(I)}$ is a natural isomorphism from $\hat{R}$ to R.

Proof.

(1) We have $T(f) \circ \varepsilon_x : L(\overline{x}) \to y$, hence the universality of $(\overline{y}, \varepsilon_y)$ yields the result.

(2) If $R : I \to \mathscr{A}$ is a functor satisfying properties (a) and (b), then for all x, y and for all I-morphisms $f : x \to y$, the formula $\varepsilon_y \circ L(R(f)) = T(f) \circ \varepsilon_x$ holds. The uniqueness now follows from the uniqueness result in (1). Define R on objects i of I by $R(i) = A_i$, as claimed by (a). If $f : x \to y$ is an I-morphism, then by (1) there is a unique $\mathscr{A}$-morphism $\overline{f} : R(x) \to R(y)$ such that $\varepsilon_y \circ L(\overline{f}) = T(f) \circ \varepsilon_x$. Define $R(f) = \overline{f}$. If R really is a functor, then property (b) is satisfied by definition.

Apparently R preserves identities. If $f : x \to y$ and $g : y \to z$ are two I-morphisms, then $R(g \circ f)$ and $R(g) \circ R(f)$ are two $\mathscr{A}$-morphisms $a : A_x \to A_z$ such that $\varepsilon_z \circ L(a) = T(g \circ f) \circ \varepsilon_x$, as can be seen in the diagram below. There is only one such morphism, therefore the claim follows.

$$\begin{tikzcd}
L(A_z) \arrow[d, "\varepsilon_z"'] & L(A_y) \arrow[l, "L(R(g))"] \arrow[d, "\varepsilon_y"'] & L(A_x) \arrow[l, "L(R(f))"] \arrow[d, "\varepsilon_x"'] \arrow[ll, bend right, "L(R(g \circ f))"'] \\
T(z) & T(y) \arrow[l, "T(g)"'] & T(x) \arrow[l, "T(f)"'] \arrow[ll, bend left, "T(g \circ f)"]
\end{tikzcd}$$

(3) Let $f : i \to j$ be an I-morphism.

$$\begin{tikzcd}
L(R(i)) \arrow[r, "L(R(f))"] \arrow[d, "\varepsilon_i"] \arrow[dd, shift right=4ex, "L(\varphi_i)"'] & L(R(j)) \arrow[d, "\varepsilon_j"'] \arrow[dd, shift left=4ex, "L(\varphi_j)"] \\
T(i) \arrow[r, "T(f)"'] & T(j) \\
L(\hat{R}(i)) \arrow[u, "\hat{\varepsilon}_i"] \arrow[r, "L(\hat{R}(f))"'] & L(\hat{R}(j)) \arrow[u, "\hat{\varepsilon}_j"']
\end{tikzcd}$$

From the diagram we extract the relations $\hat{\varepsilon}_j \circ L(\hat{R}(f) \circ \varphi_i) = \hat{\varepsilon}_j \circ L(\hat{R}(f)) \circ L(\varphi_i) = T(f) \circ \hat{\varepsilon}_i \circ L(\varphi_i) = T(f) \circ \varepsilon_i = \varepsilon_j \circ L(R(f)) = \hat{\varepsilon}_j \circ L(\varphi_j) \circ L(R(f)) = \hat{\varepsilon}_j \circ L(\varphi_j \circ R(f))$. The universality of $(\hat{R}(j), \hat{\varepsilon}_j)$ now implies $\hat{R}(f) \circ \varphi_i = \varphi_j \circ R(f)$.

□

1.4.10 Corollary (Dual of Lemma 1.4.9). *Let $T : I \to \mathscr{A}$ and $R : \mathscr{B} \to \mathscr{A}$ be functors.*

(1) Assume that $f : y \to x$ is an I-morphism. Let there exist R-final morphisms $(\overline{x}, \eta_x)$ for $T(x)$ and $(\overline{y}, \eta_y)$ for $T(y)$. There is a unique $\mathscr{B}$-morphism $\overline{f} : \overline{y} \to \overline{x}$ such that $L(\overline{f}) \circ \eta_y = \eta_x \circ T(f)$.

(2) For all $i \in \mathrm{Ob}(I)$ let (B_i, η_i) be an initial morphism for $T(i)$ with respect to R. There exists a uniquely determined functor $L : I \to \mathscr{B}$ such that the family $(\eta_i)_{i \in \mathrm{Ob}(I)}$ is a natural transformation from T to $R \circ L$ and such that, for all $i \in \mathrm{Ob}(I)$, we have $L(i) = B_i$.

(3) For all $i \in \mathrm{Ob}(I)$ let (B_i, η_i) and $(\hat{B}_i, \hat{\eta}_i)$ be initial morphisms for $T(i)$ with respect to R. Define functors L and $\hat{L}$ according to (2), corresponding to the families $(B_i, \eta_i)_{i \in \mathrm{Ob}(I)}$ and $(\hat{B}_i, \hat{\eta}_i)_{i \in \mathrm{Ob}(I)}$ respectively. For $i \in \mathrm{Ob}(I)$ let $\psi_i : B_i \to \hat{B}_i$ be the unique $\mathscr{B}$–morphism having the property that $R(\psi_i) \circ \eta_i = \hat{\eta}_i$. Then $(\eta_i)_{i \in \mathrm{Ob}(I)}$ is a natural isomorphism from L to $\hat{L}$.

1.4.11 Theorem. *Let $L : \mathscr{A} \to \mathscr{B}$ be a functor. Assume that for each $b \in \mathrm{Ob}(\mathscr{B})$ there exists a final morphism (A_b, ε_b) for b with respect to L.*

(1) There is a uniquely determined functor $R : \mathscr{B} \to \mathscr{A}$ having the properties

   (a) For each object b of $\mathscr{B}$, we have $R(b) = A_b$.

   (b) The family $(\varepsilon_b)_{b \in \mathrm{Ob}(\mathscr{B})}$ is a natural transformation from $L \circ R$ to $\mathbb{1}_{\mathscr{B}}$.

(2) There is one and only one natural transformation $\eta : \mathbb{1}_{\mathscr{A}} \to R \circ L$ such that

$$(\varepsilon \star L) \circ (L \star \eta) = \mathbb{1}_L.$$

Moreover, the following equation holds.

$$(R \star \varepsilon) \circ (\eta \star R) = \mathbb{1}_R.$$

$$\begin{tikzcd}
 & L \circ R \circ L \arrow[dr, "\varepsilon \star L"] & \\
L \arrow[ur, "L \star \eta"] \arrow[rr, "\mathbb{1}_L"'] & & L
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[rr, "\mathbb{1}_R"] \arrow[dr, "\eta \star R"'] & & R \\
 & R \circ L \circ R \arrow[ur, "R \star \varepsilon"'] &
\end{tikzcd}$$

Proof.

(1) This is a consequence of Lemma 1.4.9.

(2) For each $a \in \mathrm{Ob}(\mathscr{A})$, the pair $\big((R\circ L)(a), \varepsilon_{L(a)}\big)$ is an L-final morphism for $L(a)$. There is hence a unique $\mathscr{B}$-morphism $\eta_a : a \to (R\circ L)(a)$ such that $\varepsilon_{L(a)} \circ L(\eta_a) = \mathbb{1}_{L(a)} = \mathbb{1}_L(a)$.

$$\begin{tikzcd}
(L\circ R\circ L)(a) \arrow[dr, "\varepsilon_{L(a)}"'] & & L(a) \arrow[ll, "L(\eta_a)"'] \arrow[dl, "\mathbb{1}_{L(a)}"] \\
& L(a) &
\end{tikzcd}$$

This proves uniqueness of a natural transformation η satisfying $(\varepsilon \star L)\circ(L\star\eta) = \mathbb{1}_L$. Now define η as above. To show naturality of η, let $f : x \to y$ be an $\mathscr{A}$-morphism. We need to prove the formula $\eta_y \circ f = (R\circ L)(f)\circ\eta_x$. Since $L(f)$ is a $\mathscr{B}$-morphism, we may deduce, using the naturality of ε and the defining property of η, that $\varepsilon_{L(y)} \circ L(\eta_y\circ f) = \varepsilon_{L(y)}\circ L(\eta_y)\circ L(f) = L(f) = L(f)\circ\varepsilon_{L(x)}\circ L(\eta_x) = \varepsilon_{L(y)}\circ L((R\circ L)(f))\circ L(\eta_x) = \varepsilon_{L(y)}\circ L((R\circ L)(f)\circ\eta_x)$. Hence $\eta_y\circ f$ and $(R\circ L)(f)\circ\eta_x$ are two $\mathscr{A}$-morphisms $a : x \to (R\circ L)(y)$ satisfying the equation $\varepsilon_{L(y)}\circ L(a) = L(f)$. Since $\big((R\circ L)(y), \varepsilon_{L(y)}\big)$ is an L-final morphism for $L(y)$, we have $\eta_y\circ f = (R\circ L)(f)\circ\eta_x$.
It remains to prove that the formula $(R\star\varepsilon)\circ(\eta\star R) = \mathbb{1}_R$ holds. To this end, let $b\in\mathrm{Ob}(\mathscr{B})$. Then $\varepsilon_b : (L\circ R)(b)\to b$ is a $\mathscr{B}$-morphism, which by naturality of ε implies $\varepsilon_b\circ L(R(\varepsilon_b)) = \varepsilon_b\circ\varepsilon_{L(R(b))}$.

$$\begin{tikzcd}
(L\circ R\circ L\circ R)(b) \arrow[r, "\varepsilon_{(L\circ R)(b)}"] \arrow[d, "(L\circ R)(\varepsilon_b)"'] & (L\circ R)(b) \arrow[d, "\varepsilon_b"] \\
(L\circ R)(b) \arrow[r, "\varepsilon_b"'] & b
\end{tikzcd}$$

From this formula we deduce $\varepsilon_b\circ L(R(\varepsilon_b)\circ\eta_{R(b)}) = \varepsilon_b\circ L(R(\varepsilon_b))\circ L(\eta_{R(b)}) = \varepsilon_b\circ\varepsilon_{L(R(b))}\circ L(\eta_{R(b)}) = \varepsilon_b\circ L(R(b))$. Since $(R(b),\varepsilon_b)$ is a final morphism for b with respect to L, the equality $R(\varepsilon_b)\circ\eta_{R(b)} = R(b)$ follows. The assertion is proved. □

1.4.12 Corollary (Dual of Theorem 1.4.11). *Let $R : \mathscr{B}\to\mathscr{A}$ be a functor. Assume that for each $a\in\mathrm{Ob}(\mathscr{A})$ there exists an initial morphism (B_a,η_a) for a with respect to R.*

(1) There is a uniquely determined functor $L : \mathscr{A}\to\mathscr{B}$ such that the family $(\eta_a)_{a\in\mathrm{Ob}(\mathscr{A})}$ is a natural transformation from to $\mathbb{1}_{\mathscr{A}}$ to $R\circ L$ and such that for each object a of $\mathscr{A}$ we have $L(a) = B_a$.

(2) There is one and only one natural transformation $\varepsilon : L\circ R\to\mathbb{1}_{\mathscr{B}}$ such that $(R\star\varepsilon)\circ(\eta\star R) = \mathbb{1}_R$. We then have the formula $(\varepsilon\star L)\circ(L\star\eta) = \mathbb{1}_L$.

Note that up to variable naming, the "triangular identities" exchange in the course of dualizing.

1.4.13 Definition. Let functors $\mathscr{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{B}$ be given and let there exist natural transformations $\eta : \mathbb{1}_{\mathscr{A}}\to R\circ L$ and $\varepsilon : L\circ R\to\mathbb{1}_{\mathscr{B}}$ satisfying the *triangular identities*

$$(\varepsilon\star L)\circ(L\star\eta) = \mathbb{1}_L \qquad\text{and}\qquad (R\star\varepsilon)\circ(\eta\star R) = \mathbb{1}_R.$$

Then the pair (L,R) is called an *adjunction*. We use the notation $(\eta,\varepsilon) : L\dashv R : (\mathscr{B},\mathscr{A})$ or simply $L\dashv R$ to indicate this situation. We call L the *left adjoint* and R the *right adjoint functor*. The natural transformation η is called *unit* and ε is called *counit of the adjunction*.
A functor L is called *left adjoint functor* if there is an R such that (L,R) is an adjunction. A functor R is called *right adjoint functor* if there is an L such that (L,R) is an adjunction.

1.4.14 Remark. The dual of "$(\eta,\varepsilon) : L \dashv R : (\mathscr{B},\mathscr{A})$" with respect to $\mathscr{A}$ and $\mathscr{B}$ is "$(\varepsilon,\eta) : R \dashv L : (\mathscr{A},\mathscr{B})$". The dual with respect to $\mathscr{A}$ and $\mathscr{B}$ of "$R : \mathscr{B} \to \mathscr{A}$ is a right adjoint functor" is "$R : \mathscr{B} \to \mathscr{A}$ is a left adjoint functor".

1.4.15 Theorem. *Let $(\eta,\varepsilon) : L \dashv R : (\mathscr{B},\mathscr{A})$ be an adjunction. The functors $\mathscr{A}^{\mathrm{op}} \times \mathscr{B} \to \mathbf{Set}$ defined by*

$$\begin{aligned}\hom_{\mathscr{B}}(L\_\,,=) &: (c,d) \mapsto \hom_{\mathscr{B}}(L(c),d) \text{ and}\\ \hom_{\mathscr{A}}(\_\,,R=) &: (c,d) \mapsto \hom_{\mathscr{A}}(c,R(d))\end{aligned}$$

are naturally isomorphic. Natural isomorphisms are given by the following **Set***-morphisms, for $(a,b) \in \mathrm{Ob}(\mathscr{A}\times\mathscr{B})$:*

$$\begin{aligned}\varphi(a,b) &: \hom_{\mathscr{B}}(L(a),b) \to \hom_{\mathscr{A}}(a,R(b)), \quad f \mapsto R(f)\circ\eta(a) \quad \text{and}\\ \psi(a,b) &: \hom_{\mathscr{A}}(a,R(b)) \to \hom_{\mathscr{B}}(L(a),b), \quad g \mapsto \varepsilon_b \circ L(g).\end{aligned}$$

Proof. The functions $\varphi(a,b)$ and $\psi(a,b)$ are well-defined: $a \xrightarrow{\eta_a} R(L(a)) \xrightarrow{R(f)} R(b)$ and $L(a) \xrightarrow{L(g)} L(R(b)) \xrightarrow{\varepsilon_b} b$. Let $(c,d) : (a,b) \to (a',b')$ be an $\mathscr{A}^{\mathrm{op}} \times \mathscr{B}$-morphism, that is, $c : a' \to a$ in $\mathscr{A}$ and $d : b \to b'$ in $\mathscr{B}$. We want to establish commutativity of the following diagram.

$$\begin{tikzcd}
\hom_{\mathscr{B}}(L(a),b) \arrow[r, shift left, "\varphi(a,b)"] \arrow[d, "\hom_{\mathscr{B}}(L(c),d)"'] & \hom_{\mathscr{A}}(a,R(b)) \arrow[l, shift left, "\psi(a,b)"] \arrow[d, "\hom_{\mathscr{A}}(c,R(d))"] \\
\hom_{\mathscr{B}}(L(a'),b') \arrow[r, shift left, "\varphi(a',b')"] & \hom_{\mathscr{A}}(a',R(b')) \arrow[l, shift left, "\psi(a',b')"]
\end{tikzcd}$$

For each $\mathscr{B}$-morphism $f : L(a) \to b$, we have $\varphi(a',b')\big(\hom_{\mathscr{B}}(L(c),d)(f)\big) = \varphi(a',b')(d\circ f\circ L(c)) = R(d\circ f\circ L(c))\circ\eta_{a'} = R(d\circ f)\circ(R\circ L(c))\circ\eta_{a'} = R(d\circ f)\circ\eta_a\circ c$ and $\hom_{\mathscr{A}}(c,R(d))(\varphi(a,b)(f)) = \hom_{\mathscr{A}}(c,R(d))(R(f)\circ\eta_a) = R(d)\circ R(f)\circ\eta_a\circ c$, since η is a natural transformation.
Let $f : L(a) \to b$ be a $\mathscr{B}$-morphism. The formula $(\varepsilon\star L)\circ(L\star\eta) = \mathbb{1}_L$ yields $\psi(a,b)(\varphi(a,b)(f)) = \psi(a,b)(R(f)\circ\eta_a) = \varepsilon_b\circ L(R(f)\circ\eta_a) = \varepsilon_b\circ L(R(f))\circ L(\eta_a) = f\circ\varepsilon_{L(a)}\circ L(\eta_a) = f$. Analogously, for an $\mathscr{A}$-morphism $g : a \to R(b)$, we have $\varphi(a,b)(\psi(a,b)(g)) = \varphi(a,b)(\varepsilon_b\circ L(g)) = R(\varepsilon_b)\circ(R\circ L)(g)\circ\eta_a = R(\varepsilon_b)\circ\eta_{R(b)}\circ g = g$ according to the formula $(R\star\varepsilon)\circ(\eta\star R) = \mathbb{1}_R$.
Naturality of ψ is a straightforward consequence of the above. ☐

The bijectivity of the functions $\varphi(a,b)$ and $\psi(a,b)$ yields the following corollary.

1.4.16 Corollary. *Let $(\eta,\varepsilon) : L \dashv R : (\mathscr{B},\mathscr{A})$ be an adjunction. Let $a \in \mathrm{Ob}(\mathscr{A})$ and $b \in \mathrm{Ob}(\mathscr{B})$.*

(1) For all $b \in \mathrm{Ob}(\mathscr{B})$, the pair $(R(b),\varepsilon_b)$ is an L-final morphism for b.
If $f : L(a) \to b$ is a $\mathscr{B}$-morphism, then $R(f)\circ\eta(a)$ is the unique $\mathscr{A}$-morphism $g : a \to R(b)$ such that $\varepsilon_b\circ L(g) = f$.

(2) For all $a \in \mathrm{Ob}(\mathscr{A})$, the pair $(L(a),\eta_a)$ is an R-initial morphism for a.
If $g : a \to R(b)$ is an $\mathscr{A}$-morphism, then $\varepsilon_b\circ L(g)$ is the unique $\mathscr{B}$-morphism $f : L(a) \to b$ such that $R(f)\circ\eta_a = g$.

$$
\begin{tikzcd}
L(R(b)) \arrow[dr, "\varepsilon_b"'] & & L(a) \arrow[ll, "L(R(f)\circ\eta(a))"'] \arrow[dl, "f"] \\
& b &
\end{tikzcd}
\qquad
\begin{tikzcd}
R(L(a)) \arrow[rr, "R(\varepsilon_b \circ L(g))"] & & R(b) \\
& a \arrow[ul, "\eta_a"] \arrow[ur, "g"'] &
\end{tikzcd}
$$

The following characterization of adjoint functors is also very useful.

1.4.17 Corollary. *Let $L : \mathscr{A} \to \mathscr{B}$ be a functor. The following statements are equivalent:*

(1) L ist a left adjoint functor.

(2) For each object b of $\mathscr{B}$ there is an L-final morphism for b.

Also, a functor R is a right adjoint functor if and only if there is an R-initial morphism for each object a of $\mathscr{A}$.

1.4.18 Proposition (Uniqueness of adjoint functors). *If (L, R) and $(L, \hat{R})$ are adjunctions, then R and $\hat{R}$ are naturally isomorphic. Let $\eta, \hat{\eta}, \varepsilon, \hat{\varepsilon}, \mathscr{A}$ and $\mathscr{B}$ be chosen such that $(\eta, \varepsilon) : L \dashv R : (\mathscr{B}, \mathscr{A})$ and $(\hat{\eta}, \hat{\varepsilon}) : L \dashv \hat{R} : (\mathscr{B}, \mathscr{A})$. Then a natural isomorphism φ from $\hat{R}$ to R is given by the unique $\mathscr{A}$–morphisms $\varphi_b : \hat{R}(b) \to R(b)$ satisfying $\varepsilon_b \circ L(\varphi_b) = \hat{\varepsilon}_b$, where $b \in \mathrm{Ob}(\mathscr{B})$. Conversely, if (L, R) is an adjunction and R is naturally isomorphic to $\hat{R}$, then $(L, \hat{R})$ is an adjunction.*
If (L, R) and $(\hat{L}, R)$ are adjunctions, then L and $\hat{L}$ are naturally isomorphic. Let $\eta, \hat{\eta}, \varepsilon, \hat{\varepsilon}, \mathscr{A}$ and $\mathscr{B}$ be chosen such that $(\eta, \varepsilon) : L \dashv R : (\mathscr{B}, \mathscr{A})$ and $(\hat{\eta}, \hat{\varepsilon}) : \hat{L} \dashv R : (\mathscr{B}, \mathscr{A})$. Then a natural isomorphism ψ from L to $\hat{L}$ is given by the unique $\mathscr{B}$–morphisms $\psi_a : L(a) \to \hat{L}(a)$ satisfying $R(\psi_a) \circ \eta_a = \hat{\eta}_a$, where $a \in \mathrm{Ob}(\mathscr{A})$. Conversely, if (L, R) is an adjunction and L is naturally isomorphic to $\hat{L}$, then $(\hat{L}, R)$ is an adjunction.

Proof. It is sufficient to give a proof of the first pair of statements. By reason of Corollary 1.4.16(1), for each $\mathscr{B}$-object b the pairs $(R(b), \varepsilon_b)$ and $(\hat{R}(b), \hat{\varepsilon}_b)$ are L-final morphisms for b. Since (L, R) and $(L, \hat{R})$ are adjunctions, we have $\varepsilon : L \circ R \to \mathbb{1}_{\mathscr{B}}$ and $\hat{\varepsilon} : L \circ \hat{R} \to \mathbb{1}_{\mathscr{B}}$. For $b \in \mathrm{Ob}(\mathscr{B})$ let $\varphi_b : \hat{R}(b) \to R(b)$ be the $\mathscr{A}$-morphism having the property that $\varepsilon_b \circ L(\varphi_b) = \hat{\varepsilon}_b$. By Lemma 1.4.9(3), $b \mapsto \varphi_b$ is a natural isomorphism from $\hat{R}$ to R.
To prove the second statement, let $\mu : R \to \hat{R}$ be a natural isomorphism. Define $\hat{\eta} = (\mu \star L) \circ \eta$ and $\hat{\varepsilon} = \varepsilon \circ (L \star \mu^{-1})$.

$$
\mathbb{1}_{\mathscr{A}} \xrightarrow{\eta} R \circ L \xrightarrow{\mu \star L} \hat{R} \circ L \qquad\qquad L \circ \hat{R} \xrightarrow{L \star \mu^{-1}} L \circ R \xrightarrow{\varepsilon} \mathbb{1}_{\mathscr{B}}
$$

Then $(\hat{\eta}, \hat{\varepsilon}) : L \dashv \hat{R} : (\mathscr{B}, \mathscr{A})$ is an adjunction: The proof of this statement is a straightforward verification of the triangular identities. ☐

1.4.19 Theorem. *Let functors* $\begin{tikzcd} \mathscr{A} \arrow[r, shift left, "L"] & \mathscr{B} \arrow[l, shift left, "R"] \arrow[r, shift left, "S"] & \mathscr{C} \arrow[l, shift left, "T"] \end{tikzcd}$ *be given. If (L, R) and (S, T) are adjunctions, then $(S \circ L, R \circ T)$ is also an adjunction.*

Proof. Choose $\eta, \hat{\eta}, \varepsilon, \hat{\varepsilon}$ such that $(\eta, \varepsilon) : L \dashv R : (\mathscr{B}, \mathscr{A})$ and $(\hat{\eta}, \hat{\varepsilon}) : S \dashv T : (\mathscr{C}, \mathscr{B})$. The domains and codomains of these natural transformations are as follows.

$$
\begin{aligned}
&\eta : \mathbb{1}_{\mathscr{A}} \to R \circ L &\qquad& \varepsilon : L \circ R \to \mathbb{1}_{\mathscr{B}} \\
&\hat{\eta} : \mathbb{1}_{\mathscr{B}} \to T \circ S && \hat{\varepsilon} : S \circ T \to \mathbb{1}_{\mathscr{C}}
\end{aligned}
$$

Let c be an object of $\mathscr{C}$. The pair $\big(T(c), \hat{\varepsilon}(c)\big)$ is an S-final morphism for c and $\big((R\circ T)(c), \varepsilon_{T(c)}\big)$ is an L-final morphism for $T(c)$, by Corollary 1.4.16. By reason of Proposition 1.4.6 the pair $\big((R\circ T)(c), \hat{\varepsilon}_c \circ S(\varepsilon_{T(c)})\big) = \big((R\circ T)(c), (\hat{\varepsilon}\circ(S\star\varepsilon\star T))(c)\big)$ is an $S\circ L$-final morphism for c. But $\hat{\varepsilon}\circ(S\star\varepsilon\star T) : (S\circ L)\circ(R\circ T) \to \mathbb{1}_{\mathscr{C}}$ is a natural transformation. Feeding this into Theorem 1.4.11(1), the assertion is proved. $\square$

1.5 Limits and continuity

1.5.1 Definition.
Let $\mathscr{C}$ be a category and let J be a small one. We construct a functor $\Delta : \mathscr{C} \to \mathscr{C}^J$ as follows: For each object a of $\mathscr{C}$ we define a functor $\overline{\Delta}(a)$ by $\overline{\Delta}(a)(j) = a$ for all $j \in \mathrm{Ob}(J)$. If $g : a \to a'$ is a morphism of $\mathscr{C}$, let $\Delta(g)$ be given by the natural transformation from $\overline{\Delta}(a)$ to $\overline{\Delta}(a')$ that satisfies $\Delta(g)(j) = g$ for all objects j of J. This functor is also written as $\Delta_{\mathscr{C}}$ or $\Delta_{\mathscr{C},J}$, depending on the extent to which confusion is possible. We call $\Delta_{\mathscr{C},J}$ the *diagonal functor for $\mathscr{C}$ with respect to J*.

1.5.2 Definition. Assume that $F : J \to \mathscr{C}$ is a functor, N is an object of $\mathscr{C}$ and $\eta = (\eta_j)_{j\in\mathrm{Ob}(J)}$ is a family of $\mathscr{C}$-morphisms.

(1) A *cone over F with apex N* or a *cone from N to F* is a pair (N, ε) such that ε is a natural transformation from $\Delta(N)$ to F. That is, such that $\varepsilon_j : N \to F(j)$ for all J-objects j and $F(f)\circ\varepsilon_j = \varepsilon_{j'}$ for all J-morphisms $f : j \to j'$.

A *co-cone over F with apex N* or a *co-cone from F to N* is a pair (N, η) such that η is a natural transformation from F to $\Delta(N)$. That is, such that $\eta_j : F(j) \to N$ for all J-objects j and $\eta_{j'}\circ F(f) = \eta_j$ for all J-morphisms $f : j \to j'$.

(2) The pair (N, ε) is called a *limit of F* if (N, ε) is a cone from N to F and for each cone (M, α) there is a unique $\mathscr{C}$-morphism $\xi : M \to N$ such that $\varepsilon_j \circ \xi = \alpha_j$ for all objects j of J. By abuse of notation (morphisms of $\mathscr{C}^J$ are not natural transformations according to our definition), we may identify a limit (N, ε) of F with a Δ-final morphism for F.

The pair (N, η) is called a *colimit of F* if it is a co-cone from F to N and for each co-cone (M, β) there is a unique $\mathscr{C}$-morphism $\zeta : N \to M$ such that $\zeta\circ\eta_j = \beta_j$ for all objects j of J. A colimit (N, η) may be viewed as a Δ-initial morphism for F.

$$\begin{tikzcd} N \arrow[d, "\varepsilon_j"'] \arrow[dr, "\varepsilon_i" near start] & M \arrow[l, "\xi"'] \arrow[d, "\alpha_i"] \arrow[dl, "\alpha_j"' near start] \\ F(j) & F(i) \arrow[l, "F(f)"] \end{tikzcd} \qquad \begin{tikzcd} N \arrow[r, "\zeta"] & M \\ F(j) \arrow[u, "\eta_j"] \arrow[r, "F(f)"'] \arrow[ur, "\beta_j" near end] & F(i) \arrow[u, "\beta_i"'] \arrow[ul, "\eta_i"' near end] \end{tikzcd}$$

(N, ε_i) is a limit of F $\qquad$ (M, η_i) is a colimit of F

(3) The category $\mathscr{C}$ is called *J-complete* (resp. *J-cocomplete*) if for each functor $F : J \to \mathscr{C}$ there exists a limit (resp. colimit) of F. It is called *complete* (resp. *cocomplete* if it is J-complete (resp. J-cocomplete) for each small category J.

