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Deformation Quantization and Fedosov Construction

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Introduction

Since Planck published his work on Black-body radiation in the last century, which can be regarded as the birth of quantum physics, many intellectual achievements have been accomplished. Enormous improvements have been made in theoretical as well as experimental physics. Nowadays, quantum mechanics is one of the most accurate theories which mankind can offer. Although we believe now that in comparison with classical mechanics quantum mechanics is the more fundamental theory, it is far from being complete and many quantum mechanical effects and predictions are very unusual and hard to accept. On the other side, classical physics is well understood and its theory and motivation seem much more natural. Therefore, it is not an exaggeration to say that the argumentations in classical mechanics agree with our way of thinking. This is also the reason, why we have a much more axiomatical way of introducing the concept of classical physics than the one of quantum mechanics. Disregarding some few exceptions, all quantum mechanical interpretation of mathmatical formalism requires the understanding of classical physics. Since we feel familiar with the concept of the classical physics, we can try, in some sense, to derive the quantum mechanics from the classical one. This process is called *quantization*.

We want to make this statement more precise. In classical mechanics, a physical system is described by classical observables, which are functions on the cotangent space of a manifold, while in quantum physics, this is done by giving a set of quantum observables, which are selfadjoint operators on a particular Hilbert space. The term "quantization" can be regarded as a process in which a classical observable is identified with a quantum observable. Of course, this mapping has to respect some mathematical structure, but it is more important that it encodes a physical meaning. In general, a canonical way of quantization does not exist. In classical mechanics as well as in axiomatic quantum field theory, the algebra of observables plays a fundamental role and the idea of quantization is to construct the algebra of quantum observables from a given algebra of classical observables. From a mathematical point of view, there will be many methods of quantization and it is impossible to decide which one should be preferred in general. Therefore, physical arguments are always necessary for choosing the "correct" quantization.

In this diploma thesis, we want to discuss the so-called *deformation quantization* which is one of many possibilities of quantization. One of the advantages of deformation quantization is the separation of algebraic and analytic methods. On the one side, many restrictions to a reasonable quantization can be found by merely studying the algebraic properties of the algebra of observables. On the other side, the algebraic method of deformation is still flexible enough to provide a general framework for studying quantization on any symplectic manifold.

The main idea, discussed in [BFF$^+$78], is to regard the commutative product of classical observables as the 0-th level of a power series of bilinear maps with a variable $\hbar$. This power series itself is a bilinear map and it is called the *star product*. For a star product, one constructs an algebra such that the multiplication on it is given by said star product. One requires this algebra to be large enough to embed the set of quantum observables. The fact that one can recover the commutative product by decreasing the value of $\hbar$ is the mathemtical interpretation of the physical requirement of the existence of a classical limit. Nevertheless, one should mention that the star product is not unique. The algebraic reason for the existence of different star products is the possiblity of choosing so-called orderings. If $\mathbb{R}^{2n}$ is the manifold to be quantized, by choosing a particular ordering, called Weyl ordering, and by applying the method described above one can construct a noncommutative product on the algebra of observables. This star product is usually called the *Weyl star product*.

The mathematical theory behind the deformation quantization is called *deformation theory*. Algebraic deformation theory was invented mostly by Gerstenhaber. Since the publication of his papers [Ger64], [Ger66], [Ger68], [Ger74] and [HG88], deformation theory has developed rapidly and became extremely successful. On the physical side, Weyl and Moyal were the pioneers on this relative new field of research, while Bayen et al. used the concept of deformation theory in quantum physics in a systematic way, [BFF$^+$77] and [Lic].

Although the existence of star products was known for simple manifolds such as $\mathbb{R}^{2n}$, for many years it was unclear whether a general symplectic manifold or a Poisson manifold admits a star product. The first existence proof for a star product on symplectic manifold was given by De Wilde and Lecomte [WL83a]. In [Kon] Kontsevich conjectured that the same statement is even true for Poisson manifolds. Later, he gave its proof in [Kon97], also see [Kon03].

Meanwhile, one can verify the existence of star products on symplectic manifolds in many different ways. In this work we will present a proof first given by Fedosov (see [Fed94], [Fed85], [Fed86] and [Fed89]) in the last chapter. His proof has the advantage to be relatively elementary and very geometric.

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Chapter 1

Quantization

1.1 Classical Mechanics versus Quantum Mechanics

In this section we compare classical mechanics with quantum mechanics as well as discuss the properties they share with each other and their differences. In the following section we will regard the algebra of observables as a fundamental aspect of physics and we will introduce other terms such as states as derived objects. Later we will use the results of this section to give a precise definition of the term "quantization".

1.1.1 Classical Mechanics

Observables: In classical mechanics, the set of observables is a subset of $C^\infty(M, \mathbb{C})$, the Poisson *-algebra of complex functions on a Poisson manifold (M, π). The Poisson bracket is defined by the Poisson structure π and the $*$-involution is given by complex conjugation. A function f in $C^\infty(M, \mathbb{C})$ is called a *classical observable* if it is hermitian which in this case is equivalent to requiring that the function f is real, i.e. $f = \overline{f}$. Of course, not every function in $C^\infty(M, \mathbb{C})$ admits a physical interpretation, therefore, one often works with a particular Poisson $*$-subalgebra of $C^\infty(M, \mathbb{C})$. The choice of subalgebras is not given a priori, but depends on additional structures of the system of interest. In the following, the algebra of classical observables will often be denoted by $\mathcal{A}_C$

States: For a $*$-algebra $\mathcal{A}_C$ with unit e, a *state* is a linear functional $\psi\colon \mathcal{A}_C \to \mathbb{C}$ such that

1. ψ is positive, i.e. $\psi(x^* x) \geq 0$ for every element $x \in \mathcal{A}_C$ and
2. $\psi(e) = 1$.

According to the Riesz representation theorem, for every positive linear functional $\psi\colon C^\infty(M, \mathbb{C}) \to \mathbb{C}$, there is a unique regular Borel measure μ on M such that

$$\psi(f) = \int_M f d\mu,$$

for every f in $C^\infty(M,\mathbb{C})$. Therefore, we see that a state is nothing but a positive regular Borel measure on the Poisson manifold M. We call ψ a pure state if the unique regular Borel measure μ associated to ψ is the Dirac measure δ_x for $x \in M$ and the state will be denoted by ψ_x.

Expectation Value and Variance: The *expectation value* $E_\psi(f)$ of an observable f in the state ψ is given by

$$E_\psi(f) := \psi(f) = \int_M f d\mu.$$

In particular, we see that if ψ_x is a pure state, then we have $E_{\psi_x}(f) = \int_M f d\delta_x = f(x)$ for $x \in M$. The *variance* V_ψ of a state ψ is defined to be

$$V_\psi(f) = E_\psi(f^2) - E_\psi(f)^2$$

and for a pure state ψ_x we have $V_\psi(f) = f(x)^2 - f(x)^2 = 0$.

Dynamics of the System: The dynamics of a system is given by the time evolution of observables. Since the algebra structure has to be time independent, the time evolution has to be a $*$-algebra automorphism. More precisely, the automorphism is the flow $\Phi_t^{X_H}$ of the vector field X_H, called the Hamilton vector field associated to a Hamilton function $H \in \mathcal{C}^\infty(M)$. The time evolution is given by the evaluation of $\Phi_t^{X_H}$ on functions in $\mathcal{C}^\infty(M)$:

$$f_t = \Phi_t^{X_H *}(f_0),\ \forall f_0 \in \mathcal{C}^\infty(M).$$

The flow forms a one parameter subgroup of diffeomorphisms and is determined by two initial values: $\frac{d}{dt}\Phi_t^{X_H}(p) = X_H(p),\ \forall p \in M$ and $\Phi_0^{X_H} = \mathrm{id}$. Therefore, the infinitesimal dynamics is described by the Hamiltonian vector field X_H. Since the Poisson bracket satisfies the equation $\{f,g\} = X_g(f)$, this is equivalent to saying that the infinitesimal dynamics is given by the derivation $\{-,H\}$:

$$\frac{d}{dt} f_t = \{f_t, H\},$$

which is called the Hamiltonian equation of motion.

1.1.2 Quantum Mechanics

Observables: For a Hilbert space $\mathfrak{H}$, the set of operators defined on a dense subset of $\mathfrak{H}$ is a $*$-algebra, if the involution is given by adjunction. If $\mathcal{A}_Q(\mathfrak{H})$ denotes this $*$-algebra, then a *quantum observable* is defined to be a selfadjoint element in $\mathcal{A}_Q(\mathfrak{H})$. Compared to the case of classical mechanics, the algebra $\mathcal{A}_Q(\mathfrak{H})$ is usually noncommutative.

States: The pure states consist of elements of the projective Hilbert space $\mathbb{P}$ and the mixed states are given by density matrices ω, i.e. self-adjoint, positive semi-definite operators of trace one. If ψ denotes a pure state, then there is an associated density matrix $\omega_\psi := \frac{|\psi\rangle\langle\psi|}{\langle\psi,\psi\rangle}$. Therefore, pure states are in particular special states. One should be aware that not every pure state or mixed state admits a physical interpretation, because they do not need to lie in the intersection of the domains of physical observables.

Expectation Value and Variance: For a state ψ and an observable A, the expectation value of A in ψ is defined to be

$$E_\psi(A) := \mathrm{tr}(\psi A).$$

This implies that the expectation value of a pure state is given by $\frac{\langle\psi|A|\psi\rangle}{\langle\psi,\psi\rangle}$. As in classical mechanics, the map $A \mapsto \mathrm{tr}(\psi A)$ is a positive functional.

Dynamics of the System: In quantum physics, the Heisenberg picture corresponds to the classical case, because the dynamics is controlled by the time evolution of observables. In particular, this means the infinitesimal dynamics is controlled by a selfadjoint operator H, called Hamilton operator, and is given by a formula called the Heisenberg equation of motion:

$$\frac{d}{dt}A_t = \frac{i}{\hbar}[H, A_t], \ \forall A_0 \in \mathcal{A}_Q.$$

Contrary to classical mechanics, the Heisenberg equation of motion is an inner derivation. Again, there exists an integrated version of the equation of motion and it forms a one parameter group consisting of unitary operators. For a time independent Hamilton operator, we have

$$A_t = e^{\frac{it}{\hbar}H} A_0 e^{-\frac{it}{\hbar}H}.$$

1.1.3 Conclusion

The comparison presented here reveals that the major difference between the classical and the quantum mechanics is the structure of the algebra of observables. In the classical case the algebra is commutative and admits an additional structure given by the Poisson bracket. This Lie bracket, which respects the commutative multiplication on the algebra, comes from the Poisson structure of a given manifold M. The algebra of quantum observables on the other side is noncommutative, hence admits a canonical nontrivial Lie bracket, the commutator. This difference will play an essential role in quantization.

1.2 Quantization

Loosely speaking, quantization is the attempt to use classical theories as hints and to guess a quantum theory which describes nature more accurately. In this sense, we need this process called quantization just because we could not construct a quantum theory a priori, as we did in the classical case. As we have seen above, the algebra of observables is a fundamental concept in both the classical and the quantum theory case, while the states can be derived from the observables. Therefore, in the following *quantization* will denote the procedure of constructing the algebra of quantum observables $\mathcal{A}_Q$ from the algebra of classical observables $\mathcal{A}_C$.

For our purpose, it is more convenient to have an algebraic description of the quantum observables instead of an analytic one. This means that we interpret a quantum observable as a selfadjoint object in a unitary $*$-algebra, i.e. an algebra with a unit and an involution denoted by $*$. We refer the reader to [BR79] for mathematical properties of $*$-algebras. One can recover the analytic description of quantum observables by a representation of the $*$-algebra on a Hilbert space. In the deformation quantization one separates the algebraic from the analytic methods. Probably, the most important algebraic property of the algebra of quantum observables is its noncommutativity. The so-called canonical commutation relation is a well-known fact in quantum physics and states that the value of the commutator of the position operator Q and the momentum operator P of a point particle is given by

$$[Q, P] = i\hbar.$$

Since we are interested in algebraic properties, we will not discuss the analytic conditions of this equation and simply assume that the domain is the algebra of smooth functions with compact support. The equation abouve implies in particular that every commutator of polynomials or functions in P and Q vanishs, if the value of the physical constant $\hbar$ is zero. We will regard the canoncial commutation relation as a fundamental fact of nature and we interpret the noncommutativity of the algebra of quantum observables as a consequence of nonvanishing value of $\hbar$. Although the description of nature by quantum physics is much more accurate, classical physics also provide acceptable results for macroscopic systems. This fact allows us to regard classical physics as an approximation of the quantum mechanics and the process of approximation is called *classical limit*. We now want to discuss the meaning of the existence of such a limiting process in an algebraic context. As we have seen above, the only crucial difference between classical and quantum theory is the noncommutativity of the algebra of the quantum observables. Therefore, the mathematical interpretation of a classical limit should be the process of decreasing $\hbar$. However, since $\hbar$ admits a physical meaning, the numerical value depends on its physical dimension. One should be aware that $\hbar \to 0$ means that its numerical value becomes neglectable compared to values of other quantities of the same physical dimension. Obviously, that if the physical meaning of $\hbar$ is decreasing, then the effect of noncommutativity of observables on measurements is also decreasing simultaneously and the predictions by quantum theory can be approximated by those of the classial theory.

The classical limit is not only a set theoretical limiting process, but should also preserve the physical meaning of observables as well as respect algebraic structures. These requirements are expressed in the *correspondence principle*.

1. Physical aspects of the correspondence principle:

   The physical interpretation of a quantum observable - hence an element in $\mathcal{A}_Q$ - is given by its classical limit, i.e. a classical observable, which is an element in $\mathcal{A}_C$. For example, just knowing what the $*$-algebra $\mathcal{A}_Q$ is, becomes meaningless if one does not know which element corresponds to the momentum operator or the Hamilton operator.

2. Mathematical aspects of the correspondence principle:

   Since $\mathcal{A}_C$ is a commutative $*$-algebra and $\mathcal{A}_Q$ is not, the correspondence cannot be an isomorphism of $*$-algebras, so we impose the following conditions: For $X, Y \in \mathcal{A}_Q$ and $x, y \in \mathcal{A}_C$

   (a) $aX + bY \rightsquigarrow ax + by$

   (b) $XY \rightsquigarrow xy$

   (c) $X^* \rightsquigarrow x^*$

This requirement should indicate that describing nature by the algebra of classical observables is not totally wrong (it has the correct structure), but the derived theory is not that accurate, because the elements are just the classical limit of the "real" operators.

Moreover, the correspondence principle requires that the dynamics of a quantum system corresponds to the dynamics of a classical system. In particular, the infinitesmal interpretation should hold:

$$\frac{1}{i\hbar}[-,H] \rightsquigarrow \{-,H\},$$

where H denotes the Hamilton operator. Therefore, it is reasonable to require for all $X,Y \in \mathcal{A}_Q$ and all $x,y \in \mathcal{A}_C$

$$\frac{1}{i\hbar}[X,Y] \rightsquigarrow \{x,y\}.$$

Remark 1.2.1. At moment, the meaning of the correspondence, indicated by $\rightsquigarrow$, is still very vague. As mentioned, due to the noncommutativity of quantum observables it cannot be an algebra isomomorphism. Nevertheless, one can hope that a Lie algebra isomorphism can satisfy all these requirements. As we will see at the end of this chapter, even this is not possible. The right interpretation of $\rightsquigarrow$ is given by the deformation quantization.

The uncertainty principle is a consequence of the noncommutativity of the algebra $\mathcal{A}_Q$ and is controlled by the value of $\hbar$. By applying the classical limit, $\hbar$ goes to zero and the product becomes commutative.

Quantization is defined to be the inverse process of the classical limit $\rightsquigarrow$, i.e. one tries to reconstruct the quantum system if the classical system is known.

1.3 Example

If the manifold is $\mathbb{R}^n$, we can identify its cotangent space with $\mathbb{R}^{2n}$. One possibility of quantization is called the "canonical quantization". This term is badly choosen, because one has to choose an ordering, but it is often used in the literature. If the functions q^k and p_k, for $k = 1,...,n$, denote the coordinates of the cotangent space, then the quantization is a map given by

$$q^k \mapsto Q^k \text{ for } 1 \leq k \leq n \text{ and } p_k \mapsto P_k \text{ for } n+1 \leq k \leq 2n.$$

Here, Q^k is called the position operator and is defined by

$$Q^k\colon\ f \mapsto q^k f,$$

while P_k is called the momentum operator and it is defined by

$$P_k\colon\ f \mapsto -i\hbar\frac{\partial}{\partial q^k}f.$$

By an easy calculation, one sees that these operators satisfy the commutation relation

$$[Q^k, P_l] = i\hbar\delta^k_l$$

and hence also the last correspondence principle $\frac{1}{i\hbar}[Q^k, P_l] = \delta_l^k \rightsquigarrow \{q^k, p_l\} = \delta_l^k$. Of course, one has to consider the domains of the involved operators and make sure that the equations above hold on these domains. Since we only want to use the analytical results without proving them, we refer the reader to [BB93], [RS72] and [Thi94]. The right choice of domain will be $\mathcal{C}^\infty_{cp}(\mathbb{R}^n)$, the set of smooth functions on $\mathbb{R}^n$ with compact support. On this domain, all the equations above are defined and the operators Q^k and P_k satisfy

$$\langle f, Q^k \rangle = \langle Q^k f, g\rangle \text{ and } \langle f, P_k\rangle = \langle P_k f, g\rangle,$$

where the inner product is defined for all functions by

$$\langle f, g\rangle = \int \overline{f} g d^n q, f, g \in \mathcal{C}^\infty_{cp}(\mathbb{R}^n).$$

Since we already know the quantization of the functions q^k and p_k, we can extend it to the subalgebra $\mathrm{Pol}(\mathbb{R}^{2n})$ in $\mathcal{C}^\infty(\mathbb{R}^{2n})$ which is generated by the functions q^k and p_k, $k = 1, ..., n$. By the first requirement of the correspondence principle the extension should be linear, hence, one only needs to define the quantization of monomials. Because of the noncommutativity of the position and momentum operators, one has to choose an ordering. We will discuss different aspects of ordering in a later chapter.

1.4 Groenewold-van Hove Property

In this and the following sections, we want to explain the reason why a Lie algebra isomorphism satisfying the conditions of the correspondence principle cannot exist. This fact is a result of a mathematical theorem by Groenewold and van Hove. First we recall some mathematical terminologies we need for defining the Groenewold-van Hove property.

Definition 1.4.1. An associative algebra A over a field k is a k-vector space with a bilinear map μ: $A \otimes_k A \to A$, called multiplication, satisfying the property:

$$\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu) : A \otimes_k A \otimes_k A \to A.$$

If we want to emphasize the multiplication map, we write: (A, μ).

The associative algebra A is commutative, if

$$\mu \circ \tau = \mu, \ \forall a, b \in A,$$

where τ: $A \otimes_k A \to A \otimes_k A$ is the flipping map, i.e. $\tau(a, b) = (b, a)$

Definition 1.4.2. Let $\mathfrak{g}$ be a k-vector space. We say $\mathfrak{g}$ is a *Lie algebra*, if it is equipped with a Lie bracket, i.e. a bilinear map $[-,-]$: $\mathfrak{g} \otimes_k \mathfrak{g} \to \mathfrak{g}$ which satisfies in addition the following two properties:

1. (antisymmetry)
$$[a, b] = -[b, a],$$

2. (Jacobi identity)
$$[a,[b,c]] = [[a,b],c] + [b,[a,c]],$$
for every $a, b, c \in \mathfrak{g}$.

