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Titel der Diplomarbeit

Multi-particle S-matrix models in $1 + 1$-dimensions and associated QFTs

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Abstract

Based on the work of my advisor Gandalf Lechner [L1, L2] we extend the algebraic construction and classification of Quatum Field Theories on 1+1 dimensional Minkowski space, applying principles of inverse scattering theory with factorizing S-matrices to models with several particle species. We construct a Borchers triple $(\mathcal{M}, U, \mathcal{H})$, and show that the local net, obtained from a von Neumann algebra constructed from two different wedge-local fields, is a covariant standard right wedge algebra. Moreover, we show that its generators are polarization-free and temperate. We work out the underlying scattering theory following [H1, BBS1] and solve the inverse scattering problem on the two-particle level. This results in the matrix-valued scattering function, initially defining the symmetry of the model, to be the $(2 \rightarrow 2)$ S-matrix. A proof of asymptotic completeness of the space of $(2 \rightarrow 2)$ scattering states is given. Stating the general solution to the inverse scattering problem in form of a total expression is not contained in this work, because a complete set of solutions to the Yang-Baxter equation has not been found so far. This complication is due to the matrix character of the scattering matrix, in contrast to the scalar setting where a solution can be given. Therefore some examples e.g. Sigma models are discussed.

Zusammenfassung

In der vorliegenden Arbeit wird aufbauend auf jener meines Betreuers Gandalf Lechner [L1, L2] die algebraische Konstruktion und Klassifikation von 1+1 dimensionalen Quantenfeldtheorien mittels inverser Streutheorie auf Vielteilchensysteme erweitert. Es wird gezeigt, dass zwei zueinander keillokale Felder existieren, welche benutzt werden um ein lokales Netz von Feldalgebren zu definieren. Weiters wird gezeigt, dass das somit gewonnene Borchers Tripel eine kovariante "standard right wedge algebra" ist, und dass die Feldoperatoren temperierte polarisationsfreie Generatoren sind. Weiters wird [H1, BBS1] die Streutheorie der Felder ausgearbeitet und asymptotische Vollständigkeit des Hilbertraumes der $(2 \rightarrow 2)$ Streuzustände bewiesen. Es zeigt sich, dass die anfänglich als die Symmetrie des Modells definierende Streufunktion auch die $(2 \rightarrow 2)$ Streumatrix ist. Eine allgemeine Lösung als geschlossene Formel wurde nicht gefunden, da keine allgemeine Lösung der Yang-Baxter Gleichung bekannt ist. Es werden deshalb Beispiele (z.B. Sigma Modelle) besprochen, welche den Bedingungen an die Streumatrix genügen.

Contents

1 Introduction

In the long endeavor of finding interacting Quantum Field Theories (QFTs), a lot of intelligence has been invested in the last decades. However, it is still an open problem to find such QFTs in more than three space time dimensions.

This problem turned out to be so difficult that a tool was worked out to analyze which type of field theory the most promising would be. This mechanism, the renormalization group flow analysis, provided indications on how QFTs could be formulated to have the necessary convergence properties at least in two and three spacetime dimensions.

Before that, a description of scattering processes in terms of creation operators acting on the vacuum and representing incoming and outgoing n-particle states of increasing and decreasing rapidities, respectively has been investigated in the form factor program [BF1]. There the local quantum fields are characterized in terms of their form factors. But these form factors have a very complicated structure, and no viable one-particle generators can be obtained. Therefore the description of collision processes can not be deduced from the theory but has to be assumed.

A different approach has been initiated by Haag and Kastler [HK1], who formulated a framework for QFTs in an axiomatic setting, using the formalism of operator algebras and the work of E. Wigner [W1] for irreducible positive energy representations of the Poincaré group. By this algebraic approach it is possible to categorize and formulate general properties of QFTs without having to dwell on the analytic properties of fields for each single theory. The motivation for this algebraic approach arose from the special relativistic principle of locality, implying that no causal interaction of any kind is possible over spacelike separated distances. This principle inherently contains a Minkowski space like structure and the feature that operators, representing physical entities are localized in some region in Minkowski space. The operators make up the local operator algebras $\mathcal{A}$ which give the local net $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ for a region $\mathcal{O}$ in Minkowski space. The configuration of the local net in terms of these operators is the crucial point and contains the physical properties of a QFT.

In the present work we will follow the approach by G. Lechner [L1], motivated by B. Schroer [S1, S2], of constructing quantum field theories in $1 + 1$ spacetime dimensions by the underlying scattering matrices in an inverse scattering picture. This will be done by defining a class $\mathcal{S}$ of so called "matrix-valued scattering functions" which will be interpreted in the beginning as the fundamental objects, defining the symmetry and statistics of the theory. The work in [L2] is extended by us, allowing for an arbitrary finite number of particle species.

A Fock space is constructed, with a matrix-valued scattering function defining its symmetry, and a representation of the proper orthochronous Poincaré group is given.

In sight of the relative wedge locality in the single particle case of the two fields defined there [L2], we proceed analogously by defining a second field via a TCP reflection operator from the initial field and prove relative wedge locality of these for an arbitrary finite number of particle species.

To obtain "physical quantities", following [H1, R1] the fields are smeared with special test functions, allowing for asymptotic states and the interpretation of "incoming" and "outgoing" particles, described by a multi-particle factorizing scattering function. We will show that this scattering function coincides with the initially defined class of "matrix-valued scattering functions" $\mathcal{S}$ used for defin-

ing the structure of the underlying Fock space. Factorizing means here that the scattering matrix for arbitrary $n \to m$ scattering processes splits up into a product of $2 \to 2$ scattering matrices. In absence of particle production this is always the case [I1] in $1 + 1$ dimensions.

To illustrate the generality and wide applicability of this algebraic approach, we give known examples (e.g. Sigma models) contained in the class of matrix-valued scattering functions defined in this work, by checking whether the scattering matrices given in [J1, LM1, AA1] comply. By virtue of allowing for an arbitrary number of particle species, the matrix form of the scattering function gives rise to additional constraints on the matrix valued scattering function S. Despite these additional constraints, the lack of a complete set of solutions of the Yang-Baxter equation [PA1], makes it impossible to explicitly state the class of matrix-valued scattering functions. However, a physically interesting set of solutions has been worked out in the case of Toda systems [J1] and sigma models [AA1]. We will pick out some of them and show that they comply with the assumptions we made on $\mathcal{S}$.

2 Inverse scattering theory

In inverse scattering theory we assume a factorizing scattering function (i.e. that a function describing the scattering of arbitrary incoming and outgoing particles can be decomposed into a product of $2 \to 2$ scattering matrices) [A2], and the particle spectrum to be given.

For the construction of a class of QFTs from these given objects in the algebraic approach, we will require the fields and local net to fulfill the Haag-Kastler axioms [HK1]. Thereby the scattering function is severely constrained by the mass spectrum of the theory, Lorentz invariance, and additional constraints such as crossing symmetry, which these axioms impose (cf. 3.3).

2.1 The scattering function

We exclude particle production from our 1+1 dimensional models (which cannot be done in higher dimensions as in an interacting relativistic QFT the particle number is not conserved) and therefore the scattering matrix factorizes into a product of two particle scattering functions [I1, A2].

Our aim is to construct multi-particle quantum field theories in which we assume the particle spectrum to be given. This causes constraints on the scattering function as we will see later on when we restrain a representation of the Poincaré group to the space of symmetric wave functions. Moreover, as we want to define a model including different particle species, the two-particle scattering functions will be matrix-valued.

Let $\mathcal{K}$ be a separable Hilbert space, then we will define the *matrix-valued two-particle scattering function* to map the rapidity parameter $\theta \in \mathbb{R}$ to the space of bounded operators $\mathcal{B}(\mathcal{K} \otimes \mathcal{K})$.

The *rapidity* θ is a reparametrization of the speed of an object in a given frame of reference, given by $\theta = \tanh^{-1}(v/c)$ where v is the velocity of the object and c the speed of light. In $1 + 1$ dimensions, the rapidity gives a linear parametrization, i.e. rapidities can be simply added, even at relativistic velocities.

The Lorentz boost with rapidity $\theta \in \mathbb{R}$ acts as a velocity transformation via

$$x \longmapsto \Lambda_\theta x = \left(\begin{array}{cc} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{array} \right) x, \tag{2.1}$$

and satisfies $\Lambda_\theta \Lambda_{\theta'} = \Lambda_{\theta+\theta'} \; \forall \theta, \theta' \in \mathbb{R}$.

Before we can define the matrix valued scattering function, we introduce the **flip operator** $F \colon \mathcal{K} \otimes \mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$. On tensor products we set

$$F(\psi \otimes \varphi) := \varphi \otimes \psi, \tag{2.2}$$

and extend F by linearity and continuity to a bounded, unitary operator on $\mathcal{K} \otimes \mathcal{K}$, with $F^2 = id$.

As we are working with matrix-valued functions, index notation can be useful (in fact it is hardly avoidable). Therefore we choose an orthonormal basis $\{e_\alpha\}$ for $\mathcal{K}$ where the sub- or superscript position of the index is of no formal relevance. So, for a vector $\Psi_n \in \mathcal{K}^{\otimes n} = \mathcal{K} \underbrace{\otimes \ldots \otimes}_{\text{n times}} \mathcal{K}$ and $M \in \mathcal{B}(\mathcal{K}^{\otimes n})$ we write

$$\Psi_n^{\alpha_1 \ldots \alpha_n} := \langle e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_n}, \Psi_n \rangle$$
$$M_{\beta_1 \ldots \beta_n}^{\alpha_1 \ldots \alpha_n} := \langle e^{\alpha_1} \otimes \ldots \otimes e^{\alpha_n}, M \, e_{\beta_1} \otimes \ldots \otimes e_{\beta_n} \rangle. \tag{2.3}$$

Another short hand notation we introduce for convenience is, for $M \in \mathcal{B}(\mathcal{K}^{\otimes 2})$

$$M_k := \mathbb{1}_{k-1} \otimes M \otimes \mathbb{1}_{n-k-1} \qquad n \geq 2, \quad k = 1, ..., n-1. \qquad (2.4)$$

Here $\mathbb{1}_j$ denotes the identity on $\mathcal{K}^{\otimes j}$, and $M_k \in \mathcal{B}(\mathcal{K}^{\otimes n})$.

As we aim at the construction of multi-particle theories with mass spectrum $m_\alpha \in \{m_1, ..., m_N | m_i \geq 0, N = dim(\mathcal{K})\}$; antiparticles have to be contained in the theory. Therefore we introduce the notion of writing an antiparticle index as $\overline{\alpha}$ where $\overline{\alpha}$ is the order of the charge-conjugated particle spectrum, i.e. $\{1...N\} \ni \alpha \mapsto \overline{\alpha} \in \{1...N\}$ is a permutation that satisfies $\overline{\overline{\alpha}} = \alpha$ and $m_\alpha = m_{\overline{\alpha}}$, i.e. the antiparticle and particle have the same mass.
Moreover, we introduce the abbreviated notation $\underline{\theta} := (\theta_1, ..., \theta_n)$ and $\underline{\alpha} := (\alpha_1 ... \alpha_n)$.

Definition 2.1. *A continuous, bounded function S:* $\overline{\xi \in \mathbb{C}} =: \overline{\{S(0,\pi)} : 0 < Im(\xi) < \pi\} \to \mathcal{B}(\mathcal{K} \otimes \mathcal{K})$ *that is analytic in the open strip $S(0,\pi)$, and for a given mass spectrum $m_\alpha \in \{m_1, ..., m_N | m_i \geq 0, N = dim(\mathcal{K})\}$ satisfies,*

$$S(\theta)^* = S(\theta)^{-1} = F \, S(-\theta)F, \qquad (2.5a)$$

$$S_{\overline{\beta}_1 \overline{\beta}_2}^{\overline{\alpha}_1 \overline{\alpha}_2}(\theta) = S_{\alpha_1 \alpha_2}^{\beta_1 \beta_2}(\theta), \qquad (2.5b)$$

$$(FS(\theta) \otimes \mathbb{1})\,(\mathbb{1} \otimes FS(\theta'))\,(FS(\theta' - \theta) \otimes \mathbb{1})$$
$$= \quad (\mathbb{1} \otimes FS(\theta' - \theta))\,(FS(\theta') \otimes \mathbb{1})(\mathbb{1} \otimes FS(\theta)), \qquad (2.5c)$$

$$S_{\gamma\delta}^{\alpha\beta}(i\pi - \theta) = S_{\delta\overline{\alpha}}^{\beta\overline{\gamma}}(\theta) \qquad (2.5d)$$

and

$$S_{\beta_1 \beta_2}^{\alpha_1 \alpha_2}(\theta) = 0 \qquad if \qquad (m_{\alpha_1} \neq m_{\beta_1} \quad or \quad m_{\alpha_2} \neq m_{\beta_2}), \qquad (2.5e)$$

*will be called **matrix-valued scattering function**.*

The set of all functions fulfilling the above Definition will be denoted $\mathcal{S}$. The first equation (2.5a) and the so called Yang-Baxter equation (2.5c), are necessary to be able to construct a representation of the permutation group. This will be used later on where we will define the action of the transposition operator D_n as a representation of the permutation group, where it will be necessary to require D_n to fulfill the Yang Baxter equation [ZA1].
Condition (2.5b) will be used to define a PCT operators, it is an additional assumption which is necessary only in the multi-particle case. Equation (2.5d) is called *crossing symmetry* and is needed to construct "wedge local fields" i.e. fields localized in a wedge shaped region of spacetime.
In relativistic theories, crossing corresponds to the substitution of an incoming particle of momentum p by an outgoing antiparticle with momentum $-p$. With the Mandelstam variable $s = (p_i + p_j)^2 = m_i^2 + m_j^2 + 2m_i m_j \cosh\theta$ this substitution corresponds to $s \to 4m^2 - s$ or $\theta \to i\pi - \theta$. Hence the analyticity region of the scattering function is transformed from the real line with an interval to the region $S(0,\pi)$. Equation (2.5e) guarantees the compliance of the scattering function with the mass spectrum and can be seen after having scattering theory available, to represent momentum conservation. In our setting it will be needed to

restrict the representation of the Poincaré group to the S-symmetric Fock space. Conditions (2.5a) and (2.5c) written in index notation can be found in Appendix (A.5), (A.6).

We would like to make a remark on the point of view that is represented by the analyticity condition on S. In classical field theory, bound states are obtained from the poles of the scattering function, see e.g. [AA1] for a rich discussion of that topic. So we exclude these bound states here.

The matrix valued scattering function S could also be called two-particle scattering function, but this might be misleading since this name is usually reserved for two-particle scattering functions not being matrix valued. However, in the following we will omit the term "matrix valued" and simply call this function S the scattering function, except for section 5 where we will focus on scattering processes.

From Definition 2.1 one can see that S depends on the chosen basis of $\mathcal{K}$. In the case of $\mathcal{K} = \mathbb{C}$ we get the two-particle scattering function for a single massive field, as described by G. Lechner in [L2]. For a discussion of examples of scattering functions see section 6.

2.2 Construction of a Hilbert space

Our aim is the construction of a Hilbert space $\mathcal{H}$, and a representation of translations U' on $\mathcal{H}$. Further on, we will also define a "wedge algebra" $\mathcal{M}$, completing what is needed for a so called associated Borchers triple $(\mathcal{M}, U', \mathcal{H})_S$, providing us with the basic ingredients of a QFT.

In analogy to the usual Fock space representation, we define the Hilbert spaces

$$\mathcal{H}_1 := L^2(\mathbb{R} \to \mathcal{K}, d\theta) \simeq L^2(\mathbb{R}, d\theta) \otimes \mathcal{K}, \tag{2.6a}$$

$$\mathcal{H}_1^{\otimes n} := \underbrace{\mathcal{H}_1 \otimes ... \otimes \mathcal{H}_1}_{n \ times} \simeq L^2(\mathbb{R}^n) \otimes \mathcal{K}^{\otimes n}. \tag{2.6b}$$

The space $\mathcal{H}_1^{\otimes n}$ is much too "large", it still contains all equivalent vectors with the arguments permuted. To single out these sets of equivalent vectors we construct a projector acting on $\mathcal{H}_1^{\otimes n}$. This is done via a transposition operator $D_n(\tau_k)$ generating transpositions τ_k of components of vectors $\psi_n \in \mathcal{H}_1^{\otimes n}$. Here $\tau_k \in \mathfrak{S}_n$ flips the kth and $(k+1)^{th}$ entry of such vectors, $\mathfrak{S}_n$ is the group of permutations of n elements and $k \in \{1, 2, ...n-1\}$.

Definition 2.2. *Let $n \in \mathbb{N}$ and $k \in \{1, ..., n-1\}$. We define the transposition operator $D_n(\tau_k) \in \mathcal{B}(\mathcal{H}_1^{\otimes n})$ as, $\psi_n \in \mathcal{H}_1^{\otimes n}$*

$$[D_n(\tau_k)\psi]_n (\theta_1, ..., \theta_n) := F_k S_k(\theta_{k+1} - \theta_k) \, \psi_n(\theta_1, ..., \theta_{k+1}, \theta_k, ..., \theta_n). \tag{2.7}$$

It consists of the flip operator F which is a representation of transpositions on the Hilbert space, and the two-particle scattering function $S_k(\theta)$ which adds a term to account for the flip.

Again, this definition of the action of the transposition operator in index notation can be found in Appendix (A.4).

For D_n to be a representation of the permutation group, it must allow for any permutation π as an argument. So far we have only defined D_n for transpositions τ_k, but the extension is straightforward since every permutation can be decomposed into a product of transpositions.

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Definition 2.3. *For an arbitrary permutation* $\pi = \tau_{\alpha_1}...\tau_{\alpha_k} \in \mathfrak{S}_n$, τ_{α_i} *being transpositions, we define*

$$D_n(\pi) := D_n(\tau_{\alpha_1})...D_n(\tau_{\alpha_i}). \tag{2.8}$$

Lemma 2.1. D_n *is a unitary representation of* $\mathfrak{S}_n$ *on* $\mathcal{H}_1^{\otimes n}$.

Proof. As every permutation $\pi \in \mathfrak{S}_n$ can be written as a product of transpositions τ_k, it is sufficient to verify the following three conditions for $k, j \in \{1, ..., n-1\}$
a) $D_n(\tau_k)^2 = id$,
b) $D_n(\tau_k)D_n(\tau_j) = D_n(\tau_j)D_n(\tau_k)$ for $|k - j| \geq 2$,
c) $D_n(\tau_{k+1})D_n(\tau_k)D_n(\tau_{k+1}) = D_n(\tau_k)D_n(\tau_{k+1})D_n(\tau_k)$.

a) Let $\psi_n \in \mathcal{H}_1^{\otimes n}$.

$$[D_n(\tau_k)^2\psi]_n(\theta_1, ..., \theta_n)$$
$$= F_k S_k(\theta_{k+1} - \theta_k)[D_n(\tau_k)\psi]_n \, (\theta_1, ..., \theta_{k+1}, \theta_k..., \theta_n)$$
$$= \underbrace{F_k S_k(\theta_{k+1} - \theta_k)F_k \, S_k(\theta_k - \theta_{k+1})}_{=id \text{ since } S(\theta)^{-1}=F\, S(-\theta)F \text{ see } (2.5a)} \psi_n \, (\theta_1, ...\theta_k, \theta_{k+1}, ..., \theta_n)$$
$$= \psi_n(\theta_1, ..., \theta_n).$$

b) For $|k - j| \geq 2$, w.l.o.g. assume $k < j$

$$[D_n(\tau_k)D_n(\tau_j)\psi]_n(\theta_1, ..., \theta_n)$$
$$= F_k S_k(\theta_{k+1} - \theta_k)F_j \, S_j(\theta_{j+1} - \theta_j)\psi_n(\theta_1, ...\theta_{k+1}, \theta_k, ...\theta_{j+1}, \theta_j, ...\theta_n)$$
since $|k - j| \geq 2$, $F_k S_k(\theta_{k+1} - \theta_k)$ and $F_j S_j(\theta_{j+1} - \theta_j)$ act independently, they commute:
$$= [D_n(\tau_j)D_n(\tau_k)\psi]_n \, (\theta_1, ..., \theta_n).$$

c)

$$[D_n(\tau_{k+1})D_n(\tau_k)D_n(\tau_{k+1})\psi]_n(\theta_1, ..., \theta_n)$$
$$= (\mathbb{1}_{k-1} \otimes (\mathbb{1} \otimes FS(\theta_{k+2} - \theta_{k+1})) \otimes \mathbb{1}_{n-k-2})$$
$$\times [D_n(\tau_k)D_n(\tau_{k+1})\psi]_n(\theta_1, ..., \theta_k, \theta_{k+2}, \theta_{k+1}..., \theta_n)$$
$$= (\mathbb{1}_{k-1} \otimes (\mathbb{1} \otimes FS(\theta_{k+2} - \theta_{k+1})) (FS(\theta_{k+2} - \theta_k) \otimes \mathbb{1}) \otimes \mathbb{1}_{n-k-2})$$
$$\times [D_n(\tau_{k+1})\psi]_n(\theta_1, ..., \theta_{k+2}, \theta_k, \theta_{k+1}..., \theta_n)$$
$$= (\mathbb{1}_{k-1} \otimes (\mathbb{1} \otimes FS(\theta_{k+2} - \theta_{k+1})) (FS(\theta_{k+2} - \theta_k) \otimes \mathbb{1})$$
$$\otimes (\mathbb{1} \otimes FS(\theta_{k+1} - \theta_k)) \mathbb{1}_{n-k-2})\psi_n(\theta_1, ..., \theta_{k+2}, \theta_{k+1}, \theta_k..., \theta_n),$$
which can be rewritten, using (2.5c) as
$$= (\mathbb{1}_{k-1} \otimes (FS(\theta_{k+1} - \theta_k) \otimes \mathbb{1}) (\mathbb{1} \otimes FS(\theta_{k+2} - \theta_k))$$
$$\times (FS(\theta_{k+2} - \theta_{k+1}) \otimes \mathbb{1}) \otimes \mathbb{1}_{n-k-2})\psi_n(\theta_1, ..., \theta_{k+2}, \theta_{k+1}, \theta_k..., \theta_n)$$
$$= [D_n(\tau_k)D_n(\tau_{k+1})D_n(\tau_k)\psi]_n \, (\theta_1, ..., \theta_n).$$

Finally, unitarity of D_n is obvious since in (2.5a) we have assumed unitarity for $S(\theta)$, $F^* = F^{-1}$ since it is a transposition operator, and a composition of unitary operators is again unitary. $\qquad\square$

For the construction of an S-symmetric Hilbert space we define the projection operator P_n as

$$P_n = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} D_n(\pi), \tag{2.9}$$

with $P_n^* = P_n$, and $P_n^2 = P_n$ since $\text{card}(\mathfrak{S}_n) = n!$.

By means of this projection operator we can define the S-symmetrized Hilbert spaces

$$\mathcal{H}_n := P_n \mathcal{H}_1^{\otimes n} \text{ for } n \geq 1, \quad \text{and} \quad \mathcal{H}_0 := \mathbb{C}. \tag{2.10}$$

Now we are ready to define the **S-symmetrized Fock space** $\mathcal{H}$ containing vectors $\Psi = (\Psi_0, \Psi_1, ...)$ of arbitrary order n:

$$\mathcal{H} := \bigoplus_{n=0}^{\infty} \mathcal{H}_n = \bigoplus_{n=0}^{\infty} P_n \mathcal{H}_1^{\otimes n} \tag{2.11}$$

with P_n projecting on the S-symmetric subspace $\mathcal{H}_n$ whose elements Ψ_n satisfy the symmetry

$$\Psi_n(\theta_1, ..., \theta_n) = F_k S_k(\theta_{k+1} - \theta_k) \, \Psi_n(\theta_1, ..., \theta_{k+1}, \theta_k, ..., \theta_n). \tag{2.12}$$

Alternatively, with all the indices (cf. (2.3)):

$$\Psi_n^{\alpha_1 ... \alpha_n}(\theta_1, ..., \theta_n) = S(\theta_{k+1} - \theta_k)_{\beta_{k+1} \beta_k}^{\alpha_{k+1} \alpha_k} \, \Psi_n^{\alpha_1 ... \beta_{k+1} \beta_k ... \alpha_n}(\theta_1, ..., \theta_{k+1}, \theta_k, ..., \theta_n).$$

NB. summation convention for multiple indices is used from now on, and we use upper case Greek letters (e.g. Ψ) for elements of symmetrized Hilbert spaces, and lower case Greek (e.g. ψ) for elements of unsymmetrized ones. The scalar product and norm for elements $\Psi, \Phi \in \mathcal{H}$ are given by the underlying L^2 space scalar product and norm

$$\langle \Psi, \Phi \rangle_{\mathcal{H}} := \sum_{n=0}^{\infty} \langle \Psi_n, \Phi_n \rangle_{\mathcal{H}_1^{\otimes n}} = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} \langle \Psi_n(\theta), \Phi_n(\theta) \rangle_{\mathcal{K}^{\otimes n}} \, d^n\theta, \tag{2.13}$$

$$\|\Psi\| = \langle \Psi, \Psi \rangle_{\mathcal{H}}^{1/2}. \tag{2.14}$$

The Hilbert space $\mathcal{H}$ can be interpreted as a direct sum over all possible n-particle spaces $\mathcal{H}_n$, and $\mathcal{H}_1 = L^2(\mathbb{R}, d\theta) \otimes \mathcal{K}$ can be interpreted as the one-particle space with reparametrized on-shell Lebesgue measure $d\theta$ as we will show below in Theorem 5.1. Moreover, it will be shown in Lemma 2.3 that in $\mathcal{H}_0$ a unit vector $\Omega = \mathbb{1} \oplus 0 \oplus 0 \oplus ...$ representing the vacuum can be uniquely chosen.