1.5.3 Remark. The notions "limit" and "colimit" are dual to each other: A pair (M, a) is a limit of $F : \mathscr{A} \to \mathscr{B}$ if and only if (M, a) is a colimit of $F : \mathscr{A}^{\mathrm{op}} \to \mathscr{B}^{\mathrm{op}}$. Therefore the dual of "(M, a) is a limit of $F : \mathscr{A} \to \mathscr{B}$" with respect to $\mathscr{A}$ and $\mathscr{B}$ is "(M, a) is a colimit of $F : \mathscr{A} \to \mathscr{B}$".

1.5.4 Examples.

(1) (Limits in **Set**.) Let I be a small category and let $F : I \to \mathbf{Set}$ be a functor. Then the set

$$A_F = \Big\{ x \in \prod_{i \in \mathrm{Ob}(I)} F(i) : x_{\mathrm{cod} f} = F(f)(x_{\mathrm{dom} f}) \text{ for all } f \in \mathrm{Mor}(I) \Big\}$$

together with the projection maps $\mathrm{pr}_i : (x_i)_{i \in \mathrm{Ob}(I)} \mapsto x_i$ is a limit of F.

First, if $f : i \to j$ is an I-morphism, then for each $x \in A_F$ we have $(F(f) \circ \mathrm{pr}_i)(x) = F(f)(\mathrm{pr}_i(x)) = F(f)(x_i) = x_j = \mathrm{pr}_j(x)$. Consequently, $\big(A_F, (\mathrm{pr}_i)_{i \in \mathrm{Ob}(I)}\big)$ is a cone over F. Let $\big(B, (q_i)_{i \in \mathrm{Ob}(I)}\big)$ be another cone over F and let $f : i \to j$ be an I-morphism. Then $F(f) \circ q_i = q_j$, that is, $F(f)(q_i(b)) = q_j(b)$ for all $b \in B$. For a **Set**-morphism $g : B \to A$ to have the property that $\mathrm{pr}_i \circ g = q_i$ for all $i \in \mathrm{Ob}(I)$ it is neccessary and sufficient that $g(b) = (q_i(b))_{i \in \mathrm{Ob}(I)}$ for all $b \in B$, which proves uniqueness of such a morphism; on the other hand, defining g this way, we have $F(f)(\mathrm{pr}_i(g(b))) = F(f)(q_i(b)) = q_j(b) = \mathrm{pr}_j(g(b))$ for all $b \in B$, hence g is indeed a function from B to A. Therefore the category **Set** is complete.

Let $(I', \leq)$ be a preordered set and let I be *the dual of* the associated category (see Example 1.1.5(5)). That is, there exists a morphism from β to α if and only if $\alpha \leq \beta$. Such a morphism is necessarily unique. A functor F from I to **Set** can be thought of as an projective system of sets relative to the preordered set $(I', \leq)$: The functor F restricted to the objects of I corresponds to a family $(A_\alpha)_{\alpha \in I'}$ of sets and F itself corresponds to a family of mappings $(f_{\alpha\beta})_{\alpha \leq \beta}$ having the properties that $f_{\alpha\beta} : A_\beta \to A_\alpha$ for $\alpha \leq \beta$, $f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$ for $\alpha \leq \beta \leq \gamma$ and that $f_{\alpha\alpha}$ is the identity map on A_α for $\alpha \in I'$. The set A_F can then be written as $A_F = \big\{x \in \prod_{\alpha \in I'} A_\alpha : x_\alpha = f_{\alpha\beta}(x_\beta) \text{ for all } \alpha, \beta \in I' \text{ such that } \alpha \leq \beta\big\}$. A limit of F is the same as a projective limit of the projective system $(A_\alpha, f_{\alpha,\beta})$.

(2) (Limits in categories of modules.) Let R and S be rings. We denote by ${}_R\mathbf{M}_S$ the category of (R,S)-bimodules. Let $U : {}_R\mathbf{M}_S \to \mathbf{Set}$ be the forgetful functor. If I is a small category arising from a preordered set $(I', \leq)$ and $F : I \to {}_R\mathbf{M}_S$ is a functor, then F has a limit $(N, (\mathrm{pr}_i)_i)$. The module N is given by the set $A_{U \circ F}$ defined as in (1), where the module structure arises from pointwise addition and multiplications with scalars. The projections pr_i are given as usual, and they turn out to be linear. Similarly we may construct products in other categories such as **CRing** or **Grp**.

(3) (Equalizers and coequalizers.) If the category I is of the form $\bullet \rightrightarrows \bullet$, then a limit of a functor $F : I \to \mathscr{C}$ is called an *equalizer*. Dually, a colimit of such a functor is called a *coequalizer*.

An equalizer of two parallel arrows $f, g : A \to B$ in $\mathscr{C}$ is therefore an object E of $\mathscr{C}$ together with a pair of morphisms $e : E \to A$, $k : E \to B$ such that $f \circ e = g \circ e = k$, and if E' is another object and $e' : E' \to A$, $k' : E' \to B$ are morphisms satisfying $f \circ e' = g \circ e' = k'$, then there is a unique morphism $h : E' \to E$ such that $e \circ h = e'$ and $k \circ h = k'$. Since E and k are determined by e (and E' and k' are determined by e'), it is safe to call the morphism e an *equalizer of* f and g.

$$\begin{tikzcd}
E \arrow[d, "k"'] \arrow[dr, "e"] & E' \arrow[l, "h"'] \arrow[dl, "k'"'] \arrow[d, "e'"] \\
B & A \arrow[l, shift right, "f"'] \arrow[l, shift left, "g"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
E \arrow[dr, "e"] & & \\
& A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
E' \arrow[uu, "h"] \arrow[ur, "e'"'] & &
\end{tikzcd}$$

Equalizer as a limit — Simpler definition

An equalizer in **Set** of a pair of functions $f, g : A \to B$ is given by the inclusion map $\iota : E \to A$, where $E = \{x \in A : f(x) = g(x)\}$. Indeed, for a set C and a function $c : C \to A$ the assertion that $f \circ c = g \circ c$ is equivalent to $c(C) \subseteq E$. If c is such a function, then the function c' that arises from c by restricting its target to E is the the unique function $h : C \to E$ such that $\iota \circ h = c$.

This construction carries over to the categories **Top**, **Grp** and ${}_R\mathbf{M}_S$ by furnishing the set E with the structure of a topological subspace, subgroup and subbimodule respectively.

A *coequalizer* of two parallel arrows $f, g : A \to B$ is a morphism $c : B \to C$ such that $c \circ f = c \circ g$ and if $c' : B \to C'$ satisfies $c' \circ f = c' \circ g$ then there is a unique morphism $h : C \to C'$ having the property that $h \circ c = c'$.

$$\begin{tikzcd} & & C \arrow[d, "h"] \\ A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[ur, "c"] \arrow[dr, "c'"'] & \\ & & C' \end{tikzcd}$$

Let $f, g : A \to B$ be **Set**-morphisms. Let Q be the equivalence relation on B generated by the relation $E = \{(f(a), g(a)) : a \in A\}$. Then the quotient map $p : B \to B/Q$ is a coequalizer of f and g:
Clearly, p fulfils the equation $p \circ f = p \circ g$. If $q : B \to C$ is any map satisfying $q \circ f = q \circ g$, then $q(x_1) = q(x_2)$ for all $(x_1, x_2) \in E$, therefore the equivalence relation $\sim$ on B defined by $x \sim y \leftrightarrow q(x) = q(y)$ is coarser than Q. Consequently, there is a unique map $h : B/Q \to C$ such that $h \circ p = q$.

In the category ${}_R\mathbf{M}_S$, given (R, S)-linear functions $f, g : A \to B$, a coequalizer for f and g is given by the projection $p : B \to B/Q$, where Q is the congruence generated by the relation $\{(f(a), g(a)) : a \in A\}$. In other words, p is the projection onto the quotient bimodule $B/\mathrm{im}(f - g)$. Given another (R, S)-linear function $q : B \to C$ satisfying $q \circ f = q \circ g$, it follows that $\ker(q) \supseteq \mathrm{im}(f - g)$, hence the assertion is a consequence of the fundamental homomorphism theorem for modules.

(4) Let $F : I \to \mathscr{C}$ be a functor and let I be a small discrete category. A limit of F is called a *product* and a colimit of F is called a *coproduct* of the family $(F_i)_{i \in \mathrm{Ob}(I)}$. This definition of "product" extends the definition of a product of sets, groups, modules etc. to general categories. In the category **Set**, coproducts are given by the disjoint union of sets; in categories of (bi)modules, coproducts are given by the direct sum of modules; in the category of commutative, associative and unitary R-algebras (where R is a commutative ring), coproducts are given by the tensor product of algebras, see section 3.8 below.

(5) Assume that $(I', \leq)$ is an up-directed preordered set, that is, for all $\alpha, \beta \in I'$ there exists a $\gamma \in I'$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$. Let I be the corresponding category according to Example 1.1.5(5) and let $F : I \to \mathbf{Set}$ be a functor. Then F can be seen as a direct system of sets relative to the preordered set $(I', \leq)$: $(F_\alpha)_{\alpha \in \mathrm{Ob}(I)}$ is a family of sets and $(F_{(\beta,\alpha)})_{\alpha \leq \beta}$ is a family of functions having the properties that for all α, β, γ such that $\alpha \leq \beta \leq \gamma$ we have (1) $F_{(\beta,\alpha)} : F_\alpha \to F_\beta$, (2) $F_{(\alpha,\alpha)}$ is the identity on F_α and (3) $F_{(\gamma,\alpha)} = F_{(\gamma,\beta)} \circ F_{(\beta,\alpha)}$. A direct limit of this direct system is a colimit of the functor F. Analogously, direct systems of rings, modules, topological spaces etc. relative to $(I', \leq)$ correspond to functors and a direct limit of such a direct system is a colimit of the corresponding functor.

1.5.5 Theorem. *Let $F : I \to \mathscr{C}$ be a functor.*

(1) Suppose that $\big(N, (p_i)_{i \in \mathrm{Ob}(I)}\big)$ is a product of the family $(F_i)_{i \in \mathrm{Ob}(I)}$ and that $\big(M, (q_f)_{f \in \mathrm{Mor}(I)}\big)$ is a product of the family $(F_{\mathrm{cod}(f)})_{f \in \mathrm{Mor}(I)}$. By the universal property of products, there are unique $\mathscr{C}$-morphisms x and y from N to M such that

$$\begin{aligned} q_f \circ x &= p_{\mathrm{cod} f} && \text{and} \\ q_f \circ y &= F(f) \circ p_{\mathrm{dom} f} && \text{for all } f \in \mathrm{Mor}(I). \end{aligned}$$

If $e : E \to N$ is an equalizer of x and y, then $\big(E, (p_i \circ e)_{i \in \mathrm{Ob}(I)}\big)$ is a limit of F.

(2) *Suppose that $\big(N, (p_i)_{i \in \mathrm{Ob}(I)}\big)$ is a coproduct of the family $(F_i)_{i\in\mathrm{Ob}(I)}$ and that $\big(M, (q_f)_{f\in\mathrm{Mor}(I)}\big)$ is a coproduct of the family $(F_{\mathrm{dom}(f)})_{f\in\mathrm{Mor}(I)}$. There are unique $\mathscr{C}$-morphisms x and y from M to N such that*

$$\begin{aligned} x \circ q_f &= p_{\mathrm{dom} f} && \text{and} \\ y \circ q_f &= p_{\mathrm{cod} f} \circ F(f) && \text{for all } f \in \mathrm{Mor}(I). \end{aligned}$$

If $c : N \to C$ is a coequalizer of x and y, then $\big(C, (c \circ p_i)_{i\in\mathrm{Ob}(I)}\big)$ is a colimit of F.

Proof. It is enough to prove (1). We have $p_i \circ e : E \to F(i)$ and $F(f) \circ p_i \circ e = p_j \circ e$, consequently the pair $(E, (p_i \circ e)_i)$ is a cone over F.

Assume that $(E', (\alpha_i)_{i\in\mathrm{Ob}(I)})$ is another cone over F. There is a unique morphism $a : E' \to N$ with the property that $p_i \circ a = \alpha_i$ for all objects i of I. We have $q_f \circ x \circ a = p_{\mathrm{cod} f} \circ a = \alpha_{\mathrm{cod} f}$ and $q_f \circ y \circ a = F(f) \circ p_{\mathrm{dom} f} \circ a = F(f) \circ \alpha_{\mathrm{dom} f} = \alpha_{\mathrm{cod} f}$, therefore both of $x \circ a$ and $y \circ a$ are morphisms z from E' to M such that $q_f \circ z = \alpha_{\mathrm{cod} f}$ for all I-morphisms f. By definition of M, we have $x \circ a = y \circ a$.

$$\begin{tikzcd} M \arrow[dr, "q_f"'] & & N \arrow[ll, shift right, "x"'] \arrow[ll, shift left, "y"] \arrow[dl, "p_{\mathrm{cod} f}"'] & & E \arrow[ll, "e"'] \\ & F_{\mathrm{cod} f} & & E' \arrow[ll, "\alpha_{\mathrm{cod} f}"] \arrow[ul, "a"] \arrow[ur, "h"'] & \end{tikzcd}$$

Since e is an equalizer, there is a unique morphism $h : E' \to E$ such that $e \circ h = a$. It has the property that $p_i \circ e \circ h = p_i \circ a = \alpha_i$ for all $i \in \mathrm{Ob}(I)$. If h' is another morphism having this property, we have $p_i \circ (e \circ h') = \alpha_i = p_i \circ (e \circ h)$ for all $i \in \mathrm{Ob}(I)$, hence $e \circ h = a = e \circ h'$ by definition of a, from which $h = h'$ follows. $\square$

We define functors according to Lemma 1.4.9 and its dual, Corollary 1.4.10.

1.5.6 Corollary.

(1) *Let $F, G : J \to \mathscr{C}$ be functors and $\zeta : F \to G$ be a natural transformation. Assume that there exist limits $\big(\lim(F), p^F\big)$ of F and $\big(\lim(G), p^G\big)$ of G. Then there exists a unique $\mathscr{C}$-morphism $\lim(\zeta)$ such that $p^G_j \circ \lim(\zeta) = \zeta_j \circ p^F_j$ for all $j \in \mathrm{Ob}(J)$.*
If there exist colimits $\big(\mathrm{colim}(F), \iota^F\big)$ of F and $\big(\mathrm{colim}(G), \iota^G\big)$ of G, then there exists a unique $\mathscr{C}$-morphism $\mathrm{colim}(\zeta)$ such that $\mathrm{colim}(\zeta) \circ \iota^F_j = \iota^G_j \circ \zeta_j$ for all $j \in \mathrm{Ob}(J)$.

$$\begin{tikzcd} G(j) & \lim(G) \arrow[l, "p^G_j"'] & F_j \arrow[r, "\iota^F_j"] \arrow[d, "\zeta_j"'] & \mathrm{colim}(F) \arrow[d, "\mathrm{colim}(\zeta)"] \\ F(j) \arrow[u, "\zeta_j"] & \lim(F) \arrow[l, "p^F_j"] \arrow[u, "\lim(\zeta)"'] & G_j \arrow[r, "\iota^G_j"'] & \mathrm{colim}(G) \end{tikzcd}$$

(2) *Let the category $\mathscr{C}$ be J-complete. Then the function $\lim : \mathrm{Mor}(\mathscr{C}^J) \to \mathrm{Mor}(\mathscr{C})$ defined in (1) is a functor, and $(\Delta, \lim)$ is an adjunction. If $\mathscr{C}$ is J-cocomplete, then the function colim is a functor such that (colim, Δ) is an adjunction.*

The functor lim thus defined is often written $\lim_{\mathscr{C}}$ or even $\lim_{\mathscr{C},J}$, depending on the context. Likewise we use the notations $\mathrm{colim}_{\mathscr{C}}$ and $\mathrm{colim}_{\mathscr{C},J}$. We call the functor $\lim_{\mathscr{C},J}$ a *limit functor for $\mathscr{C}$ with respect to J* and $\mathrm{colim}_{\mathscr{C},J}$ a *colimit functor for $\mathscr{C}$ with respect to J*.

1.5.7 Remark. There are in general many functors that deserve the name "limit (colimit) functor for $\mathscr{C}$ with respect to I" since there is one for each choice of universal morphisms. For the remaining part of this text however it will not matter at all which one we take since we will only be interested in functors "up to natural isomorphism".

1.5.8 Remark. Let I, $\mathscr{A}$ and $\mathscr{B}$ be categories, where I is small. Let $F : \mathscr{A} \to \mathscr{B}$ be a functor. A functor $F^I : \mathscr{A}^I \to \mathscr{B}^I$ is defined by the claim that $F^I(\zeta) = F \star \zeta$ for morphisms ζ of $\mathscr{A}^I$. Identifying the objects of $\mathscr{A}^I$ with functors, we have $F^I(H) = F \circ H$ for each object H of $\mathscr{A}^I$. If $G : \mathscr{A} \to \mathscr{B}$ is another functor and $\varepsilon : F \to G$ is a natural transformation, then the function $\varepsilon^I : \mathrm{Ob}(\mathscr{A}^I) \to \mathrm{Mor}(\mathscr{B}^I)$ defined by $\varepsilon^I(H) = \varepsilon \star H$ (where $H : I \to \mathscr{A}$ is a functor) is a natural transformation from F^I to G^I.

1.5.9 Remark. Let I, $\mathscr{A}$ and $\mathscr{B}$ be categories, where I is small. Let $L : \mathscr{A} \to \mathscr{B}$ be a functor. Let $\Delta_{\mathscr{A}}$ and $\Delta_{\mathscr{B}}$ be the diagonal functors with respect to I for $\mathscr{A}$ and $\mathscr{B}$ respectively. Then, for each $\mathscr{A}$-morphism a, the equation $\Delta_{\mathscr{B}}(L(a)) = L \star \Delta_{\mathscr{A}}(a)$ holds. This gives us the following equality:

$$\Delta_{\mathscr{B}} \circ L = L^I \circ \Delta_{\mathscr{A}}.$$

These are functors from $\mathscr{A}$ to $\mathscr{B}^I$.

1.5.10 Lemma. *Let* $(\eta, \varepsilon) : L \dashv R : (\mathscr{B}, \mathscr{A})$ *and let* I *be a small category. Then* $(\eta^I, \varepsilon^I) : L^I \dashv R^I : (\mathscr{B}^I, \mathscr{A}^I)$.

Proof. We verify one of the triangular identities. The proof of the second is analogous. For each $\mathscr{B}^I$-object F, we have $((\varepsilon^I \star L^I) \circ (L^I \star \eta^I))(F) = (\varepsilon^I \star L^I)(F) \circ (L^I \star \eta^I)(F) = (\varepsilon \star L \star F) \circ (L \star \eta \star F) = ((\varepsilon \star L) \circ (L \star \eta))(F) = L \star F = L^I(F)$. □

1.5.11 Proposition. *Suppose that the categories* $\mathscr{A}$ *and* $\mathscr{B}$ *are* I*-complete, where* I *is a small category, and that* $R : \mathscr{B} \to \mathscr{A}$ *is a right adjoint functor. Then the functors* $R \circ \lim_{\mathscr{B}}$ *and* $\lim_{\mathscr{A}} \circ R^I$ *from* $\mathscr{B}^I$ *to* $\mathscr{A}$ *are naturally isomorphic.*

Proof. Choose L such that (L, R) is an adjunction. Then $\Delta_{\mathscr{B}} \circ L$ is left adjoint to $R \circ \lim_{\mathscr{B}}$ by Theorem 1.4.19 and $L^I \circ \Delta_{\mathscr{A}}$ is left adjoint to $\lim_{\mathscr{A}} \circ R^I$ by the same theorem and Lemma 1.5.10. By Remark 1.5.9 the functors $\Delta^I_{\mathscr{B}} \circ L$ and $L^I \circ \Delta_{\mathscr{A}}$ are equal. From Proposition 1.4.18 the claim follows. □

The proof may be visualized by the following diagrams the first of which is commutative.

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "\Delta_{\mathscr{A}}"] \arrow[d, "L"'] & \mathscr{A}^I \arrow[d, "L^I"] \\
\mathscr{B} \arrow[r, "\Delta_{\mathscr{B}}"'] & \mathscr{B}^I
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{A} \arrow[r, shift left, "\Delta_{\mathscr{A}}"] \arrow[d, shift right, "L"'] & \mathscr{A}^I \arrow[l, shift left, "\lim_{\mathscr{A}}"] \arrow[d, shift left, "L^I"] \\
\mathscr{B} \arrow[u, shift right, "R"'] \arrow[r, shift right, "\Delta_{\mathscr{B}}"'] & \mathscr{B}^I \arrow[l, shift right, "\lim_{\mathscr{B}}"'] \arrow[u, shift left, "R^I"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{A} & \mathscr{A}^I \arrow[l, "\lim_{\mathscr{A}}"'] \\
\mathscr{B} \arrow[u, "R"] & \mathscr{B}^I \arrow[l, "\lim_{\mathscr{B}}"] \arrow[u, "R^I"']
\end{tikzcd}$$

The conclusion of this proposition also holds under more general circumstances. In the course of working out the situation in detail, we are also going to give a more admissible description of the natural isomorphism whose existence is guaranteed by the theorem.

1.5.12 Definition.

(1) Let $F : I \to \mathscr{B}$ be a functor. A functor $R : \mathscr{B} \to \mathscr{A}$ is said to *preserve the limits of* F if $(R(\hat{F}), R \star \pi)$ is a limit of $R \circ F$ whenever $(\hat{F}, \pi)$ is a limit of F. The functor R is said to *preserve* I*-limits* if it

preserves the limits of each functor $F : I \to \mathscr{B}$. If R preserves I-limits for each small category I, then R is called *continuous*.

(2) Let $F : I \to \mathscr{A}$ be a functor. A functor $L : \mathscr{A} \to \mathscr{B}$ is said to *preserve the colimits of* F if $(L(\hat{F}), L \star \iota)$ is a colimit of $L \circ F$ whenever $(\hat{F}, \iota)$ is a colimit of F. The functor L is said to *preserve I-colimits* if it preserves the colimits of each functor $F : I \to \mathscr{A}$. If L preserves I-colimits for each small category I, then L is called *cocontinuous*.

To prove that R preserves the limits (resp. colimits) of F, it is sufficient to show that the image under R of one limit (resp. colimit) of F is a limit (resp. colimit) of $R \circ F$.

1.5.13 Remark. Let I be a category. A functor $G : \mathscr{C} \to \mathscr{D}$ preserves the limits of $F : I \to \mathscr{C}$ if and only if the following holds:

> If F has a limit, then there exists a limit $\big(N, (p_i)_{i \in \mathrm{Ob}(I)}\big)$ of F having the property that $\big(G(N), (G(p_i))_{i \in \mathrm{Ob}(I)}\big)$ is a limit of $G \circ F$.

The condition is clearly necessary. Conversely, if the statement holds, and if $\big(M, (q_i)_i\big)$ is any limit of F, then there exists an isomorphism $\zeta : M \to N$ such that $p_i \circ \zeta = q_i$ for all i. The morphism $G(\zeta) : G(M) \to G(N)$ is an isomorphism, hence $\big(G(M), (G(p_i) \circ G(\zeta))_i\big) = \big(G(M), (G(q_i))_i\big)$ is a limit of $G \circ F$.

1.5.14 Remark. Let G and H be functors from $\mathscr{C}$ to $\mathscr{D}$ and let $\varepsilon : G \to H$ be a natural isomorphism. Assume that $F : I \to \mathscr{C}$ is a functor, where I is a small category. The functor G preserves the limits (resp. the colimits) of F if and only H does. In particular, G is continuous (resp. cocontinuous) if and only if H is.

We prove the statement concerning limits. Assume that G preserves the limits of F and that (N, π) is a limit of F. The following diagram is commutative:

$$\begin{tikzcd} \Delta(G(N)) \arrow[r, "\Delta(\varepsilon(N))"] \arrow[d, "G \star \pi"'] & \Delta(H(N)) \arrow[d, "H \star \pi"] \\ G \circ F \arrow[r, "\varepsilon \star F"'] & H \circ F \end{tikzcd}$$

Since $\big(G(N), G \star \pi\big)$ is a limit of $G \circ F$, that is, a Δ-final morphism for $G \circ F$, by Proposition 1.4.4(5) it follows that $(H(N), H \star \pi)$ is a limit of $H \circ F$.

1.5.15 Lemma. *Suppose that the category $\mathscr{C}$ is complete (resp. cocomplete) and that the functor $G : \mathscr{C} \to \mathscr{D}$ preserves products and equalizers (resp. coproducts and coequalizers). Then G preserves limits (resp. colimits).*

Proof. Let $F : I \to \mathscr{C}$ be a functor. Choose products $\big(N, (p_i)_{i \in \mathrm{Ob}(I)}\big)$ and $\big(M, (q_f)_{f \in \mathrm{Mor}(I)}\big)$ of the families $(F_i)_{i \in \mathrm{Ob}(I)}$ and $(F_{\mathrm{cod} f})_{f \in \mathrm{Mor}(I)}$ respectively. Let $x, y : M \to N$ be chosen such that $q_f \circ x = p_{\mathrm{cod} f}$ and $q_f \circ y = F(f) \circ p_{\mathrm{dom} f}$ for all morphisms f of I and let $e : E \to N$ be an equalizer of x of y.

Now $\big(G(N), (G(p_i))_i\big)$ is a product of $(G(F_i))_i$ and $\big(G(M), (G(q_f))_f\big)$ is a product of $(G(F_{\mathrm{cod} f}))_f$. Moreover, the formulas $G(q_f) \circ G(x) = G(p_{\mathrm{cod} f})$ and $G(q_f) \circ G(y) = G(F(f)) \circ G(p_{\mathrm{dom} f})$ hold for all I-morphisms f and $G(e) : G(E) \to G(N)$ is an equalizer of $G(x)$ and $G(y)$. By Theorem 1.5.5, $(E, (p_i \circ e)_i)$ is a limit of F and $\big(G(E), (G(p_i) \circ G(e))_i\big) = \big(G(E), (G(p_i \circ e))_i\big)$ is a limit of $G \circ F$. Applying Remark 1.5.13 the claim follows. $\square$

1.5.16 Remark. Let functors $I \xrightarrow{F} \mathscr{B} \xrightarrow{R} \mathscr{A}$ be given. Assume that (M, π) is a limit of F and that (N, ρ) is a limit of $R \circ F$. There exists a unique $\mathscr{A}$-morphism $\tau_F : R(M) \to N$ such that $\rho \circ \Delta_{\mathscr{A}}(\tau_F) = R \star \pi$. In other words, τ_F is the unique $\mathscr{A}$-morphism from $R(M)$ to N such that $\rho_i \circ \tau_F = R(\pi_i)$ for all objects i of I.

$$\begin{tikzcd}
\Delta_{\mathscr{A}}(N) \arrow[dr, "\rho"'] & & \Delta_{\mathscr{A}}(R(M)) \arrow[ll, "\Delta_{\mathscr{A}}(\tau_F)"'] \arrow[dl, "R\star\pi"] \\
& R \circ F &
\end{tikzcd}
\qquad
\begin{tikzcd}
N \arrow[d, "\rho_j"'] \arrow[dr, "\rho_i"] & RM \arrow[l, "\tau_F"'] \arrow[dl, "R\pi_j"'] \arrow[d, "R\pi_i"] \\
RFj & RFi \arrow[l, "RFf"]
\end{tikzcd}$$

In the following, whenever limits of F and $R \circ F$ are given, we assume that the morphism τ_F is defined this way. The limits in question will be apparent from the context.

