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Abstract

Based upon M. Oberguggenberger's "Multiplication of distributions and applications to partial differential equations", the focus of this thesis is developing and presenting a hierarchy, describing the relations between several products of distributions.
The methods discussed in this context range from distribution theory, locally convex vector spaces and functional analysis to even microlocal analysis, with the aim of equipping the reader with the proper tools to delve deeper into the topic of nonlinear distribution theory or enabling the reader to apply them to applications in partial differential equations.
After a short introduction in the first chapter, the investigation of the topic starts in the second chapter with the problems of multiplication of distributions and several possible approaches to deal with them.
The third chapter presents the duality method and applies it to general Sobolev spaces, which are included in a separate section of this chapter.
Chapter four covers the Fourier method by first developing a convolution for tempered distributions, then defining the Fourier product and connecting it with a microlocal perspective.
The third method, described in the fifth chapter, is gained through regularization. Having several possible ways to define strict and model products, this method generates a very general concept for a multiplication of distributions.
The final chapter is dedicated to proving the relations within the hierarchy for products of distributions in the form of compatibility theorems.
In order to comprehend the content, the reader is advised to have basic knowledge of linear distribution theory. By trying to give comprehensible and full proofs in addition to an extensive bibliography, no further requirements are necessary.

Zusammenfassung

Basierend auf dem Werk "Multiplication of distributions and applications to partial differential equations" von M. Oberguggenberger, liegt der Fokus dieser Diplomarbeit auf der Entwicklung einer Hierarchie, die Zusammenhänge von verschiedenen Produkten von Distributionen beschreibt.
Die im Zuge dessen behandelten Methoden reichen von der Distributionentheorie, über lokal konvexe Vektorräume und Funktionalanalysis bis hin zu mikrolokaler Analysis. Dabei wird der Versuch unternommen, den Leser und die Leserin mit dem mathematischen Handwerkszeug auszustatten, um selbst tiefer in nicht lineare Distributionentheorie eintauchen, oder es im Bereich der partiellen Differentialgleichungen anwenden zu können.
Nach einer kurzen Einleitung im ersten Kapitel, werden im zweiten Kapitel die Probleme, die mit der Multiplikation von Distributionen einhergehen, und mehrere Lösungsansätze erörtert.
Im dritten Kapitel wird die Dualitätsmethode und deren Anwendung auf allgemeine Sobolev Räume präsentiert, welche in einem separaten Abschnitt dieses Kapitels behandelt werden.
Kapitel vier deckt die Fouriermethode ab. Zunächst wird dabei die Faltung von temperierten Distributionen entwickelt, dann das Fourierprodukt definiert und schlussendlich mit einer mikrolokalen Sichtweise verknüpft.
Die dritte Methode wird im fünften Kapitel beschrieben. Dabei wird eine Distribution auf mehreren Arten geglättet, um dann mit Hilfe eines Grenzwertprozesses, verschiedene Produkte von Distributionen zu definieren. Dies führt zu einem sehr allgemeinen Konzept für die Multiplikation von Distributionen.
Das letzte Kapitel widmet sich der Hierarchie. In Form von Kompatibilitätstheoremen werden hier die Zusammenhänge zwischen den verschiedenen Produkten bewiesen.
Um den Inhalt dieser Diplomarbeit leicht verstehen zu können, werden Kenntnisse aus linearer Distributionentheorie empfohlen. Weitere Anforderungen sind, dank einer ausführlichen Bibliographie und dem Versuch verständliche und komplette Beweise zu präsentieren, nicht notwendig.

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Contents

1 Introduction

"...is impossible", could be a common way to complete the title of this thesis. However, this way of putting it, simplifies the message, conveyed in the famous impossibility result of L. Schwartz, to a fairly large degree. Though in fact, one may beg to differ, by either accepting inconsistency with classical operations or restricting oneself in terms of generality. Mainly following the first two chapters of M. Oberguggenberger's "Multiplication of distributions and applications to partial differential equations" [22], we will choose the latter and investigate several approaches to define products of distributions. The various products we will discuss in this thesis, are considered "irregular intrinsic operations". The necessity of such products arises in certain applications. However, in general no rigorous theory treats all of them together. Nevertheless, M. Oberguggenberger presents a hierarchy in which some of them can be related to each other. Our overall goal in this thesis is to develop the necessary details of these products and then bring them together, such that we are able to prove the relations claimed in that hierarchy.
To this end, we will start by setting the scene for all our efforts in chapter 2, by outlining the problems that come with the multiplication of distributions and elaborating on possible solutions. Moreover, we will establish in detail, which properties are to be and not to be expected according to our course of action.
In chapter 3, the first of three methods to define a product of distributions is introduced. Following J. Horvath [15], we begin with collecting requirements to define the duality product in the first section. This involves spaces of distributions and locally convex topologies. The second section focuses on the definition of the duality product and its general properties. The final section of the first chapter is devoted to Sobolev spaces based on all L^p-spaces. They will serve as prominent examples for the duality method. As the discussion of these more general Sobolev spaces is often left out in standard courses of functional analysis, this section will treat them rather extensively following Robert A. Adams and John J.F. Fournier [1].
The Fourier product will be the main topic of chapter 4. This product relies on a concept based on the convolution theorem, as such the first section is dedicated to the convolution of tempered distributions. Based upon this extended convolution, we define the Fourier

product in the second section. Moreover, we will discuss a special microlocal version of it due to L. Hörmander [14].
The last and most general product is going to be discussed in chapter 5. By using the regularization method, which depends on classes of mollifiers and methods of standard distribution theory, we will define four products at once and arrange them into two types: strict products and model products.
The final chapter contains the hierarchy, which brings all of the above products together, and moreover the compatibility results leading up to it.
As already mentioned above, the main source of ideas and results throughout all chapters is [22]. However, additional references to other specific sources are made within the text.

2 Prerequisites, Problems and Perspectives

Our aim in this chapter is to shortly outline the topic of distributional products and discuss a few of the problems which naturally come with it.
To begin with we fix some basic notations, which will be used throughout our discussion and we recall some basic definitions. We will assume familiarity with distribution theory, our main reference being [8]. In particular, our notation for the concepts of linear distribution theory are more or less standard. Nevertheless, we summarize some of it in the following list without further comments.

Notation 2.1.

- $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$.
- When dealing with nets the index ε will always vary in $I := (0, 1]$.
- For arbitrary open subsets of $\mathbb{R}^n$ we will reserve the letter Ω.
- In case of derivatives, we will write both ∂^α or D^α with $\alpha \in \mathbb{N}_0^n$ being a multi-index. Also, partial derivatives will be denoted by either ∂_{x_i}, ∂_i, D_{x_i} or D_i, where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $1 \leq i \leq n$.
- The n-dimensional sphere will be denoted by S^n.
- For any $\varepsilon > 0$ we define the (open) ε-ball $B_\varepsilon = B_\varepsilon(0) := \{x \in \mathbb{R}^n \mid |x| < \varepsilon\}$.
- If we are working with a locally convex vector space (LCVS) X, we denote the respective locally convex Hausdorff topology by τ_X.
- We give a list of spaces we will work with:
  - $\mathcal{C}^k$ with $1 \leq k \leq \infty$ ... the space of k-times continuously differentiable functions.
  - $\mathcal{D} = \mathcal{C}_c^\infty$ ... the space of test functions, i.e., the space of compactly supported smooth functions.
  - L^p with $1 \leq p \leq \infty$ ... the space of (equivalence classes of) Lebesgue measurable functions for which the norm $\|.\|_p$ is bounded.

  - $\mathcal{O}_{\mathrm{M}}$ ... the space of moderate functions, i.e., the space of smooth functions such that all derivatives are of at most polynomial growth.
  - $\mathcal{S}$ ... the space of rapidly decreasing functions.
  - $\mathcal{D}'$ ... the space of distributions equipped with the strong topology $\beta(\mathcal{D}', \mathcal{D})$.
  - $\mathcal{E}'$ ... the space of distributions with compact support.
  - $\mathcal{S}'$ ... the space of temperate distributions.

- For the action of a distribution $u \in \mathcal{D}'(\Omega)$ on a test function $\varphi \in \mathcal{D}(\Omega)$ we will of course write $\langle u, \varphi \rangle$. Moreover, we will write

$$\langle u, v \rangle = \int_{\Omega} u(x)v(x)dx$$

  for any functions u, v, for which the right hand side makes sense (e.g., for u a regular distribution and v a test function).

- We define the Fourier transform of a rapidly decreasing function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ by

$$\hat{\varphi}(\xi) = \mathcal{F}(\varphi)(\xi) := \int_{\mathbb{R}^n} \varphi(x) e^{-2\pi i x \xi} dx.$$

  Furthermore, its inverse is given by

$$\mathcal{F}^{-1}(\psi)(x) = \int_{\mathbb{R}^n} \psi(\xi) e^{2\pi i x \xi} d\xi, \quad \psi \in \mathcal{S}(\mathbb{R}^n).$$

  Finally, using transposition the Fourier transform is extended to temperate distributions, i.e., we have $\mathcal{F} : \mathcal{S}' \to \mathcal{S}'$ and

$$\langle \mathcal{F}(u), \varphi \rangle = \langle \hat{u}, \varphi \rangle = \langle u, \hat{\varphi} \rangle, \quad \text{for } u \in \mathcal{S}', \varphi \in \mathcal{S}.$$

Next we define mollifiers. These test functions are an important tool in the theory of distributions. They are used to regularize distributions and are also going to play an important key role in chapter 5.

Definition 2.2. *(Mollifier)*
A test function $\rho \in \mathcal{D}(\mathbb{R}^n)$ is called *mollifier* if

- $\int_{\mathbb{R}^n} \rho(x)dx = 1$ and
- $\mathrm{supp}(\rho) \subseteq \overline{B_1}$.

We will frequently work with the following two classes of mollifiers:

(a) Let $(\rho_\varepsilon)_\varepsilon$ be a net of test functions in $\mathcal{D}(\mathbb{R}^n)$. We say $(\rho_\varepsilon)_\varepsilon$ is a *strict delta net* if

 (i) $\operatorname{supp}(\rho_\varepsilon) \to \{0\}$ for $\varepsilon \to 0$,

 (ii) $\int_{\mathbb{R}^n} \rho_\varepsilon(x)dx = 1$ for all $\varepsilon > 0$ and

 (iii) $\int_{\mathbb{R}^n} |\rho_\varepsilon(x)|dx$ is bounded independently of ε.

(b) Suppose $\rho \in \mathcal{D}(\mathbb{R}^n)$ is a mollifier. Then we consider for $\varepsilon > 0$ the scaled test functions $(\rho_\varepsilon)_\varepsilon \in \mathcal{D}(\mathbb{R}^n)$, given by $\rho_\varepsilon(x) := \varepsilon^{-n}\rho\left(\frac{x}{\varepsilon}\right)$, thereby obtaining

 (i) $\int_{\mathbb{R}^n} \rho_\varepsilon(x)dx = 1$ and

 (ii) $\operatorname{supp}(\rho_\varepsilon) \subseteq \overline{B_\varepsilon}$.

 We will refer to these kind of mollifiers as *model delta nets*.

In addition, for $u \in \mathcal{D}'(\mathbb{R}^n)$ and $(\rho_\varepsilon)_\varepsilon \in \mathcal{D}(\mathbb{R}^n)$ a net of mollifiers we say $u_\varepsilon := u * \rho_\varepsilon \in \mathcal{C}^\infty$ is the *mollification* or *regularization of* u.
For more details concerning regularization, we refer to [8, section 1.2 and section 5.2].

The starting point of our discussion of distributional products is an extension of the multiplication of functions, which is gained through transposition.

Definition 2.3. *(A first product - Multiplication with $\mathcal{C}^\infty$-functions)*
For $u \in \mathcal{D}'(\Omega)$ and $f \in \mathcal{C}^\infty(\Omega)$ we define the product $u \cdot f \in \mathcal{D}'(\Omega)$ commutatively as

$$\langle uf, \varphi\rangle = \langle fu, \varphi\rangle := \langle u, f\varphi\rangle \quad \forall \varphi \in \mathcal{D}(\Omega). \tag{2.1}$$

The domain of this product is

$$M(\Omega) := (\mathcal{D}'(\Omega) \times \mathcal{C}^\infty(\Omega)) \cup (\mathcal{C}^\infty(\Omega) \times \mathcal{D}'(\Omega)).$$

Now, one of our goals is to gain more generality so we do not have to limit ourselves just to the domain $M(\Omega)$. To see which difficulties arise, let us assume we already have a multiplication on $\mathcal{D}'(\Omega) \times \mathcal{D}'(\Omega)$. Then it is a reasonable wish to include the above given product (2.1) in the more general concept. However, already this seemingly harmless requirement prohibits the new product from being an associative operation, because already the product (2.1) has this flaw as seen in the following example. In other words, any extension of the product (2.1) would inherit the same flaw and hence could not be associative.

Example 2.4. Let us consider the function $x \in \mathcal{C}^\infty(\mathbb{R}^n)$, the delta distribution $\delta(x) \in \mathcal{D}'(\mathbb{R}^n)$ and the Cauchy principal value of $\frac{1}{x}$ which we denote by $\mathrm{vp}(\frac{1}{x}) \in \mathcal{D}'(\mathbb{R}^n)$. Then by (2.1) the following formulae hold

$$\begin{aligned} \delta(x) \cdot 1 &= \delta(x), \\ \delta(x) \cdot x &= 0, \\ x \cdot \mathrm{vp}\left(\frac{1}{x}\right) &= 1. \end{aligned}$$

With these we can conclude

$$0 = (\delta(x) \cdot x) \cdot \mathrm{vp}\left(\frac{1}{x}\right) \neq \delta(x) \cdot (x \cdot \mathrm{vp}\left(\frac{1}{x}\right)) = \delta(x) \cdot 1 = \delta(x).$$

Hence, the product (2.1) is not associative.

Unfortunately the non-associativity of a multiplication map $\mathcal{D}'(\Omega) \times \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ is not the only problem we have to deal with. As it turns out, the idea of having a multiplication map defined on the whole distribution space seems to be even more questionable, if we attempt to square the delta distribution. The details of this matter follow in our next example.

Example 2.5. In this example we try to define δ^2 as an element of $\mathcal{D}'(\mathbb{R}^n)$ by means of a regularzation process. Therefore, let $(\rho_\varepsilon)_\varepsilon \in \mathcal{D}(\mathbb{R}^n)$ be a strict delta net and additionally suppose ρ_ε is real-valued. Furthermore, we take a test function $\varphi \in \mathcal{D}(\mathbb{R}^n)$ such that $\varphi \equiv 1$ in a neighbourhood of 0. Then we have

$$\int \rho_\varepsilon^2(x)\varphi(x)dx = \int \rho_\varepsilon^2(x)dx.$$

Now, if ρ_ε^2 converged weakly in $\mathcal{D}'(\mathbb{R}^n)$ then $(\rho_\varepsilon)_\varepsilon$ would be bounded in $L^2(\mathbb{R}^n)$ and thus have a weak convergent subsequence in $L^2(\mathbb{R}^n)$. Hence, δ would need to be an element in $L^2(\mathbb{R}^n)$, which of course is not the case.
Nevertheless, the same does not hold for complex-valued regularizations of delta. One can show that such regularizations can be arranged such that their squares do converge in $\mathcal{D}'$ to a variety of distributions (cf. [22, Example 10.6, p. 97]). This led to many suggestions for a definition of δ^2, such as 0, $c \cdot \delta$, $c \cdot \delta + \frac{1}{2\pi i}\delta'$, $c \cdot \delta + c' \cdot \delta'$, with c and c' being arbitrary constants. The latter statement of course making it impossible to consistently incorporate such suggestions into a reasonable and satisfying concept of a multiplication map on $\mathcal{D}' \times \mathcal{D}' \to \mathcal{D}'$.

Finally, also observe that this example proofs that a multiplication map defined on $\mathcal{D}'$ cannot be jointly continuous.

However, the list of problems still continues, even if we let go of the idea of having all of $\mathcal{D}'(\Omega) \times \mathcal{D}'(\Omega)$ as underlying domain for a multiplication map. The next example shows, that problems also occur by applying a simple regularization process to a product of "only" locally integrable functions.

Example 2.6. Let us consider the two locally integrable functions

$$\begin{aligned} x_+ &:= xH(x) \\ x_- &:= -xH(-x). \end{aligned}$$

We look at the pointwise a.e. product of $x_+^{-\frac{1}{2}}$ and $x_-^{-\frac{1}{2}}$ which vanishes a.e.. Now, let $\varphi \in \mathcal{D}(\mathbb{R})$ be a mollifier, $\varphi_\varepsilon(x) := \frac{1}{\varepsilon}\varphi(\frac{x}{\varepsilon})$ the corresponding model delta net, and regularize one factor. Then, for any test function $\psi \in \mathcal{D}(\mathbb{R})$ we have

$$\begin{aligned} \langle (\varphi_\varepsilon * x_-^{-\frac{1}{2}}) \cdot x_+^{-\frac{1}{2}}, \psi \rangle &= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\varepsilon}\varphi\left(\frac{y}{\varepsilon}\right)(x-y)_-^{-\frac{1}{2}} x_+^{-\frac{1}{2}} \psi(x) dy dx \\ &= \int_0^\infty \int_x^\infty \varphi(y)(y-x)^{-\frac{1}{2}} x^{-\frac{1}{2}} \psi(\varepsilon x) dy dx \\ &\overset{(\varepsilon \to 0)}{\to} \psi(0) \int_0^\infty \int_0^y (y-x)^{-\frac{1}{2}} x^{-\frac{1}{2}} dx \varphi(y) dy \\ &= \psi(0)\pi \underbrace{\int_0^\infty \varphi(y) dy}_{=:c(\varphi)}. \end{aligned}$$

Hence, we have $(\varphi_\varepsilon * x_-^{-\frac{1}{2}})x_+^{-\frac{1}{2}} \to \pi c(\varphi)\delta$ $(\varepsilon \to 0)$ in $\mathcal{D}'(\mathbb{R})$, where apparently the limit depends on the mollifier chosen. However, if we regularize both factors, we obtain by a similar calculation $(\varphi_\varepsilon * x_-^{-\frac{1}{2}})(\varphi_\varepsilon * x_+^{-\frac{1}{2}}) \to \frac{\pi}{2}\delta$ for $\varepsilon \to 0$. So not only do the limits differ, but also the dependance of the mollifier disappears. We conclude that it is impossible to define the product of two locally integrable functions such that it is stable w.r.t. regularizations. Moreover, we are confronted with the same problem when applying simple perturbations instead of regularizations, to seemingly unproblematic products, locally of the form $\mathcal{C}^\infty \cdot L^1$. Indeed, for $\varepsilon \to 0$ we have

$$(x+\varepsilon)_-^{-\frac{1}{2}}(x-\varepsilon)_+^{-\frac{1}{2}} \equiv 0,$$

while

$$(x-\varepsilon)_-^{-\frac{1}{2}}(x+\varepsilon)_+^{-\frac{1}{2}} \to \pi\delta.$$

Remark 2.7. *(On Example 2.6)*
As an alternative to the product (2.1), we may define a pointwise product of two functions in $L^1(\mathbb{R})$ a.e., which ends up being an element of $L^{\frac{1}{2}}(\mathbb{R})$. However, this is rather unsatisfying within the context of distribution theory, as this space is not a space of distributions (cf. Definition 3.1 below). Indeed, there exists no continuous and injective linear map $j : L^{\frac{1}{2}}(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$, due to the fact that the existence of such an imbedding would imply by transposition $\mathcal{D}(\mathbb{R}) \subseteq (L^{\frac{1}{2}}(\mathbb{R}))' = \{0\}$, which is obviously wrong.

Having seen some of the problems that arise when one tries to (naively) define products of distributions, we try to bring some order into the matter. Following M. Oberguggenberger [22, section 3] we suggest the following three approaches:

1. Regular intrinsic operations.
2. Irregular intrinsic operations.
3. Extrinsic products and algebras containing the distributions.

As already pointed out in the introduction we will discuss the approach of irregular intrinsic operations. Nevertheless, we elaborate briefly on regular intrinsic operations, extrinsic products and algebras containing the distributions.
Working with "only" regular intrinsic operations, basically means to be satisfied with regular objects and classically defined operations. Hence, the generality of this approach is greatly compromised. For example, one has to dismiss the delta distribution. In return, one gains differential-algebraic properties and continuity of operations. For instance, the algebra of "retarded distributions" or the algebra of distributional boundary values of holomorphic functions on $\mathbb{C} \setminus \mathbb{R}$ with support in the upper half plane or algebras of functions like $L^\infty_{\text{loc}}(\mathbb{R}^n)$, $\mathcal{C}^k(\mathbb{R}^n)$ and the Sobolev spaces $H^m(\mathbb{R}^n)$ with $m > \frac{n}{2}$ (cf. Remark 3.12(v)) are representative examples of this approach. Also note, that regular intrinsic operations can be viewed as special cases of irregular intrinsic operations and therefore are included in our discussion later on.
The idea of the third suggested approach is, to enlarge the set of objects and to go beyond distributions. This leads to algebras of generalized functions. There are many advantages taking this route. More precisely, one gains associative and commutative differential algebras in which the space of distributions is imbedded linearly. Moreover, initial and boundary value problems for nonlinear partial differential equations can be formulated unrestrictedly in such algebras. However, some sacrifices have to be made. For instance, associativity of the algebra implies immediately that some classical formulae cannot hold in this context, as can be seen by Example 2.4. Also, considering commutativity, either

$H^2 = H$ or $H' = \delta$ cannot hold simultaneously (cf. [22, Example 2.6., p. 29]). In other words, compatibility with classically defined operations, such as the multiplication (2.1) or the pointwise product on $L^\infty \times L^\infty$ (cf. [12, Example 1.1.1(iii), p. 3]), is lost along this way. In addition, one has to keep the notorious impossibility result of L. Schwartz in mind, which is stated in the following remark.

Remark 2.8. *(The impossibility result of L. Schwartz (1954))*
There exists no associative and commutative algebra $(\mathcal{A}(\mathbb{R}), +, \circ)$ with unit $1 \in \mathcal{A}(\mathbb{R})$ satisfying

(i) $\mathcal{D}'(\mathbb{R})$ is linearly imbedded into $\mathcal{A}(\mathbb{R})$ such that the constant function $1 \in \mathcal{D}'(\mathbb{R})$ is mapped to $1 \in \mathcal{A}(\mathbb{R})$,

(ii) there exists a derivation $\partial : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$, i.e., a linear map satisfying the Leibniz rule,

(iii) $\partial|_{\mathcal{D}'(\mathbb{R})}$ is the usual partial derivative on distributions and

(iv) $\circ|_{\mathcal{C}(\mathbb{R})\times\mathcal{C}(\mathbb{R})}$ coincides with the pointwise product of continuous functions.

