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DISSERTATION

Titel der Dissertation

On the isospectral problem of the Camassa–Holm equation

Verfasser

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angestrebter akademischer Grad

Doktor der Naturwissenschaften (Dr. rer. nat.)

Wien, im November 2011

Studienkennzahl lt. Studienblatt:	A 791 405
Dissertationsgebiet lt. Studienblatt:	Mathematik
Betreuer:	Univ.-Prof. Dipl.-Ing. Dr. Gerald Teschl

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Introduction

Ever since its discovery by Gardner, Greene, Kruskal and Miura [27] in 1967, the inverse scattering transform has been developed into a powerful tool for solving completely integrable systems (see e.g. [1], [25] or [33] for a survey). Initially introduced in order to solve the Korteweg–de Vries equation, this method has successfully been applied to many other non-linear wave equations. The main motivation for the present thesis stems from the corresponding transform for the dispersionless Camassa–Holm equation.

The dispersionless Camassa–Holm equation is the non-linear partial differential equation

$$u_t - u_{txx} = 2u_x u_{xx} - 3uu_x + uu_{xxx}, \tag{CH}$$

where subscripts denote partial differentiation. It has first been derived as a bi-Hamiltonian equation by Fokas and Fuchssteiner [26] in 1981, using the method of recursion operators. Later on, in 1993 it has been rediscovered by Camassa and Holm [10], [11] as a model for unidirectional wave propagation on shallow water with u representing the height of the waters free surface above the flat bottom. Regarding the hydrodynamical relevance of this equation as a model for shallow water waves we also refer to [30], [15].

One of the most intriguing properties of the Camassa–Holm equation is the occurrence of wave breaking. This means that even smooth initial data may lead to blow-up in finite time, which happens in such a way that the solution stays bounded but its slope develops a singularity. Of course, this behavior attracted a lot of interest [13], [14] and criteria for wave breaking to occur may be found in [34]. However, note that it is possible to continue solutions beyond wave breaking in such a way that either the energy is conserved [8] or such that a loss of energy occurs due to wave breaking [9].

Another remarkable phenomenon of the dispersionless Camassa–Holm equation are the so-called peakon solutions. These are solutions traveling at constant speed $c \in \mathbb{R}$ and with some phase shift $\eta \in \mathbb{R}$, explicitly given by

$$u(x,t) = c\,\mathrm{e}^{-|x-ct-\eta|}, \quad x,\, t \in \mathbb{R}.$$

Of course, since these solutions have a peak at their crest, they have to be interpreted as suitable weak solutions [16]. More generally, one can also consider so-called multi-peakon solutions

$$u(x,t) = \sum_{n=1}^{N} p_n(t)\mathrm{e}^{-|x-q_n(t)|}, \quad x,\, t \in \mathbb{R},$$

which are finite sums of single peakons for each fixed time t. This function u is a weak solution, provided that the functions p, q satisfy some Hamiltonian system of ordinary differential equations [2].

Associated with the Camassa–Holm equation is the following family of isospectral problems [12], parameterized by the time t

$$\text{(ISP)} \qquad -f''(x) + \frac{1}{4} f(x) = z\omega(x,t) f(x), \quad x \in \mathbb{R},\ z \in \mathbb{C},$$

where ω is related to solutions of the Camassa–Holm equation via

$$\omega(x,t) = u(x,t) - u_{xx}(x,t), \quad x \in \mathbb{R}$$

and referred to as the weight (or momentum). Now the relevance of these weighted Sturm–Liouville problems stems from the fact that they allow us to express the flow of the Camassa–Holm equation as a linear flow of some suitable spectral quantities of these problems. In fact, if u is a given, spatially decaying solution of (CH), then the family in (ISP) is isospectral with purely discrete spectrum and the time evolution of the remaining spectral data, the so-called norming constants, is simply given by

$$\gamma_\lambda^2(t) = \gamma_\lambda^2(0) \mathrm{e}^{\frac{t}{4\lambda}},$$

for each eigenvalue λ. Now this makes it possible to, at least in principle solve the Cauchy problem of the Camassa–Holm equation as follows:

$$\begin{array}{ccc} \mathcal{S}(0) & \xrightarrow{\qquad\qquad} & \mathcal{S}(t) \\ \Big\uparrow & & \Big\downarrow \\ u(\,\cdot\,,0) & \xrightarrow{\text{Camassa–Holm flow}} & u(\,\cdot\,,t) \end{array}$$

Given the initial data at time zero, one computes the associated spectral data for which the time evolution is known explicitly. Now the solution at some time t may be recovered by determining the corresponding weight from the spectral data at time t. From this one sees that direct and inverse spectral theory for the isospectral problem (ISP) are of peculiar interest for solving the Camassa–Holm equation. Of course, the decisive part hereby is contained in solving the inverse spectral problem.

Provided that the weight is a strictly positive function, it is well known that the isospectral problem (with some suitable boundary conditions) gives rise to a self-adjoint operator in the weighted Hilbert space $L^2(\mathbb{R};\omega)$. Moreover, if the weight is smooth enough, then it is even possible to transform this operator into a Schrödinger operator and some inverse spectral conclusions may be drawn from this. However, in order to incorporate multi-peakon solutions and wave breaking, it is necessary to at least allow the weight to be an arbitrary finite signed measure. In fact, multi-peakon solutions correspond to weights which are a finite sum of weighted Dirac measures and wave breaking only occurs if the weight changes sign. Unfortunately, for this class of weights neither direct nor inverse spectral theory are very well developed. Apart from the fact that the weight is allowed to be a measure, the main difficulties arise from the fact that both endpoints are quite singular and that the weight may change sign. In particular, as a consequence of this, the associated operators are in general not self-adjoint anymore.

In this respect, the left-definite point of view seems to be more natural for the isospectral problem. This approach relies on the fact that the coefficients on the left-hand side of (ISP) are strictly positive and hence give rise to a modified Sobolev space. Now in contrast to the former setting, the isospectral problem gives rise to a self-adjoint linear relation in this modified Sobolev space. Moreover, this linear relation is even unique since no boundary conditions are needed here. This left-definite approach has been pursued by Bennewitz [3], Brown and Weikard [4]. In particular, they were able to prove a uniqueness theorem for the inverse spectral problem under some restrictions on the weight. Unfortunately, these restrictions also prohibit the case of multi-peakon solutions.

The present thesis is mainly concerned with direct and inverse spectral theory of Sturm–Liouville operators and their application to the isospectral problem of the dispersionless Camassa–Holm equation. It is comprised of my publications and preprints [18], [19], [20], [21], [22], [23] and [24], which have been written during my doctoral studies at the *University of Vienna*, partially together with my thesis supervisor Gerald Teschl. Except for the two articles [21] and [22], all of them were written with a view to the isospectral problem of the dispersionless Camassa–Holm equation.

In the first article [20], submitted to the *Proceedings of the London Mathematical Society*, we show how to generalize the notion of Sturm–Liouville operators to the case when the coefficients of the differential expression are only assumed to be real-valued Borel measures. Apart from some preliminaries about measure Sturm–Liouville equations, this includes a detailed description of all associated self-adjoint realizations in a weighted L^2 space as well as of their spectra and resolvents. Moreover, we also give a comprehensive account on Weyl–Titchmarsh theory for such realizations with separate boundary conditions. The main technical difficulty arising from this generalization is that the associated operators are in general multi-valued and hence one has to work with linear relations instead of operators. However, it turns out that for a large class of coefficients (including the isospectral problem of the Camassa–Holm equation) one still ends up with single-valued operators. In particular, this allows us to associate an operator with the isospectral problem, even when the weight is only assumed to be a real-valued Borel measure. Moreover, let us mention that these measure Sturm–Liouville operators also contain Schrödinger operators with measure potentials (as well as with δ and δ' point interactions) and Jacobi difference operators as special cases.

The following two publications [21] and [22] show how to apply the results from the previous article [20] to Sturm–Liouville operators on time scales. Time scale calculus was introduced in 1988 in order to unify differential and difference calculus. Since then this approach has had an enormous impact and we only refer to [5], [6] and the references therein. However, in [21] we show that the natural derivative on each time scale is nothing but a Radon–Nikodým derivative with respect to the measure, which is naturally associated with this time scale. This article has been published in the *Journal of Mathematical Analysis and Applications*. Consequently, these results are employed in [22], which will appear in the *Journal of Difference Equations and Applications*. There we show that Sturm–Liouville operators on time scales can be regarded as a special case of measure Sturm–Liouville operators. In particular, this approach allows us to generalize several known results for Sturm–Liouville operators on time scales.

In the remaining part of this thesis we are mainly concerned with inverse spectral theory for Sturm–Liouville operators. More precisely, the next two articles [23] and [18] are about inverse uniqueness results for Schrödinger operators with potentials, which are singular at both endpoints but still allow a scalar spectral measure. Such operators attracted quite some interest recently (we only refer to [28], [31] and the references therein) since they include for example the important case of perturbed spherical Schrödinger operators. For these operators it has been shown that it is possible to develop singular Weyl–Titchmarsh theory even when both endpoints are quite singular. Moreover, the authors in [31] were even able to proof a local Borg–Marchenko uniqueness theorem for the singular Weyl–Titchmarsh function. Unfortunately, since this function is determined by the associated spectral measure only up to some entire function, this does not yield an inverse uniqueness result for the spectral measure.

However, in the case when the operators have purely discrete spectra with finite convergence exponent, refinements of the arguments in [31] yield inverse uniqueness results for the spectral measure. In fact, this is done in article [23], which has been submitted to the *Transactions of the American Mathematical Society.* There we prove an inverse uniqueness result for this case as well as a generalization of the celebrated Hochstadt–Lieberman uniqueness result to operators with singular endpoints. In particular, these results also apply to the perturbed quantum harmonic oscillator. However, although the spectral picture of this operator is quite similar to that of the isospectral problem of the dispersionless Camassa–Holm equation, the method used does not apply in that case, since it uses a detailed knowledge of the high energy asymptotics of solutions.

In article [18], I use a quite different approach in order to prove an inverse uniqueness result for the spectral measure. More precisely, this is done by applying de Branges' subspace ordering theorem for Hilbert spaces of entire functions [17] to certain de Branges spaces associated with Schrödinger operators with strongly singular potentials. In contrast to the approach taken in [23], the method here would also apply to general Sturm–Liouville operators with three coefficients, since it only requires comparable weak high energy asymptotics of solutions. This article has been submitted to the *Journal of Functional Analysis.*

The method used in [18] would also apply to yield inverse uniqueness results for the isospectral problem of the Camassa–Holm equation in the case when the weight is a strictly positive integrable function. However, in this right-definite setting it is hard to deal with the case when the weight is just a finite measure, even if it is positive. For this case, the left-definite setting is more convenient, where we are able to allow the weight to be an arbitrary finite signed measure. The article [19], submitted to the *Journal of Differential Equations*, is concerned with direct and inverse spectral theory of such left-definite Sturm–Liouville operators. In particular, there I describe all self-adjoint realizations in a modified Sobolev space and develop singular Weyl–Titchmarsh theory for such operators. Moreover, I am able to modify the approach taken in [18] in order to prove quite general inverse uniqueness theorems for these operators.

In the last article [24], we apply the results obtained so far to the isospectral problem of the dispersionless Camassa–Holm equation. First of all, we introduce the right-definite and the left-definite operator associated with the isospectral problem and show how they are related. Consequently, we give the basic properties of their spectra and compute their resolvents. All this is done for weights belonging to the class of finite signed measures and the main result states that this weight is uniquely determined by the associated spectral measure. This last article has not been submitted yet. Actually, it is still in preparation and the final version will include an account on the existence part of the inverse spectral problem as well.

The main reason for us to consider the isospectral problem (ISP) with a finite signed measure weight is that a useful inverse spectral theory for this problem should at least include the weights corresponding to the multi-peakon solutions of the Camassa–Holm equation. In fact, this might have interesting consequences regarding long-time asymptotics for solutions of equation (CH). It is expected [35] that after a long time, solutions will split into an infinite train of peakons, each of them corresponding to an eigenvalue of the corresponding isospectral problem. Asymptotically each of them travels at constant speed, essentially given by the reciprocal of the modulus of the corresponding eigenvalue. In fact, this picture is quite reasonable in view of the long-time asymptotics of similar non-linear wave equations, which are given for example in [7], [29]. There, inverse scattering transforms and in particular their reformulation as Riemann–Hilbert problems have successfully been employed to derive long-time asymptotics. However, for the dispersionless Camassa–Holm equation this approach does not apply and the only known results regarding long-time asymptotics seem to concern multi-peakon solutions [2] (see also [32]). Now given the inverse uniqueness result from [24], the quest for long-time asymptotics of general solutions boils down to proving that the forward spectral problem is continuous with respect to some suitable topologies. In fact, assuming this the inverse spectral problem would be continuous as well, which in turn would yield long-time asymptotics for the dispersionless Camassa–Holm equation.

Acknowledgments. At this point I would like to thank my thesis supervisor Gerald Teschl. First of all for all his guidance and support during my doctoral studies, for many helpful discussions but also for providing excellent working conditions and giving me the opportunity to present my research at conferences and workshops.

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Sturm–Liouville operators with measure-valued coefficients

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Submitted to *Proc. London Math. Soc.*

Abstract. We give a comprehensive treatment of Sturm–Liouville operators with measure-valued coefficients including a full discussion of self-adjoint extensions and boundary conditions, resolvents and Weyl–Titchmarsh theory. We avoid previous technical restrictions and, at the same time, extend all results to a larger class of operators. Our operators include classical Sturm–Liouville operators, Lax operators arising in the treatment of the Camassa–Holm equation, Jacobi operators and Sturm–Liouville operators on time scales as special cases.

1. Introduction

Sturm–Liouville problems

$$-\frac{d}{dx}\left(p(x)\frac{dy}{dx}(x)\right)+q(x)y(x)=zr(x)y(x) \tag{1.1}$$

have a long tradition (see e.g. the textbooks [31], [37], [38] and the references therein) and so have their generalizations to measure-valued coefficients. In fact, extensions to the case

$$\frac{d}{d\varrho(x)}\left(-p(x)\frac{dy}{dx}(x)+\int^x y(t)d\chi(t)\right)=zy(x) \tag{1.2}$$

date back at least to Feller [15] and were also advocated in the fundamental monograph by Atkinson [4]. Here the derivative on the left-hand side has to be understood as a Radon–Nikodým derivative. We refer to the book by Mingarelli [23] for a more detailed historical discussion.

However, while this generalization has been very successful on the level of differential equations (see e.g. [4], [23], [36] and the references therein), much less is known about the associated operators in an appropriate Hilbert space. First attempts were made by Feller and later complemented by Kac [19] (cf. also Langer [21] and Bennewitz [5]). Again, a survey of these results and further information can be found in the book of Mingarelli [23].

2000 *Mathematics Subject Classification.* Prim. 34B20, 34L05; Sec. 34B24, 47A10.
Keywords. Schrödinger operators, spectral theory, strongly singular potentials.
Research supported by the *Austrian Science Fund (FWF)* under grant no. Y330.

The case where only the potential is allowed to be a measure is fairly well treated since it allows to include the case of point interactions which is an important model in physics (see e.g. the monographs [1], [2] as well as the recent results in [6] and the references therein). More recently, Shavcuk and Shkalikov [25]–[28] were even able to cover the case where the potential is the derivative of an arbitrary L^2 function. Similarly, the case where the weight is a measure is known as Kreĭn string and has also attracted considerable interest recently [33]–[35].

However, while the theory developed by Kac and extended by Mingarelli is quite general, it still does exclude some cases of interest. More precisely, the basic assumptions in Chapter 3 of Mingarelli [23] require that the corresponding measures have no weight at a finite boundary point. Unfortunately, this assumption excludes for example classical cases like Jacobi operators on a half-line. The reason for this assumption is the fact that otherwise the corresponding maximal operator will be multi-valued and one has to work within the framework of multi-valued operators. This nuisance is already visible in the case of half-line Jacobi operators where the underlying Hilbert space has to be artificially expanded in order to be able to formulate appropriate boundary conditions [30]. In our case there is no natural way of extending the Hilbert space and the intrinsic approach via multi-valued operators is more natural. Moreover, this multi-valuedness is not too severe and corresponds to an at most two dimensional space which can be removed to obtain a single-valued operator. Again, a fact well-known from Jacobi operators with finite end points.

Moreover, the fact that our differential equation is defined on a larger set than the support of the measure ϱ (which determines the underlying Hilbert space) also reflects requirements from the applications we have in mind. The most drastic example in this respect is the Sturm–Liouville problem

$$\frac{d}{d\varrho(x)}\left(-\frac{dy}{dx}(x)+\frac{1}{4}\int^x y(t)dt\right)=zy(x) \tag{1.3}$$

on $\mathbb{R}$ which arises in the Lax pair of the dispersionless Camassa–Holm equation [8], [9]. In the case of a peakon ϱ is single Dirac measure and the underlying Hilbert space is one-dimensional. However, the corresponding differential equation has to be investigated on all of $\mathbb{R}$, where the Camassa–Holm equation is defined. An appropriate spectral theory for this operator in the case where ϱ is a genuine measure (i.e. not absolutely continuous with respect to Lebesgue measure) seems to be missing and is one of the main motivations for the present paper.

Furthermore, there is of course another reason why Sturm–Liouville equations with measure-valued coefficients are of interest, namely, the unification of the continuous with the discrete case. While such a unification already was one of the main motivations in Atkinson [4] and Mingarelli [23], it has recently attracted enormous attention via the introduction of the calculus on time scales [7]. In fact, given a time scale $\mathbb{T}\subseteq\mathbb{R}$, the so-called associated Hilger (or delta) derivative is nothing but the Radon–Nikodým derivative with respect to the measure ϱ, which corresponds to the distribution function $R(x)=\inf\{y\in\mathbb{T}\,|\,y>x\}$. We refer to [13] for further details and to a follow-up publication [14], where we will provide further details on this connection.

2. Notation

Let (a, b) be an arbitrary interval and μ be a locally finite complex Borel measure on (a, b). By $AC_{\mathrm{loc}}((a, b); \mu)$ we denote the set of left-continuous functions, which are locally absolutely continuous with respect to μ. These are precisely the functions f which can be written in the form

$$f(x) = f(c) + \int_c^x h(s)d\mu(s), \quad x \in (a, b),$$

where $h \in L^1_{\mathrm{loc}}((a, b); \mu)$ and the integral has to be read as

$$\int_c^x h(s)d\mu(s) = \begin{cases} \int_{[c,x)} h(s)d\mu(s), & \text{if } x > c, \\ 0, & \text{if } x = c, \\ -\int_{[x,c)} h(s)d\mu(s), & \text{if } x < c. \end{cases} \tag{2.1}$$

The function h is the Radon–Nikodým derivative of f with respect to μ. It is uniquely defined in $L^1_{\mathrm{loc}}((a, b); \mu)$ and we write

$$\frac{df}{d\mu} = h.$$

Every function f which is locally absolutely continuous with respect to μ is locally of bounded variation and hence also the right-hand limits

$$f(x+) = \lim_{\varepsilon \downarrow 0} f(x + \varepsilon), \quad x \in (a, b)$$

of f exist everywhere. Also note that some function $f \in AC_{\mathrm{loc}}((a, b); \mu)$ can only be discontinuous in some point, if μ has mass in this point.

In this respect we also recall the integration by parts formula ([18, Theorem 21.67]) for two locally finite complex Borel measures μ, ν on (a, b)

$$\int_\alpha^\beta F(x)d\nu(x) = FG|_\alpha^\beta - \int_\alpha^\beta G(x+)d\mu(x), \quad \alpha, \beta \in (a, b), \tag{2.2}$$

where F, G are left-continuous distribution functions of μ, ν respectively.

3. Sturm–Liouville equations with measure-valued coefficients

Let (a, b) be an arbitrary interval and ϱ, ς and χ be locally finite complex Borel measures on (a, b). We want to define a linear differential expression τ which is informally given by

$$\tau f = \frac{d}{d\varrho}\left(-\frac{df}{d\varsigma} + \int f d\chi\right).$$

Up to now the only additional assumptions on our measures is that ς is supported on the whole interval, i.e. $\mathrm{supp}(\varsigma) = (a, b)$.

The maximal domain $\mathfrak{D}_\tau$ of functions such that τf makes sense consists of all functions $f \in AC_{\mathrm{loc}}((a, b); \varsigma)$ for which the function

$$-\frac{df}{d\varsigma}(x) + \int_c^x f d\chi, \quad x \in (a, b) \tag{3.1}$$

is locally absolutely continuous with respect to ϱ, i.e. there is some representative of this function lying in $AC_{\mathrm{loc}}((a,b);\varrho)$. As a consequence of the assumption $\operatorname{supp}(\varsigma) = (a,b)$, this representative is unique. We then set $\tau f \in L^1_{\mathrm{loc}}((a,b);\varrho)$ to be the Radon–Nikodým derivative of this function with respect to ϱ. One easily sees that this definition is independent of $c \in (a,b)$ since the corresponding functions (3.1) as well as their unique representatives only differ by an additive constant. As usual, we denote the Radon–Nikodým derivative with respect to ς of some function $f \in \mathfrak{D}_\tau$ by

$$f^{[1]} = \frac{df}{d\varsigma} \in L^1_{\mathrm{loc}}((a,b);|\varsigma|).$$

The function $f^{[1]}$ is called the first quasi-derivative of f.

We note that the definition of τ is consistent with classical theory. Indeed, let ϱ, ς and χ be locally absolutely continuous with respect to Lebesgue measure, and denote by r, p^{-1} and q the respective densities i.e.

$$\varrho(B) = \int_B r(x)dx, \quad \varsigma(B) = \int_B \frac{1}{p(x)}dx \quad \text{and} \quad \chi(B) = \int_B q(x)dx$$

for each Borel set B. Then some function f lies in $\mathfrak{D}_\tau$ if and only if f as well as its quasi-derivative $f^{[1]} = pf'$ are locally absolutely continuous (with respect to Lebesgue measure). In this case

$$\tau f(x) = \frac{1}{r(x)}\left(-\frac{d}{dx}\left(p(x)\frac{df}{dx}(x)\right) + q(x)f(x)\right), \quad x \in (a,b)$$

is the usual Sturm–Liouville differential expression.

Moreover, choosing the measures

$$\varrho(B) = \sum_{n\in\mathbb{Z}} \delta_n(B), \quad \varsigma(B) = \int_B \frac{1}{p_{\lfloor x\rfloor}}dx \quad \text{and} \quad \chi(B) = \sum_{n\in\mathbb{Z}} q_n\delta_n(B),$$

where $p_n \neq 0$, $q_n \in \mathbb{R}$ and δ_n is the Dirac measure in $n \in \mathbb{Z}$ we obtain the usual Jacobi difference expression. In fact, $\tau f(n)$ at some point $n \in \mathbb{Z}$ is equal to the jump of the function

$$-p_{n-1}\mathbb{1}_{(n-1,n]}(x)f'(x) + \sum_{n\leq x} q_n f(n), \quad x \in \mathbb{R}$$

in that point and hence

$$\tau f(n) = p_{n-1}(f(n) - f(n-1)) - p_n(f(n+1) - f(n)) + q_n f(n).$$

Now from the theory of linear measure differential equations (see Appendix A for the required results) we get an existence and uniqueness theorem for differential equations associated with τ.

Theorem 3.1. *Fix some arbitrary function $g \in L^1_{\mathrm{loc}}((a,b);\varrho)$. Then there is a unique solution $f \in \mathfrak{D}_\tau$ of the initial value problem*

$$(\tau - z)f = g \quad \textit{with} \quad f(c) = d_1 \quad \textit{and} \quad f^{[1]}(c) = d_2 \tag{3.2}$$

for each $z \in \mathbb{C}$, $c \in (a,b)$ and d_1, $d_2 \in \mathbb{C}$ if and only if

$$\varrho(\{x\})\varsigma(\{x\}) = 0 \quad \textit{and} \quad \chi(\{x\})\varsigma(\{x\}) \neq 1 \tag{3.3}$$

for all $x \in (a,b)$. If in addition g, d_1, d_2 and z are real, then the solution is real.

Proof. Some function $f \in \mathfrak{D}_\tau$ is a solution of $(\tau - z)f = g$ with $f(c) = d_1$ and $f^{[1]}(c) = d_2$ if and only if for each $x \in (a,b)$

$$\begin{aligned} f(x) &= d_1 + \int_c^x f^{[1]} d\varsigma, \\ f^{[1]}(x) &= d_2 + \int_c^x f d\chi - \int_c^x (zf + g)\, d\varrho. \end{aligned}$$

Now set $\omega = |\varsigma| + |\chi| + |\varrho|$ and let m_{12}, m_{21} and f_2 be the Radon–Nikodým derivatives of ς, $\chi - z\varrho$ and $g\varrho$ with respect to ω. Then these equations can for each $x \in (a,b)$ be written as

$$\begin{pmatrix} f(x) \\ f^{[1]}(x) \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} + \int_c^x \begin{pmatrix} 0 & m_{12} \\ m_{21} & 0 \end{pmatrix} \begin{pmatrix} f \\ f^{[1]} \end{pmatrix} d\omega + \int_c^x \begin{pmatrix} 0 \\ f_2 \end{pmatrix} d\omega.$$

Hence the claim follows from Theorem A.2, since (3.3) holds for all $x \in (a,b)$ if and only if

$$I + \omega(\{x\}) \begin{pmatrix} 0 & m_{12}(x) \\ m_{21}(x) & 0 \end{pmatrix} = \begin{pmatrix} 1 & \varsigma(\{x\}) \\ \chi(\{x\}) - z\varrho(\{x\}) & 1 \end{pmatrix}$$

is regular for all $z \in \mathbb{C}$ and $x \in (a,b)$. $\square$

Note that if $g \in L^1_{\mathrm{loc}}((a,b);\varrho)$ and (3.3) holds for each $x \in (a,b)$, then there is also a unique solution of the initial value problem

$$(\tau - z)f = g \quad \text{with} \quad f(c+) = d_1 \quad \text{and} \quad f^{[1]}(c+) = d_2$$

for every $z \in \mathbb{C}$, $c \in (a,b)$, d_1, $d_2 \in \mathbb{C}$ by Corollary A.3.

Because of Theorem 3.1, in the following we will always assume that the measure ς has no point masses in common with ϱ or χ, i.e.

$$\varsigma(\{x\})\varrho(\{x\}) = \varsigma(\{x\})\chi(\{x\}) = 0 \tag{3.4}$$

for all $x \in (a,b)$. This assumption is stronger than the one needed in Theorem 3.1 but we will need it for the Lagrange identity below.

For f, $g \in \mathfrak{D}_\tau$ we define the Wronski determinant

$$W(f,g)(x) = f(x)g^{[1]}(x) - f^{[1]}(x)g(x), \quad x \in (a,b). \tag{3.5}$$

This function is locally absolutely continuous with respect to ϱ with

$$\frac{dW(f,g)}{d\varrho} = g\,\tau f - f\,\tau g.$$

Indeed, this is a simple consequence of the following Lagrange identity.

Proposition 3.2. *For each f, $g \in \mathfrak{D}_\tau$ and α, $\beta \in (a,b)$ we have*

$$\int_\alpha^\beta g(x)\tau f(x) - f(x)\tau g(x) d\varrho(x) = W(f,g)(\beta) - W(f,g)(\alpha). \tag{3.6}$$

Proof. By definition g is a distribution function of the measure $g^{[1]}\varsigma$. Furthermore, the function

$$f_1(x) = -f^{[1]}(x) + \int_\alpha^x f d\chi, \quad x \in (a,b)$$

is a distribution function of $\tau f \varrho$. Hence one gets from integration by parts

$$\int_\alpha^\beta g(t)\tau f(t) d\varrho(t) = [f_1(t)g(t)]_{t=\alpha}^\beta - \int_\alpha^\beta f_1(t+)g^{[1]}(t)d\varsigma(t).$$

We can drop the right-hand limit in the integral since the discontinuities of f_1 are a null set with respect to ς by (3.4). Hence the integral becomes

$$\begin{aligned}\int_\alpha^\beta f_1(t)g^{[1]}(t)d\varsigma(t) &= \int_\alpha^\beta \int_\alpha^t f d\chi\, g^{[1]}(t)d\varsigma(t) - \int_\alpha^\beta f^{[1]}(t)g^{[1]}(t)d\varsigma(t) \\ &= g(\beta)\int_\alpha^\beta f d\chi - \int_\alpha^\beta g f d\chi - \int_\alpha^\beta f^{[1]}(t)g^{[1]}(t)d\varsigma(t),\end{aligned}$$

where we performed another integration by parts (and used again (3.4)). Now verifying the identity is an easy calculation. $\square$

As a consequence of the Lagrange identity, the Wronskian $W(u_1, u_2)$ of two solutions u_1, $u_2 \in \mathfrak{D}_\tau$ of $(\tau - z)u = 0$ is constant. Furthermore, we have

$$W(u_1, u_2) \neq 0 \quad \Leftrightarrow \quad u_1,\ u_2 \text{ linearly independent.}$$

Indeed, the Wronskian of two linearly dependent solutions vanishes obviously. Conversely $W(u_1, u_2) = 0$ means that the vectors

$$\begin{pmatrix} u_1(x) \\ u_1^{[1]}(x) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u_2(x) \\ u_2^{[1]}(x) \end{pmatrix}$$

are linearly dependent for each $x \in (a, b)$. But because of uniqueness of solutions this implies the linear dependence of u_1 and u_2.

For every $z \in \mathbb{C}$ we call two linearly independent solutions of $(\tau - z)u = 0$ a fundamental system of $(\tau - z)u = 0$. From the existence and uniqueness theorem and the properties of the Wronskian, one sees that fundamental systems always exist.

Proposition 3.3. *Let $z \in \mathbb{C}$ and u_1, u_2 be a fundamental system of the equation $(\tau - z)u = 0$. Furthermore, let $c \in (a, b)$, $d_1, d_2 \in \mathbb{C}$, $g \in L^1_{\mathrm{loc}}((a, b); \varrho)$. Then there exist c_1, $c_2 \in \mathbb{C}$ such that the solution f of*

$$(\tau - z)f = g \quad \textit{with} \quad f(c) = d_1 \quad \textit{and} \quad f^{[1]}(c) = d_2$$

is given by

$$\begin{aligned} f(x) &= c_1 u_1(x) + c_2 u_2(x) + \int_c^x \frac{u_1(x)u_2(t) - u_1(t)u_2(x)}{W(u_1, u_2)} g(t)d\varrho(t), \\ f^{[1]}(x) &= c_1 u_1^{[1]}(x) + c_2 u_2^{[1]}(x) + \int_c^x \frac{u_1^{[1]}(x)u_2(t) - u_1(t)u_2^{[1]}(x)}{W(u_1, u_2)} g(t)d\varrho(t), \end{aligned}$$

for each $x \in (a, b)$. If u_1, u_2 is the fundamental system with

$$u_1(c) = u_2^{[1]}(c) = 1 \quad \textit{and} \quad u_1^{[1]}(c) = u_2(c) = 0,$$

then $c_1 = d_1$ and $c_2 = d_2$.

Proof. We set

$$h(x) = u_1(x)\int_c^x u_2 g\, d\varrho - u_2(x)\int_c^x u_1 g\, d\varrho, \quad x \in (a, b).$$

Integration by parts shows that

$$\int_\alpha^\beta u_1^{[1]}(x)\int_c^x u_2 g\,d\varrho - u_2^{[1]}(x)\int_c^x u_1 g\,d\varrho\,d\varsigma(x) = \\ = \left[u_1(x)\int_c^x u_2 g\,d\varrho - u_2(x)\int_c^x u_1 g\,d\varrho\right]_{x=\alpha}^{\beta}$$

for all α, $\beta\in(a,b)$ with $\alpha<\beta$, hence

$$h^{[1]}(x) = u_1^{[1]}(x)\int_c^x u_2 g\,d\varrho - u_2^{[1]}(x)\int_c^x u_1 g\,d\varrho, \quad x\in(a,b).$$

Using again integration by parts we get

$$\begin{aligned}\int_\alpha^\beta u_1(x)\int_c^x u_2 g\,d\varrho\,d\chi(x) - z\int_\alpha^\beta u_1(x)\int_c^x u_2 g\,d\varrho\,d\varrho(x) = \\ &= \left[\int_c^x u_2 g\,d\varrho\left(\int_c^x u_1\,d\chi - z\int_c^x u_1\,d\varrho\right)\right]_{x=\alpha}^{\beta} \\ &\quad - \int_\alpha^\beta\left(\int_c^x u_1\,d\chi - z\int_c^x u_1\,d\varrho\right)u_2(x)g(x)d\varrho(x) \\ &= \left[\int_c^x u_2 g\,d\varrho\left(u_1^{[1]}(x) - u_1^{[1]}(c)\right)\right]_{x=\alpha}^{\beta} \\ &\quad - \int_\alpha^\beta\left(u_1^{[1]}(x) - u_1^{[1]}(c)\right)u_2(x)g(x)d\varrho(x) \\ &= u_1^{[1]}(\beta)\int_c^\beta u_2 g\,d\varrho - u_1^{[1]}(\alpha)\int_c^\alpha u_2 g\,d\varrho - \int_\alpha^\beta u_2 u_1^{[1]} g\,d\varrho\end{aligned}$$

for all $\alpha,\beta\in(a,b)$ with $\alpha<\beta$. Now an easy calculation shows that

$$\int_\alpha^\beta h\,d\chi - \int_\alpha^\beta zh + W(u_1,u_2)g\,d\varrho = h^{[1]}(\beta) - h^{[1]}(\alpha).$$

Hence h is a solution of $(\tau - z)h = W(u_1,u_2)g$ and therefore the function f given in the claim is a solution of $(\tau - z)f = g$. Now if we choose

$$c_1 = \frac{W(f,u_2)(c)}{W(u_1,u_2)(c)} \quad\text{and}\quad c_2 = \frac{W(u_1,f)(c)}{W(u_1,u_2)(c)},$$

then f satisfies the initial conditions at c. □

Another important identity for the Wronskian is the following Plücker identity.

Proposition 3.4. *For each functions f_1, f_2, f_3, $f_4\in\mathfrak{D}_\tau$ we have*

$$0 = W(f_1,f_2)W(f_3,f_4) + W(f_1,f_3)W(f_4,f_2) + W(f_1,f_4)W(f_2,f_3).$$

Proof. The right-hand side is equal to the determinant

$$\frac{1}{2}\begin{vmatrix} f_1 & f_2 & f_3 & f_4 \\ f_1^{[1]} & f_2^{[1]} & f_3^{[1]} & f_4^{[1]} \\ f_1 & f_2 & f_3 & f_4 \\ f_1^{[1]} & f_2^{[1]} & f_3^{[1]} & f_4^{[1]} \end{vmatrix}.$$

□

We say τ is regular at a, if $|\varrho|((a,c])$, $|\varsigma|((a,c])$ and $|\chi|((a,c])$ are finite for one (and hence for all) $c \in (a,b)$. Similarly one defines regularity for the right endpoint b. Finally, we say τ is regular if τ is regular at both endpoints, i.e. if $|\varrho|$, $|\varsigma|$ and $|\chi|$ are finite.

Theorem 3.5. *Let τ be regular at a, $z \in \mathbb{C}$ and $g \in L^1((a,c);\varrho)$ for each $c \in (a,b)$. Then for every solution f of $(\tau - z)f = g$ the limits*

$$f(a) := \lim_{x \to a} f(x) \quad \text{and} \quad f^{[1]}(a) := \lim_{x \to a} f^{[1]}(x)$$

exist and are finite. For each d_1, $d_2 \in \mathbb{C}$ there is a unique solution of

$$(\tau - z)f = g \quad \text{with} \quad f(a) = d_1 \quad \text{and} \quad f^{[1]}(a) = d_2.$$

Furthermore, if g, d_1, d_2 and z are real, then the solution is real. Similar results hold for the right endpoint b.

Proof. The first part of the theorem is an immediate consequence of Theorem A.4. From Proposition 3.3 we infer that all solutions of $(\tau - z)f = g$ are given by

$$f(x) = c_1 u_1(x) + c_2 u_2(x) + f_0(x), \quad x \in (a,b),$$

where c_1, $c_2 \in \mathbb{C}$, u_1, u_2 are a fundamental system of $(\tau - z)u = 0$ and f_0 is some solution of $(\tau - z)f = g$. Now since

$$W(u_1,u_2)(a) = u_1(a)u_2^{[1]}(a) - u_1^{[1]}(a)u_2(a) \neq 0,$$

there is exactly one choice for the coefficients $c_1, c_2 \in \mathbb{C}$ such that the solution f satisfies the initial values at a. If g, d_1, d_2 and z are real then u_1, u_2, and f_0 can be chosen real and hence also c_1 and c_2 are real. □

Under the assumptions of Theorem 3.5 one sees that Proposition 3.3 remains valid even in the case when $c = a$ (respectively $c = b$) with essentially the same proof.

We now turn to analytic dependence of solutions on the spectral parameter $z \in \mathbb{C}$. These results will be needed in Section 9.

Theorem 3.6. *Let $g \in L^1_{\mathrm{loc}}((a,b);\varrho)$, $c \in (a,b)$, $d_1, d_2 \in \mathbb{C}$ and for each $z \in \mathbb{C}$ let f_z be the unique solution of*

$$(\tau - z)f = g \quad \text{with} \quad f(c) = d_1 \quad \text{and} \quad f^{[1]}(c) = d_2.$$

Then $f_z(x)$ and $f_z^{[1]}(x)$ are entire functions of order at most $1/2$ in z for every point $x \in (a,b)$. Moreover, for each α, $\beta \in (a,b)$ with $\alpha < \beta$ we have

$$|f_z(x)| + |f_z^{[1]}(x)| \leq C \mathrm{e}^{B\sqrt{|z|}}, \quad x \in [\alpha,\beta], \ z \in \mathbb{C}$$

for some constants C, $B \in \mathbb{R}$.

Proof. The analyticity part follows by applying Theorem A.5 to the equivalent system from the proof of Theorem 3.1. For the remaining part note that because of Proposition 3.3 it suffices to consider the case when g vanishes identically. If we set for each $z \in \mathbb{C}$ with $|z| \geq 1$

$$v_z(x) = |z||f_z(x)|^2 + |f_z^{[1]}(x)|^2, \quad x \in (a,b),$$

an integration by parts shows that for each $x \in (a,b)$

$$\begin{aligned} v_z(x) = v_z(c) + |z| \int_c^x \left(f_z f_z^{[1]*} + f_z^{[1]} f_z^* \right) d\varsigma \\ + \int_c^x \left(f_z f_z^{[1]*} + f_z^{[1]} f_z^* \right) d\chi - \int_c^x \left(z f_z f_z^{[1]*} + z^* f_z^{[1]} f_z^* \right) d\varrho. \end{aligned}$$

Because of the elementary estimate

$$2|f_z(x) f_z^{[1]}(x)| \le \frac{|z||f_z(x)|^2 + |f_z^{[1]}(x)|^2}{\sqrt{|z|}} = \frac{v_z(x)}{\sqrt{|z|}}, \quad x \in (a,b),$$

we get an upper bound for v_z

$$v_z(x) \le v_z(c) + \int_c^x v_z(t) \sqrt{|z|} d\omega(t), \quad x \in [c,b),$$

where $\omega = |\varsigma| + |\chi| + |\varrho|$, as in the proof of Theorem 3.1. Now an application of Lemma A.1 yields

$$v_z(x) \le v_z(c) \mathrm{e}^{\int_c^x \sqrt{|z|} d\omega}, \quad x \in [c,b).$$

To the left-hand side of c we have

$$v_z(x+) \le v_z(c) + \int_{x+}^c v_z(t) \sqrt{|z|} d\omega(t), \quad x \in (a,c)$$

and hence again by the Gronwall lemma A.1

$$v_z(x+) \le v_z(c) \mathrm{e}^{\int_{x+}^c \sqrt{|z|} d\omega}, \quad x \in (a,c),$$

which is the required bound. □

Under the assumptions of Theorem 3.6 also the right-hand limits of f_z and their quasi-derivatives are entire functions in z of order at most $1/2$ with corresponding bounds. Moreover, the same analytic properties are true for the solutions f_z of the initial value problem

$$(\tau - z) f = g \quad \text{with} \quad f_z(c+) = d_1 \quad \text{and} \quad f_z^{[1]}(c+) = d_2.$$

Indeed, this fact follows for example from the remark after the proof of Theorem A.5 in Appendix A.

Furthermore, if, in addition to the assumptions of Theorem 3.6, τ is regular at a and g is integrable near a, then the functions

$$z \mapsto f_z(a) \quad \text{and} \quad z \mapsto f_z^{[1]}(a)$$

are entire of order at most $1/2$ as well and the bound in Theorem 3.6 holds for all $x \in [a, \beta]$. Indeed, this follows since the entire functions

$$z \mapsto f_z(x) \quad \text{and} \quad z \mapsto f_z^{[1]}(x)$$

are locally bounded, uniformly in $x \in (a,c)$. Moreover, in this case the assertions of Theorem 3.6 are valid even if we take $c = a$ and/or $\alpha = a$. This follows from the construction of the solution in the proof of Theorem 3.5, whereas the bound is proven as in the general case (hereby note that ω is finite near a since τ is regular there).

We gather the assumptions made on the coefficients so far and add some new which are needed in the sequel. Therefore we say that some interval

(α,β) is a gap of $\operatorname{supp}(\varrho)$ if it is contained in the complement of $\operatorname{supp}(\varrho)$ but the endpoints α and β lie in $\operatorname{supp}(\varrho)$.

Hypothesis 3.7.

(i) *The measure ϱ is positive.*
(ii) *The measure χ is real-valued.*
(iii) *The measure ς is real-valued and supported on the whole interval;*

$$\operatorname{supp}(\varsigma) = (a,b).$$

(iv) *The measure ς has no point masses in common with ϱ or χ, i.e.*

$$\varsigma(\{x\})\chi(\{x\}) = \varsigma(\{x\})\varrho(\{x\}) = 0.$$

(v) *For each gap (α,β) of $\operatorname{supp}(\varrho)$ and every function $f\in\mathfrak{D}_\tau$ with the outer limits $f(\alpha-) = f(\beta+) = 0$ we have $f(x)=0$, $x\in(\alpha,\beta)$.*
(vi) *The measure ϱ is supported on more than one point.*

As a consequence of the real-valuedness of the measures, τ is a real differential expression, i.e. $f\in\mathfrak{D}_\tau$ if and only if $f^*\in\mathfrak{D}_\tau$ and $\tau f^* = (\tau f)^*$ in this case. Moreover, ϱ has to be positive in order to obtain a definite inner product later. Furthermore, condition (v) in Hypothesis 3.7 is crucial for Proposition 3.9 and Proposition 3.10 to hold. In fact, if $(0,2\pi)$ is a gap in the support of ϱ and we choose ς and $-\chi$ to be equal to the Lebesgue measure, then the function $f(x)=\sin(x)$ for $x\in(0,2\pi)$ and $f(x)=0$ else lies in $\mathfrak{D}_\tau$ with $\tau f = 0$. However, this condition is satisfied by a large class of measures as the next lemma shows.

Lemma 3.8. *If for each gap (α,β) of $\operatorname{supp}(\varrho)$ the measures $\varsigma|_{(\alpha,\beta)}$ and $\chi|_{(\alpha,\beta)}$ are of one and the same sign, then (v) in Hypothesis (3.7) holds.*

Proof. Let (α,β) be a gap of $\operatorname{supp}(\varrho)$ and $f\in\mathfrak{D}_\tau$ with $f(\alpha-)=f(\beta+)=0$. As in the proof of Proposition 3.2, integration by parts yields

$$\begin{aligned} f(\beta)^*\tau f(\beta)\varrho(\{\beta\}) &= \int_\alpha^{\beta+} \tau f(x) f(x)^* d\varrho(x) \\ &= \int_\alpha^{\beta+} |f^{[1]}(x)|^2 d\varsigma(x) + \int_\alpha^{\beta+} |f(x)|^2\, d\chi(x). \end{aligned}$$

Now the left-hand side vanishes since either $\varrho(\{\beta\})=0$ or f is continuous in β and hence $f(\beta)=f(\beta+)=0$. Thus $f^{[1]}$ vanishes almost everywhere with respect to ς, i.e. $f^{[1]}$ vanishes in (α,β) and f is constant in (α,β). Now since $f(\beta+) = f(\beta) + f^{[1]}(\beta)\varsigma(\{\beta\})$, we infer that f vanishes in (α,β) □

The theory we are going to develop from now on is not applicable if the support of ϱ consists of not more than one point, since in this case $L^1_{\mathrm{loc}}((a,b);\varrho)$ is only one-dimensional (and hence all solutions of $(\tau-z)u=0$ are linearly dependent). In particular, the essential Proposition 3.9 does not hold in this case. Hence we have to exclude this case from now on. Nevertheless this case is important, in particular for applications to the isospectral problem of the Camassa–Holm equation. Hence we will treat the case when $\operatorname{supp}(\varrho)$ consists of only one point separately in Appendix C.

Our aim is to introduce linear operators in the Hilbert space $L^2((a,b);\varrho)$, induced by the differential expression τ. As a first step we define a linear relation T_{loc} of $L^1_{\mathrm{loc}}((a,b);\varrho)$ into $L^1_{\mathrm{loc}}((a,b);\varrho)$ by

$$T_{\mathrm{loc}} = \{(f,\tau f)\,|\, f\in\mathfrak{D}_\tau\} \subseteq L^1_{\mathrm{loc}}((a,b);\varrho)\times L^1_{\mathrm{loc}}((a,b);\varrho).$$

For a brief introduction to the theory of linear relations we refer to Appendix B and the references cited there. Now, in contrast to the classical case, in general $\mathfrak{D}_\tau$ is not embedded in $L^1_{\mathrm{loc}}((a,b);\varrho)$, i.e. T_{loc} is multi-valued. Instead we have the following result, which is important for our approach. For later use, we introduce the abbreviations

$$\alpha_\varrho = \inf\operatorname{supp}(\varrho) \quad\text{and}\quad \beta_\varrho = \sup\operatorname{supp}(\varrho)$$

for the endpoints of the convex hull of the support of ϱ.

Proposition 3.9. *The linear map*

$$\begin{array}{ccc} \mathfrak{D}_\tau & \to & T_{\mathrm{loc}} \\ f & \mapsto & (f,\tau f) \end{array}$$

is bijective.

Proof. Clearly this mapping is linear and onto T_{loc} by definition. Now let $f\in\mathfrak{D}_\tau$ such that $f=0$ almost everywhere with respect to ϱ. We will show that f is of the form

$$f(x) = \begin{cases} c_a u_a(x), & \text{if } x\in(a,\alpha_\varrho], \\ 0, & \text{if } x\in(\alpha_\varrho,\beta_\varrho], \\ c_b u_b(x), & \text{if } x\in(\beta_\varrho,b), \end{cases} \tag{3.7}$$

where c_a, $c_b\in\mathbb{C}$ and u_a, u_b are the solutions of $\tau u = 0$ with

$$u_a(\alpha_\varrho-) = u_b(\beta_\varrho+) = 0 \quad\text{and}\quad u_a^{[1]}(\alpha_\varrho-) = u_b^{[1]}(\beta_\varrho+) = 1.$$

Obviously we have $f(x)=0$ for all x in the interior of $\operatorname{supp}(\varrho)$ and points of mass of ϱ. Now if (α,β) is a gap of $\operatorname{supp}(\varrho)$, then since α, $\beta\in\operatorname{supp}(\varrho)$ at least we have $f(\alpha-)=f(\beta+)=0$ and hence $f(x)=0$, $x\in[\alpha,\beta]$ by Hypothesis 3.7. Hence all points $x\in(\alpha_\varrho,\beta_\varrho)$ for which possibly $f(x)\neq 0$, lie on the boundary of $\operatorname{supp}(\varrho)$ such that there are monotone sequences $x_{+,n}$, $x_{-,n}\in\operatorname{supp}(\varrho)$ with $x_{+,n}\downarrow x$ and $x_{-,n}\uparrow x$. Then for each $n\in\mathbb{N}$, we either have $f(x_{-,n}+)=0$ or $f(x_{-,n}-)=0$, hence

$$f(x-) = \lim_{n\to\infty} f(x_{-,n}-) = \lim_{n\to\infty} f(x_{-,n}+) = 0.$$

Similarly one shows that also $f(x+)=0$. Now since f is a solution of $\tau u=0$ outside of $[\alpha,\beta]$, it remains to show that $f(\alpha_\varrho)=f(\beta_\varrho)=0$. Therefore assume that f is not continuous in α_ϱ, i.e. that $\varsigma(\{\alpha_\varrho\})\neq 0$. Then $f^{[1]}$ is continuous in α_ϱ and hence $f^{[1]}(\alpha_\varrho)=0$. But this yields

$$f(\alpha_\varrho-) = f(\alpha_\varrho+) - f^{[1]}(\alpha_\varrho)\varsigma(\{\alpha_\varrho\}) = 0.$$

Hence f is of the claimed form. Furthermore, a simple calculation yields

$$\tau f = c_a \mathbb{1}_{\{\alpha_\varrho\}} - c_b \mathbb{1}_{\{\beta_\varrho\}}. \tag{3.8}$$

Now in order to prove that our mapping is one-to-one let $f\in\mathfrak{D}_\tau$ be such that $f=0$ and $\tau f=0$ almost everywhere with respect to ϱ. By the existence and uniqueness theorem it suffices to prove that $f(c)=f^{[1]}(c)=0$

at some point $c \in (a,b)$. But this is valid for all points between α_ϱ and β_ϱ by the first part of the proof. $\square$

In the following we will always identify the elements of the linear relation T_{loc} with functions in $\mathfrak{D}_\tau$. Hence some element $f \in T_{\mathrm{loc}}$ is always identified with some function $f \in \mathfrak{D}_\tau$, which is an $AC_{\mathrm{loc}}((a,b);\varsigma)$ representative of the first component of f (as an element of T_{loc}) and $\tau f \in L^1_{\mathrm{loc}}((a,b);\varrho)$ is the second component of f (again as an element of T_{loc}). In general the relation T_{loc} is multi-valued, i.e.

$$\mathrm{mul}(T_{\mathrm{loc}}) = \left\{ g \in L^1_{\mathrm{loc}}((a,b);\varrho) \,|\, (0,g) \in T_{\mathrm{loc}} \right\} \neq \{0\}.$$

In view of the formulation of the next result, note that ϱ has no mass in a and b by convention.

Proposition 3.10. *The multi-valued part of T_{loc} is given by*

$$\mathrm{mul}(T_{\mathrm{loc}}) = \mathrm{span}\left\{ \mathbb{1}_{\{\alpha_\varrho\}}, \mathbb{1}_{\{\beta_\varrho\}} \right\}.$$

In particular

$$\dim \mathrm{mul}(T_{\mathrm{loc}}) = \begin{cases} 0, & \text{if } \varrho \text{ has neither mass in } \alpha_\varrho \text{ nor in } \beta_\varrho, \\ 1, & \text{if } \varrho \text{ has either mass in } \alpha_\varrho \text{ or in } \beta_\varrho, \\ 2, & \text{if } \varrho \text{ has mass in } \alpha_\varrho \text{ and in } \beta_\varrho. \end{cases}$$

Hence T_{loc} is an operator if and only if ϱ has neither mass in α_ϱ nor in β_ϱ.

Proof. Let $(f, \tau f) \in T_{\mathrm{loc}}$ with $f = 0$ almost everywhere with respect to ϱ. In the proof of Proposition 3.9 we saw that such an f is of the form (3.7) and τf is a linear combination of $\mathbb{1}_{\{\alpha_\varrho\}}$ and $\mathbb{1}_{\{\beta_\varrho\}}$ by (3.8). It remains to prove that $\mathrm{mul}(T_{\mathrm{loc}})$ indeed contains $\mathbb{1}_{\{\alpha_\varrho\}}$ if ϱ has mass in α_ϱ. Therefore consider the function

$$f(x) = \begin{cases} u_a(x), & \text{if } x \in (a, \alpha_\varrho], \\ 0, & \text{if } x \in (\alpha_\varrho, b). \end{cases}$$

One easily checks that f lies in $\mathfrak{D}_\tau$ and hence $(0, \mathbb{1}_{\{\alpha_\varrho\}}) = (f, \tau f) \in T_{\mathrm{loc}}$. Similarly one shows that $\mathbb{1}_{\{\beta_\varrho\}}$ indeed lies in $\mathrm{mul}(T_{\mathrm{loc}})$ if ϱ has mass in β_ϱ. Furthermore, note that $\mathbb{1}_{\{\alpha_\varrho\}} = 0$ (respectively $\mathbb{1}_{\{\beta_\varrho\}} = 0$) as functions in $L^1_{\mathrm{loc}}((a,b);\varrho)$ provided that ϱ has no mass in α_ϱ (respectively in β_ϱ). $\square$

In contrast to the classical case one can not define a proper Wronskian for elements in $\mathrm{dom}(T_{\mathrm{loc}})$, instead we define the Wronskian of two elements f, g of the linear relation T_{loc} as

$$W(f,g)(x) = f(x) g^{[1]}(x) - f^{[1]}(x) g(x), \quad x \in (a,b).$$

The Lagrange identity then takes the form

$$W(f,g)(\beta) - W(f,g)(\alpha) = \int_\alpha^\beta g(x)\tau f(x) - f(x)\tau g(x) d\varrho(x).$$

Furthermore, note that by the existence and uniqueness theorem we have

$$\mathrm{ran}(T_{\mathrm{loc}} - z) = L^1_{\mathrm{loc}}((a,b);\varrho) \quad \text{and} \quad \dim\ker(T_{\mathrm{loc}} - z) = 2 \tag{3.9}$$

for each $z \in \mathbb{C}$.

4. Sturm–Liouville relations

In this section we will restrict the differential relation T_{loc} in order to obtain a linear relation in the Hilbert space $L^2((a,b);\varrho)$ with scalar product

$$\langle f,g\rangle = \int_a^b f(x)g(x)^* d\varrho(x).$$

First we define the maximal relation $T_{\max}$ in $L^2((a,b);\varrho)$ by

$$T_{\max} = \{(f,\tau f)\in T_{\mathrm{loc}}\,|\, f\in L^2((a,b);\varrho), \tau f\in L^2((a,b);\varrho)\}. \tag{4.1}$$

In general $T_{\max}$ is not an operator. Indeed we have

$$\mathrm{mul}(T_{\max}) = \mathrm{mul}(T_{\mathrm{loc}}),$$

since all elements of $\mathrm{mul}(T_{\mathrm{loc}})$ are square integrable with respect to ϱ. In order to obtain a symmetric relation we restrict the maximal relation $T_{\max}$ to functions with compact support

$$T_0 = \{(f,\tau f)\,|\, f\in\mathfrak{D}_\tau,\ \mathrm{supp}(f) \text{ compact in } (a,b)\}. \tag{4.2}$$

Indeed, this relation T_0 is an operator as we will see later.

Since τ is a real differential expression, the relations T_0 and $T_{\max}$ are real with respect to the natural conjugation in $L^2((a,b);\varrho)$, i.e. if $f\in T_{\max}$ (respectively $f\in T_0$), then also $f^*\in T_{\max}$ (respectively $f^*\in T_0$), where the conjugation is defined componentwise.

We say some measurable function f lies in $L^2((a,b);\varrho)$ near a (respectively near b) if f lies in $L^2((a,c);\varrho)$ (respectively in $L^2((c,b);\varrho)$) for all $c\in(a,b)$. Furthermore, we say some $f\in T_{\mathrm{loc}}$ lies in $T_{\max}$ near a (respectively near b) if f and τf both lie in $L^2((a,b);\varrho)$ near a (respectively near b). One easily sees that some $f\in T_{\mathrm{loc}}$ lies in $T_{\max}$ near a (respectively b) if and only if f^* lies in $T_{\max}$ near a (respectively b).

Proposition 4.1. *Let τ be regular at a and f lie in $T_{\max}$ near a. Then both limits*

$$f(a) := \lim_{x\to a} f(x) \quad \textit{and} \quad f^{[1]}(a) := \lim_{x\to a} f^{[1]}(x)$$

exist and are finite. A similar result holds at b.

Proof. Under this assumptions τf lies in $L^2((a,b);\varrho)$ near a and since ϱ is a finite measure near a we have $\tau f\in L^1((a,c);\varrho)$ for each $c\in(a,b)$. Hence the claim follows from Theorem 3.5. □

From the Lagrange identity we now get the following lemma.

Lemma 4.2. *If f and g lie in $T_{\max}$ near a, then the limit*

$$W(f,g^*)(a) := \lim_{\alpha\to a} W(f,g^*)(\alpha)$$

exists and is finite. A similar result holds at b. If f, $g\in T_{\max}$, then

$$\langle \tau f, g\rangle - \langle f,\tau g\rangle = W(f,g^*)(b) - W(f,g^*)(a) =: W_a^b(f,g^*). \tag{4.3}$$

Proof. If f and g lie in $T_{\max}$ near a, then the limit $\alpha\to a$ of the left-hand side in equation (3.6) exists. Hence also the limit in the claim exists. Now the remaining part follows by taking the limits $\alpha\to a$ and $\beta\to b$. □

If τ is regular at a and f and g lie in $T_{\max}$ near a, then we clearly have

$$W(f,g^*)(a) = f(a)g^{[1]}(a)^* - f^{[1]}(a)g(a)^*.$$

In order to determine the adjoint of T_0

$$T_0^* = \{(f,g) \in L^2((a,b);\varrho) \times L^2((a,b);\varrho) \,|\, \forall (u,v) \in T_0 : \langle f, v\rangle = \langle g, u\rangle\},$$

as in the classical theory, we need the following lemma (see [31, Lemma 9.3]).

Lemma 4.3. *Let V be a vector space over $\mathbb{C}$ and $F_1, \dots, F_n, F \in V^*$, then*

$$F \in \operatorname{span}\{F_1, \dots, F_n\} \quad \Leftrightarrow \quad \bigcap_{i=1}^n \ker F_i \subseteq \ker F.$$

Theorem 4.4. *The adjoint of T_0 is $T_{\max}$.*

Proof. From Lemma 4.2 one immediately gets $T_{\max} \subseteq T_0^*$. Indeed, for each $f \in T_0$ and $g \in T_{\max}$ we have

$$\langle \tau f, g\rangle - \langle f, \tau g\rangle = \lim_{\beta\to b} W(f,g^*)(\beta) - \lim_{\alpha\to a} W(f,g^*)(\alpha) = 0,$$

since $W(f,g^*)$ has compact support. Conversely let $(f,f_2) \in T_0^*$ and $\tilde{f}$ be a solution of $\tau\tilde{f} = f_2$. We expect that $(f-\tilde{f},0) \in T_{\rm loc}$. To prove this we will invoke Lemma 4.3. Therefore we consider linear functionals

$$\begin{aligned} l(g) &= \int_a^b \Big(f(x) - \tilde{f}(x)\Big)^* g(x)d\varrho(x), && g \in L_c^2((a,b);\varrho), \\ l_j(g) &= \int_a^b u_j(x)^* g(x) d\varrho(x), && g \in L_c^2((a,b);\varrho),\ j = 1,2, \end{aligned}$$

where u_j are two solutions of $\tau u = 0$ with $W(u_1,u_2) = 1$ and $L_c^2((a,b);\varrho)$ is the space of square integrable functions with compact support. For these functionals we have $\ker l_1 \cap \ker l_2 \subseteq \ker l$. Indeed let $g \in \ker l_1 \cap \ker l_2$, then the function

$$u(x) = u_1(x)\int_a^x u_2(t)g(t)d\varrho(t) + u_2(x)\int_x^b u_1(t)g(t)d\varrho(t), \quad x \in (a,b)$$

is a solution of $\tau u = g$ by Proposition 3.3 and has compact support since g lies in the kernel of l_1 and l_2, hence $u \in T_0$. Then the Lagrange identity and the definition of the adjoint yields

$$\begin{aligned} \int_a^b \Big(f(x) - \tilde{f}(x)\Big)^* \tau u(x) d\varrho(x) &= \langle \tau u, f\rangle - \int_a^b \tilde{f}(x)^* \tau u(x) d\varrho(x) \\ &= \langle u, f_2\rangle - \int_a^b \tau\tilde{f}(x)^* u(x) d\varrho(x) = 0 \end{aligned}$$

and hence $g = \tau u \in \ker l$. Now applying Lemma 4.3 there are c_1, $c_2 \in \mathbb{C}$ such that

$$(*) \qquad \int_a^b \Big(f(x) - \tilde{f}(x) + c_1 u_1(x) + c_2 u_2(x)\Big)^* g(x) d\varrho(x) = 0$$

for each function $g \in L_c^2((a,b);\varrho)$. By definition of $T_{\rm loc}$ we obviously have $(\tilde{f} + c_1u_1 + c_2u_2, f_2) \in T_{\rm loc}$. But the first component of this pair is equal to f, almost everywhere with respect to ϱ because of $(*)$. Hence we also have $(f,f_2) \in T_{\rm loc}$ and therefore $(f,f_2) \in T_{\max}$. □

By the preceding theorem T_0 is symmetric. The closure $T_{\min}$ of T_0 is called the minimal relation,

$$T_{\min} = \overline{T_0} = T_0^{**} = T_{\max}^*.$$

In order to determine $T_{\min}$ we need the following lemma on functions in the maximal relation $T_{\max}$.

Lemma 4.5. *If f_a lies in $T_{\max}$ near a and f_b lies in $T_{\max}$ near b, then there exists an $f \in T_{\max}$ such that $f = f_a$ near a and $f = f_b$ near b (regarded as functions in $\mathfrak{D}_\tau$).*

Proof. Let u_1, u_2 be a fundamental system of $\tau u = 0$ with $W(u_1, u_2) = 1$ and let α, $\beta \in (a, b)$ with $\alpha < \beta$ such that the functionals

$$F_j(g) = \int_\alpha^\beta u_j g d\varrho, \quad g \in L^2((a, b); \varrho),\ j = 1, 2$$

are linearly independent. This is possible since otherwise u_1 and u_2 would be linearly dependent in $L^2((a, b); \varrho)$ and hence also in $\mathfrak{D}_\tau$ by the identification in Lemma 3.9. First we show that there is some $u \in \mathfrak{D}_\tau$ such that

$$u(\alpha) = f_a(\alpha), \quad u^{[1]}(\alpha) = f_a^{[1]}(\alpha), \quad u(\beta) = f_b(\beta) \quad \text{and} \quad u^{[1]}(\beta) = f_b^{[1]}(\beta).$$

Indeed, let $g \in L^2((a, b); \varrho)$ and consider the solution u of $\tau u = g$ with the initial conditions

$$u(\alpha) = f_a(\alpha) \quad \text{and} \quad u^{[1]}(\alpha) = f_a^{[1]}(\alpha).$$

With Proposition 3.3 one sees that u has the desired properties if

$$\begin{pmatrix} F_2(g) \\ F_1(g) \end{pmatrix} = \begin{pmatrix} u_1(\beta) & -u_2(\beta) \\ u_1^{[1]}(\beta) & -u_2^{[1]}(\beta) \end{pmatrix}^{-1} \begin{pmatrix} f_b(\beta) - c_1 u_1(\beta) - c_2 u_2(\beta) \\ f_b^{[1]}(\beta) - c_1 u_1^{[1]}(\beta) - c_2 u_2^{[1]}(\beta) \end{pmatrix},$$

where c_1, $c_2 \in \mathbb{C}$ are the constants appearing in Proposition 3.3. But since the functionals F_1, F_2 are linearly independent, it is possible to choose a function $g \in L^2((a, b); \varrho)$ such that this equation is valid. Now the function f defined by

$$f(x) = \begin{cases} f_a(x), & \text{if } x \in (a, \alpha], \\ u(x), & \text{if } x \in (\alpha, \beta], \\ f_b(x), & \text{if } x \in (\beta, b), \end{cases}$$

has the claimed properties. □

Theorem 4.6. *The minimal relation $T_{\min}$ is given by*

$$T_{\min} = \{ f \in T_{\max} \,|\, \forall g \in T_{\max} : W(f, g)(a) = W(f, g)(b) = 0 \}. \tag{4.4}$$

Furthermore, $T_{\min}$ is an operator, i.e. $\dim \operatorname{mul}(T_{\min}) = 0$.

Proof. If $f \in T_{\min} = T_{\max}^* \subseteq T_{\max}$ we have

$$0 = \langle \tau f, g \rangle - \langle f, \tau g \rangle = W(f, g^*)(b) - W(f, g^*)(a)$$

for each $g \in T_{\max}$. Given some $g \in T_{\max}$, there is a $g_a \in T_{\max}$ such that $g_a^* = g$ in a vicinity of a and $g_a = 0$ in a vicinity of b. Therefore

$$W(f, g)(a) = W(f, g_a^*)(a) - W(f, g_a^*)(a) = 0.$$

Similarly one sees that $W(f,g)(b) = 0$ for each $g \in T_{\max}$. Conversely if $f \in T_{\max}$ such that for each $g \in T_{\max}$, $W(f,g)(a) = W(f,g)(b) = 0$, then

$$\langle \tau f, g\rangle - \langle f, \tau g\rangle = W(f,g^*)(b) - W(f,g^*)(a) = 0,$$

hence $f \in T_{\max}^* = T_{\min}$.

In order to show that $T_{\min}$ is an operator, let $f \in T_{\min}$ with $f = 0$ almost everywhere with respect to ϱ. If $\alpha_\varrho > a$ and $\varrho(\{\alpha_\varrho\}) \neq 0$, then f is of the form (3.7). From what we already proved we know that

$$W(f,u_1)(a) = W(f,u_2)(a) = 0$$

for each fundamental system u_1, u_2 of $\tau u = 0$. But $W(f,u_j)(x)$ is constant on (a,α_ϱ) and hence we infer $f(\alpha_\varrho) = f^{[1]}(\alpha_\varrho) = 0$. From this we see that f vanishes on (a,α_ϱ). Similarly one proves that f also vanishes on (β_ϱ, b), hence $f = 0$. □

For regular differential expressions we may characterize the minimal operator in terms of the boundary values of functions $f \in T_{\max}$.

Corollary 4.7. *If τ is regular at a and $f \in T_{\max}$, then we have*

$$f(a) = f^{[1]}(a) = 0 \quad \Leftrightarrow \quad \forall g \in T_{\max} : W(f,g)(a) = 0.$$

A similar result holds at b.

Proof. The claim follows from $W(f,g)(a) = f(a)g^{[1]}(a) - f^{[1]}(a)g(a)$ and the fact that one finds $g \in T_{\max}$ with prescribed initial values at a. Indeed, one can take g to coincide with some solution of $\tau u = 0$ near a. □

If the measure ϱ has no weight near some endpoint, we get another characterization for functions in $T_{\min}$ in terms of their left-hand (respectively right-hand) limit at α_ϱ (respectively at β_ϱ).

Corollary 4.8. *If $\alpha_\varrho > a$ and $f \in T_{\max}$, then we have*

$$f(\alpha_\varrho-) = f^{[1]}(\alpha_\varrho-) = 0 \quad \Leftrightarrow \quad \forall g \in T_{\max} : W(f,g)(a) = 0.$$

A similar result holds at b.

Proof. The Wronskian of two functions f, g which lie in $T_{\max}$ near a is constant on (a,α_ϱ) by the Lagrange identity. Hence we have

$$W(f,g)(a) = \lim_{x\uparrow\alpha_\varrho} f(x)g^{[1]}(x) - f^{[1]}(x)g(x).$$

Now the claim follows since we may find some g which lies in $T_{\max}$ near a, with prescribed left-hand limits at α_ϱ. Indeed, one may take g to be a suitable solution of $\tau u = 0$. □

Note that all functions in $T_{\min}$ vanish outside of $(\alpha_\varrho, \beta_\varrho)$. In general the operator $T_{\min}$ is, because of

$$\operatorname{dom}(T_{\min})^\perp = \operatorname{mul}(T_{\min}^*) = \operatorname{mul}(T_{\max}),$$

not densely defined. On the other side, $\operatorname{dom}(T_{\max})$ is always dense in the Hilbert space $L^2((a,b);\varrho)$ since

$$\operatorname{dom}(T_{\max})^\perp = \operatorname{mul}(T_{\max}^*) = \operatorname{mul}(T_{\min}) = \{0\}.$$

Next we will show that $T_{\min}$ always has self-adjoint extensions.

Theorem 4.9. *The deficiency indices of the minimal relation* $T_{\min}$ *are equal and at most two, i.e.*

$$n(T_{\min}) := \dim \operatorname{ran}(T_{\min} - \mathrm{i})^{\perp} = \dim \operatorname{ran}(T_{\min} + \mathrm{i})^{\perp} \leq 2. \tag{4.5}$$

Proof. The fact that the dimensions are less than two, is a consequence of the inclusion

$$\operatorname{ran}(T_{\min} \pm \mathrm{i})^{\perp} = \ker(T_{\max} \mp \mathrm{i}) \subseteq \ker(T_{\mathrm{loc}} \mp \mathrm{i}).$$

Now since $T_{\min}$ is real with respect to the natural conjugation in $L^2((a,b);\varrho)$, we see that the natural conjugation is a conjugate-linear isometry from the kernel of $T_{\max} + \mathrm{i}$ onto the kernel of $T_{\max} - \mathrm{i}$ and hence their dimensions are equal. □

5. Weyl's alternative

We say τ is in the limit-circle (l.c.) case at a, if for each $z \in \mathbb{C}$ all solutions of $(\tau - z)u = 0$ lie in $L^2((a,b);\varrho)$ near a. Furthermore, we say τ is in the limit-point (l.p.) case at a, if for each $z \in \mathbb{C}$ there is some solution of $(\tau - z)u = 0$ which does not lie in $L^2((a,b);\varrho)$ near a. Similarly one defines the l.c. and l.p. cases for the endpoint b. It is clear that τ is only either in the l.c. or in the l.p. case at some boundary point. The next lemma shows that τ indeed is in one of these cases at each endpoint.

Lemma 5.1. *If there is a* $z_0 \in \mathbb{C}$ *such that all solutions of* $(\tau - z_0)u = 0$ *lie in* $L^2((a,b);\varrho)$ *near* a*, then* τ *is in the l.c. case at* a*. A similar result holds at the endpoint* b*.*

Proof. Let $z \in \mathbb{C}$ and u be a solution of $(\tau - z)u = 0$. If u_1, u_2 are a fundamental system of $(\tau - z_0)u = 0$ with $W(u_1, u_2) = 1$, then u_1 and u_2 lie in $L^2((a,b);\varrho)$ near a by assumption. Therefore there is some $c \in (a,b)$ such that the function $v = |u_1| + |u_2|$ satisfies

$$|z - z_0| \int_a^c v^2 d\varrho \leq \frac{1}{2}.$$

Since u is a solution of $(\tau - z_0)u = (z - z_0)u$, we have for each $x \in (a,b)$

$$u(x) = c_1 u_1(x) + c_2 u_2(x) + (z - z_0) \int_c^x (u_1(x)u_2(t) - u_1(t)u_2(x))\, u(t) d\varrho(t)$$

for some constants c_1, $c_2 \in \mathbb{C}$ by Proposition 3.3. Therefore we have

$$|u(x)| \leq C v(x) + |z - z_0| v(x) \int_x^c v(t)|u(t)| d\varrho(t), \quad x \in (a,c),$$

where $C = \max(|c_1|, |c_2|)$ and furthermore, using Cauchy–Schwarz

$$|u(x)|^2 \leq 2C^2 v(x)^2 + 2|z - z_0|^2 v(x)^2 \int_x^c v(t)^2 d\varrho(t) \int_x^c |u(t)|^2 d\varrho(t).$$

Now an integration yields for each $s \in (a,c)$

$$\begin{aligned} \int_s^c |u|^2 d\varrho &\leq 2C^2 \int_a^c v^2 d\varrho + 2|z - z_0|^2 \left(\int_a^c v^2 d\varrho \right)^2 \int_s^c |u|^2 d\varrho \\ &\leq 2C^2 \int_a^c v^2 d\varrho + \frac{1}{2} \int_s^c |u|^2 d\varrho \end{aligned}$$

and therefore

$$\int_s^c |u|^2 d\varrho \le 4C^2 \int_a^c v^2 d\varrho < \infty.$$

Since $s \in (a,c)$ was arbitrary, this yields the claim. □

Theorem 5.2 (Weyl's alternative). *Each boundary point is either in the l.c. case or in the l.p. case.*

Proposition 5.3. *If τ is regular at a or if ϱ has no weight near a, then τ is in the l.c. case at a. A similar result holds at the endpoint b.*

Proof. If τ is regular at a each solution of $(\tau - z)u = 0$ can be continuously extended to a. Hence u is in $L^2((a,b);\varrho)$ near a, since ϱ is a finite measure near a. If ϱ has no weight near a, each solution lies in $L^2((a,b);\varrho)$ near a, since every solution is locally bounded. □

The set $\mathrm{r}(T_{\min})$ of points of regular type of $T_{\min}$ consists of all complex numbers $z \in \mathbb{C}$ such that $(T_{\min} - z)^{-1}$ is a bounded operator (not necessarily everywhere defined). Recall that $\dim \operatorname{ran}(T_{\min} - z)^{\perp}$ is constant on every connected component of $\mathrm{r}(T_{\min})$ ([37, Theorem 8.1]) and thus

$$\dim \operatorname{ran}(T_{\min} - z)^{\perp} = \dim \ker(T_{\max} - z^*) = n(T_{\min})$$

for every $z \in \mathrm{r}(T_{\min})$.

Lemma 5.4. *For each $z \in \mathrm{r}(T_{\min})$ there is a non-trivial solution of the equation $(\tau - z)u = 0$ which lies in $L^2((a,b);\varrho)$ near a. A similar result holds for the endpoint b.*

Proof. Let $z \in \mathrm{r}(T_{\min})$ and first assume that τ is regular at b. If there were no solutions of $(\tau - z)u = 0$ which lie in $L^2((a,b);\varrho)$ near a, we would have $\ker(T_{\max} - z) = \{0\}$ and hence $n(T_{\min}) = 0$, i.e. $T_{\min} = T_{\max}$. But since there is an $f \in T_{\max}$ with

$$f(b) = 1 \quad \text{and} \quad f^{[1]}(b) = 0,$$

this is a contradiction to Theorem 4.6.