The action of the projector $P_n : \mathcal{H}_1^{\otimes n} \to \mathcal{H}_n$ can be extended to $\bigoplus_{n=0}^{\infty} \mathcal{H}_1^{\otimes n}$ in the obvious way by defining

$$P := \bigoplus_{n=1}^{\infty} P_n. \tag{2.15}$$

2.3 Representation of the Poincaré group

In $1+1$ dimensions the Poincaré group can be parametrized by a *shift* $a \in \mathbb{R}^2$ and the rapidity $\lambda \in \mathbb{R}$, as a linear parametrization of a Lorentz boost.

To define a representation U of the (proper, orthochronous) Poincaré group, it is again necessary to choose an orthonormal basis $\{e_\alpha\}$ for $\mathcal{K}$, like in (2.3).

Definition 2.4. *Let $a \in \mathbb{R}^2$, $\lambda \in \mathbb{R}$. On the unsymmetrized space $\bigoplus_{n=0}^{\infty} \mathcal{H}_1^{\otimes n}$, for a family of masses $\{m_1, ..., m_N | \, m_i \geq 0, \, N = dim(\mathcal{K})\}$, the operator $U(a, \lambda)$ is defined as*

$$[U(a, \lambda)\Psi]_n^{\alpha_1 ... \alpha_n} (\underline{\theta}) := \exp(i \sum_{k=1}^{n} m_{\alpha_k} p(\theta_k) \cdot a) \, \Psi_n^{\alpha_1 ... \alpha_n} (\underline{\theta} - \lambda), \tag{2.16}$$

where $p(\theta_k) := \binom{\cosh\theta_k}{\sinh\theta_k}$, and $(\underline{\theta}-\lambda)$ is to be understood as λ being subtracted from every entry of $\underline{\theta}$.

The $\cdot$ will always denote the Minkowski inner product in the following.

Lemma 2.2. *The operator $U(a,\lambda)$ can be restricted to the S-symmetric Hilbert space $\mathcal{H}$.*

Proof. To restrict $U(a,\lambda)$ to $\bigoplus_{n=0}^{\infty} P_n\mathcal{H}_1^{\otimes n}$ we need $[U(a,\lambda),P]=0$ $\forall a \in \mathbb{R}^2$, $\lambda \in \mathbb{R}$. This can be seen as follows:

Since $P_n = \frac{1}{n!}\sum_{\pi\in\mathfrak{S}_n}D_n(\pi)$, and $\mathfrak{S}_n$ is generated by the transpositions τ_k, it is sufficient to do this calculation for $D_n(\tau_k)$ (for more details on index notation, see Appendix A).

$$[U(a,\lambda)D_n(\tau_k)\Psi]_n^{\alpha_1...\alpha_n}(\underline{\theta}) \tag{2.17}$$
$$= e^{i\sum_{k=1}^n m_{\alpha_k}p(\theta_k)\cdot a}S_{\beta_{k+1}\beta_k}^{\alpha_{k+1}\alpha_k}(\theta_{k+1}-\theta_k)$$
$$\times \Psi_n^{\alpha_1...\beta_{k+1}\beta_k...\alpha_n}(\theta_1-\lambda,...,\theta_{k+1}-\lambda,\theta_k-\lambda,...,\theta_n-\lambda)$$

which should equal

$$[D_n(\tau_k)U(a,\lambda)\Psi]_n^{\alpha_1...\alpha_n}(\underline{\theta}) \tag{2.18}$$
$$= S_{\beta_{k+1}\beta_k}^{\alpha_{k+1}\alpha_k}(\theta_{k+1}-\theta_k)\,e^{i(m_{\alpha_1}p(\theta_1)+...+m_{\beta_{k+1}}p(\theta_{k+1})+m_{\beta_k}p(\theta_k)+...+m_{\alpha_n}p(\theta_n))\cdot a}$$
$$\times \Psi_n^{\alpha_1...\beta_{k+1}\beta_k...\alpha_n}(\theta_1-\lambda,...,\theta_{k+1}-\lambda,\theta_k-\lambda,...,\theta_n-\lambda).$$

Now, comparing (2.17) and (2.18) one can see that for all α_i except α_k and α_{k+1} this condition represents no restriction on S, and it can be reduced to

$$S_{\beta_{k+1}\beta_k}^{\alpha_{k+1}\alpha_k}(\theta_{k+1}-\theta_k)e^{i(m_{\alpha_{k+1}}p(\theta_{k+1})+m_{\alpha_k}p(\theta_k))\cdot a}$$
$$= S_{\beta_{k+1}\beta_k}^{\alpha_{k+1}\alpha_k}(\theta_{k+1}-\theta_k)e^{i(m_{\beta_{k+1}}p(\theta_{k+1})+m_{\beta_k}p(\theta_k))\cdot a}, \tag{2.19}$$

which can be rewritten as the condition

$$S_{\beta_1\beta_2}^{\alpha_1\alpha_2}(\theta)=0 \qquad \text{if} \qquad (m_{\alpha_1}\neq m_{\beta_1} \quad \text{or} \quad m_{\alpha_2}\neq m_{\beta_2}). \tag{2.20}$$

Recalling Definition 2.1 we see that this property of S is included there already in equation (2.5e). $\qquad\square$

Lemma 2.3. *U as defined in (2.16) is a unitary, positive energy representation of the proper orthochronous Poincaré group $\mathcal{P}_+^{\uparrow}$ and strongly continuous for translations $U(a,0)$ with the (up to a phase) unique invariant (vacuum) vector $\Omega = 1 \oplus 0 \oplus 0 \oplus ... \in \mathcal{H}$.*

Proof. We begin by showing that U is a representation of $\mathcal{P}_+^{\uparrow}$ on $\bigoplus_{n=0}^{\infty}\mathcal{H}_1^{\otimes n}$ by verifying the multiplication law $U(a,\lambda)U(a',\lambda') = U(\Lambda_\lambda a' + a, \lambda + \lambda')$, which gives the inverse $U(a,\lambda)^{-1} = U(-\Lambda_{-\lambda}a,-\lambda)$ as well:

$$[U(a,\lambda)U(a',\lambda')\Psi]_n^{\alpha_1...\alpha_n}(\theta_1,...,\theta_n)$$
$$= \exp(i\sum_{l=1}^n m_{\alpha_l}\binom{\cosh\theta_l}{\sinh\theta_l}\cdot a)\,\exp(i\sum_{i=1}^n m_{\alpha_i}\binom{\cosh(\theta_i-\lambda)}{\sinh(\theta_i-\lambda)}\cdot a')$$
$$\times \Psi_n^{\alpha_1...\alpha_n}(\underline{\theta}-\lambda-\lambda')$$
$$= \exp(i\sum_{l=1}^n m_{\alpha_l}\underbrace{\left(\binom{\cosh\theta_l}{\sinh\theta_l}\cdot a + \binom{\cosh(\theta_l-\lambda)}{\sinh(\theta_l-\lambda)}\cdot a'\right)}_{(\Lambda_\lambda a'+a)\cdot\binom{\cosh\theta_l}{\sinh\theta_l}})\Psi_n^{\underline{\alpha}}(\underline{\theta}-\underbrace{\lambda-\lambda'}_{-(\lambda+\lambda')})$$
$$= [U(\Lambda_\lambda a'+a,\lambda+\lambda')\Psi]_n^{\alpha_1...\alpha_n}(\theta_1,...,\theta_n).$$

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Here Λ_λ denotes the boost with rapidity λ, cf. (2.1).

Unitarity of U can be seen from its definition:

$$\langle U\Psi, U\Phi \rangle_\mathcal{H} = \sum_{n=0}^\infty \int d^n\underline{\theta} \exp(-i\sum_{k=1}^n m_{\alpha_k} \begin{pmatrix} \cosh\theta_k \\ \sinh\theta_k \end{pmatrix} \cdot a) \; \overline{\Psi_n^\alpha(\underline{\theta} - \lambda)}$$

$$\times \exp(i\sum_{k=1}^n m_{\alpha_k} \begin{pmatrix} \cosh\theta_k \\ \sinh\theta_k \end{pmatrix} \cdot a) \; \Phi_n^\alpha(\underline{\theta} - \lambda)$$

$$= \sum_{n=0}^\infty \int \langle \Psi_n(\underline{\theta} - \lambda) , \; \Phi_n(\underline{\theta} - \lambda) \rangle_{\mathcal{K}^{\otimes n}} \; d^n\theta = \langle \Psi, \Phi \rangle_\mathcal{H}.$$

At this point we have shown that U is a unitary representation of $\mathcal{P}_+^\uparrow$ on $\bigoplus_{n=0}^\infty \mathcal{H}_1^{\otimes n}$. It was shown in Lemma 2.2 that it can be restricted to the S-symmetric subspace $\mathcal{H}$ already.

In order to be able to apply Stone's Theorem, i.e. writing $U(a,0)$ as an exponential, strong continuity of the translations $U(a,0)$ is necessary.

Therefore we look at

$$\lim_{a\to 0} \langle U(a,0)\,\Psi, \Phi \rangle_\mathcal{H}$$

$$= \sum_{n=0}^\infty \int_{\mathbb{R}^n} \lim_{a\to 0} \exp(-i\sum_{k=1}^n m_{\alpha_k} \begin{pmatrix} \cosh\theta_k \\ \sinh\theta_k \end{pmatrix} \cdot a) \; \overline{\Psi_n^\alpha(\underline{\theta})} \Phi_n^\alpha(\underline{\theta}) \; d^n\underline{\theta}$$

$$= \langle \Psi, \Phi \rangle_\mathcal{H}.$$

Pulling the limit into the integral is legitimated by Lebesgue's Theorem of dominated convergence, stating that for a sequence $\{f_n\}$ of measurable functions on a measure space (M,μ) converging pointwise to a function f which is dominated by some integrable function g (i.e. $|f_n| \le g \; \forall n$), the limiting function f is integrable, and $\lim_{n\to\infty} \int_s f_n d\mu = \int_s f d\mu$, since we can estimate the oscillating function $|\exp(i\sum_{l=1}^n m_{\alpha_l} \begin{pmatrix} \cosh\theta_l \\ \sinh\theta_l \end{pmatrix} \cdot a)| = 1$.

To show the existence of a unique invariant vector $\Omega \in \mathcal{H}$, invariance under U has to be shown. This corresponds to Ω fulfilling $U(a,\lambda)\Omega = \Omega \; \forall\lambda \in \mathbb{R}, \; \forall a \in \mathbb{R}^2$. Written out, what we want is

$$[U(a,\lambda)\Omega]_n^\alpha(\underline{\theta}) = \exp(i\sum_{k=1}^n m_{\alpha_k} \begin{pmatrix} \cosh\theta_k \\ \sinh\theta_k \end{pmatrix} \cdot a) \; \Omega_n^\alpha(\underline{\theta} - \lambda) \overset{!}{=} \Omega_n^\alpha(\underline{\theta}),$$

$\forall\underline{\theta},\underline{\alpha},n,\lambda,a$. This is a very strong restriction, since for a function not being altered by translations of its arguments, it has to be constant. Hence not $\in L^2$, except for the possibility of being zero everywhere. This yields $\Omega = 0$ for all n except $n = 0$. But as U acting on $\Omega \in \mathcal{H}_0 = \mathbb{C}$ leaves this constant invariant (because of the constant function having no argument, hence no subtraction of λ, and the sum in the exponential of U no phase) the statement is shown.

For $U(a,0)$ to be a positive energy representation, we need the joint spectrum of the generators of the energy and momentum operators P_0, P_1 to be contained in the closed forward light cone $\overline{V}^+ = \{p \in \mathbb{R}^2 : p^2 \ge 0, \; p_0 \ge 0\}$, which is equivalent to P_0 and $P_0^2 - P_1^2$ both being positive[1]. Therefore we note that the generators P_0 and P_1 multiply with $\sum_k(m_{\alpha_k}\cosh\theta_k)$ and $\sum_k(m_{\alpha_k}\sinh\theta_k)$ respectively on

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[1]For details on spectra see [RS2] Chap IX.8.

$\mathcal{H}_n$, and check:

$$|\sum_k m_{\alpha_k} \sinh \theta_k| \le \sum_k m_{\alpha_k} |\sinh \theta_k| = \sum_k m_{\alpha_k} \sinh |\theta_k|$$
$$\le \sum_k m_{\alpha_k} \cosh |\theta_k| = \sum_k m_{\alpha_k} \cosh \theta_k,$$

where in the first inequality the triangle inequality has been applied and we made use of the claim that $m_k \ge 0 \, \forall \, k$. So, taking into account the positivity of $\cosh \theta$, we have shown that both P_0 and $P_0^2 - P_1^2$ are positive. $\qquad \square$

2.3.1 The PCT operator

In the following, we will make use of a spacetime reflection operator I_{ST}, which can be seen as extending the representation U of $\mathcal{P}_+^\uparrow$ to the proper Poincaré group $\mathcal{P}_+$. This reflection operator will be important for our subsequent discussion of wedge-local fields. Therefore we give the following

Definition 2.5. *A spacetime reflection operator on $\mathcal{H}_n$ is given by*

$$[I_{ST}\Psi]_n^{\alpha_1...\alpha_n}(\theta_1,...,\theta_n) := \overline{\Psi_n^{\alpha_n...\alpha_1}(\theta_n,...,\theta_1)}, \qquad \Psi \in \mathcal{D}. \qquad (2.21)$$

NB. reversed order of indices and arguments. The whole *PCT* operation can be formulated by including the

Definition 2.6. *of a charge conjugation operator C,*

$$[C\Psi]_n^{\alpha_1...\alpha_n}(\theta_1,...,\theta_n) := \Psi_n^{\overline{\alpha_1}...\overline{\alpha_n}}(\theta_1,...,\theta_n), \qquad \Psi \in \mathcal{D}. \qquad (2.22)$$

Here $\overline{\alpha}$ is the charge conjugated particle spectrum, i.e. $\{1...N\} \ni \alpha \to \overline{\alpha} \in \{1...N\}$ is a permutation that satisfies $\overline{\overline{\alpha}} = \alpha$. Obviously, $C^2 = id$.

From Definition 2.1 we know that $m_\alpha = m_{\overline{\alpha}}$, i.e. the antiparticle has the same mass as the particle. Combining Definitions 2.5 and 2.6 gives the PCT operator

$$J := I_{ST} \, C = C \, I_{ST}. \qquad (2.23)$$

Lemma 2.4. *The PCT operator J is an anti-unitary involution and commutes with P. Moreover J extends U via $JU(a,\lambda)J = U(-a,\lambda)\forall a, \lambda$ to a representation of the proper Poincaré group $\mathcal{P}_+$.*

Proof. We begin with anti-unitarity of J:

$$\langle J\Psi, J\Phi \rangle_\mathcal{H} = \sum_{n=0}^\infty \int \overline{[J\Psi]_n^{\alpha_1...\alpha_n}(\underline{\theta})} [J\Phi]_n^{\alpha_1...\alpha_n}(\underline{\theta}) \, d^n\underline{\theta}$$
$$= \sum_{n=0}^\infty \int \Psi_n^{\overline{\alpha}_n...\overline{\alpha}_1}(\theta_n...\theta_1) \overline{\Phi_n^{\overline{\alpha}_n...\overline{\alpha}_1}(\theta_n...\theta_1)} d^n\underline{\theta}$$
$$= \overline{\langle \Psi, \Phi \rangle_\mathcal{H}} = \langle \Phi, \Psi \rangle_\mathcal{H}.$$

The last equation holds true because $\alpha \mapsto \overline{\alpha}$ is a bijection. Next, we will calculate the commutation relation of J and P where it is again sufficient, as

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$P_n = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} D_n(\pi)$ to do it for $D_n(\tau_k)$:

$[JD_n(\tau_k)\Psi]_n^{\alpha_1 \ldots \alpha_n}(\theta_1, \ldots, \theta_k, \theta_{k+1}, \ldots, \theta_n)$

$= \overline{[D_n(\tau_k)\Psi]_n^{\overline{\alpha}_n \ldots \overline{\alpha}_{n-k+1} \overline{\alpha}_{n-k} \ldots \overline{\alpha}_1}(\theta_n, \ldots, \theta_{n-k+1}, \theta_{n-k}, \ldots, \theta_1)}$

$= S_{\overline{\beta}_{n-k} \overline{\beta}_{n-k+1}}^{\overline{\alpha}_{n-k} \overline{\alpha}_{n-k+1}}(\theta_{n-k} - \theta_{n-k+1}) \; \overline{\Psi_n^{\overline{\alpha}_n \ldots \overline{\beta}_{n-k} \overline{\beta}_{n-k+1} \ldots \overline{\alpha}_1}(\theta_n, \ldots, \theta_{n-k}, \theta_{n-k+1}, \ldots, \theta_1)},$

using (2.5a) in the form of (A.5), and (2.5b),

$= S_{\beta_{n-k+1} \beta_{n-k}}^{\alpha_{n-k+1} \alpha_{n-k}}(\theta_{n-k+1} - \theta_{n-k}) \overline{\Psi_n^{\overline{\alpha}_n \ldots \overline{\beta}_{n-k} \overline{\beta}_{n-k+1} \ldots \overline{\alpha}_1}(\theta_n, \ldots, \theta_{n-k}, \theta_{n-k+1}, \ldots, \theta_1)}$

$= S_{\beta_{n-k+1} \beta_{n-k}}^{\alpha_{n-k+1} \alpha_{n-k}}(\theta_{n-k+1} - \theta_{n-k}) \; [J\Psi]_n^{\alpha_1 \ldots \beta_{n-k+1} \beta_{n-k} \ldots \alpha_n}(\theta_1, \ldots, \theta_{n-k+1}, \theta_{n-k}, \ldots, \theta_n)$

$= [D_n(\tau_{n-k})J\Psi]_n^{\alpha_1 \ldots \alpha_n}(\underline{\theta}),$

where from the third line on $D_n(\tau_k)$ acts as $F_{n-k}S_{n-k}$ because the kth argument is θ_{n-k}, e.g. for $k = 2$, in $\{5, (4,3), 2, 1\}$ 4 and 3 flips, hence in $\{1, 2, (3,4), 5\}$ to flip 3 and 4 the transposition has to act on the $5 - 2 = 3^{rd}$ position.

As P_n contains all permutations, these shifted transpositions generate again P_n, leading to $[J, P_n] = 0$.

It is seen that J extends U from a representation of $\mathcal{P}_+^{\uparrow}$ to $\mathcal{P}_+$ as follows

$$[JU(a, \lambda)J\Psi]_n^{\alpha_1 \ldots \alpha_n}(\underline{\theta}) \tag{2.24}$$

$$= \overline{\exp(i \sum_{k=1}^{n} m_{\overline{\alpha}_k} \begin{pmatrix} \cosh \theta_k \\ \sinh \theta_k \end{pmatrix} \cdot a) \; [J\Psi]_n^{\overline{\alpha}_n \ldots \overline{\alpha}_1}(\theta_n - \lambda, \ldots, \theta_1 - \lambda)}$$

$$= \exp(-i \sum_{k=1}^{n} m_{\alpha_k} \begin{pmatrix} \cosh \theta_k \\ \sinh \theta_k \end{pmatrix} \cdot a) \; \Psi_n^{\alpha_1 \ldots \alpha_n}(\underline{\theta} - \lambda)$$

$$= [U(-a, \lambda)\Psi]_n^{\alpha_1 \ldots \alpha_n}(\underline{\theta}), \tag{2.25}$$

where in the second equation we have used $m_{\overline{\alpha}} = m_{\alpha}$. $\qquad \square$

3 Fields

So far we have defined a S-symmetric "Fock" space $\mathcal{H}$ and a representation of the proper Poincaré group. In order to define field operators on $\mathcal{H}$ with the usually claimed properties, we start with the definition of the following creation and annihilation operators.

Definition 3.1. *We define the unsymmetrized annihilation and creation operators $a(\varphi)$ and $a^\dagger(\varphi)$ for $\varphi \in \mathcal{H}_1$, $\Psi_n \in \mathcal{D}$, on the subspace $\mathcal{D} \subset \mathcal{H}$ of finite particle number as*

$$[a(\varphi)\Psi]_n^{\alpha_1...\alpha_n}(\theta_1...\theta_n) := \sqrt{n+1} \int d\theta' \varphi_{\alpha_0}(\theta')\Psi_{n+1}^{\alpha_0\alpha_1...\alpha_n}(\theta', \theta_1...\theta_n), \qquad n \in \mathbb{N}_0,$$
(3.1a)

$$[a^\dagger(\varphi)\Psi]_n^{\alpha_1...\alpha_n}(\theta_1...\theta_n) := \sqrt{n}\; \varphi^{\alpha_1}(\theta_1)\; \Psi_{n-1}^{\alpha_2...\alpha_n}(\theta_2...\theta_n), \qquad n \neq 0,$$
(3.1b)
$$[a^\dagger(\varphi)\Psi]_0 := 0.$$

These operators obey

$$\langle \Psi, a(\varphi)\Phi \rangle_\mathcal{H} = \sum_{n=0}^\infty \int d^n\theta \langle \Psi_n(\underline{\theta}), [a(\varphi)\Phi]_n(\underline{\theta}) \rangle_{\mathcal{K}^{\otimes n}}$$

$$= \sum_{n=0}^\infty \sqrt{n+1} \int d^n\theta\; \overline{\Psi_n^{\alpha_1...\alpha_n}(\theta_1,...,\theta_n)} \int d\theta_0\; \varphi^{\alpha_0}(\theta_0)\Phi_{n+1}^{\alpha_0...\alpha_n}(\theta_0,...,\theta_n),$$

by renaming $n \to n+1$,

$$= \sum_{n=1}^\infty \sqrt{n} \int d\theta_0...d\theta_{n-1}\; \varphi^{\alpha_0}(\theta_0)\overline{\Psi_{n-1}^{\alpha_1...\alpha_{n-1}}(\theta_1,...,\theta_{n-1})}\Phi_n^{\alpha_0...\alpha_{n-1}}(\theta_0,...,\theta_{n-1})$$

$$= \langle a^\dagger(\varphi)\Psi, \Phi \rangle, \qquad \text{since } [a^\dagger(\varphi)\Psi]_0 = 0,$$
(3.2)

i.e. $a(\varphi)^* \supset a^\dagger(\overline{\varphi})$, where we use the symbol $\dagger$ if the hermitian adjoint $*$ only acts on the operator, but not on its argument.

Choosing $\varphi \in \mathscr{S}(\mathbb{R}) \otimes \mathcal{K} \subset \mathcal{H}_1$, $\mathscr{S}$ being the Schwartz space of test functions (i.e. decreasing faster than any exponential), one can rewrite these operators as operator valued distributions, by defining $a_\alpha^\#(\theta)$ as distributional integral kernels of

$$a(\varphi) =: \int a_\alpha(\theta)\varphi^\alpha(\theta)d\theta, \qquad a^\dagger(\varphi) =: \int a_\alpha^\dagger(\theta)\varphi^\alpha(\theta)d\theta.$$
(3.3)

From now on $\varphi \in \mathscr{S}(\mathbb{R}) \otimes \mathcal{K}$, which implies a restriction because $\mathscr{S}(\mathbb{R}) \otimes \mathcal{K} \subset \mathcal{H}_1 = L^2(\mathbb{R}, d\theta) \otimes \mathcal{K}$.

3.1 S-symmetric fields

So far the operators a and $a^\dagger$ applied to $\Psi \in \mathcal{D}$ do not obey the S-symmetry of $\mathcal{H}$. More exactly, only the creation operator does not. In this section we will define creation and annihilation operators adhering to the S-symmetric structure of $\mathcal{H}$.

