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On singular wave equations

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Abstract

This work is concerned with the existence theory of wave equations on Lorentzian manifolds of low regularity using the theory of generalized functions. We build upon a result by Grant, Mayerhofer and Steinbauer [GMS09] who proved an existence and uniqueness theorem for scalar linear wave equations with coefficients of low regularity (in the sense of Colombeau). From a geometrical viewpoint this proves the well-posedness of the Cauchy problem for the Laplace-Beltrami operator of a generalized Lorentzian manifold. In the context of Clarke's proposal [Cla98] such spacetimes are called $\mathcal{G}$-hyperbolic and can be considered as non-singular. In this work we extend the existence and uniqueness theory of [GMS09] to linear tensorial wave equations including lower order terms with regard to a possible application on the existence theory of quasilinear equations. The technical centrepiece are higher order energy estimates which we present in a particular clear way to derive an existence and uniqueness theorem in the generalized setting. We also discuss the relations of low regularity wave equations and symmetric hyperbolic first order systems based on the work of Hörmann and Spreitzer [HoSp11] which amounts in further existence and uniqueness results for both wave equations and first order systems.

Preface

This doctoral thesis represents my research in the field of partial differential equations and generalized functions and is, in a sense, the logical continuation of the work I have done in my master thesis [Han06].

Many thanks go to my parents Annemarie and Gerhard Hanel for their continuing support and patience. Equally, I want to thank Marianne and Nelli Hainisch, especially for their support in the last stage of this thesis project. I also want to express my gratitude to my friends, in particular Oswald Massiczek, Andreas Németh, and Evelina Erlacher.

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Contents

1. Introduction

The analysis of wave equations is a classic topic in the theory of partial differential equations, here we refer to the book of Friedlander [Fri75] for an extensive treatment of the wave equation on a curved space time. Also in recent time there has been increasing interest in wave equations, especially using a specifically geometric language [BGP07]. Apart from the classical treatment of this topic, the analysis of the Cauchy problem for linear wave equations on spacetimes of low regularity has been the source for recent publications [ViWi00, GMS09]. Precisely, one discusses the existence and uniqueness theory of the connection d'Alembertian

$$\Box_g = \sum_{\alpha,\beta=0}^{n-1} g^{\alpha\beta}\nabla_\alpha\nabla_\beta = \sqrt{|\det \mathbf{g}|}\sum_{\alpha,\beta=0}^{n-1}\frac{\partial}{\partial x^\alpha}\left(\sqrt{|\det \mathbf{g}|}g^{\alpha\beta}\frac{\partial}{\partial x^\beta}\right)$$

of a Lorentzian metric $\mathbf{g}$ of low regularity with Levi-Cività connection ∇. The motivation to study this problem is twofold: First, the well-posedness of the Cauchy problem for the wave equation was proposed as a way of measuring whether a given spacetime is (non-) singular [Cla96, Cla98]; secondly, when treating the initial value formulation of Einstein's field equations in general relativity, quasilinear wave equations appear naturally, and regularity issues are of great importance in this area. In this work we are concerned with an extension of the research done in [GMS09] towards normally hyperbolic equations with low regularity coefficients.

1.1. Motivation

An important motivation for our research on wave equations of low regularity are applications in general relativity, i. e. Einstein's theory of gravity and matter. There, *spacetime* is described by a connected, time-oriented, four-dimensional Lorentzian manifold

$(M, \mathbf{g})$. Since the appearance of the celebrated singularity theorems by Hawking and Penrose [HaPe70], the *singularities* of spacetime have been a matter of interest to many researchers. In the standard approach singularities are characterized as obstructions to the extension of geodesics, see [HaEl73, Ch. 8]. A drawback of the standard approach is that it classifies many spacetimes as singular that have been used to model physically reasonable scenarios, e. g. impulsive gravitational waves, thin cosmic strings, and shell-crossing singularities. In [Cla96] Clarke put forward an alternative approach to the analysis of spacetime singularities: He proposed that one should treat singularities as obstructions to the well-posedness of the Cauchy problem for the scalar d'Alembert operator rather than to the extension of geodesics. A scalar field can, physically speaking, be seen as a reasonable replacement for an extended test body, which is too hard to model in general relativity. Using such a scalar field to detect singularities leads to the notion of *generalized hyperbolicity* (referring to the classical concept of global hyperbolicity, cf. [Wal84, Sec. 8.3]). More precisely, a space time is called *generalized hyperbolic* and viewed as "non-singular" if the scalar wave equation can be uniquely solved locally around each point. Of course, here one has to invoke a suitable weak solution concept since the coefficients of the resulting equation will typically be of low regularity—that is usually $\mathcal{C}^{1,1}$ (i. e. the first derivative being locally Lipschitz continuous) and even below.

Another motivation to treat wave equations with low regularity coefficients concerns the evolution of spacetime itself. The gravitational field of a spacetime is linked to its curvature, represented by the *Einstein tensor* $\mathbf{G} = \mathbf{Ric} - \frac{1}{2}(\mathrm{tr}_{\mathbf{g}} \mathbf{Ric})\mathbf{g}$, and the evolution of spacetime is described by *Einstein's field equations* relating the curvature to the *energy-momentum tensor* $\mathbf{T}$:

$$\mathbf{G} = 8\pi\mathbf{T}.$$

Here $\mathbf{Ric}$ denotes the Ricci tensor of a 4-manifold $(M, \mathbf{g})$. We refer to the special case $\mathbf{T} = 0$ as the *vacuum Einstein equations*. A matter of fundamental interest is the Cauchy problem for these vacuum Einstein equations, i. e. derive the evolution of the gravitational field from an *initial data set*. More precisely, given a 3-manifold Σ, a Riemannian metric $\mathbf{h}$, and a covariant symmetric tensor field χ on Σ, we say that the triple $(\Sigma, \mathbf{h}, \chi)$ is an initial data set if $\mathbf{h}$ and χ satisfy the constraint equations. By solving the Cauchy problem, we mean finding a Lorentzian 4-manifold $(M, \mathbf{g})$ and an embedding $i : \Sigma \to M$ such that $\mathbf{h}$ is the pullback of the Lorentzian metric $\mathbf{g}$ onto Σ and χ is the pullback of the second fundamental form of the hypersurface $i(\Sigma)$ onto Σ. Thus the constraint equations for $\mathbf{h}$ and χ are the pullback of the Gauß-Codazzi equations induced on $i(\Sigma)$. Note that Einsteins Theory of Gravitation is diffeomorphism invariant, so initial data sets are considered to be unique up to diffeomorphisms only and equivalent initial data sets are supposed to give equivalent solutions [KlNi03, HaEl73]. A consequence of the diffeomorphism invariance is the free choice of local coordinates and it was proved that using, so called, *harmonic coordinates* the evolution equations of the gravitational field reduce to tensorial quasilinear wave equations [Fou52, CGP10]. In this work we provide existence and uniqueness results for linear wave equations of low regularity that will eventually be useful in the context of the initial value problem for Einstein's equations.

Deeply connected with the Cauchy problem for the Einstein equations is the weak cosmic censorship hypothesis (weak CCH), which prohibits the existence of "naked" singularities. In other words: Any singularity should be shielded from the distant observer by an event horizon. The weak cosmic censorship hypothesis is related to the strong CCH (see [Pen79]), which states that any "generic" spacetime is globally hyperbolic. Since one is still missing a good definition of some of its ingredients, there exists no precise mathematical formulation of the CCH. Singularities are one of the ingredients of the CCH and the analysis of the Cauchy problem for Einstein's equations gives a better understanding of singular spacetimes.

1.2. Recent results

In the recent past a number of results on the Cauchy problem for wave equations on singular spacetimes have been published. In [Cla98] Clarke proved generalized hyperbolicity of shell crossing singularities using a suitable weak solution concept. In [ViWi00] Vickers and Wilson proved generalized hyperbolicity of conical space times using the theory of nonlinear generalized functions of Colombeau [Col84, Col92]. More precisely, they proved unique solvability of the wave equation in this framework, invoking a refined version of higher order energy estimates (cf. [HaEl73, Ch. 7]). They also succeeded in showing that their generalized solution can be associated with a distribution that fits the expectations from physics. The next notable step was achieved by Mayerhofer in his PhD-thesis [May06]: He proved existence and uniqueness for the wave equation on static spacetimes of low regularity. Later Grant, Mayerhofer, and Steinbauer in [GMS09] generalized the work of [May06, ViWi00] to a fairly large class of "weakly singular" spacetimes, where essentially the metric was assumed to be locally bounded. Modelling such spacetime metrics in Colombeau generalized functions from the start, they proved generalized hyperbolicity. However, they did not relate their result to more classical notions.

1.3. Results of this thesis

Existence and uniqueness for wave equations

Recently, in [Han10], we achieved a significant enhancement of the work of [GMS09] by generalizing the existence and uniqueness result to a larger class of normally hyperbolic partial differential operators. More precisely, the principal symbol of the operator is given by a Lorentzian metric of low regularity. Hence—in local coordinates—the operator takes the form

$$L = \sum_{\alpha,\beta=0}^{n-1} g^{\alpha\beta}\partial_\alpha\partial_\beta + \text{lower order terms.} \tag{1.1}$$

where $g^{\alpha\beta}$ are the contravariant components of the Lorentzian metric. In order to write L in an explicit coordinate free way one introduces a smooth background metric $\hat{\mathbf{g}}$ which enables one to write L in the form

$$(L\mathbf{u})^I_J = g^{ab}\widehat{\nabla}_a\widehat{\nabla}_b u^I_J + B^{aIP}_{JQ}\widehat{\nabla}_a u^Q_P + C^{IP}_{JQ}u^Q_P, \tag{1.2}$$

with I, J, P, Q multiindices (for details of the notation see Section 2.1). Here $\widehat{\nabla}$ denotes the Levi-Cività connection with respect to the *smooth* metric $\hat{\mathbf{g}}$, and we also work in the framework of non-linear distributional geometry [KuSt02a, KuSt02b], that is generalized functions in the sense of Colombeau [Col84, Col92]. In this thesis we take one more step forward and consider also the metric $\hat{\mathbf{g}}$ to be a generalized tensor field to give an accurate and complete picture of the asymptotic conditions to be considered, when deriving the energy estimates for the generalized existence and uniqueness theory. Furthermore, this allows us develop a common existence and uniqueness theory for the settings presented in [GMS09] and [Han10].

Throughout the analysis of the methods used in [ViWi00, GMS09], where the Cauchy problem for the Laplace-Beltrami operator of a singular metric is considered, we find that the (singular) spacetime metric has to play different roles in different places: defining the Levi-Cività connection, as the principal part of the operator, and defining the main part of the energy tensor which is the essential tool in deriving the key estimates (which for causality reasons always has to be derived from the same metric as the principal part of the operator). Here, we separate these roles by using the two distinct metrics $\mathbf{g}$ and $\hat{\mathbf{g}}$, and we consider both, i. e. $\mathbf{g}$, which defines the principal part of the operator L, and $\hat{\mathbf{g}}$, which defines the Levi-Cività connection, to be of low regularity. The main benefit lies in gaining some new insight into the fine structure of the energy estimates and in improving on questions of regularity of generalized solutions. Note that an elaborate regularity theory within algebras of generalized functions does exist (see e. g. [Hor04, Obe06a, Obe06b]) and connecting to it seems necessary to relate also the results of [GMS09] to more classical function spaces. In particular, our asymptotic conditions on $\mathbf{g}$ are quite different from those of [GMS09]. As a consequence we also improve the results of [GMS09] in such a way that the method of Picard iteration [HaEl73, Sec. 7.5], [Hor97, Sec. 6.1], [Sog95, Sec. 1.4] can be employed to prove results on quasilinear equations, which eventually allows us to solve the low regularity Cauchy problem for Einstein's equations, cf. Section 1.1.

Relation to first order systems

Classically one can rephrase second order hyperbolic equations of the form

$$-\partial_t^2 u + \sum_{\alpha,\beta=1}^{n} R^{\alpha\beta}\partial_\alpha\partial_\beta u + \sum_{\alpha=0}^{n} b^\alpha \partial_\alpha u + cu - f = 0$$

(with $\mathbf{R}$ a Riemannian metric) in terms of hyperbolic systems of first order equations of the form

$$\partial_t \boldsymbol{w} + \mathbf{A}\partial_x \boldsymbol{w} + \mathbf{B}\boldsymbol{w} - \boldsymbol{F} = 0$$

(with $\mathbf{A}, \mathbf{B}$ symmetric matrices of functions in x and t). While in the smooth case this rewriting is without problems, it becomes a delicate issue if the data and hence the solution is distributional. In joint work with Christian Spreitzer, Günther Hörmann, and Roland Steinbauer we studied this problem for hyperbolic equations in Colombeau algebras. We obtained results in algebras of generalized functions that relate existence and uniqueness for the initial value problem of wave-type equations to that of first order systems and vice versa (for certain systems that admit a representation as a second order equation) such that in either way we draw essential benefit from the different methods used to prove the classical (e. g. [BeSe07, Fri75, HaEl73, HoSp11]) and generalized (e. g. [GMS09, Han10, LaOb91]) existence results. This can be seen as a first step to derive geometrical energy estimates, similar to those for wave-type equations, for hyperbolic first order systems, which enable us to tackle first order operators like the Dirac operator.

1.4. Layout of this thesis

This work is organized in the following way: In the second chapter we provide the mathematical prerequisites for this thesis. We start with a few notational remarks in Section 2.1. The following Section 2.2 gives a very short introduction into Lorentzian geometry. In Sections 2.3 and 2.4 we introduce the framework of Colombeau generalized functions on manifolds. We conclude the chapter with some remarks on spacetime foliations and Leray forms.

Chapter three can be considered the technical centre piece of the thesis. Here, we develop the geometric energy method for normally hyperbolic operators in the smooth case in such a way that we keep precise track of all dependencies on the coefficients and right hand side of the equation. Thus we can use this result in the following chapter to derive geometric energy estimates for the generalized problem.

In chapter four we build on the results the previous chapter and formulate our main theorem. We derive geometric higher order energy estimates for the generalized initial value problem, and we completely analyse the existence theory of wave equations of low regularity and derive the precise asymptotics needed for our proof of the existence and uniqueness result. In the final remark we show that the main theorem improves the existence and uniqueness result of [GMS09].

We dedicate the final chapter to the relation between wave equations and hyperbolic first order systems, which is a joint work with Günther Hörmann, Christian Spreitzer, and Roland Steinbauer. We prove an equivalence theorem that allows us to apply the existence and uniqueness theory for certain symmetric hyperbolic systems to wave equations and vice versa.

The thesis concludes with an appendix that details some of the rather lengthy calculations of chapter three.

2. Preliminaries

In this chapter we mainly cover notational conventions used in this thesis and give an introduction to Colombeau generalized functions, which are the setting used throughout the following chapters. Apart from that we give a short introduction into a few concepts from Lorentzian geometry used in the context of wave equations.

2.1. Setting and notation

Throughout this thesis we suppose M to be a separable, smooth, and orientable Hausdorff manifold of dimension $n > 1$. However, for convenience of the reader we refer to M only as *smooth manifold*. As usual, we denote the set of smooth vector fields on M by $\mathfrak{X}(M)$ and the set of smooth k-forms by $\Omega^k(M)$. We generally use semi-bold letters to denote a vector field or a k-form, e. g. $\boldsymbol{\xi} \in \mathfrak{X}(M)$ or $\boldsymbol{\omega} \in \Omega^k(M)$. The set of smooth type (k, l) tensor fields is defined in the usual way, i. e.

$$\mathcal{T}_l^k(M) = \operatorname{Lin}\Big(\prod_{i=1}^{k} \Omega^1(M) \times \prod_{i=1}^{l} \mathfrak{X}(M), \mathcal{C}^\infty(M)\Big).$$

A tensor field will be denoted by semi-bold and sans-serif letters, e. g. $\mathsf{u} \in \mathcal{T}_k^l(M)$. Furthermore, we will make extensive use of abstract index notation, cf. [PeRi84]. Thus for a tensor field u of type (k, l) we write $u^{i_1\dots i_k}_{j_1\dots j_l}$ or for short u^I_J using multiindices I and J of length $|I| = k$ resp. $|J| = l$. Especially for a vector field $\boldsymbol{\xi}$ we write ξ^a and for a one-form $\boldsymbol{\omega}$ we write ω_a. The tensor product is simply denoted by concatenating the two objects in question, i. e. for $\boldsymbol{\omega}, \boldsymbol{\nu} \in \Omega^1(M)$ we write the tensor product $\boldsymbol{\omega} \otimes \boldsymbol{\nu}$ in the form

$$(\boldsymbol{\omega} \otimes \boldsymbol{\nu})_{ab} = \omega_a \nu_b.$$

The operation of tensorial contraction is denoted by using twice the same index letter. So for $\boldsymbol{\omega} \in \Omega^1(M)$ and $\boldsymbol{\xi} \in \mathfrak{X}(M)$, we have

$$\boldsymbol{\omega}(\boldsymbol{\xi}) := \omega_a \xi^a.$$

For a metric tensor $\mathbf{e}$, by a slight abuse of the multiindex notation, we write e_{IJ} for $e_{i_1 j_1} \cdots e_{i_k j_k}$, whenever $|I| = |J| = k$. Furthermore, the inverse of some metric e_{ab} we denote by $(e^{-1})^{ab}$ with the same convention in the case of multi-indices. The Levi-Cività connection is denoted by ∇ or in abstract index notation ∇_a. For multiple derivatives, we write ∇^k with a natural number k or $\nabla_{i_1} \cdots \nabla_{i_k} = \nabla_I$ with I a multiindex of length k. To distinguish abstract index notation from calculations in a coordinate system, we will alway use greek indices for tensorial components in a coordinate system. So, e. g. the coordinates of $e^{ab}\omega_a$ will read $\sum_\alpha e^{\alpha\beta}\omega_\alpha$. When working on an n-dimensional Lorentzian manifold, the coordinate indices run from 0 to $n-1$. We denote the coordinates by (x_λ) with $0 \leq \lambda \leq n-1$, where in case of a timelike coordinate direction we also use $t = x_0$. For the partial resp. covariant derivative in the timelike coordinate, we explicitly write the index letter t, e. g. ∂_t or ∇_t. Furthermore, note that any differential operator only acts on the adjacent symbol unless otherwise indicated by parentheses, e. g. the Leibniz rule reads $\nabla(fg) = \nabla f g + f \nabla g$.

2.2. Lorentzian geometry

This section is dedicated to a short introduction into elements of Lorentzian geometry. Since in the main chapters, we are dealing with normally hyperbolic equations, the connection to Lorentzian metrics is immediate, see [BGP07, Section 1.5]. Our main reference on this topic is [ONe83].

Recall that a *symmetric bilinear form* on a vector space V over $\mathbb{R}$ is a real bilinear function $\mathbf{b} : V \times V \to \mathbb{R}$ such that for $v, w \in V : \mathbf{b}(v, w) = \mathbf{b}(w, v)$. The *index* ν of $\mathbf{b}$ is the dimension of the largest subspace $W \subseteq V$ such that $\mathbf{b}|W \times W$ is negative definitive. Obviously, we have $0 \leq \nu \leq \dim V$. For a positive definite form $\mathbf{b}$, the index ν equals 0. We call a bilinear form non-degenerate provided that $\mathbf{b}(v, w) = 0$ for all w implies $v = 0$. Thus we start with the following definition of a semi-Riemannian metric.

2.2.1. Definition (Semi-Riemannian metric): A *semi-Riemannian metric* $\mathbf{g}$ on a manifold M with $n = \dim M > 1$ is a symmetric non-degenerate $(0,2)$-tensor field of constant index $\nu(\mathbf{g})$.

So $\mathbf{g}$ assigns to each point $p \in M$ a non-degenerate bilinear form $\mathbf{g}_p$ on the tangent space $\mathrm{T}_p M$ and the index ν is the same for all p. We call a metric with index $\nu = 0$, i. e. a positive definite metric, *Riemannian* and a metric with $\nu = 1$ *Lorentzian*. In the latter case this fixes the signature of the metric to be $(1, n-1)$ or equivalently $(-,+,\ldots,+)$.

We call a pair $(M, \mathbf{g})$, where M is a smooth manifold and $\mathbf{g}$ is a Lorentzian resp. Riemannian metric, a *Lorentzian manifold* resp. *Riemannian manifold*.

Moreover, a Lorentzian metric $\mathbf{g}$ naturally induces a causality structure on a manifold. Depending on the sign of the scalar product of a vector field with itself, we classify vector fields according to the following definition.

2.2.2. Definition: On a Lorentzian manifold $(M, \mathbf{g})$ a vector field v is called

- *spacelike* if $\mathbf{g}(v,v) > 0$ or $v = 0$,
- *null* if $\mathbf{g}(v,v) = 0$ and $v \neq 0$,
- *timelike* if $\mathbf{g}(v,v) < 0$.

The terms *non-timelike* and *non-spacelike/causal* refer to the corresponding relations with $\geq$ and $\leq$.

This notion extends to curves in M by application to their tangent vector fields and in a similar way to normal forms (via duality, since $\mathbf{g}$ resp. $\mathbf{g}^{-1}$ acts as an isomorphism between the tangent bundle and the cotangent bundle). Consequently we call a hypersurface spacelike, timelike or null, if its normal vector field (or normal form) is timelike, spacelike or null.

Another important causality concept is the time orientability of a Lorentzian manifold, which allows us to distinguish between future and past directed vector fields.

2.2.3. Definition (Time orientability): A Lorentzian manifold M is called time orientable if there exists a smooth timelike vector field $\boldsymbol{\xi} \in \mathfrak{X}(M)$. We refer to such a vector field $\boldsymbol{\xi}$ as *time orientation*.

Note that locally any Lorentzian manifold is time orientable. For an equivalent notion of time orientation and additional information, see [ONe83, pp. 143–145]. Equipped with the notion of time orientability, we can now define future- and past-directed vector fields and curves.

2.2.4. DEFINITION:

- A timelike vector field $v \in \mathrm{T}M$ is called future resp. past directed if we have $\mathbf{g}(v, \boldsymbol{\xi}) < 0$ resp. $\mathbf{g}(v, \boldsymbol{\xi}) > 0$ for a time orientation $\boldsymbol{\xi}$.
- A curve is called future resp. past directed if its tangent vector field is future resp. past directed.

2.2.5. DEFINITION (Causality relations): For $p, q \in M$, we write $p \ll q$ if there exists a future-directed timelike curve in M from p to q.

2.2.6. DEFINITION (Chronological future): For $\Omega \subseteq M$, we set

$$I^+(\Omega) := \{q \in M | \exists p \in \Omega : p \ll q\}, \text{ the } \textit{chronological future}.$$

2.3. Colombeau generalized functions

In our work we consider differential equations with low regularity coefficients and initial data, i. e. differentiability is usually below $\mathcal{C}^2$ or we even consider distributions. This scenario also amounts to solutions of low regularity. As a consequence, when working with such low regularity differential equations, one has to deal with the issue of multiplying such (generalized) functions. In principle, see [Obe92], there are three different approaches to that task:

- We could choose to remain at the level of distributions and work with so called *regular intrinsic products*, i. e. we work on a restriction of a distribution space that admits a product. A rather prominent example of such a function algebra would be the Sobolev spaces $\mathcal{H}^s(\mathbb{R}^n)$ with $s > {}^n/_2$.

- Another possibility would be *irregular intrinsic products*. This usually covers methods, where one defines the product of certain pairs of distribtions. Generally this approach will lack basic properties like associativity and continuity.

- Finally—that is the approach we chose for our work—one could work with *extrinsic products*: We embed the distributions into a larger space of generalized functions, which is a differential algebra.

Preferably, for such a differential algebra $\mathcal{A}$ we would like to have a number of basic properties; for a thorough discussion of these properties we refer to [GKOS01, section 1.1]:

1. There exists a linear embedding of the distributions $\mathcal{D}'$ into $\mathcal{A}$ such that $f \equiv 1$ is the multiplicative unity in $\mathcal{A}$.

2. We have derivative operators from $\mathcal{A}$ into itself that are linear and satisfy the Leibniz rule.

3. The derivative operators on $\mathcal{A}$ restricted to the space $\mathcal{D}'$ give the usual $\mathcal{D}'$-derivative operators.

4. The restriction of multiplication to smooth functions coincides with the usual pointwise product of functions.

It is a well known fact, called the *Schwartz impossibility result*, cf. [Sch54], that there is no such differential algebra if in the last condition one replaces smooth functions with differentiability of finite order. However, differential algebras satisfying these four conditions were introduced by Jean-François Colombeau in [Col84, Col85, Col92]. A comprehensive introduction into the field of Colombeau generalized functions can be found in [GKOS01].

When working with differential equations most notably the special Colombeau algebra has proven as a convenient tool, thus we start with the following definition:

2.3.1. Definition (Special algebra): We set $\mathcal{E}(M) := \mathcal{C}^\infty(M)^{]0,1]}$, the basic space. Let us denote compact subsets of M by K, and let $\mathcal{P}(M)$ be the space of linear differential operators on M. Then

$$\begin{aligned}\mathcal{E}_M(M) := \{(u_\varepsilon)_\varepsilon \in \mathcal{E}(M) | \forall K\, \forall P \in \mathcal{P}(M)\, \exists N \in \mathbb{N}_0 :\\ \sup_{p\in K} |Pu_\varepsilon(p)| = O(\varepsilon^{-N})\},\\ \mathcal{N}(M) := \{(u_\varepsilon)_\varepsilon \in \mathcal{E}(M) | \forall K\, \forall P \in \mathcal{P}(M)\, \forall m \in \mathbb{N}_0 :\\ \sup_{p\in K} |Pu_\varepsilon(p)| = O(\varepsilon^{m})\}.\end{aligned}$$

are the spaces of moderate resp. negligible nets of smooth functions. The quotient $\mathcal{G}(M) := \mathcal{E}_M(M)/\mathcal{N}(M)$ is the special Colombeau algebra on M.

If not explicitly stated otherwise, by $\mathcal{G}(M)$ we denote only real valued generalized functions. If needed, we write $\mathcal{G}(M, \mathbb{C})$ for complex-valued functions. Elements in $\mathcal{G}(M)$ are denoted as follows,

$$u = [(u_\varepsilon)_\varepsilon] = (u_\varepsilon)_\varepsilon + \mathcal{N}(M).$$

The basic operations on $\mathcal{G}(M)$ (vector space operations, multiplication of generalized functions, and differentiation) are defined component-wise. Precisely, we have for $u, v \in \mathcal{G}(M)$, $\lambda \in \mathbb{R}$ resp. $\mathbb{C}$, and $P \in \mathcal{P}(M)$

- $\lambda u + v := [(\lambda u_\varepsilon + v_\varepsilon)_\varepsilon]$,
- $u \cdot v := [(u_\varepsilon \cdot v_\varepsilon)_\varepsilon]$, and
- $Pu := [(Pu_\varepsilon)_\varepsilon]$.

Note that generalized functions on a manifold allow a local description. To a generalized function $u \in \mathcal{G}(M)$ we assign generalized functions $u_\alpha := u \circ \psi_\alpha^{-1} \in \mathcal{G}(\psi_\alpha(U_\alpha))$ with $\{(U_\alpha, \psi_\alpha) | \alpha \in A\}$ an atlas of M. The function u_α is called *local expression* of u with respect to the chart (U_α, ψ_α), see [GKOS01, Section 3.2]. We introduce generalized numbers as the ring of constants in the special algebra $\mathcal{G}(M, \mathbb{C})$ resp. $\mathcal{G}(M)$, i. e., generalized complex or real valued functions with vanishing first derivative. We denote the generalized complex numbers by $\widetilde{\mathbb{C}}$ and the generalized reals by the symbol $\widetilde{\mathbb{R}}$. The n-dimensional free module over $\widetilde{\mathbb{R}}$ is written $\widetilde{\mathbb{R}}^n$. For a characterization using asymptotic estimates, we refer to [GKOS01, Section 1.2.4].

We obtain generalized tensor fields as tensor product of scalar Colombeau functions with smooth tensor fields, i. e. the space of generalized tensor fields of type (k,l) is

$$\mathcal{G}_l^k(M) := \mathcal{G}(M) \otimes \mathcal{T}_l^k(M).$$

Furthermore, a generalized tensor field can also be intepreted as a multilinear functional that maps generalized vector fields and one forms into a generalized function. Thus for a generalized tensor field $T \in \mathcal{G}_l^k(M)$ we have

$$\mathbf{g} : \prod_{i=1}^{l} \mathcal{G}_0^1 \times \prod_{j=1}^{k} \mathcal{G}_1^0 \to \mathcal{G}(M).$$

Apart from the definition of generalized functions given above, where we have equality in $\mathcal{G}(M)$ on the level of representatives modulo negligible nets, there exists a different, but equally valid, approach: The characterization via generalized point values which is an analogue to the point values of classical functions. In fact, considering only classical point values is insufficient as [GKOS01, Theorem 3.2.8] shows.

2.3.2. THEOREM: *Let $f \in \mathcal{G}(M)$. The following are equivalent:*

(i) $f = 0$ *in* $\mathcal{G}(M)$,

(ii) $f(\tilde{p}) = 0$ *in* $\widetilde{\mathbb{R}}$ *for each* $\tilde{p} \in \widetilde{M}_c$.

Here $\widetilde{M}_c$ denotes the set of compactly supported points in M, where a generalized point $(p_\varepsilon)_\varepsilon \in M^{]0,1]}$ is compactly supported, if there exists $K \subset\subset M$ and $\eta > 0$ such that $p_\varepsilon \in K$ for $\varepsilon < \eta$.

The following definition covers another important property: the positivity of generalized functions. Compared to classical functions, this is a more delicate issue since it is not very meaningful for a generalized function to have a representative where $f_\varepsilon > 0$ for all ε. Instead we have

2.3.3. DEFINITION (Strict positivity): Let $f \in \mathcal{G}(M)$ and denote by K a compact subset of M. Then, if there exists a representative $(f_\varepsilon)_\varepsilon$ of f such that

$$\forall K\, \exists m \in \mathbb{N} : \inf_{p \in K} f_\varepsilon(p) \geq \varepsilon^m \text{ as } \varepsilon \to 0,$$

we call f strictly positive and, for short, we write $f > 0$.

Moreover, it is straight forward to call a function f strictly negative if $-f > 0$. Furthermore, we call f strictly nonzero if $|f|$ is strictly positive.

2.4. Elements of nonlinear distributional geometry

2.4.1. DEFINITION (Symmetry and non-degenerateness): A tensor field in $\mathbf{g} \in \underset{\sim}{\mathcal{G}^0_2}(M)$ is called symmetric and nondegenerate if for any chart (U_α, ψ_α) and each $\tilde{x} \in \psi_\alpha(U_\alpha)_c$ the map $\mathbf{g}_\alpha(\tilde{x}) : \widetilde{\mathbb{R}}^n \times \widetilde{\mathbb{R}}^n \to \widetilde{\mathbb{R}}$ is symmetric and nondegenerate.

Here, by nondegenerate we mean that $\boldsymbol{\xi} \in \widetilde{\mathbb{R}}^n$, $\mathbf{g}_\alpha(\tilde{x})(\boldsymbol{\xi}, \boldsymbol{\eta}) = 0\, \forall \boldsymbol{\eta} \in \widetilde{\mathbb{R}}^n$ implies $\boldsymbol{\xi} = 0$. Apart from this generalized pointwise definition, there exist equivalent characterizations of a symmetric and non-degenerate generalized tensor field, cp. [GKOS01, Theorem 3.2.74]:

2.4.2. THEOREM: *Let $\mathbf{g} \in \mathcal{G}^0_2(M)$. The following are equivalent:*

1. $\mathbf{g}$ *is symmetric and nondegenerate.*
2. $\mathbf{g} : \mathcal{G}^1_0(M) \times \mathcal{G}^1_0(M) \to \mathcal{G}(M)$ *is symmetric and* $\det \mathbf{g}$ *is invertible in* $\mathcal{G}(M)$.
3. $\det \mathbf{g}$ *is invertible in $\mathcal{G}(M)$ and for each relatively compact open set $\Omega \subseteq M$ there exists a representative $(g_\varepsilon)_\varepsilon$ of $\mathbf{g}$ and some $\varepsilon_0 > 0$ such that $\mathbf{g}_\varepsilon|_\Omega$ is a smooth semi-Riemannian metric for $\varepsilon < \varepsilon_0$.*

REMARK: In the case of smooth ε-dependence of the generalized tensor field $\mathbf{g}$, we can equivalently replace the third statement by a global one, cf. [HKS11, Lemma 4.3 and Remark 4.4].

We can also interprete a nondegenerate tensor field $\mathbf{g} \in \mathcal{G}^0_2(M)$ as an isomorphism between generalized vector fields and one forms, i. e. $\mathbf{g} : \mathcal{G}^1_0(M) \to \mathcal{G}^0_1(M)$.

This allows us to define the generalized version of a symmetric, nondegenerate tensor field's index.

2.4.3. Definition: If for a symmetric, nondegenerate tensor field $\mathbf{g} \in \mathcal{G}^0_2(M)$, on every relatively compact set $\Omega \subseteq M$, there exists a representative $(\mathbf{g}_\varepsilon)_\varepsilon$ of $\mathbf{g}$ as in Theorem 2.4.2.3, such that the index of each $\mathbf{g}_\varepsilon$ equals j, then we call $j =: \nu(\mathbf{g})$ the index of the tensor field $\mathbf{g}$.

Completely analogous to the classical case we have

2.4.4. Definition (Generalized semi-Riemannian metric): A generalized semi-Riemannian metric is a symmetric, nondegenerate generalized tensor field $\mathbf{g} \in \mathcal{G}^0_2(M)$ with constant index $\nu(\mathbf{g})$. We call generalized metrics with index $\nu(\mathbf{g}) = 1$ Lorentzian and such with index $\nu(\mathbf{g}) = 0$ Riemannian.

We call a smooth manifold M together with a generalized semi-Riemannian metric $\mathbf{g}$ a generalized semi-Riemannian manifold, denoted by $(M, \mathbf{g})$.

We also recall the definition of generalized eigenvalues of a symmetric matrix as given in [May08, Definition 4.5].

2.4.5. Definition (Generalized eigenvalues): Let $\mathbf{A} \in M_n(\widetilde{\mathbb{R}})$ be a symmetric matrix, and let $(\mathbf{A}_\varepsilon)_\varepsilon$ be an arbitrary representative of $\mathbf{A}$. Let for any $\varepsilon \in]0,1]$, $\theta_{k,\varepsilon} := \mu_{k,\varepsilon} + i\nu_{k,\varepsilon}$ with $(1 \leq k \leq n)$ be the eigenvalues of $\mathbf{A}_\varepsilon$ ordered by the size of their real parts, i. e. $\mu_{1,\varepsilon} \geq \cdots \geq \mu_{n,\varepsilon}$. The generalized eigenvalues $\theta_k \in \widetilde{\mathbb{C}}$ $(1 \leq k \leq n)$ of $\mathbf{A}$ are defined as the classes $(\theta_{k,\varepsilon})_\varepsilon + \mathcal{N}_{\widetilde{\mathbb{C}}}$.

We conclude this section with the introduction of the generalized Levi-Cività connection on a generalized semi-Riemannian manifold. The details and proofs can be found in [KuSt02b, Section 5].