In the general case we take some $c \in (a,b)$ and consider the minimal operator T_c in $L^2((a,c);\varrho)$ induced by $\tau|_{(a,c)}$. Then z is a point of regular type of T_c. Indeed, we can extend each $f_c \in \operatorname{dom}(T_c)$ with zero and obtain a function $f \in \operatorname{dom}(T_{\min})$. For these functions and some positive constant C we have

$$\left\|(T_c - z)f_c\right\|_c = \left\|(T_{\min} - z)f\right\| \ge C\left\|f\right\| = C\left\|f_c\right\|_c,$$

where $\|\cdot\|_c$ is the norm on $L^2((a,c);\varrho)$. Now since the solutions of the equation $(\tau|_{(a,c)} - z)u = 0$ are exactly the solutions of $(\tau - z)u = 0$ restricted to (a,c), the claim follows from what we already proved. □

Corollary 5.5. *If $z \in \mathrm{r}(T_{\min})$ and τ is in the l.p. case at a, then there is a (up to scalar multiples) unique non-trivial solution of $(\tau - z)u = 0$, which lies in $L^2((a,b);\varrho)$ near a. A similar result holds for the endpoint b.*

Proof. If there were two linearly independent solutions in $L^2((a,b);\varrho)$ near a, τ would be in the l.c. case at a. □

Lemma 5.6. *τ is in the l.p. case at a if and only if*

$$W(f,g)(a) = 0, \quad f,\, g \in T_{\max}.$$

τ is in the l.c. case at a if and only if there is a $f \in T_{\max}$ such that

$$W(f,f^*)(a) = 0 \quad \text{and} \quad W(f,g)(a) \neq 0 \quad \text{for some } g \in T_{\max}.$$

Similar results hold at the endpoint b.

Proof. Let τ be in the l.c. case at a and u_1, u_2 be a real fundamental system of $\tau u = 0$ with $W(u_1,u_2) = 1$. Both, u_1 and u_2 lie in $T_{\max}$ near a. Hence there are f, $g \in T_{\max}$ with $f = u_1$ and $g = u_2$ near a and $f = g = 0$ near b. Then we have

$$W(f,g)(a) = W(u_1,u_2)(a) = 1$$

and

$$W(f,f^*)(a) = W(u_1,u_1^*)(a) = 0$$

since u_1 is real.

Now assume τ is in the l.p. case at a and regular at b. Then $T_{\max}$ is a two-dimensional extension of $T_{\min}$, since $\dim\ker(T_{\max} - \mathrm{i}) = 1$ by Corollary 5.5. Let v, $w \in T_{\max}$ with $v = w = 0$ in a vicinity of a and

$$v(b) = w^{[1]}(b) = 1 \quad \text{and} \quad v^{[1]}(b) = w(b) = 0.$$

Then

$$T_{\max} = T_{\min} + \operatorname{span}\{v,w\},$$

since v and w are linearly independent modulo $T_{\min}$ and do not lie in $T_{\min}$. Then for each f, $g \in T_{\max}$ there are f_0, $g_0 \in T_{\min}$ such that $f = f_0$ and $g = g_0$ in a vicinity of a and therefore

$$W(f,g)(a) = W(f_0,g_0)(a) = 0.$$

Now if τ is not regular at b we take some $c \in (a,b)$. Then for each $f \in T_{\max}$ the function $f|_{(a,c)}$ lies in the maximal relation induced by $\tau|_{(a,c)}$ and the claim follows from what we already proved. □

Lemma 5.7. *Let τ be in the l.p. case at both endpoints and $z \in \mathbb{C}\backslash\mathbb{R}$. Then there is no non-trivial solution of $(\tau - z)u = 0$ in $L^2((a,b);\varrho)$.*

Proof. If $u \in L^2((a,b);\varrho)$ is a solution of $(\tau - z)u = 0$, then u^* is a solution of $(\tau - z^*)u = 0$ and both, u and u^* lie in $T_{\max}$. Now the Lagrange identity yields for each α, $\beta \in (a,b)$ with $\alpha < \beta$

$$W(u,u^*)(\beta) - W(u,u^*)(\alpha) = (z - z^*)\int_\alpha^\beta u u^* d\varrho = 2\mathrm{i}\,\mathrm{Im}(z)\int_\alpha^\beta |u|^2 d\varrho.$$

As $\alpha \to a$ and $\beta \to b$, the left-hand side converges to zero by Lemma 5.6 and the right-hand side converges to $2\mathrm{i}\,\mathrm{Im}(z)\|u\|^2$, hence $\|u\| = 0$. □

Theorem 5.8. *The deficiency index of the minimal relation is given by*

$$n(T_{\min}) = \begin{cases} 0, & \textit{if } \tau \textit{ is in the l.c. case at no boundary point,} \\ 1, & \textit{if } \tau \textit{ is in the l.c. case at exactly one boundary point,} \\ 2, & \textit{if } \tau \textit{ is in the l.c. case at both boundary points.} \end{cases}$$

Proof. If τ is in the l.c. case at both endpoints, all solutions of $(\tau-\mathrm{i})u=0$ lie in $L^2((a,b);\varrho)$ and hence in T_{max}. Therefore $n(T_{\mathrm{min}}) = \dim\ker(T_{\mathrm{max}}-\mathrm{i}) = 2$. In the case when τ is in the l.c. case at exactly one endpoint, there is (up to scalar multiples) exactly one non-trivial solution of $(\tau-\mathrm{i})u=0$ in $L^2((a,b);\varrho)$, by Corollary 5.5. Now if τ is in the l.p. case at both endpoints, we have $\ker(T_{\mathrm{max}}-\mathrm{i}) = \{0\}$ by Lemma 5.7 and hence $n(T_{\mathrm{min}}) = 0$. □

6. Self-adjoint relations

We are interested in the self-adjoint restrictions of T_{max} (or equivalent the self-adjoint extensions of T_{min}). To this end recall that we introduced the convenient short-hand notation

$$W_a^b(f,g^*) = W(f,g^*)(b) - W(f,g^*)(a), \quad f,\, g\in T_{\mathrm{max}}.$$

Theorem 6.1. *Some relation S is a self-adjoint restriction of T_{max} if and only if*

$$S = \{f\in T_{\mathrm{max}} \,|\, \forall g\in S : W_a^b(f,g^*) = 0\}. \tag{6.1}$$

Proof. We denote the right-hand side by S_0. First assume S is a self-adjoint restriction of T_{max}. If $f\in S$, then

$$0 = \langle \tau f, g\rangle - \langle f, \tau g\rangle = W_a^b(f,g^*)$$

for each $g\in S$, hence $f\in S_0$. Now if $f\in S_0$, then

$$0 = W_a^b(f,g^*) = \langle \tau f, g\rangle - \langle f, \tau g\rangle$$

for each $g\in S$ and hence $f\in S^* = S$.

Conversely assume that $S = S_0$, then S is symmetric since we have $\langle \tau f, g\rangle = \langle f, \tau g\rangle$ for each $f,\, g\in S$. Now let $f\in S^*\subseteq T_{\mathrm{max}}$, then

$$0 = \langle \tau f, g\rangle - \langle f, \tau g\rangle = W_a^b(f,g^*)$$

for each $g\in S$ and hence $f\in S_0 = S$. □

The aim of this section is to determine all self-adjoint restrictions of T_{max}. If both endpoints are in the l.p. case, this is an immediate consequence of Theorem 5.8.

Theorem 6.2. *If τ is in the l.p. case at both endpoints then $T_{\mathrm{min}} = T_{\mathrm{max}}$ is a self-adjoint operator.*

Next we turn to the case when one endpoint is in the l.c. case and the other one is in the l.p. case. But before we do this, we need some more properties of the Wronskian.

Lemma 6.3. *Let $v\in T_{\mathrm{max}}$ such that $W(v,v^*)(a) = 0$ and suppose there is an $h\in T_{\mathrm{max}}$ with $W(h,v^*)(a)\neq 0$. Then for each $f,\, g\in T_{\mathrm{max}}$ we have*

$$W(f,v^*)(a) = 0 \quad \Leftrightarrow \quad W(f^*,v^*)(a) = 0 \tag{6.2}$$

and

$$W(f,v^*)(a) = W(g,v^*)(a) = 0 \quad \Rightarrow \quad W(f,g)(a) = 0. \tag{6.3}$$

Similar results hold at the endpoint b.

Proof. Choosing $f_1 = v$, $f_2 = v^*$, $f_3 = h$ and $f_4 = h^*$ in the Plücker identity, we see that also $W(h, v)(a) \neq 0$. Now let $f_1 = f$, $f_2 = v$, $f_3 = v^*$ and $f_4 = h$, then the Plücker identity yields (6.2), whereas $f_1 = f$, $f_2 = g$, $f_3 = v^*$ and $f_4 = h$ yields (6.3). □

Theorem 6.4. *Suppose τ is in the l.c. case at a and in the l.p. case at b. Then some relation S is a self-adjoint restriction of $T_{\max}$ if and only if there is a $v \in T_{\max}\backslash T_{\min}$ with $W(v, v^*)(a) = 0$ such that*

$$S = \{f \in T_{\max} \,|\, W(f, v^*)(a) = 0\}. \tag{6.4}$$

A similar result holds if τ is in the l.c. case at b and in the l.p. case at a.

Proof. Because of $n(T_{\min}) = 1$ the self-adjoint extensions of $T_{\min}$ are precisely the one-dimensional, symmetric extensions of $T_{\min}$. Hence some relation S is a self-adjoint extension of $T_{\min}$ if and only if there is some $v \in T_{\max}\backslash T_{\min}$ with $W(v, v^*)(a) = 0$ such that

$$S = T_{\min} \dot{+} \operatorname{span}\{v\}.$$

Hence we have to prove that

$$T_{\min} \dot{+} \operatorname{span}\{v\} = \{f \in T_{\max} \,|\, W(f, v^*)(a) = 0\}.$$

The subspace on the left-hand side is included in the right one because of Theorem 4.6 and $W(v, v^*)(a) = 0$. But if the subspace on the right-hand side was larger, it would be equal to $T_{\max}$ and hence would imply $v \in T_{\min}$. □

Two such self-adjoint restrictions are distinct if and only if the corresponding functions v are linearly independent modulo $T_{\min}$. Furthermore, v can always be chosen such that v is equal to some real solution of $(\tau - z)u = 0$ with $z \in \mathbb{R}$ in some vicinity of a. By Lemma 6.3 one sees that all these self-adjoint restrictions are real with respect to the natural conjugation.

In contrast to the classical theory, not all of this self-adjoint restrictions of $T_{\max}$ are operators. We will determine which of them are multi-valued in the following section.

It remains to consider the case when both endpoints are in the l.c. case.

Theorem 6.5. *Suppose τ is in the l.c. case at both endpoints. Then some relation S is a self-adjoint restriction of $T_{\max}$ if and only if there are some v, $w \in T_{\max}$, linearly independent modulo $T_{\min}$, with*

$$W_a^b(v, v^*) = W_a^b(w, w^*) = W_a^b(v, w^*) = 0, \tag{6.5}$$

such that

$$S = \{f \in T_{\max} \,|\, W_a^b(f, v^*) = W_a^b(f, w^*) = 0\}. \tag{6.6}$$

Proof. Since $n(T_{\min}) = 2$, the self-adjoint extensions of $T_{\min}$ are precisely the two-dimensional, symmetric extensions of $T_{\min}$. Hence a relation S is a self-adjoint restriction of $T_{\max}$ if and only if there are v, $w \in T_{\max}$, linearly independent modulo $T_{\min}$, with (6.5) such that

$$S = T_{\min} \dot{+} \operatorname{span}\{v, w\}.$$

Therefore we have to prove that

$$T_{\min} \dot{+} \operatorname{span}\{v, w\} = \{f \in T_{\max} \,|\, W_a^b(f, v^*) = W_a^b(f, w^*) = 0\} = T,$$

where we denote the subspace on the right-hand side by T. Indeed the subspace on the left-hand side is contained in T by Theorem 4.6 and (6.5). In order to prove that it is also not larger, consider the linear functionals F_v, F_w on $T_{\max}$ defined by

$$F_v(f) = W_a^b(f, v^*) \quad \text{and} \quad F_w(f) = W_a^b(f, w^*) \quad \text{for } f \in T_{\max}.$$

The intersection of the kernels of these functionals is precisely T. Furthermore, these functionals are linearly independent. Indeed, assume c_1, $c_2 \in \mathbb{C}$ and $c_1 F_v + c_2 F_w = 0$, then for all $f \in T_{\max}$ we have

$$0 = c_1 F_v(f) + c_2 F_w(f) = c_1 W_a^b(f, v^*) + c_2 W_a^b(f, w^*) = W_a^b(f, c_1 v^* + c_2 w^*).$$

But by Lemma 4.5 this yields

$$W(f, c_1 v^* + c_2 w^*)(a) = W(f, c_1 v^* + c_2 w^*)(b) = 0$$

for all $f \in T_{\max}$ and hence $c_1 v^* + c_2 w^* \in T_{\min}$. Now since v, w are linearly independent modulo $T_{\min}$, we get that $c_1 = c_2 = 0$. Now from Lemma 4.3 we infer that

$$\ker F_v \not\subseteq \ker F_w \quad \text{and} \quad \ker F_w \not\subseteq \ker F_v.$$

Hence there exist f_v, $f_w \in T_{\max}$ such that $W_a^b(f_v, v^*) = W_a^b(f_w, w^*) = 0$ but $W_a^b(f_v, w^*) \neq 0$ and $W_a^b(f_w, v^*) \neq 0$. Both, f_v and f_w do not lie in T and are linearly independent. Hence T is at most a two-dimensional extension of the minimal relation $T_{\min}$. $\square$

In the case when τ is in the l.c. case at both endpoints, we may divide the self-adjoint restrictions of $T_{\max}$ into two classes. Indeed, we say some relation is a self-adjoint restriction of $T_{\max}$ with separate boundary conditions if it is of the form

$$S = \{f \in T_{\max} \,|\, W(f, v^*)(a) = W(f, w^*)(b) = 0\}, \tag{6.7}$$

where v, $w \in T_{\max}$ are such that $W(v, v^*)(a) = W(w, w^*)(b) = 0$ but $W(h, v^*)(a) \neq 0 \neq W(h, w^*)(b)$ for some $h \in T_{\max}$. Conversely each relation of this form is a self-adjoint restriction of $T_{\max}$ by Theorem 6.5 and Lemma 4.5. The remaining self-adjoint restrictions are called self-adjoint restrictions of $T_{\max}$ with coupled boundary conditions.

From Lemma 6.3 one sees that all self-adjoint restrictions of $T_{\max}$ with separate boundary conditions are real with respect to the natural conjugation in $L^2((a, b); \varrho)$. In the case of coupled boundary conditions this is not the case in general. Again we will determine the self-adjoint restrictions which are multi-valued in the next section.

7. Boundary conditions

In this section let w_1, $w_2 \in T_{\max}$ with

$$W(w_1, w_2^*)(a) = 1 \quad \text{and} \quad W(w_1, w_1^*)(a) = W(w_2, w_2^*)(a) = 0, \tag{7.1a}$$

if τ is in the l.c. case at a and

$$W(w_1, w_2^*)(b) = 1 \quad \text{and} \quad W(w_1, w_1^*)(b) = W(w_2, w_2^*)(b) = 0, \tag{7.1b}$$

if τ is in the l.c. case at b. We will describe the self-adjoint restrictions of $T_{\max}$ in terms of the linear functionals BC_a^1, BC_a^2, BC_b^1 and BC_b^2 on $T_{\max}$, defined by

$$BC_a^1(f) = W(f, w_2^*)(a) \quad \text{and} \quad BC_a^2(f) = W(w_1^*, f)(a) \quad \text{for } f \in T_{\max},$$

if τ is in the l.c. case at a and

$$BC_b^1(f) = W(f, w_2^*)(b) \quad \text{and} \quad BC_b^2(f) = W(w_1^*, f)(b) \quad \text{for } f \in T_{\max},$$

if τ is in the l.c. case at b.

Note that if τ is in the l.c. case at some endpoint, such functions w_1, $w_2 \in T_{\max}$ with (7.1a) (respectively with (7.1b)) always exist. Indeed, one may take them to coincide near this endpoint with some real solutions u_1, u_2 of $(\tau - z)u = 0$ with $W(u_1, u_2) = 1$ for some $z \in \mathbb{R}$ and use Lemma 4.5.

In the regular case these functionals may take the form of point evaluations of the function and its quasi-derivative at the boundary point.

Proposition 7.1. *Suppose τ is regular at a. Then there are w_1, $w_2 \in T_{\max}$ with* (7.1a) *such that the corresponding linear functionals BC_a^1 and BC_a^2 are given by*

$$BC_a^1(f) = f(a) \quad \textit{and} \quad BC_a^2(f) = f^{[1]}(a) \quad \textit{for } f \in T_{\max}.$$

A similar result holds at the endpoint b.

Proof. Take w_1, $w_2 \in T_{\max}$ to coincide near a with the real solutions u_1, u_2 of $\tau u = 0$ with the initial conditions

$$u_1(a) = u_2^{[1]}(a) = 1 \quad \text{and} \quad u_1^{[1]}(a) = u_2(a) = 0.$$

□

Moreover, also if ϱ has no weight near some endpoint, we may choose special functionals.

Proposition 7.2. *Suppose that ϱ has no weight near a, i.e. $\alpha_\varrho > a$. Then there are w_1, $w_2 \in T_{\max}$ with* (7.1a) *such that the corresponding linear functionals BC_a^1 and BC_a^2 are given by*

$$BC_a^1(f) = f(\alpha_\varrho -) \quad \textit{and} \quad BC_a^2(f) = f^{[1]}(\alpha_\varrho -) \quad \textit{for } f \in T_{\max}.$$

A similar result holds at the endpoint b.

Proof. Take w_1, $w_2 \in T_{\max}$ to coincide near a with the real solutions u_1, u_2 of $\tau u = 0$ with the initial conditions

$$u_1(\alpha_\varrho -) = u_2^{[1]}(\alpha_\varrho -) = 1 \quad \text{and} \quad u_1^{[1]}(\alpha_\varrho -) = u_2(\alpha_\varrho -) = 0.$$

Then since the Wronskian is constant on (a, α_ϱ), we get

$$BC_a^1(f) = W(f, u_2)(\alpha_\varrho -) = f(\alpha_\varrho -)$$

and

$$BC_a^2(f) = W(u_1, f)(\alpha_\varrho -) = f^{[1]}(\alpha_\varrho -)$$

for each $f \in T_{\max}$. □

Using the Plücker identity one easily obtains the equality

$$W(f,g)(a) = BC_a^1(f)BC_a^2(g) - BC_a^2(f)BC_a^1(g), \quad f,\, g \in T_{\max}$$

for the Wronskian. Furthermore, for each $v \in T_{\max}\backslash T_{\min}$ which satisfies $W(v,v^*)(a) = 0$ but $W(h,v^*)(a) \neq 0$ for some $h \in T_{\max}$, one may show that there is a $\varphi_\alpha \in [0,\pi)$ such that for each $f \in T_{\max}$

$$W(f,v^*)(a) = 0 \quad \Leftrightarrow \quad BC_a^1(f)\cos\varphi_\alpha - BC_a^2(f)\sin\varphi_\alpha = 0. \tag{7.2}$$

Conversely, if some $\varphi_\alpha \in [0,\pi)$ is given, then there is some $v \in T_{\max}\backslash T_{\min}$ with $W(v,v^*)(a) = 0$ but $W(h,v^*)(a) \neq 0$ for some $h \in T_{\max}$ such that

$$W(f,v^*)(a) = 0 \quad \Leftrightarrow \quad BC_a^1(f)\cos\varphi_\alpha - BC_a^2(f)\sin\varphi_\alpha = 0 \tag{7.3}$$

for each $f \in T_{\max}$. Using this, Theorem 6.4 immediately yields the following characterization of the self-adjoint restrictions of $T_{\max}$ in terms of the boundary functionals.

Theorem 7.3. *Suppose τ is in the l.c. case at a and in the l.p. case at b. Then some relation S is a self-adjoint restriction of $T_{\max}$ if and only if*

$$S = \{f \in T_{\max} \,|\, BC_a^1(f)\cos\varphi_\alpha - BC_a^2(f)\sin\varphi_\alpha = 0\}$$

for some $\varphi_\alpha \in [0,\pi)$. A similar result holds if τ is in the l.c. case at b and in the l.p. case at a.

Now we will determine which self-adjoint restrictions of $T_{\max}$ are multi-valued in this case. Of course, we only have to consider the case when $\alpha_\varrho > a$ and ϱ has mass in α_ϱ.

Corollary 7.4. *Suppose ϱ has mass in α_ϱ and τ is in the l.p. case at b. Then some self-adjoint restriction S of $T_{\max}$ as in Theorem 7.3 is an operator if and only if*

$$\cos\varphi_\alpha w_2(\alpha_\varrho-) + \sin\varphi_\alpha w_1(\alpha_\varrho-) \neq 0. \tag{7.4}$$

A similar result holds for the endpoint b.

Proof. Assume (7.4) does not hold and for each $c \in \mathbb{C}$ consider the functions

$$f_c(x) = \begin{cases} c\, u_a(x), & \text{if } x \in (a,\alpha_\varrho], \\ 0, & \text{if } x \in (\alpha_\varrho, b), \end{cases} \tag{7.5}$$

where u_a is a solution of $\tau u = 0$ with $u_a(\alpha_\varrho) = 0$ and $u_a^{[1]}(\alpha_\varrho) = 1$. These functions lie in S with $\tau f_c \neq 0$, hence S is multi-valued. Conversely assume (7.4) holds and let $f \in S$ such that $f = 0$ and $\tau f = 0$ almost everywhere with respect to ϱ. Then f is of the form (7.5), but because of the boundary condition

$$c = f^{[1]}(\alpha_\varrho) = f(\alpha_\varrho)\frac{\cos\varphi_\alpha w_2^{[1]}(\alpha_\varrho)^* + \sin\varphi_\alpha w_1^{[1]}(\alpha_\varrho)^*}{\cos\varphi_\alpha w_2(\alpha_\varrho)^* + \sin\varphi_\alpha w_1(\alpha_\varrho)^*} = 0,$$

i.e. $f = 0$. $\square$

Note that in this case there is precisely one multi-valued, self-adjoint restriction S of $T_{\max}$. In terms of the boundary functionals from Proposition 7.2 it is precisely the one with $\varphi_\alpha = 0$. That means that in this case each function in S vanishes in α_ϱ. Now since ϱ has mass in this point one sees that the domain of S is not dense and hence S is not an operator. However, if we exclude the linear span of $\mathbb{1}_{\alpha_\varrho}$ from $L^2((a,b);\varrho)$ by setting

$$\mathfrak{D} = \overline{\mathrm{dom}(S)} = L^2((a,b);\varrho) \ominus \mathrm{span}\{\mathbb{1}_{\{\alpha_\varrho\}}\},$$

the linear relation $S_{\mathfrak{D}}$ in the Hilbert space $\mathfrak{D}$, given by

$$S_{\mathfrak{D}} = S \cap (\mathfrak{D} \times \mathfrak{D}),$$

is a self-adjoint operator (see (B.6) in Appendix B). Also note that if $\tilde{\tau}$ is obtained from τ by removing the point mass in α_ϱ from the measure ϱ, then $S_{\mathfrak{D}}$ is a self-adjoint restriction of the maximal relation corresponding to $\tilde{\tau}$.

Next we will give a characterization of the self-adjoint restrictions of $T_{\max}$, if τ is in the l.c. case at both endpoints.

Theorem 7.5. *Suppose τ is in the l.c. case at both endpoints. Then some relation S is a self-adjoint restriction of $T_{\max}$ if and only if there are matrices B_a, $B_b \in \mathbb{C}^{2\times 2}$ with*

$$\text{(7.6)} \quad \mathrm{rank}(B_a|B_b) = 2 \quad \textit{and} \quad B_a J B_a^* = B_b J B_b^* \quad \textit{with} \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

such that

$$\text{(7.7)} \qquad S = \left\{ f \in T_{\max} \,\middle|\, B_a \begin{pmatrix} BC_a^1(f) \\ BC_a^2(f) \end{pmatrix} = B_b \begin{pmatrix} BC_b^1(f) \\ BC_b^2(f) \end{pmatrix} \right\}.$$

Proof. If S is a self-adjoint restriction of $T_{\max}$, there exist v, $w \in T_{\max}$, linearly independent modulo $T_{\min}$, with

$$W_a^b(v,v^*) = W_a^b(w,w^*) = W_a^b(v,w^*) = 0,$$

such that

$$S = \{f \in T_{\max} \,|\, W_a^b(f,v^*) = W_a^b(f,w^*) = 0\}.$$

Let B_a, $B_b \in \mathbb{C}^{2\times 2}$ be defined by

$$B_a = \begin{pmatrix} BC_a^2(v^*) & -BC_a^1(v^*) \\ BC_a^2(w^*) & -BC_a^1(w^*) \end{pmatrix} \quad \text{and} \quad B_b = \begin{pmatrix} BC_b^2(v^*) & -BC_b^1(v^*) \\ BC_b^2(w^*) & -BC_b^1(w^*) \end{pmatrix}.$$

Then a simple computation shows that

$$B_a J B_a^* = B_b J B_b^* \quad \Leftrightarrow \quad W_a^b(v,v^*) = W_a^b(w,w^*) = W_a^b(v,w^*) = 0.$$

In order to prove $\mathrm{rank}\,(B_a|B_b) = 2$, let c_1, $c_2 \in \mathbb{C}$ and

$$0 = c_1 \begin{pmatrix} BC_a^2(v^*) \\ -BC_a^1(v^*) \\ BC_b^2(v^*) \\ -BC_b^1(v^*) \end{pmatrix} + c_2 \begin{pmatrix} BC_a^2(w^*) \\ -BC_a^1(w^*) \\ BC_b^2(w^*) \\ -BC_b^1(w^*) \end{pmatrix} = \begin{pmatrix} BC_a^2(c_1 v^* + c_2 w^*) \\ -BC_a^1(c_1 v^* + c_2 w^*) \\ BC_b^2(c_1 v^* + c_2 w^*) \\ -BC_b^1(c_1 v^* + c_2 w^*) \end{pmatrix}.$$

Hence the function $c_1 v^* + c_2 w^*$ lies in the kernel of BC_a^1, BC_a^2, BC_b^1 and BC_b^2, therefore $W(c_1 v^* + c_2 w^*, f)(a) = 0$ und $W(c_1 v^* + c_2 w^*, f)(b) = 0$ for each $f \in T_{\max}$. This means that $c_1 v^* + c_2 w^* \in T_{\min}$ and hence $c_1 = c_2 = 0$, since v^*, w^* are linearly independent modulo $T_{\min}$. This proves that $(B_a|B_b)$ has rank two. Furthermore, a calculation yields that for each $f \in T_{\max}$

$$W_a^b(f,v^*) = W_a^b(f,w^*) = 0 \quad \Leftrightarrow \quad B_a \begin{pmatrix} BC_a^1(f) \\ BC_a^2(f) \end{pmatrix} = B_b \begin{pmatrix} BC_b^1(f) \\ BC_b^2(f) \end{pmatrix},$$

which proves that S is given as in the claim.

Conversely let B_a, $B_b \in \mathbb{C}^{2\times 2}$ with the claimed properties be given. Then there are v, $w \in T_{\mathrm{max}}$ such that

$$B_a = \begin{pmatrix} BC_a^2(v^*) & -BC_a^1(v^*) \\ BC_a^2(w^*) & -BC_a^1(w^*) \end{pmatrix} \quad \text{and} \quad B_b = \begin{pmatrix} BC_b^2(v^*) & -BC_b^1(v^*) \\ BC_b^2(w^*) & -BC_b^1(w^*) \end{pmatrix}.$$

In order to prove that v and w are linearly independent modulo T_{min}, assume $c_1 v + c_2 w \in T_{\mathrm{min}}$ for some c_1, $c_2 \in \mathbb{C}$, then

$$0 = \begin{pmatrix} BC_a^2(c_1^* v^* + c_2^* w^*) \\ -BC_a^1(c_1^* v^* + c_2^* w^*) \\ BC_b^2(c_1^* v^* + c_2^* w^*) \\ -BC_b^1(c_1^* v^* + c_2^* w^*) \end{pmatrix} = c_1^* \begin{pmatrix} BC_a^2(v^*) \\ -BC_a^1(v^*) \\ BC_b^2(v^*) \\ -BC_b^1(v^*) \end{pmatrix} + c_2^* \begin{pmatrix} BC_a^2(w^*) \\ -BC_a^1(w^*) \\ BC_b^2(w^*) \\ -BC_b^1(w^*) \end{pmatrix}.$$

Now the rows of $(B_a|B_b)$ are linearly independent, hence $c_1 = c_2 = 0$. Since again we have

$$B_a J B_a^* = B_b J B_b^* \quad \Leftrightarrow \quad W_a^b(v, v^*) = W_a^b(w, w^*) = W_a^b(v, w^*) = 0,$$

the functions v, w satisfy the assumptions of Theorem 6.5. As above one sees again that for each $f \in T_{\mathrm{max}}$

$$B_a \begin{pmatrix} BC_a^1(f) \\ BC_a^2(f) \end{pmatrix} = B_b \begin{pmatrix} BC_b^1(f) \\ BC_b^2(f) \end{pmatrix} \quad \Leftrightarrow \quad W_a^b(f, w^*) = W_a^b(f, w^*) = 0.$$

Hence S is a self-adjoint restriction of T_{max} by Theorem 6.5. □

As in the preceding section, if τ is in the l.c. case at both endpoints, we may divide the self-adjoint restrictions of T_{max} into two classes.

Theorem 7.6. *Suppose τ is in the l.c. case at both endpoints. Then some relation S is a self-adjoint restriction of T_{max} with separate boundary conditions if and only if there are φ_α, $\varphi_\beta \in [0, \pi)$ such that*

$$S = \left\{ f \in T_{\mathrm{max}} \,\middle|\, \begin{array}{l} BC_a^1(f)\cos\varphi_\alpha - BC_a^2(f)\sin\varphi_\alpha = 0 \\ BC_b^1(f)\cos\varphi_\beta - BC_b^2(f)\sin\varphi_\beta = 0 \end{array} \right\}. \tag{7.8}$$

Furthermore, S is a self-adjoint restriction of T_{max} with coupled boundary conditions if and only if there are $\varphi \in [0, \pi)$ and $R \in \mathbb{R}^{2\times 2}$ with $\det R = 1$ such that

$$S = \left\{ f \in T_{\mathrm{max}} \,\middle|\, \begin{pmatrix} BC_b^1(f) \\ BC_b^2(f) \end{pmatrix} = e^{\mathrm{i}\varphi} R \begin{pmatrix} BC_a^1(f) \\ BC_a^2(f) \end{pmatrix} \right\}. \tag{7.9}$$

Proof. Using (7.2) and (7.3) one easily sees that the self-adjoint restrictions of T_{max} are precisely the ones given in (7.8). Hence we only have to prove the second claim. Let S be a self-adjoint restriction of T_{max} with coupled boundary conditions and B_a, $B_b \in \mathbb{C}^{2\times 2}$ matrices as in Theorem 7.5. Then by (7.6) either both of them have rank one or both have rank two. In the first case we had

$$B_a d = c_a^T d\, w_a \quad \text{and} \quad B_b d = c_b^T d\, w_b, \quad d \in \mathbb{C}^2$$

for some nonzero c_a, c_b, w_a, $w_b \in \mathbb{C}^2$. Since the vectors w_a and w_b are linearly independent by $\operatorname{rank}(B_a|B_b) = 2$ we have for each $f \in T_{\mathrm{max}}$

$$B_a \begin{pmatrix} BC_a^1(f) \\ BC_a^2(f) \end{pmatrix} = B_b \begin{pmatrix} BC_b^1(f) \\ BC_b^2(f) \end{pmatrix} \quad \Leftrightarrow \quad B_a \begin{pmatrix} BC_a^1(f) \\ BC_a^2(f) \end{pmatrix} = B_b \begin{pmatrix} BC_b^1(f) \\ BC_b^2(f) \end{pmatrix} = 0.$$

In particular this shows

$$B_a J B_a^* = B_b J B_b^* \quad \Leftrightarrow \quad B_a J B_a^* = B_b J B_b^* = 0.$$

Now let $v \in T_{\max}$ with $BC_a^2(v^*) = c_1$ and $BC_a^1(v^*) = -c_2$. A simple calculation yields

$$\begin{aligned} 0 = B_a J B_a^* &= W(w_1,w_2)(a)(BC_a^1(v)BC_a^2(v^*) - BC_a^2(v)BC_a^1(v^*))w_a w_a^{*T} \\ &= W(w_1,w_2)(a)W(v,v^*)(a)w_a w_a^{*T}. \end{aligned}$$

Hence $W(v,v^*)(a) = 0$ and since $(BC_a^1(v), BC_a^2(v)) = (c_2, c_1) \neq 0$, we also have $v \notin T_{\min}$. Furthermore, for each $f \in T_{\max}$ we have

$$\begin{aligned} B_a \begin{pmatrix} BC_a^1(f) \\ BC_a^2(f) \end{pmatrix} &= (BC_a^1(f)BC_a^2(v^*) - BC_a^2(f)BC_a^1(v^*))w_a \\ &= W(f,v^*)(a)w_a. \end{aligned}$$

Similarly one gets a function $w \in T_{\max}\backslash T_{\min}$ with $W(w,w^*)(b) = 0$ and

$$B_b \begin{pmatrix} BC_b^1(f) \\ BC_b^2(f) \end{pmatrix} = W(f,w^*)(b)w_b, \quad f \in T_{\max}.$$

But this shows that S is a self-adjoint restriction with separate boundary conditions.

Hence both matrices, B_a and B_b have rank two. If we set $B = B_b^{-1}B_a$, then $B = J(B^{-1})^* J^*$ and therefore $|\det B| = 1$, hence $\det B = e^{2i\varphi}$ for some $\varphi \in [0,\pi)$. If we set $R = e^{-i\varphi}B$, one sees from the equation

$$\begin{aligned} B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = J(B^{-1})^*J^* = e^{2i\varphi} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_{22}^* & -b_{21}^* \\ -b_{12}^* & b_{11}^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ = e^{2i\varphi} \begin{pmatrix} b_{11}^* & b_{12}^* \\ b_{21}^* & b_{22}^* \end{pmatrix} \end{aligned}$$

that $R \in \mathbb{R}^{2\times 2}$ with $\det R = 1$. Now because we have for each $f \in T_{\max}$

$$B_a \begin{pmatrix} BC_a^1(f) \\ BC_a^2(f) \end{pmatrix} = B_b \begin{pmatrix} BC_b^1(f) \\ BC_b^2(f) \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} BC_b^1(f) \\ BC_b^2(f) \end{pmatrix} = e^{\mathrm{i}\varphi} R \begin{pmatrix} BC_a^1(f) \\ BC_a^2(f) \end{pmatrix},$$

S has the claimed representation.

Conversely if S is of the form (7.9), then Theorem 7.5 shows that it is a self-adjoint restriction of $T_{\max}$. Now if S was a self-adjoint restriction with separate boundary conditions, we would have an $f \in S\backslash T_{\min}$, vanishing in some vicinity of a. But then, because of the boundary condition we would also have $BC_b^1(f) = BC_b^2(f) = 0$, i.e. $f \in T_{\min}$. Hence S can not be a self-adjoint restriction with separate boundary conditions. □

Now we will again determine the self-adjoint restrictions of $T_{\max}$ which are multi-valued. In the case of separate boundary conditions these are determined by whether

$$\cos\varphi_\alpha w_2(\alpha_\varrho -) + \sin\varphi_\alpha w_1(\alpha_\varrho -) \neq 0, \tag{7.10a}$$

$$\cos\varphi_\beta w_2(\beta_\varrho +) + \sin\varphi_\beta w_1(\beta_\varrho +) \neq 0, \tag{7.10b}$$

hold or not. Note that if one takes the functionals from Proposition 7.2, then (7.10a) (respectively (7.10b)) is equivalent to $\varphi_\alpha \neq 0$ (respectively $\varphi_\beta \neq 0$). We start with the case when $\dim \mathrm{mul}(T_{\max}) = 1$.

Corollary 7.7. *Suppose τ is in the l.c. case at both endpoints and ϱ has mass in α_ϱ but not in β_ϱ, i.e.* $\dim \mathrm{mul}(T_{\max}) = 1$. *Then for each self-adjoint restriction S of $T_{\max}$ with separate boundary conditions as in Theorem 7.6 we have*

$$\mathrm{mul}(S) = \begin{cases} \{0\}, & \text{if (7.10a) holds,} \\ \mathrm{span}\{\mathbb{1}_{\{\alpha_\varrho\}}\}, & \text{if (7.10a) does not hold.} \end{cases}$$

Furthermore, each self-adjoint restriction of $T_{\max}$ with coupled boundary conditions is an operator. Similar results hold if ϱ has mass in β_ϱ but no mass in α_ϱ.

Proof. If S is a self-adjoint restriction of $T_{\max}$ with separate boundary conditions, then the claim follows as in the proof of Corollary 7.4.

Now let S be a self-adjoint restriction of $T_{\max}$ with coupled boundary conditions as in Theorem 7.6 and $f \in S$ with $f = 0$ and $\tau f = 0$ almost everywhere with respect to ϱ. Then again f is of the form (7.5). But because of the boundary conditions this shows that $BC_a^1(f) = BC_a^2(f) = 0$, hence f vanishes everywhere. □

The remark after Corollary 7.4 also holds literally here under the assumptions of Corollary 7.7. It remains to determine the self-adjoint restrictions of $T_{\max}$ which are multi-valued in the case when ϱ has mass in α_ϱ and in β_ϱ.

Corollary 7.8. *Suppose ϱ has mass in α_ϱ and in β_ϱ, i.e.* $\dim \mathrm{mul}(T_{\max}) = 2$. *If S is a self-adjoint restriction of $T_{\max}$ with separate boundary conditions as in Theorem 7.6, then*

$$\mathrm{mul}(S) = \begin{cases} \{0\}, & \text{if (7.10a) and (7.10b) hold,} \\ \mathrm{span}\{\mathbb{1}_{\{\alpha_\varrho\}}\}, & \text{if (7.10b) holds and (7.10a) does not,} \\ \mathrm{span}\{\mathbb{1}_{\{\beta_\varrho\}}\}, & \text{if (7.10a) holds and (7.10b) does not,} \\ \mathrm{span}\{\mathbb{1}_{\{\alpha_\varrho\}}, \mathbb{1}_{\{\beta_\varrho\}}\}, & \text{if neither (7.10a) nor (7.10b) holds.} \end{cases}$$

If S is a self-adjoint restriction of $T_{\max}$ with coupled boundary conditions as in Theorem 7.6 and

$$\tilde{R} = \begin{pmatrix} w_2^{[1]}(\beta_\varrho+)^* & -w_2(\beta_\varrho+)^* \\ -w_1^{[1]}(\beta_\varrho+)^* & w_1(\beta_\varrho+)^* \end{pmatrix}^{-1} R \begin{pmatrix} w_2^{[1]}(\alpha_\varrho-)^* & -w_2(\alpha_\varrho-)^* \\ -w_1^{[1]}(\alpha_\varrho-)^* & w_1(\alpha_\varrho-)^* \end{pmatrix},$$

then

$$\mathrm{mul}(S) = \begin{cases} \{0\}, & \text{if } \tilde{R}_{12} \neq 0, \\ \mathrm{span}\{\mathbb{1}_{\{\alpha_\varrho\}} + e^{\mathrm{i}\varphi}\tilde{R}_{22}\mathbb{1}_{\{\beta_\varrho\}}\}, & \text{if } \tilde{R}_{12} = 0. \end{cases}$$

Proof. If S is a self-adjoint restriction of $T_{\max}$ with separate boundary conditions, the claim follows as in the proof of Corollary 7.4.

In order to prove the second part let S be a self-adjoint restriction of $T_{\max}$ with coupled boundary conditions, which can be written as

$$\begin{pmatrix} f(\beta_\varrho+) \\ f^{[1]}(\beta_\varrho+) \end{pmatrix} = e^{\mathrm{i}\varphi}\tilde{R} \begin{pmatrix} f(\alpha_\varrho-) \\ f^{[1]}(\alpha_\varrho-) \end{pmatrix}, \quad f \in S.$$

Now assume that $\tilde{R}_{12} \neq 0$ and let $f \in S$ with $f = 0$ almost everywhere with respect to ϱ. Then from this boundary condition we infer that also

$f^{[1]}(\alpha_\varrho-) = f^{[1]}(\beta_\varrho+) = 0$, i.e. $f = 0$. Otherwise, if we assume that $\tilde{R}_{12} = 0$, then the boundary condition becomes

$$f^{[1]}(\beta_\varrho+) = e^{\mathrm{i}\varphi}\tilde{R}_{22} f^{[1]}(\alpha_\varrho-), \quad f \in S.$$

Hence all functions $f \in S$ which vanish almost everywhere with respect to ϱ are of the form

$$f(x) = \begin{cases} c_a u_a(x), & \text{if } x \in (a, \alpha_\varrho], \\ 0, & \text{if } x \in (\alpha_\varrho, \beta_\varrho], \\ e^{\mathrm{i}\varphi}\tilde{R}_{22} c_a u_b(x), & \text{if } x \in (\beta_\varrho, b). \end{cases}$$

Conversely, all functions of this form lie in S, which yields the claim. □

Note that if one uses the boundary functionals of Proposition 7.2, then $\tilde{R} = R$. In contrast to Corollary 7.7, in this case there is a multitude of multi-valued, self-adjoint restrictions S of $T_{\max}$. However, if we again restrict S to the closure $\mathfrak{D}$ of the domain of S by

$$S_{\mathfrak{D}} = S \cap (\mathfrak{D} \times \mathfrak{D}),$$

we obtain a self-adjoint operator in the Hilbert space $\mathfrak{D}$.

8. Spectrum and resolvent

In this section we will compute the resolvent of the self-adjoint restrictions S of $T_{\max}$. The resolvent set $\rho(S)$ is the set of all $z \in \mathbb{C}$ such that

$$R_z = (S - z)^{-1} = \{(g, f) \in L^2((a,b);\varrho) \times L^2((a,b);\varrho) \,|\, (f, g) \in S - z\}$$

is an everywhere defined operator in $L^2((a,b);\varrho)$, i.e. $\mathrm{dom}(R_z) = L^2((a,b);\varrho)$ and $\mathrm{mul}(R_z) = \{0\}$. According to Theorem B.1, the resolvent set $\rho(S)$ is a non-empty, open subset of $\mathbb{C}$ and the resolvent $z \mapsto R_z$ is an analytic function of $\rho(S)$ into the space of bounded linear operators on $L^2((a,b);\varrho)$. Note that in general the operators R_z, $z \in \rho(S)$ are not injective, indeed we have

$$\ker(R_z) = \mathrm{mul}(S) = \mathrm{dom}(S)^\perp = \mathrm{ran}(R_z)^\perp, \quad z \in \rho(S). \tag{8.1}$$

First we deal with the case, when both endpoints are in the l.c. case.

Theorem 8.1. *Suppose τ is in the l.c. case at both endpoints and S is a self-adjoint restriction of $T_{\max}$. Then for each $z \in \rho(S)$ the resolvent R_z is an integral operator*

$$R_z f(x) = \int_a^b G_z(x,y) f(y) d\varrho(y), \quad x \in (a,b),\ f \in L^2((a,b);\varrho), \tag{8.2}$$

with a square integrable kernel G_z. For any given linearly independent solutions u_1, u_2 of $(\tau - z)u = 0$, there are coefficients $m_{ij}^\pm(z) \in \mathbb{C}$, i, $j \in \{1,2\}$ such that the kernel is given by

$$G_z(x,y) = \begin{cases} \sum_{i,j=1}^2 m_{ij}^+(z) u_i(x) u_j(y), & \text{if } y \le x, \\ \sum_{i,j=1}^2 m_{ij}^-(z) u_i(x) u_j(y), & \text{if } y > x. \end{cases} \tag{8.3}$$

Proof. Let u_1, u_2 be two linearly independent solutions of $(\tau - z)u = 0$ with $W(u_1, u_2) = 1$. If $g \in L^2_c((a,b);\varrho)$, then $(R_z g, g) \in (S - z)$, hence there is some $f \in \mathfrak{D}_\tau$ satisfying the boundary conditions with $f = R_z g$ and $(\tau - z)f = g$. From Proposition 3.3 we get for suitable constants c_1, $c_2 \in \mathbb{C}$

$$f(x) = u_1(x)\left(c_1 + \int_a^x u_2 g\, d\varrho\right) + u_2(x)\left(c_2 - \int_a^x u_1 g\, d\varrho\right) \tag{8.4}$$

for each $x \in (a,b)$. Furthermore, since f satisfies the boundary conditions

$$B_a \begin{pmatrix} BC_a^1(f) \\ BC_a^2(f) \end{pmatrix} = B_b \begin{pmatrix} BC_b^1(f) \\ BC_b^2(f) \end{pmatrix}$$

for some suitable matrices B_a, $B_b \in \mathbb{C}^{2\times 2}$ as in Theorem 7.5. Now because g has compact support, we have

$$\begin{aligned}\begin{pmatrix} BC_a^1(f) \\ BC_a^2(f) \end{pmatrix} &= \begin{pmatrix} c_1 BC_a^1(u_1) + c_2 BC_a^1(u_2) \\ c_1 BC_a^2(u_1) + c_2 BC_a^2(u_2) \end{pmatrix} = \begin{pmatrix} BC_a^1(u_1) & BC_a^1(u_2) \\ BC_a^2(u_1) & BC_a^2(u_2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= M_\alpha \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\end{aligned}$$

as well as

$$\begin{aligned}\begin{pmatrix} BC_b^1(f) \\ BC_b^2(f) \end{pmatrix} &= \begin{pmatrix} \left(c_1 + \int_a^b u_2 g d\varrho\right) BC_b^1(u_1) \\ \left(c_1 + \int_a^b u_2 g d\varrho\right) BC_b^2(u_1) \end{pmatrix} + \begin{pmatrix} \left(c_2 - \int_a^b u_1 g d\varrho\right) BC_b^1(u_2) \\ \left(c_2 - \int_a^b u_1 g d\varrho\right) BC_b^2(u_2) \end{pmatrix} \\ &= \begin{pmatrix} BC_b^1(u_1) & BC_b^1(u_2) \\ BC_b^2(u_1) & BC_b^2(u_2) \end{pmatrix} \begin{pmatrix} c_1 + \int_a^b u_2 g\, d\varrho \\ c_2 - \int_a^b u_1 g\, d\varrho \end{pmatrix} \\ &= M_\beta \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + M_\beta \begin{pmatrix} \int_a^b u_2 g\, d\varrho \\ -\int_a^b u_1 g\, d\varrho \end{pmatrix}.\end{aligned}$$

Hence we have

$$(B_a M_\alpha - B_b M_\beta) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = B_b M_\beta \begin{pmatrix} \int_a^b u_2 g\, d\varrho \\ -\int_a^b u_1 g\, d\varrho \end{pmatrix}.$$

Now if $B_a M_\alpha - B_b M_\beta$ was not invertible, we would have

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \in \mathbb{C}^2 \setminus \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \quad \text{with} \quad B_a M_\alpha \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = B_b M_\beta \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}.$$

Then the function $d_1 u_1 + d_2 u_2$ would be a solution of $(\tau - z)u = 0$ satisfying the boundary conditions of S, hence an eigenvector with eigenvalue z. But since this would contradict $z \in \rho(S)$, $B_a M_\alpha - B_b M_\beta$ has to be invertible. Now because of

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = (B_a M_\alpha - B_b M_\beta)^{-1} B_b M_\beta \begin{pmatrix} \int_a^b u_2 g\, d\varrho \\ -\int_a^b u_1 g\, d\varrho \end{pmatrix},$$

the constants c_1 and c_2 may be written as linear combinations of

$$\int_a^b u_2 g\, d\varrho \quad \text{and} \quad \int_a^b u_1 g\, d\varrho,$$

where the coefficients are independent of g. Now using equation (8.4) one sees that f has an integral-representation with a function G_z as claimed.

Moreover, the function G_z is square-integrable, since the solutions u_1 and u_2 lie in $L^2((a,b);\varrho)$ by assumption. Finally, since the operator K_z

$$K_z g(x) = \int_a^b G_z(x,y)g(y)d\varrho(y), \quad x \in (a,b),\ g \in L^2((a,b);\varrho)$$

on $L^2((a,b);\varrho)$, as well as the resolvent R_z are bounded, the claim follows since they coincide on a dense subspace. □

As in the classical case, the compactness of the resolvent implies discreteness of the spectrum.

Corollary 8.2. *Suppose τ is in the l.c. case at both endpoints and S is a self-adjoint restriction of $T_{\max}$. Then the relation S has purely discrete spectrum, i.e. $\sigma(S) = \sigma_d(S)$ with*

$$\sum_{\substack{\lambda\in\sigma(S)\\ \lambda\neq 0}} \frac{1}{\lambda^2} < \infty \quad \textit{and} \quad \dim\ker(S-\lambda) \leq 2, \quad \lambda \in \sigma(S).$$

Proof. Since the resolvent is compact, Theorem B.2 shows that the spectrum of S consists of isolated eigenvalues. Furthermore, the sum converges since the resolvent is Hilbert–Schmidt. Finally, their multiplicity is at most two because of (3.9). □

If S is a self-adjoint restriction of $T_{\max}$ with separate boundary conditions or if not both endpoints are in the l.c. case, the resolvent has a simpler form.

Theorem 8.3. *Suppose S is a self-adjoint restriction of $T_{\max}$ with separate boundary conditions (if τ is in the l.c. case at both endpoints) and $z \in \rho(S)$. Furthermore, let u_a and u_b be non-trivial solutions of $(\tau - z)u = 0$, such that*

$$u_a \begin{cases} \textit{satisfies the boundary condition at } a \textit{ if } \tau \textit{ is in the l.c. case at } a, \\ \textit{lies in } L^2((a,b);\varrho) \textit{ near } a \textit{ if } \tau \textit{ is in the l.p. case at } a, \end{cases}$$

and

$$u_b \begin{cases} \textit{satisfies the boundary condition at } b \textit{ if } \tau \textit{ is in the l.c. case at } b, \\ \textit{lies in } L^2((a,b);\varrho) \textit{ near } b \textit{ if } \tau \textit{ is in the l.p. case at } b. \end{cases}$$

Then the resolvent R_z is given by

$$R_z g(x) = \int_a^b G_z(x,y)g(y)d\varrho(y), \quad x \in (a,b),\ g \in L^2((a,b);\varrho), \tag{8.5}$$

where

$$G_z(x,y) = \frac{1}{W(u_b,u_a)} \begin{cases} u_a(y)u_b(x), & \textit{if } y \leq x, \\ u_a(x)u_b(y), & \textit{if } y > x. \end{cases} \tag{8.6}$$

Proof. The functions u_a, u_b are linearly independent, since otherwise they were an eigenvector of S corresponding to the eigenvalue z. Hence they form a fundamental system of $(\tau - z)u = 0$. Now for each $f \in L^2((a,b);\varrho)$ we define a function f_g by

$$f_g(x) = W(u_b,u_a)^{-1}\left(u_b(x)\int_a^x u_a g\, d\varrho + u_a(x)\int_x^b u_b g\, d\varrho\right), \quad x \in (a,b).$$

If $f \in L^2_c((a,b);\varrho)$, then f_g is a solution of $(\tau - z)f = g$ by Proposition 3.3. Moreover, f_g is a scalar multiple of u_a near a and a scalar multiple of u_b near b. Hence the function f_g satisfies the boundary conditions of S and therefore $(f_g, \tau f_g - z f_g) = (f_g, g) \in (S - z)$, i.e. $R_z g = f_g$. Now if $g \in L^2((a,b);\varrho)$ is arbitrary and $g_n \in L^2_c((a,b);\varrho)$ is a sequence with $g_n \to g$ as $n \to \infty$, we have, since the resolvent is bounded $R_z g_n \to R_z g$. Furthermore, f_{g_n} converges pointwise to f_g and hence $R_z g = f_g$. □

If τ is in the l.p. case at some endpoint, then Corollary 5.5 shows that there is always a, unique up to scalar multiples, non-trivial solution of $(\tau - z)u = 0$, lying in $L^2((a,b);\varrho)$ near this endpoint. Also if τ is in the l.c. case at some endpoint, there exists a, unique up to scalar multiples, non-trivial solution of $(\tau - z)u = 0$, satisfying the boundary condition at this endpoint. Hence functions u_a and u_b, as in Theorem 8.3 always exist.

Corollary 8.4. *If S is a self-adjoint restriction of $T_{\max}$ with separate boundary conditions (if τ is in the l.c. at both endpoints), then all eigenvalues of S are simple.*

Proof. Suppose $\lambda \in \mathbb{R}$ is an eigenvalue and $u_j \in S$ with $\tau u_j = \lambda u_j$ for $j = 1, 2$, i.e. they are solutions of $(\tau - \lambda)u = 0$. If τ is in the l.p. case at some endpoint, then clearly the Wronskian $W(u_1, u_2)$ vanishes. Otherwise, since both functions satisfy the same boundary conditions this follows using the Plücker identity. □

According to Theorem B.7 the essential spectrum of self-adjoint restrictions is independent of the boundary conditions, i.e. all self-adjoint restrictions of $T_{\max}$ have the same essential spectrum. We conclude this section by proving that the essential spectrum of the self-adjoint restrictions of $T_{\max}$ is determined by the behavior of the coefficients in some arbitrarily small neighborhood of the endpoints. In order to state this result we need some notation. Fix some $c \in (a,b)$ and denote by $\tau|_{(a,c)}$ (respectively by $\tau|_{[c,b)}$) the differential expression on (a,b) corresponding to the coefficients ς, χ and $\varrho|_{(a,c)}$ (respectively $\varrho|_{[c,b)}$). Furthermore, let $S_{(a,c)}$ (respectively $S_{[c,b)}$) be some self-adjoint realizations of $\tau|_{(a,c)}$ (respectively of $\tau|_{[c,b)}$).

Theorem 8.5. *For each $c \in (a,b)$ we have*

$$\sigma_e(S) = \sigma_e\left(S_{(a,c)}\right) \cup \sigma_e\left(S_{[c,b)}\right). \tag{8.7}$$

Proof. If one identifies $L^2((a,b);\varrho)$ with the orthogonal sum

$$L^2((a,b);\varrho) = L^2((a,b);\varrho|_{(a,c)}) \oplus L^2((a,b);\varrho|_{[c,b)}),$$

then the linear relation

$$S_c = S_{(a,c)} \oplus S_{[c,b)}$$

is self-adjoint in $L^2((a,b);\varrho)$. Now since S and S_c both are finite dimensional extensions of the symmetric linear relation

$$T_c = \{f \in T_{\min} \,|\, f(c) = f^{[1]}(c) = 0\},$$

an application of Theorem B.7 and Theorem B.8 yields the claim. □

As an immediate consequence one sees that the essential spectrum only depends on the coefficients in some neighborhood of the endpoints.

Corollary 8.6. *For each α, $\beta \in (a,b)$ we have*

$$\sigma_e(S) = \sigma_e\left(S_{(a,\alpha)}\right) \cup \sigma_e\left(S_{[\beta,b)}\right). \tag{8.8}$$

9. Singular Weyl–Titchmarsh functions

In this section let S be a self-adjoint restriction of $T_{\max}$ with separate boundary conditions (if τ is in the l.c. case at both endpoints). Our aim is to define a singular Weyl–Titchmarsh function as introduced recently in [16], [20] for Schrödinger operators. To this end we need a real entire fundamental system θ_z, ϕ_z, $z \in \mathbb{C}$ of $(\tau - z)u = 0$ with $W(\theta_z, \phi_z) = 1$, such that ϕ_z lies in S near a, i.e. ϕ_z lies in $L^2((a,b);\varrho)$ near a and satisfies the boundary condition at a if τ is in the l.c. case there.

Hypothesis 9.1. *There is a real entire fundamental system θ_z, ϕ_z, $z \in \mathbb{C}$ of $(\tau - z)u = 0$ with $W(\theta_z, \phi_z) = 1$, such that ϕ_z lies in S near a.*

Under the assumption of Hypothesis 9.1 we may define a complex-valued function M on $\rho(S)$ by requiring that the solutions

$$\psi_z = \theta_z + M(z)\phi_z, \quad z \in \rho(S) \tag{9.1}$$

lie in S near b, i.e. they lie in $L^2((a,b);\varrho)$ near b and satisfy the boundary condition at b, if τ is in the l.c. case at b. This function M is well-defined (use Corollary 5.5 if τ is in the l.p. case at b) and called the singular Weyl–Titchmarsh function of S. The solutions ψ_z, $z \in \rho(S)$ are referred to as the Weyl solutions of S.

Theorem 9.2. *The singular Weyl–Titchmarsh function M is analytic and furthermore satisfies*

$$M(z) = M(z^*)^*, \quad z \in \rho(S). \tag{9.2}$$

Proof. Let c, $d \in (a,b)$ with $c < d$. From Theorem 8.3 and the equation

$$W(\psi_z, \phi_z) = W(\theta_z, \phi_z) + M(z)W(\phi_z, \phi_z) = 1, \quad z \in \rho(S),$$

we get for each $z \in \rho(S)$ and $x \in [c,d)$

$$\begin{aligned} R_z \mathbb{1}_{[c,d)}(x) &= \psi_z(x) \int_c^x \phi_z \, d\varrho + \phi_z(x) \int_x^d \psi_z \, d\varrho \\ &= (\theta_z(x) + M(z)\phi_z(x)) \int_c^x \phi_z \, d\varrho + \phi_z(x) \int_x^d \theta_z + M(z)\phi_z \, d\varrho \\ &= M(z)\phi_z(x) \int_c^d \phi_z(y) d\varrho(y) + \int_c^d \tilde{G}_z(x,y) d\varrho(y), \end{aligned}$$

where

$$\tilde{G}_z(x,y) = \begin{cases} \phi_z(y)\theta_z(x), & \text{if } y \le x, \\ \phi_z(x)\theta_z(y), & \text{if } y > x, \end{cases}$$

and hence

$$\langle R_z \mathbb{1}_{[c,d)}, \mathbb{1}_{[c,d)} \rangle = M(z) \left(\int_c^d \phi_z(y) d\varrho(y) \right)^2 + \int_c^d \int_c^d \tilde{G}_z(x,y) d\varrho(y) \, d\varrho(x).$$

The left-hand side of this equation is analytic on $\rho(S)$ since the resolvent is. Furthermore, the integrals are analytic on $\rho(S)$ as well, since the integrands are analytic and locally bounded by Theorem 3.6. Hence M is analytic provided that for each $z_0 \in \rho(S)$, there are some c, $d \in (a,b)$ such that

$$\int_c^d \phi_{z_0}(y) d\varrho(y) \neq 0.$$

But this is true since otherwise ϕ_{z_0} would vanish almost everywhere with respect to ϱ. Moreover, equation (9.2) is valid since the functions

$$\theta_{z^*} + M(z)^* \phi_{z^*} = (\theta_z + M(z)\phi_z)^*, \quad z \in \rho(S)$$

lie in S near b by Lemma 6.3. □

As an immediate consequence of Theorem 9.2 we see that the functions $\psi_z(x)$ and $\psi_z^{[1]}(x)$ are analytic in $z \in \rho(S)$ for each $x \in (a,b)$.

Remark 9.3. *Note that a fundamental system as in Hypothesis 9.1 is not unique. In fact, any other such system is given by*

$$\tilde{\theta}_z = e^{-g(z)}\theta_z - f(z)\phi_z \quad \text{and} \quad \tilde{\phi}_z = e^{g(z)}\phi_z, \quad z \in \mathbb{C} \tag{9.3}$$

for some real entire functions f, g. Moreover, the corresponding singular Weyl–Titchmarsh functions are related via

$$\tilde{M}(z) = e^{-2g(z)}M(z) + e^{-g(z)}f(z), \quad z \in \rho(S). \tag{9.4}$$

In particular, the maximal domain of holomorphy or the structure of poles and singularities do not change.

We continue with the construction of a real entire fundamental system in the case when τ is in the l.c. case at a.

Theorem 9.4. *Suppose τ is in the l.c. case at a. Then there exists a real entire fundamental system θ_z, ϕ_z, $z \in \mathbb{C}$ of $(\tau - z)u = 0$ with $W(\theta_z, \phi_z) = 1$, such that ϕ_z lies in S near a and for each z_1, $z_2 \in \mathbb{C}$ we have*

$$W(\theta_{z_1}, \phi_{z_2})(a) = 1 \quad \text{and} \quad W(\theta_{z_1}, \theta_{z_2})(a) = W(\phi_{z_1}, \phi_{z_2})(a) = 0. \tag{9.5}$$

Proof. Let θ, ϕ be a real fundamental system of $\tau u = 0$ with $W(\theta, \phi) = 1$ such that ϕ lies in S near a. Now fix some $c \in (a,b)$ and for each $z \in \mathbb{C}$ let $u_{z,1}$, $u_{z,2}$ be the fundamental system of

$$(\tau - z)u = 0 \quad \text{with} \quad u_{z,1}(c) = u_{z,2}^{[1]}(c) = 1 \quad \text{and} \quad u_{z,1}^{[1]}(c) = u_{z,2}(c) = 0.$$

Then by the existence and uniqueness theorem we have $u_{z^*,j} = u_{z,j}^*$ for $j = 1$, 2. If we introduce

$$\begin{aligned} \theta_z(x) &= W(u_{z,1}, \theta)(a) u_{z,2}(x) - W(u_{z,2}, \theta)(a) u_{z,1}(x), \qquad x \in (a,b), \\ \phi_z(x) &= W(u_{z,1}, \phi)(a) u_{z,2}(x) - W(u_{z,2}, \phi)(a) u_{z,1}(x), \qquad x \in (a,b), \end{aligned}$$

then the functions ϕ_z lie in S near a since

$$W(\phi_z, \phi)(a) = W(u_{z,1}, \phi)(a) W(u_{z,2}, \phi)(a) - W(u_{z,2}, \phi)(a) W(u_{z_1}, \phi)(a) = 0.$$

Furthermore, a direct calculation shows that $\theta_{z^*} = \theta_z^*$ and $\phi_{z^*} = \phi_z^*$. The remaining equalities follow using the Plücker identity several times. It remains to prove that the functions $W(u_{z,1},\theta)(a)$, $W(u_{z,2},\theta)(a)$, $W(u_{z,1},\phi)(a)$ and $W(u_{z,2},\phi)(a)$ are entire in z. Indeed, we get from the Lagrange identity

$$W(u_{z,1},\theta)(a) = W(u_{z,1},\theta)(c) - z \lim_{x\to a} \int_x^c \theta(t) u_{z,1}(t) d\varrho(t), \quad z \in \mathbb{C}.$$

Now the integral on the right-hand side is analytic by Theorem 3.6 and in order to prove that the limit is also analytic we need to show that the integral is bounded as $x \to a$, locally uniformly in z. But the proof of Lemma 5.1 shows that for each $z_0 \in \mathbb{C}$ we have

$$\left| \int_x^c \theta(t) u_{z,1}(t) d\varrho(t) \right|^2 \leq K \int_a^c |\theta|^2 \, d\varrho \int_a^c \left(|u_{z_0,1}| + |u_{z_0,2}|\right)^2 d\varrho$$

for some constant $K \in \mathbb{R}$ and all z in some neighborhood of z_0. Analyticity of the other functions is proved similarly. $\square$

If τ is even regular at a, then one may take θ_z, ϕ_z, $z \in \mathbb{C}$ to be the solutions of $(\tau - z)u = 0$ with the initial values

$$\theta_z(a) = \phi_z^{[1]}(a) = \cos\varphi_\alpha \quad \text{and} \quad -\theta_z^{[1]}(a) = \phi_z(a) = \sin\varphi_\alpha$$

for some suitable $\varphi_\alpha \in [0,\pi)$. Furthermore, in the case when ϱ has no weight near a, one may take for θ_z, ϕ_z, $z \in \mathbb{C}$ the solutions of $(\tau - z)u = 0$ with the initial values

$$\theta_z(\alpha_\varrho-) = \phi_z^{[1]}(\alpha_\varrho-) = \cos\varphi_\alpha \quad \text{and} \quad -\theta_z^{[1]}(\alpha_\varrho-) = \phi_z(\alpha_\varrho-) = \sin\varphi_\alpha$$

for some $\varphi_\alpha \in [0,\pi)$.

Corollary 9.5. *Suppose τ is in the l.c. case at a and θ_z, ϕ_z, $z \in \mathbb{C}$ is a real entire fundamental system of $(\tau - z)u = 0$ as in Theorem 9.4. Then the singular Weyl–Titchmarsh function M is a Herglotz–Nevanlinna function.*

Proof. In order to prove the claim, we show that

$$0 < \|\psi_z\|^2 = \frac{\mathrm{Im}(M(z))}{\mathrm{Im}(z)}, \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{9.6}$$

Indeed if z_1, $z_2 \in \rho(S)$, then

$$\begin{aligned} W(\psi_{z_1},\psi_{z_2})(a) &= W(\theta_{z_1},\theta_{z_2})(a) + M(z_2) W(\theta_{z_1},\phi_{z_2})(a) \\ &\qquad + M(z_1) W(\phi_{z_1},\theta_{z_2})(a) + M(z_1)M(z_2) W(\phi_{z_1},\phi_{z_2})(a) \\ &= M(z_2) - M(z_1). \end{aligned}$$

If τ is in the l.p. case at b, then furthermore we have

$$W(\psi_{z_1},\psi_{z_2})(b) = 0,$$

since clearly ψ_{z_1}, $\psi_{z_2} \in T_{\max}$. This also holds if τ is in the l.c. case at b, since then ψ_{z_1} and ψ_{z_2} satisfy the same boundary condition at b. Now using the Lagrange identity yields

$$\begin{aligned} (z_1 - z_2) \int_a^b \psi_{z_1}(t) \psi_{z_2}(t) d\varrho(t) &= W(\psi_{z_1},\psi_{z_2})(b) - W(\psi_{z_1},\psi_{z_2})(a) \\ &= M(z_1) - M(z_2). \end{aligned}$$

In particular, for $z \in \mathbb{C}\backslash\mathbb{R}$, using $M(z^*) = M(z)^*$ as well as

$$\psi_{z^*} = \theta_{z^*} + M(z^*)\phi_{z^*} = \psi_z^*,$$

we get

$$||\psi_z||^2 = \int_a^b \psi_z(t)\psi_{z^*}(t)d\varrho(t) = \frac{M(z)-M(z^*)}{z-z^*} = \frac{\mathrm{Im}(M(z))}{\mathrm{Im}(z)}.$$

Since ψ_z is a non-trivial solution, we furthermore have $0 < ||\psi_z||^2$. □

We conclude this section with a necessary and sufficient condition for Hypothesis 9.1 to hold. To this end recall that for each $c \in (a,b)$, $S_{(a,c)}$ is some self-adjoint operator associated with the restricted differential expression $\tau|_{(a,c)}$. The proofs are the same as those for Schrödinger operators given in [20, Lemma 2.2 and Lemma 2.4].

Theorem 9.6. *The following properties are equivalent:*

(i) *Hypothesis 9.1.*
(ii) *The spectrum of $S_{(a,c)}$ is purely discrete for some $c \in (a,b)$.*
(iii) *There is a real entire solution ϕ_z, $z \in \mathbb{C}$ of $(\tau - z)u = 0$ which lies in S near a.*

10. Spectral transformation

In this section let S again be a self-adjoint restriction of $T_{\max}$ with separate boundary conditions as in the preceding section. Furthermore, we assume that there is a real entire fundamental system θ_z, ϕ_z, $z \in \mathbb{C}$ of the differential equation $(\tau - z)u = 0$ with $W(\theta_z, \phi_z) = 1$ such that ϕ_z lies in S near a. With M we denote the corresponding singular Weyl–Titchmarsh function and with ψ_z, $z \in \rho(S)$ the Weyl solutions of S.

Recall that by Lemma B.4 for all functions f, $g \in L^2((a,b);\varrho)$ there is a unique complex measure $E_{f,g}$ on $\mathbb{R}$ such that

$$\langle R_z f, g\rangle = \int_{\mathbb{R}} \frac{1}{\lambda - z} dE_{f,g}(\lambda), \quad z \in \rho(S).$$

Indeed, these measures are obtained by applying a variant of the spectral theorem to the operator part

$$S_{\mathfrak{D}} = S \cap (\mathfrak{D} \times \mathfrak{D}), \qquad \mathfrak{D} = \overline{\mathrm{dom}(S)} = \mathrm{mul}(S)^{\perp}$$

of S (see Lemma B.4 in Appendix B).

In order to obtain a spectral transformation we define for each function $f \in L^2_c((a,b);\varrho)$ the transform of f as

$$\hat{f}(z) = \int_a^b \phi_z(x) f(x) d\varrho(x), \quad z \in \mathbb{C}. \tag{10.1}$$

Next we can use this to associate a measure with M by virtue of the Stieltjes–Livšić inversion formula, following literally the proof of [20, Lemma 3].

Lemma 10.1. *There is a unique Borel measure μ on $\mathbb{R}$ defined via*

$$\mu((\lambda_1, \lambda_2]) = \lim_{\delta\downarrow 0} \lim_{\varepsilon\downarrow 0} \frac{1}{\pi} \int_{\lambda_1+\delta}^{\lambda_2+\delta} \mathrm{Im}(M(\lambda + \mathrm{i}\varepsilon)) d\lambda \tag{10.2}$$

for each λ_1, $\lambda_2 \in \mathbb{R}$ with $\lambda_1 < \lambda_2$, such that for every f, $g \in L^2_c((a,b);\varrho)$

$$E_{f,g} = \hat{f}\, \hat{g}^* \mu \tag{10.3}$$

and in particular

$$\langle R_z f, g\rangle = \int_{\mathbb{R}} \frac{\hat{f}(\lambda)\hat{g}(\lambda)^*}{\lambda - z} d\mu(\lambda), \quad z \in \rho(S). \tag{10.4}$$

In particular the preceding lemma shows that the mapping $f \mapsto \hat{f}$ is an isometry from $L^2_c((a,b);\varrho) \cap \mathfrak{D}$ into $L^2(\mathbb{R};\mu)$. In fact, for each function $f \in L^2_c((a,b);\varrho) \cap \mathfrak{D}$ we have

$$\|\hat{f}\|_\mu^2 = \int_{\mathbb{R}} \hat{f}(\lambda)\hat{f}(\lambda)^* d\mu(\lambda) = \int_{\mathbb{R}} dE_{f,f} = \|f\|^2.$$

Hence we may uniquely extend this mapping to an isometric linear operator $\mathcal{F}$ on the Hilbert space $\mathfrak{D}$ into $L^2(\mathbb{R};\mu)$ by

$$\mathcal{F}f(\lambda) = \lim_{\alpha\to a}\lim_{\beta\to b}\int_\alpha^\beta \phi_\lambda(x)f(x)d\varrho(x), \quad \lambda\in\mathbb{R},\ f\in\mathfrak{D},$$

where the limit on the right-hand side is a limit in the Hilbert space $L^2(\mathbb{R};\mu)$. Using this operator $\mathcal{F}$, it is quite easy to extend the result of Lemma 10.1 to functions f, $g \in \mathfrak{D}$. Indeed, one gets that $E_{f,g} = \mathcal{F}f\,\mathcal{F}g^*\mu$, i.e.

$$\langle R_z f, g\rangle = \int_{\mathbb{R}} \frac{\mathcal{F}f(\lambda)\mathcal{F}g(\lambda)^*}{\lambda - z} d\mu(\lambda), \quad z \in \rho(S).$$

We will see below that $\mathcal{F}$ is not only isometric, but also onto. In order to compute the inverse and the adjoint of $\mathcal{F}$, we introduce for each function $g \in L^2_c(\mathbb{R};\mu)$ the transform

$$\check{g}(x) = \int_{\mathbb{R}} \phi_\lambda(x) g(\lambda) d\mu(\lambda), \quad x \in (a,b).$$

For arbitrary α, $\beta \in (a,b)$ with $\alpha < \beta$ we have

$$\begin{aligned}\int_\alpha^\beta |\check{g}(x)|^2\, d\varrho(x) &= \int_\alpha^\beta \check{g}(x)\int_{\mathbb{R}} \phi_\lambda(x)g(\lambda)^* d\mu(\lambda)\, d\varrho(x)\\ &= \int_{\mathbb{R}} g(\lambda)^* \int_\alpha^\beta \phi_\lambda(x)\check{g}(x)d\varrho(x)\, d\mu(\lambda)\\ &\le \|g\|_\mu \left\|\mathcal{F}\left(\mathbb{1}_{[\alpha,\beta)}\check{g}\right)\right\|_\mu\\ &\le \|g\|_\mu \sqrt{\int_\alpha^\beta |\check{g}(x)|^2\, d\varrho(x)}.\end{aligned}$$

Hence $\check{g}$ lies in $L^2((a,b);\varrho)$ with $\|\check{g}\| \le \|g\|_\mu$ and we may uniquely extend this mapping to a bounded linear operator $\mathcal{G}$ on $L^2(\mathbb{R};\mu)$ into $\mathfrak{D}$.

If F is a Borel measurable function on $\mathbb{R}$, then we denote with M_F the maximally defined operator of multiplication with F in $L^2(\mathbb{R};\mu)$.

Lemma 10.2. *The isometry $\mathcal{F}$ is onto with inverse $\mathcal{F}^{-1} = \mathcal{G}$ and adjoint*

$$\mathcal{F}^* = \{(g,f) \in L^2(\mathbb{R};\mu)\times L^2((a,b);\varrho)\,|\,\mathcal{G}g - f \in \mathrm{mul}(S)\}. \tag{10.5}$$

Proof. In order to prove $\mathrm{ran}(\mathcal{G}) \subseteq \mathfrak{D}$, let $g \in L^2_c(\mathbb{R};\mu)$. If $\mathbb{1}_{\{\alpha_\varrho\}} \in \mathrm{mul}(S)$, then the solutions ϕ_z, $z \in \mathbb{C}$ vanish in α_ϱ, hence also

$$\check{g}(\alpha_\varrho) = \int_{\mathbb{R}} \phi_\lambda(\alpha_\varrho)g(\lambda)d\mu(\lambda) = 0.$$

Furthermore, if $\mathbb{1}_{\{\beta_\varrho\}} \in \mathrm{mul}(S)$, then the spectrum of S is discrete and the solutions ϕ_λ, $\lambda \in \sigma(S)$ vanish in β_ϱ. Now since μ is supported on $\sigma(S)$, we also have

$$\check{g}(\beta_\varrho) = \int_{\sigma(S)} \phi_\lambda(\beta_\varrho) g(\lambda) d\mu(\lambda) = 0.$$

From this one sees that $\check{g} \in \mathrm{mul}(S)^\perp = \mathfrak{D}$, i.e. $\mathrm{ran}(\mathcal{G}) \subseteq \mathfrak{D}$.