Definition 1.4.3. Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{gl}$ denotes the Lie algebra of the general linear group $\mathrm{Gl}(V)$ of a vector space V. A *representation* of $\mathfrak{g}$ is a Lie algebra homomorphism
$$\rho\colon\ \mathfrak{g} \to \mathfrak{gl}(V).$$
If there is no nontrivial invariant subspaces, then the representation is called *irreducible.*

Since the Lie bracket of $\mathfrak{gl}(V)$ is given by the commutator, being a Lie algebra homomorphism means the map ρ satisfies
$$\rho([a,b]) = [\rho(a), \rho(b)],$$

Definition 1.4.4 (Groenewold-van Hove property)**.** Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Lie subalgebra. We say that the pair $(\mathfrak{g}, \mathfrak{h})$ has the *Groenewold-van Hove property* if no faithful irreducible representation on $\mathfrak{h}$ can be extended to a representation on $\mathfrak{g}$.

1.5 Groenewold-van Hove Theorem

In this section we will present a mathematical reason why a canonical way of quantization cannot exist. We will follow the proof of the Groenewold-van Hove theorem given in [Wal07, Section 5.2] and discuss its consequences.

By defining the Lie bracket on the vector space $\mathcal{C}^\infty(\mathbb{R}^{2n})$ to be the Poisson bracket, the vector space $\mathcal{C}^\infty(\mathbb{R}^{2n})$ itself and its subvector space $\mathrm{Pol}(\mathbb{R}^{2n})$ become Lie algebras.

Theorem 1.5.1 (Groenewold-van Hove theorem)**.** *[Gro46][vH51] Let $\mathfrak{g}$ denote the Lie algebra $\mathrm{Pol}(\mathbb{R}^{2n})$ and let $\mathfrak{h}$ denote the Lie subalgebra generated by the functions $1, q^1, ..., q^n, p_1, ..., p_n$. Then the pair $(\mathfrak{g}, \mathfrak{h})$ has the Groenewold-van Hove property.*

Remark 1.5.2. Since the quantization provides an irreducible representation of the Lie subalgebra $\mathfrak{h}$, the Groenewold-van Hove theorem states that it is impossible to extend this quantization to the whole Lie algebra $\mathrm{Pol}(\mathbb{R}^{2n})$ in a canonical way.

Proof. We will give the proof for the case $n = 1$, the proof for $n > 1$ is similar. The main idea is to calculate the representation of quadratic and cubic terms, provided the conditions as stated in the theorem are satisfied. This can be done by a close examination of the Poisson bracket of quadratic and cubic terms. At the end of the proof, by using these results, we will write $\rho(q^2p^2)$ in two different ways, which will result in a contradiction. This fact implies that the assumption of the extensability of an irreducible representation was wrong.

Let V be a vector space and let $\mathfrak{gl}(V)$ be the Lie algebra of endomorphisms on V. By rescaling the Lie bracket of $\mathfrak{gl}(V)$, we can assume that the representation $\rho\colon\ \mathfrak{g} \to \mathfrak{gl}(V)$ satisfies the condition
$$i\hbar\rho(\{f,g\}) = [\rho(f), \rho(g)]$$
instead of the usual Lie algebra homomorphism property $\rho(\{f,g\}) = [\rho(f), \rho(g)]$.

Now, let ρ be an irreducible, faithful representation of $\mathfrak{h}$ on the vector space V. Since 1 commutes with every element in $\mathfrak{g}$, $\rho(1)$ must lie in the center of $\mathfrak{gl}(V)$. Thus, $\rho(1) = c\,\mathrm{id}$ for some $c \in \mathbb{C}$. Moreover, faithfulness implies that $c \neq 0$. If we define two operators

$$Q := \frac{1}{\sqrt{c}}\rho(q) \text{ and } P := \frac{1}{\sqrt{c}}\rho(p),$$

it follows from our assumption above that

$$[Q, P] = \frac{1}{c}[\rho(q), \rho(p)] = \frac{i\hbar}{c}\rho(\{q, p\}) = i\hbar.$$

We assume that there exists an extension of ρ to the whole Lie algebra $\mathfrak{g}$, which is again denoted by ρ. Using the Leibniz rule for Poisson bracket and for the commutator we got

$$[\rho(q^2), P] = \frac{i\hbar}{\sqrt{c}}\rho(\{q^2, p\}) = \frac{i\hbar}{\sqrt{c}}\rho(2q) = 2i\hbar Q \text{ and } [Q^2, P] = 2i\hbar Q.$$

By changing the position of Q and P, the same approach shows

$$[\rho(p^2), Q] = -2i\hbar P \text{ and } [P^2, Q] = -2i\hbar P.$$

These results and the identity $[\rho(q^2), \rho(q)] = i\hbar\rho(\{q^2, q\}) = 0 = [\rho(p^2), \rho(p)]$ imply that the elements $\rho(q^2) - Q^2$ and $\rho(p^2) - P^2$ commute with P and Q. Hence, by the irreducibility of ρ, we have

$$\rho(q^2) = Q^2 + c_q\,\mathrm{id} \text{ and } \rho(p^2) = P^2 + c_p\,\mathrm{id},$$

for some $c_q, c_p \in \mathbb{C}$. By Leibniz rule and by commutativity of the algebra $\mathrm{Pol}(\mathbb{R}^{2n})$, there are two identities of the Poisson bracket.

1. The identity $\{q^2, p^2\} = 4qp$ implies

$$\begin{aligned}\rho(qp) = \rho(\{q^2, p^2\}) &= \frac{1}{4i\hbar}[\rho(q^2), \rho(p^2)] = \frac{1}{4i\hbar}[Q^2 + c_q\,\mathrm{id}, P^2 + c_p\,\mathrm{id}] \\ &= \frac{1}{4i\hbar}[Q^2, P^2] = \frac{1}{4i\hbar}([Q^2, P]P + P[Q, P^2]) = \frac{1}{2}(QP + PQ).\end{aligned}$$

2. While the identity $\{pq, p^2\} = 2p^2$ and the previous result imply

$$\begin{aligned}2(P^2 + c_p) = 2\rho(p^2) &= \frac{1}{i\hbar}[\rho(pq), \rho(p^2)] = \frac{1}{i\hbar}[\frac{1}{2}(QP + PQ), P^2 + c_q] \\ &= \frac{1}{2i\hbar}[QP + PQ, P^2] = \frac{1}{2i\hbar}([Q, P^2]P + P[Q, P^2]) = 2P^2.\end{aligned}$$

From these calculations we infer that $c_p = 0$ and similarly $c_q = 0$. As a conclusion, we have for quadratic monomials the identities:

$$\rho(q^2) = Q^2,\ \rho(p^2) = P^2 \text{ and } \rho(qp) = \frac{1}{2}(QP + PQ).$$

Now we do the same procedure for cubic monomials. Again, by using the Leibniz rule for

Poisson bracket and for the commutator, we have

$$[\rho(q^3), P] = \frac{i\hbar}{\sqrt{c}}\rho(\{q^3, p\}) = \frac{i\hbar}{\sqrt{c}}\rho(3q^2) = \frac{3i\hbar}{\sqrt{c}}Q^2 \text{ and } [Q^3, P] = 3i\hbar Q^2.$$

Exchanging the position of Q and P, yields

$$[\rho(p^3), Q] = \frac{-3i\hbar}{\sqrt{c}}P^2 \text{ and } [P^3, Q] = -3i\hbar P^2.$$

The same arguments as in the qudratic case show that there exists $c'_q, c'_p \in \mathbb{C}$ such that

$$\sqrt{c}\rho(q^3) - Q^3 = c'_q \text{ and } \sqrt{c}\rho(p^3) - P^3 = c'_p.$$

Similar to the previous case, we will use two identities of the Poisson bracket to show that these constants are actually zero.

1. The identity $\{q^3, p^2\} = 6q^2p$ implies

$$\begin{aligned}\rho(q^2p) &= \frac{1}{6}\rho(\{q^3, p^2\}) = \frac{-i}{6\hbar}[\rho(q^3), \rho(p^2)] = \frac{-i}{6\hbar\sqrt{c}}[Q^3 + c'_q\,\mathrm{id}, P^2] \\ &= \frac{-i}{6\hbar\sqrt{c}}Q^2[Q, P^2] + [Q^2, P^2]Q = 2Q^2P + 2QPQ + 2PQ^2 = \frac{1}{2\sqrt{c}}(Q^2P + PQ^2).\end{aligned}$$

2. While the identity $\{qp, p^3\} = 3p^3$ implies

$$\begin{aligned}\frac{3}{\sqrt{c}}(P^3 + c'_p) = 2\rho(p^3) &= \frac{1}{i\hbar}[\rho(pq), \rho(p^3)] = \frac{-i}{\hbar\sqrt{c}}[\frac{1}{2}(QP + PQ), P^3 + c'_p] \\ &= \frac{1}{2i\hbar}[QP + PQ, P^3] = \frac{3}{\sqrt{c}}P^3.\end{aligned}$$

In conclusion, these calculations show that

$$\rho(q^2p) = \frac{1}{2\sqrt{c}}(Q^2P + PQ^2) \text{ and } c'_p = 0.$$

Similarly, we have

$$\rho(qp^2) = \frac{1}{2\sqrt{c}}(QP^2 + P^2Q) \text{ and } c'_q = 0.$$

For the final step, we calculate $\rho(q^2p^2)$ in two different ways.

1. On the one side, we have $\{q^3, p^3\} = 9q^2p^2$ which implies

$$9\rho(q^2p^2) = \frac{1}{i\hbar}[\rho(q^3), \rho(p^3)] = \frac{1}{ci\hbar}[Q^3, P^3].$$

2. On the other side, we have $\{q^2p, qp^2\} = 3q^2p^2$ which implies

$$9\rho(q^2p^2) = \frac{3}{i\hbar}[\rho(q^2p), \rho(qp^2)] = \frac{3}{ci\hbar}[\frac{1}{2}(Q^2P + PQ^2), \frac{1}{2}(QP^2 + P^2Q)].$$

The contradiction now arises from the fact that $[Q^3, P^3] - \frac{3}{4}[Q^2P + PQ^2, QP^2 + P^2Q] = 24\hbar$ which is nonzero. The verification of this equation is done by using commutation relation and Leibniz rule. Since it is a long but straightforward calculation, we will not present it here. $\square$

The important observation in the previous proof was that the representation of an element which lies in the center of the algebra has to be of the form $c\,\mathrm{id}$, for $c \in \mathbb{C}$. The same idea will be used in order to proof the following proposition.

Proposition 1.5.3. *There is no Lie algebra isomorphism of the form*

$$\Psi\colon (\mathrm{Pol}(\mathbb{R}^{2n}), \{-,-\}) \to (A, \frac{1}{i\hbar}[-,-]),$$

where A is an associative algebra and its Lie algebra structure is given by the commutator.

Proof. Once more, we only prove the case $n = 1$, the general case is proven similarly. Suppose, such a Lie algebra isomorphism Ψ exists, then the center of A is bijective to the center of $\mathrm{Pol}(\mathbb{R}^{2n})$, hence, the elements of the center have the form $c\Psi(1)$, for $c \in \mathbb{C}$. As in the proof of the Groenewold-van Hove theorem, we write $\Psi(1) = c\Psi(1)$, for $c \in \mathbb{C}\backslash\{0\}$. By the Lie algebra homomorphism property of Ψ and the commutation relation, we have

$$[\Psi(q), \Psi(p)] = i\hbar\Psi(\{q,p\}) = i\hbar c,$$
$$[\Psi(q), \Psi(f)] = i\hbar\Psi(\{q,f\}) = i\hbar\Psi(\frac{\partial f}{\partial p}).$$

Similarly, we have $[\Psi(p), \Psi(f)] = -i\hbar\Psi(\frac{\partial f}{\partial q})$. Let $a \in A$ be an element which lies in the center of the algebra. There exists an $f \in \mathrm{Pol}(\mathbb{R}^{2n})$ such that such that $\Psi(f) = a$. The observation above implies that f is a constant function, because it satisfies $\frac{\partial f}{\partial q} = 0 = \frac{\partial f}{\partial p}$. Hence, an element lies in the center of A if and only if it commutes with $\Psi(q)$ and $\Psi(p)$. From now on by applying the same arguments to quadratic and cubic terms as in the previous theorem proves the claim of this proposition. $\square$

In particular, the Groenewold-van Hove theorem 1.5.1 and proposition 1.5.3 squash our hopes of finding an algebra of quantum observables which is isomorphic to $\mathrm{Pol}(\mathbb{R}^{2n})$ as a Lie algebra and satisfies the correspondence principles. It is therefore impossible to extend the equation $\frac{1}{i\hbar}[X, Y] = \{x, y\}$ in a canoncial way. This fact forces us to choose an ordering which we are going to discuss excessively in a later chapter. For example it tells us what the quantization of the product qp, QP, PQ or $\frac{1}{2}(QP + PQ)$ should be. Since the commutator is controlled by $\hbar$, we have $QP = PQ + i\hbar$, which implies that the difference between the various orderings vanishes during the process of forming the classical limit. This observation leads us to the deformation quantization.

1.6 Deformation Quantization

In last section we mentioned many difficulties of quantization and now we want to show which of them can be solved by introducing a new concept, the *deformation quantization.* As has been mentioned before, the motivation for deformation quantization is the desire to separate the algebraic methods from the analytic ones. The algebraic approach allows us to formulate the concept of quantization in a most general way. Moreover, it is restrictive enough to realize what can and cannot be required.

We will introduce the mathematical theory of deformation quantization, which is called deformation theory, in the next chapter, but now we want to present some basis ideas of deformation quantization. One should be aware that the classical limit is always the idea behind all the considerations below.

One way to construct a noncommutative algebra out of a commutative algebra is to deform its product. In our case, the commutative algebra is $\mathcal{C}^\infty(M)$ and we denote its pointwise multiplication map by μ_0: $\mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$. If we embed this algebra into the algebra of formal power series $\mathcal{C}^\infty(M)[[t]]$, we can "deform" the multiplication map μ_0. Here, we use the parameter t instead of the physical constant $\hbar$ in order to derive the theory in a more general framework. The process of deformation means, we are looking for a bilinear map μ: $\mathcal{C}^\infty(M)[[t]] \times \mathcal{C}^\infty(M)[[t]] \to \mathcal{C}^\infty(M)[[t]]$, such that there exist bilinear maps μ_i for $i \geq 1$ satisfying

$$\mu(f,g) = \mu_0(f,g) + \sum_{i=1}^{\infty} t^i \mu_i(f,g), \ \forall f,g \in \mathcal{C}^\infty(M).$$

Since this map μ should be a deformed, noncommutative product, one has to require it to be associative on $\mathcal{C}^\infty(M)[[t]]$. It will turn out that this condition is difficult to be satisfied and it makes the construction of μ very complicate. We will often write $f*g$ for $\mu(f,g)$ and we call it the *star product* of f and g. The star product respects the requirements of classical limit, because if the parameter t is approaching zero, the deformed multiplication μ becomes the original product μ_0 on $\mathcal{C}^\infty(M)$.

Moreover, the definition of μ provides a precise interpretation of the correspondence principle. Namely, the map $\rightsquigarrow$ should be understood as a modulo operation. This means $\rightsquigarrow$ is an equality, if one neglects the higher orders of t, we have

1. $f * g = fg + \mathscr{O}(t)$
2. $\frac{1}{i\hbar}[f,g] = \{f,g\} + \mathscr{O}(t)$,

for all $f,g \in \mathcal{C}^\infty(M)$. Together with requirement $X^* \rightsquigarrow x^*$ we have

$$\overline{f * g} = \overline{g} * \overline{f},$$

because the involution is just complex conjugation in $\mathcal{C}^\infty(M)[[t]]$.

The physical interpretation of the star product is given by the identification $t = \hbar$. In this manner, the classical observable $f \in \mathcal{C}^\infty(M)$ is identified with the associated quantum observable $f \in \mathcal{C}^\infty(M)[[\hbar]]$. Therefore, the algebra of classical observables is embedded into the algebra of quantum observables. Of course, at this point it is necessary to think about convergence properties of the series $f * g = fg + \sum_{i=1}^{\infty} t^i \mu_i(f,g)$. It turns out, that it is impossible to require the convergence of all functions in $\mathcal{C}^\infty(M)$, but the star product of two functions may be convergent for some $*$-subalgebras.

In general, one starts with a physical system of interest and then one tries to find a $*$-subalgebra that includes all functions which have to be quantized. Then one finds a star product such that the product of two given functions in the subalgebra converges. Thus, the $*$-subalgebra as well as the star product on it depend directly on the physical input. Once the star product on a $*$-subalgebra is found, one recovers the usual quantum mechanical interpretation of the system by a representation of this $*$-subalgebra on a Hilbert space.

Chapter 2

Deformation Theory

Deformation theory is a vast field of research in mathematics. We will only need results of a specific algebraic branch of this theory, the deformation theory of algebras. The four papers of Gerstenhaber, [Ger64], [Ger66], [Ger68] and [Ger74], can be seen as the foundation of the algebraic deformation theory, but here, we will follow a more modern presentation of this subject which can be found in [DMZ09].

2.1 Algebraic Preliminary

Definition 2.1.1. A left module M over an associative k-algebra (A, μ_A) is a k-vector space M with a map $\mu^l_M\colon A \otimes_k M \to M$ satisfying the properties:

$$\mu^l_M \circ (\mu_A \otimes_k \mathrm{id}_M) = \mu^l_M \circ (\mathrm{id}_A \otimes_k \mu^l_M) : A \otimes_k A \otimes_k M \to M.$$

We sometimes write (M, μ^l_M).

A right module N over an associative k-algebra (A, μ_A) is a k-vector space N with a map $\mu^r_N\colon M \otimes_k A \to M$ satisfying the properties:

$$\mu^r_M \circ (\mathrm{id}_M \otimes_k \mu_A) = \mu^r_M \circ (\mu^r_M \otimes_k \mathrm{id}_A) : M \otimes_k A \otimes_k A \to M.$$

A bimodule M over an associative k-algebra (A, μ_A) is a module M which is both a left and a right module and its left multiplication commutes with its right multiplication:

$$\mu^l_M \circ (\mathrm{id}_A \otimes_k \mu^r_M) = \mu^r_M \circ (\mu^l_M \otimes_k \mathrm{id}_A) : A \otimes_k M \otimes_k A \to M.$$

Notation 2.1.2. We will assume from now on that our field k is a field of characteristic zero and denote the k-tensor product $\otimes$ simply by $\otimes_k$. Moreover, all algebras will be assumed to be commutative associative k-algebras unless noted otherwise.

Definition 2.1.3. Let k be a field. An *augmentation* of an algebra R which is an associative, commutative k-algebra with unit e is a homomorphism of k-algebras $\epsilon\colon R \to k$ such that $\epsilon \circ \alpha = id_k$, where α is the unique k-algebra homomorphism $k \to R$ given by $\alpha(1) = e$

Such an algebra R is called an *augmented algebra* and kernel of ϵ is called the *augmentation ideal* of R.

We will give some examples of augmented algebras which are important for the definition of the formal deformation (see 2.2.5) and we will see later that many algebras do not admit an augmentation.