2.4.6. Definition (Generalized connection): A *generalized linear connection* on a generalized semi-Riemannian manifold $(M, \mathbf{g})$ is a function

$$\nabla : \mathcal{G}_0^1(M) \times \mathcal{G}_0^1(M) \to \mathcal{G}_0^1(M)$$

such that

($\nabla 1$) $\nabla_{\boldsymbol{\xi}} \boldsymbol{\eta}$ is $\mathcal{G}(M)$-linear in $\boldsymbol{\xi}$,

($\nabla 2$) $\nabla_{\boldsymbol{\xi}} \boldsymbol{\eta}$ is $\widetilde{\mathbb{R}}$-linear in $\boldsymbol{\eta}$,

($\nabla 3$) $\nabla_{\boldsymbol{\xi}}(f\boldsymbol{\eta}) = (\boldsymbol{\xi} f)\boldsymbol{\eta} + f \nabla_{\boldsymbol{\xi}} \boldsymbol{\eta}$ for $f \in \mathcal{G}(M)$.

For a connection ∇, we call $\nabla_{\boldsymbol{\xi}} \boldsymbol{\eta}$ the *covariant derivative* of $\boldsymbol{\eta}$ with respect to $\boldsymbol{\xi}$.

The following theorem, see also [KuSt02b, Theorem 5.2 and Proposition 5.3], characterizes the Levi-Cività connection analogously to the smooth case.

2.4.7. Theorem (Generalized Levi-Cività connection): *Let $(M, \mathbf{g})$ be a generalized semi-Riemannian manifold. Then there exists a unique connection ∇ such that*

($\nabla 4$) $[\boldsymbol{\eta}, \boldsymbol{\xi}] = \nabla_{\boldsymbol{\eta}} \boldsymbol{\xi} - \nabla_{\boldsymbol{\xi}} \boldsymbol{\eta}$ and

($\nabla 5$) $\boldsymbol{\zeta} \mathbf{g}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \mathbf{g}(\nabla_{\boldsymbol{\zeta}} \boldsymbol{\xi}, \boldsymbol{\eta}) + \mathbf{g}(\boldsymbol{\xi}, \nabla_{\boldsymbol{\zeta}} \boldsymbol{\eta})$,

for all generalized vector fields $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, $\boldsymbol{\zeta}$. ∇ is called the generalized Levi-Cività connection *of M and it is characterized by the Koszul formula*

$$2\mathbf{g}(\nabla_{\boldsymbol{\xi}} \boldsymbol{\eta}, \boldsymbol{\zeta}) = \boldsymbol{\xi} \mathbf{g}(\boldsymbol{\eta}, \boldsymbol{\zeta}) + \boldsymbol{\eta} \mathbf{g}(\boldsymbol{\zeta}, \boldsymbol{\xi}) - \boldsymbol{\zeta} \mathbf{g}(\boldsymbol{\xi}, \boldsymbol{\eta}) - \mathbf{g}(\boldsymbol{\xi}, [\boldsymbol{\eta}, \boldsymbol{\zeta}]) + \mathbf{g}(\boldsymbol{\eta}, [\boldsymbol{\zeta}, \boldsymbol{\xi}]) + \mathbf{g}(\boldsymbol{\zeta}, [\boldsymbol{\xi}, \boldsymbol{\eta}]).$$

Furthermore, the Levi-Cività connection is locally given by

$$\nabla_{\boldsymbol{\xi}} \boldsymbol{\eta} := \sum_{\alpha,\beta,\gamma} \left(\xi^\beta \frac{\partial \eta^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\beta\gamma} \xi^\beta \eta^\gamma \right) \frac{\partial}{\partial x^\alpha} \tag{2.1}$$

where

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} \sum_\delta g^{\alpha\delta} \left\{ \frac{\partial g_{\gamma\delta}}{\partial x^\beta} + \frac{\partial g_{\beta\delta}}{\partial x^\gamma} - \frac{\partial g_{\beta\gamma}}{\partial x^\delta} \right\}. \tag{2.2}$$

Here $[\cdot, \cdot]$ denotes the Lie bracket of two generalized vector fields, defined in the usual way on the level of representatives, cf. [KuSt02a, Definition 10].

2.5. Foliations and Leray forms

Our method to prove existence and uniqueness for the Cauchy problem strongly relies on the concept of energy estimates and spacetime foliations. More precisely, we want to derive estimates for the energy integrals at each time t, thus we need to *foliate* the n-dimensional spacetime manifold into slices of $(n-1)$-dimensional spacelike hypersurfaces. Additionally, we would like this foliation to feature a decomposition of the volume integrals as in Fubini's theorem. Fortunately, locally for any Lorentzian manifold such a foliation exists, as the following lemmata show, for proofs see [Fri75, Section 2.9].

2.5.1. LEMMA: *Suppose that h is a smooth function on a relatively compact open set $U \subseteq M$, such that $\mathrm{d}h$ is non-vanishing on all of U. Then every point $p \in U$ has a neighbourhood that admits local coordinates x^i such that $x^0 = h(p)$.*

From the property $\mathrm{d}h \neq 0$ on U we can conclude that the sets $\Sigma_\tau = \{q \in U | h(q) = \tau\}$ are hypersurfaces unless they are empty. If we, furthermore, demand that $\mathrm{d}h$ is timelike, then by definition the hypersurfaces are spacelike.

2.5.2. LEMMA (Leray form): *Take $h \in \mathcal{C}^\infty(U)$ with $\mathrm{d}h \neq 0$ on U. Then there exists an $(n-1)$-form $\boldsymbol{\mu}_h$ such that $\mathrm{d}h \wedge \boldsymbol{\mu}_h = \boldsymbol{\mu}$, where $\boldsymbol{\mu}$ denotes the volume form on M. The restriction of $\boldsymbol{\mu}_h$ to each hypersurface Σ_τ is unique and we denote this form by $\boldsymbol{\mu}_\tau$.*

We conclude this chapter with a consequence of Fubini's theorem.

2.5.3. LEMMA: *Let f be a locally integrable function on U with compact support, then*

$$\int f\boldsymbol{\mu} = \int \int_{\Sigma_\tau} f\boldsymbol{\mu}_\tau \mathrm{d}\tau.$$

Note that when working with a volume form derived from a generalized metric, the construction above is to understand ε-wise on the level of representatives.

3. A geometric energy method

To obtain existence and uniqueness results for partial differential equations, energy estimates are a useful tool. There exist various methods to establish such estimates. For tensorial second order linear hyperbolic equations—i. e. equations, where the principal part is given by a Lorentzian metric $\mathbf{g}$—the technique of *geometric energy estimates* (see [HaEl73, Chapter 5]) is very useful since it takes causality arguments of the Lorentzian manifold structure into consideration. In this chapter, restricting ourselves to a scenario that involves only smooth functions, we aim at a construction scheme for energy estimates that allows us to keep track of all relevant parameters such that at a later stage we can easily replace smooth functions and fields by generalized ones.

In this approach an essential feature, that differs from previous work on this topic, is the use of two different Lorentzian metrics and a Riemannian background metric. On the one hand we would like to have detailed information which properties of the metrics enter at a certain point in the calculation, which allows us to retain precise control on the asymptotic behaviour of our energy estimates. Then again, when tackling certain nonlinear problems, like Einstein's equations, one is forced to use two separate metrics: A Lorentzian metric that represents the coefficients of the differential equation's principal part and a Lorentzian background metric that realizes the Levi-Cività connection resp. covariant derivative. Furthermore, for the definition of Sobolev norms, we need a Riemannian background metric to calculate a "*pointwise*" positive norm of tensor fields. Summarized, we have

- a Lorentzian metric $\mathbf{g}$ representing the principal part of the equation,
- a Lorentzian metric $\hat{\mathbf{g}}$ representing the Levi-Cività connection and the metric volume form,
- a Riemannian metric $\mathbf{e}$—constructed from $\mathbf{g}$—used with Sobolev norms and energy tensors.

3.1. The setting

For the rest of this chapter, we fix a smooth, n-dimensional Lorentzian manifold $(M, \mathbf{g})$. We are interested in higher order energy estimates for the Cauchy problem of hyperbolic linear partial differential operators with smooth coefficients, i. e.

$$(L\mathbf{u})^I_J = g^{ab}\widehat{\nabla}_a\widehat{\nabla}_b u^I_J + B^{aIP}_{JQ}\widehat{\nabla}_a u^Q_P + C^{IP}_{JQ}u^Q_P = F^I_J, \tag{3.1}$$

with smooth initial data

$$\mathbf{u}|_{\Sigma_0} = \mathbf{u}_0 \qquad\qquad \widehat{\nabla}_{\sigma^\sharp}\mathbf{u}|_{\Sigma_0} = \mathbf{u}_1. \tag{3.2}$$

Here $\widehat{\nabla}$ is the Levi-Cività connection of an eligible background metric $\hat{\mathbf{g}}$ and Σ_0 denotes some initial surface (to be detailed below). In this context eligible metric means that the metric $\hat{\mathbf{g}}$ has the same time orientation as $\mathbf{g}$ and σ is timelike with respect to $\hat{\mathbf{g}}$. Note that by σ we denote a one-form, also timelike with respect to $\mathbf{g}$, to be defined below; the symbol $\sigma^\sharp$ denotes the dual vector field. We denote by $\mathbf{g} = g^{ab}$, $\mathbf{B} = B^{aIP}_{JQ}$ and $\mathbf{C} = C^{IP}_{JQ}$ the (smooth) coefficients of L. In particular, $\mathbf{B}$ and $\mathbf{C}$ will be tensor fields of suitable type. Also the right hand side $\mathbf{F}$, and the data $\mathbf{u}_0$, and $\mathbf{u}_1$ are smooth tensor fields. Note that by σ we denote a timelike one-form to be defined below; the symbol $\sigma^\sharp$ denotes the dual vector field.

We aim at an estimate of the energy at later times in terms of the initial conditions and the right hand side. Therefore, for a point $p \in M$ we choose a relatively compact open neighbourhood and a function $h \in \mathcal{C}^\infty(U)$ such that $h(p) = 0$ and $\sigma := \mathrm{d}h$ is timelike with respect to $\mathbf{g}$. Indeed, the level surfaces $\Sigma_\tau := \{q \in U | h(q) = \tau\}$ with $\tau \in [0, \gamma]$ for some $\gamma > 0$ are spacelike hypersurfaces with respect to $\mathbf{g}$ (see Lemma 2.5.1).

To simplify notation we introduce the following abbreviations for any one-form $\boldsymbol{\omega}$, any vector field $\boldsymbol{\xi}$, and the metric $\mathbf{g}$.

3.1.1. Notation:

1. We denote by $\sharp$ and $\flat$ the musical isomorphisms $\sharp : \mathrm{T}^*M \to \mathrm{T}M$ and $\flat : \mathrm{T}M \to \mathrm{T}^*M$. Thus
   - $\boldsymbol{\omega}^\sharp := \mathbf{g}^{-1}(\boldsymbol{\omega}, \cdot)$, the vector field dual to $\boldsymbol{\omega}$,

- $\xi^\flat := \mathbf{g}(\boldsymbol{\xi}, \cdot)$, the one form dual to $\boldsymbol{\xi}$.

2. For the length of a vector field or one-form, measured in terms of the metric $\mathbf{g}$, we use the same letter in regular italic font, i., e.

   - $\omega := (-\mathbf{g}^{-1}(\boldsymbol{\omega}, \boldsymbol{\omega}))^{\frac{1}{2}} = (-\mathbf{g}(\boldsymbol{\omega}^\sharp, \boldsymbol{\omega}^\sharp))^{\frac{1}{2}}$,
   - $\xi := (-\mathbf{g}(\boldsymbol{\xi}, \boldsymbol{\xi}))^{\frac{1}{2}} = (-\mathbf{g}^{-1}(\boldsymbol{\xi}^\flat, \boldsymbol{\xi}^\flat))^{\frac{1}{2}}$.

3. We write for a unit vector field or one-form, measured in terms of the metric $\mathbf{g}$,

   - $\tilde{\boldsymbol{\omega}} := \boldsymbol{\omega}/\omega$,
   - $\tilde{\boldsymbol{\xi}} := \boldsymbol{\xi}/\xi$.

REMARK:

- This notation is exclusively used in relation to the metric $\mathbf{g}$. Whenever $\hat{\mathbf{g}}$ is used, we will write out terms in full detail.
- Since the action of raising or lowering indices in abstract index notation corresponds to the duality maps denoted by $\sharp$ and $\flat$, the previous remark also applies to this case: Raising and lowering indices will always be done with respect to $\mathbf{g}$, compare this with Remark 2 to Definition 3.2.1 below.
- In abstract index notation we denote the inverse metric of g_{ab} by g^{ab}. In general this notion is not used with any other metric, the only exception beeing $\mathbf{e}$ (see Defintion 3.2.1 and remark).

3.2. Energy estimates

In substance, the proof of existence and uniqueness theorems like [HaEl73, Proposition 7.4.7] relies on higher order energy estimates performed relatively to a foliation of U into the spacelike hypersurfaces Σ_τ, $0 \leq \tau \leq \gamma$ of Section 3.1. We will relate the energy of a solution $\mathbf{u}$ on a surface Σ_τ to the energy on the initial surface Σ_0. To this end, we introduce some more notation (see Figure 1). For any set $\Omega \subset U$, we define

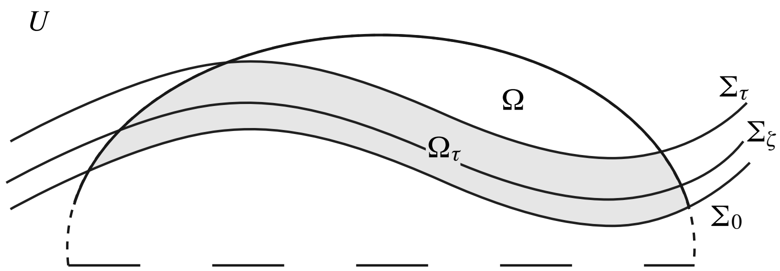

Figure 3.1.: Local foliation of space time

$$\Omega_\tau := \left(\Omega \cap \bigcup_{0\le\zeta\le\tau} \Sigma_\zeta\right)^\circ \text{ and}$$
$$S_\tau := \Sigma_\tau \cap \Omega,$$

where $^\circ$ denotes the interior of a set. Now, let $p \in \Sigma_0$ and let $\Omega \subset U$ be a relatively compact neighbourhood of p such that $\partial\Omega \cap \bigcup_{0\le\zeta} \Sigma_\zeta$ is spacelike. We denote by $\hat{\boldsymbol{\mu}}$ the volume form on M with respect to $\hat{\mathbf{g}}$ and by $\hat{\boldsymbol{\mu}}_\tau$ the restriction of the Leray form to Σ_τ.

Prior to defining the notion of energy tensors and energies, we introduce an auxiliary Riemannian metric which allows for a particularly elloquent formulation of these notions.

3.2.1. DEFINITION: For a timelike one-form $\boldsymbol{\omega}$ and the Lorentzian metric $\mathbf{g}$, we define

$$\mathbf{e}_\omega := 2\tilde{\boldsymbol{\omega}} \otimes \tilde{\boldsymbol{\omega}} + \mathbf{g}.$$

REMARK:

1. The so defined metric $\mathbf{e}_\omega$ is in fact a Riemannian metric: Locally there exist vector fields $\boldsymbol{\eta}_i$ with $1 \le i \le n-1$ such that $\tilde{\boldsymbol{\omega}}^\sharp, \boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_{n-1}$ is a frame field relative to $\mathbf{g}$,

cf. [ONe83, Chapter 3, Corollary 46]. Thus we obtain

$$\begin{aligned}
\mathbf{e}_\omega(\boldsymbol{\eta}_i,\boldsymbol{\eta}_j) &= 2\tilde{\omega}(\boldsymbol{\eta}_i)\tilde{\omega}(\boldsymbol{\eta}_j)+\mathbf{g}(\boldsymbol{\eta}_i,\boldsymbol{\eta}_j)=\delta_{ij}\\
\mathbf{e}_\omega(\tilde{\boldsymbol{\omega}}^\sharp,\boldsymbol{\eta}_i) &= 2\tilde{\omega}(\tilde{\boldsymbol{\omega}}^\sharp)\tilde{\omega}(\boldsymbol{\eta}_i)+\mathbf{g}(\tilde{\boldsymbol{\omega}}^\sharp,\boldsymbol{\eta}_i)=0\\
\mathbf{e}_\omega(\tilde{\boldsymbol{\omega}}^\sharp,\tilde{\boldsymbol{\omega}}^\sharp) &= 2\tilde{\omega}(\tilde{\boldsymbol{\omega}}^\sharp)\tilde{\omega}(\tilde{\boldsymbol{\omega}}^\sharp)+\mathbf{g}(\tilde{\boldsymbol{\omega}}^\sharp,\tilde{\boldsymbol{\omega}}^\sharp)=1,
\end{aligned}$$

where by δ we denote the Kronecker symbol, i. e. the $(n-1)$-dimensional identity matrix.

2. Note that we actually obtain the inverse metric $\mathbf{e}_\omega^{-1}$ by raising the indices of $\mathbf{e}_\omega$ with $\mathbf{g}$ as shown below. We start with the expression

$$e_\omega^{ab}e^\omega_{bc}=\Big(2\tilde{\omega}^a\tilde{\omega}^b+g^{ab}\Big)\cdot\Big(2\tilde{\omega}_b\tilde{\omega}_c+g_{cd}\Big).$$

A short calculation gives

$$e_\omega^{ab}e^\omega_{bc}=-4\tilde{\omega}^a\tilde{\omega}_c+2\tilde{\omega}^a\tilde{\omega}_c+2\tilde{\omega}^a\tilde{\omega}_c+\delta^a_c=\delta^a_c.$$

Thus $e_\omega^{ab}=(e_\omega^{-1})^{ab}$. This is certainly not true for an arbitrary Riemannian metric.

3. In particular when using $\omega=\sigma$, where $\sigma=\mathrm{d}h$ is the timelike one-form defined by the spacetime foliation, it is convenient to write

$$\mathbf{e}:=\mathbf{e}_\sigma=2\tilde{\sigma}\otimes\tilde{\sigma}+\mathbf{g}. \tag{3.3}$$

Now, we define the *"pointwise" norm* of a smooth tensor field $\mathbf{v}$ with respect to a smooth Riemannian metric $\mathbf{m}$, i. e.

$$|\mathbf{v}|^2_{\mathrm{m}}:=(m^{-1})^{IJ}m_{KL}v_I^Kv_J^L.$$

In case we use choose $\mathbf{m}=\mathbf{e}$, where $\mathbf{e}$ is the metric defined above, we write for short

$$|\mathbf{v}|^2:=|\mathbf{v}|^2_{\mathrm{e}}=e^{IJ}e_{KL}v_I^Kv_J^L.$$

This allows us to give

3.2.2. Definition (Sobolev norms): Let $\mathbf{v}$ be a smooth tensor field, and let $0\le\tau\le\gamma$, $m\in\mathbb{N}_0$; we define the Sobolev norms

$$\|\mathbf{v}\|^m_{\Omega_\tau}:=\Big(\sum_{j=0}^m\int_{\Omega_\tau}|\widehat{\nabla}^j\mathbf{v}|^2\hat{\boldsymbol{\mu}}\Big)^{1/2}\text{ and}$$

$$\|\mathbf{v}\|^m_{S_\tau}:=\Big(\sum_{j=0}^m\int_{S_\tau}|\widehat{\nabla}^j\mathbf{v}|^2\hat{\boldsymbol{\mu}}_\tau\Big)^{1/2},$$

where $|\widehat{\nabla}^j \mathbf{v}|^2 = |\widehat{\nabla}_{p_1} \cdots \widehat{\nabla}_{p_j} v_I^K|^2 = (\widehat{\nabla}_{p_1} \cdots \widehat{\nabla}_{p_j} v_I^K)(\widehat{\nabla}_{q_1} \cdots \widehat{\nabla}_{q_j} v_S^R) e^{p_1 q_1} \cdots e^{p_j q_j} e^{IS} e_{KR}$.

Note that since we work locally on compact sets our definition is in fact independent of the choice of $\mathbf{e}$. Furthermore, the $(n-1)$-dimensional Sobolev norm $\|\mathbf{v}\|^m_{S_\tau}$ is defined via the full n-dimensional derivative $\widehat{\nabla}$, i. e. derivatives are not confined to be tangential to the hypersurface S_τ. Equipped with this notion of Sobolev norms, we give the following definitions of *energy tensors* and *energy integrals*.

3.2.3. DEFINITION (Energy integrals): For a smooth tensor field $\mathbf{v}$, $m > 0$, and multi-indices K and R with $|K| = |R| = m-1$, we define the *energy tensors* $\mathbf{T}^m(\mathbf{v})$ of $\mathbf{v}$ of order m by

$$\begin{aligned} T^{ab,0}(\mathbf{v}) &:= -\frac{1}{\sigma^2} g^{ab} |\mathbf{v}|^2 \\ T^{ab,m}(\mathbf{v}) &:= \frac{1}{\sigma^2}(2g^{ac}g^{bd} - g^{ab}g^{cd}) e^{KR} e^{JQ} e_{IP} (\widehat{\nabla}_c \widehat{\nabla}_K v_J^I)(\widehat{\nabla}_d \widehat{\nabla}_R v_Q^P). \end{aligned}$$

For $0 \leq \tau \leq \gamma$ and for $m \geq 0$, we define the *energy integral* $E^m_\tau(\mathbf{v})$ of $\mathbf{v}$ of order m on S_τ by

$$E^m_\tau(\mathbf{v}) := \sum_{j=0}^{m} \int_{S_\tau} T^{ab,j}(\mathbf{v}) \sigma_a \sigma_b \hat{\boldsymbol{\mu}}_\tau.$$

REMARK: Observe that compared with previous literature on this topic (cp. [HaEl73, ViWi00, GMS09]), we slightly altered the defintions of the energy tensors $\mathbf{T}^j$ and the energy integral $E^m_\tau(\mathbf{v})$. For example, compared with [GMS09], we chose to introduce a factor of $\frac{2}{\sigma^2}$ in $\mathbf{T}^j$, we also chose the more symmetrical form $\sum_{j=0}^m \int_{S_\tau} T^{ab,j}(\mathbf{v}) \sigma_a \sigma_b \hat{\boldsymbol{\mu}}_\tau$ of the energy integral, as compared to the expression $\sum_{j=0}^m \int_{S_\tau} T^{ab,j}(\mathbf{v}) \sigma_a \tilde{\sigma}_b \tilde{\boldsymbol{\mu}}_\tau$, where $\tilde{\boldsymbol{\mu}}_\tau = \sigma \hat{\boldsymbol{\mu}}_\tau$ as used in [GMS09]; the advantage being that the squares of Sobolev norms and energies coincide (see also the following lemma).

It is essential to relate Sobolev norms and energy integrals in the proof of our energy estimate. To this end we formulate a variation of Lemma 4.1(1) in [GMS09].

3.2.4. LEMMA: *For all $m \geq 0$ and all smooth tensor fields $\mathbf{v}$, we have*

$$E^m_\tau(\mathbf{v}) = (\|\mathbf{v}\|^m_{S_\tau})^2. \tag{3.4}$$

Proof: For $m = 0$, we have

$$T^{ab,0}(\mathbf{v})\sigma_a\sigma_b = -\frac{1}{\sigma^2}g^{ab}\sigma_a\sigma_b|\mathbf{v}|^2 = |\mathbf{v}|^2. \tag{3.5}$$

So the result for $m = 0$ follows via integration over S_τ.

Now, for $m > 0$, we have

$$\begin{aligned}\frac{1}{\sigma^2}(2g^{ac}g^{bd} - g^{ab}g^{cd})\sigma_a\sigma_b =& 2\tilde{\sigma}^c\tilde{\sigma}^d - \mathbf{g}^{-1}(\tilde{\sigma},\tilde{\sigma})g^{cd}\\ =& 2\tilde{\sigma}^c\tilde{\sigma}^d + g^{cd}.\end{aligned}$$

Hence, for $1 \le j \le m$

$$\begin{aligned}T^{ab,j}(\mathbf{v})\sigma_a\sigma_b = &\left(2\tilde{\sigma}^c\tilde{\sigma}^d + g^{cd}\right)\\ &\cdot e^{KR}e^{JQ}e_{IP}(\widehat{\nabla}_c\widehat{\nabla}_K v^I_J)(\widehat{\nabla}_d\widehat{\nabla}_R v^P_Q),\end{aligned}$$

where the expression in parentheses in the first line is the inverse of the Riemannian metric (3.3). Thus

$$T^{ab,j}(\mathbf{v})\sigma_a\sigma_b = |\widehat{\nabla}^j\mathbf{v}|^2 \tag{3.6}$$

Finally, integration over S_τ and summation over $j = 0,\ldots,m$ gives the result. *q. e. d.*

A key feature in the following energy estimates is the dominant energy condition, cf. [HaEl73, Section 4.3].

3.2.5. DEFINITION: Let $\mathbf{S}$ be a symmetric tensor field of type $(2,0)$. We say $\mathbf{S}$ satisfies the dominant energy condition with respect to $\mathbf{g}$ if for any one form ω with $\mathbf{g}^{-1}(\omega,\omega) < 0$, we have

(i) $\mathbf{S}(\omega,\omega) \ge 0$, and

(ii) $\nu = \mathbf{S}(\omega,\cdot)$ is a non-spacelike vector field.

Next we prove the essential fact that the energy tensors $\mathbf{T}^j(\mathbf{v})$ satisfy the *dominant energy condition* with respect to $\mathbf{g}$. Indeed, for $\mathbf{T}^j(\mathbf{v})$, we have

3.2.6. LEMMA: *For all $j \in \mathbb{N}_0$, the energy tensors $\mathbf{T}^j(\mathbf{v})$ satisfy the dominant energy condition.*

Proof: At first, we have to prove the symmetry of $\mathbf{T}^j(\mathbf{v})$. Obviously, $\mathbf{T}^0(\mathbf{v})$ is symmetric since $\mathbf{g}$ is. For the case $j > 0$, we have to show that $T^{ab} = T^{ba}$. Now, by definition we have

$$\frac{1}{\sigma^2}(2g^{ac}g^{bd} - g^{ab}g^{cd})e^{KR}e^{JQ}e_{IP}(\widehat{\nabla}_c\widehat{\nabla}_K v^I_J)(\widehat{\nabla}_d\widehat{\nabla}_R v^P_Q).$$

The term $\bar{v}_{cd} := e^{KR}e^{JQ}e_{IP}(\widehat{\nabla}_c\widehat{\nabla}_K v^I_J)(\widehat{\nabla}_d\widehat{\nabla}_R v^P_Q)$ is symmetric due to the symmetry of the metric $\mathbf{e}$ and also the quadratic occurrence of $\widehat{\nabla}^m\mathbf{v}$. Thus, we only have to consider the following short calculation. We start with

$$\begin{aligned} T^{ab} =& \frac{1}{\sigma^2}(2g^{ac}g^{bd} - g^{ab}g^{cd})\bar{v}_{cd} \\ =& \frac{1}{\sigma^2}(2g^{ac}g^{bd} - g^{ab}g^{cd})\bar{v}_{dc}. \end{aligned}$$

Through relabelling the indices c and d, we obtain, using also the symmetry of $\mathbf{g}$,

$$\begin{aligned} T^{ab} =& \frac{1}{\sigma^2}(2g^{ad}g^{bc} - g^{ab}g^{dc})\bar{v}_{cd} \\ =& \frac{1}{\sigma^2}(2g^{bc}g^{ad} - g^{ba}g^{cd})\bar{v}_{cd} = T^{ba} \end{aligned}$$

which proves the assertion.

Let $\boldsymbol{\omega}$ be a smooth timelike one-form and let $\boldsymbol{\nu}_j := \mathbf{T}^j(\mathbf{v})(\boldsymbol{\omega}, \cdot)$ as in Definition 3.2.5. Let us consider the case $j = 0$: By definition we have $\mathbf{T}^0(\mathbf{v})(\boldsymbol{\omega}, \boldsymbol{\omega}) = -\frac{1}{\sigma^2}\mathbf{g}^{-1}(\boldsymbol{\omega}, \boldsymbol{\omega}) \geq 0$, which is condition (i). Now, for condition (ii) we need a little calculation:

$$\begin{aligned} g_{bd}\nu_0^b\nu_0^d =& \frac{1}{\sigma^4}g_{bd}g^{ab}g^{cd}\omega_a\omega_c|\mathbf{v}|^2 \\ =& \frac{1}{\sigma^4}g^{ac}\omega_a\omega_c|\mathbf{v}|^2 \\ =& \frac{1}{\sigma^4}\mathbf{g}^{-1}(\boldsymbol{\omega}, \boldsymbol{\omega})|\mathbf{v}|^2. \end{aligned}$$

Since $\boldsymbol{\omega}$ is timelike, the above expression is less or equal to zero and we are done.

For the case $j > 0$, consider the following

1. CLAIM: $\mathbf{T}^j(\mathbf{v})(\boldsymbol{\omega}, \boldsymbol{\omega}) \geq 0$.

   We obtain

$$\begin{aligned} T^{ab,j}(\mathbf{v})\omega_a\omega_b =& \frac{1}{\sigma^2}(2g^{ac}g^{bd} - g^{ab}g^{cd})\omega_a\omega_b \cdot \\ & e^{KR}e^{JQ}e_{IP}(\widehat{\nabla}_c\widehat{\nabla}_K v^I_J)(\widehat{\nabla}_d\widehat{\nabla}_R v^P_Q), \end{aligned} \tag{3.7}$$

where $(2g^{ac}g^{bd} - g^{ab}g^{cd})\omega_a\omega_b = 2\omega^c\omega^d + \omega^2 g^{cd} = \omega^2 e_\omega^{cd}$ is the inverse of a Riemannian metric (cf. with Definition 3.2.1); thus, expression (3.7) is greater or equal to zero. $\square$

2. CLAIM: $\boldsymbol{\nu}_j$ is non-spacelike.

We find for scalar $\mathbf{v}$ and $j = 1$ that

$$\begin{aligned}
T^{ab,1}(\mathbf{v})\omega_a =& \frac{1}{\sigma^2}(2g^{ac}g^{bd}\nabla_c v\nabla_d v\omega_a - g^{ab}g^{cd}\nabla_c v\nabla_d v\omega_a)\\
=& \frac{1}{\sigma^2}(2\omega^c\nabla_c v g^{bd}\nabla_d v - \omega^b g^{cd}\nabla_c v\nabla_d v)\\
g_{b\bar{b}}T^{ab,1}(\mathbf{v})\omega_a T^{\bar{a}\bar{b},1}(\mathbf{v})\omega_{\bar{a}} =& \frac{1}{\sigma^4}(2\omega^c\nabla_c v\nabla_{\bar{b}} v - \omega_{\bar{b}} g^{cd}\nabla_c v\nabla_d v)\\
&\cdot (2\omega^{\bar{c}}\nabla_{\bar{c}} v g^{\bar{b}\bar{d}}\nabla_{\bar{d}} v - \omega^{\bar{b}} g^{\bar{c}\bar{d}}\nabla_{\bar{c}} v\nabla_{\bar{d}} v)\\
=& \frac{1}{\sigma^4}(4\omega^c\nabla_c v\omega^{\bar{c}}\nabla_{\bar{c}} v g^{\bar{b}\bar{d}}\nabla_{\bar{b}} v\nabla_{\bar{d}} v\\
&- 2\omega^{\bar{d}}\nabla_{\bar{d}} v\omega^{\bar{c}}\nabla_{\bar{c}} v g^{cd}\nabla_c v\nabla_d v\\
&- 2\omega^c\nabla_c v\omega^{\bar{b}}\nabla_{\bar{b}} v g^{\bar{c}\bar{d}}\nabla_{\bar{c}} v\nabla_{\bar{d}} v\\
&+ \omega_{\bar{b}}\omega^{\bar{b}} g^{cd}\nabla_c v\nabla_d v g^{\bar{c}\bar{d}}\nabla_{\bar{c}} v\nabla_{\bar{d}} v)\\
=& \frac{1}{\sigma^4}\omega_{\bar{b}}\omega^{\bar{b}}(g^{cd}\nabla_c v\nabla_d v)^2,
\end{aligned} \tag{3.8}$$

where the last line is less or equal to zero, since $\boldsymbol{\omega}$ is timelike.

To show the general case where $\mathbf{v}$ is a tensor field and $j > 1$, we choose an orthonormal basis with respect to the metric $\mathbf{e}$. In such coordinates

$$T^{\alpha\beta,j}(\mathbf{v}) = \sum_{\substack{\mu_1,\ldots,\mu_l\\ \nu_1,\ldots,\nu_k\\ \rho_1,\ldots,\rho_{j-1}}} \frac{1}{\sigma^2}(2g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\beta}g^{\gamma\delta})\cdot$$
$$\nabla_\gamma\nabla_{\rho_{j-1}}\ldots\nabla_{\rho_1}v^{\mu_1\ldots\mu_l}{}_{\nu_1\ldots\nu_k}\nabla_\delta\nabla_{\rho_{j-1}}\ldots\nabla_{\rho_1}v^{\mu_1\ldots\mu_l}{}_{\nu_1\ldots\nu_k}.$$

We fix the indices $\mu_1,\ldots,\mu_l,\nu_1,\ldots,\nu_k,\rho_1,\ldots,\rho_{j-1}$ and apply equation (3.8) to each ${}_{\rho_{j-1}\ldots\rho_1}T^{\alpha\beta\mu_1\ldots\mu_l}{}_{\nu_1\ldots\nu_k}(\mathbf{v})\omega_\alpha\omega_\beta$ separately. By the positivity of (3.7) we know that all the ${}_{\rho_{j-1}\ldots\rho_1}T^{\alpha\beta\mu_1\ldots\mu_l,j}{}_{\nu_1\ldots\nu_k}(\mathbf{v})\omega_\alpha$ have the same time orientation, namely that of $-\omega^\beta$. Therefore, we finally obtain that

$$T^{\alpha\beta,j}(\mathbf{v})\omega_\alpha = \sum_{\substack{\mu_1,\ldots,\mu_l\\ \nu_1,\ldots,\nu_k\\ \rho_1,\ldots,\rho_{j-1}}} {}_{\rho_{j-1}\ldots\rho_1}T^{\alpha\beta\mu_1\ldots\mu_l,j}{}_{\nu_1\ldots\nu_k}(\mathbf{v})\omega_\alpha$$

is non-spacelike since the sum of non-spacelike vectors of the same time orientation is indeed non-spacelike. □

This completes the proof and so we obtain that $\mathbf{T}^j(\mathbf{v})$ satisfies the dominant energy condition. *q. e. d.*

The dominant energy condition is a key feature in the following estimates. It guarantees positivity of

$$\int\limits_{\partial\Omega_\tau\setminus(S_\tau\cup S_0)} T^{ab,j}(\mathbf{v})\sigma_a \,\mathrm{d}\Omega_b,$$

where $\mathrm{d}\Omega_a$ is the surface element on $\partial\Omega$. Hence the dominant energy condition implies via Stokes' theorem the following lemma, similar to the *divergence theorem* (cf. Lemma 4.3.1 in [HaEl73]). At this point we want to indicate once more, that it is imperative for the metric $\hat{\mathbf{g}}$ to have the same time-orientation as $\mathbf{g}$, and that $\mathbf{g}^{-1}(\sigma,\sigma) < 0$.