For an elementary (and easy) proof of this result we refer to [22, Example 2.5, p. 27]. A modification of the proof also shows that the result still holds, if (iv) is replaced by

(iv)' $\circ|_{\mathcal{C}^k(\mathbb{R})\times\mathcal{C}^k(\mathbb{R})}$ coincides with the pointwise product of k-times continuously differentiable functions ($k \in \mathbb{N}$).

However, J. F. Colombeau was able to show that an algebra satisfying conditions (i)-(iii) and

(iv)" $\circ|_{\mathcal{C}^\infty(\mathbb{R})\times\mathcal{C}^\infty(\mathbb{R})}$ coincides with the pointwise product of smooth functions,

does exist with an explicit construction.
Nevertheless, the impossibility result of L. Schwartz makes it clear, that by imbedding the distributions into an associative and commutative algebra, either the product or the derivatives of continuously differentiable functions cannot be preserved. More precisely, the proof shows, the impossibility of multiplying continuous functions, differentiating differentiable functions in the usual way and allowing for the existence of a delta distribution at the same time.

In case of irregular intrinsic operations, one assigns a product to specific pairs of distributions in a "reasonable" way and ends up with a distribution again. This grants a lot of freedom, thus a lot of methods exist. Moreover, a considerable amount of generality is

gained. Obviously, full generality cannot be achieved, as displayed in Example 2.5. Hence, not all pairs of distributions allow the assignment of a reasonable product. Like each of the above mentioned approaches to a multiplication of distributions, also the path of irregular intrinsic operations has its shortcomings. On the one hand, all products we will discuss in the following chapters, lack associativity, as they all include the non associative product (2.1) as a special case. Also, continuity often fails (cf. Example 2.6) and differentiation frequently leads out of the domain of such products. On the other hand, one benefit is that coherence with classical results can be maintained. Furthermore, within a particular context, such products work very well. However, the variety of methods often lacks a common structure, which is why they are usually not transferable to other applications than the ones they were "designed" for.
Since we aim for proving compatibility results between several such products, we actually are going to present a common structure in form of the announced hierarchy in chapter 6. Along the way, we will discuss products defined by the duality method, the Fourier method and the regularization method.
The basis for all our efforts is still the simple product (2.1). Now that we know, what cannot be expected (full generality, associativity and continuity), it is time to investigate what we can expect. In the following list, we discuss some properties, which are shared by (almost) all of the upcoming distributional products.

Remark 2.9. *(Properties for distributional products)*

(i) **Bilinearity** : Trivially, we want a product to be bilinear whenever the addition and scalar multiplication is defined on its domain $M(\Omega)$. In other words, if (u_1, v), $(u_2, v) \in M(\Omega)$ and c_1, $c_2 \in \mathbb{C}$ then

$$(c_1 u_1 + c_2 u_2, v) \in M(\Omega)$$

and

$$(c_1 u_1 + c_2 u_2) \cdot v = c_1 u_1 \cdot v + c_2 u_2 \cdot v.$$

In particular, if we have $u \in \mathcal{D}'(\Omega)$ with $(u, v) \in M(\Omega)$ for some v, then $u \cdot 0 = 0$.

(ii) **Commutativity** : In short:

$$\exists u \cdot v \Leftrightarrow \exists v \cdot u \quad \text{and} \quad u \cdot v = v \cdot u.$$

(iii) **Partial associativity** : Although associativity fails for all products extending the product (2.1), we will see that the weaker but very useful property of partial associativity holds for almost all of them. More precisely, if $u \cdot v$ exists and $f \in \mathcal{C}^\infty(\Omega)$, then $(fu) \cdot v$ and $u \cdot (fv)$ also exist and we have

$$(fu) \cdot v = u \cdot (fv) = f(u \cdot v),$$

where the multiplication with f is understood in the sense of (2.1).

(iv) **Locality properties** : Partial associativity of a product implies the following two properties:

(a) If $u \cdot v$ exists, then we have

$$\mathrm{supp}(u \cdot v) \subseteq \mathrm{supp}(u) \ \cap \ \mathrm{supp}(v).$$

(b) Let ω be an open subset of Ω, assume $u_1 \cdot v$ and $u_2 \cdot v$ exist and $u_1|_\omega = u_2|_\omega$. Then, we also have $(u_1 \cdot v)|_\omega = (u_2 \cdot v)|_\omega$.

Proof. To see (a), let $\varphi \in \mathcal{D}(\Omega)$ with $\mathrm{supp}(\varphi) \subseteq \mathrm{supp}(u)^{\mathrm{C}}$. In addition, we can have a partition of unity χ_1, $\chi_2 \in \mathcal{D}(\Omega)$ with $\mathrm{supp}(\chi_1) \subseteq \mathrm{supp}(u)^{\mathrm{C}}$, $\mathrm{supp}(\chi_2) \subseteq \mathrm{supp}(\varphi)^{\mathrm{C}}$ and $\chi_1 + \chi_2 \equiv 1$. Now, linearity and partial associativity gives

$$\begin{aligned}\langle u \cdot v, \varphi \rangle &= \langle (\chi_1 + \chi_2) \cdot u \cdot v, \varphi \rangle \\ &= \langle (\chi_1 u) \cdot v, \varphi \rangle + \langle u \cdot v, \chi_2 \varphi \rangle \\ &= \langle 0 \cdot v, \varphi \rangle + \langle u \cdot v, 0 \rangle = 0.\end{aligned}$$

Next, (b) holds true, by first observing that $\omega \subseteq \mathrm{supp}(u_1 - u_2)^{\mathrm{C}}$. Indeed, by (a) this immeadiately implies, that $\omega \nsubseteq \mathrm{supp}(u_1 - u_2) \cap \mathrm{supp}(v)$, thus $((u_1 - u_2) \cdot v)|_\omega = 0$. □

(v) **Localization** : In particular, partial associativity grants access to an useful method called localization, which is applicable to all products we are going to discuss. It allows us to globally define the product of two distributions out of localized products of distributions. More precisely, let $f_x \in \mathcal{D}(\Omega)$ be a cut-off with $f_x \equiv 1$ on Ω_x, where Ω_x is a neighbourhood of any point $x \in \Omega$. In addition, assume $(f_x u) \cdot (f_x v)$ exists. Then,

$$\langle w_x, \varphi \rangle := \langle (f_x u) \cdot (f_x v), \varphi \rangle \quad \forall \varphi \in \mathcal{D}(\Omega_x),$$

defines a distribution in $\mathcal{D}'(\Omega_x)$, called the product of u and v near x. This is well defined, i.e., independent of the choice of f_x, and moreover fulfils the compatibility condition

$$\langle w_x, \varphi\rangle = \langle w_y, \varphi\rangle,$$

for $x, y \in \Omega$ and $\varphi \in \mathcal{D}(\Omega_x \cap \Omega_y)$, which can be seen by the following simple calculation. By partial associativity we have on $\Omega_x \cap \Omega_y$

$$\begin{aligned} w_x = (f_x u) \cdot (f_x v) &= f_y^2 (f_x u) \cdot (f_x v) \\ &= (f_x f_y u) \cdot (f_x f_y v) = f_x^2 (f_y u) \cdot (f_y v) \\ &= (f_y u) \cdot (f_y v) = w_y. \end{aligned}$$

Hence, there exists a unique distribution $w \in \mathcal{D}'(\Omega)$, referred to as product of u and v, such that $w|_{\Omega_x} = w_x$ for each $x \in \Omega$, due to a distribution being uniquely determined by its localizations (cf. [8, Thm. 1.4.3, p. 12]). Taking a partition of unity $(\chi_j)_{j\in\mathbb{N}}$ subordinate to a countable cover $(\Omega_{x_j})_{j\in\mathbb{N}}$ of Ω, we can define the product of u and v by

$$u \cdot v := w = \sum_{j\in\mathbb{N}} \chi_j w_{x_j}.$$

Finally note, that the global product w inherits the partial associativity of the localized products w_{x_j}.

For our investigation of distributional products, the procedure in Remark 2.9(v) means, that properties, which are of interest or required for some product, only need to hold locally, not globally. A first application is given in the following Proposition, considering the notion of singular supports. Also, it is our first, however simple, generalization of the product (2.1).

Remark 2.10. *(Distributions with disjoint singular support)*
Let u, $v \in \mathcal{D}'(\Omega)$ such that their singular supports are disjoint, i.e.,

$$\operatorname{singsupp}(u) \ \cap \ \operatorname{singsupp}(v) = \emptyset$$

and $f_x \in \mathcal{D}(\Omega)$ with $f_x \equiv 1$ on Ω_x, where Ω_x is a neighbourhood of a given point $x \in \Omega$. Moreover, let either $f_x u$ or $f_x v$ belong to $\mathcal{C}^\infty(\Omega)$. Then the product $(f_x u) \cdot (f_x v)$ is defined by (2.1), and Remark 2.9(v) yields the product $u \cdot v$ on Ω.
In particular, a partition of unity consisting only of two elements suffices. Indeed, let χ_1, $\chi_2 \in \mathcal{D}(\Omega)$ be a partition of unity with $\operatorname{supp}(\chi_1) \subset \operatorname{singsupp}(u)^{\mathrm{C}}$ and $\operatorname{supp}(\chi_2) \subset$

$\mathrm{singsupp}(v)^{\mathrm{C}}$. Then, by (2.1) we can set

$$u \cdot v = (\chi_1 u) \cdot v + u \cdot (\chi_2 v).$$

Another consequence of Remark 2.9(v) is, that by working with products of the form $w_x = (f_x u) \cdot (f_x v)$, it suffices to investigate multiplication of distributions on $\Omega = \mathbb{R}^n$ only. Thus, the discussion in the upcoming chapters will be restricted to the domain $\mathbb{R}^n$ for the most part. Therefore, we end this chapter by adapting our notation.

Notation 2.11. Whenever we work with a space defined on $\mathbb{R}^n$, we will most of the times drop the domain in our notation, e.g., we will write L^p for $L^p(\mathbb{R}^n)$.

3 The Duality Method

3.1 Spaces of Distributions

The first product of distributions we are going to discuss is the duality product. The main idea behind the duality method is, to create a setting, such that the multiplication of two distributions can be translated into the dual action between them. This leads us to study several spaces of distributions in the first section. We start by giving the definition of these spaces and then investigate the requirements to preserve properties w.r.t. duality. As it turns out, the topologies involved play an important role in this matter. At the end of this section we will also take a glimpse on localized spaces.

Definition 3.1. *(Spaces and normal spaces of distributions)*

(i) A pair (X, j) consisting of a LCVS X and a continuous and injective linear map $j : X \to \mathcal{D}'$ will be called an *injective pair*. The image $j(X)$ will be called a *space of distributions*. Note that we will consider $j(X)$ still with the topology of X, which by assumption is finer than the one induced by the strong topology of $\mathcal{D}'$.

(ii) A triple (i, X, j) consisting of a LCVS X, a pair of continuous, injective linear maps $i : \mathcal{D} \to X$ and $j : X \to \mathcal{D}'$ will be called a *normal triple* if

- $\mathrm{Im}(i)$ is dense in X and
- $j \circ i(f) = T_f$, where $T_f : \mathcal{D} \to \mathbb{C}, \varphi \mapsto \int f\varphi$, with $f \in \mathcal{D}$ is the usual embedding of $\mathcal{D}$ into $\mathcal{D}'$.

The image $j(X)$ will be called a *normal space of distributions.*

We shall most often identify $\mathcal{D}$ with its image $i(\mathcal{D})$ and X with its image $j(X)$, thus simply say X is a (normal) space of distributions and omit the injections i and j in our notation.

On the way to defining the duality product we have to investigate the dual space of a normal space of distributions. This will be part of the next Proposition, which is going to provide us with the basic setting. However, before we formulate the Proposition, we recall some important topologies:

- $\sigma(X, X')$ ... the *weak topology* on X, i.e., the topology of uniform convergence on finite subsets of X' (similarly $\sigma(X', X)$)
- $\beta(X, X')$ ... the *strong topology* on X, i.e., the topology of uniform convergence on $\sigma(X', X)$-bounded subsets of X' (similarly $\beta(X', X)$)
- $\mu(X', X)$ ... the *Mackey topology* on X', i.e., the topology of uniform convergence on balanced, convex and $\sigma(X, X')$-compact subsets of X (cf. [15, Def. 3.5.1, p. 206])
- $\kappa(X', X)$ ... the topology of uniform convergence on balanced, convex and compact subsets of X (cf. [15, p. 235])
- $\lambda(X', X)$ ... the topology of uniform convergence on precompact subsets of X (cf. [15, Def. 3.9.2, p. 234]).

Proposition 3.2. (The dual of a normal space (cf. [15, Scholium 3.12.1, p. 259]))
Let X be a normal space of distributions.

(1) If X' is equipped with $\beta(X', X)$, then we have

(i) $i^t : X' \to \mathcal{D}'$ is continuous and injective.

(ii) $j^t : \mathcal{D} \to X'$ is continuous and injective.

(2) If X' is equipped with $\kappa(X', X)$, $\lambda(X', X)$ or $\mu(X', X)$, then (i) and (ii) remain valid. Moreover, in these cases X' is a normal space of distributions as well.

Notation 3.3. By (1) we may call, for any $u \in X'$ and $\varphi \in \mathcal{D} = \mathcal{D}''$, $i^t(u) \in \mathcal{D}'$ and $j^t(\varphi) \in X'$ the restrictions of u to $\mathcal{D}$ resp. of φ to X and denote it by $i^t(u) = u|_{\mathcal{D}}$ and $j^t(\varphi) = \varphi|_X$.

Proof. (1)(i) By [15, Cor. to Prop. 3.12.3, p. 256] $i^t : (X', \beta(X', X)) \to (\mathcal{D}', \beta(\mathcal{D}', \mathcal{D}))$ is continuous. To prove injectivity first note that again by [15, Cor. to Prop. 3.12.3, p. 256] i^t is also continuous for the weak topologies, i.e., $\sigma(X', X)$ and $\sigma(\mathcal{D}', \mathcal{D})$. Since $i(\mathcal{D}) \subseteq X$ is dense w.r.t. the locally convex Hausdorff topology τ_X of X, it is also dense for the coarser topology $\sigma(X, X')$. Moreover, by [15, Prop. 3.12.3(a), p. 256] i (which by assumption is continuous w.r.t. $\beta(\mathcal{D}, \mathcal{D}')$ and τ_X) is continuous w.r.t. $\sigma(\mathcal{D}, \mathcal{D}')$ and $\sigma(X, X')$. Hence we may apply [15, Cor. 2 to Prop. 3.12.2, p. 256], which states that $\sigma(X, X')$ denseness of $i(\mathcal{D}) \subseteq X$ is equivalent to injectivity of the transpose i^t.

(ii) Since denseness of $j(i(\mathcal{D}))$ implies denseness of $j(X) \subseteq \mathcal{D}'$ (w.r.t. $\beta(\mathcal{D}', \mathcal{D})$) we may proceed completely analogous to (i).

(2) First we show that the continuity statements of (i) and (ii) remain true if we replace $\beta(X', X)$ with $\kappa(X', X)$, $\lambda(X', X)$ or $\mu(X', X)$. Indeed for κ and λ [15, Prop. 3.12.7, p. 258] and for μ [15, Cor. to Prop. 3.12.3, p. 256] and [15, Prop. 3.12.5, p. 257] guarantee continuity of

$$\left.\begin{array}{l} i^t : (X', \kappa(X', X)) \to (\mathcal{D}', \kappa(\mathcal{D}', \mathcal{D})) \\ i^t : (X', \lambda(X', X)) \to (\mathcal{D}', \lambda(\mathcal{D}', \mathcal{D})) \\ i^t : (X', \mu(X', X)) \to (\mathcal{D}', \mu(\mathcal{D}', \mathcal{D})) \end{array}\right\} = (\mathcal{D}', \beta(\mathcal{D}', \mathcal{D})),$$

where $\kappa = \lambda = \mu = \beta$ follows from the fact that $\mathcal{D}$ is a Montel space (cf. [15, p. 235]). Turning to j^t we recall that by (ii) $j^t : (\mathcal{D}'', \beta(\mathcal{D}'', \mathcal{D})) \to (X', \beta(X', X))$ is continuous. By reflexity of $\mathcal{D}$ we have $(\mathcal{D}'', \beta(\mathcal{D}'', \mathcal{D})) = (\mathcal{D}, \tau_{\mathcal{D}})$. Moreover, we may replace $\beta(X', X)$ with one of the coarser topologies $\kappa(X', X)$, $\lambda(X', X)$ or $\mu(X', X)$ such that we obtain continuity of

$$j^t : (\mathcal{D}, \tau_{\mathcal{D}}) \to \left\{\begin{array}{l} (X', \kappa(X', X)) \\ (X', \lambda(X', X)) \\ (X', \mu(X', X)) \end{array}\right. .$$

So all statements of (1) remain valid for κ, λ and μ, since the injectivity again follows along the lines of (1)(i).

Finally we prove that X' equipped with $\kappa(X', X)$, $\lambda(X', X)$ or $\mu(X', X)$ is a normal space of distributions. To begin with observe that $i^t \circ j^t$ is the usual embedding T_f of $\mathcal{D}$ into $\mathcal{D}'$ since for any $f \in \mathcal{D}$

$$\langle i^t \circ j^t(f), \varphi\rangle = \langle f, j \circ i(\varphi)\rangle = \int f\varphi = \langle T_f, \varphi\rangle \quad \forall \varphi \in \mathcal{D}.$$

Hence we only have to show that $i^t \circ j^t(\mathcal{D}) = (j \circ i)^t(\mathcal{D}) \subseteq (\mathcal{D}', \beta(\mathcal{D}', \mathcal{D}))$ is dense as well as $j^t(\mathcal{D}) \subseteq X'$ equipped with $\kappa(X', X)$, $\lambda(X', X)$ or $\mu(X', X)$. To prove the first statement recall that $\beta(\mathcal{D}', \mathcal{D}) = \kappa(\mathcal{D}', \mathcal{D})$ and that $j \circ i$ is injective and continuous w.r.t. the weak topologies $\sigma(\mathcal{D}, \mathcal{D}')$ and $\sigma(\mathcal{D}', \mathcal{D})$. Hence we may use [15, Cor. 2 to Prop. 3.12.2, p. 256] on $(j \circ i)^t = j \circ i$ to deduce the $\sigma(\mathcal{D}', \mathcal{D})$-density of $(j \circ i)^t(\mathcal{D}) \subseteq \mathcal{D}'$. But since any topology on $\mathcal{D}'$ compatible with the pairing $\langle \mathcal{D}', \mathcal{D}\rangle$ has the same closed and convex sets (cf. [15, Prop. 3.4.3, p. 198]) and the closed, balanced and convex sets form a fundamental system of neighborhoods (cf. [15, Prop. 2.4.4, p. 87]) $(j \circ i)^t(\mathcal{D}) \subseteq \mathcal{D}'$ is dense for any such topology. In particular, this holds for $\kappa(\mathcal{D}', \mathcal{D})$.

The other density statements hold by precisely the same reasoning: since $j^t = j$ is injective, $j^t(\mathcal{D}) \subseteq X'$ is dense for any topology compatible with $\langle X', X\rangle$, hence in particular for

$\kappa(X', X)$, $\lambda(X', X)$ and $\mu(X', X)$. □

Remark 3.4. *(On Proposition 3.2)*

(i) If X is a normal space of distributions, then by (1)(i) $(X', \beta(X', X))$ is (canonically isomorphic to) a space of distributions.

(ii) If X is a normal space of distributions, then $(X', \beta(X', X))$ need not be a normal space of distributions, because $j^t(\mathcal{D})$ need not be dense in X'! Take e.g. $X = L^1$, then $(X', \beta(X', X)) = (L^1, \|.\|_1)' = (L^\infty, \|.\|_\infty)$ and $\mathcal{D} \subseteq L^\infty$ is not dense (cf. [29, Remark 28.2, p. 303]).

(iii) Also note that in contrast to the statement in [22, section 5, p. 41] X cannot be equipped with an arbitrary compatible topology to ensure that the dual space X' becomes a normal space of distributions. This can be seen by taking $X = \mathcal{D}$ equipped with the weak topology $\sigma(\mathcal{D}, \mathcal{D}')$, because then the map i^t fails to be continuous.

(iv) With the notation from the above Proposition we have the following setting

$$\mathcal{D} \overset{i}{\hookrightarrow} X \overset{j}{\hookrightarrow} \mathcal{D}'$$

$$\mathcal{D}' \overset{i^t}{\hookleftarrow} X' \overset{j^t}{\hookleftarrow} \mathcal{D}$$

where the image of i and j^t is dense in X resp. X'. Additionally, we have:

$$\underset{X'}{\langle}\, u, i(\psi) \underset{X}{\rangle} = \underset{\mathcal{D}'}{\langle}\, i^t(u), \psi \underset{\mathcal{D}}{\rangle} \quad \forall \psi \in \mathcal{D}, \forall u \in X', \tag{3.1}$$

which will be written as $\underset{X'}{\langle}\, u, \psi \underset{X}{\rangle} = \underset{\mathcal{D}'}{\langle}\, u, \psi \underset{\mathcal{D}}{\rangle}$ and

$$\underset{X'}{\langle}\, j^t(\varphi), v \underset{X}{\rangle} = \underset{\mathcal{D}}{\langle} \varphi, j(v) \underset{\mathcal{D}'}{\rangle} \quad \forall \varphi \in \mathcal{D}, \forall v \in X, \tag{3.2}$$

similarly denoted by $\underset{X'}{\langle}\, \varphi, v \underset{X}{\rangle} = \underset{\mathcal{D}}{\langle} \varphi, v \underset{\mathcal{D}'}{\rangle}$. These facts will be needed when defining the duality product.

To gain a certain degree of generality, we will also consider the notion of local spaces:

Definition 3.5. *(Local spaces of distributions)*
Let X be a space of distributions.

(i) We call

$$X_{\text{loc}} := \{u \in \mathcal{D}' \mid \varphi u \in X, \forall \varphi \in \mathcal{D}\}$$

a *local space of distributions* and equip X_{loc} with the coarsest topology such that for all $\varphi \in \mathcal{D}$ the map $X_{\text{loc}} \to X, u \mapsto \varphi u$ is continuous.