From (2.7) and (2.8) it is clear that for each $\pi \in \mathfrak{S}_n$, we have $[D_n(\pi)\Psi_n](\underline{\theta}) = S_\pi(\underline{\theta})\Psi_n(\theta_{\pi(1)},...,\theta_{\pi(n)})$ with some tensor $S_\pi(\underline{\theta}) \in \mathcal{B}(\mathcal{K}^{\otimes n})$. In the following we will need the special permutation $\sigma_k(1, 2,...,k,...n) := (k, 1, 2,...,\hat{k},...,n)$ where the hat on k indicates that this variable is omitted, repeatedly and therefore compute S_{σ_k}.

$$[D_n(\sigma_k)\Psi_n(\underline{\theta})]^{\underline{\alpha}} = [D_n(\tau_{k-1}) \cdots D_n(\tau_1)\Psi_n](\underline{\theta})^{\underline{\alpha}}$$
$$= S(\theta_k - \theta_{k-1})^{\alpha_k \alpha_{k-1}}_{\rho_k \rho_{k-1}} [D_n(\tau_{k-2}) \cdots D_n(\tau_1)\Psi_n(\theta_1, ..., \theta_k, \theta_{k-1}, ..., \theta_n)]^{\alpha_1...\rho_k\rho_{k-1}...\alpha_n}$$
$$= S(\theta_k - \theta_{k-1})^{\alpha_k \alpha_{k-1}}_{\rho_k \rho_{k-1}} S(\theta_k - \theta_{k-2})^{\rho_k \alpha_{k-2}}_{\pi_k \pi_{k-2}}$$
$$\times [D_n(\tau_{k-3}) \cdots D_n(\tau_1)\Psi_n(\theta_1, ..., \theta_k, \theta_{k-2}, \theta_{k-1}, ..., \theta_n)]^{\alpha_1...\pi_k\pi_{k-2}\rho_{k-1}...\alpha_n}$$
$$\vdots$$
$$= S(\theta_k - \theta_{k-1})^{\alpha_k \alpha_{k-1}}_{\rho_k \rho_{k-1}} S(\theta_k - \theta_{k-2})^{\rho_k \alpha_{k-2}}_{\pi_k \pi_{k-2}} ... S(\theta_k - \theta_2)^{\delta_k \alpha_2}_{\tau_k \tau_2} S(\theta_k - \theta_1)^{\tau_k \alpha_1}_{\mu_k \mu_1},$$
$$\times \Psi_n^{\mu_k \mu_1 \tau_2 ... \pi_{k-2}\rho_{k-1}\alpha_{k+1}...\alpha_n}(\theta_k, \theta_1, ..., \hat{\theta}_k, ..., \theta_n).$$

By renaming indices this product can be rewritten as follows:

$$= \sum_{\xi_1...\xi_k} \delta^{\alpha_k}_{\xi_k} \delta^{\beta_k}_{\xi_1} \prod_{l=k-1}^{1} S^{\xi_{l+1}\alpha_l}_{\xi_l \beta_l}(\theta_k - \theta_l)\Psi_n^{\beta_k \beta_1 ... \beta_{k-1}\alpha_{k+1}...\alpha_n}(\theta_k, \theta_1, ...\hat{\theta}_k, ..., \theta_n)$$
$$= [S_{\sigma_k}(\underline{\theta})]^{\alpha_k \alpha_1 ... \alpha_{k-1}}_{\beta_k \beta_1 ... \beta_{k-1}} \Psi_n^{\beta_k \beta_1 ... \beta_{k-1}\alpha_{k+1}...\alpha_n}(\theta_k, \theta_1, ...\hat{\theta}_k, ..., \theta_n). \tag{3.4}$$

The notation of the product running from $l = k-1$ to 1 is chosen to indicate the order of transpositions, albeit not of formal relevance due to index notation.
We will now state the creation and annihilation operators for the S-symmetric space $\mathcal{H}$.

Theorem 3.1. *For $\Psi \in \mathcal{D}$, and $\varphi \in \mathscr{S}(\mathbb{R})\otimes\mathcal{K}$, the operators $z^\dagger(\varphi)\Psi := Pa^\dagger(\varphi)\Psi$ and $z(\varphi)\Psi := Pa(\varphi)\Psi$ are given by*

$$[z(\varphi)\Psi]_n^{\alpha_1...\alpha_n}(\theta_1, ..., \theta_n) = \sqrt{n+1} \int d\theta'\varphi^\beta(\theta')\Psi_{n+1}^{\beta\alpha_1...\alpha_n}(\theta', \theta_1...\theta_n), \tag{3.5a}$$

$$[z^\dagger(\varphi)\Psi]_n^{\alpha_1...\alpha_n}(\theta_1, ..., \theta_n) = \frac{1}{\sqrt{n}} \sum_{k=1}^n [S_{\sigma_k}(\underline{\theta})]^{\alpha_k \alpha_1 ... \alpha_{k-1}}_{\beta_k \beta_1 ... \beta_{k-1}} \varphi^{\beta_k}(\theta_k)$$
$$\times \Psi_{n-1}^{\beta_1...\beta_{k-1}\alpha_{k+1}...\alpha_n}(\theta_1, ..., \hat{\theta}_k, ..., \theta_n) \qquad \text{for } n \geq 1, \tag{3.5b}$$
$$[z^\dagger(\varphi)\Psi]_0 = 0.$$

They obey $\langle\Psi, z(\varphi)\Phi\rangle = \langle z^\dagger(\overline{\varphi})\Psi, \Phi\rangle$, i.e. $z(\varphi)^* \supset z^\dagger(\overline{\varphi})$, and can be estimated by

$$\|z(\varphi)\Psi\| \leq \|\varphi\|\|N^{1/2}\Psi\|, \qquad \|z^\dagger(\varphi)\Psi\| \leq \|\varphi\|\|(N+1)^{1/2}\Psi\|, \tag{3.6}$$

where N is the particle number operator defined by

$$[N\Psi]_n := n \Psi_n. \tag{3.7}$$

Proof. For $z(\varphi)$ the statement is clearly fulfilled since it annihilates an argument of an element of the S-symmetric space, so $z(\varphi)\mathcal{H}_n \subset \mathcal{H}_{n-1}$. For $z^\dagger(\varphi)$ note that in $P_n(\varphi\otimes\Psi)$, $\Psi \in \mathcal{H}_{n-1}$ is already S-symmetric, and a permutation $\pi \in \mathfrak{S}_n$ can be rewritten as $\pi = \sigma_k \cdot \rho$ with $\sigma_k \in \mathfrak{S}_n$ and $\rho \in \mathfrak{S}_{n-1}$ acting on $\{2, ..., n\} \subset \{1, ..., n\}$. $\sigma_k = \tau_{k-1}\tau_{k-2} \cdot ... \cdot \tau_1$, $k \in \{1, ..., n\}$, $\sigma_1 = id$, τ_i being transpositions. Hence the projector P_n can be written as

$$P_n = \frac{1}{n!} \sum_{k=1}^n \sum_{\rho\in\mathfrak{S}_{n-1}} D_n(\sigma_k)(\mathbb{1} \otimes D_{n-1}(\rho)) = \frac{1}{n} \sum_{k=1}^n D_n(\sigma_k)(\mathbb{1} \otimes P_{n-1}).$$

The form of S_{σ_k} of $D_n(\sigma_k)$ is given in equation (3.4).

For the bounds on $z^\dagger(\varphi)$ consider

$$\|z^\dagger(\varphi)\Psi\|_{\mathcal{H}}^2 = \sum_{n=0}^{\infty} \|[z^\dagger(\varphi)\Psi]_n\|_{\mathcal{H}_n}^2 = \sum_{n=1}^{\infty} \|\sqrt{n}P_n(\varphi \otimes \Psi_{n-1})\|_{\mathcal{H}_n}^2$$

$$\leq \sum_{n=1}^{\infty} n\|\varphi \otimes \Psi_{n-1}\|^2 = \sum_{n=0}^{\infty}(n+1)\|\varphi\|_{\mathcal{H}_1}^2\|\Psi_n\|_{\mathcal{H}_n}^2$$

$$= \|\varphi\|_{\mathcal{H}}^2\|(N+1)^{1/2}\Psi\|_{\mathcal{H}}^2.$$

For $z(\varphi)$ the same calculation applies, except that in the estimate a $(n-1)$ due to the particle number enters, giving the $N^{1/2}$. The inclusion $z(\varphi)^* \supset z^\dagger(\overline{\varphi})$ can be seen from (3.2) and the fact that the projector P is selfadjoint: For $\Psi, \Phi \in \mathcal{D}$, we have

$$\langle \Psi, z^\dagger(\varphi)\Phi\rangle = \langle \Psi, Pa^\dagger(\varphi)\Phi\rangle = \langle a(\overline{\varphi})\underbrace{P\,\Psi}_{\Psi}, \Phi\rangle = \langle \underbrace{a(\overline{\varphi})\Psi}_{S-\mathrm{symmetric}}, \Phi\rangle = \langle z(\overline{\varphi})\Psi, \Phi\rangle.$$

$$\square$$

We will also work with the distributional kernels $z_\alpha^\#(\theta)$ (the wild-card $\#$ implies both the annihilation and the creation operator) in analogy to the non-symmetric case (3.3), related to the operators $z^\#(\varphi)$ for $\varphi \in \mathscr{S}(\mathbb{R}) \otimes \mathcal{K} \subset \mathcal{H}_1$ by the formal integrals

$$z^\#(\varphi) =: \int \varphi^\alpha(\theta)z_\alpha^\#(\theta)d\theta. \tag{3.8}$$

The action of $z_\alpha^\#(\theta)$ can be obtained from Theorem 3.1 by formally setting $\varphi^\beta(\theta) = \delta_\beta^\alpha\delta(\theta' - \theta)$, with α and θ fixed (implying that the particle type α is chosen). This yields

$$[z_\gamma(\theta')\Psi]_n^\alpha(\underline{\theta}) = \sqrt{n+1}\Psi_{n+1}^{\gamma\alpha_1...\alpha_n}(\theta', \theta_1...\theta_n), \tag{3.9a}$$

$$[z_\gamma^\dagger(\theta')\Psi]_n^\alpha(\underline{\theta}) = \frac{1}{\sqrt{n}}\sum_{k=1}^{n}[S_{\sigma_k}(\underline{\theta})]_{\gamma\beta_1...\beta_{k-1}}^{\alpha_k\alpha_1...\alpha_{k-1}}\delta(\theta_k - \theta')\Psi_{n-1}^{\beta_1...\hat{\alpha}_k...\alpha_n}(\theta_1,...,\hat{\theta}_k,...,\theta_n). \tag{3.9b}$$

Lemma 3.2. *The commutation relations of the operators $z_\alpha^\dagger(\theta)$ and $z_\beta(\theta)$ are given by*

$$z_\beta(\theta')z_\rho(\theta'') - S_{\alpha\gamma}^{\beta\rho}(\theta' - \theta'')z_\gamma(\theta'')z_\alpha(\theta') = 0,$$
$$z_\beta^\dagger(\theta')z_\rho^\dagger(\theta'') - S_{\beta\rho}^{\alpha\gamma}(\theta'' - \theta')z_\gamma^\dagger(\theta'')z_\alpha^\dagger(\theta') = 0, \tag{3.10}$$
$$z_\beta(\theta')z_\rho^\dagger(\theta'') - S_{\rho\alpha}^{\gamma\beta}(\theta'' - \theta')z_\gamma^\dagger(\theta'')z_\alpha(\theta') = \delta^{\beta\rho}\delta(\theta' - \theta'')\mathbb{1},$$

$\forall\, \theta', \theta'' \in \mathbb{R}$, *defining a Zamolodchikov algebra [ZA1].*

Proof. First we calculate the commutation relation of two annihilation operators. Let $\Psi \in \mathcal{D}$ be continuous wave functions.

$$[z_\beta(\theta')z_\rho(\theta'')\Psi]_n^{\alpha_1...\alpha_n}(\underline{\theta}) = \sqrt{n+1}[z_\rho(\theta'')\Psi]_{n+1}^{\beta\,\alpha_1...\alpha_n}(\theta', \theta_1, .., \theta_n)$$

$$= \sqrt{(n+1)(n+2)}\Psi_{n+2}^{\rho\beta\,\alpha_1...\alpha_n}(\theta'', \theta', \theta_1, .., \theta_n)$$

$$= \sqrt{(n+1)(n+2)}S_{\alpha\gamma}^{\beta\rho}(\theta' - \theta'')\Psi_{n+2}^{\alpha\,\gamma\,\alpha_1...\alpha_n}(\theta', \theta'', \theta_1, .., \theta_n)$$

$$= [S_{\alpha\gamma}^{\beta\rho}(\theta' - \theta'')z_\gamma(\theta'')z_\alpha(\theta')\Psi]_n^{\alpha_1...\alpha_n}(\theta_1, .., \theta_n).$$

Here no flip operator F appears due to the index notation and the order of arguments, giving it explicitly. By conjugation, one obtains (making use of (A.5)) the analogous result for the creation operators:

$$
\begin{aligned}
0 &= [z_\beta(\theta')z_\rho(\theta'') - S^{\beta\rho}_{\alpha\gamma}(\theta' - \theta'')z_\gamma(\theta'')z_\alpha(\theta')]^* \\
&= z_\rho(\theta'')^* z_\beta(\theta')^* - S^{\gamma\alpha}_{\rho\beta}(\theta'' - \theta')z_\alpha(\theta')^* z_\gamma(\theta'')^* \\
&= z^\dagger_\rho(\theta'')z^\dagger_\beta(\theta') - S^{\gamma\alpha}_{\rho\beta}(\theta'' - \theta')z^\dagger_\alpha(\theta')z^\dagger_\gamma(\theta'').
\end{aligned}
$$

In the mixed case we again compute each term of the commutator separately using (3.9b),

$$
\begin{aligned}
&[z^\dagger_\gamma(\theta'')z_{\beta_0}(\theta_0)\Psi]^\alpha_n(\theta_1,...,\theta_n) \\
&= \frac{1}{\sqrt{n}} \sum_{k=1}^n [S_{\sigma_k}(\underline{\theta})]^{\alpha_k\alpha_1...\alpha_{k-1}}_{\gamma\beta_1...\beta_{k-1}} [z_{\beta_0}(\theta_0)\Psi]^{\beta_1...\beta_{k-1}\hat{\alpha}_k...\alpha_n}_{n-1}(\theta_1,...,\hat{\theta}_k,...,\theta_n) \\
&= \sum_{k=1}^n [S_{\sigma_k}(\underline{\theta})]^{\alpha_k\alpha_1...\alpha_{k-1}}_{\gamma\beta_1...\beta_{k-1}} \delta(\theta_k - \theta'')\Psi^{\beta_0...\beta_{k-1}\alpha_{k+1}...\alpha_n}_n(\theta_0,...,\hat{\theta}_k,...,\theta_n) \\
&= \sum_{k=1}^n \sum_{\epsilon_1...\epsilon_k} \delta^{\alpha_k}_{\epsilon_k}\delta^{\epsilon_1}_\gamma \prod_{l=k-1}^1 S^{\epsilon_{l+1}\alpha_l}_{\epsilon_l\beta_l}(\theta_k - \theta_l)\delta(\theta_k - \theta'')\Psi^{\beta_0...\hat{\alpha}_k...\alpha_n}_n(\theta_0,...,\hat{\theta}_k,...,\theta_n).
\end{aligned}
\tag{3.11}
$$

The flipped situation yields

$$
\begin{aligned}
&[z_{\alpha_0}(\theta_0)z^\dagger_\rho(\theta'')\Psi]^{\alpha_1...\alpha_n}_n(\theta_1,...,\theta_n) \\
&= \sqrt{n+1}\,[z^\dagger_\rho(\theta'')\Psi]^{\alpha_0...\alpha_n}_{n+1}(\theta_0,...,\theta_n) \\
&= \sum_{k=0}^n [S_{\sigma_k}(\theta'',\underline{\theta})]^{\alpha_k\alpha_0...\alpha_{k-1}}_{\rho\beta_0...\beta_{k-1}} \Psi^{\beta_0...\beta_{k-1}\alpha_{k+1}...\alpha_n}_n(\theta_0,\theta_1,...,\hat{\theta}_k,...,\theta_n) \\
&= \sum_{k=0}^n \sum_{\epsilon_0...\epsilon_k} \delta^{\alpha_k}_{\epsilon_k}\delta^{\epsilon_0}_{\beta_k} \prod_{l=k-1}^0 S^{\epsilon_{l+1}\alpha_l}_{\epsilon_l\beta_l}(\theta_k - \theta_l)\delta^{\beta_k}_\rho\delta(\theta_k - \theta'')\Psi^{\beta_0...\hat{\alpha}_k...\alpha_n}_n(\theta_0,...,\hat{\theta}_k,...,\theta_n).
\end{aligned}
\tag{3.12}
$$

The difference between (3.12) and (3.11) is the sum, in (3.12) starting at $k = 0$ which adds $\delta^{\beta_0}_\rho\delta(\theta' - \theta'')$ (and no S-factor, see Definition (3.4)), and the product $[S_{\sigma_k}(\underline{\theta})]^{\epsilon_k\alpha_1...\alpha_{k-1}}_{\epsilon_1\beta_1...\beta_{k-1}}$ in (3.11) starting to act on the second index of Ψ, hence having one factor less in each summand than in (3.12). Therefore, by plugging (3.12) into (3.11) and re-expressing it in terms of $z(\theta)$ and $z^\dagger(\theta)$, we obtain

$$
\begin{aligned}
z_{\alpha_0}(\theta')z^\dagger_\rho(\theta'') &= \delta^{\alpha_0}_\rho\delta(\theta' - \theta'')\mathbb{1} \\
&+ S^{\alpha_k\alpha_0}_{\beta_k\beta_0}(\theta'' - \theta_0)\delta^{\alpha_k}_\gamma\delta^{\beta_k}_\rho [\sum_{k=1}^n \sum_{\epsilon_2...\epsilon_{k-1}} \prod_{l=k-1}^1 S^{\epsilon_{l+1}\alpha_l}_{\epsilon_l\beta_l}(\theta_k - \theta_l)]\delta(\theta_k - \theta'')z_{\beta_0}(\theta'),
\end{aligned}
$$

where one can read off the extra $S^{\gamma\alpha_0}_{\rho\beta_0}$. $\qquad\qquad\square$

Having now symmetric creation and annihilation operators at hand, we can define a field in the usual way as a linear combination of these creation and annihilation operators with test functions supported on the positive and negative mass shell as arguments, respectively.

Definition 3.2. *For $f \in \mathscr{S}(\mathbb{R}^2) \otimes \mathcal{K}$, we define the field as a Segal-type operator*

$$\phi(f) := z^\dagger(f^+) + z(Cf^-), \tag{3.13}$$

with the Fourier transform

$$f_\alpha^\pm(\theta) := \frac{1}{2\pi} \int f_\alpha(\pm x) e^{im_\alpha p(\theta) \cdot x} d^2 x. \tag{3.14}$$

Here $p(\theta) = \binom{\sinh\theta}{\cosh\theta}$, as defined in (2.16).

Remark: $f^{*\pm} = J f^\mp$, as we will see in the following theorem, by using $\overline{f^\pm} = \overline{f}^\mp$.

Theorem 3.3. *Properties of the field operator $\phi(f)$:*

1. *From the bounds of the creation and annihilation operators we obtain the bounds for the field, $\Psi \in \mathcal{D}$,*

$$\|\phi(f)\Psi\| \leq (\|f^+\| + \|f^-\|) \, \|(N+1)^{1/2}\Psi\|. \tag{3.15}$$

2. *For $\Psi \in \mathcal{D}$, the adjoint field is given by*

$$\phi(f)^*\Psi = \phi(f^*)\Psi, \quad \text{with the adjoint field} \quad f^*(x) := C\overline{f(x)} \tag{3.16}$$

   All vectors in $\mathcal{D}$ are entirely analytic for $\phi(f)$. If for $f \in \mathscr{S}(\mathbb{R}^2) \otimes \mathcal{K}$ $f = f^* = C\overline{f}$ holds, $\phi(f)$ is essentially selfadjoint on $\mathcal{D}$.

3. *$\phi(f)$ transforms covariantly under the representation U of $\mathcal{P}_+^\uparrow$:*

$$U(a,\lambda)\phi(f)U(a,\lambda)^* = \phi(f_{(a,\lambda)}), \tag{3.17}$$
$$f_{(a,\lambda)}^\alpha(x) = f^\alpha(\Lambda_\lambda^{-1}(x-a)), \quad x,a \in \mathbb{R}^2, \ \lambda \in \mathbb{R},$$

   where Λ_λ denotes the boost with rapidity λ.

4. *The vacuum vector Ω is cyclic for the field ϕ, i.e. for any open set $\mathcal{O} \subset \mathbb{R}^2$, the subspace*

$$\mathscr{D}_\mathcal{O} := span\{\phi(f_1)...\phi(f_n)\Omega \, : \, f_1,...,f_n \in \mathscr{S}(\mathcal{O}) \otimes \mathcal{K}, n \in \mathbb{N}_0\}$$

   is dense in $\mathcal{H}$.

5. *ϕ is local if and only if $S = 1$.*

We will see in Theorem 4.4 that ϕ is relatively "wedge-local" to another one. Nevertheless we will be able to work with this weaker locality.

Proof. 1. By inserting the previously given bounds (3.6) on $z^\dagger$ and z, this is straightforward:

$$\|\phi(f)\Psi\| \leq \|f^+\|\|(N+1)^{1/2}\Psi\| + \|Cf^-\|\|N^{1/2}\Psi\|$$
$$\leq (\|f^+\| + \|f^-\|) \, \|(N+1)^{1/2}\Psi\|.$$

2. First, we observe

$$\overline{f_\alpha^\pm(\theta)} = \frac{1}{2\pi} \int \overline{f}_\alpha(\mp x) e^{im_\alpha p(\theta) x} d^2 x = \overline{f}_\alpha^\mp(\theta),$$

and

$$f_\alpha^{*\mp}(\theta) = \frac{1}{2\pi} \int \overline{f}_{\bar\alpha}(\mp x) e^{im_\alpha p(\theta) x} d^2 x = \overline{(f)_{\bar\alpha}^\mp(\theta)} = [Jf^\pm](\theta),$$

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which we use to show $\phi(f)^*\Psi = \phi(f^*)\Psi$:

$$\phi(f)^*\Psi = (z^\dagger(f^+) + z(Cf^-))^*\Psi = (z^\dagger(C\overline{f^-}) + z(\overline{f^+}))\Psi$$
$$= (z^\dagger(C\overline{f}^+) + z(\overline{f}^-))\Psi = (z^\dagger(f^{*+}) + z(Cf^{*-})))\Psi = \phi(f^*)\Psi.$$

To see that $\phi(f)$ is essentially selfadjoint, i.e. that $\phi(f)$ has a unique self-adjoint extension, we proceed analogously to [L1]. Let $\Psi_n \in \mathcal{H}_n$, and $c_f := \|f^+\| + \|f^-\|$. In view of the bound in 1., we have the estimates $\|\phi(f)\Psi_n\| \leq \sqrt{n+1}c_f\|\Psi_n\|$ and

$$\|\phi(f)^k\Psi_n\| \leq \sqrt{n+k}\,c_f\,\|\phi(f)^{k-1}\Psi_n\| \leq \sqrt{n+k}...\sqrt{n+1}c_f^k\|\Psi_n\|, \quad k \in \mathbb{N}.$$

Thus, for arbitrary $\xi \in \mathbb{C}$ there holds (by the ratio criterion for $k \to \infty$)

$$\sum_{k=0}^\infty \frac{|\xi|^k}{k!}\|\phi(f)^k\Psi_n\| \leq \|\Psi_n\| \sum_{k=0}^\infty \sqrt{\frac{(n+k)!}{n!}}\frac{1}{k!}(|\xi|c_f)^k < \infty.$$

This shows that every $\Psi \in \mathcal{D}$ is an entirely analytic vector for $\phi(f)$. Since $\mathcal{D}$ is dense in $\mathcal{H}$, we can use Nelson's Theorem, stating that for a symmetric (i.e. $f^* = f$) operator on a Hilbert space whose domain of definition contains a total set of analytic vectors is essentially selfadjoint [RS2]. In the following, we will use the same symbol $\phi(f)$ for the operator and its selfadjoint closure.

3. Here we use the fact that U commutes with P_n from Lemma 2.2:

$$[U(a,\lambda)z^\dagger(\varphi)U(a,\lambda)^*\Psi]_n = \sqrt{n}U(a,\lambda)P_n(\varphi \otimes U(a,\lambda)^*\Psi_{n-1})$$
$$= \sqrt{n}P_n(U(a,\lambda)\varphi \otimes \Psi_{n-1}) = [z^\dagger(U(a,\lambda)\varphi)\Psi]_n.$$

Next, with the definition of U (2.16),

$$U(a,\lambda)z(\varphi)U(a,\lambda)^* = (U(a,\lambda)z^\dagger(\overline{\varphi})U(a,\lambda)^*)^*$$
$$= z(\overline{U(a,\lambda)\overline{\varphi}}) = z(U(-a,\lambda)\varphi).$$

In view of the definition of $f^\pm$ (3.14), we observe that $U(\pm a,\lambda)f^\pm = f^\pm_{(a,\lambda)}$, and conclude (using $[U,C] = 0$),

$$U(a,\lambda)\phi(f)U(a,\lambda)^{-1} = z^\dagger(U(a,\lambda)f^+) + z(CU(-a,\lambda)f^-) = \phi(f_{(a,\lambda)}).$$

4. Choosing $f \in \mathscr{S}(\mathbb{R}^2) \otimes \mathcal{K}$ such that supp $f_\alpha \cap H^-_{m_\alpha} = \emptyset\ \forall\alpha$ with $H^-_{m_\alpha}$ being the negative mass shell of the mass m_α), we have $z^\dagger(f^+) \in \mathcal{P}(\mathbb{R}^2)$, where $\mathcal{P}(\mathbb{R}^2)$ is the algebra generated by all polynomials in the field $\phi(f)$ with test functions $f \in \mathscr{S}(\mathbb{R}^2) \otimes \mathcal{K}$. Varying f gives a dense set of f^+ in $\mathcal{H}_1$, implying that Ω is cyclic for $\mathcal{P}(\mathbb{R}^2)$.

Now, to extend this construction for Ω to be cyclic for test functions $f \in \mathscr{S}(\mathcal{O}) \otimes \mathcal{K}$, $\mathcal{O}$ an open set in $\mathbb{R}^2$, we need to employ the Reeh-Schlieder Theorem [SW1, Thm. 4-2], stating that for an open set $\mathcal{O}$ of spacetime, a vector Ω is cyclic for $\mathcal{P}(\mathcal{O})$ if it is a cyclic vector for $\mathcal{P}(\mathbb{R}^2)$.