3.2.7. Lemma: *Let Ω_τ and $\mathbf{T}^j(\mathbf{v})$ be defined as before, then we have the following estimate*

$$\int\limits_{S_\tau} T^{ab,j}(\mathbf{v})\sigma_b\sigma_a\hat{\boldsymbol{\mu}}_\tau \le \int\limits_{S_0} T^{ab,j}(\mathbf{v})\sigma_b\sigma_a\hat{\boldsymbol{\mu}}_0 + \int\limits_0^\tau\int\limits_{S_\zeta} T^{ab,j}(\mathbf{v})\widehat{\nabla}_b\sigma_a\hat{\boldsymbol{\mu}}_\zeta \,\mathrm{d}\zeta + \int\limits_{\Omega_\tau} \widehat{\nabla}_a T^{ab,j}(\mathbf{v})\sigma_b\hat{\boldsymbol{\mu}}. \tag{3.9}$$

Proof: To ensure a compact notation, in this proof we write $\mathbf{T}^{ab,j}$ for $\mathbf{T}^{ab,j}(\mathbf{v})$. Let $\boldsymbol{\nu}$ denote the outward pointing unit normal form on Ω_τ. The boundary of Ω_τ consists of three parts: S_0, S_τ and $\partial\Omega_\tau\setminus(S_\tau\cup S_0) =: S_{\Omega,\tau}$. On S_τ we have that $\boldsymbol{\nu} = \hat{\boldsymbol{\sigma}}$, whereas on S_0 we have that $\boldsymbol{\nu} = -\hat{\boldsymbol{\sigma}}$. By $\hat{\boldsymbol{\sigma}}$, we mean the unit form related to $\boldsymbol{\sigma}$ via the metric $\hat{\mathbf{g}}$, i. e. we have

$$\hat{\boldsymbol{\sigma}} := \frac{\boldsymbol{\sigma}}{\sqrt{|\hat{\mathbf{g}}^{-1}(\sigma,\sigma)|}}.$$

There is no need to specify $\boldsymbol{\nu}$ on $S_{\Omega,\tau}$ but we know that on $S_{\Omega,\tau}$ both, $\boldsymbol{\nu}$ and $\hat{\boldsymbol{\sigma}}$, have the same time orientation. Now, consider the volume integral

$$I(t) = \int\limits_{\Omega_\tau} \widehat{\nabla}_b(T^{ab,j}\sigma_a)\hat{\boldsymbol{\mu}} = \int\limits_{\Omega_\tau} T^{ab,j}\widehat{\nabla}_b\sigma_a\hat{\boldsymbol{\mu}} + \int\limits_{\Omega_\tau} \widehat{\nabla}_b T^{ab,j}\sigma_a\hat{\boldsymbol{\mu}}. \tag{3.10}$$

Using Stokes' theorem, we can transform (3.10) into a surface integral over $\partial\Omega_\tau$. Hence

$$I(t) = \int_{\partial\Omega_\tau} T^{ab,j}\sigma_a \nu_b \hat{\boldsymbol{\mu}}_{\partial\Omega_\tau}.$$

Splitting the boundary of Ω_τ into the three parts mentioned above, the integral is given by

$$I(t) = -\int_{S_0} T^{ab,j}\sigma_a \hat{\sigma}_b \hat{\boldsymbol{\mu}}_{S_0} + \int_{S_\tau} T^{ab,j}\sigma_a \hat{\sigma}_b \hat{\boldsymbol{\mu}}_{S_\tau} + \int_{S_{\Omega,\tau}} T^{ab,j}\sigma_a \nu_b \hat{\boldsymbol{\mu}}_{S_{\Omega,\tau}},$$

where $\boldsymbol{\mu}_U$ denotes the area form on an $(n-1)$-dimensional subset U of M. Replacing the area forms in the first two terms by the corresponding Leray forms yields

$$I(t) = -\int_{S_0} T^{ab,j}\sigma_a \hat{\sigma}_b |(\hat{g}^{-1})^{cd}\sigma_c\hat{\sigma}_d| \hat{\boldsymbol{\mu}}_0 + \int_{S_\tau} T^{ab,j}\sigma_a \hat{\sigma}_b |(\hat{g}^{-1})^{cd}\sigma_c\hat{\sigma}_d| \hat{\boldsymbol{\mu}}_\tau + \int_{S_{\Omega,\tau}} T^{ab,j}\sigma_a \nu_b \hat{\boldsymbol{\mu}}_{S_{\Omega,\tau}}.$$

By the $\mathcal{C}^\infty$-linearity of the tensor fields this gives

$$\begin{aligned} I(t) &= -\int_{S_0} T^{ab,j}\sigma_a \sigma_b |(\hat{g}^{-1})^{cd}\hat{\sigma}_c\hat{\sigma}_d| \hat{\boldsymbol{\mu}}_0 + \int_{S_\tau} T^{ab,j}\sigma_a \sigma_b |(\hat{g}^{-1})^{cd}\hat{\sigma}_c\hat{\sigma}_d| \hat{\boldsymbol{\mu}}_\tau + \int_{S_{\Omega,\tau}} T^{ab,j}\sigma_a \nu_b \hat{\boldsymbol{\mu}}_{S_{\Omega,\tau}} \\ &= -\int_{S_0} T^{ab,j}\sigma_a \sigma_b \hat{\boldsymbol{\mu}}_0 + \int_{S_\tau} T^{ab,j}\sigma_a \sigma_b \hat{\boldsymbol{\mu}}_\tau + \int_{S_{\Omega,\tau}} T^{ab,j}\sigma_a \nu_b \hat{\boldsymbol{\mu}}_{S_{\Omega,\tau}}. \end{aligned}$$

By the dominant energy condition, we know that $\mathbf{T}^j(\boldsymbol{\sigma}, \cdot)$ is a non spacelike vector field with reversed time orientation as compared to $\boldsymbol{\sigma}$ and thus, on $S_{\Omega,\tau}$, also reversed to $\boldsymbol{\nu}$. Therefore, $\mathbf{T}^j(\boldsymbol{\sigma}, \boldsymbol{\nu}) \geq 0$ and we obtain

$$I(t) \geq -\int_{S_0} T^{ab,j}\sigma_a \sigma_b \hat{\boldsymbol{\mu}}_0 + \int_{S_\tau} T^{ab,j}\sigma_a \sigma_b \hat{\boldsymbol{\mu}}_\tau.$$

Combining the last equation with (3.10) gives

$$\int_{\Omega_\tau} T^{ab,j}\widehat{\nabla}_b \sigma_a \hat{\boldsymbol{\mu}} + \int_{\Omega_\tau} \widehat{\nabla}_b T^{ab,j}\sigma_a \hat{\boldsymbol{\mu}} \geq -\int_{S_0} T^{ab,j}\sigma_a \sigma_b \hat{\boldsymbol{\mu}}_0 + \int_{S_\tau} T^{ab,j}\sigma_a \sigma_b \hat{\boldsymbol{\mu}}_\tau.$$

The final result follows from application of Lemma 2.5.3 to the leftmost expression. *q. e. d.*

Our next task is to estimate the expression involving $\widehat{\nabla}\sigma$ in equation (3.9). By the Cauchy-Schwartz inequality on tensor bundles we obtain

$$T^{ab,j}(\mathbf{v})\widehat{\nabla}_b \sigma_a \leq |T^{ab,j}(\mathbf{v})\widehat{\nabla}_b \sigma_a| \leq |\mathbf{T}^j(\mathbf{v})| \cdot |\widehat{\nabla}\sigma| \leq A|\mathbf{T}^j(\mathbf{v})|.$$

Here $A = \sup_{\Omega_\tau} |\widehat{\nabla}\sigma|$ is a constant depending on certain components of the Christoffel symbols for $\hat{\mathbf{g}}$, denoted by $\hat{\mathbf{\Gamma}}$, and $\partial\sigma$.

Now, we aim on the estimate on the energy tensor. We split the calculation into two cases. At first, let $j = 0$; we obtain

$$|\mathbf{T}^0(\mathbf{v})|^2 = \left(\frac{1}{\sigma^2}\right)^2 e_{ab}e_{cd}g^{ac}g^{bd}|\mathbf{v}|^4 = \frac{n}{\sigma^4}|\mathbf{v}|^4 \leq \tilde{A}_0|\mathbf{v}|^4, \tag{3.11}$$

where $\tilde{A}_0$ is a constant dependent on $\mathbf{g}$ and σ. Application of equation (3.5) and integration over S_ζ gives with $A_0 = A \cdot \tilde{A}_0$

$$\int\limits_{S_\zeta} T^{ab,0}(\mathbf{v})\widehat{\nabla}_b\sigma_a\hat{\boldsymbol{\mu}}_\zeta \leq A_0(\mathbf{g},\sigma,\widehat{\nabla}\sigma)\int\limits_{S_\tau} T^{ab,0}(\mathbf{v})\sigma_a\sigma_b\hat{\boldsymbol{\mu}}_\zeta,$$

which concludes the case $j = 0$. For the remaining part of the calculation, let $j > 0$. We have that

$$\begin{aligned}|\mathbf{T}^j(\mathbf{v})|^2 =&\left(\frac{1}{\sigma^2}\right)^2 e_{aa'}e_{bb'}(2g^{ac}g^{bd} - g^{ab}g^{cd})(2g^{a'c'}g^{b'd'} - g^{a'b'}g^{c'd'})\\ &\cdot(\widehat{\nabla}_c\widehat{\nabla}_I v^K_P)(\widehat{\nabla}_d\widehat{\nabla}_J v^L_Q)(\widehat{\nabla}_{c'}\widehat{\nabla}'_I v^{K'}_{P'})(\widehat{\nabla}'_d\widehat{\nabla}_{J'} v^{L'}_{Q'})\\ &\cdot e^{IJ}e_{KL}e^{PQ}e^{I'J'}e_{K'L'}e^{P'Q'}.\end{aligned} \tag{3.12}$$

We calculate the first line of equation (3.12); thus,

$$\begin{aligned}&\left(\frac{1}{\sigma^2}\right)^2 e_{aa'}e_{bb'}(2g^{ac}g^{bd} - g^{ab}g^{cd})(2g^{a'c'}g^{b'd'} - g^{a'b'}g^{c'd'})\\ &\qquad=\left(\frac{1}{\sigma^2}\right)^2 e_{aa'}e_{bb'}(4g^{ac}g^{bd}g^{a'c'}g^{b'd'} - 2g^{ac}g^{bd}g^{a'b'}g^{c'd'}\\ &\qquad\quad - 2g^{a'c'}g^{b'd'}g^{ab}g^{cd} + g^{ab}g^{cd}g^{a'b'}g^{c'd'})\\ &\qquad=\frac{1}{\sigma^4}(4e^{cc'}e^{dd'} + (n-4)g^{cd}g^{c'd'}).\end{aligned}$$

Note that we have $\mathbf{g}^{-1}(\omega,\omega) \leq A'\mathbf{e}^{-1}(\omega,\omega)$ for all smooth one forms ω, where A' is a constant depending on $\mathbf{g}$. Together with equation (3.12) this yields

$$\begin{aligned}|\mathbf{T}^j(\mathbf{v})|^2 =&\frac{A'}{\sigma^4}(4e^{cc'}e^{dd'}+(n-4)e^{cd}e^{c'd'})\\&\cdot(\widehat{\nabla}_c\widehat{\nabla}_I v^K_P)(\widehat{\nabla}_d\widehat{\nabla}_J v^L_Q)(\widehat{\nabla}_{c'}\widehat{\nabla}'_I v^{K'}_{P'})(\widehat{\nabla}'_d\widehat{\nabla}_{J'}v^{L'}_{Q'})\\&\cdot e^{IJ}e_{KL}e^{PQ}e^{I'J'}e_{K'L'}e^{P'Q'}\\\leq&\frac{\tilde{A}_j}{\sigma^4}e^{cd}e^{c'd'}(\widehat{\nabla}_c\widehat{\nabla}_I v^K_P)(\widehat{\nabla}_d\widehat{\nabla}_J v^L_Q)(\widehat{\nabla}_{c'}\widehat{\nabla}'_I v^{K'}_{P'})(\widehat{\nabla}'_d\widehat{\nabla}_{J'}v^{L'}_{Q'})e^{IJ}e_{KL}e^{PQ}e^{I'J'}e_{K'L'}e^{P'Q'}\\\leq&\frac{\tilde{A}_j}{\sigma^4}|\widehat{\nabla}^j\mathbf{v}|^2.\end{aligned}$$

where $\tilde{A}_j$ depends on the recombination of the indices in the first line of the above equation, the dimension of the manifold, and A' from the norm inequality above. Now, we apply equation (3.6) and integrate over S_ζ such that we obtain for $j>0$

$$\int\limits_{S_\zeta} T^{ab,j}(\mathbf{v})\widehat{\nabla}_b\sigma_a\hat{\boldsymbol{\mu}}_\zeta \leq A_j(\mathbf{g},\boldsymbol{\sigma},\widehat{\nabla}\boldsymbol{\sigma})\int\limits_{S_\tau} T^{ab,j}(\mathbf{v})\sigma_a\sigma_b\hat{\boldsymbol{\mu}}_\zeta,$$

where we have $A_j = A\cdot\tilde{A}_j\inf_{\Omega_\tau}\frac{1}{\sigma^4}$. Finally, inserting (3.11) and the last result into inequality (3.9) yields for all $j\geq 0$

$$\begin{aligned}\int\limits_{S_\tau} T^{ab,j}(\mathbf{v})\sigma_b\sigma_a\hat{\boldsymbol{\mu}}_\tau \leq \int\limits_{S_0} T^{ab,j}(\mathbf{v})\sigma_b\sigma_a\hat{\boldsymbol{\mu}}_0 + A_j\int\limits_0^\tau\int\limits_{S_\zeta} T^{ab,j}(\mathbf{v})\sigma_b\sigma_a\hat{\boldsymbol{\mu}}_\zeta\,\mathrm{d}\zeta\\+\int\limits_{\Omega_\tau}\widehat{\nabla}_a T^{ab,j}(\mathbf{v})\sigma_b\hat{\boldsymbol{\mu}}.\end{aligned}$$

Summation over j for $0\leq\tau\leq\gamma$ yields the estimate

$$\begin{aligned}E^m_\tau(\mathbf{v})\leq&E^m_0(\mathbf{v})+\sum_{j=0}^m\left(P\int\limits_0^\tau\int\limits_{S_\zeta} T^{ab,j}(\mathbf{v})\sigma_a\sigma_b\hat{\boldsymbol{\mu}}_\zeta\mathrm{d}\zeta+\int\limits_{\Omega_\tau}\sigma_b\widehat{\nabla}_a T^{ab,j}(\mathbf{v})\hat{\boldsymbol{\mu}}\right)\\=&E^m_0(\mathbf{v})+P\int\limits_0^\tau E^m_\zeta(\mathbf{v})\mathrm{d}\zeta+\sum_{j=0}^m\int\limits_{\Omega_\tau}\sigma_b\widehat{\nabla}_a T^{ab,j}(\mathbf{v})\hat{\boldsymbol{\mu}}\end{aligned}\tag{3.13}$$

which will be our main tool in this chapter. The constant P depends on $\mathbf{g}$, $\boldsymbol{\sigma}$ and $\widehat{\nabla}\boldsymbol{\sigma}$. In fact, (3.13) will be used to prove the energy estimates needed to derive moderateness and negligibility of generalized solutions in chapter 4. The technical core is to provide estimates on the divergence term $\widehat{\nabla}_a T^{ab,j}(\mathbf{u})$ in (3.13) for a solution $\mathbf{u}$ of the differential equation (3.1).

3.2.8. PROPOSITION: *Let* $\mathbf{u}$ *be a solution of the initial value problem* (3.1), (3.2) *on* U. *Then, for every* $m \geq 1$, *there exist constants*

$$C'_m(\mathbf{g}, \widehat{\nabla}\mathbf{g}, \mathbf{e}, \widehat{\nabla}\mathbf{e}, \sigma, \widehat{\nabla}\sigma, \mathbf{B}),$$
$$C''_m(\mathbf{g}, \sigma), \text{ and}$$
$$C'''_m(\mathbf{g}, \dots, \widehat{\nabla}^{m-1}\mathbf{g}, \sigma, \mathbf{B}, \dots, \widehat{\nabla}^{m-1}\mathbf{B}, \mathbf{C}, \dots, \widehat{\nabla}^{m-1}\mathbf{C}, \hat{\mathbf{R}}, \dots, \widehat{\nabla}^{m-1}\hat{\mathbf{R}})$$

such that for every $0 \leq \tau \leq \gamma$,

$$\begin{aligned} E^m_\tau(\mathbf{u}) \leq & E^m_0(\mathbf{u}) + C'_m \int_0^\tau E^m_\zeta(\mathbf{u})\,\mathrm{d}\zeta \\ & + C''_m(\|\mathbf{F}\|^{m-1}_{\Omega_\tau})^2 + C'''_m \int_0^\tau E^{m-1}_\zeta(\mathbf{u})\,\mathrm{d}\zeta. \end{aligned} \tag{3.14}$$

Observe that the constant C'_m in front of the first integral in (3.14) depends only on derivatives of a fixed order (i. e. the order has no dependence on m). This is essential later on, when applying Gronwall's inequality in the course of proving the key energy estimate for generalized solutions of the wave equation (see the next section, Corollary 4.3.6).

Proof: To prove Proposition 3.2.8, we calculate the divergence expression of estimate (3.13) for the energy tensor $\mathbf{T}^j(\mathbf{u})$ in the cases $j = 0$, $j = 1$, and $j > 1$. The final result then follows via integration over Ω_τ and summation over j from $j = 0$ to m. Starting with $\sigma_b \widehat{\nabla}\mathbf{T}^0(\mathbf{u})$, we have

$$\begin{aligned} \sigma_b \widehat{\nabla}_a T^{ab,0}(\mathbf{u}) = & -\sigma_b \widehat{\nabla}_a \left(\frac{1}{\sigma^2} g^{ab} u^I_J u^P_Q e_{IP} e^{JQ} \right) \\ = & -\sigma_b \widehat{\nabla}_a \left(\frac{1}{\sigma^2} g^{ab} \right) |\mathbf{u}|^2 - 2\frac{\sigma_b}{\sigma^2} g^{ab} u^I_J \widehat{\nabla}_a u^P_Q e^{JQ} e_{IP} \\ & - \frac{\sigma_b}{\sigma^2} g^{ab} u^I_J u^P_Q \widehat{\nabla}_a (e^{JQ} e_{IP}). \end{aligned}$$

Therefore, by the Cauchy-Schwartz inequality for the inner product induced by $\mathbf{e}$ on the tensor bundle on M, we have

$$\begin{aligned}
|\sigma_b \widehat{\nabla}_a T^{ab,0}(\mathbf{u})| \leq & \left|\sigma_b \widehat{\nabla}_a \frac{g^{ab}}{\sigma^2}\right| \cdot |\mathbf{u}|^2 + \left|2\frac{\sigma_b}{\sigma^2} g^{ab} \widehat{\nabla}_a u^P_Q e^{JQ} e_{IP} u^I_J\right| \\
& + \left|\frac{\sigma_b}{\sigma^2} g^{ab} \widehat{\nabla}_a (e^{J'Q} e_{I'P}) u^P_Q e_{J'J''} e^{J''J} e^{I'I''} e_{I''I} u^I_J\right| \\
= & P_0 |\mathbf{u}|^2 + \left\langle 2\frac{\sigma_b}{\sigma^2} g^{ab} \widehat{\nabla}_a u^P_Q, u^I_J \right\rangle_{\mathrm{e}} \\
& + \left\langle \frac{\sigma_b}{\sigma^2} g^{ab} \widehat{\nabla}_a (e^{J'Q} e_{I'P}) u^P_Q, u^I_J \right\rangle_{\mathrm{e}} \\
\leq & P_0 |\mathbf{u}|^2 + \left|2\frac{\sigma_b}{\sigma^2} g^{ab} \widehat{\nabla}_a u^P_Q\right| \cdot |\mathbf{u}| \\
& + \left|\frac{\sigma_b}{\sigma^2} g^{ab} \widehat{\nabla}_a (e^{J'Q} e_{I'P}) u^P_Q\right| \cdot |\mathbf{u}| \\
\leq & P'_0 |\mathbf{u}|^2 + P''_0 |\mathbf{u}| \cdot |\widehat{\nabla}\mathbf{u}| \leq \bar{P}_0 |\mathbf{u}|^2 + \tilde{P}_0 |\widehat{\nabla}\mathbf{u}|^2
\end{aligned}$$

This yields

$$|\sigma_b \widehat{\nabla}_a T^{ab,0}(\mathbf{u})| \leq \alpha_0 (|\mathbf{u}|^2 + |\widehat{\nabla}\mathbf{u}|^2)$$

for a constant $\alpha_0 = \alpha_0(\mathbf{g}, \widehat{\nabla}\mathbf{g}, \mathbf{e}, \widehat{\nabla}\mathbf{e}, \sigma, \widehat{\nabla}\sigma) = \bar{P}_0 + \tilde{P}_0$.

Now, let $j = 1$, then $\sigma(\operatorname{div} \mathbf{T}^1(\mathbf{u}))$ reads

$$\begin{aligned}
\sigma_b \widehat{\nabla}_a T^{ab,1}(\mathbf{u}) = & \frac{\sigma_b}{\sigma^2} (2g^{ac} g^{bd} - g^{ab} g^{cd}) (\widehat{\nabla}_a \widehat{\nabla}_c u^I_J \widehat{\nabla}_d u^P_Q + \widehat{\nabla}_c u^I_J \widehat{\nabla}_a \widehat{\nabla}_d u^P_Q) e^{JQ} e_{IP} \\
& + \sigma_b \widehat{\nabla}_a \left(\frac{1}{\sigma^2} (2g^{ac} g^{bd} - g^{ab} g^{cd})\right) \widehat{\nabla}_c u^I_J \widehat{\nabla}_d u^P_Q e^{JQ} e_{IP} \\
& + \frac{\sigma_b}{\sigma^2} (2g^{ac} g^{bd} - g^{ab} g^{cd}) \widehat{\nabla}_c u^I_J \widehat{\nabla}_d u^P_Q \widehat{\nabla}_a (e^{JQ} e_{IP})
\end{aligned}$$

We obtain by the symmetry of g^{cd} and $\mathbf{e}$

$$\begin{aligned}
\sigma_b \widehat{\nabla}_a T^{ab,1}(\mathbf{u}) = & \frac{\sigma_b}{\sigma^2} 2g^{ac} g^{bd} \widehat{\nabla}_a \widehat{\nabla}_c u^I_J \widehat{\nabla}_d u^P_Q e^{JQ} e_{IP} + \frac{\sigma_b}{\sigma^2} 2g^{ac} g^{bd} \widehat{\nabla}_a \widehat{\nabla}_d u^I_J \widehat{\nabla}_c u^P_Q e^{JQ} e_{IP} \\
& - \frac{\sigma_b}{\sigma^2} 2g^{ab} g^{cd} \widehat{\nabla}_a \widehat{\nabla}_d u^I_J \widehat{\nabla}_c u^P_Q e^{JQ} e_{IP} + \sigma_b \widehat{\nabla}_a (h^{abcd}) \widehat{\nabla}_c u^I_J \widehat{\nabla}_d u^P_Q e^{JQ} e_{IP} \\
& + \sigma_b h^{abcd} \widehat{\nabla}_c u^I_J \widehat{\nabla}_d u^P_Q \widehat{\nabla}_a (e^{JQ} e_{IP})
\end{aligned}$$

where $h^{abcd} := \frac{1}{\sigma^2}(2g^{ac} g^{bd} - g^{ab} g^{cd})$. Finally, we interchange the indices a and d, to sum the second and third term

$$\begin{aligned}
\sigma_b \widehat{\nabla}_a T^{ab,1}(\mathbf{u}) = & \frac{2\sigma_b}{\sigma^2} g^{ac} g^{bd} \widehat{\nabla}_a \widehat{\nabla}_c u^I_J \widehat{\nabla}_d u^P_Q e^{JQ} e_{IP} + \frac{2\sigma_b}{\sigma^2} g^{ac} g^{bd} [\widehat{\nabla}_a, \widehat{\nabla}_d] u^I_J \widehat{\nabla}_c u^P_Q e^{JQ} e_{IP} \\
& + \widehat{\nabla}_a h^{abcd} \widehat{\nabla}_c u^I_J \widehat{\nabla}_d u^P_Q e^{JQ} e_{IP} + h^{abcd} \widehat{\nabla}_c u^I_J \widehat{\nabla}_d u^P_Q \widehat{\nabla}_a (e^{JQ} e_{IP}), \qquad (3.15)
\end{aligned}$$

where $[\cdot,\cdot]$ denotes the commutator. Our goal is to reduce the number of covariant derivatives by one. This can be achieved by manipulation of the leading order term with the differential equation (3.1), thus

$$\begin{aligned}\sigma_b \widehat{\nabla}_a T^{ab,1}(\mathbf{u}) =& \frac{2\sigma_b}{\sigma^2} g^{bd} F^I_J \widehat{\nabla}_d u^P_Q e^{JQ} e_{IP} - \frac{2\sigma_b}{\sigma^2} g^{bd} B^{aIJ'}_{JI'} \widehat{\nabla}_a u^{I'}_{J'} \widehat{\nabla}_d u^P_Q e^{JQ} e_{IP} \\ &- \frac{2\sigma_b}{\sigma^2} g^{bd} C^{IJ'}_{JI'} u^{I'}_{J'} \widehat{\nabla}_d u^P_Q e^{JQ} e_{IP} + \frac{2\sigma_b}{\sigma^2} g^{ac} g^{bd} [\widehat{\nabla}_a, \widehat{\nabla}_d] u^I_J \widehat{\nabla}_c u^P_Q e^{JQ} e_{IP} \\ &+ \widehat{\nabla}_a h^{abcd} \widehat{\nabla}_c u^I_J \widehat{\nabla}_d u^P_Q e^{JQ} e_{IP} + h^{abcd} \widehat{\nabla}_c u^I_J \widehat{\nabla}_d u^P_Q \widehat{\nabla}_a (e^{JQ} e_{IP}). \end{aligned} \tag{3.16}$$

Observe that the commutator, by Ricci's identity, is a linear combination of contractions of $\mathbf{u}$ and the Riemannian curvature tensor $\hat{\mathbf{R}}$ of the metric $\hat{\mathbf{g}}$. By repeated use of the Cauchy-Schwartz inequality and the inequality of arithmetic and geometric means, we obtain

$$\begin{aligned}\left|\frac{2\sigma_b}{\sigma^2} g^{bd} F^I_J \widehat{\nabla}_d u^P_Q e^{JQ} e_{IP}\right| \le& \sqrt{\beta_1} |\mathbf{F}| \cdot |\widehat{\nabla} \mathbf{u}| \le \beta_1 |\mathbf{F}|^2 + |\widehat{\nabla}\mathbf{u}|^2 \\ \left|\frac{2\sigma_b}{\sigma^2} g^{bd} B^{aIJ'}_{JI'} \widehat{\nabla}_a u^{I'}_{J'} \widehat{\nabla}_d u^P_Q e^{JQ} e_{IP}\right| \le& P_{1,\mathrm{B}} |\widehat{\nabla}\mathbf{u}|^2 \\ \left|\frac{2\sigma_b}{\sigma^2} g^{bd} C^{IJ'}_{JI'} u^{I'}_{J'} \widehat{\nabla}_d u^P_Q e^{JQ} e_{IP}\right| \le& P_{1,\mathrm{C}} |\mathbf{u}| \cdot |\widehat{\nabla}\mathbf{u}| \le P^2_{1,\mathrm{C}} |\mathbf{u}|^2 + |\widehat{\nabla}\mathbf{u}|^2 \\ \left|\frac{2\sigma_b}{\sigma^2} g^{ac} g^{bd} [\widehat{\nabla}_a, \widehat{\nabla}_d] u^I_J \widehat{\nabla}_c u^P_Q e^{JQ} e_{IP}\right| \le& P_{1,\mathrm{R}} |\mathbf{u}| \cdot |\widehat{\nabla}\mathbf{u}| \le P^2_{1,\mathrm{R}} |\mathbf{u}|^2 + |\widehat{\nabla}\mathbf{u}|^2 \\ \left|\widehat{\nabla}_a h^{abcd} \widehat{\nabla}_c u^I_J \widehat{\nabla}_d u^P_Q e^{JQ} e_{IP}\right| \le& P_{1,\widehat{\nabla}\mathrm{h}} |\widehat{\nabla}\mathbf{u}|^2 \\ \left|h^{abcd} \widehat{\nabla}_c u^I_J \widehat{\nabla}_d u^P_Q \widehat{\nabla}_a (e^{JQ} e_{IP})\right| \le& P_{1,\widehat{\nabla}\mathrm{e}} |\widehat{\nabla}\mathbf{u}|^2. \end{aligned}$$

Altogether

$$|\sigma_b \widehat{\nabla}_a T^{ab,1}(\mathbf{u})| \le \alpha_1 |\widehat{\nabla}\mathbf{u}|^2 + \beta_1 |\mathbf{F}|^2 + \gamma_1 |\mathbf{u}|^2, \tag{3.17}$$

where

$$\begin{aligned}\alpha_1 =& \alpha_1(\mathbf{g}, \widehat{\nabla}\mathbf{g}, \mathbf{e}, \widehat{\nabla}\mathbf{e}, \sigma, \widehat{\nabla}\sigma, \mathbf{B}) = P_{1,\mathrm{B}} + P_{1,\widehat{\nabla}\mathrm{h}} + P_{1,\widehat{\nabla}\mathrm{e}} + 3 \\ \beta_1 =& \beta_1(\mathbf{g}, \sigma), \text{ and} \\ \gamma_1 =& \gamma_1(\mathbf{g}, \sigma, \mathbf{C}, \hat{\mathbf{R}}) = (P^2_{1,\mathrm{C}} + P^2_{1,\mathrm{R}}). \end{aligned}$$

It is worth to note, that in the case of a scalar differential equation the commutator vanishes and therefore, γ_1 is independent of $\hat{\mathbf{R}}$.

For the case $j > 1$, consider the expression

$$\begin{aligned}
\sigma_b \widehat{\nabla}_a T^{ab,j}(\mathbf{u}) =& \sigma_b h^{abcd} \widehat{\nabla}_a \widehat{\nabla}_c \widehat{\nabla}_K u^I_J \widehat{\nabla}_d \widehat{\nabla}_L u^P_Q e^{KL} e^{JQ} e_{IP} \\
&+ \sigma_b h^{abcd} \widehat{\nabla}_c \widehat{\nabla}_K u^I_J \widehat{\nabla}_a \widehat{\nabla}_d \widehat{\nabla}_L u^P_Q e^{KL} e^{JQ} e_{IP} \\
&+ \sigma_b \widehat{\nabla}_a h^{abcd} \widehat{\nabla}_c \widehat{\nabla}_K u^I_J \widehat{\nabla}_d \widehat{\nabla}_L u^P_Q e^{KL} e^{JQ} e_{IP} \\
&+ \sigma_b h^{abcd} \widehat{\nabla}_c \widehat{\nabla}_K u^I_J \widehat{\nabla}_d \widehat{\nabla}_L u^P_Q \widehat{\nabla}_a (e^{KL} e^{JQ} e_{IP}).
\end{aligned}$$

Analogously to eqation (3.15) the above expression takes the form

$$\begin{aligned}
\sigma_b \widehat{\nabla}_a T^{ab,j}(u) =& \frac{2\sigma_b}{\sigma^2} g^{bd} g^{ac} \widehat{\nabla}_a \widehat{\nabla}_c \widehat{\nabla}_K u^I_J \widehat{\nabla}_d \widehat{\nabla}_L u^P_Q e^{KL} e^{JQ} e_{IP} \\
&+ \frac{2\sigma_b}{\sigma^2} g^{bd} g^{ac} [\widehat{\nabla}_a, \widehat{\nabla}_d] \widehat{\nabla}_K u^I_J \widehat{\nabla}_c \widehat{\nabla}_L u^P_Q e^{KL} e^{JQ} e_{IP} \\
&+ \sigma_b \widehat{\nabla}_a h^{abcd} \widehat{\nabla}_c \widehat{\nabla}_K u^I_J \widehat{\nabla}_d \widehat{\nabla}_L u^P_Q e^{KL} e^{JQ} e_{IP} \\
&+ \sigma_b h^{abcd} \widehat{\nabla}_c \widehat{\nabla}_K u^I_J \widehat{\nabla}_d \widehat{\nabla}_L u^P_Q \widehat{\nabla}_a (e^{KL} e^{JQ} e_{IP}).
\end{aligned} \tag{3.18}$$

Actually, the calculation follows exactly from the above case upon replacing $\mathbf{u}$ by $\widehat{\nabla}^{j-1}\mathbf{u}$. Denoting any linear combination of contractions of the $(j-i)^{\text{th}}$ $\widehat{\nabla}$-derivative of $\hat{\mathbf{R}}$ with the i^{th} $\widehat{\nabla}$-derivative of $\mathbf{u}$ by $\widehat{\mathcal{R}}^{(j,i)}\mathbf{u}$, we have

$$\begin{aligned}
\sigma_b \widehat{\nabla}_a T^{ab,j}(u) =& \frac{2\sigma_b}{\sigma^2} g^{bd} g^{ac} \widehat{\nabla}_a \widehat{\nabla}_c \widehat{\nabla}_K u^I_J \widehat{\nabla}_d \widehat{\nabla}_L u^P_Q e^{KL} e^{JQ} e_{IP} \\
&+ \frac{2\sigma_b}{\sigma^2} g^{bd} g^{ac} (\mathcal{R}^{(j-1,j-1)} \mathbf{u})^I_{adKJ} \widehat{\nabla}_c \widehat{\nabla}_L u^P_Q e^{KL} e^{JQ} e_{IP} \\
&+ \sigma_b \widehat{\nabla}_a h^{abcd} \widehat{\nabla}_c \widehat{\nabla}_K u^I_J \widehat{\nabla}_d \widehat{\nabla}_L u^P_Q e^{KL} e^{JQ} e_{IP} \\
&+ \sigma_b h^{abcd} \widehat{\nabla}_c \widehat{\nabla}_K u^I_J \widehat{\nabla}_d \widehat{\nabla}_L u^P_Q \widehat{\nabla}_a (e^{KL} e^{JQ} e_{IP}).
\end{aligned}$$

Note, that the notation $\widehat{\mathcal{R}}^{(j,i)}\mathbf{u}$ here simply resolves into application of the Ricci identity to $[\widehat{\nabla}_a, \widehat{\nabla}_d] \widehat{\nabla}_K u^I_J$. We primarily chose this notation since it is also useful in the following equations.