(ii) We say X is a *semi local space of distributions* if $X \subseteq X_{\text{loc}}$, i.e. for all $\varphi \in \mathcal{D}$:

$$u \in X \Rightarrow \varphi u \in X.$$

If, additionally for all $\varphi \in \mathcal{D}$ the map $X \to X$, $u \mapsto \varphi u$ is continuous we say X is *topologically semi local.*

Note that in both Definitions the multiplication by a fixed element of $\mathcal{D}$ is considered in the sense of (2.1). Also in [16] a slightly different definition of local spaces is used than the one above, but by [16, Prop. 4, p. 217] they coincide.

Example 3.6. (i) $\mathcal{D}_{\text{loc}} = \mathcal{C}^\infty$: Clearly if $f \in \mathcal{C}^\infty$ and $\varphi \in \mathcal{D}$, then $\varphi f \in \mathcal{D}$. Conversely, if we have $u \in \mathcal{D}'$ such that $\varphi u \in \mathcal{D}$, for all $\varphi \in \mathcal{D}$, then choosing φ to be a bump function around some point x we see that u is smooth at x, hence $u = f \in \mathcal{C}^\infty$. Furthermore for $\psi \in \mathcal{D}$ we obviously have $\psi\varphi \in \mathcal{D}$, for all $\varphi \in \mathcal{D}$ and multiplication with $\varphi \in \mathcal{D}$ is clearly continuous. Hence $\mathcal{D}$ is topologically semi local.

(ii) L^p_{loc}: Let χ_K denote the characteristic function of a set K. For $1 \leq p \leq \infty$ the space L^p_{loc} can be defined as the space of (equivalence classes of) Lebesgue-measurable functions f such that for every compact subset K of $\mathbb{R}^n$ the function $\chi_K f$ belongs to L^p (cf. [16, Example 5, p. 220]). We want to show that $f \in L^p_{\text{loc}}$ if and only if $\varphi f \in L^p$ for every $\varphi \in \mathcal{D}$, which means that L^p_{loc} can also be viewed as local space of distributions.
First let's assume that $f \in L^p_{\text{loc}}$. Then φf is measurable for any $\varphi \in \mathcal{D}$. Setting $K :=$ supp(φ) we have $|\varphi(x)| \leq \chi_K \|\varphi\|_\infty$, hence $\int_{\mathbb{R}^n} |\varphi f|^p = \int_K |\varphi f|^p \leq \|\varphi\|_\infty \int_K |f|^p < \infty$, i.e., $\varphi f \in L^p$. Conversely, assume that $\varphi f \in L^p$ for all $\varphi \in \mathcal{D}$. Given any compact set K of $\mathbb{R}^n$, there exists a cut-off $\varphi \in \mathcal{D}$ such that $\varphi \equiv 1$ on K. Then of course $\chi_K \leq \varphi$ and $\chi_K \varphi = \chi_K$. Hence $\chi_K f = \chi_K \varphi f$ is measurable and $\int \chi_K |f|^p < \infty$, i.e., $f \in L^p_{\text{loc}}$. This also shows that L^p_{loc} is equipped with the coarsest topology for which the maps $f \mapsto \varphi f$ from L^p_{loc} to L^p are continuous for all $\varphi \in \mathcal{D}$.

Next let us take a closer look on the product (2.1) when dealing with local spaces of distributions.

Lemma 3.7. *Let X be a space of distributions. Then the product map restricted to $\mathcal{C}^\infty \times X_{\text{loc}}$ takes values in X_{loc}.*

Proof. Let $f \in \mathcal{C}^\infty, u \in X_{\text{loc}} \Rightarrow \chi(fu) = (\chi f)u \in X$, for all $\chi \in \mathcal{D}$. □

For more information on local spaces of distributions we again refer to [16].

3.2 The Duality Product

Equipped with the necessary definitions and background to make the duality method work, we start this section with its very definition and sum up its basic properties.

Definition 3.8. *(The duality product)*
Let X be a topologically semi local, normal space of distributions. Let X' be its dual equipped with the topology $\kappa(X', X)$, $\lambda(X', X)$ or $\mu(X', X)$ and assume X' is topologically semi local. For $u \in X'_{\text{loc}}$ and $v \in X_{\text{loc}}$ we then define the *duality product* $u \cdot v \in \mathcal{D}'$ by

$$\underset{\mathcal{D}'}{\langle} u \cdot v, \varphi \underset{\mathcal{D}}{\rangle} := \underset{X'}{\langle} \chi u, \varphi v \underset{X}{\rangle}, \tag{3.3}$$

for $\varphi \in \mathcal{D}$. Here $\chi \in \mathcal{D}$ is an arbitrary cut-off with $\chi \equiv 1$ on $\text{supp}(\varphi)$. Symmetrically we define $v \cdot u \in \mathcal{D}'$ by

$$\underset{\mathcal{D}'}{\langle} v \cdot u, \varphi \underset{\mathcal{D}}{\rangle} := \underset{X}{\langle} \chi v, \varphi u \underset{X'}{\rangle}. \tag{3.4}$$

We collect some general properties of the duality product:

Proposition 3.9. (Properties of the duality product)
The duality product is

(i) independent of the choice of χ

(ii) commutative

(iii) partially associative

(iv) separately continuous from $X'_{\text{loc}} \times X_{\text{loc}}$ to $\mathcal{D}'$.

Proof. Let $u \in X'_{\text{loc}}$, $v \in X_{\text{loc}}$, $\varphi \in \mathcal{D}$ and $\chi \in \mathcal{D}, \chi \equiv 1$ on $\text{supp}(\varphi)$.

(i) Let $\tilde{\chi}$ be another cut-off, i.e. $\tilde{\chi} \in \mathcal{D}, \tilde{\chi} \equiv 1$ on $\text{supp}(\varphi)$. Since $\mathcal{D} \subseteq X'$ is dense $\exists\, u_\varepsilon \in \mathcal{D}$: $u_\varepsilon \overset{X'}{\to} \chi u$ and $\exists\, \tilde{u}_\varepsilon \in \mathcal{D}$: $\tilde{u}_\varepsilon \overset{X'}{\to} \tilde{\chi} u$. Then we have using (3.2)

$$\begin{aligned} \underset{X'}{\langle} \chi u, \varphi v \underset{X}{\rangle} &= \lim_\varepsilon \underset{X'}{\langle} u_\varepsilon, \varphi v \underset{X}{\rangle} = \lim_\varepsilon \underset{\mathcal{D}}{\langle} u_\varepsilon, \varphi v \underset{\mathcal{D}'}{\rangle} = \lim_\varepsilon \underset{\mathcal{D}}{\langle} \tilde{\chi} u_\varepsilon, \varphi v \underset{\mathcal{D}'}{\rangle} \\ &= \underset{X'}{\langle} \tilde{\chi}\chi u, \varphi v \underset{X}{\rangle} = \lim_\varepsilon \underset{\mathcal{D}}{\langle} \chi \tilde{u}_\varepsilon, \varphi v \underset{\mathcal{D}'}{\rangle} = \lim_\varepsilon \underset{X'}{\langle} \tilde{u}_\varepsilon, \varphi v \underset{X}{\rangle} = \underset{X'}{\langle} \tilde{\chi} u, \varphi v \underset{X}{\rangle}. \end{aligned}$$

(ii) Let u_ε be as above. We then have again using (3.2)

$$\underset{\mathcal{D}'}{\langle} u \cdot v, \varphi \underset{\mathcal{D}}{\rangle} = \underset{X'}{\langle} \chi u, \varphi v \underset{X}{\rangle} = \underset{X'}{\langle} \chi u, \chi\varphi v \underset{X}{\rangle} = \lim_{\varepsilon} \underset{\mathcal{D}}{\langle} u_\varepsilon, \chi\varphi v \underset{\mathcal{D}'}{\rangle}$$
$$= \lim_{\varepsilon} \underset{\mathcal{D}}{\langle} \varphi u_\varepsilon, \chi v \underset{\mathcal{D}'}{\rangle} = \underset{X'}{\langle} \varphi\chi u, \chi v \underset{X}{\rangle} = \underset{X'}{\langle} \varphi u, \chi v \underset{X}{\rangle} = \underset{\mathcal{D}'}{\langle} v \cdot u, \varphi \underset{\mathcal{D}}{\rangle}.$$

(iii) First observe that by Lemma 3.7 $fu \in X'_{\text{loc}}$ and $fv \in X_{\text{loc}}$, $\forall f \in \mathcal{C}^\infty, u \in X'_{\text{loc}}$, $v \in X_{\text{loc}}$. So $(fu) \cdot v$ as well as $u \cdot (fv)$ exist. To show that they are both equal to $f \cdot (uv)$ let $v_\varepsilon \in \mathcal{D}, v_\varepsilon \overset{X}{\to} \varphi v$, u_ε be as above. We have now using (3.1)

$$\underset{\mathcal{D}'}{\langle} (fu) \cdot v, \varphi \underset{\mathcal{D}}{\rangle} = \underset{X'}{\langle} \chi(fu), \varphi v \underset{X}{\rangle} = \lim_{\varepsilon} \underset{\mathcal{D}'}{\langle} \chi(fu), v_\varepsilon \underset{\mathcal{D}}{\rangle} = \lim_{\varepsilon} \underset{\mathcal{D}'}{\langle} f\chi u, v_\varepsilon \underset{\mathcal{D}}{\rangle} = \underset{X'}{\langle} f\chi u, \varphi v \underset{X}{\rangle},$$

and again using (3.2)

$$\underset{\mathcal{D}'}{\langle} u \cdot (fv), \varphi \underset{\mathcal{D}}{\rangle} = \underset{X'}{\langle} \chi u, \varphi(fv) \underset{X}{\rangle} = \lim_{\varepsilon} \underset{\mathcal{D}}{\langle} u_\varepsilon, \varphi(fv) \underset{\mathcal{D}'}{\rangle} = \lim_{\varepsilon} \underset{\mathcal{D}}{\langle} fu_\varepsilon, \varphi v \underset{\mathcal{D}'}{\rangle} = \underset{X'}{\langle} f\chi u, \varphi v \underset{X}{\rangle}.$$

Finally we have

$$\underset{\mathcal{D}'}{\langle} f \cdot (uv), \varphi \underset{\mathcal{D}}{\rangle} = \underset{\mathcal{D}'}{\langle} uv, f\varphi \underset{\mathcal{D}}{\rangle} = \underset{X'}{\langle} \chi u, (f\varphi)v \underset{X}{\rangle}$$
$$= \lim_{\varepsilon} \underset{\mathcal{D}}{\langle} u_\varepsilon, (f\varphi)v \underset{\mathcal{D}'}{\rangle} = \lim_{\varepsilon} \underset{\mathcal{D}}{\langle} fu_\varepsilon, \varphi v \underset{\mathcal{D}'}{\rangle} = \underset{X'}{\langle} f\chi u, \varphi v \underset{X}{\rangle}.$$

(iv) Let $u_\varepsilon \to u \in X'_{\text{loc}}$, then we have

$$\underset{\mathcal{D}'}{\langle} u_\varepsilon \cdot v, \varphi \underset{\mathcal{D}}{\rangle} = \underset{X'}{\langle} \chi u_\varepsilon, \varphi v \underset{X}{\rangle} \to \underset{X'}{\langle} \chi u, \varphi v \underset{X}{\rangle} = \underset{\mathcal{D}'}{\langle} u \cdot v, \varphi \underset{\mathcal{D}}{\rangle}$$

and likewise for the second factor. □

Remark 3.10. *(Compatibility with (2.1))*
The product defined in (2.1) can be viewed as a special case of the duality product by considering $X = \mathcal{D}$ and Example 3.6(i).

3.3 Sobolev Spaces

In this section we introduce Sobolev spaces of integer order on arbitrary open sets $\Omega \subseteq \mathbb{R}^n$, although later on we restrict ourselves again to $\mathbb{R}^n$ as our usual domain. These spaces serve as our main examples for the duality method and therefore we establish some of their properties, thereby mainly following [1].

Definition 3.11. *(Sobolev spaces)*
Let $m \in \mathbb{N}_0$ and $1 \leq p \leq \infty$. We define the *Sobolev spaces on* Ω and the respective *Sobolev norms* by

(i) $W^{m,p}(\Omega) := \{u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega) \text{ for } 0 \leq |\alpha| \leq m, \alpha \in \mathbb{N}_0^n\}$ with

$$\|u\|_{m,p} := \left(\sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_p^p \right)^{1/p} \quad \text{if } 1 \leq p < \infty$$
$$\|u\|_{m,\infty} := \max_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_\infty,$$

where of course $\|.\|_p$ denotes the usual norm on $L^p(\Omega)$.

(ii) $W_0^{m,p}(\Omega) := \overline{\mathcal{D}(\Omega)}^{W^{m,p}(\Omega)}$, i.e., the closure of $\mathcal{D}(\Omega)$ in $W^{m,p}(\Omega)$.

Remark 3.12. *(On Sobolev spaces)*
We collect some simple observations and facts on Sobolev spaces without giving their proofs:

(i) Clearly $W^{0,p}(\Omega) = L^p(\Omega)$.

(ii) For $1 \leq p < \infty$: $W_0^{0,p}(\Omega) = L^p(\Omega)$, because $\mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$.

(iii) For any $m \in \mathbb{N}_0$ we obviously have

$$W_0^{m,p}(\Omega) \hookrightarrow W^{m,p}(\Omega) \hookrightarrow L^p(\Omega).$$

(iv) For all $m \in \mathbb{N}_0$ and $1 \leq p \leq \infty$ the spaces $W^{m,p}(\Omega)$ are Banach spaces (cf. [1, Thm. 3.3, p. 60]). Furthermore, they are separable for $1 \leq p < \infty$, reflexive and uniformly convex for $1 < p < \infty$ (cf. Remark 3.13(i) below resp. [1, Thm. 3.6, p. 61]).

(v) The Sobolev spaces $H^m(\Omega) := W^{m,2}(\Omega)$ based on L^2 are Hilbert spaces for all $m \in \mathbb{N}_0$ with the inner product

$$\langle u \mid v \rangle_m := \sum_{0 \leq |\alpha| \leq m} \langle D^\alpha u \mid D^\alpha v \rangle_{L^2},$$

where of course $\langle . | . \rangle_{L^2}$ is the usual inner product on L^2 (cf. [1, Thm. 3.6, p. 61]).

(vi) For each $m \in \mathbb{N}_0$ the space $W_0^{m,p}(\Omega)$ is a closed subspace of $W^{m,p}(\Omega)$, thus by [1, Thm. 1.22, p. 8] and (iv) we obtain that they are Banach spaces, separable for $1 \leq p < \infty$, reflexive and uniformly convex for $1 < p < \infty$.

(vii) We may define further Sobolev spaces by

$$H^{m,p}(\Omega) := \overline{\{u \in \mathcal{C}^m(\Omega) \mid \|u\|_{m,p} < \infty\}}^{\|.\|_{m,p}},$$

i.e., the completion w.r.t. the Sobolev norm. By a result, published in 1964 by Meyers and Serrin, it turned out that $H^{m,p}(\Omega) = W^{m,p}(\Omega)$, which ended much confusion about the relationship of these spaces that existed in literature before that time. For a proof of this statement cf. [1, Thm. 3.17, p. 67].

Remark 3.13. *(Towards a Riesz Representation Theorem for $W^{m,p}$)*

(i) On the way to proving an extension of the Riesz representation theorem for $W^{m,p}(\Omega)$, it turns out to be convenient to regard $W^{m,p}(\Omega)$ as a closed subspace of $L^p(\Omega)$ in the following sense:
For $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$ let $N = N(n,m)$ be the number of multi-indices $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \leq m$ and consider for each α, Ω_α as a copy of Ω in a *different* copy of $\mathbb{R}^n$, so that the N domains Ω_α are *de facto* disjoint. Let Ω_m be the union of these N domains, i.e., $\Omega_m := \bigcup_{|\alpha| \leq m} \Omega_\alpha$. Take $u \in W^{m,p}(\Omega)$ and let U be the function on Ω_m with $U|_{\Omega_\alpha} = D^\alpha u$. The function $P : W^{m,p}(\Omega) \to L^p(\Omega_m)$, $u \mapsto U$ is clearly an isometry. As $W^{m,p}(\Omega)$ is a Banach space, the range $\operatorname{ran}(P) =: W$ is a closed subspace of $L^p(\Omega_m)$. Moreover, it is also separable for $1 \leq p < \infty$, reflexive and uniformly convex for $1 < p < \infty$ (cf. [1, Thm. 2.39, p. 45 and Thm. 1.22, p. 8]). Therefore the same holds for $W^{m,p}(\Omega) = P^{-1}(W)$, which actually proves Remark 3.12(iv) above.

(ii) We recall the Riesz representation theorem for $L^p(\Omega)$ with the above notation of domains:
Let $1 \leq p < \infty$ and let p' be its conjugate exponent (cf. Notation 3.14(i) below). For

all $L \in (L^p(\Omega_m))'$ there exists a unique $v \in L^{p'}(\Omega_m)$, such that for all $u \in L^p(\Omega_m)$ we have

$$L(u) = \int_{\Omega_m} u(x)v(x)dx = \sum_{|\alpha|\leq m} \int_{\Omega_\alpha} u_\alpha(x)v_\alpha(x)dx = \sum_{|\alpha|\leq m} \langle u_\alpha, v_\alpha \rangle,$$

where $u_\alpha := u|_{\Omega_\alpha}$ and $v_\alpha := v|_{\Omega_\alpha}$.
Moreover $\|L\|_{(L^p(\Omega_m))'} = \|v\|_{L^{p'}(\Omega_m)}$. Thus $(L^p(\Omega_m))' \cong L^{p'}(\Omega_m)$.

Notation 3.14. Motivated by Remark 3.13 we fix some notation.

(i) For given p we will always denote by p' the conjugate exponent given by

$$p' = \begin{cases} \infty & \text{if } p = 1 \\ p/(p-1) & \text{if } 1 < p < \infty \\ 1 & \text{if } p = \infty. \end{cases}$$

(ii) The restriction of a function $f \in L^p(\Omega_m)$ to Ω_α will be denoted by f_α, i.e., $f_\alpha = f|_{\Omega_\alpha}$.

Proposition 3.15. (The dual of $W^{m,p}$)
Let $1 < p < \infty$ and $m \in \mathbb{N}$. Then for all $L \in (W^{m,p}(\Omega))'$ there exists a unique $v \in L^{p'}(\Omega_m)$ such that we have for all $u \in W^{m,p}(\Omega)$

$$L(u) = \sum_{|\alpha|\leq m} \langle D^\alpha u, v_\alpha \rangle. \tag{3.5}$$

Moreover, $\|L\| = \|v\|_{L^{p'}(\Omega_m)}$.
In case $p = 1$, $v \in L^\infty(\Omega_m)$ is no longer uniquely determined but nevertheless there exist $v \in L^\infty(\Omega_m)$ sucht that (3.5) holds and $\|L\| = \min_{v\in R} \|v\|_{L^\infty(\Omega_m)}$, where $R := \{v \in L^\infty(\Omega_m) \mid$ (3.5) holds for all $u \in W^{m,1}(\Omega)\}$.

Proof. Let L be in $(W^{m,p}(\Omega))'$. Using the notation from Remark 3.13(i) we can define a linear functional L^* on W as follows:

$$L^*(Pu) := L(u) \quad \forall u \in W^{m,p}(\Omega).$$

Since P is an isometric isomorphism on W, we have $L^* \in W'$ and $\|L^*\| = \|L\|$. By the Hahn-Banach theorem (cf. [1, Thm. 1.13, p. 6]) there exists a norm preserving extension

$\hat{L}$ of L^* to $L^p(\Omega_m)$ and by Remark 3.13(ii) there exists a $v \in L^{p'}(\Omega_m)$ such that

$$\hat{L}(u) = \sum_{|\alpha| \le m} \langle u_\alpha, v_\alpha \rangle \quad \text{for } u \in L^p(\Omega_m).$$

For $u \in W^{m,p}(\Omega)$ we obtain

$$L(u) = L^*(Pu) = \hat{L}(Pu) = \sum_{|\alpha| \le m} \langle D^\alpha u, v_\alpha \rangle.$$

Moreover, for $1 < p < \infty$

$$\|L\| = \|L^*\| = \|\hat{L}\| = \|v\|_{L^{p'}(\Omega_m)}.$$

If $1 < p < \infty$, then $L^{p'}(\Omega_m)$ is uniformly convex. We show that in this case $\hat{L}$ is uniquely determined. Suppose not. Then there exist L_1 and $L_2 \in L^{p'}(\Omega_m)$ with $L_1|_W = L_2|_W$ such that $L_1(w) \neq L_2(w)$ for some $w \in L^p(\Omega_m)$. Now suppose $u \in W$ with $L_1(u) = L_2(u) = 1$. Without loss of generality, we may assume that $\|L_1\|_{L^{p'}(\Omega_m)} = \|L_2\|_{L^{p'}(\Omega_m)} = 1$, $\|u\| = 1$, $L_1(w) - L_2(w) = 2$ and finally $L_1(w) = 1$ and $L_2(w) = -1$ (this is possible by replacing w by $\tilde{w} = w + \sigma u$, with $\sigma = 1 - L_1(w)$). If $t > 0$, then $L_1(u + tw) = 1 + t$ and since $\|L_1\|_{L^{p'}(\Omega_m)} = 1$, we have $\|u + tw\|_{p'} \ge 1 + t$. Analogously $L_2(u - tw) = 1 + t$ and thus $\|u - tw\|_{p'} \ge 1 + t$. If $1 < p' \le 2$, then Clarkson's inequality (cf. [1, Thm. 2.38.(33), p. 44]) gives

$$\begin{aligned} 1 + t^{p'} \|w\|_{p'}^{p'} &= \left\| \frac{(u + tw) + (u - tw)}{2} \right\|_{p'}^{p'} + \left\| \frac{(u + tw) - (u - tw)}{2} \right\|_{p'}^{p'} \\ &\ge \frac{1}{2} \|u + tw\|_{p'}^{p'} + \frac{1}{2} \|u + tw\|_{p'}^{p'} \ge (1 + t)^{p'}, \end{aligned}$$

which is not possible for all $t > 0$. Similarly, if $2 \le p' < \infty$, then Clarkson's inequality (cf. [1, Thm. 2.38.(31), p. 44]) gives

$$\begin{aligned} 1 + t^{p} \|w\|_{p'}^{p} &= \left\| \frac{(u + tw) + (u - tw)}{2} \right\|_{p'}^{p} + \left\| \frac{(u + tw) - (u - tw)}{2} \right\|_{p'}^{p} \\ &\ge \left(\frac{1}{2} \|u + tw\|_{p'}^{p'} + \frac{1}{2} \|u + tw\|_{p'}^{p'} \right)^{p-1} \ge (1 + t)^{p}, \end{aligned}$$

which is also not possible for all $t > 0$. Hence no such w can exist and therefore $L_1 = L_2$. In case $p = 1$, v need not be unique, nevertheless every $v \in R$ corresponds to an extension $\hat{L}$ of L^* and thus we have to resort to the minimum over all these. □

Remark 3.16. *(The relation between $(W^{m,p})'$ and $\mathcal{D}'$)*
By (3.5) we see that L is given by the sum of weak derivatives of a regular distribution. To actually calculate this distribution we introduce the following notation:

$$\langle T_{v_\alpha}, \varphi\rangle := \langle \varphi, v_\alpha\rangle \quad \forall \varphi \in \mathcal{D} \text{ and some } v \in L^{p'}(\Omega_m) \tag{3.6}$$

$$T := \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha T_{v_\alpha}. \tag{3.7}$$

With this, we now have the following statement:
Any functional $L \in (W^{m,p}(\Omega))'$ is given by the extension of the distribution $T \in \mathcal{D}'(\Omega)$ to $W^{m,p}(\Omega)$, i.e., $L|_{\mathcal{D}(\Omega)} = T$.
Indeed we have for all $\varphi \in \mathcal{D}(\Omega)$

$$\langle T, \varphi\rangle = \sum_{|\alpha|\leq m} \langle (-1)^{|\alpha|} D^\alpha T_{v_\alpha}, \varphi\rangle = \sum_{|\alpha|\leq m} \langle T_{v_\alpha}, D^\alpha \varphi\rangle = \sum_{|\alpha|\leq m} \langle D^\alpha \varphi, v_\alpha\rangle = L(\varphi).$$

Observe that the space of all such T is in general not the dual $(W^{m,p}(\Omega))'$ but it turns out to be the dual $(W_0^{m,p}(\Omega))'$! Before proving the latter statement we give the following definition:

Definition 3.17. *(Sobolev spaces of negative integer orders)*
For $m \in \mathbb{N}$ and $1 \leq p \leq \infty$ we define

$$W^{-m,p}(\Omega) := \left\{ T \in \mathcal{D}'(\Omega) \;\middle|\; T = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha T_{v_\alpha} \text{ for some } v \in L^p(\Omega_m) \right\}$$

and equip it with the norm $\|T\| = \|v\|_{L^p(\Omega_m)}$ (resp. the minimum over all v in case $p = \infty$).