1.5.17 Theorem. *Let $F : I \to \mathscr{B}$ and $R : \mathscr{B} \to \mathscr{A}$ be functors.*

(1) Let the functor F have a limit and let R preserve the limits of F. For each limit of $R \circ F$, the morphism τ_F is an isomorphism.

(2) Let F and $R \circ F$ possess limits. Then R preserves the limits of F if and only if τ_F is an isomorphism.

(3) For each functor $F : I \to \mathscr{B}$, let $\big(\lim_{\mathscr{B}}(F), \pi^F\big)$ be a limit of F, and let $\big(\lim_{\mathscr{A}}(R \circ F), \rho^F\big)$ be a limit of $R \circ F$. Extend $\lim_{\mathscr{A}} \circ R^I$ and $\lim_{\mathscr{B}}$ to functors according to Lemma 1.4.9. That is, for each $\mathscr{B}^I$-morphism $\zeta : F \to G$ choose $\lim_{\mathscr{B}}(\zeta)$ and $\lim_{\mathscr{A}}(R \star \zeta)$ in such a way that the following diagrams are commutative.

$$\begin{tikzcd}
\Delta_{\mathscr{B}} \lim_{\mathscr{B}} F \arrow[r, "\pi^F"] \arrow[d, "\Delta_{\mathscr{B}} \lim_{\mathscr{B}} \zeta"'] & F \arrow[d, "\zeta"] \\
\Delta_{\mathscr{B}} \lim_{\mathscr{B}} G \arrow[r, "\pi^G"'] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
R \circ F \arrow[d, "R\star\zeta"'] & \Delta_{\mathscr{A}} \lim_{\mathscr{A}}(R \circ F) \arrow[l, "\rho^F"'] \arrow[d, "\Delta_{\mathscr{A}} \lim_{\mathscr{A}}(R\star\zeta)"] \\
R \circ G & \Delta_{\mathscr{A}} \lim_{\mathscr{A}}(R \circ G) \arrow[l, "\rho^G"]
\end{tikzcd}$$

The family $\tau = (\tau_F)_{F \in \mathrm{Ob}(\mathscr{B}^I)}$ is a natural transformation from $R \circ \lim_{\mathscr{B}}$ to $\lim_{\mathscr{A}} \circ R^I$. If in addition the functor R preserves I-limits, then τ is a natural isomorphism.

Proof.

(1) Let $(\hat{F}, \pi)$ be a limit of F. The pair $(R(\hat{F}), R \star \pi)$ is a limit of $R \circ F$, hence the claim follows from Proposition 1.4.4(1).

(2) Let τ_F be an isomorphism and let $(\hat{F}, \pi)$ be a limit of F. By Proposition 1.4.4(2) $\big(R(\hat{F}), R \star \pi\big)$ is a limit of $R \circ F$.

(3) Putting together the definitions and using Remark 1.5.9, we obtain the following diagram.

$$\begin{tikzcd}
& R\circ F \arrow[dd, "R\star\zeta"] & \\
\Delta_{\mathscr{A}} R\lim_{\mathscr{B}} F \arrow[ur, "R\star\pi^F"] \arrow[rr, "\Delta_{\mathscr{A}}(\tau_F)"] \arrow[dd, "\Delta_{\mathscr{A}} R\lim_{\mathscr{B}}(\zeta)"'] & & \Delta_{\mathscr{A}} \lim_{\mathscr{A}}(R\circ F) \arrow[ul, "\rho^F"'] \arrow[dd, "\Delta_{\mathscr{A}} \lim_{\mathscr{A}}(R\star\zeta)"] \\
& R\circ G & \\
\Delta_{\mathscr{A}} R\lim_{\mathscr{B}} G \arrow[ur, "R\star\pi^G"] \arrow[rr, "\Delta_{\mathscr{A}}(\tau_G)"'] & & \Delta_{\mathscr{A}} \lim_{\mathscr{A}}(R\circ G) \arrow[ul, "\rho^G"']
\end{tikzcd}$$

Each of $\lim_{\mathscr{A}}(R\star\zeta)\circ\tau_F$ and $\tau_G\circ R(\lim_{\mathscr{B}}(\zeta))$ is an $\mathscr{A}$-morphism $a : R(\lim_{\mathscr{B}}(F)) \to \lim_{\mathscr{A}}(R\circ G)$ having the property that $\rho^G\circ\Delta_{\mathscr{A}}(a) = (R\star\zeta)\circ(R\star\pi^F)$. Since $\big(\lim_{\mathscr{A}}(R\circ G), \rho^G\big)$ is a limit, the claim follows. $\square$

We turn to the dual of Theorem 1.5.17.

1.5.18 Remark. Let functors $I \xrightarrow{F} \mathscr{A} \xrightarrow{L} \mathscr{B}$ be given. Assume that (M,ι) is a colimit of F and that (N,κ) is a colimit of $L\circ F$. There exists a unique $\mathscr{B}$-morphism $\mu_F : N \to L(M)$ such that $\Delta_{\mathscr{B}}(\mu_F)\circ\kappa = L\star\iota$. In other words, μ_F is the unique $\mathscr{B}$-morphism from N to $L(M)$ such that $\mu_F\circ\kappa_i = L(\iota_i)$ for all objects i of I.

$$\begin{tikzcd}
\Delta_{\mathscr{B}}(N) \arrow[rr, "\Delta_{\mathscr{B}}(\mu_F)"] & & \Delta_{\mathscr{B}}(L(M)) \\
& L\circ F \arrow[ul, "\kappa"] \arrow[ur, "L\star\iota"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
N \arrow[r, "\mu_F"] & LM \\
LFj \arrow[u, "\kappa_j"] \arrow[r, "LFf"'] \arrow[ur, "L\iota_j"' near end] & LFi \arrow[u, "L\iota_i"'] \arrow[ul, "\kappa_i" near end]
\end{tikzcd}$$

1.5.19 Corollary. *Let $F : I \to \mathscr{A}$ and $L : \mathscr{A} \to \mathscr{B}$ be functors.*

(1) Let the functor F have a colimit and let L preserve the colimits of F. For each colimit of $L\circ F$, the morphism μ_F is an isomorphism.

(2) Let F and $L\circ F$ possess colimits. Then L preserves the colimits of F if and only if μ_F is an isomorphism.

(3) For each functor $F : I \to \mathscr{A}$, let $\big(\operatorname{colim}_{\mathscr{A}}(F), \iota^F\big)$ be a colimit of F, and $\big(\operatorname{colim}_{\mathscr{B}}(L\circ F), \kappa^F\big)$ be a colimit of $L\circ F$. Extend $\operatorname{colim}_{\mathscr{B}}\circ L^I$ and $\operatorname{colim}_{\mathscr{A}}$ to functors according to Corollary 1.4.10. That is, for each $\mathscr{A}^I$-morphism $\zeta : G \to F$ choose $\operatorname{colim}_{\mathscr{A}}(\zeta)$ and $\operatorname{colim}_{\mathscr{B}}(L\star\zeta)$ in such a way that the following diagrams are commutative.

$$\begin{tikzcd}
\Delta_{\mathscr{A}} \operatorname{colim}_{\mathscr{A}} F & F \arrow[l, "\iota^F"'] \\
\Delta_{\mathscr{A}} \operatorname{colim}_{\mathscr{A}} G \arrow[u, "\Delta_{\mathscr{A}} \operatorname{colim}_{\mathscr{A}} \zeta"] & G \arrow[l, "\iota^G"] \arrow[u, "\zeta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
L\circ F \arrow[r, "\kappa^F"] & \Delta_{\mathscr{B}} \operatorname{colim}_{\mathscr{B}}(L\circ F) \\
L\circ G \arrow[u, "L\star\zeta"] \arrow[r, "\kappa^G"'] & \Delta_{\mathscr{B}} \operatorname{colim}_{\mathscr{B}}(L\circ G) \arrow[u, "\Delta_{\mathscr{B}} \operatorname{colim}_{\mathscr{B}}(L\star\zeta)"']
\end{tikzcd}$$

The family $\mu = (\mu_F)_{F\in\operatorname{Ob}(\mathscr{A}^I)}$ is a natural transformation from $\operatorname{colim}_{\mathscr{B}}\circ L^I$ to $L\circ\operatorname{colim}_{\mathscr{A}}$. If in addition R preserves I-colimits, then μ is a natural isomorphism.

There are many continuous functors: Right adjoint functors and hom-functors are examples.

1.5.20 Theorem. *Right adjoint functors are continuous. Left adjoint functors are cocontinuous.*

Proof. Let $(\varepsilon, \eta) : L \dashv R : (\mathscr{B}, \mathscr{A})$ be an adjunction. For a $\mathscr{B}$-morphism $f : L(a) \to b$, we write $f^\sharp_a$ for the $\mathscr{A}$-morphism $R(f) \circ \eta_a$. Analogously, if $g : a \to R(b)$ is an $\mathscr{A}$-morphism, we write $g^\flat_b$ for the $\mathscr{B}$-morphism $\varepsilon_b \circ L(g)$. By Corollary 1.4.16, the morphism $f^\sharp_a$ is the unique $\mathscr{A}$-morphism x from a to $R(b)$ with the property that $\varepsilon_b \circ L(x) = f$ and $g^\flat_b$ is the unique $\mathscr{B}$-morphism x from $L(a)$ to b with the property that $R(x) \circ \eta_a = g$. Also, $(f^\sharp_a)^\flat_b = f$ and $(g^\flat_b)^\sharp_a = g$.

Assume that J is a category, that $T : J \to \mathscr{B}$ is a functor and that (X, τ) is a limit of T. We want to prove that $(R(X), R \star \tau)$ is a limit of $R \circ T$. Since $R \star \tau$ is a natural transformation from $R \circ \Delta_{\mathscr{B}}(X) = \Delta_{\mathscr{A}}(R(X))$ to $R \circ T$, it is a cone over $R \circ T$ with apex $R(X)$. Let (Y, σ) be another cone over $R \circ T$. We have to prove existence and uniqueness of a $\mathscr{B}$-morphism $x : Y \to R(X)$ such that $R(\tau_j) \circ x = \sigma_j$ for all J-objects j.

The pair $\big(L(Y), ((\sigma_j)^\flat_{T(j)})_j\big)$ is a cone over T: If $u : i \to j$ is a J-morphism, we have $(\sigma_j)^\flat_{T(j)} = \big(R(T(u)) \circ \sigma_i\big)^\flat_{T_j} = \varepsilon_{T(j)} \circ L\big(R(T(u)) \circ \sigma_i\big) = \varepsilon_{T(j)} \circ (L \circ R)(T(u)) \circ L(\sigma_i) = T(u) \circ \varepsilon_{T_i} \circ L(\sigma_i) = T(u) \circ (\sigma_i)^\flat_{T(i)}$.

Therefore there is a unique $\mathscr{B}$-morphism $h : L(Y) \to X$ such that $\tau_j \circ h = (\sigma_j)^\flat_{T(j)}$ for all j. Now the $\mathscr{A}$-morphism $h^\sharp_Y : Y \to R(X)$ has the property that $R(\tau_j) \circ h^\sharp_Y = R(\tau_j) \circ R(h) \circ \eta_Y = G(\tau_j \circ h) \circ \eta_Y = (\tau_j \circ h)^\sharp_Y = ((\sigma_j)^\flat_{T(j)})^\sharp_Y = \sigma_j$. If z is another, we have $\sigma_j = R(\tau_j) \circ (z^\flat_X)^\sharp_Y = R(\tau_j) \circ G(z^\flat_X) \circ \eta_Y = R(\tau_j \circ z^\flat_X) \circ \eta_Y$, that is, $\tau_j \circ z^\flat_X = (\sigma_j)^\flat_{T(j)}$ for all j. But h is the only one with this property, hence $h = z^\flat_X$. It follows that $z = h^\sharp_Y$. □

1.5.21 Theorem (hom-functors are continuous). *Assume that $\mathscr{C}$ is a category and that I is a small one.*

(1) Let $F : I \to \mathscr{C}$ be a functor and let $E \in \mathrm{Ob}(\mathscr{C})$. If $\big(\tilde{F}, (\varepsilon_i)_i\big)$ is a limit of F, then $\big(\hom(E, \tilde{F}), (\hom(E, \varepsilon_i))_i\big)$ is a limit of $\hom(E, \_\,) \circ F : I \to \mathbf{Set}$.

(2) Let $F : I^{\mathrm{op}} \to \mathscr{C}$ be a functor and let E be an object of $\mathscr{C}$. If $\big(\tilde{F}, (\eta_i)_i\big)$ is a colimit of F, then $\big(\hom(\tilde{F}, E), (\hom(\eta_i, E))_i\big)$ is a limit of $\hom(\_\,, E) \circ F : I \to \mathbf{Set}$.

Proof. The second assertion follows from the first: The pair $(\tilde{F}, \eta_i)$ is a limit of $F : I \to \mathscr{C}^{\mathrm{op}}$, hence $\big(\hom_{\mathscr{C}^{\mathrm{op}}}(E, \tilde{F}), (\hom_{\mathscr{C}^{\mathrm{op}}}(E, \eta_i))_i\big) = \big(\hom_{\mathscr{C}}(\tilde{F}, E), (\hom_{\mathscr{C}}(\eta_i, E))_i\big)$ is a limit of $\hom_{\mathscr{C}^{\mathrm{op}}}(E, \_\,) \circ F = \hom_{\mathscr{C}}(\_\,, E) \circ F : I \to \mathbf{Set}$. We turn to the proof of the first statement. Since $\hom(E, F(f))(x_{\mathrm{dom} f}) = F(f) \circ x_{\mathrm{dom} f}$ for all I-morphisms f, by example 1.5.4(1) the set $A = \big\{x \in \prod_{i \in \mathrm{Ob}(I)} \hom(E, F(i)) : x_{\mathrm{cod} f} = F(f) \circ x_{\mathrm{dom} f} \text{ for all } f \in \mathrm{Mor}(I)\big\}$ together with the usual projection maps $\mathrm{pr}_i : A \to \hom(E, F(i))$ is a limit of $\hom(E, \_\,) \circ F$. Define a function $\zeta : \hom(E, \tilde{F}) \to A$ by $\zeta(g)(i) = \varepsilon_i \circ g$. As $\varepsilon_j \circ g = (F(f) \circ \varepsilon_i) \circ g = F(f) \circ (\varepsilon_i \circ g)$ for I-morphisms $f : i \to j$ this is well-defined. The function is bijective: The elements of $\mathscr{A}$ are the cones over F with apex E, hence for each $k \in A$ there is exactly one $g : E \to \tilde{F}$ such that $\varepsilon_i \circ g = k_i$ for all $i \in \mathrm{Ob}(I)$. Finally, we have $(\mathrm{pr}_i \circ \zeta)(g) = \mathrm{pr}_i(\zeta(g)) = \varepsilon_i \circ g = \hom(E, \varepsilon_i)(g)$ for $g : E \to \tilde{F}$. By Proposition 1.4.4 the claim follows. □

1.6 Associativity of the limit functor

1.6.1 Remark. Assume that $F : I \times J \to \mathscr{C}$ is a functor and that $f : x \to y$ is an I-morphism. By $F(f, \_\,)$ we denote the natural transformation $j \mapsto F(f, j)$ (where $j \in \mathrm{Ob}(J)$) from $F(x, \_\,)$ to $F(y, \_\,)$. The natural transformation $F(\_\,, g)$, for J-morphisms g, is defined analogously.

1.6.2 Theorem. *Let $\mathscr{C}$ be a category and let I and J be small categories. Then a functor $\Lambda : \mathscr{C}^{I \times J} \to$*

$\left(\mathscr{C}^I\right)^J$ is defined by $\Lambda(\sigma)(j)(i) = \sigma(i,j)$, where

$$\begin{aligned} &\sigma \in \mathrm{Ob}\big([I \times J, \mathscr{C}]\big),\ j \in \mathrm{Ob}(J) \text{ and } i \in \mathrm{Mor}(I) \qquad \text{or} \\ &\sigma \in \mathrm{Ob}\big([I \times J, \mathscr{C}]\big),\ j \in \mathrm{Mor}(J) \text{ and } i \in \mathrm{Ob}(I) \qquad \text{or} \\ &\sigma \in \mathrm{Mor}\big([I \times J, \mathscr{C}]\big),\ j \in \mathrm{Ob}(J) \text{ and } i \in \mathrm{Mor}(I). \end{aligned}$$

It is an isomorphism. Similarly, an isomorphism $\Xi : \mathscr{C}^{I\times J} \to \left(\mathscr{C}^J\right)^I$ is defined by the assignment $\Xi(\sigma)(i)(j) = \sigma(i,j)$.

Proof. For the first statement, see Herrlich, 1973 [7, Theorem 15.9]. The proof of the second is analogous. □

For the rest of this section, let $\mathscr{C}$, I and J be categories, where I and J are small ones. Let Λ and Ξ be the isomorphisms defined in Theorem 1.6.2 and let Γ and Φ be their inverses.

1.6.3 Lemma. *The following diagrams are commutative.*

$$\begin{tikzcd} \mathscr{C}^{I\times J} \arrow[r, shift left, "\Lambda"] & \left(\mathscr{C}^I\right)^J \arrow[l, shift left, "\Gamma"] \\ \mathscr{C} \arrow[u, "\Delta_{\mathscr{C},I\times J}"] \arrow[r, "\Delta_{\mathscr{C},I}"'] & \mathscr{C}^I \arrow[u, "\Delta_{\mathscr{C}^I,J}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathscr{C}^{I\times J} \arrow[r, shift left, "\Xi"] & \left(\mathscr{C}^J\right)^I \arrow[l, shift left, "\Phi"] \\ \mathscr{C} \arrow[u, "\Delta_{\mathscr{C},I\times J}"] \arrow[r, "\Delta_{\mathscr{C},J}"'] & \mathscr{C}^J \arrow[u, "\Delta_{\mathscr{C}^J,I}"'] \end{tikzcd}$$

Proof. We only prove the first part. For $c \in \mathrm{Ob}(\mathscr{C})$, $j \in \mathrm{Ob}(J)$ and $i \in \mathrm{Mor}(I)$, we have $\Lambda(\Delta_{\mathscr{C},I\times J}(c))(j)(i) = \Delta_{\mathscr{C},I\times J}(c)(i,j) = c = \Delta_{\mathscr{C},I}(c)(i) = \Delta_{\mathscr{C}^I,J}(\Delta_{\mathscr{C},I}(c))(j)(i)$. The same calculation is valid for $c \in \mathrm{Ob}(\mathscr{C})$, $j \in \mathrm{Mor}(J)$ and $i \in \mathrm{Ob}(I)$ and also for $c \in \mathrm{Mor}(\mathscr{C})$, $j \in \mathrm{Ob}(J)$ and $i \in \mathrm{Ob}(I)$. □

1.6.4 Theorem (Associativity of limits). *Let $F : I \times J \to \mathscr{C}$ be a functor. Assume that for all I-objects i the functor $F(i, \_\,)$ possesses a limit $\big(\tilde{F}(i), \varepsilon^i\big)$. Let $\tilde{F} : I \to \mathscr{C}$ be the continuation of $\tilde{F}$ to a functor, as defined in Lemma 1.4.9(2). For each $j \in \mathrm{Ob}(J)$, the family $\varepsilon_j = (\varepsilon^i_j)_{i\in\mathrm{Ob}(I)}$ is a natural transformation from $\tilde{F}$ to $F(\_\,, j) = \Lambda(F)(j)$ and $\big(\tilde{F}, (\varepsilon_j)_{j\in\mathrm{Ob}(J)}\big)$ is a limit of $\Lambda(F)$.*
If $\big(\bar{F}, (\kappa^i)_{i\in\mathrm{Ob}(I)}\big)$ is a limit of the functor $\tilde{F}$, then $\big(\bar{F}, (\varepsilon^i_j \circ \kappa^i)_{(i,j)\in\mathrm{Ob}(I\times J)}\big)$ is a limit of F. Conversely, if F has a limit, then $\tilde{F}$ has one.

Proof. Let $H : I \to \mathscr{C}$ be a functor and let $\eta^i_j : H(i) \to F(i,j)$ be a $\mathscr{C}$-morphism for $i \in \mathrm{Ob}(I)$ and $j \in \mathrm{Ob}(J)$. For $\big(H, ((\eta^i_j)_i)_j\big)$ to be a cone over $\Lambda(F)$ it is necessary und sufficient that the following diagram be commutative, for all I-morphisms $f : i \to i'$ and for all J-morphisms $g : j \to j'$.

$$\begin{tikzcd} & F(i,j) \arrow[dr, "F(i,g)"] \arrow[dd, "F(f,j)"] & \\ H(i) \arrow[ur, "\eta^i_j"] \arrow[rr, "\eta^i_{j'}"] \arrow[dd, "H(f)"'] & & F(i,j') \arrow[dd, "F(f,j')"] \\ & F(i',j) \arrow[dr, "F(i',g)"] & \\ H(i') \arrow[ur, "\eta^{i'}_j"] \arrow[rr, "\eta^{i'}_{j'}"'] & & F(i',j') \end{tikzcd}$$

Setting $H = \tilde{F}$ and $\eta^i_j = \varepsilon^i_j$, the left parallelogram is commutative by the definition of the functor $\tilde{F}$; the triangles are commutative since each $\big(\tilde{F}(i), \varepsilon^i\big)$ is a cone. Therefore $\big(\tilde{F}, (\varepsilon_j)_{j\in \mathrm{Ob}(J)}\big)$ is a cone over $\Lambda(F)$. Assume that $\big(K, (\kappa_j)_j\big)$ is another cone over $\Lambda(F)$. Then for all objects i of I, $\big(K(i), (\kappa_j(i))_{j\in\mathrm{Ob}(J)}\big)$ is a cone over $F(i, \_\,)$, and so is $\big(\tilde{F}(i), \varepsilon^i\big)$.

For $i \in \mathrm{Ob}(I)$, there is then exactly one $\mathscr{C}$-morphism $x(i) : K(i) \to \tilde{F}(i)$ such that $\varepsilon^i_j \circ x(i) = \kappa_j(i)$ for all $j \in \mathrm{Ob}(J)$. This proves uniqueness of a $\mathscr{C}^I$-morphism $h : K \to \tilde{F}$ with the property that $\varepsilon_j \circ h = \kappa_j$ for all objects j of J. To finish the proof that $\big(\tilde{F}, (\varepsilon_j)_j\big)$ is a limit of $\Lambda(F)$, we have to show that the function x is a natural transformation from K to $\tilde{F}$. For this, let $f : i \to i'$ be an I-morphism.

First, $\big(K(i), (F(f,j) \circ \kappa_j(i))_{j\in\mathrm{Ob}(J)}\big)$ is a cone over $F(i', \_\,)$, as can be seen in the following diagram which is commutative for each J-morphism $g : j \to j'$.

$$\begin{tikzcd}
 & F(i,j) \arrow[r, "F(f,j)"] \arrow[d, "F(i,g)"] & F(i',j) \arrow[d, "F(i',g)"] \\
K(i) \arrow[ur, "\kappa_j(i)"] \arrow[dr, "\kappa_{j'}(i)"'] & & \\
 & F(i,j') \arrow[r, "F(f,j')"'] & F(i',j')
\end{tikzcd}$$

The two triangles and the left and right parallelograms of the following diagram have already been shown to be commutative. Therefore $\tilde{F}(f) \circ x(i)$ and $x(i') \circ K(f)$ are two $\mathscr{C}$-morphisms s from $K(i)$ to $\tilde{F}(i')$ having the property that $\varepsilon^{i'}_j \circ s = F(f,j) \circ \kappa_j(i)$ for all J-objects j.

$$\begin{tikzcd}
 & F(i,j) \arrow[dd, "F(f,j)"] & \\
K(i) \arrow[ur, "\kappa_j(i)"] \arrow[rr, "x(i)"] \arrow[dd, "K(f)"'] & & \tilde{F}(i) \arrow[ul, "\varepsilon^i_j"'] \arrow[dd, "\tilde{F}(f)"] \\
 & F(i',j) & \\
K(i') \arrow[ur, "\kappa_j(i')"] \arrow[rr, "x(i')"'] & & \tilde{F}(i') \arrow[ul, "\varepsilon^{i'}_j"']
\end{tikzcd}$$

Since $\tilde{F}(i')$ is a limit, it follows that $\tilde{F}(f) \circ x(i) = x(i') \circ K(f)$. This proves that x is natural, and therefore $\big(\tilde{F}, (\varepsilon_j)_{j\in\mathrm{Ob}(J)}\big)$ is really a limit of $\Lambda(F)$.
The next statement follows from composition of final morphisms, Proposition 1.4.6(1). Finally, the existence of a limit of F implies, by means of Proposition 1.4.6(2), the existence of a limit of $\tilde{F}$.

□

1.6.5 Corollary (Associativity of colimits). *Let $F : I \times J \to \mathscr{C}$ be a functor. Assume that the functor $F(i, \_\,)$ possesses a colimit $\big(\mathrm{colim}_{\mathscr{C},J}(F(i,\_\,)), \eta^i\big)$ for all $i \in \mathrm{Ob}(I)$.*
Let $\tilde{F} : I \to \mathscr{C}$ be the functor defined by $f \mapsto \mathrm{colim}_{\mathscr{C},J}(F(f, \_\,))$. For each $j \in \mathrm{Ob}(J)$, the family $(\eta^i_j)_{i\in\mathrm{Ob}(I)}$ is a natural transformation, η_j, from $F(\_\,, j) = \Lambda(F)(j)$ to $\tilde{F}$ and $\big(\tilde{F}, (\eta_j)_{j\in\mathrm{Ob}(J)}\big)$ is a colimit of $\Lambda(F)$.
If $\big(\bar{F}, (\lambda^i)_{i\in\mathrm{Ob}(I)}\big)$ is a colimit of the functor $\tilde{F}$, then $\big(\bar{F}, (\lambda^i \circ \eta^i_j)_{(i,j)\in\mathrm{Ob}(I\times J)}\big)$ is a colimit of F. Conversely, if F has a colimit, then $\tilde{F}$ has one.

Theorem 1.6.4 and Corollary 1.6.5 involve the isomorphism Λ from Theorem 1.6.2. Of course, there are analogous theorems using the isomorphism Ξ instead. The statement of these is left to the reader.

1.6.6 Corollary. *Let I and J be small categories.*

(1) *(Pointwise computation of a limit). If the category $\mathscr{C}$ is J-complete, then $\mathscr{C}^I$ is J-complete. For each functor $G : J \to \mathscr{C}^I$ the $\mathscr{C}^I$-object $i \mapsto \lim_{\mathscr{C},J}(G(\_)(i))$ (where $i \in \mathrm{Mor}(I)$) is the object part of a limit of G.*

Define a functor $M : \mathscr{C}^{I\times J} \to \mathscr{C}^I$ by $M(K)(i) = \lim_{\mathscr{C},J}(K(i,\_))$, where K is an object (resp. a morphism) of $[I \times J, \mathscr{C}]$ and i is a morphism (resp. an object) of I, that is, $M = \lim^I_{\mathscr{C},K} \circ \Xi$. Then $\lim_{\mathscr{C}^I,J} \circ \Lambda$ and M are naturally isomorphic.

$$\begin{tikzcd}
\mathscr{C}^{I\times J} \arrow[r, "\Lambda"] \arrow[d, "\Xi"'] & \left(\mathscr{C}^I\right)^J \arrow[d, "\lim_{\mathscr{C}^I,J}"] \\
\left(\mathscr{C}^J\right)^I \arrow[r, "\lim^I_{\mathscr{C},J}"'] & \mathscr{C}^I
\end{tikzcd}$$

(2) *If $\mathscr{C}$ is I-and J-complete, then $\mathscr{C}$ is also $I \times J$-complete and the functors $\lim_{\mathscr{C},I\times J}$ and $\lim_{\mathscr{C},I} \circ \lim_{\mathscr{C}^I,J} \circ \Lambda$ are naturally isomorphic.*

If $\mathscr{C}$ is I-and J-cocomplete, then $\mathscr{C}$ is $I \times J$-cocomplete and the functors $\mathrm{colim}_{\mathscr{C},I\times J}$ and $\mathrm{colim}_{\mathscr{C},I} \circ \mathrm{colim}_{\mathscr{C}^I,J} \circ \Lambda$ are naturally isomorphic.