Example 2.1.4. The algebra of formal power series $k[[t]]$ has a unit 1 and an augmentation ϵ: $k[[t]] \to k$ which is the projection on k:

$$\epsilon(\sum_k a_k t^k) := a_0.$$

Similarly, the algebra $k[t]$ of polynomials is augmented by $\epsilon(\sum_k^n a_k t^k) := a_0$. Since the field k is assumed to be of characteristic zero, ϵ is just the evaluation map at $t = 0$. For every $n \in \mathbb{N}$, the augmentation of $k[t]$ immediately induces an augmentation on the algebra of truncated polynomials $k[t]/(t^n)$.

Example 2.1.5. Every nontrivial field extension of k is an example of a k-algebra which does not admit an augmentation. If $k \hookrightarrow k'$ is a nontrivial field extension, then the augmentation ideal of an augmentation ϵ: $k' \to k$ is an ideal of the field k', which therefore has to be either 0 or all of k', whence the augmentation map ϵ is either injective or the zero-map. Both cases are impossible, because the field extension was nontrivial and the map ϵ has to satisfy the equation $\epsilon \circ \alpha = id_k$.

This example shows in particular that $\mathbb{C}$ cannot be an augmentation of the field of real numbers $\mathbb{R}$.

2.2 Deformation of Algebras

Definition 2.2.1. For a k-algebra R and a left R-module M, the *reduction* of M is defined to be the left k-module $k \otimes_R M$. We will denote the reduction of M by $\bar{M}$.

Definition 2.2.2. Let R be an augmented algebra and let A be an associative k-algebra. An *R-deformation* of A is a pair (B, φ), where B is an associative R-algebra and φ is an isomorphism of k-algebra: $\bar{B} \to A$.

Definition 2.2.3. Two R-deformations of A, (B, φ) and (B', φ') are said to be *equivalent*, if there exists an R-algebra isomorphism ϕ: $B \to B'$ which extends the k-algebra isomorphism $\varphi \circ {\varphi'}^{-1}$.

Remark 2.2.4. In addition to the definition of an R-deformation, we will also assume that the R-module B is a free R-module which implies that there exists an R-module isomorphism ψ: $B \to R \otimes A$. Therefore, the algebra A can be identified via ψ with the k-subspace $1 \otimes A$ in B and $A \otimes A$ can be identified with $(1 \otimes A) \otimes (1 \otimes A)$ in $B \otimes B$.

If we denote the multiplication in an R-deformation of a k-algebra A by μ', then the following diagram commutes:

$$\begin{tikzcd} B \otimes_R B \arrow[rr, "\mu"] \arrow[d, "\psi\otimes\psi"'] & & B \arrow[d, "\psi"] \\ (R\otimes A)\otimes_R (R\otimes A) \arrow[r, "\cong"] & R\otimes(A\otimes A) \arrow[r, "\mu'"] & (R\otimes A) \end{tikzcd} ,$$

where the homomorphism $(R \otimes A) \otimes_R (R \otimes A) \cong R \otimes (A \otimes A)$ is the canoncial R-algebra isomorphism.

This diagram shows that the multiplication μ' induces an R-algebra structure on $R \otimes A$, hence the multiplication in B is totally determined by its restriction to the subspace $A \otimes A$. Moreover, since an equivalence of deformations ϕ is an R-algebra homomorphism which extends a k-isomorphism, the same argument shows that ϕ is also determined by its restriction to A.

Given an R-deformation (B, φ) and a k-module isomorphism φ: $\bar{B} \to A$, the R-module isomorphism ψ: $B \to R \otimes A$ induces a k-isomorphism $\bar{\psi}$: $\bar{B} = k \otimes_R B \to k \otimes_R R \otimes A$. We have the following diagram:

$$\begin{tikzcd} \bar{B} \arrow[rr, "\bar{\psi}", "\cong"'] \arrow[dr, "\cong", "\varphi"'] & & k\otimes_R R\otimes A \arrow[dl, "\varphi'"] \\ & A & \end{tikzcd} ,$$

where the map φ' is induced by the two k-isomorphisms φ and $\bar{\psi}$.

If the map γ: $k \otimes_R R \otimes A \to A$ denotes the canonical isomorphism, then the diagram above can be extended to

$$\begin{tikzcd} \bar{B} \arrow[r, "\bar{\psi}"] \arrow[dr, "\varphi"'] & k\otimes_R R\otimes A \arrow[r, "\varphi''"] \arrow[d, "\varphi'"'] & k\otimes_R R\otimes A \arrow[dl, "\gamma"] \\ & A & \end{tikzcd}$$

The map φ'' is induced by the k-isomorphisms γ and φ' and can be extended to an R-algebra isomorphism. This observation shows that any R-deformation $(B, \varphi : \bar{B} \to A)$ is equivalent to $(R \otimes A, \gamma : k \otimes_R R \otimes A \to A)$. If we are only interested in equivalence classes of deformations, we can assume that our R-deformation B is the R-module $R \otimes A$ and we have the canonicial identification map γ.

Definition 2.2.5. A *formal deformation* is a deformation over the complete local augmented algebra $k[[t]]$.

In the remark above we saw that the multiplication μ in B is determined by the objects in A. For $a, b \in A$, we can write

$$\mu(a, b) = \sum_{n=0}^{\infty} t^n \mu_n(a, b),$$

where μ_n are k-bilinear maps from $A \otimes A$ to A. Since we identify A with the subspace $1 \otimes A$ of B, the map μ_0 is just the multiplication in A. By the definition of multiplication in $k[[t]]$, we see that the map μ is associative in B if and only if for every $k \in \mathbb{N}$ and every $a, b, c \in A$,

$$\sum_{n+m=k} \mu_n(\mu_m(a,b),c) = \sum_{n+m=k} \mu_n(a,\mu_m(b,c)).$$

Since we assumed that every deformation is of the form $(R \otimes A, \gamma)$, two deformations B and B' are equivalent if and only if there exists an isomorphism ψ in $\mathrm{Hom}_R(R \otimes A, R \otimes A)$ such that $\psi \circ \mu = \mu' \circ \psi \otimes \psi$. In case of a formal deformation, we have $\mathrm{Hom}_{k[[t]]}(k[[t]] \otimes A, k[[t]] \otimes A) \cong \{\psi = \mathrm{id}_A + t\psi_1 + t^2\psi_2 + t^3\psi_3 + ..., \ \psi_i \in \mathrm{Hom}(A,A)\}$.

We want to summarize this observation in the lemma below.

Lemma 2.2.6. *If (B,μ) and (B',μ') are two formal deformations of an associative k-algebra A, then they are equivalent if and only if there exists a map $\psi = \mathrm{id}_A + t\psi_1 + t^2\psi_2 + t^3\psi_3 + ...$, where ψ_i are maps in $\mathrm{Hom}(A,A)$, such that*

$$\psi \circ \mu = \mu'(\psi \otimes \psi).$$

Definition 2.2.7. A *formal n-deformation* or just *n-deformation* is a deformation over the local Artinian algebra $k[t]/(t^{n+1})$.

By the discussion above, the multiplication of an n-deformation of an associative k-algebra is given by the set $\{\mu_i \colon A \otimes A \to A | 1 \leq i \leq n\}$, where μ_i satisfy the condition $\sum_{i+j=k} \mu_i(\mu_j(a,b),c) = \sum_{i+j=k} \mu_i(a,\mu_j(b,c))$ for every $k \leq n$.

By the definition of n-deformation and lemma 2.2.6, we see that two n-deformations are equivalent if and only if there exists a family of maps $\psi_1, \psi_2, ..., \psi_n$ such that $\psi = \mathrm{id}_A + t\psi_1 + t^2\psi_2 + ... + t^n\psi_n$ maps one deformation to the other.

Lemma 2.2.8. *If μ and μ' are two deformations which are equivalent as n-deformations, then there is a deformation $\tilde{\mu}$ which equals μ' as n-deformations and is equivalent to μ.*

Proof. One defines the map ψ to be $\mathrm{id}_A + t\psi_1 + t^2\psi_2 + ... + t^n\psi_n + ...$, where $\psi_1, \psi_2, ..., \psi_n$ are given by the equivalence as n-deformations and the maps ψ_m are arbitrary for $m \geq n+1$. By definition, μ is equivalent to $\psi \circ \mu \circ (\psi^{-1} \otimes \psi^{-1}) =: \tilde{\mu}$ and by assumption, $\tilde{\mu}$ equals μ' as n-deformations. ☐

Definition 2.2.9. An *extension* of an n-deformation given by $\{\mu_i \colon A \otimes A \to A | 1 \leq i \leq n\}$ is an $(n+1)$-deformation given by $\{\mu'_i \colon A \otimes A \to A | \mu'_i = \mu_i$ for $1 \leq i \leq n\}$.

Lemma 2.2.8 says that if we are asking for existence of extensions of n-deformations, without loss of generality, we can assume that equivalences of deformations are already equal as n-deformations.

2.3 Hochschild Complex and Gerstenhaber Bracket

In order to understand the behaviour of the extensions of an n-deformation, we now introduce the concept of the Hochschild complex and the Hochschild cohomology.

Definition 2.3.1. Let A be an associative k-algebra and M an A-bimodule (see definition 2.1.1). The *Hochschild cochain complex* $C^*_H(A,M)$ is the cochain complex

$$0 \longrightarrow M \xrightarrow{{d_H}^0} C^1_H(A,M) \xrightarrow{{d_H}^1} \ldots \xrightarrow{{d_H}^{n-1}} C^n_H(A,M) \xrightarrow{{d_H}^n} \ldots ,$$

where $C^n_H(A,M) := \mathrm{Hom}(\bigotimes^n A, M)$. The differential or coboundary operator ${d_H}^n\colon C^n_H(A,M) \to C^{n+1}_H(A,M)$ is defined by:

$$\begin{aligned} {d_H}^n f(a_0 \otimes \ldots \otimes a_n) := a_0 f(a_1 \otimes \ldots \otimes a_n) &+ (-1)^{n-1} f(a_0 \otimes \ldots \otimes a_{n-1}) a_n \\ &+ \sum_{i=0}^{n-1} (-1)^{i+1} f(a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n). \end{aligned}$$

Definition 2.3.2. If $(C^*_H(A,M), d_H)$ is a Hochschild complex with A and M as in definition 2.3.1, then the *Hochschild cohomology* with coefficient in M is the cohomology module

$$\{H^n_H(A,M)\}_{\{n\in\mathbb{N}\}} := \frac{\ker {d_H}^n}{\mathrm{im}\, {d_H}^{n-1}}.$$

The Hochschild cohomology will allow us to answer the question whether an $(n+1)$-extension for a given n-deformation exists, but before we can present the answer, we have to introduce some more terminology.

Definition 2.3.3. A $\mathbb{Z}$-graded algebra A is a direct sum of algebras A^k, i.e.

$$A = \bigoplus_{k\in\mathbb{Z}} A^k,$$

with a multiplication μ: $A^k \otimes A^l \to A^{k+l}$, for every $k, l \in \mathbb{Z}$. An object a in A is said to be homogenous of degree k if it is an object in A^k. Its degree is often denoted by $|a|$.

A degree k linear map f between two graded algebras A and B is a linear map such that $f(a) \in B^{k+|a|}$ for every homogeneous object $a \in A$.

A prominent example of graded algebras are graded Lie algebras.

Definition 2.3.4. A *graded Lie algebra* $\mathfrak{g}$ is a $\mathbb{Z}$-graded vector space $\mathfrak{g}$ with a degree 0 bilinear map, called the *Lie bracket*:

$$[-,-]\colon \mathfrak{g}^i \otimes \mathfrak{g}^j \to \mathfrak{g}^{i+j},\ \forall i,j \in \mathbb{Z}$$

satisfying the following properties:

1. (graded antisymmetry)
$$[a,b] = -(-1)^{|a||b|}[b,a],$$
2. (graded Jacobi identity)
$$[a,[b,c]] = [[a,b],c] + (-1)^{|a||b|}[b,[a,c]].$$

Example 2.3.5. Given an associative $\mathbb{Z}$-algebra A, then a *graded commutator* $[-,-]$ is defined by

$$[a,b] = ab - (-1)^{|a||b|}ba.$$

The graded commutator is antisymmetric and satisfies the graded Jacobi identity, therefore the graded algebra $(A,[-,-])$ is a graded Lie algebra.

A Gerstenhaber algebra is a modified version of a graded Lie algebra and it turns out that the Hochschild cohomology, which is one of the most important algebraic subject in the deformation theory, is indeed a Gerstenhaber algebra. (see theorem 2.3.16.)

Definition 2.3.6. A *Gerstenhaber algebra* A is an associative, graded commutative, $\mathbb{Z}$-graded algebra with a bracket

$$[[-,-]]$$

of degree -1:

$$[[-,-]] : A^i \otimes A^j \to A^{i+j-1},\ \forall i,j \in \mathbb{Z}.$$

Moreover, for homogeneous elements $a,b,c \in A$, the bracket satisfies the following properties:

1. (antisymmetry)
$$[[a,b]] = -(-1)^{(|a|-1)(|b|-1)}[[b,a]],$$

2. (Jacobi identity)
$$[[a,[[b,c]]]] = [[[[a,b]],c]] + (-1)^{(|a|-1)(|b|-1)}[[b,[[a,c]]]],$$

3. (Leibniz rule)
$$[[a,bc]] = [[a,b]]c + (-1)^{(|a|-1)|b|}b[[a,c]].$$

Remark 2.3.7. If A is a Gerstenhaber algebra and $|-|$ denotes the degree map of A, then the bracket $[[-,-]]$ defines a Lie algebra structure with respect to a new degree map $||-||$ defined by $||a|| := |a|-1$ for every homogeneous element a in A. Therefore, if $(A,|-|)$ is a Gerstenhaber algebra then $(A,||-||)$ is a Lie algebra.

Example 2.3.8. Let $\mathfrak{X}^*(M)$ denote the graded vector space of multivector fields on a manifold M. Equipped with the wedge product, $\mathfrak{X}^*(M)$ is a graded commutative, $\mathbb{Z}$-graded algebra. If the bracket $[[-,-]]$ is defined on the generators by:

$$\begin{aligned}
&[[f,g]] = 0\ \forall f,g \in \mathfrak{X}^0(M) = \mathcal{C}^\infty(M)\\
&[[X,f]] = X(f) = -[[f,X]]\ \forall f \in \mathfrak{X}^0(M), X \in \mathfrak{X}^1(M)\\
&[[X,Y]] = [X,Y]\ \forall X,Y \in \mathfrak{X}^1(M),
\end{aligned}$$

then $(\mathfrak{X}^*(M),[[-,-]])$ is a Gerstenhaber algebra and the graded bracket $[[-,-]]$ is called the *Schouten Nijenhuis bracket.* According to remark 2.3.7, the Schouten Nijenhuis bracket is a Lie bracket with respect to the shifted degree map. For calculations we often need explicit formulas. Given a general multivector field $X = X_1 \wedge ... \wedge X_n$ and a multivector field $Y = Y_1 \wedge ... \wedge Y_m$, we can apply the Leibniz rule (see definition 2.3.6) and we have:

$$[[X, Y]] = \sum_{i=1}^{n} \sum_{j=1}^{m} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge ... \wedge \hat{X}_i \wedge ... \wedge X_n \wedge Y_1 \wedge ... \wedge \hat{Y}_j \wedge ... \wedge Y_m.$$

The notation "$\hat{X}_i$" indicates that the vector field X_i is omitted in the expression. For an $f \in \mathcal{C}^\infty(M)$, we have:

$$[[f, X]] = -i_{df} X = \sum_{i=1}^{n} (-1)^i X_i(f) X_1 \wedge ... \wedge \hat{X}_i \wedge ... \wedge X_n,$$

where $i_{df}X$ denotes the insertion operation.

Definition 2.3.9. If A is a $\mathbb{Z}$-graded algebra with multiplication μ: $A \otimes A \to A$, then a *derivation* of degree k is a degree k linear map d: $A \to A$ which satisfies the graded Leibniz rule:

$$d \circ \mu = \mu(d \otimes \mathrm{id}) + \mu(\mathrm{id} \otimes d).$$

A derivation d of degree 1 is often called a *differential derivation*, or just a *differential*, if $d \circ d = d^2 = 0$.

As one easily verifies, the vector space of the derivations of a graded algebra is a graded Lie algebra whose Lie bracket is given by the graded commutator.

Remark 2.3.10. If we are working with graded algebraic objects, we always use the *Koszul sign convention* which states that one has to multiply a factor of $(-1)^{pq}$ each time one commutes two objects of degree p and q respectively. This means that for two homogeneous maps of graded vector spaces f: $V \to V'$ and g: $W \to W'$, the evalutation of $f \otimes g$ on tensor product of homogeneous elements is given by

$$(f \otimes g)(v \otimes w) := (-1)^{deg(g)deg(v)} f(v) \otimes g(w).$$

Using the Koszul sign convention we can specialize the definition of a degree k derivation. A degree k linear map of a graded algebra A is a derivation if and only if for every homogeneous object a in A and for every object b in A, we have the equality:

$$d(ab) = d(a)b + (-1)^{k|a|} a d(b).$$

Definition 2.3.11. A *differential graded Lie algebra* $\mathfrak{g}$ is a graded Lie algebra with a differential derivation satisfying the graded Leibniz rule:

$$d[a, b] = [da, b] + (-1)^{|a|} [a, db].$$

Given a Hochschild complex with coefficients in the algebra A, i.e. $\{C^n_H(A, A)\}_{\{n \in \mathbb{N}\}}$, we define the shifted Hochschild complex via:

$$(C^{*+1}_H(A, A))_n := (C_H(A, A))_{(n-1)}.$$

The shifted Hochschild complex allows us to define a composition $f \circ_i g \in C_H^{(n+m+1)}(A, A)$ for every object $f \in C_H^{(n+1)}(A, A)$, $g \in C_H^{(m+1)}(A, A)$ and every i by

1. For every $i \leq |f|$:

$$(f \circ_i g)(a_0 \otimes ... \otimes a_{n+m}) = f(a_0 \otimes ... \otimes a_{i-1} \otimes g(a_i \otimes ... \otimes a_{i+m}) \otimes a_{i+m+1} \otimes ... \otimes a_{n+m}).$$

2. For every $i > |f|$: $f \circ_i g = 0$.

We can extend this notion to

$$f \circ g := \sum_{i=0}^{|f|} (-1)^{i|g|} f \circ_i g,$$

for homogeneous objects f and g. Since every object in a graded algebra is a finite sum of homogeous objects, the operation $\circ$ is even defined on the whole Hochschild complex.

Moreover, we note that $|f \circ_i g| = |f| + |g|$ and therefore also $|f \circ g| = |f| + |g|$.

We now summarise some useful equations which can be obtained by evaluating both sides on objects in $A^{\otimes n}$ for an appropriate n. For homogeneous objects $f, g, h \in C_H^{*+1}(A, A)$, we have:

$$\begin{aligned}(f \circ_m g) \circ_n h &= (f \circ_n h) \circ_{m+|h|} g, \text{ if } n < m \\ &= f \circ_m (g \circ_{n-m} h), \text{ if } m \leq n \leq m + |g| \\ &= (f \circ_{n-|g|} h) \circ_m g, \text{ if } m + |g| < n.\end{aligned}$$

The map $\circ$: $(C_H^{*+1}(A, A))_n \otimes (C_H^{*+1}(A, A))_m \to (C_H^{*+1}(A, A))_{n+m}$ is not associative, but it satisfies the following equation:

$$(f \circ g) \circ h - f \circ (g \circ h) = (-1)^{|g||h|}((f \circ h) \circ g - f \circ (h \circ g)).$$

As we will see in the definition below the operation $\circ$ allows us to define a new structure on the shifted Hochschild complex, which plays an important role in deformation theory.