Finally, to bring the divergence of the energy tensor into a form, such that we can use the differential equation, we have to interchange the covariant derivatives once more. In fact, this means that we have to transform the term $\widehat{\nabla}_a \widehat{\nabla}_c \widehat{\nabla}_K u^I_J$ into $\widehat{\nabla}_K \widehat{\nabla}_a \widehat{\nabla}_c u^I_J$ to obtain

$$\begin{aligned}
\sigma_b \widehat{\nabla}_a T^{ab,j}(\mathbf{u}) =& \frac{2\sigma_b}{\sigma^2} g^{bd} g^{ac} \widehat{\nabla}_K \widehat{\nabla}_a \widehat{\nabla}_c u^I_J \widehat{\nabla}_d \widehat{\nabla}_L u^P_Q e^{KL} e^{JQ} e_{IP} \\
&+ \frac{2\sigma_b}{\sigma^2} g^{bd} g^{ac} \Big(\sum_{i=0}^{j-1} (\mathcal{R}^{(j-1,i)} \mathbf{u})^I_{acKJ} \Big) \widehat{\nabla}_d \widehat{\nabla}_L u^P_Q e^{KL} e^{JQ} e_{IP} \\
&+ \frac{2\sigma_b}{\sigma^2} g^{bd} g^{ac} (\mathcal{R}^{(j-1,j-1)} \mathbf{u})^I_{adKJ} \widehat{\nabla}_c \widehat{\nabla}_L u^P_Q e^{KL} e^{JQ} e_{IP} \\
&+ \sigma_b \widehat{\nabla}_a h^{abcd} \widehat{\nabla}_c \widehat{\nabla}_K u^I_J \widehat{\nabla}_d \widehat{\nabla}_L u^P_Q e^{KL} e^{JQ} e_{IP} \\
&+ \sigma_b h^{abcd} \widehat{\nabla}_c \widehat{\nabla}_K u^I_J \widehat{\nabla}_d \widehat{\nabla}_L u^P_Q \widehat{\nabla}_a (e^{KL} e^{JQ} e_{IP}).
\end{aligned} \tag{3.19}$$

The details of this calculation can be found in Appendix A.1. In the next step (for a thorough calculation see Appendix A.2), using the differential equation (3.1) on the first line of (3.19), we reduce the order of derivatives by one, and so we obtain that $\sigma_b \widehat{\nabla}_a T^{ab,j}(\mathbf{u}) = \sum_{i=1}^{8} I_i$ with

$$\begin{aligned}
I_1 =& \frac{2\sigma_b}{\sigma^2} g^{bd} \widehat{\nabla}_K F^I_J \widehat{\nabla}_d \widehat{\nabla}_L u^P_Q e^{KL} e^{JQ} e_{IP}, \\
I_2 =& -\frac{2\sigma_b}{\sigma^2} g^{bd} \Big(\sum_{i=2}^{j} \mathcal{G}^{(j+1,i)} \mathbf{u})^I_{KJ} \Big) \widehat{\nabla}_d \widehat{\nabla}_L u^P_Q e^{KL} e^{JQ} e_{IP}, \\
I_3 =& -\frac{2\sigma_b}{\sigma^2} g^{bd} \Big(\sum_{i=1}^{j} \mathcal{B}^{(j,i)} \mathbf{u})^I_{KJ} \Big) \widehat{\nabla}_d \widehat{\nabla}_L u^P_Q e^{KL} e^{JQ} e_{IP}, \\
I_4 =& -\frac{2\sigma_b}{\sigma^2} g^{bd} \Big(\sum_{i=0}^{j-1} \mathcal{C}^{(j-1,i)} \mathbf{u})^I_{KJ} \Big) \widehat{\nabla}_d \widehat{\nabla}_L u^P_Q e^{KL} e^{JQ} e_{IP}, \\
I_5 =& \frac{2\sigma_b}{\sigma^2} g^{bd} g^{ac} \Big(\sum_{i=0}^{j-1} (\widehat{\mathcal{R}}^{(j-1,i)} \mathbf{u})^I_{acKJ} \Big) \widehat{\nabla}_d \widehat{\nabla}_L u^P_Q e^{KL} e^{JQ} e_{IP}, \\
I_6 =& \frac{2\sigma_b}{\sigma^2} g^{bd} g^{ac} (\widehat{\mathcal{R}}^{(j-1,j-1)} \mathbf{u})^I_{adKJ} \widehat{\nabla}_c \widehat{\nabla}_L u^P_Q e^{KL} e^{JQ} e_{IP}, \\
I_7 =& \sigma_b \widehat{\nabla}_a h^{abcd} \widehat{\nabla}_c \widehat{\nabla}_K u^I_J \widehat{\nabla}_d \widehat{\nabla}_L u^P_Q e^{KL} e^{JQ} e_{IP}, \\
I_8 =& \sigma_b h^{abcd} \widehat{\nabla}_c \widehat{\nabla}_K u^I_J \widehat{\nabla}_d \widehat{\nabla}_L u^P_Q \widehat{\nabla}_a (e^{KL} e^{JQ} e_{IP}).
\end{aligned} \tag{3.20}$$

Here $\mathcal{G}^{(j,i)}\mathbf{u}$, $\mathcal{B}^{(j,i)}\mathbf{u}$, and $\mathcal{C}^{(j,i)}\mathbf{u}$ denote linear combinations of contractions of the $(j-i)^{th}$ $\widehat{\nabla}$-derivative of $\mathbf{g}$, $\mathbf{B}$, resp. $\mathbf{C}$, resp. with the i^{th} $\widehat{\nabla}$-derivative of $\mathbf{u}$.

In the remaining part of the proof we study the terms $I_1, \ldots, I_8$. The following manipulations make repeatedly use of the Cauchy-Schwartz inequality and the inequality of arithmetic and geometric means. Proceeding in the same way as for the term of order $j = 0$ and $j = 1$, we obtain

$$
\begin{aligned}
|I_1| \leq& P_{j,1}(|\widehat{\nabla}^j \mathbf{u}|^2 + |\widehat{\nabla}^{j-1}\mathbf{F}|^2) \\
|I_2| \leq& P_{j,2}\Big((j-1)|\widehat{\nabla}^j \mathbf{u}|^2 + g_j^2 \sum_{i=2}^{j-1} |\widehat{\nabla}^i \mathbf{u}|^2\Big) \\
|I_3| \leq& P_{j,3}\Big(j|\widehat{\nabla}^j \mathbf{u}|^2 + b_j^2 \sum_{i=1}^{j-1} |\widehat{\nabla}^i \mathbf{u}|^2\Big) \\
|I_4| \leq& P_{j,4}\Big(j|\widehat{\nabla}^j \mathbf{u}|^2 + c_j^2 \sum_{i=0}^{j-1} |\widehat{\nabla}^i \mathbf{u}|^2\Big) \\
|I_5| \leq& P_{j,5}\Big(j|\widehat{\nabla}^j \mathbf{u}|^2 + \hat{r}_j^2 \sum_{i=0}^{j-1} |\widehat{\nabla}^i \mathbf{u}|^2\Big) \\
|I_6| \leq& P_{j,6}(|\widehat{\nabla}^j \mathbf{u}|^2 + \tilde{r}_j^2 |\widehat{\nabla}^{j-1}\mathbf{u}|^2) \\
|I_7| \leq& P_{j,7}|\widehat{\nabla}^j \mathbf{u}|^2 \\
|I_8| \leq& P_{j,8}|\widehat{\nabla}^j \mathbf{u}|^2.
\end{aligned}
$$

with constants

$$
\begin{aligned}
P_{j,1} =& P_{j,1}(\mathbf{g}, \sigma) & P_{j,2} =& P_{j,2}(\mathbf{g}, \widehat{\nabla}\mathbf{g}, \sigma) \\
P_{j,3} =& P_{j,3}(\mathbf{g}, \sigma, \mathbf{B}) & P_{j,4} =& P_{j,4}(\mathbf{g}, \sigma) \\
P_{j,5} =& P_{j,5}(\mathbf{g}, \sigma) & P_{j,6} =& P_{j,6} \\
P_{j,7} =& P_{j,7}(\mathbf{g}, \widehat{\nabla}\mathbf{g}, \sigma, \widehat{\nabla}\sigma) & P_{j,8} =& P_{j,8}(\mathbf{g}, \mathbf{e}, \widehat{\nabla}\mathbf{e}, \sigma, \widehat{\nabla}\sigma) \\
g_j =& g_j(\widehat{\nabla}^2\mathbf{g}, \dots, \widehat{\nabla}^{j-1}\mathbf{g}) & b_j =& b_j(\widehat{\nabla}\mathbf{B}, \dots, \widehat{\nabla}^{j-1}\mathbf{B}) \\
c_j =& c_j(\mathbf{C}, \dots, \widehat{\nabla}^{j-1}\mathbf{C}) & \hat{r}_j =& \hat{r}_j(\mathbf{g}, \hat{\mathbf{R}}, \dots, \widehat{\nabla}^{j-1}\hat{\mathbf{R}}) \\
\tilde{r}_j =& \tilde{r}_j(\mathbf{g}, \sigma, \hat{\mathbf{R}}).
\end{aligned}
$$

Note that we do not specify the exact polynomial dependence in this notation. For the full details of this calculation, we refer to Appendix A.3. Summing up by ordering terms according to the order of derivatives of u, we obtain that for $j > 1$, there exist positive constants

$\alpha_j(\mathbf{g}, \widehat{\nabla}\mathbf{g}, \mathbf{e}, \widehat{\nabla}\mathbf{e}, \sigma, \widehat{\nabla}\sigma, \mathbf{B})$,

$\beta_j(\mathbf{g}, \sigma)$, and

$\gamma_j(\mathbf{g}, \dots, \widehat{\nabla}^{j-1}\mathbf{g}, \sigma, \mathbf{B}, \dots, \widehat{\nabla}^{j-1}\mathbf{B}, \mathbf{C}, \dots, \widehat{\nabla}^{j-1}\mathbf{C}, \hat{\mathbf{R}}, \dots, \widehat{\nabla}^{j-1}\hat{\mathbf{R}})$

such that

$$|\sigma_b \widehat{\nabla}_a T^{ab,j}(\mathbf{u})| \leq \alpha_j |\widehat{\nabla}^j \mathbf{u}|^2 + \beta_j |\widehat{\nabla}^{j-1} \mathbf{F}|^2 + \gamma_j \sum_{i=0}^{j-1} |\widehat{\nabla}^i \mathbf{u}|^2.$$

Altogether, combining the last equation with the results of the cases $j = 0$ and $j = 1$ we have

$$\begin{aligned}
\sum_{j=0}^{m} \sigma_b \widehat{\nabla}_a T^{ab,j}(\mathbf{u}) \leq & \sum_{j=0}^{m} |\sigma_b \widehat{\nabla}_a T^{ab,j}(\mathbf{u})| \\
= & \alpha_0 (|\mathbf{u}|^2 + |\widehat{\nabla} \mathbf{u}|^2) \\
& + \alpha_1 |\widehat{\nabla} \mathbf{u}|^2 + \beta_1 |\mathbf{F}|^2 + \gamma_1 |\mathbf{u}|^2 \\
& + \sum_{j=2}^{m} \alpha_j |\widehat{\nabla}^j \mathbf{u}|^2 + \sum_{j=2}^{m} \beta_j |\widehat{\nabla}^{j-1} \mathbf{F}|^2 + \sum_{j=2}^{m} \gamma_j \sum_{i=0}^{j-1} |\widehat{\nabla}^i \mathbf{u}|^2 \\
\leq & \tilde{\alpha}_m \sum_{j=0}^{m} |\widehat{\nabla}^j \mathbf{u}|^2 + \tilde{\beta}_m \sum_{j=0}^{m-1} |\widehat{\nabla}^j \mathbf{F}|^2 + \tilde{\gamma}_m \sum_{j=0}^{m-1} |\widehat{\nabla}^j \mathbf{u}|^2
\end{aligned}$$

with $\tilde{\alpha}_m := \max(\alpha_0, \alpha_0 + \alpha_1, \alpha_2, \ldots, \alpha_m)$, $\tilde{\beta}_m := \max_{1 \leq j \leq m}(\beta_j)$, and $\tilde{\gamma}_m := \sum_{j=1}^{m} \gamma_j$. Note the slight irregularity in the construction of the constant $\tilde{\alpha}_m$: The numbers α_0 and $\alpha_0 + \alpha_1$ reflect the fact, that the divergence of the zero order energy tensor includes derivatives of order 1, as opposed to the divergences of the j^{th} order energy tensor that are of order j and not $j + 1$. Integration over Ω_τ yields

$$\begin{aligned}
\sum_{j=0}^{m} \int_{\Omega_\tau} \sigma_b \widehat{\nabla}_a T^{ab,j}(\mathbf{u}_\varepsilon) \hat{\mu} \leq & \tilde{\alpha}_m \int_0^\tau E_\zeta^m(\mathbf{u}) \, \mathrm{d}\zeta + \tilde{\beta}_m (\|\mathbf{F}\|_{\Omega_\tau}^{m-1})^2 \\
& + \tilde{\gamma}_m \int_0^\tau E_\zeta^{m-1}(\mathbf{u}) \, \mathrm{d}\zeta,
\end{aligned}$$

where we used Lemma 3.2.4 to translate Sobolev norms into energy integrals. Substitution of the last estimate into (3.13) gives inequality (3.14), with the following constants (remember that the constant P from (3.13) depends on $\mathbf{g}$, $\boldsymbol{\sigma}$, and $\widehat{\nabla} \boldsymbol{\sigma}$)

$$\begin{aligned}
C'_m &= P + \tilde{\alpha}_m \\
C''_m &= \tilde{\beta}_m \\
C'''_m &= \tilde{\gamma}_m
\end{aligned}$$

This concludes the proof. *q. e. d.*

REMARK: As mentioned before, observe that for $j = 1$ the commutator in equations (3.16) and (3.18) vanishes if $\mathbf{u}$ is a scalar field. In a scenario involving generalized covariant derivatives this allows us to demand less regularity for $\hat{\mathbf{R}}$.

4. A low regularity Cauchy problem

This chapter is dedicated to an existence and uniqueness theorem for wave equations with generalized coefficients. We fix a generalized Lorentzian manifold $(M, \mathbf{g})$, i. e. a manifold M equipped with a Lorentzian metric $\mathbf{g}$ modelled in $\mathcal{G}^0_2(M)$, cf. Definition 2.4.4. We are interested in the local forward-in-time Cauchy problem for hyperbolic linear partial differential operators of with coefficients of low regularity; thus we consider the differential equation

$$(L\mathbf{u})^I_J = g^{ab}\widehat{\nabla}_a\widehat{\nabla}_b u^I_J + B^{aIP}_{JQ}\widehat{\nabla}_a u^Q_P + C^{IP}_{JQ} u^Q_P = F^I_J \tag{4.1}$$

with initial data

$$\mathbf{u}|_{\Sigma_0} = \mathbf{u}_0 \qquad \widehat{\nabla}_{\sigma^\sharp}\mathbf{u}|_{\Sigma_0} = \mathbf{u}_1, \tag{4.2}$$

composed of generalized coefficients, a generalized connection $\widehat{\nabla}$ which is the Levi-Cività connection of a generalized metric $\hat{\mathbf{g}}$ (cf. Definition 2.4.6 and Theorem 2.4.7), and a generalized right hand side—all of them modelled in $\mathcal{G}$.

In particular, $\mathbf{B}$ and $\mathbf{C}$ will be generalized tensor fields of suitable type, subject to additional conditions to be specified later. Also, the data $\mathbf{F}$, $\mathbf{u}_0$, and $\mathbf{u}_1$ are allowed to be generalized, that is $\mathbf{F} \in \mathcal{G}^k_l(M)$ and $\mathbf{u}_0, \mathbf{u}_1 \in \mathcal{G}^k_l(\Sigma_0)$. We require the metric $\hat{\mathbf{g}}$ to have the same time orientation as $\mathbf{g}$. We look for solutions $\mathbf{u} \in \mathcal{G}^k_l$ at least locally.

Before we go into detail we undertake a comparison of this situation with [GMS09] and [Han10].

- The most striking difference concerns the rôles of the metric. In partial differential operators of the form (4.1) a metric can enter at two positions: The Levi-Cività connection and the coefficient matrix of the operator's principal part. In this thesis we consider the problem with two different metrics, a connection induced by $\hat{\mathbf{g}}$

and a principal part given by $\mathbf{g}$, whereas in [GMS09, Section 3], there is one common generalized metric for both. The immediate consequence of this approach is that we end up with a normally hyperbolic operator for each ε instead of the connection d'Alembertian (see also Section 1.2 and [BGP07, Section 1.5]).

- Another significant difference is the extension from scalar equations to tensor equations, which amounts in extensive technical difficulties, when calculating the energy estimates (of which most are resolved in chapter 3).
- In [Han10] the background metric $\hat{\mathbf{g}}$ was considered to be smooth. Here we extend the setting to a generalized Lorentzian background metric. This allows us to treat simultaneously the situations in [GMS09] and [Han10].

4.1. The setting

Our first task is to specify a class of generalized Lorentzian metrics suitable to act as a principal part of L. We want to make sure that there exists a suitable foliation of M. To this end, we chose a relatively compact open neighbourhood U of $p \in M$ and ask for the existence of a function $h \in \mathcal{C}^\infty(U)$ such that $\sigma := \mathrm{d}h$ is timelike with respect to $\mathbf{g}$ and $\hat{\mathbf{g}}$. This will be implied by the existence of a constant $M_0 > 0$ such that for all ε

$$\frac{1}{M_0} \leq -\mathbf{g}_\varepsilon^{-1}(\sigma, \sigma) \leq M_0 \tag{4.3}$$

on U. Similar to the classical setting considered in chapter 3, the level surfaces $\Sigma_\tau := \{q \in U | h(q) = \tau\}$ with $\tau \in [0, \gamma]$ for some $\gamma > 0$ are spacelike hypersurfaces with respect to all the $(\mathbf{g}_\varepsilon)_\varepsilon$ and therefore with respect to the generalized metric $\mathbf{g}$. By Σ_0 we denote the initial surface needed in (4.2).

Next we specify a number of asymptotic conditions on $\mathbf{g}$, $\hat{\mathbf{g}}$, $\mathbf{B}$, and $\mathbf{C}$. To this end, we make use of a smooth Riemannian metric $\mathbf{m}$ on M to define the *"pointwise" norm* of a smooth tensor field $\mathbf{v}$ with respect to $\mathbf{m}$, i. e.

$$|\mathbf{v}|_{\mathrm{m}}^2 := (m^{-1})^{IJ} m_{KL} v_I^K v_J^L.$$

Note that since we work locally, our conditions are in fact independent of the choice of $\mathbf{m}$. We now suppose:

(i) For every representative $(\mathbf{g}_\varepsilon)_\varepsilon$, $(\hat{\mathbf{g}}_\varepsilon)_\varepsilon$, $(\mathbf{B}_\varepsilon)_\varepsilon$, and $(\mathbf{C}_\varepsilon)_\varepsilon$ of $\mathbf{g}$, $\hat{\mathbf{g}}$, $\mathbf{B}$, resp. $\mathbf{C}$, we demand for all K compact in U

$$
\begin{aligned}
\sup_K |\mathbf{g}_\varepsilon^{-1}|_{\mathrm{m}} &= O(1) & \sup_K |\widehat{\nabla}_\varepsilon \mathbf{g}_\varepsilon^{-1}|_{\mathrm{m}} &= O(1)\\
\sup_K |\mathbf{g}_\varepsilon|_{\mathrm{m}} &= O(1) & \sup_K |\mathbf{B}_\varepsilon|_{\mathrm{m}} &= O(1)\\
\sup_K |\mathbf{C}_\varepsilon|_{\mathrm{m}} &= O(1) & \sup_K |\hat{\mathbf{g}}_\varepsilon|_{\mathrm{m}} &= O(1)\\
\sup_K |\hat{\mathbf{R}}_\varepsilon|_{\mathrm{m}} &= O(1)
\end{aligned}
$$

as ε tends to zero (Observe that $\widehat{\nabla}$ denotes the Levi-Cività connection associated with $\hat{\mathbf{g}}$ and not with $\mathbf{g}$.).

(ii) For every representative $(\mathbf{g}_\varepsilon)_\varepsilon$ on U, the level set Σ_0 is a past compact spacelike hypersurface such that $\partial I_\varepsilon^+(\Sigma_0) = \Sigma_0$. Moreover, there exists a nonempty open set $A \subset M$ and some $\varepsilon_0 > 0$ such that $A \subset \bigcap_{\varepsilon \leq \varepsilon_0} \overline{I_\varepsilon^+(\Sigma_0)}$.

(iii) The metrics $\mathbf{g}$ and $\hat{\mathbf{g}}$ have the same time orientation and $-\hat{\mathbf{g}}_\varepsilon(\sigma, \sigma) \geq \widehat{M}_0$ for some constant $\widehat{M}_0$.

For a better understanding, we discuss these conditions more thoroughly.

REMARK:

- Condition (i) gives bounds on the coefficients $\mathbf{g}$, $\mathbf{B}$, and $\mathbf{C}$, independent of ε, which turn out to allow to control the asymptotic behaviour of the energy integrals. Observe that condition (i) also implies $\sup_K |\widehat{\nabla}\mathbf{g}|_{\mathrm{m}} = O(1)$ for all compact K.
- In case we have $\mathbf{g} = \hat{\mathbf{g}}$ (such that we obtain the problem posed in [GMS09, Section 3]), the expression $\widehat{\nabla}\mathbf{g}^{-1}$ vanishes. Therefore, we do not have any estimates on the Christoffel symbols $\hat{\mathbf{\Gamma}}$. This can be avoided by the following additional condition:

  (i') For every representative $(\hat{\mathbf{g}}_\varepsilon)_\varepsilon$ of $\hat{\mathbf{g}}$, we demand for all K compact in U

  $$\sup_K |\widehat{\nabla}_\varepsilon \sigma|_{\mathrm{m}} = O(1) \tag{4.4}$$

  as ε tends to zero.

Note that this condition also realizes bounds on the second fundamental form of the hypersurfaces Σ_τ, cf. [HKS11, Section 5.1]. For any smooth vector fields $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathfrak{X}(\Sigma_\tau)$, we have

$$\widehat{\nabla}_\varepsilon \boldsymbol{\sigma}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \boldsymbol{\eta}(\boldsymbol{\sigma}(\boldsymbol{\xi})) - \boldsymbol{\sigma}(\widehat{\nabla}^\varepsilon_{\boldsymbol{\eta}} \boldsymbol{\xi}) = -\boldsymbol{\sigma}(\widehat{\nabla}^\varepsilon_{\boldsymbol{\eta}} \boldsymbol{\xi}) = -\boldsymbol{\sigma}(\widehat{\nabla}^\varepsilon_{\boldsymbol{\eta}} \boldsymbol{\xi})^\perp = -\boldsymbol{\sigma}(\mathrm{II}_\varepsilon(\boldsymbol{\eta}, \boldsymbol{\xi})).$$

Thus we obtain $\sup_K |\mathrm{II}_\varepsilon|_{\mathsf{m}} = O(1)$ as $\varepsilon \to 0$.

- Condition (i) implies that the components of the (now generalized) Riemannian metric $\mathbf{e} := 2\tilde{\boldsymbol{\sigma}} \otimes \tilde{\boldsymbol{\sigma}} + \mathbf{g}$ and its inverse are locally bounded, i. e. we have $|\mathbf{e}|_{\mathsf{m}}, |\mathbf{e}^{-1}|_{\mathsf{m}} = O(1)$.

- We chose to formulate the asymptotic conditions (i) and (i') in terms of the norm $|\cdot|_{\mathsf{m}}$ with the smooth Riemannian metric $\mathbf{m}$, instead of a formulation in terms of $|\cdot|_{\mathsf{e}_\varepsilon}$. Actually, the conditions are independent of the choice of $\mathbf{m}$ since we work locally on compact sets. Together with the local boundedness of $|\mathbf{e}|_m$ and $|\mathbf{e}^{-1}|_{\mathsf{m}}$, we obtain that the smooth metric $\mathbf{m}$ and the generalized metric $\mathbf{e}$ induce equivalent norms on compact sets.

- Condition (iii) is sufficient to guarantee the existence of classical solutions on the level of representatives on a common domain.

- Actually we can choose any background metric to represent a Levi-Cività connection. Consider the following: Let $\widetilde{\nabla}$ and $\widehat{\nabla}$ be two different connections given by background metrics $\tilde{\mathbf{g}}$ and $\hat{\mathbf{g}}$, then we have

$$\begin{aligned}\widehat{\nabla}_b u^{i_1 \dots i_k}_{j_1 \dots j_l} =& \widetilde{\nabla}_b u^{i_1 \dots i_k}_{j_1 \dots j_l} + \sum_{s=1}^k (\tilde{\Gamma}^{i_s}_{db} - \hat{\Gamma}^{i_s}_{db}) u^{i_1 \dots d \dots i_k}_{j_1 \dots j_l} - \sum_{s=1}^l (\tilde{\Gamma}^d_{j_s b} - \hat{\Gamma}^d_{j_s b}) u^{i_1 \dots i_k}_{j_1 \dots d \dots j_l} \\ =& \widetilde{\nabla}_b u^{i_1 \dots i_k}_{j_1 \dots j_l} + \sum_{s=1}^k G^{i_s}_{db} u^{i_1 \dots d \dots i_k}_{j_1 \dots j_l} - \sum_{s=1}^l G^d_{j_s b} u^{i_1 \dots i_k}_{j_1 \dots d \dots j_l},\end{aligned}$$

where the tensor field $\mathbf{G}$ denotes the difference of the Christoffel symbols. Now, we obtain for the second derivative of $\mathbf{u}$

$$
\begin{aligned}
\widehat{\nabla}_a \widehat{\nabla}_b u^{i_1\ldots i_k}_{j_1\ldots j_l} =& \widetilde{\nabla}_a \widetilde{\nabla}_b u^{i_1\ldots i_k}_{j_1\ldots j_l} + \sum_{s=1}^{k} (\widetilde{\nabla}_a G^{i_s}_{db} u^{i_1\ldots d\ldots i_k}_{j_1\ldots j_l} + G^{i_s}_{db} \widetilde{\nabla}_a u^{i_1\ldots d\ldots i_k}_{j_1\ldots j_l}) \\
& - \sum_{s=1}^{l} (\widetilde{\nabla}_a G^{d}_{j_s b} u^{i_1\ldots i_k}_{j_1\ldots d\ldots j_l} + G^{d}_{j_s b} \widetilde{\nabla}_a u^{i_1\ldots i_k}_{j_1\ldots d\ldots j_l}) \\
& + \sum_{r=1}^{k} G^{i_r}_{ca} \left(\widetilde{\nabla}_b u^{i_1\ldots c\ldots i_k}_{j_1\ldots j_l} + \sum_{s=1}^{k} G^{i_s}_{db} u^{i_1\ldots c\ldots d\ldots i_k}_{j_1\ldots j_l} - \sum_{s=1}^{l} G^{d}_{j_s b} u^{i_1\ldots c\ldots i_k}_{j_1\ldots d\ldots j_l} \right) \\
& + \sum_{r=1}^{l} G^{c}_{j_r a} \left(\widetilde{\nabla}_b u^{i_1\ldots i_k}_{j_1\ldots c\ldots j_l} + \sum_{s=1}^{k} G^{i_s}_{db} u^{i_1\ldots d\ldots i_k}_{j_1\ldots c\ldots j_l} - \sum_{s=1}^{l} G^{d}_{j_s b} u^{i_1\ldots i_k}_{j_1\ldots c\ldots d\ldots j_l} \right) \\
& + G^{c}_{ba} (\widetilde{\nabla}_c u^{i_1\ldots i_k}_{j_1\ldots j_l} + \sum_{s=1}^{k} G^{i_s}_{dc} u^{i_1\ldots d\ldots i_k}_{j_1\ldots j_l} - \sum_{s=1}^{l} G^{d}_{j_s c} u^{i_1\ldots i_k}_{j_1\ldots d\ldots j_l}).
\end{aligned}
$$

With the exception of the first term on the right hand side there are no more second order derivatives involved. Thus, when choosing another Levi-Cività connection, we can obviously adjust the lower order coefficients in such a way that the equation is of the same form.

- Since our energy estimates rely on ε-wise application of Proposition 3.2.8, which we proved by application of Lemma 3.2.7, we need condition (iii).

Also in this chapter, we use the abbreviations from Section 4.1:

- $\boldsymbol{\sigma}^\sharp := \mathbf{g}^{-1}(\boldsymbol{\sigma}, \cdot)$, the vector field dual to $\boldsymbol{\sigma}$.
- $\sigma := (-\mathbf{g}^{-1}(\boldsymbol{\sigma}, \boldsymbol{\sigma}))^{\frac{1}{2}} = (-\mathbf{g}(\boldsymbol{\sigma}^\sharp, \boldsymbol{\sigma}^\sharp))^{\frac{1}{2}}$, the norm of $\boldsymbol{\sigma}$ measured in terms of $\mathbf{g}$.
- $\tilde{\boldsymbol{\sigma}} := \boldsymbol{\sigma}/\sigma$, the unit one-form to $\boldsymbol{\sigma}$ measured with respect to $\mathbf{g}$.

Note that the expressions $\boldsymbol{\sigma}^\sharp$, σ, and $\tilde{\boldsymbol{\sigma}}$ are now generalized tensor fields, resp. functions, whereas $\boldsymbol{\sigma}$ is still a smooth one form!

At this point, we compare once more with [GMS09]. In Section 2 we find three conditions similar to those in this thesis that allow to prove the energy estimate of the connection d'Alembert operator (Note that for reasons of notational consistence we translated their conditions into the notation used herein.):

(A) For all K compact in U, for all orders of derivative $k \in \mathbb{N}_0$, for all k-tuples of vector fields $\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_n \in \mathfrak{X}(U)$, and for any representative $(\mathbf{g}_\varepsilon)_\varepsilon$, we have:

$$\sup_K |\mathcal{L}_{\eta_1} \cdots \mathcal{L}_{\eta_n} \mathbf{g}|_{\mathsf{m}} = O(\varepsilon^{-k})$$
$$\sup_K |\mathcal{L}_{\eta_1} \cdots \mathcal{L}_{\eta_n} \mathbf{g}^{-1}|_{\mathsf{m}} = O(\varepsilon^{-k})$$

as ε tends to zero.

(B) For all K compact in U, we have

$$\sup_K |\nabla_\varepsilon \boldsymbol{\sigma}_\varepsilon^\sharp|_{\mathsf{m}} = O(1)$$

as ε tends to zero, where ∇_ε denotes the Levi-Cività connection with respect to $\mathbf{g}_\varepsilon$.

(C) For every representative $(\mathbf{g}_\varepsilon)_\varepsilon$ on U, the level set Σ_0 is a past compact spacelike hypersurface such that $\partial I_\varepsilon^+(\Sigma_0) = \Sigma_0$. Moreover, there exists a nonempty open set $A \subset M$ and some $\varepsilon_0 > 0$ such that $A \subset \bigcap_{\varepsilon \le \varepsilon_0} \overline{I_\varepsilon^+(\Sigma_0)}$.

Obviously condition (C) is the precise analogue of condition (ii), both guarantee the existence and uniqueness of smooth solutions for each fixed ε.

A quick look at condition (B) reveals us that this is in fact the same statment as in condition (i')—remember that condition (i') only applies in the case $\mathbf{g} = \hat{\mathbf{g}}$. We have

$$\nabla_\varepsilon \boldsymbol{\sigma}_\varepsilon^\sharp = g_\varepsilon^{ab} \nabla_c^\varepsilon \sigma_b$$

since $\nabla_\varepsilon \mathbf{g}_\varepsilon = 0$. By the Cauchy-Schwartz inequality we have

$$\sup_K |\nabla_\varepsilon \boldsymbol{\sigma}^\sharp|_{\mathsf{m}} \le \sup_K |\mathbf{g}|_{\mathsf{m}} \cdot \sup_K |\nabla_\varepsilon \boldsymbol{\sigma}|_{\mathsf{m}} = O(1) \quad \text{as } \varepsilon \to 0.$$

The converse direction follows analogously by $\nabla_\varepsilon \boldsymbol{\sigma} = \nabla_c^\varepsilon (g_{ab}^\varepsilon \sigma_\varepsilon^b)$.

However, condition (A) significantly differs from condition (i). First, observe that in contrast to [GMS09] we also consider lower order terms; hence, we have estimates on $\mathbf{B}$ and $\mathbf{C}$. Furthermore, the estimate on $\hat{\mathbf{R}}$ is only a consequence of the generalization to tensor fields and can be omitted in the scalar case. The main difference to [GMS09] is the fact that in this work we eliminated the asymptotic conditions on higher order derivatives of the metric tensor $\mathbf{g}$ such that only local uniform boundedness with respect to ε of $\mathbf{g}$, $\mathbf{g}^{-1}$, and $\widehat{\nabla}\mathbf{g}$ remains (cf. paragraph before 4.3.4), which in the case $\mathbf{g} = \hat{\mathbf{g}}$ and for a scalar wave equation allows an improvement of the existence and uniqueness theorem in [GMS09, Section 3]. Furthermore, local uniform boundedness also applies to $\hat{\mathbf{g}}$, which is a consequence of the separation of the rôles of the metrics.

4.2. The main theorem

Following the general strategy for solving differential equations on a space of Colombeau generalized functions, we have to pursue the following tasks: We start by writing out the initial value problem in terms of representatives, i. e.

$$L_\varepsilon \mathbf{u}_\varepsilon = \mathbf{F}_\varepsilon \qquad \mathbf{u}_\varepsilon|_{\Sigma_0} = \mathbf{u}_{0,\varepsilon} \qquad \widehat{\nabla}^\varepsilon_{\sigma^\sharp_\varepsilon} \mathbf{u}_\varepsilon|_{\Sigma_0} = \mathbf{u}_{1,\varepsilon}. \tag{4.5}$$

Then, using classical theory, we solve separately for each ε, obtaining a net $(\mathbf{u}_\varepsilon)_\varepsilon$ which is a candidate for a generalized solution. However, to obtain existence in generalized functions, first, we have to ensure that the $\mathbf{u}_\varepsilon$ are defined on a common domain (which is guaranteed by condition (ii) in Section 4.1), and then we have to prove moderateness of the net $(\mathbf{u}_\varepsilon)_\varepsilon$. To obtain uniqueness of solutions, we have to show independence of the class $[(\mathbf{u}_\varepsilon)_\varepsilon]$ of the choice of representatives of the data $\mathbf{F}$, $\mathbf{u}_0$, and $\mathbf{u}_1$. Observe that the latter statement amounts to proving a stability property of the problem. Note that we do not have to proof independence of the representatives of the coefficients of L since $\mathcal{G}$ is a differential algebra. In this way, we will provide a proof of

4.2.1. Theorem: *Let $(M, \mathbf{g})$ be a generalized Lorentzian manifold, and let L be a second-order partial differential operator of the form* (4.1) *with coefficients $\mathbf{g}$, $\mathbf{B}$, $\mathbf{C}$, where $\hat{\mathbf{g}}$ is another generalized Lorentzian metric, and $\widehat{\nabla}$ is the Levi-Cività connection of $\hat{\mathbf{g}}$. Furthermore, let $\mathbf{g}$, $\mathbf{B}$, $\mathbf{C}$, and $\hat{\mathbf{g}}$ be subject to conditions (i), (ii), and (iii) above, as well as (i') in the case $\mathbf{g} = \hat{\mathbf{g}}$. Let Σ_0 be a spacelike hypersurface, locally described by $h^{-1}(\{0\})$ for a function $h \in \mathcal{C}^\infty(U)$ such that* (4.3) *holds for $\sigma := \mathrm{d}h$, where U is a neighbourhood of some point $p \in \Sigma_0$. Then, for any such point $p \in \Sigma_0$, there exists an open neighbourhood $V \subseteq U$ of p such that the initial value problem* (4.1), (4.2) *has a unique solution $\mathbf{u} \in \mathcal{G}^k_l(V)$.*

4.3. Generalized energy estimates

For our energy estimates in the Colombeau case, we consider an analogue situation to Sections 3.1 and 3.2: We have a foliation of U into spacelike hypersurfaces Σ_τ and define Ω_τ and S_τ as before. Note that the volume form on M, derived from $\hat{\mathbf{g}}$, is now a generalized volume form, and we have for all ε that $\boldsymbol{\sigma} \wedge \hat{\boldsymbol{\mu}}^\varepsilon_{\Sigma_\tau} = \hat{\boldsymbol{\mu}}^\varepsilon$ for some representatives of $\hat{\boldsymbol{\mu}}_{\Sigma_\tau}$ and $\hat{\boldsymbol{\mu}}$.