Proof.

(1) Each functor $G : J \to \mathscr{C}^I$ is of the form $\Lambda(F)$ for some $F : I \times J \to \mathscr{C}$. By the theorem, the object part of a limit of G is given by the object part of a limit of the functor $i \mapsto \lim_{\mathscr{C},J}(F(i,\_)) = \lim_{\mathscr{C},J}(G(\_)(i))$. We show that the functor $M \circ \Gamma : (\mathscr{C}^I)^J \to \mathscr{C}^I$ is right adjoint to $\Delta_{\mathscr{C}^I,J}$. Let $G : J \to \mathscr{C}^I$ be a functor. For $i \in \mathrm{Ob}(I)$ and $j \in \mathrm{Ob}(J)$, let $\varepsilon^i_j : \lim_{\mathscr{C},J}(G(\_)(i)) \to G(j)(i)$ be the j-th projection. By the above and the theorem, the $\mathscr{C}^I$-object $i \mapsto \lim_{\mathscr{C},J}(G(\_)(i))$ (where $i \in \mathrm{Mor}(I)$) together with $((\varepsilon^i_j)_i)_j$ is a limit of G. Also, if $\xi : G \to G'$ is a natural transformation, where G and G' are functors from J to $\mathscr{C}^I$, and if $\varepsilon^i_j : \lim_{\mathscr{C},J}(G(\_)(i)) \to G(j)(i)$ and $\varepsilon'^i_j : \lim_{\mathscr{C},J}(G'(\_)(i)) \to G'(j)(i)$ are the projections, the following diagram is commutative for all $i \in \mathrm{Ob}(I)$ and $j \in \mathrm{Ob}(J)$.

$$\begin{tikzcd}
\lim(G(_)(i)) \arrow[r, "\lim(\xi(_)(i))"] \arrow[d, "\varepsilon^i_j"'] & \lim(G'(_)(i)) \arrow[d, "\varepsilon'^i_j"] \\
G(j)(i) \arrow[r, "\xi(j)(i)"'] & G'(j)(i)
\end{tikzcd}$$

This shows that $i \mapsto \lim(\xi(\_)(i))$ is the (uniquely determined) $\mathscr{C}^I$-morphism x from $(M \circ \Gamma)(G)$ to $(M \circ \Gamma)(G')$ having the property that $\xi(j) \circ (\varepsilon^i_j)_{i\in \mathrm{Ob}(I)} = (\varepsilon'^i_j)_{i\in\mathrm{Ob}(I)} \circ x$ for all $j \in \mathrm{Ob}(J)$, which is the definition of a limit of ξ. Therefore $\left(M \circ \Gamma, \Delta_{\mathscr{C}^I,J}\right)$ is an adjunction. Hence both of $M \circ \Gamma$ and $\lim_{\mathscr{C}^I,J}$ are right adjoint to the functor $\Delta_{\mathscr{C}^I,J}$. Proposition 1.4.18 yields the result.

(2) The theorem implies the existence of a limit for each functor $F : I \times J \to \mathscr{C}$. According to Theorem 1.4.11, the four pairs $\left(\Delta_{\mathscr{C},I\times J}, \lim_{\mathscr{C},I\times J}\right)$, $\left(\Delta_{\mathscr{C}^I,J}, \lim_{\mathscr{C}^I,J}\right)$, $\left(\Delta_{\mathscr{C},I}, \lim_{\mathscr{C},I}\right)$ and $\left(\Gamma, \Lambda\right)$ are adjunctions. Applying Theorem 1.4.19, the formula $\Delta_{\mathscr{C},I\times J} = \Gamma \circ \Delta_{\mathscr{C}^I,J} \circ \Delta_{\mathscr{C},I}$ and Proposition 1.4.18 the claim follows. The second statement follows analogously: $\left(\mathrm{colim}_{\mathscr{C},I\times J}, \Delta_{\mathscr{C},I\times J}\right)$, $\left(\mathrm{colim}_{\mathscr{C}^I,J}, \Delta_{\mathscr{C}^I,J}\right)$, $\left(\mathrm{colim}_{\mathscr{C},I}, \Delta_{\mathscr{C},I}\right)$ and $\left(\Lambda, \Gamma\right)$ are adjunctions.

□

1.6.7 Corollary. *Let the category $\mathscr{C}$ be I-and J-complete. The following diagram is commutative up to natural isomorphisms.*

$$\begin{tikzcd}
& \left(\mathscr{C}^I\right)^J \arrow[r, "\lim^J_{\mathscr{C},I}"] \arrow[dr, "\lim\limits_{\mathscr{C}^I,J}" near end] & \mathscr{C}^J \arrow[dr, "\lim_{\mathscr{C},J}"] & \\
\mathscr{C}^{I\times J} \arrow[ur, "\Lambda"] \arrow[dr, "\Xi"'] & & & \mathscr{C} \\
& \left(\mathscr{C}^J\right)^I \arrow[r, "\lim^I_{\mathscr{C},J}"'] \arrow[ur, "\lim\limits_{\mathscr{C}^J,I}"' near end] & \mathscr{C}^I \arrow[ur, "\lim_{\mathscr{C},I}"'] &
\end{tikzcd}$$

□

1.7 Naturally isomorphic bifunctors

1.7.1 Lemma. *Assume that $\mathscr{A}_1,\dots,\mathscr{A}_n$ are categories. Let H and K be functors from $\mathscr{A}_1\times\cdots\times\mathscr{A}_n$ to $\mathscr{C}$. Suppose that there exists a function $\varepsilon:\mathrm{Ob}(\mathscr{A}_1\times\cdots\times\mathscr{A}_n)\to\mathrm{Mor}(\mathscr{C})$ such that for all i in $\{1,\dots,n\}$ and for all $(a_j)_{j\neq i}\in\prod_{j\neq i}\mathrm{Ob}(\mathscr{A}_j)$ the function $z\mapsto\varepsilon(a_1,\dots,a_{i-1},z,a_{i+1},\dots,a_n)$ from $\mathrm{Ob}(\mathscr{A}_i)$ to $\mathrm{Mor}(\mathscr{C})$ is a natural transformation from $H(a_1,\dots,a_{i-1},\_\,,a_{i+1},\dots,a_n)$ to $K(a_1,\dots,a_{i-1},\_\,,a_{i+1},\dots,a_n)$. Then ε is a natural transformation from H to K.*

Proof. Let $f_j : x_j \to y_j$ be an $\mathscr{A}_i$-morphism for $1\le j\le n$. We have $\varepsilon(y_1,\dots,y_n)\circ H(f_1,\dots,f_n) = \varepsilon(y_1,\dots,y_n)\circ H(f_1,y_2,\dots,y_n)\circ H(x_1,f_2,\dots,f_n) = K(f_1,y_2,\dots,y_n)\circ\varepsilon(x_1,y_2,\dots,y_n)\circ H(x_1,f_2,\dots,f_n)$ and $K(f_1,\dots,f_{i-1},y_i,\dots,y_n)\circ\varepsilon(x_1,\dots,x_{i-1},y_i,\dots,y_n)\circ H(x_1,\dots,x_{i-1},f_i,\dots,f_n) =$
$K(f_1,\dots,f_{i-1},y_i,\dots,y_n)\circ\varepsilon(x_1,\dots,x_{i-1},y_i,\dots,y_n)\circ H(x_1,\dots,x_{i-1},f_i,y_{i+1},\dots,y_n)\circ$
$H(x_1,\dots,x_i,f_{i+1},\dots,f_n) = K(f_1,\dots,f_{i-1},y_i,\dots,y_n)\circ K(x_1,\dots,x_{i-1},f_i,y_{i+1},\dots,y_n)\circ$
$\varepsilon(x_1,\dots,x_i,f_{i+1},\dots,f_n)\circ H(x_1,\dots,x_i,f_{i+1},\dots,f_n) = K(f_1,\dots,f_i,y_{i+1},\dots,y_n)\circ$
$\varepsilon(x_1,\dots,x_i,f_{i+1},\dots,f_n)\circ H(x_1,\dots,x_i,f_{i+1},\dots,f_n)$ for $2\le i\le n$. Induction on i finishes the proof. □

1.7.2 Theorem. *Assume that $\mathscr{A}$ is I-complete, that $\mathscr{B}$ is J-complete and that $\mathscr{C}$ is I- and J-complete. Let $H:\mathscr{A}\times\mathscr{B}\to\mathscr{C}$ be a functor with the property that $H(a,\_\,):\mathscr{B}\to\mathscr{C}$ preserves I-limits for all $a\in\mathrm{Ob}(\mathscr{A})$ and that $H(\_\,,b):\mathscr{A}\to\mathscr{C}$ preserves J-limits for all $b\in\mathrm{Ob}(\mathscr{B})$.*

Define a functor $X:\mathscr{A}^I\times\mathscr{B}^J\to(\mathscr{A}\times\mathscr{B})^{I\times J}$ by $X(F,G)(f,g)=(F(f),G(g))$ for functors $F: I\to\mathscr{A}$ and $G:J\to\mathscr{B}$ and morphisms f of I and g of J, and $X(\varphi,\gamma)(i,j)=(\varphi(i),\gamma(j))$ for natural transformations φ, γ and objects i of I and j of J.

The functors $H\circ(\lim\limits_{\mathscr{A},I}\times\lim\limits_{\mathscr{B},J})$ and $\lim\limits_{\mathscr{C},I\times J}\circ H^{I\times J}\circ X$ from $\mathscr{A}^I\times\mathscr{B}^J$ to $\mathscr{C}$ are naturally isomorphic.

$$\begin{tikzcd}
\mathscr{A}^I\times\mathscr{B}^J \arrow[r, "X"] \arrow[d, "\lim\limits_{\mathscr{A},I}\times\lim\limits_{\mathscr{B},J}"'] & (\mathscr{A}\times\mathscr{B})^{I\times J} \arrow[r, "H^{I\times J}"] & \mathscr{C}^{I\times J} \arrow[d, "\lim\limits_{\mathscr{C},I\times J}"] \\
\mathscr{A}\times\mathscr{B} \arrow[rr, "H"'] & & \mathscr{C}
\end{tikzcd}$$

A natural isomorphism is given as follows: For all functors $F:I\to\mathscr{A}$ and $G:J\to\mathscr{B}$, let $(p_i)_{i\in\mathrm{Ob}(I)}$ be the family of projections for $\lim\limits_{\mathscr{A},I}F$, let $(q_j)_{j\in\mathrm{Ob}(J)}$ be the family of projections for $\lim\limits_{\mathscr{B},J}G$ and let $(r_{i,j})_{(i,j)\in\mathrm{Ob}(I\times J)}$ be the family of projections for $\lim\limits_{\mathscr{C},I\times J}\big((i,j)\mapsto H(F_i,G_j)\big)$. There exists a unique morphism $\varphi_{F,G}$ from $H\big(\lim\limits_{\mathscr{A},I}F,\lim\limits_{\mathscr{B},J}G\big)$ to $\lim\limits_{\mathscr{C},I\times J}\big((i,j)\mapsto H(F_i,G_j)\big)$ satisfying $r_{i,j}\circ\varphi_{F,G}=H(p_i,q_j)$ for *all* $(i,j) \in \mathrm{Ob}(I \times J)$. *The family* $(F,G) \mapsto \varphi_{F,G}$ *is a natural isomorphism from* $H \circ (\lim\limits_{\mathscr{A},I} \times \lim\limits_{\mathscr{B},J})$ *to* $\lim\limits_{\mathscr{C},I\times J} \circ H^{I\times J} \circ X$.

Proof. Let $F : I \to \mathscr{A}$ and $G : J \to \mathscr{B}$ be functors. Let $(\pi_i)_{i\in\mathrm{Ob}(I)}$ be the family of projections for $\lim(i \mapsto H(F_i, \lim G))$, let $(\kappa_i)_{i\in\mathrm{Ob}(I)}$ be the family of projections for $\lim(i \mapsto \lim(j \mapsto H(F_i, G_j)))$ and for for all $i \in \mathrm{Ob}(I)$, let ε^i be the family of projections for $\lim(j \mapsto H(F_i, G_j))$. We want to prove the existence of isomorphisms ψ_i (for $i \in \mathrm{Ob}(I)$), φ_1, φ_2 and φ_3 such that the following diagram is commutative, for all $(i,j) \in \mathrm{Ob}(I \times J)$.

$$\begin{tikzcd}
 & \lim\big((i,j) \mapsto H(F_i,G_j)\big) \arrow[rd, bend left, "r_{i,j}"] & \\
\lim(i \mapsto \lim(j \mapsto H(F_i,G_j))) \arrow[ru, bend left, "\varphi_1"] \arrow[r, "\kappa_i"] & \lim(j \mapsto H(F_i,G_j)) \arrow[r, "\varepsilon^i_j"] & H(F_i,G_j) \\
\lim(i \mapsto H(F_i,\lim G)) \arrow[u, "\varphi_2"] \arrow[r, "\pi_i"'] & H(F_i,\lim G) \arrow[u, "\psi_i"] \arrow[ru, "H(F_i,q_j)"] & H(\lim F,\lim G) \arrow[u, "H(p_i,q_j)"'] \arrow[l, "H(p_i,\lim G)"] \arrow[ll, bend left, "\varphi_3"]
\end{tikzcd}$$

Let $i \in \mathrm{Ob}(I)$. Since $\big(H(F_i, \lim G), (H(F_i, q_j))_{j\in\mathrm{Ob}(J)}\big)$ is a cone over the functor $j \mapsto H(F_i, G_j)$, there exists a unique isomorphism $\psi_i : H(F_i, \lim G) \to \lim(j \mapsto H(F_i, G_j))$ such that $\varepsilon^i_j \circ \psi_i = H(F_i, q_j)$ for all j. The function $i \mapsto \psi_i$ is a natural transformation: Let $f : i \to i'$ be an I–morphism.

$$\begin{tikzcd}
 & H(F_i,G_j) \arrow[dd, "H(F_f,G_j)"] & \\
H(F_i,\lim G) \arrow[ru, "H(F_i,q_j)"] \arrow[rr, "\psi_i"] \arrow[dd, "H(F_f,\lim G)"'] & & \lim(j \mapsto H(F_i,G_j)) \arrow[lu, "\varepsilon^i_j"'] \arrow[dd, "\lim(j \mapsto H(F_f,G_j))"] \\
 & H(F_{i'},G_j) & \\
H(F_{i'},\lim G) \arrow[ru, "H(F_{i'},q_j)"] \arrow[rr, "\psi_{i'}"'] & & \lim(j \mapsto H(F_{i'},G_j)) \arrow[lu, "\varepsilon^{i'}_j"']
\end{tikzcd}$$

Then $\varepsilon^{i'}_j \circ \psi_{i'} \circ H(F_f, \lim G) = \varepsilon^{i'}_j \circ \lim(j \mapsto H(F_f, G_j)) \circ \psi_i$ for all j, hence $\psi_{i'} \circ H(F_f, \lim G) = \lim(j \mapsto H(F_f, G_j)) \circ \psi_i$ because $\lim(j \mapsto H(F_{i'}, G_j))$ together with $(\varepsilon^{i'}_j)_{j\in\mathrm{Ob}(J)}$ is a limit. It follows that $\lim(i \mapsto H(F_i, \lim G))$ together with $(\psi_i \circ \pi_i)_{i\in\mathrm{Ob}(I)}$ is a cone; therefore there exists a unique isomorphism φ_2 such that $\kappa_i \circ \varphi_2 = \psi_i \circ \pi_i$ for all $i \in \mathrm{Ob}(I)$.
By the theorem on the associativity of limits, Theorem 1.6.4, there exists a unique isomorphism φ_1 from $\lim(i \mapsto \lim(j \mapsto H(F_i, G_j)))$ to $\lim((i,j) \mapsto H(F_i, G_j))$ such that $r_{i,j} \circ \varphi_1 = \varepsilon^i_j \circ \kappa_i$ for all $(i,j) \in \mathrm{Ob}(I \times J)$.

Since $H(\lim F, \lim G)$ together with the family $(H(p_i, \lim G))_{i \in \mathrm{Ob}(I)}$ is a cone, there exists a unique isomorphism φ_3 such that $\pi_i \circ \varphi_3 = H(p_i, \lim G)$ for all $i \in \mathrm{Ob}(I)$.
Putting these pieces together (as seen in the first diagram above) it follows that the isomorphism $\varphi_{F,G} = \varphi_1 \circ \varphi_2 \circ \varphi_3$ is the unique morphism a from $H(\lim F, \lim G)$ to $\lim\big((i,j) \mapsto H(F_i, G_j)\big)$ having the property that that $r_{i,j} \circ a = H(p_i, q_j)$ for all $(i,j) \in \mathrm{Ob}(I \times J)$.
It remains to show that the the isomorphisms $\varphi_{F,G}$ define a natural isomorphism from $H \circ \big(\lim_{\mathscr{A}} \times \lim_{\mathscr{B}}\big)$ to $\lim_{\mathscr{C}} \circ H^{I \times J} \circ X$. First, note that $H(\lim F, \lim G) = \big(H \circ (\lim_{\mathscr{A}} \times \lim_{\mathscr{B}})\big)(F, G)$ and that $\lim\big((i,j) \mapsto H(F_i, G_j)\big) = \big(\lim_{\mathscr{C}} \circ H^{I \times J} \circ X\big)(F, G)$. Let $T = H^{I \times J} \circ X$ for abbreviation. The object $\lim\big((i,j) \mapsto H(F_i, G_j)\big)$ together with $(r_{i,j})_{(i,j) \in \mathrm{Ob}(I \times J)}$ is a limit if $T(F,G)$, that is, a $\Delta_{\mathscr{C}, I \times J}$-final morphism for $T(F,G)$. Since $\varphi_{F,G}$ is an isomorphism, by Proposition 1.4.4 (2) $H(\lim F, \lim G)$ together with the morphisms $\varphi_{F,G}^{-1} \circ r_{i,j} = H(p_i, q_j)$ is a $\Delta_{\mathscr{C}, I \times J}$- final morphism for $T(F,G)$ too. These two limits are related via the equation $r_{i,j} \circ \varphi_{F,G} = H(p_i, q_j)$. By Lemma 1.4.9(3), $(F,G) \mapsto \varphi_{F,G}$ is a natural isomorphism. This fact can also be seen more directly in the following diagram.

$$
\begin{tikzcd}
 & H(F_i, G_j) \arrow[dd, "H(\zeta_i, \eta_j)"] & \\
H(\lim F, \lim G) \arrow[ur, "H(p_i, q_j)"] \arrow[rr, "\varphi_{F,G}"] \arrow[dd, "H(\lim \zeta, \lim \eta)"'] & & \lim((i,j) \mapsto H(F_i, G_j)) \arrow[ul, "r_{i,j}"'] \arrow[dd, "\lim((i,j) \mapsto H(\zeta_i, \eta_j))"] \\
 & H(\hat{F}_i, \hat{G}_j) & \\
H(\lim \hat{F}, \lim \hat{G}) \arrow[ur, "H(\hat{p}_i, \hat{q}_i)"] \arrow[rr, "\varphi_{\hat{F}, \hat{G}}"'] & & \lim((i,j) \mapsto H(\hat{F}_i, \hat{G}_j)) \arrow[ul, "\hat{r}_{i,j}"']
\end{tikzcd}
$$

$\square$

2 Isomorphisms in categories of bimodules

All rings are supposed to have an identity element but are not necessarily commutative. Since in the category ${}_R\mathbf{M}_S$ there exist small products, equalizers, small coproducts and coequalizers, by Theorem 1.5.5 categories of bimodules are complete and cocomplete.

2.0.3 Corollary. *Let R and S be rings. Then the category ${}_R\mathbf{M}_S$ is complete and cocomplete.*

2.1 Hom-functors

2.1.1 Remark. Let R and S be rings. We denote by ${}_R\mathbf{M}$ the category of left R-modules, by $\mathbf{M}_S$ the category of right S-modules and by ${}_R\mathbf{M}_S$ the category of (R,S)-bimodules. Let E and F be left R-modules. By ${}_R\mathrm{Hom}(E,F)$ we denote the Abelian group of R-module homomorphisms from E to F. If $f : E' \to E$ and $g : F \to F'$ are morphisms of left R-modules, then $\mathrm{Hom}(f,g) : {}_R\mathrm{Hom}(E,F) \to {}_R\mathrm{Hom}(E',F')$, defined by $\mathrm{Hom}(f,g)(x) = g \circ x \circ f$, is a homomorphism of Abelian groups, denoted by ${}_R\mathrm{Hom}(f,g)$. Similarly, we write $\mathrm{Hom}_R(E,F)$ for the Abelian group of R-module homomorphisms in the case that both of E and F are right R-modules and $\mathrm{Hom}_R(f,g)$ for the morphism $\mathrm{Hom}(f,g)$ of Abelian groups, given that f and g are morphisms of right R-modules. Finally, for (R,S)-bimodules E and F, ${}_R\mathrm{Hom}_S(E,F)$ denotes the Abelian group of (R,S)-bimodule homomorphisms $h : E \to F$ and, for (R,S)-linear maps f and g, ${}_R\mathrm{Hom}_S(f,g)$ denotes the morphism $\mathrm{Hom}(f,g)$ of Abelian groups.

If in addition module structures are defined on one of these Abelian groups that are compatible with each other, we sometimes indicate this by adding the corresponding rings as upper left or right indices. For example, ${}^S_R\mathrm{Hom}^T(E,F)$ denotes an (S,T)-bimodule of R-linear maps from the left R-module E to the left R-module F. Should the associated group homomorphisms $\mathrm{Hom}(f,g)$ turn out to be linear with respect to the rings in question, we indicate this by adding the rings as upper indices to the notation.

We will however only be interested in module structures on hom-sets that are defined according to the following proposition. Since every left R-module is an $(R,\mathbb{Z})$-bimodule and every right S-module is a $(\mathbb{Z},S)$-bimodule, it suffices to examine only bimodules.

2.1.2 Proposition.

(1) *Let E be an (R,S)-bimodule and F be a (R,T)-bimodule. For $\alpha \in S$, $\beta \in T$ and $f \in {}_R\mathrm{Hom}(E,F)$, define $(\alpha \cdot f)(x) = f(x\alpha)$ and $(f \cdot \beta)(x) = f(x)\beta$. With respect to these operations, ${}_R\mathrm{Hom}(E,F)$ is an (S,T)-bimodule. If $f : E' \to E$ and $g : F \to F'$ are (R,S)- and (R,T)-linear respectively, then ${}_R\mathrm{Hom}(f,g)$ is (S,T)-linear. A functor*

$$ {}^S_R\mathrm{Hom}^T(E, \_\,) : {}_R\mathbf{M}_T \to {}_S\mathbf{M}_T $$

is thus defined.

(2) *Let E be an (S,T)-bimodule and let F be an (R,T)-bimodule. For $\alpha \in R$, $\beta \in S$ and $f \in \mathrm{Hom}_T(E,F)$, define $(\alpha \cdot f)(x) = \alpha f(x)$ and $(f \cdot \beta)(x) = f(\beta x)$. With respect to these operations, $\mathrm{Hom}_T(E,F)$ is an (R,S)-bimodule. If $f : E' \to E$ and $g : F \to F'$ are (S,T)- and (R,T)-linear*

respectively, then $\mathrm{Hom}_T(f,g)$ *is* (R,S)*-linear. A functor*

$$^R\mathrm{Hom}_T^S(E,\_\,) : {}_R\mathbf{M}_T \to {}_R\mathbf{M}_S$$

is thus defined.

The proof is by straightforward calculation.

2.1.3 Corollary. *Suppose that* R, S *and* T *are rings. If* $U_1 : {}_R\mathbf{M}_S \to {}_R\mathbf{M}$, $U_2 : {}_R\mathbf{M}_T \to {}_R\mathbf{M}$ *and* $V_1 : {}_S\mathbf{M}_T \to \mathbf{Set}$ *are the forgetful functors, then the following diagram is commutative.*

$$\begin{tikzcd} {}_R\mathbf{M}_S^{\mathrm{op}} \times {}_R\mathbf{M}_T \arrow[r, "{}_R\mathrm{Hom}"] \arrow[d, "U_1 \times U_2"'] & {}_S\mathbf{M}_T \arrow[d, "V_1"] \\ {}_R\mathbf{M}^{\mathrm{op}} \times {}_R\mathbf{M} \arrow[r, "\mathrm{hom}"'] & \mathbf{Set} \end{tikzcd}$$

If $U_3 : {}_S\mathbf{M}_T \to \mathbf{M}_T$, $U_4 : {}_R\mathbf{M}_T \to \mathbf{M}_T$ *and* $V_2 : {}_R\mathbf{M}_S \to \mathbf{Set}$ *are the forgetful functors, then the following diagram is commutative.*

$$\begin{tikzcd} {}_S\mathbf{M}_T^{\mathrm{op}} \times {}_R\mathbf{M}_T \arrow[r, "\mathrm{Hom}_T"] \arrow[d, "U_3 \times U_4"'] & {}_R\mathbf{M}_S \arrow[d, "V_2"] \\ \mathbf{M}_T^{\mathrm{op}} \times \mathbf{M}_T \arrow[r, "\mathrm{hom}"'] & \mathbf{Set} \end{tikzcd}$$

2.1.4 Lemma. *Let* R *and* S *be rings. The forgetful functors from* ${}_R\mathbf{M}_S$ *to* ${}_R\mathbf{M}$ *and* $\mathbf{M}_S$ *respectively are continuous and cocontinuous. Furthermore, the forgetful functor from* ${}_R\mathbf{M}_S$ *to* $\mathbf{Set}$ *is continuous.*

Proof. Note first that categories of bimodules are complete and cocomplete. We want to apply Lemma 1.5.15. If $A = (A_i)_i$ is a family of (R,S)-bimodules, then a product of A is obtained from a product $(N,(p_i)_i)$ in $\mathbf{Set}$ by equipping N with a bimodule structure. The functions p_i are linear with respect to this structure. The forgetful functor returns the original product. Likewise for equalizers, so that the forgetful functor is indeed continuous.
The remaining statements are proved with the help of the same argument: Products, equalizers, Coproducts and Coequalizers in categories of bimodules are obtained from the respective constructions in the category $\mathbf{Ab}$ of Abelian groups by adding scalar multiplications. $\square$

2.1.5 Lemma. *Let* A, B *and* C *be* (R,S)*-bimodules. Suppose that* I *is a set and that* $(f_i)_{i\in I}$ *and* $(g_i)_{i\in I}$ *are families of* (R,S)*-linear maps, where* $f_i : A \to C$ *and* $g_i : B \to C$ *for* $i \in I$. *Suppose moreover that there exists a uniquely determined map* $l : B \to A$ *with the property that* $f_i \circ l = g_i$ *for all* $i \in I$. *Then* l *is* (R,S)*-linear.*

Proof. For all $x \in B$, the set $l(x)$ is the unique element z of A such that $f_i(z) = g_i(x)$ for all i. Assume that x, $y \in B$, $\alpha \in R$ and $\beta \in S$. Then $f_i(\alpha l(x)\beta + l(y)) = f_i(\alpha l(x)\beta) + f_i(l(y)) = \alpha g_i(x)\beta + g_i(y) = g_i(\alpha x\beta) + g_i(y) = g_i(\alpha x\beta + y) = f_i(l(\alpha x\beta + y))$ for all i. By uniqueness, the claim follows. $\square$

2.1.6 Lemma. *Assume that* R *and* S *are rings. The forgetful functor* $U : {}_R\mathbf{M}_S \to \mathbf{Set}$ reflects limits, *which means the following:*

If I is a small category, $F : I \to {}_R\mathbf{M}_S$ is a functor and $\big(N, (p_i)_i\big)$ is a cone over F such that $\big(U(N), (U(p_i))_i\big)$ is a limit of $U \circ F$, then $\big(N, (p_i)_i\big)$ is a limit of F.