Definition 2.3.12. Given a shifted Hochschild complex $C_H^{*+1}(A, A)$, the bracket defined by:

$$[f, g] := f \circ g - (-1)^{|f||g|} g \circ f$$

is called the *Gerstenhaber bracket.*

A straightforward calculation using the equations above shows that this bracket is graded antisymmetric and satisfies the graded Jacobi identity. Thus, the shifted Hochschild complex $C_H^{*+1}(A, A)$ is a Lie algebra with respect to the Gerstenhaber bracket.

If we adapt the definition of the coboundary map d_H to the shifted Hochschild complex $C_H^{*+1}(A, A)$, we have:

$$\begin{aligned}d_H f(a_0 \otimes ... \otimes a_n) := a_0 f(a_1 \otimes ... \otimes a_n) + (-1)^{|f|} f(a_0 \otimes ... \otimes a_{n-1}) a_n \\ + \sum_{i=0}^{|f|} (-1)^{i+1} f(a_0 \otimes ... \otimes a_i a_{i+1} \otimes ... \otimes a_n),\end{aligned}$$

Denoting multiplication in the algebra A by μ, the formula reads:

$$d_H(f) = -f \circ \mu + (-1)^{|f|} \mu \circ f.$$

Obviously, this is exactly $-[f, \mu] = (-1)^{|f|}[\mu, f]$, since $|\mu| = 1$ in the shifted Hochschild complex. Thus, the differential d_H is the same as $-[-, \mu]$. This observation is important, because now the Jacobi identity implies that

$$d_H[f, g] = -[[f, g], \mu] = -[f, [g, \mu]] - (-1)^{|g|}[[f, \mu], g] = [f, d_H g] + (-1)^{|g|}[d_H f, g],$$

which is the graded Leibniz rule for the shifted Hochschild complex.

Moreover, it shows that the cohomology is well defined with respect to the Gerstenhaber bracket.

We summarize the observations in the following theorem.

Theorem 2.3.13. *Let A be an associative k-algebra. The shifted Hochschild complex $(C_H^{*+1}(A, A), d_H, [-, -])$ is a graded differential Lie algebra, where the Lie bracket is given by the Gerstenhaber bracket.*

Now, one can ask whether the shifted Hochschild complex is even a Gerstenhaber algebra. Before we can investigate this question, we need to define a graded commutative product on the Hochschild complex. We will see in theorem 2.3.16 that the complex does not admit a Gerstenhaber algebra structure, but its Hochschild cohomology does.

Definition 2.3.14. For an associative algebra A, the *cup product* $\cup$ of a Hochschild complex is defined by its evaluation

$$f \cup g(a_1 \otimes ... \otimes a_{i+j}) := f(a_1 \otimes ...a_i)g(a_{i+1} \otimes ... \otimes a_{i+j}),$$

for all $f \in C^i(A, A)$, $g \in C^j(A, A)$ and $a_1, ..., a_{i+j} \in A$.

The cup product is compatible with the differential of a Hochschild complex:

Proposition 2.3.15. *If $C^*(A, A)$ is a Hochschild complex, then the cup product defines an associative algebra structure on it. Moreover, the cup product is graded commutative and we have the graded Leibniz rule:*

$$d_H(f \cup g) = d_H f \cup g + (-1)^{|f|} f \cup d_H g.$$

Proof. The proof is technical and we refer the reader to [Wal07, Satz 6.2.16.] □

This theorem shows that cohomology classes of the Hochschild cohomology are in particular compatible with the cup product. Although the Hochschild complex with the cup product does not admit a Gerstenhaber algebra structure, Gerstenhaber [Ger63] proved that at least the Hochschild cohomology is a Gerstenhaber algebra.

Theorem 2.3.16. *For an associative k-algebra A, the Hochschild cohomology is a Gerstenhaber algebra with product induced by the cup product and with the graded Lie bracket defined by the Gerstenhaber bracket (see definition 2.3.12).*

As we will see later, the first three Hochschild cohomology groups are important for deformation theory. By definition $H^0_{hoch}(A, A)$ is given by the set of cocycles, which is the set $\{a \in A : d_H(a)(b) = \mu(b, a) - \mu(a, b) = 0, \ \forall b \in A$. Hence, the first Hochschild cohomology group is the center of the algebra A and we write:

$$H^0_{hoch}(A, A) = Z(A).$$

For $f \in \operatorname{Hom}(A, A)$ and $a, b \in A$, the evaluation $d_H(f)(a, b) = \mu(a, f(b)) - f(\mu(a, b)) + \mu(f(a), b)$ shows that f is a cocycle if and only if f is a derivation. The image of $d_H : A \to C^1_H(A, A)$ is called the *inner derivations* of A, they are maps of the form $d_H(a) = f_a$ such that $f_a(b) = \mu(b, a) - \mu(a, b)$. It follows that

$$H^1_{hoch}(A, A) = \operatorname{Der}(A, A)/\operatorname{Ider}(A, A).$$

Lemma 2.3.17. *An object μ in $C^2_{hoch}(A, A)$ defines an associative multiplication if and only if $[\mu, \mu] = 0$.*

Proof. In the shifted Hochschild complex the map μ has degree 1, therefore

$$[\mu, \mu] = (\mu \circ \mu - (-1)\mu \circ \mu) = 2\mu \circ \mu = 2(\mu \circ_0 \mu - \mu \circ_1 \mu) = 2(\mu(\mu \otimes \mathrm{id}_A) - \mu(\mathrm{id}_A \otimes \mu)).$$

Since we assumed that the field k is of characteristic zero, we have $[\mu, \mu] = 0$ if and only if $\mu(\mu \otimes \mathrm{id}_A) - \mu(\mathrm{id}_A \otimes \mu) = 0$ which is equivalent to defining an associative multiplication. □

2.4 Extension of Deformations

With a better understanding of the Hochschild cohomology, we are now able to describe the relation between the second cohomology group and extensions of a given n-deformation.

Proposition 2.4.1. *Let A be an associative k-algebra with a multiplication μ_0. Let an n-deformation of A be given by $\mu^{(n)} = \mu_0 + \mu_1 + ... + \mu_n$, then there exists an $(n+1)$-deformation of $\mu^{(n)}$ if and only if there exists a map μ_{n+1} such that*

$$d_H \mu_{n+1} = \frac{1}{2} \sum_{i=1}^{n} [\mu_i, \mu_{n+1-i}].$$

If this condition is satisfied, then an $(n+1)$-deformation is given by $\mu^{(n+1)} = \mu^{(n)} + t^{n+1}\mu_{n+1}$.

Proof. The n-deformation $\mu^{(n)}$ is a map from $k[t]/(t^{(n+1)}) \otimes A \times k[t]/(t^{(n+1)}) \otimes A$ to $k[t]/(t^{(n+1)}) \otimes A$ and by lemma 2.3.17 the map $[\mu^{(n)}, \mu^{(n)}]$ can be viewed as a map from $k[t] \otimes A \times k[t] \otimes A$ to $k[t] \otimes A$ given by $0 + ... + 0 + t^{(n+1)} f$, for some $f \colon A \times A \to A$. If there exists an $(n+1)$-extension of $\mu^{(n)}$, i.e. $\mu^{(n+1)} = \mu^{(n)} + t^{(n+1)}\mu_{(n+1)}$, then

$$[\mu^{(n)} + t^{(n+1)}\mu_{(n+1)}, \mu^{(n)} + t^{(n+1)}\mu_{(n+1)}] =$$

$$0 + ... + 0 + t^{n+1}([\mu_0, \mu_{n+1}] + [\mu_{n+1}, \mu_0] + \sum_{i=1}^{n} [\mu_i, \mu_{n-i+1}]) + \mathscr{O}(t^{n+2}).$$

By noting that the maps μ_i have degree 1 in the shifted Hochschild complex, we have that $[\mu_0, \mu_{n+1}] = [\mu_{n+1}, \mu_0]$ and $\mu^{(n+1)} = \mu^{(n)} + t^{(n+1)}\mu_{(n+1)}$ defines an $(n+1)$-deformation if and only if $\sum_{i=1}^{n}[\mu_i, \mu_{n-i+1}] = -2[\mu_{n+1}, \mu_0] = 2d_H\mu_{n+1}$. $\square$

In general, the term $\sum_{i=1}^{n}[\mu_i, \mu_{n+1-i}]$ is not a coboundary, but nevertheless it is always closed. This can be seen by using the graded Jacobi identity. If we once more use the fact that $\mu^{(n+1)}$ is a map of degree 1, then the graded Jacobi identity provides the equality of the following chain of equations:

$$\begin{aligned} 0 &= [\mu^{(n+1)}, [\mu^{(n+1)}, \mu^{(n+1)}]] \\ &= [\mu^{(n+1)}, t^{n+1}(-2d_H\mu_{n+1} + \sum_{i=1}^{n}[\mu_i, \mu_{n-i+1}]) + \mathscr{O}(t^{n+2})] \\ &= t^{n+1}([\mu_0, -2d_H\mu_{n+1} + \sum_{i=1}^{n}[\mu_i, \mu_{n-i+1}]]) + \mathscr{O}(t^{n+2}). \end{aligned}$$

Since $d_H^2 = 0$, we have $0 = [\mu_0, \sum_{i=1}^{n}[\mu_i, \mu_{n-i+1}]] = -d_H(\sum_{i=1}^{n}[\mu_i, \mu_{n-i+1}])$.

Every closed object defines an element in the cohomology group, as an immediate consequence of this fact is the corollary below.

Corollary 2.4.2. *If the third Hochschild cohomology group $H^3_{hoch}(A, A)$ of an associative k-algebra is trivial, then there exists an $(n+1)$-extension for any given n-deformation.*

As mentioned before, we are interested in equivalence classes of deformations, therefore it is natural to ask whether an $(n+1)$-extension respects the equivalence relation between n-deformations. Lemma 2.2.8 tells us that we only have to study the case of the equivalent deformations μ and μ' which are also equal as n-deformations. An answer of this question is given in the proposition below.

Proposition 2.4.3. *If μ and μ' are two deformations which are equal as n-deformations, then there exists an $(n+1)$-extension ψ of μ and μ' as n-deformations such that $\psi = id + t^{n+1}\psi_{n+1} + ...$ if and only if*

$$\mu_{n+1} - \mu'_{n+1} = d_H\psi_{n+1}.$$

Proof. By definition, the two deformations are equivalent if and only if $\psi \circ \mu = \mu' \circ (\psi \otimes \psi)$ as maps in $k[t]/(t^{n+2})$. By remark 2.2.4 we have only to check the equality for objects in A. For every elements a, b in A, let $\mu(a, b)$ be $\mu_0(a, b) + t\mu_1(a, b) + ... + t^n\mu_n(a, b) + t^{n+1}\mu_{n+1}(a, b) + ...$ and let $\mu'(a, b)$ be $\mu_0(a, b) + t\mu_1(a, b) + ... + t^n\mu_n(a, b) + t^{n+1}\mu'_{n+1}(a, b) + ...$. Note that μ and μ' only differ if the degree is greater than n.

Therefore, we have $\psi\mu(a, b) = \mu_0(a, b) + ... + t^n\mu_n(a, b) + t^{n+1}(\psi_{n+1}(\mu_0(a, b)) + \mu_{n+1}(a, b)) + \mathscr{O}(t^{n+2})$ and similarly,

$$\mu'(\psi(a), \psi(b)) = \mu_0(a, b) + ... + t^n\mu_n(a, b) + t^{n+1}(\mu_0(\psi_{n+1}(a), b) + \mu_0(a, \psi_{n+1}(b)) + \mu'_{n+1}(a, b)) + \mathscr{O}(t^{n+2}).$$

Hence, the two deformation are equivalent if and only if $\mu_{n+1}(a, b) - \mu'_{n+1}(a, b) = \mu_0(\psi_{n+1}(a), b) +$

$\mu_0(a, \psi_{n+1}(b)) - \psi_{n+1}(\mu_0(a,b))$, which is equivalent to

$$\mu_{n+1} - \mu'_{n+1} = -[\psi_{n+1}, \mu_0] = d_H \psi_{n+1}.$$

□

As in the case of extensions, we would like to have a relationship between the map ψ_{n+1} as in the proposition above and Hochschild cohomology groups. We can reformulate the statement in the previous proposition: A map $\psi = id + t^{n+1}\psi_{n+1} + ...$ extends the equivalence of two n-deformations if and only if $\mu_{n+1} - \mu'_{n+1}$ is a coboundary. If we could show that it is always closed, then the extensions of equivalences are described by the second Hochschild cohomology group $H^2_{hoch}(A, A)$.

This is indeed the case. First we observe that $t^{n+1} d_H(\mu_{n+1} - \mu'_{n+1}) = -t^{n+1}[\mu_{n+1} - \mu'_{n+1}, \mu_0]$ is the $n+1$-degree term of $[\mu, \mu - \mu']$, because μ and μ' are assumed to be equal as n-deformations. As deformations μ and μ' are in particular associative and therefore $[\mu, \mu] = 0 = [\mu', \mu']$. This also shows that

$$[\mu_{n+1} - \mu'_{n+1}, \mu_0] = -[\mu', \mu] = \frac{1}{2}[\mu - \mu', \mu - \mu'].$$

Now the claim follows by observing that the first nontrivial term of $[\mu - \mu', \mu - \mu']$ has degree $2(n+1)$.

Corollary 2.4.4. *[HG88] If the second Hochschild cohomology group $H^2_{hoch}(A, A)$ of an associative k-algebra is trivial, then every two formal deformations are equivalent.*

Remark 2.4.5. We want to note that not every equivalence between μ and μ' as in proposition 2.4.3 has to be of the form $\psi = id + t^{n+1}\psi_{n+1} + ...$. In general ψ has the form as in 2.2.6 and it is possible that μ and μ' are equivalent even if $\mu_{n+1} - \mu'_{n+1}$ is not a coboundary.

Nevertheless, in the special case of $n = 1$ it is easily seen that this problem as just described cannot occur.

Corollary 2.4.6. *The map $\mu = \mu_0 + t\mu_1$ defines a 1-deformation if and only if $d_H\mu_1 = 0$. Two deformations μ and μ' are equivalent if and only if $\mu_1 - \mu'_1 = d_H\psi_1$, for a ψ_1 in $C^1(A, A)$.*

2.5 Hochschild-Kostant-Rosenberg Theorem

The results of previous sections reveal that, for a manifold M, the Hochschild cohomology of the algebra $\mathcal{C}^\infty(M)$ controls its deformation theory. If one wants to know about existence of extensions or equivalences, then one has to study certain Hochschild cohomology groups. In this section we want to discuss the Hochschild-Kostant-Rosenberg theorem, because it allows us to calculate the Hochschild cohomology explicitly and provides a better understanding of $H^*_{hoch}(M)$. In the original paper [HKR62] the authors, Hochschild, Kostant and Rosenberg, proved this theorem in a purely algebraic way for the algebra of polynomials $\mathrm{Pol}(\mathbb{R}^n)$. Since not every function in $\mathcal{C}^\infty(M)$ is usually of interest, one often restricts oneself to the special family of functions which are continuous with respect to the Fréchet topology (see [Hir76] for its definition and properties).

Definition 2.5.1. Let M be a smooth manifold, then

1. $C^*_{cont}(\mathcal{C}^\infty(M))$ denotes the Hochschild cochain complex such that $C^n_{cont}(\mathcal{C}^\infty(M)) := \{f \in C^n_H(\mathcal{C}^\infty(M), \mathcal{C}^\infty(M)) | f$ is continuous with respect to the Fréchet topology $\}$.
2. $C^*_{loc}(\mathcal{C}^\infty(M))$ denotes the Hochschild cochain complex such that $C^n_{loc}(\mathcal{C}^\infty(M)) := \{f \in C^n_H(\mathcal{C}^\infty(M), \mathcal{C}^\infty(M)) | f$ is local $\}$.
3. $C^*_{diff}(\mathcal{C}^\infty(M))$ denotes the Hochschild cochain complex such that $C^n_{diff}(\mathcal{C}^\infty(M)) := \{f \in C^n_H(\mathcal{C}^\infty(M), \mathcal{C}^\infty(M)) | f$ is differential $\}$.
4. $C^*_{diff,n.c.}(\mathcal{C}^\infty(M))$ denotes the Hochschild cochain complex such that $C^n_{diff,n.c.}(\mathcal{C}^\infty(M)) := \{f \in C^n_H(\mathcal{C}^\infty(M), \mathcal{C}^\infty(M)) | f$ vanishes on constant functions $\}$.

These special Hochschild cochain complexes are even subcochain complexes, therefore there exist corresponding cohomology modules.

Definition 2.5.2. Let M be a smooth manifold, then

1. $H^*_{cont,H}(\mathcal{C}^\infty(M))$ denotes the cohomology module of the Hochschild cochain complex $C^*_{cont}(\mathcal{C}^\infty(M))$.
2. $H^*_{loc,H}(\mathcal{C}^\infty(M))$ denotes the cohomology module of the Hochschild cochain complex $C^*_{loc}(\mathcal{C}^\infty(M))$.
3. $H^*_{diff,H}(\mathcal{C}^\infty(M))$ denotes the cohomology module of the Hochschild cochain complex $C^*_{diff}(\mathcal{C}^\infty(M))$.
4. $H^*_{diff,n.c.,H}(\mathcal{C}^\infty(M))$ denotes the cohomology module of the Hochschild cochain complex $C^*_{diff,n.c.}(\mathcal{C}^\infty(M))$.

For the study of star products, $H^*_{diff,H}(\mathcal{C}^\infty(M))$ and $H^*_{diff,n.c.,H}(\mathcal{C}^\infty(M))$ are particularly important. Before, we can give the definition of the Hochschild-Kostant-Rosenberg map, we need a lemma.

Lemma 2.5.3. *Let A be a commutative associative algebra and let the alternation map* Alt*: $C^*(A) \to C^*(A)$ be defined by its evaluation on homogeneous element $f \in C^n(A)$ by*

$$\operatorname{Alt} f(a_1, ..., a_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) f(a_{\sigma(1)}, ..., a_{\sigma(n)}) \text{ if } n \geq 1,$$
$$\operatorname{Alt} f = 0 \text{ if } n = 0.$$

Then

$$\operatorname{Alt} \circ \delta_{hoch} = 0.$$

Therefore, if f is a totally antisymmetric Hochschild cocyle, it is a Hochschild coboundary if and only if $f = 0$.