Using again the (now generalized) Riemannian metric $\mathbf{e} := 2\tilde{\boldsymbol{\sigma}} \otimes \tilde{\boldsymbol{\sigma}} + \mathbf{g}$, we can define the following ε-dependent Sobolev norms on Ω_τ and S_τ. Note that again $|\cdot| = |\cdot|_{\mathbf{e}}$. Analogous to Defintion 3.2.2 we have

4.3.1. Definition (Sobolev norms): Let $\mathbf{v}$ be a smooth tensor field, let $0 \leq \tau \leq \gamma$, and let $m \in \mathbb{N}_0$. We define the Sobolev norms with respect to $\widehat{\nabla}$ and with respect to the partial derivatives by

1.
$$ {}^{\widehat{\nabla}}\|\mathbf{v}\|^m_{\Omega_\tau,\varepsilon} := \Big(\sum_{j=0}^{m} \int_{\Omega_\tau} |\widehat{\nabla}^j_\varepsilon \mathbf{v}|^2 \hat{\boldsymbol{\mu}}^\varepsilon \Big)^{1/2} \text{ and} $$
$$ {}^{\partial}\|\mathbf{v}\|^m_{\Omega_\tau,\varepsilon} := \Big(\sum_{j=0}^{m} \int_{\Omega_\tau} |\partial^j \mathbf{v}|^2 \hat{\boldsymbol{\mu}}^\varepsilon \Big)^{1/2}, $$

2.
$$ {}^{\widehat{\nabla}}\|\mathbf{v}\|^m_{S_\tau,\varepsilon} := \Big(\sum_{j=0}^{m} \int_{S_\tau} |\widehat{\nabla}^j_\varepsilon \mathbf{v}|^2 \hat{\boldsymbol{\mu}}^\varepsilon_\tau \Big)^{1/2} \text{ and} \tag{4.6} $$
$$ {}^{\partial}\|\mathbf{v}\|^m_{S_\tau,\varepsilon} := \Big(\sum_{j=0}^{m} \int_{S_\tau} |\partial^j \mathbf{v}|^2 \hat{\boldsymbol{\mu}}^\varepsilon_\tau \Big)^{1/2}. \tag{4.7} $$

Here, compared to the smooth case of chapter 3, we also defined Sobolev norms with respect to the partial derivative such that in the proof of Lemma 4.4.2 we can apply the Sobolev imbedding theorem.

The definition of the energy integrals is completely analogous to section 3.2.

4.3.2. Definition (Generalized Energy integrals): For a smooth tensor field $\mathbf{v}$, some number $m > 0$, and multiindices K and R with $|K| = |R| = m - 1$, we define the *energy tensors* $\mathbf{T}^m_\varepsilon(\mathbf{v})$ of $\mathbf{v}$ of order m by

$$T^{ab,0}_\varepsilon(\mathbf{v}) := -\frac{1}{\sigma_\varepsilon^2} g^{ab}_\varepsilon |\mathbf{v}|^2$$
$$T^{ab,m}_\varepsilon(\mathbf{v}) := \frac{1}{\sigma_\varepsilon^2}(2g^{ac}_\varepsilon g^{bd}_\varepsilon - g^{ab}_\varepsilon g^{cd}_\varepsilon) e^{KR} e^{JQ} e_{IP} (\widehat{\nabla}^\varepsilon_c \widehat{\nabla}^\varepsilon_K v^I_J)(\widehat{\nabla}^\varepsilon_d \widehat{\nabla}^\varepsilon_R v^P_Q).$$

For $0 \le \tau \le \gamma$ and for $m \ge 0$, we define the *energy integral* $E^m_\tau(\mathbf{v})$ of $\mathbf{v}$ of order m on S_τ by

$$E^m_{\tau,\varepsilon}(\mathbf{v}) := \sum_{j=0}^{m} \int_{S_\tau} T^{ab,j}_\varepsilon(\mathbf{v}) \sigma_a \sigma_b \hat{\boldsymbol{\mu}}^\varepsilon_\tau.$$

To proceed further, additionally to Lemma 3.2.4, we need the following lemma comparing the Sobolev norms (4.6) and (4.7), which is a slight variation of Lemma 4.5.2 in [GMS09].

4.3.3. Lemma: *For a smooth tensor field $\mathbf{v}$ and each $m \ge 1$, there exist constants B_m such that*

$$({}^{\partial}\|\mathbf{v}\|^k_{S_\tau,\varepsilon})^2 \le Q_m \sum_{j=1}^{m} \varepsilon^{-N_j} \cdot ({}^{\widehat{\nabla}}\|\mathbf{v}\|^j_{S_\tau,\varepsilon})^2, \tag{4.8}$$

where the N_j are natural numbers. For $m = 0$, obviously ${}^{\partial}\|\mathbf{v}\|^0_{S_\tau,\varepsilon} = {}^{\widehat{\nabla}}\|\mathbf{v}\|^0_{S_\tau,\varepsilon}$.

For the proof of this lemma, it is essential to note that on compact subsets of a local coordinate patch the Euclidean metric δ and the metric $\mathbf{e}$ are equivalent. In other words, there exist constants $P_{l+k,1}$ and $P_{l+k,2}$ such that for any tensor field $\mathbf{v}$ of type (l,k) we have

$$\begin{aligned} P_{l+k,1}\delta_{\mu_1\rho_1} \cdots \delta_{\mu_l\rho_l} \delta^{\nu_1\sigma_1} \cdots \delta^{\nu_k\sigma_k} v^{\mu_1\ldots\mu_l}_{\nu_1\ldots\nu_k} v^{\rho_1\ldots\rho_l}_{\sigma_1\ldots\sigma_k} & \\ \le e_{\mu_1\rho_1} \cdots e_{\mu_l\rho_l} e^{\nu_1\sigma_1} \cdots e^{\nu_k\sigma_k} v^{\mu_1\ldots\mu_l}_{\nu_1\ldots\nu_k} v^{\rho_1\ldots\rho_l}_{\sigma_1\ldots\sigma_k} & \\ \le P_{l+k,2}\delta_{\mu_1\rho_1} \cdots \delta_{\mu_l\rho_l} \delta^{\nu_1\sigma_1} \cdots \delta^{\nu_k\sigma_k} v^{\mu_1\ldots\mu_l}_{\nu_1\ldots\nu_k} v^{\rho_1\ldots\rho_l}_{\sigma_1\ldots\sigma_k} & \end{aligned} \tag{4.9}$$

Proof of the Lemma: Let, without loss of generality, S_τ be contained in one local chart. The case $m = 0$ is trivial. For $m = 1$, we can write the partial derivative of $\mathbf{v}$ locally as

$$\begin{aligned}\partial_\alpha v^{\mu_1\ldots\mu_l}_{\nu_1\ldots\nu_k} =& \widehat{\nabla}_\alpha v^{\mu_1\ldots\mu_l}_{\nu_1\ldots\nu_k} - \sum_\beta (\hat{\Gamma}^{\mu_1}_{\alpha\beta} v^{\beta\ldots\mu_l}_{\nu_1\ldots\nu_k} - \cdots - \hat{\Gamma}^{\mu_l}_{\alpha\beta} v^{\mu_1\ldots\beta}_{\nu_1\ldots\nu_k} \\ &+ \hat{\Gamma}^{\beta}_{\alpha\nu_1} v^{\mu_1\ldots\mu_l}_{\beta\ldots\nu_k} + \cdots + \hat{\Gamma}^{\beta}_{\alpha\nu_k} v^{\mu_1\ldots\mu_l}_{\nu_1\ldots\beta}).\end{aligned}$$

Thus, since $\mathbf{g}$ is moderate, we have (where summation is meant with regard to all the free indices)

$$\sum |\partial_\alpha v^{\mu_1\ldots\mu_l}_{\nu_1\ldots\nu_k}|^2 \le \sum |\widehat{\nabla}_\alpha v^{\mu_1\ldots\mu_l}_{\nu_1\ldots\nu_k}|^2 + Q_1'''\varepsilon^{-N_1} \sum |v^{\mu_1\ldots\mu_l}_{\nu_1\ldots\nu_k}|^2.$$

with a constant Q_1'''. Here, we also repeatedly applied the triangle inequality, the inequality of arithmetic and geometric means, and the Cauchy-Schwartz inequality. Together with equation (4.9) this yields

$$\begin{aligned}|\partial\mathbf{v}|^2 \le& P_{l+k,2}\big(\sum |\widehat{\nabla}_\alpha v^{\mu_1\ldots\mu_l}_{\nu_1\ldots\nu_k}|^2 + Q_1'''\varepsilon^{-N_1}\sum |v^{\mu_1\ldots\mu_l}_{\nu_1\ldots\nu_k}|^2\big) \\ \le& Q_1''\varepsilon^{-N_1}\big(\sum |\widehat{\nabla}_\alpha v^{\mu_1\ldots\mu_l}_{\nu_1\ldots\nu_k}|^2 + \sum |v^{\mu_1\ldots\mu_l}_{\nu_1\ldots\nu_k}|^2\big) \\ \le& Q_1'\varepsilon^{-N_1}(|\widehat{\nabla}\mathbf{v}|^2 + |\mathbf{v}|^2) \\ |\mathbf{v}|^2 + |\partial\mathbf{v}|^2 \le& 2Q_1'\varepsilon^{-N_1}(|\widehat{\nabla}\mathbf{v}|^2 + |\mathbf{v}|^2)\end{aligned}$$

and the claim for $m = 1$ follows by integration and by setting $Q_1 := 2Q_1'$. For $m > 1$, one proceeds in a similar way. Then the calculation involves derivatives of the Christoffel symbols, which can be estimated in the same way, since $\mathbf{g}$ is moderate. *q. e. d.*

REMARK: Actually, if one looks at condition (i) in Section 4.1, it is clear that, whenever $\mathbf{g} \neq \hat{\mathbf{g}}$ the Christoffel symbols $\hat{\mathbf{\Gamma}}$ are $\mathcal{O}(1)$, and therefore in the case $m = 1$ we obtain the better result $({}^{\partial}\|\mathbf{v}\|^1_{S_\tau,\varepsilon})^2 \le Q_1 \cdot ({}^{\widehat{\nabla}}\|\mathbf{v}\|^1_{S_\tau,\varepsilon})^2$.

We aim at a generalized version of Proposition 3.2.8 that already includes the asymptotic estimates of Section 4.1. The key property of the energy tensors is the *dominant energy condition*. Arguing for fixed ε, it is evident that the energy tensors $\mathbf{T}^j_\varepsilon(\mathbf{v})$ satisfy the dominant energy condition with respect to $\mathbf{g}_\varepsilon$ for all ε. As mentioned in section 3.2, the dominant energy condition allows us to formulate the energy estimate (3.14), which in full detail reads for all $m > 0$

$$\begin{aligned} E^m_\tau(\mathbf{u}) \leq & E^m_0(\mathbf{u}) + C'_m \int_0^\tau E^m_\zeta(\mathbf{u})\,\mathrm{d}\zeta \\ & + C''_m(\|\mathbf{F}\|^{m-1}_{\Omega_\tau})^2 + C'''_m \int_0^\tau E^{m-1}_\zeta(\mathbf{u})\,\mathrm{d}\zeta. \end{aligned}$$

In Proposition 3.2.8 we have obtained the following dependencies for the constants:

$$\begin{aligned} C'_m =& C'_m(\mathbf{g}, \widehat{\nabla}\mathbf{g}, \mathbf{B}, \widehat{\nabla}\mathbf{e}) \\ C''_m =& C''_m(\mathbf{g}) \\ C'''_m =& C'''_m(\mathbf{g}, \dots, \widehat{\nabla}^{m-1}\mathbf{g}, \mathbf{B}, \dots, \widehat{\nabla}^{m-1}\mathbf{B}, \mathbf{C}, \dots, \widehat{\nabla}^{m-1}\mathbf{C}, \hat{\mathbf{R}}, \dots, \widehat{\nabla}^{m-1}\hat{\mathbf{R}}). \end{aligned} \tag{4.10}$$

Now, we apply this result ε-wise to the energies $E^m_{\tau,\varepsilon}$. Using conditions (i) and (ii) of Section 4.1, we immediately see that C'_m and C''_m are in fact independent of ε since the terms $|\mathbf{g}|$, $|\widehat{\nabla}\mathbf{g}|$, $|\mathbf{B}|$, and $|\widehat{\nabla}\mathbf{e}|$ are bounded in ε, whereas we write $C'''_m = \tilde{C}'''_m\varepsilon^{-N_m}$ for natural numbers N_m with $\tilde{C}'''_m$ independent of ε. Thus, we proved

4.3.4. Proposition: *Let $\mathbf{u}_\varepsilon$ be a solution of the differential equation* (4.5) *on U. Then, for every $m \geq 1$, there exist constants C'_m, C''_m, and $\tilde{C}'''_m$ such that for every $0 \leq \tau \leq \gamma$,*

$$\begin{aligned} E^m_{\tau,\varepsilon}(\mathbf{u}_\varepsilon) \leq & E^m_{0,\varepsilon}(\mathbf{u}_\varepsilon) + C'_m \int_0^\tau E^m_{\zeta,\varepsilon}(\mathbf{u}_\varepsilon)\,\mathrm{d}\zeta \\ & + C''_m({}^{\widehat{\nabla}}\|\mathbf{F}_\varepsilon\|^{m-1}_{\Omega_{\tau,\varepsilon}})^2 + \tilde{C}'''_m\varepsilon^{-N_m} \int_0^\tau E^{m-1}_{\zeta,\varepsilon}(\mathbf{u}_\varepsilon)\,\mathrm{d}\zeta. \end{aligned} \tag{4.11}$$

Observe that the coefficient in front of the first integral in (4.11) does not depend on ε: This is essential later on, when applying Gronwall's inequality in the course of proving moderateness resp. negligibility of the nets $E^m_{\tau,\varepsilon}(\mathbf{u}_\varepsilon)$.

Due to the fact that C'''_1 depends only on $\mathbf{g}$, $\mathbf{B}$, $\mathbf{C}$, and $\hat{\mathbf{R}}$, in the case $m = 1$, see equation (4.10), by condition (i) of Section 4.1 we have $N_1 = 0$. Hence, we obtain an even better result in the case $m = 1$.

4.3.5. COROLLARY: *For $m = 1$, we can improve inequality* (3.14) *to*

$$E^1_{\tau,\varepsilon}(\mathbf{u}_\varepsilon) \leq E^1_{0,\varepsilon}(\mathbf{u}_\varepsilon) + (C'_1 + \tilde{C}'''_1) \int_0^\tau E^1_{\zeta,\varepsilon}(\mathbf{u}_\varepsilon)\, \mathrm{d}\zeta + C''_1 (\|^{\widehat{\nabla}} \mathbf{F}_\varepsilon\|^0_{\Omega_\tau,\varepsilon})^2. \tag{4.12}$$

REMARK: We undertake a comparison with [GMS09].

- When looking at [GMS09, Proposition 5.1] and its proof, one sees that the inclusion of lower order terms would give rise to an additional term for $j = 0$ in the sum. This term, since dependent on ε, would obstruct a successful application of Gronwall's inequality in the case $m = 1$, cp. also with estimate (3.17). To compensate for this ε-dependence, we had to introduce bounds on $\mathbf{B}$ and $\mathbf{C}$ to obtain the vital Corollary 4.3.5.

- Since, in contrast to [GMS09], we do not necessarily work with the Levi-Cività connection of the "coefficient metric" $\mathbf{g}$, we had to sharpen the condition on first order derivatives of $\mathbf{g}_\varepsilon$ to $\sup_K |\widehat{\nabla}\mathbf{g}_\varepsilon^{-1}| = O(1)$, see condition (i) in Section 4.1.

We may now apply Gronwall's inequality to (4.11) and (4.12) to immediately obtain

4.3.6. COROLLARY: *Let $\mathbf{u}_\varepsilon$ be a solution of the differential equation* (4.5) *on U. Then, for every $m > 1$, there exist constants C'_m, C''_m, and $\tilde{C}'''_m$ such that for* $0 \leq \tau \leq \gamma$,

$$E^m_{\tau,\varepsilon}(\mathbf{u}_\varepsilon) \leq \left(E^m_{0,\varepsilon}(\mathbf{u}_\varepsilon) + C''_m (\|^{\widehat{\nabla}} \mathbf{F}_\varepsilon\|^{m-1}_{\Omega_\tau,\varepsilon})^2 + \tilde{C}'''_m \varepsilon^{-N_m} \int_0^\tau E^{m-1}_{\zeta,\varepsilon}(\mathbf{u}_\varepsilon)\, \mathrm{d}\zeta \right) e^{C'_m \tau}. \tag{4.13}$$

For $m = 1$, we have

$$E^1_{\tau,\varepsilon}(\mathbf{u}_\varepsilon) \leq \left(E^1_{0,\varepsilon}(\mathbf{u}_\varepsilon) + C''_1 (\|^{\widehat{\nabla}} \mathbf{F}_\varepsilon\|^0_{\Omega_\tau,\varepsilon})^2 \right) e^{(C'_1 + \tilde{C}'''_1)\tau}. \tag{4.14}$$

The consequence of (4.13) and (4.14) is that by iterating m, we obtain that moderate resp. negligible initial energy integrals $E^m_{0,\varepsilon}(\mathbf{u}_\varepsilon)$ and right hand side $\mathbf{F}_\varepsilon$ imply moderate resp. negligible energy integrals $E^m_{\tau,\varepsilon}(\mathbf{u}_\varepsilon)$ at later times τ.

4.3.7. COROLLARY: *Let $\mathbf{u}_\varepsilon$ be a solution of the initial value problem* (4.5) *on U. Then, for all $m \geq 1$,*

$$\sup_{0\leq\tau\leq\gamma} E^m_{\tau,\varepsilon}(\mathbf{u}_\varepsilon)_\varepsilon \tag{4.15}$$

is a moderate net of real numbers if the initial energy integrals are moderate.

Likewise, (4.15) *is negligible if the initial energy integrals are and, additionally, $(\mathbf{F}_\varepsilon)_\varepsilon$ is negligible.*

Proof: We take equation (4.13) for m and successively insert the same estimate for lower orders until we reach the innermost integral $\int_0^\tau E^1_{\zeta,\varepsilon}(\mathbf{u}_\varepsilon)\,\mathrm{d}\zeta$, which we substitute by estimate (4.14), i. e.

$$\begin{aligned}
E^m_{\tau,\varepsilon}(\mathbf{u}_\varepsilon) \leq & e^{C'_m\tau}\Bigg(E^m_{0,\varepsilon}(\mathbf{u}_\varepsilon) + C''_m(\widehat{\nabla}\|\mathbf{F}_\varepsilon\|^{m-1}_{\Omega_{\tau,\varepsilon}})^2 + \tilde{C}'''_m\varepsilon^{-N_m} \\
& \cdot\int_0^\tau e^{C'_{m-1}\tau}\Bigg(E^{m-1}_{0,\varepsilon}(\mathbf{u}_\varepsilon) + C''_{m-1}(\widehat{\nabla}\|\mathbf{F}_\varepsilon\|^{m-2}_{\Omega_{\tau,\varepsilon}})^2 + \tilde{C}'''_{m-1}\varepsilon^{-N_{m-1}} \\
& \cdot\int_0^\tau e^{C'_{m-2}\tau}\Bigg(\cdots + \tilde{C}'''_2\varepsilon^{-N_2} \\
& \cdot\int_0^\tau e^{(C'_1+\tilde{C}'''_1)\tau}\Bigg(E^1_{0,\varepsilon}(\mathbf{u}_\varepsilon) + C''_1(\widehat{\nabla}\|\mathbf{F}_\varepsilon\|^{0}_{\Omega_{\tau,\varepsilon}})^2\Bigg)\,\mathrm{d}\zeta_2\cdots\Bigg)\,\mathrm{d}\zeta_{m-2}\Bigg)\,\mathrm{d}\zeta_{m-1}\Bigg).
\end{aligned}$$

It is now easy to see that $E^m_{\tau,\varepsilon}(\mathbf{u}_\varepsilon)$ can be estimated solely by the initial energies $E^j_{0,\varepsilon}(\mathbf{u}_\varepsilon)$ and Sobolev norms of the right hand side $(\widehat{\nabla}\|\mathbf{F}_\varepsilon\|^{j-1}_{\Omega_{\tau,\varepsilon}})^2$ as well as the constants C'_j, C''_j and $\tilde{C}'''_j$ for $1 \leq j \leq m$ such that the result follows immediately from the moderateness resp. negligibility assumptions. *q. e. d.*

We will make use of this corollary in the proof of the main theorem.

4.4. Proof of the main theorem

Before we start the actual proof of the main theorem, we need two more estimates. First, we translate bounds on initial data into bounds on initial energy integrals and bounds on energy integrals into bounds on solutions. Afterwards, together with Corollary 4.3.7, we will establish existence and uniqueness of generalized solutions to the generalized initial value problem (4.1), (4.2).

Bounds on initial energy integrals from bounds on initial data

The following Lemma, similar to [GMS09, Lemma 6.1], gives bounds on initial energy integrals.

4.4.1. Lemma: *Let $(\mathbf{u}_\varepsilon)_\varepsilon$ be a solution of (4.5). If $(\mathbf{u}_{0,\varepsilon})_\varepsilon$, $(\mathbf{u}_{1,\varepsilon})_\varepsilon$ and $(\mathbf{F}_\varepsilon)_\varepsilon$ are moderate resp. negligible, then the initial energy integrals $(E^m_{0,\varepsilon}(\mathbf{u}_\varepsilon))_\varepsilon$ for each $m \geq 0$ are moderate resp. negligible nets of real numbers.*

Proof: We have to consider the energy integral

$$E^m_{0,\varepsilon}(\mathbf{u}_\varepsilon) = \sum_{j=0}^{m} \int_{S_0} T^{ab,j}_\varepsilon(\mathbf{u}_\varepsilon)\sigma_a\sigma_b\boldsymbol{\mu}_\tau.$$

Obviously, one can see that the moderateness resp. negligibility of the initial energy integral $E^1_{0,\varepsilon}(\mathbf{u}_\varepsilon)$ is equivalent to the moderateness resp. negligibility of $\mathbf{u}_\varepsilon$ and its first order derivatives on S_0; thus, immediately follows from moderateness resp. negligibility of the data.

To deal with the higher order initial energy integrals, we need further arguments: The higher order initial energy integrals include time derivatives of order greater than one. Thus, we cannot directly replace these terms by the initial data, which are functions of the spatial variables only. Therefore, we choose a coordinate system $(t, x^\alpha) = (x^0, x^\alpha)$, we have

$$\begin{aligned} \mathbf{u}_\varepsilon(0, x^\alpha) =& \mathbf{u}_{0,\varepsilon}(x^\alpha) \\ \partial_t\mathbf{u}_\varepsilon(0, x^\alpha) =& \tilde{\mathbf{u}}_{1,\varepsilon}(x^\alpha), \end{aligned}$$

where

$$\tilde{\mathbf{u}}_{1,\varepsilon} := \frac{1}{\sigma^0_\varepsilon}\Big(\mathbf{u}_{1,\varepsilon} - \sum_{\lambda=1}^{n-1}\sigma^\lambda_\varepsilon\widehat{\nabla}_\lambda\mathbf{u}_{0,\varepsilon} - \sigma^0_\varepsilon(\widehat{\nabla}_t - \partial_t)\mathbf{u}_{0,\varepsilon}\Big) \tag{4.16}$$

and $(\widehat{\nabla}_t - \partial_t)\mathbf{u}_{0,\varepsilon}$ denotes the difference between the covariant and the partial derivative of $\mathbf{u}_{0,\varepsilon}$, which may be expressed in terms of the Christoffel symbols $\hat{\mathbf{\Gamma}}$. Now, terms of the energy tensors including first order time-derivatives are by (4.16) rewritten in terms of the (known) spatial derivatives of the data. For higher order time-derivatives, we inductively use the differential equation in the form

$$\partial_t^2 u^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_l,\varepsilon} = -\frac{1}{g^{00}_\varepsilon}\Big(\sum_{\mu,\nu=1}^{n-1} g^{\mu\nu}_\varepsilon \partial_\mu\partial_\nu u^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_l,\varepsilon} + \text{terms with less than 2 time derivatives}\Big)$$

to reduce the order of time-derivatives on $\mathbf{u}$ to 1. For the moderateness part, we have by assumption that all the terms occurring in the differential equation are moderate and not contributing more than a factor ε^{-N} for some natural number N. For the negligibility part, we have that each term of the differential equation is either some moderate function multiplied with the negligible spatial derivatives of $\mathbf{u}_{0,\varepsilon}$ or $\tilde{\mathbf{u}}_{1,\varepsilon}$ or it consists solely of F_ε, which is negligible anyway, thus contributing only a factor asymptotically decreasing faster than any positive power of ε. *q. e. d.*

Bounds on solutions from bounds on energy integrals

We obtain bounds on solutions by application of the following Lemma, which is similar to [GMS09, Lemma 6.2].

4.4.2. LEMMA: *Let α be a multi-index with $|\alpha| = m$. For $s > (n-1)/2$ an integer, there exists a constant C and number N such that for all $\mathbf{u} \in \mathcal{T}^k_l(\Omega_\tau)$ and for all $\zeta \in [0, \tau]$, we have*

$$\sup_{p\in\Omega_\tau} |\partial^\alpha \mathbf{u}(x)| \le K \sup_{0\le\zeta\le\tau} (E^{s+m}_{\zeta,\varepsilon}(\mathbf{u}))^{1/2}.$$

Proof: By the Sobolev embedding theorem on S_τ and the local uniform boundedness of the volume form $\hat{\boldsymbol{\mu}}_\tau$, we obtain for $s > (n-1)/2$,

$$\sup_{p\in S_\zeta} |\mathbf{u}(x)| \le C\, {}^{\partial}\|\mathbf{u}\|^s_{S_\zeta,\varepsilon}. \tag{4.17}$$

We apply (4.8) and obtain

$$\begin{aligned}
\sup_{p\in S_\zeta}|\mathbf{u}(x)| \leq & C\sqrt{Q_s\sum_{j=1}^{s}\varepsilon^{-N_j}(\,^{\widehat{\nabla}}\|\mathbf{u}\|^{s}_{S_\zeta,\varepsilon})^2}\\
\leq & C\sqrt{Q_s}\sum_{j=1}^{s}\sqrt{\varepsilon^{-N_j}(\,^{\widehat{\nabla}}\|\mathbf{u}\|^{s}_{S_\zeta,\varepsilon})^2}\\
\leq & C\sqrt{Q_s}\sum_{j=1}^{s}\varepsilon^{-N_j/2}\cdot\,^{\widehat{\nabla}}\|\mathbf{u}\|^{s}_{S_\zeta,\varepsilon}\\
\leq & K\sum_{j=1}^{s}\varepsilon^{-N}\cdot\,^{\widehat{\nabla}}\|\mathbf{u}\|^{s}_{S_\zeta,\varepsilon},
\end{aligned}$$

where $K = C\sqrt{Q_s}$ and $N = \max_j N_j$. Application of (3.4) for fixed ε yields

$$\sup_{p\in S_\zeta}|\mathbf{u}(x)| \leq K\varepsilon^{-N}(E^{s}_{\zeta,\varepsilon}(\mathbf{u}))^{1/2}.$$

We take the supremum over $\zeta \in [0,\tau]$ on the right hand side and obtain the result for $m = 0$.

Since time derivatives are transversal to S_τ, they are not covered by the Sobolev embedding theorem. Therefore, to show the general result, we replace $\mathbf{u}$ by the respective derivatives, i. e. we replace (4.17) by

$$\sup_{p\in S_\zeta}|\partial_{\rho_1}\cdots\partial_{\rho_i}\partial_t^j\mathbf{u}| \leq C\,^{\partial}\|\partial_t^j\mathbf{u}\|^{s+i}_{S_\zeta,\varepsilon} \leq C\,^{\partial}\|\mathbf{u}\|^{s+i+j}_{S_\zeta,\varepsilon},$$

where $m = i + j$ *q. e. d.*

Proof of the Main Theorem and conclusion

Before we are ready to prove Theorem 4.2.1, we cite a classical existence and uniqueness result for wave equations, [Fri75], Theorem 5.3.2:

4.4.3. THEOREM: *Let Σ_0 be a past-compact spacelike hypersurface with the property $\partial I^+(\Sigma_0) = \Sigma_0$. Suppose that $\mathbf{F}$ is smooth and that smooth Cauchy data are given on Σ_0. Then the Cauchy problem for $L\mathbf{u} = \mathbf{F}$ has a unique solution $\mathbf{u}$ in $\overline{I^+(\Sigma_0)}$ such that $\mathbf{u}$ is smooth.*

Finally, we give the proof of Theorem 4.2.1.

Proof of the main theorem:

Step 1: Existence of classical solutions.
Theorem 4.4.3 together with assumption (ii) in Section 4.1 guarantees existence of a unique smooth solution $\mathbf{u}_\varepsilon$ of (4.5) for each ε on a domain $A \subset \bigcap_{\varepsilon<\varepsilon_0} \overline{I_\varepsilon^+(\Sigma)}$. Without loss of generality we may assume, that $\Omega_\gamma \subset A$.

Step 2: Existence of generalized solutions.
We show that the net obtained in step 1 is moderate on Ω_γ. By assumption $\mathbf{u}_0$ and $\mathbf{u}_1$ are moderate, so by Lemma 4.4.1 we obtain moderate initial energy integrals $E^m_{0,\varepsilon}(\mathbf{u}_\varepsilon)$ for $m \geq 1$ and $0 \leq \tau \leq \gamma$. Now, Corollary 4.3.7 ensures moderateness of the energy integrals $E^m_{\tau,\varepsilon}(\mathbf{u}_\varepsilon)$ for $m \geq 1$ and $0 \leq \tau \leq \gamma$. Finally, Lemma 4.4.2 implies moderateness of $\mathbf{u}_\varepsilon$; hence, $\mathbf{u} := [(\mathbf{u}_\varepsilon)_\varepsilon]$ is a generalized solution to the initial value problem (4.1), (4.2) on Ω_γ.

Step 3: Uniqueness of generalized solutions.
The proof follows the line of arguments already used in step 2. Since L is a linear differential operator, it suffices to show that the solution $\mathbf{u}$ of an equation with negligible $\mathbf{F}$, $\mathbf{u}_0$, and $\mathbf{u}_1$ is negligible as well. To establish this result, we proceed as before using the negligibility parts of Lemma 4.4.1 and Corollary 4.3.7. Thus, the solution $\mathbf{u} = [(\mathbf{u}_\varepsilon)_\varepsilon]$ is unique, and we are done.

q. e. d.

REMARK: As the present analysis shows, the assumptions in condition (A) of [GMS09] on the derivatives of $\mathbf{g}$ are not necessary to prove the existence and uniqueness result [GMS09, Theorem 3.1]. In fact, for the scalar wave equation derived from the connection d'Alembert operator, Theorem 4.2.1 guarantees solutions for the following asymptotic conditions, cf. equation (4.10) and the remark at the end of chapter 3.

(i) For every representative $(\mathbf{g}_\varepsilon)_\varepsilon$ of $\mathbf{g}$, we demand for all K compact in U

$$\sup_K |\mathbf{g}_\varepsilon^{-1}|_\mathsf{m} = O(1) \qquad \sup_K |\mathbf{g}_\varepsilon|_\mathsf{m} = O(1)$$

as ε tends to zero.

(i′) For every representative $(\mathbf{g}_\varepsilon)_\varepsilon$ of $\mathbf{g}$, we demand for all K compact in U

$$\sup_K |\nabla_\varepsilon \sigma|_{\mathsf{m}} = O(1) \tag{4.18}$$

as ε tends to zero.

This is as significant improvement of the main theorem in [GMS09]. On the other hand, the original condition (A) can obviously be used to derive the precise asymptotics of the solutions of [GMS09, Theorem 3.1], hence to prove an additional regularity result.

5. Wave equations and first order systems

A classically well-known problem is the question whether a wave equation is equivalent to a first order hyperbolic system of partial differential equations. While for smooth equations with smooth coefficients and data, the process of reformulating wave equations in terms of hyperbolic systems and vice versa is well understood, in the case low regularity situations, such as distributional equations, this is a delicate issue. In this chapter we analyse this problem in a setting of Colombeau generalized functions.

Symmetric hyperbolic first order systems play a major role in the analysis of wave propagation in highly heterogeneous media, where certain physical properties such as sound speed and density are modelled by functions of non smooth variation. Thus connecting this research to the Cauchy problem for second order wave equations of low regularity seems to be an important issue. The theory of first order systems used herein is based on [HoSp11, Theorems 3.1, 3.2, and 3.4], the theory for wave equations is based on Theorem 4.2.1 in this thesis. Details on the motivations from natural sciences for first order systems can be found in the following papers of closely related research [CoOb90, GaOb11a, GaOb11b, Hor04, HoHo01, KmHo01, Obe09]; second order equations have been discussed in [Gra93, GMS09, Han10, ViWi00].

We also adopt the notational conventions from the previous chapters to the setting in $\mathbb{R}^{n+1}$ resp. $\widetilde{\mathbb{R}}^{n+1}$: A point (t, x) in $\mathbb{R}^n$ will be denoted by (x_λ) with $0 \leq \lambda \leq n$. We use semi-bold and sans-serif letters to denote matrix- and tensor-valued functions, e. g. R; we use semi-bold italic letters to denote vector valued functions, e. g. $\boldsymbol{v}$ and rows, e. g. $\boldsymbol{R}_{\alpha\cdot}$ or columns, e. g. $\boldsymbol{R}_{\cdot\alpha}$ of a matrix, and we use italic letters for all scalar values like components of a vector, e. g. v_α or matrix, e. g. $R_{\alpha\beta}$. Furthermore, the euclidean scalar product for vector valued functions is written $\langle \boldsymbol{v}, \boldsymbol{w} \rangle$. The gradient of a scalar u shall be the vector $\boldsymbol{u}'$; the Hessian shall be the matrix u''. Note that in this chapter we take the gradient, e. g. $\operatorname{grad} u = \boldsymbol{u}'$, and the divergence, e. g. $\operatorname{div} \boldsymbol{v}$, only with respect to the space coordinates x_λ $(1 \le \lambda \le n)$. For time derivatives, we explicitly write ∂_t. The full derivative of a Matrix valued function $\mathbf{A}$ is denoted by $\mathrm{d}\mathbf{A}$; the spatial derivative is denoted by $\mathbf{A}'$; the spatial derivative of a vector valued function v we denote by $\mathbf{v}'$.