Proposition 3.18. (The dual of $W_0^{m,p}$ – #1)
If $1 \leq p < \infty$ and $m \geq 1$, the dual space $(W_0^{m,p}(\Omega))'$ is isometrically isomorphic to $W^{-m,p'}(\Omega)$.

Proof. Since any functional T in $(W_0^{m,p}(\Omega))'$ has a unique norm preserving extension to $W^{m,p}(\Omega)$, the calculation stated in Remark 3.16 also remains valid for $(W_0^{m,p}(\Omega))'$ in place of $(W^{m,p}(\Omega))'$ and due to Definition 3.17 $T \in W^{-m,p'}(\Omega)$. Thus it remains to show that for any $T \in W^{-m,p'}(\Omega)$ given by some $v \in L^{p'}(\Omega_m)$ there exists a unique continuous extension to $(W_0^{m,p}(\Omega))'$. Since $\mathcal{D}(\Omega)$ is dense in $W_0^{m,p}(\Omega)$ we may choose $\varphi_n \in \mathcal{D}(\Omega)$ a sequence converging to $u \in W_0^{m,p}(\Omega)$. Then we have

$$\begin{aligned}|\langle T, \varphi_n\rangle - \langle T, \varphi_k\rangle| &= \left| \sum_{|\alpha|\le m} \langle T_{v_\alpha}, D^\alpha \varphi_n\rangle - \langle T_{v_\alpha}, D^\alpha \varphi_k\rangle \right| \\ &\le \sum_{|\alpha|\le m} |\langle T_{v_\alpha}, D^\alpha\varphi_n - D^\alpha\varphi_k\rangle| = \sum_{|\alpha|\le m} |\langle D^\alpha\varphi_n - D^\alpha\varphi_k, v_\alpha\rangle| \\ &\le \sum_{|\alpha|\le m} \|D^\alpha(\varphi_n - \varphi_k)\|_p \|v_\alpha\|_{p'} \\ &\le \|\varphi_n - \varphi_k\|_{m,p} \|v\|_{L^{p'}(\Omega_m)} \to 0 \quad \text{as } k,\, n \to \infty.\end{aligned}$$

Hence $(\langle T, \varphi_n\rangle)_n$ is a Cauchy sequence in $\mathbb{C}$ and therefore converges to a limit. We hence may define a functional $L \in (W_0^{m,p}(\Omega))'$ by

$$L(u) := \lim_{n\to\infty} \langle T, \varphi_n\rangle.$$

This is indeed well defined since for another sequence $\psi_n \in \mathcal{D}(\Omega)$, also with $\|\psi_n - u\|_{m,p} \to 0$, we have by an analogous calculation to the one above that $\langle T, \varphi_n\rangle - \langle T, \psi_n\rangle \to 0$, as $n \to \infty$. The functional L is obviously linear and belongs to $(W_0^{m,p}(\Omega))'$, since again by a similar calculation we obtain

$$|L(u)| = \lim_{n\to\infty} |\langle T, \varphi_n\rangle| \le \lim_{n\to\infty} \|\varphi_n\|_{m,p} \|v\|_{L^{p'}(\Omega_m)} = \|u\|_{m,p} \|v\|_{L^{p'}(\Omega_m)}.$$

Finally, again by Remark 3.16 and by Proposition 3.15 we have

$$\|T\| = \|v\|_{L^{p'}(\Omega_m)} = \|L\|,$$

resp. the minimum over all such v in case $p = 1$. □

Remark 3.19. *(On $W^{-m,p}$)*

(i) The spaces $W^{-m,p}(\Omega)$ $(m \in \mathbb{N})$ are Banach spaces, their completeness follows, via the isometric isomorphism of Proposition 3.18.

(ii) The spaces $W^{-m,p}(\Omega)$ are reflexive and separable for $1 < p < \infty$, since each of them is isomorphic to the dual space of a reflexive and separable space (cf. Remark 3.12(vi) and [1, Thm. 1.15, p. 7]).

If $1 < p < \infty$ there is another way of characterizing the dual space of $W_0^{m,p}(\Omega)$ avoiding the domains Ω_m for the L^p-spaces which also leads to the definition of an alternative norm.

We will now investigate this approach. Each element $v \in L^{p'}(\Omega)$ determines an element $L_v \in (W_0^{m,p}(\Omega))'$ by means of $L_v(u) = \langle u, v\rangle$, because

$$|L_v(u)| = |\langle u, v\rangle| \le \|u\|_p \|v\|_{p'} \le \|u\|_{m,p}\|v\|_{p'}.$$

This gives rise to the following definition:

Definition 3.20. *(The $(-m,p')$-norm on $L^{p'}$)*
For $1 < p < \infty$ and $m \ge 1$ we define the $(-m,p')$-norm of $v \in L^{p'}(\Omega)$ to be the norm of the functional L_v, i.e.,

$$\|v\|_{-m,p'} := \|L_v\|_{(W_0^{m,p}(\Omega))'} = \sup_{u\in W_0^{m,p}(\Omega),\ \|u\|_{m,p}\le 1} |\langle u, v\rangle|.$$

Remark 3.21. *(On the $(-m,p')$-norm)*
Clearly $\|v\|_{-m,p'} \le \|v\|_{p'}$ and for any $u \in W_0^{m,p}(\Omega)$ and $v \in L^{p'}(\Omega)$ we have

$$|\langle u, v\rangle| = \|u\|_{m,p} \left|\left\langle \frac{u}{\|u\|_{m,p}}, v\right\rangle\right| \le \|u\|_{m,p}\|v\|_{-m,p'}, \tag{3.8}$$

which is a generalization of Hölder's inequality.

Proposition 3.22. (The dual of $W_0^{m,p}$ – #2)
Let $1 < p < \infty$ and $m \in \mathbb{N}$, then $V := \{L_v \in (W_0^{m,p}(\Omega))' \mid v \in L^{p'}(\Omega)\}$ is dense in $(W_0^{m,p}(\Omega))'$.

Proof. To prove the denseness of V it is sufficient to show that if $U \in (W_0^{m,p}(\Omega))''$ satisfies $U(L_v) = 0$ for every $L_v \in V$, then $U = 0$ in $(W_0^{m,p}(\Omega))''$. Since $W_0^{m,p}(\Omega)$ is reflexive, there exists $u \in W_0^{m,p}(\Omega)$ corresponding to $U \in (W_0^{m,p}(\Omega))''$ such that

$$\langle u, v\rangle = L_v(u) = U(L_v) = 0, \quad \forall v \in L^{p'}(\Omega).$$

But then u must be zero a.e., hence we have $u = 0$ in $W_0^{m,p}(\Omega)$ and $U = 0$ in $(W_0^{m,p}(\Omega))''$. □

Definition 3.23. *(The Sobolev spaces $H^{-m,p}$)*
Motivated by Proposition 3.22, we define for $m \in \mathbb{N}$ and $1 < p < \infty$ the Sobolev spaces $H^{-m,p}(\Omega)$ as the completion of the spaces $L^p(\Omega)$ w.r.t. the norm $\|.\|_{-m,p}$.

Remark 3.24. *(The relation between $W^{-m,p'}$ and $H^{-m,p'}$)*

(i) Proposition 3.18 and 3.22 combined, tell us that

$$H^{-m,p'}(\Omega) \cong (W_0^{m,p}(\Omega))' \cong W^{-m,p'}(\Omega).$$

In detail, this means that on the one hand for each $v \in H^{-m,p'}(\Omega)$ there exists a distribution $T_v \in W^{-m,p'}(\Omega)$ such that $\langle T_v, \varphi \rangle = \lim_{n\to\infty} \langle \varphi, v_n \rangle$, for every $\varphi \in \mathcal{D}(\Omega)$ and every sequence $v_n \in L^{p'}(\Omega)$ with $\lim_{n\to\infty} \|v_n - v\|_{-m,p'} = 0$ and on the other hand any $T \in W^{-m,p'}(\Omega)$ satisfies $T = T_v$, for some such v. Moreover, by (3.8) we have $|\langle T_v, \varphi \rangle| \leq \|\varphi\|_{m,p} \|v\|_{-m,p'}$.

(ii) The proof of Proposition 3.22 is exclusively based on reflexivity of $W_0^{m,p}(\Omega)$, thus we can apply an analogous argument to $W^{m,p}(\Omega)$ and characterize its dual $(W^{m,p}(\Omega))'$ for $1 < p < \infty$ by the completion of $L^{p'}(\Omega)$ w.r.t. the adapted norm

$$\|v\|^*_{-m,p'} = \|L_v\|_{(W^{m,p}(\Omega))'} = \sup_{u \in W^{m,p}(\Omega),\ \|u\|_{m,p} \leq 1} |\langle u, v \rangle|.$$

Remark 3.25. *(Normality of Sobolev spaces)*
Having investigated the duals of Sobolev spaces, we yet need to know whether $\mathcal{D}(\Omega)$ is dense in $W^{m,p}(\Omega)$ or not. As our main focus lies in the special case of $\Omega = \mathbb{R}^n$, an easy answer can be given by [1, Cor. 3.23, p. 70], which tells us that for $1 \leq p < \infty$

$$W^{m,p}(\mathbb{R}^n) = W_0^{m,p}(\mathbb{R}^n).$$

This holds due to the fact that the domain $\mathbb{R}^n$ satisfies the so called *segment condition* (cf. [1, 3.21 and Thm. 3.22, p. 68]), which is not satisfied by an arbitrary domain $\Omega \subset \mathbb{R}^n$! This leaves us with the following relation between Sobolev spaces

$$L^{p'}(\mathbb{R}^n_m) \cong (W^{m,p}(\mathbb{R}^n))' = (W_0^{m,p}(\mathbb{R}^n))' \cong W^{-m,p'}(\mathbb{R}^n) \cong H^{-m,p'}(\mathbb{R}^n).$$

Next we deal with the dual space of $W^{m,\infty}$, which we have excluded in our investigation so far. Note that $W^{m,\infty}$ is not a normal space of distributions since $\mathcal{D}$ is not dense in it and also $W^{-m,1}$ is not its dual. However, we now show that $W^{-m,1}$ is included in $(W^{m,\infty})'$, which suffices for our purpose.

Proposition 3.26. (Embedding of $W^{-m,1}$ into the dual of $W^{m,\infty}$)

For $m \in \mathbb{N}_0$ *the map* $\iota : W^{-m,1}(\mathbb{R}^n) \to (W^{m,\infty}(\mathbb{R}^n))'$ *given by*

$$\langle \iota \left(\sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha T_{v_\alpha} \right), \psi \rangle = \sum_{|\alpha| \le m} \langle D^\alpha \psi, v_\alpha \rangle \quad \text{for } \psi \in W^{m,\infty} \text{ and some } v \in L^1(\mathbb{R}^n_m)$$

is welldefined, injective and continuous.

Proof. We first show that ι is welldefined, i.e., ι does not depend on the choice of v. To do so, let $\psi \in W^{m,\infty}, \varphi_\varepsilon \in \mathcal{D}$ be a mollifier and $\psi_\varepsilon := \psi * \varphi_\varepsilon$. By [6, Appendix D Thm. 6(iv)] we have that $D^\alpha \psi_\varepsilon \to D^\alpha \psi$ in L^p_{loc} for all $p < \infty$ and $|\alpha| \le m$. Thus we obtain up to a subsequence that $D^\alpha \psi_\varepsilon \to D^\alpha \psi$ almost everywhere for $|\alpha| \le m$ (cf. [1, Cor. 2.17, p. 30]). Now, suppose

$$\sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha T_{v_\alpha} = 0 \text{ in } \mathcal{D}',$$

then we have

$$\langle \iota \left(\sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha T_{v_\alpha} \right), \psi \rangle = 0$$

just by approximating ψ by ψ_ε.
Since $\mathcal{D} \subset W^{m,\infty}$ this argument also gives injectivity.
Finally,

$$\langle \iota \left(\sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha T_{v_\alpha} \right), \psi \rangle \le \|v\|_{L^1(\mathbb{R}^n_m)} \|D^\alpha \psi\|_{m,\infty}$$

gives the continuity of ι. □

Before we are finally able to apply the duality method to Sobolev spaces, we need one more result concerning $W^{1,p}$. It will allow us to consider representatives of elements of $W^{m,p}$ as generalizations of absolutely continuous functions on $\mathbb{R}$ (cf. [31, Thm. 2.1.4, p. 44]).

Theorem 3.27. (Absolute continuity along lines)
Let $u \in L^p$ *for* $1 \le p < \infty$*, then* $u \in W^{1,p}$ *iff* u *has a representative* $\bar{u}$ *that is absolutely continuous on almost all lines parallel to the coordinate axes and whose (classical) partial derivatives belong to* L^p*.*

Proof. To begin with suppose $u \in W^{1,p}$. First, we show the existence of a representative with the required properties on a rectangular cell in $\mathbb{R}^n$ denoted by

$$R := [a_1, b_1] \times \cdots \times [a_n, b_n],$$

where all side lengths are rational. For an element $x \in R$ we write $x = (\bar{x}, x_i)$, where $\bar{x} \in \mathbb{R}^{n-1}$ and $x_i \in [a_i, b_i], 1 \leq i \leq n$. Since the mollifiers u_ε of u converge in the $W^{1,p}_{\text{loc}}$-norm to u (cf. [31, Lemma 2.1.3, p. 43]) and because of Fubini's Theorem, there exists a subsequence of u_ε, again denoted by u_ε, such that

$$\lim_{\varepsilon \to 0} \int_{a_i}^{b_i} |u_\varepsilon(\bar{x}, x_i) - u(\bar{x}, x_i)|^p + |Du_\varepsilon(\bar{x}, x_i) - Du(\bar{x}, x_i)|^p dx_i = 0$$

for almost all $\bar{x}$. Applying the fundamental theorem of calculus we obtain

$$|u_\varepsilon(\bar{x}, \xi) - u_\varepsilon(\bar{x}, a_i)| \leq \int_{a_i}^{b_i} |Du_\varepsilon(\bar{x}, x_i)| dx_i \leq \int_{a_i}^{b_i} |Du(\bar{x}, x_i)| dx_i + \eta$$

for all $\bar{x}$, $\eta > 0$, some $\xi \in [a_i, b_i]$ and all ε small enough. Hence the sequence u_ε is uniformly bounded on $[a_i, b_i]$ (since without loss of generality we can choose another subsequence, such that $(u_\varepsilon(\bar{x}, a_i))_\varepsilon$ converges). Moreover, as a function of x_i, the u_ε are absolutely continuous and even uniformly continuous w.r.t. ε, which is implied by the convergence of Du_ε to Du in L^1. This allows us to apply the Arzelà-Ascoli Theorem to conclude that the u_ε converge to an absolutely continuous function $\bar{u}$ with $u = \bar{u}$ almost everywhere on R. Using a diagonalization argument we obtain the general result.
To show the converse direction, we now assume we already have such a representative $\bar{u}$ of u. Then $\bar{u}\varphi$ is also absolutely continuous for any $\varphi \in \mathcal{D}$. Thus, on almost every line whose end-points belong to $\mathbb{R}^n \setminus \operatorname{supp}(\varphi)$ and in addition is parallel to the i^{th} coordinate axis we can write

$$\int \bar{u} D_i \varphi dS = -\int D_i \bar{u} \varphi dS.$$

Finally, Fubini's Theorem implies that $D_i u$ has $D_i \bar{u}$ as a representative. □

At long last we are now ready to use the duality method on Sobolev spaces. Indeed, by Proposition 3.18 we are able to consider $\langle W^{m,p}, W^{-m,p'} \rangle$ as dual pairing for $1 \leq p < \infty$ and $m \in \mathbb{N}_0$. Furthermore, we have that the Sobolev spaces are normal for the same scale of indices p and m by Remark 3.25. Thus we can equip them with a topology according to Proposition 3.2(2) to make sure their dual space is normal as well. Although not normal, even in the extremal case $p = \infty$ ($m \in \mathbb{N}$) we are still left with a dual pairing by Proposition 3.26. The form of the elements in the dual spaces, is described by (3.7) and in Remark 3.24(i). Now, all of this together gives to the following general result:

Theorem 3.28. (The duality product on Sobolev spaces)
Let $l, m \in \mathbb{Z}$, $1 \leq p,q \leq \infty$ with $l+m \geq 0$ and $\frac{1}{p}+\frac{1}{q} \leq 1$. Define $k := \min(l,m)$ and r by $\frac{1}{r} = \frac{1}{p}+\frac{1}{q}$. Then the duality method gives a continuous, bilinear multiplication map

$$W^{m,q}_{\text{loc}} \times W^{l,p}_{\text{loc}} \to W^{k,r}_{\text{loc}}.$$

Proof. To begin with, we make sure that every element of $W^{m,q}$ with compact support is in $(W^{l,p})'$, because then, we can consider the dual pairing $\langle W^{-l,p'}, W^{l,p}\rangle$ and are able to apply the duality method. If $l \leq 0$, everything is fine since $0 \leq -l \leq m$ and every element in $W^{m,q}$ with compact support is also in $W^{-l,p'}$ since by $\frac{1}{p}+\frac{1}{q} \leq 1$ we have $p' \leq q$. For $l > 0$ and $u \in W^{m,q}$ with compact support by (3.8) and Definition 3.20 we have

$$\|u\|_{-l,p'} = \sup_{v \in W^{l,p}_0,\ \|v\|_{l,p} \leq 1} |\langle u, v\rangle| \leq \|u\|_{m,q}\|v\|_{-m,q'} \leq \|u\|_{m,q}\|v\|_{l,p} \leq \|u\|_{m,q}$$

since $-m \leq l$.
Now, let K be a relatively compact, open subset of $\mathbb{R}^n$, $\chi \in \mathcal{D}$ a cut-off with $\chi \equiv 1$ on K, $u \in W^{m,q}_{\text{loc}}$, $v \in W^{l,p}_{\text{loc}}$ and $\varphi \in \mathcal{D}(K)$. Then according to (3.3) we have

$$\underset{\mathcal{D}'}{\langle} u \cdot v, \varphi \underset{\mathcal{D}}{\rangle} = \underset{W^{-l,p'}}{\langle} \chi u, \varphi v \underset{W^{l,p}}{\rangle} = \underset{W^{-l,p'}}{\langle} \chi u, \varphi \chi v \underset{W^{l,p}}{\rangle}.$$

To show continuity, we distinguish between three cases. First of all, if $m = l = 0$, then $\underset{L^{p'}}{\langle} \chi u, \varphi \chi v \underset{L^p}{\rangle} = \int \chi u \chi v \varphi dx$ and $\chi u \chi v|_K$ is just the usual product, which belongs to $L^r(K)$.
If $l \geq 1$ and $m \geq 1$, by Theorem 3.27 both $u \in W^{m,q}$, $v \in W^{l,p}$ have an absolutely continuous representative. Thus integration by parts yields $D(\chi u \chi v) = \chi u(D\chi v)+(D\chi u)\chi v \in L^r$, hence $\chi u \chi v|_K \in W^{1,r}(K)$. Taking differentials up to order $k = \min(l,m)$ establishes the result for all $m \geq 0$ and $l \geq 0$.
Finally, let $m < 0$ with $l+m \geq 0$ and write for $u \in W^{m,q}_{\text{loc}}$, $u = \sum_{|\alpha| \leq |m|}(-1)^{|\alpha|} D^\alpha T_{u_\alpha}$, for

some $u_\alpha \in L^q_{\text{loc}}(\mathbb{R}^n_\alpha)$. Now using the Leibniz formula we have

$$\begin{aligned}\langle u \cdot v, \varphi\rangle &= \langle \chi u, \varphi\chi v\rangle \\ &= \sum_{|\alpha|\le|m|} \langle (-1)^{|\alpha|} D^\alpha(\chi T_{u_\alpha}), \varphi\chi v\rangle = \sum_{|\alpha|\le|m|} \langle \chi T_{u_\alpha}, D^\alpha(\varphi\chi v)\rangle \\ &= \sum_{|\alpha|\le|m|} \langle D^\alpha(\varphi\chi v), \chi u_\alpha\rangle = \sum_{|\alpha|\le|m|} \int \chi u_\alpha D^\alpha(\varphi\chi v)dx \\ &= \sum_{|\alpha|\le|m|}\sum_{\beta\le\alpha} \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \int \chi u_\alpha D^\beta(\chi v) D^{\alpha-\beta}\varphi dx.\end{aligned}$$

With this we can represent $u \cdot v$ on K by the transposed Leibniz formula, more precisely by

$$u\cdot v = \sum_{|\alpha|\le|m|}\sum_{\beta\le\alpha} (-1)^{|\alpha-\beta|} \begin{pmatrix}\alpha\\ \beta\end{pmatrix} D^{\alpha-\beta}(\chi T_{u_\alpha} D^\beta(\chi v)).$$

Since each product $\chi u_\alpha D^\beta(\chi v)$ belongs to $L^r(K)$, we are done. □

To conclude this section we look at two examples.