In the case of a narrower localization area (we will see in Theorem 4.4 that the fields are wedge-local), it is seen that the Reeh-Schlieder theorem can still be applied, as follows: The Fourier transform of the tempered distribution F defined there [SW1, Thm. 4-2] as $F(-x_1, x_1 - x_2, ...x_{n-1} - x_n) = (\Psi, \phi(x_1), ...\phi(x_n)\Omega)$ vanishes unless each x_i lies in the closed forward light cone. This statement is independent of the type of localization of the

fields. Therefore F can be analytically continued to a holomorphic function in the tube $\mathscr{T} = \{x + iy \; : \; y \in \mathbb{R}^+\}$, and by the Edge of the Wedge Theorem it vanishes. This shows that for Ψ orthogonal to all vectors of the form $\sum_j \phi(f_1), ... \phi(f_j)\Omega$ with $\text{supp} f_{k,\alpha} \subset \mathcal{O}$, it is also orthogonal to such vectors with $\text{supp} f_{k,\alpha} \subset \mathbb{R}$, i.e. Ψ is orthogonal to $\mathcal{P}(\mathbb{R}^2)\Omega$ and hence $\Psi = 0$.

5. For $S = 1$, $\phi(f)$ is the local free field by definition (2.11), as $\mathcal{H}$ is the Bose-Fock space then. To see the non-locality of $\phi(f)$ for f, $g \in \mathscr{S}(\mathbb{R}^2) \otimes \mathcal{K}$ in the case $S \neq 1$ we take a look at

$$[P_2[\phi(f), \phi(g)]\Omega]^{\alpha\beta}(\theta_1, \theta_2) = \frac{1}{\sqrt{2}}(f_\alpha^+(\theta_1)g_\beta^+(\theta_2) + S_{\epsilon\gamma}^{\beta\alpha}(\theta_2 - \theta_1)g_\epsilon^+(\theta_1)f_\gamma^+(\theta_2)$$
$$- g_\alpha^+(\theta_2)f_\beta^+(\theta_1) - S_{\epsilon\gamma}^{\beta\alpha}(\theta_2 - \theta_1)f_\epsilon^+(\theta_1)g_\gamma^+(\theta_2)).$$

This expression vanishes for arbitrary spacelike separated test functions f and g if and only if $S = \mathbb{1}$.

$\square$

We will see later in Theorem 4.5 that ϕ and ϕ' are so called polarization-free generators (this term was introduced by Schroer, and further investigated by Borchers, Buchholz and Schroer [BBS1]), see Definition 4.3.

4 Finding a local algebra

Our interest lies in the localization properties of fields. Therefore the following subset of Minkowski space is defined, and will be used extensively.

Definition 4.1. *In $1 + 1$ dimensional Minkowski space, the region*

$$W_R := \{x \in \mathbb{R}^2 | \, x_1 > |x_0|\} \tag{4.1}$$

will be called right wedge. This also defines a left wedge, by $W_L := -W_R$.

We will denote the commutant of a set $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ as $\mathcal{A}' := \{B \in \mathcal{B}(\mathcal{H}) \; : \; [A, B] = 0 \; \forall A \in \mathcal{A}\}$, which enables us to state regions in Minkowski space in a rather implicit way, and by taking the double commutant providing a convenient notion for the closure in the weak operator topology [BR1].

The previously defined field ϕ does not have the usual locality properties but, as was shown by Schroer [S1, S2], is localized in a wedge shaped region W_L in Minkowski space. But with the PCT operator J (2.23), it is possible to define a second "reflected" field that will be localized in the opposite wedge W_R.

4.1 A second field

The main purpose of introducing the TCP Operator J was not to get a representation of the proper Poincaré group, but we use it now to define another field:

Definition 4.2.

$$\phi'(f) := J\phi(f^j)J, \qquad f^j(x) := \overline{f(-x)}, \quad or, \; equivalently \quad f^{j\pm} = \overline{f^\pm}. \tag{4.2}$$

It is relatively wedge-local to the previously defined ϕ, as we will show in Theorem 4.4.
In the definition the field ϕ' we did not make use of the creation and annihilation operators for this new field, but as we will need them to investigate the locality properties of ϕ', we state them here:

$$z(\psi)' := Jz(\psi)J, \qquad z^\dagger(\psi)' := Jz^\dagger(\psi)J. \tag{4.3}$$

These operators again form a Zamolodchikov algebra like the one generated by z and $z^\dagger$ in Lemma 3.2. The reflected field can be written as a linear combination of creation and annihilation operators

$$\phi'(f) = J(z^\dagger(f^{j+}) + z(Cf^{j-}))J = J(z^\dagger(\overline{f^+}) + z(C\overline{f^-}))J = z^\dagger(\overline{f^+})' + z(C\overline{f^-})'.$$

It will become clear that this field differs from the previously defined one when we state the relative localization properties of ϕ and ϕ'. There we will see that this new field is localized in the opposite wedge, and $\phi' = \phi \Leftrightarrow S = 1$.

Lemma 4.1. *For $\Psi \in \mathcal{D}$, we have*

$$(z^\dagger(\varphi)')^* \Psi = z(\overline{\varphi})' \Psi. \tag{4.4}$$

Proof. By making us of (3.2), this is seen easily:

$$(z^\dagger(\varphi)')^* \Psi = J^*(z^\dagger(\overline{\varphi}))^* J^* \Psi = Jz(\overline{\varphi})J\Psi = z(\overline{\varphi})'\Psi.$$

$\square$

Theorem 4.2. *The field ϕ' has the same properties as ϕ stated in Theorem 3.3:*

1. *The bounds for the field are*

$$\|\phi'(f)\Psi\| \leq (\|f^+\| + \|f^-\|) \|(N+1)^{1/2}\Psi\|. \tag{4.5}$$

2. *For $\Psi \in \mathcal{D}$, the adjoint field is given by*

$$\phi'(f)^* = \phi'(f^*). \tag{4.6}$$

All vectors in $\mathcal{D}$ are entirely analytic for $\phi(f)$. If, for $f \in \mathscr{S}(\mathbb{R}^2) \otimes \mathcal{K}$ $f^* = f$, $\phi(f)$ is essentially selfadjoint on $\mathcal{D}$.

3. *$\phi'(f)$ transforms covariantly under the representation U of $\mathcal{P}_+^\uparrow$ (2.16)*

$$U(a,\lambda)\phi'(f)U(a,\lambda)^{-1} = \phi(f_{(a,\lambda)}), \tag{4.7}$$

$$f_{(a,\lambda)}^\alpha(x) = f^\alpha(\Lambda_\lambda^{-1}(x-a)), \quad x \in \mathbb{R}^2, \ \lambda \in \mathbb{R}, \tag{4.8}$$

where Λ_λ denotes the boost with rapidity λ.

4. *The vacuum vector Ω is cyclic for the field ϕ', i.e. for an open set $\mathcal{O} \subset \mathbb{R}^2$, the subspace*

$$\mathscr{D}_{\mathcal{O}} := span\{\phi(f_1)...\phi(f_n)\Omega \ : \ f_1,...,f_n \in \mathscr{S}(\mathcal{O}) \otimes \mathcal{K}, n \in \mathbb{N}_0\} \tag{4.9}$$

is dense in $\mathcal{H}$.

5. *ϕ' is local if and only if $S = 1$.*

Proof. As the statements have been shown for ϕ in the proof of Theorem 3.3 already, we will only focus on the differences. Let again $\Psi \in \mathcal{D}$.

1. Here we have the operator J appearing, which has to be estimated, and f^j instead of f. But from its action (2.23) it is clear that $\|J\| = 1$, as well as $\|f^{j\pm}\| = \|\overline{f^\pm}\| = \|f^\pm\|$.

2. In this case J does not alter the argument as it is selfadjoint:

$$\phi'(f)^*\Psi = J^*(\phi(f^j))^*J^*\Psi = J\phi(f^{j*})J\Psi = \phi'(f^{*j})\Psi = \phi'(f^*)\Psi.$$

The proof of $\phi'(f)$ being essentially selfadjoint is the same as the one given in Theorem 3.3 as the bounds on $\phi'(f)$ are the same as shown in 1.

3. Here we use the fact that $U(a,\lambda)J\Psi = JU(-a,\lambda)\Psi$ (Lemma 2.4):

$$[U(a,\lambda)\, z^\dagger(\varphi)'\, U(a,\lambda)^*\Psi]_n = [U(a,\lambda)\, Jz^\dagger(\varphi)J\, U(a,\lambda)^*\Psi]_n$$
$$= [JU(-a,\lambda)\, z^\dagger(\varphi)\, U(-a,\lambda)^*J\Psi]_n = [z^\dagger(U(-a,\lambda)\varphi)'\Psi]_n.$$

For the annihilation operator, this gives

$$U(a,\lambda)\, z(\varphi)'\, U(a,\lambda)^*\Psi = (U(a,\lambda)z^\dagger(\overline{\varphi})'U(a,\lambda)^*)^*\Psi = (z^\dagger(U(-a,\lambda)\overline{\varphi})')^*\Psi$$
$$= z(U(a,\lambda)\varphi)'\Psi.$$

Recalling the definition of $f^\pm$ (3.14), we observe that $U(\pm a,\lambda)f^\pm = f_{(a,\lambda)}^\pm$, and obtain

$$U(a,\lambda)\phi(f)'U(a,\lambda)^{-1} = z^\dagger(U(-a,\lambda)\overline{f^+})' + z(CU(a,\lambda)\overline{f^-})' = \phi(f_{(a,\lambda)})'.$$

4. and 5. Again, the arguments of the proof of Theorem 3.3 can be used in the same way in this context.

$\square$

In Lemma 3.2 we stated the Zamolodchikov algebra generated by z and $z^\dagger$. In order to show the relative wedge-locality of ϕ and ϕ', we need the mixed commutator relations of $z^\#$ and $z'^\#$ as well. In view of the relations needed then, we will state these in a rather peculiar way including the charge conjugation operator.

Lemma 4.3. *For $\psi_1, \psi_2 \in \mathcal{H}_1$, the commutation relations of the "reflected" Zamolodchikov operators with the original ones are given by,*

$$[z(\psi_1)',\ z(\psi_2)] = 0, \qquad [z^\dagger(\psi_1)',\ z^\dagger(\psi_2)] = 0, \tag{4.10a}$$

$$[[z(C\overline{\psi_1})',\ z^\dagger(\psi_2)]\Psi]_n \ =: \ B_n^{C\psi_1\,\psi_2}\Psi_n,$$
$$[B_n^{C\psi_1\,\psi_2}(\underline{\theta})]^{\alpha_1...\alpha_n}_{\beta_1...\beta_n} = \int \psi_1^{\alpha_{n+1}}(\theta')[S_{\sigma_{n+1}}(\underline{\theta},\theta')]^{\alpha_{n+1}\alpha_1...\alpha_n}_{\beta_{n+1}\beta_1...\beta_n}\psi_2^{\beta_{n+1}}(\theta')\,d\theta', \tag{4.10b}$$

$$[[z^\dagger(C\overline{\psi_1})',\ z(\psi_2)]\Psi]_n \ =: \ G_n^{C\psi_1\psi_2}\Psi_n,$$
$$[G_n^{C\psi_1\psi_2}(\underline{\theta})]^{\alpha_1...\alpha_n}_{\beta_1...\beta_n} = -\int \psi_1^{\alpha_{n+1}}(\theta')\,[S_{\sigma_{n+1}^{-1}}(\underline{\theta},\theta')]^{\alpha_1...\alpha_n\alpha_{n+1}}_{\beta_1...\beta_n\beta_{n+1}}\,\psi_2^{\beta_{n+1}}(\theta')\,d\theta'. \tag{4.10c}$$

From this definition one can see that $B_n^{C\psi_1\psi_2}$ and $G_n^{C\psi_1\psi_2}$ act as multiplication operators on Ψ. The explicit form of $S_{\sigma_{n+1}^{-1}}$ is given in equation (4.18).

Proof. Before showing the commutator relations, we calculate the action of the reflected annihilation operator:

$$[z(C\psi)'\Psi]_n^{\alpha_1...\alpha_n}(\underline{\theta})$$
$$= \overline{[z(C\psi)J\Psi]_n^{\overline{\alpha_n}...\overline{\alpha_1}}(\theta_n, ..., \theta_1)}$$
$$= \sqrt{n+1}\int \overline{\psi^{\overline{\beta}}(\theta')[J\Psi]_{n+1}^{\beta\overline{\alpha_n}...\overline{\alpha_1}}(\theta', \theta_n, ..., \theta_1)}\,d\theta'$$
$$= \sqrt{n+1}\int \overline{\psi^{\overline{\beta}}(\theta')}\Psi_{n+1}^{\alpha_1...\alpha_n\overline{\beta}}(\theta_1, ..., \theta_n, \theta')\,d\theta'$$
$$= \sqrt{n+1}\int \overline{\psi^{\beta}(\theta')}\Psi_{n+1}^{\alpha_1...\alpha_n\beta}(\theta_1, ..., \theta_n, \theta')\,d\theta', \tag{4.11}$$

where in the last equation, we used that the map $\alpha \mapsto \overline{\alpha}$ is bijective, hence $\sum_\alpha \phi^{\overline{\alpha}}\psi^{\overline{\alpha}} = \sum_\alpha \phi^\alpha\psi^\alpha$. So no $\overline{\alpha}$ appears in the index notation of $z(C\psi)'\Psi$. Like in (3.8), one can integrate out $\overline{\psi^\beta(\theta')}$ by choosing $\overline{\psi^\beta(\theta')} = \delta^{\alpha_{n+1}}_\beta\delta(\theta' - \theta)$, with fixed α_{n+1} and θ:

$$[z_\beta(\theta')\Psi]_n^\alpha(\underline{\theta}) = \sqrt{n+1}\,\Psi_{n+1}^{\alpha_1...\alpha_n\beta}(\theta_1, ..., \theta_n, \theta'). \tag{4.12}$$

To calculate the action of the creation operator is not necessary for the proof as we will get the other commutators from adjoining the first ones. However, since it might be interesting though, it can be found in Appendix (B.3).

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Now to the first commutation relation:

$$[[z_\beta(\theta)', \ z_\gamma(\theta_0)]\Psi]_n^{\alpha_1...\alpha_n}(\underline{\theta})$$
$$= \sqrt{n+1}[z_\gamma(\theta_0)\Psi]_{n+1}^{\alpha_1...\alpha_n\beta}(\theta_1,...,\theta_n,\theta)$$
$$\quad - \sqrt{n+1}[z_\beta(\theta)'\Psi]_{n+1}^{\gamma\alpha_1...\alpha_n}(\theta_0,\theta_1,...,\theta_n)$$
$$= \sqrt{n+1}\sqrt{n+2}\Psi_{n+2}^{\gamma\alpha_1...\alpha_n\beta}(\theta_0,\theta_1,...,\theta_n,\theta)$$
$$\quad - \sqrt{n+1}\sqrt{n+2}\Psi_{n+2}^{\gamma\alpha_1...\alpha_n\beta}(\theta_0,\theta_1,...,\theta_n,\theta)$$
$$= 0. \tag{4.13}$$

For the action of the creation operators we take the adjoint of the commutator of the annihilation operators (4.13):

$$0 = -[z(\overline{\psi_1})', \ z(\overline{\psi_2})]^* = [Jz(\overline{\psi_1})^*J, \ z(\overline{\psi_2})^*]$$
$$= [Jz^\dagger(\psi_1)J, z^\dagger(\psi_2)] = [z^\dagger(\psi_1)', z^\dagger(\psi_2)]. \tag{4.14}$$

In the third commutator $B_n^{C\psi_1\psi_2}\Psi_n$, the notation defined in (3.4) will be used for the k-fold products of matrix valued functions.

$$[[z(C\overline{\psi_1})', \ z^\dagger(\psi_2)]\Psi]_n^\alpha(\underline{\theta})$$
$$= \sqrt{n+1}\int d\theta_{n+1}\psi_1^{\alpha_{n+1}}(\theta_{n+1})[z^\dagger(\psi_2)\Psi]_{n+1}^{\alpha_1...\alpha_{n+1}}(\theta_1,...,\theta_n,\theta_{n+1})$$
$$\quad - \frac{1}{\sqrt{n}}\sum_{k=1}^n[S_{\sigma_k}(\underline{\theta})]_{\beta_k\beta_1...\beta_{k-1}}^{\alpha_k\alpha_1...\alpha_{k-1}}\psi_2^{\beta_k}(\theta_k)[z(C\overline{\psi_1})'\Psi]_{n-1}^{\beta_1...\beta_{k-1}\alpha_{k+1}...\alpha_n}(\theta_1,...,\hat\theta_k,...,\theta_n)$$
$$= \int d\theta_{n+1}\psi_1^{\alpha_{n+1}}(\theta_{n+1})\sum_{k=1}^{n+1}[S_{\sigma_k}(\underline{\theta})]_{\beta_k\beta_1...\beta_{k-1}}^{\alpha_k\alpha_1...\alpha_{k-1}}\psi_2^{\beta_k}(\theta_k)$$
$$\qquad\qquad\qquad\qquad\times \Psi_n^{\beta_1...\beta_{k-1}...\alpha_{n+1}}(\theta_1,...,\hat\theta_k,...,\theta_{n+1})$$
$$\quad - \sum_{k=1}^n[S_{\sigma_k}(\underline{\theta})]_{\beta_k\beta_1...\beta_{k-1}}^{\alpha_k\alpha_1...\alpha_{k-1}}\psi_2^{\beta_k}(\theta_k)\int d\theta_{n+1}\psi_1^{\alpha_{n+1}}(\theta_{n+1})$$
$$\qquad\qquad\qquad\qquad\times \Psi_n^{\beta_1...\beta_{k-1}...\alpha_{n+1}}(\theta_1,...,\hat\theta_k,...,\theta_{n+1})$$
$$= \int d\theta_{n+1}\,\psi_1^{\alpha_{n+1}}(\theta_{n+1})[S_{\sigma_{n+1}}(\underline{\theta})]_{\beta_{n+1}\beta_1...\beta_n}^{\alpha_{n+1}\alpha_1...\alpha_n}\psi_2^{\beta_{n+1}}(\theta_{n+1})\,\Psi_n^{\beta_1...\beta_n}(\underline{\theta})$$
$$= [B_n^{C\psi_1\psi_2}(\underline{\theta})]_{\underline\beta}^{\underline\alpha}\Psi_n^{\underline\beta}(\underline{\theta}) \tag{4.15}$$

where from the second to the third equation all terms except the $(n+1)^{th}$ canceled out. So this commutator acts via annihilation in the last, and creation in the first position. Moreover, it also does not involve a flip of any arguments of Ψ and therefore acts as a multiplication operator, multiplying with a tensor.
To obtain the fourth commutator we adjoin equation (4.10b):

$$\left(-B^{C\overline{\psi_1}\,\overline{\psi_2}}\right)^* = -[z(C\psi_1)', \ z^\dagger(\overline{\psi_2})]^* = [Jz(C\psi_1)^*J, \ z^\dagger(\overline{\psi_2})^*]$$
$$= [Jz^\dagger(C\overline{\psi_1})J, \ z(\psi_2)] = [z^\dagger(\overline{C\psi_1})', \ z(\psi_2)] = G^{C\psi_1\psi_2}. \tag{4.16}$$

So we have $\left(-B_n^{C\overline{\psi_1}\,\overline{\psi_2}}\right)^* = G_n^{C\psi_1\psi_2}$, giving an important hint for later on, how $B_n^{C\psi_1\psi_2}$ and $G_n^{C\psi_1\psi_2}$ are connected, which provides us the fourth commutator,

completing the proof:

$$[G_n^{C\psi_1\,\psi_2}(\underline{\theta})]_{\underline{\beta}}^{\underline{\alpha}} = -[\left(B_n^{C\overline{\psi_1}\,\overline{\psi_2}}(\underline{\theta})\right)^*]_{\underline{\beta}}^{\underline{\alpha}} = -\overline{[B_n^{C\overline{\psi_1}\,\overline{\psi_2}}(\underline{\theta})]_{\underline{\alpha}}^{\underline{\beta}}}$$

$$= -\int d\theta'\, \psi_1^{\beta_{n+1}}(\theta') \sum_{\xi_1...\xi_{n+1}} \delta_{\xi_1}^{\alpha_{n+1}} \delta_{\xi_{n+1}}^{\beta_{n+1}} \prod_{l=n}^{1} S_{\beta_l\xi_{l+1}}^{\alpha_l\xi_l}(\theta$$

$$(4.17)$$

where unitarity (2.5a) in the notation of (A.5) has been used in the second line. The product of scattering functions in the last line of the above equation is unhandy, which is why we will denote it from now on as

$$[S_{\sigma_{n+1}^{-1}}(\underline{\theta}, \theta')]_{\beta_1...\beta_n\beta_{n+1}}^{\alpha_1...\alpha_n\alpha_{n+1}} = \sum_{\xi_1...\xi_{n+1}} \delta_{\xi_1}^{\alpha_{n+1}} \delta_{\xi_{n+1}}^{\beta_{n+1}} \prod_{l=n}^{1} S_{\beta_l\xi_{l+1}}^{\alpha_l\xi_l}(\theta_l - \theta'). \qquad (4.18)$$

Note that $S_{\sigma_k^{-1}}$ corresponds to $D_n(\sigma_k^{-1})$. This can be seen by writing down the product of scattering functions corresponding to $D_n(\sigma_k^{-1})$, with $\sigma_k^{-1}(1, 2, ..., n)$ $:= (2, ..., k - 1, 1, k, ..., n)$. A more detailed argument for the shape of $S_{\sigma_{n+1}^{-1}}$ can be found in Appendix (B.1).
So (4.17) can be rewritten as

$$[G_n^{C\psi_1\,\psi_2}]_{\underline{\beta}}^{\underline{\alpha}} = -\int d\theta'\, \psi_1^{\beta_{n+1}}(\theta') \sum_{\xi_1...\xi_{n+1}} \delta_{\xi_1}^{\alpha_{n+1}} \delta_{\xi_{n+1}}^{\beta_{n+1}} [S_{\sigma_{n+1}^{-1}}(\underline{\theta}, \theta')]_{\beta_1...\beta_n\xi_{n+1}}^{\alpha_1...\alpha_n\xi_{n+1}} \psi_2^{\alpha_{n+1}}(\theta'),$$

$$\square$$

Alternatively, one can get $S_{\sigma_{n+1}^{-1}}$ as follows: D_n is a unitary representation of $\mathfrak{S}_n$. This carries over to $(F_{\sigma_{n+1}} S_{\sigma_n})^*$ as it was constructed from unitary matrices (2.2).
We highlight this fact by adding a $^{-1}$ to σ, emphasizing the inverse action:

$$(F_{\sigma_{n+1}} S_{\sigma_{n+1}})^* = S_{\sigma_{n+1}^{-1}}. \qquad (4.19)$$

4.2 Relative wedge-locality of the fields and a local net

Having the mixed commutation relations of Lemma 4.3 at hand, we can proceed in analyzing the locality properties of ϕ and ϕ', essentially by proving the following Theorem 4.4.
Before doing so, we recall that in Definition 2.1 we have claimed the matrix-valued scattering function to be analytic in the region $\{\xi \in \mathbb{C} \,:\, 0 < \mathrm{Im}(\xi) < \pi\}$, and that it has the so called crossing symmetry $S_{\gamma\delta}^{\alpha\beta}(i\pi - \theta) = S_{\delta\alpha}^{\beta\overline{\gamma}}(\theta)$. These two assumptions are essential to prove

Theorem 4.4. *The field operators ϕ and ϕ' are relatively wedge-local to each other, i.e for $f \in \mathscr{S}(W_R) \otimes \mathcal{K}$, $g \in \mathscr{S}(W_L) \otimes \mathcal{K}$,*

$$[\phi'(f), \phi(g)]\Psi = 0, \qquad\qquad \Psi \in \mathcal{D}. \qquad (4.20)$$

One should keep in mind that for $f \in \mathscr{S}(W_L) \otimes \mathcal{K}$, $g \in \mathscr{S}(W_R) \otimes \mathcal{K}$ this statement is in general wrong (except for $S = \pm 1$). So the localization region of the fields can not be chosen arbitrarily, but is already contained in their definition to be either in the left or right wedge. Therefore Theorem 4.4 implies that the field ϕ' is located in W_R and ϕ in W_L.

Proof. By plugging in definitions and observing that $\overline{f^\pm} = \overline{f}^\mp$ and $\overline{g^\pm} = \overline{g}^\mp$, what needs to be shown reduces by Lemma 4.3 to

$$
\begin{aligned}
[\phi'(f), \phi(g)]\Psi_n &= [z^\dagger(f^{j+})' + z(Cf^{j-})', z^\dagger(g^+) + z(Cg^-)]\Psi_n \\
&= [z^\dagger(\overline{f^+})' + z(C\overline{f^-})', z^\dagger(g^+) + z(Cg^-)]\Psi_n \\
&= [z^\dagger(\overline{f^+})', z(Cg^-)]\Psi_n + [z(C\overline{f^-})', z^\dagger(g^+)]\Psi_n \\
&= (G_n^{f^+ \, Cg^-} + B_n^{Cf^- \, g^+})\Psi_n.
\end{aligned} \tag{4.21}
$$

In order to show $(G_n^{f^+ \, Cg^-} + B_n^{Cf^- \, g^+})\Psi_n = 0$, we compare the commutators of Lemma 4.3. There we noted that $B_n^{C\psi_1 \, \psi_2}$ and $G_n^{C\psi_1 \, \psi_2}$ are multiplication operators, not altering Ψ_n itself. So we can proceed without paying attention to the action on Ψ. The charge conjugation operator C, in Lemma 4.3 acting on ψ_1 in $G_n^{C\psi_1 \, \psi_2}$ represents a difference tough.