In this chapter we consider scalar wave equations of the form

$$-\partial_t^2 u + 2\sum_{\lambda=1}^{n} h_\lambda \partial_\lambda \partial_t u + \sum_{\lambda,\mu,\nu=1}^{n} R_{\lambda\mu}\partial_\lambda\partial_\mu u + a\partial_t u + \sum_{\lambda=1}^{n} b_\lambda \partial_\lambda u + cu = f, \tag{5.1}$$

where $\boldsymbol{b}$, $\boldsymbol{h}$, a, c, and f are generalized (vector valued) functions on $\mathbb{R}^{n+1}$, and $\mathbf{R}$ is a Riemannian metric on $\mathbb{R}^n$ for each generalized time value $\tilde{t}$. Note that, locally, we can transform any wave equation into this form, which suits best for our purpose of rewriting it into a hyperbolic first order system. For further details, see the following section 5.1. We aim at relating the above equation to a first order system of the form

$$-\partial_t \boldsymbol{w} + \sum_{\lambda=1}^{n} \mathbf{A}_\lambda \partial_\lambda \boldsymbol{w} + \mathbf{B}\boldsymbol{w} = \boldsymbol{F}, \tag{5.2}$$

with $\mathbf{A}$, $\mathbf{B}$ and $\boldsymbol{F}$ matrix/vector valued functions of the form

$$\mathbf{A}_\lambda = \begin{pmatrix} 0 & 0 & 0_{1\times n} \\ 0 & 2h_\lambda & S_{\lambda\cdot} \\ 0_{n\times 1} & S_{\cdot\lambda} & 0_{n\times n} \end{pmatrix} \tag{5.3a}$$

$$\mathbf{B} = \begin{pmatrix} 0 & 1 & 0_{1\times n} \\ c & a & (\operatorname{div}\mathbf{S})^t + (\boldsymbol{b} - \operatorname{div}\mathbf{S}^2)^t\mathbf{S}^{-1} \\ 0_{n\times 1} & 0_{n\times 1} & (\partial_t\mathbf{S})\mathbf{S}^{-1} \end{pmatrix}, \tag{5.3b}$$

$$\boldsymbol{F} = \begin{pmatrix} 0 \\ f \\ 0_{n\times 1} \end{pmatrix}. \tag{5.3c}$$

Here, the functions $\boldsymbol{b}$, $\boldsymbol{h}$, a, c, and f are as described above and $\mathbf{S}^2 = \mathbf{R}$. The solutions of (5.1) and (5.2) shall be related via $\boldsymbol{w} = (u, \partial_t u, \mathbf{S}\boldsymbol{u}')$. For any symmetric matrix $\mathbf{S} \in M_k(\mathcal{G}(\mathbb{R}^{n+1}))$, we set $(\operatorname{div}\mathbf{S})_\lambda = \sum_{\mu=1}^{k} \partial_\mu S_{\lambda\mu}$, i.e. $\operatorname{div}\mathbf{S}$ is meant to be a vector, whose λ-th entry is the divergence of the λ-th row (or column) of the matrix $\mathbf{S}$.

In the first sections we study a important properties of generalized Lorentzian metrics on Euclidean space. The chapter continues with proofs of the equivalence theorems that allow relate second order equations and first order systems. We conclude the chapter with an extension to the existence and uniqueness theory for both wave equations and first order systems using the results in [HoSp11] and Chapter 4 in combination with the equivalence theorems.

The results of this chapter originate in collaborative work with Günther Hörmann, Christian Spreitzer, and Roland Steinbauer.

5.1. On generalized Lorentzian metrics

To rewrite a wave equation into a first order system, we require the wave equation to have a suitable algebraic layout. On the one hand, we need the equation to be in the form 5.1 such that we can rewrite it into a first order system. On the other hand, we would like to handle a preferably large class of wave equations. Therefore, in this section we show that a any semi-Riemannian metric $\mathbf{g}$, defined on a level of generalized point values by

$$\mathbf{g}(\tilde{t},\tilde{x}) = \begin{pmatrix} -1 & \boldsymbol{h}_{\tilde{t}}^T(\tilde{x}) \\ \boldsymbol{h}_{\tilde{t}}(\tilde{x}) & \mathbf{R}_{\tilde{t}}(\tilde{x}) \end{pmatrix} \text{ with } (\tilde{t},\tilde{x}) \in \widetilde{\mathbb{R}}_c^{n+1} \tag{5.4}$$

with $\mathbf{R}_{\tilde{t}}$ for each $\tilde{t} \in \widetilde{\mathbb{R}}_c$ a generalized Riemannian metric on $\mathbb{R}^n$ and $\boldsymbol{h}_{\tilde{t}}$ for each $\tilde{t} \in \widetilde{\mathbb{R}}_c$ some generalized vector field on $\mathbb{R}^n$, is in fact Lorentzian, and that locally we can transform any scalar wave equation

$$\sum_{\lambda,\mu=0}^{n} \bar{g}^{\lambda\mu}\partial_\lambda\partial_\mu u + \sum_{\lambda=0}^{n} \bar{b}_\lambda\partial_\lambda u + \bar{c}u = \bar{f}$$

into a form, where the principal part of the differential operator is generated by such a Lorentzian metric $\mathbf{g}$.

Before we actually prove our assertion on $\mathbf{g}$, let us recall a significant result on the properties of linear mappings in a finite dimensional free module of generalized numbers. Note that $M_n(\widetilde{\mathbb{R}})$ denotes the module of n-dimensional matrices with entries in $\widetilde{\mathbb{R}}$.

5.1.1. Lemma: *Let $\mathbf{A} \in M_n(\widetilde{\mathbb{R}})$. The following are equivalent:*

(i) *$\mathbf{A}$ is nondegenerate, i. e. $\boldsymbol{\xi} \in \widetilde{\mathbb{R}}^n$, $\boldsymbol{\xi}^t\mathbf{A}\boldsymbol{\eta} = 0 \,\forall \boldsymbol{\eta} \in \widetilde{\mathbb{R}}^n$ implies $\boldsymbol{\xi} = 0$.*

(ii) *$\mathbf{A} : \widetilde{\mathbb{R}}^n \to \widetilde{\mathbb{R}}^n$ is injective.*

(iii) $\mathbf{A} : \widetilde{\mathbb{R}}^n \to \widetilde{\mathbb{R}}^n$ *is bijective.*

(iv) $\det \mathbf{A}$ *is invertible.*

For a proof, see [KuSt02b, Lemma 2.6]. We should also note [May08, Lemma 4.3] which characterizes symmetric matrices over $\widetilde{\mathbb{R}}$.

5.1.2. LEMMA: *Let* $\mathbf{A} \in M_n(\widetilde{\mathbb{R}})$*. The following are equivalent:*

(i) $\mathbf{A}$ *is symmetric, that is* $\mathbf{A} = \mathbf{A}^t$ *in* $M_n(\widetilde{\mathbb{R}})$*.*

(ii) There exists a symmetric representative $(\mathbf{A}_\varepsilon)_\varepsilon := (A_{ij,\varepsilon})_\varepsilon$ *of* $\mathbf{A}$*.*

Now, we proceed with the proof of

5.1.3. LEMMA (Lorentzian metrics): *Let* $\mathbf{g}$ *be defined as in* (5.4)*, then* $\mathbf{g}$ *is a generalized Lorentzian metric on* $\mathbb{R}^{n+1}$*.*

Proof: According to the definition of a generalized Lorentzian metric (see Definition 2.4.4, we choose all of $\mathbb{R}^{n+1}$ with the identity map as chart), we split the proof into two parts. At first, we show that $\mathbf{g}(\tilde{t}, \tilde{x})$ is a symmetric, nondegenerate map in $\mathrm{End}(\widetilde{\mathbb{R}}^{n+1})$ for each generalized point $(\tilde{t}, \tilde{x}) \in \widetilde{\mathbb{R}}_c^{n+1}$. Hence, $\mathbf{g}$ is a symmetric, nondegenerate generalized $(0, 2)$-tensor field on $\mathbb{R}^{n+1}$. In the second part we show that $\mathbf{g}$ has constant index 1.

1. We choose an arbitrary compactly supported generalized point $(\tilde{t}, \tilde{x})$. For convenience of the reader, we omit the variables $\tilde{t}$ and $\tilde{x}$ in the following. Thus let $\tilde{\boldsymbol{h}} = \boldsymbol{h}_{\tilde{t}}(\tilde{x})$ and $\tilde{\mathbf{R}} = \mathbf{R}_{\tilde{t}}(\tilde{x})$ and therefore

$$\tilde{\mathbf{g}} = \begin{pmatrix} -1 & \tilde{\boldsymbol{h}}^T \\ \tilde{\boldsymbol{h}} & \tilde{\mathbf{R}} \end{pmatrix} := \mathbf{g}(\tilde{t}, \tilde{x}).$$

Obviously, the matrix $\tilde{\mathbf{g}}$ is symmetric by definition. By Lemma 5.1.1 the nondegeneracy of any map in $\mathrm{End}(\widetilde{\mathbb{R}}^{n+1})$ is equivalent to its injectivity; thus, it suffices to show that $\tilde{\mathbf{g}} : \widetilde{\mathbb{R}}^{n+1} \to \widetilde{\mathbb{R}}^{n+1}$ is injective. So we assume for some vector

$\tilde{v} = (\tilde{\tau}, \tilde{\boldsymbol{\xi}}) \in \widetilde{\mathbb{R}}^{n+1}$ that $\tilde{\mathbf{g}}\tilde{v} = 0$, and we show that this implies $\tilde{v} = 0$. We have

$$\tilde{\mathbf{g}}\tilde{v} = \begin{pmatrix} -\tilde{\tau} + \langle \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{\xi}} \rangle \\ \tilde{\tau}\tilde{\boldsymbol{h}} + \tilde{\mathbf{R}}\tilde{\boldsymbol{\xi}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

note that $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product. Multiplying the second equation with $\tilde{\mathbf{R}}^{-1}$—which is possible since $\tilde{\mathbf{R}}$ is Riemannian and therefore invertible—and solving for $\tilde{\boldsymbol{\xi}}$ yields $\tilde{\boldsymbol{\xi}} = -\tilde{\tau}\tilde{\mathbf{R}}^{-1}\tilde{\boldsymbol{h}}$. Substitution of $\tilde{\boldsymbol{\xi}}$ in the first equation leads to $-\tilde{\tau}(1 + \langle \tilde{\boldsymbol{h}}, \tilde{\mathbf{R}}^{-1}\tilde{\boldsymbol{h}} \rangle) = 0$. Since $\tilde{\mathbf{R}}$ and thus also $\tilde{\mathbf{R}}^{-1}$ are positive definite generalized matrices, there exists a representative $(\tilde{\mathbf{R}}_\varepsilon^{-1})_\varepsilon$ of $\tilde{\mathbf{R}}^{-1}$ and $(\tilde{\boldsymbol{h}}_\varepsilon)_\varepsilon$ of $\tilde{\boldsymbol{h}}$ such that $\langle \tilde{\boldsymbol{h}}_\varepsilon, \tilde{\mathbf{R}}_\varepsilon^{-1}\tilde{\boldsymbol{h}}_\varepsilon \rangle \geq 0$ for all ε. Hence the expression $s := -1 - \langle \tilde{\boldsymbol{h}}, \tilde{\mathbf{R}}^{-1}\tilde{\boldsymbol{h}} \rangle$ is strictly negative and therefore no zero divisor. We divide by s and this yields $\tilde{\tau} = 0$ in $\widetilde{\mathbb{R}}$. As a consequence, the second equation reduces to $\tilde{\mathbf{R}}\tilde{\boldsymbol{\xi}} = 0$ and so we have $\tilde{\boldsymbol{\xi}} = 0$ in $\widetilde{\mathbb{R}}^n$ by the injectivity of $\tilde{\mathbf{R}}$.

2. To determine the index of $\tilde{\mathbf{g}}$, we evaluate the bilinear form related to $\tilde{\mathbf{g}}$ for the representative $(\tilde{\mathbf{g}}_\varepsilon)_\varepsilon$ defined by

$$\tilde{\mathbf{g}}_\varepsilon = \begin{pmatrix} -1 & \tilde{\boldsymbol{h}}_\varepsilon^T \\ \tilde{\boldsymbol{h}}_\varepsilon & \tilde{\mathbf{R}}_\varepsilon \end{pmatrix},$$

where $\tilde{\boldsymbol{h}}_\varepsilon = \boldsymbol{h}_\varepsilon(t_\varepsilon, x_\varepsilon)$ is a representative of $\tilde{\boldsymbol{h}}$ and $\tilde{\mathbf{R}}_\varepsilon = \mathbf{r}_\varepsilon(t_\varepsilon, x_\varepsilon)$ is a positive definite representative of $\tilde{\mathbf{R}}$. Take the (classical) vectors $\boldsymbol{v} = (\tau, \boldsymbol{\xi})$ and $\boldsymbol{w} = (\sigma, \boldsymbol{\eta})$, then $\langle \boldsymbol{v}, \tilde{\mathbf{g}}_\varepsilon \boldsymbol{w} \rangle = -\tau\sigma + \tau\langle \tilde{\boldsymbol{h}}_\varepsilon, \boldsymbol{\eta} \rangle + \sigma\langle \tilde{\boldsymbol{h}}_\varepsilon, \boldsymbol{\xi} \rangle + \langle \boldsymbol{\xi}, \tilde{\mathbf{R}}_\varepsilon \boldsymbol{\eta} \rangle$. It obviously follows for each ε that $\langle \boldsymbol{v}, \tilde{\mathbf{g}}_\varepsilon \boldsymbol{w} \rangle$ is negative for nonzero vectors $\boldsymbol{v}, \boldsymbol{w}$ with vanishing space entries v_λ, w_λ ($\lambda > 0$) and positive for nonzero vectors with vanishing time entries v_0, w_0 since $\tilde{\mathbf{R}}_\varepsilon$ is positive definite. Thus, by Sylvester's law of inertia we found a one-dimensional invariant subspace for which the bilinear mapping $\tilde{\mathbf{g}}_\varepsilon$ is negative and an n-dimensional subspace for which the mappings are positive definite for each ε, and so we have a representative of $\mathbf{g}$ that has constant index $\nu(\mathbf{g}) = 1$. By [GKOS01, Lemma 3.2.76] the index is well defined; hence, together with the result from the first part, $\mathbf{g}$ is a generalized Lorentzian metric on $\mathbb{R}^{n+1}$, and we are done.

q. e. d.

Furthermore, we need Lemma 4.6 from [May08] that allows us to diagonalize a symmetric matrix over $\widetilde{\mathbb{R}}$.

5.1.4. LEMMA (Generalized diagonalization): *Let* $\mathbf{A} \in M_n(\widetilde{\mathbb{R}})$ *be a symmetric matrix. Then the eigenvalues* λ_k *of* $\mathbf{A}$ *with* $(1 \leq k \leq n)$ *as introduced in Definition 2.4.5 are well defined elements of* $\widetilde{\mathbb{R}}$*. Furthermore, there exists an orthogonal matrix* $\mathbf{U} \in M_n(\widetilde{\mathbb{R}})$ *such that*

$$\mathbf{U}\mathbf{A}\mathbf{U}^t = \operatorname{diag}(\lambda_1, \ldots, \lambda_n). \tag{5.5}$$

The matrix $\mathbf{A}$ *is nondegenerate if and only if all generalized eigenvalues are invertible.*

The last find of this section shows that a certain class of second order partial differential equations has in fact a principal part given by a Lorentzian metric and that any of these equations may be transformed into the form (5.1).

5.1.5. COROLLARY: *Consider the equation*

$$\bar{g}_{00}\partial_t^2 u + 2\sum_{\lambda=1}^{n} \bar{g}_{0\lambda}\partial_\lambda\partial_t u + \sum_{\lambda,\mu=1}^{n} \bar{g}_{\lambda\mu}\partial_\lambda\partial_\mu u + \bar{a}\partial_t u + \sum_{\lambda=1}^{n} \bar{b}_\lambda\partial_\lambda u + \bar{c}u = \bar{f} \tag{5.6}$$

with coefficients in the special Colombeau algebra. Assume that $\bar{g}_{00}$ *is strictly negative, and* $(\bar{g}_{\lambda\mu})_{1\leq\lambda,\mu\leq n}$ *is positive definite. Then the generalized matrix* $\bar{\mathbf{g}}$ *with components* $(\bar{g}_{\lambda\mu})_{0\leq\lambda,\mu\leq n}$ *is a generalized Lorentzian metric on* $\mathbb{R}^{n+1}$*, and* (5.6) *is equivalent to*

$$-\partial_t^2 u + 2\sum_{i=1}^{n} h_\lambda\partial_\lambda\partial_t u + \sum_{\lambda,\mu,\nu=1}^{n} S_{\lambda\nu}S_{\nu\mu}\partial_\lambda\partial_\mu u + a\partial_t u + \sum_{\lambda=1}^{n} b_\lambda\partial_\lambda u + cu = f, \tag{5.7}$$

where $\mathbf{S}$ *is the square root of a Riemannian metric* $\mathbf{R}$*.*

Note that the lower order terms are completely irrelevant in this calculation, we only specified them for the sake of completeness.

Proof: We start by dividing equation (5.6) by $|\bar{g}_{00}|$ and obtain

$$-\partial_t^2 u + 2\sum_{\lambda=1}^{n} g_{0\lambda}\partial_\lambda\partial_t u + \sum_{\lambda,\mu}^{n} g_{\lambda\mu}\partial_\lambda\partial_\mu u + a\partial_t u + \sum_{\lambda=1}^{n} b_\lambda\partial_\lambda u + cu = f,$$

where $g_{\lambda\mu} = -\bar{g}_{\lambda\mu}/\bar{g}_{00}$, $a = -\bar{a}/\bar{g}_{00}$, $b_\lambda = -\bar{b}_\lambda/\bar{g}_{00}$, $c = -\bar{c}/\bar{g}_{00}$, and $f = -\bar{f}/\bar{g}_{00}$. By lemma 5.1.3 we know that $\mathbf{g} = (g_{\lambda\mu})_{0\leq\lambda,\mu\leq n}$ is Lorentzian. Since we have that $\bar{\mathbf{g}} = (\bar{g}_{\lambda\mu})_{0\leq\lambda,\mu\leq n} = -\bar{g}_{00}(g_{\lambda\mu})_{0\leq\lambda,\mu\leq n}$ with $\bar{g}_{00} < 0$, $\bar{\mathbf{g}}$ is also Lorentzian.

Now, consider the positive definite matrix of point values at $(\tilde{t}, \tilde{x}) \in \widetilde{\mathbb{R}}_c^{n+1}$ defined by $\tilde{\mathbf{R}} := (g_{\lambda\mu}(\tilde{t}, \tilde{x}))_{1 \leq \lambda,\mu \leq n}$. Therefore, by lemma 5.1.4, there exists some orthogonal matrix $\mathbf{U} \in M_n(\widetilde{\mathbb{R}})$ such that $\mathbf{U}\tilde{\mathbf{R}}\mathbf{U}^t = \operatorname{diag}(\lambda_1, ..., \lambda_n)$ with λ_k strictly positive. Putting $\tilde{\mathbf{R}} := \mathbf{U}^t \cdot \operatorname{diag}(\sqrt{\lambda_1}, ..., \sqrt{\lambda_n}) \cdot \mathbf{U}$, we have found a positive definite matrix $\tilde{\mathbf{S}}$ such that $\tilde{\mathbf{S}}^2 = \tilde{\mathbf{R}}$. *q. e. d.*

Thus, we proved that any partial differential equation on $\mathbb{R}^{n+1}$ with principal part induced by a generalized metric of the form

$$\mathbf{g} = \begin{pmatrix} \bar{g}_{00} & (\bar{g}_{\lambda 0})_{1 \leq \lambda,\mu \leq n} \\ (\bar{g}_{0\lambda})_{1 \leq \lambda,\mu \leq n} & (\bar{g}_{\lambda\mu})_{1 \leq \lambda,\mu \leq n} \end{pmatrix}$$

with $\bar{g}_{00}$ strictly negative and $\bar{\mathbf{R}} = (\bar{g}_{\lambda\mu})_{1 \leq \lambda,\mu \leq n}$ a generalized Riemannian metric can be transformed into one of type (5.7).

5.2. Generalized equivalence

From now on we only consider equations of type (5.7) and refer to them without loss of generality as wave equations. We start with

$$-\partial_t^2 u + 2\sum_{\lambda=1}^{n} h_\lambda \partial_\lambda \partial_t u + \sum_{\lambda,\mu,\nu=1}^{n} S_{\lambda\nu} S_{\nu\mu} \partial_\lambda \partial_\mu u + a\partial_t u + \sum_{\lambda=1}^{n} b_\lambda \partial_\lambda u + cu = f,$$

and a first order real symmetric hyperbolic system of type (5.2)

$$\partial_t \boldsymbol{w} + \sum_{\lambda=1}^{n} \mathbf{A}_\lambda \partial_\lambda \boldsymbol{w} + \mathbf{B}\boldsymbol{w} = \boldsymbol{F}$$

with properties as in (5.3).

REMARK: It is worth to note, that the matrix $\mathbf{S}$ needs not be positive definite since the square of any symmetric, invertible matrix is positive definite by application Lemmata 5.1.2 and 5.1.4. Precisely, suppose we diagonalize $\mathbf{S}$ with an orthogonal matrix $\mathbf{U}$, then $\mathbf{U}^t\mathbf{S}\mathbf{U} = \mathbf{D}$ for some nondegenerate diagonal matrix $\mathbf{D}$. Consequently, $\mathbf{R} = \mathbf{S}^2 = \mathbf{U}\mathbf{D}^2\mathbf{U}^t$. Since $\mathbf{D}$ has strictly nonzero, i. e. strictly negative or strictly positive, diagonal entries, we know that $\mathbf{D}^2$ has strictly positive diagonal entries, namely the squares of the entries of $\mathbf{D}$.

We want to relate the existence of a solution to (5.7) to the existence of a solution to (5.2) and vice versa. The process of rewriting the equation resp. system and relating the corresponding solutions is completely constructive, the key feature being the algebraic properties of the special Colombeau algebra, in particular, the fact that $\mathcal{G}$ is a differential algebra.

Wave equation to first order system

Assume that we have a solution u to the wave equation (5.7). In the proof of the following Lemma we will construct a solution to the first order system (5.2). Basically, we define a vector $\boldsymbol{w} = (u, \partial_t u, \mathbf{S}\boldsymbol{u}')^t$ and rewrite the wave equation in terms of the three components of $\boldsymbol{w}$.

5.2.1. LEMMA: *Consider the second-order hyperbolic equation* (5.7) *with initial condition* $(u, \partial_t u)|_{t=0} = (u_0, u_1)$. *If* $u \in \mathcal{G}(\mathbb{R}^{n+1})$ *is a solution of* (5.7)*, then the vector* $\boldsymbol{w} = (u, \partial_t u, \mathbf{S}\boldsymbol{u}')^t$ *is a solution to the first-order system* (5.2) *with initial condition* $\boldsymbol{w}|_{t=0} = (u_0, u_1, \mathbf{S}\boldsymbol{u}_0')^t$.

Proof: First we bring equation (5.7) into divergence form, i.e.

$$-\partial_t^2 u + 2\langle \boldsymbol{h}, \partial_t \boldsymbol{u}' \rangle + \operatorname{div}(\mathbf{S}^2 \boldsymbol{u}') + a\partial_t u + \langle \boldsymbol{b} - \operatorname{div} \mathbf{S}^2, \boldsymbol{u}' \rangle + cu = f.$$

Here, we have used the fact that $\operatorname{tr}(\mathbf{S}^2 \mathbf{u}'') = \operatorname{div}(\mathbf{S}^2 \boldsymbol{u}') - \langle \operatorname{div} \mathbf{S}^2, \boldsymbol{u}' \rangle$. We also have used $\langle \boldsymbol{h}, \partial_t \boldsymbol{u}' \rangle = \sum_{\lambda=1}^n h_\lambda \partial_\lambda \partial_t u$ and $\langle \boldsymbol{b}, \boldsymbol{u}' \rangle = \sum_{\lambda=1}^n b_\lambda \partial_\lambda u$. Introducing new variables $z := \partial_t u$ and $\boldsymbol{v} := \mathbf{S}\boldsymbol{u}'$, we may write

$$-\partial_t z + 2\langle \boldsymbol{h}, \boldsymbol{z}' \rangle + \operatorname{div}(\mathbf{S}\boldsymbol{v}) + az + \langle \boldsymbol{b} - \operatorname{div} \mathbf{S}^2, \mathbf{S}^{-1}\boldsymbol{v} \rangle + cu = f.$$

In the next step we collect the variables u, z, and $\boldsymbol{v}$ in a vector $\boldsymbol{w} = (u, z, \boldsymbol{v})^t$ and rewrite the equation in terms of $\boldsymbol{w}$. Hence, we have that

$$-\partial_t \boldsymbol{w} + \sum_{\lambda=1}^{n} \begin{pmatrix} 0 & 0 & 0_{1\times n} \\ 0 & 2h_\lambda & S_{\lambda\cdot} \\ 0_{n\times 1} & S_{\cdot\lambda} & 0_{n\times n} \end{pmatrix} \partial_\lambda \boldsymbol{w}$$
$$+ \begin{pmatrix} 0 & 1 & 0_{1\times n} \\ c & a & (\operatorname{div} \mathbf{S})^t + (\boldsymbol{b} - \operatorname{div} \mathbf{S}^2)^t \mathbf{S}^{-1} \\ 0_{n\times 1} & 0_{n\times 1} & (\partial_t \mathbf{S})\mathbf{S}^{-1} \end{pmatrix} \boldsymbol{w} = \begin{pmatrix} 0 \\ f \\ 0_{n\times 1} \end{pmatrix},$$

where we have used that $\operatorname{div}(\mathbf{S}\boldsymbol{v}) = \operatorname{tr}(\mathbf{S}\mathbf{v}') + (\operatorname{div} \mathbf{S})^t \boldsymbol{v}$. The first line in the above system reads

$$-\partial_t u + z = 0$$

and is therefore just the variable transformation for z. The second line of the system is the wave equation itself written in the new variables u, z and $\boldsymbol{v}$

$$-\partial_t z + 2\langle \boldsymbol{h}, \boldsymbol{z}' \rangle + \operatorname{tr}(\mathbf{S}\mathbf{v}') + cu + az + \langle \operatorname{div} \mathbf{S}, v \rangle + \langle \boldsymbol{b} - \operatorname{div} \mathbf{S}^2, \mathbf{S}^{-1}\boldsymbol{v} \rangle = f.$$

For the third line, we have

$$-\partial_t \boldsymbol{v} + \mathbf{S}\boldsymbol{z}' + (\partial_t \mathbf{S})\mathbf{S}^{-1}\boldsymbol{v} = 0.$$

If we apply the variable transformations $z = \partial_t u$ and $\boldsymbol{u}' = \mathbf{S}^{-1}\boldsymbol{v}$, we obtain

$$-\partial_t \boldsymbol{v} + \mathbf{S}\partial_t \boldsymbol{u}' + (\partial_t \mathbf{S})\boldsymbol{u}' = -\partial_t \boldsymbol{v} + \partial_t(\mathbf{S}\boldsymbol{u}') = 0,$$

which is just the derivative of the variable transformation $\boldsymbol{v} = \mathbf{S}\boldsymbol{u}'$ and thus obviously true. Finally, evaluating the system's solution $\boldsymbol{w}$ at time $t = 0$ yields the initial condition $\boldsymbol{w}|_{t=0} = (u_0, u_1, \mathbf{S}\boldsymbol{u}')^t$. *q. e. d.*

First order system to wave equation

Now, we look at the converse situation: Given a solution to the first order system (5.2), we would like to prove existence of solutions for wave-type equations. Again let $\mathbf{S}$ be a symmetric and invertible n-dimensional matrix with entries in $\mathcal{G}(\mathbb{R}^{n+1})$; let $\boldsymbol{g}$ and $\boldsymbol{b}$ be generalized vector fields in $\mathbb{R}^{n+1}$, and let a, c be generalized functions.

5.2.2. Lemma: *If $\boldsymbol{w} = (u, z, \boldsymbol{v})^t \in \mathcal{G}(\mathbb{R}^{n+1})^{n+2}$ is a solution to the first-order system* (5.2) *with initial condition $\boldsymbol{w}|_{t=0} = (u_0, u_1, \mathbf{S}\boldsymbol{u}_0')^t$, then u is a solution to* (5.7) *with initial condition $(u, \partial_t u)|_{t=0} = (u_0, u_1)$.*

Proof: Observe that the first line of our system just translates to $z = \partial_t u$. Furthermore, the last n equations read

$$-\partial_t \boldsymbol{v} + \mathbf{S} z' + (\partial_t \mathbf{S})\mathbf{S}^{-1}\boldsymbol{v} = 0. \tag{5.8}$$

By using Leibniz' rule we obtain

$$\partial_t \boldsymbol{v} = \partial_t(\mathbf{S}\mathbf{S}^{-1}\boldsymbol{v}) = \mathbf{S}\partial_t(\mathbf{S}^{-1}\boldsymbol{v}) + (\partial_t \mathbf{S})\mathbf{S}^{-1}\boldsymbol{v}. \tag{5.9}$$

We substitute (5.9) into (5.8) and this yields

$$\mathbf{S} z' = \mathbf{S}\partial_t(\mathbf{S}^{-1}\boldsymbol{v}).$$

Multiplication of the last equation with $\mathbf{S}^{-1}$ from the left hand side gives $\boldsymbol{z}' = \partial_t(\mathbf{S}^{-1}\boldsymbol{v})$, and using that $z = \partial_t u$ (thus $\boldsymbol{z}' = \partial_t \boldsymbol{u}'$), we find

$$\partial_t(\boldsymbol{u}' - \mathbf{S}^{-1}\boldsymbol{v}) = 0. \tag{5.10}$$

Equation (5.10) is a simple ordinary differential equation that we can solve globally uniquely by integration taking the initial condition $\boldsymbol{u}'|_{t=0} = \mathbf{S}^{-1}\boldsymbol{v}|_{t=0}$ into account. Thus, we have $\boldsymbol{u}' = \mathbf{S}^{-1}\boldsymbol{v}$ for all t, which is the variable transformation for $\boldsymbol{v}$ and $\boldsymbol{u}'$. This leaves us with the second line of the system. Replacing z by $\partial_t u$ as well as $\boldsymbol{v}$ by $\mathbf{S}\boldsymbol{u}'$, we obtain

$$\begin{aligned} -\partial_t^2 u + 2\sum_{\lambda=1}^{n} h_\lambda \partial_\lambda \partial_t u + a\partial_t u + \sum_{\lambda=1}^{n} b_\lambda \partial_\lambda u + cu \\ + \sum_{\lambda,\mu,\nu=1}^{n} \Big(S_{\lambda\mu}\partial_\lambda(S_{\mu\nu}\partial_\nu u) + (\partial_\lambda S_{\lambda\mu})S_{\mu\nu}\partial_\nu u - \partial_\lambda(S_{\lambda\mu}S_{\mu\nu})\partial_\nu u \Big) = f. \end{aligned} \tag{5.11}$$

We take a closer look on the second line and apply Leibniz' rule. Thus, we obtain

$$\begin{aligned} &\sum_{\lambda,\mu,\nu=1}^{n} \Big(S_{\lambda\nu}\partial_\lambda(S_{\nu\mu}\partial_\mu u) + \partial_\lambda S_{\lambda\nu} S_{\nu\mu}\partial_\mu u - \partial_\lambda(S_{\lambda\nu}S_{\nu\mu})\partial_\mu u \Big) = \\ &\sum_{\lambda,\mu,\nu=1}^{n} \Big(S_{\lambda\nu}\partial_\lambda S_{\nu\mu}\partial_\mu u + S_{\lambda\nu}S_{\nu\mu}\partial_\lambda\partial_\mu u + \partial_\lambda S_{\lambda\nu}S_{\nu\mu}\partial_\mu u - \partial_\lambda S_{\lambda\nu}S_{\nu\mu}\partial_\mu u - S_{\lambda\nu}\partial_\lambda S_{\nu\mu}\partial_\mu u \Big) = \\ &\sum_{\lambda,\mu,\nu=1}^{n} S_{\lambda\nu}S_{\nu\mu}\partial_\lambda\partial_\mu u, \end{aligned}$$

and therefore, equation (5.11) equals (5.7). Thus, we translated the system (5.2) into the wave equation (5.7).

Moreover, the condition $(u, \partial_t u)|_{t=0} = (u_0, u_1)$ is a direct consequence of the initial condition for the system. *q. e. d.*

REMARK: When looking at the matrix $\mathbf{B}$, one could get the impression that the structure of the term $(\operatorname{div} \mathbf{S})^t + (\boldsymbol{b} - \operatorname{div} \mathbf{S}^2)^t \mathbf{S}^{-1}$ restricts the choice of this specific coefficient. However, this is not the case. Let $\tilde{\boldsymbol{b}}^t = (\operatorname{div} \mathbf{S})^t + (\boldsymbol{b} - \operatorname{div} \mathbf{S}^2)^t \mathbf{S}^{-1}$. Multiplication with $\mathbf{S}$ from the right hand side gives

$$\tilde{\boldsymbol{b}}^t \mathbf{S} = (\operatorname{div} \mathbf{S})^t \mathbf{S} + (\boldsymbol{b} - \operatorname{div} \mathbf{S}^2)^t.$$

Bringing all terms except the one containing $\boldsymbol{b}$ to the other side results in

$$\tilde{\boldsymbol{b}}^t \mathbf{S} - (\operatorname{div} \mathbf{S})^t \mathbf{S} + (\operatorname{div} \mathbf{S}^2)^t = \boldsymbol{b}^t.$$

Finally, transposition gives an equation for $\boldsymbol{b}$ entirely in terms of $\mathbf{S}$ and an arbitrarily chosen coefficient $\tilde{\boldsymbol{b}}$:

$$\boldsymbol{b} = \mathbf{S}^t \tilde{\boldsymbol{b}} - \mathbf{S}^t \operatorname{div} \mathbf{S} + (\operatorname{div} \mathbf{S}^2).$$

Equivalence theorems

The content of Lemmata 5.2.1 and 5.2.2 can be summarized as follows: The problem of finding a solution to the Cauchy problem for the wave equation (5.7) is equivalent to the problem of finding a solution to the corresponding Cauchy problem for a first order system (5.2) and vice versa. This equivalency will be made more precise in the following theorem.