Proof. If $\big(M, (q_i)_{i\in I}\big)$ is another cone over F, then $\big(U(F), (U(q_i))_i\big)$ is a cone over $U \circ F$, hence there exists a unique **Set**-morphism $l : U(M) \to U(N)$ such that $U(p_i) \circ l = U(q_i)$ for all i. By Lemma 2.1.5 there is an ${}_R\mathbf{M}_S$-morphism $k : M \to N$ such that $U(k) = l$. The morphism k is the only one with this property, since U is faithful. On the other hand, each morphism c with this property satisfies $U(f_i \circ c) = U(f_i) \circ k = U(g_i)$ for all i, hence $f_i \circ c = g_i$ for all i because U is faithful. □

2.1.7 Proposition. *Assume that R, S and T are rings, that E is an (R, S)-bimodule and that F is an (R, T)-bimodule. Then the functors*

$$\begin{aligned} {}_R\mathrm{Hom}(E, \_\,) :&\quad {}_R\mathbf{M}_T \to {}_S\mathbf{M}_T \quad \textit{and} \\ {}_R\mathrm{Hom}(\_\,, F) :&\quad {}_R\mathbf{M}_S^{\mathrm{op}} \to {}_S\mathbf{M}_T \quad \textit{are continuous.} \end{aligned}$$

Proof. The following more general statement holds: Assume that $K : \mathscr{A} \to \mathscr{B}$, $G : \mathscr{B} \to \mathscr{C}$ and $H : \mathscr{A} \to \mathscr{C}$ are functors and that $H = G \circ K$. If H preserves limits and G reflects limits, then K preserves limits.

If (N, p) is a limit of $F : I \to \mathscr{A}$, then $(K(N), K \star p)$ is a cone over $K \circ F$ and $(H(N), H \star p) = \big(G(K(N)), G \star (K \star p)\big)$ is a limit of $H \circ F = G \circ (K \circ F)$. Since G reflects limits, the claim is proved.

Now the functors $\hom_{{}_R\mathbf{M}}(E, \_\,)$ and $\hom_{{}_R\mathbf{M}}(\_\,, F) = \hom_{{}_R\mathbf{M}^{\mathrm{op}}}(F, \_\,)$ are continuous by Theorem 1.5.21. Moreover, by Lemma 2.1.4 the forgetful functor from ${}_R\mathbf{M}_S$ to ${}_R\mathbf{M}$ is cocontinuous and consequently continous as a functor from ${}_R\mathbf{M}_S^{\mathrm{op}}$ to ${}_R\mathbf{M}^{\mathrm{op}}$. Also, the forgetful functor from ${}_R\mathbf{M}_T$ to ${}_R\mathbf{M}$ is continuous. Since the composition of continuous functors is continuous, the original statements follow from Lemma 2.1.6 and Corollary 2.1.3. □

Theorem 1.7.2 gives us the following natural isomorphism.

2.1.8 Theorem. *Let R, S and T be rings and let I and J be small categories. The functors from $({}_R\mathbf{M}_S^{\mathrm{op}})^I \times ({}_R\mathbf{M}_T)^J$ to ${}_S\mathbf{M}_T$ defined by*

$$\begin{aligned} &(F, G) \mapsto \lim_{I\times J}\big((i, j) \mapsto {}_R\mathrm{Hom}(F_i, G_j)\big) \quad \textit{and} \\ &(F, G) \mapsto {}_R\mathrm{Hom}\big(\lim_I(F), \lim_J(G)\big) \qquad \textit{are naturally isomorphic.} \end{aligned}$$

2.2 Multilinear maps

2.2.1 Definition.

(1) Let $E_1, \dots, E_n$ and F be $\mathbb{Z}$-modules. A function $f : \prod_{1\le i\le n} E_i \to F$ is called *distributive* if for all $i \in \{1, \dots, n\}$ and all $(x_i)_{i\neq j} \in \prod_{i\neq j} E_i$ the function $x \mapsto f(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n)$ from E_i to F is $\mathbb{Z}$-linear.

(2) For $1 \le i \le n$ let E_i be an (R_{i-1}, R_i)-bimodule and let F be an (R_0, R_n)-bimodule. A function $f : \prod_{1\le i\le n} E_i \to F$ is called $(R_0, \dots, R_n)$-*multilinear* if the following properties are satisfied:

(a) f is distributive.

(b) For $1 < i \le n$, $\alpha \in R_{i-1}$ and $x \in \prod_{1\le i\le n} E_i$, we have
$f(x_1, \dots, x_{i-2}, x_{i-1}\alpha, x_i, \dots, x_n) = f(x_1, \dots, x_{i-1}, \alpha x_i, x_{i+1}, \dots, x_n)$.

(c) For all $\beta \in R_0$ and all $\gamma \in R_n$, the formula $f(\beta x_1, x_2, \dots, x_{n-1}, x_n\gamma) = \beta f(x_1, \dots, x_n)\gamma$ holds.

We denote the set of $(R_0,\dots,R_n)$-multilinear maps $f:\prod\limits_{1\le i\le n}E_i\to F$ by $\mathscr{L}(E_1,\dots,E_n,F)$ or, unambiguously, by $\underset{R_0,\dots,R_n}{\mathscr{L}}(E_1,\dots,E_n,F)$.

Proposition 2.1.2 admits the following generalization.

2.2.2 Proposition. *Let $R_0,\dots,R_n$ and S be rings. For $1\le i\le n$, let E_i be an (R_{i-1},R_i)-bimodule.*

(1) If F is an (R_0,R_n)-bimodule, then the set $\underset{R_0,\dots,R_n}{\mathscr{L}}(E_1,\dots,E_n,F)$ of provided with the usual addition of functions is an Abelian group. If $f_i:E_i'\to E_i$ is (R_{i-1},R_i)-linear for $1\le i\le n$ and if $g:F\to F'$ is (R_0,R_n)-linear, a homomorphism

$$\underset{R_0,\dots,R_n}{\mathscr{L}}(f_1,\dots,f_n,g):\underset{R_0,\dots,R_n}{\mathscr{L}}(E_1,\dots,E_n,F)\to\underset{R_0,\dots,R_n}{\mathscr{L}}(E_1',\dots,E_n',F')$$

of Abelian groups is defined by the rule $\underset{R_0,\dots,R_n}{\mathscr{L}}(f_1,\dots,f_n,g)(u)=g\circ u\circ(f_1,\dots,f_n)$. We get a functor $\underset{R_0,\dots,R_n}{\mathscr{L}}$ from ${}_{R_0}\mathbf{M}_{R_1}^{\mathrm{op}}\times\cdots\times{}_{R_{n-1}}\mathbf{M}_{R_n}^{\mathrm{op}}\times{}_{R_0}\mathbf{M}_{R_n}$ to $\mathbf{Ab}$.

(2) Assume that F is an (S,R_n)-bimodule. The Abelian group $\underset{\mathbb{Z},R_1,\dots,R_n}{\mathscr{L}}(E_1,\dots,E_n,F)$ is an (S,R_0)-bimodule by virtue of the scalar multiplications defined by the formulas

$$\begin{aligned}(\alpha u)(e_1,\dots,e_n)&=\alpha(u(e_1,\dots,e_n))\text{ and}\\(u\beta)(e_1,\dots,e_n)&=u(\beta e_1,\dots,e_n).\end{aligned}$$

If $f_i:E_i'\to E_i$ is (R_{i-1},R_i)-linear for $1\le i\le n$ and if $g:F\to F'$ is (S,R_n)-linear, an (S,R_0)-linear function

$$\underset{\mathbb{Z},R_1,\dots,R_n}{\mathscr{L}}(f_1,\dots,f_n,g):\underset{\mathbb{Z},R_1,\dots,R_n}{\mathscr{L}}(E_1,\dots,E_n,F)\to\underset{\mathbb{Z},R_1,\dots,R_n}{\mathscr{L}}(E_1',\dots,E_n',F')$$

is defined by the rule $\underset{\mathbb{Z},R_1,\dots,R_n}{\mathscr{L}}(f_1,\dots,f_n,g)(u)=g\circ u\circ(f_1,\dots,f_n)$. We get a functor ${}^{S}\underset{\mathbb{Z},R_1,\dots,R_n}{\mathscr{L}}{}^{R_0}$ from ${}_{R_0}\mathbf{M}_{R_1}^{\mathrm{op}}\times\cdots\times{}_{R_{n-1}}\mathbf{M}_{R_n}^{\mathrm{op}}\times{}_{S}\mathbf{M}_{R_n}$ to ${}_{S}\mathbf{M}_{R_0}$.

(3) Assume that F is an (R_0,S)-bimodule. The Abelian group $\underset{R_0,\dots,R_{n-1},\mathbb{Z}}{\mathscr{L}}(E_1,\dots,E_n,F)$ is an (R_n,S)-bimodule by virtue of the scalar multiplications defined by the formulas

$$\begin{aligned}(\alpha u)(e_1,\dots,e_n)&=u(e_1,\dots,e_n\alpha)\text{ and}\\(u\beta)(e_1,\dots,e_n)&=(u(e_1,\dots,e_n))\beta.\end{aligned}$$

If $f_i:E_i'\to E_i$ is (R_{i-1},R_i)-linear for $1\le i\le n$ and if $g:F\to F'$ is (R_0,S)-linear, an (R_n,S)-linear function

$$\underset{R_0,\dots,R_{n-1},\mathbb{Z}}{\mathscr{L}}(f_1,\dots,f_n,g):\underset{R_0,\dots,R_{n-1},\mathbb{Z}}{\mathscr{L}}(E_1,\dots,E_n,F)\to\underset{R_0,\dots,R_{n-1},\mathbb{Z}}{\mathscr{L}}(E_1',\dots,E_n',F')$$

is defined by the rule $\underset{R_0,\dots,R_{n-1},\mathbb{Z}}{\mathscr{L}}(f_1,\dots,f_n,g)(u)=g\circ u\circ(f_1,\dots,f_n)$. A functor ${}^{R_n}\underset{R_0,\dots,R_{n-1},\mathbb{Z}}{\mathscr{L}}{}^{S}$ from ${}_{R_0}\mathbf{M}_{R_1}^{\mathrm{op}}\times\cdots\times{}_{R_{n-1}}\mathbf{M}_{R_n}^{\mathrm{op}}\times{}_{R_0}\mathbf{M}_{S}$ to ${}_{R_n}\mathbf{M}_{S}$ is thus obtained.

Proof. The difference of two $(R_0,\dots,R_n)$-multilinear functions is again $(R_0,\dots,R_n)$-multilinear, hence the first statement in (1). It is also easy to show that the prospective scalar multiplications in (2) and (3) are well-defined and that they define bimodule structures.

Let $f_i:E_i'\to E_i$ be (R_{i-1},R_i)-linear for $1\le i\le n$ and let $g:F\to F'$ be a homomorphism of Abelian groups. Then for all $(\mathbb{Z},R_1,\dots,R_{n-1},\mathbb{Z})$-multilinear maps $u:E_1\times\cdots\times E_n\to F$, the function

$g \circ u \circ (f_1, \dots, f_n)$ is also $(\mathbb{Z}, R_1, \dots, R_{n-1}, \mathbb{Z})$-multilinear: It is distributive since each f_i is $\mathbb{Z}$-linear, u is distributive and g is $\mathbb{Z}$-linear; for $1 \le i < n$, $\beta \in R_i$ and $(e_1, \dots, e_n) \in E_1 \times \dots \times E_n$, we have

$$g(u(f_1(e_1), \dots, f_{i-1}(e_{i-1}), f_i(e_i\beta), f_{i+1}(e_{i+1}), \dots, f_n(e_n))) = \\ = g(u(f_1(e_1), \dots, f_i(e_i), f_{i+1}(\beta e_{i+1}), f_{i+2}(e_{i+2}), \dots, f_n(e_n))).$$

If moreover $g : F \to F'$ is a homomorphism of right R_n-modules and u is $(\mathbb{Z}, R_1, \dots, R_n)$-multilinear, we have $g(u(f_1(e_1), \dots, f_{n-1}(e_{n-1}), f_n(e_n\gamma))) = g(u(f_1(e_1), \dots, f_n(e_n))\gamma) = g(u(f_1(e_1), \dots, f_n(e_n)))\gamma$ for all $\gamma \in R_n$ and $(e_1, \dots, e_n) \in E_1 \times \dots \times E_n$. If $g : F \to F'$ is a homomorphism of left R_0-modules and u is $(R_0, \dots, R_{n-1}, \mathbb{Z})$-multilinear, for all $\alpha \in R_0$ and $(e_1, \dots, e_n) \in E_1 \times \dots \times E_n$ we have $g(u(f_1(\alpha e_1), f_2(e_2), \dots, f_n(e_n))) = g(\alpha u(f_1(e_1), \dots, f_n(e_n))) = \alpha g(u(f_1(e_1), \dots, f_n(e_n)))$. Therefore the functions are well-defined.

Let $f_i : E_i' \to E_i$ be (R_{i-1}, R_i)-linear for $1 \le i \le n$ and let $g : F \to F'$ be a homomorphism of Abelian groups. Let $u, v \in \underset{\mathbb{Z}, R_1, \dots, R_{n-1}, \mathbb{Z}}{\mathscr{L}}(E_1, \dots, E_n, F)$. Then

$$\mathscr{L}(f_1, \dots, f_n, g)(u + v) = g \circ (u + v) \circ (f_1, \dots, f_n) = g \circ ((u \circ (f_1, \dots, f_n)) + (v \circ (f_1, \dots, f_n))) = \\ = g \circ u \circ (f_1, \dots, f_n) + g \circ v \circ (f_1, \dots, f_n) = \mathscr{L}(f_1, \dots, f_n, g)(u) + \mathscr{L}(f_1, \dots, f_n, g)(v).$$

In particular, if $g : F \to F'$ is even (R_0, R_n)-linear, $\underset{R_0, \dots, R_n}{\mathscr{L}}(f_1, \dots, f_n, g)$ is $\mathbb{Z}$-linear.

Let $g : F \to F'$ be a homomorphism of (S, R_n)-bimodules. For all $u \in \underset{\mathbb{Z}, R_1, \dots, R_n}{\mathscr{L}}(E_1, \dots, E_n, F)$, $\alpha \in S$, $\beta \in R_0$ and for all $e_i \in E_i$ $(1 \le i \le n)$, we have

$$\underset{\mathbb{Z}, R_1, \dots, R_n}{\mathscr{L}}(f_1, \dots, f_n, g)(\alpha u)(e_1, \dots, e_n) = g((\alpha u)(f_1(e_1), \dots, f_n(e_n))) = \\ = g(\alpha(u(f_1(e_1), \dots, f_n(e_n)))) = \alpha g(u(f_1(e_1), \dots, f_n(e_n))) = \\ = \alpha\Big(\underset{\mathbb{Z}, R_1, \dots, R_n}{\mathscr{L}}(f_1, \dots, f_n, g)(u)(e_1, \dots, e_n)\Big) = \Big(\alpha \underset{\mathbb{Z}, R_1, \dots, R_n}{\mathscr{L}}(f_1, \dots, f_n, g)(u)\Big)(e_1, \dots, e_n)$$

and

$$\underset{\mathbb{Z}, R_1, \dots, R_n}{\mathscr{L}}(f_1, \dots, f_n, g)(u\beta)(e_1, \dots, e_n) = g((u\beta)(f_1(e_1), \dots, f_n(e_n))) = \\ = g(u(\beta f_1(e_1), f_2(e_2), \dots, f_n(e_n))) = g(u(f_1(\beta e_1), f_2(e_2), \dots, f_n(e_n))) = \\ = \underset{\mathbb{Z}, R_1, \dots, R_n}{\mathscr{L}}(f_1, \dots, f_n, g)(u)(\beta e_1, e_2, \dots, e_n) = \Big(\underset{\mathbb{Z}, R_1, \dots, R_n}{\mathscr{L}}(f_1, \dots, f_n, g)(u)\beta\Big)(e_1, \dots, e_n).$$

Therefore $\underset{\mathbb{Z}, R_1, \dots, R_n}{\mathscr{L}}(f_1, \dots, f_n, g)$ is (S, R_0)-linear. An analogous calculation gives (R_n, S)-linearity of the function $\underset{R_0, \dots, R_{n-1}, \mathbb{Z}}{\mathscr{L}}(f_1, \dots, f_n, g)$, where $g : F \to F'$ is a homomorphism of (R_0, S)-bimodules.

Finally, if $f_i' : E_i'' \to E_i'$ and $f_i : E_i' \to E_i$ are (R_{n-1}, R_n)-linear for $1 \le i \le n$, $g : F \to F'$ and $g' : F' \to F''$ are $\mathbb{Z}$-linear and $u : E_1 \times \dots \times E_n \to F$ is $(\mathbb{Z}, R_1, \dots, R_{n-1}, \mathbb{Z})$-multilinear, we have $\underset{\mathbb{Z}, R_1, \dots, R_{n-1}, \mathbb{Z}}{\mathscr{L}}((f_1', \dots, f_n', g') \circ (f_1, \dots, f_n, g))(u) = \underset{\mathbb{Z}, R_1, \dots, R_{n-1}, \mathbb{Z}}{\mathscr{L}}(f_1 \circ f_1', \dots, f_n \circ f_n', g' \circ g)(u) = (g' \circ g) \circ u \circ (f_1 \circ f_1', \dots, f_n \circ f_n') = g' \circ (g \circ u \circ (f_1, \dots, f_n)) \circ (f_1', \dots, f_n') = \Big(\underset{\mathbb{Z}, R_1, \dots, R_{n-1}, \mathbb{Z}}{\mathscr{L}}(f_1', \dots, f_n', g') \circ \mathscr{L}(f_1, \dots, f_n, g)\Big)(u)$. Hence functors $\underset{R_0, \dots, R_n}{\mathscr{L}}$, ${}^{S}\underset{\mathbb{Z}, R_1, \dots, R_n}{\mathscr{L}}{}^{R_0}$ and ${}^{R_n}\underset{R_0, \dots, R_{n-1}, \mathbb{Z}}{\mathscr{L}}{}^{S}$ are defined. $\square$

2.2.3 Proposition. *Let rings $R_0, \dots, R_n$ be given, where $n \ge 2$.*

(1) *Assume that F is an (S, R_n)-bimodule and that E_i is an (R_{i-1}, R_i)-bimodule for $1 \le i \le n$. The function*

$$\delta_{E_1,\dots,E_n,F} : {}^{S}_{\mathbb{Z},R_1,\dots,R_n}\mathscr{L}^{R_0}(E_1,\dots,E_n,F) \to {}^{S}\mathrm{Hom}^{R_0}_{R_1}\Big(E_1, {}^{S}_{\mathbb{Z},R_2,\dots,R_n}\mathscr{L}^{R_1}(E_2,\dots,E_n,F)\Big)$$

defined by $\delta_{E_1,\dots,E_n,F}(u)(e_1)(e_2,\dots,e_n) = u(e_1,\dots,e_n)$ is an isomorphism of (S, R_0)-bimodules.

If $S = R_0$, restricting $\delta_{E_1,\dots,E_n,F}$ to the set of $(R_0,\dots,R_n)$-multilinear maps induces an isomorphism

$$\delta'_{E_1,\dots,E_n,F} : \mathscr{L}_{R_0,\dots,R_n}(E_1,\dots,E_n,F) \to {}_{R_0}\mathrm{Hom}_{R_1}\Big(E_1, {}^{R_0}_{\mathbb{Z},R_2,\dots,R_n}\mathscr{L}^{R_1}(E_2,\dots,E_n,F)\Big)$$

of Abelian groups. Moreover, these isomorphisms are natural in all variables: The functors ${}^{S}_{\mathbb{Z},R_1,\dots,R_n}\mathscr{L}^{R_0}(\underset{1}{-},\underset{2}{-},\dots,\underset{n}{-},=)$ and ${}^{S}\mathrm{Hom}^{R_0}_{R_1}\Big(\underset{1}{-}, {}^{S}_{\mathbb{Z},R_2,\dots,R_n}\mathscr{L}^{R_1}(\underset{2}{-},\dots,\underset{n}{-},=)\Big)$ from ${}_{R_0}\mathbf{M}^{\mathrm{op}}_{R_1} \times \cdots \times {}_{R_{n-1}}\mathbf{M}^{\mathrm{op}}_{R_n} \times {}_{S}\mathbf{M}_{R_n}$ to ${}_{S}\mathbf{M}_{R_0}$ are naturally isomorphic, and the functors $\mathscr{L}_{R_0,\dots,R_n}(\underset{1}{-},\underset{2}{-},\dots,\underset{n}{-},=)$ and ${}_{R_0}\mathrm{Hom}_{R_1}\Big(\underset{1}{-}, {}^{R_0}_{\mathbb{Z},R_2,\dots,R_n}\mathscr{L}^{R_1}(\underset{2}{-},\dots,\underset{n}{-},=)\Big)$ from ${}_{R_0}\mathbf{M}^{\mathrm{op}}_{R_1} \times \cdots \times {}_{R_{n-1}}\mathbf{M}^{\mathrm{op}}_{R_n} \times {}_{R_0}\mathbf{M}_{R_n}$ to $\mathbf{Ab}$ are also naturally isomorphic.

(2) *Assume that E_i is an (R_{i-1}, R_i)-bimodule for $1 \le i \le n$ and that F is an (R_0, S)-bimodule. The function*

$$\sigma_{E_1,\dots,E_n,F} : {}^{R_n}_{R_0,\dots,R_{n-1},\mathbb{Z}}\mathscr{L}^{S}(E_1,\dots,E_n,F) \to {}^{R_n}_{R_{n-1}}\mathrm{Hom}^{S}\Big(E_n, {}^{R_{n-1}}_{R_0,\dots,R_{n-2},\mathbb{Z}}\mathscr{L}^{S}(E_1,\dots,E_{n-1},F)\Big)$$

defined by $\sigma_{E_1,\dots,E_n,F}(u)(e_n)(e_1,\dots,e_{n-1}) = u(e_1,\dots,e_n)$ is an isomorphism of (R_n, S)-bimodules.

If $S = R_n$, restricting $\sigma_{E_1,\dots,E_n,F}$ to the set of $(R_0,\dots,R_n)$-multilinear maps induces an isomorphism

$$\sigma'_{E_1,\dots,E_n,F} : \mathscr{L}_{R_0,\dots,R_n}(E_1,\dots,E_n,F) \to {}_{R_{n-1}}\mathrm{Hom}_{R_n}\Big(E_n, {}^{R_{n-1}}_{R_0,\dots,R_{n-2},\mathbb{Z}}\mathscr{L}^{R_n}(E_1,\dots,E_{n-1},F)\Big).$$

of Abelian groups. These isomorphisms are natural in all variables.

Proof. We give the proof of (1), that of (2) being similar. We have to show the well-definedness of $\delta_{E_1,\dots,E_n,F}$, which we simply call δ for the sake of brevity. Let $u : E_1 \times \cdots \times E_n \to F$ be $(\mathbb{Z}, R_1,\dots,R_n)$-multilinear. It is easy to see that for each $e_1 \in E_1$ the function $u^{e_1} : \prod_{2\le i\le n} E_i \to F$, $(e_2,\dots,e_n) \mapsto u(e_1,e_2,\dots,e_n)$ is $(\mathbb{Z}, R_2,\dots,R_n)$-multilinear, so that it is enough to prove right R_1-linearity of the function $e \mapsto u^e$. Equality of $u^{e+e'}$ and $u^e + u^{e'}$ is obvious from the distributivity of u. For $\beta \in R_1$, $e \in E_1$ and $(e_2,\dots,e_n) \in E_2 \times \cdots \times E_n$, we have $u^{e\beta}(e_2,\dots,e_n) = u(e\beta, e_2,\dots,e_n) = u(e, \beta e_2, e_3,\dots,e_n) = u^e(\beta e_2, e_3,\dots,e_n) = (u^e\beta)(e_2,\dots,e_n)$ by multilinearity of u and the definition of right multiplication in ${}^{S}_{\mathbb{Z},R_2,\dots,R_n}\mathscr{L}^{R_1}(E_2,\dots,E_n,F)$. The $\mathbb{Z}$-linearity of δ is very easy to see. Using the definitions of the scalar multiplications on the participating sets of multilinear maps, for $\alpha \in S$, $u \in \mathscr{L}_{\mathbb{Z},R_1,\dots,R_n}(E_1,\dots,E_n,F)$ and $(e_1,\dots,e_n) \in E_1 \times \cdots \times E_n$ we have $(\alpha\delta(u))(e_1)(e_2,\dots,e_n) = \big(\alpha(\delta(u)(e_1))\big)(e_2,\dots,e_n) = \alpha\big(\delta(u)(e_1)(e_2,\dots,e_n)\big) = \alpha(u(e_1,\dots,e_n)) = (\alpha u)(e_1,\dots,e_n) = \delta(\alpha u)(e_1)(e_2,\dots,e_n)$ and for $\beta \in R_0$, we have $(\delta(u)\beta)(e_1)(e_2,\dots,e_n) = \delta(u)(\beta e_1)(e_2,\dots,e_n) = u(\beta e_1, e_2,\dots,e_n) = (u\beta)(e_1,\dots,e_n) = \delta(u\beta)(e_1)(e_2,\dots,e_n)$.

We want to show that the inverse of δ is given by the function τ from $\mathrm{Hom}_{R_1}\big(E_1, \mathscr{L}(E_2,\dots,E_n,F)\big)$ to $\mathscr{L}(E_1,\dots,E_n,F)$ defined by the rule $\tau(u)(e_1,\dots,e_n) = u(e_1)(e_2,\dots,e_n)$. The function τ is well defined: Let u be a right R_1-linear function from E_1 to $\mathscr{L}(E_2,\dots,E_n,F)$. We have to show that the function $(e_1,\dots,e_n) \mapsto u(e_1)(e_2,\dots,e_n)$ is $(\mathbb{Z}, R_1,\dots,R_n)$-multilinear. First, it is distributive, since u is $\mathbb{Z}$-linear and $u(e_1)$ is $(\mathbb{Z}, R_2,\dots,R_n)$-multilinear for all $e_1 \in E_1$.

Let $e_i \in E_i$ for $1 \le i \le n$. If $1 < i < n$, then for $\beta \in R_i$ and $e_i \in E_i$ for all i we have $u(e_1)(e_2, \ldots, e_{i-1}, e_i\beta, e_{i+1}, \ldots, e_n) = u(e_1)(e_2, \ldots, e_i, \beta e_{i+1}, e_{i+2}, \ldots, e_n)$ since $u(e_1)$ is $(\mathbb{Z}, R_2, \ldots, R_n)$-multilinear for all $e_1 \in E_1$. Moreover, if $\beta \in R_1$ and $e_i \in E_i$ for all i, we have $u(e_1\beta)(e_2, \ldots, e_n) = (u(e_1)\beta)(e_2, \ldots, e_n) = u(e_1)(\beta e_2, e_3, \ldots, e_n)$ by right R_1-linearity of u and the definition of right multiplication in $\mathscr{L}(E_2, \ldots, E_n, F)$. If $\gamma \in R_n$, we have $u(e_1)(e_2, \ldots, e_n\gamma) = (u(e_1)(e_2, \ldots, e_n))\gamma$ since $u(e)$ is $(\mathbb{Z}, R_2, \ldots, R_n)$-multilinear, so that τ is a well-defined function. Obviously, δ and τ are inverse to each other.

Now assume that $S = R_0$. To prove the well-definedness of δ', it is sufficient to show that for each $(R_0, \ldots, R_n)$-multilinear function u, the function $e \mapsto u^e$, where $u^e(e_2, \ldots, e_n) = u(e, e_2, \ldots, e_n)$, is left R_0-linear. For $\alpha \in R_0$, $e, e' \in E_1$ and $(e_2, \ldots, e_n) \in E_2 \times \cdots \times E_n$ we have $u^{\alpha e + e'}(e_2, \ldots, e_n) = u(\alpha e + e', e_2, \ldots, e_n) = \alpha(u(e, e_2, \ldots, e_n)) + u(e', e_2, \ldots, e_n) = \alpha(u^e(e_2, \ldots, e_n)) + u^{e'}(e_2, \ldots, e_n) = (\alpha u^e)(e_2, \ldots, e_n) + u^{e'}(e_2, \ldots, e_n) = (\alpha u^e + u^{e'})(e_2, \ldots e_n)$ by multilinearity of u and the definition of left multiplication in ${}^{R_0}\mathscr{L}^{R_1}_{\mathbb{Z}, R_2, \ldots, R_n}(E_2, \ldots, E_n, F)$. Conversely, let $u : E_1 \to \mathscr{L}(E_2, \ldots, E_n, F)$ be (R_0, R_1)-linear. We have already shown that $\tau(u)$ is $(\mathbb{Z}, R_1, \ldots, R_n)$-multilinear, where τ is the inverse of δ. Let $e_i \in E_i$ for $1 \le i \le n$. For $\alpha \in R_0$, we have $u(\alpha e_1)(e_2, \ldots, e_n) = (\alpha u(e))(e_2, \ldots, e_n) = \alpha(u(e_1)(e_1, \ldots, e_n))$ by left R_0-linearity of u and the definition of left multiplication in $\mathscr{L}(E_2, \ldots, E_n, F)$, therefore $\tau(u)$ is $(R_0, \ldots, R_n)$-multilinear.