Next we prove $\mathcal{G}\mathcal{F}f = f$ for each $f \in \mathfrak{D}$. Indeed, if f, $g \in L^2_c((a,b);\varrho) \cap \mathfrak{D}$, then we have

$$\begin{aligned} \langle f, g\rangle &= \int_{\mathbb{R}} dE_{f,g} = \int_{\mathbb{R}} \hat{f}(\lambda)\hat{g}(\lambda)^* d\mu(\lambda) \\ &= \lim_{n\to\infty} \int_{(-n,n]} \hat{f}(\lambda) \int_a^b \phi_\lambda(x) g(x)^* d\varrho(x)\, d\mu(\lambda) \\ &= \lim_{n\to\infty} \int_a^b g(x)^* \int_{(-n,n]} \hat{f}(\lambda)\phi_\lambda(x) d\mu(\lambda)\, d\varrho(x) \\ &= \lim_{n\to\infty} \langle \mathcal{G}\mathrm{M}_{\mathbb{1}_{(-n,n]}}\mathcal{F}f, g\rangle = \langle \mathcal{G}\mathcal{F}f, g\rangle. \end{aligned}$$

Now since $\mathrm{ran}(\mathcal{G}) \subseteq \mathfrak{D}$ and $L^2_c((a,b);\varrho) \cap \mathfrak{D}$ is dense in $\mathfrak{D}$ we infer that $\mathcal{G}\mathcal{F}f = f$ for all $f \in \mathfrak{D}$. In order to prove that $\mathcal{G}$ is the inverse of $\mathcal{F}$, it remains to show that $\mathcal{F}$ is onto, i.e. $\mathrm{ran}(\mathcal{F}) = L^2(\mathbb{R};\mu)$. Therefore pick some f, $g \in \mathfrak{D}$ and let F, G be bounded measurable functions on $\mathbb{R}$. Since $E_{f,g}$ is the spectral measure of the operator part $S_{\mathfrak{D}}$ of S (see the proof of Lemma B.4) we get

$$\langle \mathrm{M}_G \mathcal{F} F(S_{\mathfrak{D}}) f, \mathcal{F}g\rangle_\mu = \langle G(S_{\mathfrak{D}})F(S_{\mathfrak{D}})f, g\rangle = \langle \mathrm{M}_G \mathrm{M}_F \mathcal{F}f, \mathcal{F}g\rangle_\mu.$$

Now if we set $h = F(S_{\mathfrak{D}})f$, we get from this last equation

$$\int_{\mathbb{R}} G(\lambda)\mathcal{F}g(\lambda)^* \left(\mathcal{F}h(\lambda) - F(\lambda)\mathcal{F}f(\lambda)\right) d\mu(\lambda) = 0.$$

Since this holds for each bounded measurable function G, we infer

$$\mathcal{F}g(\lambda)^* \left(\mathcal{F}h(\lambda) - F(\lambda)\mathcal{F}f(\lambda)\right) = 0$$

for almost all $\lambda \in \mathbb{R}$ with respect to μ. Furthermore, for each $\lambda_0 \in \mathbb{R}$ we can find a $g \in L^2_c((a,b);\varrho) \cap \mathfrak{D}$ such that $\hat{g} \neq 0$ in a vicinity of λ_0. Hence we even have $\mathcal{F}h = F\mathcal{F}f$ almost everywhere with respect to μ. But this shows that $\mathrm{ran}(\mathcal{F})$ contains all characteristic functions of intervals. Indeed, let $\lambda_0 \in \mathbb{R}$ and choose $f \in L^2_c((a,b);\varrho) \cap \mathfrak{D}$ such that $\hat{f} \neq 0$ in a vicinity of λ_0. Then for each interval J, whose closure is contained in this vicinity one may choose

$$F(\lambda) = \begin{cases} \hat{f}(\lambda)^{-1}, & \text{if } \lambda \in J, \\ 0, & \text{if } \lambda \in \mathbb{R}\backslash J, \end{cases}$$

and gets $\mathbb{1}_J = \mathcal{F}h \in \mathrm{ran}(\mathcal{F})$. Thus we have obtained $\mathrm{ran}(\mathcal{F}) = L^2(\mathbb{R};\mu)$. Finally the fact that the adjoint is given as in the claim follows from the equivalence

$$\mathcal{G}g - f \in \mathrm{mul}(S) \quad \Leftrightarrow \quad \forall u \in \mathfrak{D} : 0 = \langle \mathcal{G}g - f, u\rangle = \langle g, \mathcal{F}u\rangle_\mu - \langle f, u\rangle,$$

which holds for every $f \in L^2((a,b);\varrho)$ and $g \in L^2(\mathbb{R};\mu)$. □

Note that $\mathcal{F}$ is a unitary map from $L^2((a,b);\varrho)$ onto $L^2(\mathbb{R};\mu)$ if and only if S is an operator.

Theorem 10.3. *The self-adjoint relation S is given by $S = \mathcal{F}^*\mathrm{M}_{\mathrm{id}}\mathcal{F}$.*

Proof. First note that for each $f \in \mathfrak{D}$ we have

$$\begin{aligned} f \in \mathrm{dom}(S) \quad &\Leftrightarrow \quad \int_{\mathbb{R}} |\lambda|^2 dE_{f,f}(\lambda) < \infty \quad \Leftrightarrow \quad \int_{\mathbb{R}} |\lambda|^2 |\mathcal{F}f(\lambda)|^2 d\mu(\lambda) < \infty \\ &\Leftrightarrow \quad \mathcal{F}f \in \mathrm{dom}(\mathrm{M}_{\mathrm{id}}) \qquad \Leftrightarrow \quad f \in \mathrm{dom}(\mathcal{F}^*\mathrm{M}_{\mathrm{id}}\mathcal{F}). \end{aligned}$$

Furthermore, if $(f, f_\tau) \in S$, then from Lemma B.4 and Lemma 10.1 we infer

$$\begin{aligned} \langle f_\tau, g\rangle &= \int_{\mathbb{R}} \lambda\, dE_{f,g}(\lambda) = \int_{\mathbb{R}} \lambda \mathcal{F}f(\lambda)\mathcal{F}g(\lambda)^* d\mu(\lambda) \\ &= \int_{\mathbb{R}} \mathrm{M}_{\mathrm{id}}\mathcal{F}f(\lambda)\mathcal{F}g(\lambda)^* d\mu(\lambda) = \langle \mathcal{G}\mathrm{M}_{\mathrm{id}}\mathcal{F}f, g\rangle, \quad g \in \mathfrak{D} \end{aligned}$$

and hence $\mathcal{G}\mathrm{M}_{\mathrm{id}}\mathcal{F}f = Pf_\tau$, where P is the orthogonal projection onto $\mathfrak{D}$. This and Lemma 10.2 show that $(\mathrm{M}_{\mathrm{id}}\mathcal{F}f, f_\tau) \in \mathcal{F}^*$, which is equivalent to $(f, f_\tau) \in \mathcal{F}^*\mathrm{M}_{\mathrm{id}}\mathcal{F}$. Now if we conversely assume that $(g, g_\tau) \in \mathcal{F}^*\mathrm{M}_{\mathrm{id}}\mathcal{F}$, then $(\mathrm{M}_{\mathrm{id}}\mathcal{F}g, g_\tau) \in \mathcal{F}^*$ (note that $g \in \mathrm{dom}(S)$). Hence $\mathcal{G}\mathrm{M}_{\mathrm{id}}\mathcal{F}g - g_\tau$ lies in $\mathrm{mul}(S)$ and since $(g, \mathcal{G}\mathrm{M}_{\mathrm{id}}\mathcal{F}g) \in S$, we also get $(g, g_\tau) \in S$. □

Note that the self-adjoint operator $S_{\mathfrak{D}}$ is unitarily equivalent to the operator of multiplication M_{id}. In fact, $\mathcal{F}$ is unitary as an operator from $\mathfrak{D}$ onto $L^2(\mathbb{R};\mu)$ and maps $S_{\mathfrak{D}}$ onto multiplication with the independent variable. Now the spectrum can be read off from the boundary behavior of the singular Weyl–Titchmarsh function M in the usual way.

Corollary 10.4. *The spectrum of S is given by*

$$\sigma(S) = \sigma(S_{\mathfrak{D}}) = \mathrm{supp}(\mu) = \overline{\{\lambda \in \mathbb{R} \,|\, 0 < \limsup_{\varepsilon\downarrow 0} \mathrm{Im}(M(\lambda + \mathrm{i}\varepsilon))\}}. \tag{10.6}$$

Moreover,

$$\begin{aligned} \sigma_p(S_{\mathfrak{D}}) &= \{\lambda \in \mathbb{R} \,|\, 0 < \lim_{\varepsilon\downarrow 0} \varepsilon \mathrm{Im}(M(\lambda + \mathrm{i}\varepsilon))\}, \\ \sigma_{ac}(S_{\mathfrak{D}}) &= \overline{\{\lambda \in \mathbb{R} \,|\, 0 < \limsup_{\varepsilon\downarrow 0} \mathrm{Im}(M(\lambda + \mathrm{i}\varepsilon)) < \infty\}}^{ess}, \end{aligned}$$

where $\overline{\Omega}^{ess} = \{\lambda \in \mathbb{R} \,|\, |(\lambda - \varepsilon, \lambda + \varepsilon) \cap \Omega| > 0$ for all $\varepsilon > 0\}$, is the essential closure of a Borel set $\Omega \subseteq \mathbb{R}$, and

$$\Sigma_s = \{\lambda \in \mathbb{R} \,|\, \limsup_{\varepsilon\downarrow 0} \mathrm{Im}(M(\lambda + \mathrm{i}\varepsilon)) = \infty\}$$

is a minimal support for the singular spectrum (singular continuous plus pure point spectrum) of $S_{\mathfrak{D}}$.

Proof. Since the operator part $S_{\mathfrak{D}}$ of S is unitary equivalent to M_{id} we infer from Lemma B.3 that $\sigma(S) = \sigma(\mathrm{M}_{\mathrm{id}}) = \mathrm{supp}(\mu)$. Now the remaining part of the claim follows as in [20, Corollary 3.5]. □

Proposition 10.5. *If $\lambda \in \sigma(S)$ is an eigenvalue, then*

$$\mu(\{\lambda\}) = \|\phi_\lambda\|^{-2}. \tag{10.7}$$

Proof. Under this assumptions ϕ_λ is an eigenvector, i.e. $(\phi_\lambda, \lambda\phi_\lambda) \in S$. Hence we get from the proof of Theorem 10.3 that $\mathrm{M_{id}}\mathcal{F}\phi_\lambda = \lambda\mathcal{F}\phi_\lambda$. But this shows that $\mathcal{F}\phi_\lambda(z)$ vanishes for almost all $z \neq \lambda$ with respect to μ. Now from this we get

$$\begin{aligned}\|\phi_\lambda\|^2 = \|\mathcal{F}\phi_\lambda\|_\mu^2 &= \int_{\{\lambda\}} |\mathcal{F}\phi_\lambda(z)|^2\, d\mu(z) \\ &= \mu(\{\lambda\})\left(\int_a^b \phi_\lambda(x)^2 d\varrho(x)\right)^2 = \mu(\{\lambda\})\,\|\phi_\lambda\|^4.\end{aligned}$$

□

With P we denote the orthogonal projection from $L^2((a,b);\varrho)$ onto $\mathfrak{D}$. If S is an operator, P is simply the identity.

Lemma 10.6. *For every $z \in \rho(S)$ and all $x \in (a,b)$ the transform of the Green function $G_z(x,\cdot\,)$ and its quasi-derivative $\partial_x^{[1]} G_z(x,\cdot\,)$ are given by*

$$\mathcal{F}PG_z(x,\cdot\,)(\lambda) = \frac{\phi_\lambda(x)}{\lambda - z} \quad \textit{and} \quad \mathcal{F}P\partial_x^{[1]}G_z(x,\cdot\,)(\lambda) = \frac{\phi_\lambda^{[1]}(x)}{\lambda - z}, \quad \lambda \in \mathbb{R}.$$

Proof. First note that $G_z(x,\cdot\,)$ and $\partial_x^{[1]}G_z(x,\cdot\,)$ both lie in $L^2((a,b);\varrho)$. Then, using Lemma 10.1 we get for each $f \in L^2_c((a,b);\varrho)$ and $g \in L^2_c(\mathbb{R};\mu)$

$$\langle R_z\breve{g}, f\rangle = \int_{\mathbb{R}} \frac{g(\lambda)\hat{f}(\lambda)^*}{\lambda - z} d\mu(\lambda) = \int_a^b \int_{\mathbb{R}} \frac{\phi_\lambda(x)}{\lambda - z} g(\lambda) d\mu(\lambda)\, f(x)^* d\varrho(x).$$

Hence we have

$$R_z\breve{g}(x) = \int_{\mathbb{R}} \frac{\phi_\lambda(x)}{\lambda - z} g(\lambda) d\mu(\lambda)$$

for almost all $x \in (a,b)$ with respect to ϱ. Using Theorem 8.3 one gets

$$\langle \mathcal{F}PG_z(x,\cdot\,), g^*\rangle_\mu = \langle G_z(x,\cdot\,), \breve{g}^*\rangle = \int_{\mathbb{R}} \frac{\phi_\lambda(x)}{\lambda - z} g(\lambda) d\mu(\lambda)$$

for almost all $x \in (a,b)$ with respect to ϱ. Since all three terms are absolutely continuous with respect to ς, this equality is true for all $x \in (a,b)$, which proves the first part of the claim. The second equality follows from

$$\langle \mathcal{F}P\partial_x G_z(x,\cdot\,), g^*\rangle_\mu = \langle \partial_x G_z(x,\cdot\,), \breve{g}^*\rangle = R_z\breve{g}^{[1]}(x) = \int_{\mathbb{R}} \frac{\phi_\lambda^{[1]}(x)}{\lambda - z} g(\lambda) d\mu(\lambda).$$

□

Note that $\mathcal{F}P$ is the unique extension to $L^2((a,b);\varrho)$ of the bounded linear mapping defined in (10.1) on $L^2_c((a,b);\varrho)$.

Lemma 10.7. *Suppose τ is in the l.c. case at a and θ_z, ϕ_z, $z \in \mathbb{C}$ is a fundamental system as in Theorem 9.4. Then for each $z \in \rho(S)$ the transform of the Weyl solution ψ_z is given by*

$$\mathcal{F}P\psi_z(\lambda) = \frac{1}{\lambda - z}, \quad \lambda \in \mathbb{R}. \tag{10.8}$$

Proof. From Lemma 10.6 we obtain for each $x \in (a,b)$

$$\mathcal{F}P\tilde{\psi}_z(x,\cdot)(\lambda) = \frac{W(\theta_z,\phi_\lambda)(x)}{\lambda - z}, \quad \lambda \in \mathbb{R},$$

where

$$\tilde{\psi}_z(x,y) = \begin{cases} M(z)\phi_z(y), & \text{if } y < x, \\ \psi_z(y), & \text{if } y \geq x. \end{cases}$$

Now the claim follows by letting $x \to a$, using Theorem 9.4. □

Under the assumptions of Lemma 10.7, M is a Herglotz–Nevanlinna function. Hence we have

$$M(z) = c_1 + c_2 z + \int_{\mathbb{R}} \frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} d\mu(\lambda), \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{10.9}$$

where the constants c_1, c_2 are given by

$$c_1 = \mathrm{Re}(M(\mathrm{i})) \quad \text{and} \quad c_2 = \lim_{\eta\uparrow\infty} \frac{M(\mathrm{i}\eta)}{\mathrm{i}\eta} \geq 0.$$

Corollary 10.8. *Suppose τ is in the l.c. case at a and θ_z, ϕ_z, $z \in \mathbb{C}$ is a fundamental system as in Theorem 9.4. Then the second constant in* (10.9) *is given by*

$$c_2 = \lim_{\eta\uparrow\infty} \frac{M(i\eta)}{i\eta} = \begin{cases} \theta_z(\alpha_\varrho)^2 \varrho(\{\alpha_\varrho\}), & \text{if } \mathbb{1}_{\{\alpha_\varrho\}} \in \mathrm{mul}(S), \\ 0, & \text{else.} \end{cases}$$

Proof. Taking imaginary parts in (10.9) yields for each $z \in \mathbb{C}\backslash\mathbb{R}$

$$\begin{aligned} \mathrm{Im}(M(z)) &= c_2 \,\mathrm{Im}(z) + \int_{\mathbb{R}} \mathrm{Im}\left(\frac{1}{\lambda - z}\right) d\mu(\lambda) \\ &= c_2 \,\mathrm{Im}(z) + \int_{\mathbb{R}} \frac{\mathrm{Im}(z)}{|\lambda - z|^2} d\mu(\lambda). \end{aligned}$$

From this we get, using Lemma 10.7 and (9.6)

$$\begin{aligned} c_2 + \int_{\mathbb{R}} \frac{1}{|\lambda - z|^2} d\mu(\lambda) = \frac{\mathrm{Im}(M(z))}{\mathrm{Im}(z)} &= \|\psi_z\|^2 = \|(I-P)\psi_z\|^2 + \|\mathcal{F}P\psi_z\|_\mu^2 \\ &= \|(I-P)\psi_z\|^2 + \int_{\mathbb{R}} \frac{1}{|\lambda - z|^2} d\mu(\lambda). \end{aligned}$$

Hence we have (note that $\psi_z(\beta_\varrho) = 0$ if $\mathbb{1}_{\{\beta_\varrho\}} \in \mathrm{mul}(S)\backslash\{0\}$)

$$c_2 = \|(I-P)\psi_z\|^2 = \begin{cases} |\psi_z(\alpha_\varrho)|^2 \varrho(\{\alpha_\varrho\}), & \text{if } \mathbb{1}_{\{\alpha_\varrho\}} \in \mathrm{mul}(S), \\ 0, & \text{else.} \end{cases}$$

Now assume $\mathbb{1}_{\{\alpha_\varrho\}} \in \mathrm{mul}(S)\backslash\{0\}$, then $\phi_z(\alpha_\varrho) = 0$ and hence

$$c_2 = |\theta_z(\alpha_\varrho) + M(z)\phi_z(\alpha_\varrho)|^2 \varrho(\{\alpha_\varrho\}) = |\theta_z(\alpha_\varrho)|^2 \varrho(\{\alpha_\varrho\}), \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

Finally, since θ_z is a real entire function, this proves the claim. □

Remark 10.9. *Given another singular Weyl–Titchmarsh function $\tilde{M}$ as in Remark 9.3, the corresponding spectral measures are related by*

$$\tilde{\mu} = e^{-2g}\mu, \tag{10.10}$$

where g is the real entire function appearing in Remark 9.3. Hence the measures are mutually absolutely continuous and the associated spectral transformations just differ by a simple rescaling with the positive function e^{-2g}. Also note that the spectral measure does not depend on the second solution.

11. Spectral transformation II

In this section let S be a self-adjoint restriction of $T_{\max}$ with separate boundary conditions as in the preceding section. We now want to consider the case where none of the endpoints satisfies the requirements of the previous section. In such a situation the spectral multiplicity of S could be two and hence we will need to work with a matrix-valued transformation.

In the following we will fix some $x_0 \in (a,b)$ and consider the real entire fundamental system of solutions θ_z, ϕ_z, $z \in \mathbb{C}$ with the initial conditions

$$\phi_z(x_0) = -\theta_z^{[1]}(x_0) = -\sin(\varphi_\alpha) \quad \text{and} \quad \phi_z^{[1]}(x_0) = \theta_z(x_0) = \cos(\varphi_\alpha)$$

for some fixed $\alpha \in [0,\pi)$. The Weyl solutions are given by

$$\psi_{z,\pm}(x) = \theta_z(x) \pm m_\pm(z)\phi_z(x), \quad x \in (a,b),\ z \in \mathbb{C}\backslash\mathbb{R}, \tag{11.1}$$

such that ψ_- lies in $L^2((a,b);\varrho)$ near a and ψ_+ lies in $L^2((a,b);\varrho)$ near b. Here $m_\pm$ are the Weyl–Titchmarsh functions of the operators $S_\pm$ obtained by restricting S to (a,x_0) and (x_0,b) with a boundary condition

$$\cos(\varphi_\alpha)f(x_0) + \sin(\varphi_\alpha)f^{[1]}(x_0) = 0,$$

respectively. According to Corollary 9.5 the functions $m_\pm$ are Herglotz–Nevanlinna functions. Now we introduce the 2×2 Weyl–Titchmarsh matrix

$$M(z) = \begin{pmatrix} -\frac{1}{m_-(z)+m_+(z)} & \frac{1}{2}\frac{m_-(z)-m_+(z)}{m_-(z)+m_+(z)} \\ \frac{1}{2}\frac{m_-(z)-m_+(z)}{m_-(z)+m_+(z)} & \frac{m_-(z)m_+(z)}{m_-(z)+m_+(z)} \end{pmatrix}, \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{11.2}$$

In particular, note that we have $\det(M(z)) = -\frac{1}{4}$. The function M is a matrix Herglotz–Nevanlinna function with representation

$$M(z) = M_0 + \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right) d\Omega(\lambda), \quad z \in \mathbb{C}\backslash\mathbb{R},$$

where M_0 is a self-adjoint matrix and Ω is a symmetric matrix-valued measure given by the Stieltjes inversion formula

$$\Omega((\lambda_1,\lambda_2]) = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} \frac{1}{\pi}\int_{\lambda_1+\delta}^{\lambda_2+\delta} \mathrm{Im}(M(\lambda+\mathrm{i}\varepsilon))d\lambda, \quad \lambda_1,\lambda_2 \in \mathbb{R},\ \lambda_1 < \lambda_2.$$

Moreover, the trace $\Omega^{\mathrm{tr}} = \Omega_{1,1} + \Omega_{2,2}$ of Ω is a nonnegative measure and the components of Ω are absolutely continuous with respect to Ω^{tr}. The respective densities are denoted by $R_{i,j}$, $i,j \in \{1,2\}$, and are given by

$$R_{i,j}(\lambda) = \lim_{\varepsilon\downarrow 0} \frac{\mathrm{Im}(M_{i,j}(\lambda+\mathrm{i}\varepsilon))}{\mathrm{Im}(M_{1,1}(\lambda+\mathrm{i}\varepsilon) + M_{2,2}(\lambda+\mathrm{i}\varepsilon))}, \tag{11.3}$$

where the limit exists almost everywhere with respect to Ω^{tr}. Note that R is non-negative and has trace equal to one. In particular, all entries of R are bounded; $0 \leq R_{1,1}, R_{2,2} \leq 1$ and $|R_{1,2}| = |R_{2,1}| \leq 1/2$.

Furthermore, the corresponding Hilbert space $L^2(\mathbb{R};\Omega)$ is associated with the inner product

$$\langle \hat{f}, \hat{g} \rangle_\Omega = \int_{\mathbb{R}} \hat{f}(\lambda)\hat{g}(\lambda)^* d\Omega(\lambda) = \int_{\mathbb{R}} \sum_{i,j=0}^{1} \hat{f}_i(\lambda) R_{i,j}(\lambda) \hat{g}_j(\lambda)^* d\Omega^{\mathrm{tr}}(\lambda).$$

Now for each $f \in L^2_c((a,b);\varrho)$ we define the transform of f as

$$\hat{f}(z) = \begin{pmatrix} \int_a^b \theta_z(x) f(x) d\varrho(x) \\ \int_a^b \phi_z(x) f(x) d\varrho(x) \end{pmatrix}, \quad z \in \mathbb{C}. \tag{11.4}$$

In the following lemma we will relate the 2×2 matrix-valued measure Ω to the operator-valued spectral measure E of S. If F is a measurable function on $\mathbb{R}$, we denote with M_F the maximally defined operator of multiplication with F in the Hilbert space $L^2(\mathbb{R};\Omega)$.

Lemma 11.1. *If f, $g \in L^2_c((a,b);\varrho)$, then we have*

$$\langle E((\lambda_1, \lambda_2]) f, g \rangle = \langle \mathrm{M}_{\mathbb{1}_{(\lambda_1,\lambda_2]}} \hat{f}, \hat{g} \rangle_\Omega \tag{11.5}$$

for all λ_1, $\lambda_2 \in \mathbb{R}$ with $\lambda_1 < \lambda_2$.

Proof. This follows by evaluating Stone's formula

$$\langle E((\lambda_1, \lambda_2]) f, g \rangle = \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{\lambda_1 + \delta}^{\lambda_2 + \delta} \mathrm{Im}\left(\langle R_{\lambda + \mathrm{i}\varepsilon} f, g \rangle \right) d\lambda,$$

using our formula for the resolvent (8.5) together with Stieltjes inversion formula, literally following the proof of [16, Theorem 2.12]. □

Lemma 11.1 shows that the transformation defined in (11.4) uniquely extends to an isometry $\mathcal{F}$ from $L^2((a,b);\varrho)$ into $L^2(\mathbb{R};\Omega)$.

Theorem 11.2. *The operator $\mathcal{F}$ is unitary with inverse given by*

$$\mathcal{F}^{-1} g(x) = \lim_{N \to \infty} \int_{[-N,N)} g(\lambda) \begin{pmatrix} \theta_\lambda(x) \\ \phi_\lambda(x) \end{pmatrix} d\Omega(\lambda), \quad g \in L^2(\mathbb{R};\Omega), \tag{11.6}$$

where the limit exists in $L^2((a,b);\varrho)$. Moreover, $\mathcal{F}$ maps S onto M_{id}.

Proof. By our previous lemma it remains to show that $\mathcal{F}$ is onto. Since it is straightforward to verify that the integral operator on the right-hand side of (11.6) is the adjoint of $\mathcal{F}$, we can equivalently show $\ker(\mathcal{F}^*) = \{0\}$. To this end let $g \in L^2(\mathbb{R},\Omega)$, $N \in \mathbb{N}$ and $z \in \rho(S)$, then

$$(S - z) \int_{-N}^{N} \frac{1}{\lambda - z} g(\lambda) \begin{pmatrix} \theta_\lambda(x) \\ \phi_\lambda(x) \end{pmatrix} d\Omega(\lambda) = \int_{-N}^{N} g(\lambda) \begin{pmatrix} \theta_\lambda(x) \\ \phi_\lambda(x) \end{pmatrix} d\Omega(\lambda),$$

since interchanging integration with the Radon–Nikodým derivatives can be justified using Fubini's theorem. Taking the limit $N \to \infty$ we conclude

$$\mathcal{F}^* \frac{1}{\cdot - z} g = R_z \mathcal{F}^* g, \quad g \in L^2(\mathbb{R},\Omega).$$

By Stone–Weierstraß we even conclude $\mathcal{F}^* \mathrm{M}_F g = F(S) \mathcal{F}^* g$ for any continuous function F vanishing at infinity and by a consequence of the spectral theorem (e.g. the last part of [31, Theorem 3.1]) we can further extend this to characteristic functions of intervals I. Hence, for $g \in \ker(\mathcal{F}^*)$ we conclude

$$\int_I g(\lambda) \begin{pmatrix} \theta_\lambda(x) \\ \phi_\lambda(x) \end{pmatrix} d\Omega(\lambda) = 0$$

for any compact interval I. Moreover, after taking Radon–Nikodým derivatives we also have

$$\int_I g(\lambda) \begin{pmatrix} \theta_\lambda^{[1]}(x) \\ \phi_\lambda^{[1]}(x) \end{pmatrix} d\Omega(\lambda) = 0.$$

Choosing $x = x_0$ we see

$$\int_I g(\lambda) \begin{pmatrix} \cos(\varphi_\alpha) \\ -\sin(\varphi_\alpha) \end{pmatrix} d\Omega(\lambda) = \int_I g(\lambda) \begin{pmatrix} \sin(\varphi_\alpha) \\ \cos(\varphi_\alpha) \end{pmatrix} d\Omega(\lambda) = 0$$

for any compact interval I and thus $g = 0$ as required. □

Next, there is a measurable unitary matrix $U(\lambda)$ which diagonalizes $R(\lambda)$, that is,

$$R(\lambda) = U(\lambda)^* \begin{pmatrix} r_1(\lambda) & 0 \\ 0 & r_2(\lambda) \end{pmatrix} U(\lambda), \tag{11.7}$$

where $0 \leq r_1(\lambda) \leq r_2(\lambda) \leq 1$ are the eigenvalues of $R(\lambda)$. Also note that $r_1(\lambda) + r_2(\lambda) = 1$ by $\mathrm{tr}(R(\lambda)) = 1$. The matrix $U(\lambda)$ provides a unitary operator $L^2(\mathbb{R};\Omega) \to L^2(\mathbb{R}; r_1 d\Omega^{\mathrm{tr}}) \oplus L^2(\mathbb{R}; r_2 d\Omega^{\mathrm{tr}})$ which leaves M_{id} invariant. From this observation we immediately obtain the analog of Corollary 10.4.

Corollary 11.3. *Introduce the Herglotz–Nevanlinna function*

$$M^{\mathrm{tr}}(z) = \mathrm{tr}(M(z)) = \frac{m_-(z)m_+(z) - 1}{m_-(z) + m_+(z)}, \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{11.8}$$

associated with the measure Ω^{tr}. Then the spectrum of S is given by

$$\sigma(S) = \mathrm{supp}(\Omega^{\mathrm{tr}}) = \overline{\{\lambda \in \mathbb{R} \,|\, 0 < \limsup_{\varepsilon\downarrow 0} \mathrm{Im}(M^{\mathrm{tr}}(\lambda + \mathrm{i}\varepsilon))\}}. \tag{11.9}$$

Moreover,

$$\sigma_p(S) = \{\lambda \in \mathbb{R} \,|\, 0 < \lim_{\varepsilon\downarrow 0} \varepsilon \mathrm{Im}(M^{\mathrm{tr}}(\lambda + \mathrm{i}\varepsilon))\},$$
$$\sigma_{ac}(S) = \overline{\{\lambda \in \mathbb{R} \,|\, 0 < \limsup_{\varepsilon\downarrow 0} \mathrm{Im}(M^{\mathrm{tr}}(\lambda + \mathrm{i}\varepsilon)) < \infty\}}^{ess},$$

and

$$\Sigma_s = \{\lambda \in \mathbb{R} \,|\, \limsup_{\varepsilon\downarrow 0} \mathrm{Im}(M^{\mathrm{tr}}(\lambda + \mathrm{i}\varepsilon)) = \infty\}$$

is a minimal support for the singular spectrum (singular continuous plus pure point spectrum) of S.

Furthermore, this allows us to investigate the spectral multiplicity of S.

Lemma 11.4. *If we define*

$$\Sigma_1 = \{\lambda \in \mathrm{supp}(\Omega^{tr}) \,|\, \det R(\lambda) = r_1(\lambda) r_2(\lambda) = 0\},$$
$$\Sigma_2 = \{\lambda \in \mathrm{supp}(\Omega^{tr}) \,|\, \det R(\lambda) = r_1(\lambda) r_2(\lambda) > 0\},$$

then $\mathrm{M}_{\mathrm{id}} = \mathrm{M}_{\mathrm{id}\cdot\mathbb{1}_{\Sigma_1}} \oplus \mathrm{M}_{\mathrm{id}\cdot\mathbb{1}_{\Sigma_2}}$ and the spectral multiplicity of $\mathrm{M}_{\mathrm{id}\cdot\mathbb{1}_{\Sigma_1}}$ is one and the spectral multiplicity of $\mathrm{M}_{\mathrm{id}\cdot\mathbb{1}_{\Sigma_2}}$ is two.

Proof. It is easy to see that $\mathrm{M}_{\mathrm{id}\cdot\mathbb{1}_{\Sigma_1}}$ is unitary equivalent to multiplication with λ in $L^2(\mathbb{R};\mathbb{1}_{\Sigma_1}\Omega^{\mathrm{tr}})$. Moreover, since $r_j\mathbb{1}_{\Sigma_2}\Omega^{\mathrm{tr}}$ and $\mathbb{1}_{\Sigma_2}\Omega^{\mathrm{tr}}$ are mutually absolutely continuous, $\mathrm{M}_{\mathrm{id}\cdot\mathbb{1}_{\Sigma_2}}$ is unitary equivalent to M_{id} in the Hilbert space $L^2(\mathbb{R};\mathbb{1}_{\Sigma_1}\Omega^{\mathrm{tr}}I_2)$. ☐

Combining (11.2) with (11.3) we see that

$$\det R(\lambda) = \lim_{\varepsilon\downarrow 0} \frac{\mathrm{Im}(m_+(\lambda+\mathrm{i}\varepsilon))\mathrm{Im}(m_-(\lambda+\mathrm{i}\varepsilon))}{|m_+(\lambda+\mathrm{i}\varepsilon)+m_-(\lambda+\mathrm{i}\varepsilon)|^2}\frac{1}{\mathrm{Im}(M^{\mathrm{tr}}(\lambda+\mathrm{i}\varepsilon))^2},$$

where the first factor is bounded by $1/4$. Now Lemma 11.4 immediately gives the following result.

Lemma 11.5. *The singular spectrum of S has spectral multiplicity one. The absolutely continuous spectrum of S has multiplicity two on the subset $\sigma_{ac}(S_+)\cap\sigma_{ac}(S_-)$ and multiplicity one on $\sigma_{ac}(S)\backslash(\sigma_{ac}(S_+)\cap\sigma_{ac}(S_-))$. Here $S_\pm$ are the restrictions of S to (a,x_0) and (x_0,b), respectively.*

Proof. Using the fact that Σ_s is a minimal support for the singular part of S we obtain $S_s = S_{pp}\oplus S_{sc} = E(\Sigma_s)S$ and $S_{ac} = (1-E(\Sigma_s))S$. So we see that the singular part has multiplicity one by Lemma 11.4.

For the absolutely continuous part use that

$$\Sigma_{ac,\pm} = \{\lambda\in\mathbb{R}\,|\,0<\lim_{\varepsilon\downarrow 0}\mathrm{Im}(m_\pm(\lambda+\mathrm{i}\varepsilon))<\infty\}$$

are minimal supports for the absolutely continuous spectra of $S_\pm$. Again the remaining result follows from Lemma 11.4. ☐

Appendix A. Linear measure differential equations

In this appendix we collect some necessary facts from linear differential equations with measure coefficients. We refer to Bennewitz [5], Persson [24], Volkmer [36], Atkinson [4] or Schwabik, Tvrdý and Vejvoda [29] for further information. In order to make our presentation self-contained we have included proofs for all results.

Let (a,b) be a finite or infinite interval and ω a positive Borel measures on (a,b). Furthermore, let M be a $\mathbb{C}^{n\times n}$ valued measurable function on (a,b) and F a $\mathbb{C}^n$ valued measurable function on (a,b), such that $\|M(\cdot)\|$ and $\|F(\cdot)\|$ are locally integrable with respect to ω. Here $\|\cdot\|$ denotes some norm on $\mathbb{C}^n$ as well as the corresponding operator norm on $\mathbb{C}^{n\times n}$.

For $c\in(a,b)$ and $Y_c\in\mathbb{C}^n$, some $\mathbb{C}^n$ valued function Y on (a,b) is a solution of the initial value problem

$$\frac{dY}{d\omega} = MY+F, \qquad Y(c)=Y_c, \tag{A.1}$$

if the components of Y are locally absolutely continuous with respect to ω, their Radon–Nikodým derivatives satisfy (A.1) almost everywhere with respect to ω and $Y(c)=Y_c$. An integration shows that some function Y is a solution of the initial value problem (A.1) if and only if it solves the vector integral equation

$$Y(x) = Y_c + \int_c^x (M(t)Y(t)+F(t))\,d\omega(t), \quad x\in(a,b). \tag{A.2}$$

Before we prove existence and uniqueness for solutions of this initial value problem, we need a Gronwall lemma. The proof follows [5, Lemma 1.2 and Lemma 1.3] (see also Atkinson [4, page 455]).

Lemma A.1. *Let $c \in (a,b)$ and $v \in L^1_{\mathrm{loc}}((a,b);\omega)$ be real-valued such that*

$$0 \le v(x) \le K + \int_c^x v(t)d\omega(t), \quad x \in [c,b) \tag{A.3}$$

for some constant $K \ge 0$, then v can be estimated by

$$v(x) \le K \mathrm{e}^{\int_c^x d\omega}, \quad x \in [c,b). \tag{A.4}$$

Proof. First of all note that the function $F(x) = \int_c^x d\omega$, $x \in [c,b)$, satisfies

$$F(x)^{n+1} \ge (n+1) \int_c^x F(t)^n d\omega(t), \quad x \in [c,b), \tag{A.5}$$

by a variant of the substitution rule for Lebesgue–Stieltjes integrals [32, Corollary 5.3]. Now we will prove that

$$v(x) \le K \sum_{k=0}^n \frac{F(x)^k}{k!} + \frac{F(x)^n}{n!} \int_c^x v(t)d\omega(t), \quad x \in [c,b)$$

for each $n \in \mathbb{N}_0$. For $n = 0$ this is just the assumption of our lemma. Otherwise we get inductively

$$\begin{aligned} v(x) &\le K + \int_c^x v(t)d\omega(t) \\ &\le K + \int_c^x \left(K \sum_{k=0}^n \frac{F(t)^k}{k!} + \frac{F(t)^n}{n!} \int_c^t v\, d\omega \right) d\omega(t) \\ &\le K \left(1 + \sum_{k=0}^n \int_c^x \frac{F(t)^k}{k!} d\omega(t) \right) + \int_c^x \frac{F(t)^n}{n!} d\omega(t) \int_c^x v\, d\omega \\ &\le K \sum_{k=0}^{n+1} \frac{F(x)^k}{k!} + \frac{F(x)^{n+1}}{(n+1)!} \int_c^x v\, d\omega, \quad x \in [c,b), \end{aligned}$$

where we used (A.5) twice in the last step. Now taking the limit $n \to \infty$ yields the claim. □

Because of the definition of our integral the assertion of this lemma is only true to the right of c. However, a simple reflection proves that

$$0 \le v(x) \le K + \int_{x+}^{c+} v(t)d\omega(t), \quad x \in (a,c] \tag{A.6}$$

for some constant $K \ge 0$, implies

$$v(x) \le K e^{\int_{x+}^{c+} d\omega}, \quad x \in (a,c]. \tag{A.7}$$

We are now ready to prove the basic existence and uniqueness result.

Theorem A.2. *The initial value problem* (A.1) *has a unique solution for each $c \in (a,b)$ and $Y_c \in \mathbb{C}^n$ if and only if the matrix*

$$I + \omega(\{x\})M(x) \quad \textit{is regular} \tag{A.8}$$

for all $x \in (a,b)$. In this case solutions are real if M, F and Y_c are real.

Proof. First assume that the initial value problem (A.1) has a unique solution for each $c \in (a,b)$ and $Y_c \in \mathbb{C}^n$. Now if the matrix (A.8) was not regular for some $x_0 \in (a,b)$, we would have two distinct solutions Y_1, Y_2 such that $Y_1(x_0) \neq Y_2(x_0)$ but $Y_1(x_0+) = Y_2(x_0+)$. Indeed, one only had to take solutions with different initial conditions at x_0 such that

$$\begin{aligned} Y_1(x_0+) + \omega(\{x_0\})F(x_0) &= (I + \omega(\{x_0\})M(x_0))\, Y_1(x_0) \\ &= (I + \omega(\{x_0\})M(x_0))\, Y_2(x_0) \\ &= Y_2(x_0+) + \omega(\{x_0\})F(x_0). \end{aligned}$$

But then one had

$$\begin{aligned} \|Y_1(x) - Y_2(x)\| &= \left\| \int_{x_0+}^{x} M(t)\,(Y_1(t) - Y_2(t))\, d\omega(t) \right\| \\ &\leq \int_{x_0+}^{x} \|M(t)\|\, \|Y_1(t) - Y_2(t)\|\, d\omega(t), \quad x \in (x_0, b) \end{aligned}$$

and hence by Lemma A.1, $Y_1(x) = Y_2(x)$ for all $x \in (x_0, b)$. But this is a contradiction since now Y_1 and Y_2 are two different solutions of the initial value problem with $Y_c = Y(c)$ for some $c \in (x_0, b)$.

Now assume (A.8) holds for all $x \in (a,b)$ and let α, $\beta \in (a,b)$ with $\alpha < c < \beta$. It suffices to prove that there is a unique solution of (A.2) on (α, β). In order to prove uniqueness, take a solution Y of the homogenous system, i.e. $Y_c = 0$ and $F = 0$. We get

$$\|Y(x)\| \leq \int_{c}^{x} \|M(t)\|\, \|Y(t)\|\, d\omega(t), \quad x \in [c, \beta)$$

and hence $Y(x) = 0$, $x \in [c, \beta)$, by Lemma A.1. To the left-hand side of the point c we have

$$\begin{aligned} Y(x) &= -\int_{x}^{c} M(t)Y(t)d\omega(t) = -\int_{x+}^{c} M(t)Y(t)d\omega(t) - \omega(\{x\})M(x)Y(x) \\ &= -\left(I + \omega(\{x\})M(x)\right)^{-1} \int_{x+}^{c} M(t)Y(t)d\omega(t), \quad x \in (\alpha, c) \end{aligned}$$

and hence

$$\|Y(x)\| \leq \left\| \left(I + \omega(\{x\})M(x)\right)^{-1} \right\| \int_{x+}^{c+} \|M(t)\|\, \|Y(t)\|\, d\omega(t), \quad x \in (\beta, c].$$

Now the function in front of the integral is bounded. Indeed, since M is locally integrable, we have $\omega(\{x\})\|M(x)\| < \frac{1}{2}$ for all but finitely many $x \in [\beta, c]$. Moreover, for those x we have

$$\left\| \left(I + \omega(\{x\})M(x)\right)^{-1} \right\| = \left\| \sum_{n=0}^{\infty} \left(-\omega(\{x\})M(x)\right)^n \right\| \leq 2.$$

Therefore estimate (A.7) applies and yields $Y(x) = 0$, $x \in (\beta, c]$.

Next we will construct the solution by successive approximation. To this end we define

$$Y_0(x) = Y_c + \int_{c}^{x} F(t)d\omega(t), \quad x \in [c, \beta) \tag{A.9}$$

and inductively for each $n \in \mathbb{N}$

$$Y_n(x) = \int_c^x M(t)Y_{n-1}(t)d\omega(t), \quad x \in [c,\beta). \tag{A.10}$$

These functions are bounded by

$$\|Y_n(x)\| \leq \sup_{t\in[c,x)} \|Y_0(t)\| \frac{\left(\int_c^x \|M(t)\|\, d\omega(t)\right)^n}{n!}, \quad x \in [c,\beta). \tag{A.11}$$

Indeed, for $n = 0$ this is obvious, for $n > 0$ we get inductively, using (A.5),

$$\begin{aligned}
\|Y_n(x)\| &\leq \int_c^x \|M(t)\|\, \|Y_{n-1}(t)\|\, d\omega(t) \\
&\leq \sup_{t\in[c,x)} \|Y_0(t)\| \int_c^x \|M(t)\| \frac{\left(\int_c^t \|M(s)\|\, d\omega(s)\right)^n}{n!} d\omega(t) \\
&\leq \sup_{t\in[c,x)} \|Y_0(t)\| \frac{\left(\int_c^x \|M(t)\|\, d\omega(t)\right)^{n+1}}{(n+1)!}.
\end{aligned}$$

Hence the sum $Y(x) = \sum_{n=0}^\infty Y_n(x)$, $x \in [c,\beta)$ converges absolutely and uniformly. Moreover, we have

$$\begin{aligned}
Y(x) &= Y_0(x) + \sum_{n=1}^\infty \int_c^x M(t)Y_{n-1}(t)d\omega(t) \\
&= Y_c + \int_c^x M(t)Y(t) + F(t)d\omega(t), \quad x \in [c,\beta).
\end{aligned}$$

In order to extend the solution to the left of c, pick some points $x_k \in [\alpha, c]$, $k = -N, \dots, 0$ with

$$\alpha = x_{-N} < x_{-N+1} < \cdots < x_0 = c,$$

such that

$$\int_{(x_k,x_{k+1})} \|M(t)\| d\omega(t) < \frac{1}{2}, \quad -N \leq k < 0, \tag{A.12}$$

which is possible since M is locally integrable. More precisely, first take all points $x \in (\alpha, c)$ with $\omega(\{x\})\|M(x)\| \geq \frac{1}{2}$ (these are at most finitely many because $\|M(\cdot)\|$ is locally integrable). Then divide the remaining subintervals such that (A.12) is valid. Now let $-N < k \leq 0$ and assume Y is a solution on $[x_k, \beta)$. We will show that Y can be extended to a solution on $[x_{k-1}, \beta)$. To this end we define

$$Z_0(x) = Y(x_k) + \int_{x_k}^x F(t)d\omega(t), \quad x \in (x_{k-1}, x_k] \tag{A.13}$$

and inductively for each $n \in \mathbb{N}$

$$Z_n(x) = \int_{x_k}^x M(t)Z_{n-1}(t)d\omega(t), \quad x \in (x_{k-1}, x_k]. \tag{A.14}$$

Using (A.12) it is not hard to prove inductively that for each $n \in \mathbb{N}$ and $x \in (x_{k-1}, x_k]$ these functions are bounded by

$$\|Z_n(x)\| \leq \left(\|Y(x_k)\| + \int_{[x_{k-1},x_k]} \|F(t)\| \, d\omega(t) \right) \frac{1}{2^n}. \tag{A.15}$$

Hence we may extend Y onto (x_{k-1}, x_k) by

$$Y(x) = \sum_{n=0}^{\infty} Z_n(x), \quad x \in (x_{k-1}, x_k),$$

where the sum converges absolutely and uniformly. As above one shows that Y is a solution of (A.2) on (x_{k-1}, β). Now if we set $Y(x_{k-1})$ to be equal to (note that the right-hand limit exists because of (A.2))

$$(I - \omega(\{x_{k-1}\})M(x_{k-1}))^{-1} \left(Y(x_{k-1}+) + \omega(\{x_{k-1}\})F(x_{k-1})\right), \tag{A.16}$$

it is easy to show that Y satisfies (A.2) for all $x \in [x_{k-1}, \beta)$. After finitely many steps we arrive at a solution Y, satisfying (A.2) for all $x \in (\alpha, \beta)$.

Finally, if the data M, F and Y_c are real, one easily sees that all quantities in the construction stay real. □

The proof of Theorem A.2 shows that condition (A.8) is actually only needed for all points x to the left of the initial point c. Indeed, it is always possible to extend solutions uniquely to the right of the initial point but not to the left. For a counterexample take $n = 1$, the interval $(-2, 2)$, $y_0 \in \mathbb{C}$ and $\omega = -\delta_{-1} - \delta_1$, where $\delta_{\pm 1}$ are the Dirac measures in ± 1. Then one easily checks that the integral equation

$$y(x) = y_0 + \int_0^x y(t) d\omega(t), \quad x \in (-2, 2)$$

has the solutions

$$y_d(x) = \begin{cases} d, & \text{if } x \in (-2, -1], \\ y_0, & \text{if } x \in (-1, 1], \\ 0, & \text{if } x \in (1, 2), \end{cases}$$

for each $d \in \mathbb{C}$. Hence we see that the solutions are not unique to the left of the initial point $c = 0$.

Corollary A.3. *Assume* (A.8) *holds for each* $x \in (a, b)$. *Then for each* $c \in (a, b)$ *and* $Y_c \in \mathbb{C}^n$, *the initial value problem*

$$\frac{dY}{d\omega} = MY + F \quad \textit{with} \quad Y(c+) = Y_c \tag{A.17}$$

has a unique solution. If M, F *and* Y_c *are real, then the solution is real.*

Proof. Some function Y is a solution of this initial value problem if and only if it is a solution of

$$\frac{dY}{d\omega} = MY + F \quad \text{with} \quad Y(c) = (I + \omega(\{c\})M(c))^{-1} \left(Y_c - \omega(\{c\})F(c)\right).$$

□

Theorem A.4. *Assume $\|M(\cdot)\|$ and $\|F(\cdot)\|$ are integrable near a and Y is a solution of the initial value problem* (A.1)*. Then the limit*

$$Y(a) := \lim_{x\to a} Y(x) \tag{A.18}$$

exists and is finite. A similar result holds for the endpoint b.

Proof. By assumption there is a $c \in (a,b)$ such that

$$\int_a^c \|M(t)\|\, d\omega(t) \le \frac{1}{2}.$$

We first prove that $\|Y(\cdot)\|$ is bounded near a. Indeed if it was not, we had a monotone sequence $x_n \in (a,c)$ with $x_n \downarrow a$ such that $\|Y(x_n)\| \ge \|Y(x)\|$, $x \in [x_n, c]$. From the integral equation which Y satisfies we would get

$$\begin{aligned}\|Y(x_n)\| &\le \|Y(c)\| + \int_{x_n}^c \|M(t)\|\|Y(t)\|d\omega(t) + \int_{x_n}^c \|F(t)\|d\omega(t)\\ &\le \|Y(c)\| + \|Y(x_n)\| \int_{x_n}^c \|M(t)\|d\omega(t) + \int_a^c \|F(t)\|d\omega(t)\\ &\le \|Y(c)\| + \int_a^c \|F(t)\|d\omega(t) + \frac{1}{2}\|Y(x_n)\|.\end{aligned}$$

Hence $\|Y(\cdot)\|$ has to be bounded near a by some constant K. Now the claim follows because we have

$$\begin{aligned}\|Y(x_1) - Y(x_2)\| &= \left\| \int_{x_2}^{x_1} M(t)Y(t) + F(t)d\omega(t) \right\|\\ &\le K \int_{x_1}^{x_2} \|M(t)\|\, d\omega(t) + \int_{x_1}^{x_2} \|F(t)\|d\omega(t)\end{aligned}$$

for each x_1, $x_2 \in (a,c)$, $x_1 < x_2$ i.e. $Y(x)$ is a Cauchy-sequence as $x \to a$. □

Under the assumption of Theorem A.4 one can show that there is always a unique solution of the initial value problem

$$\frac{dY}{d\omega} = MY + F \quad \text{with} \quad Y(a) = Y_a$$

with essentially the same proof as for Theorem A.2. If $\|M(\cdot)\|$ and $\|F(\cdot)\|$ are integrable near b, then one furthermore has to assume that (A.8) holds for all $x \in (a,b)$ in order to get unique solutions of the initial value problem

$$\frac{dY}{d\omega} = MY + F \quad \text{with} \quad Y(b) = Y_b.$$

In the following let M_1, M_2 be $\mathbb{C}^{n\times n}$ valued measurable functions on (a,b) such that $\|M_1(\cdot)\|$, $\|M_2(\cdot)\| \in L^1_{\mathrm{loc}}((a,b);\omega)$. We are interested in the analytic dependence on $z \in \mathbb{C}$ of solutions to the initial value problems

$$\frac{dY}{d\omega} = (M_1 + zM_2)\, Y + F \quad \text{with} \quad Y(c) = Y_c. \tag{A.19}$$

Theorem A.5. *Assume* (A.8) *holds for each $x \in (a,b)$. If for each $z \in \mathbb{C}$, Y_z is the unique solution of* (A.19)*, then $Y_z(x)$ is analytic for each $x \in (a,b)$.*

Proof. We show that the construction in the proof of Theorem A.2 yields analytic solutions. Indeed, let α, $\beta \in (a,b)$ with $\alpha < c < \beta$ as in the proof of Theorem A.2. Then the now z dependent functions $Y_{z,n}(x)$, $n \in \mathbb{N}_0$ (defined as in (A.9) and (A.10)) are polynomials in z for each fixed point $x \in (c,\beta)$. Furthermore, the sum $\sum_{n=0}^{\infty} Y_{z,n}(x)$ converges locally uniformly in z by (A.11) which proves that $Y_z(x)$ is analytic. Now in order to prove analyticity to the left of c fix some $R \in \mathbb{R}^+$. Then there are some points $x_k \in [\alpha, c]$, $k = -N, \dots, 0$ as in the proof of Theorem A.2 such that (A.12) holds for all $M = M_1 + zM_2$, $|z| < R$. It suffices to prove that if $Y_z(x_k)$ is analytic for some $-N < k \leq 0$ then $Y_z(x)$ is analytic for all $x \in [x_{k-1}, x_k)$. Indeed, for each $n \in \mathbb{N}_0$ and $x \in (x_{k-1}, x_k)$ the functions $Z_{z,n}(x)$ (defined as in (A.13) and (A.14)) are analytic and locally bounded in $|z| < R$. From the bound (A.15) one sees that $\sum_{n=0}^{\infty} Z_{z,n}(x)$ converges locally uniformly in $|z| < R$. Hence $Y_z(x)$ is analytic in $\mathbb{C}$. Furthermore, $Y_z(x_{k-1})$ is analytic by (A.16) (note that $Y_z(x_{k-1}+)$ is also analytic since $Y_z(x)$ is bounded locally uniformly in z to the right of x_{k-1}). □

Under the assumptions of the last theorem we even see that the right-hand limit $Y_z(x+)$ is analytic for each $x \in (a,b)$. In fact, this follows since

$$Y_z(x+) = \lim_{\xi \downarrow x} Y_z(\xi), \quad z \in \mathbb{C}$$

and $Y_z(x)$ is bounded locally uniformly in x and z. Furthermore, one can show (see the proof of Corollary A.3) that if for each $z \in \mathbb{C}$, Y_z is the solution of the initial value problem

$$\frac{dY}{d\omega} = (M_1 + zM_2)\,Y + F \quad \text{with} \quad Y(c+) = Y_c,$$

then $Y_z(x)$ as well as $Y_z(x+)$ are analytic in $z \in \mathbb{C}$ for each $x \in (a,b)$.

Appendix B. Linear relations in Hilbert spaces

Let X and Y be linear spaces over $\mathbb{C}$. A linear relation of X into Y is a linear subspace of $X \times Y$. The space of all linear relations of X into Y is denoted by $\mathrm{LR}(X,Y)$. Linear relations generalize the notion of linear operators. Indeed, if D is a linear subspace of X and $T : D \to Y$ is a linear operator, then we may identify T with its graph, which is a linear relation of X into Y. In this way any operator can be regarded as a linear relation. Motivated by this point of view, we define the domain, range, kernel and multi-valued part of some linear relation $T \in \mathrm{LR}(X,Y)$ as

$$\begin{aligned}
\mathrm{dom}(T) &= \{x \in X \,|\, \exists y \in Y : (x,y) \in T\},\\
\mathrm{ran}(T) &= \{y \in Y \,|\, \exists x \in X : (x,y) \in T\},\\
\ker(T) &= \{x \in X \,|\, (x,0) \in T\},\\
\mathrm{mul}(T) &= \{y \in Y \,|\, (0,y) \in T\}.
\end{aligned}$$

Note that some relation T is (the graph of) an operator if and only if $\mathrm{mul}(T) = \{0\}$. In this case these definitions are consistent with the usual definitions for operators.

Again motivated by an operator theoretic viewpoint, we define the following algebraic operations. For T, $S \in \mathrm{LR}(X,Y)$ and $\lambda \in \mathbb{C}$ we set

$$T + S = \{(x,y) \in X \times Y \,|\, \exists y_1, y_2 \in Y : (x,y_1) \in T, (x,y_2) \in S, y = y_1 + y_2\}$$

and

$$\lambda T = \{(x,y) \in X \times Y \,|\, \exists y_0 \in Y : (x,y_0) \in T, y = \lambda y_0\}.$$

It is simple to check that both, $T + S$ and λT are linear relations of X into Y. Moreover, we can define the composition of two linear relations. If $T \in \mathrm{LR}(X,Y)$ and $S \in \mathrm{LR}(Y,Z)$ for some linear space Z, then

$$ST = \{(x,z) \in X \times Z \,|\, \exists y \in Y : (x,y) \in T, (y,z) \in S\}$$

is a linear relation of X into Z. One may even define an inverse of a linear relation $T \in \mathrm{LR}(X,Y)$ by

$$T^{-1} = \{(y,x) \in Y \times X \,|\, (x,y) \in T\},$$

as a linear relation of Y into X. For further properties of these algebraic operations of linear relations er refer to [3, 2.1 Theorem], [11, Chapter 1] or [17, Appendix A].

From now on assume X and Y are Hilbert spaces with inner products $\langle \cdot\,, \cdot \rangle_X$ and $\langle \cdot\,, \cdot \rangle_Y$. The adjoint of a linear relation $T \in \mathrm{LR}(X,Y)$, given by

$$T^* = \{(y,x) \in Y \times X \,|\, \forall (u,v) \in T : \langle u,x \rangle_X = \langle v,y \rangle_Y\},$$

is a linear relation of Y into X. The adjoint of a linear relation is always closed, i.e. T^* is closed with respect to the product topology on $Y \times X$. Moreover, one has

$$T^{**} = \overline{T}, \quad \ker(T^*) = \mathrm{ran}(T)^\perp \quad \text{and} \quad \mathrm{mul}(T^*) = \mathrm{dom}(T)^\perp. \tag{B.1}$$

If $S \in \mathrm{LR}(X,Y)$ is another linear relation we also have

$$T \subseteq S \quad \Rightarrow \quad T^* \supseteq S^*. \tag{B.2}$$

All these properties of adjoints may be found for example in [3, Section 3] or in [17, Proposition C.2.1].

Now let T be a closed linear relation of X into X. The resolvent set $\rho(T)$ of T consists of all numbers $z \in \mathbb{C}$ such that $R_z = (T - z)^{-1}$ is an everywhere defined operator. Here $T - z$ is short-hand for the relation $T - zI$, where I is the identity operator on X. The mapping $z \mapsto R_z$ on $\rho(T)$, called the resolvent of T, has the following properties (see e.g. [11, Section VI.1] or [17, Proposition A.2.3]).

Theorem B.1. *The resolvent set $\rho(T)$ is open and the resolvent identity*

$$R_z - R_w = (z - w) R_z R_w, \quad z,\, w \in \rho(T) \tag{B.3}$$

holds. Moreover, the resolvent is analytic as a mapping into the space of everywhere defined, bounded linear operators on X, equipped with the operator norm.

The spectrum $\sigma(T)$ of a closed linear relation T is the complement of the resolvent set. One may divide the spectrum into three disjoint parts.

$$\sigma_p(T) = \{\lambda \in \sigma(T) \,|\, \ker(T-\lambda) \neq \{0\}\},$$

$$\sigma_c(T) = \{\lambda \in \sigma(T) \,|\, \ker(T-\lambda) = \{0\},\, \operatorname{ran}(T-\lambda) \neq X,\, \overline{\operatorname{ran}(T-\lambda)} = X\},$$

$$\sigma_r(T) = \{\lambda \in \sigma(T) \,|\, \ker(T-\lambda) = \{0\},\, \overline{\operatorname{ran}(T-\lambda)} \neq X\}.$$

The set $\sigma_p(T)$ is called the point spectrum, $\sigma_c(T)$ is the continuous spectrum and $\sigma_r(T)$ is the residual spectrum of T. Elements of the point spectrum are called eigenvalues. The spaces $\ker(T-\lambda)$ corresponding to some eigenvalue λ are called eigenspaces, the non zero elements of the eigenspaces are called eigenvectors.

We need a variant of the spectral mapping theorem for the resolvent (see e.g. [11, Section VI.4] or [17, Proposition A.3.1]).

Theorem B.2. *For each $z \in \rho(T)$ we have*

$$\sigma(R_z) \backslash \{0\} = \left\{ \frac{1}{\lambda - z} \,\middle|\, \lambda \in \sigma(T) \right\}. \tag{B.4}$$

Some linear relation T is said to be symmetric provided that $T \subseteq T^*$. If T is a closed symmetric linear relation, we have $\rho(T) \subseteq \mathrm{r}(T)$ and $\mathbb{C}\backslash\mathbb{R} \subseteq \mathrm{r}(T)$, where

$$\mathrm{r}(T) = \{z \in \mathbb{C} \,|\, (T-z)^{-1} \text{ is a bounded operator}\}$$

denotes the points of regular type of T. Moreover, some linear relation S is said to be self-adjoint, if $S = S^*$. In this case S is closed, the spectrum of S is real and from (B.1) one sees that

$$\operatorname{mul}(S) = \operatorname{dom}(S)^\perp \quad \text{and} \quad \ker(S) = \operatorname{ran}(S)^\perp. \tag{B.5}$$

In particular S is an operator if and only if its domain is dense. Furthermore,

$$S_{\mathfrak{D}} = S \cap (\mathfrak{D} \times \mathfrak{D}) \tag{B.6}$$

is a self-adjoint operator in the Hilbert space $\mathfrak{D}$, where $\mathfrak{D}$ is the closure of the domain of S. These properties of symmetric and self-adjoint linear relations may be found in [12, Theorem 3.13, Theorem 3.20 and Theorem 3.23]. Moreover, the following result shows that S and $S_{\mathfrak{D}}$ (as an operator in the Hilbert space $\mathfrak{D}$) have many spectral properties in common.

Lemma B.3. *S and $S_{\mathfrak{D}}$ have the same spectrum and*

$$R_z f = (S_{\mathfrak{D}} - z)^{-1} P f, \quad f \in X,\ z \in \rho(S), \tag{B.7}$$

where P is the orthogonal projection onto $\mathfrak{D}$. Moreover, the eigenvalues as well as the corresponding eigenspaces coincide.

Proof. From the equalities

$$\operatorname{ran}(S_{\mathfrak{D}} - z) = \operatorname{ran}(S-z) \cap \mathfrak{D} \quad \text{and} \quad \ker(S_{\mathfrak{D}} - z) = \ker(S-z), \quad z \in \mathbb{C}$$

one sees that S and $S_{\mathfrak{D}}$ have the same spectrum as well as the same point spectrum and corresponding eigenspaces. Now let $z \in \rho(S)$, $f \in X$ and set $g = (S-z)^{-1} f$, i.e. $(g,f) \in S - z$. If $f \in \mathfrak{D}$, then since $g \in \mathfrak{D}$, we have $(g,f) \in S_{\mathfrak{D}} - z$, i.e. $(S_{\mathfrak{D}} - z)^{-1} f = g$. Finally note that if $f \in \mathfrak{D}^\perp$, then $g = 0$ since we have $\operatorname{mul}(S-z) = \operatorname{mul}(S) = \operatorname{dom}(S)^\perp$. ☐

Applying a variant of the spectral theorem to $S_{\mathfrak{D}}$, we get the following result for the self-adjoint relation S.

Lemma B.4. *For each f, $g \in X$ there is a unique complex Borel measure $E_{f,g}$ on $\mathbb{R}$ such that*

$$\langle R_z f, g\rangle_X = \int_{\mathbb{R}} \frac{1}{\lambda - z} dE_{f,g}(\lambda), \quad z \in \rho(S). \tag{B.8}$$

Moreover,

$$\langle Pf, g\rangle_X = \int_{\mathbb{R}} dE_{f,g}, \quad f,\, g \in X \tag{B.9}$$

and for each $f \in X$, $E_{f,f}$ is a positive measure with

$$Pf \in \operatorname{dom}(S) \quad \Leftrightarrow \quad \int_{\mathbb{R}} |\lambda|^2 dE_{f,f}(\lambda) < \infty. \tag{B.10}$$

In this case

$$\langle f_S, Pg\rangle_X = \int_{\mathbb{R}} \lambda dE_{f,g}(\lambda), \tag{B.11}$$

whenever $(Pf, f_S) \in S$.

Proof. Since $S_{\mathfrak{D}}$ is a self-adjoint operator in $\mathfrak{D}$, there is an operator-valued spectral measure E such that for all f, $g \in \mathfrak{D}$

$$\langle (S_{\mathfrak{D}} - z)^{-1} f, g\rangle_X = \int_{\mathbb{R}} \frac{1}{\lambda - z} dE_{f,g}(\lambda), \quad z \in \rho(S_{\mathfrak{D}}),$$

where $E_{f,g}$ is the complex measure given by $E_{f,g}(B) = \langle E(B)f, g\rangle_X$, for each Borel set B. For arbitrary f, $g \in X$ we set $E_{f,g} = E_{Pf,Pg}$. Because of Lemma B.3 we get for each $z \in \rho(S)$ the claimed equality

$$\begin{aligned}\langle R_z f, g\rangle_X = \langle R_z Pf, Pg\rangle_X = \langle (S_{\mathfrak{D}} - z)^{-1} Pf, Pg\rangle_X &= \int_{\mathbb{R}} \frac{1}{\lambda - z} dE_{Pf,Pg}(\lambda) \\ &= \int_{\mathbb{R}} \frac{1}{\lambda - z} dE_{f,g}(\lambda),\end{aligned}$$

where we used $R_z = PR_zP$ (see (8.1)). Uniqueness of these measures follows from the Stieltjes inversion formula. The remaining claims follow from the fact that E is the spectral measure of $S_{\mathfrak{D}}$. □

We are interested in self-adjoint extensions of symmetric relations in X. Therefore let T be a closed symmetric linear relation in $X \times X$. The linear relations

$$M_\pm(T) = \{(x, y) \in T^* \,|\, y = \pm \mathrm{i} x\}$$

are called deficiency spaces of T. Note that $M_\pm(T)$ are operators with

$$\operatorname{dom}(M_\pm(T)) = \operatorname{ran}(T \mp \mathrm{i})^\perp = \ker(T^* \pm \mathrm{i}).$$

Furthermore, one has an analog of the first von Neumann formula (see e.g. [12, Theorem 6.1])

$$T^* = T \oplus M_+(T) \oplus M_-(T), \tag{B.12}$$

where the sums are orthogonal with respect to the usual inner product in $X \times X$. Now the existence of self-adjoint extensions of T is determined by these subspaces (see e.g. [10, Theorem 15] or [12, Corollary 6.4]).

Theorem B.5. *The closed symmetric linear relation T has a self-adjoint extension if and only if the dimensions of the deficiency subspaces are equal. In this case all self-adjoint extensions S of T are of the form*

$$S = T \oplus (I - V)M_+(T), \tag{B.13}$$

where V is an isometry from $M_+(T)$ onto $M_-(T)$. Conversely, for each such isometry V the linear relation S given by (B.13) *is self-adjoint.*

In particular, we are interested in the case when the deficiency subspaces are finite-dimensional, i.e.

$$n_\pm(T) = \dim M_\pm(T) < \infty.$$

The numbers $n_\pm(T)$ are called deficiency indices of T.

Corollary B.6. *If T has equal and finite deficiency indices, i.e.*

$$n_-(T) = n_+(T) = n \in \mathbb{N},$$

then the self-adjoint extensions of T are precisely the n-dimensional symmetric extensions of T.

Proof. By Theorem B.5 each self-adjoint extension is an n-dimensional symmetric extension of T, since $\dim(I-V)M_+(T) = n$. Conversely assume that S is an n-dimensional symmetric extension of T, i.e. $S = T \dot{+} M$ for some n-dimensional symmetric subspace M. Then since

$$\dim \operatorname{ran}(M \mp \mathrm{i}) = n = \dim X / \operatorname{ran}(T \mp \mathrm{i}),$$

(note that $(M \mp \mathrm{i})^{-1}$ is an operator) we get

$$\operatorname{ran}(S \mp \mathrm{i}) = \operatorname{ran}(T \mp \mathrm{i}) \dot{+} \operatorname{ran}(M \mp \mathrm{i}) = X.$$

Hence we have $\dim M_\pm(S) = 0$ and therefore $S^* = S$ in view of (B.12). □

The essential spectrum $\sigma_e(S)$ of a self-adjoint linear relation S consists of all eigenvalues of infinite multiplicity and all accumulation points of the spectrum. Moreover, the discrete spectrum $\sigma_d(S)$ of S consists of all eigenvalues of S with finite multiplicity which are isolated points of the spectrum of S. From Lemma B.3 one immediately sees that

$$\sigma_e(S) = \sigma_e(S_{\mathfrak{D}}) \quad \text{and} \quad \sigma_d(S) = \sigma_d(S_{\mathfrak{D}}).$$

Using this equality, we get the following two theorems on the stability of the essential spectrum.

Theorem B.7. *Let T be a symmetric relation with equal and finite deficiency indices n and S_1, S_2 be self-adjoint extensions of T. Then the operators*

$$(S_1 \pm \mathrm{i})^{-1} - (S_2 \pm \mathrm{i})^{-1} \tag{B.14}$$

are at most n-dimensional. In particular we have

$$\sigma_e(S_1) = \sigma_e(S_2). \tag{B.15}$$

Proof. Because of $\dim\operatorname{ran}(T \pm \mathrm{i})^{\perp} = n$ and

$$(S_1 \pm \mathrm{i})^{-1} f = (T \pm \mathrm{i})^{-1} f = (S_2 \pm \mathrm{i})^{-1} f, \quad f \in \overline{\operatorname{ran}(T \pm \mathrm{i})},$$

the difference of the resolvents is at most n-dimensional. Now the remaining claim follows from Lemma B.3 and [31, Theorem 6.19]. □

Theorem B.8. *Let X_1, X_2 be closed subspaces of X such that $X = X_1 \oplus X_2$. If S_1 is a self-adjoint linear relation in X_1 and S_2 is a self-adjoint linear relation in X_2, then $S_1 \oplus S_2$ is a self-adjoint linear relation in x with*

$$\sigma_e\left(S_1 \oplus S_2\right) = \sigma_e\left(S_1\right) \cup \sigma_e\left(S_2\right). \tag{B.16}$$

Proof. A simple calculation shows that $(S_1 \oplus S_2)^* = S_1^* \oplus S_2^* = S_1 \oplus S_2$. Since

$$\mathfrak{D} = \overline{\operatorname{dom}(S_1 \oplus S_2)} = \overline{\operatorname{dom}(S_1)} \oplus \overline{\operatorname{dom}(S_2)} = \mathfrak{D}_1 \oplus \mathfrak{D}_2$$

and

$$(S_1 \oplus S_2)_{\mathfrak{D}} = S_{1\mathfrak{D}_1} \oplus S_{2\mathfrak{D}_2},$$

the claim follows from the corresponding result for operators. □

Appendix C. One dimensional Sturm–Liouville problems

For the sake of completeness in this section we consider the case when ϱ is not necessarily supported on more than one point, i.e. we only assume that (i) to (v) of Hypothesis 3.7 hold. Because of the lack of the identification of Proposition 3.9 in this case, we make the following definition. Some linear subspace $S \subseteq \mathfrak{D}_\tau$ is said to give rise to a self-adjoint relation if the map

$$\begin{array}{ccc} S & \to & L^2((a,b);\varrho) \times L^2((a,b);\varrho) \\ f & \mapsto & (f, \tau f) \end{array} \tag{C.1}$$

is well-defined, injective and its range is a self-adjoint relation of $L^2((a,b);\varrho)$ into $L^2((a,b);\varrho)$. By the identification of Proposition 3.9 one sees that we already determined all linear subspaces of $\mathfrak{D}_\tau$ which give rise to a self-adjoint relation if ϱ is supported on more than one point. Hence we need only consider the case when ϱ is supported on only one point. Indeed, we will do this by proving a version of Theorem 7.6 (note that τ is in the l.c. case at both endpoints). Therefore assume in the following $\varrho = \varrho_0 \delta_{x_0}$ for some $\varrho_0 \in \mathbb{R}^+$ and $x_0 \in (a,b)$. In this case each function $f \in \mathfrak{D}_\tau$ is of the form

$$f(x) = \begin{cases} u_a(x), & \text{if } x \in (a, x_0], \\ u_b(x), & \text{if } x \in (x_0, b), \end{cases}$$

where u_a and u_b are solutions of $\tau u = 0$ with $u_a(x_0-) = u_b(x_0+)$, i.e. f is continuous in x_0 but in general the quasi-derivative $f^{[1]}$ is not. In this case τf is given by

$$\tau f(x_0) = \frac{1}{\varrho_0}\left(-f^{[1]}(x_0+) + f^{[1]}(x_0-) + f(x_0)\chi(\{x_0\})\right). \tag{C.2}$$

Furthermore, for two functions f, $g \in \mathfrak{D}_\tau$, the limits

$$W(f,g)(a) := \lim_{x\to a} W(f,g)(x) \quad \text{and} \quad W(f,g)(b) := \lim_{x\to b} W(f,g)(x)$$

exist and are finite. In fact, the Wronskian is constant away from x_0. Now as in Section 7 let w_1, $w_2 \in \mathfrak{D}_\tau$ with

$$W(w_1, w_2^*)(a) = 1 \qquad \text{and} \qquad W(w_1, w_1^*)(a) = W(w_2, w_2^*)(a) = 0,$$
$$W(w_1, w_2^*)(b) = 1 \qquad \text{and} \qquad W(w_1, w_1^*)(b) = W(w_2, w_2^*)(b) = 0,$$

and define the linear functionals BC_a^1, BC_a^2, BC_b^1 and BC_b^2 on $\mathfrak{D}_\tau$ by

$$BC_a^1(f) = W(f, w_2^*)(a) \quad \text{and} \quad BC_a^2(f) = W(w_1^*, f)(a) \quad \text{for } f \in \mathfrak{D}_\tau,$$
$$BC_b^1(f) = W(f, w_2^*)(b) \quad \text{and} \quad BC_b^2(f) = W(w_1^*, f)(b) \quad \text{for } f \in \mathfrak{D}_\tau.$$

Again one may choose special functions w_1, w_2 as in Proposition 7.2.