Proof. Since Alt is a projection, i.e. $\operatorname{Alt} \circ \operatorname{Alt} = \operatorname{Alt}$, f is totally antisymmetric if and only if $\operatorname{Alt} f = f$. For $a_0, ..., a_n \in A$, if we calculate $\operatorname{Alt} \delta_{hoch} f(a_0, ..., a_n)$, then by the definition of δ_{hoch} (see definition 2.3.1) and by the commutativity of A, we have

$$\operatorname{Alt} \sum_{i=0}^{n-1} (-1)^{i+1} f(a_0 \otimes ... \otimes a_i a_{i+1} \otimes ... \otimes a_n) = 0.$$

For the other two terms, one observes that by commutativity the following holds

$$\begin{aligned}
&\operatorname{Alt}(a_0 f(a_1 \otimes ... \otimes a_n) + (-1)^{n-1} f(a_0 \otimes ... \otimes a_{n-1}) a_n) \\
&= \operatorname{Alt} a_{\sigma(n)} f(a_{\sigma(0)} \otimes ... \otimes a_{\sigma(n-1)}) + \operatorname{Alt}(-1)^{n-1} f(a_0 \otimes ... \otimes a_{n-1}) a_n) \\
&= (-1)^n \operatorname{Alt} f(a_0 \otimes ... \otimes a_{n-1}) a_n + \operatorname{Alt}(-1)^{n-1} f(a_0 \otimes ... \otimes a_{n-1}) a_n) = 0,
\end{aligned}$$

where σ denotes the particular circular permutation $(0, 1, ..., n) \mapsto (1, ..., n, 0)$ which has sign $\sigma = (-1)^n$. If f is a totally antisymmetric Hochschild cocyle, then $\operatorname{Alt} f = f$ and $\delta_{hoch} f = 0$. Therefore, f is a coboundary if and only if $f = \delta_{hoch} g = \operatorname{Alt} \circ \delta_{hoch} g = 0$. □

Definition 2.5.4 (Hochschild-Kostant-Rosenberg map)**.** The Hochschild-Kostant-Rosenberg map $\mathscr{F}_1 \colon \mathfrak{X}^* \to C^*_{diff,n.c.}(\mathcal{C}^\infty(M))$ is defined by

$$\mathscr{F}_1(X)(f_1, ..., f_n) := \frac{1}{n!} i_{df_n} ... i_{df_1} X,$$

for $X \in \mathfrak{X}^*$ and $f_1, ..., f_n \in \mathcal{C}^\infty(M)$.

For a multivector field X of the form $X = X_1 \wedge ... \wedge X_n$, the definition implies

$$\mathscr{F}_1(X)(f_1, ..., f_n) = \frac{1}{n!} \sum_\sigma \operatorname{sign}(\sigma) X_{\sigma(1)}(f_1) .... X_{\sigma(n)}(f_n).$$

Lemma 2.5.5. *The image of the Hochschild-Kostant-Rosenberg map $\mathscr{F}_1$ consists of Hochschild cocycles, i.e.*

$$\delta_{hoch} \circ \mathscr{F}_1 = 0.$$

Proof. Since d is a derivation we have

$$i_{d(fg)} = i_{df \wedge g} + i_{f \wedge dg} = i_{df} \circ i_g + i_f \circ i_{dg},$$

where we used the identity $i_{a \wedge b} = i_a \circ i_b$. By $i_f(\omega) = f\omega$ and by commutativity of the algebra $\mathcal{C}^\infty(M)$, for functions $f_0, ..., f_n \in \mathcal{C}^\infty(M)$ and $0 \le i \le n-1$, we get

$$\begin{aligned}
&\mathscr{F}_1(X)(f_0, ..., f_i f_{i+1}, ..., f_n) = \\
&\mathscr{F}_1(X)(f_0, ..., f_i, f_{i+2}, ..., f_n) f_{i+1} + \mathscr{F}_1(X)(f_0, ..., f_{i-1}, f_{i+1}, ..., f_n) f_i.
\end{aligned}$$

Therefore, by definition 2.3.1, we have

$$\begin{aligned}
&(\delta_{hoch} \mathscr{F}_1 X)(f_0, ..., f_n) = f_0 (\mathscr{F}_1 X)(f_1, ..., f_n) + (-1)^{n-1} (\mathscr{F}_1 X)(f_0 \otimes ... \otimes f_{n-1}) f_n \\
&+ \sum_{i=0}^{n-1} (-1)^{i+1} ((\mathscr{F}_1 X)(f_0, ..., f_i, f_{i+2}, ..., f_n) f_{i+1} + \mathscr{F}_1(X)(f_0, ..., f_{i-1}, f_{i+1}, ..., f_n) f_i) = 0.
\end{aligned}$$

This shows that $\delta_{hoch} \mathscr{F}_1 X$ is always a Hochschild cocycle. □

It is obvious that $\delta_{hoch} \mathscr{F}_1 X$ is totally antisymmetric, hence the following two corollaries are an immediate consequence of lemma 2.5.3 and lemma 2.5.5.

Corollary 2.5.6. *If $X \in \mathfrak{X}^*$, then $\mathscr{F}_1 X$ is exact if and only if $X = 0$. In particular the Hochschild-Kostant-Rosenberg map $\mathscr{F}_1\colon \mathfrak{X}^* \to C^*_{diff,n.c.}(\mathcal{C}^\infty(M))$ is injective.*

Corollary 2.5.7. *The Hochschild-Kostant-Rosenberg map $\mathscr{F}_1$ induces an injective map*

$$\mathfrak{X}^*(M) \to H^*_{hoch}(\mathcal{C}^\infty(M)).$$

Notation 2.5.8. By abuse of notation, we will also denote $\mathscr{F}_1$ the induced map $\mathfrak{X}^*(M) \to H^*_{hoch}(\mathcal{C}^\infty(M))$ by $\mathscr{F}_1$.

In general the map $\mathscr{F}_1\colon \mathfrak{X}^*(M) \to H^*_{hoch}(\mathcal{C}^\infty(M))$ is not surjective, but on particular subcochains, such as $C^*_{loc}(\mathcal{C}^\infty(M))$, $C^*_{diff}(\mathcal{C}^\infty(M))$ or $C^*_{diff,n.c.}(\mathcal{C}^\infty(M))$, it is a bijection. The following theorem proves this statement for $C^k_{loc}(\mathcal{C}^\infty(M))$ and a slight modification shows that it is also true for $C^*_{diff}(\mathcal{C}^\infty(M))$ and $C^k_{diff,n.c.}(\mathcal{C}^\infty(M))$.

Theorem 2.5.9. *If $f \in C^n_{loc}(\mathcal{C}^\infty(M))$ such that $\delta f = 0$, then there exists a unique n-vector field $X \in \mathfrak{X}^n(M)$ and $g \in C^{n-1}_{loc}(\mathcal{C}^\infty(M))$, such that f decomposes as decomposition*

$$f = \mathscr{F}_1 X + \delta g.$$

Proof. A proof of this theorem can be found in [CGD80]. □

By using other methods one can even show that $\mathscr{F}_1$ is bijective for the subcochain $C^*_{cont}(\mathcal{C}^\infty(M))$. For a discussion, we refer the reader to [Gut97], [Nad99] and [Pfl98].

In the theorem below we will see that $\mathscr{F}_1$ does not only provide a set theoretical bijection, it also respects the Gerstenhaber algebra structure on $\mathfrak{X}^*(M)$ and $H^*_{diff,H}(\mathcal{C}^\infty(M))$.

Theorem 2.5.10 (Hochschild-Kostant-Rosenberg Theorem)**.** *For a manifold M, the Hochschild-Kostant-Rosenberg map*

$$\mathscr{F}_1\colon \mathfrak{X}^*(M) \to H^*_{diff,H}(\mathcal{C}^\infty(M))$$

is an isomorphism of Gerstenhaber algebras. The same statement is true, if one replaces "diff" by "cont", "loc" or "diff, n.c.".

Proof. In order to prove the statement, we have to verify that the map is a bijection and a homomorphism of Gerstenhaber algebras. Corollary 2.5.7 guarantees that the map is injective, while theorem 2.5.9 states that it is also surjective. Thus, we only need to verify the homomorphism properties, which means that for any $X, Y \in \mathfrak{X}^*(M)$, we have

$$\mathscr{F}_1(X \wedge Y) = \mathscr{F}_1(X) \cup \mathscr{F}_1(Y) \text{ and } \mathscr{F}_1([[X,Y]]) = [\mathscr{F}_1(X), \mathscr{F}_1(Y)],$$

where $[[-,-]]$ denotes the Schouten-Nijenhuis bracket in $\mathfrak{X}^*(M)$, while $[-,-]$ denotes the Gerstenhaber bracket in $H^*_{diff,H}(\mathcal{C}^\infty(M))$. By linearity we only consider the homogeneous case, i.e. $X = X_1 \wedge ... \wedge X_k$ for some vector fields $X_i \in \mathfrak{X}(M)$. The definition of the Hochschild-Kostant-Rosenberg map implies that $\mathscr{F}_1(X_1 \wedge ... \wedge X_k) = \mathrm{Alt}(\mathscr{F}_1(X_1) \cup ... \cup \mathscr{F}_1(X_k))$ and by graded commutativity (2.3.15) we have the equality $\mathrm{Alt}(\mathscr{F}_1(X_1) \cup ... \cup \mathscr{F}_1(X_k)) = \mathscr{F}_1(X_1) \cup ... \cup \mathscr{F}_1(X_k)$ in the cohomology.

Since both the Schouten-Nijenhuis bracket as well as the Gerstenhaber bracket satisfy the Leibniz rule, we only need to verify the second claim for the generators, i.e. vector fields and functions, but this obviously follows from the definition of the Hochschild-Kostant-Rosenberg map. $\square$

Chapter 3

Ordering and Star Products

In this chapter we want to apply the theory of deformation of algebras to geometric objects. Since the case of symplectic manifolds is particular interesting for us, we will first recall its definition and properties. We will often state propositions without giving their proofs, they are standard results and one can easily find them in books about differential geometry such as [Lan99] or [GS84].

3.1 Differential Geometric Preliminary

Definition 3.1.1. A *symplectic manifold* is a pair (M, ω), where M is a manifold and ω is a closed nondegenerated two form, called the *symplectic form*.

Given a symplectic manifold (M, ω), then its symplectic form induces an isomorphism:

$$i\_\omega : TM \to T^*M,$$

where i_ω is defined by $i\_\omega(X) = i_X\omega = \omega(X, -)$. In a local coordinate ω has the form

$$\omega = \frac{1}{2}\omega_{ij}dx^i \wedge dx^j$$

and

$$i_X\omega = \omega_{ij}X^i dx^j.$$

If the inverse of the matrix ω_{ij} is denoted by ω^{ij}, then for every one form $\alpha = \alpha_i dx^i$ the inverse map of i_ω is given by:

$$i\_\omega^{-1} : \alpha_i dx^i \to \omega^{ij}\alpha_i \frac{\partial}{\partial x^j}.$$

Definition 3.1.2. Let (M, ω) be symplectic manifold and let $\mathscr{L}_X$ denote the Lie derivative along a vector field X.

1. A vector field X is *symplectic*, if $\mathscr{L}_X\omega = 0$.
2. For a smooth function $H \in \mathcal{C}^\infty(M)$, the vector field X_H is a *Hamilton vector field*, if $i_{X_H}\omega = dH$.

3. A diffeomorphism f between two symplectic manifolds (M, ω) and (M', ω') is a *symplectomorphism*, if ω is a pullback of ω' along f, i.e. $f^*\omega' = \omega$.

The symplectic form ω allows us to define the Poisson bracket.

Definition 3.1.3. For a symplectic manifold (M, ω) and $f, g \in \mathcal{C}^\infty$, the *Poisson bracket* $\{-,-\}\colon \mathcal{C}^\infty \times \mathcal{C}^\infty \to \mathcal{C}^\infty$ is defined by

$$\{f, g\} := \omega(X_f, X_g),$$

where X_f and X_g denote the Hamilton vector fields associated to f and g, respectively.

Note that the Poisson bracket is sometimes defined with an additional minus sign.

A generalization of the concept of symplectic manifolds are Poisson manifolds.

Definition 3.1.4. Let M be a manifold and let $[[-,-]]$ denotes the Schouten-Nijenhuis-bracket (see definition 2.3.8) of the graded Lie algebra $\mathfrak{X}^*(M)$. The manifold M is a *Poisson manifold*, if it admits a Poisson structure π, i.e. there exists a bivector field $\pi \in \mathfrak{X}^2(M)$ such that $[[\pi, \pi]] = 0$.

Since the Poisson structure π does not need to be nondegenerated anymore, the induced homomorphism

$$-i_{-}\pi\colon T^*M \to TM$$

is in general not an isomorphism.

Definition 3.1.5. Let (M, π) be Poisson manifold and let $\mathscr{L}_X$ denote the Lie derivative along a vector field X.

1. A vector field X is a *Poisson vector field*, if $\mathscr{L}_X \pi = 0$.

2. For a smooth function $H \in \mathcal{C}^\infty(M)$, the vector field X_H is a *Hamilton vector field*, if $X_H = [[H, \pi]]$.

3. A diffeomorphism f between two Poisson manifolds (M, π) and (M', π') is a *Poisson diffeomorphism*, if π is a pullback of π' along f, i.e. $f^*\pi' = \pi$.

Definition 3.1.6. If (M, π) is a Poisson manifold and $f, g \in \mathcal{C}^\infty(M)$, then the Poisson bracket on it is defined by

$$\{f, g\} = i_{dg} i_{df} \pi = -[[[[f, \pi]], g]].$$

Locally the Poisson structure has the form $\pi = \frac{1}{2}\pi^{ij}\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$. Hence, locally the Poisson bracket is given by

$$\{f, g\} = \pi^{ij}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}.$$

In local coordinates, the Poisson bracket of a symplectic manifold has the form $\{f, g\} = -\omega^{ij}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}$. Therefore, every symplectic manifold is particularly a Poisson manifold with the Poisson structure $\pi = -\omega^{-1}$. If the Poisson structure comes from a symplectic form, then it is clear that a Poisson vector field is a symplectic vector field and that a Poisson diffeomorphism is a symplectomorphism.

In the following proposition we want to emphasize the relation between Poisson manifolds and symplectic manifolds.

Proposition 3.1.7. *If (M, π) is a symplectic manifold, then the Poisson structure π comes from a symplectic form. More precise, $\pi = -\omega^{-1}$ and ω is a closed nondegenerated two-form, if and only if π is pointwise nondegenerated.*

Moreover, the closeness of ω is equivalent to the condition $[[\pi, \pi]] = 0$.

This proposition implies that for a Poisson structure π_0 which comes from a symplectic form ω_0, the evaluation of one forms in the formal bivector field $\pi = \sum_{k=0}^{\infty} t^k \pi_k$, $\pi_k \in \mathfrak{X}^2(M)$ induces a homomorphism of algebras over the algebra $\mathcal{C}^\infty(M)[[t]]$:

$$F\colon \Gamma^\infty(T^*M)[[t]] \to \Gamma^\infty(TM)[[t]].$$

Since π_0 is nondegenerated, the zero-th level of F is an isomorphism, thus F is an isomorphism.

If one tries to define a map $d_\pi\colon \mathfrak{X}^*(M) \to \mathfrak{X}^{*+1}(M)$ by

$$d_\pi X = [[\pi, X]],$$

where π is an arbitrary bivector field, then the graded Jacobi identity of Schouten-Nijenhuis bracket implies that this map is a differential if and only if it is a Poisson structure, i.e. $d_\pi^2 = 0$ if and only if $[[\pi, \pi]] = 0$. Therefore, the condition that π is a Poisson structure is essential in the following theorem.

Theorem 3.1.8. *If (M, π) is a Poisson manifold, then there exists a Poisson cohomology (H_π^*, d_π) on its Gerstenhaber algebra $(\mathfrak{X}^*(M), \wedge, [[-,-]])$ (see example 2.3.8).*

The proposition below shows that in case of a symplectic manifold the Poisson cohomology is nothing new.

Proposition 3.1.9. *If (M, ω) is a symplectic manifold, then the isomorphism $i\_\omega\colon T^*M \to TM$ induces an isomorphism between the Poisson cohomology (H_π^*, d_π) the de Rham cohomology $(H_{dR}^*(M), d)$.*

As in the undeformed case, a deformation $\omega = \sum_{i=0}^{\infty} t^i \omega_i$ of a symplectic form ω_0 is closed if and only if the associated Poisson structure $\pi = \sum_{i=0}^{\infty} t^i \pi_i$ satisfies the condition $[[\pi, \pi]] = 0$.

Proposition 3.1.10. *Let (M, ω_0) be a symplectic manifold. Every formal deformation of Poisson structure $\pi = \sum_i^\infty t^i \pi_i$ is induced by a formal deformation of a symplectic form $\omega = \sum_i^\infty t^i \omega_i$, where every ω_i is a closed form.*

Every two deformations π and π' are equivalent if and only if their associated symplectic forms ω and ω' are equivalent.

Proof. It is a special case of 2.4.3. □

Proposition 3.1.11. *Let (M, ω_0) be a symplectic manifold and let π_0 be the associated Poisson structure. Two deformations π and π' are equivalent if and only if the difference $\pi - \pi'$ is exact.*

Moreover, for each n-deformation there exists an $(n+1)$-extension.

3.2 Formal Star Product

Definition 3.2.1. Let (M, π) be a Poisson manifold and let f and g be two maps in $\mathcal{C}^\infty(M)$. A *formal star product* on (M, π) is a formal $\mathbb{C}[[t]]$-deformation of the $\mathbb{C}$-algebra $\mathcal{C}^\infty(M)$.

In particular, it means that the formal star product

$$*\colon \mathcal{C}^\infty(M)[[t]] \otimes_{\mathbb{C}[[t]]} \mathcal{C}^\infty(M)[[t]] \to \mathcal{C}^\infty(M)[[t]]$$

is given by

$$f * g = \sum_{i=0}^{\infty} t^i F_i(f, g),$$

where $F_i\colon \mathcal{C}^\infty(M) \otimes_{\mathbb{C}} \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ are $\mathbb{C}$-bilinear maps. By definition of a formal deformation it is clear, that the map F_0 is the multiplication of the $\mathbb{C}$-algebra $\mathcal{C}^\infty(M)$.

In addition, we require that the formal star product satisfies the following properties:

1. $F_1(f, g) - F_1(g, f) = i\{f, g\}$,

2. there exists a unit 1 with respect to the formal star product, i.e. $1 * f = f = f * 1$.

Remark 3.2.2. Since a formal star product is an associative formal deformation, the formula in 2.2.5 reads explicitly that for every $k \in \mathbb{N}$ and every $f, g, h \in \mathcal{C}^\infty(M)$ we have

$$\sum_{n+m=k} F_n(F_m(f, g), h) = \sum_{n+m=k} F_n(f, F_m(g, h)).$$

Since the definition of a formal star product is very general, it is no surprise that there are many different star products. We now want to list the most important types of star products.

Definition 3.2.3. Let $(M, \pi, *)$ be a Poisson manifold with a formal star product $*$. We say the star product $*$ is

1. *local*, if the maps F_i are local for every i, i.e. $\operatorname{supp} F_i(f, g) \subseteq \operatorname{supp} f \cap \operatorname{supp} g$.

2. *differential*, if the maps F_i are bidifferentialoperators for every i.

3. *natural* or *of Vey-type*, if the maps F_i are bidifferential operator of order at most i in each argument.

4. *Hermitian*, if $\overline{f * g} = \overline{g} * \overline{f}$, where the $\overline{(-)}$ denotes the complex conjugation of the algebra $\mathbb{C}[[t]]$.

5. *of Weyl-type*, if the formal star product is Hermitian and $F_i(f, g) = (-1)^i F_i(g, f)$.

Remark 3.2.4. In [GR03] the definition of a natural star product slightly differs from ours. There, the first property of a formal star product in definition 3.2.1 is replaced by $F_1(f, g) - F_1(g, f) = 2\{f, g\}$.