There remains to be proven that these isomorphisms are natural. Let $f_i : E'_i \to E_i$ be (R_{i-1}, R_i)-linear for $1 \le i \le n$ and let $g : F \to F'$ be (S, R_n)-linear. If $u \in \mathscr{L}_{\mathbb{Z}, R_1, \ldots, R_n}(E_1, \ldots, E_n, F)$ and $e_i \in E'_i$ for $1 \le i \le n$, we have

$$
\begin{aligned}
\Big(\delta_{E'_1, \ldots, E'_n, F'} \circ \mathscr{L}_{\mathbb{Z}, R_1, \ldots, R_n}(f_1, \ldots, f_n, g)\Big)(u)(e_1)(e_2, \ldots, e_n) &= \\
= \delta_{E'_1, \ldots, E'_n, F'}\big(g \circ u \circ (f_1, \ldots, f_n)\big)(e_1)(e_2, \ldots, e_n) &= \\
= g\big(u((f_1, \ldots, f_n)(e_1, \ldots, e_n))\big) &= g\big(u(f_1(e_1), \ldots, f_n(e_n))\big)
\end{aligned}
$$

and

$$
\begin{aligned}
\Big(\mathrm{Hom}_{R_1}\Big(f_1, \mathscr{L}_{\mathbb{Z}, R_2, \ldots, R_n}(f_2, \ldots, f_n, g)\Big) \circ \delta_{E_1, \ldots, E_n, F}\Big)(u)(e_1)(e_2, \ldots, e_n) &= \\
= \Big(\mathscr{L}_{\mathbb{Z}, R_2, \ldots, R_n}(f_2, \ldots, f_n, g) \circ (\delta_{E_1, \ldots, E_n, F}(u)) \circ f_1\Big)(e_1)(e_2, \ldots, e_n) &= \\
= \mathscr{L}_{\mathbb{Z}, R_2, \ldots, R_n}(f_2, \ldots, f_n, g)\big(\delta_{E_1, \ldots, E_n, F}(u)(f_1(e_1))\big)(e_2, \ldots, e_n) &= \\
= g\big(\delta_{E_1, \ldots, E_n, F}(u)(f_1(e_1))(f_2(e_2), \ldots, f_n(e_n))\big) &= g\big(u(f_1(e_1), \ldots, f_n(e_n))\big).
\end{aligned}
$$

□

2.2.4 Remark. For a given $(R_0, \ldots, R_n)$-multilinear map $u : E_1 \times \cdots \times E_n \to F$, we will sometimes call the (R_0, R_1)-linear map $\hat{u} : E_1 \to \mathscr{L}_{\mathbb{Z}, R_2, \ldots, R_n}(E_2, \ldots, E_n, F)$ defined by $\hat{u}(e_1)(e_2, \ldots, e_n) = u(e_1, \ldots, e_n)$ the *associated* (R_0, R_1)-linear map for u, and the (R_{n-1}, R_n)-linear map $\check{u} : E_n \to \mathscr{L}_{R_0, \ldots, R_{n-2}, \mathbb{Z}}(E_1, \ldots, E_{n-1}, F)$ defined by $\check{u}(e_n)(e_1, \ldots, e_{n-1}) = u(e_1, \ldots, e_n)$ the *associated* (R_{n-1}, R_n)-linear map for u. Similarly, u is called *associated* to $\hat{u}$ and also to $\check{u}$.

2.3 The tensor product

2.3.1 DEFINITION. Let $R_0, \dots, R_n$ be rings. For $1 \le i \le n$ let E_i be an (R_{i-1}, R_i)-bimodule. The *tensor product of* $E_1, \dots, E_n$ is an (R_0, R_n)-bimodule, written as $\bigotimes\limits_{1\le i\le n} E_i$, together with an $(R_0, \dots, R_n)$-multilinear map $\otimes : \prod\limits_{1\le i\le n} E_i \to \bigotimes\limits_{1\le i\le n} E_i$ such that the following property is satisfied.

> For any (R_0, R_n)-bimodule G and any $(R_0, \dots, R_n)$-multilinear map $g : \prod\limits_{1\le i\le n} E_i \to G$ there is a unique (R_0, R_n)-linear map $f : \bigotimes\limits_{1\le i\le n} E_i \to G$ such that $f(x_1 \otimes \cdots \otimes x_n) = h(x_1, \dots, x_n)$ for all $x \in \prod\limits_{1\le i\le n} E_i$.

By abuse of language, we also call the module $\bigotimes\limits_{1\le i\le n} E_i$ (without the map $\otimes$) a tensor product of the modules $E_1, \dots, E_n$. The multilinear mapping $\otimes$ is however always understood to be defined.

For $1 \le i \le n$ let E_i and F_i be (R_{i-1}, R_i)-bimodules and $u_i : E_i \to F_i$ be an (R_{i-1}, R_i)-linear map. Suppose that $\bigotimes\limits_{1\le i\le n} E_i$ is a tensor product of $E_1, \dots, E_n$ and that $\bigotimes\limits_{1\le i\le n} F_i$ is a tensor product of $F_1, \dots, F_n$. Then the function $(x_1, \dots, x_n) \mapsto u_1(x_1) \otimes \cdots \otimes u_n(x_n)$ from $\prod\limits_{1\le i\le n} E_i$ to $\bigotimes\limits_{1\le i\le n} F_i$ is $(R_0, \dots, R_n)$-multilinear. The unique (R_0, R_n)-linear map u from $\bigotimes\limits_{1\le i\le n} E_i$ to $\bigotimes\limits_{1\le i\le n} F_i$ that satisfies

$$u(x_1 \otimes \cdots \otimes x_n) = u_1(x_1) \otimes \cdots \otimes u_n(x_n) \text{ for all } x \in \prod_{1\le i\le n} E_i$$

is called the *tensor product of the family* $(u_1, \dots, u_n)$ and is written as $u_1 \times \cdots \times u_n$ or $\bigtimes\limits_{1\le i\le n} u_i$.

It is well known that each family $(E_i)_{1\le i\le n}$ of (R_{i-1}, R_i)-bimodules has a tensor product. Furthermore, forming the tensor product of families of linear maps is functorial: If E_i, F_i and G_i are (R_{i-1}, R_i)-bimodules and if $u_i : E_i \to F_i$ and $v_i : F_i \to G_i$ is (R_{i-1}, R_i)-linear for $1 \le i \le n$, then $(v \circ u)(x_1 \otimes \cdots \otimes x_n) = v(u_1(x_1) \otimes \cdots \otimes u_n(x_n)) = (v_1 \circ u_1)(x_1) \otimes \cdots \otimes (v_n \circ u_n)(x_n)$ for all $x \in \prod\limits_{1\le i\le n} E_i$, therefore $\bigtimes\limits_{1\le i\le n} v_i \circ \bigtimes\limits_{1\le i\le n} u_i = \bigtimes\limits_{1\le i\le n} (v_i \circ u_i)$. For each choice of tensor products of modules we obtain a *tensor product functor*

$$\bigtimes_{R_0,\dots,R_n} : {}_{R_0}\mathbf{M}_{R_1} \times \cdots \times {}_{R_{n-1}}\mathbf{M}_{R_n} \to {}_{R_0}\mathbf{M}_{R_n}.$$

Since we are not interested in the particular choice of tensor products but only in functors up to natural isomorphism, we will speak of *the* tensor product functor.

2.3.2 PROPOSITION. *Let E_i be an (R_{i-1}, R_i)-bimodule for $1 \le i \le n$, where $n \ge 2$ is a natural number. Let H and K be (R_0, R_n)-bimodules and let $h : \prod\limits_{1\le i\le n} E_i \to H$ and $k : \prod\limits_{1\le i\le n} E_i \to K$ be $(R_0, \dots, R_n)$-multilinear. Assume that $\hat{h} : E_1 \to \mathop{\mathscr{L}}\limits_{\mathbb{Z},R_2,\dots,R_n}(E_2, \dots, E_n, H)$ and $\hat{k} : E_1 \to \mathop{\mathscr{L}}\limits_{\mathbb{Z},R_2,\dots,R_n}(E_2, \dots, E_n, K)$ are the associated (R_0, R_1)-linear functions for h and k respectively and that $\check{h} : E_n \to \mathop{\mathscr{L}}\limits_{R_0,\dots,R_{n-2},\mathbb{Z}}(E_1, \dots, E_{n-1}, H)$ and $\check{k} : E_n \to \mathop{\mathscr{L}}\limits_{R_0,\dots,R_{n-2},\mathbb{Z}}(E_1, \dots, E_{n-1}, K)$ are the associated (R_{n-1}, R_n)-linear functions for h and k respectively. For all (R_0, R_n)-linear maps $\varphi : H \to K$, we have*

$$\begin{aligned} \varphi \circ h = k \quad & \textit{if and only if} \\ \mathop{\mathscr{L}}_{\mathbb{Z},R_2,\dots,R_n}(E_2, \dots, E_n, \varphi) \circ \hat{h} = \hat{k} \quad & \textit{if and only if} \\ \mathop{\mathscr{L}}_{R_0,\dots,R_{n-2},\mathbb{Z}}(E_1, \dots, E_{n-1}, \varphi) \circ \check{h} = \check{k}. & \end{aligned}$$

In particular,

$$
\begin{aligned}
&(H,h) \text{ is a tensor product of } E_1,\dots,E_n \quad \textit{if and only if}\\
(H,\hat{h}) \text{ is an } &\underset{\mathbb{Z},R_2,\dots,R_n}{\mathscr{L}}(E_2,\dots,E_n,\_)\text{-initial morphism for } E_1 \quad \textit{if and only if}\\
(H,\check{h}) \text{ is an } &\underset{R_0,\dots,R_{n-2},\mathbb{Z}}{\mathscr{L}}(E_1,\dots,E_{n-1},\_)\text{-initial morphism for } E_n.
\end{aligned}
$$

$$
\begin{tikzcd}
 & H \arrow[dd, "\varphi"] \\
E_1\times\cdots\times E_n \arrow[ur, "h"] \arrow[dr, "k"'] & \\
 & K
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \underset{\mathbb{Z},R_2,\dots,R_n}{\mathscr{L}}(E_2,\dots,E_n,H) \arrow[dd, "\mathscr{L}(E_2,\dots,E_n,\varphi)"] \\
E_1 \arrow[ur, "\hat{h}"] \arrow[dr, "\hat{k}"'] & \\
 & \underset{\mathbb{Z},R_2,\dots,R_n}{\mathscr{L}}(E_2,\dots,E_n,K)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \underset{R_0,\dots,R_{n-2},\mathbb{Z}}{\mathscr{L}}(E_1,\dots,E_{n-1},H) \arrow[dd, "\mathscr{L}(E_1,\dots,E_{n-1},\varphi)"] \\
E_n \arrow[ur, "\check{h}"] \arrow[dr, "\check{k}"'] & \\
 & \underset{R_0,\dots,R_{n-2},\mathbb{Z}}{\mathscr{L}}(E_1,\dots,E_{n-1},K)
\end{tikzcd}
$$

Proof. The statement that $\underset{\mathbb{Z},R_2,\dots,R_n}{\mathscr{L}}(E_2,\dots,E_n,\varphi)\circ\hat{h}=\hat{k}$ is equivalent to saying that $k(e_1,\dots,e_n) = \hat{k}(e_1)(e_2,\dots,e_n) = \Big(\underset{\mathbb{Z},R_2,\dots,R_n}{\mathscr{L}}(E_2,\dots,E_n,\varphi)\circ\hat{h}\Big)(e_1)(e_2,\dots,e_n) = \big(\varphi\circ(\hat{h}(e_1))\big)(e_2,\dots,e_n) = \varphi(\hat{h}(e_1)(e_2,\dots,e_n)) = \varphi(h(e_1,\dots,e_n)) = (\varphi\circ h)(e_1,\dots,e_n)$ for all $(e_1,\dots,e_n)\in E_1\times\cdots\times E_n$. Also, $\underset{R_0,\dots,R_{n-2},\mathbb{Z}}{\mathscr{L}}(E_1,\dots,E_{n-1},\varphi)\circ\check{h}=\check{k}$ if and only if $k(e_1,\dots,e_n)=\check{k}(e_n)(e_1,\dots,e_{n-1}) = \Big(\underset{R_0,\dots,R_{n-2},\mathbb{Z}}{\mathscr{L}}(E_1,\dots,E_{n-1},\varphi)\circ\check{k}\Big)(e_n)(e_1,\dots,e_{n-1}) = \big(\varphi\circ(\check{h}(e_n))\big)(e_1,\dots,e_{n-1}) = \varphi(\check{h}(e_n)(e_1,\dots,e_{n-1})) = \varphi(h(e_1,\dots,e_n)) = (\varphi\circ h)(e_1,\dots,e_n)$ for all $e_i\in E_i$ $(1\le i\le n)$.

□

We have seen that the tensor products of modules can be seen as initial morphisms. Also, the tensor product $f\times\mathrm{id}_F$ of a linear map $f: E\to E'$ with a module coincides with the "mediating morphism" from $E\otimes F$ to $E'\otimes F$, as the following remark shows.

2.3.3 Remark. Assume that $n\ge 2$ is a natural number. Let E_i be an (R_{i-1},R_i)-bimodule for $2\le i\le n$ and let $f: E_1\to E_1'$ be a homomorphism of (R_0,R_1)-bimodules.

Let $\otimes: E_1\times\cdots\times E_n\to E_1\underset{R_1}{\otimes}\cdots\underset{R_{n-1}}{\otimes}E_n$ and $\otimes': E_1'\times E_2\times\cdots\times E_n\to E_1'\underset{R_1}{\otimes}E_2\underset{R_2}{\otimes}\cdots\underset{R_{n-1}}{\otimes}E_n$ be the canonical $(R_0,\dots,R_n)$-multilinear mappings. The (R_0,R_n)-linear map $f\times\mathrm{id}_{E_1}\times\cdots\times\mathrm{id}_{E_n}$ from $E_1\underset{R_1}{\otimes}\cdots\underset{R_{n-1}}{\otimes}E_n$ to $E_1'\underset{R_1}{\otimes}E_2\underset{R_2}{\otimes}\cdots\underset{R_{n-1}}{\otimes}E_n$ is the only one with the property that $f\times\mathrm{id}_{E_1}\times\cdots\times\mathrm{id}_{E_n}(e_1\otimes\cdots\otimes e_n) = f(e_1)\otimes' e_2\otimes'\cdots\otimes' e_n$ for all $e_i\in E_i$ $(1\le i\le n)$.

$$
\begin{tikzcd}
E_1\underset{R_1}{\otimes}E_2\underset{R_2}{\otimes}\cdots\underset{R_{n-1}}{\otimes}E_n \arrow[rr, "f\times\mathrm{id}_{E_2}\times\cdots\times\mathrm{id}_{E_n}"] & & E_1'\underset{R_1}{\otimes}E_2\underset{R_2}{\otimes}\cdots\underset{R_{n-1}}{\otimes}E_n \\
E_1\times E_2\times\cdots\times E_n \arrow[u, "\otimes"] \arrow[rr, "(f,\mathrm{id}_{E_1},\dots,\mathrm{id}_{E_n})"'] & & E_1'\times E_2\times\cdots\times E_n \arrow[u, "\otimes'"']
\end{tikzcd}
$$

If we denote by η and η' the (R_0,R_1)-linear mappings $\eta: E_1\to\underset{\mathbb{Z},R_2,\dots,R_n}{\mathscr{L}}(E_2,\dots,E_n,E_1\underset{R_1}{\otimes}\cdots\underset{R_{n-1}}{\otimes}E_n)$ and $\eta': E_1'\to\underset{\mathbb{Z},R_2,\dots,R_n}{\mathscr{L}}(E_2,\dots,E_n,E_1'\underset{R_1}{\otimes}E_2\underset{R_2}{\otimes}\cdots\underset{R_{n-1}}{\otimes}E_n)$ associated to $\otimes$ and $\otimes'$ respectively, then $\eta'\circ f$ is the (R_0,R_1)-linear map associated to $(e_i)_{1\le i\le n}\mapsto f(e_1)\otimes' e_2\otimes'\cdots\otimes' e_n$, as is easy to see.

From Proposition 2.3.2 it follows that $f\times\mathrm{id}_{E_2}\times\cdots\times\mathrm{id}_{E_n}$ is also the unique (R_0,R_n)-linear mapping $u: E_1\underset{R_1}{\otimes}\cdots\underset{R_{n-1}}{\otimes}E_n\to E_1'\underset{R_1}{\otimes}E_2\underset{R_2}{\otimes}\cdots\underset{R_{n-1}}{\otimes}E_n$ with the property that $\underset{\mathbb{Z},R_2,\dots,R_n}{\mathscr{L}}(E_2,\dots,E_n,u)\circ\eta=\eta'\circ f$.

$$\begin{tikzcd}
\underset{\mathbb{Z},R_2,\dots,R_n}{\mathscr{L}}(E_2,\dots,E_n,E_1 \underset{R_1}{\otimes} \cdots \underset{R_{n-1}}{\otimes} E_n) \arrow[r, "{\mathscr{L}(E_2,\dots,E_n,u)}"] & \underset{\mathbb{Z},R_2,\dots,R_n}{\mathscr{L}}(E_2,\dots,E_n,E_1' \underset{R_1}{\otimes} E_2 \underset{R_2}{\otimes} \cdots \underset{R_{n-1}}{\otimes} E_n) \\
E \arrow[u, "\eta"] \arrow[r, "f"'] & E' \arrow[u, "\eta'"']
\end{tikzcd}$$

Similarly, if E_i is an (R_{i-1},R_i)-bimodule for $1 \le i \le n-1$ and if $g : E_n \to E_n'$ is a homomorphism of (R_{n-1},R_n)-bimodules, the (R_0,R_n)-linear map $u = \mathrm{id}_{E_1} \times \cdots \times \mathrm{id}_{E_{n-1}} \times g : E_1 \underset{R_1}{\otimes} \cdots \underset{R_{n-2}}{\otimes} E_{n-1} \underset{R_{n-1}}{\otimes} E_n \to E_1 \underset{R_1}{\otimes} \cdots \underset{R_{n-2}}{\otimes} E_{n-1} \underset{R_{n-1}}{\otimes} E_n'$ is the only one with the property that $\underset{R_0,\dots,R_{n-2},\mathbb{Z}}{\mathscr{L}}(E_1,\dots,E_{n-1},u) \circ \theta = \theta' \circ g$, where θ and θ' are the associated (R_{n-1},R_n)-linear maps for the canonical $(R_0,\dots,R_n)$-multilinear maps $\otimes$ and $\otimes'$.

The tensor product as it is usually defined therefore gives rise to the following adjunctions.
Assume that E_i is an (R_{i-1},R_i)-bimodule for $2 \le i \le n$. For each (R_0,R_1)-bimodule E_1, let $\eta(E_1) : E_1 \to \underset{\mathbb{Z},R_2,\dots,R_n}{\mathscr{L}}(E_2,\dots,E_n,E_1 \underset{R_1}{\otimes} \cdots \underset{R_{n-1}}{\otimes} R_n)$ be the associated (R_0,R_1)-linear map for $\otimes : E_1 \times \cdots \times E_n \to E_1 \underset{R_1}{\otimes} \cdots \underset{R_{n-1}}{\otimes} E_n$. Then η is a natural transformation from the identity functor on ${}_{R_0}\mathbf{M}_{R_1}$ to $\underset{\mathbb{Z},R_2,\dots,R_n}{\mathscr{L}}(E_2,\dots,E_n,\_ \underset{R_1}{\otimes} E_2 \underset{R_2}{\otimes} \cdots \underset{R_{n-1}}{\otimes} R_n)$. By Corollary 1.4.12, η is the unit of an adjunction $\_ \underset{R_1}{\otimes} E_2 \underset{R_2}{\otimes} \cdots \underset{R_{n-1}}{\otimes} E_n \dashv \underset{\mathbb{Z},R_2,\dots,R_n}{\mathscr{L}}(E_2,\dots,E_n,\_)$.

If E_1 is an (R_{i-1},R_i)-bimodule for $1 \le i \le n-1$, the canonical bilinear maps $\otimes : E_1 \times \cdots \times E_n \to E_1 \underset{R_1}{\otimes} \cdots \underset{R_{n-1}}{\otimes} E_n$ (where E_n is an (R_{n-1},R_n)-bimodule) give rise to a natural transformation θ from the identity functor on ${}_{R_{n-1}}\mathbf{M}_{R_n}$ to $\underset{R_0,\dots,R_{n-2},\mathbb{Z}}{\mathscr{L}}(E_1,\dots,E_{n-1},E_1 \underset{R_1}{\otimes} \cdots \underset{R_{n-2}}{\otimes} E_{n-1} \underset{R_{n-1}}{\otimes} \_)$ such that it is the unit of an adjunction $E_1 \underset{R_1}{\otimes} \cdots \underset{R_{n-2}}{\otimes} E_{n-1} \underset{R_{n-1}}{\otimes} \_ \dashv \underset{R_0,\dots,R_{n-2},\mathbb{Z}}{\mathscr{L}}(E_1,\dots,E_{n-1},\_)$.

2.3.4 Theorem. *Let R_i be a ring for $0 \le i \le n$, where $n \ge 2$ is a natural number. For all (R_{i-1},R_i)-bimodules E_i (where $1 \le i \le n$) and for all (R_0,R_n)-bimodules F, define a function $\mu_{E_1,\dots,E_n,F}$ by*

$$\mu_{E_1,\dots,E_n,F} : {}_{R_0}\mathrm{Hom}_{R_n}\Big(\bigotimes_{1\le i\le n} E_i, F\Big) \to \underset{R_0,\dots,R_n}{\mathscr{L}}(E_1,\dots,E_n,F),$$
$$\mu_{E_1,\dots,E_n,F}(u)(e_1,\dots,e_n) = u(e_1 \otimes \cdots \otimes e_n).$$

The functions $\mu_{E_1,\dots,E_n,F}$ are isomorphisms of $\mathbb{Z}$-modules. Furthermore, they define a natural isomorphism between functors from ${}_{R_0}\mathbf{M}^{\mathrm{op}}_{R_1} \times \cdots \times {}_{R_{n-1}}\mathbf{M}^{\mathrm{op}}_{R_n} \times {}_{R_0}\mathbf{M}_{R_n}$ to the category $\mathbf{Ab}$ of Abelian groups:

$$\mu : {}_{R_0}\mathrm{Hom}_{R_n}\big(\underset{1}{\_} \underset{R_1}{\otimes} \cdots \underset{R_{n-1}}{\otimes} \underset{n}{\_}, =\big) \to \underset{R_0,\dots,R_n}{\mathscr{L}}\big(\underset{1}{\_},\dots,\underset{n}{\_}, =\big).$$

Proof. The $\mathbb{Z}$-linearity of $\mu_{E_1,\dots,E_n,F}$ is obvious as soon as well-definedness has been established. By Remark 2.3.3, we have the adjunction $\_ \underset{R_1}{\otimes} E_2 \underset{R_2}{\otimes} \cdots \underset{R_{n-1}}{\otimes} E_n \dashv \underset{\mathbb{Z},R_2,\dots,R_n}{\mathscr{L}}(E_2,\dots,E_n,\_)$ having the unit η defined by $\eta(E_1)(e_1)(e_2,\dots,e_n) = e_1 \otimes \cdots \otimes e_n$ for (R_0,R_1)-bimodules E_1 and $e_i \in E_i$ for $1 \le i \le n$. For (R_0,R_n)-linear maps $u : \bigotimes_{1\le i\le n} E_i \to F$, define $\varphi_{E_1,\dots,E_n,F}(u) = \underset{\mathbb{Z},R_2,\dots,R_n}{\mathscr{L}}(E_2,\dots,E_n,u)\circ\eta(E_1)$. According to Theorem 1.4.15, the function $(E_1,F) \mapsto \varphi_{E_1,\dots,E_n,F}$ is a natural isomorphism from ${}_{R_0}\mathrm{Hom}_{R_n}\big(\_ \underset{R_1}{\otimes} E_2 \underset{R_2}{\otimes} \cdots \underset{R_{n-1}}{\otimes} E_n, =\big)$ to ${}_{R_0}\mathrm{Hom}_{R_1}\Big(\_, \underset{\mathbb{Z},R_2,\dots,R_n}{\mathscr{L}}(E_2,\dots,E_n,=)\Big)$. We show that φ is natural in the remaining variables too. The definition of $\varphi_{E_1,\dots,E_n,F}$ reads as $\varphi_{E_1,\dots,E_n,F}(u)(e_1)(e_2,\dots,e_n) = u(e_1 \otimes \cdots \otimes e_n)$, where $u : \bigotimes_{1\le i\le n} E_i \to F$ is (R_0,R_n)-linear and $e_i \in E_i$ for $1 \le i \le n$.

Let E_1 be an (R_0, R_1)-bimodule, let F be an (R_0, R_n)-bimodule and let $f_i : E_i' \to E_i$ be a morphism of (R_{i-1}, R_i)-bimodules for $2 \le i \le n$. Let $f_1 : E_1 \to E_1$ be the identity. Then for all $e_i \in E_i$ and $1 \le i \le n$, we have $\big(\varphi_{E_1,E_2',\dots,E_n',F} \circ \mathrm{Hom}(f_1 \times \cdots \times f_n, F)\big)(u)(e_1)(e_2,\dots,e_n) = \varphi_{E_1,E_2',\dots,E_n',F}(\mathrm{Hom}(f_1 \times \cdots \times f_n, F)(u))(e_1)(e_2,\dots,e_n) = \mathrm{Hom}(f_1 \times \cdots \times f_n, F)(u)(e_1 \otimes \cdots \otimes e_n) = (u \circ (f_1 \times \cdots \times f_n))(e_1 \otimes \cdots \otimes e_n) = u(f_1(e_1) \otimes \cdots \otimes f_n(e_n))$ and

$$\begin{aligned}
&\big(\mathrm{Hom}\big(E_1, \mathscr{L}(f_2,\dots,f_n,F)\big) \circ \varphi_{E_1,\dots,E_n,F}\big)(u)(e_1)(e_2,\dots,e_n) = \\
&\qquad = \mathrm{Hom}\big(E_1, \mathscr{L}(f_2,\dots,f_n,F)\big)(\varphi_{E_1,\dots,E_n,F}(u))(e_1)(e_2,\dots,e_n) = \\
&= \mathscr{L}(f_2,\dots,f_n,F)(\varphi_{E_1,\dots,E_n,F}(u)(e_1))(e_2,\dots,e_n) = (\varphi_{E_1,\dots,E_n,F}(u)(e_1) \circ (f_2,\dots,f_n))(e_2,\dots,e_n) = \\
&\qquad = \varphi_{E_1,\dots,E_n,F}(u)(e_1)(f_2(e_2),\dots,f_n(e_n)) = u(f_1(e_1) \otimes \cdots \otimes f_n(e_n)).
\end{aligned}$$

By Lemma 1.7.1, φ is a natural isomorphism. The claim now follows from the associativity of multilinear maps, Proposition 2.2.3. □

2.4 Associativity of the tensor product

We give a proof of the associativity of the tensor product in terms of initial morphisms. Let $n \ge 3$ and assume that E_i is an (R_{i-1}, R_i)-bimodule for $1 \le i \le n$. Set $F_2 = E_1 \underset{R_1}{\otimes} E_2$ and $F_i = E_i$ for $3 \le i \le n$. We have the canonical mappings

$$\begin{aligned}
&\varphi_1 : E_1 \to \mathrm{Hom}_{R_2}(E_2, E_1 \underset{R_1}{\otimes} E_2), && \varphi_1(e_1)(e_2) = e_1 \otimes e_2 \\
&\varphi_2 : E_1 \underset{R_1}{\otimes} E_2 \to \underset{\mathbb{Z},R_3,\dots,R_n}{\mathscr{L}}(E_3,\dots,E_n, \bigotimes_{2\le i\le n} F_i), && \varphi_2(x)(e_3,\dots,e_n) = x \otimes e_3 \otimes \cdots \otimes e_n \\
&\varphi_3 : E_1 \to \underset{\mathbb{Z},R_2,\dots,R_n}{\mathscr{L}}\big(E_2,\dots,E_n, \bigotimes_{1\le i\le n} E_i\big), && \varphi_3(e_1)(e_2,\dots,e_n) = e_1 \otimes \cdots \otimes e_n.
\end{aligned}$$

By Proposition 2.3.2, the pair $\big(E_1 \underset{R_1}{\otimes} E_2, \varphi_1\big)$ is an initial morphism for E_1 with respect to $\mathrm{Hom}_{R_2}(E_2, \_)$, the pair $\Big(\bigotimes_{2\le i\le n} F_i, \varphi_2\Big)$ is an initial morphism for $E_1 \underset{R_1}{\otimes} E_2$ with respect to $\underset{\mathbb{Z},R_3,\dots,R_n}{\mathscr{L}}(E_3,\dots,E_n,\_)$ and the pair $\Big(\bigotimes_{1\le i\le n} E_i, \varphi_3\Big)$ is an initial morphism for E_1 with respect to $\underset{\mathbb{Z},R_2,\dots,R_n}{\mathscr{L}}(E_2,\dots,E_n,\_)$. By composition of initial morphisms, the (R_0, R_n)-bimodule $(E_1 \underset{R_1}{\otimes} E_2) \underset{R_2}{\otimes} \cdots \underset{R_{n-1}}{\otimes} E_n$ together with the (R_0, R_1)-linear map $\mathrm{Hom}_{R_2}(E_2, \varphi_2) \circ \varphi_1$ is an initial morphism for E_1 with respect to $\mathrm{Hom}_{R_2}\big(E_2, \underset{\mathbb{Z},R_3,\dots,R_n}{\mathscr{L}}(E_3,\dots,E_n,\_)\big)$.