By the bounds on ϕ and ϕ' from (3.15) and (4.5), for $\Psi, \Phi \in \mathcal{D}$, $(f, g) \mapsto \langle \Phi, [\phi(f), \phi'(g)]\Psi \rangle$ is a tempered distribution in f and g. Hence it is sufficient to treat the case of $f \in C_0^\infty(W_R) \otimes \mathcal{K}$, $g \in C_0^\infty(W_L) \otimes \mathcal{K}$ because if the product vanishes on $\mathscr{S}(W_R) \otimes \mathcal{K} \times \mathscr{S}(W_L) \otimes \mathcal{K}$, it also vanishes on $C_0^\infty(W_R) \otimes \mathcal{K} \times C_0^\infty(W_L) \otimes \mathcal{K}$.

We note that in $f_\alpha^\pm(\theta + i\pi)$ the hyperbolic trigonometric functions change sign due to the added $i\pi$ in the argument:

$$
\begin{aligned}
f_\alpha^\pm(\theta + i\pi) &= \frac{1}{2\pi} \int f_\alpha(x) \exp\{\pm im_\alpha \begin{pmatrix} \cosh(\theta + i\pi) \\ \sinh(\theta + i\pi) \end{pmatrix} \cdot x\} \, dx \\
&= \frac{1}{2\pi} \int f_\alpha(x) \exp\{\mp im_\alpha \begin{pmatrix} \cosh\theta \\ \sinh\theta \end{pmatrix} \cdot x\} \, dx \\
&= f_\alpha^\mp(\theta), \tag{4.22}
\end{aligned}
$$

and since g is defined the same way, this argument holds for it as well. This equality is what brings up the idea of proving Theorem 4.4 by analytic continuation to the complex plane and shifting the integration path from the real axis to $\theta + i\pi$.

Therefore we have to check the analyticity of f and g when continued to the complex plane, in order to employ Cauchy's theorem. For analyticity of $B_n^{Cf^- \, g^+}$, the real part of the exponential of f is the crucial object. For $0 < \eta < \pi$ we have

$$
f_\alpha^\pm(\theta + i\eta) = \frac{1}{2\pi} \int f_\alpha(x) \exp\{\pm im_\alpha \cos\eta \begin{pmatrix} \cosh\theta \\ \sinh\theta \end{pmatrix} \cdot x \mp m_\alpha \sin\eta \begin{pmatrix} \sinh\theta \\ \cosh\theta \end{pmatrix} \cdot x\} \, dx, \tag{4.23}
$$

where $\cosh(x+iy) = \cosh x \cos y + i \sinh x \sin iy$, and $\sinh(x+iy) = \sinh x \cos y + i \cosh x \sin y$ has been used.

Now we have to consider the real part of the exponent of f^- and g^+:

First, for f^- to be bounded it is necessary to have a negative product $\begin{pmatrix} \sinh\theta \\ \cosh\theta \end{pmatrix} \cdot x$ in the second term of the exponent in (4.23), since $\sin\eta > 0$ for $0 < \eta < \pi$. The first term is oscillating and does not disturb convergence. $\begin{pmatrix} \sinh\theta \\ \cosh\theta \end{pmatrix} \in W_R$ because $\cosh\theta > |\sinh\theta|$. Therefore we need $x \in W_R$, which implies $f^\alpha \in \mathscr{S}(W_R)$.

Secondly, for g^+, by applying the same line of thought, we demand a positive product $\begin{pmatrix} \cosh\theta \\ \sinh\theta \end{pmatrix} \cdot x$ leading to $x \in W_L$ which implies $g^\beta \in \mathscr{S}(W_L)$. This legitimates our choice of $f \in \mathscr{S}(W_R) \otimes \mathcal{K}$, $g \in \mathscr{S}(W_L) \otimes \mathcal{K}$.

Because supp g and supp f were chosen to be compact, it follows (cf. [RS1, Thm. IX.14]) that g^+ and f^- are bounded on $S(0, \pi)$, and $|g_\alpha^+(R + i\eta)|$ and $|f_\alpha^-(R + i\eta)|$ converge exponentially to zero for fixed η as $R \to \pm\infty$.

With crossing symmetry (2.5d), analyticity in $\{\xi \mid 0 < \text{Im}(\xi) < \pi\}$ and boundedness of S, f and g, we can proceed by shifting integration variables from $\mathbb{R}$ to $\mathbb{R} + i\eta$ in

$$[B_n^{Cf^- \, g^+}(\underline{\theta})]_{\beta_1...\beta_n}^{\alpha_1...\alpha_n}$$

$$= \int d\theta' \, (f^-)^{\alpha_{n+1}}(\theta' + i\eta) \delta_{\xi_{n+1}}^{\alpha_{n+1}} [S_{\sigma_{n+1}}(\underline{\theta} + i\eta)]_{\xi_1 \beta_1...\beta_n}^{\xi_{n+1}\alpha_1...\alpha_n} \delta_{\xi_1}^{\beta_{n+1}} (g^+)^{\beta_{n+1}}(\theta' + i\eta)$$

$$= \int d\theta' \, (f^-)^{\alpha_{n+1}}(\theta' + i\eta) \sum_{\xi_1...\xi_{n+1}} \delta_{\xi_{n+1}}^{\alpha_{n+1}} \delta_{\xi_1}^{\beta_{n+1}} \prod_{l=1}^{n} S_{\xi_l \beta_l}^{\xi_{l+1} \alpha_l}(\theta' - \theta_l + i\eta)$$

$$\times (g^+)^{\beta_{n+1}}(\theta' + i\eta).$$

By using crossing symmetry (2.5d), (4.22), and setting $\eta = \pi$, this gives

$$= \int d\theta' \, (f^+)^{\alpha_{n+1}}(\theta') \sum_{\xi_1...\xi_{n+1}} \delta_{\xi_{n+1}}^{\alpha_{n+1}} \delta_{\xi_1}^{\beta_{n+1}} \prod_{l=n}^{1} S_{\beta_l \overline{\xi_{l+1}}}^{\alpha_l \overline{\xi_l}}(\theta_l - \theta') \, (g^-)^{\beta_{n+1}}(\theta')$$

$$= \int d\theta' \, (f^+)^{\overline{\alpha_{n+1}}}(\theta') [S_{\sigma_{n+1}^{-1}}(\underline{\theta}, \theta')]_{\beta_1...\beta_n \beta_{n+1}}^{\alpha_1...\alpha_n \alpha_{n+1}} \, (g^-)^{\overline{\beta_1}}(\theta')$$

$$= -[G_n^{f^+ \, Cg^-}(\underline{\theta})]_{\alpha_1..\alpha_n}^{\beta_1...\beta_n}, \tag{4.24}$$

and arrive at the desired result. In the last equation we used the fact that $\overline{\overline{\alpha}} = \alpha$. $\qquad\qquad\square$

In view of bound states in the spectra in some Sigma models and the poles in $S(0, \pi)$ they originate from, it would be pleasing to have an extension of Theorem 4.4 for *meromorphic scattering functions*. Also regarding the symmetries we claimed in the beginning, one might think this could be possible to prove. But as it turns out this is not the case. We will briefly outline the arguments here.
To extend the proof of Theorem 4.4 to meromorphic scattering functions, one would make use of the Residue Theorem. Then, the residues of the poles of S in the strip $S(0, \pi)$ might cancel out each other due to one of the symmetries of S. One finds that crossing symmetry does not help as it represents a crossover reflection in the strip $S(0, \pi)$, hence it does not reverse the orientation of the integration path around the reflected pole.
The second symmetry $S(-\overline{\theta}) = S(\theta)$ for which one has to additionally claim $\overline{S(\theta)} = S(\overline{\theta})$ in the multi-particle case, does the job pretty well as it only reflects about the imaginary axis, hence reverses the orientation of the integration path around the symmetric pole. Therefore all residues would cancel.
But as the essence is to show that $(G_n^{f^+ \, Cg^-} + B_n^{Cf^- \, g^+})\Psi_n = 0$, which is done by continuation to the complex plane in the proof of Theorem 4.4, not only S needs to be symmetric, but f^- and g^+ as well. Which they are not, as one can see by considering the definition of $f^\pm$ (3.2). There, the Fourier transform acts via $p(\theta) = (\sinh\theta, \cosh\theta)$ and the hyperbolic trigonometric functions do not possess this symmetry for $\text{Re}\,\theta \neq 0$. In the case of $\text{Re}\,\theta = 0$ the residua do not cancel as there are only single poles without any symmetric partner.
Also rewriting the product of the Fourier transforms of f and g in $B_n^{Cf^- \, g^+}$ as convolution does not provide a compensation of the non-symmetric sinh terms in the p_1 component of the exponential.

In the following, the set of all wedges will be denoted by

$$\mathcal{W} := \mathcal{P}W_R = \{W_L + x \, : \, x \in \mathbb{R}^2\} \cup \{W_R + x \, : \, x \in \mathbb{R}^2\}, \tag{4.25}$$

with a Poincaré transformation $g \in \mathcal{P}$. The second equation holds true because W_R and W_R are invariant under the action of a boost $\Lambda(\theta)$ (2.1), since the eigenvectors are light like.

Definition 4.3. *A polarization-free generator G is a closed operator, satisfying the following conditions:*

a) *G is affiliated with a wedge algebra $\mathcal{A}(W)$ for $W \in \mathcal{W}$.*

b) *The vacuum vector Ω is contained in the domains of G and G^*.*

c) *The vectors $G\Omega$ and $G^*\Omega$ lie in the one-particle space.*

A polarization-free generator is called temperate if there exists a dense, translation invariant subspace $\mathcal{D}$ of its domain such that the functions $x \mapsto GU(x)\Psi$ are strongly continuous and polynomially bounded for $\Psi \in \mathcal{D}$, and the same holds true for its adjoint G^*.

So far we have developed fields for two-dimensional quantum field theories by means of inverse scattering methods. Now we are going to complete the so called Borchers triple $(\mathcal{A}, U, \mathcal{H})_S$ by defining a local net $W \to \mathcal{A}(W)$. With the representation of the proper orthochronous Poincaré group U (2.3) and the Hilbert space $\mathcal{H}$ (2.11), we already have two of the elements of the triple.

Definition 4.4. *We will call*

$$\mathcal{A}(W_L + x) := \{e^{i\phi(f)} \ : \ f \in \mathscr{S}(W_L + x) \otimes \mathcal{K} \, f = f^*\}'', \qquad (4.26a)$$

a left wedge algebra and

$$\mathcal{A}(W_R + x) := \{e^{i\phi'(f)} \ : \ f \in \mathscr{S}(W_R + x) \otimes \mathcal{K} \, f = f^*\}'', \qquad (4.26b)$$

a right wedge algebra. A local net is defined via the von Neumann algebras

$$W \mapsto \mathcal{A}(W), \qquad (4.27)$$

acting on the S symmetric Fock space $\mathcal{H}$.

Taking the fields as exponentials will be legitimated in Theorem 4.5 by showing that ϕ and ϕ' are temperate polarization-free generators. Moreover, Theorem 4.5 states that the constructed triple $(\mathcal{A}, U, \mathcal{H})_S$ has all the necessary properties the Haag Kastler axioms [HK1] demand.

Theorem 4.5. *Properties of the Borchers triple $(\mathcal{A}(W_R), U, \mathcal{H})$.*

a) *The map $W \mapsto \mathcal{A}(W)$ with $W \in \mathcal{W}$ is a local net of von Neumann algebras which transforms covariantly under the action of the representation U of the proper Poincaré group.*

b) *The triple $(\mathcal{A}(W), U, \mathcal{H})$ is a standard right wedge algebra, i.e. there holds*

    1) *U is strongly continuous and unitary, the joint spectrum is contained in the closed forward light cone, and there exists a unique (up to a phase) vector $\Omega \in \mathcal{H}$ which is invariant under the action of U.*

    2) *Ω is cyclic and separating for $\mathcal{A}$.*

    3) *For each $x \in \overline{W_R}$ the adjoint action of $U(x)$ induces endomorphisms on $\mathcal{A}$:*

$$\mathcal{A}(x) := U(x)\mathcal{A}U(x)^{-1} \subset \mathcal{A}, \qquad x \in \overline{W_R}. \qquad (4.28)$$

c) With respect to the net $W \to \mathcal{A}(W)$, the fields $\phi(f)$ and $\phi'(f)$ are temperate polarization-free generators in the sense of Definition 4.3, affiliated with $\mathcal{A}((W_L + supp\, f)'')$ and $\mathcal{A}((W_R + supp\, f)'')$, respectively.

Proof. We will only prove this for analytic vectors f.

a) Regarding the covariance properties (3.17) of ϕ, the exponentiated field operator transforms as $U(a, \lambda)e^{i\phi(f)}U(a, \lambda)^{-1} = e^{i\phi(f_{(a,\lambda)})}$. This can be seen by expanding the exponential

$$U(a, \lambda)e^{i\phi(f)}U(a, \lambda)^{-1} = U(a, \lambda)\sum_{k=0}^{\infty}\frac{i^k}{k!}\phi(f)^k U(a, \lambda)^{-1},$$

where in $U(a, \lambda)\phi(f)^k U(a, \lambda)^{-1}$ we can, for all k, insert $U(a, \lambda)^{-1}U(a, \lambda) = 1$, giving $U(a, \lambda)\phi(f)U(a, \lambda)^{-1}U(a, \lambda)\phi(f)U(a, \lambda)^{-1}...U(a, \lambda)\phi(f)U(a, \lambda)^{-1}$, hence $\phi(f_{(a,\lambda)})^k$. Moreover, for $f^* = f$, for the PCT reflection operator J the field transforms according to $J\phi(f)J = \phi'(f_{-1})$, $f_{-1}(x) = f(-x)$. The same holds true for $e^{i\phi'(f)}$, as it transforms covariantly as well, cf. Theorem 4.2. As the algebras $\mathcal{A}(W)$ are generated by $e^{i\phi(f)}$ and since for $g \in \mathcal{P}_+$, $supp(f_g) = g\,supp(f)$, they transform accordingly, i.e. $U(g)\mathcal{A}(W)U(g)^{-1} = \mathcal{A}(gW)$.

To show locality of the algebras we refer to Theorem 4.4 where we have proven the relative wedge-locality of ϕ and ϕ'. Here we need to extend this to the exponentiated fields. Therefore we apply Theorem 3.1 et seq. from [DF1] stating that for selfadjoint operators ϕ, ϕ' fulfilling

$$\|\phi(f)(1 + H)^{-1}\| < \infty \qquad \text{and} \qquad \|\phi'(g)(1 + H)^{-1}\| < \infty,$$

all bounded functions commute.

Bounds on ϕ and ϕ' were already given in Theorem 3.3 and Theorem 4.2 respectively. Regarding the Hamiltonian, we have the general bound $H \leq \sup_\alpha(m_\alpha \cdot N)$, hence $\|(1 + H)^{-1}\| < \infty$. This implies $\mathcal{A}(W_R) \subset \mathcal{A}(W_L)'$, and so locality of the net follows by covariance.

b) 1) has been shown in the proof of Lemma 2.3.

2) has been shown for $f \in \mathscr{S}(W)$ in [L1] and poses no difference to the present situation which is why we are brief about it. The argumentation there follows the arguments of [BW1]. For $f_1...f_n \in \mathscr{S}(W_L)$ and the spectral projection $E_k(t)$ of the selfadjoint operator $\phi(f_k)$ corresponding to spectral values in $[-t, t]$, we define $F_k(f) := E_k(t)\phi(f_k) \in \mathcal{A}(W_L)$ for all $t \in \mathbb{R}$. $F_k(t) \to \phi(f_k)$ strongly on the domain of ϕ as $t \to \infty$. Hence $F_1(t)...F_n(t)\Omega \to \phi(f_1)...\phi(f_n)\Omega$ as $t \to \infty$, and we conclude the cyclicity of Ω for $\mathcal{A}(W_L)$ from the cyclicity of Ω for ϕ from Theorem 3.3. The same argument can be applied to ϕ' as well, yielding the cyclicity of Ω for $\mathcal{A}(W_R)$. Moreover, as $\mathcal{A}(W_L)$ and $\mathcal{A}(W_R)$ commute, it follows that Ω is cyclic and separating for these algebras. By covariance of $\mathcal{A}$ and the invariance of Ω under U, this statement carries over to all wedge algebras.

3) has already been shown in a).

c) By the Theorem of Driessler and Fröhlich, already used for the proof of 1) it follows that $\phi(f)$ commutes with all $\mathcal{A}(W_R + x)$ if $W_R + x$ and $(W_L + supp\, f)''$ are space-like separated, i.e. $\phi(f)$ and its adjoint $\phi(f)^*$ are affiliated with $\mathcal{A}((W_L + supp\, f)'')$. By construction, $\phi(f)\Omega = f^+$ and $\phi(f)^*\Omega = f^{*+}$ are single particle states, implying that $\phi(f)$ is a polarization-free generator. Temperateness is seen as follows: The bounds on ϕ have been stated in

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Theorem 3.3 already, and the fact that $\mathcal{D}$ is dense in $\mathcal{H}$ follows from cyclicity of Ω from b)1). Finally, as the bounds in Theorem 4.2 hold for $\phi'(f)$ as well as for $\phi'(f)^*$ if f is selfadjoint, i.e. $C\overline{f} = f$, the same holds true for $\phi'(f)$.

$\square$

5 Scattering Theory

So far we understood the matrix-valued scattering function S as the defining quantity of the symmetry of the theory. In this chapter, the aim is to construct scattering states and formulate a scattering theory providing a *scattering matrix* $\hat{S}$ as well. As it turns out [L1] that in the scalar case (by "scalar" we mean $N = 1$, i.e. only one uncharged particle species), the scattering matrix is strictly determined by the scattering function. Our interest lies in proving this result for $N > 1$ as well.

The construction of scattering states will be done by employing Haag-Ruelle scattering theory [H2, R1, A2] in the special case of ϕ, ϕ' being polarization-free generators as worked out in [BBS1]. This will give us the necessary time evolution parameters to define asymptotic incoming and outgoing states. We will proceed analogously to [L1].

5.1 Two-particle scattering states

We start this chapter with the construction of two-particle states. This suggests itself as from the wedge-shaped localization regions of the so far defined fields ϕ and ϕ' it is clear that it is not possible to causally separate more than two wedges. If one shows the existence of such compactly loclized states, it is indeed possible to describe $(n \to m)$ scattering states for $n, m \in \mathbb{N}_0$. See the discussion in section 5.2 and [L2] for more details on that topic. For the definition of a time evolution of the fields it is necessary to introduce certain time-dependent momentum-space wave functions:

$$f_t^\alpha(x) := \frac{1}{2\pi} \int \tilde{f}^\alpha(p_0, p_1) e^{i(p_0 - \omega_{p,\alpha})t} e^{-ipx} \, d^2p, \qquad \omega_{p,\alpha} := (m_\alpha^2 + p_1^2)^{1/2}, \quad (5.1)$$

with $f \in \mathscr{S}(\mathbb{R}^2) \otimes \mathcal{K}$, $t \in \mathbb{R}$.

For describing the asymptotic behavior of the fields $\phi'(f_t), \phi(f_t)$ it is helpful to formally define the *velocity support V of a test function $f^\alpha \in \mathscr{S}(\mathbb{R}^2)$* as

$$V(f^\alpha) := \{(1, p_1/\omega_{p,\alpha}) \; : \; (p_0, p_1) \in \mathrm{supp}(\tilde{f}^\alpha)\}. \qquad (5.2)$$

As shown in [H3, BBS1], $\mathrm{supp}(f_t^\alpha)$ is essentially contained in $t\, V(f^\alpha)$ for $t \to \pm\infty$. This can be seen by defining the function

$$\chi(x) := \begin{cases} 1 & \text{for } x \in V(f^\alpha), \\ 0 & \text{for } x \notin \text{ a slightly larger region than } V(f^\alpha), \end{cases} \qquad (5.3)$$

and smooth in between. Then $\hat{f}_t^\alpha(x) := \chi(x/t) f_t^\alpha(x)$ is the asymptotically dominant part of f_t^α, i.e. $f_t^\alpha - \hat{f}_t^\alpha \to 0$ for $t \to \pm\infty$ in the topology of $\mathscr{S}(\mathbb{R}^2)$.

Via the velocity support (5.2) we can define the "precursor" called ordering relation

$$f^\alpha \prec g^\beta \qquad \text{if} \qquad V(g^\beta) - V(f^\alpha) \subset \{0\} \times (0, \infty), \qquad (5.4a)$$

as well as

$$f \prec g \qquad \text{if} \qquad f^\alpha \prec g^\beta \quad \forall \alpha, \beta \in \{1, ..., N\}. \qquad (5.4b)$$

For momentum space wave functions $\tilde{f}, \tilde{g}$ with disjoint support on all the mass shells $H_{m_\alpha}^+$ and $f \prec g$, the operators $\phi(f_t)$ and $\phi'(g_t)$ are essentially localized in $W_L + t\, V(f)$ and $W_R + t\, V(g)$, respectively. For $t \to +\infty$, the spatial distance of

the velocity supports of f_t and g_t increases. Thus, the outgoing two-particle state $(f^+ \times g^+)_{out}$ is defined via the limit

$$\lim_{t \to \infty} \phi(f_t)\phi'(g_t)\Omega =: (f^+ \times g^+)_{out}, \qquad f \prec g. \qquad (5.5)$$

Here, the symbol $\times$ introduced by R. Haag in [H1] is used in the sense of a product of incoming or outgoing particle *states*, in contrast to the symbol $\otimes$ defined as tensor product of *Hilbert spaces*, both being bilinear operations.

For the incoming state $(f \times g)_{in}$, the precursor relation between f_t and g_t has to be reversed since for $t \to -\infty$ the spatial distance of the velocity supports of f_t and g_t only then increases, as $\phi(g_t)$ and $\phi'(f_t)$ are essentially localized in $W_L + t V(g)$ and $W_R + t V(f)$, respectively. Hence we can define the incoming state

$$\lim_{t \to -\infty} \phi(f_t)\phi'(g_t)\Omega =: (f^+ \times g^+)_{in} \qquad g \prec f. \qquad (5.6)$$

These incoming and outgoing states are asymptotically symmetric in the limit. For smooth one-particle functions ψ_1, $\psi_2 \in \mathcal{H}_1$ with compact support and $\text{supp}(\psi_2^\beta) - \text{supp}(\psi_1^\alpha) \subset (0, \infty)$, we can find $f, g \in \mathscr{S}(\mathbb{R}^2) \otimes \mathcal{K}$ such that $f^+ = \psi_1$, $g^+ = \psi_2$ and with $V(f^\alpha)$, $V(g^\beta)$ replaced by $\text{supp}\Psi_1^\alpha$, $\text{supp}\Psi_2^\beta$ respectively, such that $f \prec g$. Therefore Definition (5.4a) will be used for wave functions ψ_1, $\psi_2 \in \mathcal{H}_1$ as well. The important result that the matrix-valued scattering function S introduced as a factor defining the symmetry of the theory can be interpreted as $(2 \to 2)$ *scattering matrix* is formulated in the following

Theorem 5.1. *Given a matrix-valued scattering function $\theta \mapsto S(\theta) \in \mathcal{B}(\mathcal{K} \otimes \mathcal{K})$, the associated quantum field theory defined in terms of the wedge local fields ϕ (3.2), ϕ' (4.2) and S is asymptotically complete at the two-particle level, i.e. there exist total sets of incoming and outgoing two-particle states in the subspace $\mathcal{H}_2 \subset \mathcal{H}$.*
These states have the explicit forms

$$(\psi_1 \times \psi_2)_{out} = \sqrt{2}P_2(\psi_1 \otimes \psi_2), \qquad \psi_1 \prec \psi_2, \qquad (5.7a)$$

$$(\psi_1 \times \psi_2)_{in} = \sqrt{2}P_2(\psi_2 \otimes \psi_1), \qquad \psi_1 \prec \psi_2. \qquad (5.7b)$$

The two-particle S-matrix $S_{2,2}$ is given by the underlying matrix-valued scattering function.

$$[S_{2,2}\Psi_2^+]^{\alpha_1\alpha_2}(\theta_1, \theta_2) = \begin{cases} S_{\beta_2\beta_1}^{\alpha_2\alpha_1}(\theta_2 - \theta_1) & : \quad \theta_1 \leq \theta_2 \\ S_{\beta_1\beta_2}^{\alpha_1\alpha_2}(\theta_1 - \theta_2) & : \quad \theta_1 > \theta_2 \end{cases} \Psi_2^{+\beta_1\beta_2}(\theta_1, \theta_2), \quad \Psi_2^+ \in \mathcal{H}_2^+.$$

$$(5.8)$$

The interpretation of $\mathcal{H}_2$ as "two-particle space" is justified by Theorem 5.1 because any $\Psi_2 \in \mathcal{H}_2$ can be written as a superposition of incoming and outgoing two-particle collision states.