In [BFF+78] a Hermitian star product is also called *symmetric*.

According to the Peetre theorem (e.g. see [KMS93]) every local star product is local differential, every natural star product is differential and hence also local. Therefore, for an open neighbourhood U, one can work with the restriction of the star product on U and the restricted star product inherits the properties of the global star product.

Since a formal star product is nothing but a formal deformation, the following definition is a natural one.

Definition 3.2.5. Let $*$ and $*'$ be two formal star products on a Poisson manifold (M, π). These two star products are said to be *local*, *differential*, *natural* or *Hermitian* equivalent if and only if they are equivalent in sense of definitions 2.2.3 and the equivalence map ϕ is local, differential, natural or Hermitian, respectively.

Later we will give many examples of star product and discuss their different properties, but from a different point of view all these star products have similar structure. More precisely, all the star products defined in the sections below can be constructed in a simple way. The method we present here was invented by M. Gerstenhaber and proved in his paper [Ger68, Theorem 8].

Definition 3.2.6. Let A be an algebra. For a set of pairwise commuting derivations D_i and D'_i, $i \in \{1, ..., n\}$, define an operator ρ: $A \otimes A \to A \otimes A$ to be

$$\rho := \sum_{i=1}^{n} D_i \otimes D'_i.$$

Theorem 3.2.7. *[Ger68, Theorem 8] Let (A, μ) be an associative k-algebra and let ρ be defined as above, then the bilinear map $*$: $A[[t]] \otimes A[[t]] \to A[[t]]$ defined by*

$$f * g = \mu \circ e^{t\rho}(f \otimes g)$$

is an associative formal deformation of μ.

Example 3.2.8. The algebra $\mathcal{C}^\infty(\mathbb{R}^{2n})$ with the pointwise multiplication is an associative algebra and, for a global coordinate system $(p_1, ..., p_n, q^1, ..., q^n)$, the derivatives $\frac{\partial}{\partial p_k}$ and $\frac{\partial}{\partial q^k}$ are commuting with each other. Therefore, we can apply the theorem 3.2.7 and there exists an associative formal star product on the manifold $\mathbb{R}^{2n}$.

3.3 t-Ordering

In the first chapter we saw that the algebra of classical observable on a manifold M contains the algebra of polynomials $\mathrm{Pol}(T^*M)$. Later, we mentioned how to quantize this algebra by methods of deformation quantization and we introduced the concept of formal star products in the previous section. In order to provide an explicit formula, we are going to defind t-ordering and discuss their properties in this section. For simplicity, we will restrict ourselves to the simple case, where the underlying manifold is $\mathbb{R}^n$ and its cotangent bundle is identified with $\mathbb{R}^{2n}$.

Definition 3.3.1. Let p: $E \to M$ be a vector bundle with a total space E and a base space M. The graded vector space of polynomials $\mathrm{Pol}^*(E)$ is defined to be

$$\mathrm{Pol}^*(E) := \bigoplus_{k=0}^{\infty} \mathrm{Pol}^k(E),$$

where $\mathrm{Pol}^k(E)$ consists of maps $f \in \mathcal{C}^\infty(E)$ such that the restriction $f|_{E_p}$, $p \in M$, is a polynomial of degree k. Endowed with the usual multiplication of polynomials, this vector space has an algebra structure.

For the vector bundle p: $\mathrm{T}^*\mathbb{R}^n \to \mathbb{R}^n$, the algebra $\mathrm{Pol}^*(\mathrm{T}^*\mathbb{R}^n)$ are the algebra of classical observables. Now, we try to define a bijective map from $\mathrm{Pol}^*(\mathrm{T}^*\mathbb{R}^n)$ to the algebra of the quantum mechanical observables $\mathrm{Diff}(\mathbb{R}^n)$. The reason therefor is, once the bijection is constructed, one can define a multiplication on $\mathrm{Pol}^*(\mathrm{T}^*\mathbb{R}^n)$ by the multiplication on $\mathrm{Diff}(\mathbb{R}^n)$. This forces the bijection to be an algebra homomorphism and as we will see, the new defined multiplication is actually a star product.

The so-called *standard ordering* S is one of the easierst way to define a map from $\mathrm{Pol}(\mathrm{T}^*\mathbb{R}^n)$ to the algebra of differential operators $\mathrm{Diff}(\mathbb{R})$ with smooth coefficients.

Definition 3.3.2. Let f be a polynomial in $\mathrm{Pol}(\mathrm{T}^*\mathbb{R}^n)$ and let (q^l, p_l) denotes the canonical coordinates on $\mathrm{Pol}(\mathrm{T}^*\mathbb{R}^n)$. The *standard ordering* S: $\mathrm{Pol}(\mathrm{T}^*\mathbb{R}^n) \to \mathrm{Diff}(\mathbb{R}^n)$ is given by

$$S(f) = \sum_{k=0}^{\infty} \frac{1}{k!}(-i\hbar)^k \sum_{l_1,\dots,l_k} \frac{\partial^k f}{\partial p_{l_1} \dots \partial p_{l_k}}|_{p=0} \frac{\partial}{\partial q^{l_1}} \cdots \frac{\partial}{\partial q^{l_k}}.$$

The inspiration of defining such a map comes from the idea of separating the canonical coordinates and map each of them to the corresponding operators. This means one defines a map S: $\mathrm{Pol}(\mathbb{R}^{2n}) \to \mathrm{Diff}(\mathbb{R}^n)$ by giving the value on the monomials:

$$S(\prod_{l_1,\dots,l_m} q^{l_k} \prod_{j_1,\dots,j_n} p_{j_k}) = \prod_{l_1,\dots,l_m} q^{l_k} \prod_{j_1,\dots,j_n} (-i\hbar) \frac{\partial}{\partial q^{l_k}}.$$

If one extend this formula to all polynomials, then the equation given in 3.3.2 describes the general case. One also easily verifies that one can identify $\mathbb{R}^{2n}$ with its cotangent space $\mathrm{T}^*\mathbb{R}^n$. Therefore, the definition of the standard ordering also makes sense, if we extend domain of S to the set $\mathrm{Pol}(\mathrm{T}^*\mathbb{R}^n)$.

Since the differentials of the form $\frac{\partial}{\partial q}$ generate the vector space $\mathrm{Diff}(\mathbb{R}^n)$, it is clear, that the standard ordering is bijective. We introduce S^{-1}: $\mathrm{Diff}(\mathbb{R}^n) \to \mathrm{Pol}(\mathrm{T}^*\mathbb{R}^n)$ which is given by

$$S^{-1}(d) = e^{\frac{1}{i\hbar}pq} d(e^{-\frac{1}{i\hbar}pq}),$$

where pq denotes $\sum_k p_k q^k$.

Let us check $S^{-1} \circ S(f) = f$ for all $f \in \mathrm{Pol}(\mathrm{T}^*\mathbb{R}^n)$, i.e. the map S^{-1} is really the inverse map of S. Again, by using the linearity, we only need to check the equation for a generator $f(q,p) = \varphi(q) p_{j_1} \dots p_{j_n}$:

$$(S^{-1} \circ S(f))(p,q) = e^{\frac{1}{i\hbar}pq} \varphi(q)(-i\hbar)^n (\frac{\partial}{\partial q^{j_1}} \cdots \frac{\partial}{\partial q^{j_n}})(e^{-\frac{1}{i\hbar}pq}) = \varphi(q) p_{l_1} \dots p_{l_n} = f(q,p).$$

A major disadvantage of the standard ordering is the fact that the image of symmetric operators does not need to be symmetric anymore, which implies that the image of observables may not be observables. One way to repair this flaw is to define a map N_t.

Definition 3.3.3. We define $\Delta = \sum_k \frac{\partial^2}{\partial q^k \partial p_k}\colon \mathcal{C}^\infty(\mathbb{R}^{2n}) \to \mathcal{C}^\infty(\mathbb{R}^{2n})$ and

$$N(-,-)\colon \mathbb{R} \times \mathrm{Pol}(\mathrm{T}^*\mathbb{R}^\mathrm{n}) \to \mathrm{Pol}(\mathrm{T}^*\mathbb{R}^\mathrm{n})$$

by $N(t,(q,p)) = e^{-it\hbar\Delta}(q,p)$. We will often write $N_t(-)$ instead of $N(t,-)$.

It is obvious from the definition of N_t that, for every $t \in \mathbb{R}$, N_t is bijective and its inverse is N_{-t}. It is also easily verified that $N_t \circ N_{t'} = N_{t+t'}$ and $\overline{N_t(f)} = N_{-t}(\overline{f})$ for every $f \in \mathrm{Pol}(\mathrm{T}^*\mathbb{R}^\mathrm{n})$.

Definition 3.3.4 (t-ordering)**.** For every $t \in \mathbb{R}$ the map $S_t\colon \mathrm{Pol}(\mathrm{T}^*\mathbb{R}^\mathrm{n}) \to \mathrm{Diff}(\mathbb{R}^n)$ is defined by the composition $S_t := S \circ N_t$. The map S_t is called the *t-ordering.*

Obviously, for $t = 0$ the map $N_0 = \mathrm{id}$ and $S(f) = S_0(f)$, therefore the definition above generates the standard ordering S.

The case, where $t = \frac{1}{2}$ is of particular importance.

Definition 3.3.5. The *Weyl ordering* S_{Weyl} is defined by

$$S_{Weyl} = S \circ N_{\frac{1}{2}}.$$

By the explicite formula for the standard ordering given in definition 3.3.2, we have for $f \in \mathrm{Pol}(\mathrm{T}^*\mathbb{R}^\mathrm{n})$:

$$S_{Weyl}(f) = \sum_{k=0}^{\infty} \frac{1}{k!}(-i\hbar)^k \sum_{l_1,\ldots,l_k} \frac{\partial^k(N_{\frac{1}{2}} f)}{\partial p_{l_1} \ldots \partial p_{l_k}}|_{p=0} \frac{\partial}{\partial q^{l_1}} \cdots \frac{\partial}{\partial q^{l_k}}.$$

A calculation shows that

$$S(f)^\dagger = S \circ N_1(\overline{f}).$$

This inconspicuous equation repairs the flaw of the standard ordering mentioned earlier. Since $N_{\frac{1}{2}}^{-1} = N_{-\frac{1}{2}}$, we have $\overline{N_{\frac{1}{2}}(f)} = N_{-\frac{1}{2}}(\overline{f}) = N_{\frac{1}{2}}^{-1}(\overline{f})$. Furthermore, using the equation above, we have

$$S_{Weyl}(f)^\dagger = S(N_{\frac{1}{2}}(f))^\dagger = S \circ N_1(\overline{N_{\frac{1}{2}}(f)}) = S \circ N_1(N_{\frac{1}{2}}^{-1}(\overline{f})) = S_{Weyl}(\overline{f}).$$

Therefore, the Weyl ordering has the nice property to take symmetric elements in $\mathrm{Pol}(\mathrm{T}^*\mathbb{R}^\mathrm{n})$ to symmetric elements in $\mathrm{Diff}(\mathbb{R}^n)$. In other words, for an observable $f = \overline{f}$ in $\mathrm{Pol}(\mathrm{T}^*\mathbb{R}^\mathrm{n})$, we have $S_{Weyl}(f)^\dagger = S_{Weyl}(f)$.

Another application of $S(f)^\dagger = S \circ N_1(\overline{f})$ is the following one. For a $t \in \mathbb{R}$ we have

$$S_t(f)^\dagger = S \circ N_t(f)^\dagger = S \circ N_1(\overline{N_t(f)}) = S \circ N_1(N_{-t}\overline{f}) = S_{1-t}(\overline{f}).$$

Equivalently, we can write this equation also in this form: $(-)^\dagger \circ S_t = S_{1-t} \circ \overline{(-)}$, where $\overline{(-)}$ denotes the complex conjugation. Since all maps involved are isomorphisms, it also holds $S_{1-t}^{-1} \circ (-)^\dagger = \overline{(-)} \circ S_t^{-1}$.

We also have $S_1(f) = S_0(\overline{f})^\dagger$. Using this equation, one shows that if $t = 1$, then

$$\langle \phi, S_1(f)\psi \rangle = \langle S_0(\overline{f})\phi, \psi \rangle = \langle \phi, \sum_{k=0}^{\infty} \frac{1}{2}(-i\hbar)^k \sum_{l_1,\ldots,l_k} \frac{\partial}{\partial q^{l_1}} \cdots \frac{\partial}{\partial q^{l_k}} \frac{\partial^k f}{\partial p_{l_1} \ldots \partial p_{l_k}}|_{p=0}\psi \rangle$$

Since this equality holds for all $\phi, \psi \in \mathcal{C}^\infty(M)$ and the scalar product is nondegenerated, both sides of the scalar product are equal. This justifies the following definition.

Definition 3.3.6. For $t = 1$ the map S_1 is called the *antistandard ordering* and it is given by

$$S_1(f) = \sum_{k=0}^{\infty} \frac{1}{k!}(-i\hbar)^k \sum_{l_1,\dots,l_k} \frac{\partial}{\partial q^{l_1}} \cdots \frac{\partial}{\partial q^{l_k}} \frac{\partial^k f}{\partial p_{l_1} \dots \partial p_{l_k}}|_{p=0}.$$

3.4 t-ordered Star Product

We have seen that a t-ordering S_t provides a bijection between the vector space $\mathrm{Pol}(\mathrm{T}^*\mathbb{R}^\mathrm{n})$ and the vector space $\mathrm{Diff}(\mathbb{R}^n)$. Although both vector spaces admit an algebra structure, a t-ordering does not respect this structure. Nevertheless, a t-ordering allows us to define a new multiplication on $\mathrm{Pol}(\mathrm{T}^*\mathbb{R}^\mathrm{n})$ by requiring the bijection to be an algebra homomorphism.

Definition 3.4.1. For $t \in \mathbb{R}$, let $S_t\colon \mathrm{Pol}(\mathrm{T}^*\mathbb{R}^\mathrm{n}) \to \mathrm{Diff}(\mathbb{R}^n)$ be the t-ordering. The *t-ordered star product* $*_t\colon \mathrm{Pol}(\mathrm{T}^*\mathbb{R}^\mathrm{n}) \otimes \mathrm{Pol}(\mathrm{T}^*\mathbb{R}^\mathrm{n}) \to \mathrm{Pol}(\mathrm{T}^*\mathbb{R}^\mathrm{n})$ is defined by

$$f *_t g = {S_t}^{-1}(S_t(f)S_t(g)),$$

for f and g in $\mathcal{C}(\mathbb{R}^n)$.

1. $*_0$ is called the *standard ordered star product.*
2. $*_{\frac{1}{2}}$ is called the *Weyl ordered star product* or *Weyl-Moyal star product.*
3. $*_1$ is called the *antistandard ordered star product.*

It follows directly from the definition that the map S_t is an algebra isomorphism

$$S_t\colon (\mathrm{Pol}(\mathrm{T}^*\mathbb{R}^\mathrm{n}), *_t) \cong \mathrm{Diff}(\mathbb{R}^n).$$

It is also obvious that the star product is associative and that the constant function 1 is the unit in the algebra. The manifold $\mathrm{T}^*\mathbb{R}^\mathrm{n}$ admits a canonical symplectic form, hence, it is in particular a Poisson manifold.

By the definitions of the map S_t and the star products, we have for polynomials f and g

$$f *_t g = {S_t}^{-1}(S_t(f)S_t(g)) = {N_t}^{-1}S^{-1}(SN_t(f)SN_t(g)) = {N_t}^{-1}(N_t(f) *_0 N_t(g)).$$

If one compares the map N_t in definition 3.3.3 and the map ϕ in definition 2.2.3, then this equation shows that the map N_t induces an equivalence between different star products. Therefore, all star products we have seen so far are equivalent.

The equation $S_{1-t}^{-1} \circ (-)^\dagger = \overline{(-)} \circ S_t^{-1}$ from the previous section shows that

$$\overline{f *_t g} = \overline{(-)} \circ S_t^{-1}(S_t(f) \otimes S_t(g)) = S_{1-t}^{-1} \circ (-)^\dagger (S_t(f) \otimes S_t(g)) = S_{1-t}^{-1}(S_{1-t}(\overline{g})S_{1-t}(\overline{f})) = \overline{g} *_{1-t} \overline{f}.$$

In particular, we see that

$$\overline{f *_{\frac{1}{2}} g} = \overline{g} *_{\frac{1}{2}} \overline{f}.$$

Thus, for the Weyl-Moyal product $*_{\frac{1}{2}}$ complex conjugation is an involution.

We know from corollary 2.4.4 that, if the second Hochschild cohomology group is trivial, then every deformation are equivalent to each other. Gutt and Lichnerowicz proved in [Gut79] and [Lic79] that in case of a symplectic manifold, one only has to verify this condition for the second de Rham cohomology group.

Theorem 3.4.2. *[Gut79][Lic79] For a symplectic manifold (M, ω) whose second de Rham cohomology group $H^2_{dR}(M)$ is trivial, all star products on M are equivalent.*

By this theorem, it is clear that all star products on the contractible symplectic manifold $\mathbb{R}^2$ must be equivalent to each other.

Now we have to verify that the star products as defined in 3.4.1 are indeed star products in the sense of definition 3.2.1. We only need to prove that $F_1(f,g) - F_1(g,f) = i\{f,g\}$, where maps F_k are the coefficient maps in the expression $f * g = \sum_{i=0}^{\infty} t^i F_i(f,g)$. We will prove this by a careful studying of star products.

We first try to find an explicit formula for the standard star product. Starting with the manifold $T^*\mathbb{R}$, every $f, g \in \mathrm{Pol}(T^*\mathbb{R})$ have the form $f(q,p) = \alpha(q)p_k$ and $g(q,p) = \beta(q)p^l$. A calculation shows:

$$\begin{aligned}
f *_0 g = S^{-1}(S(f)S(g)) &= S^{-1}((-i\hbar)^{k+l}\alpha(q)\frac{\partial^k}{\partial q^k}(\beta(q)\frac{\partial^l}{\partial q^l})) \\
&= S^{-1}((-i\hbar)^{k+l}\alpha(q)\sum_{j=0}^{k}\binom{k}{j}\frac{\partial^j\beta(q)}{\partial q^j}\frac{\partial^{l+k-j}}{\partial q^{l+k-j}}) \\
&= S^{-1}(S(\sum_{j=0}^{k}(-i\hbar)^j\binom{k}{j}\alpha(q)p^{k-j}\frac{\partial^j\beta(q)}{\partial q^j}p^l)) \\
&= \sum_{j=0}^{k}\frac{(-i\hbar)^j}{j!}\alpha(q)\frac{\partial^j p_k}{\partial p^j}\frac{\partial^j\beta(q)}{\partial q^j}p^l \\
&= \sum_{j=0}^{\infty}\frac{(-i\hbar)^j}{j!}\frac{\partial^j f}{\partial p^j}\frac{\partial^j g}{\partial q^j}.
\end{aligned}$$

Here, we used the explicit formula of the standard ordering in definition 3.3.2 in the forth equality and we replaced k by ∞ in the last equation, since f has finite degree. The general formula for arbitrary polynomials follows from the calculation above, because derivatives with respect to different p_i commute with each other. Hence, we have for $f, g \in (\mathrm{T}^*\mathbb{R}^n, *_0)$:

$$f *_0 g = \sum_{k=0}^{\infty}\frac{(-i\hbar)^k}{k!}\sum_{l_1,\ldots,l_k}\prod_{j_1,\ldots,j_k}\frac{\partial f}{\partial p_{l_k}}\frac{\partial g}{\partial q^{l_k}}.$$

The coefficient of $(\hbar)^k$ is the function we are searching for:

$$F_k(f,g) = \frac{(-i)^k}{k!}\sum_{l_1,\ldots,l_k}\prod_{j_1,\ldots,j_k}\frac{\partial f}{\partial p_{l_k}}\frac{\partial g}{\partial q^{l_k}}.$$

In particular, we have

$$F_1(f,g) - F_1(g,f) = -i\sum_{k=1}^{n}(\frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q^k} - \frac{\partial g}{\partial p_k}\frac{\partial f}{\partial q^k}) = i\{f,g\}.$$

Thus, the standard star product is indeed a formal star product in the sense of definition 3.2.1.