If $\sigma : \underset{\mathbb{Z},R_2,\dots,R_n}{\mathscr{L}}\big(E_2,\dots,E_n, \bigotimes_{1\le i\le n} E_i\big) \to \mathrm{Hom}_{R_2}\Big(E_2, \underset{\mathbb{Z},R_3,\dots,R_n}{\mathscr{L}}\big(E_3,\dots,E_n, \bigotimes_{1\le i\le n} E_i\big)\Big)$ is the isomorphism given by $\sigma(u)(e_2)(e_3,\dots,e_n) = u(e_2,\dots,e_n)$, then by Proposition 2.2.3 and Corollary 1.4.5 also the (R_0, R_n)-bimodule $\bigotimes_{1\le i\le n} E_i$ together with the (R_0, R_1)-linear map $\sigma \circ \varphi_3$ is an initial morphism for E_1 with respect to $\mathrm{Hom}_{R_2}\big(E_2, \underset{\mathbb{Z},R_3,\dots,R_n}{\mathscr{L}}(E_3,\dots,E_n,\_)\big)$.

There is therefore a unique isomorphism $\psi_{E_1,\dots,E_n}$ from $\bigotimes_{1\le i\le n} E_i$ onto $(E_1 \underset{R_1}{\otimes} E_2) \underset{R_2}{\otimes} \cdots \underset{R_{n-1}}{\otimes} E_n$ with the property that $\mathrm{Hom}_{R_2}(E_2, \varphi_2) \circ \varphi_1 = \mathrm{Hom}_{R_2}\Big(E_2, \underset{\mathbb{Z},R_2,\dots,R_n}{\mathscr{L}}(E_2,\dots,E_n, \psi_{E_1,\dots,E_n})\Big) \circ \sigma \circ \varphi_3$, in other words, such that for all $(e_1, \ldots, e_n) \in E_1 \times \cdots \times E_n$

$$\begin{aligned}
(e_1 \otimes e_2) \otimes e_3 \otimes \cdots \otimes e_n &= \varphi_2(e_1 \otimes e_2)(e_3, \ldots, e_n) = \varphi_2(\varphi_1(e_1)(e_2))(e_3, \ldots, e_n) = \\
&= (\varphi_2 \circ (\varphi_1(e_1)))(e_2)(e_3, \ldots, e_n) = \big(\mathrm{Hom}_{R_2}(E_2, \varphi_2) \circ \varphi_1\big)(e_1)(e_2)(e_3, \ldots, e_n) = \\
&= \Big(\mathrm{Hom}_{R_2}\Big(E_2, \underset{\mathbb{Z}, R_2, \ldots, R_n}{\mathscr{L}}(E_2, \ldots, E_n, \psi_{E_1, \ldots, E_n})\Big) \circ \sigma \circ \varphi_3\Big)(e_1)(e_2)(e_3, \ldots, e_n) = \\
&= \Big(\underset{\mathbb{Z}, R_2, \ldots, R_n}{\mathscr{L}}(E_2, \ldots, E_n, \psi_{E_1, \ldots, E_n}) \circ (\sigma(\varphi_3(e_1)))\Big)(e_2)(e_3, \ldots, e_n) = \\
&= \big(\psi_{E_1, \ldots, E_n} \circ (\sigma(\varphi_3(e_1))(e_2))\big)(e_3, \ldots, e_n) = \psi_{E_1, \ldots, E_n}(\sigma(\varphi_3(e_1))(e_2))(e_3, \ldots, e_n) = \\
&= \psi_{E_1, \ldots, E_n}(\sigma(\varphi_3(e_1))(e_2)(e_3, \ldots, e_n)) = \psi_{E_1, \ldots, E_n}(\varphi_3(e_1)(e_2, \ldots, e_n)) = \psi_{E_1, \ldots, E_n}(e_1 \otimes \ldots \otimes e_n).
\end{aligned}$$

Furthermore, this isomorphism is natural in all n variables: Let $u_i : E_i \to E_i'$ be (R_{i-1}, R_i)-linear for $1 \leq i \leq n$. Then $((u_1 \times u_2) \times u_3 \times \cdots \times u_n) \circ \psi_{E_1, \ldots, E_n}$ and $\psi_{E_1', \ldots, E_n'} \circ (u_1 \times \cdots \times u_n)$ are two (R_0, R_n)-linear functions s from $E_1 \underset{R_1}{\otimes} \cdots \underset{R_{n-1}}{\otimes} E_n$ to $(E_1 \underset{R_1}{\otimes} E_2) \underset{R_2}{\otimes} E_3 \underset{R_3}{\otimes} \cdots \underset{R_{n-1}}{\otimes} E_n$ with the property that $s(e_1 \otimes \cdots \otimes e_n) = (u_1(e_1) \otimes u_2(e_2)) \otimes u_3(e_3) \otimes \cdots \otimes u_n(e_n)$ for all $(e_1, \ldots, e_n) \in \prod_{1 \leq i \leq n} E_i$. Since the function $(e_1, \ldots, e_n) \mapsto (u_1(e_1) \otimes u_2(e_2)) \otimes u_3(e_3) \otimes \cdots \otimes u_n(e_n)$ is $(R_0, \ldots, R_n)$- multilinear, the universal property of the tensor product gives us $((u_1 \times u_2) \times u_3 \times \cdots \times u_n) \circ \psi_{E_1, \ldots, E_n} = \psi_{E_1', \ldots, E_n'} \circ (u_1 \times \cdots \times u_n)$. We have therefore proved the first part of the following theorem. The second part follows from an analogous argument.

2.4.1 Theorem. *Let $n \geq 3$ be a natural number and let $R_0, \ldots, R_n$ be rings.*

(1) *For each family $(E_i)_{1 \leq i \leq n}$, where E_i is an (R_{i-1}, R_i)-bimodule for $1 \leq i \leq n$, there is a unique (R_0, R_n)-linear function $\psi_{E_1, \ldots, E_n} : \bigotimes_{1 \leq i \leq n} E_i \to (E_1 \underset{R_1}{\otimes} E_2) \underset{R_2}{\otimes} E_3 \underset{R_3}{\otimes} \cdots \underset{R_{n-1}}{\otimes} E_n$ such that $\psi_{E_1, \ldots, E_n}(e_1 \otimes \cdots \otimes e_n) = (e_1 \otimes e_2) \otimes e_3 \otimes \cdots \otimes e_n$ for all $(e_1, \ldots, e_n) \in E_1 \times \cdots \times E_n$. The function $(E_1, \ldots, E_n) \mapsto \psi_{E_1, \ldots, E_n}$ is a natural isomorphism from $\underset{1}{\_} \times \underset{2}{\_} \times \underset{3}{\_} \times \cdots \times \underset{n}{\_}$ to $(\underset{1}{\_} \times \underset{2}{\_}) \times \underset{3}{\_} \times \cdots \times \underset{n}{\_}$, both of which are functors from ${}_{R_0}\mathbf{M}_{R_1} \times \cdots \times {}_{R_{n-1}}\mathbf{M}_{R_n}$ to ${}_{R_0}\mathbf{M}_{R_n}$.*

$$\begin{tikzcd}
E_1 \underset{R_1}{\otimes} \cdots \underset{R_{n-1}}{\otimes} E_n \arrow[r, "\psi_{E_1, \ldots, E_n}"] \arrow[d, "u_1 \times \cdots \times u_n"'] & (E_1 \underset{R_1}{\otimes} E_2) \underset{R_2}{\otimes} E_3 \underset{R_3}{\otimes} \cdots \underset{R_{n-1}}{\otimes} E_n \arrow[d, "(u_1 \times u_2) \times u_3 \times \cdots \times u_n"] \\
E_1' \underset{R_1}{\otimes} \cdots \underset{R_{n-1}}{\otimes} E_n' \arrow[r, "\psi_{E_1', \ldots, E_n'}"'] & (E_1' \underset{R_1}{\otimes} E_2') \underset{R_2}{\otimes} E_3' \underset{R_3}{\otimes} \cdots \underset{R_{n-1}}{\otimes} E_n'
\end{tikzcd}$$

(2) *For each family $(E_i)_{1 \leq i \leq n}$, where E_i is an (R_{i-1}, R_i)-bimodule for $1 \leq i \leq n$, there is a unique (R_0, R_n)-linear function $\psi_{E_1, \ldots, E_n} : \bigotimes_{1 \leq i \leq n} E_i \to E_1 \underset{R_1}{\otimes} \cdots \underset{R_{n-3}}{\otimes} E_{n-2} \underset{R_{n-2}}{\otimes} (E_{n-1} \underset{R_{n-1}}{\otimes} E_n)$ such that $\psi_{E_1, \ldots, E_n}(e_1 \otimes \cdots \otimes e_n) = e_1 \otimes \cdots \otimes e_{n-2} \otimes (e_{n-1} \otimes e_n)$ for all $(e_1, \ldots, e_n) \in E_1 \times \cdots \times E_n$. The function $(E_1, \ldots, E_n) \mapsto \psi_{E_1, \ldots, E_n}$ is a natural isomorphism from $\underset{1}{\_} \times \cdots \times \underset{n-2}{\_} \times \underset{n-1}{\_} \times \underset{n}{\_}$ to $\underset{1}{\_} \times \cdots \times \underset{n-2}{\_} \times (\underset{n-1}{\_} \times \underset{n}{\_})$, both of which are functors from ${}_{R_0}\mathbf{M}_{R_1} \times \cdots \times {}_{R_{n-1}}\mathbf{M}_{R_n}$ to ${}_{R_0}\mathbf{M}_{R_n}$.*

2.4.2 Corollary. *Suppose that $n \geq 3$ is a natural number and that R_i is a ring for $0 \leq i \leq n$. Let $2 \leq i \leq n-1$. There is a natural isomorphism ψ from the functor $\underset{1}{\_} \times \cdots \times \underset{n}{\_}$ to the functor $(\underset{1}{\_} \times \cdots \times \underset{i}{\_}) \times \underset{i+1}{\_} \times \cdots \times \underset{n}{\_}$ having the property that for all (R_{i-1}, R_i) modules E_i and $e_i \in E_i$ (where $1 \leq i \leq n$), we have $\psi_{E_1, \ldots, E_n}(e_1 \otimes \cdots \otimes e_n) = (e_1 \otimes \cdots \otimes e_i) \otimes e_{i+1} \otimes \cdots \otimes e_n$.*

Also, there is a natural isomorphism ψ from the functor $\underset{1}{\_} \times \cdots \times \underset{n}{\_}$ to the functor $\underset{1}{\_} \times \cdots \times \underset{i-1}{\_} \times$

$(\underset{i}{\_} \times \cdots \times \underset{n}{\_})$ *having the property that for all* (R_{i-1}, R_i) *modules* E_i *and* $e_i \in E_i$ *(where* $1 \leq i \leq n$*), we have* $\psi_{E_1,\ldots,E_n}(e_1 \otimes \cdots \otimes e_n) = e_1 \otimes \cdots \otimes e_{i-1} \otimes (e_i \otimes \cdots \otimes e_n)$.

Proof. By induction on n, it is easy to show that the statement holds for $i = 2$ and $i = n-1$. As a second step, apply induction on i for fixed n to obtain the general result. We leave the details to the reader. $\square$

2.5 Tensor products preserve colimits

Let $n \geq 2$ and let R_i be a ring for $1 \leq i \leq n$. By Remark 2.3.3 we have the following adjunctions: $\_ \underset{R_1}{\otimes} E_2 \underset{R_2}{\otimes} \cdots \underset{R_{n-1}}{\otimes} E_n \dashv \underset{\mathbb{Z},R_2,\ldots,R_n}{\mathscr{L}}(E_2, \ldots, E_n, \_)$ for all (R_{i-1}, R_i)-bimodules E_i $(2 \leq i \leq n)$ and $E_1 \underset{R_1}{\otimes} \cdots \underset{R_{n-2}}{\otimes} E_{n-1} \underset{R_{n-1}}{\otimes} \_ \dashv \underset{R_0,\ldots,R_{n-2},\mathbb{Z}}{\mathscr{L}}(E_1, \ldots, E_{n-1}, \_)$ for all (R_{i-1}, R_i)-bimodules E_i $(1 \leq i \leq n-1)$. From Theorem 1.5.20 it follows that the functors $\_ \underset{R_1}{\otimes} E_2 \underset{R_2}{\otimes} \cdots \underset{R_{n-1}}{\otimes} E_n$ and $E_1 \underset{R_1}{\otimes} \cdots \underset{R_{n-2}}{\otimes} E_{n-1} \underset{R_{n-1}}{\otimes} \_$ are cocontinuous. (Note that these adjunctions also give an alternative proof of the fact that the functors ${}^S_R\mathrm{Hom}^T(E, \_) : {}_R\mathbf{M}_T \to {}_S\mathbf{M}_T$ for an (R,S)-bimodule E and ${}^R\mathrm{Hom}^S_T(E, \_) : {}_R\mathbf{M}_T \to {}_R\mathbf{M}_S$ for an (S,T)-bimodule F are continuous.) From Theorem 1.7.2 we may deduce the following corollary.

2.5.1 COROLLARY. *Let* I *and* J *be small categories. If* R*,* S *and* T *are rings, then the functors from* $({}_R\mathbf{M}_S)^I \times ({}_S\mathbf{M}_T)^J$ *to* ${}_R\mathbf{M}_T$ *defined by*

$$(F,G) \mapsto \operatorname*{colim}_{I \times J}\big((i,j) \mapsto F_i \underset{S}{\otimes} G_j\big) \quad \text{and}$$
$$(F,G) \mapsto \operatorname*{colim}_{I}(F) \underset{S}{\otimes} \operatorname*{colim}_{J}(G) \qquad \text{are naturally isomorphic.}$$

With the aid of Corollary 2.4.2 we see that the tensor product is cocontinuous not only in the leftmost and rightmost argument, but in each argument.

2.5.2 COROLLARY. *Let* $R_0, \ldots, R_n$ *be rings and let* E_i *be an* (R_{i-1}, R_i)*-bimodule for* $1 \leq i \leq n$*. Assume that* $1 \leq k \leq n$*. The functor*

$$E_1 \underset{R_1}{\otimes} \cdots \underset{R_{k-2}}{\otimes} E_{k-1} \underset{R_{k-1}}{\otimes} \_ \underset{R_k}{\otimes} E_{k+1} \underset{R_{k+1}}{\otimes} \cdots \underset{R_{n-1}}{\otimes} E_n : \quad {}_{R_{k-1}}\mathbf{M}_{R_k} \to {}_{R_0}\mathbf{M}_{R_n}$$

is cocontinuous.

Proof. The case that $k = 1$ or $k = n$ is clear from the above.
Let $2 \leq k \leq n-1$. The functors $\_ \underset{R_k}{\otimes} E_{k+1} \underset{R_{k+1}}{\otimes} \cdots \underset{R_{n-1}}{\otimes} E_n$ and $E_0 \underset{R_1}{\otimes} \cdots \underset{R_{k-2}}{\otimes} E_{k-1} \underset{R_{k-1}}{\otimes} \_$ are cocontinuous, and so is the composite functor $E_0 \underset{R_1}{\otimes} \cdots \underset{R_{k-2}}{\otimes} E_{k-1} \underset{R_{k-1}}{\otimes} (\_ \underset{R_k}{\otimes} \cdots \underset{R_{n-1}}{\otimes} E_n)$. The latter is naturally isomorphic to the functor $E_1 \underset{R_1}{\otimes} \cdots \underset{R_{k-2}}{\otimes} E_{k-1} \underset{R_{k-1}}{\otimes} \_ \underset{R_k}{\otimes} E_{k+1} \underset{R_{k+1}}{\otimes} \cdots \underset{R_{n-1}}{\otimes} E_n$ by Corollary 2.4.2. The claim now follows from Remark 1.5.14. $\square$

3 Special cases

3.1 Associativity of products

Let I and J be sets, that is, small discrete categories, and let $(A_{i,j})_{(i,j)\in I\times J}$ be a family of sets. Then $A : (i,j) \mapsto A_{i,j}$ is a functor from $I \times J$ to **Set**. The product $\prod\limits_{(i,j)\in I\times J} A_{i,j}$ together with the family $(r_{i,j})_{(i,j)\in I\times J}$ of projections is a limit of A. By the theorem on the associativity of limits, Theorem 1.6.4, a limit of A can also be computed the following way: For $j \in J$, we have the product $\prod\limits_{i\in I} A_{i,j}$ of the family $(A_{i,j})_{i\in I}$ together with the family $(p_i^j)_{i\in I}$ of projections, that is, a limit of the functor $A(\_\,, j)$. Also, the product $\prod\limits_{j\in J}\prod\limits_{i\in I} A_{i,j}$ together with the family $(q_j)_{j\in J}$ of projections is a limit of $j \mapsto \prod\limits_{i\in I} A_{i,j}$. Consequently there exists a unique function $\varphi : \prod\limits_{j\in J}\prod\limits_{i\in I} A_{i,j} \to \prod\limits_{i\in I, j\in J} A_{i,j}$ such that $r_{i,j} \circ \varphi = p_i^j \circ q_j$ for all $(i,j) \in I \times J$. It is a bijection.

$$\begin{tikzcd}
\prod\limits_{i\in I} A_{i,j} \arrow[d, "p_i^j"'] & \prod\limits_{j\in J}\prod\limits_{i\in I} A_{i,j} \arrow[l, "q_j"'] \arrow[d, "\varphi"] \\
A_{i,j} & \prod\limits_{j\in J, i\in I} A_{i,j} \arrow[l, "r_{i,j}"]
\end{tikzcd}$$

Following the usual construction of products, the bijection φ has the property that $\varphi\big(((a_{i,j})_{i\in I})_{j\in J}\big) = (a_{i,j})_{(i,j)\in I\times J}$ for all $a_{i,j} \in A_{i,j}$ and $(i,j) \in I \times J$.

3.2 Associativity of quotients

Assume that N and K are normal subgroups of the group L (written multiplicatively) and that $N \subseteq K$. It is well known that K/N is a normal subgroup of L/N and that $(L/N)/(K/N)$ is isomorphic to L/K. We show that this result can be understood as a special case of the associativity of colimits.

Let $I = J$ be the category $a \underset{g}{\overset{f}{\rightrightarrows}} b$. Define a functor F from $I \times J$ to the category **Grp** of groups according to the following table, where ι_A^B denotes the embedding of A into B for groups A and B such that $A \subseteq B$ and 1_A^B denotes the homomorphism $a \mapsto 1$, for all groups A and B.

F	a	b	f	g
a	N	K	1_N^K	ι_N^K
b	N	L	1_N^L	ι_N^L
f	1_N^N	1_K^L	1_K^L	1_K^L
g	ι_N^N	ι_K^L	1_K^L	ι_N^L

A colimit of F can be computed in two different ways.

(1) First, we form colimits of the functors $F(a,\_)$ and $F(b,\_)$. The group K/N together with the functions $\eta_a^a = 1_N^{K/N}$ and $\eta_b^a = \mathrm{pr}_K^{K/N}$ is a colimit of $F(a,\_)$ and the group L/N together with $\eta_a^b = 1_N^{L/N}$ and $\eta_b^b = \mathrm{pr}_L^{L/N}$ is a colimit of $F(b,\_)$, see Example 1.5.4(3). Now, according to Corollary 1.5.6, $\operatorname{colim} F(f,\_)$ is the unique homomorphism c from K/N to L/N having the properties that $c\circ\eta_a^a = \eta_a^b \circ F(f,a)$ and $c\circ\eta_b^a = \eta_b^b\circ F(f,b)$, that is, $c\circ 1_N^{K/N} = 1_N^{L/N}\circ 1_N^N$ and $c\circ \mathrm{pr}_K^{K/N} = \mathrm{pr}_L^{L/N}\circ 1_K^L$. The homomorphism $1_{K/N}^{L/N}$ has this property. By analogous reasoning, we get $\operatorname{colim} F(g,\_) = \iota_{K/N}^{L/N}$. Employing Example 1.5.4(3) again, it follows that a colimit of the functor $k\mapsto \operatorname{colim} F(k,\_)$ (where $k\in \operatorname{Mor}(I) = \{a,b,f,g\}$) is given by the group $A = (L/N)/(K/N)$ together with $1_{K/N}^A$ and $\mathrm{pr}_{L/N}^A$. Now, by Corollary 1.6.5, the group A together with the four maps 1_N^A, 1_K^A, 1_N^A and $\mathrm{pr}_{L/N}^A\circ\mathrm{pr}_L^{L/N}$ is a colimit of F.

(2) The group N/N together with $\varphi_a^a = 1_N^{N/N}$ and $\varphi_b^a = 1_N^{N/N}$ is a colimit of $F(\_,a)$ and L/K together with $\varphi_a^b = 1_K^{L/K}$ and $\varphi_b^b = \mathrm{pr}_L^{L/K}$ is a colimit of $F(\_,b)$. The only homomorphism from N/N to L/K is $1_{N/N}^{L/K}$, hence $\operatorname{colim} F(\_,f) = \operatorname{colim} F(\_,g) = 1_{N/N}^{L/K}$. A colimit of the functor $k\mapsto\operatorname{colim} F(\_,k)$ (where $k\in\{a,b,f,g\}$) is L/K together with $1_{N/N}^{L/K}$ and $\mathrm{id}_{L/K}$. By Corollary 1.6.5, the group L/K together with the four maps $1_N^{L/K}$, $1_K^{L/K}$, $1_N^{L/K}$ and $\mathrm{pr}_L^{L/K}$ is a colimit of F.

There is therefore a unique isomorphism $\psi : L/K \to (L/N)/(K/N)$ such that $\psi\circ\mathrm{pr}_L^{L/K} = \mathrm{pr}_{L/N}^A\circ\mathrm{pr}_L^{L/N}$, that is, such that $\psi(aK) = (aN)(K/N)$ for all $a\in L$.

3.3 Direct sums and quotients commute

Let I be a set (considered as a discrete category) and let J be the category $a \overset{f}{\underset{g}{\rightrightarrows}} b$. Let $\mathscr{C}$ be a category of (bi)modules. Assume that $(E_i)_{i\in I}$ and $(F_i)_{i\in I}$ are families of $\mathscr{C}$-objects and that F_i is a submodule of E_i for $i\in I$. A functor H from $I\times J$ to $\mathscr{C}$ is defined by $H(i,a) = F_i$, $H(i,b) = E_i$, $H(i,f) = 0_{F_i}^{E_i}$ and $H(i,g) = \iota_{F_i}^{E_i}$ for $i\in I$. For each $i\in I$, the module E_i/F_i together with the linear maps $0_{F_1}^{E_i/F_i} : F_i \to E_i/F_i$ and $\mathrm{pr}_{E_i}^{E_i/F_i} : E_i\to E_i/F_i$ is a colimit of $H(i,\_)$. Moreover, since the direct sum of modules is the same as a coproduct (a colimit with respect to a discrete category), the module $G = \bigoplus_{i\in I}(E_i/F_i)$ together with the embeddings ι_{E_i/F_i}^G $(i\in I)$ is a colimit of the functor $i\mapsto \operatorname{colim} H(i,\_)$. It follows that G together with the linear maps $0_{F_i}^G$ and $\iota_{E_i/F_i}^G\circ\mathrm{pr}_{E_i}^{E_i/F_i}$ (where $i\in I$) is a colimit of G.
On the other hand, the module $A = \bigoplus_{i\in I} F_i$ together with the family $(\iota_{F_i}^A)_{i\in I}$ is a colimit of $G(\_,a)$ and the module $B = \bigoplus_{i\in I} E_i$ together with the family $(\iota_{E_i}^B)_{i\in I}$ is a colimit of $G(\_,b)$. The colimit of the natural transformation $G(\_,f)$ is the map 0_A^B, since $0_A^B\circ\iota_{F_i}^A = \iota_{E_i}^B\circ 0_{F_i}^{E_i}$ for all $i\in I$, and a colimit of $G(\_,g)$ is the map ι_A^B, since $\iota_A^B\circ\iota_{F_i}^A = \iota_{E_i}^B\circ\iota_{F_i}^{E_i}$ for all $i\in I$. Hence, by associativity of colimits, B/A together with the linear maps $0_{F_i}^{B/A}$ and $\mathrm{pr}_B^{B/A}\circ\iota_{E_i}^B$ (for $i\in I$) is a colimit of G.
It follows that there exists a unique isomorphism $\varphi : \bigoplus_{i\in I}(E_i/F_i) \to \big(\bigoplus_{i\in I}E_i\big)/\big(\bigoplus_{i\in I}F_i\big)$ having the property that $\varphi\Big(\iota_{E_i/F_i}^{\bigoplus(E_i/F_i)}(x+F_i)\Big) = \iota_{E_i}^{\bigoplus E_i}(x) + \bigoplus_{i\in I}F_i$ for all $i\in I$ and $x\in E_i$.