Proof. As f is chosen such that the support of $\tilde{f}^\alpha$ does not intersect with the lower mass shell $H_{m_\alpha}^-$ for all α, we have $f_t^- = 0$, $f = f^*$, $f_t - \hat{f}_t \to 0$ in $\mathscr{S}(\mathbb{R}^2) \otimes \mathcal{K}$ as $t \to \pm\infty$ and from the bounds in (3.15) on f it follows that $f \mapsto \phi(f)\Psi$ is a vector valued distribution. Hence we have

$$\phi(f)\Psi = \phi(f_t)\Psi = \lim_{t \to \infty} \phi(\hat{f}_t)\Psi, \qquad \Psi \in \mathcal{D}. \qquad (5.9)$$

Due to the strong convergence $\phi(\hat{f}_t)\Psi \to \phi(f)\Psi$ for $t \to \infty$ the time dependence drops out and the incoming and outgoing states are

$$(f^+ \times g^+)_{in} = \lim_{t \to -\infty} \phi(g_t)\phi'(f_t)\Omega = \lim_{t \to -\infty} z^\dagger(g_t^+)z^\dagger(f_t^+)'\Omega = \sqrt{2}P_2(g^+ \otimes f^+),$$
$$f \prec g,$$
$$(f^+ \times g^+)_{out} = \lim_{t \to +\infty} \phi(f_t)\phi'(g_t)\Omega = \lim_{t \to +\infty} z^\dagger(t_t^+)z^\dagger(g_t^+)'\Omega = \sqrt{2}P_2(f^+ \otimes g^+).$$
$$(5.10a)$$

Varying f and g within the limitations given by the claimed support properties and the condition $f \prec g$, we obtain total sets of incoming and outgoing two-particle states on $\mathcal{H}_1 \otimes \mathcal{H}_1$.

Moreover, every smooth function with compact support in $R := \{(\theta_1, \theta_2) : \theta_1 \leq \theta_2\}$ can be approximated by linear combinations of functions of the form $\psi_1^\alpha \otimes \psi_2^\beta$ with $\psi_1^\alpha \prec \psi_2^\beta$. But the projection $P_2 : L^2(R, d\theta_1 d\theta_2) \otimes \mathcal{K} \to \mathcal{H}_2$ is linear, continuous and onto, implying that the scattering states (5.7a) both form total sets in $\mathcal{H}_2$.

For the two-particle scattering matrix $S_{2,2}$, we start by constructing Møller operators $V_{in}^{(2)}, V_{out}^{(2)} : \mathcal{H}_2^+ \to \mathcal{H}_2$, $S_{2,2} := V_{out}^{(2)*}V_{in}^{(2)} : \mathcal{H}_2^+ \to \mathcal{H}_2^+$. $\mathcal{H}_2^+$ denotes the symmetric two-particle subspace of the Bose-Fock space $\mathcal{H}^+$ over $L^2(\mathbb{R}) \otimes \mathcal{K}$, (i.e. $S = \mathbb{1}$).

The bosonic scattering states (5.7a) are represented in $\mathcal{H}_2^+$ by the vectors (cf. [H1, Ch.II, Sec. 3])

$$V_{out}^{(2)*}(\psi_1 \times \psi_2)_{out} = \sqrt{2}P_2^+(\psi_1 \otimes \psi_2), \qquad \psi_1 \prec \psi_2,$$
$$V_{in}^{(2)*}(\psi_1 \times \psi_2)_{in} = \sqrt{2}P_2^+(\psi_1 \otimes \psi_2), \qquad \psi_2 \prec \psi_1.$$

Plugging (5.7a) into the left hand side, and as on $\mathcal{H}_2^+$ $S = \mathbb{1}$, $V_{out}^{(2)*}$ acts according to

$$V_{out}^{(2)*}(\psi_1 \otimes \psi_2 + D_2(\tau_1)(\psi_1 \otimes \psi_2)) = (\psi_1 \otimes \psi_2 + \psi_2 \otimes \psi_1), \qquad \psi_1 \prec \psi_2.$$

Writing out this action in index notation gives, $\theta_1 > \theta_2$

$$(V_{out}^{(2)*})_{\beta_1\beta_2}^{\alpha_1\alpha_2}(\theta_1, \theta_2) \, (D_2(\tau_1)(\psi_1 \otimes \psi_2))^{\beta_1\beta_2}(\theta_1, \theta_2) = (\psi_2 \otimes \psi_1)^{\alpha_1\alpha_2}(\theta_1, \theta_2)$$

$$(V_{out}^{(2)*})_{\beta_1\beta_2}^{\alpha_1\alpha_2}(\theta_1, \theta_2) \, S_{\gamma_2\gamma_1}^{\beta_2\beta_1}(\theta_2 - \theta_1)\psi_2^{\gamma_2}(\theta_1)\psi_1^{\gamma_1}(\theta_2) = \psi_2^{\alpha_1}(\theta_1)\psi_1^{\alpha_2}(\theta_2).$$

Therefore, and from the support properties of $\psi_1 \prec \psi_2$ as seen above, we deduce that $V_{out}^{(2)*}$ is a multiplication operator, multiplying with

$$V_{out}^{(2)*}(\theta_1, \theta_2) = \begin{cases} S(\theta_1 - \theta_2) & : \quad \theta_1 > \theta_2 \\ \mathbb{1} & : \quad \theta_1 \leq \theta_2, \end{cases} \qquad (5.11)$$

where we have used $S(\theta)^{-1} = F\, S(-\theta)F$ (2.5a). In index notation, we have $(V_{out}^{(2)*})_{\beta_1\beta_2}^{\alpha_1\alpha_2}(\theta_1, \theta_2) = (S^{-1}(\theta_2 - \theta_1))_{\beta_1\beta_2}^{\alpha_1\alpha_2} = S_{\alpha_2\alpha_1}^{\beta_2\beta_1}(\theta_1 - \theta_2)$. Analogously, from the action of $V_{in}^{(2)*}$

$$V_{in}^{(2)*}(\psi_2 \otimes \psi_1 + D_2(\tau_1)(\psi_2 \otimes \psi_1)) = (\psi_2 \otimes \psi_1 + \psi_1 \otimes \psi_2), \qquad \psi_1 \prec \psi_2, \quad (5.12)$$

we infer the action of $V_{in}^{*(2)}$ to be given by

$$V_{in}^{*(2)}(\theta_1, \theta_2) = \begin{cases} S(\theta_1 - \theta_2) & : \quad \theta_1 \leq \theta_2 \\ \mathbb{1} & : \quad \theta_1 > \theta_2. \end{cases} \qquad (5.13)$$

Here we have used $S(\theta)^* = S(\theta)^{-1}$ (2.5a) (twice), first to take the inverse and then adjoin $V_{in}^{(2)*}$ to get $V_{in}^{(2)}$. Combining (5.13) and (5.11), we see that the two-particle S-matrix $S_{2,2} = V_{out}^{(2)*} V_{in}^{(2)}$ is given by the symmetric matrix-valued function $(\theta_1, \theta_2) \mapsto S(|\theta_1 - \theta_2|)$, $\theta_1, \theta_2 \in \mathbb{R}$

$$[S_{2,2}\Psi_2^+]^{\alpha_1\alpha_2}(\theta_1, \theta_2) = \begin{cases} S_{\beta_2\beta_1}^{\alpha_2\alpha_1}(\theta_2 - \theta_1) & : \ \theta_1 \leq \theta_2 \\ S_{\beta_1\beta_2}^{\alpha_1\alpha_2}(\theta_1 - \theta_2) & : \ \theta_1 > \theta_2 \end{cases} \Psi_2^{+\beta_1\beta_2}(\theta_1, \theta_2), \quad \Psi_2^+ \in \mathcal{H}_2^+,$$

and it is determined by the matrix-valued scattering function defined at the beginning. $\qquad\square$

5.2   General scattering states

In general it is not possible to describe n-particle scattering states with the theory developed so far because of the wedge-localization of the fields ϕ and ϕ'. In order to have n space like separated fields, they have to be localized in double cones, that is regions in Minkowski space consisting of intersections of two overlapping opposite wedges, (cf. Definition 5.15). By proving the existence of observables compactly localized in double cones, it is possible to define multi-particle scattering states. Therefore one has to show that the wedge Algebras that generate the Algebra of fields localized in double cones have the so called split property [L1, Sec. 2.2]. An inclusion $\mathcal{M}_1 \subset \mathcal{M}_2$ of two von Neumann algebras $\mathcal{M}_1$, $\mathcal{M}_2$, is called split, if there exists an type I factor $\mathcal{N}$ such that $\mathcal{M}_1 \subset \mathcal{N} \subset \mathcal{M}_2$
Parts of the result of Theorem 5.1 can be generalized to n-particle scattering states if the domain of localization of the fields can be narrowed into double cones. The existence of such fields is not shown here, but could probably be achieved by proving the wedge algebra to obey the modular nuclearity condition [L2].

Recalling Section 2.2, where in equation (2.9) we defined $P_n = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} D_n(\pi)$, we will call the scattering function for an arbitrary permutation π S_n^π.

$$[P_n\Psi]^{\underline{\alpha}}(\underline{\theta}) = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} [D_n(\pi)\Psi_n]^{\underline{\alpha}}(\underline{\theta}) = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} S_n^\pi(\underline{\theta})_{\underline{\beta}}^{\underline{\alpha}} \Psi_n^{\underline{\beta}}(\theta_{\pi(1)}, ..., \theta_{\pi(n)}). \ (5.14)$$

An explicit form of S_n^π would be nice to have, but as $\pi \in \mathfrak{S}_n$ is arbitrary, the summation indices in the product stated explicitly would be rather confusing. This lack of an explicit expression is the reason why we object from stating the $n \to n$ scattering matrix explicitly.

Definition 5.1. *We will call the intersection of two opposite wedges W_L, and W_R a double cone*

$$\mathcal{O} := (W_R + x) \cap (W_L + y), \qquad\qquad (5.15)$$

for some $x - y \in W_L$.

Theorem 5.2. *Let S be a matrix valued scattering function and assume that Ω is cyclic for a double cone algebra $\mathcal{A}(\mathcal{O}) := \mathcal{A}(W_R + x) \cap \mathcal{A}(W_L + y)$. Then the associated quantum field theory is asymptotically complete, i.e. there exist total sets of incoming and outgoing states in the subspace $\mathcal{H}_n \subset \mathcal{H}$, given by*

$$(\psi_1 \times ... \times \psi_n)_{out} = z^\dagger(\psi_1)...z^\dagger(\psi_n)\Omega, \qquad \psi_1 \prec ... \prec \psi_n, \qquad (5.16a)$$

$$(\psi_1 \times ... \times \psi_2)_{in} = z^\dagger(\psi_1)...z^\dagger(\psi_n)\Omega, \qquad \psi_n \prec ... \prec \psi_1. \qquad (5.16b)$$

The statement that the theory is asymptotically complete is a very strong one as asymptotic completeness has to be taken as a prerequisite in most theories and does not follow from initial assumptions.

Sketch of a proof:

Here we can build on the work we have already done in the proof of Theorem 5.1. In order to extend it to the case of n particles we chose the method of induction in n. For the induction hypothesis for $n = 1$, we have

$$\phi(f)\Omega = f^+ = f^+_{out} = f^+_{in},$$

since f^+ is a single-particle state. For the induction step from n to $n + 1$, let $A_1, ..., A_n \in \mathcal{A}(\mathcal{O})$ be operators localized in $\mathcal{O}$. We aim at establishing commutation relations between $\phi(f)$ and creation operators $A_k(g_k)_{out}$, where $f \prec g_1 \prec ... \prec g_k$. From equation (5.9) we obtain the form of the product of n creation operators with a field $\phi(f)$:

$$\langle \phi(f)^* \Psi, (A_1(g_1)\Omega \times ... \times A_n(g_n)\Omega)_{out} \rangle = \lim_{t \to \infty} \langle \phi(\hat{\bar{f}}_t^*)\Psi, A_1(\hat{g}_{1,t})\Omega ... A_n(\hat{g}_{n,t})\Omega \rangle.$$

Analogously to the case of $n = 2$, $\hat{f}_t^\alpha$ and $\hat{g}_{k,t}^\beta$ shall have for $t \to \pm\infty$ supports for all α, β, in small neighborhoods of $tV(f^\alpha)$ and $tV(g_k^\beta)$ respectively. Hence for large enough $t > 0$, these regions are spacelike separated and their distance increases linearly with t. As $\phi(\hat{\bar{f}}_t)$ is affiliated with $\mathcal{A}(W_L(f_t))$, it follows that this operator commutes with $A_k(\hat{g}_{k,t})$, $k = 1, ..., n$. Hence

$$\langle \phi(\bar{f})\Psi, (A_1(g_1)\Omega \times ... \times A_n(g_n)\Omega)_{out} \rangle = \lim_{t \to \infty} \langle \Psi, A_1(\hat{g}_{1,t}) ... A_n(\hat{g}_{n,t})\phi(\hat{f}_t)\Omega \rangle$$

$$= \lim_{t \to \infty} \langle \Psi, A_1(\hat{g}_{1,t}) ... A_n(\hat{g}_{n,t})\hat{f}_t^+ \rangle,$$

with $\hat{f}_t^+ \to f^+$, and $\hat{A}_{k,t} \to A_k$ by the arguments of [L2, eqn. (6.1.11) et seq.]

$$= \langle \Psi, A_1(g_1)_{out} ... A_n(g_n)_{out} f^+ \rangle$$

$$= \langle \Psi, (A_1(g_1)\Omega \times ... \times A_n(g_n)\Omega \times f^+)_{out} \rangle$$

$$= \langle \Psi, (f^+ \times A_1(g_1)\Omega \times ... \times A_n(g_n)\Omega)_{out} \rangle,$$

where in the last equation Bose symmetry of the scattering states has been used. By the Reeh-Schlieder property of $\mathcal{A}(\mathcal{O})$, f_k^+ can be approximated with $A_k(g_k)\Omega$, i.e. $\exists \epsilon$: $\| f_k^+ - A_k(g_k)\Omega \| < \epsilon$, $k = 1, ... n$. Hence

$$\langle \Psi(\bar{f})\Psi, (f_1^+ \times ... \times f_n^+)_{out} \rangle = \langle \Psi, (f^+ \times f_1^+ \times ... \times f_n^+)_{out} \rangle.$$

As $\Psi \in \mathcal{D}$ was arbitrary and $\mathcal{D} \subset \mathcal{H}$ dense, this implies via the induction hypothesis that for $f \prec f_1 \prec ... \prec f_n$

$$\phi(f)\phi(f_1)...\phi(f_n)\Omega = \phi(f_n)(f_1^+ \times ... \times f_n^+)_{out} = (f^+ \times f_1^+ \times ... \times f_n^+)_{out}.$$

In the case of incoming particle states the same argument holds true except for the order of the velocity supports of $f_1, ..., f_n$ since $W_L + tV(f_1)$ becomes spacelike separated from $\mathcal{O} + tV(f_k)$ for $t \to -\infty$ if $f_k \prec f$.

By continuity of (5.16a), and (5.16b) in $f_1^+, ..., f_n^+$, we get n-particle scattering states of the form

$$(\psi_1 \times ... \times \psi_n)_{out} = z^\dagger(\psi_1)...z^\dagger(\psi_n)\Omega = \sqrt{n!}P_n(\psi_1 \otimes ... \otimes \psi_n), \quad \psi_1 \prec ... \prec \psi_n,$$
$$(5.17a)$$

$$(\psi_1 \times ... \times \psi_n)_{in} = z^\dagger(\psi_n)...z^\dagger(\psi_1)\Omega = \sqrt{n!}P_n(\psi_n \otimes ... \otimes \psi_1), \quad \psi_1 \prec ... \prec \psi_n.$$
$$(5.17b)$$

To prove that both incoming and outgoing states form total sets in $\mathcal{H}$, we first note that the functions $\psi_1 \otimes ... \otimes \psi_n$, $\psi_1 \prec ... \prec \psi_n$, form a total set in $L^2(E_n, d^n\underline{\theta}) \otimes \mathcal{K}^{\otimes n}$ with $E_n = \{(\theta_1, ..., \theta_n) \in \mathbb{R}^n : \theta_1 \leq ... \leq \theta_n\}$. Since $P_n : L^2(E_n, d^n\underline{\theta})^{\otimes n} \to \mathcal{H}_n$ is linear, continuous and surjective, totality of the constructed outgoing state follows. To show that the incoming states form a total set as well, the same arguments can be applied. Moreover, as $\mathcal{D}$ is dense in $\mathcal{H}$, and any vector $\Psi \in \mathcal{H}$ can be approximated by linear combinations of in- and outgoing states the theory is asymptotically complete.

It would be pleasing to have an analogous result as in the one-particle case [L1], where an explicit form for the general $n \to n$ scattering matrix S_n^π can be formulated, and the scattering matrix $\hat{S}$ reads $\hat{S}(\underline{\theta}) = \prod_{1 \leq l < k \leq n} S_2(|\theta_k - \theta_l|)$. We give here the necessary steps for such a proof as far as one gets.
Let $\mathcal{H}^+ = \bigoplus_{n=0}^\infty \mathcal{H}_n^+$ again be the symmetric Bose-Fock space over $\mathcal{H}_1 = L^2(\mathbb{R}, d\theta) \otimes \mathcal{K}$ spanned by the asymptotic states, and let P_n^+ be the orthogonal projection onto $\mathcal{H}_n^+$. From (5.17a) and (5.17b), we deduce that the Møller operators have the form

$$V_{out} P_n^+ (\psi_1 \otimes ... \otimes \psi_n) = P_n(\psi_1 \otimes ... \otimes \psi_n), \qquad \psi_1 \prec ... \prec \psi_n, \qquad (5.18a)$$

$$V_{in} P_n^+ (\psi_n \otimes ... \otimes \psi_1) = P_n(\psi_n \otimes ... \otimes \psi_1), \qquad \psi_1 \prec ... \prec \psi_n. \qquad (5.18b)$$

These equations determine V_{in} and V_{out} uniquely, as $\|P_n^+(\psi_1 \otimes ... \otimes \psi_n)\| = \|P_n(\psi_1 \otimes ... \otimes \psi_n)\| = \sqrt{n!}\|\psi_1\|...\|\psi_n\|$ are isometries, mapping $\mathcal{H}^+$ onto $\mathcal{H}$. Inserting (5.14) into (5.17a) yields

$$V_{out}^* \sum_{\pi \in \mathfrak{S}_n} (\psi_{\pi^{-1}(1)} \otimes ... \otimes \psi_{\pi^{-1}(n)}) = \sum_{\pi \in \mathfrak{S}_n} [S_n^\pi (\psi_{\pi^{-1}(1)} \otimes ... \otimes \psi_{\pi^{-1}(n)})],$$

where F^π acts as $F^\pi(e_1 \otimes ... \otimes e_n) := e_{\pi(1)} \otimes ... \otimes e_{\pi(n)}$. As above in the two-particle case, from the support properties of $\psi_1 \prec ... \prec \psi_n$, we see that the action of V_{out}^* is given by a multiplication with the tensor

$$V_{out}^{(n)*}(\theta_1, ..., \theta_n) = S^\pi(\theta_1, ..., \theta_n)^{-1}, \qquad \theta_{\pi(1)} \leq ... \leq \theta_{\pi(n)}. \qquad (5.19a)$$

Analogously, we have for V_{in}

$$V_{in}^{(n)}(\theta_1, ..., \theta_n) = S^{\pi\iota}(\theta_1, ..., \theta_n), \qquad \theta_{\pi(1)} \leq ... \leq \theta_{\pi(n)}, \qquad (5.19b)$$

where $\iota(k) := n - k + 1$ is the total inversion permutation due to the inverse order of rapidities in (5.19b). Therefore the $n \to n$ scattering matrix $\hat{S}_n = V_{out}^{*(n)} V_{in^{(n)}}$ also acts as a multiplication operator S_n on $\mathcal{H}_n^+$, according to (5.19a), (5.19b):

$$\hat{S}_n(\underline{\theta}) = S_n^\pi(\underline{\theta})^{-1} \cdot S_n^{\pi\iota}(\underline{\theta}), \qquad \theta_{\pi(1)} \leq ... \leq \theta_{\pi(n)}. \qquad (5.20)$$

From Definition 2.8, one can see that we can rewrite

$$S_n^{\pi\iota}(\underline{\theta})\Psi_n(\theta_{\pi\iota(1)}, ..., \theta_{\pi\iota(n)}) = [D_n(\pi\iota)\Psi_n](\underline{\theta}) = [D_n(\pi)D_n(\iota)\Psi_n](\underline{\theta})$$
$$= S_n^\pi(\underline{\theta}) S_n^\iota(\theta_{\pi(1)}, ..., \theta_{\pi(n)})\Psi_n(\theta_{\pi\iota(1)}, ..., \theta_{\pi\iota(n)}).$$

Therefore, equation (5.20) can be reduced to, $\theta_{\pi(1)} \leq ... \leq \theta_{\pi(n)}$,

$$\hat{S}_n(\underline{\theta}) = S_n^\pi(\theta_1, ..., \theta_n)^{-1} \cdot S_n^{\pi\iota}(\theta_1, ..., \theta_n) = S^\iota(\theta_{\pi(1)}, ..., \theta_{\pi(n)}). \qquad (5.21)$$

In order to give $\hat{S}_n$ for arbitrary n, we would need to compute the product of $n(n+1)/2$ scattering functions. This would give a rather confusing index appellation. Instead we state

$$\hat{S}_3(\theta_1, \theta_2, \theta_3)^{\alpha_3 \alpha_2 \alpha_1}_{\beta_3 \beta_2 \beta_1} = S^{\gamma_2 \gamma_1}_{\alpha_2 \alpha_1}(\theta_{21}) \, S^{\gamma_3 \beta_1}_{\alpha_3 \gamma_1}(\theta_{31}) \, S^{\beta_3 \beta_2}_{\gamma_3 \gamma_2}(\theta_{32}), \qquad (5.22)$$

with $\theta_1 \leq \theta_2 \leq \theta_3$, $\theta_{ij} = \theta_i - \theta_j$, which is the same expression as in the Yang-Baxter equation (A.6).

6 Examples of scattering functions

In the following we will give examples of scattering matrices from various sources [AA1, J1, L2, LM1, ZA1] and show that they comply with the constraints we imposed on the scattering function in Definition 2.1.

6.1 Scalar scattering functions

The case of dim $\mathcal{K} = N = 1$ and $\overline{\alpha} = \alpha$, i.e. of only one uncharged particle species, is the setting worked out in [L2]. There, it is possible to state the general form of the class of scattering functions $\mathcal{S}_0$ (not being matrix-valued due to $N = 1$), and the multi-particle scattering matrix explicitly.

$$\mathcal{S}_0 = \left\{ \xi \mapsto \epsilon \cdot e^{ia \sinh \xi} \cdot \prod_k \frac{\sinh \beta_k - \sinh \xi}{\sinh \beta_k + \sinh \xi} \ : \ \epsilon = \pm 1, a \geq 0, \{\beta_k\} \in \mathscr{Z} \right\}, \quad (6.1)$$

where the family $\mathscr{Z}$ consists of the finite or infinite sequences $\{\beta_k\} \subset \mathbb{C}$ satisfying the following conditions:

i) $0 < \text{Im}\beta_k \leq \pi/2$,

ii) β_k and $-\overline{\beta}_k$ appear the same (finite) number of times in the sequence $\{\beta_k\}$,

iii) $\{\beta_k\}$ has no finite limit point,

iv) $\sum_k \text{Im}\frac{1}{\sinh \beta_k} < \infty$.

The $2 \to 2$ scattering matrix $\hat{S}_{2,2} : \mathcal{H}_2^+ \to \mathcal{H}_2^+$ acts as a multiplication operator and has the form

$$[\hat{S}_{2,2}\Psi_2](\theta_1, \theta_2) = S(|\theta_1 - \theta_2|) \cdot \Psi_2(\theta_1, \theta_2), \qquad \Psi_2^+ \in \mathcal{H}_2^+. \qquad (6.2)$$

It is even possible to prove the existence of $n \to n$ asymptotic states [L2] by restricting the allowed set of scattering functions, which is achieved by claiming that they must be bounded and analytic in a larger region $\overline{S(-\kappa, \pi + \kappa)}$, compare the remarks in the beginning of section 5.2. Here $\kappa(S) := \inf \{\text{Im}\,\xi \ : \ \xi \in S(0, \pi/2), \quad S(\xi) = 0\} > 0$. The now smaller family of so called regular scattering functions has the form

$$S(\xi) = \pm \prod_{k=1}^M \frac{\sinh \beta_k - \sinh \xi}{\sinh \beta_k + \sinh \xi}, \qquad 0 < \text{Im}\beta_1, ..., \text{Im}\beta_M \leq \frac{\pi}{2}.$$

The $n \to n$ scattering matrix $S_{n,n} : \mathcal{H}_n^+ \to \mathcal{H}_n^+$ again acts as a multiplication operator, and has the form

$$[\hat{S}_{n,n}\Psi_n^+](\theta_1, ..., \theta_n) = \prod_{1 \leq l < k \leq n} S(|\theta_l - \theta_k|) \cdot \Psi_n^+(\theta_1, ..., \theta_n). \qquad (6.3)$$

We refrain from proving that this class of scattering functions obeys Definition 2.1 because for the whole class it is a rather lengthy task and we are focusing on matrix-valued scattering functions in this work. However, in [L1] this is worked out thoroughly.