Theorem C.1. *Let $S \subseteq \mathfrak{D}_\tau$ be a linear subspace of the form*

$$S = \left\{ f \in \mathfrak{D}_\tau \;\middle|\; \begin{array}{l} BC_a^1(f)\cos\varphi_\alpha - BC_a^2(f)\sin\varphi_\alpha = 0 \\ BC_b^1(f)\cos\varphi_\beta - BC_b^2(f)\sin\varphi_\beta = 0 \end{array} \right\} \tag{C.3}$$

for some φ_α, $\varphi_\beta \in [0,\pi)$. Then S gives rise to a self-adjoint relation if and only if one of the following inequalities

$$w_2(x_0-)\cos\varphi_\alpha + w_1(x_0-)\sin\varphi_\alpha \neq 0, \tag{C.4a}$$
$$w_2(x_0+)\cos\varphi_\beta + w_1(x_0+)\sin\varphi_\beta \neq 0, \tag{C.4b}$$

holds. This relation is an operator if and only if (C.4a) *and* (C.4b) *hold.*

Proof. The boundary conditions can by written as

$$W(f, w_2^*\cos\varphi_\alpha + w_1^*\sin\varphi_\alpha)(x_0-) = 0,$$
$$W(f, w_2^*\cos\varphi_\beta + w_1^*\sin\varphi_\beta)(x_0+) = 0.$$

From this one sees that the mapping (C.1) is injective if and only if one of the inequalities (C.4a) or (C.4b) holds. Hence for the first part it remains to show that in this case the range of the mapping (C.1) is a self-adjoint relation. First consider the case when both inequalities hold. Then we get from the boundary conditions

$$f^{[1]}(x_0-) = f(x_0)\frac{\cos\varphi_\alpha w_2^{[1]}(x_0-)^* + \sin\varphi_\alpha w_1^{[1]}(x_0-)^*}{\cos\varphi_\alpha w_2(x_0-)^* + \sin\varphi_\alpha w_1(x_0-)^*}, \quad f \in S$$

and similar for the right-hand limit. A simple calculation shows that the imaginary part of this fraction as well as the imaginary part of the corresponding fraction for the right-hand limit vanish. Hence from (C.2) we infer that the range of the mapping (C.1) is a self-adjoint operator (multiplication with a real scalar). Now in the case when one inequality, say (C.4a) does not hold, we get $f(x_0) = 0$ for each $f \in S$ from the boundary condition at a. Hence it suffices to prove that $\tau f(x_0)$ takes each value in $\mathbb{C}$ if f runs through S, i.e. S corresponds to the self-adjoint, multi-valued relation $\{0\} \times L^2((a,b);\varrho)$. But this follows since all functions of the form

$$f(x) = \begin{cases} u_a(x), & \text{if } x \in (a, x_0], \\ 0, & \text{if } x \in (x_0, b), \end{cases}$$

where u_a is a solution of $\tau u = 0$ with $u_a(x_0) = 0$, lie in S. □

The preceding theorem corresponds to separate boundary conditions. Next we discuss the case of coupled boundary conditions.

Theorem C.2. *Let $S \subseteq \mathfrak{D}_\tau$ be a linear subspace of the form*

$$S = \left\{ f \in \mathfrak{D}_\tau \,\middle|\, \begin{pmatrix} BC_b^1(f) \\ BC_b^2(f) \end{pmatrix} = e^{i\varphi} R \begin{pmatrix} BC_a^1(f) \\ BC_a^2(f) \end{pmatrix} \right\} \tag{C.5}$$

for some $\varphi \in [0, \pi)$ and $R \in \mathbb{R}^{2\times 2}$ with $\det R = 1$ and set

$$\tilde{R} = \begin{pmatrix} w_2^{[1]}(x_0+)^* & -w_2(x_0+)^* \\ -w_1^{[1]}(x_0+)^* & w_1(x_0+)^* \end{pmatrix}^{-1} R \begin{pmatrix} w_2^{[1]}(x_0-)^* & -w_2(x_0-)^* \\ -w_1^{[1]}(x_0-)^* & w_1(x_0-)^* \end{pmatrix}.$$

Then S gives rise to a self-adjoint relation if and only if

$$\tilde{R}_{12} \neq 0 \qquad \text{or} \qquad e^{i\varphi}\tilde{R}_{11} \neq 1 \neq e^{i\varphi}\tilde{R}_{22}.$$

This relation is an operator if and only if $\tilde{R}_{12} \neq 0$.

Proof. The boundary conditions can be written as

$$\begin{pmatrix} f(x_0+) \\ f^{[1]}(x_0+) \end{pmatrix} = e^{i\varphi}\tilde{R} \begin{pmatrix} f(x_0-) \\ f^{[1]}(x_0-) \end{pmatrix}.$$

First of all note that $\tilde{R}$ is a real matrix. Indeed, since for each $j = 1, 2$, w_j and w_j^* are solutions of $\tau u = 0$ on (a, x_0) we see that they must be linearly dependent, hence we get $w_j(x) = w_j(x)^*$, $x \in (a, x_0)$. Of course the same holds to the right of x_0 and since R is real also $\tilde{R}$ is real. If $\tilde{R}_{12} \neq 0$, then the boundary conditions show that the mapping (C.1) is injective. Furthermore, using (C.2) one gets

$$\tau f(x_0)\varrho_0 = f(x_0)\frac{1 - e^{i\varphi}\left(\tilde{R}_{11} + \tilde{R}_{22}\right) + e^{2i\varphi}\det\tilde{R}}{e^{i\varphi}\tilde{R}_{12}} + f(x_0)\chi(\{x_0\}) \quad f \in S.$$

A simple calculation shows that $\det\tilde{R} = \det R = 1$ and that the fraction is real. Hence we see that S gives rise to a self-adjoint, single-valued relation.

Now assume $\tilde{R}_{12} = 0$ and $e^{i\varphi}\tilde{R}_{11} \neq 1 \neq e^{i\varphi}\tilde{R}_{22}$, then again the boundary conditions show that the mapping (C.1) is injective. Furthermore, they show that each function $f \in S$ satisfies $f(x_0) = 0$. Hence it suffices to show that $\tau f(x_0)$ takes every value as f runs through S. But this is true since all functions

$$f_c(x) = \begin{cases} cu_a(x), & \text{if } x \in (a, x_0], \\ ce^{i\varphi}\tilde{R}_{22}u_b(x), & \text{if } x \in (x_0, b), \end{cases} \tag{$*$}$$

where $c \in \mathbb{C}$ and u_a, u_b are solutions of $\tau u = 0$ with $u_a(x_0-) = u_b(x_0+) = 0$ and $u_a^{[1]}(x_0-) = u_b^{[1]}(x_0+) = 1$, lie in S. If $\tilde{R}_{12} = 0$ but $e^{i\varphi}\tilde{R}_{22} = 1$, then the mapping (C.1) is not injective. Indeed, all functions of the form $(*)$ are mapped onto zero. Finally, if $\tilde{R}_{12} = 0$ and $e^{i\varphi}\tilde{R}_{11} = 1 \neq e^{i\varphi}\tilde{R}_{22}$, then since S is two-dimensional it does not give rise to a self-adjoint relation. ☐

Note that if we choose for BC_a^1, BC_a^2, BC_b^1 and BC_b^2 the functionals from Proposition 7.2, then we get $\tilde{R} = R$.

The resolvent of the self-adjoint relations given in Theorem C.1 and Theorem C.2 can be written as in Section 8. In fact, Theorem 8.1 and Corollary 8.2 are obviously valid since the resolvents are simply multiplication by some scalar. Moreover, Theorem 8.3 and Corollary 8.4 for self-adjoint relations as in Theorem C.1 may be proven along the same lines as in the general case. The remaining theorems of Section 8 are void of meaning here, since all self-adjoint relations have purely discrete spectrum. Finally, the results of Sections 9 and 10 are also valid for self-adjoint relations as in Theorem C.1 since all proofs in these sections also apply in this simple case.

Acknowledgments. We thank Fritz Gesztesy and Andrei Shkalikov for help with respect to the literature.

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On the connection between the Hilger and Radon–Nikodým derivatives

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Published in *J. Math. Anal. Appl.* **385** (2012), no. 2, 1184–1189

Abstract. We show that the Hilger derivative on time scales is a special case of the Radon–Nikodým derivative with respect to the natural measure associated with every time scale. Moreover, we show that the concept of delta absolute continuity agrees with the one from measure theory in this context.

1. Introduction

Time scale calculus was introduced by Hilger in 1988 as a means of unifying differential and difference calculus. Since then this approach has had an enormous impact and developed into a new field of mathematics (see e.g. [3], [4] and the references therein). However, the aim to unify discrete and continuous calculus is of course much older and goes back at least to the introduction of the Riemann–Stieltjes integral, which unifies sums and integrals, by Stieltjes in 1894. Of course these ideas have also been used to unify differential and difference equations and we refer to the seminal work of Atkinson [2] or the book by Mingarelli [15]. The inverse operation to the Lebesgue–Stieltjes integral is the Radon–Nikodým derivative and it is of course natural to ask in what sense this old approach is related to the new time scale calculus. Interestingly this question has not attained much attention and is still not fully answered to the best of our knowledge. It is the aim of the present paper to fill this gap by showing that the Hilger derivative equals the Radon–Nikodým derivative with respect to the measure which is naturally associated with every time scale. It can be defined in several equivalent ways, for example via its distribution function, which is just the forward shift function (cf. (2.1)), or as the image of Lebesgue measure under the backward shift function (cf. (2.3)). This measure was first introduced by Guseinov in [13] and it was shown by Bohner and Guseinov in Chapter 5 of [4] that the delta integral on time scales is a special case of the Lebesgue–Stieltjes integral associated with this measure (see also [6], [9], [16] for further results in this direction).

2010 *Mathematics Subject Classification.* Prim. 26E70, 28A15; Sec. 34N05, 39A12.
Keywords. Time scale calculus, Hilger derivative, Radon–Nikodým derivative.
Research supported by the *Austrian Science Fund (FWF)* under grant no. Y330.

Moreover, Cabada and Vivero [5] introduced the concept of absolutely continuous functions on time scales and proved a corresponding fundamental theorem of calculus. Again the natural question arises, in what sense this new concept is related to the usual concept of absolute continuity with respect to the natural measure associated with the time scale. Of course this is also related to the concept of weak derivatives introduced by Agarwal, Otero-Espinar, Perera, and Vivero [1] (see also the alternative approach by Davidson and Rynne [8], [17] via completion of continuous functions).

Finally, our result also generalizes the work of Chyan and Fryszkowski [7] who showed that every increasing function on a time scale has a right derivative almost everywhere.

2. The Hilger derivative as a Radon–Nikodým derivative

To set the stage we recall a few definitions and facts from time scale calculus [3], [4]. Let $\mathbb{T}$ be a time scale, that is, a nonempty closed subset of $\mathbb{R}$. We define the forward shift on $\mathbb{R}$ via

$$\sigma(t) = \begin{cases} \inf\{s \in \mathbb{T} \,|\, t < s\}, & t < \sup \mathbb{T}, \\ \sup \mathbb{T}, & t \geq \sup \mathbb{T}, \end{cases}$$

and the backward shift via

$$\rho(t) = \begin{cases} \sup\{s \in \mathbb{T} \,|\, t > s\}, & t > \inf \mathbb{T}, \\ \inf \mathbb{T}, & t \leq \inf \mathbb{T}, \end{cases}$$

in the usual way. Note that σ is nondecreasing right continuous and ρ is nondecreasing left continuous. The quantity

$$\mu(t) = \sigma(t) - t, \quad t \in \mathbb{T}$$

is known as the graininess. A point $t \in \mathbb{T}$ is called right scattered if $\sigma(t) > t$ and left scattered if $\rho(t) < t$. Since a nondecreasing function can have at most countably many discontinuities there are only countably many right or left scattered points.

Associated with $\mathbb{T}$ is a unique Borel measure which is defined via its distribution function σ (this procedure is standard and we refer to, e.g. [18, Section A.1] for a brief and concise account). For notational simplicity we denote this measure by the same letter σ and hence have

$$\sigma(A) = \begin{cases} \sigma_+(b) - \sigma_+(a), & A = (a,b], \\ \sigma_+(b) - \sigma_-(a), & A = [a,b], \\ \sigma_-(b) - \sigma_+(a), & A = (a,b), \\ \sigma_-(b) - \sigma_-(a), & A = [a,b). \end{cases} \tag{2.1}$$

Here we use the short-hand notation

$$f_\pm(t) = \lim_{\varepsilon \downarrow 0} f(t \pm \varepsilon), \quad t \in \mathbb{R}$$

for functions $f : \mathbb{R} \to \mathbb{C}$ which are locally of bounded variation (such that the limits always exist). Note that since $\sigma_-(t) = t$ for $t \in \mathbb{T}$ we have

$$\sigma(\{t\}) = \mu(t), \quad t \in \mathbb{T}.$$

The topological support of σ is given by

$$\operatorname{supp}(\sigma) = \mathbb{T}^\kappa, \qquad \mathbb{T}^\kappa = \overline{\mathbb{T}\backslash\{\sup \mathbb{T}\}}.$$

Note that $\mathbb{T}^\kappa = \mathbb{T}$ if $\mathbb{T}$ does not have a left scattered maximum and otherwise $\mathbb{T}^\kappa$ is $\mathbb{T}$ without this left scattered maximum.

The Riemann–Stieltjes integral with respect to this measure is known as the delta integral

$$\int_a^b f(t)\Delta t := \int_{[a,b)} f(t) d\sigma(t), \qquad a,b \in \mathbb{T}.$$

There is also an alternate way [17] of defining the integral (and thus the measure) using

$$\int_{[a,b)} f(t) d\sigma(t) = \int_a^b f(\rho(t)) dt, \qquad a,b \in \mathbb{T}. \tag{2.2}$$

Indeed this equality is due to the fact that σ is the image measure of the Lebesgue measure λ under the function ρ, i.e.

$$\sigma(A) = \lambda(\rho^{-1}(A)) \tag{2.3}$$

for each Borel set A (which is proved readily for intervals). Furthermore, this shows that some measurable function f is integrable with respect to σ if and only if $f \circ \rho$ is integrable with respect to Lebesgue measure.

A function f on $\mathbb{T}$ is said to be delta (or Hilger) differentiable at some point $t \in \mathbb{T}$ if there is a number $f^\Delta(t)$ such that for every $\varepsilon > 0$ there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \le \varepsilon |\sigma(t) - s|, \quad s \in U.$$

If $\mu(t) = 0$ then f is differentiable at t if and only if it is continuous at t and

$$f^\Delta(t) = \lim_{s \to t} \frac{f(s) - f(t)}{s - t} \tag{2.4}$$

exists (the limit has to be taken for $s \in \mathbb{T}\backslash\{t\}$). Similarly, if $\mu(t) > 0$ then f is differentiable at t if and only if it is continuous at t and

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} \tag{2.5}$$

in this case.

Every function $f : \mathbb{T} \to \mathbb{C}$ can be extended to all of $\mathbb{R}$ via

$$\bar{f}(t) = f(\sigma(t)), \qquad t \notin \mathbb{T}. \tag{2.6}$$

Note that if the original function f is continuous at $t \in \mathbb{T}$, then the extension will satisfy $\bar{f}_-(t) = f(t)$ and $\bar{f}_+(t) = f(\sigma(t))$. In particular, $\bar{f}$ will be left continuous if f is continuous.

Next we briefly review the concept of the derivative of a function on $\mathbb{R}$ with respect to the Borel measure σ. As already pointed out above, if $\nu : \mathbb{R} \to \mathbb{C}$ is locally of bounded variation we have an associated measure (denoted by the same letter for notational simplicity) and we can consider the Radon–Nikodým derivative

$$\frac{d\nu}{d\sigma}(t),$$

which is defined almost everywhere with respect to σ. We recall (see e.g. [12, Section 1.6]) that

$$\frac{d\nu}{d\sigma}(t) = \lim_{\varepsilon\downarrow 0} \frac{\nu((t-\varepsilon,t+\varepsilon))}{\sigma((t-\varepsilon,t+\varepsilon))} = \lim_{\varepsilon\downarrow 0} \frac{\nu_-(t+\varepsilon)-\nu_+(t-\varepsilon)}{\sigma_-(t+\varepsilon)-\sigma_+(t-\varepsilon)}, \tag{2.7}$$

where the limit exists almost everywhere with respect to σ. The function ν is said to be absolutely continuous with respect to σ on some interval $[a,b)$ if the associated measure, restricted to this interval is absolutely continuous with respect to σ, i.e. if

$$\nu_-(x) - \nu_-(a) = \int_{[a,x)} \frac{d\nu}{d\sigma}(t) d\sigma(t), \quad x \in [a,b).$$

Furthermore, ν is locally absolutely continuous with respect to σ if it is absolutely continuous on each such interval. Note that in this case the only possible discontinuities of ν are the right scattered points.

Lemma 2.1. *Suppose $f : \mathbb{T} \to \mathbb{C}$ is delta differentiable in some point $t \in \mathbb{T}^\kappa$ and $\bar{f}$ is locally of bounded variation. Then the limit in* (2.7) *exists and satisfies*

$$\frac{d\bar{f}}{d\sigma}(t) = f^\Delta(t). \tag{2.8}$$

Proof. First suppose that $\mu(t) = 0$ and note that (2.4) implies

$$\lim_{\varepsilon\downarrow 0} \frac{\bar{f}_-(t+\varepsilon) - f(t) - f^\Delta(t)\eta_+(\varepsilon)}{\eta_+(\varepsilon)} = 0, \qquad \eta_+(\varepsilon) = \sigma_-(t+\varepsilon) - t.$$

Indeed this follows because of $\bar{f}_-(t+\varepsilon) = f(\sigma_-(t+\varepsilon))$ and since $\sigma_-(t+\varepsilon) \to t$ as $\varepsilon \downarrow 0$ (also note that $\sigma_-(t+\varepsilon) \in \mathbb{T}$). Furthermore, if t is left dense we similarly obtain, using $\bar{f}_+(t-\varepsilon) = f(\sigma_+(t-\varepsilon))$ and $\sigma_+(t-\varepsilon) \to t$ as $\varepsilon \downarrow 0$ (also note that $\sigma_+(t-\varepsilon) \in \mathbb{T}$)

$$\lim_{\varepsilon\downarrow 0} \frac{f(t) - \bar{f}_+(t-\varepsilon) - f^\Delta(t)\eta_-(\varepsilon)}{\eta_-(\varepsilon)} = 0, \qquad \eta_-(\varepsilon) = t - \sigma_+(t-\varepsilon).$$

Now observe that for each $\varepsilon > 0$ we have

$$\begin{aligned}\frac{\bar{f}_-(t+\varepsilon) - \bar{f}_+(t-\varepsilon)}{\sigma_-(t+\varepsilon) - \sigma_+(t-\varepsilon)} - f^\Delta(t) = \\ &= \frac{\bar{f}_-(t+\varepsilon) - f(t) - f^\Delta(t)\eta_+(\varepsilon)}{\eta_+(\varepsilon)} \frac{\eta_+(\varepsilon)}{\eta_+(\varepsilon)+\eta_-(\varepsilon)} \\ &+ \frac{f(t) - \bar{f}_+(t-\varepsilon) - f^\Delta(t)\eta_-(\varepsilon)}{\eta_-(\varepsilon)} \frac{\eta_-(\varepsilon)}{\eta_+(\varepsilon)+\eta_-(\varepsilon)}.\end{aligned}$$

If t is left scattered, then for small enough ε the second term vanishes, since then $f(t) = \bar{f}_+(t-\varepsilon)$ as well as $\eta_-(\varepsilon) = 0$. Hence the claim follows since the first term converges to zero. Otherwise if t is left dense, both terms converge to zero and the claim again follows (also note that the fractions stay bounded since $\eta_+(\varepsilon)$ and $\eta_-(\varepsilon)$ are positive).

Now if $\mu(t) > 0$, i.e. t is right scattered, we have for small enough $\varepsilon > 0$

$$\frac{\bar{f}_-(t+\varepsilon) - \bar{f}_+(t-\varepsilon)}{\sigma_-(t+\varepsilon) - \sigma_+(t-\varepsilon)} = \frac{f(\sigma(t)) - \bar{f}_+(t-\varepsilon)}{\mu(t) + t - \sigma(t-\varepsilon)} \to \frac{f(\sigma(t)) - f(t)}{\mu(t)}$$

as $\varepsilon \downarrow 0$ and the claim follows from (2.5). $\square$

Example: It might be interesting to note that the extension $\bar{f}$ need not be differentiable (in the usual sense) at some point $t \in \mathbb{T}$ if f is delta differentiable at t even not if t is dense. Indeed, consider the time scale

$$\mathbb{T} = \{0\} \cup \{\, \pm t_n \mid n \in \mathbb{N}\}, \qquad t_n = \frac{1}{n!}, \quad n \in \mathbb{N}$$

and the function

$$f(0) = 0, \qquad f\,(\pm t_n) = \frac{\pm 1}{(n+1)!}, \quad n \in \mathbb{N}.$$

Then f is delta differentiable at zero since

$$f^{\Delta}(0) = \lim_{n\to\infty} \frac{f\,(\pm t_n) - f(0)}{\pm t_n} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = 0.$$

However, the extension $\bar{f}$ is not even right differentiable there, since

$$\lim_{n\to\infty} \frac{\bar{f}\,(c\,t_n) - f(0)}{c\,t_n} = \lim_{n\to\infty} \frac{f\,(t_{n-1}) - f(0)}{c\,t_n} \lim_{n\to\infty} \frac{n!}{c\,n!} = \frac{1}{c} \neq f^{\Delta}(0)$$

for each positive constant $c > 1$.

As an immediate consequence of this lemma we obtain our main result:

Theorem 2.2. *Suppose f is delta differentiable for all $t \in \mathbb{T}^{\kappa}$ and $\bar{f}$ is locally of bounded variation. Then the Radon–Nikodým derivative of $\bar{f}$ and the Hilger derivative of f coincide at every point in $\mathbb{T}^{\kappa}$.*

Concerning applications of this result we emphasize that it makes several results from measure theory directly available to time scale calculus. For example, this result shows that the theory of generalized differential equations with measure-valued coefficients as developed in the book by Mingarelli [15] contains differential equations on time scales as a special case. We will use this in a follow up publication [10] to prove some new results about Sturm–Liouville equations on time scales based on some recent extensions for Sturm–Liouville equations with measure-valued coefficients [11].

3. Absolute continuity

Absolutely continuous functions on time scales were introduced in [5]. Here we will denote them by delta absolutely continuous functions to distinguish them from absolutely continuous functions in the usual measure theoretic definition.

Let a, $b \in \mathbb{T}$ with $a < b$ and $[a,b]_{\mathbb{T}} = [a,b] \cap \mathbb{T}$ be a subinterval of $\mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{C}$ is said to be delta absolutely continuous on $[a,b]_{\mathbb{T}}$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\{[a_k, b_k) \cap \mathbb{T}\}_{k=1}^{n}$, with a_k, $b_k \in [a,b]_{\mathbb{T}}$ is a finite pairwise disjoint family of subintervals of $[a,b]_{\mathbb{T}}$ with $\sum_{k=1}^{n}(b_k - a_k) < \delta$, then $\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$.

For functions which are delta absolutely continuous on $[a,b]_{\mathbb{T}}$, we have a variant of the fundamental theorem of calculus.

Theorem 3.1 ([5, Theorem 4.1])**.** *A function $f : \mathbb{T} \to \mathbb{C}$ is delta absolutely continuous on $[a,b]_{\mathbb{T}}$ if and only if f is delta differentiable almost everywhere with respect to σ on $[a,b)_{\mathbb{T}}$, $f^{\Delta} \in L^1([a,b)_{\mathbb{T}};\sigma)$ and*

$$f(x) = f(a) + \int_{[a,x)_{\mathbb{T}}} f^{\Delta}(t)\,d\sigma(t), \quad x \in [a,b]_{\mathbb{T}}. \tag{3.1}$$

Note that if f is delta absolutely continuous on $[a,b]_{\mathbb{T}}$, then the extension satisfies

$$\bar{f}(x) = \bar{f}(a) + \int_{[a,x)} f^{\Delta}(t) d\sigma(t), \quad x \in [a,b]. \tag{3.2}$$

The next lemma of Lebesgue is well known (e.g. Corollary 1 in Section 1.7.1 of [12] or Theorem A.34 in [18]).

Lemma 3.2. *Let $g \in L^1(\mathbb{R})$, then*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} |g(t) - g(x)| \, dt = 0 \tag{3.3}$$

for almost all $x \in \mathbb{R}$ with respect to Lebesgue measure.

Theorem 3.3. *Some function $f : \mathbb{T} \to \mathbb{C}$ is delta absolutely continuous on $[a,b]_{\mathbb{T}}$ if and only if $\bar{f}$ is left continuous on $[a,b]$ and absolutely continuous with respect to σ on $[a,b)$. In this case*

$$f^{\Delta}(t) = \frac{d\bar{f}}{d\sigma}(t) \tag{3.4}$$

for almost all $t \in [a,b)$ with respect to σ.

Proof. If f is delta absolutely continuous on $[a,b]_{\mathbb{T}}$, then f is continuous on $[a,b]_{\mathbb{T}}$, hence $\bar{f}$ is left continuous on $[a,b]$. Furthermore, $\bar{f}$ is absolutely continuous with respect to σ on the interval $[a,b)$ by Theorem 3.1. Conversely assume the extension $\bar{f}$ is left continuous on $[a,b]$ and absolutely continuous with respect to σ on $[a,b)$, i.e. $d\bar{f}/d\sigma \in L^1([a,b);\sigma)$ and

$$\bar{f}(x) = \bar{f}(a) + \int_{[a,x)} \frac{d\bar{f}}{d\sigma}(t) d\sigma(t), \quad x \in [a,b].$$

Then for each $t \in [a,b)_{\mathbb{T}}$ there are four cases:

(i) t is an isolated point. In this case f is Hilger differentiable, with

$$f^{\Delta}(t) = \frac{\bar{f}(\sigma(t)) - \bar{f}(t)}{\sigma(t) - t} = \frac{\int_{[t,\sigma(t))} \frac{d\bar{f}}{d\sigma}(s) d\sigma(s)}{\sigma(t) - t} = \frac{d\bar{f}}{d\sigma}(t).$$

(ii) t is right scattered and left dense. In this case for each small enough $\varepsilon > 0$ (σ has no mass to the right of t) with $t - \varepsilon \in \mathbb{T}$ we have

$$\begin{aligned}\left| \frac{f(\sigma(t)) - f(t-\varepsilon)}{\sigma(t) - (t-\varepsilon)} - \frac{d\bar{f}}{d\sigma}(t) \right| &= \left| \frac{\int_{[t-\varepsilon,\sigma(t))} \frac{d\bar{f}}{d\sigma}(s) d\sigma(s)}{\sigma([t-\varepsilon,t])} - \frac{\int_{[t-\varepsilon,t]} \frac{d\bar{f}}{d\sigma}(t) d\sigma(s)}{\sigma([t-\varepsilon,t])} \right| \\ &\le \frac{1}{\sigma([t-\varepsilon,t])} \int_{[t-\varepsilon,t]} \left| \frac{d\bar{f}}{d\sigma}(s) - \frac{d\bar{f}}{d\sigma}(t) \right| d\sigma(s).\end{aligned}$$

Now the right-hand side converges to zero as $\varepsilon \downarrow 0$, since the denominator is bounded from below by $\sigma(\{t\}) > 0$.

(iii) t is left scattered and right dense. These points are a null set with respect to σ.

(iv) t is dense. By redefining the Radon–Nikodým derivative on a null set we may assume that

$$\frac{d\bar{f}}{d\sigma}(s) = \frac{d\bar{f}}{d\sigma}(\rho(s)), \quad s \notin \mathbb{T}.$$

From (2.2) we see that this function is integrable over $[a,b)$ with respect to the Lebesgue measure and that

$$\bar{f}(x) = \bar{f}(a) + \int_{[a,x)} \frac{d\bar{f}}{d\sigma}(s)d\sigma(s) = \bar{f}(a) + \int_a^x \frac{d\bar{f}}{d\sigma}(s)ds, \quad x \in [a,b]_{\mathbb{T}}.$$

Now let $\varepsilon > 0$ with $t - \varepsilon \in \mathbb{T}$, then

$$\begin{aligned} \left| \frac{f(\sigma(t)) - f(t-\varepsilon)}{\sigma(t) - (t-\varepsilon)} - \frac{d\bar{f}}{d\sigma}(t) \right| &= \left| \frac{1}{\varepsilon} \int_{t-\varepsilon}^t \frac{d\bar{f}}{d\sigma}(s)ds - \frac{1}{\varepsilon} \int_{t-\varepsilon}^t \frac{d\bar{f}}{d\sigma}(t)ds \right| \\ &\le \frac{1}{\varepsilon} \int_{t-\varepsilon}^t \left| \frac{d\bar{f}}{d\sigma}(s) - \frac{d\bar{f}}{d\sigma}(t) \right| ds \\ &\le \frac{1}{\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} \left| \frac{d\bar{f}}{d\sigma}(s) - \frac{d\bar{f}}{d\sigma}(t) \right| ds. \end{aligned}$$

Similar one obtains for each $\varepsilon > 0$ with $t + \varepsilon \in \mathbb{T}$ the estimate

$$\left| \frac{f(\sigma(t)) - f(t+\varepsilon)}{\sigma(t) - (t+\varepsilon)} - \frac{d\bar{f}}{d\sigma}(t) \right| \le \frac{1}{\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} \left| \frac{d\bar{f}}{d\sigma}(s) - \frac{d\bar{f}}{d\sigma}(t) \right| ds.$$

Now Lemma 3.2 shows that the Hilger derivative exists for almost all dense t with respect to Lebesgue measure and coincides with the Radon–Nikodým derivative. But since a Lebesgue null set of dense points is also a null set with respect to σ, this and Theorem 3.1 prove that f is delta absolutely continuous on $[a,b]_{\mathbb{T}}$. $\square$

Of course, absolutely continuous functions have derivatives in the weak sense as introduced in [1]. This follows from the rule of integration by parts for functions of bounded variation [14, Theorem 21.67].

Acknowledgments. We thank Gusein Guseinov and Martin Bohner for helpful discussions and hints with respect to the literature. We also thank one of the referees for pointing out further references.

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Sturm–Liouville operators on time scales

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To appear in *J. Difference Equ. Appl.*

Abstract. We establish the connection between Sturm–Liouville equations on time scales and Sturm–Liouville equations with measure-valued coefficients. Based on this connection we generalize several results for Sturm–Liouville equations on time scales which have been obtained by various authors in the past.

1. Introduction

Time scale calculus was introduced by Hilger in 1988 as a means of unifying differential and difference calculus. Since then this approach has had an enormous impact and developed into a new field of mathematics (see e.g. [5], [6] and the references therein). In particular, Sturm–Liouville equations on time scales have attracted substantial interest (see e.g. [1, 2, 3, 9, 10, 11, 14, 15, 16, 17, 18, 19, 20, 21, 23, 28] and the references therein) since it contains both, continuous Sturm–Liouville equations as well as their discrete analog, Jacobi equations, as special cases. However, efforts to unify these two cases go at least back to the seminal work of Atkinson [4] or the book by Mingarelli [22]. The approach chosen there is at first sight somewhat different and is based on Sturm–Liouville operators with measure-valued coefficients. In this setting, derivatives have to be understood as Radon–Nikodým derivatives and we have shown only recently [12] that the Hilger derivative, the natural derivative on a time scale, can in fact be viewed as a special Radon–Nikodým derivative.

The purpose of the present paper is to show that Sturm–Liouville equations on time scales can in fact be obtained as a special case of Sturm–Liouville equations with measure-valued coefficients. Based on this we will show how some recent results for Sturm–Liouville operators with measure-valued coefficients can be used to extend several results for Sturm–Liouville operators on time scales which have been obtained by various authors in the past.

2010 *Mathematics Subject Classification.* Prim. 34B20, 26E70; Sec. 34N05, 39A12.
Keywords. Time scale calculus, Sturm–Liouville operators, Weyl–Titchmarsh theory.
Research supported by the *Austrian Science Fund (FWF)* under grant no. Y330.

2. Sturm–Liouville differential expressions on time scales

To set the stage we recall a few definitions and facts from time scale calculus [5], [6]. Let $\mathbb{T}$ be a time scale, that is, a nonempty closed subset of $\mathbb{R}$ and let a, b be the left and right endpoint of $\mathbb{T}$;

$$a = \inf \mathbb{T} \quad \text{and} \quad b = \sup \mathbb{T}.$$

We define the forward and backward shifts on $\mathbb{R}$ via

$$\sigma(t) = \begin{cases} \inf\{s \in \mathbb{T} \,|\, t < s\}, & t < b, \\ b, & t \geq b, \end{cases} \qquad \rho(t) = \begin{cases} \sup\{s \in \mathbb{T} \,|\, t > s\}, & t > a, \\ a, & t \leq a, \end{cases}$$

in the usual way. Note that σ is nondecreasing right continuous and ρ is nondecreasing left continuous. A point $t \in \mathbb{T}$ is called right scattered if $\sigma(t) > t$ and left scattered if $\rho(t) < t$. Associated with each time scale $\mathbb{T}$ are two Borel measure which are defined via their distribution functions σ and ρ (this procedure is standard and we refer to, e.g. [26, Section A.1] for a brief and concise account). For notational simplicity we denote these measures by the same letters σ and ρ. Furthermore, we define

$$\mathbb{T}_\kappa = \begin{cases} \mathbb{T}\backslash\{a\}, & \text{if } \sigma(a) > a, \\ \mathbb{T}, & \text{else}, \end{cases} \qquad \mathbb{T}^\kappa = \begin{cases} \mathbb{T}\backslash\{b\}, & \text{if } \rho(b) < b, \\ \mathbb{T}, & \text{else}. \end{cases}$$

The forward shift on the time scale $\mathbb{T}_\kappa$ is denoted by σ_κ, i.e.

$$\sigma_\kappa(x) = \begin{cases} \sigma(x), & \text{if } x > a, \\ \sigma(a), & \text{if } x \leq a. \end{cases}$$

Again for notational simplicity the corresponding measure is also denoted by σ_κ.

Some function f on $\mathbb{T}$ is said to be Δ-differentiable at some point $t \in \mathbb{T}$ if there is a number $f^\Delta(t)$ such that for every $\varepsilon > 0$ there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|, \quad s \in U.$$

Analogously one may define the notion of ∇-differentiability of some function using the backward shift ρ. One can show ([16, Theorem 2.1]) that

$$f^\Delta(t) = f^\nabla(\sigma(t)) \quad \text{and} \quad f^\nabla(t) = f^\Delta(\rho(t)), \tag{2.1}$$

for continuously differentiable functions f.

A function f on $\mathbb{T}$ is locally Δ-absolutely continuous if f is Δ-absolutely continuous on each bounded subinterval $[\alpha,\beta]_\mathbb{T} = [\alpha,\beta] \cap \mathbb{T}$ with α, $\beta \in \mathbb{T}$. For the notion of Δ-absolute continuity on time scales, we refer to [7]. If f is locally Δ-absolutely continuous, then f is continuous on $\mathbb{T}$, f^Δ exists almost everywhere on $\mathbb{T}^\kappa$ with respect to σ and $f^\Delta \in L^1_{\mathrm{loc}}(\mathbb{T}^\kappa;\sigma)$, i.e. f^Δ is integrable over each bounded subinterval of $\mathbb{T}^\kappa$ with respect to σ. Furthermore, note that if $\mathbb{T}$ is a bounded time scale, then f is Δ-absolutely continuous on $\mathbb{T}$, i.e. $f^\Delta \in L^1(\mathbb{T}^\kappa;\sigma)$. Of course, similar statements are true for locally ∇-absolutely continuous functions on time scales.

Now let r, p and q be some complex-valued functions on $\mathbb{T}_\kappa$ such that $r, q \in L^1_{\mathrm{loc}}(\mathbb{T}_\kappa; \sigma)$ and $1/p \in L^1_{\mathrm{loc}}(\mathbb{T}_\kappa; \rho)$. Consider the differential expression

$$\ell u = \frac{1}{r}\left(- \left(p u^\nabla\right)^\Delta + q u \right), \quad u \in \mathfrak{D}_\ell, \tag{2.2}$$

where $\mathfrak{D}_\ell$ is the maximal set of functions for which this expression makes sense. It consists of all functions u on $\mathbb{T}$ which are locally ∇-absolutely continuous and for which pu^∇ is locally Δ-absolutely continuous on $\mathbb{T}_\kappa$. Consequently, we have $\ell u \in L^1_{\mathrm{loc}}(\mathbb{T}_\kappa; |r|\sigma_\kappa)$ for $u \in \mathfrak{D}_\ell$, where $|r|\sigma_\kappa$ is the measure σ_κ weighted with the function $|r|$. Also note that for $u \in \mathfrak{D}_\ell$, u^∇ is only locally integrable with respect to ρ, whereas the function

$$u^{[1]}(x) = p(x) u^\nabla(x), \quad x \in \mathbb{T}^\kappa \tag{2.3}$$

is locally Δ-absolutely continuous. Therefore we will mainly work with this function, referred to as the first quasi-derivative of u.

We remark that in particular in the case $p = r \equiv 1$ some authors also work with

$$\tilde{\ell} u = -u^{\Delta\Delta} + q\, u^\sigma, \qquad u^\sigma = u \circ \sigma,$$

which is equivalent to the formulation we have chosen here by virtue of (2.1). However, this formulation does not play nice with our scalar product and does not give rise to a self-adjoint operator in general as pointed out in [9, Section 5].

In the remaining part of this section we will show that the differential expression ℓ may be identified with a Sturm–Liouville differential expression on $\mathbb{R}$, whose coefficients are measures. Given a function f on $\mathbb{T}_\kappa$ we define its extension to all of $\mathbb{R}$ by

$$\bar{f}(t) = \begin{cases} f(t), & t \in \mathbb{T}_\kappa, \\ f(\sigma_\kappa(t)), & t \notin \mathbb{T}_\kappa. \end{cases} \tag{2.4}$$

Now we are able to define locally finite complex Borel measures on $\mathbb{R}$ by

$$\varrho(B) = \int_{B\cap\mathbb{T}_\kappa} r(t) d\sigma_\kappa(t), \quad \varsigma(B) = \int_B \frac{1}{\overline{p}(t)} dt, \quad \chi(B) = \int_{B\cap\mathbb{T}_\kappa} q(t) d\sigma_\kappa(t),$$

for each Borel set B. Let $\mathfrak{D}_\tau$ be the set of all locally absolutely continuous functions f on $\mathbb{R}$ such that $\overline{p} f'$ is locally absolutely continuous with respect to σ_κ and consider the differential expression

$$\tau f = -\frac{d}{d\varrho}\frac{df}{d\varsigma} + \frac{q}{r} f, \qquad f \in \mathfrak{D}_\tau. \tag{2.5}$$

Here the derivatives have to be interpreted as Radon–Nikodým derivatives, hence the right-hand side exists almost everywhere on $\mathbb{T}_\kappa$ with respect to $|r|\sigma_\kappa$. Consequently, we have $\tau f \in L^1_{\mathrm{loc}}(\mathbb{R}; |r|\sigma_\kappa)$ for every $f \in \mathfrak{D}_\tau$. Next we want to show that one may identify the spaces $\mathfrak{D}_\ell$ and $\mathfrak{D}_\tau$ and the corresponding differential expressions. Therefore, for each function $u \in \mathfrak{D}_\ell$ let $\hat{u}$ be the extension

$$\hat{u}(x) = \begin{cases} u(x), & \text{if } x \in \mathbb{T}, \\ u(\sigma_\kappa(x)) + u^\nabla(\sigma_\kappa(x))\,(x - \sigma_\kappa(x)), & \text{if } x \notin \mathbb{T}, \end{cases} \tag{2.6}$$

to all of $\mathbb{R}$.

Lemma 2.1. *If $u \in \mathfrak{D}_\ell$, then $\hat{u} \in \mathfrak{D}_\tau$ with*

$$\overline{p}\hat{u}'(x) = \overline{pu^\nabla}(x) \tag{2.7}$$

for almost all $x \in \mathbb{R}$ with respect to Lebesgue measure and

$$\frac{d\overline{p}\hat{u}'}{d\sigma_\kappa}(x) = \left(pu^\nabla\right)^\Delta(x) \tag{2.8}$$

for almost all $x \in \mathbb{R}$ with respect to σ_κ.

Proof. If $u \in \mathfrak{D}_\ell$, then [7, Corollary 3.1] shows that $\hat{u}$ is locally absolutely continuous and [7, Lemma 4.1] shows that the first equality holds. Since by assumption pu^∇ is locally Δ-absolutely continuous on $\mathbb{T}_\kappa$, the results in [12, Theorem 3.3] prove that $\overline{p}\hat{u}'$ is locally absolutely continuous with respect to σ_κ and also that the second equality holds. □

The next lemma shows that extending functions of $\mathfrak{D}_\ell$ yields all functions of $\mathfrak{D}_\tau$.

Lemma 2.2. *If $f \in \mathfrak{D}_\tau$, then the restriction $f|_\mathbb{T}$ lies in $\mathfrak{D}_\ell$ with $\widehat{f|_\mathbb{T}} = f$. In particular, the map (2.6) is a bijection between $\mathfrak{D}_\ell$ and $\mathfrak{D}_\tau$.*

Proof. Since f is continuous and linear outside of $\mathbb{T}$, we see that f really is the extension of $f|_\mathbb{T}$, as claimed. Hence because of [7, Corollary 3.1], the restriction $f|_\mathbb{T}$ is locally ∇-absolutely continuous. Furthermore, [7, Lemma 4.1] shows that

$$\overline{f|_\mathbb{T}^\nabla}(x) = f'(x)$$

for almost all $x \in \mathbb{R}$. Now since $\overline{pf|_\mathbb{T}^\nabla} = \overline{p}f'$ is locally absolutely continuous with respect to σ_κ, the results in [12, Theorem 3.3] show that $pf|_\mathbb{T}^\nabla$ is locally Δ-absolutely continuous on $\mathbb{T}_\kappa$ and hence $f|_\mathbb{T} \in \mathfrak{D}_\ell$. □

Now from these lemmas one sees that the time scale Sturm–Liouville differential expression ℓ is essentially equal to the Sturm–Liouville expression τ, which's coefficients are measures.

Theorem 2.3. *For each $u \in \mathfrak{D}_\ell$ and $f \in \mathfrak{D}_\tau$ we have*

$$\ell u(x) = \tau\hat{u}(x) \quad \text{and} \quad \tau f(x) = \ell f|_\mathbb{T}(x), \tag{2.9}$$

for almost all $x \in \mathbb{R}$ with respect to $|r|\sigma_\kappa$.

Proof. For $u \in \mathfrak{D}_\ell$ we get from Lemma 2.1 that

$$\frac{d}{d\varrho}\frac{d\hat{u}}{d\varsigma} = \frac{1}{r}\frac{d}{d\sigma}\overline{p}\hat{u}' = \frac{1}{r}\left(pu^\nabla\right)^\Delta$$

almost everywhere with respect to σ_κ, which proves the first claim. The second claim immediately follows from this and Lemma 2.2. □

Using the preceding theorem and the results in [13, Theorem 3.1] one readily obtains the following existence and uniqueness theorem.

Theorem 2.4. *Let $g \in L^1_{\mathrm{loc}}(\mathbb{T}_\kappa; |r|\sigma_\kappa)$ and $z \in \mathbb{C}$. Then for each $c \in \mathbb{T}_\kappa$, d_1, $d_2 \in \mathbb{C}$ there is a unique solution $u \in \mathfrak{D}_\ell$ of*

$$(\ell - z)u = g \quad \text{with} \quad u(c) = d_1 \quad \text{and} \quad u^{[1]}(c) = d_2.$$

Moreover, [13, Theorem 3.6] shows that the solutions from the preceding theorem depend analytically on $z \in \mathbb{C}$. Given two functions $f,\ g \in \mathfrak{D}_\ell$ we define the Wronski determinant $W(f,g)$ as

$$W(f,g)(x) = f(x)g^{[1]}(x) - f^{[1]}(x)g(x), \quad x \in \mathbb{T}_\kappa.$$

This function is locally Δ-absolutely continuous on $\mathbb{T}_\kappa$ with derivative

$$W(f,g)^\Delta(x) = \left(g(x)\ell f(x) - f(x)\ell g(x)\right) r(x)$$

for almost all $x \in \mathbb{T}_\kappa$ with respect to σ_κ. In particular if $u_1,\ u_2 \in \mathfrak{D}_\ell$ are two solutions of $(\ell - z)u = 0$, then the Wronskian $W(u_1,u_2)$ is constant with

$$W(u_1,u_2) \neq 0 \quad \Leftrightarrow \quad u_1,\ u_2 \text{ linearly independent.}$$

In order to obtain self-adjoint operators we make the following additional assumptions on our coefficients.

Hypothesis 2.5.

(i) r *is positive almost everywhere with respect to* σ_κ.
(ii) q *is real valued almost everywhere with respect to* σ_κ.
(iii) p *is real-valued and nonzero almost everywhere with respect to* ρ.
(iv) $\mathbb{T}$ *consists of more than four points.*

Note that under these assumptions, the measures ϱ, ς and χ satisfy the requirements of [13, Hypothesis 3.7]. The last assumption in Hypothesis 2.5 is necessary in order to work in Hilbert spaces which are at least two-dimensional.

3. Sturm–Liouville operators on bounded time scales

In this section we consider the case when $\mathbb{T}$ is a bounded time scale. The results are special cases of the more general results in the next section. However, since this case has attracted considerable interest in the past [1, 2, 10, 15, 16, 21], we want to single out the corresponding results. The most general results seem to be the ones in [10], where the case $p \in H^1$ and $q \in L^2$ is treated.

In order to obtain an operator in $L^2(\mathbb{T}_\kappa; r\sigma_\kappa)$ we first restrict $\mathfrak{D}_\ell$ to the subspace

$$\mathfrak{D}_\ell^2 = \{f \in \mathfrak{D}_\ell \,|\, f,\ \ell f \in L^2(\mathbb{T}_\kappa; r\sigma_\kappa)\}.$$

Since this subspace does not give rise to a self-adjoint operator we have to restrict it further. Moreover, since the map (2.6) acts as the identity on $\mathbb{T}$, the operators associated with ℓ and τ are identical by virtue of Theorem 2.3. Consequently, the following theorem is an immediate consequence of the results in [13, Section 7].

Theorem 3.1. *Let* $\varphi_\alpha,\ \varphi_\beta \in [0,\pi)$ *and suppose that* $\varphi_\alpha \neq 0$, *if* $\sigma(a)$ *is right scattered and* $p(b)\sin\varphi_\beta \neq (b - \rho(b))\cos\varphi_\beta$, *if* $\rho(b)$ *is right scattered. Then the linear operator* S *in* $L^2(\mathbb{T}_\kappa; r\sigma_\kappa)$ *given by*

$$\mathfrak{D}\,(S) = \left\{ f \in \mathfrak{D}_\ell^2 \;\middle|\; \begin{array}{l} 0 = f(\sigma(a))\cos\varphi_\alpha - f^{[1]}(\sigma(a))\sin\varphi_\alpha \\ 0 = f(b)\cos\varphi_\beta - f^{[1]}(b)\sin\varphi_\beta \end{array} \right\} \tag{3.1}$$

and $Sf = \ell f$ *for* $f \in \mathfrak{D}\,(S)$ *is well-defined and self-adjoint.*

Note that here well-defined means that each function in $\mathfrak{D}(S)$ has a unique representative in $\mathfrak{D}_\ell^2$ satisfying the boundary conditions. In the cases excluded by the condition on the parameters φ_α and φ_β this fails and $\mathfrak{D}(S)$ (as defined in (3.1)) will no longer be dense. Explicitly, if $\sigma(a)$ is right scattered and $\varphi_\alpha = 0$, the boundary condition reads $f(\sigma(a)) = 0$ and $\mathfrak{D}(S)$ lacks the corresponding one-dimensional subspace. Moreover, $f^{[1]}(\sigma(a)) = p(\sigma(a))\frac{f(\sigma(a))-f(a)}{\sigma(a)-a}$ does not enter the boundary condition and the value of $f(a)$ cannot be determined in terms of $f(\sigma(a))$. Consequently, different values of $f(a)$ for one and the same element in $L^2(\mathbb{T}_\kappa; r\sigma_\kappa)$ will give rise to different values of ℓf and S becomes a multi-valued operator, which, however, is still self-adjoint (cf. [13, Theorem 7.6]). One can obtain a single-valued operator by removing $\sigma(a)$ from the Hilbert space. In particular, this case is covered by starting with the time scale $\mathbb{T}_\kappa$ from the outset. Similarly if $\rho(b)$ is left scattered.

The self-adjoint boundary conditions given in Theorem 3.1 above are separate, i.e. they are given for each endpoint separately. As in the classical theory of Sturm–Liouville operators there are also coupled self-adjoint boundary conditions.

Theorem 3.2. *Let $\varphi \in [0,\pi)$, $R \in \mathbb{R}^{2\times 2}$ with $\det R = 1$ and suppose that*

$$p(b)R_{12} \neq (b-\rho(b))R_{22}, \tag{3.2}$$

if $\sigma(a)$ and $\rho(b)$ are right scattered. Then the linear operator S in $L^2(\mathbb{T}_\kappa; r\sigma_\kappa)$ given by

$$\mathfrak{D}(S) = \left\{ f \in \mathfrak{D}_\ell^2 \,\middle|\, \begin{pmatrix} f(b) \\ f^{[1]}(b) \end{pmatrix} = e^{i\varphi} R \begin{pmatrix} f(\sigma(a)) \\ f^{[1]}(\sigma(a)) \end{pmatrix} \right\} \tag{3.3}$$

and $Sf = \ell f$ for $f \in \mathfrak{D}(S)$ is well-defined and self-adjoint.

As before, the cases excluded by (3.2) give rise to a multi-valued, self-adjoint linear operator in $L^2(\mathbb{T}_\kappa; r\sigma_\kappa)$.

Finally, we also mention that the results from Volkmer [27] for measure-valued Sturm–Liouville equations apply to our situation. To this end we need the Lebesgue decomposition of the measure σ which follows from [8, Theorem 5.2].

Lemma 3.3. *The Lebesgue decomposition of σ with respect to the Lebesgue measure λ is given by*

$$\sigma = \chi_{\mathbb{T}}\lambda + \sum_{t_n \in \mathbb{T}} \mu(t_n)\delta_{t_n}, \tag{3.4}$$

where δ_t is the Dirac measure in the point $t \in \mathbb{T}$, $\chi_{\mathbb{T}}$ is the characteristic function of the set $\mathbb{T}$, and $\mu(t) = \sigma(t) - t$ is the graininess of the time scale.

Proof. It suffices to show that the measures on the left and right-hand side of (3.4) agree on every interval $I = [\alpha,\beta)$ whose endpoints are in $\mathbb{T}$. For such an interval we have $\sigma(I) = \sigma_-(\beta) - \sigma_-(\alpha) = \beta - \alpha$. So let us turn to the other side. Since $\mathbb{T}$ is closed, it can be written as $[a,b]$ minus a countable union of disjoint open intervals $(t_n, \sigma(t_n))$. Hence $\int_{[\alpha,\beta)} \chi_{\mathbb{T}}(t)dt$ gives $\beta - \alpha$ minus the intervals missing in $\mathbb{T} \cap [\alpha,\beta)$ and the sum over the Dirac measures just makes up for this missing part. □

Theorem 3.4. *Let S be a self-adjoint operator from Theorem 3.1 and suppose $p > 0$. Then S has purely discrete spectrum and if $L^2(\mathbb{T}_\kappa; r\sigma_\kappa)$ is infinite dimensional, the eigenvalues $E_0 < E_1 < E_2 < \cdots$ have the following asymptotics*

$$\lim_{n\to\infty} \frac{E_n}{n^2} = \frac{\pi^2}{L^2}, \qquad L = \int_{\mathbb{T}} \sqrt{\frac{r(t)}{p(t)}} dt. \tag{3.5}$$

Proof. By our assumptions, S satisfies the hypothesis of [27, Theorem 5.5] and the above claim holds with $L = GM(\varrho, \varsigma)$, where $GM(\varrho, \varsigma)$ is the geometric mean of the two measures ϱ and ς (see also the summary on page 14 in [27]). As pointed out on page 11 in [27] this geometric mean is given by $GM(\varrho, \varsigma) = \sqrt{g} d\varsigma$, where $g = r\, p\, \chi_{\mathbb{T}}$ is the Radon–Nikodým derivative of ϱ with respect to ς. □

The above theorem shows that the leading asymptotics comes from the continuous part of the time scale. If the time scale has Lebesgue measure zero (i.e. $L = 0$), the leading asymptotics will change and will no longer be captured by the above theorem. See [2], [3] for some results in this direction.

4. Weyl's alternative

In this section we will allow our time scale $\mathbb{T}$ to be unbounded. The crucial step in order to determine the associated self-adjoint operators is the classification of the endpoints into two cases following the original ideas of Weyl. In the special case where $r = p \equiv 1$ and continuous q this was first investigated in [28] using the original approach via Weyl circles. Further results can be found in [19].

We say ℓ is in the limit-circle (l.c.) case at the left endpoint a if for each $z \in \mathbb{C}$ every solution of $(\ell - z)u = 0$ lies in $L^2(\mathbb{T}_\kappa; r\sigma_\kappa)$ near a, i.e. is square integrable near a with respect to $r\sigma_\kappa$. Furthermore, ℓ is said to be in the limit-point (l.p.) case at a if for each $z \in \mathbb{C}$ there is some solution of $(\ell - z)u = 0$ which does not lie in $L^2(\mathbb{T}_\kappa; r\sigma_\kappa)$ near a. Similarly one defines the limit-circle and limit-point cases for the right endpoint b. For example note that each finite endpoint is in the l.c. case. Now [13, Theorem 5.2] yields Weyl's alternative.

Theorem 4.1. *At each endpoint, τ is either in the l.c. or in the l.p. case.*

As in the case of bounded time scales we consider the subspace

$$\mathfrak{D}_\ell^2 = \{f \in \mathfrak{D}_\ell \,|\, f,\ \ell f \in L^2(\mathbb{T}_\kappa; r\sigma_\kappa)\}$$

of $\mathfrak{D}_\ell$. Given two functions f, $g \in \mathfrak{D}_\ell^2$ the limits

$$W(f,g)(a) = \lim_{\substack{x\to a\\ x\in\mathbb{T}}} W(f,g)(\sigma(x)) \quad \text{and} \quad W(f,g)(b) = \lim_{\substack{x\to b\\ x\in\mathbb{T}}} W(f,g)(\sigma(x))$$

exist and are finite. According to [13, Lemma 5.6] it is possible to characterize the l.c. and the l.p. case in terms of the Wronskian at this endpoint.

Lemma 4.2. *ℓ is in the l.p. case at a if and only if*

$$W(f,g)(a) = 0, \quad f,\ g \in \mathfrak{D}_\ell^2.$$

ℓ is in the l.c. case at a if and only if there is a $f \in \mathfrak{D}_\ell^2$ such that

$$W(f, f^*)(a) = 0 \quad \text{and} \quad W(f, g)(a) \neq 0 \quad \text{for some } g \in \mathfrak{D}_\ell^2.$$

Similar results hold at the right endpoint b.

Concerning self-adjointness, the case when both endpoints are in the l.p. case is the simplest, as [13, Theorem 6.2] shows.

Theorem 4.3. *If ℓ is in the l.p. case at both endpoints, then the linear operator S given by $Sf = \ell f$ for $f \in \mathfrak{D}(S) = \mathfrak{D}_\ell^2$ is well-defined and self-adjoint.*

In order to determine self-adjoint operators in the remaining cases we introduce functionals on $\mathfrak{D}_\ell^2$, given by

$$BC_a^1(f) = W(f, w_2)(a) \quad \text{and} \quad BC_a^2(f) = W(w_1, f)(a), \quad f \in \mathfrak{D}_\ell^2,$$

if ℓ is in the l.c. case at a and

$$BC_b^1(f) = W(f, w_2)(b) \quad \text{and} \quad BC_b^2(f) = W(w_1, f)(b), \quad f \in \mathfrak{D}_\ell^2,$$

if ℓ is in the l.c. case at b. Here, w_1 and w_2 are some real functions in $\mathfrak{D}_\ell^2$ with

$$W(w_1, w_2)(a) = W(w_1, w_2)(b) = 1. \tag{4.1}$$

Note that point evaluations in a finite endpoint are a special case of these functionals, as [13, Proposition 7.1] shows.

Proposition 4.4. *One may choose real w_1, $w_2 \in \mathfrak{D}_\ell^2$ satisfying* (4.1) *such that the corresponding functionals satisfy*

$$BC_a^1(f) = f(\sigma(a)) \quad \text{and} \quad BC_a^2(f) = f^{[1]}(\sigma(a)), \quad f \in \mathfrak{D}_\ell^2,$$

if $\mathbb{T}$ is bounded from below and

$$BC_b^1(f) = f(b) \quad \text{and} \quad BC_b^2(f) = f^{[1]}(b), \quad f \in \mathfrak{D}_\ell^2,$$

if $\mathbb{T}$ is bounded from above.

Now the self-adjoint operators in the case when one endpoint is in the l.c. case and the other is in the l.p. case may be obtained from [13, Theorem 7.3].

Theorem 4.5. *Suppose ℓ is in the l.c. case at a, in the l.p. case at b and let $\varphi_\alpha \in [0, \pi)$ such that*

$$\cos\varphi_\alpha w_2(\sigma(a)) + \sin\varphi_\alpha w_1(\sigma(a)) \neq 0, \tag{4.2}$$

if $a \in \mathbb{R}$ and $\sigma(a)$ is right scattered. Then the linear operator S in $L^2(\mathbb{T}_\kappa; r\sigma_\kappa)$, given by

$$\mathfrak{D}(S) = \{f \in \mathfrak{D}_\ell^2 \,|\, BC_a^1(f)\cos\varphi_\alpha - BC_a^2(f)\sin\varphi_\alpha = 0\} \tag{4.3}$$

and $Sf = \ell f$ for $f \in \mathfrak{D}(S)$ is well-defined and self-adjoint.

Again the excluded case gives rise to a multi-valued, self-adjoint linear operator. If ℓ is in the l.p. case at a and in the l.c. case at b, self-adjoint operators may be given similarly in terms of the functionals BC_b^1 and BC_b^2. We now turn to the case when both endpoints are in the l.c. case. As in the case of bounded time scales, the self-adjoint operators may be divided into two classes. The case of separate boundary conditions may be obtained from [13, Theorem 7.6].

Theorem 4.6. *Suppose ℓ is in the l.c. case at both endpoints and let some φ_α, $\varphi_\beta \in [0,\pi)$ such that*

$$\cos\varphi_\alpha w_2(\sigma(a)) + \sin\varphi_\alpha w_1(\sigma(a)) \neq 0, \tag{4.4}$$

if $a \in \mathbb{R}$ and $\sigma(a)$ is right scattered and

$$\cos\varphi_\beta w_2(b) + \sin\varphi_\beta w_1(b) \neq 0, \tag{4.5}$$

if $b \in \mathbb{R}$ and $\rho(b)$ is right scattered. Then the linear operator S in $L^2(\mathbb{T}_\kappa; r\sigma_\kappa)$, given by

$$\mathfrak{D}\left(S\right) = \left\{ f \in \mathfrak{D}_\ell^2 \;\middle|\; \begin{matrix} 0 = BC_a^1(f)\cos\varphi_\alpha - BC_a^2(f)\sin\varphi_\alpha \\ 0 = BC_b^1(f)\cos\varphi_\beta - BC_b^2(f)\sin\varphi_\beta \end{matrix} \right\} \tag{4.6}$$

and $Sf = \ell f$ for $f \in \mathfrak{D}\left(S\right)$ is well-defined and self-adjoint.

Now the second class of self-adjoint operators in this case is defined via coupled boundary conditions. The corresponding results may be found in [13, Theorem 7.6].

Theorem 4.7. *Suppose ℓ is in the l.c. case at both endpoints and let some $\varphi \in [0,\pi)$, $R \in \mathbb{R}^{2\times 2}$ with $\det R = 1$. Furthermore, if a, $b \in \mathbb{R}$ and $\sigma(a)$, $\rho(b)$ are right scattered, set*

$$\tilde{R} = \begin{pmatrix} w_2^{[1]}(b) & -w_2(b) \\ -w_1^{[1]}(b) & w_1(b) \end{pmatrix}^{-1} R \begin{pmatrix} w_2^{[1]}(\sigma(a)) & -w_2(\sigma(a)) \\ -w_1^{[1]}(\sigma(a)) & w_1(\sigma(a)) \end{pmatrix} \tag{4.7}$$

and assume $\tilde{R}_{12} \neq 0$. Then the linear operator S in $L^2(\mathbb{T}_\kappa; r\sigma_\kappa)$, given by

$$\mathfrak{D}\left(S\right) = \left\{ f \in \mathfrak{D}_\ell^2 \;\middle|\; \begin{pmatrix} BC_b^1(f) \\ BC_b^2(f) \end{pmatrix} = e^{i\varphi} R \begin{pmatrix} BC_a^1(f) \\ BC_a^2(f) \end{pmatrix} \right\} \tag{4.8}$$

and $Sf = \ell f$ for $f \in \mathfrak{D}\left(S\right)$ is well-defined and self-adjoint.

5. Spectrum and resolvent

In this section we will provide the resolvents of self-adjoint operators given in the preceding section. As the results in [13, Section 8] show, they turn out to be integral operators. We start with the case when both endpoints are in the l.c. case.

Theorem 5.1. *Suppose ℓ is in the l.c. case at both endpoints and S is a self-adjoint operator as in Theorem 4.6 or Theorem 4.7. Then for each $z \in \rho(S)$ the resolvent R_z is an integral operator*

$$R_z f(x) = \int_{\mathbb{T}_\kappa} G_z(x,y) f(y) r(y) d\sigma_\kappa(y), \quad x \in \mathbb{T}_\kappa,\ f \in L^2(\mathbb{T}_\kappa; r\sigma_\kappa), \tag{5.1}$$

with some square integrable kernel G_z on $\mathbb{T}_\kappa \times \mathbb{T}_\kappa$. If u_1, u_2 are two linearly independent solutions of $(\ell - z)u = 0$, then there are coefficients $m_{ij}^{\pm}(z) \in \mathbb{C}$, i, $j \in \{1,2\}$ such that the kernel is given by

$$G_z(x,y) = \begin{cases} \sum_{i,j=1}^2 m_{ij}^+(z) u_i(x) u_j(y), & \text{if } y \leq x, \\ \sum_{i,j=1}^2 m_{ij}^-(z) u_i(x) u_j(y), & \text{if } y > x. \end{cases} \tag{5.2}$$

Since the resolvents are clearly Hilbert–Schmidt operators, as in [13, Corollary 8.2] we obtain some information about the spectrum in this case.

Corollary 5.2. *Suppose ℓ is in the l.c. case at both endpoints and S is a self-adjoint operator as in Theorem 4.6 or Theorem 4.7. Then S has purely discrete spectrum, i.e. $\sigma(S) = \sigma_d(S)$ with*

$$\sum_{\substack{\lambda\in\sigma(S)\\ \lambda\neq 0}} \frac{1}{\lambda^2} < \infty \quad \textit{and} \quad \dim\ker(S-\lambda) \leq 2, \quad \lambda \in \sigma(S).$$

If S is a self-adjoint operator with separate boundary conditions as in Theorem 4.3, Theorem 4.5 or Theorem 4.6, then the resolvent has a simpler form, as [13, Theorem 8.3] shows.

Theorem 5.3. *Suppose S is a self-adjoint operator as in Theorem 4.3, Theorem 4.5 or Theorem 4.6. Furthermore, let $z \in \rho(S)$ and u_a, u_b be non-trivial solutions of $(\ell - z)u = 0$ such that*

$$u_a \begin{cases} \textit{satisfies the boundary condition at } a \textit{ if } \ell \textit{ is l.c. at } a, \\ \textit{lies in } L^2(\mathbb{T}_\kappa; r\sigma_\kappa) \textit{ near } a \textit{ if } \ell \textit{ is l.p. at } a, \end{cases}$$

and

$$u_b \begin{cases} \textit{satisfies the boundary condition at } b \textit{ if } \ell \textit{ is l.c. at } b, \\ \textit{lies in } L^2(\mathbb{T}_\kappa; r\sigma_\kappa) \textit{ near } b \textit{ if } \ell \textit{ is l.p. at } b. \end{cases}$$

Then the resolvent R_z is given by

$$R_z f(x) = \int_{\mathbb{T}_\kappa} G_z(x,y) f(y) r(y) d\sigma_\kappa(y), \quad x \in \mathbb{T}_\kappa,\ f \in L^2(\mathbb{T}_\kappa; r\sigma_\kappa), \tag{5.3}$$

where

$$G_z(x,y) = \begin{cases} W(u_b,u_a)^{-1} u_a(x) u_b(y), & \textit{if } x < y, \\ W(u_b,u_a)^{-1} u_a(y) u_b(x), & \textit{if } x \geq y. \end{cases} \tag{5.4}$$

Furthermore if S is a self-adjoint operator as in Theorem 5.3, then [13, Corollary 8.4] shows that all eigenvalues of S are simple.

6. Weyl–Titchmarsh theory

The associated eigenfunction expansion was considered in [11], [16], [20] for the case of bounded time scales and for semi-unbounded time scales in [17], [18]. Here we will obtain it via classical Weyl–Titchmarsh theory, thereby generalizing the presently best result from [18] where the case $r = 1$, p differentiable, and q continuous is treated. For further generalizations of Weyl–Titchmarsh theory to time scale systems see [24], [25] and the references therein.

In this section assume that our time scale $\mathbb{T}$ is bounded from below and let S be a self-adjoint operator as in Theorem 4.5 or Theorem 4.6. In particular there is some $\varphi_\alpha \in [0,\pi)$ such that the boundary condition at the point $\sigma(a)$ reads

$$f(\sigma(a)) \cos\varphi_\alpha - f^{[1]}(\sigma(a)) \sin\varphi_\alpha = 0.$$

Now for each $z \in \mathbb{C}$ consider the linearly independent solutions θ_z, ϕ_z of $(\ell - z)u = 0$ with the initial conditions

$$\theta_z(\sigma(a)) = \phi_z^{[1]}(\sigma(a)) = \cos\varphi_\alpha \quad \text{and} \quad -\theta_z^{[1]}(\sigma(a)) = \phi_z(\sigma(a)) = \sin\varphi_\alpha.$$

Note that the solution ϕ_z satisfies the boundary condition at $\sigma(a)$. Given these solutions one may define a function m on the resolvent set of S by requiring that the solutions

$$\psi_z(x) = \theta_z(x) + m(z)\phi_z(x), \quad x \in \mathbb{T},\ z \in \rho(S)$$

lie in $L^2(\mathbb{T}_\kappa; r\sigma_\kappa)$ near b and satisfy the boundary condition at b if S is self-adjoint operator as in Theorem 4.6. The function m is called the Weyl–Titchmarsh m-function of S, the solutions ψ_z, $z \in \rho(S)$ are called the Weyl solutions. Now the results in [13, Section 9 and Section 10] readily yield the following properties of m.

Theorem 6.1. *The Weyl–Titchmarsh m-function is a Herglotz–Nevanlinna function. In particular, there is a unique Borel measure μ on $\mathbb{R}$ such that*

$$m(z) = \mathrm{Re}(m(\mathrm{i})) + \int_{\mathbb{R}} \frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} d\mu(\lambda), \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{6.1}$$

The measure μ is called the spectral measure of S. Indeed the next theorem, obtained from [13, Section 10] justifies this name.

Theorem 6.2. *The mapping $\mathcal{F}$ given by*

$$\mathcal{F}f(\lambda) = \lim_{\substack{\beta\to b\\ \beta\in\mathbb{T}}} \int_{(a,\beta]\cap\mathbb{T}_\kappa} \phi_\lambda(x) f(x) r(x) d\sigma_\kappa(x), \quad \lambda \in \mathbb{R},\ f \in L^2(\mathbb{T}_\kappa; r\sigma_\kappa),$$

where the limit exists in $L^2(\mathbb{R};\mu)$, is unitary from $L^2(\mathbb{T}_\kappa; r\sigma_\kappa)$ onto $L^2(\mathbb{R};\mu)$ and maps S onto multiplication with the independent variable in $L^2(\mathbb{R};\mu)$.

As a consequence of this theorem we may read off the spectrum of S from the boundary behavior of the Weyl–Titchmarsh m-function in the usual way.

Corollary 6.3. *The spectrum of S is given by*

$$\sigma(S) = \mathrm{supp}(\mu) = \overline{\{\lambda \in \mathbb{R} \,|\, 0 < \limsup_{\varepsilon\downarrow 0} \mathrm{Im}(m(\lambda + \mathrm{i}\varepsilon))\}}.$$

Acknowledgments. We thank Gusein Guseinov for helpful discussions and hints with respect to the literature.

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Uniqueness results for Schrödinger operators on the line with purely discrete spectra

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Submitted to *Trans. Amer. Math. Soc.*

Abstract. We provide an abstract framework for singular one-dimensional Schrödinger operators with purely discrete spectra to show when the spectrum plus norming constants determine such an operator completely. As an example we apply our findings to prove a new uniqueness results for perturbed quantum mechanical harmonic oscillators. In addition, we also show how to establish a Hochstadt–Lieberman type result for these operators. Our approach is based on the singular Weyl–Titchmarsh theory which is extended to cover the present situation.

1. Introduction

The present paper is concerned with uniqueness results for one-dimensional Schrödinger operators

$$H = -\frac{d^2}{dx^2} + q(x), \quad x \in (a, b)$$

in the Hilbert space $L^2(a, b)$ with a real-valued potential $q \in L^1_{\text{loc}}(a, b)$. We are particularly interested in the case when H has purely discrete spectrum. Of course this problem is well understood in the case where the operator is regular, that is (a, b) is compact and $q \in L^1(a, b)$, but for singular operators there are still many open questions. One of the prime examples in this respect are perturbations of the quantum mechanical oscillator and in particular its isospectral class [24]. In particular, perturbations

$$H = -\frac{d^2}{dx^2} + x^2 + q(x), \quad x \in \mathbb{R} \tag{1.1}$$

of the harmonic oscillator have attracted much interest recently, see Chelkak, Kargaev and Korotyaev, [2], [3], [4] and the references therein.

Moreover, it has been shown by Kodaira [17], Kac [15] and more recently by Fulton [6], Gesztesy and Zinchenko [10], Fulton and Langer [7], Kurasov and Luger [22], and Kostenko, Sakhnovich and Teschl [18], [19], [20], [21] that, for a large class of singularities at a, it is still possible to define a

2010 *Mathematics Subject Classification.* Prim. 34B20, 34L05; Sec. 34B24, 47A10.
Keywords. Schrödinger operators, inverse spectral theory, discrete spectra.
Research supported by the *Austrian Science Fund (FWF)* under grant no. Y330.

singular Weyl function at the base point a. While in these previous works the main focus was on applications to spherical Schrödinger operators

$$H = -\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2} + q(x), \quad x \in (0, \infty) \tag{1.2}$$

(also known as Bessel operators), our interest here will be to apply these techniques to operators of the form (1.1).

The outline of our paper is as follows. In Section 2 we will fix our notation and recall some basic facts from singular Weyl–Titchmarsh theory. In [20] the authors have proven a local Borg–Marchenko theorem in the case where the spectrum created by the singular endpoint has convergence exponent less than one. While this was sufficient to cover Bessel-type operators (1.2) it is not good enough for (1.1) where the convergence exponent will be one. Hence our first aim will be to extend the results from [20] to arbitrary (finite) exponential growth orders in Section 3 such that we can provide an associated local Borg–Marchenko theorem in Section 4. In Section 5 we will then use this to prove inverse uniqueness results for operators which have purely discrete spectra with finite convergence exponent. We will provide a general result which shows that the spectrum together with the norming constants uniquely determines the operator. As a special case we will obtain a (slight) generalization of the main result from [3].

The local Borg–Marchenko uniqueness theorem is of course also the main ingredient in a vast number of other uniqueness results in inverse spectral theory. One of these results which had particular impact is the celebrated Hochstadt–Lieberman theorem [11]. Hence we will try to use this direction as a test case for our results and prove a powerful generalization of this famous theorem to singular operators which's spectra have finite convergence exponents. In fact, while many extensions are known to date, we refer to [25], [13], [14] for recent accounts, most of them concern regular operators (including the case where the potentials are distributions) and we are only aware of two references dealing with singular operators. First of all the work by Gesztesy and Simon [9], who considered the case of operators which grow faster than the harmonic operator and satisfy $q(-x) \geq q(x)$. Secondly Khodakovsky [16], who improved their result and removed the growth restriction. However, there are many interesting physical examples which are not covered by this result. For example, Pöschl–Teller type potentials, which have non-integrable singularities near both (finite) endpoints, or perturbations of the harmonic oscillator (1.1). We will show in Sections 5 and 6 that our result is able to cover these examples.

2. Singular Weyl–Titchmarsh theory

Our fundamental ingredient will be singular Weyl–Titchmarsh theory and hence we begin by recalling the necessary facts from [20]. To set the stage, we will consider one-dimensional Schrödinger differential expressions on an interval (a, b) of the form

$$\tau = -\frac{d^2}{dx^2} + q(x), \quad x \in (a, b), \tag{2.1}$$

with $-\infty \le a < b \le \infty$, where the potential q is real-valued satisfying

$$q \in L^1_{\mathrm{loc}}(a,b).$$

We will use τ to denote the formal differential expression and H to denote a corresponding self-adjoint operator in $L^2(a,b)$ given by τ with separated boundary conditions at a and/or b.

We will fix a point $c \in (a,b)$ and also consider the operators

$$H^D_{(a,c)},\ H^D_{(c,b)},$$

which are obtained by restricting H to (a,c), (c,b), respectively, with a Dirichlet boundary condition at c. The corresponding operators with a Neumann boundary condition will be denoted by

$$H^N_{(a,c)},\ H^N_{(c,b)}.$$

Moreover, let $c(z,x)$, $s(z,x)$ be the solutions of $\tau u = z\,u$ corresponding to the initial conditions $c(z,c)=1$, $c'(z,c)=0$ and $s(z,c)=0$, $s'(z,c)=1$. Now we define the Weyl functions $m_\pm(z)$ (corresponding to the given base point c) such that the functions

$$u_-(z,x) = c(z,x) - m_-(z)s(z,x), \quad z \in \rho(H^D_{(a,c)}) \tag{2.2a}$$

are square integrable near a and satisfy the boundary condition there (if any) and such that the functions

$$u_+(z,x) = c(z,x) + m_+(z)s(z,x), \quad z \in \rho(H^D_{(c,b)}) \tag{2.2b}$$

are square integrable near b and satisfy the boundary condition there (if any). The solutions $u_\pm(z,x)$ (as well as their multiples) are called Weyl solutions at a, b. For further background we refer to [26, Chapter 9] or [27].

To define an analogous singular Weyl function at the, in general singular, endpoint a we will first need the analog of the system of solutions $c(z,x)$ and $s(z,x)$. Hence our first goal is to find a system of entire solutions $\theta(z,x)$ and $\phi(z,x)$ such that $\phi(z,x)$ lies in the domain of H near a and such that the Wronskian $W(\theta(z),\phi(z)) = 1$. To this end we start with a hypothesis which will turn out necessary and sufficient for such a system of solutions to exist.

Hypothesis 2.1. *Suppose that the spectrum of $H^D_{(a,c)}$ is purely discrete for one (and hence for all) $c \in (a,b)$.*

Note that this hypothesis is for example satisfied if $q(x) \to +\infty$ as $x \to a$ (cf. Problem 9.7 in [26]).