Example 3.29. (i) Denote the variables in $\mathbb{R}^4$ by (x,t), where $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$. We look at the fundamental solution of the wave operator with support in the forward light cone, which is the functional

$$\varphi \mapsto \frac{1}{4\pi}\iiint \frac{1}{|x|}\varphi(x,|x|)dx.$$

Usually, it is denoted by $\frac{1}{4\pi|x|}\delta(t-|x|)$. Now, using the above Theorem, this expression may be seen as the product of the function $\frac{1}{4\pi|x|}$, which belongs to $W^{1,p}_{\text{loc}}(\mathbb{R}^4)$ for $1 \le p < \frac{3}{2}$ and the distribution $\delta(t-|x|) = \partial_t H(t-|x|)$, which belongs to $W^{-1,\infty}_{\text{loc}}(\mathbb{R}^4)$. This product belongs to $W^{-1,p}_{\text{loc}}(\mathbb{R}^4)$ and can be calculated via integration by parts. More precisely, we have

$$\begin{aligned}\left\langle \frac{1}{4\pi|x|}\delta(t-|x|), \varphi \right\rangle &= \iiiint H(t-|x|)\frac{1}{4\pi|x|}\partial_t\varphi(x,t)dtdx \\ &= \frac{1}{4\pi}\iiint \frac{1}{|x|}\varphi(x,|x|)dx.\end{aligned}$$

(ii) Let us consider a *gt-regular metric*[1] g, i.e., a symmetric section of the bundle $T^0_2(M)$

[1] This notion refers to Geroch and Traschen, who isolated this class of metrics (cf. [9]).

which additionally is of regularity $W^{1,2}_{\text{loc}} \cap L^\infty_{\text{loc}}$, with M an oriented, smooth manifold. Such metrics appear in the context of general relativity, where one is interested in their nondegeneracy. For that matter one wants to calculate the determinant of g and has to deal with products of distributions. In particular, for u and $v \in W^{1,2}_{\text{loc}} \cap L^\infty_{\text{loc}}$ one question is whether the product $u \cdot v$ is also in $W^{1,2}_{\text{loc}} \cap L^\infty_{\text{loc}}$ or not. The more or less simple answer is that $W^{1,2}_{\text{loc}} \cap L^\infty_{\text{loc}}$ actually forms an algebra. Indeed, we obviously have that $u \cdot v \in L^\infty_{\text{loc}} \subseteq L^2_{\text{loc}}$. Moreover, using the Leibniz rule (which is proven in $W^{1,2}_{\text{loc}}$ using approximation by $\mathcal{C}^\infty$-functions) we can write $\partial_j(u \cdot v) = (\partial_j u)v + (\partial_j v)u$ which is a sum of products in $L^2_{\text{loc}} \times L^\infty_{\text{loc}}$. Hence $\partial_j(u \cdot v)$ is in L^2_{loc} and thus $u \cdot v$ is in $W^{1,2}_{\text{loc}}$.
However, if we alternatively apply Theorem 3.28 in this situation, we only get $u \cdot v \in W^{1,1}_{\text{loc}}$. Hence, this example shows the limitations of Theorem 3.28, which is the loss of regularity in r.
For more information on the context of this example see [28].

4 The Fourier Method

4.1 Convolution of Tempered Distributions

The underlying idea for the definition of the Fourier product on $\mathcal{S}' \times \mathcal{S}'$ is the convolution theorem, which basically tells us, that the Fourier transform of a convolution is the point-wise product of the Fourier transforms. In standard courses on distribution theory, the Fourier transformation, on the one hand, is naturally considered on the space of tempered distributions $\mathcal{S}'$, where as on the other hand, convolution is defined on $\mathcal{E}' \times \mathcal{D}'$. To make things work in the setting of temperate distributions, we first need to extend the usual definition of convolution to $\mathcal{S}' \times \mathcal{S}$, which is done in the following Proposition (also cf. [5]) and then investigate $\mathcal{S}'$-convolution. Before we start, we fix some notation.

Notation 4.1. For a function f on $\mathbb{R}^n$ we denote the usual translation for a vector $h \in \mathbb{R}^n$ by τ_h, i.e., $\tau_h f(x) = f(x-h)$. Furthermore, we denote by $\check{f}$ the function $x \mapsto f(-x)$. Obviously, we have $\check{\check{f}} = f$. These notions combined give for instance

$$\tau_h \check{f}(x) = \check{f}(x-h) = f(h-x) = \tau_{-h} f(-x) = (\tau_{-h} f)\check{}.$$

Proposition 4.2. (Convolution on $\mathcal{S}' \times \mathcal{S}$)
Let $u \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$. We define the convolution of u and φ by

$$(u * \varphi)(x) := \langle u, \tau_x \check{\varphi} \rangle.$$

Then the function $u * \varphi$ belongs to $\mathcal{O}_\mathrm{M}$. Moreover, the map $\mathcal{S} \to \mathcal{O}_\mathrm{M}$, $\varphi \mapsto u * \varphi$ is continuous for each $u \in \mathcal{S}'$.

Remark 4.3. *(Compatibility of convolutions)*
If either $u \in \mathcal{S}'$ or $\varphi \in \mathcal{S}$ has compact support, then by [8, Thm. 5.2.1, p. 53] the above definition of the convolution coincides with the usual definition of the convolution on $\mathcal{E}' \times \mathcal{D}'$ which for $u \in \mathcal{E}'$ and $v \in \mathcal{D}'$ is given by

$$\langle u * v, \varphi \rangle = \langle u \otimes v, \rho(x)\varphi(x+y) \rangle, \quad \forall \varphi \in \mathcal{D} \tag{4.1}$$

where $\rho \in \mathcal{D}$ is some cut-off with $\rho \equiv 1$ on $\operatorname{supp}(u)$. Note that (4.1) is independent of the choice of ρ (cf. [8, section 5.1]).

Proof. To begin with, $u * \varphi$ is a $\mathcal{C}^\infty$-function. This can easily be seen by using convergence of directional derivatives in $\mathcal{S}$ (cf. [15, Lemma 4.11.2, p. 421]). To see the moderateness of $u * \varphi$, we simply use continuity of the temperate distribution u and obtain for some N, C, and for all $\varphi \in \mathcal{S}$

$$\begin{aligned}|(u * \varphi)(x)| &= |\langle u, \tau_x \check{\varphi} \rangle| \\ &\leq C \sum_{|\alpha| \leq N} \|(1 + |y|^2)^N D^\alpha \check{\varphi}(y - x)\|_\infty \\ &= C \sum_{|\alpha| \leq N} \|(1 + |y + x|^2)^N D^\alpha \check{\varphi}(y)\|_\infty.\end{aligned}$$

Since $1 + |y + x|^2 \leq (1 + |y|^2)(1 + |x|^2)$, we have

$$|(u * \varphi)(x)| \leq (1 + |x|^2)^N C \sum_{|\alpha| \leq N} \|(1 + |y|^2)^N D^\alpha \check{\varphi}(y)\|_\infty,$$

which finishes the proof. □

Remark 4.4. *(Towards the $\mathcal{S}'$-convolution)*
For the definition of the $\mathcal{S}'$-convolution it turns out to be convenient, to have several notions of convolution at hand. To this end, consider u, $v \in L^1$ and $\varphi \in \mathcal{S}$. Then by a straight forward estimate $(\check{u} * \varphi)v$ belongs to L^1 and we obtain

$$\begin{aligned}\langle u * v, \varphi \rangle = \langle u \otimes v, \varphi(x + y) \rangle &= \iint u(y) v(x - y) \varphi(x) dy dx \\ &= \iint u(-y) \varphi(x - y) v(x) dy dx = \langle (\check{u} * \varphi) v, 1 \rangle.\end{aligned}$$

In Definition 4.6 below, we will use this formula to obtain the $\mathcal{S}'$-convolution.
To motivate another approach, let $\psi_j \in \mathcal{D}$ be a sequence converging to δ in $\mathcal{D}'$. Then we have

$$\begin{aligned}\langle (\check{u} * \varphi) v, 1 \rangle &= \int (\check{u} * \varphi)(x) v(x) dx = \lim_{j \to \infty} \int (\check{u} * \varphi)(x) \cdot (v * \psi_j)(x) dx \\ &= \lim_{j \to \infty} \langle (u * v) * \psi_j, \varphi \rangle = \langle (u * v) * \delta, \varphi \rangle = \langle u * v, \varphi \rangle.\end{aligned}$$

This formula will be further discussed in Proposition 4.11 below. However, next we give the definition of two spaces, as we will need them in the definition thereafter.

Definition 4.5. *(The spaces $\mathcal{D}_{L^\infty}$ and $\mathcal{D}'_{L^1}$)*
We define the locally convex spaces

(i) $\mathcal{D}_{L^\infty}(\mathbb{R}^n) := \bigcap_{m\in\mathbb{N}_0} W^{m,\infty}(\mathbb{R}^n)$ the space of smooth functions with bounded derivatives (equipped with the locally convex topology given in [15, Example 17, p. 91])

(ii) $\mathcal{D}'_{L^1}(\mathbb{R}^n) := \bigcup_{m\in\mathbb{N}_0} W^{-m,1}(\mathbb{R}^n)$ the space of *integrable distributions* (equipped withthe locally convex inductive limit topology w.r.t. the injections $W^{-m,1} \to \mathcal{D}'_{L^1}$, $m \geq 0$).

Note that $\mathcal{D}_{L^\infty}$ (denoted by $\mathcal{S}_0$ or $\mathcal{B}_0$ in [15]) is a normal space of distributions (cf. [15, Prop. 4.11.6, p. 419]). Also by Proposition 3.26 $\langle \mathcal{D}'_{L^1}, \mathcal{D}_{L^\infty}\rangle$ forms a dual pairing and in particular an element in $\mathcal{D}'_{L^1}$ can be evaluated at the constant function $1 \in \mathcal{D}_{L^\infty}$.

Definition 4.6. *(The $\mathcal{S}'$-convolution)*
Let u, $v \in \mathcal{S}'$. We say the $\mathcal{S}'$-*convolution* of u and v exists, if $(\check{u} * \varphi)v \in \mathcal{D}'_{L^1}$ for every $\varphi \in \mathcal{S}$. In this case we define the $\mathcal{S}'$-convolution $u * v$ by

$$\langle u * v, \varphi\rangle := \langle(\check{u} * \varphi)v, 1\rangle.$$

Remark 4.7. *(On the $\mathcal{S}'$-convolution)*

(i) The concept of $\mathcal{S}'$-convolution was first introduced by Hirata and Ogata in [13] following the notion of convolution of C. Chevalley (cf. [3]). In fact, they defined for u, $v \in \mathcal{S}'$ the $\mathcal{S}'$-convolution $u * v$ by

$$\langle (u * v) * \psi, \varphi\rangle = \int (\check{u} * \varphi)(x) \cdot (v * \psi)(x) dx$$

if

$$\check{u} * \varphi \cdot v * \psi \in L^1, \quad \forall \varphi, \psi \in \mathcal{S}. \tag{4.2}$$

However, R. Shiraishi proved in [26, Thm. 3, p. 26] the equivalence of (4.2) with our defining condition in Definition 4.6 and also to the following condition due to L. Schwartz (cf. [25]) (u, $v \in \mathcal{S}'$)

$$(u \otimes v)\varphi(x + y) \in \mathcal{D}'_{L^1}(\mathbb{R}^{2n}), \quad \forall \varphi \in \mathcal{S}.$$

(ii) In [25] L. Schwartz actually introduced a more general concept of convolution. He said the convolution of two distributions u, $v \in \mathcal{D}'$ exists if

$$(u \otimes v)\varphi(x+y) \in \mathcal{D}'_{L^1}(\mathbb{R}^{2n}), \quad \forall \varphi \in \mathcal{D}. \tag{4.3}$$

R. Shiraishi proved, that if two distributions u, $v \in \mathcal{D}'$, both not zero, satisfy condition (4.3) for all $\varphi \in \mathcal{S}$, then both distributions are tempered (cf. [26, Rem. 1, p. 27]).
Moreover, in [26, Rem. 2, p. 28] he observed that the concept of $\mathcal{S}'$-convolution would be contained in L. Schwartz concept, if in addition for two tempered distributions u, $v \in \mathcal{S}'$, which satisfy (4.3), the convolution was tempered. However, this possibility was ruled out by Dierolf and Voigt in [4] giving a counterexample. They also gave a further collection of equivalent conditions for the existence of the $\mathcal{S}'$-convolution in [4, Thm. 2.3, p. 193].

Our next step is to show, that the $\mathcal{S}'$-convolution of two tempered distributions, is again tempered. For this purpose, we will follow the original idea of Hirata and Ogata and use condition (4.2) to give a rather elementary proof. For a proof using locally convex spaces and a suitable version of the closed graph theorem we refer to [22, Lemma 6.6, p. 54].
To begin with we derive a result concerning translation-invariant operators and convolution operators. Using the Schwartz Kernel Theorem, which we recall first, we give a proof along the lines of [2, Prop. 1 and Lemma 1, p. 169].

Theorem 4.8. (The Schwartz Kernel Theorem)
Let $K : \mathcal{D} \to \mathcal{D}'$ be a linear map. Then K is sequentially continuous iff it is generated by a kernel distribution $k \in \mathcal{D}'(\mathbb{R}^{2n})$, i.e.,

$$\langle K\varphi, \psi \rangle = \langle k, \psi \otimes \varphi \rangle, \quad \forall \varphi, \psi \in \mathcal{D}.$$

The kernel k is uniquely determined by K. Moreover, if $K : \mathcal{S} \to \mathcal{S}'$, we have the same result with $k \in \mathcal{S}'(\mathbb{R}^{2n})$.

Proof. (cf. [8, Thm. 6.1.1, p. 70] and [11, Thm. 14.3.4]) □

Definition 4.9. *(Translation-invariant operators)*
We say a linear and continuous map $K : \mathcal{S} \to \mathcal{S}'$ is *translation-invariant*, if

$$\tau_h K\varphi = K(\tau_h \varphi), \quad \forall \varphi \in \mathcal{S} \text{ and } \forall h \in \mathbb{R}^n.$$

This obviously is the same as writing

$$\langle K\varphi, \tau_{-h}\psi\rangle = \langle K(\tau_h\varphi), \psi\rangle, \quad \forall\varphi, \psi \in \mathcal{S} \text{ and } \forall h \in \mathbb{R}^n.$$

Proposition 4.10. (Translation-invariant operators are convolution operators)
Let $K : \mathcal{S} \to \mathcal{S}'$ be linear and continuous with $k \in \mathcal{S}'(\mathbb{R}^{2n})$ its kernel distribution. Then K is translation-invariant iff there exists a unique $u \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$k(x, y) = u(x - y).$$

We then have $K\varphi = u * \varphi$ for $\varphi \in \mathcal{S}$ and $K : \mathcal{S} \to \mathcal{O}_{\mathrm{M}}$.

Proof. We first prove translation-invariance, which translates into

$$\langle K\varphi, \tau_{-h}\psi\rangle = \langle K(\tau_h\varphi), \psi\rangle, \quad \forall\varphi, \psi \in \mathcal{S} \text{ and } \forall h \in \mathbb{R}^n.$$

Suppose first $k(x, y) = u(x - y)$ for $u \in \mathcal{S}'$, then we have

$$\begin{aligned}\langle \tau_h K\varphi, \psi\rangle &= \langle k(x-h, y), \psi(x) \otimes \varphi(y)\rangle = \langle\langle u(x - (y+h)), \varphi(y)\rangle, \psi(x)\rangle \\ &= \langle\langle u(x-y), \varphi(y-h)\rangle, \psi(x)\rangle = \langle k(x, y), \psi(x) \otimes \varphi(y-h)\rangle = \langle K(\tau_h\varphi), \psi\rangle.\end{aligned}$$

To show the converse direction, we first observe that, if K is translation-invariant, the above calculation yields

$$\langle k(x-h, y), \psi(x) \otimes \varphi(y)\rangle = \langle\langle k(x, y), \varphi(y-h)\rangle, \psi(x)\rangle = \langle k(x, y+h), \psi(x) \otimes \varphi(y)\rangle.$$

Hence we have $k(x-h, y) = k(x, y+h)$, which implies $k(x, y) = k(x+h, y+h)$ and thus

$$\langle k(x, y), \phi(x-h, y-h)\rangle = \langle k(x, y), \phi(x, y)\rangle, \quad \forall\phi \in \mathcal{S}(\mathbb{R}^{2n}) \text{ and } \forall h \in \mathbb{R}^n. \tag{4.4}$$

Next, we show uniqueness: let $\chi \in \mathcal{S}$ with $\int_{\mathbb{R}^n} \chi = 1$ and assume we already have some $u \in \mathcal{S}'$ with $k(x, y) = u(x - y)$. Then, we obtain $u(s) = k(s + t, t)$, i.e., u does not depend on t, so we can write

$$\langle u, \psi\rangle = \langle k(s+t, t), \psi(s)\chi(t)\rangle.$$

Hence u is uniquely determined by k.
To prove existence, we define $u \in \mathcal{S}'$ in the above way, i.e.,

$$\langle u, \psi\rangle := \langle k(s+t, t), \psi(s)\chi(t)\rangle, \quad \forall\psi \in \mathcal{S},$$

and for some fixed $\chi \in \mathcal{S}$ with $\int_{\mathbb{R}^n} \chi = 1$. Then, we have

$$\begin{aligned}\langle u(x-y), \psi(x)\varphi(y)\rangle &= \langle u(x), \psi(x+y)\varphi(y)\rangle \\ &= \langle k(s+t,t), \psi(s+y)\varphi(y)\chi(t)\rangle.\end{aligned}$$

Applying the changed coordinates $(s,t,y) \mapsto (s-t, t, y+t)$ and using (4.4) we finally obtain

$$\begin{aligned}\langle k(s,t), \psi(s+y)\varphi(y+t)\chi(t)\rangle &= \langle k(s,t), \psi(s)\varphi(t)\chi(t-y)\rangle \\ &= \langle k(s,t), \psi(s)\varphi(t)\rangle.\end{aligned}$$

Thus we have $k(x,y) = u(x-y)$.
To finish the proof we still need to see that K is actually a convolution operator. Indeed, we have

$$\begin{aligned}\langle K\varphi, \psi\rangle &= \langle\langle k(x,y), \varphi(y)\rangle, \psi(x)\rangle = \langle\langle u(x-y), \varphi(y)\rangle, \psi(x)\rangle \\ &= \langle\langle u(z), \varphi(x-z)\rangle, \psi(x)\rangle = \langle\langle u(z), \tau_x\check{\varphi}(z)\rangle, \psi(x)\rangle = \langle (u*\varphi)(x), \psi(x)\rangle.\end{aligned}$$

□

Proposition 4.11. (The $\mathcal{S}'$-convolution of tempered distributions is tempered)
Suppose u, $v \in \mathcal{S}'$ satisfy condition (4.2). Then the $\mathcal{S}'$-convolution $u * v$ belongs to $\mathcal{S}'$ and moreover it is uniquely given by

$$\langle (u*v)*\psi, \varphi\rangle = \int (\check{u}*\varphi)(x)\cdot(v*\psi)(x)dx, \quad \forall\varphi,\psi \in \mathcal{S}.$$

Proof. We first consider the bilinear form defined on $\mathcal{S}\times\mathcal{S}$ by

$$B(\psi,\varphi) = \int (\check{u}*\varphi)(x)\cdot(v*\psi)(x)dx.$$

B is separately continuous and since $\mathcal{S}$ is a Fréchet space, it is even continuous on $\mathcal{S}\times\mathcal{S}$.
Now, let K be the linear and continuous map $\mathcal{S} \to \mathcal{S}'$ given by

$$\langle K\psi, \varphi\rangle = B(\psi,\varphi),$$

which is translation-invariant. Indeed, for any $h \in \mathbb{R}^n$ we have

$$\begin{aligned}\langle \tau_h K\psi, \varphi\rangle &= B(\psi, \tau_{-h}\varphi) = \int (\check{u} * \tau_{-h}\varphi)(x) \cdot (v * \psi)(x)dx \\ &= \int (\check{u} * \varphi)(x) \cdot (v * \tau_h\psi)(x)dx = B(\tau_h\psi, \varphi) = \langle K(\tau_h\psi), \varphi\rangle, \quad \forall \varphi, \psi \in \mathcal{S}.\end{aligned}$$

Therefore, by Proposition 4.10 there exists a unique tempered distribution $U \in \mathcal{S}'$ such that $K\psi = U * \psi$, hence by setting $u * v := U$ we are done. □

Furthermore, the $\mathcal{S}'$-convolution is commutative whenever it exists. More precisely, we have for u, $v \in \mathcal{S}'$ that $u * v$ also exists if $u(\check{v} * \varphi) \in \mathcal{D}'_{L^1}$ for every $\varphi \in \mathcal{S}$ and

$$\langle u * v, \varphi\rangle = \langle (\check{u} * \varphi)v, 1\rangle = \langle u(\check{v} * \varphi), 1\rangle = \langle v * u, \varphi\rangle.$$

To see the difficult proof of this fact, we again have to refer to [26, Thm. 3, p. 26].
In addition to the commutativity, the $\mathcal{S}'$-convolution also has the property of partial associativity, but only for every $f \in \mathcal{S}$ instead of $\mathcal{C}^\infty$. This is discussed in the following Lemma.