If the pointwise multiplication of the algebra $\mathcal{C}^\infty(\mathbb{R}^n)$ is denoted by μ, then the explicit formula for the standard star product has the form

$$*_0(-,-) = \mu \circ e^{-i\hbar\rho}(-,-),$$

where ρ is a linear map as in theorem 3.2.7. As already mentioned in example 3.2.8 ρ has in this particular case the form $\rho = \sum_{k=1}^{n} \frac{\partial}{\partial p_k} \otimes \frac{\partial}{\partial q^k}$. The operator Δ in definition 3.3.3 has the form $\Delta = \sum_k \frac{\partial^2}{\partial q^k \partial p_k}$. For f and g we have $\frac{\partial^2 fg}{\partial q^k \partial p_k} = f\frac{\partial^2 g}{\partial q^k \partial p_k} + g\frac{\partial^2 f}{\partial q^k \partial p_k} + \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q^k} + \frac{\partial g}{\partial p_k}\frac{\partial f}{\partial q^k}$. If we define the map $\rho' := \tau \circ \rho \circ \tau$, where τ is the flipping map as defined in 1.4.1, it holds $\Delta \circ \mu = \mu \circ (\Delta \otimes \mathrm{id} + \rho + \rho' + \mathrm{id} \otimes \Delta)$. The operator N_t was defined to be $e^{-it\hbar\Delta}$, therefore, the calculation above implies that

$$N_t \circ \mu = \mu \circ e^{-it\hbar(\Delta\otimes\mathrm{id} + \rho + \rho' + \mathrm{id}\otimes\Delta)} = \mu \circ e^{-it\hbar(\rho+\rho')} \circ e^{-it\hbar(\Delta\otimes\Delta)}.$$

The second equality follows from the observation that all partial derivatives commute with each other, therefore $e^{D1+D2} = e^{D1} \circ e^{D2}$, where $D1$ and $D2$ stand for sums and products of partial derivatives. Now, we use these equations to write the t-star product in another form.

$$\begin{aligned}
*_t &= N_t^{-1} \circ *_0 \circ (N_t \otimes N_t) = e^{it\hbar\Delta} \circ \mu \circ e^{-it\hbar\rho} \circ (e^{-it\hbar\Delta} \otimes e^{-it\hbar\Delta}) \\
&= \mu \circ e^{it\hbar(\rho+\rho')} \circ e^{-it\hbar\rho} \circ e^{-it\hbar(\Delta\otimes\Delta)} \circ (e^{-it\hbar\Delta} \otimes e^{-it\hbar\Delta}) \\
&= \mu \circ e^{i\hbar((t-1)(\rho)+t\rho')}.
\end{aligned}$$

The following corollary is an immediate consequence of this calculation.

Corollary 3.4.3. *For $t \in \mathbb{R}$ and $f, g \in \mathrm{Pol}(\mathrm{T}^*\mathbb{R}^n)$ it holds:*

1. $f *_t g = \mu \circ e^{i\hbar((t-1)(\rho)+t\rho')}(f \otimes g)$,
2. $f *_{\frac{1}{2}} g = \mu \circ e^{-\frac{i\hbar}{2}(\rho-\rho')}(f \otimes g)$,
3. $f *_1 g = \mu \circ e^{i\hbar\rho'}(f \otimes g)$,

3.5 $\tilde{t}$- Ordering

Up to now, we are working with real symplectic manifolds. If there exist a complex structure, then other useful orderings arise. Some of the orderings are inspired by the quantum mechanics. In particular, the so-called Wick ordering is tightly related with physics. As a motivation, let us recall the quantization of a harmonic oszillator. The manifold of interest is $T^*\mathbb{R}^n$ which we identify with $\mathbb{R}^{2n}$. Instead of using its canonical coordinates $x^1, ..., x^n, p_{x^1}, ..., p_{x^n}$, one usually rescale them

$$q^k := (m\omega)^{\frac{1}{2}}x^k \quad p_k := (m\omega)^{-\frac{1}{2}}p_{x^k}, 1 \leq k \leq n$$

in order to have a nice physical dimension. Here, m is the mass and ω is the frequency of a harmonic oszillator. The procedure of a quantization is defining a map $\tilde{S}$ which sends q^k to the quantum mechanic position operator Q^k and p_k to the quantum mechanic momentum operator P_k. These allow us to define the creation and annihilation operators:

$$A_k := \frac{1}{\sqrt{2\hbar}}(Q^k + iP_k) \text{ and } {A_k}^\dagger := \frac{1}{\sqrt{2\hbar}}(Q^k - iP_k).$$

These quantum mechanic operators are dimensionless operators satisfy the commutation relation

$$[A_k, A_l] = 0 = [{A_k}^\dagger, {A_l}^\dagger] \quad \text{and} \quad [A_k, {A_l}^\dagger] = \delta_{kl}.$$

If we treat the real manifold $T^*\mathbb{R}^n$ as $\mathbb{C}^n$, we can mimic the quantum mechanic procedure from above. The corresponding complex coordinates of (q^k, p_k) have the form:

$$z^k := q^k + ip_k \text{ and } \overline{z}^k := q^k - ip_k.$$

Obviously, the change of coordinates, (q^k, p_k) to $(z^k, \overline{z}^k)$ is invertible. As a second step we want to find a map which sends $(z^k, \overline{z}^k)$ to operators.

The behaviour of coordinates z and $\overline{z}$ with respect to the Poisson bracket can be seen as an analogon to the classical commutation relation of creation and annihilation operators. For $k, l \in \{1, ..., n\}$, it holds

$$\{z^k, z^l\} = 0 = \{\overline{z}^k, \overline{z}^l\} \qquad \{z^k, \overline{z}^l\} = -2i\delta^{k\bar{l}}.$$

It seems that it is reasonable to define $\tilde{S}$ as following (we choose the notation "$\tilde{S}$", because, as will we see later, it has similarity with the map S as defined in 3.3.2):

$$\begin{aligned}\tilde{S}(\overline{z}^k) &= a_k := \sqrt{2\hbar}\overline{z}^k(\sqrt{2\hbar}\overline{z}^k : f \mapsto \sqrt{2\hbar}\overline{z}^k f)\\ \tilde{S}(z^k) &= {a_k}^\dagger := \sqrt{2\hbar}\frac{\partial}{\partial \overline{z}^k}(\sqrt{2\hbar}\frac{\partial}{\partial \overline{z}^k} : f \mapsto \sqrt{2\hbar}\frac{\partial}{\partial \overline{z}^k} f).\end{aligned}$$

As in the case of the standard ordering or the t-ordering in general, the map $\tilde{S}$ here has $\mathrm{Pol}(\mathbb{R}^{2n})$ as domain and the space of differential operators as codomain. Since the manifold is complex, we have to find a new algebra to replace the algebra $\mathrm{Diff}(M)$. The question about the right codomain is nontrivial, because one requires in addition the equality of operators $(\overline{z}^k)^\dagger = 2\hbar\frac{\partial}{\partial \overline{z}^k}$. A nice suggestion is to work with differential operators on a certain subspace of the Hilbert space $L^2(\mathbb{C}^n, d\sigma)$, where the measure $d\sigma$ is defined by

$$d\sigma(z, \overline{z}) = \frac{1}{(2\pi\hbar)^n} e^{-\frac{z\overline{z}}{2\hbar}} dz d\overline{z}.$$

The scalar product on this Hilbert space is $\langle -, - \rangle$ given by

$$\langle \phi, \psi \rangle = \int \overline{\phi(z, \overline{z})}\psi(z, \overline{z}) d\sigma(z, \overline{z}).$$

The subspace we are interested in is the so-called *Bargmann Fock space*

Definition 3.5.1. The *Bargmann Fock space BF* consists of the antiholomorphic functions in $L^2(\mathbb{C}^n, d\sigma)$.

Bargmann proved in his paper [Bar61] some useful properties of the Bargmann Fock space and we summarize them in the theorem below.

Theorem 3.5.2. *The Bargmann Fock space is a closed subspace of the $L^2(\mathbb{C}^n, d\sigma)$, endowed with the restricted scalar product it is a Hilbert space. It admits a set of orthonormal basis $\{f_{k_1 \dots k_n}\}$, consisting of elements of the form*

$$f_{k_1 \dots k_n}(z) = \prod_{k_i} \frac{(\overline{z}^i)^{k_i}}{\sqrt{(2\hbar)^{k_i} k_i!}}$$

The restriction of the scalar product to the Bargmann Fock space has the form

$$\langle \phi, \psi \rangle|_{BF} = \sum_{m=0}^{\infty} \sum_{k_1, \dots k_m} \frac{(2\hbar)^m}{m!} \overline{\frac{\partial^m \phi}{\partial \overline{z}^{k_1}, \dots \overline{z}^{k_m}}}|_{z=0} \frac{\partial^m \phi}{\partial \overline{z}^{k_1}, \dots \overline{z}^{k_m}}|_{z=0}$$

Now, one defines a map $\tilde{S}$ from the space of antiholomorphic polynomials $\mathbb{C}[\overline{z}^1, \dots, \overline{z}^n]$ to the space of differential operators on the Bargmann Fock space Diff_{BF}. As in the case of the standard ordering, the map $\tilde{S}$ is totally determined by the image of monomials.

Definition 3.5.3. The *Wick ordering* is a map $\tilde{S}$ defined by

$$\tilde{S}(\prod_{l_1, \dots, l_m} z^{l_i} \prod_{j_1, \dots, j_n} \overline{z}^{j_i}) = \prod_{j_1, \dots, j_n} \sqrt{2\hbar} \overline{z}^{j_i} \prod_{l_1, \dots, l_m} \sqrt{2\hbar} \frac{\partial}{\partial \overline{z}^{l_i}}.$$

If we extend this expression to all polynomials, we have

$$\tilde{S}(f) = \sum_{m,n=0}^{\infty} \frac{1}{m!n!} \sum_{k_1, \dots, k_n, l_1, \dots, l_m} \frac{\partial^{m+n} f}{\partial z^{k_1} \dots \partial z^{k_n} \partial \overline{z}^{l_1} \dots \partial \overline{z}^{l_m}}(0) \prod_{l_1, \dots, l_m} \sqrt{2\hbar} \overline{z}^{l_i} \prod_{k_1, \dots, k_n} \sqrt{2\hbar} \frac{\partial}{\partial \overline{z}^{k_i}}.$$

The following equations, which are consequences of partial integration, states that these two operators are adjoint to each other.

$$\langle \phi, a_k \psi \rangle|_{BF} = \langle {a_k}^\dagger \phi, \psi \rangle|_{BF}$$

It also follows directly from the definition of a_k and ${a_k}^\dagger$ that they satisfy the commutation relation as their quantum mechanic analogons: $[a_k, a_l] = 0 = [{a_k}^\dagger, {a_l}^\dagger]$ and $[a_k, {a_l}^\dagger] = 2\hbar \delta_{k\overline{l}}$. Using the terminology of canonical quantization, we would say that the Wick ordering is nothing but writing all creation operators to the left while writing all annihilation operators to the right.

A consequence of these two observations is that for every polynomial f

$$\tilde{S}(f)^\dagger = \tilde{S}(\overline{f}).$$

It is enough to check the equality for monomials:

$$\begin{aligned}(\tilde{S}(\prod_{l_1,\dots,l_m} z^{l_i} \prod_{j_1,\dots,j_n} \overline{z}^{j_i}))^\dagger &= (\prod_{j_1,\dots,j_n} \sqrt{2\hbar}\overline{z}^{j_i} \prod_{l_1,\dots,l_m} \sqrt{2\hbar}\frac{\partial}{\partial \overline{z}^{l_i}})^\dagger \\ &= (\prod_{j_1,\dots,j_n} a_{j_i} \prod_{l_1,\dots,l_m} a^\dagger_{l_i})^\dagger = \prod_{l_1,\dots,l_m} a_{l_i} \prod_{j_1,\dots,j_n} a^\dagger_{j_i} \\ &= \tilde{S}(\prod_{j_1,\dots,j_n} z^{j_i} \prod_{l_1,\dots,l_m} \overline{z}^{l_i}) = \tilde{S}(\overline{\prod_{l_1,\dots,l_m} z^{l_i} \prod_{j_1,\dots,j_n} \overline{z}^{j_i}})\end{aligned}$$

Here, we used the adjointness property and the fact that $[a_k, a_l] = 0 = [{a_k}^\dagger, {a_l}^\dagger]$ in the third equality.

Starting from the standard ordering, we defined the so-called t-ordering for a $t \in \mathbb{R}$ and we have seen that for special values, such as $t = \frac{1}{2}$ or $t = 1$, the ordering map S_t has nice properties. We will see that the Wick ordering also arises as a special case of a $\tilde{t}$-ordering.

Definition 3.5.4. We define $\tilde{\Delta} = \sum_k \frac{\partial^2}{\partial z^k \partial \overline{z}^k}\colon \mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n)$ and

$$\tilde{N}(-,-)\colon \mathbb{R} \times \mathrm{Pol}(\mathbb{R}^{2n}) \to \mathrm{Pol}(\mathbb{R}^{2n})$$

by $\tilde{N}(t,(z,\overline{z})) = e^{t\hbar\tilde{\Delta}}$. For brevity, we will write $\tilde{N}_t(-)$ instead of $\tilde{N}(t,-)$.

As in the case of N_t, it is obvious that $\tilde{N}_t \circ \tilde{N}_{t'} = \tilde{N}_{t+t'}$

Definition 3.5.5 ($\tilde{t}$-ordering)**.** For every $t \in \mathbb{R}$ the map $\tilde{S}_t\colon \mathrm{Pol}(\mathbb{R}^{2n}) \to \mathrm{Diff}_{BF}$ is defined to be the composition $\tilde{S}_t := \tilde{S} \circ \tilde{N}_{1-t}$. We will call this map the *$\tilde{t}$-ordering*.

This new concept includes the Wick ordering, since for $t = 1$ the map $\tilde{S}_1(f) = \tilde{S} \circ \tilde{N}_{1-1} = \tilde{S}(f)$.

The map $\tilde{\Delta}$ is a real operator, thus we have $\overline{\tilde{N}_t} = \tilde{N}_t$ and $\overline{\tilde{N}_t(f)} = \tilde{N}_t(\overline{f})$. Therefore, for a polynomial f, the equation $\tilde{S}(f)^\dagger = \tilde{S}(\overline{f})$ implies that

$$(\tilde{S}_t(f))^\dagger = (\tilde{S} \circ \tilde{N}_{1-t}(f))^\dagger = \tilde{S}(\overline{\tilde{N}_{1-t}(f)}) = \tilde{S} \circ \tilde{N}_{1-t}(\overline{f}) = \tilde{S}_t(\overline{f}),$$

for every $t \in \mathbb{R}$. Thus, this shows that for every $t \in \mathbb{R}$ $\tilde{S}_t$ maps symmetric elements to symmetric elements.

Definition 3.5.6. The map $\tilde{S}_{-1}$ as defined above is called the *antiwick ordering*.

An explicit calculation reveals that $\tilde{S}_{-1}$ has the following form

$$\tilde{S}_{-1}(\prod_{l_1,\dots,l_m} z^{l_i} \prod_{j_1,\dots,j_n} \overline{z}^{j_i}) = \prod_{l_1,\dots,l_m} \sqrt{2\hbar}\frac{\partial}{\partial \overline{z}^{l_i}} \prod_{j_1,\dots,j_n} \sqrt{2\hbar}\overline{z}^{j_i}.$$

In other words, applying the antiwick ordering means to write all annihilator operators to the left while write all creation operators to the right. Therefore, it is clear why this operation is called the *anti*wick ordering. Although $\tilde{S}_{-1}$ is determined by the formula given above, we want to provide an explicit expression for an arbitrary polynomial f:

$$\tilde{S}_{-1}(f) = \sum_{m,n=0}^{\infty} \sum_{k_1,\ldots,k_n,l_1,\ldots,l_m} \frac{\partial^{m+n} f}{\partial z^{k_1} \ldots \partial z^{k_n} \partial \overline{z}^{l_1} \ldots \partial \overline{z}^{l_m}}(0) \prod_{k_1,\ldots,k_n} \sqrt{2\hbar} \frac{\partial}{\partial \overline{z}^{k_i}} \prod_{l_1,\ldots,l_m} \sqrt{2\hbar} \overline{z}^{l_i}.$$

3.6 $\tilde{t}$-ordered Star Products

Using similar methods as discussed for the case of $\mathrm{Pol}(\mathrm{T}^*\mathbb{R}^n)$, one can define star products on $\mathrm{Pol}(\mathbb{R}^{2n})$. The properties of a $\tilde{t}$-ordered star product resembles those of t-ordered star product. Therefore, the proofs below are similar to those in the previous chapter.

Definition 3.6.1. For $t \in \mathbb{R}$, let $\tilde{S}_t$: $\mathrm{Pol}(\mathbb{R}^{2n}) \to \mathrm{Diff}_{BF}$ be the $\tilde{t}$-ordering. The *$\tilde{t}$-ordered star product* $\tilde{*}_t$: $\mathrm{Pol}(\mathbb{R}^{2n}) \otimes \mathrm{Pol}(\mathbb{R}^{2n}) \to \mathrm{Pol}(\mathbb{R}^{2n})$ is defined by

$$f \tilde{*}_t g = \tilde{S}_t^{-1}(\tilde{S}_t(f)\tilde{S}_t(g)),$$

for f and g in $\mathcal{C}(\mathbb{R}^n)$.

1. $\tilde{*}_1$ is called the *Wick ordered star product*
2. $\tilde{*}_0$ is called the *Weyl ordered star product* or *Weyl-Moyal star product*
3. $\tilde{*}_{-1}$ is called the *antiwick ordered star product.*

Remark 3.6.2. In lemma 3.6.4 we will see that $\tilde{*}_0$ equals $*_{\frac{1}{2}}$. Thus, the Weyl ordered star product as just defined is same Weyl-Moyal star product as defined in 3.4.1.

By the definition of the $\tilde{t}$-ordered star product, we see that every $\tilde{t}$-ordered star products are equivalent. Similar to the situation of t-ordered star product, for polynomials f and g we have

$$f *_t g = \tilde{S}_t^{-1}(\tilde{S}_t(f)\tilde{S}_t(g)) = \tilde{N}_t^{-1}\tilde{S}^{-1}(\tilde{S}\tilde{N}_t(f)\tilde{S}\tilde{N}_t(g)) = \tilde{N}_t^{-1}(\tilde{N}_t(f)\tilde{*}_1\tilde{N}_t(g)).$$

As in the case of a t-ordered star product, we want to apply theorem 3.2.7 in order to prove the associativity of the $\tilde{t}$-ordered star products. Therefore, we first write the Wick ordered star product in an explicit form, see 3.6.3. As a second step, we have to find the right replacement of the map ρ and by using the explicit formula, we realize that the Wick ordered star product can be written in the form as in theorem 3.2.7.