Lemma 2.2 ([20])**.** *The following properties are equivalent:*

(i) *The spectrum of $H^D_{(a,c)}$ is purely discrete for some $c \in (a,b)$.*
(ii) *There is a real entire solution $\phi(z,x)$, which is non-trivial and lies in the domain of H near a for each $z \in \mathbb{C}$.*
(iii) *There are real entire solutions $\phi(z,x)$, $\theta(z,x)$ with $W(\theta,\phi) = 1$, such that $\phi(z,x)$ is non-trivial and lies in the domain of H near a for each $z \in \mathbb{C}$.*

Remark 2.3. *It is important to point out that such a fundamental system is not unique and any other such system is given by*

$$\tilde{\theta}(z,x) = \mathrm{e}^{-g(z)}\theta(z,x) - f(z)\phi(z,x), \qquad \tilde{\phi}(z,x) = \mathrm{e}^{g(z)}\phi(z,x), \tag{2.3}$$

where $g(z)$, $f(z)$ are real entire functions. The singular Weyl functions are related via

$$\tilde{M}(z) = \mathrm{e}^{-2g(z)}M(z) + \mathrm{e}^{-g(z)}f(z). \tag{2.4}$$

We will need the following simple lemma on the high energy asymptotics of the solution $\phi(z,x)$. Note that we always use the principal square root with branch cut along the negative real axis.

Lemma 2.4. *If $\phi(z,x)$ is a real entire solution which lies in the domain of H near a, then for every x_0, $x \in (a,b)$*

$$\phi(z,x) = \phi(z,x_0)\mathrm{e}^{(x-x_0)\sqrt{-z}}\left(1 + \mathcal{O}\left(1/\sqrt{-z}\right)\right), \tag{2.5}$$

as $|z| \to \infty$ along any nonreal ray.

Proof. Using

$$\phi(z,x) = \phi(z,c)(c(z,x) - m_-(z)s(z,x)), \quad x \in (a,b), \ z \in \mathbb{C}\backslash\mathbb{R}$$

and the well-known asymptotics of the solutions $c(z,x)$, $s(z,x)$ and $m_-(z)$ (cf. [26, Lemma 9.18 and Lemma 9.19]) we see (2.5) for $x_0 = c$ and $x > x_0$. The case $x < x_0$ follows after reversing the roles of x_0 and x. Since c is arbitrary the proof is complete. □

Given a system of real entire solutions $\phi(z,x)$ and $\theta(z,x)$ as in Lemma 2.2 we can define the singular Weyl function

$$M(z) = -\frac{W(\theta(z), u_+(z))}{W(\phi(z), u_+(z))} \tag{2.6}$$

such that the solution which is in the domain of H near b (cf. (2.2a)) is given by

$$u_+(z,x) = a(z)\big(\theta(z,x) + M(z)\phi(z,x)\big),$$

where $a(z) = -W(\phi(z), u_+(z))$. By construction we obtain that the singular Weyl function $M(z)$ is analytic in $\mathbb{C}\backslash\mathbb{R}$ and satisfies $M(z) = M(z^*)^*$. Rather than $u_+(z,x)$ we will use

$$\psi(z,x) = \theta(z,x) + M(z)\phi(z,x). \tag{2.7}$$

Recall also from [20, Lem. 3.2] that associated with $M(z)$ is a corresponding spectral measure by virtue of the Stieltjes–Livšić inversion formula

$$\frac{1}{2}\left(\rho\big((x_0,x_1)\big) + \rho\big([x_0,x_1]\big)\right) = \lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{x_0}^{x_1}\mathrm{Im}\big(M(x+\mathrm{i}\varepsilon)\big)dx. \tag{2.8}$$

Theorem 2.5 ([10])**.** *Define*

$$\hat{f}(\lambda) = \lim_{c\uparrow b}\int_a^c \phi(\lambda,x)f(x)dx, \tag{2.9}$$

where the right-hand side has to be understood as a limit in $L^2(\mathbb{R}, d\rho)$. Then the map

$$U : L^2(a,b) \to L^2(\mathbb{R}, d\rho), \qquad f \mapsto \hat{f}, \tag{2.10}$$

is unitary and its inverse is given by

$$f(x) = \lim_{r\to\infty} \int_{-r}^{r} \phi(\lambda, x)\hat{f}(\lambda) d\rho(\lambda), \tag{2.11}$$

where again the right-hand side has to be understood as a limit in $L^2(a,b)$. Moreover, U maps H to multiplication with λ.

Remark 2.6. *We have seen in Remark 2.3 that $M(z)$ is not unique. However, given $\tilde{M}(z)$ as in Remark 2.3, the spectral measures are related by*

$$d\tilde{\rho}(\lambda) = \mathrm{e}^{-2g(\lambda)} d\rho(\lambda). \tag{2.12}$$

Hence the measures are mutually absolutely continuous and the associated spectral transformation just differ by a simple rescaling with the positive function $\mathrm{e}^{-2g(\lambda)}$.

Finally, $M(z)$ can be reconstructed from ρ up to an entire function via the following integral representation.

Theorem 2.7 ([20])**.** *Let $M(z)$ be a singular Weyl function and ρ its associated spectral measure. Then there exists an entire function $g(z)$ such that $g(\lambda) \ge 0$ for $\lambda \in \mathbb{R}$ and $\mathrm{e}^{-g(\lambda)} \in L^2(\mathbb{R}, d\rho)$.*

Moreover, for any entire function $\hat{g}(z)$ such that $\hat{g}(\lambda) > 0$ for $\lambda \in \mathbb{R}$ and $(1+\lambda^2)^{-1}\hat{g}(\lambda)^{-1} \in L^1(\mathbb{R}, d\rho)$ (e.g. $\hat{g}(z) = \mathrm{e}^{2g(z)}$) we have the integral representation

$$M(z) = E(z) + \hat{g}(z) \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right) \frac{d\rho(\lambda)}{\hat{g}(\lambda)}, \quad z \in \mathbb{C}\backslash\sigma(H), \tag{2.13}$$

where $E(z)$ is a real entire function.

Remark 2.8. *Choosing an entire function $g(z)$ such that $\exp(-2g(\lambda))$ is integrable with respect to ρ we see that*

$$M(z) = \mathrm{e}^{2g(z)} \int_{\mathbb{R}} \frac{1}{\lambda - z} \mathrm{e}^{-2g(\lambda)} d\rho(\lambda) - E(z). \tag{2.14}$$

Hence if we choose $f(z) = \exp(-g(z))E(z)$ and switch to a new system of solutions as in Remark 2.3, we see that the new singular Weyl function is a Herglotz–Nevanlinna function

$$\tilde{M}(z) = \int_{\mathbb{R}} \frac{1}{\lambda - z} \mathrm{e}^{-2g(\lambda)} d\rho(\lambda). \tag{2.15}$$

3. Exponential growth rates

While a real entire fundamental system $\theta(z,x)$, $\phi(z,x)$ as in Section 2 is good enough to define a singular Weyl function and an associated spectral measure, it does not suffice for the proof of our uniqueness results. For them we will need information on the exponential growth rate of the solutions $\theta(\,\cdot\,,x)$ and $\phi(\,\cdot\,,x)$. In the case where a is finite with a repelling potential the growth rate will be $1/2$ and this case was dealt with in [20, Section 6]. The aim of the present section is to extend these results to cover arbitrary (finite) growth rates. Therefore we will say a real entire solution $\phi(z,x)$ is of exponential growth order at most $s > 0$ if both $\phi(\,\cdot\,,x)$ and $\phi'(\,\cdot\,,x)$ are of growth order at most s for all $x \in (a,b)$.

Our first aim is to extend Lemma 2.2 and to show how the growth order of $\phi(\,\cdot\,, x)$ is connected with the convergence exponent of the spectrum. To this end we begin by recalling some basic notation. We refer to the classical book by Levin [23] for proofs and further background. Given some discrete set $S \subseteq \mathbb{C}$, the number

$$\inf\left\{ s \geq 0 \;\middle|\; \sum_{\mu \in S} \frac{1}{1+|\mu|^s} < \infty \right\} \in [0,\infty]$$

is referred to as the convergence exponent of S. Moreover, the smallest integer $p \in \mathbb{N}$ for which

$$\sum_{\mu \in S} \frac{1}{1+|\mu|^{p+1}} < \infty$$

will be called the genus of S. Introducing the elementary factors

$$E_p(\zeta, z) = \left(1 - \frac{z}{\zeta}\right) \exp\left(\sum_{k=1}^{p} \frac{1}{k} \frac{z^k}{\zeta^k} \right), \quad z \in \mathbb{C},$$

if $\zeta \neq 0$ and $E_p(z, 0) = z$, we recall that the product $\prod_{\mu \in S} E_p(\mu, z)$ converges uniformly on compact sets to an entire function of growth order s, where s and p are the convergence exponent and genus of S, respectively.

Furthermore, we will denote the spectrum of $H^D_{(a,c)}$ and $H^N_{(a,c)}$ (provided they are discrete) by

$$\sigma(H^D_{(a,c)}) = \{\mu_n(c)\}_{n \in N} \quad \text{and} \quad \sigma(H^N_{(a,c)}) = \{\nu_{n-1}(c)\}_{n \in N},$$

where the index set N is either $\mathbb{N}$ or $\mathbb{Z}$. The eigenvalues $\mu_n(c)$, $\nu_n(c)$ are precisely the zeros of $\phi(\,\cdot\,, c)$ and $\phi'(\,\cdot\,, c)$, respectively. Recall that both spectra are interlacing

$$\nu_{n-1}(c) < \mu_n(c) < \nu_n(c), \quad n \in N, \tag{3.1}$$

and that Kreĭn's theorem [23, Theorem 27.2.1] states

$$m_-(z) = C \prod_{j \in N} \frac{E_0(\nu_{n-1}(c), z)}{E_0(\mu_n(c), z)}, \quad C \neq 0. \tag{3.2}$$

Note that in general the products in the numerator and denominator will not converge independently but only jointly since due to the interlacing properties of the eigenvalues the sum

$$\sum_{n \in N} \left(\frac{1}{\nu_{n-1}(c)} - \frac{1}{\mu_n(c)} \right)$$

will converge.

Theorem 3.1. *For each $s > 0$ the following properties are equivalent:*

(i) *The spectrum of $H^D_{(a,c)}$ is purely discrete and has convergence exponent at most s.*
(ii) *There is a real entire solution $\phi(z, x)$ of exponential growth order at most s which is non-trivial and lies in the domain of H near a for each $z \in \mathbb{C}$.*

In this case $s \geq \frac{1}{2}$.

Proof. Suppose the spectrum of $H^D_{(a,c)}$ is purely discrete and has convergence exponent at most s. The same then holds true for the spectrum of $H^N_{(a,c)}$ and according to [20, Lemma 6.3] s is at least $1/2$. Denote by $p \in \mathbb{N}_0$ the genus of these sequences and consider the real entire functions

$$\alpha(z) = \prod_{n\in N} E_p\left(\mu_n(c), z\right) \quad \text{and} \quad \tilde{\beta}(z) = \prod_{n\in N} E_p\left(\nu_{n-1}(c), z\right).$$

Then α and $\tilde{\beta}$ are of exponential growth order at most s by Borel's theorem (see [23, Theorem 4.3.3]). Next note that

$$m_-(z) = \mathrm{e}^{h(z)} \frac{\tilde{\beta}(z)}{\alpha(z)}$$

for some entire function $h(z)$ since the right-hand side has the same poles and zeros as $m_-(z)$. Comparing this with Kreĭn's formula (3.2) we obtain that $h(z)$ is in fact a polynomial of degree at most p:

$$h(z) = \sum_{k=1}^{p} \frac{z^k}{k} \sum_{n\in N} \left(\frac{1}{\mu_n(c)^k} - \frac{1}{\nu_{n-1}(c)^k} \right) + \ln(C).$$

Observe that the sums converge absolutely by our interlacing assumption. In particular, the function

$$\beta(z) = -m_-(z)\alpha(z) = -\mathrm{e}^{h(z)}\tilde{\beta}(z) \tag{3.3}$$

is of exponential growth order at most s as well. Hence the solutions

$$\phi(z,x) = \alpha(z)s(z,x) + \beta(z)c(z,x), \quad x \in (a,b),\ z \in \mathbb{C}$$

lie in the domain of H near a and are of growth order at most s by [26, Lemma 9.18]. Conversely let $\phi(z,x)$ be a real entire solution of exponential growth order at most s which lies in the domain of H near a. Then since $m_-(z) = -\phi'(z,c)/\phi(z,c)$, the spectrum of $H^D_{(a,c)}$ is purely discrete and coincides with the zeros of $\phi(\,\cdot\,,c)$. Now since $\phi(\,\cdot\,,c)$ is of growth order at most s, its zeros are of convergence exponent at most s. □

Note that because of the interlacing property of eigenvalues it is irrelevant which boundary condition we chose at the point c. Moreover, the preceding theorem also shows that the convergence exponent of $\sigma(H^D_{(a,c)})$ is independent of $c \in (a,b)$. Finally note that for (i) to hold it suffices that there is some real entire solution $\phi(z,x)$ such that $\phi(\,\cdot\,,x)$ is of growth order at most s for some $x \in (a,b)$.

Unfortunately, given a real entire solution $\phi(z,x)$ of exponential growth order $s > 0$ we are not able to prove the existence of a second solution of the same growth order. However, at least under some additional assumptions we get a second solution $\theta(z,x)$ of growth order arbitrarily close to s. For the proof we will need the following version of the corona theorem for entire functions.

Theorem 3.2 ([12])**.** *Let $R_s(\mathbb{C})$, $s > 0$, be the ring of all entire functions f for which there are constants A, $B > 0$ such that*

$$|f(z)| \le B\mathrm{e}^{A|z|^s}, \quad z \in \mathbb{C}. \tag{3.4}$$

Then $f_j \in R_s(\mathbb{C})$, $j = 1, \dots, n$ generate $R_s(\mathbb{C})$ if and only if

$$|f_1(z)| + \cdots + |f_n(z)| \ge b\mathrm{e}^{-a|z|^s}, \quad z \in \mathbb{C} \tag{3.5}$$

for some constants a, $b > 0$.

As an immediate consequence of this result we obtain the following necessary and sufficient criterion for existence of a second solution $\theta(z,x)$ of the required growth.

Lemma 3.3. *Suppose $\phi(z,x)$ is a real entire solution of exponential growth order $s > 0$ and let $\varepsilon > 0$. Then there is a real entire second solution $\theta(z,x)$ with $W(\theta,\phi) = 1$ and $\theta(\,\cdot\,,x)$, $\theta'(\,\cdot\,,x) \in R_{s+\varepsilon}(\mathbb{C})$ if and only if*

$$|\phi(z,c)| + |\phi'(z,c)| \ge b\mathrm{e}^{-a|z|^{s+\varepsilon}}, \quad z \in \mathbb{C} \tag{3.6}$$

for some constants a, $b > 0$.

Proof. If $\theta(z,x)$ is a second solution which lies in $R_{s+\varepsilon}(\mathbb{C})$, then

$$\theta(z,c)\phi'(z,c) - \theta'(z,c)\phi(z,c) = 1, \quad z \in \mathbb{C}$$

implies that the functions $\phi(\,\cdot\,,c)$ and $\phi'(\,\cdot\,,c)$ generate $R_{s+\varepsilon}(\mathbb{C})$ and (3.6) follows from Theorem 3.2. Conversely if (3.6) holds, then $\phi(\,\cdot\,,c)$ and $\phi'(\,\cdot\,,c)$ generate $R_{s+\varepsilon}(\mathbb{C})$. Thus there are real entire functions γ, $\delta \in R_{s+\varepsilon}(\mathbb{C})$ with

$$\gamma(z)\phi'(z,c) - \delta(z)\phi(z,c) = 1, \quad z \in \mathbb{C}.$$

Now take $\theta(z,x)$ to be the solutions with initial conditions $\theta(z,c) = \gamma(z)$ and $\theta'(z,c) = \delta(z)$. ☐

We are also able to provide a sufficient condition for a second solution of exponential order to exist, in terms of the zeros of $\phi(\,\cdot\,,c)$ and $\phi'(\,\cdot\,,c)$. For the proof we need the following lemma on the minimal modulus of an entire function of finite exponential growth order.

Lemma 3.4. *Suppose F is an entire function of growth order s with zeros ζ_j, $j \in \mathbb{N}$. Then for each δ, $\varepsilon > 0$ there are constants A, $B > 0$ such that*

$$|F(z)| \ge B\mathrm{e}^{-A|z|^{s+\varepsilon}}, \tag{3.7}$$

except possibly when z belongs to one of the discs $|z - \zeta_j| < |\zeta_j|^{-\delta}$.

Proof. This follows from Hadamard's factorization theorem and the estimates in [1, Lemma 2.6.18]. ☐

Lemma 3.5. *Suppose $\phi(z,x)$ is a real entire solution of growth order $s > 0$ and that for some $r > 0$ all but finitely many of the circles given by*

$$|z - \mu_n(c)| < |\mu_n(c)|^{-r} \quad \textit{and} \quad |z - \nu_{n-1}(c)| < |\nu_{n-1}(c)|^{-r}, \quad n \in N \tag{3.8}$$

are disjoint. Then for every $\varepsilon > 0$ there is a real entire second solution $\theta(z,x)$ of exponential growth at most $s + \varepsilon$ and with $W(\theta,\phi) = 1$.

Proof. Lemma 3.4 implies that we have (3.7) for either $\phi(\,\cdot\,,c)$ or $\phi'(\,\cdot\,,c)$ and hence in particular for the sum of both. ☐

Remark 3.6. *By the Hadamard product theorem* [23, Theorem 4.2.1]*, a solution $\phi(z,x)$ of exponential growth order $s>0$ is unique up to a factor $\mathrm{e}^{g(z)}$, for some real polynomial $g(z)$ of degree at most p, where $p\in\mathbb{N}_0$ is the genus of the eigenvalues of $H^D_{(a,c)}$. A solution $\theta(z,x)$ of growth order at most s is unique only up to $f(z)\phi(z,x)$, where $f(z)$ is an entire function of growth order at most s.*

Finally, note that under the assumptions in this section one can use the function $\hat{g}(z)=\exp(z^{2\lceil (p+1)/2\rceil})$ in Theorem 2.7. If in addition H is bounded from below, then one can also use $\hat{g}(z)=\exp(z^{p+1})$.

4. A local Borg–Marchenko uniqueness result

The purpose of the present section is again to extend the corresponding results from [20, Section 7] to the case of arbitrary growth orders.

Lemma 4.1 ([20])**.** *For each $x\in(a,b)$ the singular Weyl function $M(z)$ and the Weyl solution $\psi(z,x)$ defined in* (2.7) *have the asymptotics*

$$M(z) = -\frac{\theta(z,x)}{\phi(z,x)} + \mathcal{O}\left(\frac{1}{\sqrt{-z}\phi(z,x)^2}\right), \tag{4.1}$$

$$\psi(z,x) = \frac{1}{2\sqrt{-z}\phi(z,x)}\left(1+\mathcal{O}\left(\frac{1}{\sqrt{-z}}\right)\right), \tag{4.2}$$

as $|z|\to\infty$ in any sector $|\mathrm{Im}(z)|\ge\delta\,|\mathrm{Re}(z)|$ with $\delta>0$.

In particular, (4.1) shows that asymptotics of $M(z)$ immediately follow once one has corresponding asymptotics for the solutions $\theta(z,x)$ and $\phi(z,x)$. Moreover, the leading asymptotics depend only on the values of q near the endpoint a (and on the choice of $\theta(z,x)$ and $\phi(z,x)$). The following Borg–Marchenko type uniqueness result shows that the converse is also true.

In order to state this theorem let q_0 and q_1 be two potentials on intervals (a,b_0) and (a,b_1), respectively. By H_0 and H_1 we denote some corresponding self-adjoint operators with separate boundary conditions. Furthermore, for $j=0,1$ let $\theta_j(z,x)$, $\phi_j(z,x)$ be some real entire fundamental system of solutions with $W(\theta_j,\phi_j)=1$ such that $\phi_j(z,x)$ lies in the domain of H_j near a. The associated singular Weyl functions are denoted by $M_0(z)$ and $M_1(z)$. We will also use the common short-hand notation $\phi_1(z)\sim\phi_2(z)$ to abbreviate the asymptotic relation $\phi_1(z)=\phi_2(z)(1+o(1))$ (or equivalently $\phi_2(z)=\phi_1(z)(1+o(1))$) as $|z|\to\infty$ in some specified manner.

Theorem 4.2. *Suppose $\theta_0(z,x)$, $\theta_1(z,x)$, $\phi_0(z,x)$, $\phi_1(z,x)$ are of growth order at most s for some $s>0$ and $\phi_1(z,x)\sim\phi_0(z,x)$ for one (and hence by* (2.5) *for all) $x\in(a,b_0)\cap(a,b_1)$ as $|z|\to\infty$ along some nonreal rays dissecting the complex plane into sectors of length less than π/s. Then for each $c\in(a,b_0)\cap(a,b_1)$ the following properties are equivalent:*

(i) *$W(\phi_0,\phi_1)(a)=0$ and $q_0(x)=q_1(x)$ for almost all $x\in(a,c)$.*

(ii) *For each $\delta>0$ there is an entire function f of exponential growth order at most s such that*

$$M_1(z)-M_0(z) = f(z) + \mathcal{O}\left(\frac{1}{\sqrt{-z}\phi_0(z,c)^2}\right),$$

as $|z|\to\infty$ in the sector $|\mathrm{Im}(z)|\ge\delta\,|\mathrm{Re}(z)|$.

(iii) *For each $x \in (a,c)$ there is an entire function f of exponential growth order at most s such that*

$$M_1(z) - M_0(z) = f(z) + \mathcal{O}\left(\frac{1}{\phi_0(z,x)^2}\right),$$

as $|z| \to \infty$ along our nonreal rays.

Proof. If (i) holds, then by Remark 3.6 the solutions are related by

$$\phi_1(z,x) = \phi_0(z,x)\mathrm{e}^{g(z)}, \quad x \in (a,c],\ z \in \mathbb{C} \tag{4.3}$$

and

$$\theta_1(z,x) = \theta_0(z,x)\mathrm{e}^{-g(z)} - f(z)\phi_1(z,x), \quad x \in (a,c],\ z \in \mathbb{C} \tag{4.4}$$

for some real polynomial g of degree at most s and some real entire function f of growth order at most s. From the asymptotic behavior of the solutions ϕ_0, ϕ_1 we infer that $g = 0$. Now the asymptotics in Lemma 4.1 show that

$$\begin{aligned} M_1(z) - M_0(z) &= \frac{\theta_0(z,c)}{\phi_0(z,c)} - \frac{\theta_1(z,c)}{\phi_1(z,c)} + \mathcal{O}\left(\frac{1}{\sqrt{-z}\phi_0(z,c)^2}\right) \\ &= f(z) + \mathcal{O}\left(\frac{1}{\sqrt{-z}\phi_0(z,c)^2}\right), \end{aligned}$$

as $|z| \to \infty$ in any sector $|\mathrm{Im}(z)| \geq \delta\,|\mathrm{Re}(z)|$. The implication (ii) $\Rightarrow$ (iii) is obvious. Now suppose property (iii) holds and for fixed $x \in (a,c)$ consider the entire function G_x for each $z \in \mathbb{C}$ given by

$$G_x(z) = \phi_1(z,x)\theta_0(z,x) - \phi_0(z,x)\theta_1(z,x) - f(z)\phi_0(z,x)\phi_1(z,x). \tag{4.5}$$

Since away from the real axis this function may be written as

$$\begin{aligned} G_x(z) = \phi_1(z,x)\psi_0(z,x) &- \phi_0(z,x)\psi_1(z,x) \\ &+ (M_1(z) - M_0(z) - f(z))\phi_0(z,x)\phi_1(z,x), \quad z \in \mathbb{C}\backslash\mathbb{R}, \end{aligned}$$

it vanishes as $|z| \to \infty$ along our nonreal rays. For the first two terms this follows from (4.2) together with our hypothesis that $\phi_0(\,\cdot\,,x)$ and $\phi_1(\,\cdot\,,x)$ have the same asymptotics. The last term tends to zero because of our assumption on the difference of the Weyl functions. Moreover, by our hypothesis G_x is of growth order at most s and thus we can apply the Phragmén–Lindelöf theorem (e.g., [23, Section 6.1]) in the angles bounded by our rays. This shows that G_x is bounded on all of $\mathbb{C}$. By Liouville's theorem it must be constant and since it vanishes along a ray, it must be zero; that is,

$$\phi_1(z,x)\theta_0(z,x) - \phi_0(z,x)\theta_1(z,x) = f(z)\phi_0(z,x)\phi_1(z,x), \quad z \in \mathbb{C}$$

for all $x \in (a,c)$. Dividing both sides of this identity by $\phi_0(z,x)\phi_1(z,x)$, differentiating with respect to x, and using $W(\theta_j, \phi_j) = 1$ shows that we have $\phi_1(z,x)^2 = \phi_0(z,x)^2$. Taking the logarithmic derivative further gives $\phi_1'(z,x)/\phi_1(z,x) = \phi_0'(z,x)/\phi_0(z,x)$, which shows $W(\phi_0,\phi_1)(a) = 0$. Differentiating once more shows $\phi_1''(z,x)/\phi_1(z,x) = \phi_0''(z,x)/\phi_0(z,x)$. This finishes the proof since $q_j(x) = z + \phi_j''(z,x)/\phi_j(z,x)$. □

Note that the implication (iii) $\Rightarrow$ (i) could also be proved under somewhat weaker conditions. First of all the assumption on the growth of the entire functions f is only due to the use of the Phragmén–Lindelöf principle. Hence it would also suffice that for each $\varepsilon > 0$ we have

$$\sup_{|z|=r_n} |f(z)| \le B\mathrm{e}^{Ar_n^{s+\varepsilon}}, \quad n \in \mathbb{N} \tag{4.6}$$

for some increasing sequence of positive numbers $r_n \uparrow \infty$ and constants A, $B \in \mathbb{R}$. Furthermore, for this implication to hold it would also suffice that the solutions have the same order of magnitude as $|z| \to \infty$ along our nonreal rays instead of the same asymptotics.

While at first sight it might look like the condition on the asymptotics of the solutions $\phi_j(z,x)$ requires knowledge about them, this is not the case, since the high energy asymptotics will only involve some qualitative information on the kind of the singularity at a as we will show in Section 6. Next, the appearance of the additional freedom of the function f just reflects the fact that we only ensure the same normalization for the solutions $\phi_0(z,x)$ and $\phi_1(z,x)$ but not for $\theta_0(z,x)$ and $\theta_1(z,x)$ (cf. Remark 3.6).

Corollary 4.3. *Suppose $\theta_0(z,x)$, $\theta_1(z,x)$, $\phi_0(z,x)$, $\phi_1(z,x)$ are of exponential growth order at most s for some $s > 0$ and that $\phi_1(z,x) \sim \phi_0(z,x)$ for some $x \in (a,b_0) \cap (a,b_1)$ as $|z| \to \infty$ along some nonreal rays dissecting the complex plane into sectors of length less than π/s. If*

$$M_1(z) - M_0(z) = f(z), \quad z \in \mathbb{C}\backslash\mathbb{R} \tag{4.7}$$

for some entire function f of growth order at most s, then $H_0 = H_1$.

Proof. Without loss of generality suppose that $b_0 \le b_1$. Theorem 4.2 shows that $q_0 = q_1$ on (a,b_0) and that the boundary condition at a (if any) is the same. Moreover, the relations (4.3) and (4.4) hold, hence

$$\begin{aligned}\psi_1(z,x) &= \theta_1(z,x) + M_1(z)\phi_1(z,x)\\ &= \theta_0(z,x) - f(z)\phi_1(z,x) + (M_0(z) + f(z))\phi_0(z,x)\\ &= \theta_0 + M_0(z)\phi_0(z,x) = \psi_0(z,x)\end{aligned}$$

for each $x \in (a,b_0)$, $z \in \mathbb{C}\backslash\mathbb{R}$. If $b_0 < b_1$, then the right endpoint b_0 of H_0 would be regular as q_1 is integrable over $[c,b_0]$. Thus $\psi_0(z,x)$ and hence also $\psi_1(z,x)$ would satisfy some boundary condition at b_0. Since this is not possible we necessarily have $b_0 = b_1$. Finally since $\psi_0(z,x) = \psi_1(z,x)$ for nonreal z, H_0 and H_1 have the same boundary condition at b (if any). □

Note that instead of (4.7) it would also suffice to assume that for each fixed value $c \in (a,b_0) \cap (a,b_1)$

$$M_0(z) - M_1(z) = f(z) + \mathcal{O}\left(\frac{1}{\phi_0(z,c)^2}\right),$$

as $|z| \to \infty$ along our nonreal rays and that $M_1(z_0) = M_0(z_0) + f(z_0)$ for some nonreal $z_0 \in \mathbb{C}\backslash\mathbb{R}$.

5. Uniqueness results for operators with discrete spectra

Now we are finally able to investigate when the spectral measure determines the potential for operators with purely discrete spectrum. In this respect observe that the uniqueness results for the singular Weyl function from the previous sections do not immediately yield such results. In fact, if

$\rho_0 = \rho_1$ then the difference of the corresponding singular Weyl functions is an entire function by Theorem 2.7. However, in order to apply Corollary 4.3 we would need some bound on the exponential growth of this function. Fortunately, in the case of purely discrete spectrum with finite convergence exponent, a refinement of the arguments in the proof of Theorem 4.2 shows that the growth condition is not necessary.

Corollary 5.1. *Suppose $\phi_0(z,x)$, $\phi_1(z,x)$ are of growth order at most s for some $s > 0$ and $\phi_1(z,x) \sim \phi_0(z,x)$ for an $x \in (a,b_0) \cap (a,b_1)$ as $|z| \to \infty$ along some nonreal rays dissecting the complex plane into sectors of length less than π/s. Furthermore, assume that H_0 and H_1 have purely discrete spectrum with convergence exponent at most s. If*

$$M_1(z) - M_0(z) = f(z), \quad z \in \mathbb{C}\backslash\mathbb{R} \tag{5.1}$$

for some entire function f, then $H_0 = H_1$.

Proof. It suffices to show that the functions G_x, $x \in (a,b_0) \cap (a,b_1)$, defined by (4.5) in Theorem 4.2 satisfy a growth restriction as in (4.6). Because of Theorem 3.1 there are real entire solutions $\chi_0(z,x)$, $\chi_1(z,x)$ of growth order at most s, which are square integrable near the right endpoint and satisfy the boundary condition there if necessary. These solutions are related to the Weyl solutions $\psi_0(z,x)$, $\psi_1(z,x)$ by

$$\psi_j(z,x) = \frac{\chi_j(z,x)}{W(\chi_j,\phi_j)(z)}, \quad x \in (a,b_j),\ z \in \rho(H_j),\ j = 0,1.$$

Using the definition of the singular Weyl–Titchmarsh functions $M_j(z)$ we get for each $x \in (a,b_0) \cap (a,b_1)$

$$\begin{aligned} G_x(z) &= \phi_1(z,x)\theta_0(z,x) - \phi_0(z,x)\theta_1(z,x) - f(z)\phi_0(z,x)\phi_1(z,x) \\ &= \phi_1(z,x)\psi_0(z,x) - \phi_0(z,x)\psi_1(z,x) \\ &= \frac{\phi_1(z,x)\chi_0(z,x)}{W(\chi_0,\phi_0)(z)} - \frac{\phi_0(z,x)\chi_1(z,x)}{W(\chi_1,\phi_1)(z)}, \quad z \in \mathbb{C}\backslash\mathbb{R}. \end{aligned}$$

Now since the numerators and denominators in the last line are of growth order at most s, Lemma 3.4 shows that there is some increasing sequence $r_n \uparrow \infty$ and positive constants A_x, B_x such that

$$\sup_{|z|=r_n} |G_x(z)| \le B_x \mathrm{e}^{A_x r_n^{\tilde{s}}}, \quad n \in \mathbb{N},$$

where $\tilde{s} > s$ such that our nonreal rays dissect the complex plane into sectors of length less than $\pi/\tilde{s}$. Now we may use the remark after Theorem 4.2. □

Now the lack of a growth restriction in Corollary 5.1 implies that it immediately translates into a corresponding inverse uniqueness result for the spectral measure.

Theorem 5.2. *Suppose that $\phi_0(z,x)$, $\phi_1(z,x)$ are of growth order at most s for some $s > 0$ and that $\phi_1(z,x) \sim \phi_0(z,x)$ for an $x \in (a,b_0) \cap (a,b_1)$ as $|z| \to \infty$ along some nonreal rays dissecting the complex plane into sectors of length less than π/s. Furthermore, assume that H_0 and H_1 have purely discrete spectrum with convergence exponent at most s. If the corresponding spectral measures ρ_0 and ρ_1 are equal, then we have $H_0 = H_1$.*

Proof. Since the spectral measures are the same, Theorem 2.7 shows that the difference of the corresponding singular Weyl functions is an entire function and Corollary 5.1 is applicable. □

It should be emphasized that a similar inverse uniqueness result has been proven in [5] using the theory of de Branges spaces. However, the assumptions on $\phi_j(z,x)$ are of a different nature and we are not aware how to verify the assumptions on $\phi_j(z,x)$ to apply [5, Theorem 4.1] to the examples envisaged in the present paper. Nevertheless, in some sense our assumptions here are stronger since they exclude (in the case $(a,b)=\mathbb{R}$) the possibility that one potential is a translation of the other (which clearly would leave the spectral measure unchanged).

Note that in the case of discrete spectra, the spectral measure is uniquely determined by the eigenvalues λ_n and the corresponding norming constants

$$\gamma_n^2 = \int_a^b \phi(\lambda_n, x)^2 dx, \tag{5.2}$$

since in this case we have

$$\rho = \sum_n \gamma_n^{-2} \delta_{\lambda_n}, \tag{5.3}$$

where δ_λ is the Dirac measure in the point λ.

As another application we are also able to proof a generalization of Hochstadt–Lieberman type uniqueness results. To this end let us consider an operator H whose spectrum is purely discrete and has convergence exponent (at most) s. Since the operator

$$H_c^D = H_{(a,c)}^D \oplus H_{(c,b)}^D,$$

with an additional Dirichlet boundary condition at the point c is a rank one perturbation of H we conclude that the convergence exponents of both $H_{(a,c)}^D$ and $H_{(c,b)}^D$ are at most s and hence by Theorem 3.1 there are real entire solutions $\phi(z,x)$ and $\chi(z,x)$ of growth order at most s which are in the domain of H near a and b, respectively.

Theorem 5.3. *Suppose H_0 is an operator with purely discrete spectrum of finite convergence exponent s. Let $\phi_0(z.x)$ and $\chi_0(z,x)$ be entire solutions of growth order at most s which lie in the domain of H_0 near a and b, respectively, and suppose there is an $c \in (a,b)$ such that*

$$\frac{\chi_0(z,c)}{\phi_0(z,c)} = \mathcal{O}(1), \tag{5.4}$$

as $|z| \to \infty$ along some nonreal rays dissecting the complex plane into sectors of length less than π/s. Then every other isospectral operator H_1 for which $q_1 = q_0$ almost everywhere on (a,c) and which is associated with the same boundary condition at a (if any) is equal to H_0.

Proof. Start with some solutions $\phi_j(z,x)$, $\chi_j(z,x)$ of growth order at most s and note that we can choose $\phi_1(z,x) = \phi_0(z,x)$ for $x \le c$ since H_1 and H_0 are associated with the same boundary condition at a (if any). Moreover, note that we have $\phi_0(z,x) \sim \phi_1(z,x)$ as $|z| \to \infty$ along every nonreal ray even for fixed $x > c$ by Lemma 2.5. Next note that the zeros of the Wronskian $W(\phi_j,\chi_j)$ are precisely the eigenvalues of H_j and thus, by assumption, are equal. Hence by the Hadamard factorization theorem $W(\phi_1,\chi_1)=\mathrm{e}^g W(\phi_0,\chi_0)$ for some polynomial g of degree at most s. Since we can absorb this factor in $\chi_1(z,x)$, we can assume $g=0$ without loss of generality. Hence we have

$$\begin{aligned} 1=\frac{W(\phi_0,\chi_0)}{W(\phi_1,\chi_1)} &= \frac{\phi_0(z,x)\chi_0'(z,x)-\phi_0'(z,x)\chi_0(z,x)}{\phi_1(z,x)\chi_1'(z,x)-\phi_1'(z,x)\chi_1(z,x)} \\ &= \frac{\phi_0(z,x)}{\phi_1(z,x)}\frac{\chi_0(z,x)}{\chi_1(z,x)}\left(\frac{\chi_0'(z,x)}{\chi_0(z,x)}-\frac{\phi_0'(z,x)}{\phi_0(z,x)}\right)\left(\frac{\chi_1'(z,x)}{\chi_1(z,x)}-\frac{\phi_1'(z,x)}{\phi_1(z,x)}\right)^{-1} \end{aligned}$$

and by virtue of the well-known asymptotics (see [26, Lemma 9.19])

$$\frac{\chi_j'(z,x)}{\chi_j(z,x)}=-\sqrt{-z}+\mathcal{O}(1) \quad \text{and} \quad \frac{\phi_j'(z,x)}{\phi_j(z,x)}=\sqrt{-z}+\mathcal{O}(1), \quad j=0,1,$$

as $|z|\to\infty$ along any nonreal rays we conclude $\chi_1(z,x)\sim\chi_0(z,x)$ as well. Furthermore, equality of the Wronskians implies

$$\chi_1(z,x)=\chi_0(z,x)+F(z)\phi_0(z,x), \quad x\le c$$

for some entire function $F(z)$ of growth order at most s. Moreover, our assumption (5.4) implies that the function

$$F(z)=\frac{\chi_1(z,c)-\chi_0(z,c)}{\phi_0(z,c)}=\frac{\chi_0(z,c)}{\phi_0(z,c)}\left(\frac{\chi_1(z,c)}{\chi_0(z,c)}-1\right)$$

vanishes along our rays and thus it must be identically zero by the Phragmén–Lindelöf theorem. Finally, choosing second solutions $\theta_j(z,x)$ such that $\theta_1(z,x)=\theta_0(z,x)$ for $x\le c$ this implies that the associated singular Weyl functions are equal and the claim follows from Corollary 5.1. □

Note that by (2.5) the growth of $\phi_0(\,\cdot\,,c)$ will increase as c increases while (by reflection) the growth of $\chi_0(\,\cdot\,,c)$ will decrease. In particular, if (5.4) holds for some c it will hold for any other $c'>c$ as well.

As a first example we give a generalization of the Hochstadt–Lieberman result from [11] to operators on the interval $[0,1]$ with Bessel-type singularities at both endpoints. Note that the case $k=l=0$ and $q_0\in L^1(0,1)$ is the classical Hochstadt–Lieberman result.

Theorem 5.4. *Let l, $k\ge-\frac{1}{2}$ and consider an operator of the form*

$$H_0=-\frac{d^2}{dx^2}+q_0(x), \qquad q_0(x)=\frac{l(l+1)}{x^2}+\frac{k(k+1)}{(1-x)^2}+\tilde{q}_0(x), \quad x\in(0,1),$$

with $\tilde{q}_0$ satisfying

$$f_l(x)f_k(1-x)\tilde{q}_0(x)\in L^1(0,1), \qquad f_l(x)=\begin{cases} x, & \text{if } l>-\frac{1}{2}, \\ (1-\ln(x))x, & \text{if } l=-\frac{1}{2}. \end{cases}$$

If $l<\frac{1}{2}$ we choose the boundary condition

$$\lim_{x\to 0} x^l((l+1)\phi(x)-x\phi'(x))=0,$$

associated with the Friedrichs extension at 0 and similarly at 1 if $k<\frac{1}{2}$. Suppose H_1 satisfies $q_1(x)=q_0(x)$ for almost all $x\in(0,1/2+\varepsilon)$ and has the *same boundary condition at* 0 *if* $l < \frac{1}{2}$, *where* $\varepsilon = 0$ *if* $k \geq l$ *and* $\varepsilon > 0$ *if* $k < l$. *Then,* $H_0 = H_1$ *if both have the same spectrum.*

Proof. Immediate from Theorem 5.3 together with the asymptotics of solutions of H_0 given in Lemma 2.2 (see in particular (2.24)) of [19]. □

In particular, this applies for example to the Pöschl–Teller operator

$$H = -\frac{d^2}{dx^2} + \frac{\nu(\nu+1)}{\sin(\pi x)^2}, \quad \nu \geq 0,$$

whose spectrum is given by $\sigma(H) = \{\pi^2(n+\nu)^2\}_{n\in\mathbb{N}}$ which plays an important role as an explicitly solvable model in physics.

As another result we can generalize Theorem 1.4 from [9].

Corollary 5.5 ([9])**.** *Let* H_0 *be an operator on* $(-a, a)$ *with purely discrete spectrum which is bounded from below and has convergence exponent* $s < 1$. *If* $q_0(x) \geq q_0(-x)$ *for* $x > 0$, *then* q_0 *on* $(-a, 0)$ *and the spectrum uniquely determine* H_0.

Proof. Since the convergence exponent is less than one there are solutions given by

$$\phi_0(z,0) = \prod_{n\in\mathbb{N}} E_0(\mu_{-,n}, z) \quad \text{and} \quad \chi_0(z,0) = \prod_{n\in\mathbb{N}} E_0(\mu_{+,n}, z), \quad z \in \mathbb{C},$$

where $\mu_{\pm,n}$ are the Dirichlet eigenvalues on $(0, \pm a)$, respectively. By our assumption the min-max principle implies $\mu_{-,n} \geq \mu_{+,n}$ and (5.4) follows from monotonicity of $E_0(\mu, \mathrm{i}y)$ as $y \to \infty$. □

Note that, as pointed out in [9, Proposition 5.1], the convergence exponent will satisfy $s \leq 1 - \frac{\varepsilon}{4+2\varepsilon}$ provided $q(x) \geq C|x|^{2+\varepsilon} - D$ for some $C, D, \varepsilon > 0$ and our result is indeed a generalization of [9, Theorem 1.4]. In fact, it was conjectured in [9] and later proven in [16] that the restriction on the convergence exponent is indeed superfluous. We will show how to replace the condition $q_0(x) \geq q_0(-x)$ by an asymptotic condition in the next section.

6. Perturbed harmonic oscillators

Let q be a real-valued function on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} \frac{|q(t)|}{1+|t|} dt < \infty \tag{6.1}$$

and consider the Schrödinger operator

$$H = -\frac{d^2}{dx^2} + x^2 + q(x), \quad x \in \mathbb{R} \tag{6.2}$$

in $L^2(\mathbb{R})$. We will see shortly that H is bounded from below and hence in the limit-point case at both endpoints by a Povzner–Wienholtz type argument (cf. [8, Lemma C.1]). Hence no boundary conditions are needed and the associated operator is unique. Moreover, the case $q = 0$ is of course the famous harmonic oscillator which can be solved explicitly in terms of Weber functions D_ν (or parabolic cylinder functions) on $\mathbb{R}$. We refer the reader to (e.g.) §16.5 in [28] for basic properties of these functions. In particular, two linearly independent entire solutions of the unperturbed equation

$$-\phi_0''(z,x) + x^2\phi_0(z,x) = z\phi_0(z,x), \quad x \in \mathbb{R},\ z \in \mathbb{C}$$

are given by

$$\phi_{0,\pm}(z,x) = D_{\frac{z-1}{2}}(\pm\sqrt{2}x), \quad x \in \mathbb{R},\ z \in \mathbb{C}.$$

They are known to have the spatial asymptotics

$$\phi_{0,\pm}(z,x) \sim (\pm\sqrt{2}x)^{\frac{z-1}{2}} \mathrm{e}^{-\frac{x^2}{2}}, \tag{6.3a}$$

$$\phi_{0,\pm}'(z,x) \sim \mp x(\pm\sqrt{2}x)^{\frac{z-1}{2}} \mathrm{e}^{-\frac{x^2}{2}}, \tag{6.3b}$$

as $x \to \pm\infty$, uniformly for all z in bounded domains. In particular, this guarantees that $\phi_{0,\pm}(z,\cdot\,)$ are square integrable near $\pm\infty$.

By virtue of the usual perturbation techniques one can show that (6.1) has solutions which asymptotically look like the unperturbed ones. Details can be found in [2, Section 2] or [3, Section 3]. We collect the relevant results plus some necessary extensions in the following theorem.

Theorem 6.1. *There are unique solutions $\phi_\pm(z,x)$ of*

$$-\phi_\pm''(z,x) + \left(x^2 + q(x)\right)\phi_\pm(z,x) = z\phi_\pm(z,x), \quad x \in \mathbb{R},\ z \in \mathbb{C} \tag{6.4}$$

such that for each $z \in \mathbb{C}$ we have the asymptotics

$$\phi_\pm(z,x) \sim D_{\frac{z-1}{2}}(\pm\sqrt{2}x) \quad \text{and} \quad \phi_\pm'(z,x) \sim \pm\sqrt{2}D_{\frac{z-1}{2}}'(\pm\sqrt{2}x), \tag{6.5}$$

as $x \to \pm\infty$. Moreover, for each $x \in \mathbb{R}$ the functions $\phi_\pm(\,\cdot\,,x)$ and $\phi_\pm'(\,\cdot\,,x)$ are real entire of exponential growth order at most one.

Proof. Existence and analyticity of these solutions is proved in [2, Section 2] respectively in [3, Section 3]. Uniqueness follows from the required asymptotic behavior and it remains to show that these solutions are of growth order at most one. First of all note that

$$\phi_\pm(z,x) = \phi_\pm(z,0)c(z,x) + \phi_\pm'(z,0)s(z,x), \quad x \in \mathbb{R},\ z \in \mathbb{C},$$

where s and c are the solutions with the initial conditions

$$s(z,0) = c'(z,0) = 0 \quad \text{and} \quad s'(z,0) = c(z,0) = 1, \quad z \in \mathbb{C}.$$

Hence it suffices to show that the entire functions $\phi_\pm(\,\cdot\,,0)$ and $\phi_\pm'(\,\cdot\,,0)$ are of growth order at most one. Therefore we will need the estimate

$$1 + |z|^{\frac{1}{12}} + \left|t^2 - z\right|^{\frac{1}{4}} \geq \frac{\sqrt{|t|+1}}{\sqrt{|z|}}, \quad t \in \mathbb{R},\ z \in \mathbb{C},\ |z| \geq 2. \tag{$*$}$$

Note that by the reverse triangle inequality we only have to check this for all real $z \in [2,\infty)$. If

$$|t| \leq z(z - z^{-1})^{-\frac{1}{2}},$$

then from $z \geq 2$ one sees that $|t| + 1 \leq 2z$ and hence $(*)$ holds. Otherwise, one ends up with

$$z|t^2 - z|^{\frac{1}{2}} \geq |t|,$$

and again $(*)$ holds. Using inequality $(*)$ we get the following bound for the functions

$$\begin{aligned}\beta_{\pm}(z) &= \pm\int_0^{\pm\infty} \frac{|q(t)|}{\left(1+|z|^{\frac{1}{12}}+|t^2-z|^{\frac{1}{4}}\right)^2}dt \\ &\leq \pm|z|\int_0^{\pm\infty} \frac{|q(t)|}{|t|+1}dt, \quad z\in\mathbb{C},\ |z|\geq 2.\end{aligned}$$

Now the estimates given in [3, Lemma 3.2] or in [2, Corollary 2.6] show

$$|\phi_{\pm}(z,0)|,\ |\phi_{\pm}'(z,0)| \leq B\mathrm{e}^{A|z|\ln|z|}, \quad z\in\mathbb{C},\ |z|\geq 2$$

for some constants A, $B\in\mathbb{R}$, which proves the claim. □

In particular, the asymptotics in (6.3a) show that the solution $\phi_{\pm}(z,\cdot\,)$ is square integrable near $\pm\infty$.

Corollary 6.2. *The spectrum of H is purely discrete, bounded from below and has convergence exponent at most one.*

Proof. The existence of real entire solutions which lie in $L^2(\mathbb{R})$ near $\pm\infty$ guarantees the discreteness of the spectrum. Since the eigenvalues of H are the zeros of the entire function of growth order at most one

$$W(\phi_+,\phi_-)(z) = \phi_+(z,0)\phi_-'(z,0) - \phi_+'(z,0)\phi_-(z,0), \quad z\in\mathbb{C},$$

the spectrum has convergence exponent at most one. To see that the operator is bounded from below note that by the spatial asymptotics the underlying differential equation is non-oscillatory. □

In order to apply our uniqueness result Theorem 5.2, we need high energy asymptotics of the solutions $\phi_{\pm}(z,x)$.

Lemma 6.3. *For each $x\in\mathbb{R}$ the solutions $\phi_{\pm}(z,x)$ have the asymptotics*

$$\phi_{\pm}(z,x) \sim D_{\frac{z-1}{2}}(\pm\sqrt{2}x), \tag{6.6}$$

as $|z|\to\infty$ along each nonreal ray.

Proof. First of all note that because of the asymptotics in Lemma 2.4, it suffices to consider the case $x=0$. The estimates in [3, Lemma 3.2] or in [2, Corollary 2.6] show that

$$\left|\phi_{\pm}(z,0) - D_{\frac{z-1}{2}}(0)\right| \leq B\,\Phi(z)\beta_{\pm}(z)\mathrm{e}^{A\beta_{\pm}(z)}, \quad z\in\mathbb{C}$$

for some constants B, $A\in\mathbb{R}$ and the function

$$\Phi(z) = \left|\frac{z}{2\mathrm{e}}\right|^{\frac{\mathrm{Re}z}{4}} \mathrm{e}^{\frac{\pi-\alpha}{4}\mathrm{Im}z}(1+|z|)^{-\frac{1}{4}}, \quad z=|z|e^{\mathrm{i}\alpha},\ \alpha\in[0,2\pi).$$

Moreover, the dominated convergence theorem shows that $\beta_{\pm}(z)$ converges to zero as $|z|\to\infty$ along each ray except the positive real axis. Using

$$D_{\frac{z-1}{2}}(0) = 2^{\frac{z-1}{4}}\frac{\sqrt{\pi}}{\Gamma\left(\frac{3-z}{4}\right)}, \quad z\in\mathbb{C}$$

and Stirling's formula for the Gamma function

$$\Gamma(z) \sim \mathrm{e}^{-z} z^z \sqrt{\frac{2\pi}{z}},$$

as $|z| \to \infty$ along all rays except the negative real axis, we get

$$(*) \qquad \left| \frac{\Phi(z)}{D_{\frac{z-1}{2}}(0)} \right| = \mathcal{O}\left(\mathrm{e}^{\frac{\mathrm{Re}\, z}{4}(\ln|z| - \ln|3-z|)} \mathrm{e}^{\frac{\mathrm{Im}\, z}{4}\left(\pi - \alpha + \mathrm{Im} \ln \frac{3-z}{4}\right)} \right),$$

as $|z| \to \infty$ along each ray with angle $\alpha \in (0, 2\pi)$. Now since

$$\ln|3-z| - \ln|z| = \ln\left|1 - \frac{3}{z}\right| = \mathcal{O}\left(z^{-1}\right),$$

as well as

$$\begin{aligned} \pi - \alpha + \mathrm{Im} \ln\left(\frac{3-z}{4}\right) &= \mathrm{Im}\left(\ln(3-z) - \ln(-z)\right) = \mathrm{Im} \ln\left(1 - \frac{3}{z}\right) \\ &= \mathcal{O}\left(z^{-1}\right), \end{aligned}$$

as $|z| \to \infty$ along each ray except the positive real axis, the fraction in $(*)$ is bounded along these rays, which proves the claim. □

Denote by λ_n, $n \in \mathbb{N}$, the eigenvalues of H in increasing order. Associated to each eigenvalue is a left and right norming constant

$$(6.7) \qquad \gamma_{n,\pm}^2 = \int_{\mathbb{R}} \phi_\pm(\lambda_n, x)^2 dx, \quad n \in \mathbb{N}.$$

Now an application of Theorem 5.2 yields the following uniqueness theorem.

Theorem 6.4. *Let q be a real-valued function satisfying* (6.1). *Then the eigenvalues together with the left or right norming constants determine the function q uniquely.*

Proof. Let q_1, q_2 be two functions satisfying (6.1). From Theorem 6.1 and Lemma 6.3 we infer that the corresponding solutions $\phi_{1,\pm}(z,x)$, $\phi_{2,\pm}(z,x)$ are of growth order at most one and have the same asymptotics on each nonreal ray. Since the spectra of these operators have convergence exponent at most one, the claim follows from Theorem 5.2. □

This result is very close to [3, Theorem 1.1] with the main advantage that our class of perturbations is somewhat more explicit. There, uniqueness is proved for real-valued perturbations in a Banach space B with

$$\int_{\mathbb{R}} \frac{|q(t)|}{(1 + |z|^{\frac{1}{12}} + |t^2 - z|^{\frac{1}{4}})^2} dt \le b(|z|) \|q\|_B, \quad z \in \mathbb{C},\ q \in B$$

for some function b which decreases to zero as $|z| \to \infty$. Clearly, (6.1) must hold for the left-hand side to be finite. Also note that the norming constants ν_n, $n \in \mathbb{N}$, which are used in [3] are related to ours via

$$\gamma_{\pm,n}^2 = (-1)^n \mathrm{e}^{\mp\nu_n} \dot{W}(\lambda_n), \quad n \in \mathbb{N},$$

where W is the Wronskian

$$W(z) = \phi_-(z,x)\phi_+'(z,x) - \phi_-'(z,x)\phi_+(z,x), \quad x \in \mathbb{R},\ z \in \mathbb{C}$$

and the dot indicates differentiation with respect to z.

Now an application of Theorem 5.3 yields a Hochstadt–Lieberman type uniqueness result for the perturbed quantum harmonic oscillator.

Theorem 6.5. *Let q be a real-valued function satisfying* (6.1). *Then q on $\mathbb{R}_+$ or $\mathbb{R}_-$ together with the eigenvalues determine the function q uniquely.*

Proof. By Lemma 6.3 we have $\phi_+(z,0) \sim \phi_-(z,0)$ as $|z| \to \infty$ along nonreal rays. Hence the claim immediately follows from Theorem 5.3. □

Acknowledgments. We thank Rostyslav Hryniv for hints with respect to the literature. G. T. gratefully acknowledges the stimulating atmosphere at the Isaac Newton Institute for Mathematical Sciences in Cambridge during October 2011 where parts of this paper were written as part of the international research program on Inverse Problems.

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Schrödinger operators with strongly singular potentials and de Branges spaces

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Submitted to *J. Funct. Anal.*

Abstract. We utilize the theory of de Branges spaces to show that certain Schrödinger operators with strongly singular potentials are uniquely determined by their associated spectral measure. The results are applied in order to obtain a uniqueness theorem for perturbed Bessel operators.

1. Introduction

We consider Schrödinger operators H (with separate boundary conditions), associated with the differential expression

$$\tau = -\frac{d^2}{dx^2} + q(x)$$

on some interval (a, b), where $q \in L^1_{\mathrm{loc}}(a, b)$ is a real-valued potential. It has been shown by Kodaira [11], Kac [10] and more recently by Fulton [7], Gesztesy and Zinchenko [9], Fulton and Langer [8], Kurasov and Luger [15], and Kostenko, Sakhnovich and Teschl [12], [13], [14] that, even when the potential is quite singular at a, it is still possible to introduce a singular Weyl–Titchmarsh function as well as a scalar spectral measure. In fact, this only requires some nontrivial real entire solution ϕ of

$$-\phi''(z, x) + q(x)\phi(z, x) = z\phi(z, x), \quad x \in (a, b),\ z \in \mathbb{C},$$

which lies in $L^2(a, b)$ near a and satisfies the boundary condition at a if τ is in the limit-circle case there. Here, by a real entire solution we mean that the functions

$$z \mapsto \phi(z, c) \quad \text{and} \quad z \mapsto \phi'(z, c)$$

are real entire for one (and hence for all) $c \in (a, b)$. For example, if τ is in the limit-circle case at a, then such a solution is known to exist and a Weyl–Titchmarsh theory has been developed e.g. in [6], [3], analogously to the regular case. In general, for such a solution ϕ to exist it is necessary and sufficient that the operator H_c has purely discrete spectrum (see e.g. [9,

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2010 *Mathematics Subject Classification.* Prim. 34L05, 46E22; Sec. 34L40, 34B24.
Keywords. Schrödinger operators, de Branges spaces, strongly singular potentials.
Research supported by the *Austrian Science Fund (FWF)* under grant no. Y330.

Lemma 3.2], [14, Lemma 2.2]) for some $c \in (a, b)$, where H_c is the restriction of H to $L^2(a, c)$ with some boundary conditions at the point c.

Regarding inverse spectral theory, Kostenko, Sakhnovich and Teschl [14] were able to prove a local Borg–Marchenko uniqueness result for the singular Weyl–Titchmarsh function under restrictions on the exponential growth of solutions. Their proof follows the simple proof of Bennewitz [2], which covers the case of regular left endpoints. However, since the spectral measure determines the singular Weyl–Titchmarsh function only up to some real entire function, their Borg–Marchenko theorem does not immediately yield an inverse uniqueness result for the associated spectral measure. In fact, all one would need is some growth restriction on the difference of two singular Weyl–Titchmarsh functions, corresponding to the same spectral measure. Actually, this has been done in [5], in the case when the spectra of the operators are assumed to be purely discrete with finite convergence exponent.

The present paper pursues a different approach. We utilize de Branges' theory of Hilbert spaces of entire functions in order to obtain an inverse uniqueness theorem for the spectral measure. In particular, we apply de Branges' subspace ordering theorem to conclude that the de Branges spaces associated with Schrödinger operators with a common spectral measure are equal. Therefore, we will first provide a brief review of the theory of de Branges spaces in Section 2. For a detailed discussion we refer to de Branges' book [4]. The following section introduces de Branges spaces associated with a self-adjoint Schrödinger operator as above. The core of this section is quite similar to [19, Section 3] (see also [20]) with the only difference that we do not assume the left endpoint to be regular. Section 4 is devoted to our uniqueness theorem for the spectral measure. Finally, we apply our results to the case of perturbed Bessel (or spherical Schrödinger) operators.

As a last remark let us mention that our uniqueness result still holds when the potential is not a function but only a real-valued Borel measure on (a, b) with essentially the same proofs. Moreover, the approach taken here equally well applies to general Sturm–Liouville operators associated with the differential expression

$$\tau = \frac{1}{r(x)}\left(-\frac{d}{dx}p(x)\frac{d}{dx} + q(x)\right)$$

on some interval (a, b). In that case the associated operators are determined by the spectral measure only up to a so-called Liouville transform as in [1].

2. Hilbert spaces of entire functions

First of all recall that an analytic function N in the upper complex half-plane $\mathbb{C}^+$ is said to be of bounded type if it can be written as the quotient of two bounded analytic functions. For such a function the number

$$\operatorname{mt} N := \limsup_{y\to\infty} \frac{\ln |N(\mathrm{i}y)|}{y} \in [-\infty, \infty)$$

is referred to as the mean type of N.

A de Branges function is an entire function E, which satisfies the estimate

$$|E(z)| > |E(z^*)|, \quad z \in \mathbb{C}^+.$$

The de Branges space B associated with such a function consists of all entire functions F such that

$$\int_{\mathbb{R}} \frac{|F(\lambda)|^2}{|E(\lambda)|^2} d\lambda < \infty$$

and such that F/E and $F^{\#}/E$ are of bounded type in $\mathbb{C}^+$ with nonpositive mean type. Here $F^{\#}$ is the entire function given by

$$F^{\#}(z) = F(z^*)^*, \quad z \in \mathbb{C}.$$

Equipped with the inner product

$$[F, G] = \frac{1}{\pi} \int_{\mathbb{R}} \frac{F(\lambda)G(\lambda)^*}{|E(\lambda)|^2} d\lambda, \quad F,\, G \in B,$$

the vector space B turns into a Hilbert space (see [4, Theorem 21]). For each $\zeta \in \mathbb{C}$, point evaluation in ζ is a continuous linear functional on B, i.e.

$$F(\zeta) = [F, K(\zeta, \cdot)], \quad F \in B,$$

where the reproducing kernel K is given by (see [4, Theorem 19])

$$K(\zeta, z) = \frac{E(z)E^{\#}(\zeta^*) - E(\zeta^*)E^{\#}(z)}{2\mathrm{i}(\zeta^* - z)}, \quad \zeta,\, z \in \mathbb{C}. \tag{2.1}$$

Hereby note that though there is a multitude of de Branges functions giving rise to the same de Branges space (including norms), the reproducing kernel K is independent of the actual de Branges function.

Our uniqueness result relies on the subspace ordering theorem due to de Branges; [4, Theorem 35]. In order to state it let E_1, E_2 be two de Branges functions and B_1, B_2 be the corresponding de Branges spaces.

Theorem 2.1. *Suppose B_1, B_2 are isometrically embedded in $L^2(\mathbb{R}; \rho)$, for some Borel measure ρ on $\mathbb{R}$. If E_1/E_2 is of bounded type in the upper complex half-plane and has no real zeros or singularities, then B_1 contains B_2 or B_2 contains B_1.*

Moreover, one has the following simple converse statement.

Lemma 2.2. *If B_1 contains B_2 or B_2 contains B_1, then E_1/E_2 is of bounded type in the upper complex half-plane.*

Proof. For each $F \in B_1 \cap B_2$ the functions

$$\frac{F(z)}{E_1(z)} \quad \text{and} \quad \frac{F(z)}{E_2(z)}, \quad z \in \mathbb{C}^+$$

are of bounded type by definition, which proves the claim. □

3. Schrödinger operators and de Branges spaces

In this section let (a, b) be some bounded or unbounded interval, q a real-valued, locally integrable function on (a, b) and τ the differential expression

$$\tau = -\frac{d^2}{dx^2} + q(x)$$

on (a, b). With H we denote some associated self-adjoint Schrödinger operator in $L^2(a, b)$ with separate boundary conditions (if τ is in the limit-circle case at both endpoints). Concerning the regularity of the τ near the endpoint a, we will only assume that there is some real entire solution ϕ of

$$-\phi''(z,x) + q(x)\phi(z,x) = z\phi(z,x), \quad x \in (a,b),\ z \in \mathbb{C},$$

such that for each $z \in \mathbb{C}$, $\phi(z,\cdot\,)$ is not identically zero, lies in $L^2(a,b)$ near a and satisfies the boundary condition at a if τ is in the limit-circle case there. Here by real entire we mean that for some (and hence for all) $c \in (a,b)$ the functions

$$z \mapsto \phi(z,c) \quad \text{and} \quad z \mapsto \phi'(z,c)$$

are real entire. For the proof of our inverse uniqueness result we will need the following simple lemma on the high energy asymptotics of the solution ϕ. Note that we always use the principal square root with branch cut along the negative real axis.

Lemma 3.1. *For each x, $\tilde{x} \in (a,b)$ we have the asymptotics*

$$\frac{\phi(z,x)}{\phi(z,\tilde{x})} = \mathrm{e}^{(x-\tilde{x})\sqrt{-z}}\left(1 + o(1)\right), \tag{3.1}$$

as $|z| \to \infty$ along the imaginary axis.

Proof. For each $z \in \mathbb{C}$ let $c(z,\cdot\,)$ and $s(z,\cdot\,)$ be the solutions of $(\tau - z)u = 0$ with the initial conditions

$$c(z,\tilde{x}) = s'(z,\tilde{x}) = 1 \quad \text{and} \quad c'(z,\tilde{x}) = s(z,\tilde{x}) = 0.$$

Now if $x \geq \tilde{x}$ the claim follows from

$$\phi(z,x) = \phi(z,\tilde{x})\left(c(z,x) + \frac{\phi'(z,\tilde{x})}{\phi(z,\tilde{x})}s(z,x)\right), \quad z \in \mathbb{C}\backslash\mathbb{R}$$

and the well-known asymptotics of the quotient on the right-hand side (see [22, Lemma 9.19]) and the solutions c and s (see [22, Lemma 9.18]). The case when $x < \tilde{x}$ follows by reversing the roles of x and $\tilde{x}$. $\square$

For each $c \in (a,b)$ we denote with $L^2(a,c)$ the closed linear subspace of $L^2(a,b)$ consisting of all functions which vanish almost everywhere outside of (a,c). Now as in the case of regular left endpoints, one may define the transform of a function $f \in L^2(a,c)$ as

$$\hat{f}(z) = \int_a^b \phi(z,x)f(x)dx, \quad z \in \mathbb{C}. \tag{3.2}$$

Given this, it is known (see e.g. [9, Section 3], [14, Section 3]) that there is some Borel measure ρ on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} |\hat{f}(\lambda)|^2 d\rho(\lambda) = \int_a^b |f(x)|^2 dx, \quad f \in L^2(a,c) \tag{3.3}$$

holds for each $c \in (a,b)$. Moreover, this transformation uniquely extends to a unitary map from $L^2(a,b)$ onto $L^2(\mathbb{R};\rho)$ and the operator H is mapped onto multiplication with the independent variable in $L^2(\mathbb{R};\rho)$. Note that the measure ρ is uniquely determined by these properties and hence referred to as the spectral measure of H associated with the solution ϕ.

From these results one sees that the space of transforms of all functions $f \in L^2(a,c)$, equipped with the norm inherited from the space $L^2(\mathbb{R};\rho)$, forms a Hilbert space. In order to show that it is even a de Branges space, fix some $c \in (a,b)$ and consider the entire function

$$E(z,c) = \phi(z,c) + \mathrm{i}\phi'(z,c), \quad z \in \mathbb{C}. \tag{3.4}$$

Using the Lagrange identity and the fact that the Wronskian of two solutions satisfying the same boundary condition at a (if any) vanishes in a, one gets

$$\frac{E(z,c)E^{\#}(\zeta^*,c) - E(\zeta^*,c)E^{\#}(z,c)}{2\mathrm{i}(\zeta^* - z)} = \int_a^c \phi(\zeta,x)^*\phi(z,x)dx, \quad \zeta,\, z \in \mathbb{C}^+.$$

In particular, taking $\zeta = z$ this shows that $E(\,\cdot\,,c)$ is a de Branges function. Moreover, note that $E(\,\cdot\,,c)$ does not have any real zero λ, since otherwise both, $\phi(\lambda,c)$ and $\phi'(\lambda,c)$ would vanish. With $B(c)$ we denote the de Branges space associated with the de Branges function $E(\,\cdot\,,c)$ endowed with the inner product

$$[F,G]_{B(c)} = \frac{1}{\pi}\int_{\mathbb{R}} \frac{F(\lambda)G(\lambda)^*}{|E(\lambda,c)|^2}d\lambda = \frac{1}{\pi}\int_{\mathbb{R}} \frac{F(\lambda)G(\lambda)^*}{\phi(\lambda,c)^2 + \phi'(\lambda,c)^2}d\lambda$$

for F, $G \in B(c)$. Now using (2.1) and a similar calculation as above, one shows that the reproducing kernel $K(\,\cdot\,,\,\cdot\,,c)$ of this space is given by

$$K(\zeta,z,c) = \int_a^c \phi(\zeta,x)^*\phi(z,x)dx, \quad \zeta,\, z \in \mathbb{C}. \tag{3.5}$$

Theorem 3.2. *For every $c \in (a,b)$ the transformation $f \mapsto \hat{f}$ is unitary from $L^2(a,c)$ onto $B(c)$, in particular*

$$B(c) = \big\{\hat{f} \,\big|\, f \in L^2(a,c)\big\}. \tag{3.6}$$

Proof. For each $\lambda \in \mathbb{R}$ consider the function

$$f_\lambda(x) = \begin{cases} \phi(\lambda,x), & \text{if } x \in (a,c], \\ 0, & \text{if } x \in (c,b). \end{cases}$$

The transforms of these functions are given by

$$\hat{f}_\lambda(z) = \int_a^c \phi(\lambda,x)\phi(z,x)dx = K(\lambda,z,c), \quad z \in \mathbb{C}.$$

In particular, this shows that the transforms of the functions f_λ, $\lambda \in \mathbb{R}$ lie in $B(c)$. Moreover, we have for all λ_1, $\lambda_2 \in \mathbb{R}$

$$\begin{aligned}\langle f_{\lambda_1}, f_{\lambda_2}\rangle &= \int_a^c \phi(\lambda_1,x)\phi(\lambda_2,x)dx = K(\lambda_1,\lambda_2,c) \\ &= [K(\lambda_1,\,\cdot\,,c), K(\lambda_2,\,\cdot\,,c)]_{B(c)}.\end{aligned}$$

Hence our transform is an isometry on the linear span D of all functions f_λ, $\lambda \in \mathbb{R}$. But this span is dense in $L^2(a,c)$ since it contains the eigenfunctions of the operator H_c. Moreover, the linear span of all transforms $K(\lambda,\,\cdot\,,c)$, $\lambda \in \mathbb{R}$ is dense in $B(c)$. Indeed, each $F \in B(c)$ such that

$$0 = [F, K(\lambda,\,\cdot\,,c)]_{B(c)} = F(\lambda), \quad \lambda \in \mathbb{R}$$

vanishes identically. Thus our transformation restricted to D uniquely extends to a unitary map V from $L^2(a,c)$ onto $B(c)$. In order to identify V with our transformation note that for each fixed $z \in \mathbb{C}$, both $f \mapsto \hat{f}(z)$ and $f \mapsto Vf(z)$ are continuous on $L^2(a,c)$. $\square$

As an immediate consequence of Theorem 3.2 and the fact that our transformation from (3.2) extends to a unitary map from $L^2(a,b)$ onto $L^2(\mathbb{R};\rho)$, we get the following corollary.

Corollary 3.3. *For each $c \in (a,b)$ the de Branges space $B(c)$ is isometrically embedded in $L^2(\mathbb{R};\rho)$, that is*

$$\int_{\mathbb{R}} |F(\lambda)|^2 d\rho(\lambda) = \|F\|^2_{B(c)}, \quad F \in B(c). \tag{3.7}$$

Moreover, the union of the spaces $B(c)$, $c \in (a,b)$ is dense in $L^2(\mathbb{R};\rho)$, i.e.

$$\overline{\bigcup_{c\in(a,b)} B(c)} = L^2(\mathbb{R};\rho). \tag{3.8}$$

The following corollary shows that the de Branges spaces $B(c)$, $c \in (a,b)$ are totally ordered, strictly increasing and continuous in some sense.

Corollary 3.4. *If c_1, $c_2 \in (a,b)$ with $c_1 < c_2$, then $B(c_1)$ is isometrically embedded in, but not equal to $B(c_2)$. Moreover, for each $c \in (a,b)$ we have*

$$\overline{\bigcup_{x\in(a,c)} B(x)} = B(c) = \bigcap_{x\in(c,b)} B(x). \tag{3.9}$$

Proof. The embedding is clear from Theorem 3.2 and Corollary 3.3. Moreover, Theorem 3.2 shows that $B(c_2) \ominus B(c_1)$ is unitarily equivalent to the space $L^2(c_1,c_2)$, hence $B(c_1)$ is not equal to $B(c_2)$. The second claim follows from the similar fact that

$$\overline{\bigcup_{x\in(a,c)} L^2(a,x)} = L^2(a,c) = \bigcap_{x\in(c,b)} L^2(a,x).$$

□

As a final remark, note that the solution ϕ is not uniquely determined. In fact, [14, Corollary 2.3] shows that any other solution with the same properties as ϕ is given by

$$\tilde{\phi}(z,x) = \mathrm{e}^{g(z)}\phi(z,x), \quad x \in (a,b),\ z \in \mathbb{C},$$

where g is some real entire function. Furthermore [14, Remark 3.8] shows that the corresponding spectral measures are related by

$$\tilde{\rho} = \mathrm{e}^{-2g}\rho.$$

In particular they are mutually absolutely continuous. Using Theorem 3.2 it is easily seen that for each $c \in (a,b)$, multiplication with the entire function e^g maps $B(c)$ isometrically onto the corresponding de Branges space $\tilde{B}(c)$.

4. An inverse uniqueness result

In this section we will prove our inverse uniqueness result. Therefore let q_1, q_2 be two real-valued, locally integrable functions on intervals (a_1,b_1) respectively (a_2,b_2) and H_1, H_2 two associated self-adjoint Schrödinger operators with separate boundary conditions. Suppose there are nontrivial real entire solutions ϕ_1, ϕ_2 which are square integrable near the left endpoint and satisfy the boundary condition there, if any. As in the previous section we denote with ρ_1, ρ_2 the corresponding spectral measures, with E_1, E_2 the corresponding de Branges functions, with B_1, B_2 the corresponding de Branges spaces and with K_1, K_2 the corresponding reproducing kernels. We say H_1 and H_2 are equal up to some shift if there is a linear function η with $\eta' = 1$, mapping (a_1, b_1) onto (a_2, b_2) such that $q_1 = q_2 \circ \eta$ and

$$H_1 = U^{-1} H_2 U,$$

where U is the unitary map from $L^2(a_1, b_1)$ onto $L^2(a_2, b_2)$ induced by η.

Theorem 4.1. *Suppose there is some real entire function g such that*

$$\mathrm{e}^{g(z)} \frac{E_1(z, x_1)}{E_2(z, x_2)}, \quad z \in \mathbb{C}^+ \tag{4.1}$$

is of bounded type for some $x_1 \in (a_1, b_1)$ and $x_2 \in (a_2, b_2)$. If $\rho_1 = \mathrm{e}^{-2g} \rho_2$, then H_1 and H_2 are equal up to some shift.