Lemma 4.12. (Partial associativity of the $\mathcal{S}'$-convolution)
The $\mathcal{S}'$-convolution is partially associative, i.e., if the $\mathcal{S}'$-convolution of $u \in \mathcal{S}'$ and $v \in \mathcal{S}'$ exists, then for any $f \in \mathcal{S}$ the $\mathcal{S}'$-convolution of $(f * u)$ and v exists as well and

$$(f * u) * v = f * (u * v).$$

Proof. First, let $\varphi \in \mathcal{S}$ and observe that $\check{f} * \varphi$ again belongs to $\mathcal{S}$. Using the associativity of the usual convolution when at least two members belong to $\mathcal{S}$ and Definition 4.6 we obtain

$$\begin{aligned}\langle (f * u) * v, \varphi\rangle &= \langle ((f * u)\check{} * \varphi)v, 1\rangle = \langle ((\check{f} * \check{u}) * \varphi)v, 1\rangle = \langle (\check{u} * (\check{f} * \varphi))v, 1\rangle \\ &= \langle u * v, \check{f} * \varphi\rangle = \langle (\check{f} * \varphi) \cdot (u * v), 1\rangle = \langle f * (u * v), \varphi\rangle.\end{aligned}$$

□

We finish our discussion of the $\mathcal{S}'$-convolution with a short list of examples.

Example 4.13. In the following cases the $\mathcal{S}'$-convolution exists,

(i) $u \in L^p_m$ and $v \in L^q_l$ for $1 \leq p, q \leq \infty$, $m, l \in \mathbb{Z}$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $m + l \geq 0$, where

$$L^p_m(\mathbb{R}^n) := \{u : \mathbb{R}^n \to \mathbb{C} \text{ measurable} \mid (1 + |x|)^m u(x) \in L^p(\mathbb{R}^n)\}$$

are the *spaces of weighted L^p-functions.*

(ii) $u \in \mathcal{S}'$ and $v \in \mathcal{E}'$ or $v \in \mathcal{O}'_{\mathrm{C}}$, where $\mathcal{O}'_{\mathrm{C}}$ is the *space of rapidly decreasing distributions* given by

$$\mathcal{O}'_{\mathrm{C}}(\mathbb{R}^n) := \{T \in \mathcal{D}'(\mathbb{R}^n) \mid \forall k \in \mathbb{Z} : (1+|x|^2)^k T \in \mathcal{D}'_{L^1}(\mathbb{R}^n)\}.$$

(iii) u and v belong to $\mathcal{S}'$, u has its support in a closed, convex acute cone[1] Γ with axial vector n and v has its support in a closed half space with n as its interior normal. Moreover, the tempered distributions with support in Γ form an algebra with convolution.

4.2 The Fourier product

To begin with this section, we give the definition of the Fourier product and establish its partial associativity, again restricted to Schwartz functions.

Definition 4.14. *(The Fourier product)*
Let u, $v \in \mathcal{S}'$ and assume that the $\mathcal{S}'$-convolution of their Fourier transforms $\hat{u}$, $\hat{v}$ exists. Then we define the *Fourier product* of u and v by

$$u \cdot v := \mathcal{F}^{-1}(\hat{u} * \hat{v}) \tag{4.5}$$

Proposition 4.15. (Partial associativity of the Fourier product)
The Fourier product is partially associative, i.e., if the Fourier product of $u \in \mathcal{S}'$ and $v \in \mathcal{S}'$ exists, then for any $f \in \mathcal{S}$ the Fourier products of (fu) and v as well as u and (fv) exist as well and we have

$$f \cdot (uv) = (fu) \cdot v = u \cdot (fv).$$

Proof. With Lemma 4.12 and (4.5) we easily obtain

$$\begin{aligned} f \cdot (uv) &= \mathcal{F}^{-1}(\hat{f} * (uv)\hat{\ }) = \mathcal{F}^{-1}(\hat{f} * \mathcal{F}(\mathcal{F}^{-1}(\hat{u} * \hat{v}))) = \mathcal{F}^{-1}((\hat{f} * \hat{u}) * \hat{v}) \\ &= \mathcal{F}^{-1}(\mathcal{F}(\mathcal{F}^{-1}(\hat{f} * \hat{u})) * \hat{v}) = \mathcal{F}^{-1}((fu)\hat{\ } * \hat{v}) = (fu) \cdot v. \end{aligned}$$

By commutativity of the $\mathcal{S}'$-convolution the second equality follows. □

Remark 4.16. *(The localized Fourier product)*
Recall, that the Fourier transform is defined on all of $\mathbb{R}^n$ and thus is a global concept.

[1] Also see Definition 4.17 below.

Nevertheless, the Fourier product can be localized via the localization procedure of Remark 2.9(v) since partial associativity of the Fourier product has been established above. Simply, define the Fourier product of $u \in \mathcal{S}'$ and $v \in \mathcal{S}'$ near $x \in \mathbb{R}^n$ as

$$w_x := \mathcal{F}^{-1}((f_x u)\hat{} * (f_x v)\hat{}),$$

where $f_x \in \mathcal{D}$ is a cut-off, with $f_x \equiv 1$ on a neighbourhood Ω_x of x and proceed along the lines of Remark 2.9(v).

Next, we discuss a product which was introduced by L. Hörmander in [14]. He defined the product of two distributions in $\mathcal{D}'$, if their wave front set is in favorable position. On the one hand, this product can be considered as a generalization of the product in Remark 2.10, while on the other hand, it can be seen as an important special case of the Fourier product. We are going to prove the latter statement in Theorem 4.19 below. However, we recall the notion of conic sets and wave front sets first.

Definition 4.17. *(Conic sets and the wave front set)*

(i) A set $\Gamma \subseteq \mathbb{R}^n \setminus \{0\}$ with the zero-section removed, is *conic*, if $\xi \in \Gamma$ implies $\lambda\xi \in \Gamma$ for all $\lambda > 0$. Similarly, a set in the cotangent bundle $\Gamma \subseteq T^*\mathbb{R}^n \setminus \{0\} \cong \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ is *conic*, if $(x, \xi) \in \Gamma$ implies $(x, \lambda\xi) \in \Gamma$ for all $\lambda > 0$. Moreover, a *conic neighourhood* of a point, is an open, conic set containing the point.

(ii) Suppose $u \in \mathcal{D}'$. We say $(x_0, \xi_0) \in T^*\mathbb{R}^n \setminus \{0\}$ is not in the *wave front set of u*, denoted by $\mathrm{WF}(u)$, iff

- $\exists \chi \in \mathcal{D}$ with $\chi(x_0) = 1$ and
- $\exists \Gamma$ a conic neighbourhood of ξ_0 such that $(1 + |\xi|)^m (\chi u)\hat{}$ is bounded for all $m \in \mathbb{N}$ on Γ.

In other words, (x_0, ξ_0) does not belong to the wave front set of u, if there exists a cut-off χ near x_0 such that $\mathcal{F}(\chi u)$ is rapidly decreasing in a conic neighbourhood of ξ_0.

Remark 4.18. *(On Definition 4.17)*

(i) In the context of constructions in conic sets, it is sometimes better to use the cosphere bundle $S^*\mathbb{R}^n \cong \mathbb{R}^n \times S^{n-1}$ instead of the cotangent bundle $T^*\mathbb{R}^n$. $S^*\mathbb{R}^n$ is given as a quotient of $T^*\mathbb{R}^n \setminus \{0\}$ w.r.t. the equivalence relation

$$(x_0, \xi_0) \sim (y_0, \eta_0) \Leftrightarrow x_0 = y_0 \text{ and } \exists \lambda > 0 : \xi_0 = \lambda\eta_0.$$

Let π denote the canonical projection $\pi : T^*\mathbb{R}^n \setminus \{0\} \to S^*\mathbb{R}^n$, then a set $\Gamma \subseteq T^*\mathbb{R}^n \setminus \{0\}$ is conic iff $\pi^{-1}(\pi(\Gamma)) = \Gamma$. Hence a conic neighbourhood Γ of $(x_0, \xi_0) \in T^*\mathbb{R}^n \setminus \{0\}$ can be given by a product of a neighbourhood U of x_0 in $\mathbb{R}^n$ with a neighbourhood V of $\frac{\xi_0}{|\xi_0|}$ in S^{n-1}, i.e., $\Gamma = U \times V$.

(ii) If $(x, \xi) \notin \mathrm{WF}(u)$ we also say that u is *microlocally regular at* (x, ξ).

(iii) By definition the wave front set is a closed and conic subset of $T^*\mathbb{R}^n \setminus \{0\}$.

(iv) The notion of wave front sets refines the notion of singular supports. More precisely, the projection of the wave front set on the first component is the singular support, i.e.,

$$\mathrm{singsupp}(u) = \{x \in \mathbb{R}^n \mid \exists \xi \neq 0 \text{ with } (x, \xi) \in \mathrm{WF}(u)\}.$$

For a proof see [8, Prop. 11.1.1, p. 146].

(v) For more information on the wave front set and microlocal analysis we again refer to [8]. Moreover, an advanced but still comprehensible approach using pseudodifferential operators can be found in [7].

Theorem 4.19. (Distributions with wave front set in favorable position)
Let u, $v \in \mathcal{D}'$ and assume that for every $(x, \xi) \in S^*\mathbb{R}^n$ we have

$$(x, \xi) \in \mathrm{WF}(u) \Rightarrow (x, -\xi) \notin \mathrm{WF}(v).$$

Then the Fourier product of u and v exists.

Proof. We need to show that the $\mathcal{S}'$-convolution of the Fourier transforms $\hat{u}$ and $\hat{v}$ exists. More precisely, we are going to prove that for every $x_0 \in \mathbb{R}^n$ there is a cut-off $\rho \in \mathcal{D}$ near x_0 such that

$$(\mathcal{F}(\rho u)\check{\ } * \varphi)\mathcal{F}(\rho v) \in L^1, \quad \forall \varphi \in \mathcal{S}.$$

We start by recalling that the Fourier transform of a distribution with compact support is a moderate function, i.e., $\mathcal{F} : \mathcal{E}' \to \mathcal{O}_{\mathrm{M}}$. Our next step is to prove that if a function $w(\xi) \in \mathcal{O}_{\mathrm{M}}$ is rapidly decreasing in some cone Γ, then for $\psi \in \mathcal{S}$, $(w * \psi)(\xi)$ is rapidly decreasing in every smaller cone $\Gamma' \subseteq \Gamma$. To this end, we note that for $\xi \in \Gamma'$ and $\eta \in \Gamma^{\mathrm{C}}$ we have $|\xi - \eta| \geq C|\xi|$. Now, for $\xi \in \Gamma'$, all $r > 0$, some m, C and using Peetre's inequality we obtain

$$\begin{aligned}
|(w*\psi)(\xi)| &\le \int_\Gamma |w(\eta)||\psi(\xi-\eta)|d\eta + \int_{\Gamma^C} |w(\eta)||\psi(\xi-\eta)|d\eta \\
&\le \int_\Gamma C(1+|\eta|^2)^{-\frac{s}{2}}(1+|\xi-\eta|^2)^{-\frac{r}{2}}d\eta \\
&+ \int_{\Gamma^C} C(1+|\eta|^2)^{\frac{m}{2}} \underbrace{(1+|\xi-\eta|^2)^{-\frac{2m+2t+r}{2}}}_{(1+|\xi-\eta|^2)^{-\frac{m+t}{2}}(1+|\xi-\eta|^2)^{-\frac{m+t+r}{2}}} d\eta \\
&\le \int_\Gamma C(1+|\eta|^2)^{-\frac{s}{2}}(1+|\eta|^2)^{\frac{r}{2}}(1+|\xi|^2)^{-\frac{r}{2}}d\eta \\
&+ \int_{\Gamma^C} C(1+|\eta|^2)^{\frac{m}{2}}(1+|\eta|^2)^{-\frac{m+t}{2}}(1+|\xi|^2)^{\frac{m+t}{2}}(1+|\xi|^2)^{-\frac{m+t+r}{2}}d\eta \\
&\le C(1+|\xi|^2)^{-\frac{r}{2}}\int_\Gamma (1+|\eta|^2)^{\frac{r-s}{2}}d\eta + C(1+|\xi|^2)^{-\frac{r}{2}}\int_{\Gamma^C}(1+|\eta|^2)^{-\frac{t}{2}}d\eta.
\end{aligned}$$

If we now choose s and t large enough the claim follows.
Next, let $x_0 \in \mathbb{R}^n$ and $\varphi \in \mathcal{S}$. We distinguish for every $\xi_0 \in S^{n-1}$ between two cases:

- If $(x_0, \xi_0) \notin \mathrm{WF}(v)$, then there exists a cut-off $\chi \in \mathcal{D}$ near x_0 and a conic neighbourhood Γ of ξ_0 such that $\mathcal{F}(\chi v)$ is rapidly decreasing in Γ.
- If $(x_o, \xi_0) \in \mathrm{WF}(v)$ then by assumption we have that $(x_0, -\xi_0) \notin \mathrm{WF}(u)$. Hence, in a similar manner there exists a cut-off $\chi \in \mathcal{D}$ near x_0 and a conic neighbourhood Γ of ξ_0 such that $\mathcal{F}(\chi u)$ is rapidly decreasing in Γ. Moreover, by the observation above we obtain the same for $\mathcal{F}(\chi u)\check{} * \varphi$.

Since S^{n-1} is compact, we can cover $\mathbb{R}^n \setminus \{0\}$ by finitely many such cones $\Gamma_1, \dots, \Gamma_m$. Considering the corresponding cut-offs $\chi_1, \dots, \chi_m \in \mathcal{D}$, which are all identically one near x_0, we obtain

$$(\mathcal{F}(\chi_j u)\check{} * \varphi)\mathcal{F}(\chi_j v) \in L^1(\Gamma_j), \quad \forall 1 \le j \le m.$$

Finally, to obtain the result, choose a cut-off $\rho \in \mathcal{D}$ near x_0 with $\rho = \chi_j\rho$ for all $1 \le j \le m$. Then $\mathcal{F}(\rho u) = \mathcal{F}(\rho) * \mathcal{F}(\chi_j u)$ resp. $\mathcal{F}(\rho v) = \mathcal{F}(\rho) * \mathcal{F}(\chi_j v)$ are still rapidly decreasing in some smaller cones Γ'_j, which may be chosen to cover $\mathbb{R}^n \setminus \{0\}$ nevertheless. □

To end this section we look at two examples.

Example 4.20. (i) We begin with an example which is covered by both methods we have discussed so far. Let $u \in H^m_{\mathrm{loc}} = W^{m,2}_{\mathrm{loc}} \subseteq L^2_{\mathrm{loc}}$ and $v \in H^l_{\mathrm{loc}} = W^{l,2}_{\mathrm{loc}} \subseteq L^2_{\mathrm{loc}}$ for m, $l \in \mathbb{Z}$ with $m + l \ge 0$. Then their product $u \cdot v$ exists as a duality product (cf. Theorem 3.28) and as a Fourier product. To see the latter, we first assume without loss of generality that u and v are in L^2 (otherwise just apply a cut-off and Remark 4.16). Then by Plancherel's theorem (cf. [8, Thm. 9.2.2, p. 118]) the Fourier transforms $\hat{u}$ and $\hat{v}$ are also in L^2. Thus by Example 4.13(i) the $\mathcal{S}'$-convolution of $\hat{u}$ and $\hat{v}$ exists, hence the Fourier product $u \cdot v$ exists as well. Whether both methods give the same result or not, cannot be answered yet. We still need to establish some sort of relation or compatibility between them. This will be done in chapter 6, where the above question will be answered positively.

(ii) We consider the sign function

$$\operatorname{sgn}(x) = \frac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

Apparently this function belongs to $W^{m,\infty}_{\text{loc}}(\mathbb{R})$ for all m. Thus we can apply the duality method. Then Theorem 3.28 gives $\operatorname{sgn}^2(x) = 1 \in W^{m,\infty}_{\text{loc}}(\mathbb{R})$.
Next, let us take a look at the wave front set of sgn. Since $\operatorname{sgn}'(x) = 2\delta(x)$ and $\frac{\partial}{\partial x}$ is elliptic we have

$$\operatorname{WF}(\operatorname{sgn}(x)) = \operatorname{WF}(\delta(x)) = \{(0,1), (0,-1)\}.$$

Obviously the wave front set criterion of Theorem 4.19 is not met.
However, the square of the sign function does exist as a Fourier product. Indeed $\mathcal{F}\operatorname{sgn}(\xi) = \frac{1}{i\pi}\operatorname{vp}(\frac{1}{\xi})$ (cf. [8, (8.3.17), p. 101]), where $\operatorname{vp}(\frac{1}{\xi})$ denotes the Cauchy principal value of $\frac{1}{\xi}$. Now $\operatorname{vp}(\frac{1}{.}) * \operatorname{vp}(\frac{1}{.}) = (\ln|.| * \ln|.|)''$ and $\ln|.| * \ln|.|$ exists locally as L^1-convolution. On the other hand, to calculate $\operatorname{sgn}^2(x) = 1$ via the Fourier product is more work. We would have to prove that

$$\delta(\xi) = (\frac{1}{i\pi}\operatorname{vp}\left(\frac{1}{.}\right) * \frac{1}{i\pi}\operatorname{vp}\left(\frac{1}{.}\right))(\xi). \tag{4.6}$$

This can be done using the Hilbert transform and its inverse. Recall that the Hilbert transform of a function $f \in \mathcal{D}(\mathbb{R})$ is given by

$$\mathcal{H}(f)(\xi) := \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(x)}{\xi - x} dx$$

or by

$$\mathcal{H}(f)(\xi) = (f * \operatorname{vp}\left(\frac{1}{\pi\,\cdot\,.}\right))(\xi).$$

Now we see that

$$\mathcal{H}^2(f)(\xi) = (f * \mathrm{vp}\left(\frac{1}{\cdot}\right) * \mathrm{vp}\left(\frac{1}{\cdot}\right))(\xi) = -f \Leftrightarrow \mathrm{vp}\left(\frac{1}{\cdot}\right) * \mathrm{vp}\left(\frac{1}{\cdot}\right) = \delta,$$

so that (4.6) becomes equivalent to the inversion formula for the Hilbert transform, i.e., $\mathcal{H}^{-1}(f)(\xi) = -\mathcal{H}(f)(\xi)$. For more details on the Hilbert transform we refer to [10, section 4.1].

Finally, note that if we already know $\mathrm{sgn}^2(x) = 1$ is valid, we can easily prove (4.6) by using the convolution theorem (cf. Theorem 6.3 below), which actually is the more common way to proceed. However, this example shows, that it is important and useful to have several approaches to distributional products and statements on their consistency at hand.

5 The Regularization Method

This chapter is deals with products of distributions obtained via regularization. More precisely, we will use smooth approximations of distributions such that we are able to apply either the pointwise product of $\mathcal{C}^\infty$-functions or the simple product (2.1) and then try to take the limit. If it exists, it will serve as the definition of a regularization product. Such a regularization product will of course depend on how we obtained the smooth approximation of distributions. This will be done by convolution with mollifiers. But then on the one hand, we have to keep Example 2.6 in mind, hence it is absolutely not enough, if a limit only exists for one particular mollifier! On the other hand, we cannot ask for the existence of a limit for arbitrary smooth approximations, because this for example, allows nets of smooth functions $(u_\varepsilon)_\varepsilon$ with $u_\varepsilon \to 0$ in $\mathcal{D}'$ and $u_\varepsilon(0) \to \infty$ and this would yield that a product of such regularizations with δ could not exist.
These observations lead us to consider certain classes of mollifiers. To be more precise, we will work with strict delta nets, a large class of mollifiers already mentioned in Definition 2.2(a), and establish corresponding strict products. In the end of this chapter we will also consider the smaller class of mollifiers, already mentioned in Definition 2.2(b), i.e., model delta nets and the corresponding model products.
We start by recalling strict delta nets and then consider four possible ways to define a strict product.
Strict delta nets are nets of test functions $(\rho_\varepsilon)_\varepsilon \in \mathcal{D}$ with

- $\operatorname{supp}(\rho_\varepsilon) \to \{0\}$ for $\varepsilon \to 0$,
- $\int_{\mathbb{R}^n} \rho_\varepsilon(x)dx = 1$ for all $\varepsilon > 0$ and
- $\int_{\mathbb{R}^n} |\rho_\varepsilon(x)|dx$ is bounded independently of ε.

Definition 5.1. *(Strict products)*

Let u, $v \in \mathcal{D}'$. We define the following 4 products of u and v

$$u \cdot [v] := \lim_{\varepsilon \to 0} u \cdot (v * \rho_\varepsilon) \tag{5.1}$$

$$[u] \cdot v := \lim_{\varepsilon \to 0} (u * \rho_\varepsilon) \cdot v \tag{5.2}$$

$$[u] \cdot [v] := \lim_{\varepsilon \to 0} (u * \rho_\varepsilon) \cdot (v * \sigma_\varepsilon) \tag{5.3}$$

$$[u \cdot v] := \lim_{\varepsilon \to 0} (u * \rho_\varepsilon) \cdot (v * \rho_\varepsilon), \tag{5.4}$$

if the limits in $\mathcal{D}'$ exist for all strict delta nets $(\rho_\varepsilon)_\varepsilon$ and $(\sigma_\varepsilon)_\varepsilon$. We call all four types *strict product* of u and v.

Remark 5.2. *(On Definition 5.1)*

(i) Observe, that we can replace condition (ii) in Definition 2.2(a) by

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \rho_\varepsilon(x) dx = 1$$

to obtain products equivalent to the ones in Definition 5.1.

(ii) We can also replace condition (iii) in Definition 2.2(a) by requiring that the $(\rho_\varepsilon)_\varepsilon$ are all nonnegative without changing the definitions of the strict products in Definition 5.1. However, this is not obvious and we refer to [21] and [23].

(iii) Requiring the existence of the limits in Definition 5.1 for the whole class of strict delta nets and not just for one mollifier, avoids the problems displayed in Example 2.6.

(iv) Based on the way to define the strict products in Definition 5.1, the immediate question arises, whether the products (5.1) – (5.4) are welldefined or not. To see this, first observe that the interlaced net of two strict delta nets, is again a strict delta net. More precisely, for two strict delta nets $(\rho_\varepsilon)_\varepsilon$ and $(\sigma_\varepsilon)_\varepsilon$ the interlaced net is given by

$$\tau_\varepsilon := \begin{cases} \rho_\varepsilon & \text{if} \quad \frac{1}{2n+1} \leq \varepsilon < \frac{1}{2n} \\ \sigma_\varepsilon & \text{if} \quad \frac{1}{2n} \leq \varepsilon < \frac{1}{2n-1} \end{cases}, \quad \forall n \in \mathbb{N}.$$

Obviously $(\tau_\varepsilon)_\varepsilon$ is again a strict delta net (just recall the requirements in Definition 2.2(a)).