6.2 Diagonal examples

A class of non-scalar solutions to the constraints originating from Toda systems can be found in [J1]. These solutions also fit into our framework, but are tedious to compute explicitly, as there the work is performed in a purely algebraic setting and the Lie algebraic solution spaces have a high number of dimensions.

A less high dimensional but similar solution is presented in the following, inspired by the one given in [LM1, page 293 eqn. (51)].

The setting of the model is given as follows: $N \in \mathbb{N}_0$, a neutral $(\overline{\alpha} = \alpha)$ particle spectrum $m_\alpha = m$, and functions $\sigma_{\alpha\beta}$: $\overline{S(0, \pi)} = \{\xi \in \mathbb{C} \ : \ 0 < \mathrm{Im}(\xi) < \pi\} \to \mathbb{C}$ bounded on $\overline{S(0, \pi)}$ and analytic in $S(0, \pi)$, and obeying, $\theta \in \mathbb{R}$

$$|\sigma_{\alpha\beta}(\theta)|^2 = 1 \tag{6.4a}$$

$$\sigma_{\alpha\beta}(\theta) = \sigma_{\beta\alpha}(i\pi - \theta) \tag{6.4b}$$

$$\overline{\sigma_{\alpha\beta}(\theta)} = \sigma_{\alpha\beta}(-\theta). \tag{6.4c}$$

The scattering matrix is given by

$$S_{LM}(\theta)^{\alpha\beta}_{\gamma\epsilon} = \sigma_{\alpha\beta}(\theta)\delta^{\alpha\gamma}\delta^{\beta\epsilon}. \tag{6.5}$$

Condition (6.4a) ensures unitarity, and condition (6.4b) (added by us) crossing symmetry. The explicit form of $\sigma_{\alpha\beta}$ is not given in [LM1].

The S-matrix (6.5) fulfills the assumptions of Definition 2.1 in this setting, as we will show now.
The Yang-Baxter equation (2.5c) is clearly fulfilled by $\delta^\alpha_\gamma \delta^\beta_\epsilon$, as can be seen in detail in the next section 6.3.
Crossing symmetry (2.5d) which for this model is given by,

$$\sigma_{\alpha\beta}(\theta)\delta^{\alpha\gamma}\delta^{\beta\epsilon} = \sigma_{\beta\alpha}(i\pi - \theta)\,\delta^{\beta\epsilon}\delta^{\alpha\gamma}$$
$$= \delta^{\beta\epsilon}\delta^{\alpha\gamma},$$

hold by (6.4b), and unitarity

$$\sigma_{\alpha\beta}(\theta)\,\delta^{\alpha\xi}\delta^{\beta\zeta}\,\overline{\sigma_{\gamma\epsilon}(\theta)}\,\delta^{\gamma\xi}\delta^{\epsilon\zeta} = \sigma_{\alpha\beta}(\theta)\,\overline{\sigma_{\alpha\beta}(\theta)}\,\delta^{\alpha\gamma}\delta^{\beta\epsilon}$$
$$\Leftrightarrow (6.4a),$$

as well as $S^* = FS(-\theta)F$ (2.5a)

$$\overline{\sigma_{\alpha\beta}(\theta)}\,\delta^{\alpha\gamma}\delta^{\beta\epsilon} = F^{\alpha\gamma}_{\tau\tau'}\,\sigma_{\tau\xi}(-\theta)\,\delta^{\tau\tau'}\delta^{\xi\xi'}\,F^{\xi\xi'}_{\beta\epsilon}$$
$$= \sigma_{\alpha\beta}(-\theta)\,\delta^{\gamma\alpha}\delta^{\epsilon\beta} \Leftrightarrow (6.4c)$$

are fulfilled by (6.4a) – (6.4c).

The condition of $\sigma_{\alpha\beta}$ being bounded on $\overline{S(0, \pi)}$ is not easy to fulfill as the example

$$\sigma_{\alpha\beta}(\theta) = e^{a\sinh(\theta)}$$

shows. In view of crossing symmetry this special choice yields

$$|\sigma_{\alpha\beta}(i\eta - \theta)| = |e^{ia(\cos\eta\sinh\theta + i\sin\eta\cosh\theta)}| = e^{-a(\sin\eta\cosh\theta)},$$

which is not bounded for negative a.

The simplest example is to chose a scalar function σ of the form (6.1) in section 6.1, and put $\sigma_{\alpha\beta}(\theta) := \sigma(\theta)$ for all α, β. Then one gets $S(\theta) = \sigma(\theta) \cdot \mathbb{1}$.

6.3 Sigma models

The last model we discuss is an $O(N)$ (i.e. $(M^T \otimes M^T)S(\theta)(M \otimes M) = S(\theta)$, $M \in O(N)$) invariant (non-linear) Sigma model. In [ZA1], Zamolodchikov and Zamolodchikov state the following $O(N)$ invariant solution of the conditions they impose on S.

For a given number of neutral $(\overline{\alpha} = \alpha, \ \alpha = 1, ..., N)$ particle species $N > 2$, all having the same mass $m_\alpha = m$, the scattering matrix is given by

$$S_Z(\theta)^{c_1 c_2}_{d_1 d_2} := \sigma_1(\theta)\delta^{c_1 c_2}\delta^{d_1 d_2} + \sigma_2(\theta)\delta^{c_1 d_1}\delta^{c_2 d_2} + \sigma_3(\theta)\delta^{c_1 d_2}\delta^{d_1 c_2}, \qquad (6.6)$$

where the upper or lower position of indices at Kronecker delta symbols is of no formal relevance, and the functions σ_1, σ_2 and σ_3 are defined as

$$\sigma_2(\theta) := Q(\theta)Q(i\pi - \theta), \quad \text{with} \quad Q(\theta) := \frac{\Gamma(\frac{1}{N-2} - i\frac{\theta}{2\pi})\Gamma(\frac{1}{2} - i\frac{\theta}{2\pi})}{\Gamma(\frac{1}{2} + \frac{1}{N-2} - i\frac{\theta}{2\pi})\Gamma(-i\frac{\theta}{2\pi})}, \quad (6.7a)$$

$$\sigma_1(\theta) := -\frac{i\lambda}{i\pi - \theta}\sigma_2(\theta), \qquad\qquad \lambda := \frac{2\pi}{N-2}, \qquad\qquad (6.7b)$$

$$\sigma_3(\theta) := \sigma_1(i\pi - \theta) = -\frac{i\lambda}{\theta}\sigma_2(\theta). \qquad\qquad (6.7c)$$

The conditions (except for $O(N)$ symmetry) Zamolodchikov and Zamolodchikov impose on S coincide with (2.5a, 2.5c, 2.5d, 2.5e) albeit being motivated by a different, more heuristic approach to scattering theory.
We regard this as a nice aspect of the rigorous inverse scattering approach we undertake here. It shows that the rather mathematically motivated constraints on S we imposed (originating from the definition of the representation of the permutation group via S, crossing symmetry from the proof of relative wedge locality of ϕ and ϕ', and unitarity) can also be physically motivated in a more intuitive way.
We want to stress that this is a solution for the constraints imposed on S *and* $O(N)$ symmetry, and remark that Zamolodchikov and Zamolodchikov state a second solution, differing from the one above by a CCD pole, [AW1]

$$\sigma_2^{CCD}(\theta) = \frac{\sinh\theta + i\sin\lambda}{\sinh\theta - i\sin\lambda}\sigma_2(\theta).$$

But as this solution has poles in $S(0, \pi)$, it does not comply with our Definition 2.1 of S. For the case of S not being analytic in $S(0, \pi)$, see the discussion following the proof of Theorem 4.4.

Proposition 6.1. *S_Z, as defined above, is a matrix-valued scattering function according to Definition 2.1.*

Proof. We will omit the index Z and the argument θ of S in the calculation where confusion is unlikely to arise.
First we show unitarity (A.7):

$$[S(\theta)^*]^{c_1 c_2}_{d_1 d_2} S(\theta)^{d_1 d_2}_{k_1 k_2} = \overline{S(\theta)^{d_1 d_2}_{c_1 c_2}} S(\theta)^{d_1 d_2}_{k_1 k_2}$$

$$= \delta^{c_1 c_2}\delta^{k_1 k_2}(|\sigma_1|^2 N + \overline{\sigma_1}\sigma_2 + \overline{\sigma_1}\sigma_3 + \overline{\sigma_2}\sigma_1 + \overline{\sigma_3}\sigma_1)$$

$$+ \delta^{c_1 k_1}\delta^{c_2 k_2}(|\sigma_2^2| + |\sigma_3^2|)$$

$$+ \delta^{c_1 k_2}\delta^{c_2 k_1}(\overline{\sigma_2}\sigma_3 + \overline{\sigma_3}\sigma_2).$$

For this expression to be unitary it is necessary that $(|\sigma_1|^2 N + \overline{\sigma_1}\sigma_2 + \overline{\sigma_1}\sigma_3 + \overline{\sigma_2}\sigma_1 + \overline{\sigma_3}\sigma_1) = 0$, $(|\sigma_2|^2 + |\sigma_3|^2) = 1$, and $(\overline{\sigma_2}\sigma_3 + \overline{\sigma_3}\sigma_2) = 0$. This is checked by plugging in (6.7a)– (6.7b) where we make use of the general relation $\overline{\Gamma(z)} = \Gamma(\overline{z})$, $z \in \mathbb{C}$ for the Gamma function, providing the relation $\overline{\sigma_i(-\theta)} = \sigma_i(\overline{\theta})$, $\theta \in S(0, \pi)$, for $i = 1, 2, 3$.

$$|\sigma_2|^2 + |\sigma_3|^2 = 1 \Longleftrightarrow |\sigma_2|^2 = \frac{1}{1 + |\frac{2i\pi}{N-2}|^2 \frac{1}{\theta^2}} = \frac{\theta^2}{\theta^2 + \lambda^2},$$

$$|\sigma_1|^2 N + \overline{\sigma_1}\sigma_2 + \overline{\sigma_2}\sigma_1 + \overline{\sigma_1}\sigma_3 + \overline{\sigma_3}\sigma_1$$
$$= |\sigma_2|^2 \left(N - \frac{i\pi - \theta}{\lambda} + \frac{-i\pi - \theta}{\lambda} + \frac{i\pi - \theta}{\theta} + \frac{-i\pi - \theta}{\theta} \right)$$
$$= |\sigma_2|^2 \left(N - \frac{2\pi}{\lambda} - 2 \right) = 0$$

$$\overline{\sigma_2}\sigma_3 + \overline{\sigma_3}\sigma_2 = |\sigma_2|^2 \left(\frac{2i\pi}{(N-2)\theta} - \frac{2i\pi}{(N-2)\theta} \right) = 0.$$

The first equation imposes a condition on σ_2 and the remaining two hold true independently of σ_2.

In the following we are going to check whether this condition is fulfilled by σ_2, where we will omit those functions in the product of $|Q(\theta)Q(i\pi - \theta)|^2$ that cancel out immediately.

$$|\sigma_2(\theta)|^2 = |Q(\theta)Q(i\pi - \theta)|^2$$
$$= \frac{\Gamma(+\frac{\lambda}{2\pi} - \frac{i\theta}{2\pi})\Gamma(1 + \frac{i\theta}{2\pi})\Gamma(+\frac{\lambda}{2\pi} + \frac{i\theta}{2\pi})\Gamma(1 - \frac{i\theta}{2\pi})}{\Gamma(-\frac{i\theta}{2\pi})\Gamma(1 + \frac{\lambda}{2\pi} + \frac{i\theta}{2\pi})\Gamma(\frac{i\theta}{2\pi})\Gamma(1 + \frac{\lambda}{2\pi} - \frac{i\theta}{2\pi})}$$
$$= \frac{\left(\frac{i\theta}{2\pi}\right)\left(-\frac{i\theta}{2\pi}\right)}{\left(\frac{\lambda}{2\pi} + \frac{i\theta}{2\pi}\right)\left(\frac{\lambda}{2\pi} - \frac{i\theta}{2\pi}\right)} = \frac{\theta^2}{\theta^2 + \lambda^2}.$$

Condition (2.5b), $S_{\overline{\beta_1}\overline{\beta_2}}^{\overline{\alpha_1}\overline{\alpha_2}}(\theta) = S_{\alpha_1\alpha_2}^{\beta_1\beta_2}(\theta)$ is fulfilled as S_Z is defined via Kronecker delta symbols always flipping both upper and lower indices symmetrically, and the fact that $\overline{\alpha} = \alpha$ in this model. The same holds true for the second equation in (2.5a): S_Z is invariant under a left-right and up-down flip of indices, and from $\overline{\Gamma(z)} = \Gamma(\overline{z})$ we get $\overline{\sigma_2(\theta)} = \sigma_2(-\overline{\theta})$.

Now to the Yang-Baxter equation: The upper line of (A.6) reads

$$S_{\alpha_2\alpha_1}^{\gamma_2\gamma_1}(\theta_{ij}) \, S_{\alpha_3\gamma_1}^{\gamma_3\beta_1}(\theta_{ik}) \, S_{\gamma_3\gamma_2}^{\beta_3\beta_2}(\theta_{jk})$$
$$= [\sigma_1(\theta_{ij})\delta^{\gamma_1\gamma_2}\delta^{\alpha_1\alpha_2} + \sigma_2(\theta_{ij})\delta^{\gamma_1\alpha_1}\delta^{\gamma_2\alpha_2} + \sigma_3(\theta_{ij})\delta^{\gamma_1\alpha_2}\delta^{\alpha_1\gamma_2}]$$
$$\times [\sigma_1(\theta_{ik})\delta^{\beta_1\gamma_3}\delta^{\gamma_1\alpha_3} + \sigma_2(\theta_{ik})\delta^{\beta_1\gamma_1}\delta^{\gamma_3\alpha_3} + \sigma_3(\theta_{ik})\delta^{\beta_1\alpha_3}\delta^{\gamma_1\gamma_3}]$$
$$\times [\sigma_1(\theta_{jk})\delta^{\beta_2\beta_3}\delta^{\gamma_2\gamma_3} + \sigma_2(\theta_{jk})\delta^{\beta_2\gamma_2}\delta^{\beta_3\gamma_3} + \sigma_3(\theta_{jk})\delta^{\beta_2\gamma_3}\delta^{\gamma_2\beta_3}],$$

and the lower line

$$S_{\alpha_3\alpha_2}^{\gamma_3\gamma_2}(\theta_{jk}) \, S_{\gamma_3\alpha_1}^{\beta_3\gamma_1}(\theta_{ik}) \, S_{\gamma_2\gamma_1}^{\beta_2\beta_1}(\theta_{ij})$$
$$= [\sigma_1(\theta_{jk})\delta^{\gamma_3\gamma_2}\delta^{\alpha_3\alpha_2} + \sigma_2(\theta_{jk})\delta^{\alpha_2\gamma_2}\delta^{\alpha_3\gamma_3} + \sigma_3(\theta_{jk})\delta^{\gamma_3\alpha_2}\delta^{\gamma_2\alpha_3}]$$
$$\times [\sigma_1(\theta_{ik})\delta^{\beta_3\gamma_1}\delta^{\gamma_3\alpha_1} + \sigma_2(\theta_{ik})\delta^{\beta_3\gamma_3}\delta^{\alpha_1\gamma_3} + \sigma_3(\theta_{ik})\delta^{\alpha_1\beta_1}\delta^{\gamma_1\alpha_3}]$$
$$\times [\sigma_1(\theta_{ij})\delta^{\beta_2\beta_1}\delta^{\gamma_2\gamma_1} + \sigma_2(\theta_{ij})\delta^{\beta_1\gamma_1}\delta^{\gamma_2\beta_2} + \sigma_3(\theta_{ij})\delta^{\beta_2\gamma_1}\delta^{\beta_1\gamma_2}].$$

Multiplying out these two equations gives 15 terms (for 15 possible, non-identical permutations of the indices on the delta symbols) each. Comparing coefficients

gives

$$\delta^{\alpha_1\beta_1}\delta^{\alpha_2\beta_3}\delta^{\alpha_3\beta_2}: \quad \sigma_2(\theta)\sigma_2(\theta-\theta')\sigma_3(\theta') = \sigma_3(\theta')\sigma_2(\theta-\theta')\sigma_2(\theta),$$

$$\delta^{\alpha_1\beta_1}\delta^{\alpha_2\beta_2}\delta^{\alpha_3\beta_3}: \quad \sigma_2\sigma_2\sigma_2 = \sigma_2\sigma_2\sigma_2,$$

$$\delta^{\alpha_1\beta_2}\delta^{\alpha_2\beta_1}\delta^{\alpha_3\beta_3}: \quad \sigma_3\sigma_2\sigma_2 = \sigma_2\sigma_2\sigma_3,$$

$$\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\beta_2}\delta^{\beta_1\beta_3}: \quad \sigma_2\sigma_1\sigma_2 = \sigma_2\sigma_1\sigma_2,$$

$$\delta^{\alpha_1\beta_2}\delta^{\alpha_2\alpha_3}\delta^{\beta_1\beta_3}: \quad \sigma_3\sigma_1\sigma_2 = \sigma_1\sigma_1\sigma_2 + \sigma_1\sigma_2\sigma_3, \qquad\qquad (a)$$

$$\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\beta_3}\delta^{\beta_1\beta_2}: \quad \sigma_2\sigma_1\sigma_3 = \sigma_2\sigma_1\sigma_1 + \sigma_3\sigma_2\sigma_1,$$

$$\delta^{\alpha_1\beta_3}\delta^{\alpha_2\beta_1}\delta^{\alpha_3\beta_2}: \quad \sigma_3\sigma_2\sigma_3 = \sigma_3\sigma_3\sigma_2 + \sigma_2\sigma_3\sigma_3, \qquad\qquad (6.8a)$$

$$\delta^{\alpha_1\alpha_2}\delta^{\alpha_3\beta_2}\delta^{\beta_1\beta_3}: \quad \sigma_1\sigma_1\sigma_2 + \sigma_1\sigma_2\sigma_3 = \sigma_3\sigma_1\sigma_2,$$

$$\delta^{\alpha_1\beta_2}\delta^{\alpha_2\beta_3}\delta^{\alpha_3\beta_1}: \quad \sigma_3\sigma_3\sigma_2 + \sigma_2\sigma_3\sigma_3 = \sigma_3\sigma_2\sigma_3,$$

$$\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\beta_1}\delta^{\beta_2\beta_3}: \quad \sigma_2\sigma_1\sigma_1 + \sigma_3\sigma_2\sigma_1 = \sigma_3\sigma_1\sigma_2, \qquad\qquad (6.8b)$$

$$\delta^{\alpha_1\beta_1}\delta^{\alpha_2\alpha_3}\delta^{\beta_2\beta_3}: \quad \sigma_3\sigma_1\sigma_1 + \sigma_2\sigma_2\sigma_1 = \sigma_1\sigma_1\sigma_3 + \sigma_1\sigma_3\sigma_3,$$

$$\delta^{\alpha_1\alpha_2}\delta^{\alpha_3\beta_3}\delta^{\beta_1\beta_2}: \quad \sigma_1\sigma_2\sigma_2 + \sigma_1\sigma_1\sigma_3 = \sigma_3\sigma_1\sigma_1 + \sigma_2\sigma_2\sigma_1,$$

$$\delta^{\alpha_1\beta_3}\delta^{\alpha_2\beta_2}\delta^{\alpha_3\beta_1}: \quad \sigma_3\sigma_3\sigma_3 + \sigma_2\sigma_3\sigma_2 = \sigma_2\sigma_3\sigma_2 + \sigma_3\sigma_3\sigma_3,$$

$$\delta^{\alpha_1\alpha_2}\delta^{\alpha_3\beta_1}\delta^{\beta_2\beta_3}: \quad \sigma_1\sigma_1\sigma_1 + \sigma_1\sigma_2\sigma_1 + N\sigma_1\sigma_3\sigma_1 + \sigma_2\sigma_3\sigma_1 + \sigma_3\sigma_3\sigma_1$$
$$+ \sigma_1\sigma_3\sigma_2 + \sigma_1\sigma_3\sigma_3 = \sigma_3\sigma_1\sigma_3,$$

$$\delta^{\alpha_1\beta_3}\delta^{\alpha_2\alpha_3}\delta^{\beta_1\beta_2}: \quad \sigma_3\sigma_1\sigma_3 = \sigma_1\sigma_1\sigma_1 + \sigma_1\sigma_2\sigma_1 + N\sigma_1\sigma_3\sigma_1 + \sigma_2\sigma_3\sigma_1$$
$$+ \sigma_3\sigma_3\sigma_1 + \sigma_1\sigma_3\sigma_2 + \sigma_1\sigma_3\sigma_3, \qquad\qquad (6.8c)$$

where from the second line on, the arguments of σ_i have been omitted, as they are the same in all lines. N (the number of different particle species) originates from summation over Kronecker delta symbols.

The first four equations are trivially fulfilled as the arguments of the left hand side of each function σ_i are the same as on the right hand side. Of the remaining eleven equations many are redundant except for (6.8a), (6.8b) and (6.8c). As an example we consider the equivalence of (a) to (6.8b):

The left side of (a) is a cyclic permutation of the right side of (6.8b) and hence they are equal. The same holds true for the reversed sides of (a) and (6.8b). In all other cases the situation is analogous.

For the verification of (6.8a), (6.8b) and (6.8c) we give the algebraic manipulations Zamolodchikov and Zamolodchikov have performed in [ZA1] to solve them, instead of just plugging in and multiplying out:

For (6.8a) it is convenient to introduce the ratio $h(\theta) = \sigma_2(\theta)/\sigma_3(\theta)$, giving

$$h(\theta) + h(\theta') = h(\theta + \theta'),$$

hence

$$\sigma_3(\theta) = -i\frac{\lambda}{\theta}\sigma_2(\theta), \qquad\qquad (6.9)$$

where λ is an arbitrary real parameter. Substitution of (6.9) into (6.8b) leads to

$$\rho(\theta + \theta')\rho(\theta') = \frac{i\lambda}{\theta}[\rho(\theta') - \rho(\theta - \theta')], \qquad\qquad (6.10)$$

where $\rho(\theta) = \sigma_1(\theta)/\sigma_2(\theta)$. The solution of (6.10) is

$$\rho(\theta) = -\frac{i\lambda}{i\kappa - \theta},$$

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where κ is another real parameter. Now equation (6.8c) leads to the restriction

$$\kappa = i\frac{N-2}{2}\lambda,$$

giving (6.7b) and (6.7c).

Concerning crossing symmetry, we plug (6.6) into (2.5d), giving

$$S_{d_1 d_2}^{c_1 c_2}(i\pi - \theta) = \sigma_3(\theta)\delta^{c_1 c_2}\delta^{d_1 d_2} + \sigma_2(\theta)\delta^{c_1 d_1}\delta^{c_2 d_2} + \sigma_1(\theta)\delta^{c_1 d_2}\delta^{d_1 c_2}$$

$$\overset{!}{=} S_{d_2 c_1}^{c_2 d_1}(\theta) = \sigma_1(\theta)\delta^{c_2 d_1}\delta^{d_2 c_1} + \sigma_2(\theta)\delta^{c_2 d_2}\delta^{d_1 c_1} + \sigma_3(\theta)\delta^{c_2 c_1}\delta^{d_1 d_2}.$$

If θ in (6.7b) is replaced by $i\pi - \theta$ it gives (6.7c), where crossing symmetry for $\sigma_2(\theta)$ can be seen from its Definition (6.7a).

What remains to be shown is analyticity in the open strip $S(0, \pi)$ $= \{\xi \in \mathbb{C} : 0 < \text{Im}(\xi) < \pi\}$, and boundedness on $\overline{S(0, \pi)}$.

The fractions appearing in the definitions of σ_1 and σ_3 are analytic for $\theta \neq i\pi$ and $\theta \neq 0$ respectively, as $N > 2$. As σ_2 is defined via the meromorphic[1] Gamma function it is analytic as well, as in $\{\xi \in \mathbb{C} : 0 < \text{Im}(\xi) < \pi\}$ the argument can not be zero or have vanishing imaginary part. This can be seen from investigating the poles of σ_2, which are poles of the Gamma functions in the numerator and zeros in the denominator. As the Gamma function has no zeros we only need to look at the argument of the two Gamma functions in the numerator by setting them to a negative integer $-k$, $k \in \mathbb{N}_0$,

$$\frac{1}{N-2} - i\frac{\theta}{2\pi} = -k \quad \Leftrightarrow \quad \theta = -2\pi i\left(\frac{1}{N-2} + k\right) \notin S(0, \pi),$$

$$\frac{1}{2} - i\frac{\theta}{2\pi} = -k \quad \Leftrightarrow \quad \theta = -2\pi i\left(\frac{1}{2} + k\right) \notin S(0, \pi).$$

The case $\theta \to i\pi - \theta$ does not give any poles as for $\theta \in S(0, \pi)$ as it is already covered by the preceding discussion by the symmetry of σ_2.