Proof. First of all note that without loss of generality we may assume that g vanishes identically, since otherwise we replace ϕ_1 with $\mathrm{e}^g \phi_1$. Moreover, because of Lemma 2.2 the function in (4.1) is of bounded type for all points $x_1 \in (a_1, b_1)$ and $x_2 \in (a_2, b_2)$. Now fix some arbitrary $x_1 \in (a_1, b_1)$. Since for each $x_2 \in (a_2, b_2)$, both $B_1(x_1)$ and $B_2(x_2)$ are isometrically contained in $L^2(\mathbb{R}; \rho_1)$, we infer from Theorem 2.1 (note that (4.1) has no real zeros or singularities because $E_1(\,\cdot\,, x_1)$ and $E_2(\,\cdot\,, x_2)$ do not have real zeros) that $B_1(x_1)$ is contained in $B_2(x_2)$ or $B_2(x_2)$ is contained in $B_1(x_1)$. We claim that the infimum $\eta(x_1)$ of all $x_2 \in (a_2, b_2)$ such that $B_1(x_1) \subseteq B_2(x_2)$ lies in (a_2, b_2). Indeed, otherwise we either had $B_2(x_2) \subseteq B_1(x_1)$ for all $x_2 \in (a_2, b_2)$ or $B_1(x_1) \subseteq B_2(x_2)$ for all $x_2 \in (a_2, b_2)$. In the first case this would mean that $B_1(x_1)$ is dense in $L^2(\mathbb{R}; \rho)$, which is not possible in view of Corollary 3.4. The second case would imply that for every function $F \in B_1(x_1)$ and $\zeta \in \mathbb{C}$ we have

$$\begin{aligned} |F(\zeta)| &\leq [F, K_2(\zeta, \cdot\,, x_2)]_{B_2(x_2)} \\ &\leq \|F\|_{B_2(x_2)} [K_2(\zeta, \cdot\,, x_2), K_2(\zeta, \cdot\,, x_2)]_{B_2(x_2)} \\ &= \|F\|_{B_1(x_1)} K_2(\zeta, \zeta, x_2) \end{aligned}$$

for each $x_2 \in (a_2, b_2)$. But since $K_2(\zeta, \zeta, x_2) \to 0$ as $x_2 \to a_2$ by (3.5), we then had $B_1(x_1) = \{0\}$, contradicting Theorem 3.2. Now from (3.9) we infer that

$$B_2(\eta(x_1)) = \overline{\bigcup_{x_2 < \eta(x_1)} B_2(x_2)} \subseteq B_1(x_1) \subseteq \bigcap_{x_2 > \eta(x_1)} B_2(x_2) = B_2(\eta(x_1))$$

and hence $B_1(x_1) = B_2(\eta(x_1))$, including norms.

The function $\eta : (a_1, b_1) \to (a_2, b_2)$ defined this way is strictly increasing because of Corollary 3.4 and continuous by (3.9). Moreover, since for each $\zeta \in \mathbb{C}$ we have

$$K_2(\zeta, \zeta, \eta(x_1)) = K_1(\zeta, \zeta, x_1) \to 0,$$

as $x_1 \to a$, we infer that $\eta(x_1) \to a_2$ as $x_1 \to a_1$. Finally, (3.8) shows that η actually has to be a bijection. Using the equation for the reproducing kernels (3.5) once more we get for each $z\in\mathbb{C}$

$$\int_{a_1}^{x_1} |\phi_1(z,x)|^2 dx = \int_{a_2}^{\eta(x_1)} |\phi_2(z,x)|^2 dx, \quad x_1\in(a_1,b_1).$$

Now by the implicit function theorem η is differentiable (note that the integrand does not vanish if $z\in\mathbb{C}\backslash\mathbb{R}$) with

$$(*) \qquad |\phi_1(z,x_1)|^2 = \eta'(x_1)|\phi_2(z,\eta(x_1))|^2, \quad x_1\in(a_1,b_1).$$

Using Lemma 3.1 twice we get for all x_1, $\tilde{x}_1\in(a_1,b_1)$ the asymptotics

$$\begin{aligned} \mathrm{e}^{2(x_1-\tilde{x}_1)\mathrm{Re}\sqrt{-z}}\,(1+o(1)) = \left|\frac{\phi_1(z,x_1)}{\phi_1(z,\tilde{x}_1)}\right|^2 &= \frac{\eta'(x_1)}{\eta'(\tilde{x}_1)}\left|\frac{\phi_2(z,\eta(x_1))}{\phi_2(z,\eta(\tilde{x}_1))}\right|^2 \\ &= \frac{\eta'(x_1)}{\eta'(\tilde{x}_1)}\mathrm{e}^{2(\eta(x_1)-\eta(\tilde{x}_1))\mathrm{Re}\sqrt{-z}}\,(1+o(1))\,, \end{aligned}$$

as $|z|\to\infty$ along the imaginary axis. But this shows

$$\eta(x_1)-\eta(\tilde{x}_1) = x_1-\tilde{x}_1, \quad x_1,\,\tilde{x}_1\in(a_1,b_1),$$

i.e. η is linear with gradient one.

Using $(*)$ once more, we get for each $\lambda\in\mathbb{R}$

$$\phi_1(\lambda,x_1)^2 = \phi_2(\lambda,\eta(x_1))^2, \quad x_1\in(a_1,b_1).$$

Taking logarithmic derivatives we obtain

$$(**) \qquad \frac{\phi_1'(\lambda,x_1)}{\phi_1(\lambda,x_1)} = \frac{\phi_2'(\lambda,\eta(x_1))}{\phi_2(\lambda,\eta(x_1))}$$

for almost all $x_1\in(a_1,b_1)$. Differentiating this equation once more, we get

$$\frac{\phi_1''(\lambda,x_1)}{\phi_1(\lambda,x_1)} = \frac{\phi_2''(\lambda,\eta(x_1))}{\phi_2(\lambda,\eta(x_1))}$$

for almost all $x_1\in(a_1,b_1)$ and thus also

$$q_1(x_1) = \lambda + \frac{\phi_1''(\lambda,x_1)}{\phi_1(\lambda,x_1)} = \lambda + \frac{\phi_2''(\lambda,\eta(x_1))}{\phi_2(\lambda,\eta(x_1))} = q_2(\eta(x_1))$$

for almost all $x_1\in(a_1,b_1)$.

Finally note that $(**)$ implies that $\phi_1(\lambda,\cdot\,)$ and $\phi_2(\lambda,\eta(\cdot))$ are linearly dependent for each $\lambda\in\mathbb{R}$. In particular, if τ_1 (and hence also τ_2) is in the limit-circle case at the left endpoint this shows that the boundary conditions of H_1 and H_2 are the same there. Furthermore, if τ_1 (and hence also τ_2) is in the limit-circle case at the right endpoint, then H_1 and H_2 have some common eigenvalue $\lambda\in\mathbb{R}$. Now the fact that $\phi_1(\lambda,\cdot\,)$ and $\phi_2(\lambda,\cdot\,)$ satisfy the respective boundary condition at the right endpoint shows that H_1 is equal to H_2 up to some shift. $\square$

Note that even if one fixes the left endpoint, the operator is determined by the spectral measure in general only up to some shift. This is due to the fact that we allowed the left endpoint to possibly be infinite. In fact, if one takes finite fixed left endpoints, the operators are uniquely determined by the spectral measure.

Corollary 4.2. *Suppose that $-\infty < a_1 = a_2$ and that there is some real entire function g such that*

$$\mathrm{e}^{g(z)} \frac{E_1(z, x_1)}{E_2(z, x_2)}, \quad z \in \mathbb{C}^+$$

is of bounded type for some $x_1 \in (a_1, b_1)$ and $x_2 \in (a_2, b_2)$. If $\rho_1 = \mathrm{e}^{-2g} \rho_2$, then $b_1 = b_2$, $q_1 = q_2$ and $H_1 = H_2$.

Proof. This follows from Theorem 4.1 and $\lim_{x_1 \to a_1} \eta(x_1) = a_1$. $\square$

Below we will see that some kind of growth restriction on the solutions ϕ_1 and ϕ_2 suffices to guarantee that (4.1) is of bounded type. However, note that this condition in Theorem 4.1 can not be dropped and that some assumption has to be imposed on the solutions ϕ_1 and ϕ_2. As an example consider the interval $(0, \pi)$, the potential $q_1 = 0$ and let H_1 be the associated Schrödinger operator with Dirichlet boundary conditions. As our real entire solution ϕ_1 we choose

$$\phi_1(z, x) = \frac{\sin \sqrt{z} x}{\sqrt{z}}, \quad x \in (0, \pi), \ z \in \mathbb{C}.$$

The associated spectral measure ρ_1 is given by

$$\rho_1 = \frac{2}{\pi} \sum_{n \in \mathbb{N}} n^2 \delta_{n^2},$$

where for each $n \in \mathbb{N}$, δ_{n^2} is the Dirac measure in the point n^2. Now choose some sequence κ_n, $n \in \mathbb{N}$ of positive reals such that all but finitely many of these numbers are equal to one. From the solution of the inverse spectral problem in the regular case it is known (see e.g. [17], [18]) that there is some potential $q_2 \in L^2(0, \pi)$ and a corresponding operator H_2 with Dirichlet boundary conditions such that the spectral measure ρ_2 associated with the real entire solution ϕ_2 of

$$-\phi_2''(z, x) + q_2(x) \phi_2(z, x) = z \phi_2(z, x), \quad x \in (0, \pi), \ z \in \mathbb{C}$$

with the initial conditions

$$\phi_2(z, 0) = 0 \quad \text{and} \quad \phi_2'(z, 0) = 1, \quad z \in \mathbb{C}$$

is given by

$$\rho_2 = \frac{2}{\pi} \sum_{n \in \mathbb{N}} \kappa_n n^2 \delta_{n^2}.$$

Now pick some real entire function g such that

$$g(n^2) = \frac{\ln \kappa_n}{2}, \quad n \in \mathbb{N}$$

and switch to the real entire solution

$$\tilde{\phi}_2(z, x) = \mathrm{e}^{g(z)} \phi_2(z, x), \quad x \in (0, \pi), \ z \in \mathbb{C}.$$

Then the spectral measure associated with this solution is equal to ρ_1, but the corresponding operators H_1 and H_2 are different (at least if not all κ_n are equal to one). However, also note that in this case (4.1) seems to fail to be of bounded type rather badly, since the function e^g is not even of finite exponential order.

We conclude this section by showing that condition (4.1) in Theorem 4.1 holds if the solutions ϕ_1, ϕ_2 satisfy some growth condition. Therefore recall that an entire function F belongs to the Cartwright class $\mathcal{C}$ if it is of finite exponential type and the logarithmic integral

$$\int_{\mathbb{R}} \frac{\ln^+ |F(\lambda)|}{1+\lambda^2} d\lambda < \infty$$

exists, where $\ln^+$ is the positive part of the natural logarithm. In particular note that the class $\mathcal{C}$ contains all entire functions of exponential order less than one. Now a theorem of Kreĭn [21, Theorem 6.17], [16, Section 16.1] states that the class $\mathcal{C}$ consists of all entire functions which are of bounded type in the upper and in the lower complex half-plane. Since the quotient of two functions of bounded type is of bounded type itself, this immediately yields the following uniqueness result.

Corollary 4.3. *Suppose that $E_1(\,\cdot\,, x_1)$ and $E_2(\,\cdot\,, x_2)$ belong to the Cartwright class $\mathcal{C}$ for some $x_1 \in (a_1, b_1)$ and $x_2 \in (a_2, b_2)$. If $\rho_1 = \rho_2$, then H_1 and H_2 are equal up to some shift.*

Again, as in Corollary 4.2, if one takes finite fixed left endpoints, the operator is uniquely determined by the spectral measure. In particular, as a special case one recovers the classical result due to Marchenko that the spectral measure uniquely determines the operator, if the left endpoint is regular. However, our result covers a larger class of potentials, as we will show in the next section. There we will apply our results in order to obtain a uniqueness theorem for perturbed Bessel operators.

5. Application to perturbed Bessel operators

Consider the differential expression

$$\tau = -\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2} + q(x)$$

on some interval $(0, b)$, where $l \in [-\frac{1}{2}, \infty)$ and q is some real-valued, locally integrable function on $(0, b)$. We will assume that the function

$$\overline{q}(x) = \begin{cases} |q(x)|x, & \text{if } l > -\frac{1}{2}, \\ |q(x)|x(1-\ln x), & \text{if } l = -\frac{1}{2}, \end{cases} \tag{5.1}$$

is integrable near zero. According to [13, Theorem 2.4], τ is in the limit-circle case at zero if and only if $l \in [-\frac{1}{2}, \frac{1}{2})$. With H we denote some associated self-adjoint operator with the boundary condition

$$\lim_{x\to 0} x^l((l+1)f(x) - xf'(x)) = 0 \tag{5.2}$$

at zero, if necessary. In [13, Lemma 2.2] it has been shown that under these assumptions there is a nontrivial real entire solution ϕ of exponential order one half which lies in $L^2(0, b)$ near zero and satisfies the boundary condition (5.2) there if $l \in [-\frac{1}{2}, \frac{1}{2})$. Note that this solution is unique up to scalar multiples because of the growth restriction, as it has been shown in [14, Lemma 6.4]. Consequently, also the spectral measure ρ associated with this solution ϕ is unique up to a scalar multiple.

In order to state our inverse uniqueness theorem let l_1, $l_2 \in [-\frac{1}{2}, \infty)$ and q_1, q_2 be two potentials on intervals $(0, b_1)$ respectively $(0, b_2)$, such that the functions $\overline{q}_1$, $\overline{q}_2$ defined as in (5.1) are integrable near zero. Furthermore, let H_1, H_2 be two corresponding self-adjoint operators with the boundary condition (5.2) at zero, if necessary. With ϕ_1, ϕ_2 we denote some real entire solutions of exponential order one half which are square integrable near zero and satisfy the boundary condition there, if any. Finally, let ρ_1, ρ_2 be the associated spectral measures. Our uniqueness results from the preceding section now yield the following theorem.

Theorem 5.1. *If $\rho_1 = \rho_2$, then $l_1 = l_2$, $b_1 = b_2$, $q_1 = q_2$ and $H_1 = H_2$.*

Proof. Since the solutions are of exponential order one half, we may immediately apply Corollary 4.3 and obtain $b_1 = b_2$,

$$\frac{l_1(l_1+1)}{x^2} + q_1(x) = \frac{l_2(l_2+1)}{x^2} + q_2(x)$$

for almost all $x \in (0, b_1)$ and $H_1 = H_2$. Now since the function

$$xq_1(x) - xq_2(x) = \frac{l_2(l_2+1) - l_1(l_1+1)}{x}, \quad x \in (0, b_1)$$

is integrable near zero we infer $l_1(l_1+1) = l_2(l_2+1)$ and hence $l_1 = l_2$. □

Acknowledgments. I thank Fritz Gesztesy, Gerald Teschl and Harald Woracek for helpful discussions and hints with respect to the literature.

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Direct and inverse spectral theory of singular left-definite Sturm–Liouville operators

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Submitted to *J. Differential Equations*

Abstract. We give a comprehensive treatment of Sturm–Liouville operators in the left-definite setting. In particular, we describe all self-adjoint realizations in a modified Sobolev space and develop Weyl–Titchmarsh theory for these operators (with separate boundary conditions and strongly singular coefficients). Finally, we apply de Branges' subspace ordering theorem to obtain inverse uniqueness results for the associated spectral measure. The results can be applied to the inverse spectral problem associated with the Camassa–Holm equation.

1. Introduction

Consider the left-definite Sturm–Liouville problem

$$-\frac{d}{dx}\left(p(x)\frac{dy}{dx}(x)\right) + q(x)y(x) = zr(x)y(x) \tag{1.1}$$

on some interval (a,b). Here, by left-definite we mean that the real-valued function r is allowed to change sign but p and q are assumed to be non-negative. In the case of a regular left endpoint, Bennewitz [5], Brown and Weikard [7] recently developed Weyl–Titchmarsh theory for such problems, analogously to the right-definite case. Moreover, they were also able to prove that the associated spectral measure uniquely determines the left-definite Sturm–Liouville problem up to a so-called Liouville transform.

In the present paper we give an alternative proof of this result, using de Branges' subspace ordering theorem for Hilbert spaces of entire functions. In fact, this approach allows us to deal with a larger class of problems. For instance, we allow the left endpoint to be quite singular and the weight function r to be a real-valued Borel measure. However, at a second glance our approach is not too different from the approach taken in [5] and [7]. The authors there prove Paley–Wiener type results to describe the spectral transforms of functions with compact support in order to obtain an appropriate Liouville transform. We will show that these spaces of transforms are

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2010 *Mathematics Subject Classification.* Prim. 34L05, 34B20; Sec. 46E22, 34B40.
Keywords. Sturm–Liouville theory, left-definite problems, spectral theory.
Research supported by the *Austrian Science Fund (FWF)* under grant no. Y330.

actually hyperplanes in some de Branges spaces associated with our left-definite Sturm–Liouville problem. This will allow us to apply de Branges' subspace ordering theorem to obtain a suitable Liouville transform.

As in [5] and [7], the main motivation for this work is the Camassa–Holm equation, an integrable, non-linear wave equation which models unidirectional propagation of waves on shallow water. Due to its many remarkable properties, this equation has gotten a lot of attention recently and we only refer to e.g. [8], [9], [10], [11], [23] for further information. Associated with the Camassa–Holm equation is the left-definite Sturm–Liouville problem

$$-y''(x) + \frac{1}{4}y(x) = z\omega(x)y(x) \tag{1.2}$$

on the real line. Direct, and in particular inverse spectral theory of this weighted Sturm–Liouville problem are of peculiar interest for solving the Camassa–Holm equation. Provided ω is strictly positive and smooth enough, it is possible to transform this problem into a Sturm–Liouville problem in potential form and some inverse spectral theory may be drawn from this. However, in order to incorporate the main interesting phenomena (wave breaking [10] and multi-peakon solutions [2], [12]) of the dispersionless Camassa–Holm equation, it is necessary to allow ω at least to be an arbitrary finite signed Borel measure on $\mathbb{R}$. In [5], [7] the authors were able to prove an inverse uniqueness result under some restrictions on the measure ω, which for example prohibits the case of multi-peakon solutions of the Camassa–Holm equation. Using the results of the present paper we are able to avoid these restrictions and to cover the case of arbitrary real finite measures ω; see [17].

As mentioned above, the application we have in mind requires us to consider our Sturm–Liouville problem (1.1) with measure coefficients. For further information on measure Sturm–Liouville equations see e.g. [4] or [18] and the references therein. Moreover, the fact that we allow the weight measure to vanish on arbitrary sets, makes it necessary to work with linear relations instead of operators. Regarding the notion of linear relations, we refer to e.g. [1], [13], [15], [16], [21] or for a brief review [18, Appendix B].

The paper is organized as follows. After some preliminaries about measure Sturm–Liouville equations, we give a comprehensive treatment of associated linear relations in a modified Sobolev space. More precisely, this includes a description of all self-adjoint realizations of left-definite Sturm–Liouville problems in this Hilbert space as well as their resolvents in Section 4. In the following two sections we develop Weyl–Titchmarsh theory for such self-adjoint realization with separate boundary conditions. This part is modeled after the singular Weyl–Titchmarsh theory, recently introduced in [20] and [25] for Schrödinger operators. In Section 7 we introduce some de Branges spaces associated with a left-definite self-adjoint Sturm–Liouville problem. Moreover, we provide some crucial properties of these spaces, which are needed for the proof of our inverse uniqueness result. This proof is carried out in the last section. In particular, this section provides an inverse uniqueness result, which applies to the isospectral problem of the Camassa–Holm equation. Finally, in the appendix we give a brief review of de Branges' theory of Hilbert spaces of entire functions as far as it is needed for the proof of our inverse uniqueness result.

Before we start, let us recall some facts about functions which are absolutely continuous with respect to some measure. Therefore let (a,b) be an arbitrary interval and μ be a complex-valued Borel measure on (a,b). With $AC_{\mathrm{loc}}((a,b);\mu)$ we denote the set of all left-continuous functions, which are locally absolutely continuous with respect to the measure μ. These are precisely the functions f which can be written in the form

$$f(x) = f(c) + \int_c^x h(s)d\mu(s), \quad x \in (a,b)$$

for some $h \in L^1_{\mathrm{loc}}((a,b);\mu)$, where the integral has to be read as

$$\int_c^x h(s)d\mu(s) = \begin{cases} \int_{[c,x)} h(s)d\mu(s), & \text{if } x > c, \\ 0, & \text{if } x = c, \\ -\int_{[x,c)} h(s)d\mu(s), & \text{if } x < c. \end{cases}$$

The function h is the Radon–Nikodým derivative of f with respect to μ. It is uniquely defined in $L^1_{\mathrm{loc}}((a,b);\mu)$ and we write

$$\frac{df}{d\mu} = h.$$

Every function f which is locally absolutely continuous with respect to μ is locally of bounded variation and hence also its right-hand limits

$$f(x+) = \lim_{\varepsilon \downarrow 0} f(x+\varepsilon), \quad x \in (a,b)$$

of f exist everywhere. Furthermore, note that such a function can only be discontinuous in points of mass of μ.

In this paper we will repeatedly use the following integration by parts formula for complex-valued Borel measures μ, ν on (a,b) ([22, Theorem 21.67])

$$\int_\alpha^\beta F(x)d\nu(x) = FG|_\alpha^\beta - \int_\alpha^\beta G(x+)d\mu(x), \quad \alpha,\, \beta \in (a,b), \tag{1.3}$$

where F, G are left-continuous distribution functions of μ, ν respectively.

2. Measure Sturm–Liouville equations

Let (a,b) be an arbitrary interval and ϱ, ς and χ be complex-valued Borel measures on (a,b). We are going to define a linear differential expression τ which is informally given by

$$\tau f = \frac{d}{d\varrho}\left(-\frac{df}{d\varsigma} + \int f d\chi\right).$$

In the rest of this paper we will always assume that our measures satisfy the following four properties.

Hypothesis 2.1.

(i) *The measure ϱ is real-valued.*
(ii) *The measure ς is positive and supported on the whole interval.*
(iii) *The measure χ is positive but not identically zero.*
(iv) *The measure ς has no point masses in common with χ or ϱ, i.e.*

$$\varsigma(\{x\})\chi(\{x\}) = \varsigma(\{x\})\varrho(\{x\}) = 0, \quad x \in (a,b).$$

The maximal domain $\mathfrak{D}_\tau$ of functions such that the expression τf makes sense consists of all functions $f \in AC_{\mathrm{loc}}((a,b);\varsigma)$ for which the function

$$-\frac{df}{d\varsigma}(x) + \int_c^x f d\chi, \quad x \in (a,b) \tag{2.1}$$

is locally absolutely continuous with respect to ϱ, i.e. there is some representative of this function lying in $AC_{\mathrm{loc}}((a,b);\varrho)$. As a consequence of the assumption on the support of ς, this representative is unique. We then set $\tau f \in L^1_{\mathrm{loc}}((a,b);\varrho)$ to be the Radon–Nikodým derivative of this function with respect to ϱ. One easily sees that this definition is independent of $c \in (a,b)$ since the corresponding functions (2.1) as well as their unique representatives only differ by an additive constant. As usual, the Radon–Nikodým derivative with respect to ς of some $f \in \mathfrak{D}_\tau$ is denoted with

$$f^{[1]} = \frac{df}{d\varsigma} \in L^1_{\mathrm{loc}}((a,b);\varsigma) \tag{2.2}$$

and referred to as the quasi-derivative of f.

It is easily seen that this definition of τ is consistent with classical theory. Indeed, if ϱ, ς and χ are locally absolutely continuous with respect to the Lebesgue measure with densities r, p^{-1} and q respectively, then τ becomes

$$\tau f(x) = \frac{1}{r(x)}\left(-\frac{d}{dx}\left(p(x)\frac{df}{dx}(x)\right) + q(x)f(x)\right), \quad x \in (a,b).$$

The following existence and uniqueness theorem for solutions of measure Sturm–Liouville equations may be found in [18, Theorem 3.1].

Theorem 2.2. *For each $g \in L^1_{\mathrm{loc}}((a,b);\varrho)$, $c \in (a,b)$, d_1, $d_2 \in \mathbb{C}$ and $z \in \mathbb{C}$ there is a unique solution of the initial value problem*

$$(\tau - z)f = g \quad \text{with} \quad f(c) = d_1 \quad \text{and} \quad f^{[1]}(c) = d_2.$$

We say that τ is regular at an endpoint if the measures ϱ, ς and χ are finite near this endpoint. In this case, if g is integrable near this endpoint, then each solution of the equation $(\tau - z)f = g$ may be continuously extended to this endpoint. Moreover, the initial point c in Theorem 2.2 may be chosen as this endpoint (see e.g. [18, Theorem 3.5]).

Associated with our differential expression τ is a linear relation T_{loc} in the space $AC_{\mathrm{loc}}((a,b);\varsigma)$ defined by

$$T_{\mathrm{loc}} = \{(f, f_\tau) \in AC_{\mathrm{loc}}((a,b);\varsigma)^2 \,|\, f \in \mathfrak{D}_\tau,\ \tau f = f_\tau \text{ in } L^1_{\mathrm{loc}}((a,b);\varrho)\}.$$

Regarding notation we will make the following convention. Given some pair $f \in T_{\mathrm{loc}}$ we will denote its first component also with f and the second component with f_τ. Moreover, if $g \in AC_{\mathrm{loc}}((a,b);\varsigma)$ and f is a solution of the equation $(\tau - z)f = g$ for some $z \in \mathbb{C}$ then this solution f will often be identified with the pair $(f, g + zf) \in T_{\mathrm{loc}}$.

In the right-definite theory, a crucial role is played by the Wronskian of two functions and the associated Lagrange identity. The corresponding quantity in the left-definite case is the function

$$V(f,g^*)(x) = f_\tau(x) g^{[1]}(x)^* - f^{[1]}(x) g_\tau(x)^*, \quad x \in (a,b), \tag{2.3}$$

defined for each pair f, $g \in T_{\mathrm{loc}}$. For this modified Wronskian the following Lagrange identity holds.

Proposition 2.3. *For each f, $g \in T_{\mathrm{loc}}$ and α, $\beta \in (a,b)$ we have*

$$V(f,g^*)(\beta) - V(f,g^*)(\alpha) = \int_\alpha^\beta f_\tau(x)g(x)^* - f(x)g_\tau(x)^* d\chi(x) + \int_\alpha^\beta f_\tau^{[1]}(x)g^{[1]}(x)^* - f^{[1]}(x)g_\tau^{[1]}(x)^* d\varsigma(x).$$

Proof. For each $h \in AC_{\mathrm{loc}}((a,b);\varsigma)$ an integration by parts shows that

$$\int_\alpha^\beta fh^* d\chi + \int_\alpha^\beta f^{[1]}h^{[1]*}d\varsigma = \int_\alpha^\beta f_\tau h^* d\varrho + [f^{[1]}g^*]_\alpha^\beta \tag{2.4}$$

holds. Hereby note that the right-hand limit appearing in the integration by parts formula (1.3) may be omitted because of assumption (iv) in Hypothesis 2.1. In particular, choosing $h = g_\tau$ and subtracting the corresponding equation with the roles of f and g reversed yields the claim. □

As a consequence of the Lagrange identity one sees that for each $z \in \mathbb{C}$ the modified Wronskian $V(u_1,u_2)$ of two solutions u_1, u_2 of $(\tau - z)u = 0$ is constant. Furthermore, for $z \neq 0$ we have

$$V(u_1,u_2) \neq 0 \quad \Leftrightarrow \quad u_1,\ u_2 \text{ linearly independent.}$$

Indeed, the modified Wronskian of two linearly dependent solutions vanishes obviously. Conversely, $V(u_1,u_2) = 0$ implies that for each $c \in (a,b)$ there is a $K \in \mathbb{C}$ such that

$$Ku_1(c) = u_2(c) \quad \text{and} \quad Ku_1^{[1]}(c) = u_2^{[1]}(c),$$

where, without loss of generality we assumed that u_1 is a non-zero solution. Now uniqueness of solutions of the initial value problem implies the linear dependence of u_1 and u_2. Another useful identity for the modified Wronskian is the following Plücker identity.

Proposition 2.4. *For every f_1, f_2, f_3, $f_4 \in T_{\mathrm{loc}}$ we have*

$$0 = V(f_1,f_2)V(f_3,f_4) + V(f_1,f_3)V(f_4,f_2) + V(f_1,f_4)V(f_2,f_3).$$

Proof. The right-hand side is equal to the determinant

$$\frac{1}{2}\begin{vmatrix} f_{1,\tau} & f_{2,\tau} & f_{3,\tau} & f_{4,\tau} \\ f_1^{[1]} & f_2^{[1]} & f_3^{[1]} & f_4^{[1]} \\ f_{1,\tau} & f_{2,\tau} & f_{3,\tau} & f_{4,\tau} \\ f_1^{[1]} & f_2^{[1]} & f_3^{[1]} & f_4^{[1]} \end{vmatrix}.$$

□

3. Left-definite Sturm–Liouville relations

In order to obtain a linear relation in a Hilbert space we introduce a modified Sobolev space $H^1(a,b)$. It consists of all functions f on (a,b) which are locally absolutely continuous with respect to ς such that f is square integrable with respect to χ and the Radon–Nikodým derivative $df/d\varsigma$ is square integrable with respect to ς. The space $H^1(a,b)$ is equipped with the inner product

$$\langle f,g\rangle = \int_a^b f(x)g(x)^* d\chi(x) + \int_a^b f^{[1]}(x)g^{[1]}(x)^* d\varsigma(x), \quad f,\ g \in H^1(a,b).$$

Hereby note that f and g are always continuous in points of mass of χ in virtue of property (iv) in Hypothesis 2.1.

Proposition 3.1. *The space $H^1(a,b)$ is a reproducing kernel Hilbert space.*

Proof. Because of our assumptions on the measures ς and χ it remains to prove completeness in order to see that $H^1(a,b)$ is a Hilbert space. Therefore let $f_n \in H^1(a,b)$, $n \in \mathbb{N}$ be some Cauchy sequence. The quasi-derivatives of this sequence converge in $L^2((a,b);\varsigma)$ to say f^1. Moreover, $f_n(c)$ converges to say $F \in \mathbb{C}$ for some point $c \in (a,b)$ since $\chi \neq 0$. Now the function

$$f(x) = F + \int_c^x f^1(t) d\varsigma(t), \quad x \in (a,b)$$

is locally absolutely continuous with respect to ς with quasi-derivative f^1. Since f is the pointwise limit of the functions f_n, which converge in the space $L^2((a,b);\chi)$, one sees that f is also the limit of f_n in $L^2((a,b);\chi)$. Hence $f \in H^1(a,b)$ and the sequence f_n converges to f in $H^1(a,b)$. In order to prove that point evaluations are continuous fix some $c \in (a,b)$ and choose α, $\beta \in (a,b)$ with $\alpha < c < \beta$ such that $\chi([\alpha,\beta)) \neq 0$. Clearly, we have for each function $f \in H^1(a,b)$

$$f(c) = f(x) + \int_x^c f^{[1]} d\varsigma, \quad x \in (a,b).$$

Now, integration with respect to χ yields

$$\begin{aligned} |f(c)|\chi([\alpha,\beta)) &= \left| \int_\alpha^\beta f(x) d\chi(x) + \int_\alpha^\beta \int_x^c f^{[1]} d\varsigma \, d\chi(x) \right| \\ &\leq \chi([\alpha,\beta))^{\frac{1}{2}} \|f\| + \chi([\alpha,\beta))\varsigma([\alpha,\beta))^{\frac{1}{2}} \|f\|, \end{aligned}$$

where we used the Cauchy–Schwarz inequality twice. □

In order to obtain the maximal relation $T_{\max}$ in $H^1(a,b)$ associated with our differential expression τ we restrict T_{loc} by

$$T_{\max} = \left\{ (f, f_\tau) \in H^1(a,b) \times H^1(a,b) \,|\, (f, f_\tau) \in T_{\mathrm{loc}} \right\}.$$

The following characterization of elements of $T_{\max}$ as certain weak solutions will be quite useful. Henceforth we will denote with $H^1_c(a,b)$ the linear subspace of $H^1(a,b)$ consisting of all functions with compact support.

Proposition 3.2. *Some pair $(f, f_\tau) \in H^1(a,b) \times H^1(a,b)$ lies in the maximal relation $T_{\max}$ if and only if*

$$\int_a^b f(x) g(x)^* d\chi(x) + \int_a^b f^{[1]}(x) g^{[1]}(x)^* d\varsigma(x) = \int_a^b f_\tau(x) g(x)^* d\varrho(x) \tag{3.1}$$

for each $g \in H^1_c(a,b)$.

Proof. If $(f, f_\tau) \in T_{\max}$, then integration by parts as in (2.4) shows that equation (3.1) is valid for each $g \in H^1_c(a,b)$. In order to prove the converse let c, $d \in (a,b)$ with $c < d$, $K \in \mathbb{C}$ and integrate (3.1) by parts to obtain

$$0 = \int_a^b g^{[1]}(x)^* \left(f^{[1]}(x) - \int_c^x f d\chi + \int_c^x f_\tau d\varrho + K \right) d\varsigma(x), \quad g \in H^1_c(a,b).$$

In particular we may choose

$$g(x)^* = \begin{cases} 0, & \text{if } x \in (a,c], \\ \int_c^x f^{[1]}(t) - \int_c^t f d\chi + \int_c^t f_\tau d\varrho + K d\varsigma(t), & \text{if } x \in (c,d], \\ 0, & \text{if } x \in (d,b), \end{cases}$$

where

$$K = -\frac{1}{\varsigma[c,d)} \int_c^d \left(f^{[1]}(t) - \int_c^t f d\chi + \int_c^t f_\tau d\varrho \right) d\varsigma(t).$$

Now from this

$$\int_c^d \left| f^{[1]}(x) - \int_c^x f d\chi + \int_c^x f_\tau d\varrho + K \right|^2 d\varsigma(x) = 0$$

and we infer that the integrand is locally constant. But this shows that f lies in $\mathfrak{D}_\tau$ and that $\tau f = f_\tau$ almost everywhere with respect to $|\varrho|$, which yields the claim. □

Now Proposition 3.2 allows us to describe the multi-valued part of $T_{\max}$.

Corollary 3.3. *Some $h \in H^1(a,b)$ lies in the multi-valued part of $T_{\max}$ if and only if $h = 0$ almost everywhere with respect to $|\varrho|$.*

Proof. If $h \in \operatorname{mul}(T_{\max})$, then the definition of $T_{\max}$ shows that $h = 0$ almost everywhere with respect to $|\varrho|$. The converse immediately follows from Proposition 3.2. □

We say some function $f \in AC_{\mathrm{loc}}((a,b);\varsigma)$ lies in $H^1(a,b)$ near an endpoint if f is square integrable with respect to χ near this endpoint and its quasi-derivative is square integrable with respect to ς near this endpoint. Moreover, we say some pair $f \in T_{\mathrm{loc}}$ lies in $T_{\max}$ near an endpoint if both components f and f_τ lie in $H^1(a,b)$ near this endpoint. Clearly some $f \in T_{\mathrm{loc}}$ lies in $T_{\max}$ if and only if it lies in $T_{\max}$ near a and near b.

Lemma 3.4. *If f and g lie in $T_{\max}$ near a, then the limit*

$$V(f,g^*)(a) := \lim_{\alpha \to a} V(f,g^*)(\alpha)$$

exists and is finite. A similar result holds for the endpoint b. If f and g even lie in $T_{\max}$, then

$$\langle f_\tau, g \rangle - \langle f, g_\tau \rangle = V(f,g^*)(b) - V(f,g^*)(a) =: V_a^b(f,g^*). \tag{3.2}$$

Moreover, $V(\,\cdot\,,\cdot\,)(a)$ and $V(\,\cdot\,,\cdot\,)(b)$ are continuous bilinear forms on $T_{\max}$ with respect to the product topology.

Proof. Letting α tend to a in the Lagrange identity in Proposition 2.3 shows that the limit exists and is finite. Similarly one sees that the limit $V(f,g^*)(\beta)$ as $\beta \to b$ exists and is finite if f and g lie in $T_{\max}$ near b. Now if f, $g \in T_{\max}$, then equation (3.2) follows by letting $\alpha \to a$ and $\beta \to b$ in the Lagrange identity. In order to prove continuity fix some $c \in (a,b)$ and let α, $\beta \in (a,b)$ with $\alpha < c < \beta$. Clearly for each $f \in T_{\max}$ we have

$$f^{[1]}(c) = f^{[1]}(x) - \int_c^x f d\chi + \int_c^x f_\tau d\varrho, \quad x \in (a,b).$$

Integration with respect to ς and using the Cauchy–Schwarz inequality twice yields the estimate

$$|f^{[1]}(c)| \leq \varsigma([\alpha,\beta))^{-\frac{1}{2}}\|f\| + \chi([\alpha,\beta))^{\frac{1}{2}}\|f\| + |\varrho|([\alpha,\beta))C_{\alpha,\beta}\|f_\tau\|,$$

where the constant $C_{\alpha,\beta} \in \mathbb{R}$ is chosen such that $|g(x)| \leq C_{\alpha,\beta}\|g\|$ for all $x \in [\alpha,\beta)$ and $g \in H^1(a,b)$. Hence the mapping $f \mapsto f^{[1]}(c)$ is continuous on $T_{\max}$. Since $f \mapsto f_\tau(c)$ is also continuous in virtue of Proposition 3.1, we infer that $V(\,\cdot\,,\cdot\,)(c)$ is a continuous bilinear form on $T_{\max}$. Now the claim is easily deduced from the Lagrange identity, using the Cauchy–Schwarz inequality. $\square$

If τ is regular at an endpoint, say a, then for each f which lies in $T_{\max}$ near a the limits

$$f(a) = \lim_{\alpha\to a} f(\alpha), \quad f^{[1]}(a) = \lim_{\alpha\to a} f^{[1]}(\alpha) \quad \text{and} \quad f_\tau(a) = \lim_{\alpha\to a} f_\tau(\alpha)$$

exist and are finite. Indeed, for the first and the last limit this follows from the fact that in this case $f^{[1]}$ and $f_\tau^{[1]}$ are integrable near a with respect to ς. Moreover, the existence of the second limit may be seen from

$$-f^{[1]}(\beta) + f^{[1]}(\alpha) + \int_\alpha^\beta f\,d\chi = \int_\alpha^\beta f_\tau d\varrho, \quad \alpha,\, \beta \in (a,b),\ \alpha < \beta.$$

Now the modified Wronskian near a regular endpoint a is given by

$$V(f,g)(a) = f_\tau(a)g^{[1]}(a) - f^{[1]}(a)g_\tau(a),$$

if f and g lie in $T_{\max}$ near a. Of course similar results hold if the right endpoint b is regular.

In the remaining part of this section we will collect some more properties of the modified Sobolev space $H^1(a,b)$ and the maximal relation $T_{\max}$. Therefore we need the following result on solutions of our differential equation which has been proven in [6, Theorem 3].

Theorem 3.5. *For each $z \in \mathbb{C}$ all solutions of $(\tau - z)u = 0$ lie in $H^1(a,b)$ near a if and only if ς, χ are finite near a and the function*

$$z\int_x^c d\varrho, \quad x \in (a,b)$$

is square integrable with respect to ς near a for some $c \in (a,b)$. A similar result holds for the endpoint b.

In the following $H_0^1(a,b)$ will denote the closure of $H_c^1(a,b)$ in $H^1(a,b)$.

Proposition 3.6. *We have $H^1(a,b) = H_0^1(a,b) \oplus \ker(T_{\max})$, with*

$$\dim\ker(T_{\max}) = \begin{cases} 0, & \textit{if } \varsigma + \chi \textit{ is infinite near both endpoints,} \\ 1, & \textit{if } \varsigma + \chi \textit{ is finite near precisely one endpoint,} \\ 2, & \textit{if } \varsigma + \chi \textit{ is finite.} \end{cases}$$

Proof. Proposition 3.2 shows that $\ker(T_{\max}) = H_c^1(a,b)^\perp$. Now suppose that $\varsigma + \chi$ is infinite near both endpoints and take some $u \in \ker(T_{\max})$. An integration by parts as in (2.4) shows that for each α, $\beta \in (a,b)$ with $\alpha < \beta$

$$\int_\alpha^\beta |u|^2 d\chi + \int_\alpha^\beta |u^{[1]}|^2 d\varsigma = \mathrm{Re}(u^{[1]}(\beta)u(\beta)^*) - \mathrm{Re}(u^{[1]}(\alpha)u(\alpha)^*). \tag{3.3}$$

From this we see that the function $\mathrm{Re}(u^{[1]}u^*)$ is increasing on (a,b). Further integration by parts yield

$$|u(\beta)|^2 - |u(\alpha)|^2 = 2\int_\alpha^\beta \mathrm{Re}(u^{[1]}(x)u(x)^*)d\varsigma(x) \tag{3.4}$$

as well as

$$|u^{[1]}(\beta)|^2 - |u^{[1]}(\alpha)|^2 = 2\int_\alpha^\beta \mathrm{Re}(u^{[1]}(x)u(x)^*)d\chi(x) \tag{3.5}$$

for each α, $\beta \in (a,b)$ with $\alpha < \beta$. Now if there was some $c \in (a,b)$ such that $\mathrm{Re}(u^{[1]}(c)u(c)^*) > 0$, then the equations (3.4), (3.5) and the monotonicity of the function $\mathrm{Re}(u^{[1]}u^*)$ would show that the modulus of u and its quasi-derivative are non-decreasing to the right of c. Moreover, if there was some $c \in (a,b)$ with $\mathrm{Re}(u^{[1]}(c)u(c)^*) < 0$ one would similarly show that the modulus of u and its quasi-derivative are non-increasing to the left of c. Hence both cases would imply that u does not lie in $H^1(a,b)$ since for each endpoint at least one, ς or χ is infinite. But from this we conclude that u vanishes identically.

Now suppose $\varsigma + \chi$ is finite near exactly one endpoint, say a. Then it is easily seen that for each $f \in H^1(a,b)$ the limit of $f(x)$ as $x \to a$ exists and is finite. Moreover, point evaluation in a is continuous on $H^1(a,b)$. Indeed, this may be shown literally as in the proof of Proposition 3.1. Since there are functions in $H^1(a,b)$ which do not vanish in a, the dimension of the kernel of $T_{\max}$ is not zero. Hence the claim follows, since Theorem 3.5 shows that the dimension is at most one. The remaining case immediately follows from Theorem 3.5. □

Corollary 3.7. *There is a (up to scalar multiples) unique non-trivial real solution w_a of $\tau u = 0$ which lies in $H^1(a,b)$ near a and satisfies*

$$\lim_{\alpha\to a} g(\alpha)w_a^{[1]}(\alpha) = 0, \quad g \in H^1(a,b). \tag{3.6}$$

Similarly, there is a (up to scalar multiples) unique non-trivial real solution w_b of $\tau u = 0$ with corresponding properties near the endpoint b. The solutions w_a and w_b are linearly independent.

Proof. If $\varsigma+\chi$ is finite near a, the solution w_a of $\tau u = 0$ with initial conditions

$$w_a(a) = 1 \quad \text{and} \quad w_a^{[1]}(a) = 0$$

has the claimed properties. Any other such solution has to vanish at a in view of (3.6) and thus is a scalar multiple of w_a. Now if $\varsigma + \chi$ is not finite near a, Proposition 3.6 shows that there is a (up to scalar multiples) unique solution of $\tau u = 0$ which lies in $H^1(a,b)$ near a. Moreover, we may suppose that w_a is real since otherwise one could take the real or imaginary part of w_a. Next we show that the mapping

$$g \mapsto \lim_{\alpha\to a} g(\alpha)w_a^{[1]}(\alpha)$$

is continuous on $H^1(a,b)$. Indeed, integration by parts as in (2.4) shows

$$\int_\alpha^c g w_a\, d\chi + \int_\alpha^c g^{[1]} w_a^{[1]} d\varsigma = g(c)w_a^{[1]}(c) - g(\alpha)w_a^{[1]}(\alpha), \quad g \in H^1(a,b),$$

hence this limit exists. Moreover, it may be estimated by

$$|\lim_{\alpha\to a} g(\alpha)w_a^{[1]}(\alpha)| \leq |w_a^{[1]}(c)||g(c)| + \|g\|\sqrt{\int_a^c w_a^2\, d\chi + \int_a^c w_a^{[1]2}\, d\varsigma}$$

for each $g\in H^1(a,b)$ and thus is continuous. Since obviously (3.6) is valid for $g\in H^1_c(a,b)$ we infer that it is also valid for $g\in H^1_0(a,b)$. If $\ker(T_{\max})\neq\{0\}$, i.e. when $\varsigma+\chi$ is finite near b, then $\ker(T_{\max}) = \operatorname{span}\{w_a\}$. Hence we have to prove that (3.6) holds for $g = w_a$ in this case. Moreover, note that the proof of Proposition 3.6 shows that the function

$$x \mapsto w_a(x)w_a^{[1]}(x)$$

is strictly positive and increasing on (a,b). Now if the limit as $x\to a$ was strictly positive, one would obtain a contradiction from (3.4) or from (3.5). But this guarantees (3.6) for all $g\in H^1(a,b)$ because of the decomposition $H^1(a,b) = \ker(T_{\max})\oplus H^1_0(a,b)$. Finally, if w_a and w_b were linearly dependent, then (3.3) would show that their norm vanishes. □

Note that the functions

$$x\mapsto w_a(x)w_a^{[1]}(x) \quad\text{and}\quad x\mapsto w_b(x)w_b^{[1]}(x)$$

are increasing on (a,b) in view of (3.3) and strictly positive (respectively negative). Now for each fixed $c\in(a,b)$ we define the function

$$\delta_c(x) = \frac{1}{W(w_b,w_a)}\begin{cases} w_a(x)w_b(c), & \text{if } x\in(a,c],\\ w_a(c)w_b(x), & \text{if } x\in(c,b),\end{cases} \tag{3.7}$$

with

$$W(w_b,w_a) = w_b(x)w_a^{[1]}(x) - w_b^{[1]}(x)w_a(x), \quad x\in(a,b),$$

where the right-hand side is independent of $x\in(a,b)$ and non-zero since w_a and w_b are linearly independent solutions of $\tau u = 0$. With this definition the point evaluation in c is given by

$$f(c) = \langle f,\delta_c\rangle, \quad f\in H^1(a,b).$$

Indeed, this follows from splitting the integrals on the right-hand side, integrating by parts as in (2.4) twice and using the properties from Corollary 3.7. Moreover, if the measures ς and χ are finite near an endpoint, say a, then $f(x)$ has a finite limit as $x\to a$ for each $f\in H^1(a,b)$ and

$$f(a) = \lim_{\alpha\to a} f(\alpha) = \langle f,\delta_a\rangle, \quad f\in H^1(a,b),$$

where the function δ_a is given by

$$\delta_a(x) = -\frac{w_b(x)}{w_b^{[1]}(a)}, \quad x\in(a,b). \tag{3.8}$$

In fact, this follows from a simple integration by parts using the properties of Corollary 3.7. Of course, a similar result holds for the right endpoint b if ς and χ are finite near b. As a consequence of this we see that some function $f\in H^1(a,b)$ lies in $H^1_0(a,b)$ if and only if f vanishes in each endpoint near which ς and χ are finite.

We are now turning back to our maximal relation $T_{\max}$. Since we are interested in self-adjoint restrictions of $T_{\max}$, we will first compute its adjoint relation.

Theorem 3.8. *The maximal relation $T_{\max}$ is closed with adjoint given by*

$$T_{\max}^* = \{f \in T_{\max} \,|\, \forall g \in T_{\max} : V(f,g)(a) = V(f,g)(b) = 0\}. \tag{3.9}$$

Proof. Let $T_0 \subseteq T_{\max}$ consist of all $f \in T_{\max}$ such that $f_\tau \in H_c^1(a,b)$, f is a scalar multiple of w_a near a and a scalar multiple of w_b near b. Then the range of T_0 is actually equal to $H_c^1(a,b)$. Indeed, if $g \in H_c^1(a,b)$ is given, then the function

$$f(x) = W(w_b, w_a)^{-1} \left(w_b(x) \int_a^x w_a g \, d\varrho + w_a(x) \int_x^b w_b g \, d\varrho \right), \quad x \in (a,b)$$

is a solution of $\tau f = g$ (see [18, Proposition 3.3]) which is a scalar multiple of w_a, w_b near the respective endpoints and hence $g \in \operatorname{ran}(T_0)$. Moreover, for each $f \in T_0$, $g \in T_{\max}$ the limits of $V(f,g)(x)$ as $x \to a$ and as $x \to b$ vanish in view of Corollary 3.7. Hence Lemma 3.4 shows that $T_{\max} \subseteq T_0^*$. Conversely, if $(f_1, f_2) \in T_0^*$, then integration by parts as in (2.4) and using Corollary 3.7 shows that

$$\langle f_1, g_\tau \rangle = \langle f_2, g \rangle = \int_a^b f_2(x) g_\tau(x)^* d\varrho(x)$$

for each $g \in T_0$. Now since $\operatorname{ran}(T_0) = H_c^1(a,b)$ we infer that $(f_1, f_2) \in T_{\max}$ in view of Proposition 3.2. Thus $T_{\max}$ is the adjoint of T_0 and hence closed. Finally we obtain

$$T_{\max}^* = \overline{T_0} \subseteq \{f \in T_{\max} \,|\, \forall g \in T_{\max} : V(f,g)(a) = V(f,g)(b) = 0\} \subseteq T_{\max}^*,$$

where we used Lemma 3.4 and the fact that $V(\,\cdot\,,\cdot\,)(a)$ and $V(\,\cdot\,,\cdot\,)(b)$ are continuous on $T_{\max} \times T_{\max}$. □

The adjoint of $T_{\max}$ is referred to as the minimal relation $T_{\min}$. This linear relation is obviously symmetric with adjoint $T_{\max}$. Moreover, the following corollary shows that its deficiency indices are equal. Hence there are always self-adjoint restrictions of $T_{\max}$.

Corollary 3.9. *The deficiency indices of the minimal relation $T_{\min}$ are equal and at most two, i.e.*

$$n(T_{\min}) := \dim \operatorname{ran}(T_{\min} - \mathrm{i})^\perp = \dim \operatorname{ran}(T_{\min} + \mathrm{i})^\perp \leq 2.$$

Proof. The dimensions are less than two because of

$$\operatorname{ran}(T_{\min} \pm \mathrm{i})^\perp = \ker(T_{\max} \mp \mathrm{i}) \subseteq \ker(T_{\mathrm{loc}} \mp \mathrm{i}),$$

and since there are at most two linearly independent solutions of $(\tau \pm \mathrm{i})u = 0$. Moreover, note that the solutions of the equation $(\tau - \mathrm{i})u = 0$ are precisely the complex conjugates of the solutions of $(\tau + \mathrm{i})u = 0$. From this it is easily seen that the complex conjugation is a conjugate-linear isometry from the kernel of the relation $T_{\max} + \mathrm{i}$ onto the kernel of $T_{\max} - \mathrm{i}$ and hence their dimensions are equal. □

4. Self-adjoint restrictions

In this section we will determine the self-adjoint restrictions of $T_{\max}$. Therefore we will distinguish two different cases at each endpoint. We say τ is in the limit-circle (l.c.) case at a, if for each $z \in \mathbb{C}^\times$ (the cross indicates that zero is removed from this set) all solutions of $(\tau - z)u = 0$ lie in $H^1(a,b)$ near a. Furthermore, we say τ is in the limit-point (l.p.) case at a if for each $z \in \mathbb{C}^\times$ there is some solution of $(\tau - z)u = 0$ which does not lie in $H^1(a,b)$ near a. Similarly one defines the l.c. and l.p. cases for the right endpoint b. It is obvious that τ is only either in the l.c. or in the l.p. case at each endpoint. The next lemma shows that τ indeed is in one of these cases at each endpoint.

Lemma 4.1. *If there is a $z_0 \in \mathbb{C}^\times$ such that all solutions of $(\tau - z_0)u = 0$ lie in $H^1(a,b)$ near a, then for each $z \in \mathbb{C}^\times$ all solutions of $(\tau - z)u = 0$ lie in $H^1(a,b)$ near a. A similar result holds at the endpoint b.*

This lemma has been proven by Bennewitz and Brown in [6, Lemma 4]. In contrast to the right-definite theory, Theorem 3.5 shows that there are precise criteria for the l.c. case to prevail. In particular, if τ is in the l.c. case at some endpoint, then all solutions of $\tau u = 0$ lie in $H^1(a,b)$ near this endpoint. However, note that it might happen that τ is in the l.p. case at an endpoint although all solutions of $\tau u = 0$ lie in $H^1(a,b)$ near this endpoint.

The next aim is to determine the deficiency index of $T_{\min}$, depending on whether or not the endpoints are in the l.c. or l.p. case. We denote the set of all points of regular type of $T_{\min}$ with $\mathrm{r}(T_{\min})$.

Lemma 4.2. *Given $z \in \mathrm{r}(T_{\min})$ there is a non-trivial solution of $(\tau - z)u = 0$ which lies in $H^1(a,b)$ near a. A similar result holds for the endpoint b.*

Proof. We may assume that the measures ϱ, ς and χ are finite near the endpoint b. If there were no solutions of $(\tau - z)u = 0$ which lie in $H^1(a,b)$ near a, we would have $\ker(T_{\max} - z) = \{0\}$. Since the mapping

$$z \mapsto \dim \operatorname{ran}(T_{\min} - z)^\perp$$

is constant on each connected component of $\mathrm{r}(T_{\min})$ (see [16, Theorem 2.4] or [15, Theorem 6.1]) and thus

$$0 = \dim \ker(T_{\max} - z^*) = \dim \operatorname{ran}(T_{\min} - z)^\perp = n(T_{\min}), \quad z \in \mathrm{r}(T_{\min}),$$

we would infer that $T_{\min} = T_{\max}$. Let $g \in H^1(a,b)$ such that g vanishes near a and is equal to w_a near b. Now the solution h of $\tau f = g$ which is equal to w_a near a lies in $H^1(a,b)$ since h and its quasi-derivative are bounded near the endpoint b. From this we get

$$V(h, w_a)(b) = g(b) w_a^{[1]}(b) = w_a(b) w_a^{[1]}(b) > 0,$$

contradicting Theorem 3.8. □

Corollary 4.3. *If $z \in \mathrm{r}(T_{\min})^\times$ and τ is in the l.p. case at a, then there is a (up to scalar multiples) unique non-trivial solution of $(\tau - z)u = 0$, which lies in $H^1(a,b)$ near a. A similar result holds for the endpoint b.*

Proof. If there were two linearly independent solutions which lie in $H^1(a,b)$ near a, τ were in the l.c. case at a. □

The following result shows that one might tell from the modified Wronskian whether τ is in the l.c. or in the l.p. case.

Proposition 4.4. *τ is in the l.p. case at a if and only if*

$$V(f,g)(a) = 0, \quad f,\, g \in T_{\max}. \tag{4.1}$$

τ is in the l.c. case at a if and only if there is a $v \in T_{\max}$ such that

$$V(v,v^*)(a) = 0 \quad \text{and} \quad V(f,v^*)(a) \neq 0 \quad \text{for some } f \in T_{\max}. \tag{4.2}$$

Similar results hold at the endpoint b.

Proof. Assume that τ is in the l.p. case at a and that the measures ϱ, ς and χ are finite near b. Then $T_{\max}$ is a two-dimensional extension of $T_{\min}$, since $\dim\ker(T_{\max} - \mathrm{i}) = 1$ by Corollary 4.3. Now let $h \in T_{\max}$ be defined as in the proof of Lemma 4.2. Then w_a and h are linearly independent modulo $T_{\min}$. Indeed, if $c_1 w_a + c_2 h \in T_{\min}$ for some c_1, $c_2 \in \mathbb{C}$ then

$$c_2 w_a(b) g^{[1]}(b) - c_2 h^{[1]}(b) g_\tau(b) - c_1 w_a^{[1]}(b) g_\tau(b) = 0, \quad g \in T_{\max}.$$

Now choosing $g = w_a$, $g_\tau = 0$ yields $c_2 = 0$ and from $g = f$, $g_\tau = f_\tau$ we obtain $c_1 = 0$. Hence

$$T_{\max} = T_{\min} \dot{+} \operatorname{span}\{w_a, h\},$$

and a simple calculation shows that $V(f,g)(a) = 0$ for all f, $g \in T_{\max}$. If the measures ϱ, ς and χ are not finite near b, then pick some $c \in (a,b)$. For each f, $g \in T_{\max}$ the restrictions $f|_{(a,c)}$ and $g|_{(a,c)}$ lie in the maximal relation induced by our differential expression restricted to the interval (a,c). Since this restriction does not change the l.p. classification at a as well as the modified Wronskian near a, the claim follows from the case which we just proved.

Now suppose τ is in the l.c. case at a and in the l.p. case at b. According to Corollary 4.3, $T_{\max}$ is a two-dimensional extension of $T_{\min}$. In particular, there is some $v \in T_{\max}$ which does not lie in $T_{\min}$. We may assume that $V(v,v^*)(a) = 0$, since otherwise replace v by $v + v^*$ or $v - v^*$. Now from Theorem 3.8 and the first part of the proof we infer that

$$T_{\min} = \{f \in T_{\max} \,|\, \forall g \in T_{\max} : V(f,g)(a) = 0\}.$$

Since v does not lie in $T_{\min}$ there is some $f \in T_{\max}$ such that $V(f,v^*)(a) \neq 0$. Finally consider the case when both endpoints are in the l.c. case. Then we have u_1, $u_2 \in T_{\max}$, where u_1, u_2 are two linearly independent real solutions of $\tau u = u$. Now if one chooses $v = u_1$, then the function $f = u_2$ has the claimed property. $\square$

The results obtained so far allow us to determine the deficiency index of the minimal relation.

Theorem 4.5. *The deficiency index of $T_{\min}$ is given by*

$$n(T_{\min}) = \begin{cases} 0, & \textit{if } \tau \textit{ is in the l.c. case at no endpoint,} \\ 1, & \textit{if } \tau \textit{ is in the l.c. case at precisely one endpoint,} \\ 2, & \textit{if } \tau \textit{ is in the l.c. case at both endpoints.} \end{cases}$$

Proof. If τ is in the l.c. case at both endpoints, all solutions of $(\tau - \mathrm{i})u = 0$ lie in $H^1(a,b)$ and hence in $T_{\max}$. Therefore $n(T_{\min}) = \dim\ker(T_{\max} - \mathrm{i}) = 2$. Provided τ is in the l.c. case at exactly one endpoint, there is (up to scalar multiples) exactly one non-trivial solution of $(\tau - \mathrm{i})u = 0$ in $H^1(a,b)$, by Corollary 4.3. Now suppose τ is in the l.p. case at both endpoints and $u \in H^1(a,b)$ is a solution of $(\tau - \mathrm{i})u = 0$. Then u^* is a solution of $(\tau + \mathrm{i})u = 0$ and both u and u^* lie in $T_{\max}$. Now the Lagrange identity yields

$$V(u,u^*)(\beta) - V(u,u^*)(\alpha) = 2\mathrm{i}\left(\int_\alpha^\beta |u|^2 d\chi + \int_\alpha^\beta |u^{[1]}|^2 d\varsigma\right), \quad \alpha,\, \beta \in (a,b).$$

As $\alpha \to a$ and $\beta \to b$, the left-hand side converges to zero by Proposition 4.4 and the right-hand side converges to $2\mathrm{i}\|u\|^2$. Hence u vanishes identically and $n(T_{\min}) = 0$. □

We will now characterize the self-adjoint restrictions of $T_{\max}$. Therefore recall that these are precisely the $n(T_{\min})$-dimensional symmetric extensions of $T_{\min}$ (see e.g. [18, Corollary B.6]). Clearly, the simplest case is when both endpoints are in the l.p. case. The following result is an immediate consequence of Theorem 4.5.

Theorem 4.6. *If τ is in the l.p. case at both endpoints then $T_{\min} = T_{\max}$ is a self-adjoint relation.*

Next we will turn to the case when precisely one endpoint is in the l.p. case. But before, we need some more properties of the modified Wronskian.

Lemma 4.7. *Suppose τ is in the l.c. case at a and let $v \in T_{\max}$ such that* (4.2) *holds. Then for each f, $g \in T_{\max}$ we have*

$$V(f,v^*)(a) = 0 \quad \Leftrightarrow \quad V(f^*,v^*)(a) = 0 \tag{4.3}$$

and

$$V(f,v^*)(a) = V(g,v^*)(a) = 0 \quad \Rightarrow \quad V(f,g)(a) = 0. \tag{4.4}$$

Similar results hold at the endpoint b.

Proof. Pick some $h \in T_{\max}$ such that $V(h,v^*)(a) \neq 0$. Choosing $f_1 = v$, $f_2 = v^*$, $f_3 = h$ and $f_4 = h^*$ in the Plücker identity in Proposition 2.4, we see that also $V(h,v)(a) \neq 0$. Now choosing $f_1 = f$, $f_2 = v$, $f_3 = v^*$ and $f_4 = h$ in the Plücker identity yields (4.3), whereas the choice $f_1 = f$, $f_2 = g$, $f_3 = v^*$ and $f_4 = h$ yields (4.4). □

Theorem 4.8. *Suppose τ is in the l.c. case at a and in the l.p. case at b. Then some relation S is a self-adjoint restriction of $T_{\max}$ if and only if there is a $v \in T_{\max}$ with* (4.2) *such that*

$$S = \{f \in T_{\max} \,|\, V(f,v^*)(a) = 0\}. \tag{4.5}$$

A similar result holds if τ is in the l.c. case at b and in the l.p. case at a.

Proof. Since $n(T_{\min}) = 1$, the self-adjoint restrictions of $T_{\max}$ are precisely the one-dimensional, symmetric extensions of $T_{\min}$. Hence some relation S is a self-adjoint restriction of $T_{\max}$ if and only if there is a $v \in T_{\max}\backslash T_{\min}$ with $V(v,v^*)(a) = 0$ such that

$$S = T_{\min} \dot{+} \operatorname{span}\{v\}.$$

Thus we have to prove that

$$T_{\min} \dotplus \operatorname{span}\{v\} = \{f \in T_{\max} \,|\, V(f, v^*)(a) = 0\}.$$

The subspace on the left-hand side is included in the right one because of Theorem 3.8 and $V(v, v^*)(a) = 0$. If the subspace on the right-hand side were larger, it were equal to $T_{\max}$ and hence would imply $v \in T_{\min}$ in view of Theorem 3.8. □

It remains to consider the case when both endpoints are in the l.c. case. To this end recall that we introduced the convenient short hand notation

$$V_a^b(f, g^*) = V(f, g^*)(b) - V(f, g^*)(a), \quad f,\, g \in T_{\max}.$$

Theorem 4.9. *Suppose τ is in the l.c. case at both endpoints. Then some relation S is a self-adjoint restriction of $T_{\max}$ if and only if there are some v, $w \in T_{\max}$, linearly independent modulo $T_{\min}$, with*

$$V_a^b(v, v^*) = V_a^b(w, w^*) = V_a^b(v, w^*) = 0, \tag{4.6}$$

such that

$$S = \{f \in T_{\max} \,|\, V_a^b(f, v^*) = V_a^b(f, w^*) = 0\}. \tag{4.7}$$

Proof. Because of $n(T_{\min}) = 2$, the self-adjoint restrictions of $T_{\max}$ are precisely the two-dimensional, symmetric extensions of $T_{\min}$. Hence a relation S is a self-adjoint restriction of $T_{\max}$ if and only if there are v, $w \in T_{\max}$, linearly independent modulo $T_{\min}$, with (4.6) such that

$$S = T_{\min} \dotplus \operatorname{span}\{v, w\}.$$

Therefore, we have to prove that

$$T_{\min} \dotplus \operatorname{span}\{v, w\} = \{f \in T_{\max} \,|\, V_a^b(f, v^*) = V_a^b(f, w^*) = 0\} = T,$$

where we denote the subspace on the right-hand side by T. In fact, the subspace on the left-hand side is contained in T by Theorem 3.8 and (4.6). In order to prove that it is not larger indeed, consider the linear functionals F_v, F_w on $T_{\max}$ defined by

$$F_v(f) = V_a^b(f, v^*) \quad \text{and} \quad F_w(f) = V_a^b(f, w^*) \quad \text{for } f \in T_{\max}.$$

The intersection of the kernels of these functionals is precisely T. Furthermore, these functionals are linearly independent. In fact, if c_1, $c_2 \in \mathbb{C}$ such that $c_1 F_v + c_2 F_w = 0$, then for all $f \in T_{\max}$ we had

$$0 = c_1 F_v(f) + c_2 F_w(f) = c_1 V_a^b(f, v^*) + c_2 V_a^b(f, w^*) = V_a^b(f, c_1 v^* + c_2 w^*).$$

But this shows that $c_1 v^* + c_2 w^* \in T_{\min} = T_{\max}^*$ in view of Lemma 3.4. and since v, w are linearly independent modulo $T_{\min}$ we infer $c_1 = c_2 = 0$. Now the linear independence of our functionals guarantees that

$$\ker F_v \not\subseteq \ker F_w \quad \text{and} \quad \ker F_w \not\subseteq \ker F_v.$$

Hence there are some f_v, $f_w \in T_{\max}$ such that $V_a^b(f_v, v^*) = V_a^b(f_w, w^*) = 0$ but $V_a^b(f_v, w^*) \neq 0$ and $V_a^b(f_w, v^*) \neq 0$. Since both functions, f_v and f_w do not lie in T and are linearly independent, T is at most a two-dimensional extension of $T_{\min}$. □

In the rest of this paper we will only be interested in self-adjoint restrictions of $T_{\max}$ with separate boundary conditions. These are precisley the relations of the form

$$S = \{f \in T_{\max} \,|\, V(f,v^*)(a) = V(f,w^*)(b) = 0\}, \tag{4.8}$$

where v, $w \in T_{\max}$ such that

$$V(v,v^*)(a) = 0 \quad \text{and} \quad V(f,v^*)(a) \neq 0 \quad \text{for some } f \in T_{\max}, \tag{4.9a}$$

if τ is in the l.c. case at a and

$$V(w,w^*)(b) = 0 \quad \text{and} \quad V(f,w^*)(b) \neq 0 \quad \text{for some } f \in T_{\max}, \tag{4.9b}$$

if τ is in the l.c. case at b. Hence some $f \in T_{\max}$ lies in S if and only if it satisfies the boundary condition $V(f,v^*)(a) = 0$ at a and the boundary condition $V(f,w^*)(b) = 0$ at b. Note that boundary conditions at l.p. endpoints are actually superfluous, since each $f \in T_{\max}$ satisfies the boundary condition there in view of Proposition 4.4. Finally, let us mention that each relation of the form (4.8) is self-adjoint indeed. If τ is not at both endpoints in the l.c. case, this is evident from Theorem 4.6 and Theorem 4.8. Actually, in these cases all self-adjoint restrictions of $T_{\max}$ have separated boundary conditions. For the case when τ is in the l.c. case at both endpoints, note that S is symmetric in view of Lemma 4.7 and Lemma 3.4 and that S is at most a two-dimensional restriction of $T_{\max}$.

When τ is regular at an endpoint, say a, then the boundary condition at this endpoint may be given in a simpler form. In fact, if $v \in T_{\max}$ such that (4.9a) holds then it may be shown that there is some $\varphi_\alpha \in [0,\pi)$ such that for each $f \in T_{\max}$

$$V(f,v^*)(a) = 0 \quad \Leftrightarrow \quad f_\tau(a)\cos\varphi_\alpha - f^{[1]}(a)\sin\varphi_\alpha = 0. \tag{4.10}$$

Conversely, if some $\varphi_\alpha \in [0,\pi)$ is given, then there is a $v \in T_{\max}$ with (4.9a) such that (4.10) holds for each $f \in T_{\max}$. In this case, the boundary conditions corresponding to $\varphi_\alpha = 0$ are called Dirichlet boundary conditions, whereas the ones corresponding to $\varphi_\alpha = \pi/2$ are called Neumann boundary conditions. Moreover, note that if $z \in \mathbb{C}$ and u is a solution of $(\tau - z)u = 0$, the boundary condition at the regular endpoint a takes the form

$$zu(a)\cos\varphi_\alpha - u^{[1]}(a)\sin\varphi_\alpha = 0.$$

Proposition 4.10. *If S is a self-adjoint restriction of $T_{\max}$ with separate boundary conditions, then all non-zero eigenvalues of S are simple.*

Proof. Suppose $\lambda \in \mathbb{R}$ is an eigenvalue and u_j, $j = 1,2$ are solutions of $(\tau - \lambda)u = 0$. If τ is in the l.p. case at a, then clearly $V(u_1,u_2)$ vanishes. Otherwise, the same is true because of (4.4) in Lemma 4.7 and hence u_1, u_2 are linearly dependent. □

It might happen that zero is a double eigenvalue indeed. This is due to the fact that there are cases where each solution of $\tau u = 0$ lies in $H^1(a,b)$ near both endpoints and furthermore, satisfies the boundary condition at each endpoint which is in the l.c. case. For example, this happens for Dirichlet boundary conditions at regular endpoints or if ς and χ are finite near an endpoint which is in the l.p. case.

Next we will determine the resolvent of self-adjoint restrictions of $T_{\max}$ with separate boundary conditions.

Theorem 4.11. *Suppose S is a self-adjoint restriction of $T_{\max}$ with separate boundary conditions and let $z \in \rho(S)^\times$. Furthermore, let u_a and u_b be non-trivial solutions of $(\tau - z)u = 0$ such that*

$$u_a \begin{cases} \text{satisfies the boundary condition at } a \text{ if } \tau \text{ is in the l.c. case at } a, \\ \text{lies in } H^1(a,b) \text{ near } a \text{ if } \tau \text{ is in the l.p. case at } a, \end{cases}$$

and

$$u_b \begin{cases} \text{satisfies the boundary condition at } b \text{ if } \tau \text{ is in the l.c. case at } b, \\ \text{lies in } H^1(a,b) \text{ near } b \text{ if } \tau \text{ is in the l.p. case at } b. \end{cases}$$

Then the resolvent R_z is given by

$$R_z g(x) = \langle g, G_z(x,\cdot\,)^* \rangle, \quad x \in (a,b),\ g \in H^1(a,b), \tag{4.11}$$

where

$$G_z(x,y) + \frac{\delta_x(y)}{z} = \frac{1}{V(u_b,u_a)} \begin{cases} u_a(y)u_b(x), & \text{if } y \leq x, \\ u_a(x)u_b(y), & \text{if } y > x. \end{cases} \tag{4.12}$$

Proof. First of all note, that the solutions u_a, u_b are linearly independent, since otherwise z were an eigenvalue of S. Now if $g \in H^1_c(a,b)$, then the function f_g given by

$$f_g(x) = \frac{z}{V(u_b,u_a)} \left(u_b(x) \int_a^x u_a g \, d\varrho + u_a(x) \int_x^b u_b g \, d\varrho \right), \quad x \in (a,b)$$

is a solution of $(\tau - z)f = g$ by [18, Proposition 3.3]. Moreover, f_g is a scalar multiple of u_a near a and a scalar multiple of u_b near b. As a consequence $f_g \in T_{\max}$ satisfies the boundary conditions of S and therefore $R_z g = f_g$. Now an integration by parts as in (2.4) shows that $R_z g$ is given by (4.11). Furthermore, by continuity this holds for all $g \in H^1_0(a,b)$. Hence it remains to consider $R_z w$ when $w \in \ker(T_{\max})$. In this case integration by parts yields

$$\langle w, G_z(x,\cdot\,)^* \rangle = \frac{1}{z} \frac{V(u_b,w)(b)}{V(u_b,u_a)(b)} u_a(x) + \frac{1}{z} \frac{V(w,u_a)(a)}{V(u_b,u_a)(a)} u_b(x) - \frac{w(x)}{z}$$

for each $x \in (a,b)$. Obviously, this function is a solution of $(\tau - z)f = w$, since w is a solution of $\tau u = 0$. Moreover, if τ is in the l.p. case at a, then the second term vanishes in view of Proposition 4.4. For the same reason the first term vanishes if τ is in the l.p. case at b and hence this function even lies in $H^1(a,b)$. Finally, using the Plücker identity it is easily verified that this function also satisfies the boundary conditions at each endpoint which is in the l.c. case. $\square$

If $z \in \rho(S)^\times$ and τ is in the l.p. case at some endpoint, then Corollary 4.3 shows that there is always a (up to scalar multiples) unique non-trivial solution of $(\tau - z)u = 0$, lying in $H^1(a,b)$ near this endpoint. Moreover, if τ is in the l.c. case at some endpoint, then there exists a (up to scalar multiples) unique non-trivial solution of $(\tau - z)u = 0$, satisfying the boundary condition at this endpoint. Hence functions u_a and u_b, as in Theorem 4.11 always exist.

5. Weyl–Titchmarsh theory

In the rest of this paper S will always denote some self-adjoint restriction of $T_{\max}$ with separate boundary conditions as in (4.8). Our aim in this section is to construct a singular Weyl–Titchmarsh function as it has been introduced recently in [20], [25] and [18] in the right-definite case. To this end we first need a non-trivial real analytic solution ϕ_z, $z \in \mathbb{C}^\times$ of $(\tau - z)u = 0$ such that ϕ_z lies in S near a, i.e. ϕ_z lies in $H^1(a,b)$ near a and satisfies the boundary condition at a if τ is in the l.c. case there.

Hypothesis 5.1. *For every $z \in \mathbb{C}^\times$ there is a non-trivial solution ϕ_z of $(\tau - z)u = 0$ such that ϕ_z lies in S near a and the functions*

$$z \mapsto \phi_z(c) \quad \text{and} \quad z \mapsto \phi_z^{[1]}(c) \tag{5.1}$$

are real analytic in $\mathbb{C}^\times$ with at most poles at zero for each $c \in (a,b)$.

In order to introduce a singular Weyl–Titchmarsh function we furthermore need a second real analytic solution θ_z, $z \in \mathbb{C}^\times$ of $(\tau - z)u = 0$ such that $V(\theta_z, \phi_z) = 1$.

Lemma 5.2. *If Hypothesis 5.1 holds, then for each $z \in \mathbb{C}^\times$ there is a solution θ_z of $(\tau - z)u = 0$ such that $V(\theta_z, \phi_z) = 1$ and the functions*

$$z \mapsto \theta_z(c) \quad \text{and} \quad z \mapsto \theta_z^{[1]}(c) \tag{5.2}$$

are real analytic in $\mathbb{C}^\times$ with at most poles at zero for each $c \in (a,b)$.

Proof. Following literally the proof of [25, Lemma 2.4] there is a real analytic solution u_z, $z \in \mathbb{C}^\times$ of $(\tau - z)u = 0$ such that the usual Wronskian satisfies

$$W(\phi_z, u_z) = \phi_z(x)u_z^{[1]}(x) - \phi_z^{[1]}(x)u_z(x) = 1, \quad x \in (a,b),\ z \in \mathbb{C}^\times.$$

Now the solutions $\theta_z = z^{-1}u_z$, $z \in \mathbb{C}^\times$ have the claimed properties. □

Given a real analytic fundamental system θ_z, ϕ_z, $z \in \mathbb{C}^\times$ of $(\tau - z)u = 0$ as in Hypothesis 5.1 and Lemma 5.2 we may define a complex valued function M on $\rho(S)^\times$ by requiring that the solutions

$$\psi_z = \theta_z + M(z)\phi_z, \quad z \in \rho(S)^\times \tag{5.3}$$

lie in S near b, i.e. they lie in $H^1(a,b)$ near b and satisfy the boundary condition at b if τ is in the l.c. case there. By Corollary 4.3 and the fact that there is up to scalar multiples precisely one solution of $(\tau - z)u = 0$ satisfying the boundary condition at b if τ is in the l.c. case there, M is well-defined. This function M is referred to as the singular Weyl–Titchmarsh function of S, associated with the fundamental system θ_z, ϕ_z, $z \in \mathbb{C}^\times$. The solutions ψ_z, $z \in \rho(S)^\times$ are called the Weyl solutions.