Now, we only show that for instance the strict product (5.1) is independent of the chosen strict delta net, as we can apply the same argument to the products (5.2) –

(5.4). Therefore, we take two distributions u, $v \in \mathcal{D}'$ and suppose

$$\begin{matrix} (u * \rho_\varepsilon) \cdot v \to \alpha \\ (u * \sigma_\varepsilon) \cdot v \to \beta \end{matrix} \quad \text{as } \varepsilon \to 0 \text{ and for some } \alpha, \beta \in \mathcal{D}'.$$

Then, by the above and the existence of the limits for all strict delta nets, we have for the interlaced strict delta net $(\tau_\varepsilon)_\varepsilon$ and some $\gamma \in \mathcal{D}'$

$$(u * \tau_\varepsilon) \cdot v \to \gamma.$$

Moreover, by the construction of τ_ε we have that there exist subsequences such that

$$\begin{matrix} (u * \tau_{\varepsilon_k}) \cdot v \to \alpha \\ (u * \tau_{\varepsilon_{k'}}) \cdot v \to \beta \end{matrix} \quad (k, k' \in \mathbb{N}).$$

Hence, by the uniqueness of the limit we obtain $\alpha = \beta = \gamma$.

Since we have a variety of definitions for strict products at hand, the following question immediately comes to mind: In which relation do the strict products stand to one another? The answer is given in the next theorem, which is also the main result of this chapter.

Theorem 5.3. (The relationship between the strict products in Definition 5.1)
Suppose u, $v \in \mathcal{D}'$, then we have that conditions (5.1) – (5.3) *are equivalent. Moreover, these products are equivalent to the following condition:*

$$\left.\begin{matrix} \forall \varphi \in \mathcal{D} \ \exists \text{ a neighbourhood } \Omega \text{ of } 0 \text{ such that} \\ (\varphi u) * \check{v} \in L^\infty(\Omega) \text{ and is continuous at } 0. \end{matrix}\right\} \tag{5.5}$$

In addition, for all $\varphi \in \mathcal{D}$ we obtain

$$\langle u[v], \varphi \rangle = \langle [u]v, \varphi \rangle = \langle [u][v], \varphi \rangle = ((\varphi u) * \check{v})(0).$$

Proof. We start by giving the proof of (5.3) ⇒ (5.1) and (5.3) ⇒ (5.2) at the same time. For this, we only need to show the existence of the double limit $\lim_{\varepsilon \to 0,\ \eta \to 0} (u * \rho_\varepsilon)(v * \sigma_\eta)$ and that it equals $[u][v]$ for all strict delta nets $(\rho_\varepsilon)_\varepsilon$ and $(\sigma_\eta)_\eta$.
Suppose the limit does not equal $[u][v]$, then there exists a neighbourhood U of $[u][v]$ in $\mathcal{D}'$ such that for some subsequences ρ_{ε_k} and σ_{η_k} we have

$$(u * \rho_{\varepsilon_k})(v * \sigma_{\eta_k}) \notin U, \quad \forall k.$$

Now let us define

$$\begin{matrix} \tilde{\rho}_\varepsilon := \rho_{\varepsilon_k} \\ \tilde{\sigma}_\varepsilon := \sigma_{\eta_k} \end{matrix} \quad , \quad \text{for } \frac{1}{k+1} \leq \varepsilon < \frac{1}{k}.$$

Then $\tilde{\rho}_\varepsilon$ and $\tilde{\sigma}_\varepsilon$ are obviously again strict delta nets. Hence (5.3) gives that

$$\lim_{\varepsilon \to 0} (u * \tilde{\rho}_\varepsilon)(v * \tilde{\sigma}_\varepsilon) = [u][v],$$

which is a contradiction.

We go on with proving the converse direction (5.1) $\Rightarrow$ (5.3). First, without loss of generality we can assume u, $v \in \mathcal{E}'$, because otherwise we may choose a cut-off $\chi \in \mathcal{D}$ for a given $\varphi \in \mathcal{D}$ such that $\chi \equiv 1$ on $\operatorname{supp}(\varphi)$. Then we can write for the strict product $u \cdot [v]$ resp. $[u][v]$ and ε small enough

$$\langle (\chi u)((\chi v) * \rho_\varepsilon), \varphi \rangle = \langle u(v * \rho_\varepsilon), \varphi \rangle$$

resp.

$$\langle ((\chi u) * \rho_\varepsilon)((\chi v) * \sigma_\varepsilon), \varphi \rangle = \langle (u * \rho_\varepsilon)(v * \sigma_\varepsilon), \varphi \rangle.$$

Let $\varphi \in \mathcal{D}$ and without loss of generality $Q := [-\pi, \pi]^n$ be a cube containing the supports of u, v and φ. Now the idea is to rewrite the strict product (5.3) using a Fourier series expansion to obtain an expression which only contains a single strict delta net. This new strict delta net will of course depend on the strict delta nets $(\rho_\varepsilon)_\varepsilon$ and $(\sigma_\varepsilon)_\varepsilon$. Thus we start by expanding φ into its Fourier series, i.e.,

$$\varphi = \sum_{m \in \mathbb{Z}^n} c_m e^{imx},$$

where $c_m \in \mathbb{C}$ with

$$\sum_{m \in \mathbb{Z}^n} |c_m|(1 + |m|)^k < \infty, \quad \forall k \in \mathbb{N}.$$

To do the rewriting of the strict product in a transparent way, we are going to use integration notation and then apply Fubini's Theorem. Since the Fourier series of φ converges to φ in $\mathcal{C}^\infty(Q)$ we then have

$$\begin{aligned}\langle (u*\rho_\varepsilon)(v*\sigma_\varepsilon),\varphi\rangle &= \sum_{m\in\mathbb{Z}^n} c_m \iiint u(x-y)\rho_\varepsilon(y)v(z)\sigma_\varepsilon(x-z)e^{imx}dzdydx \\ &= \sum_{m\in\mathbb{Z}^n} c_m \iiint u(x)v(z)\rho_\varepsilon(-y)\sigma_\varepsilon(x-y-z)e^{im(x-y)}dydzdx \\ &= \sum_{m\in\mathbb{Z}^n} c_m\langle u\cdot(v*(\check{\rho}_\varepsilon e^{-im\cdot}*\sigma_\varepsilon)),e^{im\cdot}\rangle.\end{aligned} \tag{5.6}$$

This formula already resembles the strict product (5.1), but we still need to check that $\tau_\varepsilon := (\check{\rho}_\varepsilon e^{-im\cdot})*\sigma_\varepsilon$ is a strict delta net. Indeed, $\operatorname{supp}(\tau_\varepsilon)\to\{0\}$ as $\varepsilon\to 0$. Also,

$$\begin{aligned}\int|\tau_\varepsilon(x)|dx &= \int\left|\int\rho_\varepsilon(-y)e^{-imy}\sigma_\varepsilon(x-y)dy\right|dx \\ &\le \int|\rho_\varepsilon(y)|dy\int|\sigma_\varepsilon(x)|dx\end{aligned}$$

is bounded independently of ε. Finally, we have

$$\begin{aligned}\left|\int\tau_\varepsilon(x)dx-1\right| &= \left|\iint\rho_\varepsilon(-y)\sigma_\varepsilon(x-y)e^{-imy}-1dydx\right| \\ &\le C\sup_{y\in\operatorname{supp}(\rho_\varepsilon)}|e^{-imy}-1|\to 0,\quad \text{as } \varepsilon\to 0.\end{aligned}$$

Therefore, considering Remark 5.2(i) and the compact supports of u and v we have that

$$\lim_{\varepsilon\to 0} u(v*\tau_\varepsilon) = u[v] \quad \text{in } \mathcal{D}' \text{ as well as in } \mathcal{E}'.$$

Moreover, any subsequence $(u(v*\tau_{\varepsilon_j}))_j$ is a bounded subset of $\mathcal{E}'$ and due to $\mathcal{E}'$ being a barrelled space, this implies by [15, Prop. 3.6.2, p. 212] that $(u(v*\tau_{\varepsilon_j}))_j$ is also equicontinuous. Hence, there are $C>0$ and $k\in\mathbb{N}$ such that

$$|\langle u(v*\tau_{\varepsilon_j}),\psi\rangle| \le C\sup_{|\alpha|\le k}\sup_{x\in Q}|\partial^\alpha\psi(x)|,\quad \forall\psi\in\mathcal{C}^\infty \text{ and } \forall j\in\mathbb{N}.$$

More specifically, for some $C>0$ we obtain

$$|\langle u(v*\tau_{\varepsilon_j}),e^{im\cdot}\rangle \le C(1+|m|)^k,\quad \forall j\in\mathbb{N} \text{ and } \forall m\in\mathbb{Z}^n,$$

which implies that the series (5.6) converges uniformly in ε_j. However, this yields

$$\begin{aligned}\lim_{j\to\infty}\langle (u*\rho_{\varepsilon_j})(v*\sigma_{\varepsilon_j}),\varphi\rangle &= \sum_{m\in\mathbb{Z}^n} c_m \lim_{j\to\infty}\langle u(v*\tau_{\varepsilon_j}), e^{im\cdot}\rangle\\ &= \sum_{m\in\mathbb{Z}^n} c_m\langle u[v], e^{im\cdot}\rangle\\ &= \langle u[v],\varphi\rangle.\end{aligned}$$

Since, this holds for any subsequence, the strict product $[u][v]$ exists and equals the strict product $u[v]$.
Analogously, one can prove (5.2) $\Rightarrow$ (5.3).

Next, we prove (5.5) $\Rightarrow$ (5.1). For that matter, observe that by the definition of convolution we have

$$\langle u(v*\rho_\varepsilon),\varphi\rangle = \langle \varphi u, v*\rho_\varepsilon\rangle = \langle (\varphi u)*\check{v},\rho_\varepsilon\rangle.$$

By (5.5) we know that $f := (\varphi u)*\check{v}|_\Omega \in L^\infty(\Omega)$ and is continuous at 0. Thus, we obtain

$$\langle (\varphi u)*\check{v},\rho_\varepsilon\rangle - ((\varphi u)*\check{v})(0) = \int (f(x)-f(0))\rho_\varepsilon(x)dx \to 0, \quad \text{as } \varepsilon\to 0,$$

because of the support properties of $(\rho_\varepsilon)_\varepsilon$ (cf. Definition 2.2(a)(i)). So (5.1) holds and $u\cdot[v] = ((\varphi u)*\check{v})(0)$.

Finally, to finish the proof we show the converse direction, i.e., (5.1) $\Rightarrow$ (5.5). Suppose for any $\varphi\in\mathcal{D}$ that

$$c := \lim_{\varepsilon\to 0}\langle (\varphi u)*\check{v},\rho_\varepsilon\rangle$$

exists for all strict delta nets $(\rho_\varepsilon)_\varepsilon$. Then we define

$$\begin{aligned} g &:= (\varphi u)*\check{v} - c\\ U_\varepsilon &:= \left\{\psi\in\mathcal{D}\ \middle|\ \int |\psi(x)|dx \le 1 \text{ and } \operatorname{supp}(\psi)\subset B_\varepsilon\right\}.\end{aligned}$$

Now, to prove the assertion, we only need to see that g is bounded near 0 and continuous at 0. To begin with we claim

$$\forall\mu>0\,\exists\varepsilon>0 \text{ such that } |\langle g,\psi\rangle|\le\mu, \quad \forall\psi\in U_\varepsilon.$$

Let us assume the opposite, i.e., there exists a $\mu>0$ and a sequence $\psi_j\in U_{\varepsilon_j}$ $(\varepsilon_j\to 0)$, such that

$$|\langle g, \psi \rangle| > \mu, \quad \forall j \in \mathbb{N}.$$

Since $|\int \psi_j(x)dx| \leq 1$, we can choose a subsequence, again denoted by ψ_j, such that

$$\lim_{j\to\infty} \int \psi(x)dx \to \alpha, \quad \text{with some } \alpha \in \mathbb{C}.$$

This yields the following two possibilities:

- If $\alpha \neq 0$, then by Remark 5.2(i) the sequence $(\frac{\psi_j}{\alpha})_j$ is a strict delta net and so by assumption

  $$\lim_{j\to\infty} \langle g, \frac{\psi_j}{\alpha} \rangle = 0,$$

  which contradicts the construction of ψ_j.

- If $\alpha = 0$, then for any strict delta sequence $(\sigma_j)_j$ also $(\psi_j + \sigma_j)_j$ is a strict delta sequence. This implies

  $$\lim_{j\to\infty} \langle g, \sigma_j \rangle = \lim_{j\to\infty} \langle g, \psi_j + \sigma_j \rangle,$$

  and thus we obtain $\lim_{j\to\infty} \langle g, \psi_j \rangle = 0$, again a contradiction.

Moreover, for some $\eta > 0$ we even have

$$\langle g, \psi \rangle \leq 1, \quad \forall \psi \in U_\eta.$$

In other words, $g|_{B_\eta}$ is a functional on $\mathcal{D}(B_\eta)$ and continuous w.r.t. the L^1-norm, hence $g|_{B_\eta} \in L^\infty(B_\eta)$. In addition, observe that

$$\|g\|_{L^\infty(B_\varepsilon)} = \sup_{\psi \in U_\varepsilon} |\langle g, \psi \rangle| \to 0, \quad \text{as } \varepsilon \to 0.$$

Now, this implies that on B_η, g is equal a.e. to a function continuous at 0 and $g(0) = 0$, hence condition (5.5) is satisfied and we are done. □

An immediate consequence of the equivalencies in Theorem 5.3 is the partial associativity of the strict products (5.1), (5.2) and (5.3).

Corollary 5.4. (Partial associativity of strict products)
If the strict product $u[v]$ exists for u, $v \in \mathcal{D}'$, then it is partially associative, i.e., for

$f \in \mathcal{C}^\infty$ the following products exist and coincide:

$$f(u[v]) = (fu)[v] = u[fv].$$

Proof. The first equality simply follows by the continuity of the multiplication in the sense of (2.1), i.e.,

$$f(u[v]) = f \cdot \lim_{\varepsilon \to 0} u(v * \rho_\varepsilon) = \lim_{\varepsilon \to 0} fu(v * \rho_\varepsilon) = (fu)[v].$$

Furthermore, the second equality is seen via the equivalence of (5.1) and (5.2) in Theorem 5.3 and the above. Indeed, we simply have

$$f(u[v]) = f([u]v) = [u](fv) = u[fv].$$

□

Let us turn to the strict product (5.4). First, note that we obviously have that condition (5.3) implies condition (5.4). Actually, it turns out that the strict product (5.4) is more general than the others, which means that the converse implication is not true, though it lacks partial associativity. We will further discuss both issues in Example 6.6(ii) below. Nevertheless, we are able to state a similar result to Theorem 5.3 (condition (5.5)) for the strict product (5.4).

Theorem 5.5. (On the strict product (5.4))
Let u, $v \in \mathcal{D}'$, then the following are equivalent:

(i) *The strict product* (5.4) *of u and v exists.*

(ii) $\lim_{\varepsilon \to 0}(u(v * \rho_\varepsilon) + (u * \rho_\varepsilon)v)$ *exists for all strict delta nets $(\rho_\varepsilon)_\varepsilon$.*

(iii) *For all $\varphi \in \mathcal{D}$ there exists a neighbourhood Ω of* 0 *such that $(\varphi u) * \check{v} + \check{u} * (\varphi v)$ belongs to $L^\infty(\Omega)$ and is continuous at* 0.

Moreover, in this case we have

$$\langle [uv], \varphi \rangle = \frac{1}{2} \lim_{\varepsilon \to 0} \langle u(v * \rho_\varepsilon) + (u * \rho_\varepsilon)v, \varphi \rangle = \frac{1}{2}((\varphi u) * \check{v} + \check{u} * (\varphi v))(0).$$

Proof. For all strict delta nets $(\rho_\varepsilon)_\varepsilon$ and $(\sigma_\varepsilon)_\varepsilon$ we claim that

$$2[uv] = \lim_{\varepsilon \to 0}((u * \rho_\varepsilon)(v * \sigma_\varepsilon) + (u * \sigma_\varepsilon)(v * \rho_\varepsilon)) \tag{5.7}$$

is equivalent to (i). It suffices to prove this, since it enables us to proceed along the lines of the proof of Theorem 5.3 to see the above stated equivalence of (i), (ii) and (iii).
Indeed, just by setting $\rho_\varepsilon := \sigma_\varepsilon$ we already obtain (5.7) $\Rightarrow$ (i). To see the converse direction, suppose the strict product (5.4) exists. Then we also have

$$\begin{aligned} 4[uv] &= \lim_{\varepsilon\to 0}(u*(\rho_\varepsilon+\sigma_\varepsilon))(v*(\rho_\varepsilon+\sigma_\varepsilon)) \\ &= \underbrace{\lim_{\varepsilon\to 0}(u*\rho_\varepsilon)(v*\rho_\varepsilon)}_{=[uv]} + \lim_{\varepsilon\to 0}((u*\rho_\varepsilon)(v*\sigma_\varepsilon)+(u*\sigma_\varepsilon)(v*\rho_\varepsilon)) + \underbrace{\lim_{\varepsilon\to 0}(u*\sigma_\varepsilon)(v*\sigma_\varepsilon)}_{=[uv]}, \end{aligned}$$

which yields $\lim_{\varepsilon\to 0}((u*\rho_\varepsilon)(v*\sigma_\varepsilon)+(u*\sigma_\varepsilon)(v*\rho_\varepsilon)) = 2[uv]$. This is (5.7), hence (5.7) $\Leftrightarrow$ (i) and thus we are done. ☐

As announced in the beginning of this chapter we now consider other classes of mollifiers. To be more precise, we are going to discuss products obtained by regularization with the smaller class of model delta nets. This gives us a more general product of distributions, which is obtained in similar ways as the strict product.
To begin with, recall model delta nets from Definition 2.2(b). Let $\varphi \in \mathcal{D}$, then a model delta net consists of scaled test functions $(\varphi_\varepsilon)_\varepsilon \in \mathcal{D}$, with the following properties for all $\varepsilon > 0$:

- $\int_{\mathbb{R}^n} \varphi_\varepsilon(x)dx = 1$ and
- $\mathrm{supp}(\varphi_\varepsilon) \subseteq \overline{B_\varepsilon}$.

Definition 5.6. *(Model products)*
For $u,\ v \in \mathcal{D}'$ we define similarly to Definition 5.1 *model products* of u and v as the following limits, if they exist in $\mathcal{D}'$ for all model delta nets $(\varphi_\varepsilon)_\varepsilon$ and $(\psi_\varepsilon)_\varepsilon$:

$$u\cdot[v] := \lim_{\varepsilon\to 0} u\cdot(v*\varphi_\varepsilon) \tag{5.8}$$

$$[u]\cdot v := \lim_{\varepsilon\to 0}(u*\varphi_\varepsilon)\cdot v \tag{5.9}$$

$$[u]\cdot[v] := \lim_{\varepsilon\to 0}(u*\varphi_\varepsilon)\cdot(v*\psi_\varepsilon) \tag{5.10}$$

$$[u\cdot v] := \lim_{\varepsilon\to 0}(u*\varphi_\varepsilon)\cdot(v*\varphi_\varepsilon). \tag{5.11}$$

Observe that in contrast to Definition 5.1 we require the existence of the above limits independently of the chosen model delta nets, since for two given model delta nets, the interlaced net does not need be a model delta net again. Hence, we are not able to give a similar argument as in Remark 5.2(iv), which implied that the strict products are always welldefined.

Remark 5.7. *(On Definition 5.6)*

(i) Since every model delta net is also a strict delta net, the existence of a strict product implies the existence of a model product and since the converse obviously does not hold, model products are more general than strict products.

(ii) Similarly to Remark 5.2(ii) also the model products are not changed, if we require the existence of the limits only for all nonnegative model delta nets. To see this, assume for instance (5.11) exists for all nonnegative model delta nets. Moreover, if $(\chi_\varepsilon)_\varepsilon \in \mathcal{D}$ with $\chi_\varepsilon \geq 0$ and $\int \chi_\varepsilon(x)dx = c$ we have $\lim_{\varepsilon\to 0}(u * \chi_\varepsilon)(v * \chi_\varepsilon) = c^2[uv]$. In particular, for any model delta net $(\varphi_\varepsilon)_\varepsilon$ we can choose a nonnegative model delta net $(\chi_\varepsilon)_\varepsilon$ such that $\varphi + \chi \geq 0$. Thus, we obtain

$$\begin{aligned}(u * \varphi_\varepsilon)(v * \varphi_\varepsilon) &= 2(u * \chi_\varepsilon)(v * \chi_\varepsilon) + 2(u * (\varphi_\varepsilon + \chi_\varepsilon))(v * (\varphi_\varepsilon + \chi_\varepsilon)) \\ &\quad - (u * (\varphi_\varepsilon + 2\chi_\varepsilon))(v * (\varphi_\varepsilon + 2\chi_\varepsilon)) \\ &\quad \to (2c^2 + 2(1+c)^2 - (1+2c)^2)[uv] = [uv], \quad \text{as } \varepsilon \to 0.\end{aligned}$$

Hence, the limit exists for all model delta nets.

We aim at stating an analogous result to Theorem (5.3) about the relationship of model products. However, we first need to establish a notion introduced by Lojasiewicz in [20], which will then enable us to formulate the theorem.

Definition 5.8. *(Point-values of distributions)*
A distribution $u \in \mathcal{D}'$ has the *point-value* $c \in \mathbb{C}$ *in the sense of Lojasiewicz* at a point $x_0 \in \mathbb{R}^n$, if

$$\lim_{\varepsilon\to 0}\langle u(x_0 + \varepsilon x), \varphi(x)\rangle = c \int \varphi(x)dx, \quad \forall \varphi \in \mathcal{D}.$$

Observe, that this is the same as

$$\lim_{\varepsilon\to 0}\langle u(x), \varphi_\varepsilon(x - x_0)\rangle = c,$$

for all model delta nets $(\varphi_\varepsilon)_\varepsilon$.

Example 5.9. (i) $\delta(x)$ and $H(x)$ do not have point-values at 0.

(ii) Every $f \in \mathcal{C}$ has point-values, but not conversely. To see the latter, we show that the distribution

$$v_r(x) := \sum_{m=1}^{\infty} \frac{1}{m^r} \delta\left(x - \frac{1}{m}\right) \quad (r > 1)$$

has a point-value at 0 in the sense of Lojasiewicz iff $r > 2$.