5.2.3. THEOREM: *Assume we are given a wave equation* (5.7) *and the corresponding first order system* (5.2). *Then for a function $u \in \mathcal{G}(\mathbb{R}^{n+1})$ and a vector valued function $\boldsymbol{w} \in (\mathcal{G}(\mathbb{R}^{n+1}))^{n+2}$ such that $\boldsymbol{w} = (u, \partial_t u, \mathbf{S}\boldsymbol{u}')^t$, the following are equivalent:*

(i) The function u is a solution to the wave equation (5.7) *with initial condition $(u, \partial_t u)|_{t=0} = (u_0, u_1)$.*

(ii) The vector field $\boldsymbol{w}$ is a solution to the first order system (5.2) *with initial condition* $\boldsymbol{w}|_{t=0} = (u_0, u_1, \mathbf{S}\boldsymbol{u}_0')^t$.

Proof: The proof is straight forward from the lemmata. We apply Lemma 5.2.2 to prove (ii)⇒(i) and Lemma 5.2.1 for the converse direction. *q. e. d.*

5.2.4. COROLLARY: *A solution u of* (5.7) *is unique if and only if the corresponding solution $\boldsymbol{w}$ of* (5.2) *is unique.*

Proof: By Lemma 5.2.1, two distinct solutions to the initial value problem (5.7) would give rise to two distinct solutions to (5.2) since $u \mapsto \boldsymbol{w} = (u, \partial_t u, \mathbf{S}\boldsymbol{u}')^t$ is injective, thus contradicting uniqueness of (5.2).

Suppose there were two distinct solutions $\boldsymbol{w}, \widetilde{\boldsymbol{w}}$ of (5.2) with

$$\boldsymbol{w}|_{t=0} = \widetilde{\boldsymbol{w}}|_{t=0} = (u_0, u_1, \mathbf{S}\boldsymbol{u}_0')^t.$$

Then, the first component of $\boldsymbol{w}$ and $\widetilde{\boldsymbol{w}}$ would imply two distinct solutions u, $\tilde{u}$ to (5.7). But since the solution of (5.7) is unique, i. e. $\tilde{u} = u$, the first component of $\widetilde{\boldsymbol{w}}$ must be equal to the first component of $\boldsymbol{w}$. However, from the variable transformations in the proof of Lemma 5.2.2 it is clear that $\tilde{z} = \partial_t \tilde{u} = \partial_t u = z$ and $\tilde{\boldsymbol{v}} = \mathbf{S}\tilde{\boldsymbol{u}}' = \mathbf{S}\boldsymbol{u}' = \boldsymbol{v}$, hence $\widetilde{\boldsymbol{w}} = \boldsymbol{w}$. *q. e. d.*

REMARK: Observe that the usage of Lemmata 5.2.1 and 5.2.2 in the proofs of the existence of solutions, Theorem 5.2.3, and the uniqueness of solutions, Corollary 5.2.4, is reversed!

5.3. Existence and uniqueness theory

The Equivalence Theorem 5.2.3 and Corollary 5.2.4 allow us to adapt known existence and uniqueness results for wave equations of type (5.7) to hyperbolic first order systems of type (5.2) and vice versa.

Solutions for wave equations

In the following theorem we will derive conditions on the coefficients of the second order equation equation (5.7) in $\mathcal{G}$ that guarantee the existence of a unique solution to the corresponding first-order problem, and hence, by Theorem 5.2.3 and Corollary 5.2.4, the existence of a unique solution to the wave equation (5.7). To this end, for the rest of the chapter we denote by $U_T :=]0, T[\times \mathbb{R}^n$.

We will also use generalizations of the special Colombeau algebra. These topologically constructed Colombeau algebras were introduced by Claudia Garetto in [Gar05]. We can inherit essential properties of the special algebra, such as the point value characterization or strict positivity, to these new algebras.

5.3.1. Definition: Let E be a locally convex topological vector space with a topology given by a family of semi-norms $\{p_j\}$ with j in some index set J. We define

$$\mathcal{M}_E := \{(u_\varepsilon)_\varepsilon \in E^{]0,1]} | \forall j \in J \, \exists N \in \mathbb{N}_0 : p_j(u)| = O(\varepsilon^{-N})\},$$
$$\mathcal{N}_E := \{(u_\varepsilon)_\varepsilon \in E^{]0,1]} | \forall j \in J \, \forall m \in \mathbb{N}_0 : p_j(u)| = O(\varepsilon^{m})\},$$

the moderate resp. negligible subsets of $E^{]0,1]}$. Operations are induced from E by ε-wise application, so we have the (vector space) inclusion $\mathcal{N}_E \subseteq \mathcal{M}_E \subseteq E^{]0,1]}$. The generalized functions based on E are defined as the quotient space $\mathcal{G}_E := \mathcal{M}_E / \mathcal{N}_E$. If E is a differential algebra, then $\mathcal{N}_E$ is an ideal in $\mathcal{M}_E$ and therefore $\mathcal{G}_E$ is a differential algebra as well, called the Colombeau algebra based on E.

Let now U be an open subset of $\mathbb{R}^n$. If we choose $E = \mathcal{C}^\infty(U)$ with the topology of uniform convergence of all derivatives on compact sets, we obtain the special Colombeau algebra on U, i. e. $\mathcal{G}_{\mathcal{C}^\infty(U)} = \mathcal{G}(U)$.

Throughout the remaining part we will also use the following three Sobolev spaces in this construction:

- $E = H^\infty(U) = \{u \in \mathcal{C}^\infty(\overline{U}) : \partial^\alpha u \in L^2(U) \, \forall \alpha \in \mathbb{N}_0^n\}$ with the family of semi-norms
$$\|u\|_{H^k} = \Big(\sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^2} \Big)^{1/2} \quad k \in \mathbb{N}_0,$$

- $E = W^{\infty,\infty}(U) = \{u \in \mathcal{C}^\infty(\overline{U}) : \partial^\alpha u \in L^\infty(U)\,\forall\alpha \in \mathbb{N}_0^n\}$ with the family of semi-norms
$$\|u\|_{W^{k,\infty}} = \max_{|\alpha|\le k} \|\partial^\alpha u\|_{L^\infty} \quad k \in \mathbb{N}_0,$$
- $E = \mathcal{C}^\infty(\bar{I} \times \mathbb{R}^n)$, where I is an open, relatively compact interval, equipped with the family of semi-norms
$$\|u\|_{k,K} = \max_{|\alpha|\le k} \|\partial^\alpha u\|_{L^\infty(\bar{I}\times K)},$$
where K is compact $\mathbb{R}^n$ and $k \in \mathbb{N}_0$.

To simplify notation, we denote the corresponding Colombeau algebras as in [Hor11]:

$$\mathcal{G}_{L^2}(U) := G_{H^\infty(U)} \qquad \mathcal{G}_{L^\infty}(U) := G_{W^{\infty,\infty}(U)} \qquad \mathcal{G}(\bar{I} \times \mathbb{R}^n) := \mathcal{G}_{\mathcal{C}^\infty(\bar{I}\times\mathbb{R}^n)}.$$

Before we actually get into the details of the existence theorems for symmetric hyperbolic systems, we introduce the mixed $L^{1,\infty}$-norm.

5.3.2. Definition: For $\mathbf{A} \in M_m(\mathcal{C}^\infty(\overline{U}_T))$, denote
$$\|\mathbf{A}\|_{L^{\infty,1}(\Omega_T)} := \int_0^T \|\mathbf{A}(s,\cdot)\|_{L^\infty(\mathbb{R}^n)}\, \mathrm{d}s.$$

Furthermore, we would like to note several asymptotic conditions for generalized functions that we use throughout this section.

5.3.3. Definition: A generalized function

- $f \in \mathcal{G}(U)$ is said to be of *local L^∞-log-type* if it admits a representative $(f_\varepsilon)_\varepsilon$ such that for all K compact in U, we have $\|f_\varepsilon\|_{L^\infty(K)} = O(\log\frac{1}{\varepsilon})$ as $\varepsilon \to 0$ (see [Obe88, Definition 1.1]),
- $f \in \mathcal{G}(U)$ is said to be of *L^∞-log-type* if it admits a representative $(f_\varepsilon)_\varepsilon$ such that $\|f_\varepsilon\|_{L^\infty(U)} = O(\log\frac{1}{\varepsilon})$ as $\varepsilon \to 0$ (see [GKOS01, Definition 1.5.1]),
- $f \in \mathcal{G}_{L^\infty}(U_T))$ is said to be of *$L^{1,\infty}$-log-type* if it admits a representative $(f_\varepsilon)_\varepsilon$ such that $\|f_\varepsilon\|_{L^{1,\infty}} = O(\log\frac{1}{\varepsilon})$ as $\varepsilon \to 0$ (cp. [CoOb90, Definition 2.1]).

The existence and uniqueness theorem for wave equations, introduced in this section, is based on work of Günther Hörmann and Christian Spreitzer on symmetric hyperbolic first order systems, cf. [HoSp11, Theorems 3.1, 3.2, and 3.4]. Solutions candidates to the Cauchy problem for symmetric hyperbolic systems with Colombeau generalized coefficients (5.2) are obtained as a net of solutions to the classical equations $\partial_t \boldsymbol{w}_\varepsilon + \sum_{\lambda=1}^{n} \mathbf{A}_{\lambda,\varepsilon}\partial_\lambda \boldsymbol{w}_\varepsilon + \mathbf{B}_\varepsilon \boldsymbol{w}_\varepsilon = \boldsymbol{F}_\varepsilon$. By imposing some additional asymptotic growth conditions in ε on the coefficient matrices, a Gronwall-type argument can be used to prove the moderateness of the family of smooth solutions. For convenience of the reader, we combine these results, adjusted to the situation at hand, in the following theorem.

5.3.4. THEOREM: *Let $\mathbf{A}_\lambda$, $\mathbf{B} \in M_{n+2}(\mathcal{G}_{L^\infty}(U_T))$, where A_λ is symmetric. Then we have the following three results.*

The Cauchy problem for the system (5.2) *has a unique solution $\boldsymbol{w} \in (\mathcal{G}(U_T))^{n+2}$ if*

A) *(i) the spatial derivatives $\mathbf{A}'_\lambda$ as well as $\frac{1}{2}(\mathbf{B} + \mathbf{B}^t)$, the symmetric part of the matrix $\mathbf{B}$, are of local L^∞-log-type,*

(ii) the initial data $\boldsymbol{w}_0 \in (\mathcal{G}(\mathbb{R}^n))^{n+2}$ and the right hand side $\boldsymbol{F} \in (\mathcal{G}(\overline{U}_T))^{n+2}$,

(iii) there exists some constant $R_{\mathsf{A}} > 0$ such that we have $\sup_{(t,x)} |\mathbf{A}_{\lambda,\varepsilon}| = O(1)$ on $]0, T[\times\{x \in \mathbb{R}^n : |x| > R_{\mathsf{A}}$ as $\varepsilon \to 0$,

or

B) *(i) the spatial derivatives of $\mathbf{A}'_\lambda$ as well as the symmetric part of the matrix $\mathbf{B}$ are of L^∞-log-type,*

(ii) the initial data $\boldsymbol{w}_0 \in (\mathcal{G}_{L^\infty}(\mathbb{R}^n))^{n+2}$ and the right hand side $\boldsymbol{F} \in (\mathcal{G}_{L^\infty}(U_T))^{n+2}$.

C) The Cauchy problem for the system (5.2) *has a unique solution $\boldsymbol{w} \in (\mathcal{G}_{L^2}(U_T))^{n+2}$ if*

(i) the spatial derivatives of $\mathbf{A}'_\lambda$ as well as the symmetric part of the matrix $\mathbf{B}$ are of $L^{1,\infty}$-log-type,

(ii) the initial data $\boldsymbol{w}_0 \in (\mathcal{G}_{L^2}(\mathbb{R}^n))^{n+2}$ and the right hand side $\boldsymbol{F} \in (\mathcal{G}_{L^2}(U_T))^{n+2}$.

REMARK: When considering case B, the situation occurs that the initial data w_0 is an element of the algebra $(\mathcal{G}_{L^\infty}(\mathbb{R}^n))^{n+2}$, whereas the restriction of the solution w to the inital surface, i. e. $w|_{t=0}$ is in $(\mathcal{G}(\mathbb{R}^n))^{n+2}$. This issue can be resolved in the following way: Every representative $(f_\varepsilon)_\varepsilon$ of a generalized function $f \in \mathcal{G}_{L^\infty}(\mathbb{R}^n)$ is also moderate in the sense of $\mathcal{G}(\mathbb{R}^n)$; thus, $\mathcal{G}_{L^\infty}(\mathbb{R}^n)$ can be interpreted as a subset of $\mathcal{G}(\mathbb{R}^n)$ if we allow the difference of two representatives of f to be in the ideal $\mathcal{N}(\mathbb{R}^n)$ instead of $\mathcal{N}_{L^\infty}(\mathbb{R}^n)$. So, we have that $w|_{t=0} - w_0 \in (\mathcal{N}(\mathbb{R}^n))^{n+2}$ but not necessarily in $(\mathcal{N}_{L^\infty}(\mathbb{R}^n))^{n+2}$. In other words, we consider the initial data to be in the algebra $(\mathcal{G}(\mathbb{R}^n))^{n+2}$ but additionally satisfying the moderateness estimates of $(\mathcal{G}_{L^\infty}(\mathbb{R}^n))^{n+2}$.

Finally, we are able to formulate an existence and uniqueness theorem for wave equations based on Theorem 5.3.4.

5.3.5. THEOREM: *Consider the initial value problem*

$$-\partial_t^2 u + 2\sum_{\lambda=1}^{n} h_\lambda \partial_\lambda \partial_t u + \sum_{\lambda,\mu,\nu=1}^{n} R_{\lambda\mu} \partial_\lambda \partial_\mu u + a\partial_t u + \sum_{\lambda=1}^{n} b_\lambda \partial_\lambda u + cu = f$$

and

$$(u, \partial_t u)|_{t=0} = (u_0, u_1)$$

with coefficients $R_{\lambda\mu}, h_\lambda, a, b_\lambda, c$ in $\mathcal{G}_{L^\infty}(U_T)$. Let, furthermore, $\mathbf{S} = \mathbf{R}^{1/2}$, where we take the square root via diagonalization of $\mathbf{R}$, cf. Lemma 5.1.4. Then, we have the following results. The initial value problem has a unique solution $u \in \mathcal{G}(U_T)$ if

A) *(i) the lower order coefficients a, c, $\boldsymbol{b}$, as well as $\mathbf{S}$, the derivative $\mathrm{d}\mathbf{S}$, the inverse $\mathbf{S}^{-1}$ and $\mathbf{h}'$ are of of local L^∞-log-type,*

(ii) the initial data $u_0, u_1 \in \mathcal{G}(\mathbb{R}^n)$ and the right-hand side $f \in \mathcal{G}(U_T)$,

(iii) there exists $R_{\mathbf{S},\boldsymbol{h}} > 0$ such that $\sup_{(t,x)} |\boldsymbol{h}_\varepsilon| = O(1)$ and $\sup_{(t,x)} |\mathbf{S}_\varepsilon| = O(1)$ on $]0, T[\, \times \{x \in \mathbb{R}^n | |x| > R_{\mathbf{S},\boldsymbol{h}}\}$,

or

B) *(i) the lower order coefficients a, c, $\boldsymbol{b}$, as well as $\mathbf{S}$, the derivative $\mathrm{d}\mathbf{S}$, the inverse $\mathbf{S}^{-1}$ and $\mathbf{h}'$ are of L^∞-log-type,*

(ii) the initial data $u_0, u_1 \in \mathcal{G}_{L^\infty}(\mathbb{R}^n)$ and the right-hand side $f \in \mathcal{G}_{L^\infty}(U_T)$.

C) The initial value problem has a unique solution $u \in \mathcal{G}_{L^2}(\Omega_T)$ *if*

(i) the lower order coefficients a, c, $\boldsymbol{b}$, *as well as* $\mathbf{S}$, *the derivative* $\mathrm{d}\mathbf{S}$, *the inverse* $\mathbf{S}^{-1}$ *and* $\mathbf{h}'$ *are of* $L^{1,\infty}$*-log-type,*

(ii) the initial data $u_0, u_1 \in \mathcal{G}_{L^2}(\mathbb{R}^n)$ *and the right-hand side* $f \in \mathcal{G}_{L^2}(U_T)$.

Proof: We start with the proof for case A. We rewrite the wave equation into the corresponding symmetric hyperbolic system with $\mathbf{A}_\lambda$, $\mathbf{B}$ and $\boldsymbol{F}$ as in (5.3), clearly the coefficients of the hyperbolic system are in $\mathcal{G}_{L^\infty}(\mathbb{R})$ since the coefficients of the wave equation are. From condition (i) and the structure of (5.3) we obtain that $\mathbf{A}'_\lambda$ and $\frac{1}{2}(\mathbf{B} + \mathbf{B}^t)$—the symmetric part of $\mathbf{B}$—are locally of L^∞-log-type. Since by condition (ii) u_0 and u_1 are in $\mathcal{G}(\mathbb{R}^n)$ and $\mathbf{S}$ has entries in $\mathcal{G}_{L^\infty}(U_T)$, the initial data for the system $\boldsymbol{w}_0 = (u_0, u_1, \mathbf{S}u_0')^t$ is in $(\mathcal{G}(\mathbb{R}^n))^{n+2}$. Furthermore, $f \in \mathcal{G}(U_T)$ thus $F = (0, f, 0) \in (\mathcal{G}(U_T))^{n+2}$. The matrix valued function $\mathbf{S}$ and the vector valued function $\boldsymbol{h}$ satisfy condition (iii). Thus, there exists a constant $R_{\mathbf{S},\boldsymbol{h}} > 0$ such that $\mathbf{A}$, which depends only on $\mathbf{S}$ and $\boldsymbol{h}$, is $O(1)$ on $]0, T[\times\{x \in \mathbb{R}^n \| x| > R_{\mathbf{S},\boldsymbol{h}}\}$. Summing up, all conditions of Theorem 5.3.4, case A are satisfied, and we can apply the theorem to obtain a solution $\boldsymbol{w}$ to the initial value problem of the hyperbolic system. Theorem 5.2.3 and Corollary 5.2.4 guarantee a unique solution $\boldsymbol{u}$ to the Cauchy problem for the wave equation.

The proofs for cases B [and C] are completely analogous. Again, we rewrite the wave equation into its corresponding hyperbolic system and obtain matrices $\mathbf{A}_\lambda$ and $\mathbf{B}$ in $M_{n+2}(\mathcal{G}_{L^\infty}(U_T))$. By condition (i) we have that $\mathbf{A}'_\lambda$ and the symmetric part of $\mathbf{B}$ are L^∞-log-type [resp. $L^{1,\infty}$-log-type]. Since by condition (ii) the initial data u_0 and u_1 are in $\mathcal{G}_{L^\infty}(\mathbb{R}^n)$ [resp. $\mathcal{G}_{L^2}(\mathbb{R}^n)$], and $\mathbf{S}$ has entries in $\mathcal{G}_{L^\infty}(U_T)$, the initial data for the system $\boldsymbol{w}_0 = (u_0, u_1, \mathbf{S}u_0')^t$ is in $\in (\mathcal{G}_{L^\infty}(\mathbb{R}^n))^{n+2}$ [resp. $\in (\mathcal{G}_{L^2}(\mathbb{R}^n))^{n+2}$]. We also have that $f \in \mathcal{G}_{L^\infty}(U_T)$ [resp. $\mathcal{G}_{L^2}(U_T)$], thus $F = (0, f, 0) \in (\mathcal{G}_{L^\infty}(U_T))^{n+2}$ [resp. $(\mathcal{G}_{L^2}(U_T))^{n+2}$]. Altogether we can apply Theorem 5.3.4, case B [resp. C] and obtain a solution $\boldsymbol{w}$ to the initial value problem of the hyperbolic system. Finally, Theorem 5.2.3 and Corollary 5.2.4 guarantee a unique solution $\boldsymbol{u}$ to the Cauchy problem for the wave equation, and we are done. *q. e. d.*

Solutions for hyperbolic first order systems

In this subsection we treat the converse situation: We start with a first order system (5.2) and transform it into a wave equation (5.7). This allows us to apply existence and uniqueness theory for hyperbolic second order equations. Given a Colombeau solution u to the corresponding wave equation, we can derive the solution to the system by differentiation, i. e. $w = (u, \partial_t u, \mathbf{S}u')^t$.

We use the following notation to specify the entries of the $\mathbf{A}_\lambda \in M_{n+2}(\mathcal{G}(\mathbb{R}^{n+1}))$ with $1 \leq \lambda \leq n$: We reference the element in the μ^{th}-row and ν^{th}-column of $\mathbf{A}$ by $(A_\lambda)_{\mu,\nu}$, thus the indices μ and ν run from 1 to $n+2$.

5.3.6. Theorem: *Let $\mathbf{A}_\lambda \in M_{n+2}(\mathcal{G}(\mathbb{R}^{n+1}))$ be symmetric; let $\mathbf{B} \in M_{n+2}(\mathcal{G}(\mathbb{R}^{n+1}))$, and let $\boldsymbol{F} \in (\mathcal{G}(\mathbb{R}^{n+1}))^{n+2}$. If we have*

(i) $\mathbf{A}_\lambda$, $\mathbf{B}$*, and* $\boldsymbol{F}$ *as in* (5.3)*;*

(ii) $\mathbf{S}$ *is symmetric and nondegenerate for* $S_{\lambda,\mu-2} := (A_\lambda)_{2,\mu}$ *with* $1 \leq \lambda \leq n$ *and* $3 \leq \mu \leq n+2$*;*

(iii) for all representatives of $\mathbf{A}_\lambda$, $\mathbf{B}$*, and all compact sets* $K \subseteq U$

$$\sup_K |\mathbf{A}_{\lambda,\varepsilon}|_\delta = O(1) \qquad \sup_K |\mathrm{d}\mathbf{A}_{\lambda,\varepsilon}|_\delta = O(1)$$
$$\sup_K |\mathbf{B}_\varepsilon|_\delta = O(1)$$

as ε tends to 0;

(iv) for all representatives of $\mathbf{A}_\lambda$, $\mathbf{B}$*, and all compact sets* $K \subseteq U$

$$\sup_K |(\det \mathbf{g}_\varepsilon)^{-1}| = O(1)$$

as $\varepsilon \to 0$*, where* $\mathbf{g}$ *is defined by*

$$\mathbf{g} := \begin{pmatrix} 1 & \frac{(A_1)_{2,2}}{2} & \cdots & \frac{(A_n)_{2,2}}{2} \\ \frac{(A_1)_{2,2}}{2} & (A_1)_{2,3} & \cdots & (A_1)_{2,n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{(A_n)_{2,2}}{2} & (A_n)_{2,3} & \cdots & (A_n)_{2,n+2} \end{pmatrix};$$

(v) *for every representative* $(\mathbf{g}_\varepsilon)_\varepsilon$ *of* $\mathbf{g}$ *on* $\mathbb{R}^{n+1}$*, the level set* Σ_0*, i. e the set of all points in* $\mathbb{R}^{n+1}$ *with* $t = 0$*, is a past compact spacelike hypersurface such that* $\partial I_\varepsilon^+(\Sigma_0) = \Sigma_0$*. Moreover, there exists a nonempty open set* $A \subset \mathbb{R}^{n+1}$ *and some* $\varepsilon_0 > 0$ *such that* $A \subset \bigcap_{\varepsilon \leq \varepsilon_0} \overline{I_\varepsilon^+(\Sigma_0)}$*,*

then, for any point $(0, x_\lambda) \in \Sigma_0$*, there exists an open neighbourhood* V *of* $(0, x_\lambda)$ *such that the symmetric hyperbolic system*

$$\partial_t w + \sum_{\lambda=1}^{n} \mathbf{A}_\lambda \partial_\lambda w + \mathbf{B} w = F \tag{5.12}$$

with initial condition $w|_{t=0} = (u_0, u_1, \mathbf{S}u_0')^t$ *has a unique solution* $w \in (G(V))^{n+2}$.

Proof: Conditions (i) and (ii) allow us to rewrite the symmetric hyperbolic system (5.2) into a wave equation (5.7). Now, let us look at conditions (iii) and (iv): From condition (iii) we immediately obtain that the metric $\mathbf{g}$, constructed via $\mathbf{S}$ and $\boldsymbol{h}$, is of type $O(1)$ on compact sets. Furthermore, the same is valid for the derivative d$\mathbf{g}$. This is implied by the estimates on $\mathbf{A}$ and d$\mathbf{A}$. The estimate on $\mathbf{B}$ similarly implies an $O(1)$-condition on the coefficients a, b_λ and c of the wave equation. Additionally, by condition (iv) we obtain that also $\mathbf{g}^{-1}$ is of type $O(1)$. More precisely, since by condition (iii) the components of $\mathbf{g}$ are $O(1)$ and by condition (iv) the inverse of the determinant is $O(1)$, we obtain via the cofactor-formula $\mathbf{g}^{-1} = (\det \mathbf{g})^{-1} \operatorname{adj} \mathbf{g}$ that $\mathbf{g}^{-1}$ is $O(1)$. Together with condition (v) and the fact that the level sets $\Sigma_\tau = \{(t, x_\lambda) | t = \tau\}$ are spacelike hypersurfaces of $\mathbf{g}$ on $\mathbb{R}^{n+1}$ all the prerequisites of Theorem 4.2.1 are satisfied.

Thus, by Theorem 4.2.1, for each point $(0, x_\lambda) \in \mathbb{R}^{n+1}$ there exists an open neighbourhood V of $(0, x_\lambda)$ such that the wave equation (5.7) with its initial condition $(u, \partial_t u)|_{t=0} = (u_0, u_1)$ has a unique solution u. Application of Theorem 5.2.3 and Corollary 5.2.4 guarantees a solution $w := (u, \partial_t u, \mathbf{S}u')^t$ of the system (5.2) on V which concludes the proof. *q. e. d.*

Conclusion

Finally, we outline the different situation in our existence and uniqueness theorems. We start with Theorems 4.2.1 and 5.3.5 for wave equations: The most important difference concerns the nature of the Colombeau algebras used for the coefficient functions. Since

Theorem 4.2.1 is a local existence and uniqueness result, for the coefficient functions we use the special Colombeau algebra $\mathcal{G}$ which is based local moderateness estimates, i. e. on a compact set K. On the contrary, Theorem 5.3.5 gives solutions on all of $\mathbb{R}^n$ for arbitrary large times T, and we use the Colombeau algebra $\mathcal{G}_{L^\infty}$. The asymptotic estimates for the coefficient functions in Theorem 4.2.1 are quite restrictive, we need local boundedness uniformly in ε for all coefficient functions and additionally for the first derivative of the Lorentzian metric (also for the second derivative in the case of a tensorial equation). By rewriting the wave equation into a first order system, we have the advantage to allow the coefficient functions to be of (local) L^∞-log-type (cases A, B) or of $L^{1,\infty}$-log-type (case C). Also here, we additionally have to impose the same asymptotics for the derivative of the metric and its inverse. The main benefit of Theorem 4.2.1 is therefore the geometric formulation that allows us to analyse a large class of Lorentzian manifolds, but we pay a substantial price: the higher regularity of the coefficient functions. Summarized, Theorem 5.3.5 is a reasonable extension of the existence and uniqueness theory for wave equations of low regularity in Euclidean space.

We encounter quite the converse situation when considering the setting for symmetric hyperbolic first order systems. Translating the first order system into a wave equation and application of Theorem 5.3.6 is only possible for systems with coefficients that are locally bounded uniformly in ε. Furthermore, verification of the conditions imposed in Theorem 5.3.6 is much more complicated than in Theorem 5.3.4. Also, we require the first order system to be specifically in the layout (5.3), which rules out application of the theorem for most generic first order systems. However, the translation process gives a first clue how one might develop a geometric theory for symmetric hyperbolic first order systems of low regularity.

A. Appendix

A.1. Ricci's identity for higher order derivatives

We prove a Lemma on the recursive application of Ricci's identity, used in the proof of Proposition 3.2.8, equation (3.19).

A.1.1. LEMMA: *Let $\hat{\mathbf{g}}$ be a pseudo-Riemannian metric with associated covariant derivative $\widehat{\nabla}$, then the following generalization of the Ricci identity holds for any (k,l)-tensor field $\mathbf{v}$:*

$$\widehat{\nabla}_K \widehat{\nabla}_a \widehat{\nabla}_c v^I_J = \widehat{\nabla}_a \widehat{\nabla}_c \widehat{\nabla}_K v^I_J + \sum_{i=0}^{j-1} (\widehat{\mathcal{R}}^{(j-1,i)} \mathbf{v})^I_{acKJ}, \tag{A.1}$$

where $|K| = j-1$.

Here, the expression $\widehat{\mathcal{R}}^{(j-1,i)}\mathbf{v}$ denotes any linear combination of contractions of the $(j-i)^{\text{th}}$ $\widehat{\nabla}$-derivative of $\hat{\mathbf{R}}$ with the i^{th} $\widehat{\nabla}$-derivative of $\mathbf{v}$ by $\widehat{\mathcal{R}}^{(j,i)}\mathbf{v}$.

Proof: At first, we write down the left hand side of equation (A.1) without multi-indices. This reads

$$\widehat{\nabla}_{k_{j-1}} \ldots \widehat{\nabla}_{k_1} \widehat{\nabla}_a \widehat{\nabla}_c v^{i_1 \ldots i_k}{}_{j_1 \ldots j_l},$$

where we chose $K = k_{j-1} \ldots k_1$, $I = i_1 \ldots i_k$, and $J = j_1 \ldots j_l$. Then, we use the Ricci identity on $\widehat{\nabla}_a$ and $\widehat{\nabla}_{k_1}$. This yields

$$\begin{aligned}
\widehat{\nabla}_{k_{j-1}} \ldots \widehat{\nabla}_{k_1} \widehat{\nabla}_a \widehat{\nabla}_c v^{i_1 \ldots i_k}{}_{j_1 \ldots j_l} =& \widehat{\nabla}_{k_{j-1}} \ldots \widehat{\nabla}_{k_2} \widehat{\nabla}_a \widehat{\nabla}_{k_1} \widehat{\nabla}_c v^{i_1 \ldots i_k}{}_{j_1 \ldots j_l} \\
&+ \sum_{r=1}^{l} \widehat{\nabla}_{k_{j-1}} \ldots \widehat{\nabla}_{k_2} (\widehat{\nabla}_c v^{i_1 \ldots i_k}{}_{j_1 \ldots d \ldots j_l} \hat{R}^d{}_{j_r k_1 a}) \\
&+ \widehat{\nabla}_{k_{j-1}} \ldots \widehat{\nabla}_{k_2} (\widehat{\nabla}_d v^{i_1 \ldots i_k}{}_{j_1 \ldots j_l} \hat{R}^d{}_{c k_1 a}) \\
&- \sum_{r=1}^{k} \widehat{\nabla}_{k_{j-1}} \ldots \widehat{\nabla}_{k_2} (\widehat{\nabla}_c v^{i_1 \ldots d \ldots i_k}{}_{j_1 \ldots j_l} \hat{R}^{i_r}{}_{d k_1 a}).
\end{aligned}$$

After shifting $\widehat{\nabla}_a$ to the left by consecutive application of Ricci's identity, we obtain

$$\begin{aligned}
&\widehat{\nabla}_{k_{j-1}} \ldots \widehat{\nabla}_{k_1} \widehat{\nabla}_a \widehat{\nabla}_c v^{i_1 \ldots i_k}{}_{j_1 \ldots j_l} = \\
&\qquad \widehat{\nabla}_a \widehat{\nabla}_{k_{j-1}} \ldots \widehat{\nabla}_{k_1} \widehat{\nabla}_c u^{i_1 \ldots i_k}{}_{j_1 \ldots j_l} \\
&\qquad + \sum_{r=1}^{l} \widehat{\nabla}_{k_{j-2}} \ldots \widehat{\nabla}_{k_1} \widehat{\nabla}_c u^{i_1 \ldots i_k}{}_{j_1 \ldots d \ldots j_l} \hat{R}^d{}_{j_r k_{j-1} a} \\
&\qquad + \sum_{r=1}^{j-2} \widehat{\nabla}_{k_{j-2}} \ldots \widehat{\nabla}_d \ldots \widehat{\nabla}_{k_1} \widehat{\nabla}_c u^{i_1 \ldots i_k}{}_{j_1 \ldots j_l} \hat{R}^d{}_{k_r k_{j-1} a} \\
&\qquad + \widehat{\nabla}_{k_{j-2}} \ldots \widehat{\nabla}_{k_1} \widehat{\nabla}_d u^{i_1 \ldots i_k}{}_{j_1 \ldots j_l} \hat{R}^d{}_{c k_{j-1} a} \\
&\qquad - \sum_{r=1}^{k} \widehat{\nabla}_{k_{j-2}} \ldots \widehat{\nabla}_{k_1} \widehat{\nabla}_c u^{i_1 \ldots d \ldots i_k}{}_{j_1 \ldots j_l} \hat{R}^{i_r}{}_{d k_{j-1} a} \\
&\qquad + \sum_{r=1}^{l} \widehat{\nabla}_{k_{j-1}} (\widehat{\nabla}_{k_{j-3}} \ldots \widehat{\nabla}_{k_1} \widehat{\nabla}_c u^{i_1 \ldots i_k}{}_{j_1 \ldots d \ldots j_l} \hat{R}^d{}_{j_r k_{j-2} a}) \\
&\qquad + \sum_{r=1}^{j-3} \widehat{\nabla}_{k_{j-1}} (\widehat{\nabla}_{k_{j-3}} \ldots \widehat{\nabla}_d \ldots \widehat{\nabla}_{k_1} \widehat{\nabla}_c u^{i_1 \ldots i_k}{}_{j_1 \ldots j_l} \hat{R}^d{}_{k_r k_{j-2} a}) \\
&\qquad + \widehat{\nabla}_{k_{j-1}} (\widehat{\nabla}_{k_{j-3}} \ldots \widehat{\nabla}_{k_1} \widehat{\nabla}_d u^{i_1 \ldots i_k}{}_{j_1 \ldots j_l} \hat{R}^d{}_{c k_{j-2} a}) \\
&\qquad - \sum_{r=1}^{k} \widehat{\nabla}_{k_{j-1}} (\widehat{\nabla}_{k_{j-3}} \ldots \widehat{\nabla}_{k_1} \widehat{\nabla}_c u^{i_1 \ldots d \ldots i_k}{}_{j_1 \ldots j_l} \hat{R}^{i_r}{}_{d k_{j-2} a}) \pm \ldots \\
&\qquad + \sum_{r=1}^{l} \widehat{\nabla}_{k_{j-1}} \ldots \widehat{\nabla}_{k_2} (\widehat{\nabla}_c u^{i_1 \ldots i_k}{}_{j_1 \ldots d \ldots j_l} \hat{R}^d{}_{j_r k_1 a}) + \widehat{\nabla}_{k_{j-1}} \ldots \widehat{\nabla}_{k_2} (\widehat{\nabla}_d u^{i_1 \ldots i_k}{}_{j_1 \ldots j_l} \hat{R}^j{}_{c k_1 a}) \\
&\qquad - \sum_{r=1}^{k} \widehat{\nabla}_{k_{j-1}} \ldots \widehat{\nabla}_{k_2} (\widehat{\nabla}_c u^{i_1 \ldots d \ldots i_k}{}_{j_1 \ldots j_l} \hat{R}^{i_r}{}_{d k_1 a}).
\end{aligned}$$