Theorem 5.3. *The singular Weyl–Titchmarsh function M is analytic with*

$$M(z) = M(z^*)^*, \quad z \in \rho(S)^\times. \tag{5.4}$$

Proof. From Theorem 4.11 we have for each $c \in (a,b)$

$$R_z\delta_c(x) = \langle \delta_c, G_z(x, \cdot)^* \rangle = G_z(x,c), \quad x \in (a,b),\ z \in \rho(S)^\times$$

and hence

$$\langle R_z\delta_c, \delta_c \rangle = G_z(c,c), \quad z \in \rho(S)^\times.$$

Moreover, because of $V(\psi_z,\phi_z)=1$, $z\in\rho(S)^\times$ the diagonal of the kernel G_z is given by

$$\begin{aligned}\langle R_z\delta_c,\delta_c\rangle &= G_z(c,c)=\psi_z(c)\phi_z(c)-\frac{\delta_c(c)}{z}\\ &= M(z)\phi_z(c)^2+\theta_z(c)\phi_z(c)-\frac{w_a(c)w_b(c)}{zW(w_b,w_a)},\quad z\in\rho(S)^\times.\end{aligned}\tag{5.5}$$

But this shows that M is analytic since for each $z\in\mathbb{C}^\times$ there is some $c\in(a,b)$ such that $\phi_z(c)\neq 0$. Moreover, equation (5.4) is valid because the solutions

$$\theta_{z^*}+M(z)^*\phi_{z^*}=(\theta_z+M(z)\phi_z)^*,\quad z\in\rho(S)^\times$$

lie in S near b in view of Lemma 4.7. □

Remark 5.4. *Note that solutions as in Hypothesis 5.1 and Lemma 5.2 are not unique. In fact, any other such solutions are given by*

$$\tilde\theta_z=\mathrm{e}^{-g(z)}z^{-k}\theta_z-f(z)\phi_z\quad\textit{and}\quad\tilde\phi_z=\mathrm{e}^{g(z)}z^k\phi_z,\quad z\in\mathbb{C}^\times$$

for some $k\in\mathbb{Z}$, some real entire function g and some function f which is real analytic in $\mathbb{C}^\times$ and has at most a pole at zero. The corresponding singular Weyl–Titchmarsh functions are related via

$$\tilde M(z)=\mathrm{e}^{-2g(z)}z^{-2k}M(z)+\mathrm{e}^{-g(z)}z^{-k}f(z),\quad z\in\rho(S)^\times.$$

In particular, the maximal domain of holomorphy in $\mathbb{C}^\times$ or the structure of poles and singularities in $\mathbb{C}^\times$ do not change.

Next we will construct a real analytic fundamental system θ_z, ϕ_z, $z\in\mathbb{C}^\times$ of $(\tau-z)u=0$ as in Hypothesis 5.1 and Lemma 5.2, provided that τ is in the l.c. case at the endpoint a.

Theorem 5.5. *Suppose τ is in the l.c. case at a. Then there is a real analytic fundamental system θ_z, ϕ_z, $z\in\mathbb{C}^\times$ of $(\tau-z)u=0$ as in Hypothesis 5.1 and Lemma 5.2 such that for all z_1, $z_2\in\mathbb{C}^\times$ we have additionally*

$$V(\theta_{z_1},\phi_{z_2})(a)=1\quad\textit{and}\quad V(\theta_{z_1},\theta_{z_2})(a)=V(\phi_{z_1},\phi_{z_2})(a)=0.$$

Proof. Let θ, ϕ be a real fundamental system of $\tau u=u$ with $V(\theta,\phi)=1$ such that ϕ lies in S near a. Fix some $c\in(a,b)$ and let for each $z\in\mathbb{C}^\times$, $u_{z,1}$, $u_{z,2}$ be the real fundamental system of $(\tau-z)u=0$ with initial conditions

$$zu_{z,1}(c)=u_{z,2}^{[1]}(c)=1\quad\text{and}\quad u_{z,1}^{[1]}(c)=zu_{z,2}(c)=0.$$

We introduce the solutions

$$\phi_z(x)=V(u_{z,1},\phi)(a)u_{z,2}(x)-V(u_{z,2},\phi)(a)u_{z,1}(x),\quad x\in(a,b),\ z\in\mathbb{C}^\times$$

and the second solutions

$$\theta_z(x)=V(u_{z,1},\theta)(a)u_{z,2}(x)-V(u_{z,2},\theta)(a)u_{z,1}(x),\quad x\in(a,b),\ z\in\mathbb{C}^\times.$$

From this definition it is obvious that $V(\phi_z,\phi)(a)=V(\theta_z,\theta)(a)=0$ for each $z\in\mathbb{C}^\times$ and hence also $V(\phi_{z_1},\phi_{z_2})(a)=V(\theta_{z_1},\theta_{z_2})(a)=0$ for all z_1, $z_2\in\mathbb{C}^\times$ in view of Lemma 4.7. Moreover, using the Plücker identity one obtains $V(\theta_z,\phi)(a)=V(\theta,\phi_z)(a)=1$ for each $z\in\mathbb{C}^\times$ and thus also the remaining equality. Furthermore, a direct calculation shows that $\theta_{z^*}=\theta_z^*$ and $\phi_{z^*}=\phi_z^*$ for $z\in\mathbb{C}^\times$. Hence it remains to prove that the functions

$V(u_{z,1},\theta)(a)$, $V(u_{z,2},\theta)(a)$, $V(u_{z,1},\phi)(a)$ and $V(u_{z,2},\phi)(a)$ are analytic in $z\in\mathbb{C}^\times$. Indeed, from the Lagrange identity we obtain for each $z\in\mathbb{C}^\times$

$$V(u_{z,1},\theta)(a) = V(u_{z,1},\theta)(c) - (z-1)\lim_{\alpha\to a}\int_\alpha^c \theta u_{z,1}\,d\chi + \int_\alpha^c \theta^{[1]}u_{z,1}^{[1]}\,d\varsigma.$$

The integrals on the right-hand side are analytic in $\mathbb{C}^\times$ since the integrands are bounded locally uniformly in $\mathbb{C}^\times$ (see [18, Theorem 3.6]). Hence in order to prove that the limit is also analytic we need to show that the integrals are bounded as $\alpha\to a$, locally uniformly in $z\in\mathbb{C}^\times$. But this holds since

$$\int_a^c |u_{z,1}(x)|^2 d\chi(x) + \int_a^c |u_{z,1}^{[1]}(x)|^2 d\varsigma(x), \quad z\in\mathbb{C}^\times$$

is locally uniformly bounded as an inspection of the proof of [6, Lemma 4] shows. Analyticity of the remaining functions may be proved similarly. ☐

If τ is regular at a and the boundary condition there is given by

$$f_\tau(a)\cos\varphi_\alpha - f^{[1]}(a)\sin\varphi_\alpha = 0, \quad f\in S$$

for some $\varphi_\alpha\in[0,\pi)$, then a real analytic fundamental system θ_z, ϕ_z, $z\in\mathbb{C}^\times$ of $(\tau-z)u=0$ is given for example by the initial conditions

$$z\phi_z(a) = -\theta_z^{[1]}(a) = \sin\varphi_\alpha \quad\text{and}\quad \phi_z^{[1]}(a) = z\theta_z(a) = \cos\varphi_\alpha, \quad z\in\mathbb{C}^\times.$$

It is easily verified that this fundamental system satisfies the properties claimed in Theorem 5.5.

Corollary 5.6. *If τ is in the l.c. case at a and θ_z, ϕ_z, $z\in\mathbb{C}^\times$ is a real analytic fundamental system of $(\tau-z)u=0$ as in Theorem 5.5, then the singular Weyl–Titchmarsh function M is a Herglotz–Nevanlinna function.*

Proof. In order to prove the claim we will show that

$$0 < \|\psi_z\|^2 = \frac{\mathrm{Im}(M(z))}{\mathrm{Im}(z)}, \quad z\in\mathbb{C}\backslash\mathbb{R}. \tag{5.6}$$

Indeed, if z_1, $z_2\in\mathbb{C}\backslash\mathbb{R}$, then

$$\begin{aligned} V(\psi_{z_1},\psi_{z_2})(a) &= V(\theta_{z_1},\theta_{z_2})(a) + M(z_2)V(\theta_{z_1},\phi_{z_2})(a) \\ &\qquad + M(z_1)V(\phi_{z_1},\theta_{z_2})(a) + M(z_1)M(z_2)V(\phi_{z_1},\phi_{z_2})(a) \\ &= M(z_2) - M(z_1). \end{aligned}$$

If τ is in the l.p. case at b, then furthermore we have $V(\psi_{z_1},\psi_{z_2})(b)=0$, since clearly ψ_{z_1}, $\psi_{z_2}\in T_{\max}$. But this also holds if τ is in the l.c. case at b, since then ψ_{z_1} and ψ_{z_2} satisfy the same boundary condition at b. Now using the Lagrange identity from Proposition 2.3 yields

$$(z_1 - z_2^*)\langle\psi_{z_1},\psi_{z_2}\rangle = V(\psi_{z_1},\psi_{z_2}^*)(b) - V(\psi_{z_1},\psi_{z_2}^*)(a) = M(z_1) - M(z_2^*).$$

In particular for $z\in\mathbb{C}\backslash\mathbb{R}$, using $M(z^*) = M(z)^*$ as well as

$$\psi_{z^*} = \theta_{z^*} + M(z^*)\phi_{z^*} = \psi_z^*,$$

we get

$$0 < \|\psi_z\|^2 = \langle\psi_z,\psi_z\rangle = \frac{M(z)-M(z^*)}{z-z^*} = \frac{\mathrm{Im}(M(z))}{\mathrm{Im}(z)}.$$

☐

The relation between two fundamental systems in Remark 5.4 can be strengthened if they satisfy the additional properties of Theorem 5.5. In fact, it is not hard to see that in this case f, g are constant and $k = 0$. Moreover, from the construction in the proof of Theorem 5.5 it follows that for such fundamental systems the functions in (5.1) and (5.2) have at most simple poles at zero.

As in the right-definite case (see [20, Lemma 3.2], [25, Lemma 2.2], [18, Theorem 9.6]) we may give a necessary and sufficient condition for Hypothesis 5.1 to hold. Therefore fix some $c \in (a, b)$ such that $\chi((a, c)) \neq 0$ and consider the maximal relation in $H^1(a, c)$ associated with our differential expression restricted to (a, c). With S_c we denote the self-adjoint restriction of this relation with Dirichlet boundary conditions at c and the same boundary conditions as S near a.

Theorem 5.7. *Hypothesis 5.1 holds if and only if the self-adjoint relation S_c has purely discrete spectrum.*

Proof. Consider the real analytic fundamental system $\theta_{c,z}$, $\phi_{c,z}$, $z \in \mathbb{C}^\times$ of $(\tau - z)u = 0$ with initial conditions

$$z\phi_{c,z}(c) = \theta_{c,z}^{[1]}(c) = 0 \quad \text{and} \quad \phi_{c,z}^{[1]}(c) = z\theta_{c,z}(c) = 1, \quad z \in \mathbb{C}^\times.$$

Now define the function M_c on $\mathbb{C}\backslash\mathbb{R}$ by requiring that the solutions

$$\psi_{c,z} = \theta_{c,z} - M_c(z)\phi_{c,z}, \quad z \in \mathbb{C}\backslash\mathbb{R}$$

lie in S_c near a. Of course M_c is the restriction of a Weyl–Titchmarsh function of S_c with the roles of the left and the right endpoint reversed. Now if we suppose that S_c has purely discrete spectrum then the function M_c turns out to be a meromorphic Herglotz–Nevanlinna function. By the Weierstraß product theorem there is a real entire function G which has simple zeros at precisely all poles of M_c which are not removable. Thus the real analytic solution ϕ_z, $z \in \mathbb{C}^\times$, given by

$$\phi_z(x) = G(z)\psi_{c,z}(x) = G(z)\theta_{c,z}(x) - G(z)M_c(z)\phi_{c,z}(x), \quad x \in (a, b),$$

has the properties claimed in Hypothesis 5.1. Conversely, if Hypothesis 5.1 holds then $\psi_{c,z}$ and ϕ_z are linearly dependent for each $z \in \mathbb{C}\backslash\mathbb{R}$. Therefore

$$0 = V(\psi_{c,z}, \phi_z) = V(\theta_{c,z}, \phi_z) - M_c(z)V(\phi_{z,c}, \phi_z), \quad z \in \mathbb{C}\backslash\mathbb{R},$$

and hence M_c is actually meromorphic in $\mathbb{C}$. From (5.5) we infer that for each f in a dense subspace of $H^1(a, c)$ the functions

$$\langle (S_c - z)^{-1}f, f\rangle_{H^1(a,c)}, \quad z \in \mathbb{C}\backslash\mathbb{R}$$

have an analytic continuation to $\mathbb{C}^\times\backslash\Omega$, where Ω is the set of poles of M_c. But from this we infer that the operator-valued spectral measure E_c of S_c (see [18, Lemma B.3 and Lemma B.4]) satisfies $E_c(\mathbb{R}^\times\backslash\Omega) = 0$, i.e. the spectrum of S_c is purely discrete. □

If Hypothesis 5.1 holds at both endpoints then it is easily seen that all associated singular Weyl–Titchmarsh functions are meromorphic in $\mathbb{C}$. Now a similar argument as in the proof of Theorem 5.7 shows that in this case the spectrum of S is purely discrete. In particular, S has purely discrete spectrum provided that τ is in the l.c. case at both endpoints.

6. Spectral transformation

In this section let S again be a self-adjoint restriction of $T_{\max}$ with separate boundary conditions such that Hypothesis 5.1 holds. Moreover, we will assume that zero is not an eigenvalue of S. For example this excludes the case where ς and χ are finite near an endpoint which is in the l.p. case. Another case excluded hereby are Dirichlet boundary conditions at regular endpoints.

By the spectral theorem, for all functions f, $g \in H^1(a,b)$ there is a unique complex Borel measure $E_{f,g}$ on $\mathbb{R}$ such that

$$\langle R_z f, g\rangle = \int_{\mathbb{R}} \frac{1}{\lambda - z} dE_{f,g}(\lambda), \quad z \in \rho(S). \tag{6.1}$$

Indeed, these measures are obtained by applying a variant of the spectral theorem to the operator part

$$S_{\mathfrak{D}} = S \cap (\mathfrak{D} \times \mathfrak{D}), \quad \mathfrak{D} = \overline{\mathrm{dom}(S)} = \mathrm{mul}(S)^{\perp}$$

of S (see e.g. [18, Lemma B.4]). In this section we will introduce a spectral transformation for our self-adjoint linear relation S. Therefore we first need to construct our spectral measure.

Lemma 6.1. *There is a unique Borel measure μ on $\mathbb{R}^{\times}$ such that*

$$E_{\delta_\alpha,\delta_\beta}(B) = \int_{B^{\times}} \phi_\lambda(\alpha)\phi_\lambda(\beta) d\mu(\lambda) \tag{6.2}$$

for all α, $\beta \in (a,b)$ and each Borel set $B \subseteq \mathbb{R}$. In particular,

$$\langle R_z \delta_\alpha, \delta_\beta\rangle = \int_{\mathbb{R}^{\times}} \frac{\phi_\lambda(\alpha)\phi_\lambda(\beta)}{\lambda - z} d\mu(\lambda), \quad z \in \rho(S). \tag{6.3}$$

Proof. As in the proof of Theorem 5.3 one sees that for α, $\beta \in (a,b)$

$$\langle R_z \delta_\alpha, \delta_\beta\rangle = M(z)\phi_z(\alpha)\phi_z(\beta) + H_{\alpha,\beta}(z), \quad z \in \rho(S)^{\times}, \tag{6.4}$$

where $H_{\alpha,\beta}$ is a real analytic function on $\mathbb{C}^{\times}$. In particular, this shows that

$$\langle R_{z^*} \delta_\alpha, \delta_\beta\rangle = \langle R_z \delta_\alpha, \delta_\beta\rangle^*, \quad z \in \rho(S)^{\times}$$

and hence $E_{\delta_\alpha,\delta_\beta}$ is a real-valued measure, in view of equation (6.1) and the Stieltjes inversion formula. Moreover, the Stieltjes inversion formula shows that for each continuous function F on $\mathbb{R}$

$$(*) \qquad \int_{\lambda_1}^{\lambda_2} F(\lambda) dE_{\delta_\alpha,\delta_\beta}(\lambda) = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} \frac{1}{\pi} \int_{\lambda_1-\delta}^{\lambda_2-\delta} F(\lambda)\, \mathrm{Im}\langle R_{\lambda+\mathrm{i}\varepsilon}\delta_\alpha, \delta_\beta\rangle \, d\lambda,$$

where λ_1, $\lambda_2 \in \mathbb{R}$ with $\lambda_1 < \lambda_2$. Now if $[\lambda_1, \lambda_2] \subseteq \mathbb{R}^{\times}$, then we define

$$\mu([\lambda_1,\lambda_2)) = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} \frac{1}{\pi} \int_{\lambda_1-\delta}^{\lambda_2-\delta} \mathrm{Im}\, M(\lambda + \mathrm{i}\varepsilon)\, d\lambda. \tag{6.5}$$

Given $\lambda_0 \in \mathbb{R}^{\times}$, pick α, $\beta \in (a,b)$ such that $\phi_{\lambda_0}(\alpha) \neq 0 \neq \phi_{\lambda_0}(\beta)$ and set

$$F(\lambda) = \begin{cases} \phi_{\lambda_-}(\alpha)^{-1}\phi_{\lambda_-}(\beta)^{-1}, & \text{if } \lambda \in (-\infty, \lambda_-), \\ \phi_{\lambda}(\alpha)^{-1}\phi_{\lambda}(\beta)^{-1}, & \text{if } \lambda \in [\lambda_-, \lambda_+], \\ \phi_{\lambda_+}(\alpha)^{-1}\phi_{\lambda_+}(\beta)^{-1}, & \text{if } \lambda \in (\lambda_+, \infty), \end{cases}$$

where $[\lambda_-,\lambda_+]\subseteq\mathbb{R}^\times$ is such that $\lambda_0\in(\lambda_-,\lambda_+)$ and $\phi_\lambda(\alpha)\neq 0\neq\phi_\lambda(\beta)$ for all $\lambda\in[\lambda_-,\lambda_+]$. With this definition, from (6.4) and $(*)$ we get

$$\begin{aligned}\mu([\lambda_1,\lambda_2)) &= \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\lambda_1-\delta}^{\lambda_2-\delta}\phi_\lambda(\alpha)^{-1}\phi_\lambda(\beta)^{-1}\operatorname{Im}\langle R_{\lambda+\mathrm{i}\varepsilon}\delta_\alpha,\delta_\beta\rangle\,d\lambda \\ &= \int_{\lambda_1}^{\lambda_2}\phi_\lambda(\alpha)^{-1}\phi_\lambda(\beta)^{-1}\,dE_{\delta_\alpha,\delta_\beta}(\lambda)\end{aligned}\tag{6.6}$$

for all λ_1, $\lambda_2\in(\lambda_-,\lambda_+)$ with $\lambda_1<\lambda_2$. Hereby we used that

$$\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\int_{\lambda_1-\delta}^{\lambda_2-\delta}\varepsilon|\langle R_{\lambda+\mathrm{i}\varepsilon}\delta_\alpha,\delta_\beta\rangle|d\lambda=0,$$

which holds since the function $z\mapsto\langle R_z\delta_\alpha,\delta_\beta\rangle$ actually is the difference of two Herglotz–Nevanlinna functions (see e.g. [19, Theorem 2.3]). In particular, equation (6.6) shows that the limit in (6.5) actually exists. Moreover, choosing $\alpha=\beta$ it follows that the distribution function $\lambda\mapsto\mu([\lambda_-,\lambda))$ is non-decreasing on (λ_-,λ_+), hence μ uniquely extends to a positive Borel measure on $\mathbb{R}^\times$. Equation (6.6) furthermore shows that (6.2) holds for all Borel sets $B\subseteq\mathbb{R}^\times$ not containing zeros of the analytic functions $z\mapsto\phi_z(\alpha)$ and $z\mapsto\phi_z(\beta)$. Moreover, if $\lambda_0\in\mathbb{R}^\times$ was such that $\phi_{\lambda_0}(\alpha)\phi_{\lambda_0}(\beta)=0$ and $E(\{\lambda_0\})\neq 0$, then we would obtain the contradiction

$$E_{\delta_\alpha,\delta_\beta}(\{\lambda_0\})=\langle E(\{\lambda_0\})\delta_\alpha,\delta_\beta\rangle=\langle\delta_\alpha,\phi_{\lambda_0}\rangle\langle\phi_{\lambda_0},\delta_\beta\rangle=\phi_{\lambda_0}(\alpha)\phi_{\lambda_0}(\beta)=0,$$

since ϕ_{λ_0} would be an eigenfunction in that case. Hence we see that (6.2) holds for all Borel sets $B\subseteq\mathbb{R}$ (also note that $E_{\delta_\alpha,\delta_\beta}(\{0\})=0$ since zero is no eigenvalue of S). □

We are now able to define our spectral transform. If F is a Borel measurable function on $\mathbb{R}^\times$, then we denote with M_F the maximally defined operator of multiplication with F in $L^2(\mathbb{R}^\times;\mu)$.

Lemma 6.2. *There is a unique bounded linear operator*

$$\mathcal{F}:\ H^1(a,b)\to L^2(\mathbb{R}^\times;\mu)$$

such that for each $c\in(a,b)$ we have $\mathcal{F}\delta_c(\lambda)=\phi_\lambda(c)$ for almost all $\lambda\in\mathbb{R}^\times$ with respect to μ. The operator $\mathcal{F}$ is a surjective partial isometry with initial subspace $\mathfrak{D}$. Its adjoint is given by

$$\mathcal{F}^*g(x)=\int_{\mathbb{R}^\times}\phi_\lambda(x)g(\lambda)d\mu(\lambda),\quad x\in(a,b),\ g\in L^2(\mathbb{R}^\times;\mu),$$

its (in general multi-valued) inverse is given by

$$\mathcal{F}^{-1}=\{(g,f)\in L^2(\mathbb{R}^\times;\mu)\times H^1(a,b)\,|\,\mathcal{F}^*g-f\in\operatorname{mul}(S)\}.$$

Proof. First of all note that the functions δ_c, $c\in(a,b)$ are linearly independent since otherwise there were distinct points c, $c_1,\dots,c_N\in(a,b)$ and $a_n\in\mathbb{C}$, $n=1,\dots,N$ such that

$$g(c)=\sum_{n=1}^N a_n g(c_n),\quad g\in H^1(a,b).$$

But this gives a contradiction since there are functions in $H^1(a,b)$, vanishing in the points $c_1,\dots,c_N$ but not in c. Hence there is a unique linear operator

$\mathcal{F}$ on the linear span of all functions δ_c, $c \in (a,b)$ such that for all $c \in (a,b)$ one has $\mathcal{F}\delta_c(\lambda) = \phi_\lambda(c)$ for almost all $\lambda \in \mathbb{R}^\times$ with respect to μ. Thereby, note that the functions $z \mapsto \phi_\lambda(c)$, $c \in (a,b)$ are square integrable with respect to μ in view of Lemma 6.1. Furthermore, from Lemma 6.1 one sees that for each c_1, $c_2 \in (a,b)$

$$\langle P\delta_{c_1}, \delta_{c_2}\rangle = E_{\delta_{c_1},\delta_{c_2}}(\mathbb{R}) = \int_{\mathbb{R}^\times} \phi_\lambda(c_1)\phi_\lambda(c_2) d\mu(\lambda) = \langle \mathcal{F}\delta_{c_1}, \mathcal{F}\delta_{c_2}\rangle_\mu,$$

where P is the orthogonal projection from $H^1(a,b)$ onto $\mathfrak{D}$. Moreover, from linearity we also have

$$(*) \qquad \langle \mathcal{F}f_1, \mathcal{F}f_2\rangle_\mu = \langle Pf_1, f_2\rangle, \quad f_1,\ f_2 \in \operatorname{span}\{\delta_c \,|\, c \in (a,b)\}.$$

In particular, $\mathcal{F}$ is bounded on the linear span of all functions δ_c, $c \in (a,b)$ and since this span is dense in $H^1(a,b)$, $\mathcal{F}$ uniquely extends to a bounded linear operator on $H^1(a,b)$. Moreover, equation $(*)$ still holds for all functions f_1, $f_2 \in H^1(a,b)$ and hence the operator $\mathcal{F}$ is a partial isometry with initial subspace $\mathfrak{D}$. Also note that (6.2) from Lemma 6.1 now extends to all of $H^1(a,b)$ by continuity, i.e.

$$(6.7) \qquad E_{f,g}(B) = \int_{B^\times} \mathcal{F}f(\lambda)\, \mathcal{F}g(\lambda)^*\, d\mu(\lambda), \quad f,\ g \in H^1(a,b)$$

for each Borel set $B \subseteq \mathbb{R}$. In order to prove that $\mathcal{F}$ is onto, let $c \in (a,b)$ and F, G be some bounded measurable functions on $\mathbb{R}^\times$. Since E is the spectral measure of the operator part $S_{\mathfrak{D}}$ of S (see e.g. [18, Lemma B.4]) we get

$$\begin{aligned} GF(S_{\mathfrak{D}})P\delta_c(x) = \langle GF(S_{\mathfrak{D}})P\delta_c, \delta_x\rangle &= \int_{\mathbb{R}} G(\lambda)F(\lambda) dE_{\delta_c,\delta_x}(\lambda) \\ &= \int_{\mathbb{R}^\times} G(\lambda)F(\lambda)\phi_\lambda(c)\phi_\lambda(x) d\mu(\lambda) \end{aligned}$$

for each $x \in (a,b)$. On the other side a similar calculation yields

$$\begin{aligned} GF(S_{\mathfrak{D}})P\delta_c(x) &= G(S_{\mathfrak{D}})F(S_{\mathfrak{D}})P\delta_c(x) \\ &= \int_{\mathbb{R}^\times} G(\lambda)\mathcal{F}F(S_{\mathfrak{D}})P\delta_c(\lambda)\phi_\lambda(x) d\mu(\lambda). \end{aligned}$$

Since G was arbitrary, we infer that $F(\lambda)\phi_\lambda(c)\phi_\lambda(x) = \mathcal{F}F(S_{\mathfrak{D}})P\delta_c(\lambda)\phi_\lambda(x)$ for almost all $\lambda \in \mathbb{R}^\times$ with respect to μ. Moreover, since for each given $\lambda_0 \in \mathbb{R}^\times$ there is some point $x \in (a,b)$ such that $\phi_{\lambda_0}(x) \neq 0$ we even get that $F(\lambda)\phi_\lambda(c) = \mathcal{F}F(S_{\mathfrak{D}})P\delta_c(\lambda)$ for almost all $\lambda \in \mathbb{R}^\times$ with respect to μ. But this shows that the range of $\mathcal{F}$ contains all characteristic functions of intervals in $\mathbb{R}^\times$. More precisely, let $\lambda_0 \in \mathbb{R}^\times$ and choose some $c \in (a,b)$ such that $\phi_{\lambda_0}(c) \neq 0$. Then for each small enough interval $J \subseteq \mathbb{R}^\times$ around λ_0 the function

$$F(\lambda) = \begin{cases} \phi_\lambda(c)^{-1}, & \text{if } \lambda \in J, \\ 0, & \text{if } \lambda \in \mathbb{R}^\times \backslash J, \end{cases}$$

is bounded and $\mathcal{F}F(S_{\mathfrak{D}})P\delta_c = \mathrm{M}_F\mathcal{F}\delta_c = \mathbb{1}_J$ lies in the range of $\mathcal{F}$, which proves surjectivity. Now a simple calculation shows that the adjoint of $\mathcal{F}$ is given by

$$\mathcal{F}^* g(x) = \langle \mathcal{F}^* g, \delta_x \rangle = \langle g, \mathcal{F}\delta_x \rangle_\mu = \int_{\mathbb{R}^\times} \phi_\lambda(x) g(\lambda) d\mu(\lambda), \quad x \in (a,b)$$

for each $g \in L^2(\mathbb{R}^\times;\mu)$. Furthermore, the inverse is given as in the claim because of the equivalence

$$\begin{aligned} \mathcal{F}^* g - f \in \mathrm{mul}(S) \quad &\Leftrightarrow \quad \forall h \in L^2(\mathbb{R}^\times;\mu) : \langle f, \mathcal{F}^* h\rangle = \langle \mathcal{F}\mathcal{F}^* g, h\rangle_\mu \\ &\Leftrightarrow \quad (f,g) \in \mathcal{F} = \mathcal{F}^{**}, \end{aligned}$$

which holds for every $f \in H^1(a,b)$ and $g \in L^2(\mathbb{R}^\times;\mu)$. Here we used that $\mathcal{F}\mathcal{F}^*$ is the identity since $\mathcal{F}$ is a surjective partial isometry. □

There is also another way of defining the linear operator $\mathcal{F}$ as some integral transform, which may be more appealing. The approach taken here simplifies the proofs but it has the main disadvantage that the transforms are not given very explicitly. Hence we will show next that these two ways of introducing the spectral transform are actually the same. Indeed, provided that τ is in the l.c. case at b, we obviously have

$$\mathcal{F}\delta_c(\lambda) = \phi_\lambda(c) = \langle \phi_\lambda, \delta_c \rangle, \quad \lambda \in \mathbb{R}^\times$$

and hence by continuity also

$$\mathcal{F}f(\lambda) = \langle \phi_\lambda, f^* \rangle = \int_a^b \phi_\lambda(x) f(x) d\chi(x) + \int_a^b \phi_\lambda^{[1]}(x) f^{[1]}(x) d\varsigma(x), \quad \lambda \in \mathbb{R}^\times$$

for each $f \in H^1(a,b)$. Otherwise, if τ is in the l.p. case at b, then this is not possible since ϕ_λ does not lie in $H^1(a,b)$ unless λ is an eigenvalue. However, we still have the following general result.

Proposition 6.3. *If $f \in H^1(a,b)$ vanishes near b, then*

$$\mathcal{F}f(\lambda) = \int_a^b \phi_\lambda(x) f(x) d\chi(x) + \int_a^b \phi_\lambda^{[1]}(x) f^{[1]}(x) d\varsigma(x)$$

for almost all $\lambda \in \mathbb{R}^\times$ with respect to μ.

Proof. First of all note that for $\lambda \in \mathbb{R}^\times$ and $c \in (a,b)$, integration by parts shows that

$$(*) \quad \int_a^x \phi_\lambda \delta_c \, d\chi + \int_a^x \phi_\lambda^{[1]} \delta_c^{[1]} d\varsigma = \phi_\lambda(c) + \frac{w_a(c) w_b^{[1]}(x)}{W(w_b, w_a)} \phi_\lambda(x), \quad x \in (c,b).$$

Now pick some $\beta \in (a,b)$ such that f vanishes on $[\beta,b)$ and consider the space H_β of functions in $H^1(a,b)$ which are equal to a scalar multiple of w_b on $[\beta,b)$. It is not hard to see that this space is closed and that it contains all functions δ_c, $c \leq \beta$. Moreover, the linear span of these functions is even dense in H_β, i.e. f lies in the closure of $\mathrm{span}\{\delta_c \,|\, c \leq \beta\}$. Now for each $k \in \mathbb{N}$ let $N(k) \in \mathbb{N}$ and $a_n^k \in \mathbb{C}$, $c_n^k \in (a,\beta)$ for $n = 1, \ldots N(k)$ such that the functions

$$f_k(x) = \sum_{n=1}^{N(k)} a_n^k \delta_{c_n^k}(x), \quad x \in (a,b), \; k \in \mathbb{N}$$

converge to f in $H^1(a,b)$ as $k\to\infty$. Using equation $(*)$ we may estimate for each $\lambda\in\mathbb{R}^\times$

$$\begin{aligned}&\left|\int_a^b \phi_\lambda(x)f(x)d\chi(x)+\int_a^b \phi_\lambda^{[1]}(x)f^{[1]}(x)d\varsigma(x)-\sum_{n=1}^{N(k)} a_n^k\phi_\lambda(c_n^k)\right|\\ &\qquad\le\left|\int_a^\beta \phi_\lambda(f-f_k)d\chi+\int_a^\beta \phi_\lambda^{[1]}(f^{[1]}-f_k^{[1]})d\varsigma\right|+\left|\phi_\lambda(\beta)\frac{w_b^{[1]}(\beta)}{w_b(\beta)}f_k(\beta)\right|.\end{aligned}$$

The first term converges to zero since f_k converges to f in $H^1(a,b)$ as $k\to\infty$. Moreover, the second term converges to zero since $f_k(\beta)$ converges to zero as $k\to\infty$. But this proves the claim since $\mathcal{F}f_k(\lambda)$ converges to $\mathcal{F}f(\lambda)$ for almost all $\lambda\in\mathbb{R}^\times$ with respect to μ. □

We are now ready to prove the main theorem of this section.

Theorem 6.4. *The self-adjoint relation S is given by $S=\mathcal{F}^{-1}\mathrm{M}_{\mathrm{id}}\mathcal{F}$.*

Proof. First of all note that for each $f\in\mathfrak{D}$ we have (see e.g. [18, Lemma B.4])

$$\begin{aligned} f\in\operatorname{dom}(S)\quad&\Leftrightarrow\quad\int_{\mathbb{R}}|\lambda|^2dE_{f,f}(\lambda)<\infty\quad\Leftrightarrow\quad\int_{\mathbb{R}^\times}|\lambda|^2|\mathcal{F}f(\lambda)|^2d\mu(\lambda)<\infty\\ &\Leftrightarrow\quad\mathcal{F}f\in\operatorname{dom}(\mathrm{M}_{\mathrm{id}})\qquad\Leftrightarrow\quad f\in\operatorname{dom}(\mathcal{F}^{-1}\mathrm{M}_{\mathrm{id}}\mathcal{F}).\end{aligned}$$

Furthermore, if $(f,f_\tau)\in S$, then from Lemma 6.1 and [18, Lemma B.4] we infer that for each $x\in(a,b)$

$$\begin{aligned}Pf_\tau(x)=\langle Pf_\tau,\delta_x\rangle&=\int_{\mathbb{R}}\lambda dE_{f,\delta_x}(\lambda)=\int_{\mathbb{R}^\times}\lambda\mathcal{F}f(\lambda)\phi_\lambda(x)d\mu(\lambda)\\ &=\int_{\mathbb{R}^\times}\mathrm{M}_{\mathrm{id}}\mathcal{F}f(\lambda)\phi_\lambda(x)d\mu(\lambda)=\mathcal{F}^*\mathrm{M}_{\mathrm{id}}\mathcal{F}f(x).\end{aligned}$$

This and Lemma 6.2 show that $(\mathrm{M}_{\mathrm{id}}\mathcal{F}f,f_\tau)\in\mathcal{F}^{-1}$, which is equivalent to $(f,f_\tau)\in\mathcal{F}^{-1}\mathrm{M}_{\mathrm{id}}\mathcal{F}$. Conversely, if we assume that $(g,g_\tau)\in\mathcal{F}^{-1}\mathrm{M}_{\mathrm{id}}\mathcal{F}$, then $(\mathrm{M}_{\mathrm{id}}\mathcal{F}g,g_\tau)\in\mathcal{F}^{-1}$ (also note that g lies in the domain of S). From this we infer that $\mathcal{F}^*\mathrm{M}_{\mathrm{id}}\mathcal{F}g-g_\tau\in\operatorname{mul}(S)$ and because of $(g,\mathcal{F}^*\mathrm{M}_{\mathrm{id}}\mathcal{F}g)\in S$, we furthermore get $(g,g_\tau)\in S$. □

Note that all of the multi-valuedness of S is only contained in the inverse of our spectral transform. Moreover, the self-adjoint operator part $S_{\mathfrak{D}}$ of S is unitarily equivalent to the operator of multiplication M_{id} in $L^2(\mathbb{R}^\times;\mu)$. In fact, $\mathcal{F}$ is unitary as an operator from $\mathfrak{D}$ onto $L^2(\mathbb{R}^\times;\mu)$ and maps the operator part of S onto multiplication with the independent variable. Now the spectrum of S can be read off from the boundary behavior of the singular Weyl–Titchmarsh function M in the usual way.

Corollary 6.5. *The spectrum of S is given by*

$$\sigma(S)^\times=\operatorname{supp}(\mu)=\overline{\{\lambda\in\mathbb{R}^\times\,|\,0<\limsup_{\varepsilon\downarrow0}\operatorname{Im}M(\lambda+\mathrm{i}\varepsilon)\,\}}.$$

Proof. Since the operator part $S_{\mathfrak{D}}$ of S is unitarily equivalent to M_{id} we infer that (see e.g. [18, Lemma B.3]) $\sigma(S)^\times=\sigma(\mathrm{M}_{\mathrm{id}})^\times=\operatorname{supp}(\mu)$. Now let $[\lambda_1, \lambda_2] \subseteq \mathbb{R}^\times$ and $c \in (a,b)$ such that $\phi_\lambda(c) \neq 0$ for $\lambda \in [\lambda_1, \lambda_2]$. Then equation (6.4) shows that

$$\begin{aligned}\{\lambda \in \mathbb{R}^\times \,|\, 0 < \limsup_{\varepsilon\downarrow 0} \operatorname{Im} M(\lambda + \mathrm{i}\varepsilon)\,\} \cap [\lambda_1, \lambda_2] \\ = \{\lambda \in \mathbb{R} \,|\, 0 < \limsup_{\varepsilon\downarrow 0} \operatorname{Im}\langle R_{\lambda+\mathrm{i}\varepsilon}\delta_c, \delta_c\rangle\,\} \cap [\lambda_1, \lambda_2].\end{aligned}$$

Moreover, the measures μ and E_{δ_c,δ_c} restricted to $[\lambda_1, \lambda_2]$ are mutually absolutely continuous, hence $\operatorname{supp}(\mu) \cap [\lambda_1, \lambda_2] = \operatorname{supp}(E_{\delta_c,\delta_c}) \cap [\lambda_1, \lambda_2]$. Now the claim follows from standard results (see e.g. [31, Lemma 3.14]). □

With similar arguments as in the proof of the previous corollary one may show that the set of eigenvalues of S is given by

$$\sigma_p(S) = \{\lambda \in \mathbb{R}^\times \mid \lim_{\varepsilon\downarrow 0} \varepsilon \operatorname{Im} M(\lambda + \mathrm{i}\varepsilon) > 0\}.$$

The value of μ at such an eigenvalue may be computed explicitly.

Proposition 6.6. *If $\lambda \in \sigma(S)$ is an eigenvalue, then*

$$\mu(\{\lambda\}) = \|\phi_\lambda\|^{-2}.$$

Proof. Under this assumptions ϕ_λ is an eigenfunction, i.e. $(\phi_\lambda, \lambda\phi_\lambda) \in S$ and $\mathcal{F}f(\lambda) = \langle f, \phi_\lambda\rangle$, $f \in H^1(a,b)$. Now from equation (6.7) we get

$$\|\phi_\lambda\|^2 = E_{\phi_\lambda,\phi_\lambda}(\{\lambda\}) = \mathcal{F}\phi_\lambda(\lambda)\, \mathcal{F}\phi_\lambda(\lambda)^* \mu(\{\lambda\}) = \|\phi_\lambda\|^4 \mu(\{\lambda\}),$$

since $E(\{\lambda\})$ is the orthogonal projection onto the space spanned by ϕ_λ. □

Note that the measure μ is uniquely determined by the property that the mapping $\delta_c \mapsto \phi_\lambda(c)$, $c \in (a,b)$ uniquely extends to a partial isometry onto $L^2(\mathbb{R}^\times; \mu)$, which maps S onto multiplication with the independent variable. Because of this, the measure μ is referred to as the spectral measure of S associated with the real analytic solutions ϕ_z, $z \in \mathbb{C}^\times$.

Remark 6.7. *Given another real analytic solution as in Remark 5.4, the corresponding spectral measures are related by*

$$\tilde{\mu}(B) = \int_B \mathrm{e}^{-2g(\lambda)} \lambda^{-2k} d\mu(\lambda)$$

for each Borel set $B \subseteq \mathbb{R}^\times$, where $k \in \mathbb{Z}$ and g is the real entire function from Remark 5.4. In particular, the measures are mutually absolutely continuous and the associated spectral transforms just differ by a simple rescaling with a positive function.

7. Associated de Branges spaces

As in the previous sections let S be some self-adjoint restriction of $T_{\max}$ (with separate boundary conditions) which does not have zero as an eigenvalue. The aim of the present section is to describe the spaces of transforms of functions in $H^1(a,b)$ with compact support. It will turn out that these spaces are hyperplanes in some de Branges spaces associated with our left-definite Sturm–Liouville problem, at least if we somewhat strengthen Hypothesis 5.1. In fact, in this section we will assume that for each $z \in \mathbb{C}$

there is a non-trivial solution ϕ_z of $(\tau - z)u = 0$ such that ϕ_z lies in S near a and the functions

$$z \mapsto \phi_z(c) \quad \text{and} \quad z \mapsto \phi_z^{[1]}(c)$$

are real entire for each $c \in (a, b)$. In particular, note that the solution ϕ_0 is always a scalar multiple of the solution w_a (due to the assumption that zero is not an eigenvalue of S). For example, if τ is regular at a and the boundary condition at a is given by (4.10) for some $\varphi_\alpha \in (0, \pi)$, then such a real entire solution ϕ_z, $z \in \mathbb{C}$ of $(\tau - z)u = 0$ is given by the initial conditions

$$\phi_z(a) = \sin\varphi_\alpha \quad \text{and} \quad \phi_z^{[1]}(a) = z\cos\varphi_\alpha, \quad z \in \mathbb{C}.$$

Furthermore, we will assume that the measure ς is absolutely continuous with respect to the Lebesgue measure. This will guarantee that our de Branges spaces are continuous in some sense, which simplifies the discussion to some extend. However, we do not have to impose additional assumptions on the measures χ and ϱ.

First of all we will introduce the de Branges spaces associated with S and our real entire solution ϕ_z, $z \in \mathbb{C}$. For a brief review of de Branges' theory of Hilbert spaces of entire functions see Appendix A, whereas for a detailed account we refer to de Branges' book [14]. Now fix some $c \in (a, b)$ and consider the entire function

$$E(z, c) = z\phi_z(c) + \mathrm{i}\phi_z^{[1]}(c), \quad z \in \mathbb{C}. \tag{7.1}$$

Then this function is a de Branges function, i.e. it satisfies

$$|E(z, c)| > |E(z^*, c)|, \quad z \in \mathbb{C}^+,$$

where $\mathbb{C}^+$ is the upper open complex half-plane. Indeed, a simple calculation, using the Lagrange identity from Proposition 2.3 shows that

$$\begin{aligned} \frac{E(z,c)E^{\#}(\zeta^*,c) - E(\zeta^*,c)E^{\#}(z,c)}{2\mathrm{i}(\zeta^* - z)} &= \frac{\zeta^*\phi_\zeta(c)^*\phi_z^{[1]}(c) - z\phi_z(c)\phi_\zeta^{[1]}(c)^*}{\zeta^* - z} \\ &= \int_a^c \phi_\zeta^*\phi_z \, d\chi + \int_a^c \phi_\zeta^{[1]*}\phi_z^{[1]} d\varsigma \end{aligned}$$

for each ζ, $z \in \mathbb{C}^+$. In particular, choosing $\zeta = z$ this equality shows that our function $E(\,\cdot\,, c)$ is a de Branges function. Hence it gives rise to a de Branges space $B(c)$ equipped with the inner product

$$[F, G]_{B(c)} = \frac{1}{\pi}\int_{\mathbb{R}} \frac{F(\lambda)G(\lambda)^*}{|E(\lambda, c)|^2} d\lambda, \quad F,\, G \in B(c).$$

Moreover, note that $E(\,\cdot\,, c)$ does not have any real zeros λ. Indeed, if $\lambda \neq 0$ this would mean that both, ϕ_λ and its quasi-derivative vanish in c and if $\lambda = 0$ this would contradict the fact that ϕ_0 is a scalar multiple of w_a.

The reproducing kernel $K(\,\cdot\,, \cdot\,, c)$ of the de Branges space $B(c)$ is given as in equation (A.1). A similar calculation as above, using the Lagrange identity shows that for all ζ, $z \in \mathbb{C}$ it may be written as

$$K(\zeta, z, c) = \int_a^c \phi_\zeta(x)^*\phi_z(x) d\chi(x) + \int_a^c \phi_\zeta^{[1]}(x)^*\phi_z^{[1]}(x) d\varsigma(x). \tag{7.2}$$

In the following, the function $K(0,\cdot\,,c)$ will be of particular interest. An integration by parts as in (2.4) shows that this function may as well be written as

$$K(0,z,c) = \phi_0^{[1]}(c)\phi_z(c), \quad z \in \mathbb{C}, \tag{7.3}$$

where the boundary term at a vanishes since ϕ_0 is a scalar multiple of w_a.

We want to link the de Branges space $B(c)$ to our generalized Fourier transform $\mathcal{F}$, using Proposition 6.3. Therefore consider the modified Sobolev space $H^1(a,c)$ and define the transform of a function $f \in H^1(a,c)$ as

$$\hat{f}(z) = \int_a^c \phi_z(x) f(x) d\chi(x) + \int_a^c \phi_z^{[1]}(x) f^{[1]}(x) d\varsigma(x), \quad z \in \mathbb{C}. \tag{7.4}$$

We will now identify the de Branges space $B(c)$ with the space of transforms of functions from the subspace

$$D(c) = \overline{\operatorname{span}\{\phi_z|_{(a,c)} \,|\, z \in \mathbb{C}\}}$$

of $H^1(a,c)$, equipped with the norm inherited from $H^1(a,c)$.

Theorem 7.1. *The transformation $f \mapsto \hat{f}$ is a partial isometry from the modified Sobolev space $H^1(a,c)$ onto $B(c)$ with initial subspace $D(c)$*

Proof. For all $\zeta \in \mathbb{C}$, the transform of the function $f_\zeta = \phi_\zeta^*|_{(a,c)}$ is given by

$$\hat{f}_\zeta(z) = \int_a^c \phi_\zeta(x)^* \phi_z(x) d\chi(x) + \int_a^c \phi_\zeta^{[1]}(x)^* \phi_z^{[1]}(x) d\varsigma(x) = K(\zeta, z, c)$$

for each $z \in \mathbb{C}$ and hence lies in the de Branges space $B(c)$. Moreover, for some given ζ_1, $\zeta_2 \in \mathbb{C}$ we have

$$\begin{aligned}\langle f_{\zeta_1}, f_{\zeta_2}\rangle_{H^1(a,c)} &= \int_a^c \phi_{\zeta_1}(x)^* \phi_{\zeta_2}(x) d\chi(x) + \int_a^c \phi_{\zeta_1}^{[1]}(x)^* \phi_{\zeta_2}^{[1]}(x) d\varsigma(x) \\ &= K(\zeta_1,\zeta_2,c) = [K(\zeta_1,\cdot\,,c), K(\zeta_2,\cdot\,,c)]_{B(c)} = [\hat{f}_{\zeta_1}, \hat{f}_{\zeta_2}]_{B(c)}.\end{aligned}$$

Now, since the functions $K(\zeta,\cdot\,,c)$, $\zeta \in \mathbb{C}$ are dense in $B(c)$, our transformation uniquely extends to a unitary linear map V from $D(c)$ onto $B(c)$. Moreover, because the functionals $f \mapsto \hat{f}(z)$ and $f \mapsto Vf(z)$ are continuous on $D(c)$ for each $z \in \mathbb{C}$, we conclude that V is nothing but our transform restricted to $D(c)$. Finally, it is easily seen that transforms of functions which are orthogonal to $D(c)$ vanish identically. □

In the following, the closed linear subspace

$$B^\circ(c) = \{F \in B(c) \,|\, F(0) = 0\}$$

of functions in $B(c)$ which vanish at zero will be of particular interest. This subspace consists precisely of all transforms of functions in $H^1(a,c)$ which vanish in c. In fact, an integration by parts shows that

$$\hat{f}(0) = \phi_0^{[1]}(c) f(c), \quad f \in H^1(a,c),$$

where the boundary term at a vanishes since ϕ_0 is a scalar multiple of w_a. Moreover, the orthogonal complement of $B^\circ(c)$ consists of all scalar multiples of the function $K(0,\cdot\,,c)$. Hence it corresponds to the one-dimensional subspace of $D(c)$ spanned by the function $\phi_0|_{(a,c)}$, in view of our unitary transformation on $D(c)$ from Theorem 7.1.

The crucial properties of the de Branges spaces $B(c)$, $c \in (a,b)$ only hold if c lies in the support

$$\operatorname{supp}(\varrho) = \overline{\{x \in (a,b) \,|\, \forall \varepsilon > 0 : |\varrho|((x-\varepsilon, x+\varepsilon)) > 0\}}$$

of ϱ. However, for the proof of our inverse uniqueness result a modified set Σ instead of $\operatorname{supp}(\varrho)$ will be more convenient. This set $\Sigma \subseteq \operatorname{supp}(\varrho) \cup \{a,b\}$ is defined as follows. Take $\operatorname{supp}(\varrho)$ and add a if τ is regular at a, there are no Neumann boundary conditions at a and $|\varrho|$ has no mass near a. Under similar conditions one adds the endpoint b. Moreover, if a has not been added, then remove the point $a_\varrho = \inf \operatorname{supp}(\varrho)$ unless $|\varrho|((a_\varrho, c)) = 0$ for some $c \in (a_\varrho, b)$. Similarly, if b has not been added, then remove the point $b_\varrho = \sup \operatorname{supp}(\varrho)$ unless $|\varrho|((c, b_\varrho)) = 0$ for some $c \in (a, b_\varrho)$. The following lemma gives a hint why this definition might be useful.

Lemma 7.2. *The closure of the domain of S is given by*

$$\mathfrak{D} = \overline{\operatorname{span}\{\delta_c \,|\, c \in \Sigma\}}.$$

Proof. The multi-valued part of S is given by

$$(*) \qquad \operatorname{mul}(S) = \{h \in \operatorname{mul}(T_{\max}) \,|\, V((0,h), v^*)(a) = V((0,h), w^*)(b) = 0\}.$$

Now if $c \in \Sigma \cap (a,b)$, then $\delta_c \perp \operatorname{mul}(S)$ since each $h \in \operatorname{mul}(T_{\max})$ vanishes almost everywhere with respect to $|\varrho|$. Moreover, if $a \in \Sigma$ then τ is regular at a and there are no Neumann boundary conditions at a. Thus, each $h \in \operatorname{mul}(S)$ vanishes in a in view of $(*)$ and hence $\delta_a \perp \operatorname{mul}(S)$. Similarly one shows that $\delta_b \perp \operatorname{mul}(S)$ provided that $b \in \Sigma$. Hence the closure of the linear span of all functions δ_c, $c \in \Sigma$ is orthogonal to $\operatorname{mul}(S)$ and hence contained in $\mathfrak{D}$. In order to prove the converse let

$$h \in \overline{\operatorname{span}\{\delta_c \,|\, c \in \Sigma\}}^{\perp}.$$

Since h is continuous this implies that h vanishes on $\operatorname{supp}(\varrho)$, hence h lies in $\operatorname{mul}(T_{\max})$. Now suppose that

$$(**) \qquad V((0,h), v^*)(a) = \lim_{\alpha \to a} h(\alpha) v^{[1]}(\alpha)^* \neq 0,$$

then τ is necessarily in the l.c. case at a. If ϱ had mass near a, we would infer that $h(a) = 0$ since h vanishes on $\operatorname{supp}(\varrho)$. Hence τ is even regular at a and $(**)$ implies that there are no Neumann boundary conditions at a. Therefore a lies in Σ and hence $h(a) = \langle h, \delta_a \rangle = 0$, contradicting $(**)$. A similar argument for the right endpoint b shows that h lies in $\operatorname{mul}(S)$, which finishes the proof. $\square$

Also note that functions in $\mathfrak{D}$ are uniquely determined by their values on Σ. In fact, if f_1, $f_2 \in \mathfrak{D}$ such that $f_1(c) = f_2(c)$, $c \in \Sigma$, then $f_1 - f_2$ lies in the orthogonal complement of $\mathfrak{D}$ in view of Lemma 7.2 and hence $f_1 = f_2$.

Now before we state our main embedding theorem, it remains to introduce the de Branges spaces $B(a)$ if $a \in \Sigma$ and $B(b)$ if $b \in \Sigma$. First of all if $a \in \Sigma$, then let $B(a)$ be the one-dimensional space spanned by the entire function $z \mapsto \phi_z(a)$. It does not matter which inner product this space is equipped with; each one turns $B(a)$ into a de Branges space as is easily seen from [14, Theorem 23]. In particular, note that $B^\circ(a) = \{0\}$. Finally if $b \in \Sigma$, then let $B(b)$ be the de Branges space associated with the de Branges function

$$E(z,b) = z\phi_z(b) + \mathrm{i}\phi_z^{[1]}(b), \quad z \in \mathbb{C}.$$

The space $B(b)$ has the same properties as the other de Branges spaces $B(c)$, $c \in (a,b)$. For example the reproducing kernel is given as in (7.2) and Theorem 7.1 holds with c replaced by b.

The following result is basically a consequence of Theorem 7.1 and Proposition 6.3, linking our transformation with the generalized Fourier transform $\mathcal{F}$. In the following, μ will denote the spectral measure associated with the real analytic solutions ϕ_z, $z \in \mathbb{C}^\times$ as constructed in the previous section. However, note that in the present case we may extend μ to a Borel measure on $\mathbb{R}$ by setting $\mu(\{0\}) = 0$.

Theorem 7.3. *For each $c \in \Sigma$ the de Branges space $B(c)$ is a closed subspace of $L^2(\mathbb{R};\mu)$ with*

$$\langle F, G\rangle_\mu = [P^\circ F, P^\circ G]_{B(c)} + \frac{F(0)G(0)^*}{|\phi_0(c)|^2}\|\delta_c\|^2_{H^1(a,b)}, \quad F,\, G \in B(c), \tag{7.5}$$

where P° is the orthogonal projection from $B(c)$ onto $B^\circ(c)$.

Proof. First of all note that for $z \in \mathbb{C}$ and $h \in \mathrm{mul}(S) \subseteq \mathrm{mul}(T_{\max})$ we have

$$(*) \qquad \int_a^c \phi_z(x)h(x)^* d\chi(x) + \int_a^c \phi_z^{[1]}(x)h^{[1]}(x)^* d\varsigma(x) = \lim_{x\to a} \phi_z^{[1]}(x)h(x),$$

since h vanishes almost everywhere with respect to $|\varrho|$ (in particular note that $h(c) = 0$). Moreover, the limit on the right-hand side is zero since

$$\lim_{x\to a} \phi_z^{[1]}(x)h(x) = V\left((0,h),(\phi_z, z\phi_z)\right)(a) = 0$$

and both, $(0,h)$ and $(\phi_z, z\phi_z)$ lie in S near a. Now, given some arbitrary functions f, $g \in \mathrm{span}\{\phi_z|_{(a,c)} \,|\, z \in \mathbb{C}\}$, let

$$f_0(x) = \frac{f(c)}{\phi_0(c)}\phi_0(x) \quad \text{and} \quad f_1(x) = f(x) - f_0(x), \quad x \in (a,c)$$

and similarly for the function g. The extensions $\bar{f}_1$, $\bar{g}_1$ of f_1, g_1, defined by

$$\bar{f}_1(x) = \begin{cases} f_1(x), & \text{if } x \in (a,c], \\ 0, & \text{if } x \in (c,b), \end{cases}$$

and similarly for $\bar{g}_1$, lie in $H^1(a,b)$ since $f_1(c) = g_1(c) = 0$. Moreover, these extensions even lie in $\mathfrak{D}$, because $(*)$ shows that they are orthogonal to $\mathrm{mul}(S)$. Now we get the identity

$$\begin{aligned} \langle \hat{f}_1, \hat{g}_1\rangle_\mu &= \langle \mathcal{F}\bar{f}_1, \mathcal{F}\bar{g}_1\rangle_\mu = \langle \bar{f}_1, \bar{g}_1\rangle_{H^1(a,b)} = \langle f_1, g_1\rangle_{H^1(a,c)} \\ &= [\hat{f}_1, \hat{g}_1]_{B(c)}, \end{aligned} \tag{7.6}$$

where we used Proposition 6.3, Lemma 6.2 and Theorem 7.1. Moreover, from (7.3) (also note that $\delta_c \in \mathfrak{D}$) we get

$$\langle \hat{f}_0, \hat{g}_0\rangle_\mu = f_0(c)g_0(c)^* \left|\frac{\phi_0^{[1]}(c)}{\phi_0(c)}\right|^2 \int_{\mathbb{R}} |\phi_\lambda(c)|^2 d\mu(\lambda) = \frac{\hat{f}_0(0)\hat{g}_0(0)^*}{|\phi_0(c)|^2}\|\delta_c\|^2_{H^1(a,b)}.$$

Furthermore,

$$\langle \hat{f}_1, \hat{g}_0\rangle_\mu = g_0(c)^* \frac{\phi_0^{[1]}(c)}{\phi_0(c)} \int_{\mathbb{R}} \phi_\lambda(c)\hat{f}_1(\lambda)d\mu(\lambda) = g_0(c)^* \frac{\phi_0^{[1]}(c)}{\phi_0(c)} f_1(c) = 0,$$

i.e. the function $\hat{g}_0$ is orthogonal to $\hat{f}_1$ not only in $B(c)$ but also in $L^2(\mathbb{R};\mu)$. Using these properties, we finally obtain

$$\langle \hat{f}, \hat{g}\rangle_\mu = \langle \hat{f}_1, \hat{g}_1\rangle_\mu + \langle \hat{f}_0, \hat{g}_0\rangle_\mu = \langle P^\circ \hat{f}, P^\circ \hat{g}\rangle_{B(c)} + \frac{\hat{f}(0)\hat{g}(0)^*}{|\phi_0(c)|^2} \|\delta_c\|^2_{H^1(a,b)}.$$

Hence (7.5) holds for all F, G in a dense subspace of $B(c)$. Now it is quite easy to see that $B(c)$ is actually continuously embedded in $L^2(\mathbb{R};\mu)$ and that (7.5) holds for all F, $G \in B(c)$. Moreover, $B(c)$ is a closed subspace of $L^2(\mathbb{R};\mu)$ since the norms $\|\cdot\|_{B(c)}$ and $\|\cdot\|_\mu$ are equivalent on $B(c)$. $\square$

In particular, note that under the assumption of Theorem 7.3 the subspace $B^\circ(c)$ is isometrically embedded in $L^2(\mathbb{R};\mu)$. Moreover, the embedding $B(c) \to L^2(\mathbb{R};\mu)$ preserves orthogonality and simple calculation shows that for functions F in the orthogonal complement of $B^\circ(c)$ we have

$$\|F\|^2_{B(c)} = \frac{|F(0)|^2}{\phi_0^{[1]}(c)\phi_0(c)} = \left(1 - \frac{w_b^{[1]}(c)}{w_b(c)} \frac{w_a(c)}{w_a^{[1]}(c)}\right) \|F\|^2_\mu, \tag{7.7}$$

at least if $c \neq a$. This difference between $B^\circ(c)$ and its orthogonal complement stems from the fact that the functions in $H^1(a,c)$ corresponding to $B^\circ(c)$ are isometrically embedded in $H^1(a,b)$, whereas the functions corresponding to its orthogonal complement are not.

The following results contain further properties of our de Branges spaces which are needed for the inverse uniqueness theorem in the next section. First of all, we will show that they are totally ordered and strictly increasing.

Proposition 7.4. *If c_1, $c_2 \in \Sigma$ with $c_1 < c_2$, then*

$$B(c_1) \subsetneq B(c_2).$$

Moreover, if $|\varrho|((c_1,c_2)) = 0$ then $B(c_1)$ has codimension one in $B(c_2)$.

Proof. If $\delta \in H^1(a,c_2)$ is such that

$$\langle f, \delta\rangle_{H^1(a,c_2)} = f(c_1), \quad f \in H^1(a,c_2),$$

then the modified Sobolev space $H^1(a,c_2)$ may be decomposed into

$$H^1(a,c_2) = H^1_-(a,c_2) \oplus \operatorname{span}\{\delta\} \oplus H^1_+(a,c_2).$$

Here $H^1_-(a,c_2)$ is the subspace of functions in $H^1(a,c_2)$ vanishing on (c_1,c_2) and $H^1_+(a,c_2)$ is the subspace of functions in $H^1(a,c_2)$ vanishing on (a,c_1). Now the transforms of functions in $H^1_-(a,c_2)$ are precisely the transforms of functions in $H^1(a,c_1)$ which vanish in c_1, i.e. $B^\circ(c_1)$. The transform of the subspace $\operatorname{span}\{\delta\}$ is precisely the orthogonal complement of $B^\circ(c_1)$. Hence one sees that $B(c_1)$ is contained in $B(c_2)$. In order to prove that $B(c_2)$ is larger indeed, suppose that the function $z \mapsto \phi_z(c_2)$ belongs to $B(c_1)$. Since this function is orthogonal to $B^\circ(c_2)$ it is also orthogonal to $B^\circ(c_1)$ by Theorem 7.3. Thus we infer that the functions $z \mapsto \phi_z(c_1)$ and $z \mapsto \phi_z(c_2)$ are linearly dependent. Now from Lemma 6.2 (hereby also note that δ_{c_1} and

δ_{c_2} lie in $\mathfrak{D}$) one sees that δ_{c_1} and δ_{c_2} are also linearly dependent, which gives a contradiction.

It remains to prove that the space of transforms of functions in $H^1_+(a,c_2)$ is at most one-dimensional provided that $|\varrho|((c_1,c_2)) = 0$. Indeed, for each function $f \in H^1_+(a,c_2)$ an integration by parts shows that

$$\hat{f}(z) = \phi_z^{[1]}(c_2)f(c_2) - \phi_z^{[1]}(c_1)f(c_1) + z\int_{c_1}^{c_2} \phi_z f\, d\varrho = \phi_z^{[1]}(c_2)f(c_2), \quad z \in \mathbb{C},$$

since f vanishes on $(a,c_1]$ and $|\varrho|((c_1,c_2)) = 0$. □

The following result shows that our de Branges spaces are continuous in some sense. This is due to the assumption that the measure ς is absolutely continuous with respect to the Lebesgue measure. Otherwise, there would be jumps of dimension one in point masses of ς.

Proposition 7.5. *If c, α_n, $\beta_n \in \operatorname{supp}(\varrho)$, $n \in \mathbb{N}$ are such that $\alpha_n \uparrow c$ and $\beta_n \downarrow c$ as $n \to \infty$, then*

$$\overline{\bigcup_{n\in\mathbb{N}} B(\alpha_n)} = B(c) = \bigcap_{n\in\mathbb{N}} B(\beta_n), \tag{7.8}$$

where the closure is taken in $L^2(\mathbb{R};\mu)$.

Proof. From Proposition 7.4 it is clear that

$$\overline{\bigcup_{n\in\mathbb{N}} B(\alpha_n)} \subseteq B(c) \subseteq \bigcap_{n\in\mathbb{N}} B(\beta_n).$$

If $F \in B^\circ(c)$, then there is an $f \in H^1(a,c)$ with $f(c) = 0$ such that $\hat{f} = F$. Now choose a sequence $f_k \in H^1(a,c)$, $k \in \mathbb{N}$ of functions which vanish near c, such that $f_k \to f$ as $k \to \infty$. By our assumptions the transform of each of these functions lies in $B(\alpha_n)$, provided that $n \in \mathbb{N}$ is large enough, i.e.

$$\hat{f}_k \in \bigcup_{n\in\mathbb{N}} B(\alpha_n), \quad k \in \mathbb{N}.$$

Consequently the transform of f lies in the closure of this union. Moreover, for each $n \in \mathbb{N}$ the entire function $z \mapsto \phi_z(\alpha_n)$ lies in $B(\alpha_n)$. Now since $\delta_{\alpha_n} \to \delta_c$ in $H^1(a,b)$, Lemma 6.2 shows that the entire function $z \mapsto \phi_z(c)$ lies in the closure of our union which proves the first equality in (7.8).

Next, if $F \in B(\beta_n)$ for each $n \in \mathbb{N}$, then there are $f_n \in D(\beta_n)$ such that

$$F(z) = \int_a^{\beta_n} \phi_z(x)f_n(x)d\chi(x) + \int_a^{\beta_n} \phi_z^{[1]}(x)f_n^{[1]}(x)d\varsigma(x), \quad z \in \mathbb{C},\ n \in \mathbb{N}.$$

Moreover, from Theorem 7.1 and Theorem 7.3 we infer

$$\|f_n\|^2_{H^1(a,\beta_n)} = \|F\|^2_{B(\beta_n)} \leq \left(1 + \left|\frac{w_b^{[1]}(\beta_n)}{w_b(\beta_n)}\frac{w_a(\beta_n)}{w_a^{[1]}(\beta_n)}\right|\right)\|F\|^2_\mu, \quad n \in \mathbb{N},$$

where the coefficient on the right-hand side is bounded uniformly for all $n \in \mathbb{N}$ by the properties of the solutions w_a and w_b. Hence there is some subsequence of $f_n|_{(a,c)}$, $n \in \mathbb{N}$ converging weakly in $H^1(a,c)$ to say f. Now this yields for all $z \in \mathbb{C}$

$$F(z) = \hat{f}(z) + \chi(\{c\})\phi_z(c)f(c) + \lim_{n\to\infty} \int_{(c,\beta_n)} \phi_z f_n \, d\chi + \int_{(c,\beta_n)} \phi_z^{[1]} f_n^{[1]} d\varsigma,$$

where the limit is actually zero. In fact, for each $z \in \mathbb{C}$ and $n \in \mathbb{N}$ we have

$$\left| \int_{(c,\beta_n)} \phi_z f_n d\chi + \int_{(c,\beta_n)} \phi_z^{[1]} f_n^{[1]} d\varsigma \right| \leq C_z^2 \|f_n\|_{H^1(a,\beta_n)} (\chi((c,\beta_n)) + \varsigma((c,\beta_n))),$$

where $C_z \in \mathbb{R}$ is such that the moduli of ϕ_z and $\phi_z^{[1]}$ on (c,β_1) are bounded by C_z. But this shows that F actually is the transform of a function in $H^1(a,b)$ and hence lies in $B(c)$ which finishes the proof. □

Finally we will prove that our de Branges spaces decrease to zero near a and fill the whole space $L^2(\mathbb{R};\mu)$ near b.

Proposition 7.6. *The de Branges spaces $B(c)$, $c \in \Sigma$ satisfy*

$$\bigcap_{c\,\in\,\Sigma} B^\circ(c) = \{0\} \quad \text{and} \quad \overline{\bigcup_{c\,\in\,\Sigma} B(c)} = L^2(\mathbb{R};\mu). \tag{7.9}$$

Proof. First suppose that $\operatorname{supp}(\varrho) \cap (a,c) \neq \emptyset$ for each $c \in (a,b)$ and pick some $F \in \bigcap_{c\in\Sigma} B^\circ(c)$. Then for each $\zeta \in \mathbb{C}$ we have

$$\begin{aligned} |F(\zeta)| &\leq [F, K(\zeta,\cdot\,,c)]_{B(c)} \leq \|F\|_{B(c)} [K(\zeta,\cdot\,,c), K(\zeta,\cdot\,,c)]_{B(c)} \\ &\leq \|F\|_\mu K(\zeta,\zeta,c) \end{aligned}$$

for each $c \in \operatorname{supp}(\varrho)$. Now from (7.2) we infer that $K(\zeta,\zeta,c) \to 0$ as $c \to a$ and hence that $F = 0$. Otherwise, if $\alpha_\varrho = \inf \operatorname{supp}(\varrho) > a$, then the subspace

$$D^\circ(\alpha_\varrho) = \{f \in D(\alpha_\varrho) \,|\, f(\alpha_\varrho) = 0\},$$

corresponding to $B^\circ(\alpha_\varrho)$, is at most one-dimensional. In fact, this is because each function $\phi_z|_{(a,\alpha_\varrho)}$, $z \in \mathbb{C}$ is a solution of $\tau u = 0$ on (a,α_ϱ) in this case. Consequently, the functions in $D^\circ(\alpha_\varrho)$ are also solutions of $\tau u = 0$ on (a,α_ϱ). Moreover, if $\varsigma+\chi$ is infinite near a, then each $f \in D^\circ(\alpha_\varrho)$ is a scalar multiple of w_a on (a,α_ϱ) with $f(\alpha_\varrho) = 0$ and hence vanishes identically. Also if $\varsigma + \chi$ is finite near a and there are Neumann boundary conditions at a, one sees that f is a scalar multiple of w_a and hence identically zero. We conclude that the first equality in the claim holds in these cases. Finally, if $\varsigma + \chi$ is finite near a and there are no Neumann boundary conditions at a, then $a \in \Sigma$ and hence clearly $B^\circ(a) = \{0\}$. For the second equality note that the linear span of functions $z \mapsto \phi_z(c)$, $c \in \Sigma$ is dense in $L^2(\mathbb{R};\mu)$ in view of Lemma 7.2 and Lemma 6.2. □

8. Inverse uniqueness results

The present section is devoted to our inverse uniqueness result. We will prove that the spectral measure determines a left-definite Sturm–Liouville operator up to some Liouville transformation (see e.g. [7] or [3] for the right-definite case). Therefore let S_1 and S_2 be two self-adjoint left-definite

Sturm–Liouville relations (with separate boundary conditions), both satisfying the assumptions made in the previous section, i.e. zero is not an eigenvalue of S_1 and S_2 and there are real entire solutions satisfying the boundary condition at the left endpoint. Moreover, again we assume that the measures ς_1 and ς_2 are absolutely continuous with respect to the Lebesgue measure. All remaining quantities corresponding to S_1 respectively S_2 are denoted with an additional subscript.

We will first state a part of the proof of our inverse uniqueness result as a separate lemma. Note that the equality in the claim of this lemma has to be read as sets of entire functions and not as de Branges spaces. In general the norms of these spaces will differ from each other.

Lemma 8.1. *Suppose that the function*

$$\frac{E_1(z,x_1)}{E_2(z,x_2)}, \quad z \in \mathbb{C}^+ \tag{8.1}$$

is of bounded type for some $x_1 \in \Sigma_1$ and $x_2 \in \Sigma_2$. If $\mu_1 = \mu_2$, then there is an increasing continuous bijection η from Σ_1 onto Σ_2 such that

$$B_1(x_1) = B_2(\eta(x_1)), \quad x_1 \in \Sigma_1.$$

Proof. First of all note that by the definition of de Branges spaces and Proposition 7.4 the function in (8.1) is of bounded type for all $x_1 \in \Sigma_1$ and $x_2 \in \Sigma_2$. We will first consider the case when Σ_1 consists of finitely many (strictly increasing) points $x_{1,n}$, $n = 1, \dots, N$ separately. In this case $\mu_1 = \mu_2$ is supported on N points, since $\mathcal{F}_1$ is a unitary map from $\mathfrak{D}_1$ onto $L^2(\mathbb{R};\mu_1)$. Hence, Σ_2 also consists of finitely many (strictly increasing) points $x_{2,n}$, $n = 1, \dots, N$. Now let η be the unique strictly increasing bijection from Σ_1 onto Σ_2, i.e. $\eta(x_{1,n}) = x_{2,n}$, $n = 1, \dots, N$. Using the properties of our de Branges spaces it is quite simple to see that

$$\dim B_1(x_{1,n}) = \dim B_2(x_{2,n}) = n, \quad n = 1, \dots, N,$$

and therefore the claim follows from Theorem 7.3 and Theorem A.1.

Now suppose that Σ_1 consists of infinitely many points and fix some arbitrary $x_1 \in \Sigma_1 \backslash \{\inf \Sigma_1, \sup \Sigma_1\}$. Then from Theorem 7.3 and Theorem A.1 we infer that for each $x_2 \in \Sigma_2$ either $B_1(x_1) \subseteq B_2(x_2)$ or $B_1(x_1) \supseteq B_2(x_2)$ and hence also $B_1^\circ(x_1) \subseteq B_2^\circ(x_2)$ or $B_1^\circ(x_1) \supseteq B_2^\circ(x_2)$. In order to define $\eta(x_1) \in (a_2, b_2)$ we are first going to show that both of the sets

$$J_- = \{x_2 \in \Sigma_2 \,|\, B_2(x_2) \subsetneq B_1(x_1)\},$$
$$J_+ = \{x_2 \in \Sigma_2 \,|\, B_1(x_1) \subsetneq B_2(x_2)\},$$

are non-empty. Indeed, if J_- was empty, then $B_1^\circ(x_1) \subseteq B_2^\circ(x_2)$ for each $x_2 \in \Sigma_2$ and hence

$$B_1^\circ(x_1) \subseteq \bigcap_{x_2 \in \Sigma_2} B_2^\circ(x_2) = \{0\},$$

in view of Proposition 7.6. Thus we obtained the contradiction $x_1 = \inf \Sigma_1$, since otherwise there would be some $\tilde{x}_1 \in \Sigma_1$ with $\tilde{x}_1 < x_1$ such that $B_1(\tilde{x}_1) \subsetneq B_1(x_1)$. Furthermore, if J_+ was empty, then $B_2(x_2) \subseteq B_1(x_1)$

for each $x_2 \in \Sigma_2$ and hence

$$L^2(\mathbb{R};\mu_1) = \overline{\bigcup_{x_2\in\Sigma_2} B_2(x_2)} \subseteq B_1(x_1) \subseteq L^2(\mathbb{R};\mu_1).$$

But from this we infer the contradiction $x_1 = \sup\Sigma_1$, since otherwise there would be an $\tilde{x}_1 \in \Sigma_1$ with $\tilde{x}_1 > x_1$ such that $B_1(x_1) \subsetneq B_1(\tilde{x}_1) \subseteq L^2(\mathbb{R};\mu_1)$. Hence we showed that J_- and J_+ are non-empty. Now, if $J_- = \{a_2\}$ then the space $B_2(\alpha_{\varrho_2})$ is two-dimensional and α_{ϱ_2} does not lie in J_+ since otherwise

$$B_2(a_2) \subsetneq B_1(x_1) \subsetneq B_2(\alpha_{\varrho_2}).$$

Thus in this case we may set $\eta(x_1) = \alpha_{\varrho_2}$ and obtain $B_1(x_1) = B_2(\eta(x_1))$. Furthermore, if $J_+ = \{b_2\}$ then the space $B_2(\beta_{\varrho_2})$ has codimension one in $L^2(\mathbb{R};\mu)$ and β_{ϱ_2} does not lie in J_- since otherwise

$$B_2(\beta_{\varrho_2}) \subsetneq B_1(x_1) \subsetneq B_2(b_2).$$

Again, we may define $\eta(x_1) = \beta_{\varrho_2}$ and get $B_1(x_1) = B_2(\eta(x_1))$. Now in the remaining cases J_- is bounded from above in (a_2,b_2) with supremum

$$\eta_-(x_1) = \sup J_- \in (a_2,b_2),$$

and J_+ is bounded from below in (a_2,b_2) with infimum

$$\eta_+(x_1) = \inf J_+ \in (a_2,b_2).$$

Moreover, we have $\eta_\pm(x_1) \in \operatorname{supp}(\varrho_2)$ since $J_\pm\backslash\{a_2,b_2\}$ is contained in $\operatorname{supp}(\varrho_2)$. Now Proposition 7.5 shows that

$$B_2(\eta_-(x_1)) \subseteq B_1(x_1) \subseteq B_2(\eta_+(x_1)).$$

If $B_1(x_1) = B_2(\eta_-(x_1))$, set $\eta(x_1) = \eta_-(x_1)$ and if $B_1(x_1) = B_2(\eta_+(x_1))$, set $\eta(x_1) = \eta_+(x_1)$ to obtain $B_1(x_1) = B_2(\eta(x_1))$. Otherwise we have

$$B_2(\eta_-(x_1)) \subsetneq B_1(x_1) \subsetneq B_2(\eta_+(x_1)),$$

and hence $\operatorname{supp}(\varrho_2) \cap (\eta_-(x_1),\eta_+(x_1)) \neq \emptyset$ in view of Proposition 7.4. Now we may choose $\eta(x_1)$ in this intersection and get $B_1(x_1) = B_2(\eta(x_1))$ since $\eta(x_1)$ neither lies in J_- nor in J_+.