First we prove that v_r has the point-value 0 at 0, if $r > 2$. W.l.o.g. let $\varphi \in \mathcal{D}$ with $\mathrm{supp}(\varphi) \subseteq [-1, 1]$. Then for $\varepsilon > 0$ we have $-\frac{1}{\varepsilon m} \in \mathrm{supp}(\varphi) \Leftrightarrow \frac{1}{\varepsilon m} < 1 \Leftrightarrow m > \frac{1}{\varepsilon}$ hence we obtain

$$\begin{aligned} \langle v_r, \varphi_\varepsilon \rangle &= \sum_{m=1}^{\infty} \frac{1}{m^r \varepsilon} \varphi\left(-\frac{1}{m\varepsilon}\right) = \sum_{m \geq \lfloor \frac{1}{\varepsilon} \rfloor + 1} \frac{1}{m^r \varepsilon} \varphi\left(-\frac{1}{m\varepsilon}\right) \\ &\leq \|\varphi\|_\infty \sum_{m \geq \lfloor \frac{1}{\varepsilon} \rfloor + 1} \frac{1}{m^r \varepsilon} \leq \|\varphi\|_\infty \sum_{m \geq \lfloor \frac{1}{\varepsilon} \rfloor + 1} \frac{1}{m^{r-1}} < \infty. \end{aligned}$$

Now suppose $r = 2 + \eta$ with $\eta > 0$ then

$$\langle v_r, \varphi_\varepsilon \rangle \leq \|\varphi\|_\infty \sum_{m \geq \lfloor \frac{1}{\varepsilon} \rfloor + 1} \frac{1}{m^{1+\eta}} \leq \|\varphi\|_\infty \varepsilon^{\frac{\eta}{2}} \sum_{m \geq \lfloor \frac{1}{\varepsilon} \rfloor + 1} \frac{1}{m^{1+\frac{\eta}{2}}} \to 0, \quad \text{as } \varepsilon \to 0.$$

Next we want to show that v_r has no point-value at 0, if $1 < r \leq 2$. However, we only prove it for $r = 2$ explicitly as the other cases follow similarly. On the one hand, we can have a test function $\varphi \in \mathcal{D}$ with $\varphi \geq 0$ and $\varphi \equiv 1$ on $[-1, 1]$ such that by omitting terms with $0 \leq \varphi(-\frac{1}{m\varepsilon}) < 1$ we obtain

$$\langle v_r, \varphi_\varepsilon \rangle \geq \sum_{m \geq \lfloor \frac{1}{\varepsilon} \rfloor + 1} \frac{1}{m^2 \varepsilon}.$$

Choosing a subsequence ε_k with $\frac{1}{\varepsilon_k} \in \mathbb{N}$ (denoting the subsequence with ε again instead of ε_k) we can write

$$\begin{aligned} \langle v_r, \varphi_\varepsilon \rangle &\geq \frac{1}{\varepsilon} \sum_{k=1}^{\infty} \frac{1}{(\frac{1}{\varepsilon} + k)^2} = \frac{1}{\varepsilon} \sum_{k=1}^{\infty} \frac{\varepsilon^2}{(1 + k\varepsilon)^2} = \varepsilon \sum_{k=1}^{\infty} \frac{1}{(1 + k\varepsilon)^2} \\ &\geq \varepsilon \left(\frac{1}{(1+\varepsilon)^2} + \frac{1}{(1 + 2\varepsilon)^2} + \cdots + \frac{1}{(1 + \frac{1}{\varepsilon}\varepsilon)^2} \right) \\ &\geq \varepsilon \cdot \frac{1}{\varepsilon} \cdot \frac{1}{4} = \frac{1}{4} \nrightarrow 0. \end{aligned}$$

On the other hand a test function $\varphi \in \mathcal{D}$ with $\mathrm{supp}(\varphi) \subseteq (0, \infty)$ gives $\varphi(-\frac{1}{m\varepsilon}) = 0$

for all m and ε, so $\langle v_r, \varphi_\varepsilon \rangle = 0$. Hence v_r has no point-value at 0, if $r = 2$.

Theorem 5.10. (The relationship between the model products in Definition 5.6)
Suppose u, $v \in \mathcal{D}'$, then we have that conditions (5.8) – (5.10) *are equivalent. Moreover, these products are equivalent to the following condition:*

$$\left.\begin{array}{l}\text{For all } \psi \in \mathcal{D} \text{ the distribution } (\psi u) * \check{v} \\ \text{has a point} - \text{value at } 0 \text{ in the sense of Lojasiewicz.}\end{array}\right\} \tag{5.12}$$

In this case we obtain for all $\psi \in \mathcal{D}$

$$\langle u[v], \psi \rangle = \langle [u]v, \psi \rangle = \langle [u][v], \psi \rangle = ((\psi u) * \check{v})(0).$$

Proof. The theorem can be proven similarly to Theorem 5.3. However, one has to apply some modifications due to the instability of the class of model delta nets w.r.t. interlacing and the usage of point-values in the sence of Lojasiewicz. In particular, new classes of mollifiers have to be introduced and further characterizations for the existence of point-values are needed. For details, we therefore refer to [17], [19] and [27]. □

Like in the case of strict products, the model product (5.11) is more general. Obviously we have that condition (5.10) implies condition (5.11). In addition we refer to [30] or [18], for an analogous statement to Theorem 5.5.
Examples for strict products will be given at the end of the next chapter, thus we conclude this chapter here.

6 Compatibility Results

In the preceding chapters we have investigated several ways to define products of distributions. Each of them was defined independently and with completely different methods. Now, in this chapter our final aim is, to compare these products with one another. In the end, we will be able to present a hierarchy describing their relationship. Note, that the desire to have more than one approach to distributional products, demands to investigate questions of consistency resp. compatibility. This, for instance, was brought up in Example 4.20(i). However, we already know about some relations between distributional products, e.g. the product (2.1) can be considered as a special case of the duality product (cf. Remark 3.10) and the product of two distributions with disjoint support (cf. Remark 2.10) is a particular case of the product of two distributions with their wave front set in favorable position (cf. Theorem 4.19). Moreover, we have already clarified in Remark 5.7(i) that the model product (cf. Definition 5.6 and Theorem 5.10) is a generalization of the strict product.

Basically, the results in this chapter are therefore *compatibility theorems*, concerning the remaining relations between the duality product (cf. Definition 3.8 or in case of Sobolev spaces Theorem 3.28), the Fourier product (cf. Definition 4.14) and the strict product (cf. Definition 5.1 and Theorem 5.3).

We start by connecting the duality product with the strict product.

Theorem 6.1. (Compatibility Theorem – #1)
Let the duality product in the Sobolev sense of Theorem 3.28 exist for two distributions u and $v \in \mathcal{D}'$. Then the strict product (5.1) – (5.3) *exists as well and coincides with it.*

Proof. Since by Theorem 5.3 the strict products (5.1) – (5.3) are equivalent, we only consider (5.1). Suppose $v \in W^{l,p}_{\text{loc}}$ for $l \in \mathbb{Z}$ and $1 \leq p \leq \infty$. Actually, without loss of generality we may assume $v \in W^{l,p}_{\text{loc}}$ for $1 \leq p < \infty$. This can be done due to the inclusion relationships between the localized Sobolev spaces, i.e., $W^{l,\infty}_{\text{loc}} \subset W^{l,p}_{\text{loc}}$, for all $p < \infty$. Now, regularized with a strict delta net $(\rho_\varepsilon)_\varepsilon$ we have $\lim_{\varepsilon \to 0} v * \rho_\varepsilon = v$ in $W^{l,p}_{\text{loc}}$ but also at least in $\mathcal{D}'$. Then for $u \in W^{m,q}_{\text{loc}}$ ($m \in \mathbb{Z}$, $1 \leq q \leq \infty$ with $l + m \geq 0$ and $\frac{1}{p} + \frac{1}{q} \leq 1$ according to

Theorem 3.28) the continuity of the multiplication map in Theorem 3.28 yields

$$u \cdot [v] = \lim_{\varepsilon \to 0} u(v * \rho_\varepsilon) = u(\lim_{\varepsilon \to 0} v * \rho_\varepsilon) = u \cdot v,$$

where the product on the right hand side is understood as a Sobolev product. □

Remark 6.2. *(On Theorem 6.1)*
The above compatibility result is of course not restricted to the Sobolev duality. Similar statements can be given for further products obtained by the duality method, whenever the arguments in the above proof are applicable.

Moving on to the next compatibility theorem, we aim at connecting the Fourier product to the strict product. Since the Fourier product is basically defined through the $\mathcal{S}'$-convolution, we first take a look at how the strict product ties in with the $\mathcal{S}'$-convolution. Moreover, the following theorem includes the general version of the convolution theorem for the $\mathcal{S}'$-convolution.

Theorem 6.3. (Compatibility Theorem – #2)
Suppose u, $v \in \mathcal{S}'$ and let their $\mathcal{S}'$-convolution exist. Then the strict products (5.1) – (5.3) *of $\hat{u}$ and $\hat{v}$ exist. In addition, the exchange formula holds, i.e.,*

$$\mathcal{F}(u * v) = \hat{u} \cdot [\hat{v}].$$

Proof. Recall from Definition 3.17 and 4.5(ii) that an element $T \in \mathcal{D}'_{L^1}$ has the form $T = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha T_{w_\alpha}$ with $w \in L^1(\mathbb{R}^n_m)$ for $m \in \mathbb{N}_0$. Since the Fourier transform maps $L^1 \to \mathcal{C}$ we observe that all $\mathcal{F}^{-1} T_{w_\alpha}$ and also $\mathcal{F}^{-1} T$ are continuous. Moreover, we have $\mathcal{F}^{-1}(D^\alpha T_{w_\alpha})(0) = 0$ if $|\alpha| \geq 1$, since derivatives translate to multiplication with polynomials under the Fourier transform. Hence, we can write

$$\langle T, 1 \rangle = \int T_{w_0}(x) dx = \mathcal{F}^{-1} T_{w_0}(0) = \mathcal{F}^{-1} T(0).$$

With this formula and by the definition of the $\mathcal{S}'$-convolution (cf. Definition 4.6) we obtain for all $\varphi \in \mathcal{S}$

$$\langle \mathcal{F}(u * v), \varphi \rangle = \langle u * v, \hat{\varphi} \rangle = \langle (\check{u} * \hat{\varphi}) v, 1 \rangle = \mathcal{F}^{-1}((\check{u} * \hat{\varphi}) v)(0).$$

Observe that by Proposition 4.2 $\check{u} * \hat{\varphi} \in \mathcal{O}_\mathrm{M}$. Thus, we are able to use the convolution theorem for the multiplication $\mathcal{O}_\mathrm{M} \cdot \mathcal{S}'$ (cf. [15, below Thm. 4.11.3, p. 424]). Together with applying the Fourier product this yields

$$\mathcal{F}^{-1}((\check{u} * \hat{\varphi})v) = \mathcal{F}^{-1}(\check{u} * \hat{\varphi}) * \mathcal{F}^{-1}v = (\hat{u} \cdot \varphi) * \check{\hat{v}}.$$

According to the observation in the beginning of the proof, we know that $(\hat{u}\varphi) * \check{\hat{v}}$ is continuous for every $\varphi \in \mathcal{S}$, thus by (5.5) and Theorem 5.3 the strict product $\hat{u}[\hat{v}]$ exists and we have

$$\langle \hat{u}[\hat{v}], \varphi \rangle = ((\hat{u}\varphi) * \check{\hat{v}})(0) = \mathcal{F}^{-1}((\check{u} * \hat{\varphi})v)(0) = \langle \mathcal{F}(u * v), \varphi \rangle.$$

□

Remark 6.4. *(On Theorem 6.3)*

(i) Replacing the Fourier transform $\mathcal{F}$ with the Fourier inverse $\mathcal{F}^{-1}$ in Theorem 6.3 (and its proof), the result still remains valid.

(ii) The exchange formula in Theorem 6.3 is not symmetric. In other words, if for two tempered distributions u, $v \in \mathcal{S}'$ the strict products (5.1) – (5.3) exist, then the $\mathcal{S}'$-convolution of $\hat{u}$ and $\hat{v}$ does not necessarily have to exist as well. This can be seen by taking the delta distribution at two different points and applying [24, Prop. 8] to the corresponding Fourier transforms.

We are now ready to prove the third and final compatibility theorem.

Theorem 6.5. (Compatibility Theorem – #3)
If the Fourier product of two distributions u, $v \in \mathcal{D}'$ exists, then their strict products (5.1) – (5.3) *exist as well and conincide with it.*

Proof. The proof basically relies on Theorem 6.3. Using the notation of Remark 4.16 the assumption says, that the $\mathcal{S}'$-convolution of $(f_x u)\hat{}$ and $(f_x v)\hat{}$ exists. Now, applying Theorem 6.3 to this $\mathcal{S}'$-convolution and using Remark 6.4(i) gives

$$\begin{aligned} w_x &= \mathcal{F}^{-1}((f_x u)\hat{} * (f_x v)\hat{}) \\ &= \mathcal{F}^{-1}(\mathcal{F}(f_x u)) \cdot [\mathcal{F}^{-1}(\mathcal{F}(f_x v))] = (f_x u) \cdot [(f_x v)]. \end{aligned}$$

Moreover, for any $\varphi \in \mathcal{D}(\Omega_x)$ and any strict delta net $(\rho_\varepsilon)_\varepsilon$ we have

$$\begin{aligned} \langle (f_x u)[(f_x v)], \varphi \rangle &= \lim_{\varepsilon \to 0} \langle f_x u, ((f_x v) * \rho_\varepsilon)\varphi \rangle \\ &= \lim_{\varepsilon \to 0} \langle u, (v * \rho_\varepsilon)\varphi \rangle, \end{aligned}$$

hence an appropriate partition of unity leaves us with

$$\langle u \cdot [v], \varphi \rangle = \lim_{\varepsilon \to 0} \langle u, (v * \rho_\varepsilon)\varphi \rangle.$$

Thus, the Fourier product and the strict product of u and v coincide globally, by partial associativity and the localization procedure in Remark 2.9(v). □

We end our discussion of compatibility among the distributional products by giving two examples. The first one proves that the converse of Theorem 6.5 does not hold true. The second, was already mentioned in chapter 5 and clarifies the relation between the strict products (5.1) − (5.3) and the strict product (5.4) and shows that the latter is not partially associative.

Example 6.6. (i) Suppose $u = \delta \in \mathcal{D}'(\mathbb{R})$ and $v \in L^\infty(\mathbb{R})$ continuous at 0 and discontinuous in every neighbourhood of 0. Then obviously condition (5.5) is satisfied, as for all $\varphi \in \mathcal{D}(\mathbb{R})$ we have

$$(\varphi\delta) * \check{v} = \varphi(0)\check{v}.$$

Thus by Theorem 5.3 the strict products (5.1) − (5.3) of δ and v exist. However, considering any $f \in \mathcal{D}(\mathbb{R})$ with $f(0) = 1$, the $\mathcal{S}'$-convolution of the Fourier transforms $\mathcal{F}(f\delta)$ and $\mathcal{F}(fv)$ does not exist. Indeed, assuming it does exist, we may write for any $\varphi \in \mathcal{S}$

$$(\underbrace{\mathcal{F}(f\delta)\check{}}_{=\mathcal{F}(f(0)\delta)=\hat{\delta}=1} * \varphi)\mathcal{F}(fv) = \underbrace{(1 * \varphi)}_{=\text{const.}}\mathcal{F}(fv) \in \mathcal{D}'_{L^1}.$$

But this implies by the observation in the beginning of the proof of Theorem 6.3, that fv is continuous everywhere, hence a contradiction. This shows that the strict product of two distributions can exist, while their Fourier product does not.

(ii) In chapter 5 we have already established that (5.3) ⇒ (5.4). Now, we want to show that the converse direction does not hold true and thus prove, that the strict product (5.4) indeed is more general. Along the way, we are going to show the existence of several strict products and in addition the lack of partial associativity of the strict product (5.4). For that matter, we first consider the two distributions δ_+ and $\delta_- \in \mathcal{D}'(\mathbb{R})$ given by

$$\delta_+(x) := \mathrm{vp}\left(\frac{1}{x}\right) - i\pi\delta(x)$$
$$\delta_-(x) := \mathrm{vp}\left(\frac{1}{x}\right) + i\pi\delta(x)$$

and then take a look at the Fourier transform of δ_+. This is

$$\hat{\delta}_+(\xi) = \hat{\text{vp}}\left(\frac{1}{\xi}\right) - i\pi\hat{\delta}(\xi) = -i\pi\,\text{sgn}(\xi) - i\pi = \left\{\begin{array}{rl} 0 & \xi < 0 \\ -2i\pi & \xi > 0 \end{array}\right\} = -2i\pi H(\xi).$$

Moreover, δ_+^2 exists as a Fourier product since

$$(H * H)(x) = \int_{-\infty}^{\infty} H(y)H(x-y)dy = \left\{\begin{array}{rl} 0 & x < 0 \\ \int_0^x dy & x > 0 \end{array}\right\} = x_+ \in \mathcal{S}'(\mathbb{R}),$$

where x_+ denotes the kink function. To calculate its value we observe that $\hat{\delta}_+^2 = -4\pi^2\xi_+$ and

$$\mathcal{F}(\partial_x\delta_+^2) = 2\pi i\xi\hat{\delta}_+ = 4\pi^2\xi H(\xi) = 4\pi^2\xi_+.$$

Hence, we have

$$\delta_+^2(x) = -\partial_x\delta_+(x) = \text{pv}\left(\frac{1}{x^2}\right) + i\pi\delta'(x),$$

where $\text{pv}(\frac{1}{x^2})$ denotes the principal value of $\frac{1}{x^2}$. Taking Theorem 6.5 into account, the above implies that the strict product $[\delta_+]\cdot[\delta_+]$, and thus also the strict product $[\delta_+\cdot\delta_+]$, exist. By similar conclusions we obtain the existence of the square of δ_- in the sense of the strict product (5.4). More precisely, we analogously have

$$\delta_-^2(x) = \text{pv}\left(\frac{1}{x^2}\right) - i\pi\delta'(x).$$

Finally, for every strict delta net $(\rho_\varepsilon)_\varepsilon$ we calculate the difference of the squares above

$$\begin{aligned}
2i\pi\delta' &= \text{pv}\left(\frac{1}{x^2}\right) - i\pi\delta' - \text{pv}\left(\frac{1}{x^2}\right) - i\pi\delta' \\
&= [\delta_-\cdot\delta_-] - [\delta_+\cdot\delta_+] \\
&= \lim_{\varepsilon\to 0}((\text{vp}\left(\frac{1}{x}\right) * \rho_\varepsilon + i\pi\rho_\varepsilon)(\text{vp}\left(\frac{1}{x}\right) * \rho_\varepsilon + i\pi\rho_\varepsilon)) \\
&- \lim_{\varepsilon\to 0}((\text{vp}\left(\frac{1}{x}\right) * \rho_\varepsilon - i\pi\rho_\varepsilon)(\text{vp}\left(\frac{1}{x}\right) * \rho_\varepsilon - i\pi\rho_\varepsilon)) \\
&= 4i\pi\lim_{\varepsilon\to 0}(\text{vp}\left(\frac{1}{x}\right) * \rho_\varepsilon)\rho_\varepsilon \\
&= 4i\pi[\text{vp}\left(\frac{1}{x}\right)\cdot\delta].
\end{aligned}$$

With other words, the strict product (5.4) of $\mathrm{vp}(\frac{1}{x})$ and $\delta(x)$ exists and we have the following formula

$$[\mathrm{vp}\left(\frac{1}{x}\right) \cdot \delta] = -\frac{1}{2}\delta'.$$

Now choosing $f = x \in \mathcal{C}^\infty(\mathbb{R})$ and considering Example 2.4 we have

$$x[\mathrm{vp}\left(\frac{1}{x}\right) \cdot \delta] \neq [(x\,\mathrm{vp}\left(\frac{1}{x}\right)) \cdot \delta] \neq [\mathrm{vp}\left(\frac{1}{x}\right) \cdot (x\delta)]$$

and thus see that the strict product (5.4) fails to be partially associative.
To end our chain of thoughts we observe that by condition (5.5) the strict product $\mathrm{vp}(\frac{1}{x}) \cdot [\delta(x)]$ does not exist. Indeed, for all $\varphi \in \mathcal{D}(\mathbb{R})$ we have

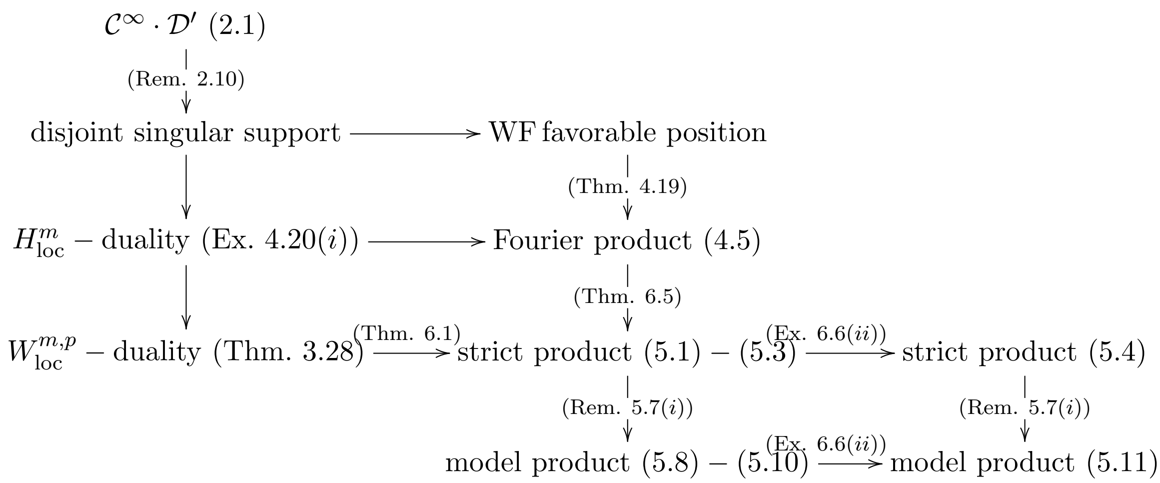
$$((\varphi\,\mathrm{vp}\left(\frac{1}{\cdot}\right)) * \check{\delta})(0) = (\varphi\,\mathrm{vp}\left(\frac{1}{\cdot}\right))(0) \notin L^\infty(\mathbb{R}).$$

Hence, we have proven that

$$(5.4) \not\Rightarrow (5.3).$$

Finally we are able to give an overview, over all distributional products and their relationships discussed here, in the form of the hierarchy given by M. Oberguggenberger in [22, section 7, p. 69], in which each product is considered as a consistent special case of its successor. Thus the model product (5.11) is the most genereal of the products presented here. Also observe, that none of the following arrows can be reversed (e.g., Example 6.6(i)).

Hierarchy for Products of Distributions



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