3.4 Pullbacks and products commute

Asume that $\mathscr{C}$ is a category and that the category J is of the form $\bullet \longrightarrow \bullet \longleftarrow \bullet$. A limit of a functor $F : J\to\mathscr{C}$ is called a *pullback*. By abuse of notation, we call a $\mathscr{C}$-object P together with morphisms

$p_1 : P \to A$ and $p_2 : P \to B$ a *pullback* of a pair (f_1, f_2) of morphisms such that $f_1 : A \to C$ and $f_2 : B \to C$ if the following property is satisfied:

> $f_1 \circ p_1 = f_2 \circ p_2$ and if Q is a $\mathscr{C}$ object and $q_1 : Q \to A$ and $q_2 : Q \to B$ are morphisms satisfying $f_1 \circ q_1 = f_2 \circ q_2$, there is a unique $\mathscr{C}$-morphism $\varphi : Q \to P$ such that $p_1 \circ \varphi = q_1$ and $p_2 \circ \varphi = q_2$.

$$\begin{tikzcd}
P \arrow[d, "p_1"'] \arrow[drr, "p_2" near start] & & Q \arrow[ll, "\varphi"'] \arrow[dll, "q_1"' near start] \arrow[d, "q_2"] \\
A \arrow[r, "f_1"'] & C & B \arrow[l, "f_2"]
\end{tikzcd}$$

We use the notation $\mathrm{PB}(f_1, f_2)$ to denote the object part P of the pullback of f_1 and f_2.
Let A, B, C and P be objects of $\mathscr{C}$ and let $f_1 : A \to C$, $f_2 : B \to C$, $p_1 : P \to A$ and $p_2 : P \to B$. Let a functor F from I to $\mathscr{C}$ be defined by the diagram $A \xrightarrow{f_1} C \xleftarrow{f_2} B$. Then the following equivalence holds: The object P together with (p_1, p_2) is a pullback of (f_1, f_2) if and only if P together with $(p_1, p_2, f_1 \circ p_1)$ (or, equivalently, $(p_1, p_2, f_2 \circ p_2)$) is a limit of F.
Using this fact and Example 1.5.4(1), we see that in the category **Set** a pullback of $f_1 : A \to C$ and $f_2 : B \to C$ is given by the set $P = \{(a, b) \in A \times B : f_1(a) = f_2(b)\}$ together with the projections $p_1 : P \to A$ and $p_2 : P \to B$.
Let $(A_i)_{i \in I}$, $(B_i)_{i \in I}$ and $(C_i)_{i \in I}$ be families of sets and let $f_i : A_i \to C_i$ and $g_i : B_i \to C_i$ be functions for $i \in I$. For $i \in I$, let $p_i : \mathrm{PB}(f_i, g_i) \to A_i$, $q_i : \mathrm{PB}(f_i, g_i) \to B_i$, $\pi_i : \prod\limits_{i \in I} \mathrm{PB}(f_i, g_i) \to \mathrm{PB}(f_i, g_i)$, $\varepsilon_i : \prod\limits_{i \in I} A_i \to A_i$ and $\eta_i : \prod\limits_{i \in I} B_i \to B_i$ be the canonical projections. Moreover, let $r : \mathrm{PB}\big(\prod\limits_{i \in I} f_i, \prod\limits_{i \in I} g_i\big) \to \prod\limits_{i \in I} A_i$ and $s : \mathrm{PB}\big(\prod\limits_{i \in I} f_i, \prod\limits_{i \in I} g_i\big) \to \prod\limits_{i \in I} B_i$ be the projections.

$$\begin{tikzcd}
 & \mathrm{PB}\big(\prod\limits_{i \in I} f_i, \prod\limits_{i \in I} g_i\big) \arrow[dl, "r"'] \arrow[d, "\varphi"] \arrow[dr, "s"] & \\
\prod\limits_{i \in I} A_i \arrow[d, "\varepsilon_i"'] & \prod\limits_{i \in I} \mathrm{PB}(f_i, g_i) \arrow[d, "\pi_i"] & \prod\limits_{i \in I} B_i \arrow[d, "\eta_i"] \\
A_i & \mathrm{PB}(f_i, g_i) \arrow[l, "p_i"] \arrow[r, "q_i"'] & B_i
\end{tikzcd}$$

By associativity of limits, there exists a unique function $\varphi : \mathrm{PB}\big(\prod\limits_{i \in I} f_i, \prod\limits_{i \in I} g_i\big) \to \prod\limits_{i \in I} \mathrm{PB}(f_i, g_i)$ such that $p_i \circ \pi_i \circ \varphi = \varepsilon_i \circ r$ and $q_i \circ \pi_i \circ \varphi = \eta_i \circ s$ for all $i \in I$. It is a bijection.

3.5 The free product of free groups is a free group

We generalize the result from Example 1.4.8(1). The forgetful functor V from the category **Grp** of groups to the category **Set** of sets has a left adjoint functor L: For a set A, the free group F_A with generating set A together with the "insertion of generators" map $f_A : A \to F_A$ (which assigns to each $a \in A$ the set of words in A that reduce to a) constitutes an initial morphism for A with respect to the functor V.

Moreover, for each discrete small category I the categories **Set** and **Grp** are I–cocomplete. For each functor $K : I \to \mathbf{Set}$ (a family $(K_i)_{i\in I}$ of sets), the disjoint union $\bigsqcup_{i\in I} K_i$ of the sets K_i together with the embeddings $K_i \hookrightarrow \bigsqcup_{i\in i} K_i$ $(i \in I)$ is a colimit of K, and for each functor $G : I \to \mathbf{Grp}$ (a family $(G_i)_{i\in I}$ of groups), the free product $\mathop{*}_{i\in I} G_i$ of the groups G_i together with the canonical injective homomorphisms $G_i \hookrightarrow \mathop{*}_{i\in I} G_i$ $(i \in I)$ is a colimit of G.

$$\begin{tikzcd}
\mathbf{Set} \arrow[d, "L"'] & \mathbf{Set}^I \arrow[l, "\bigsqcup_{i\in I}"'] \arrow[d, "L^I"] \\
\mathbf{Grp} & \mathbf{Grp}^I \arrow[l, "\mathop{*}_{i\in I}"]
\end{tikzcd}$$

According to Corollary 1.5.19(3), the functors $\mathop{*}_{i\in I} \circ L^I$ and $L \circ \bigsqcup_{i\in I}$ from $\mathbf{Set}^I$ to **Grp** are naturally isomorphic. More specifically, assume that $A = (A_i)_{i\in I}$ is a family of sets. Let $M = \bigsqcup_{i\in I} A_i$ be their disjoint union and let $\iota_i : A_i \to M$ be the map $x \mapsto x$ $(x \in A_i)$. Also, assume that $N = \mathop{*}_{i\in I} F_{A_i}$ is the free product of the (free) groups F_{A_i} and that $\kappa_i : F_{A_i} \to N$ $(i \in I)$ are the canonical homomorphisms. Then $\big(M, (\iota_i)_{i\in I}\big)$ is a colimit of A and $\big(N, (\kappa_i)_{i\in I}\big)$ is a colimit of $L \circ A$.

$$\begin{tikzcd}
\mathop{*}_{i\in I} F_{A_i} \arrow[r, "\mu"] & F_{\bigsqcup_{i\in I} A_i} \\
 & F_{A_i} \arrow[ul, "\kappa_i"] \arrow[u, "F_{\iota_i}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
V\big(F_{\bigsqcup_{i\in I} A_i}\big) & \bigsqcup_{i\in I} A_i \arrow[l, "f_{\bigsqcup_{i\in I} A_i}"'] \\
V\big(F_{A_i}\big) \arrow[u, "V(F_{\iota_i})"'] & A_i \arrow[l, "f_{A_i}"] \arrow[u, "\iota_i"']
\end{tikzcd}$$

There is a unique homomorphism μ from $\mathop{*}_{i\in I} F_{A_i}$ to $F_{\bigsqcup_{i\in I} A_i}$ having the property that $\mu \circ \kappa_i = F_{\iota_i}$ for all $i \in I$. It is an isomorphism, which means that a free product of free groups is a free group.

3.6 Inductive limits and the ring of fractions

We recall the notation from Example 1.4.3(5). All rings are assumed to be commutative and unitary. For a given ring R, we denote by $R^\times$ the group of invertible elements of R. Let **CRing** be the category of rings with unitary ring homomorphisms as morphisms. Let $\mathscr{A}$ be the category whose objects are pairs (R, S), where R is a ring and S is a submonoid of $(R, \cdot)$ and whose morphisms $(R, S) \to (R', S')$ are the ring homomorphisms $f : R \to R'$ having the property that $f(S) \subseteq S'$. Let $G : \mathbf{CRing} \to \mathscr{A}$ be the functor defined by $G(R) = (R, R^\times)$ for rings R and $G(f) = f$ for morphisms $f : R \to R'$.
In the example it was shown that for each object (R, S) of $\mathscr{A}$ the ring of fractions T of R with denominators in S, denoted $R[S^{-1}]$, together with the canonical embedding $\varepsilon_S : R \to T$ is an F-initial morphism for (R, S). Therefore there exists a "ring of fractions-functor" H from $\mathscr{A}$ to **CRing**. It is left adjoint to G and consequently it preserves colimits.
Let $(I', \leq)$ be a nonempty up-directed set and let I be the corresponding category. Let R be a ring and for $i \in I$ let S_i be a submonoid of the multiplicative structure of R. For all $i, j \in I$ such that $i \leq j$, let $S_i \subseteq S_j$. Let $F(i) = (R, S_i)$ for $i \in I' = \mathrm{Ob}(I)$ and if f is the unique I-morphism from i to j, let $F(f) : (R, S_i) \to (R, S_j)$ be given by the identity map on R. A functor F from I to $\mathscr{A}$ is then defined and

the $\mathscr{A}$-object $\big(R, \bigcup_{i\in I} S_i\big)$ together with the morphisms $(R, S_i) \to \big(R, \bigcup_{i\in I} S_i\big)$ given by the identity maps is a colimit of F. This is the case because $\bigcup_{i\in I} S_i$ together with the embeddings $S_i \to \bigcup_{i\in I} S_i$ is an inductive limit of the inductive system $\Big((S_i)_{i\in I}, (\iota_{S_i}^{S_j})_{i\le j}\Big)$, where ι_A^B denotes the canonical inclusion map of a subset A of B into B.

Let $S = \bigcup_{i\in I} S_i$. Since formation of the ring of fractions preserves colimits, the ring of fractions $R[S^{-1}]$ of R with denominators in S together with the maps $\varepsilon_{S,S_i} : R[S_i^{-1}] \to R[S^{-1}]$ defined by the formula $\varepsilon_{S,S_i} \circ \varepsilon_{S_i} = \varepsilon_S$ is a colimit of the functor $H \circ F : I \to \mathbf{CRing}$. Therefore, if we denote by $\kappa_i : R[S_i^{-1}] \to \varinjlim R[S_i^{-1}]$ the canonical homomorphism for $i \in I$, there exists a unique homomorphism $\varphi : \varinjlim R[S_i^{-1}] \to R[S^{-1}]$ such that $\varphi \circ \kappa_i = \varepsilon_{S,S_i}$ for all $i \in I$. It is an isomorphism.

3.7 Products as projective limits of finite products

Assume that $(A_i)_{i\in I}$ is a family of sets, where $I \neq \emptyset$. Let F be the set of finite nonempty subsets of I supplied with the partial order $\le$ defined by $J \le K \Leftrightarrow J \subseteq K$. Let $\mathscr{F}$ be the dual of the category arising from the ordered set $(F, \le)$ according to Example 1.1.5(5). That is, for finite nonempty subsets J and K of of I there is a morphism from J to K if and only if $J \supseteq K$. Define a functor G from the category $\mathscr{F} \times I$ to **Set** as follows. If $J \in F$ and $i \in I$, define $G(J,i) = \begin{cases} \{\emptyset\} & \text{if } i \notin J \\ A_i & \text{if } i \in J \end{cases}$. If $f : J \to K$ in $\mathscr{F}$, that is, if $J \in F$, $K \in F$ and $K \subseteq J$, define $G(f,i) = \begin{cases} \emptyset_i & \text{if } i \in J \setminus K \\ \mathrm{id}_{A_i} & \text{if } i \in K \\ \mathrm{id}_{\{\emptyset\}} & \text{if } i \notin J \end{cases}$, where $\emptyset_i$ is the function from A_i to $\{\emptyset\}$.

We compute a limit of G in two ways. First, for each $J \in F$, the set $\prod_{j\in J} A_j$ together with the functions $\mathrm{pr}_i^J : \prod_{j\in J} A_j \to A_i$, $(a_j)_{j\in J} \mapsto a_i$ for $i \in J$ and $\mathrm{pr}_i^J : \prod_{j\in J} A_j \to \{\emptyset\}$, $(a_j)_{j\in J} \mapsto \emptyset$ for $i \notin J$ is a limit of $G(J, \_\,)$. For each $\mathscr{F}$-morphism $f : J \to K$, the limit of $G(J,f)$ is the function $h_{K,J} : (a_j)_{j\in J} \mapsto (a_k)_{k\in K}$ from $\prod_{j\in J} A_j$ to $\prod_{k\in K} A_k$.

$$
\begin{tikzcd}
\prod_{k\in K} A_k \arrow[d, "\mathrm{pr}_i^K"'] & \prod_{j\in J} A_j \arrow[l, "(a_j)_{j\in J} \mapsto (a_k)_{k\in K}"'] \arrow[d, "\mathrm{pr}_i^J"] \\
A_i & A_i \arrow[l, "\mathrm{id}_{A_i}"]
\end{tikzcd}
$$

By the theorem on the associativity of limits, the object part of a limit of G if given by the object part of a limit of the functor $f \mapsto \lim G(J,f)$, that is, by the projective limit of the projective system given by the sets $\prod_{j\in J} A_j$ for all nonempty finite subsets J of I and the functions $h_{K,J} : \prod_{j\in J} A_j \to \prod_{k\in K} A_k$ for nonempty finite subsets K and J of I satisfying $K \subseteq J$.

On the other hand, let $i \in I$. A limit of $G(\_\,,i)$ is the set A_i together with the maps $q_J^i = \begin{cases} \mathrm{id}_{A_i} & \text{if } i \in J \\ \emptyset_i & \text{if } i \notin J \end{cases}$ for $J \in F$. To see this, note first that the pair $\big(A_i, (q_J^i)_{J\in F}\big)$ is a cone over $G(\_\,,i)$. Let $(B, (r_J)_{J\in F})$ be another cone over $G(\_\,,i)$. If $f : J \to K$ is an $\mathscr{F}$-morphism and $i \in K$, we have $r_K = G(f,i) \circ r_J = \mathrm{id}_{A_i} \circ r_J = r_J$. Since for all $J, K \in F$ there exists a set $L \in F$ such that $J \subseteq L$ and $K \subseteq L$, there exists a function a from B to A_i such that $r_J = a$ for all $J \in F$ satisfying $i \in J$. The function a has the property that $q_J^i \circ a = r_J$ for all $J \in F$, and it is obviously the only one with this property. Therefore the set A_i is indeed the object part of a limit of $G(\_\,, i)$. By associativity of limits, the set $\prod\limits_{i\in I} A_i$ is the object part of a limit of G. Combining this with the result from above, we see that a product of sets is a projective limit of finite products.

3.8 The tensor product of commutative algebras

Suppose that R is a commutative ring and let ${}_R\mathbf{Alg}$ be the category of associative, unitary and commutative R-algebras, where the morphisms are given by unitary R-algebra homomorphisms. Let $(E_i)_{1\le i\le n}$ be a finite family of ${}_R\mathbf{Alg}$-objects and let A be its tensor product. (The underlying module structure of A is given by the tensor product of the *modules* E_i; multiplication is defined to be the unique function $f : A \times A \to A$ satisfying $f(x_1 \otimes \cdots \otimes x_n, y_1 \otimes \cdots \otimes y_n) = x_1y_1 \otimes \cdots \otimes x_ny_n$ for all $x_i, y_i \in E_i$ and $1 \le i \le n$.) For $1 \le i \le n$, assume that e_i is the unit of E_i. Define functions $u_i : E_i \to A$ by $u_i(x_i) = e_1 \otimes \cdots \otimes e_{i-1} \otimes x_i \otimes e_{i+1} \otimes \cdots \otimes e_n$. By multilinearity of the function $(x_i)_{1\le i\le n} \mapsto \bigotimes_{1\le i\le n} x_i$ and by the definition of multiplication in A, the function u_i is an R-algebra homomorphism.

We show that the pair $\big(A, (u_i)_{1\le i\le n}\big)$ is a coproduct of the family $(E_i)_{1\le i\le n}$. To this end, note first that $\prod\limits_{1\le i\le n} u_i(x_i) = x_1 \otimes \cdots \otimes x_n$ for all $x_i \in E_i$ and $1 \le i \le n$. Let B be any (associative, unitary and commutative) R-algebra and $v_i : E_i \to B$ an unitary R-algebra homomorphism for $1 \le i \le n$. Then it is easy to see that the function $\prod\limits_{1\le i\le n} v_i$ from $\prod\limits_{1\le i\le n} E_i$ to B is R-multilinear. There exists therefore a unique R-linear map $w : A \to B$ such that $w(x_1 \otimes \cdots \otimes x_n) = v_1(x_1) \cdots v_n(x_n)$ for all $x_i \in E_i$ and $1 \le i \le n$, which implies $w \circ u_i = v_i$ for $1 \le i \le n$. The function w is an R-algebra homomorphism: Since A (as an R-module) is generated by elements of the form $\bigotimes\limits_{1\le i\le n} x_i$, it is sufficient to show the formula $w(xy) = w(x)w(y)$ for such x and y. We have $w\big(\bigotimes\limits_{1\le i\le n} x_i \cdot \bigotimes\limits_{1\le i\le n} y_i\big) = w\big(\bigotimes\limits_{1\le i\le n}(x_iy_i)\big) = \prod\limits_{1\le i\le n} v_i(x_iy_i) = \prod\limits_{1\le i\le n} v_i(x_i) \prod\limits_{1\le i\le n} v_i(y_i) = w\big(\bigotimes\limits_{1\le i\le n} x_i\big) \cdot w\big(\bigotimes\limits_{1\le i\le n} y_i\big)$ for all $x_i, y_i \in E_i$ and $1 \le i \le n$. Now if $\hat{w} : A \to B$ is another R-algebra homomorphism such that $\hat{w} \circ u_i = v_i$ for $1 \le i \le n$, we have $\hat{w}(x_1 \otimes \cdots \otimes x_n) = \hat{w}\big(\prod\limits_{1\le i\le n} u_i(x_i)\big) = \prod\limits_{1\le i\le n} \hat{w}(u_i(x_i)) = \prod\limits_{1\le i\le n} v_i(x_i)$ and therefore $w = \hat{w}$. Summarizing: In the category ${}_R\mathbf{Alg}$ finite coproducts are given by the tensor product.

Let $m \ge 1$ and $n \ge 1$ be integers and let $(E_i)_{0\le i<mn}$ be a family of ${}_R\mathbf{Alg}$-objects. For $0 \le i < mn$, let e_i be the identity of E_i. For $0 \le k < m$ and $0 \le i < n$, let $u_i^k : E_{kn+i} \to \bigotimes\limits_{j=kn}^{kn+n-1} E_j$ be the R-algebra homomorphism defined by the rule $x \mapsto e_{kn} \otimes \cdots \otimes e_{kn+i-1} \otimes x \otimes e_{kn+i+1} \otimes \cdots \otimes e_{kn+n-1}$. Let $v_k : \bigotimes\limits_{j=kn}^{kn+n-1} E_j \to \bigotimes\limits_{k=0}^{m-1} \bigotimes\limits_{j=kn}^{kn+n-1} E_j$ be the homomorphism $x \mapsto \hat{e}_0 \otimes \cdots \otimes \hat{e}_{k-1} \otimes x \otimes \hat{e}_{k+1} \otimes \cdots \otimes \hat{e}_{m-1}$, where $\hat{e}_l$ is the unity of $\bigotimes\limits_{j=ln}^{ln+n-1} E_j$ for $0 \le l < m$, that is, $\hat{e}_l = e_{ln} \otimes \cdots \otimes e_{ln+n-1}$. Moreover, let $w_i : E_i \to \bigotimes\limits_{j=0}^{mn-1} E_j$ be the homomorphism $x \mapsto e_0 \otimes \cdots \otimes e_{i-1} \otimes x \otimes e_{i+1} \otimes \cdots \otimes e_{mn-1}$ for $0 \le i < mn$.

$$\begin{tikzcd}
\bigotimes\limits_{j=kn}^{kn+n-1} E_j \arrow[r, "v_k"] & \bigotimes\limits_{k=0}^{m-1} \bigotimes\limits_{j=kn}^{kn+n-1} E_j \\
E_{kn+i} \arrow[u, "u_i^k"] \arrow[r, "w_{kn+i}"'] & \bigotimes\limits_{j=0}^{mn-1} E_j \arrow[u, "\varphi"']
\end{tikzcd}$$

By associativity of colimits, there exists a unique R-algebra homomorphism $\varphi : \bigotimes\limits_{j=0}^{mn-1} E_j \to \bigotimes\limits_{k=0}^{m-1} \bigotimes\limits_{j=kn}^{kn+n-1} E_j$ such that $\varphi \circ w_{kn+i} = v_k \circ u_i^k$ for all $i \in \{0, \ldots, n-1\}$ and all $k \in \{0, \ldots, m-1\}$. It is an isomorphism.

Expanding the definitions, we see that the isomorphism φ has the property that $\varphi(x_0 \otimes \cdots \otimes x_{mn-1}) = (x_0 \otimes \cdots \otimes x_{n-1}) \otimes (x_n \otimes \cdots \otimes x_{2n-1}) \otimes \cdots \otimes (x_{(m-1)n} \otimes \cdots \otimes x_{mn-1})$ for all $x_i \in E_i$ and $0 \leq i < mn$.

3.9 The tensor product and direct sums

Assume that R, S and T are (not necessarily commutative) rings and let I and J be sets. Let $(E_i)_{i \in I}$ be a family of (R, S)-bimodules and let $(F_j)_{j \in J}$ be a family of (S, T)-bimodules. For $i \in I$, let $\iota_i : E_i \to \bigoplus_{i \in I} E_i$ be the canonical (R, S)-linear map and for $j \in J$, let $\kappa_j : F_j \to \bigoplus_{j \in J} F_j$ be the canonical (S, T)-linear map. Moreover, for $(i, j) \in I \times J$, let $\eta_{i,j} : E_i \underset{S}{\otimes} F_j \to \bigoplus_{(i,j) \in I \times J} \big(E_i \underset{S}{\otimes} F_j\big)$ be the canonical (R, T)-linear map. The tensor product of two modules is cocontinuous in each argument (see Corollary 2.5.2), therefore we can apply the dual of Theorem 1.7.2: There exists a unique (R, T)-linear map $\psi : \bigoplus_{(i,j) \in I \times J} \big(E_i \underset{S}{\otimes} F_j\big) \to \bigoplus_{i \in I} E_i \underset{S}{\otimes} \bigoplus_{j \in J} F_j$ such that $\psi \circ \eta_{i,j} = \iota_i \times \kappa_j$ for all $(i, j) \in I \times J$. It is an isomorphism.

$$\begin{tikzcd}
\bigoplus_{(i,j) \in I \times J} \big(E_i \underset{S}{\otimes} F_j\big) \arrow[r, "\psi"] & \bigoplus_{i \in I} E_i \underset{S}{\otimes} \bigoplus_{j \in J} F_j & \bigoplus_{i \in I} E_i \times \bigoplus_{j \in J} F_j \arrow[l, "\otimes'"'] \\
 & E_i \underset{S}{\otimes} F_j \arrow[ul, "\eta_{i,j}"] \arrow[u, "\iota_i \times \kappa_j"'] & E_i \times F_j \arrow[l, "\otimes"] \arrow[u, "(\iota_i, \kappa_j)"']
\end{tikzcd}$$

3.10 The Hom-functor, products and direct sums

Assume that R, S and T are rings and let $(E_i)_{i \in I}$ and $(F_j)_{j \in J}$ be families of (R, S)- and (R, T)-modules respectively. For $i \in I$, let $\iota_i : E_i \to \bigoplus_{i \in I} E_i$ be the canonical (R, S)-linear map and for $j \in J$, let $\mathrm{pr}_j : \prod_{j \in J} F_j \to F_j$ be the canonical (R, T)-linear map. Moreover, for $i \in I$ and $j \in J$ let $r_{i,j} : \prod_{(i,j) \in I \times J} {}^S_R\mathrm{Hom}^T(E_i, F_j) \to {}^S_R\mathrm{Hom}^T(E_i, F_j)$ be the canonical (S, T)-linear map. Then the module $\bigoplus_{i \in I} E_i$ together with the maps ι_i is a colimit of the functor $i \mapsto E_i$ from the category I to ${}_R\mathbf{M}_S$, therefore a *limit* of the same functor, considered as a functor from I^{op} to ${}_R\mathbf{M}_S^{\mathrm{op}}$. Since $I^{\mathrm{op}} = I$ and the functor ${}_R\mathrm{Hom}$ from ${}_R\mathbf{M}_S^{\mathrm{op}} \times {}_R\mathbf{M}_T$ to ${}_S\mathbf{M}_T$ is continuous in each argument, by Theorem 1.7.2 we have an (S, T)-module isomorphism $\varphi : {}_R\mathrm{Hom}\big(\bigoplus_{i \in I} E_i, \prod_{j \in J} F_j\big) \to \prod_{(i,j) \in I \times J} {}_R\mathrm{Hom}(E_i, F_j)$ such that $r_{i,j} \circ \varphi = {}_R\mathrm{Hom}(\iota_i, \mathrm{pr}_j)$ for all $i \in I$ and $j \in J$.

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Abstract

The language of category theory was initially developed by Samuel Eilenberg and Saunders Mac Lane, in order to be able to talk about the concept of "natural transformation" in a precise manner. Since then, it has proven to be a convenient means to talk about topics as diverse as algebraic topology and computer science.

In the present work, we introduce some basic concepts and theorems of category theory and relate them to mathematical results, mainly from the field of algebra. Application of category theoretic results to mathematics may help to see similarities between areas that seem unrelated at first sight. Also, in the same process, it may yield shorter proofs for theorems. On the other hand, the process of translating mathematical statements into a form on which category theoretic tools can be applied, may introduce tedious passages that obfuscate the main argument.

In the first chapter, we introduce a number of important terms and theorems of category theory. Starting from the definition of the term *category*, we investigate the important concepts of *adjunction* and *limit of a functor*. Central results are the theorem on the associativity of limits and the fact that right adjoint functors preserve limits.

The second chapter covers multilinear maps and the tensor product. These concepts are connected by means of an adjunction. Starting from this fact we prove the theorem on the associativity of the tensor product of bimodules using category theoretic means.

In the third chapter we show some applications of the theorems in the first chapter. These results are well-known; but the reduction to category theory reveals similarities between the examples that are not as easy to see without this theory.

Zusammenfassung

Die Sprache der Kategorientheorie wurde ursprünglich von Samuel Eilenberg und Saunders Mac Lane entwickelt, um präzise über den Begriff der natürlichen Transformation sprechen zu können. Seither hat sie sich in vielen Themengebieten, zum Beispiel in der algebraischen Topologie oder in der Informatik, als praktisches und vielseitiges sprachliches Mittel erwiesen.

In der vorliegenden Arbeit führen wir grundlegende Konzepte und Sätze aus der Kategorientheorie ein und behandeln damit bekannte mathematische Tatsachen hauptsächlich algebraischer Natur. Die Kategorientheorie auf mathematische Fragestellungen anzuwenden kann Ähnlichkeiten zwischen Gebieten sichtbar machen, die auf den ersten Blick sehr verschieden aussehen. Außerdem werden so manchmal kürzere Beweise von Sätzen erzielt. Auf der anderen Seite können durch die Übersetzung mathematischer Aussagen in eine Form, auf die man kategorientheoretische Mittel anwenden kann, umständliche Formulierungen entstehen, die das eigentliche Argument verschleiern.

Im ersten Kapitel werden einige wichtige Begriffe und Sätze aus der Kategorientheorie behandelt. Bei der Definition einer Kategorie beginnend, behandeln wir die wichtigen Begriffe *Adjunktion* und *Limes eines Funktors*. Zentrale Resultate sind die Assoziativität von Limiten und die Tatsache, dass rechtsadjungierte Funktoren Limiten bewahren.

Das zweite Kapitel behandelt multilineare Abbildungen und das Tensorprodukt. Diese beiden Begriffe sind vermöge einer Adjunktion miteinander verbunden. Ausgehend von dieser Tatsache beweisen wir die Assoziativität des Tensorproduktes von Bimoduln mit kategorientheoretischen Mitteln.

Im dritten Kapitel werden einige Anwendungen der Sätze aus dem ersten Kapitel aufgezeigt. Es sind das Resultate, die wohlbekannt sind; durch die Zurückführung auf die Kategorientheorie werden jedoch Gemeinsamkeiten zwischen den Beispielen sichtbar, die man ohne sie nicht so leicht erkennt.

Curriculum vitæ

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