Up to now we constructed a function $\eta : \Sigma_1\backslash\{\inf\Sigma_1,\sup\Sigma_1\} \to \Sigma_2$ such that $B_1(x_1) = B_2(\eta(x_1))$ for each $x_1 \in \Sigma_1\backslash\{\inf\Sigma_1,\sup\Sigma_1\}$. Now if $\inf\Sigma_1$ lies in Σ_1 and we set $x_1 = \inf\Sigma_1\backslash\{\inf\Sigma_1\}$, then $B_1(x_1) = B_2(\eta(x_1))$ is two-dimensional and from Proposition 7.6 we infer that there is an $x_2 \in \Sigma_2$ with

$$\{0\} \subsetneq B_2(x_2) \subsetneq B_2(\eta(x_1)) = B_1(x_1).$$

Hence we may set $\eta(\inf\Sigma_1) = x_2$ and obtain $B_1(\inf\Sigma_1) = B_2(\eta(\inf\Sigma_1))$. Similarly, if $\sup\Sigma_1$ lies in Σ_1 and we set $x_1 = \sup\Sigma_1\backslash\{\sup\Sigma_1\}$, then the space $B_1(x_1) = B_2(\eta(x_1))$ has codimension one in $B_1(\sup\Sigma_1) = L^2(\mathbb{R};\mu_1)$. But because of Proposition 7.6 there is an $x_2 \in \Sigma_2$ such that

$$B_2(\eta(\beta_{\varrho_1})) \subsetneq B_2(x_2) \subseteq L^2(\mathbb{R};\mu_1).$$

Again, we may define $\eta(\sup\Sigma_1) = x_2$ and get $B_1(\sup\Sigma_1) = B_2(\eta(\sup\Sigma_1))$. Thus, we extended our function η to all of Σ_1 and are left to prove the remaining claimed properties.

The fact that η is increasing is a simple consequence of Proposition 7.4. Now if $x_2 \in \Sigma_2$, then the first part of the proof with the roles of Σ_1 and Σ_2 reversed shows that there is an $x_1 \in \Sigma_1$ with $B_1(x_1) = B_2(x_2) = B_1(\eta(x_1))$.

In view of Proposition 7.4 this yields $\eta(x_1) = x_2$ and hence η is a bijection. Finally, continuity follows from Proposition 7.5. Indeed, if c, $c_n \in \Sigma_1$, $n \in \mathbb{N}$ such that $c_n \uparrow c$ as $n \to \infty$, then

$$B_2\left(\lim_{n\to\infty} \eta(c_n)\right) = \overline{\bigcup_{n\in\mathbb{N}} B_2(\eta(c_n))} = \overline{\bigcup_{n\in\mathbb{N}} B_1(c_n)} = B_1(c) = B_2(\eta(c))$$

and hence $\eta(c_n) \to \eta(c)$ as $n \to \infty$. Similarly, if $c_n \downarrow c$ as $n \to \infty$, then

$$B_2\left(\lim_{n\to\infty} \eta(c_n)\right) = \bigcap_{n\in\mathbb{N}} B_2(\eta(c_n)) = \bigcap_{n\in\mathbb{N}} B_1(c_n) = B_1(c) = B_2(\eta(c))$$

and hence again $\eta(c_n) \to \eta(c)$ as $n \to \infty$. □

Note that the condition that the function in (8.1) is of bounded type is actually equivalent to the function

$$\frac{\phi_1(z, x_1)}{\phi_2(z, x_2)}, \quad z \in \mathbb{C}^+$$

being of bounded type for some $x_1 \in \Sigma_1$ and $x_2 \in \Sigma_2$. Unfortunately, these conditions are somewhat inconvenient in view of applications. However, note that this assumption is for example fulfilled if for some $x_1 \in \Sigma_1$ and $x_2 \in \Sigma_2$ the entire functions $\phi_1(\,\cdot\,, x_1)$, $\phi_2(\,\cdot\,, x_2)$ are of finite exponential type such that the logarithmic integrals

$$\int_{\mathbb{R}} \frac{\ln^+ |\phi_j(\lambda, x_j)|}{1 + \lambda^2} d\lambda < \infty, \quad j = 1, 2$$

are finite. Here $\ln^+$ is the positive part of the natural logarithm. Indeed, a theorem of Kreĭn [27, Theorem 6.17], [33, Section 16.1] states that in this case the functions $\phi_1(\,\cdot\,, x_1)$, $\phi_2(\,\cdot\,, x_2)$ (and hence also their quotient) are of bounded type in the upper and in the lower complex half-plane. Moreover, note that the conclusion of Lemma 8.1 is also true if for some (and hence all) $x_1 \in (a_1, b_1)$ and $x_2 \in (a_2, b_2)$ the functions $E_1(\,\cdot\,, x_1)$, $E_2(\,\cdot\,, x_2)$ are of exponential type zero, i.e.

$$\ln^+ |E_j(z, x_j)| = o(|z|), \quad j = 1, 2$$

as $|z| \to \infty$ in $\mathbb{C}$. The proof therefore is literally the same, except that one has to apply Theorem A.2 instead of Theorem A.1.

With all the work done in Lemma 8.1 it is now quite simple to show that the spectral measure determines our self-adjoint Sturm–Liouville relation up to a Liouville transform. Here, a Liouville transform $\mathcal{L}$ is a unitary mapping from $\mathfrak{D}_2$ onto $\mathfrak{D}_1$ given by

$$\mathcal{L} f_2(x_1) = \kappa(x_1) f_2(\eta(x_1)), \quad x_1 \in \Sigma_1,\ f_2 \in \mathfrak{D}_2, \tag{8.2}$$

where η is an increasing continuous bijection from Σ_1 onto Σ_2 and κ is a non-vanishing real function on Σ_1. We say that the Liouville transform $\mathcal{L}$ maps S_1 onto S_2 if

$$S_2 = \mathcal{L}^* S_1 \mathcal{L},$$

where $\mathcal{L}^*$ is the adjoint of $\mathcal{L}$ regarded as a relation in $H^1(a_2, b_2) \times H^1(a_1, b_1)$. Note that in this case the operator parts $S_{1,\mathfrak{D}}$, $S_{2,\mathfrak{D}}$ of S_1, S_2 are unitarily equivalent. In fact, the Liouville transforms $\mathcal{L}$ maps $S_{2,\mathfrak{D}}$ onto $S_{1,\mathfrak{D}}$.

Theorem 8.2. *Suppose that the function*

$$\frac{E_1(z,x_1)}{E_2(z,x_2)}, \quad z\in\mathbb{C}^+$$

is of bounded type for some $x_1\in\Sigma_1$ and $x_2\in\Sigma_2$. If $\mu_1=\mu_2$, then there is a Liouville transform $\mathcal{L}$ mapping S_1 onto S_2.

Proof. By Lemma 8.1 there is an increasing continuous bijection η from Σ_1 onto Σ_2 such that $B_1(x_1)=B_2(\eta(x_1))$ and hence also $B_1^\circ(x_1)=B_2^\circ(\eta(x_1))$ for each $x_1\in\Sigma_1$. According to Theorem 7.3, for each fixed $x_1\in\Sigma_1$ the entire functions

$$z\mapsto\phi_{1,z}(x_1) \quad\text{and}\quad z\mapsto\phi_{2,z}(\eta(x_1))$$

are orthogonal to $B_1^\circ(x_1)=B_2^\circ(\eta(x_1))$ in $L^2(\mathbb{R};\mu_1)$. From this we infer that

$$\phi_{1,z}(x_1)=\kappa(x_1)\phi_{2,z}(\eta(x_1)), \quad z\in\mathbb{C} \tag{8.3}$$

for some $\kappa(x_1)\in\mathbb{R}^\times$ and hence also

$$\mathcal{F}_1\delta_{1,x_1}=\kappa(x_1)\mathcal{F}_2\delta_{2,\eta(x_1)}. \tag{$*$}$$

Now the linear relation

$$\mathcal{L}=\mathcal{F}_1^*\mathcal{F}_2|_{\mathfrak{D}_2}$$

is a unitary mapping from $\mathfrak{D}_2$ onto $\mathfrak{D}_1$ by Lemma 6.2 and moreover, equation $(*)$ shows that

$$(\delta_{1,x_1},\kappa(x_1)\delta_{2,\eta(x_1)})\in\mathcal{L}^*=\mathcal{F}_2^{-1}\mathcal{F}_1, \quad x_1\in\Sigma_1.$$

From this one sees that the transform of some function $f_2\in\mathfrak{D}_2$ is given by

$$\begin{aligned}\mathcal{L}f_2(x_1)&=\langle\mathcal{L}f_2,\delta_{1,x_1}\rangle_{H^1(a_1,b_1)}=\kappa(x_1)\langle f_2,\delta_{2,\eta(x_1)}\rangle_{H^1(a_2,b_2)}\\&=\kappa(x_1)f_2(\eta(x_1))\end{aligned}$$

at each point $x_1\in\Sigma_1$. Finally, we conclude that

$$\begin{aligned}S_2=\mathcal{F}_2^{-1}\mathrm{M}_{\mathrm{id}}\mathcal{F}_2|_{\mathfrak{D}_2}&=\mathcal{F}_2^{-1}\mathcal{F}_1\mathcal{F}_1^{-1}\mathrm{M}_{\mathrm{id}}\mathcal{F}_1\mathcal{F}_1^*\mathcal{F}_2|_{\mathfrak{D}_2}=\mathcal{L}^*\mathcal{F}_1^{-1}\mathrm{M}_{\mathrm{id}}\mathcal{F}_1\mathcal{L}\\&=\mathcal{L}^*S_1\mathcal{L},\end{aligned}$$

from Theorem 6.4. □

We will now show to which extend the spectral measure determines the coefficients. For the proof we need a result on the high energy asymptotics of solutions of our differential equation (see e.g. [4, Section 6]). Henceforth we will denote with r_j, $j=1,2$ the densities of the absolute continuous parts of ϱ_j with respect to the Lebesgue measure and with p_j^{-1}, $j=1,2$ the densities of ς_j with respect to the Lebesgue measure.

Lemma 8.3. *For each $j=1,2$ and all points x_j, $\tilde{x}_j\in(a_j,b_j)$ we have the asymptotics*

$$\sqrt{\frac{2}{y}}\ln\frac{|\phi_{j,\mathrm{i}y}(x_j)|}{|\phi_{j,\mathrm{i}y}(\tilde{x}_j)|}\to\int_{\tilde{x}_j}^{x_j}\sqrt{\frac{|r_j(x)|}{p_j(x)}}dx,$$

as $y\to\infty$ in $\mathbb{R}^+$.

Proof. By our assumptions, the Lebesgue decomposition of the measure ϱ_j with respect to ς_j is given by

$$\varrho_j = r_j p_j \varsigma_j + \varrho_{j,s},$$

where $\varrho_{j,s}$ is the singular part of ϱ_j with respect to the Lebesgue measure. Now the results in [4, Section 6] show that (the square root is the principal one with branch cut along the negative real axis)

$$\begin{aligned}\ln \frac{|\phi_{j,\mathrm{i}y}(x_j)|}{|\phi_{j,\mathrm{i}y}(\tilde{x}_j)|} &= \mathrm{Re}\left(\int_{\tilde{x}_j}^{x_j} \sqrt{-\mathrm{i}y r_j(x) p_j(x)}\, d\varsigma_j(x) + o(\sqrt{y})\right) \\ &= \sqrt{\frac{y}{2}} \int_{\tilde{x}_j}^{x_j} \sqrt{\frac{|r_j(x)|}{p_j(x)}}\, dx + o(\sqrt{y}),\end{aligned}$$

as $y \to \infty$ in $\mathbb{R}^+$, which yields the claim. □

We are now able to establish a relation between the measure coefficients. However, this is only possible on sets where the support of the weight measure has enough density. Otherwise there would be to much freedom for the remaining coefficients. For example, at each point of our interval one could insert a subinterval on which the weight measure vanishes but which does not change the spectral measure.

Corollary 8.4. *Let α_1, $\beta_1 \in (a_1, b_1)$ with $\alpha_1 < \beta_1$ such that $r_1 \neq 0$ almost everywhere on (α_1, β_1) and $r_2 \neq 0$ almost everywhere on $(\eta(\alpha_1), \eta(\beta_1))$ with respect to the Lebesgue measure. If the function*

$$\frac{E_1(z, x_1)}{E_2(z, x_2)}, \quad z \in \mathbb{C}^+$$

is of bounded type for some $x_1 \in (a_1, b_1)$, $x_2 \in (a_2, b_2)$ and $\mu_1 = \mu_2$, then the functions η and κ from the Liouville transform of Theorem 8.2 satisfy

$$\eta' = \sqrt{\frac{p_2 \circ \eta}{p_1} \frac{|r_1|}{|r_2 \circ \eta|}} \quad \text{and} \quad \kappa^2 = \sqrt{\frac{p_2 \circ \eta}{p_1} \frac{|r_2 \circ \eta|}{|r_1|}}$$

almost everywhere on (α_1, β_1) with respect to the Lebesgue measure and for the measure coefficients we have

$$\varsigma_2 \circ \eta = \kappa^{-2} \varsigma_1, \quad \varrho_2 \circ \eta = \kappa^2 \varrho_1 \quad \text{and} \quad \chi_2 \circ \eta = \kappa^2 \chi_1 - \kappa \kappa^{[1]\prime},$$

as measures on (α_1, β_1).

Proof. From equation (8.3) and the asymptotics in Lemma 8.3 we infer that

$$\int_{\tilde{x}_1}^{x_1} \sqrt{\frac{|r_1(x)|}{p_1(x)}}\, dx = \int_{\eta(\tilde{x}_1)}^{\eta(x_1)} \sqrt{\frac{|r_2(x)|}{p_2(x)}}\, dx, \quad x_1,\, \tilde{x}_1 \in (\alpha_1, \beta_1).$$

In view of the Banach–Zareckiĭ theorem (see e.g. [28, Chapter IX; Theorem 4], [22, Theorem 18.25]) this shows that η is locally absolutely continuous on (α_1, β_1) with derivative given as in the claim. More precisely, this follows from an application of [28, Chapter IX; Exercise 13] and [28, Chapter IX; Theorem 5]. Furthermore, since $\phi_{1,0}$, $\phi_{2,0}$ are scalar multiples of $w_{1,a}$, $w_{2,a}$ respectively, we also have

$$(*) \qquad w_{1,a}(x_1) = C_a \kappa(x_1) w_{2,a}(\eta(x_1)), \quad x_1 \in (\alpha_1, \beta_1)$$

for some constant $C_a \in \mathbb{R}^\times$. In particular, this shows that κ is locally absolutely continuous on (α_1, β_1). In fact, the substitution rule for Lebesgue–Stieltjes integrals (see [32, Corollary 5.3]) shows that

$$\begin{aligned} w_{2,b}(\eta(x_1)) - w_{2,b}(\eta(\tilde{x}_1)) &= \int_{\eta(\tilde{x}_1)}^{\eta(x_1)} w_{2,b}^{[1]} \, d\varsigma_2 \\ &= \int_{\tilde{x}_1}^{x_1} w_{2,b}^{[1]} \circ \eta \, d\varsigma_2 \circ \eta, \quad x_1, \, \tilde{x}_1 \in (\alpha_1, \beta_1) \end{aligned}$$

and hence the function $x_1 \mapsto w_{2,b}(\eta(x_1))$ is locally of bounded variation on (α_1, β_1). Therefore, from [28, Chapter IX; Theorem 5] we infer that this function is even locally absolutely continuous on (α_1, β_1) and hence so is κ. Moreover, in view of Lemma 6.2, equation (8.3) yields

$$\kappa(x_1)^2 = \frac{\|\delta_{1,x_1}\|^2_{H^1(a_1,b_1)}}{\|\delta_{2,\eta(x_1)}\|^2_{H^1(a_2,b_2)}} = \frac{W(w_{2,b}, w_{2,a})}{W(w_{1,b}, w_{1,a})} \frac{w_{1,a}(x_1) w_{1,b}(x_1)}{w_{2,a}(\eta(x_1)) w_{2,b}(\eta(x_1))}$$

for each $x_1 \in (\alpha_1, \beta_1)$. Inserting $(*)$ we get from this equation

$$w_{1,b}(x_1) = C_a^{-1} \frac{W(w_{1,b}, w_{1,a})}{W(w_{2,b}, w_{2,a})} \kappa(x_1) w_{2,b}(\eta(x_1)), \quad x_1 \in (\alpha_1, \beta_1).$$

Plugging this expression and equation $(*)$ into the definition of the Wronskian $W(w_{1,b}, w_{1,a})$ one obtains

$$1 = \frac{\kappa(x_1)^2 \eta'(x_1) p_1(x_1)}{p_2(\eta(x_1))}, \quad x_1 \in (\alpha_1, \beta_1),$$

which shows that κ is given as in the claim. Next, differentiating equation (8.3) yields

$$\kappa(x_1) \phi_{1,z}^{[1]}(x_1) = \kappa^{[1]}(x_1) \phi_{1,z}(x_1) + \phi_{2,z}^{[1]}(\eta(x_1)), \quad x_1 \in (\alpha_1, \beta_1)$$

for each $z \in \mathbb{C}$. From this we get for all α, $\beta \in (\alpha_1, \beta_1)$

$$\begin{aligned} \int_\alpha^\beta \phi_{1,z} \kappa \, d\chi_1 - z \int_\alpha^\beta \phi_{1,z} \kappa \, d\varrho_1 = \\ = \int_\alpha^\beta \phi_{1,z} \, d\kappa^{[1]} + \int_\alpha^\beta \phi_{1,z} \kappa^{-1} d\chi_2 \circ \eta - z \int_\alpha^\beta \phi_{1,z} \kappa^{-1} d\varrho_2 \circ \eta, \end{aligned}$$

where we used the integration by parts formula (1.3), the differential equation and the substitution rule. In particular, choosing $z = 0$ this shows that the coefficients χ_1 and χ_2 are related as in the claim (note that $\phi_{1,0}$ does not have any zeros). Using this relation, one sees from the previous equation that for each $z \in \mathbb{C}^\times$ and α, $\beta \in (\alpha_1, \beta_1)$ we actually have

$$\int_\alpha^\beta \phi_{1,z} \kappa \, d\varrho_1 = \int_\alpha^\beta \phi_{1,z} \kappa^{-1} d\varrho_2 \circ \eta.$$

Now since for each $x_1 \in (\alpha_1, \beta_1)$ there is some $z \in \mathbb{C}^\times$ such that $\phi_{1,z}(x_1) \neq 0$, this shows that the coefficients ϱ_1 and ϱ_2 are related as in the claim. □

In particular, note that these relations among our measures show that under the assumptions of Corollary 8.4, for every $z \in \mathbb{C}$ and each solution u_2 of $(\tau_2 - z)u = 0$, the function

$$u_1(x_1) = \kappa(x_1)u_2(\eta(x_1)), \quad x_1 \in (\alpha_1, \beta_1)$$

is a solution of $(\tau_1 - z)u = 0$ on (α_1, β_1). Moreover, linear independence is preserved under this transformation.

In the remaining part of this section we will prove one more inverse uniqueness result, taylor made to fit the requirements of the isospectral problem of the Camassa–Holm equation. There, we do not want the measures ϱ_1 and ϱ_2 to necessarily have dense support; hence we can not apply Corollary 8.4. However, we will assume that the intervals and the coefficients on the left-hand side of the differential equation are fixed, i.e.

$$a := a_1 = a_2, \quad b := b_1 = b_2, \quad \varsigma := \varsigma_1 = \varsigma_2 \quad \text{and} \quad \chi := \chi_1 = \chi_2,$$

and that τ_1 and τ_2 are in the l.p. case at both endpoints. Another crucial additional assumption we will make for this inverse uniqueness result is that the norms of point evaluations (note that the modified Sobolev spaces are the same for both relations) $\|\delta_c\|_{H^1(a,b)}$ are independent of $c \in (a, b)$. For example this is the case when ς and χ are scalar multiples of the Lebesgue measure, as it is the case for the isospectral problem of the Camassa–Holm equation. Moreover, we suppose that our real entire solutions $\phi_{1,z}$ and $\phi_{2,z}$ coincide at $z = 0$, i.e.

$$\phi_{1,0}(x) = \phi_{2,0}(x), \quad x \in (a, b). \tag{8.4}$$

As a consequence of these assumptions, the coefficient of the second term on the right-hand side of (7.5) in Theorem 7.3 is the same for both problems. Now the weight measure on the right-hand side of our differential equation is uniquely determined by the spectral measure. In view of application to the isospectral problem of the Camassa–Holm equation we state this result with the assumption that our de Branges functions are of exponential type zero. Of course the same result holds if their quotient is of bounded type in the upper complex half-plane.

Theorem 8.5. *Suppose that $E_1(\,\cdot\,, c)$ and $E_2(\,\cdot\,, c)$ are of exponential type zero for some $c \in (a, b)$. If $\mu_1 = \mu_2$, then we have $\varrho_1 = \varrho_2$ and $S_1 = S_2$.*

Proof. The (remark after the) proof of Lemma 8.1 shows that there is an increasing continuous bijection η from Σ_1 onto Σ_2 such that

$$B_1(x_1) = B_2(\eta(x_1)), \quad x_1 \in \Sigma_1.$$

Moreover, the proof of Theorem 8.2 (see equation 8.3) shows that

$$\phi_{1,z}(x_1) = \kappa(x_1)\phi_{2,z}(\eta(x_1)), \quad z \in \mathbb{C},\ x_1 \in \Sigma_1$$

for some non-zero real function κ on Σ_1. In particular, from Lemma 6.2 we infer for each $x_1 \in \Sigma_1$

$$\|\delta_{x_1}\|^2_{H^1(a,b)} = \|\mathcal{F}_1\delta_{x_1}\|^2_{\mu_1} = \|\kappa(x_1)\mathcal{F}_2\delta_{\eta(x_1)}\|^2_{\mu_1} = \kappa(x_1)^2\|\delta_{\eta(x_1)}\|^2_{H^1(a,b)}$$

and hence $\kappa(x_1)^2 = 1$ in view of our additional assumptions. Moreover, Theorem 7.3 shows that $B_1(x_1)$ and $B_2(\eta(x_1))$ actually have the same norm and hence

$$\phi_{1,0}^{[1]}(x_1)\phi_{1,0}(x_1) = K_1(0,0,x_1) = K_2(0,0,\eta(x_1)) = \phi_{1,0}^{[1]}(\eta(x_1))\phi_{1,0}(\eta(x_1)).$$

Now since the function $\phi_{1,0}^{[1]}\phi_{1,0}$ is strictly increasing on (a,b) we infer that $\eta(x_1) = x_1$, $x_1 \in \Sigma_1$ and in particular $\Sigma_1 = \Sigma_2$. Hence we even have (note that (8.4) prohibits $\kappa(x_1) = -1$ for some $x_1 \in \Sigma_1$)

$$(*) \qquad \phi_{1,z}(x_1) = \phi_{2,z}(x_1), \quad x_1 \in \Sigma_1, \ z \in \mathbb{C}.$$

Moreover, if (α,β) is a gap of Σ_1, i.e. α, $\beta \in \Sigma_1$ but $(\alpha,\beta)\cap\Sigma_1 = \emptyset$, then both of this functions are solutions to the same differential equation which coincide on the boundary of the gap. Since their difference is a solution of $\tau_1 u = 0$ which vanishes on the boundary of the gap, we infer that $(*)$ holds for all x_1 in the convex hull of Σ_1 in view of (3.3). Now if $\underline{x} = \inf\Sigma_1 > a$, then $\varsigma + \chi$ is infinite near a and for each $z \in \mathbb{C}$ the solutions $\phi_{1,z}$ and $\phi_{2,z}$ are scalar multiples of w_a on $(a,\underline{x})$. Since they are equal in the point $\underline{x}$ we infer that $(*)$ also holds for all x_1 below $\underline{x}$. Similarly, if $\overline{x} = \sup\Sigma_1 < b$, then the spectrum of S_1 (and hence also of S_2) is purely discrete. Indeed, the solutions $\psi_{1,b,z}$, $z \in \mathbb{C}$ of $(\tau_1 - z)u = 0$ which are equal to w_b near b are real entire and lie in S_1 near b. Now for each eigenvalue $\lambda \in \mathbb{R}^\times$ the solutions $\phi_{1,\lambda}$ and $\phi_{2,\lambda}$ are scalar multiples of w_b on $(\overline{x},b)$. As before we infer that $(*)$ holds for $z = \lambda$ and all $x_1 \in (a,b)$. Finally, from the differential equation we get for each α, $\beta \in (a,b)$ with $\alpha < \beta$

$$\begin{aligned}\lambda\int_\alpha^\beta \phi_{1,\lambda}\,d\varrho_1 = -\phi_{1,\lambda}^{[1]}(\beta) + \phi_{1,\lambda}^{[1]}(\alpha) + \int_\alpha^\beta \phi_{1,\lambda}\,d\chi &= \lambda\int_\alpha^\beta \phi_{2,\lambda}\,d\varrho_2 \\ &= \lambda\int_\alpha^\beta \phi_{1,\lambda}\,d\varrho_2\end{aligned}$$

for each $\lambda \in \sigma(S_1)$. But this shows $\varrho_1 = \varrho_2$ and hence also $S_1 = S_2$. Hereby note that for each $x \in (a,b)$ there is an eigenvalue $\lambda \in \mathbb{R}$ such that $\phi_{1,\lambda}(x) \neq 0$. Indeed, otherwise we had $f(x) = 0$ for each $f \in \mathfrak{D}_1$, which is not possible unless $\Sigma_1 = \emptyset$. □

Note that the condition that the differential expressions are in the l.p. case may be relaxed. For example it is sufficient to assume that τ_j, $j = 1,2$ are in the l.p. case at a unless $\inf\Sigma_j = a$ and in the l.p. case at b unless $\sup\Sigma_j = b$. The proof therefore is essentially the same.

Appendix A. Hilbert spaces of entire functions

In this appendix we will briefly summarize some results of de Branges' theory of Hilbert spaces of entire functions as far as it is needed for the proof of our inverse uniqueness theorem. For a detailed discussion we refer to de Branges' book [14]. First of all recall that an analytic function N in the upper open complex half-plane $\mathbb{C}^+$ is said to be of bounded type if it can be written as the quotient of two bounded analytic functions. For such a function the number

$$\limsup_{y\to\infty} \frac{\ln|N(\mathrm{i}y)|}{y} \in [-\infty,\infty)$$

is referred to as the mean type of N.

Now a de Branges function is an entire function E, which satisfies the estimate

$$|E(z)| > |E(z^*)|, \quad z \in \mathbb{C}^+.$$

Associated with such a function is a de Branges space B. It consists of all entire functions F such that

$$\int_{\mathbb{R}} \frac{|F(\lambda)|^2}{|E(\lambda)|^2} d\lambda < \infty$$

and such that F/E and $F^{\#}/E$ are of bounded type in $\mathbb{C}^+$ with nonpositive mean type. Here $F^{\#}$ is the entire function given by

$$F^{\#}(z) = F(z^*)^*, \quad z \in \mathbb{C}.$$

Equipped with the inner product

$$[F, G] = \frac{1}{\pi} \int_{\mathbb{R}} \frac{F(\lambda)G(\lambda)^*}{|E(\lambda)|^2} d\lambda, \quad F,\, G \in B,$$

the vector space B turns into a reproducing kernel Hilbert space (see [14, Theorem 21]). For each $\zeta \in \mathbb{C}$, the point evaluation in ζ can be written as

$$F(\zeta) = [F, K(\zeta, \cdot\,)], \quad F \in B,$$

where the reproducing kernel K is given by (see [14, Theorem 19])

$$K(\zeta, z) = \frac{E(z)E^{\#}(\zeta^*) - E(\zeta^*)E^{\#}(z)}{2\mathrm{i}(\zeta^* - z)}, \quad \zeta,\, z \in \mathbb{C}. \tag{A.1}$$

Note that though there is a multitude of de Branges functions giving rise to the same de Branges space (including norms), the reproducing kernel K is independent of the actual de Branges function.

One of the main results in de Branges' theory is the subspace ordering theorem; [14, Theorem 35]. For our application we need to slightly weaken the assumptions of this theorem. In order to state it let E_1, E_2 be two de Branges functions with no real zeros and B_1, B_2 be the associated de Branges spaces.

Theorem A.1. *Suppose B_1, B_2 are homeomorphically embedded in $L^2(\mathbb{R};\mu)$ for some Borel measure μ on $\mathbb{R}$. If E_1/E_2 is of bounded type in the upper complex half-plane, then B_1 contains B_2 or B_2 contains B_1.*

Proof. If a de Branges space B is homeomorphically embedded in $L^2(\mathbb{R};\mu)$, then B equipped with the inner product inherited from $L^2(\mathbb{R};\mu)$ is a de Branges space itself. In fact, this is easily verified using the characterization of de Branges spaces in [14, Theorem 23]. Hence, without loss of generality we may assume that B_1, B_2 are isometrically embedded in $L^2(\mathbb{R};\mu)$ and thus apply [14, Theorem 35]. Therefore, also note that F_1/F_2 is of bounded type in the upper complex half-plane for all $F_1 \in B_1$, $F_2 \in B_2$ and hence so is the quotient of any corresponding de Branges functions. □

Note that the isometric embedding in [14, Theorem 35] is only needed to deduce that the smaller space is actually a de Branges subspace of the larger one. The inclusion part is valid under much more general assumptions; see [29, Theorem 5] or [30, Theorem 3.5].

Adapting the proof of [14, Theorem 35], one gets a version of de Branges' ordering theorem, where the bounded type condition is replaced by the assumption that the functions E_1, E_2 are of exponential type zero. Actually this has been done in [26] with the spaces B_1, B_2 being isometrically embedded in some $L^2(\mathbb{R};\mu)$. Again this last assumption can be weakened.

Theorem A.2. *Suppose B_1, B_2 are homeomorphically embedded in $L^2(\mathbb{R};\mu)$ for some Borel measure μ on $\mathbb{R}$. If E_1, E_2 are of exponential type zero, then B_1 contains B_2 or B_2 contains B_1.*

Proof. As in the proof of Theorem A.1, the claim can be reduced to the case where the de Branges spaces are isometrically embedded in $L^2(\mathbb{R};\mu)$. Therefore, also note that a de Branges function is of exponential type zero if and only if all functions in the corresponding de Branges space are (see e.g. [24, Theorem 3.4]). □

Acknowledgments. I thank Harald Woracek for helpful discussions and hints with respect to the literature.

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On the isospectral problem of the Camassa–Holm equation

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Not yet submitted

Abstract. We discuss spectral theory for the isospectral problem of the Camassa–Holm equation, where the weight is allowed to be a finite signed measure. In particular, we prove that this weight is uniquely determined by the spectral data.

1. Introduction

The Camassa–Holm equation is an integrable, non-linear wave equation which models unidirectional wave propagation on shallow water. Due to its many remarkable properties, this equation has gotten quite a lot of attention in the recent past and we only refer to e.g. [4], [5], [6], [8], [9] for further information. In the present paper we are concerned with the weighted Sturm–Liouville problem

$$-f''(x) + \frac{1}{4}f(x) = z\omega(x)f(x), \quad x \in \mathbb{R},\ z \in \mathbb{C} \tag{1.1}$$

on the real line, which arises as the isospectral problem in the Lax pair of the Camassa–Holm equation. Direct, and in particular inverse spectral theory for this Sturm–Liouville problem are of peculiar interest for solving the Cauchy problem of the Camassa–Holm equation.

Provided that the weight ω is a strictly positive function, it is well known that the spectral problem (1.1) (with some suitable boundary conditions) gives rise to a self-adjoint operator in the weighted Hilbert space $L^2(\mathbb{R};\omega)$. Moreover, if ω is smooth enough it is even possible to transform this problem into a (in general singular) Sturm–Liouville problem in potential form and some inverse spectral conclusions may be drawn from this. However, in order to incorporate the main interesting phenomena (wave breaking [7] and multi-peakon solutions [1], [9]) of the dispersionless Camassa–Holm equation, it is necessary to at least allow ω to be an arbitrary finite signed Borel measure on $\mathbb{R}$. In fact, multi-peakon solutions of the Camassa–Holm equation correspond to weights which are a finite sum of weighted Dirac measures and

2010 *Mathematics Subject Classification.* Prim. 37K15, 34B40; Sec. 35Q35, 34L05.
Keywords. Camassa–Holm equation, isospectral problem, inverse spectral theory.
Research supported by the *Austrian Science Fund (FWF)* under grant no. Y330.

wave breaking only occurs if the weight changes sign. In the present paper we will discuss some direct and inverse spectral theory for this isospectral problem when the weight ω is only assumed to be an arbitrary finite signed measure. Hereby note that up to now (apart from the pure multi-peakon case [1]) the best results which deal with this problem seem to be due to Bennewitz [2], Brown and Weikard [3]. However, they have to impose some restrictions on the weight, which for example exclude the multi-peakon case from their inverse uniqueness result.

The outline of this paper is as follows. In Section 2 we introduce the isospectral operator in the weighted Hilbert space $L^2(\mathbb{R};\omega)$, associated with the Sturm–Liouville problem (1.1) and discuss its basic properties. In the following section we describe the spectrum of this operator and determine its resolvent. The last section contains our main result; an inverse uniqueness theorem, which shows that the measure ω is uniquely determined by the spectral data. This is done by verifying the assumptions of an inverse uniqueness result in [10, Theorem 8.5]. However, since this uniqueness result applies to so-called left-definite operators, we introduce this operator in Appendix A and show how it is related to our isospectral operator.

2. The isospectral operator

Let ω be some arbitrary finite signed Borel measure on $\mathbb{R}$. The maximal domain $\mathfrak{D}_\tau$ of functions for which (1.1) makes sense consists of all locally absolutely continuous functions f for which the function

$$-f'(x) + f'(c) + \frac{1}{4}\int_c^x f(s)ds, \quad x \in \mathbb{R} \tag{2.1}$$

is locally absolutely continuous with respect to the measures ω. Note that this is the case if and only if there is some $g \in L^1_{\mathrm{loc}}(\mathbb{R};\omega)$ such that

$$-f'(x) + f'(c) + \frac{1}{4}\int_c^x f(s)ds = \int_c^x g(s)d\omega(s)$$

for almost all $x \in \mathbb{R}$ with respect to the Lebesgue measure. Here and in the following, integrals with respect to the measure ω have to be read as

$$\int_c^x g(s)d\omega(s) = \begin{cases} \int_{[c,x)} g(s)d\omega(s), & \text{if } x > c, \\ 0, & \text{if } x = c, \\ -\int_{[x,c)} g(s)d\omega(s), & \text{if } x \le c. \end{cases}$$

In other words, the function g is the Radon–Nikodým derivative of the function in (2.1) with respect to the measure ω. In this case, i.e. for functions $f \in \mathfrak{D}_\tau$ we set $\tau f = g$ and hence (1.1) becomes $(\tau - z)f = 0$.

Associated with this differential expression τ is a linear operator T in the weighted Hilbert space $L^2(\mathbb{R};\omega)$, equipped with the inner product

$$\langle f, g\rangle_{|\omega|} = \int_{\mathbb{R}} f(x)g(x)^* d|\omega|(x), \quad f,\, g \in L^2(\mathbb{R};\omega).$$

In order to define our linear differential operator T, we say a function $f \in \mathfrak{D}_\tau$ with $f,\, \tau f \in L^2(\mathbb{R};\omega)$ satisfies the boundary condition at $\pm\infty$ if

$$BC_{\pm\infty}(f) := \lim_{x\to\pm\infty} f(x)\mathrm{e}^{\mp\frac{x}{2}} \pm f'(x)\frac{1}{2}\mathrm{e}^{\mp\frac{x}{2}} = 0. \tag{2.2}$$

Hereby note that these limits are known to exist; see [11, Lemma 4.2]. Moreover, let us mention that the boundary conditions at $\pm\infty$ are actually superfluous, provided that ω does not decay to fast at $\pm\infty$. In fact, they are automatically satisfied if and only if

$$\int_{\mathbb{R}} \mathrm{e}^{\pm x} d|\omega|(x) = \infty.$$

This difference stems from the fact that τ is either in the limit-circle or in the limit-point case at each endpoint. Now the linear operator T is given by

$$\mathrm{dom}(T) = \{f \in \mathfrak{D}_\tau \mid f,\, \tau f \in L^2(\mathbb{R};\omega),\ BC_{\pm\infty}(f) = 0\}$$

and $Tf = \tau f$ for $f \in \mathrm{dom}(T)$. Hereby note that each function in $\mathrm{dom}(T)$ has a unique representative $f \in \mathfrak{D}_\tau$ with $f,\, \tau f \in L^2(\mathbb{R};\omega)$ and $BC_{\pm\infty}(f) = 0$; see [11, Section 7]. Hence the differential operator T is well-defined.

Unfortunately this operator T is not self-adjoint unless the measure ω is of one sign. However, it is self-adjoint in the Kreĭn space $L^2(\mathbb{R};\omega)$, equipped with the (in general) indefinite inner product

$$\langle f, g\rangle_\omega = \int_{\mathbb{R}} f(x) g(x)^* d\omega(x), \quad f,\, g \in L^2(\mathbb{R};\omega).$$

In fact, if J is the unitary operator of multiplication with the sign of ω, i.e.

$$Jf(x) = \frac{d\omega}{d|\omega|}(x)\, f(x), \quad x \in \mathbb{R},\ f \in L^2(\mathbb{R};\omega),$$

then the results in [11, Section 6] show that the operator JT is self-adjoint in the Hilbert space $L^2(\mathbb{R};\omega)$ since JT is nothing but the differential operator associated with the measure $|\omega|$. Now from this we infer that T is self-adjoint with respect to the indefinite inner product. In particular, this guarantees that T is a closed operator in $L^2(\mathbb{R};\omega)$.

Finally, we want to emphasize that we only assumed ω to be a finite signed measure. In particular, ω is allowed to vanish on arbitrary sets and may even be equal to zero at all. Of course, this last case leads to a degenerate Hilbert space $L^2(\mathbb{R};\omega)$ and a degenerate operator T. Moreover, also the case when ω is supported on only one point is still quite degenerate, since in this case all solutions of (1.1) are linearly dependent in $L^2(\mathbb{R};\omega)$. Nevertheless, it still gives rise to a (one-dimensional) self-adjoint linear operator T in $L^2(\mathbb{R};\omega)$; see [11, Appendix C].

3. Spectrum and resolvent

As a first step we construct solutions of $(\tau - z)u = 0$ which satisfy the boundary condition at $\pm\infty$ respectively and depend analytically on $z \in \mathbb{C}$.

Theorem 3.1. *For each $z \in \mathbb{C}$ there are unique solutions $\phi_\pm(z,\cdot\,)$ of the equation $(\tau - z)u = 0$ with the spatial asymptotics*

$$\phi_\pm(z,x) \sim \mathrm{e}^{\mp\frac{x}{2}} \quad \text{and} \quad \phi_\pm'(z,x) \sim \mp\frac{1}{2}\mathrm{e}^{\mp\frac{x}{2}} \tag{3.1}$$

as $x \to \pm\infty$. Moreover, the functions

$$z \mapsto \phi_\pm(z,x) \quad \text{and} \quad z \mapsto \phi_\pm'(z,x) \tag{3.2}$$

are real entire and of finite exponential type for each $x \in \mathbb{R}$.

Proof. First of all we show that for each $z \in \mathbb{C}$, the integral equations

$$m_\pm(z,x) = 1 \pm z \int_x^{\pm\infty} \left(\mathrm{e}^{\pm(x-s)} - 1\right) m_\pm(z,s) d\omega(s), \quad x \in \mathbb{R} \tag{3.3}$$

have unique bounded continuous solutions $m_\pm(z,\cdot\,)$. Therefore consider the integral operators $K_\pm$ on $C_b(\mathbb{R})$

$$K_\pm f(x) = \pm \int_x^{\pm\infty} \left(\mathrm{e}^{\pm(x-s)} - 1\right) f(s) d\omega(s), \quad x \in \mathbb{R},\ f \in C_b(\mathbb{R}),$$

where $C_b(\mathbb{R})$ is the space of bounded continuous functions on $\mathbb{R}$. Note that for each $f \in C_b(\mathbb{R})$ the functions $K_\pm f$ are continuous since the integrands are bounded and continuous. Moreover, for each $n \in \mathbb{N}$ we have the estimate

$$\sup_{s<x} \left|K_-^n f(s)\right| \leq \frac{1}{n!} \left(\int_{-\infty}^x d|\omega|\right)^n \sup_{s<x} |f(s)|, \quad x \in \mathbb{R}.$$

In fact, the case when $n = 1$ is easily verified. Otherwise we get inductively

$$\begin{aligned} \sup_{s<x} \left|K_-^n f(s)\right| &\leq \sup_{s<x} \int_{-\infty}^s \left|\mathrm{e}^{r-s} - 1\right| \left|K_-^{n-1} f(r)\right| d|\omega|(r) \\ &\leq \frac{1}{(n-1)!} \int_{-\infty}^x \left(\int_{-\infty}^r d|\omega|\right)^{n-1} d|\omega|(r) \sup_{s<x} |f(s)|, \quad x \in \mathbb{R}. \end{aligned}$$

Now an application of the substitution rule for Lebesgue–Stieltjes integrals [13] yields the claim. Similarly one obtains a corresponding estimate for $K_+ f$ which ensures that $K_\pm f \in C_b(\mathbb{R})$. Moreover, we even get the bound

$$\|K_\pm^n\| \leq \frac{1}{n!} |\omega|(\mathbb{R})^n, \quad n \in \mathbb{N} \tag{$*$}$$

and hence the Neumann series

$$m_\pm(z,x) = \sum_{n=0}^\infty z^n K_\pm^n 1(x) = (I - zK_\pm)^{-1} 1(x), \quad x \in \mathbb{R},\ z \in \mathbb{C} \tag{3.4}$$

converge absolutely, uniformly in $x \in \mathbb{R}$ and even locally uniformly in $z \in \mathbb{C}$. In particular, these functions $m_\pm(z,\cdot\,)$ are the unique solutions in $C_b(\mathbb{R})$ of the integral equations in (3.3). Moreover, integrating the right-hand sides of (3.3) by parts shows that these functions are locally absolutely continuous with derivatives given by

$$m_\pm'(z,x) = z \int_x^{\pm\infty} \mathrm{e}^{\pm(x-s)} m_\pm(z,s) d\omega(s), \quad x \in \mathbb{R},\ z \in \mathbb{C}. \tag{3.5}$$

Therefore, we have the spatial asymptotics

$$m_\pm(z,x) \to 1 \quad \text{and} \quad m_\pm'(z,x) \to 0$$

as $x \to \pm\infty$ for each $z \in \mathbb{C}$. Indeed, this follows from the integral equations (and their spatial derivatives) and the fact that the functions $m_\pm(z,\cdot\,)$ are uniformly bounded. Now equation (3.3) shows that the functions

$$\phi_\pm(z,x) = \mathrm{e}^{\mp\frac{x}{2}} m_\pm(z,x), \quad x \in \mathbb{R},\ z \in \mathbb{C}$$

satisfy the integral equations

$$\phi_\pm(z,x) = \mathrm{e}^{\mp\frac{x}{2}} \pm z \int_x^{\pm\infty} \left(\mathrm{e}^{\pm\frac{x-s}{2}} - \mathrm{e}^{\mp\frac{x-s}{2}}\right) \phi_\pm(z,s) d\omega(s), \quad x \in \mathbb{R},\ z \in \mathbb{C}.$$

From this it is easily verified that $\phi_\pm(z,\cdot\,)$ are solutions of $(\tau - z)u = 0$ (see e.g. [11, Proposition 3.3]). The spatial asymptotics of these solutions near $\pm\infty$ easily follow from the corresponding results for the functions $m_\pm(z,\cdot\,)$. Also note that these asymptotics uniquely determine the solutions $\phi_\pm(z,\cdot\,)$. Finally, the Neumann series and the estimates in $(*)$ guarantee that $m_\pm(\,\cdot\,,x)$ are real entire and of finite exponential type, uniformly for all $x\in\mathbb{R}$. Hence we see from (3.5) that $m_\pm'(\,\cdot\,,x)$ are also real entire with finite exponential type for each $x\in\mathbb{R}$. Of course, this proves that the functions in (3.2) are real entire and of finite exponential type for each $x\in\mathbb{R}$. □

From the spatial asymptotics of the solutions $\phi_\pm(z,\cdot\,)$, $z\in\mathbb{C}$ it is easily seen that they lie in $L^2(\mathbb{R};\omega)$ near $\pm\infty$ respectively and satisfy the boundary condition (2.2) there. In particular, this guarantees that the spectrum of T is purely discrete and simple. More precisely, from [11, Theorem 8.5] and [11, Theorem 9.6] it follows that JT and hence also T have purely discrete spectra. The fact that the spectrum is simple follows literally as in the proof of [11, Corollary 8.4]. Consequently, some $\lambda\in\mathbb{C}$ is an eigenvalue of T if and only if the solutions $\phi_-(\lambda,\cdot\,)$ and $\phi_+(\lambda,\cdot\,)$ are linearly dependent, that is their Wronskian

$$W(z) = \phi_+(z,x)\phi_-'(z,x) - \phi_+'(z,x)\phi_-(z,x), \quad z\in\mathbb{C}$$

vanishes in λ. In this case there is some $c_\lambda\in\mathbb{R}^\times$ such that

$$\phi_+(\lambda,x) = c_\lambda\phi_-(\lambda,x), \quad x\in\mathbb{R}.$$

Moreover, the quantity

$$\gamma_\lambda^2 = \int_{\mathbb{R}} |\phi_-(\lambda,x)|^2 d\omega(x)$$

is finite and referred to as the norming constant associated with the eigenvalue λ. A simple integration by parts, using the spatial asymptotics of the solutions $\phi_-(\lambda,\cdot\,)$ and $\phi_+(\lambda,\cdot\,)$ shows that

$$\lambda\gamma_\lambda^2 = \frac{1}{4}\int_{\mathbb{R}} |\phi_-(\lambda,x)|^2 dx + \int_{\mathbb{R}} |\phi_-'(\lambda,x)|^2 dx > 0. \tag{3.6}$$

In particular, this guarantees that λ and hence the spectrum of T are real. Moreover, from this equation one also sees that the spectrum is positive (respectively negative), provided that the measure ω is positive (respectively negative). The following lemma relates all these spectral quantities.

Lemma 3.2. *For each $\lambda\in\sigma(T)$ we have*

$$\dot W(\lambda) = \int_{\mathbb{R}} \phi_-(\lambda,x)\phi_+(\lambda,x)d\omega(x) = c_\lambda\gamma_\lambda^2 \neq 0, \tag{3.7}$$

where the dot denotes differentiation with respect to the spectral parameter.

Proof. We set

$$(*) \qquad W_\pm(z,x) = \dot\phi_\pm(z,x)\phi_\mp'(z,x) - \dot\phi_\pm'(z,x)\phi_\mp(z,x), \quad x\in\mathbb{R},\ z\in\mathbb{C},$$

where the spatial differentiation is done first. Now, using the differential equations for the solutions $\phi_\pm(z,\cdot\,)$ one gets

$$W_\pm(z,\beta) - W_\pm(z,\alpha) = \int_\alpha^\beta \phi_+(z,s)\phi_-(z,s)d\omega(s), \quad \alpha,\,\beta\in\mathbb{R}.$$

More precisely, this follows by differentiating $(*)$ with respect to the spatial variable, where the derivatives are in general Borel measures. Now differentiating the integral equations in (3.3) and their spatial derivatives in (3.5) with respect to the spectral variable we get

$$(I - zK_\pm)^{-1}\dot{m}_\pm(z,\cdot\,) = K_\pm m_\pm(z,\cdot\,), \quad z \in \mathbb{C}$$

as well as

$$\dot{m}_\pm'(z,x) = \int_x^{\pm\infty} \mathrm{e}^{\pm(x-s)}\left(m_\pm(z,s) + z\dot{m}_\pm(z,s)\right)d\omega(s), \quad x \in \mathbb{R},\ z \in \mathbb{C}.$$

In particular, this shows that

$$\dot{m}_\pm(z,x) \to 0 \quad \text{and} \quad \dot{m}_\pm'(z,x) \to 0$$

as $x \to \pm\infty$ for each $z \in \mathbb{C}$. If $\lambda \in \sigma(T)$ is an eigenvalue, then we furthermore know that

$$m_\pm(\lambda,x) = \mathrm{e}^{\pm\frac{x}{2}}\phi_\pm(\lambda,x) = c_\lambda^{\pm1}\mathrm{e}^{\pm\frac{x}{2}}\phi_\mp(\lambda,x), \quad x \in \mathbb{R}$$

and hence $m_\pm(\lambda,x)$ and $m_\pm'(\lambda,x)$ are bounded as $x \to \pm\infty$. A calculation shows that for each $x \in \mathbb{R}$ we have

$$W_\pm(\lambda,x) = \dot{m}_\pm(\lambda,x)m_\mp(\lambda,x) + \dot{m}_\pm(\lambda,x)m_\mp'(\lambda,x) - \dot{m}_\pm'(\lambda,x)m_\mp(\lambda,x),$$

which tends to zero as $x \to \pm\infty$. Therefore we conclude

$$W_\pm(\lambda,x) = \int_x^{\pm\infty} \phi_+(\lambda,s)\phi_-(\lambda,s)d\omega(s), \quad x \in \mathbb{R}$$

and hence finally

$$\dot{W}(\lambda) = W_+(\lambda,x) - W_-(\lambda,x) = \int_{\mathbb{R}} \phi_+(\lambda,s)\phi_-(\lambda,s)d\omega(s),$$

which is the claimed identity. □

We will now determine the resolvent of our operator T. Therefore note that from the Neumann series (3.4) we get

$$\phi_\pm(0,x) = \mathrm{e}^{\mp\frac{x}{2}}, \quad x \in \mathbb{R}.$$

Furthermore, this series yields an expansion of $\phi_\pm(\,\cdot\,,x)$ near zero which will be needed later on.

Proposition 3.3. *The operator T is invertible with inverse given by*

$$T^{-1}g(x) = \int_{\mathbb{R}} \mathrm{e}^{-\frac{|x-s|}{2}}g(s)d\omega(s), \quad x \in \mathbb{R},\ g \in L^2(\mathbb{R};\omega). \tag{3.8}$$

Moreover, this inverse is a trace class operator with

$$\sum_{\lambda\in\sigma(T)} \frac{1}{\lambda} = \int_{\mathbb{R}} d\omega. \tag{3.9}$$

Proof. Since the solutions $\phi_-(0,\cdot\,)$ and $\phi_+(0,\cdot\,)$ are linearly independent, JT is invertible with

$$(JT)^{-1}g(x) = \int_{\mathbb{R}} \mathrm{e}^{-\frac{|x-s|}{2}}g(s)d|\omega|(s), \quad x \in \mathbb{R},\ g \in L^2(\mathbb{R};\omega),$$

in view of [11, Theorem 8.3]. Thus T is invertible as well with inverse given as in the claim. Moreover, since the spectrum of JT is positive in view of equation (3.6), $(JT)^{-1}$ is positive as well and we infer from a lemma in [12, Section XI.4] that $(JT)^{-1}$ is even a trace class operator with trace norm

$$\|(JT)^{-1}\|_{\mathrm{tr}} = \mathrm{tr}\ (JT)^{-1} = \int_{\mathbb{R}} d|\omega|.$$

But this shows that T^{-1} is also a trace class operator with the same trace norm as $(JT)^{-1}$. In order to compute the trace of T^{-1} consider the positive integral operators

$$R_\pm g(x) = \int_{\mathbb{R}} \mathrm{e}^{-\frac{|x-s|}{2}} g(s) d\omega_\pm(s), \quad x \in \mathbb{R},\ g \in L^2(\mathbb{R};\omega_\pm)$$

in the Hilbert spaces $L^2(\mathbb{R};\omega_\pm)$, where $\omega = \omega_+ - \omega_-$ is the Hahn–Jordan decomposition of ω. Now, if we identify $L^2(\mathbb{R};\omega)$ with the orthogonal sum of the spaces $L^2(\mathbb{R};\omega_+)$ and $L^2(\mathbb{R};\omega_-)$, then we get

$$\mathrm{tr}\ T^{-1} = \mathrm{tr}\ R_+ - \mathrm{tr}\ R_- = \int_{\mathbb{R}} d\omega_+ - \int_{\mathbb{R}} d\omega_- = \int_{\mathbb{R}} d\omega,$$

in view of the previously mentioned lemma in [12, Section XI.4]. □

More generally, for each $z \in \rho(T)$ the resolvent is given by

$$(T-z)^{-1} g(x) = \int_{\mathbb{R}} G(z,x,y) g(y) d\omega(y), \quad x \in \mathbb{R},\ g \in L^2(\mathbb{R};\omega),$$

where G is the Green function

$$G(z,x,y) = W(z)^{-1} \begin{cases} \phi_-(z,y)\phi_+(z,x), & \text{if } y \le x, \\ \phi_-(z,x)\phi_+(z,y), & \text{if } y > x. \end{cases}$$

In fact, this can be shown following literally the proof of [11, Theorem 8.3] since the solutions $\phi_\pm(z,\cdot\,)$ lie in the domain of T near $\pm\infty$ respectively. Moreover, the measure ω can be read off from the expansion of the Green function near zero on the diagonal.

Lemma 3.4. *For every $x \in \mathbb{R}$ we have*

$$G(z,x,x) = 1 + z \int_{\mathbb{R}} \mathrm{e}^{-|x-s|} d\omega(s) + \mathcal{O}\left(z^2\right), \tag{3.10}$$

as $|z| \to 0$ in $\mathbb{C}$.

Proof. For each $x \in \mathbb{R}$ we get from the Neumann series (3.4)

$$m_\pm(z,x) = 1 \pm z \int_x^{\pm\infty} \left(\mathrm{e}^{\pm(x-s)} - 1\right) d\omega(s) + \mathcal{O}\left(z^2\right),$$

as $z \to 0$ in $\mathbb{C}$ and hence

$$\phi_-(z,x)\phi_+(z,x) = 1 + z \int_{\mathbb{R}} \mathrm{e}^{-|x-s|} d\omega(s) - z \int_{\mathbb{R}} d\omega(s) + \mathcal{O}\left(z^2\right),$$

as $z \to 0$ in $\mathbb{C}$. Since this first expansion holds uniformly for all $x \in \mathbb{R}$, we get from equation (3.5)

$$m_\pm'(z,x) = z \int_x^{\pm\infty} \mathrm{e}^{\pm(x-s)} d\omega(s) + \mathcal{O}\left(z^2\right),$$

for every $x \in \mathbb{R}$ as $z \to 0$ in $\mathbb{C}$. Therefore we have

$$\begin{aligned} W(z) &= m_+(z,x)m_-'(z,x) - m_+'(z,x)m_-(z,x) + m_-(z,x)m_+(z,x) \\ &= 1 - z\int_{\mathbb{R}} d\omega(s) + \mathcal{O}\left(z^2\right), \end{aligned}$$

as $z \to 0$ in $\mathbb{C}$ and hence also

$$W(z)^{-1} = 1 + z\int_{\mathbb{R}} d\omega(s) + \mathcal{O}\left(z^2\right),$$

as $z \to 0$ in $\mathbb{C}$. Thus we finally get

$$\frac{\phi_-(z,x)\phi_+(z,x)}{W(z)} = 1 + z\int_{\mathbb{R}} \mathrm{e}^{-|x-s|} d\omega(s) + \mathcal{O}\left(z^2\right),$$

for every $x \in \mathbb{R}$ as $z \to 0$ in $\mathbb{C}$. □

Note that the quantity

$$u(x) = \int_{\mathbb{R}} \mathrm{e}^{-|x-s|} d\omega(s), \quad x \in \mathbb{R}$$

is important in view of applications to the Camassa–Holm equation, since it is the unique solution of $u - u_{xx} = \omega$ in $H^1(\mathbb{R})$.

4. Exponential growth of solutions

In order to apply the inverse uniqueness result from [10] we need to show that our solutions $\phi_\pm$ actually are of exponential type zero, i.e. that

$$\ln^+ |\phi_\pm(z,c)| + \ln^+ |\phi_\pm'(z,c)| = o(|z|),$$

as $|z| \to \infty$ in $\mathbb{C}$ for some (and hence all) $c \in \mathbb{R}$. Therefore we will denote with $M_\pm(c) \subseteq \mathbb{C}$ the sets of all roots of the entire functions $\phi_\pm(\,\cdot\,,c)$.

Theorem 4.1. *The solutions $\phi_\pm$ are of exponential type zero and given by*

$$\phi_\pm(z,x) = \mathrm{e}^{\mp\frac{x}{2}} \prod_{\mu \in M_\pm(x)} \left(1 - \frac{z}{\mu}\right), \quad z \in \mathbb{C},\ x \in \mathbb{R}. \tag{4.1}$$

Proof. Fix some $c \in \mathbb{R}$ and consider the differential operator T_c in the Hilbert space $L^2((-\infty,c);\omega)$, given by

$$\mathrm{dom}(T_c) = \{f \in \mathfrak{D}_\tau \,|\, BC_{-\infty}(f) = f(c) = 0\}$$

and $T_c f = \tau f$ for $f \in \mathrm{dom}(T_c)$. Similarly as for T, the operator J_cT_c is self-adjoint in $L^2((-\infty,c);\omega)$, where J_c is the unitary operator of multiplication with the sign of ω in $L^2((-\infty,c);\omega)$. In particular, [11, Theorem 8.3] shows that its inverse is given by

$$(J_cT_c)^{-1}g(x) = \int_{-\infty}^{c} \left(\mathrm{e}^{-\frac{|x-s|}{2}} - \mathrm{e}^{\frac{x+s}{2}-c}\right) g(s)d|\omega|(s), \quad x \in (-\infty,c)$$

for each $g \in L^2((-\infty,c);\omega)$. Now as in the proof of Proposition 3.3 one shows that $(J_cT_c)^{-1}$ and hence also T_c^{-1} are actually trace class operators with trace given by

$$\sum_{\mu \in M_-(c)} \frac{1}{\mu} = \mathrm{tr}\, T_c^{-1} = \int_{-\infty}^{c} \left(1 - \mathrm{e}^{s-c}\right) d\omega(s).$$

Hereby note that $M_-(c)$ actually is the spectrum of T_c. Moreover, since the real entire function $\phi_-(\,\cdot\,,c)$ is of finite exponential type with summable zeros, the Hadamard factorization shows that

$$(*)\qquad \phi_-(z,c) = \phi_-(0,c)\mathrm{e}^{A_c z} \prod_{\mu\in M_-(c)} \left(1 - \frac{z}{\mu}\right), \quad z \in \mathbb{C}$$

for some $A_c \in \mathbb{R}$. Using the Neumann series (3.4) near zero and the representation $(*)$ on the other side we get

$$\int_{-\infty}^{c} \left(1 - \mathrm{e}^{c-s}\right) d\omega(s) = -\frac{\dot{\phi}_\pm(0,c)}{\phi_\pm(0,c)} = \sum_{\mu\in M_-(c)} \frac{1}{\mu} - A_c.$$

Since the integral on the left-hand side is equal to the trace of T_c^{-1} we conclude that $A_c = 0$, which yields the claimed representation for $\phi_-(\,\cdot\,,c)$. In particular, this shows that $\phi_-(\,\cdot\,,c)$ is of exponential type zero.

In much the same manner one shows the claimed properties for $\phi_-'(\,\cdot\,,c)$ by considering a similar operator with Neumann boundary conditions at c. Finally, the corresponding claim for ϕ_+ essentially follows by reflection. □

Also note that the derivatives of the solutions $\phi_\pm$ are given by

$$(4.2)\qquad \phi_\pm'(z,x) = \mp\frac{1}{2}\mathrm{e}^{\mp\frac{x}{2}} \prod_{\nu\in N_\pm(x)} \left(1 - \frac{z}{\nu}\right), \quad z \in \mathbb{C},\ x \in \mathbb{R},$$

where $N_\pm(c) \subseteq \mathbb{C}$ are the sets of roots of the entire functions $\phi_\pm'(\,\cdot\,,c)$. From these results we also get a product representation for the Wronskian W.

Corollary 4.2. *The Wronskian W has the product representation*

$$(4.3)\qquad W(z) = \prod_{\lambda\in\sigma(T)} \left(1 - \frac{z}{\lambda}\right), \quad z \in \mathbb{C}.$$

Proof. Since W is of exponential type zero with $W(0) = 1$ and summable roots, this follows from the Hadamard factorization. □

From the product representations of the functions in Theorem 4.1 and Corollary 4.2 one sees that it is possible to express $u(x)$ at a point $x \in \mathbb{R}$ in terms of the spectra $\sigma(T)$ and $M_\pm(x)$, using the equation in Lemma 3.4.

Finally, as a simple consequence of the results in [10] we get the following inverse uniqueness theorem for the spectral measure ρ of T given by

$$\rho = \sum_{\lambda\in\sigma(T)} \gamma_\lambda^{-2}\delta_\lambda,$$

where δ_λ is the Dirac measure in the point λ.

Theorem 4.3. *The spectral measure ρ uniquely determines the measure ω.*

Proof. This follows by applying [10, Theorem 8.5] to the left-definite linear relation S from Appendix A. The assumptions of this theorem are readily verified. Also note that the positive discrete measure

$$\sum_{\lambda\in\sigma(T)} \lambda^{-1}\gamma_\lambda^{-2}\delta_\lambda$$

is the spectral measure associated with S. □

Appendix A. The left-definite operator

In this appendix we will introduce the left-definite operator S, associated with the isospectral problem (1.1). Actually S is a self-adjoint linear relation in the Sobolev space $H^1(\mathbb{R})$, equipped with the modified inner product

$$\langle f, g\rangle_{H^1(\mathbb{R})} = \frac{1}{4}\int_{\mathbb{R}} f(x)g(x)^* dx + \int_{\mathbb{R}} f'(x)g'(x)^* dx, \quad f,\, g \in H^1(\mathbb{R}),$$

given by

$$S = \{(f, f_\tau) \in H^1(\mathbb{R}) \times H^1(\mathbb{R}) \,|\, f \in \mathfrak{D}_\tau,\ \tau f = f_\tau \text{ in } L^1_{\mathrm{loc}}(\mathbb{R};\omega)\}.$$

The fact that S is self-adjoint immediately follows from results in [10, Theorem 4.6], because here τ is in the limit-point case at both endpoints. Since S is a self-adjoint realization of the differential equation (1.1) it is not surprising that the spectral properties of S are very similar to the ones of T.

Proposition A.1. *The linear relation S has the same spectrum as T and its inverse is given by*

$$S^{-1}g(x) = \int_{\mathbb{R}} \mathrm{e}^{-\frac{|x-s|}{2}} g(s)d\omega(s), \quad x \in \mathbb{R},\ g \in H^1(\mathbb{R}). \tag{A.1}$$

Proof. From Theorem 3.1 and the remark after [10, Theorem 5.7] we infer that S has purely discrete spectrum. Moreover, the fact that zero is not an eigenvalue follows from [10, Proposition 3.6]. Now since for each $\lambda \in \mathbb{R}$ the solutions $\phi_\pm(\lambda, \cdot\,)$ lie in the domain of S near $\pm\infty$ respectively, we infer that λ is an eigenvalue of S if and only if these solutions are linearly dependent. Hence the spectra of S and T are equal. Finally, if $g \in H^1(\mathbb{R})$ has compact support, then $S^{-1}g$ is given as in the claim since the function on the right-hand side of (A.1) is a solution of $\tau f = g$ which lies in $H^1(\mathbb{R})$. The general case, when $g \in H^1(\mathbb{R})$ follows from continuity of both sides of (A.1). □

Although, this is all we need in order to apply the inverse uniqueness result from [10, Theorem 8.5] to the operator T, we will furthermore show how S and T are related. Therefore recall that with each strictly positive self-adjoint operator in a Kreĭn space one can associate a so-called left-definite operator in some Hilbert space; see e.g. [14, Section 11.4] for a discussion which is close to our situation. The left-definite Hilbert space H_1 associated with T is the domain of $\sqrt{JT}$, equipped with the inner product

$$\langle f, g\rangle_1 = \langle \sqrt{JT}f, \sqrt{JT}g\rangle_{|\omega|}, \quad f,\, g \in H_1.$$

Furthermore, the left-definite operator S_1 is obtained by restricting T to the space H_1. More precisely, S_1 is given by

$$\mathrm{dom}(S_1) = \mathrm{ran}(T^{-1}|_{H_1})$$

and $S_1 f = Tf$ for $f \in \mathrm{dom}(S_1)$. It turns out that this operator S_1 is self-adjoint in H_1. In particular, note that its domain and hence also the domain of T are dense in H_1. Moreover, it is known that the spectra of T and S_1 are the same. We will now show that one may identify H_1 with a closed subspace of $H^1(\mathbb{R})$ and that the operator S_1 is essentially the same as the linear relation S defined above.

Proposition A.2. *The operator part of S is unitarily equivalent to S_1.*

Proof. First of all note that $\mathrm{dom}(T)$ may be regarded as a subset of $H^1(\mathbb{R})$. In fact, each $f \in \mathrm{dom}(T)$ can be written as

$$f(x) = \int_{\mathbb{R}} \mathrm{e}^{-\frac{|x-s|}{2}} g(s) d\omega(s), \quad x \in \mathbb{R}$$

for some $g \in L^2(\mathbb{R};\omega)$. It is not hard to show that this function actually lies in $H^1(\mathbb{R})$. Moreover, an integration by parts shows that

$$\begin{aligned}\langle f, g\rangle_1 = \langle f, JTg\rangle_{|\omega|} &= \int_{\mathbb{R}} f(x)\tau g(x)^* d\omega(x) \\ &= \frac{1}{4}\int_{\mathbb{R}} f(x)g(x)^* dx + \int_{\mathbb{R}} f'(x)g'(x)^* dx = \langle f, g\rangle_{H^1(\mathbb{R})}\end{aligned}$$

for each f, $g \in \mathrm{dom}(T)$. Hence $\mathrm{dom}(T)$ is even isometrically embedded in $H^1(\mathbb{R})$ and thus the Hilbert space H_1 can be identified with a closed subspace of $H^1(\mathbb{R})$. Now given some $f \in \mathrm{dom}(T)$ and $g \in \mathrm{mul}(S)$ we have

$$\langle f, g\rangle_{H^1(\mathbb{R})} = \langle Tf, g\rangle_\omega = 0,$$

since g vanishes almost everywhere with respect to $|\omega|$ (see [10, Corollary 3.3]). As a consequence $\mathrm{dom}(T)$ and hence also the space H_1 are contained in the closure of $\mathrm{dom}(S)$. On the other side, given some $f \in \mathrm{dom}(S)$ there is some function $g \in H^1(\mathbb{R})$ such that

$$f(x) = \int_{\mathbb{R}} \mathrm{e}^{-\frac{|x-s|}{2}} g(s) d\omega(s), \quad x \in \mathbb{R}.$$

But since g also lies in $L^2(\mathbb{R};\omega)$ we infer from Proposition 3.3 that f also lies in $\mathrm{dom}(T)$. Thus we see that H_1 actually is the closure of the domain of S. Finally, since the inverses of S_1 and the operator part of S are given as in Proposition 3.3 and Proposition A.1 we infer that they are equal. □

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Abstract

The Camassa–Holm equation is an integrable, non-linear partial differential equation which models waves on shallow water. Associated with this equation is a weighted Sturm–Liouville problem on the real line; the isospectral problem. In order to incorporate the main interesting phenomena (wave breaking and peakon solutions) of the dispersionless Camassa–Holm equation, one has to allow this weight to be an arbitrary finite signed Borel measure. The main difficulties arising hereby are that both endpoints are quite singular and that the weight is just a finite measure, allowed to vanish on arbitrary sets and moreover, to change sign. In the present thesis we are concerned with direct and inverse spectral theory of Sturm–Liouville operators of this kind. First, we introduce self-adjoint operators associated with general Sturm–Liouville problems with measure coefficients and develop singular Weyl–Titchmarsh theory for such operators. This is done in both, the right-definite and in the left-definite setting. Regarding inverse spectral theory, first of all we prove some uniqueness results for Schrödinger operators with potentials, which are singular at both endpoints. In particular, we show when the spectral measure uniquely determines the potential and prove a Hochstadt–Lieberman type uniqueness result. Subsequently, we prove a quite general inverse uniqueness theorem for Sturm–Liouville operators with measure coefficients in the left-definite setting. Finally, these results are applied to the isospectral problem of the Camassa–Holm equation. In particular, this shows that the associated spectral measure uniquely determines the finite signed measure appearing in this weighted Sturm–Liouville problem.

Zusammenfassung

Die Camassa–Holm Gleichung ist eine integrable, nichtlineare partielle Differentialgleichung, die Wellen auf seichtem Wasser modelliert. Das dazugehörige Spektralproblem ist ein gewichtetes Sturm–Liouville Problem auf der reellen Achse. Um die interessantesten Phänomene (brechende Wellen und Peakon Lösungen) der dispersionslosen Camassa–Holm Gleichung abzudecken, ist es notwendig, dass das dabei auftretende Gewicht ein endliches signiertes Maß sein kann. Die wesentlichen Schwierigkeiten die sich daraus ergeben sind, dass beide Endpunkte ziemlich singulär sind und dass das Gewicht ein endliches Maß ist, welches auf beliebigen Teilmengen verschwinden und sogar das Vorzeichen ändern kann. In der vorliegenden Arbeit beschäftigen wir uns mit direkter und inverser Spektraltheorie von Sturm–Liouville Differentialoperatoren dieser Art. Als erstes behandeln wir allgemeine selbstadjungierte Sturm–Liouville Operatoren deren Koeffizienten Maße sind und entwickeln singuläre Weyl–Titchmarsh Theorie für solche Operatoren. All das wird sowohl im rechts-definiten, als auch im links-definiten Fall gemacht. Betreffend inverser Spektraltheorie zeigen wir zunächst einige Eindeutigkeitssätze für Schrödingeroperatoren mit Potentialen, die an beiden Endpunkten singulär sind. Insbesondere zeigen wir, wann das zugehörige Spektralmaß das Potential eindeutig bestimmt und beweisen ein verallgemeinertes Hochstadt–Lieberman Eindeutigkeitsresultat. Darauf folgend beweisen wir einen ziemlich allgemeinen inversen Eindeutigkeitssatz für Sturm–Liouville Operatoren mit Maß Koeffizienten im links-definiten Fall. Schließlich wenden wir diese Resultate auf das Spektralproblem der Camassa–Holm Gleichung an. Insbesondere zeigen wir, dass das in diesem gewichteten Sturm–Liouville Problem auftretenden Maß eindeutig durch das Spektralmaß bestimmt ist.

Curriculum vitae

Personal information

Name	Jonathan Eckhardt
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Education

Since 02/2010	**University of Vienna** Doctoral studies in mathematics
10/2005 – 01/2010	**Vienna University of Technology** Diploma studies in technical mathematics Graduation with distinction Diploma thesis: Direct and inverse spectral theory of Sturm–Liouville differential operators
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Academic employment

Since 02/2010	**Research assistant at the University of Vienna** Fully supported by the FWF Start project Y330

Awards

January 2011	**Diplomarbeitspreis der Stadt Wien** Award of the city of Vienna for excellent diploma theses at the Vienna University of Technology

Publications and preprints

- *On the isospectral problem of the Camassa–Holm equation*, together with G. Teschl, not yet submitted.
- *Direct and inverse spectral theory of singular left-definite Sturm–Liouville operators*, submitted.

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- *Sturm–Liouville operators on time scales*, together with G. Teschl, to appear in *J. Difference Equ. Appl.*
- *Schrödinger operators with strongly singular potentials and de Branges spaces*, submitted.
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- *On the connection between the Hilger and Radon–Nikodym derivatives*, together with G. Teschl, *J. Math. Anal. Appl.* **385** (2012), 1184–1189.

Talks

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- *Measure Sturm–Liouville operators*, at the joint mathematical conference of the *Austrian Mathematical Society* in September 2011, Krems an der Donau, Austria.
- *Inverse spectral theory for Schrödinger operators with strongly singular potentials*, at the annual conference of the *German Mathematical Society* in September 2011, Cologne, Germany.
- *The isospectral problem of the Camassa–Holm equation*, at the ESF conference on *Completely Integrable Systems and Applications* in July 2011, Vienna, Austria.