For the bounds on S_Z on $\overline{S(0, \pi)}$ we have to consider $\text{Im}\,\theta \in [0, \pi]$. In view of the crossing symmetry implemented in σ_2 it is sufficient to consider $Q(\theta)$. From [O1, eqns. 5.6.7, and 5.6.9] we have the following lower and upper bounds for the Gamma function $\Gamma(z)$, $z = x + iy$

$$(\text{sech}(\pi y))^{1/2}\,\Gamma(x) \leq |\Gamma(z)| \leq (2\pi)^{1/2}|z|^{x-1/2}e^{-\pi|y|/2}\exp\left(\frac{1}{6}|z|^{-1}\right), \qquad x \geq \frac{1}{2},$$

$$(6.11)$$

where $\text{sech}\,z = \frac{1}{\cosh z}$ is the hyperbolic secant, and the identity [O1, eqn. 5.4.3]

$$|\Gamma(iy)| = \sqrt{\frac{\pi}{y\sinh(\pi y)}}.$$

We now use the lower and upper bounds (6.11) to estimate the denominator and numerator of Q respectively, $x := 1/(N-2)$, $y := -\theta/(2\pi)$.

$$|Q(\theta)| = \left|\frac{\Gamma(x+iy)\Gamma(\frac{1}{2}+iy)}{\Gamma(\frac{1}{2}+x+iy)\Gamma(iy)}\right|$$

$$\leq 2\pi\frac{(x^2+y^2)^{x/2-1/4}\exp(-\frac{\pi|y|}{2} + \frac{1}{6}(x^2+y^2)^{-1/2})\,\exp(-\frac{\pi|y|}{2} + \frac{1}{6}(\frac{1}{4}+y^2)^{-1/2})}{\sqrt{\text{sech}(\pi y)}\Gamma(\frac{1}{2}+x)\sqrt{\frac{\pi}{y\sinh(\pi y)}}}.$$

$$(6.12)$$

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[1] i.e. holomorphic except in a countable set of points which are the non-positive integers.

Using the lower bound [O1, eqn. 5.6.1] $\Gamma(x) > \sqrt{2\pi}\,x^{x-\frac{1}{2}}\,e^{-x}$ for $\Gamma(\frac{1}{2}+x)$ in the denominator of (6.12) gives for $|Q(\theta)|$ the upper bound

$$\sqrt{2\pi e}\,\frac{(x^2+y^2)^{x/2-1/4}\exp(-\pi|y|+\frac{1}{6}(x^2+y^2)^{-1/2}+\frac{1}{6}(\frac{1}{4}+y^2)^{-1/2})\sqrt{y\sinh(\pi y)}}{\sqrt{\operatorname{sech}(\pi y)}\,(\frac{1}{2}+x)^x\,e^{-\frac{1}{2}-x}\,\sqrt{\pi}}.$$

(6.13)

With $\sqrt{\frac{\sinh(\pi y)}{\pi\operatorname{sech}(\pi y)}} = \frac{\sqrt{\sinh(2\pi y)}}{\sqrt{2\pi}}$, and estimating the exponential term in the numerator $\leq e^{3/2}$, we have

$$|Q(\theta)| \leq \sqrt{\frac{e}{2\pi}}\,\frac{(x^2+y^2)^{x-1/2}\,\sqrt{\sinh(2\pi y)}\sqrt{y}}{(\frac{1}{2}+x)^x}\,e^{3/2}.$$

Now, regarding the parameter N we observe that $x = 1/(N-2)$ has its largest value for small N, i.e $N = 3$ and vice versa $x \to 0$ as $N \to \infty$. As we are interested in bounds for fixed N we use this fact, setting $x = 1$ in the numerator, and $x = 0$ in the denominator, giving

$$|Q(\theta)| \leq \frac{e^2}{\sqrt{2\pi}}\,(1+y^2)^{1/4}\,\sqrt{\sinh(2\pi y)}\sqrt{y},$$

(6.14)

where we have used that $(\frac{1}{2}+x)^x \to 1$ as $x \to 0$. This expression is monotonously increasing proportional to $y^{3/2}$. It would be sufficient to be polynomially bounded, since by the maximum principle (cf. [R1, Thm. 12.9]) a function
$|f(z)| < \exp(A\,e^{\alpha|x|})$ with $\alpha < 1$, $A < \infty$, and continuous on a strip $\overline{S(0,\pi)} = \{x+iy : y \leq \pi\}$ that is bounded on the closure of that strip by $|f(x \pm i\pi)| \leq 1$ is bounded for all $z \in S(0,\pi)$ by $|f(z)| \leq 1$.
Hence $|Q(\theta)| \leq 1\ \forall\theta \in S(0,\pi)$. Regarding σ_3, we have $|\sigma_3| \leq c$ as in the limit $\theta \to 0$, the Gamma function in σ_2 compensates the diverging $1/\theta$, as $\lim\limits_{\theta \to 0} 1/(\theta\,\Gamma(\theta)) \to 1$.
For $\theta = i\pi$ in σ_1, the same argument as for σ_3 can be applied as the $1/(i\pi - \theta)$ term is compensated by $\Gamma(-i(i\pi - \theta)/(2\pi))$ from $Q(i\pi - \theta)$ in σ_2. $\qquad\square$

7 Conclusions

We have constructed a Borchers triple $(\mathcal{A}, U, \mathcal{H})$, and given the (incomplete) proof that the net $W \mapsto \mathcal{A}(W)$ is a standard right wedge algebra. The defining quantity for that was the scattering function S. We have proven in Theorem 5.1 that the class of scattering functions from Definition 2.1 allows for an exact solution of the inverse scattering problem for two-particle scattering states. Moreover we could show asymptotic completeness of the space of two-particle scattering states.

The discussion of general n-particle scattering states showed that it is in principle possible to state the $(n \to m)$ S-matrix, and that, given field operators localized in spatially separated double cones, the space of n-particle scattering states is complete. These are to our knowledge new results as it was not possible in the form factor program to give expedient one-particle generators in the multi-particle case. The strong indication that the space of scattering states is asymptotic complete is new as well, as to our knowledge this was not shown so far at all for interacting multi-particle theories.

The presented examples of S-matrices represent interesting elements of the family of scattering functions and show that many of the heuristically motivated model theories fit in this framework.

Appendices

A Index notation

Recalling section 2 where we chose an orthonormal basis $\{e_\alpha\}$ for $\mathcal{K}$ and wrote a vector $\Psi_n(\underline{\theta}) \in \mathcal{K} \otimes ... \otimes \mathcal{K}$ and a matrix-valued function $S(\theta) \in \mathcal{B}(\mathcal{K} \otimes \mathcal{K})$ as

$$\Psi_n^{\alpha_1...\alpha_n}(\theta_1,...,\theta_n) := \langle e_{\alpha_1} \otimes ... \otimes e_{\alpha_n} | \Psi_n(\theta_1,...,\theta_n) \rangle$$
$$S_{\beta_1\beta_2}^{\alpha_1\alpha_2}(\theta) := \langle e^{\alpha_1} \otimes e^{\alpha_2} | S(\theta) \, e_{\beta_1} \otimes e_{\beta_2} \rangle, \qquad (A.1)$$

we can state the defining conditions of S as follows:

$$[S(\theta)^*]_{\beta_1\beta_2}^{\alpha_1\alpha_2} = \langle e^{\alpha_1} \otimes e^{\alpha_2} | S(\theta)^* \, e_{\beta_1} \otimes e_{\beta_2} \rangle = \langle S(\theta) \, e^{\alpha_1} \otimes e^{\alpha_2} | e_{\beta_1} \otimes e_{\beta_2} \rangle = \overline{S_{\alpha_1\alpha_2}^{\beta_1\beta_2}(\theta)}, \tag{A.2}$$

where the complex conjugation is due to the hermiticity of the scalar product since $\mathcal{K}$ is a complex vector space. For the index flip operator F, we get

$$F_{\beta_1\beta_2}^{\alpha_1\alpha_2} = \langle e^{\alpha_1} \otimes e^{\alpha_2} | \underbrace{F \, e_{\beta_1} \otimes e_{\beta_2}}_{=e_{\beta_2} \otimes e_{\beta_1}} \rangle = \delta_{\beta_2}^{\alpha_1} \delta_{\beta_1}^{\alpha_2}. \tag{A.3}$$

The transposition operator D_n with transposition τ_k and scattering matrix $S(\theta)$ acts on $\mathcal{H}_1^{\otimes n} \ni \psi_n : \mathbb{R}^n \to \mathcal{K}^{\otimes n}$ as

$$[D_n(\tau_k)\psi]_n^{\alpha_1...\alpha_n}(\underline{\theta}) = S_{\beta_{k+1}\beta_k}^{\alpha_{k+1}\alpha_k}(\theta_{k+1} - \theta_k)\psi_n^{\alpha_1...\beta_{k+1}\beta_k...\alpha_n}(\theta_1,...,\theta_{k+1},\theta_k,...,\theta_n). \tag{A.4}$$

Here it is important to note that the order of the indices on S is flipped. This stems from the flipped arguments of Ψ, keeping "their" indices, which is accounted for in the notation without indices by the flip operator F.

Now we will state equations (2.5a) − (2.5c) from Definition 2.1 in index notation where we will denote $(\theta_i - \theta_j)$ as θ_{ij}, first $S(\theta)^* = F\,S(-\theta)F$ (2.5a):

$$[S(\theta)^*]_{\beta_1\beta_2}^{\alpha_1\alpha_2} = \overline{S_{\alpha_1\alpha_2}^{\beta_1\beta_2}(\theta)} \qquad = \qquad [FS(-\theta)F]_{\beta_1\beta_2}^{\alpha_1\alpha_2} = S_{\beta_2\beta_1}^{\alpha_2\alpha_1}(-\theta). \tag{A.5}$$

Expressing the left hand side of the Yang-Baxter equation (2.5c) in index notation gives

$$\langle e^{\alpha_1} \otimes e^{\alpha_2} \otimes e^{\alpha_3} | \, F_1 S_1(\theta_{21}) F_2 S_2(\theta_3 - \theta_1) F_1 S_1(\theta_{32}) \, | e^{\beta_1} \otimes e^{\beta_2} \otimes e^{\beta_3} \rangle$$
$$= S_{\beta_2\beta_1}^{\gamma_2\gamma_1}(\theta_{21}) \, S_{\beta_3\gamma_1}^{\gamma_3\alpha_1}(\theta_{31}) \, S_{\gamma_3\gamma_2}^{\alpha_3\alpha_2}(\theta_{32}).$$

Analogously the right hand side of (2.5c)

$$\langle e^{\alpha_1} \otimes e^{\alpha_2} \otimes e^{\alpha_3} | \, F_2 S_2(\theta_{32}) F_1 S_1(\theta_{31}) F_2 S_2(\theta_{21}) \, | e_{\beta_1} \otimes e_{\beta_2} \otimes e_{\beta_3} \rangle$$
$$= S_{\beta_3\beta_2}^{\gamma_3\gamma_2}(\theta_{32}) \, S_{\gamma_3\beta_1}^{\alpha_3\gamma_1}(\theta_{31}) \, S_{\gamma_2\gamma_1}^{\alpha_2\alpha_1}(\theta_{21}),$$

giving together the Yang-Baxter equation in index notation

$$S_{\beta_2\beta_1}^{\gamma_2\gamma_1}(\theta_{21}) \, S_{\beta_3\gamma_1}^{\gamma_3\alpha_1}(\theta_{31}) \, S_{\gamma_3\gamma_2}^{\alpha_3\alpha_2}(\theta_{32}) = S_{\beta_3\beta_2}^{\gamma_3\gamma_2}(\theta_{32}) \, S_{\gamma_3\beta_1}^{\alpha_3\gamma_1}(\theta_{31}) \, S_{\gamma_2\gamma_1}^{\alpha_2\alpha_1}(\theta_{21}). \tag{A.6}$$

Unitarity of S in index notation has the following form:

$$[S(\theta)^*]_{\gamma_1\gamma_2}^{\alpha_1\alpha_2} \, S_{\beta_2\beta_1}^{\gamma_1\gamma_2}(\theta) = \overline{S(\theta)_{\alpha_1\alpha_2}^{\gamma_1\gamma_2}} \, S_{\beta_2\beta_1}^{\gamma_1\gamma_2}(\theta) = \delta_{\beta_2}^{\alpha_1} \delta_{\beta_1}^{\alpha_2}. \tag{A.7}$$

B   Action of the S-matrix

In this section we give calculations which were not needed in the proof of Theorem 4.4 but for the sake of completeness might be interesting.

The conjugation of $F_{\sigma_k} S_{\sigma_k}$ follows the proof of Lemma 2.1 using (A.5) and the fact that $F^* = F$ (2.2). It is sufficient to just conjugate the product formula (3.4) when the order of transpositions is reversed as well, i.e. for $\mathfrak{S}_n \ni \pi = \tau_{\alpha_1}...\tau_{\alpha_i}$, τ_{α_i} being transpositions, $\pi^{-1} = \tau_{\alpha_i}...\tau_{\alpha_1}$. We write this out in more detail:

$$
[D_n(\sigma_k)^* \Phi(\underline{\theta})]_n^{\alpha_1...\alpha_n}
$$
$$
= [D_n(\tau_1)...D_n(\tau_{k-1})\Phi(\underline{\theta})]_n^{\alpha_1...\alpha_n}
$$
$$
= [(F_{\sigma_k} S_{\sigma_k})^* \Phi(\theta_2, ..., \theta_{k-1}, \theta_1, \theta_k, ..., \theta_n)]_n^{\alpha_2...\alpha_1...\alpha_n}
$$
$$
= \sum_{\xi_1...\xi_{n+1}} \delta_{\xi_k}^{\alpha_k} \delta_{\xi_1}^{\beta_k} \prod_{l=k-1}^{1} S_{\beta_l \xi_{l+1}}^{\alpha_l \xi_l}(\theta_l - \theta') \Phi_n^{\beta_1...\beta_k...\alpha_n}(\theta_2, ..., \theta_{k-1}, \theta_1, \theta_k, ..., \theta_n)
$$
$$
= \sum_{\xi_1...\xi_{n+1}} [S_{\sigma_{n+1}^{-1}}(\underline{\theta}, \theta')]_{\beta_1...\beta_n \beta_{n+1}}^{\alpha_1...\alpha_n \alpha_{n+1}} \Phi_n^{\beta_1...\beta_k...\alpha_n}(\theta_2, ..., \theta_{k-1}, \theta_1, \theta_k, ..., \theta_n)
$$
$$
= [D_n(\sigma_k^{-1})\Phi(\underline{\theta})]_n^{\xi_1 \alpha_2...\alpha_n}. \tag{B.1}
$$

From this equation one can see that if $F_{\sigma_k} S_{\sigma_k}$ makes up for pulling the kth argument to the first position, it is $(F_{\sigma_k} S_{\sigma_k})^*$ which compensates for pushing the first argument to the kth position. The matrices are again unitary which is clear since D_n is a unitary representation of the permutation group $\mathfrak{S}_n$, defined via unitary matrices S and F.

We will highlight this fact by adding a $^{-1}$ to sigma, emphasizing the inverse action.

$$
[F_{\sigma_k} S_{\sigma_k}(\theta_k - \underline{\theta})]^* = S_{\sigma_k^{-1}}^* F_{\sigma_k^{-1}}^*(\underline{\theta} - \theta_k). \tag{B.2}
$$

The action of the reflected creation operator in the proof of Theorem 4.4 which was not explicitly needed to calculate the commutation relation $[z^\dagger(\overline{\psi_1})', z(\psi_2)]$ (4.10b) is presented in the following:

$$
[z^\dagger(\psi)'\Psi]_n^{\alpha_1...\alpha_n}(\underline{\theta})
$$
$$
= \overline{[z^\dagger(\psi) J\Psi]_n^{\overline{\alpha}_n...\overline{\alpha}_1}(\theta_n, ..., \theta_{n-k+1}, ..., \theta_1)} \qquad l := n - k + 1
$$
$$
= \overline{\frac{1}{\sqrt{n}} \sum_{l=n}^{1} \sum_{\xi_1...\xi_n} \delta_{\xi_n}^{\alpha_l} \delta_{\xi_l}^{\beta_l} \prod_{j=l}^{n-1} S_{\xi_j \beta_j}^{\xi_{j+1} \alpha_j}(\theta_j - \theta') \psi^{\overline{\beta}_l}(\theta_l) [J\Psi]_{n-1}^{\overline{\beta}_n...\hat{\overline{\beta}}_l...\overline{\alpha}_1}(\theta_n, ..., \hat{\theta}_l, ..., \theta_1)},
$$

now, using (A.5) gives,

$$
= \frac{1}{\sqrt{n}} \sum_{l=n}^{1} \sum_{\xi_1...\xi_n} \delta_{\xi_n}^{\alpha_l} \delta_{\xi_l}^{\beta_l} \prod_{j=l}^{n-1} S_{\alpha_j \xi_{j+1}}^{\beta_j \xi_j}(\theta' - \theta_j) \overline{\psi^{\overline{\beta}_l}(\theta_l)} \Psi_{n-1}^{\alpha_1...\alpha_{l-1}\beta_{l+1}...\beta_n}(\theta_1, ..., \hat{\theta}_l, ..., \theta_n)
$$
$$
= \frac{1}{\sqrt{n}} \sum_{l=1}^{n} [S_{\sigma_l^{-1}}(\underline{\theta})]_{\beta_{l+1}...\beta_n \beta_l}^{\alpha_{l+1}...\alpha_n \alpha_l} \overline{\psi^{\overline{\beta}_l}(\theta_l)} \Psi_{n-1}^{\alpha_1...\alpha_{l-1}\beta_{l+1}...\beta_n}(\theta_1, ..., \hat{\theta}_l, ..., \theta_n). \tag{B.3}
$$

The second commutator in Lemma 4.3, calculated explicitly (again $l = n - k + 1$) reads

$$
[z^\dagger(C\overline{\psi_1})',\ z(\psi_2)]\Psi_n^{\alpha_1...\alpha_n}(\underline{\theta})
$$

$$
= \frac{1}{\sqrt{n}} \sum_{l=1}^{n} [S_{\sigma_l^{-1}}(\underline{\theta})]_{\beta_{l+1}...\beta_n\beta_l}^{\alpha_{l+1}...\alpha_n\alpha_l}\ \psi_1^{\beta_l}(\theta_l)[z(\psi_2)\Psi]_{n-1}^{\alpha_1...\alpha_{l-1}\beta_{l+1}...\beta_n}(\theta_1,...,\hat{\theta}_l,...,\theta_n)
$$

$$
\quad - \sqrt{n+1} \int d\theta_0 \psi_2^{\alpha_0}(\theta_0)[z^\dagger(C\psi_1)'\Psi]_{n+1}^{\alpha_0...\alpha_n}(\theta_0,...,\theta_n)
$$

$$
= \sum_{l=1}^{n} [S_{\sigma_l^{-1}}(\underline{\theta})]_{\beta_{l+1}...\beta_n\beta_l}^{\alpha_{l+1}...\alpha_n\alpha_l}\ \psi_1^{\beta_l}(\theta_l) \int d\theta_0 \psi_2^{\alpha_0}(\theta_0)\Psi_n^{\alpha_1...\alpha_{l-1}\beta_{l+1}...\beta_n}(\theta_0,...,\hat{\theta}_l,...,\theta_n)
$$

$$
\quad - \int d\theta_0 \psi_2^{\alpha_0}(\theta_0) \sum_{l=0}^{n} [S_{\sigma_l^{-1}}(\underline{\theta})]_{\beta_{l+1}...\beta_n\beta_l}^{\alpha_{l+1}...\alpha_n\alpha_l}\ \psi_1^{\beta_l}(\theta_l)\Psi_n^{\alpha_1...\hat{\beta}_l...\beta_n}(\theta_0,...,\hat{\theta}_l,...,\theta_n)
$$

$$
= - \int d\theta_0 \psi_2^{\alpha_0}(\theta_0)[S_{\sigma_0^{-1}}(\theta_0,\underline{\theta})]_{\beta_1...\beta_n\beta_0}^{\alpha_1...\alpha_n\alpha_0}\ \psi_1^{\beta_0}(\theta_0)\Psi_n^{\beta_1...\beta_n}(\theta_1...\theta_n)
$$

$$
= [G_n^{C\psi_1\psi_2}(\underline{\theta})]_{\beta_1...\beta_n}^{\alpha_1...\alpha_n}\ \Psi_n^{\underline{\beta}}(\underline{\theta}). \tag{B.4}
$$

This commutator annihilates the first entry of the field Ψ and creates one in the last position, where again all terms except for the $(n+1)^{th}$ cancel.

References

[AA1]   Abdalla, E., Abdalla, M., Rothe, K.: 2 dimensional quantum field theory. Singapore: World Scientific, 1991

[A2]   Åks, S.: Proof that scattering implies production in quantum Feld theory, J.Math. Phys. 6 (1965), 516-532.

[A3]   Alzer, H., Sharp upper and lower bounds for the gamma function. Proc. Royal Soc. Edinburgh: section A Mathematics (2009), 139: 709-718

[A4]   Araki, H.: Mathematical theory of quantum fields, Oxford Scientific Publications, 1999.

[AW1]   Atkinson, D., Warnock, R.L.: Persistence of the Castillejo-Dalitz-Dyson Ambiguity in Relativistic Crossing-Symmetric Amplitudes, Phys Rev. Vol 188 N5 p2098 − 2111.

[BBS1]   Borchers, H.-J., Buchholz, D., and Schroer B.: Polarization-free generators and the S-matrix, Commun. Math. Phys. 219 (2001),125-140, arXiv: hep-th/0003243.

[BF1]   Babujian H. M., Foerster A., Karowski M.: The form factor program: A review and new Results The nested $SU(N)$ off-shell Bethe Ansatz and the $1/N$ expansion. Theoretical and Mathematical Physics, 155(1): 512-522 (2008)

[BR1]   Bratteli, O., Robinson, D.: Operator Algebras and Quantum Statistical Mechanics 1, C*- and W*-Algebras Symmetry Groups Decomposition of States, 2nd edition. New York: Springer 1987.

[BW1]   J. J. Bisognano and E. H. Wichmann, On the duality condition for a hermitian scalar field, J. Math. Phys. 16 (1975), 985-1007.

[DF1]   W. Driessler and J. Fröhlich, The reconstruction of local observable algebras from the Euclidean Green's functions of relativistic quantum field theory, Ann. Inst. Henri Poincaré 27 (1977), no. 33 221-236.

[H1]   Haag, R.: Local quantum physics, 2nd edition. New York: Springer, 1992.

[H2]   Haag, R.: Quantum field theories with composite particles and asymptotic conditions, Phys. Rev. 112 (1958), 669-673.

[HK1]   Haag R., Kastler D., An algebraic approach to quantum field theory , Journal of Mathematical Physics, Bd.5, 1964, S.848-861.

[H3]   Hepp, K.: On the connection between Wightman and LSZ quantum field theory, Brandeis University Summer Institute in Theoretical Physics 1965 "Axiomatic Field Theory" (M. Chretien and S. Deser, eds.), vol. 1, Gordon and Breach, 1966, pp. 135-246.

[I1]   Iagolnitzer D., Factorization of the multiparticle S matrix in two-dimensional spacetime models. Phys. Rev. D18 (1978) 1275.

[J1]   Jimbo, M.: Quantum R Matrix for the Generalized Toda System. Commun. Math. Phys. 102, 537-547 (t986).

[L1]   Lechner, G.: Construction of Quantum Field Theories with Factorizing S-Matrices, Commun. Math. Phys. 277, 821-860 (2008).

[L2]   Lechner, G.: Polarization-Free Quantum Fields and Interaction, Lett.Math.Phys. 64 (2003) 137-154 arXiv:hep-th/0303062v1 7 Mar 2003.

[LM1]   Liguori, A., Mintchev, M.: Fock spaces with generalized statistics, Lett. Math. Phys. 33: 283-295, 1995.

[O1]   Olver, F.W.J. et al. NIST Handbook of Mathematical functions. Came-bridge University Press 2010.

[PA1]   Perk, J. H. H. and Au-Yang, H.: Yang-Baxter Equation, in Encyclopedia of Mathematical Physics, eds. J. P. Franoise, G.L. Naber and Tsou S.T., Oxford: Elsevier, 2006, Vol. 5, pp.465-473. See also its extended arXiv preprint math-ph/0606053.

[RS1]   Reed, S., Simon, B.: Methods of modern mathematical physics, I Func-tional analysis. San Diego, Ca: Academic Press Inc, 1975.

[RS2]   Reed, S., Simon, B.: Methods of modern mathematical physics, II Fourier analysis, self adjointness. San Diego, Ca: Academic Press Inc, 1975.

[R1]   Ruelle, D.: On asymptotic condition in quantum field theory, Helv. Phys. Acta 35 (1962), 147-163.

[S1]   B. Schroer, Modular localization and the bootstrap-formfactor program, Nucl. Phys. B499 (1997), 547-568, [hep-th/9702145].

[S2]   B. Schroer, Modular wedge localization and the d =1+1 formfactor pro-gram, Ann. Phys. 275 (1999), 190-223, [hep-th/9712124].

[SW1]   R. F. Streater and A. S. Wightman, PCT, spin and statistics, and all that, 3rd ed., Princeton University Press, 1980.

[SW2]   R. Shankar and E. Witten.: S matrix of the supersymmetric nonlinear σ model, Phys. Rev. D17 (1978) 2134.

[W1]   E. Wigner, On Unitary Representations of the Inhomogeneous Lorentz Group, Ann. Math. 40 (1939), 149.

[ZA1]   Zamolodchikov, A.B.: Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field models. Ann. Phys. 120, 253-291 (1979).

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