In a next step we proceed in the same way with $\widehat{\nabla}_c$ until we arrive at the final result

$$
\begin{aligned}
&\widehat{\nabla}_{k_{j-1}} \dots \widehat{\nabla}_{k_1} \widehat{\nabla}_a \widehat{\nabla}_c v^{i_1 \dots i_k}{}_{j_1 \dots j_l} = \\
&\qquad \widehat{\nabla}_a \widehat{\nabla}_c \widehat{\nabla}_{k_{j-1}} \dots \widehat{\nabla}_{k_1} u^{i_1 \dots i_k}{}_{j_1 \dots j_l} - \sum_{r=1}^{k} \widehat{\nabla}_a (\widehat{\nabla}_{k_{j-2}} \dots \widehat{\nabla}_{k_1} u^{i_1 \dots d \dots i_k}{}_{j_1 \dots j_l} \hat{R}^{i_r}{}_{d k_{j-1} c}) \\
&\qquad + \widehat{\nabla}_a \Bigg(\sum_{r=1}^{j-2} \widehat{\nabla}_{k_{j-2}} \dots \widehat{\nabla}_d \dots \widehat{\nabla}_{k_1} u^{i_1 \dots i_k}{}_{j_1 \dots j_l} \hat{R}^d{}_{k_r k_{j-1} c} \\
&\qquad + \sum_{r=1}^{l} \widehat{\nabla}_{k_{j-2}} \dots \widehat{\nabla}_{k_1} u^{i_1 \dots i_k}{}_{j_1 \dots d \dots j_l} \hat{R}^d{}_{j_r k_{j-1} c} \Bigg) \\
&\qquad \pm \dots + \sum_{r=1}^{l} \widehat{\nabla}_a \widehat{\nabla}_{k_{j-1}} \dots \widehat{\nabla}_{k_2} (u^{i_1 \dots i_k}{}_{j_1 \dots d \dots j_l} \hat{R}^d{}_{j_r k_1 c}) \\
&\qquad - \sum_{r=1}^{k} \widehat{\nabla}_a \widehat{\nabla}_{k_{j-1}} \dots \widehat{\nabla}_{k_2} (u^{i_1 \dots d \dots i_k}{}_{j_1 \dots j_l} \hat{R}^{i_r}{}_{d k_1 c}) \\
&\qquad + \sum_{r=1}^{l} \widehat{\nabla}_{k_{j-2}} \dots \widehat{\nabla}_{k_1} \widehat{\nabla}_c u^{i_1 \dots i_k}{}_{j_1 \dots d \dots j_l} \hat{R}^d{}_{j_r k_{j-1} a} \\
&\qquad + \sum_{r=1}^{j-2} \widehat{\nabla}_{k_{j-2}} \dots \widehat{\nabla}_d \dots \widehat{\nabla}_{k_1} \widehat{\nabla}_c u^{i_1 \dots i_k}{}_{j_1 \dots j_l} \hat{R}^d{}_{k_r k_{j-1} a} \\
&\qquad + \widehat{\nabla}_{k_{j-2}} \dots \widehat{\nabla}_{k_1} \widehat{\nabla}_d u^{i_1 \dots i_k}{}_{j_1 \dots j_l} \hat{R}^d{}_{c k_{j-1} a} \\
&\qquad - \sum_{r=1}^{k} \widehat{\nabla}_{k_{j-2}} \dots \widehat{\nabla}_{k_1} \widehat{\nabla}_c u^{i_1 \dots d \dots i_k}{}_{j_1 \dots j_l} \hat{R}^{i_r}{}_{d k_{j-1} a} \\
&\qquad \pm \dots + \widehat{\nabla}_{k_{j-1}} \dots \widehat{\nabla}_{k_2} \left(\sum_{r=1}^{l} \widehat{\nabla}_c u^{i_1 \dots i_k}{}_{j_1 \dots d \dots j_l} \hat{R}^d{}_{j_r k_1 a} + \widehat{\nabla}_d u^{i_1 \dots i_k}{}_{j_1 \dots j_l} \hat{R}^d{}_{c k_1 a} \right) \\
&\qquad - \sum_{r=1}^{k} \widehat{\nabla}_{k_{j-1}} \dots \widehat{\nabla}_{k_2} (\widehat{\nabla}_c u^{i_1 \dots d \dots i_k}{}_{j_1 \dots j_l} \hat{R}^{i_r}{}_{d k_1 a}).
\end{aligned}
\tag{A.2}
$$

As can be easily seen by counting the number of derivatives acting on $\hat{\mathbf{R}}$ and $\mathbf{u}$, we conclude that there are at most $j-1$ derivatives acting on various contractions of $\hat{\mathbf{R}} \otimes \mathbf{u}$. Thus, we can write

$$
\widehat{\nabla}_K \widehat{\nabla}_a \widehat{\nabla}_c u^I_J = \widehat{\nabla}_a \widehat{\nabla}_c \widehat{\nabla}_K u^I_J + \sum_{l=0}^{k-1} (\widehat{\mathcal{R}}^{(k-1,l)} \mathbf{u})^I_{acKJ},
$$

which is equation (A.1) and we are done. *q. e. d.*

A.2. Derivatives of the differential equation

Here, we give an expression for the $(j-1)^{\text{th}}$ derivative of equation (3.1), needed in the proof of Proposition 3.2.8, equation (3.20). Basically, we follow the notation of the previous section. We have

$$\begin{aligned}\widehat{\nabla}_{k_{j-1}}\dots\widehat{\nabla}_{k_1}(g^{ac}\widehat{\nabla}_a\widehat{\nabla}_c u^{i_1\dots i_l}{}_{j_1\dots j_k}) &= \widehat{\nabla}_{k_{j-1}}\dots\widehat{\nabla}_{k_1}F^{i_1\dots i_l}{}_{j_1\dots j_k}\\ &- \widehat{\nabla}_{k_{j-1}}\dots\widehat{\nabla}_{k_1}(B^{aq_1\dots q_k i_1\dots i_l}{}_{p_1\dots p_l j_1\dots j_k}\widehat{\nabla}_a u^{p_1\dots p_l}{}_{q_1\dots q_k})\\ &- \widehat{\nabla}_{k_{j-1}}\dots\widehat{\nabla}_{k_1}(C^{q_1\dots q_k i_1\dots i_l}{}_{p_1\dots p_l j_1\dots j_k}u^{p_1\dots p_l}{}_{q_1\dots q_k}).\end{aligned}$$

Since in this thesis we operate with separate metrics for the principal part of the differential equation and the Levi-Cività connection, $\widehat{\nabla}$-derivatives of the metric $\mathbf{g}$ are involved compared to earlier published results like [GMS09]. Additionally, more difficulties arise by inclusion of lower order terms in the differential equation.

Thus, after carrying out all the differentiations by use of the Leibniz rule, we obtain

$$\begin{aligned}g^{ac}\widehat{\nabla}_{k_{j-1}}\dots\widehat{\nabla}_{k_1}\widehat{\nabla}_a\widehat{\nabla}_c u^{i_1\dots i_l}{}_{j_1\dots j_k} &= \widehat{\nabla}_{k_{j-1}}\dots\widehat{\nabla}_{k_1}F^{i_1\dots i_l}{}_{j_1\dots j_k}\\ &-\sum_{\alpha=0}^{j-2}\sum_{r\in \mathrm{K}_{j-1}^{\alpha}}\Big(\prod_{\beta=1}^{j-1}\widehat{\nabla}_{k_\beta}^{(1-r_\beta)}\Big)g^{ac}\Big(\prod_{\gamma=1}^{j-1}\widehat{\nabla}_{k_\gamma}^{r_\gamma}\Big)\widehat{\nabla}_a\widehat{\nabla}_c u^{i_1\dots i_l}{}_{j_1\dots j_k}\\ &-\sum_{\alpha=0}^{j-1}\sum_{r\in \mathrm{K}_{j-1}^{\alpha}}\Big(\prod_{\beta=1}^{j-1}\widehat{\nabla}_{k_\beta}^{(1-r_\beta)}\Big)B^{aq_1\dots q_k i_1\dots i_l}{}_{p_1\dots p_l j_1\dots j_k}\Big(\prod_{\gamma=1}^{j-1}\widehat{\nabla}_{k_\gamma}^{r_\gamma}\Big)\widehat{\nabla}_a u^{i_1\dots i_l}{}_{j_1\dots j_k}\\ &-\sum_{\alpha=0}^{j-1}\sum_{r\in \mathrm{K}_{j-1}^{\alpha}}\Big(\prod_{\beta=1}^{j-1}\widehat{\nabla}_{k_\beta}^{(1-r_\beta)}\Big)C^{q_1\dots q_k i_1\dots i_l}{}_{p_1\dots p_l j_1\dots j_k}\Big(\prod_{\gamma=1}^{j-1}\widehat{\nabla}_{k_\gamma}^{r_\gamma}\Big)u^{i_1\dots i_l}{}_{j_1\dots j_k},\end{aligned}\tag{A.3}$$

where $\mathrm{K}_{j-1}^{\alpha} := \{r\in\{0,1\}^{j-1} \,|\, \sum_{\beta=1}^{j-1} r_\beta = \alpha\}$, i.e. the set of combinations without repetition. Counting the number of derivatives acting on the tensor fields $\mathbf{g}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{u}$ gives

$$g^{ac}\widehat{\nabla}_K\widehat{\nabla}_a\widehat{\nabla}_c u^I_J = \widehat{\nabla}_K F^I_J - \sum_{i=2}^{j}\mathcal{A}^{(j+1,i)}\mathbf{u})^I_{KJ} - \sum_{i=1}^{j}\mathcal{B}^{(j,i)}\mathbf{u})^I_{KJ} - \sum_{i=0}^{j-1}\mathcal{C}^{(j-1,i)}\mathbf{u})^I_{KJ}.$$

Again, we denote linear combinations of contractions of the $(j-i)^{th}$ $\widehat{\nabla}$-derivative of $\mathbf{g}$, $\mathbf{B}$, and $\mathbf{C}$ with the i^{th} $\widehat{\nabla}$-derivative of $\mathbf{u}$ by $\mathcal{A}^{(j,i)}\mathbf{u}$, $\mathcal{B}^{(j,i)}\mathbf{u}$ resp. $\mathcal{C}^{(j,i)}\mathbf{u}$.

A.3. Derivation of the higher order energy estimate

Using the Cauchy-Schwartz inequality, we derive the estimates for the terms $I_1 \dots, I_8$ in the proof of Proposition 3.2.8, equation (3.20).

$$
\begin{aligned}
|I_1| =& \Big| \frac{2\sigma_b}{\sigma^2} \widehat{\nabla}_d \widehat{\nabla}_L u^P_Q e^{KL} e^{JQ} e_{IP} \widehat{\nabla}_K F^I_J \Big| \\
=& \Big\langle \frac{2\sigma_b}{\sigma^2} \widehat{\nabla}_d \widehat{\nabla}_L u^P_Q, \widehat{\nabla}_K F^I_J \Big\rangle_{\mathrm{e}} \\
\le& \Big| \frac{2\sigma_b}{\sigma^2} \widehat{\nabla}_d \widehat{\nabla}_L u^P_Q \Big| \cdot |\widehat{\nabla}_K F^I_J| \\
\le& \tilde{P}_{j,1} |\widehat{\nabla}^j \mathbf{u}| \cdot |\widehat{\nabla}^{j-1} \mathbf{F}| \\
\le& P_{j,1} (|\widehat{\nabla}^j \mathbf{u}|^2 + |\widehat{\nabla}^{j-1} \mathbf{F}|^2),
\end{aligned}
$$

where $P_{j,1}(\mathbf{g}, \sigma)$ is a constant.

$$
\begin{aligned}
|I_2| =& \Big| \frac{2\sigma_b}{\sigma^2} \widehat{\nabla}_d \widehat{\nabla}_L u^P_Q e^{KL} e^{JQ} e_{IP} \Big(\sum_{i=2}^{j} \mathcal{G}^{(j+1,i)} \mathbf{u} \Big)^I_{KJ} \Big| \\
=& \Big\langle \frac{2\sigma_b}{\sigma^2} \widehat{\nabla}_d \widehat{\nabla}_L u^P_Q, \Big(\sum_{i=2}^{j} \mathcal{G}^{(j+1,i)} \mathbf{u} \Big)^I_{KJ} \Big\rangle_{\mathrm{e}} \\
\le& \Big| \frac{2\sigma_b}{\sigma^2} \widehat{\nabla}_d \widehat{\nabla}_L u^P_Q \Big| \cdot \Big| \Big(\sum_{i=2}^{j} \mathcal{G}^{(j+1,i)} \mathbf{u} \Big)^I_{KJ} \Big| \\
\le& \tilde{P}_{j,2} |\widehat{\nabla}^j \mathbf{u}| \cdot \Big(g_j \sum_{i=2}^{j-1} |\widehat{\nabla}^i \mathbf{u}| + \tilde{g}_j |\widehat{\nabla}^j \mathbf{u}| \Big) \\
\le& P_{j,2} \Big((j-1) |\widehat{\nabla}^j \mathbf{u}|^2 + g_j^2 \sum_{i=2}^{j-1} |\widehat{\nabla}^i \mathbf{u}|^2 \Big),
\end{aligned}
$$

where $P_{j,2}(\mathbf{g},\widehat{\nabla}\mathbf{g},\sigma)$, $g_j(\widehat{\nabla}^2\mathbf{g},\dots,\widehat{\nabla}^{j-1}\mathbf{g})$, and $\tilde{g}_j(\widehat{\nabla}\mathbf{g})$ are constants.

$$\begin{aligned}
|I_3| =&\left|\frac{2\sigma_b}{\sigma^2}\widehat{\nabla}_d\widehat{\nabla}_L u^P_Q e^{KL} e^{JQ} e_{IP}\Big(\sum_{i=1}^{j}\mathcal{B}^{(j,i)}\mathbf{u}\Big)^I_{KJ}\right|\\
=&\left\langle\frac{2\sigma_b}{\sigma^2}\widehat{\nabla}_d\widehat{\nabla}_L u^P_Q, \Big(\sum_{i=1}^{j}\mathcal{B}^{(j,i)}\mathbf{u}\Big)^I_{KJ}\right\rangle_{\mathrm{e}}\\
\leq&\left|\frac{2\sigma_b}{\sigma^2}\widehat{\nabla}_d\widehat{\nabla}_L u^P_Q\right|\cdot\left|\Big(\sum_{i=1}^{j}\mathcal{B}^{(j,i)}\mathbf{u}\Big)^I_{KJ}\right|\\
\leq&\tilde{P}_{j,3}|\widehat{\nabla}^j\mathbf{u}|\cdot\Big(b_j\sum_{i=1}^{j-1}|\widehat{\nabla}^i\mathbf{u}|+\tilde{b}_j|\widehat{\nabla}^j\mathbf{u}|\Big)\\
\leq&P_{j,3}\Big(j|\widehat{\nabla}^j\mathbf{u}|^2+b_j^2\sum_{i=1}^{j-1}|\widehat{\nabla}^i\mathbf{u}|^2\Big),
\end{aligned}$$

where $P_{j,3}(\mathbf{g},\sigma,\mathbf{B})$, $b_j(\widehat{\nabla}\mathbf{B},\dots,\widehat{\nabla}^{j-1}\mathbf{B})$, and $\tilde{b}_j(\mathbf{B})$ are constants.

$$\begin{aligned}
|I_4| =&\left|\frac{2\sigma_b}{\sigma^2}\widehat{\nabla}_d\widehat{\nabla}_L u^P_Q e^{KL} e^{JQ} e_{IP}\Big(\sum_{i=0}^{j-1}\mathcal{C}^{(j-1,i)}\mathbf{u}\Big)^I_{KJ}\right|\\
=&\left\langle\frac{2\sigma_b}{\sigma^2}\widehat{\nabla}_d\widehat{\nabla}_L u^P_Q, \Big(\sum_{i=0}^{j-1}\mathcal{C}^{(j-1,i)}\mathbf{u}\Big)^I_{KJ}\right\rangle_{\mathrm{e}}\\
\leq&\left|\frac{2\sigma_b}{\sigma^2}\widehat{\nabla}_d\widehat{\nabla}_L u^P_Q\right|\cdot\left|\Big(\sum_{i=0}^{j-1}\mathcal{C}^{(j-1,i)}\mathbf{u}\Big)^I_{KJ}\right|\\
\leq&\tilde{P}_{j,4}|\widehat{\nabla}^j\mathbf{u}|\cdot\Big(c_j\sum_{i=0}^{j-1}|\widehat{\nabla}^i\mathbf{u}|\Big)\\
\leq&P_{j,4}\Big(j|\widehat{\nabla}^j\mathbf{u}|^2+c_j^2\sum_{i=0}^{j-1}|\widehat{\nabla}^i\mathbf{u}|^2\Big),
\end{aligned}$$

where $P_{j,4}(\mathbf{g},\sigma)$ and $c_j(\mathbf{C},\ldots,\widehat{\nabla}^{j-1}\mathbf{C})$ are constants.

$$
\begin{aligned}
|I_5| =& \left|\frac{2\sigma_b}{\sigma^2}\widehat{\nabla}_d\widehat{\nabla}_L u^P_Q e^{KL} e^{JQ} e_{IP} g^{ac}\Big(\sum_{i=0}^{j-1}\widehat{\mathcal{R}}^{(j-1,i)}\mathbf{u}\Big)^I_{acKJ}\right| \\
=& \left\langle \frac{2\sigma_b}{\sigma^2}\widehat{\nabla}_d\widehat{\nabla}_L u^P_Q, g^{ac}\Big(\sum_{i=0}^{j-1}\widehat{\mathcal{R}}^{(j-1,i)}\mathbf{u}\Big)^I_{acKJ}\right\rangle_{\mathsf{e}} \\
\leq& \left|\frac{2\sigma_b}{\sigma^2}\widehat{\nabla}_d\widehat{\nabla}_L u^P_Q\right| \cdot \left|g^{ac}\Big(\sum_{i=0}^{j-1}\widehat{\mathcal{R}}^{(j-1,i)}\mathbf{u}\Big)^I_{KJ}\right| \\
\leq& \tilde{P}_{j,5}|\widehat{\nabla}^j\mathbf{u}| \cdot \Big(\hat{r}_j\sum_{i=0}^{j-1}|\widehat{\nabla}^i\mathbf{u}|\Big) \\
\leq& P_{j,5}\Big(j|\widehat{\nabla}^j\mathbf{u}|^2 + \hat{r}_j^2\sum_{i=0}^{j-1}|\widehat{\nabla}^i\mathbf{u}|^2\Big),
\end{aligned}
$$

where $P_{j,5}(\mathbf{g},\sigma)$ and $\hat{r}_j(\mathbf{g},\hat{\mathbf{R}},\ldots,\widehat{\nabla}^{j-1}\hat{\mathbf{R}})$ are constants.

$$
\begin{aligned}
|I_6| =& \left|\widehat{\nabla}_c\widehat{\nabla}_L u^P_Q e^{KL} e^{JQ} e_{IP}\frac{2\sigma_b}{\sigma^2} g^{ac}(\widehat{\mathcal{R}}^{(j-1,j-1)}\mathbf{u})^I_{adKJ}\right| \\
=& \left|\widehat{\nabla}_c\widehat{\nabla}_L u^P_Q e^{KL} e^{JQ} e_{IP} e^{fc} e_{ef}\frac{2\sigma_b}{\sigma^2} g^{ae}(\widehat{\mathcal{R}}^{(j-1,j-1)}\mathbf{u})^I_{adKJ}\right| \\
=& \left\langle \widehat{\nabla}_c\widehat{\nabla}_L u^P_Q, \frac{2\sigma_b}{\sigma^2} g^{ae}(\widehat{\mathcal{R}}^{(j-1,j-1)}\mathbf{u})^I_{adKJ}\right\rangle_{\mathsf{e}} \\
\leq& |\widehat{\nabla}_c\widehat{\nabla}_L u^P_Q| \cdot \left|\frac{2\sigma_b}{\sigma^2} g^{ae}(\mathcal{R}^{(j-1,j-1)}\mathbf{u})^I_{adKJ}\right| \\
\leq& \tilde{P}_{j,6}|\widehat{\nabla}^j\mathbf{u}| \cdot \tilde{r}_j|\widehat{\nabla}^{j-1}\mathbf{u}| \\
\leq& P_{j,6}(|\widehat{\nabla}^j\mathbf{u}|^2 + \tilde{r}_j^2|\widehat{\nabla}^{j-1}\mathbf{u}|^2),
\end{aligned}
$$

where $P_{j,6}$ and $\tilde{r}_j(\mathbf{g},\sigma,\hat{\mathbf{R}},)$ are constants.

$$
\begin{aligned}
|I_7| =& |\widehat{\nabla}_d\widehat{\nabla}_L u^P_Q e^{KL} e^{JQ} e_{IP}\sigma_b\widehat{\nabla}_a h^{abcd}\widehat{\nabla}_c\widehat{\nabla}_K u^I_J| \\
=& |\widehat{\nabla}_d\widehat{\nabla}_L u^P_Q e^{KL} e^{JQ} e_{IP} e^{df} e_{fe}\sigma_b\widehat{\nabla}_a h^{abce}\widehat{\nabla}_c\widehat{\nabla}_K u^I_J| \\
=& \langle\widehat{\nabla}_d\widehat{\nabla}_L u^P_Q, \sigma_b\widehat{\nabla}_a h^{abce}\widehat{\nabla}_c\widehat{\nabla}_K u^I_J\rangle_{\mathsf{e}} \\
\leq& |\widehat{\nabla}_d\widehat{\nabla}_L u^P_Q| \cdot |\sigma_b\widehat{\nabla}_a h^{abce}\widehat{\nabla}_c\widehat{\nabla}_K u^I_J| \\
\leq& P_{j,7}|\widehat{\nabla}^j\mathbf{u}|^2,
\end{aligned}
$$

where $P_{j,7}(\mathbf{g}, \widehat{\nabla}\mathbf{g}, \sigma, \widehat{\nabla}\sigma)$ is a constant.

$$\begin{aligned}
|I_8| =&|\widehat{\nabla}_d \widehat{\nabla}_L u^P_Q \sigma_b h^{abcd} \widehat{\nabla}_a (e^{KL} e^{JQ} e_{IP}) \widehat{\nabla}_c \widehat{\nabla}_K u^I_J| \\
=&|\widehat{\nabla}_d \widehat{\nabla}_L u^P_Q e^{df} e_{fe} e^{LL''} e_{L''L'} e^{QQ''} e_{Q''Q'} e_{PP''} e_{P''P'} \\
&\sigma_b h^{abce} \widehat{\nabla}_a (e^{KL'} e^{JQ'} e_{IP'}) \widehat{\nabla}_c \widehat{\nabla}_K u^I_J \\
=&\langle \widehat{\nabla}_d \widehat{\nabla}_L u^P_Q, \sigma_b h^{abce} \widehat{\nabla}_a (e^{KL'} e^{JQ'} e_{IP'}) \widehat{\nabla}_c \widehat{\nabla}_K u^I_J \rangle_{\mathsf{e}} \\
\leq&|\widehat{\nabla}_d \widehat{\nabla}_L u^P_Q| \cdot |\sigma_b h^{abce} \widehat{\nabla}_a (e^{KL'} e^{JQ'} e_{IP'}) \widehat{\nabla}_c \widehat{\nabla}_K u^I_J| \\
\leq&P_{j,8} |\widehat{\nabla}^j \mathbf{u}|^2,
\end{aligned}$$

where $P_{j,8}(\mathbf{g}, \mathbf{e}, \widehat{\nabla}\mathbf{e}, \sigma, \widehat{\nabla}\sigma)$ is a constant.

Index

References

[BGP07] Christian BÄR, Nicolas GINOUX, and Frank PFÄFFLE: *Wave equations on Lorentzian manifolds and quantization*. ESI Lectures in Mathematics and Physics, European Mathematical Society (EMS), Zürich, 2007. 1, 10, 44

[BeSe07] Sylvie BENZONI-GAVAGE and Denis SERRE: *Multidimensional hyperbolic partial differential equations*. Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, Oxford, 2007, first-order systems and applications. 6

[CGP10] Piotr T. CHRUŚCIEL, Gregory J. GALLOWAY, and Daniel POLLACK: *Mathematical general relativity: a sampler*. Bull. Amer. Math. Soc. (N.S.), **47**(4):567–638, 2010. 3

[Cla96] Chris J. S. CLARKE: *Singularities: boundaries or internal points*. In Pankaj S. JOSHI (editor) *Singularities, Black Holes and Cosmic Censorship*, 24–32, IUCCA, 1996. 1, 2

[Cla98] Chris J. S. CLARKE: *Generalized hyperbolicity in singular spacetimes*. Classical Quantum Gravity, **15**(4):975–984, 1998. i, 1, 3

[CoOb90] J.-F. COLOMBEAU and M. OBERGUGGENBERGER: *On a hyperbolic system with a compatible quadratic term: generalized solutions, delta waves, and multiplication of distributions*. Comm. Partial Differential Equations, **15**(7):905–938, 1990. 61, 74

[Col84] Jean-François COLOMBEAU: *New generalized functions and multiplication of distributions*, volume 84 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1984, notas de Matemática [Mathematical Notes], 90. 3, 4, 13

[Col85] Jean-François COLOMBEAU: *Elementary introduction to new generalized functions*, volume 113 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1985, notes on Pure Mathematics, 103. 13

[Col92] Jean-François COLOMBEAU: *Multiplication of distributions*, volume 1532 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1992. 3, 4, 13

[Fou52] Y. FOURÈS-BRUHAT: *Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires*. Acta Math., **88**:141–225, 1952. 3

[Fri75] Friedrich G. FRIEDLANDER: *The wave equation on a curved space-time*. Cambridge Monographs on Mathematical Physics, No. 2, Cambridge University Press, Cambridge, 1975. 1, 6, 19, 58

[Gar05] Claudia GARETTO: *Topological structures in Colombeau algebras: topological $\tilde{\mathbb{C}}$-modules and duality theory*. Acta Appl. Math., **88**(1):81–123, 2005. 73

[GaOb11a] Claudia GARETTO and Michael OBERGUGGENBERGER: *Generalised fourier integral operator methods for hyperbolic equations with singularities*, 2011, arXiv:1104.3035v1. 61

[GaOb11b] Claudia GARETTO and Michael OBERGUGGENBERGER: *Symmetrisers and generalised solutions for strictly hyperbolic systems with singular coefficients*, 2011, arXiv:1104.2281v1. 61

[Gra93] Todor GRAMCHEV: *Classical solutions to singular hyperbolic systems modelling acoustic wave propagation*. In *Qualitative aspects and applications of nonlinear evolution equations (Trieste, 1993)*, 143–148, World Sci. Publ., River Edge, NJ, 1994. 61

[GMS09] James D. E. GRANT, Eberhard MAYERHOFER, and Roland STEINBAUER: *The wave equation on singular space-times*. Comm. Math. Phys., **285**(2):399–420, 2009. i, 1, 4, 5, 6, 7, 26, 43, 44, 45, 47, 48, 49, 51, 54, 56, 57, 59, 60, 61, 84, 103

[GKOS01] Michael GROSSER, Michael KUNZINGER, Michael OBERGUGGENBERGER, and Roland STEINBAUER: *Geometric theory of generalized functions with applications to general relativity*, volume 537 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 2001. 13, 14, 15, 16, 65, 74

[Han06] Clemens HANEL: *Linear hyperbolic second order partial differential equations on space time*. Diploma thesis, Universität Wien, 2006. iii

[Han10] Clemens HANEL: *Wave-type equations of low regularity*. Appl. Anal., 2010, to appear. 4, 5, 6, 43, 44, 61

[HaEl73] Stephen W. HAWKING and George F. R. ELLIS: *The large scale structure of space-time*. Cambridge Monographs on Mathematical Physics, No. 1, Cambridge University Press, London, 1973. 2, 3, 5, 6, 21, 23, 26, 27, 30

[HaPe70] Steven W. HAWKING and Roger PENROSE: *The singularities of gravitational collapse and cosmology*. Proc. Roy. Soc. London Ser. A, **314**:529–548, 1970. 2

[Hor97] Lars HÖRMANDER: *Lectures on nonlinear hyperbolic differential equations*, volume 26 of *Mathématiques & Applications*. Springer-Verlag, Berlin, 1997. 5

[HoHo01] G. HÖRMANN and Maarten V. DE HOOP: *Microlocal analysis and global solutions of some hyperbolic equations with discontinuous coefficients*. Acta Appl. Math., **67**(2):173–224, 2001. 61

[Hor04] Günther HÖRMANN: *Hölder-Zygmund regularity in algebras of generalized functions*. Z. Anal. Anwendungen, **23**(1):139–165, 2004. 5, 61

[Hor11] Günther HÖRMANN: *The cauchy problem for schrödinger-type partial differential operators with generalized functions in the principial part and as data*. Monatsh. Math., 2011, to appear. 74

[HKS11] Günther HÖRMANN, Michael KUNZINGER, and Roland STEINBAUER: *Wave equations on non-smooth space-times*. In *Asymptotic Properties of Solutions to Hyperbolic Equations*, Progress in Mathematics, Birkhäuser, Boston, 2011, to appear. 16, 46

[HoSp11] Günther HÖRMANN and Christian SPREITZER: *Symmetric hyperbolic systems in algebras of generalized functions and distributional limits*. J. Math. Anal. Appl., 2011, to appear. i, 6, 61, 63, 75, 103

[KlNi03] Sergiu KLAINERMAN and Francesco NICOLÒ: *The evolution problem in general relativity*, volume 25 of *Progress in Mathematical Physics*. Birkhäuser Boston Inc., Boston, MA, 2003. 3

[KmHo01] I. KMIT and G. HÖRMANN: *Semilinear hyperbolic systems with singular non-local boundary conditions: reflection of singularities and delta waves*. Z. Anal. Anwendungen, **20**(3):637–659, 2001. 61

[KuSt02a] Michael KUNZINGER and Roland STEINBAUER: *Foundations of a nonlinear distributional geometry*. Acta Appl. Math., **71**(2):179–206, 2002. 4, 18

[KuSt02b] Michael KUNZINGER and Roland STEINBAUER: *Generalized pseudo-Riemannian geometry*. Trans. Amer. Math. Soc., **354**(10):4179–4199 (electronic), 2002. 4, 17, 18, 64

[LaOb91] Frederic LAFON and Michael OBERGUGGENBERGER: *Generalized solutions to symmetric hyperbolic systems with discontinuous coefficients: the multidimensional case*. J. Math. Anal. Appl., **160**(1):93–106, 1991. 6

[May06] Eberhard MAYERHOFER: *The wave equation on singular space-times*. Ph. D. thesis, Universität Wien, 2006. 4

[May08] Eberhard MAYERHOFER: *On Lorentz geometry in algebras of generalized functions*. Proc. Roy. Soc. Edinburgh Sect. A, **138**(4):843–871, 2008. 17, 64, 65

[Obe88] Michael OBERGUGGENBERGER: *Hyperbolic systems with discontinuous coefficients: examples*. In *Generalized functions, convergence structures, and their applications (Dubrovnik, 1987)*, 257–266, Plenum, New York, 1988. 74

[Obe92] Michael OBERGUGGENBERGER: *Multiplication of distributions and applications to partial differential equations*, volume 259 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, 1992. 12

[Obe06b] Michael OBERGUGGENBERGER: *Notes on regularity results in Colombeau algebras*. Integral Transforms Spec. Funct., **17**(2-3):101–107, 2006. 5

[Obe06a] Michael OBERGUGGENBERGER: *Regularity theory in Colombeau algebras*. Bull. Cl. Sci. Math. Nat. Sci. Math., (31):147–162, 2006. 5

[Obe09] Michael OBERGUGGENBERGER: *Hyperbolic systems with discontinuous coefficients: generalized wavefront sets*. In *New developments in pseudo-differential operators*, volume 189 of *Oper. Theory Adv. Appl.*, 117–136, Birkhäuser, Basel, 2009. 61

[ONe83] Barrett O'NEILL: *Semi-Riemannian geometry*, volume 103 of *Pure and Applied Mathematics*. Academic Press Inc., New York-London, 1983, with applications to relativity. 10, 12, 25

[Pen79] Roger PENROSE: *Singularities and time-asymmetry*. In Stephen W. HAWKING and Israel WERNER (editors) *General Relativity, An Einstein Centenary Survey*, 581–683, Cambridge University Press, Cambridge, 1979. 3

[PeRi84] Roger PENROSE and Wolfgang RINDLER: *Spinors and space-time. Vol. 1*. Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 1984. 9

[Sch54] Laurent SCHWARTZ: *Sur l'impossibilité de la multiplication des distributions*. C. R. Acad. Sci. Paris, **239**:847–848, 1954. 13

[Sog95] Christopher D. SOGGE: *Lectures on nonlinear wave equations*, volume II of *Monographs in Analysis*. International Press, Boston, MA, 1995. 5

[ViWi00] James A. VICKERS and Jonathan P. WILSON: *Generalized hyperbolicity in conical spacetimes*. Classical Quantum Gravity, **17**(6):1333–1360, 2000. 1, 3, 4, 5, 26, 61

[Wal84] Robert M. WALD: *General relativity*. University of Chicago Press, Chicago, IL, 1984. 2

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Zusammenfassung

Diese Arbeit beschäftigt sich mit der Existenzetheorie für Lösungen von Wellengleichungen niedriger Regularität auf Lorentzmannigfaltigkeiten unter Verwendung der Theory verallgemeinerter Funktionen. Wir bauen auf einem Existenzrsultat von Grant, Mayerhofer und Steinbauer für lineare Wellengleichungen mit Koeffizienten niedriger Regularität (im Sinne von Colombeau) auf. Von einem geometrischen Standpunkt aus, zeigt dieses Resultat, daß das Cauchyproblem für den Laplace-Beltrami-Operator einer verallgemeinerten Lorentzmannigfaltigkeit korrekt gestellt ist. Im Kontext des Vorschlags von Clarke werden solche Raumzeiten $\mathcal{G}$-hyperbolisch genannt und können als nichtsingulär betrachtet werden. In dieser Arbeite erweitern wir die Existenz- und Eindeutkeitsresultate aus [GMS09] für lineare, tensorielle Wellengleichungen und schließen, in Hinblick auf die mögliche Anwendung für die Existenztheorie quasilinearer Gleichungen, auch den Fall der Anwesenheit von Termen niedriger Differentiationsordnung mit ein. Das technische Kernstück sind Energieabschätzungen höherer Ordnung, die wir versuchen besondern klar und übersichtlich darzustellen, um ein Existenz- und Eindeutigkeitsresultat für Gleichungen mit verallgemeinerten Koeffizienten zu erhalten. Wir diskutieren auch den Zusammenhang von Wellengleichungen mit niedriger Regularität und symmetrischen, hyperbolischen Differentialgleichungssystemen erster Ordnung basierend auf den Existenzresulateten von Hörmann und Spreitzer [HoSp11] und erhalten so weitere Existenz- und Eindeutigkeitssätze für sowohl Wellengleichungen als auch hyperbolische Systeme erster Ordnung.