Lemma 3.6.3. *The Wick ordered star product as defined in 3.6.1 has the following explicit form:*

$$f \tilde{*}_1 g = \sum_n \sum_{r_1,\ldots,r_m} \frac{(2\hbar)^m}{m!} \prod_{r_1,\ldots,r_m} \frac{\partial}{\partial z^{r_s}} f \prod_{r_1,\ldots,r_m} \frac{\partial}{\partial \overline{z}^{r_s}} g,$$

for $f, g \in \mathrm{Pol}(\mathbb{R}^{2n})$.

Proof. Since partial derivatives with respect to different coordinates commute with each other, we only need to prove the identity for monomials and for $\mathbb{R}^2$. Let $f = z^i\overline{z}^j$ and $g = z^k\overline{z}^l$, then we have

$$
\begin{aligned}
f\tilde{*}_1 g &= \tilde{S}^{-1}(\tilde{S}(f)\tilde{S}(g)) = \tilde{S}^{-1}((\sqrt{2\hbar})^{i+j+k+l}\overline{z}^j\frac{\partial^i}{\partial z^i}\overline{z}^l\frac{\partial^k}{\partial z^k})) \\
&= \tilde{S}^{-1}((\sqrt{2\hbar})^{i+j+k+l}\sum_{s=0}^{i}\binom{i}{s}\overline{z}^j\frac{\partial^s\overline{z}^l}{\partial\overline{z}^s}\frac{\partial^{i+k-s}}{\partial\overline{z}^{i+k-s}}) \\
&= \tilde{S}^{-1}((\sqrt{2\hbar})^{i+j+k+l}(\sum_{s=0}^{i}\binom{i}{s}\frac{l!}{(l-s)!}\overline{z}^{j+l-s}\frac{\partial^{i+k-s}}{\partial\overline{z}^{i+k-s}}) \\
&= \tilde{S}^{-1}\circ\tilde{S}(\sum_{s=0}^{i}\frac{(2\hbar)^s}{s!}\overline{z}^j\frac{\partial^s z^i}{z^s}z^k\frac{\partial^s\overline{z}^l}{\partial\overline{z}^s}) \\
&= \sum_{s=0}^{\infty}\frac{(2\hbar)^s}{s!}\frac{\partial^s f}{\partial z^s}\frac{\partial^s g}{\partial\overline{z}^s}.
\end{aligned}
$$

The sum over infinite many numbers is justified by the fact that the degree of polynomials are finite. □

The explicit formula from above shows that the Wick product can be written in the form

$$f\tilde{*}_1 g = \mu\circ e^{2\hbar\tilde{\rho}}(f\otimes g),$$

where μ is the multiplication map on $\mathrm{Pol}(\mathbb{R}^{2n})$ and $\tilde{\rho}$: $\mathrm{Pol}(\mathbb{R}^{2n})\to\mathrm{Pol}(\mathbb{R}^{2n})$ is defined by

$$\tilde{\rho} := \sum_k \frac{\partial}{\partial z^k}\otimes\frac{\partial}{\partial\overline{z}^k}$$

As one can imagine, $\tilde{\rho}$ plays here the role as ρ in the previous section. Of course, there also exists a map $\tilde{\rho}'$:

$$\tilde{\rho}' := \sum_k \frac{\partial}{\partial\overline{z}^k}\otimes\frac{\partial}{\partial z^k}.$$

It is easy to see that $\tilde{\Delta}\circ\mu = \mu\circ(\tilde{\Delta}\otimes\mathrm{id}+\mathrm{id}\otimes\tilde{\Delta}+\tilde{\rho}+\tilde{\rho}')$. A straightforward calculation as in the case of corollary 3.4.3 reveals that the $\tilde{t}$-star product can also by written as a composition of the multiplication map μ and the exponential map:

$$\tilde{*}_t = \mu\circ e^{\hbar((t+1)\tilde{\rho}+(t-1)\tilde{\rho}')}.$$

If $\overline{(-)}$ denotes the complex conjugation, then $\overline{\frac{\partial}{\partial z}} = \frac{\partial\overline{f}}{\overline{z}}$ implies that

$$\tilde{\rho}\circ(\overline{(-)}\otimes\overline{(-)}) = (\overline{(-)}\otimes\overline{(-)})\circ\tilde{\rho}' \text{ and } \tilde{\rho}'\circ(\overline{(-)}\otimes\overline{(-)}) = (\overline{(-)}\otimes\overline{(-)})\circ\tilde{\rho}.$$

As consequence, we have:

$$\begin{aligned}\overline{(-)} \circ \tilde{*}_t &= \overline{(-)} \circ \mu \circ e^{\hbar((t+1)\tilde{\rho}+(t-1)\tilde{\rho}')} \\ &= \mu \circ \tau \circ (\overline{(-)} \otimes \overline{(-)}) \circ e^{\hbar((t+1)\tilde{\rho}+(t-1)\tilde{\rho}')} \\ &= \mu \circ \tau \circ e^{\hbar((t+1)\tilde{\rho}'+(t-1)\tilde{\rho})} \circ (\overline{(-)} \otimes \overline{(-)}) \\ &= \mu \circ e^{\hbar((t+1)\tilde{\rho}+(t-1)\tilde{\rho}')} \circ \tau \circ (\overline{(-)} \otimes \overline{(-)}).\end{aligned}$$

The last equality is justified by $\tau \circ \tilde{\rho} = \tilde{\rho}' \circ \tau$ and $\tau \circ \tilde{\rho}' = \tilde{\rho} \circ \tau$. Therefore, for $f, g \in \mathrm{Pol}(\mathbb{R}^{2n})$, it holds $\overline{f \tilde{*}_t g} = \overline{g} \tilde{*}_t \overline{f}$. In other words, for every $t \in \mathbb{R}$ the complex conjugation is an involution on $(\mathrm{Pol}(\mathbb{R}^{2n}), \tilde{*}_t)$.

The equations

$$\frac{\partial}{\partial z^k} = \frac{1}{2}(\frac{\partial}{\partial q^k} - i\frac{\partial}{\partial p_k}) \text{ and } \frac{\partial}{\partial \overline{z}^k} = \frac{1}{2}(\frac{\partial}{\partial q^k} + i\frac{\partial}{\partial p_k})$$

imply that

$$\tilde{\rho} - \tilde{\rho}' = \frac{1}{2i}(\rho - \rho').$$

Hence, it holds:

$$\tilde{*}_0 = \mu \circ e^{\hbar(\tilde{\rho}-\tilde{\rho}')} = \mu \circ e^{\hbar \frac{1}{2i}(\rho-\rho')} = *_{\frac{1}{2}}.$$

We summarize this result in the lemma below.

Lemma 3.6.4. *For $t = \frac{1}{2}$, the $\tilde{t}$-ordered star product equals the t-ordered star product which is the Weyl-Moyal star product in this case.*

Chapter 4

Fedosov Construction

In the previous chapter we introduced the concept of star products and discussed their properties, but one central question remains unanswered. Up to now, except for some trivial cases, we do not know whether a star product even exist for a symplectic manifold we are interested in. The main difficulty is to verify that it is always possible in the symplectic case to find functions $F_i, i \in \mathbb{N}$ such that the associativity condition in 3.2.2 is satisfied. Fortunately, the question can be answered affirmatively.

Neroslavski and Vlassov were the first who proved the existence of star products under the condition of vanishing the third de Rham cohomology $H^3_{dR}(M)$ [NV81]. In the paper [WL83b] De Wilde and Lecomte provided an existence proof of star products on tangent and cotangent spaces over a manifold. These results extended the results of Cahen and Gutt [CG82]. Shortly later De Wilde and Lecomte generalized the results to all symplectic manifolds [WL83a]. The first proof used massively sophisticated homological tools which are simplified in [GR99]. In [Kon] Kontsevich formulated the so-called formality conjecture which states that every Poisson manifolds admits a star product. Shortly later, he provided a proof in [Kon97].

In this chapter we want to present a geometric proof given by Fedosov [Fed94], [Fed85], [Fed86] and [Fed89]. Here, we follow the outline of [Wal07]. In the last section, we discuss some ideas of Kontsevich's proof of the formality theorem.

4.1 The Mixed Algebra

Definition 4.1.1. If M is a manifold, then the *symmetric $\mathbb{Z}$-graded algebra $\mathcal{S}$* is defined to be the direct sum of vector spaces of symmetric forms, i.e.

$$\mathcal{S}(M) := \bigoplus_{k=0}^{\infty} \Gamma(\bigvee^{k} T^*M),$$

endowed with a multiplication $\vee$ such that the algebra becomes associative and commutative. The completion of $\mathcal{S}(M)$ with respect to the toplogy induced by the symmetric degree map is the algebra

$$\Sigma(M) := \prod_{k=0}^{\infty} \Gamma(\bigvee^{k} T^*M).$$

The multiplication $\vee$ on the symmetric algebra can be extended in a canonical way to the algebra $\Sigma(M)$ and we will keep the notation for the multiplication map.

The $\mathbb{Z}$-graded algebra of antisymmetric forms are called the *Grassmann algebra* and it is defined to be

$$\Lambda(M) := \bigoplus_{k=0}^{\infty} \Gamma(\bigwedge^{k} T^*M).$$

The multiplication is denoted by $\wedge$.

The constant function 1 is the unit for both algebras.

Notation 4.1.2. The map $|-|_s\colon \mathcal{S}(M) \to \mathbb{Z}$ denotes the degree map of symmetric algebra $\mathcal{S}(M)$ and we will use the same notation for the degree map on $\Sigma(M) \to \mathbb{Z}$. Similarly, $|-|_a\colon \Lambda(M) \to \mathbb{Z}$ denotes the degree map of the Grassmann algebra $\Lambda(M)$.

Definition 4.1.3. Let M be a manifold. As a vector space the *mixed algebra* $\mathcal{M}(M)$ is defined to be

$$\mathcal{M}(M) := \Sigma(M) \otimes_{\mathcal{C}^\infty(M)} \Lambda(M).$$

The multiplication μ is given by evaluation on elements $a, b \in \Sigma(M)$ and $\alpha, \beta \in \Lambda(M)$:

$$\mu(a \otimes \alpha, b \otimes \beta) := (a \vee b) \otimes (\alpha \wedge \beta).$$

Since $\Sigma(M)$ as well as $\Lambda(M)$ are associative, the mixed algebra $\mathcal{M}(M)$ is also associative and the constant functin $1 \otimes_{\mathcal{C}^\infty(M)} 1 = 1$ is the unit of this algebra. It is also clear that for elements $\alpha, \beta \in \Lambda(M)$ with $|\alpha|_a = i$ and $|\beta|_a = j$, we have

$$\mu(a \otimes \alpha, b \otimes \beta) = (-1)^{ij} \mu(b \otimes \beta, a \otimes \alpha).$$

Hence, $\mathcal{M}(M)$ is a graded commutative with respect to the antisymmetric grading.

For a multivector field X, the insertion operator $i_s(X)$ is a derivation on $\Sigma(M)$ of degree -1. The same is true for the insertion operator $i_a(X)$ on $\Lambda(M)$. One can extend these operators to two insertion operators on the mixed algebra $\mathcal{M}(M)$ (we will use the same notation) by evaluation on homogeneous elements:

$$i_s(X)(a \otimes \alpha) := i_s(X)a \otimes \alpha \text{ and } i_a(X)(a \otimes \alpha) := a \otimes i_a(X)\alpha.$$

The lemma below summarizes some useful properties, they easily follow from properties of the derivations $i_s(X)$ and $i_a(X)$.

Lemma 4.1.4. *Let $X \in \mathfrak{X}^*(M)$ be a multivector field on a manifold M. The insertion maps $i_s(X)$ and $i_a(X)$ as defined above are derivations.*

$$i_s(X) \circ \mu(a \otimes \alpha, b \otimes \beta) = \mu \circ (i_s \otimes \mathrm{id} + \mathrm{id} \otimes i_s)(a \otimes \alpha, b \otimes \beta)$$

$$i_a(X) \circ \mu(a \otimes \alpha, b \otimes \beta) = \mu \circ (i_a \otimes \mathrm{id} + (-1)^{|a \otimes \alpha|_a} \mathrm{id} \otimes i_a)(a \otimes \alpha, b \otimes \beta)$$

Definition 4.1.5. Given a mixed algebra $\mathcal{M}(M)$, we define two different degree maps $\deg_s$ and $\deg_a$: $\mathcal{M}(M) \to \mathcal{M}(M)$ by their evaluations on homogeneous elements:

1. $\deg_s(a \otimes \alpha) := |a \otimes \alpha|_s a \otimes \alpha$, if a is homogeneous.
2. $\deg_a(a \otimes \alpha) := |a \otimes \alpha|_a a \otimes \alpha$, if α is homogeneous.

It follows from the definition that $\deg_s \circ \mu(a \otimes \alpha, b \otimes \beta)) = \deg_s(a \vee b \otimes \alpha \wedge \beta) = |a \vee b|_s (a \vee b \otimes \alpha \wedge \beta) = (|a|_s + |b|_s)\mu(a \otimes \alpha, b \otimes \beta)$

Using the bilinearity of the multiplication map μ, we see that

$$\deg_s \circ \mu = \mu \circ (\deg_s \otimes \mathrm{id} + \mathrm{id} \otimes \deg_s).$$

Hence, $\deg_s$ is a graded derivation of degree 0 with respect to the symmetric grading as well as the antisymmetric grading. Here, we want to remark that for brevity we prefer the term "graded derivation" to "formal graded derivation", although the algebra $\Sigma(M)$ is not a graded vector space,

Since we will work in local coordinates later, we want to have a local description of these maps. Given a homogeneous element $a = f dx^1 \vee ... \vee dx^k \in \Sigma(M)$ and an element $\alpha \in \Lambda(M)$, $(dx^i \otimes \mathrm{id}) \circ i_s(\frac{\partial}{\partial x^i})(a \otimes \alpha) = k a \otimes \alpha$. This shows that on homogeneous elements the maps $\deg_s$ and $(dx^i \otimes \mathrm{id}) \circ i_s(\frac{\partial}{\partial x^i})$ have the same value, therefore, they are equal on a local chart. By a similar observation we see that locally $\deg_a$ can also be described by simple functions. The lemma below is a conclusion of these observations.

Lemma 4.1.6. *The following equalities hold locally:*

$$\deg_s = (dx^i \otimes \mathrm{id}) \circ i_s(\frac{\partial}{\partial x^i})$$

$$\deg_a = (\mathrm{id} \otimes dx^i) \circ i_a(\frac{\partial}{\partial x^i})$$

The de Rham differential d and the insertion operators allow us to define two endomorphisms on $\mathcal{M}(M)$. They will play an essential role in the Fedosov construction later.

Definition 4.1.7. Let M be a manifold and let $\{x^1, ..., x^n\}$ be a local coordinate. The maps δ, δ^*: $\mathcal{M}(M) \to \mathcal{M}(M)$ are defined locally by

$$\delta := (\mathrm{id} \otimes dx^i) \circ i_s(\frac{\partial}{\partial x^i}) \text{ and } \delta^* := (dx^i \otimes \mathrm{id}) \circ i_a(\frac{\partial}{\partial x^i})$$

These maps are chart independent, therefore they are also globally defined. Moreover, δ and δ^* have symmetric degrees -1 and 1, respectively, while they have antisymmetric degree 1 and -1, respectively. The following calculation shows that δ is even a derivation with respect to the antisymmetric grading. For homogeneous elements $a \otimes \alpha$ and $b \otimes \beta$, we have

$$\begin{aligned}\delta \circ \mu(a \otimes \alpha, b \otimes \beta) &= (\mathrm{id} \otimes dx^i) \circ i_s(\frac{\partial}{\partial x^i})(a \vee b \otimes \alpha \wedge \beta) \\ &= (i_s(\frac{\partial}{\partial x^i})a \vee b + a \vee i_s(\frac{\partial}{\partial x^i})b) \otimes dx^i \wedge \alpha \wedge \beta \\ &= i_s(\frac{\partial}{\partial x^i})a \vee b \otimes dx^i \wedge \alpha \wedge \beta + (-1)^{|a\otimes\alpha|_a} a \vee i_s(\frac{\partial}{\partial x^i})b \otimes \alpha \wedge dx^i \wedge \beta \\ &= \mu(\delta a \otimes \alpha, b \otimes \beta) + (-1)^{|a\otimes\alpha|_a} \mu(a \otimes \alpha, \delta b \otimes \beta).\end{aligned}$$

A similar calculation reveals that δ^* is also a graded derivation with respect to the antisymmetric grading.

Lemma 4.1.8. *The maps δ and δ^* as defined above are differentials, i.e.*

$$\delta^2 = 0 = \delta^{*2},$$

and

$$\delta\delta^* + \delta^*\delta = \deg_s + \deg_a .$$

Proof. The first claim for δ follows from the observation that

$$\delta^2 = (\mathrm{id} \otimes dx^i) \circ i_s(\frac{\partial}{\partial x^i}) \circ (\mathrm{id} \otimes dx^j) \circ i_s(\frac{\partial}{\partial x^j}) = (\mathrm{id} \otimes dx^i \wedge dx^j) \circ i_s(\frac{\partial}{\partial x^i}) \circ i_s(\frac{\partial}{\partial x^j}) = 0$$

The last equality follows from the fact that operators $i_s(\frac{\partial}{\partial x^i})$ commute with each other, while dx^i anticommutes with each other. A similar argumentation shows that δ^* is also a differential.

As a preparation for proving the second claim, we note that

$$i_s(\frac{\partial}{\partial x^i}) \circ dx^j - dx^j \circ i_s(\frac{\partial}{\partial x^i}) = \delta_{ij}\,\mathrm{id}$$

and

$$i_a(\frac{\partial}{\partial x^i}) \circ dx^j + dx^j \circ i_a(\frac{\partial}{\partial x^i}) = \delta_{ij}\,\mathrm{id}\,.$$

Therefore, we have a chain of equalities:

$$\begin{aligned}\delta\delta^* + \delta^*\delta &= (\mathrm{id} \otimes dx^i) \circ i_s(\frac{\partial}{\partial x^j}) \circ (dx^i \otimes \mathrm{id}) \circ i_a(\frac{\partial}{\partial x^i}) + (dx^i \otimes \mathrm{id}) \circ i_a(\frac{\partial}{\partial x^i}) \circ (\mathrm{id} \otimes dx^j) \circ i_s(\frac{\partial}{\partial x^j}) \\ &= i_s(\frac{\partial}{\partial x^j}) \circ dx^j \otimes dx^i \circ i_a(\frac{\partial}{\partial x^j}) + dx^i \circ i_s(\frac{\partial}{\partial x^j}) \otimes i_a(\frac{\partial}{\partial x^i}) \otimes dx^j \\ &= (\delta_{ij}\,\mathrm{id} + dx^j \circ i_s(\frac{\partial}{\partial x^i})) \otimes dx^i \circ i_a(\frac{\partial}{\partial x^j}) + dx^i \circ i_s(\frac{\partial}{\partial x^j}) \otimes (\delta_{ij}\,\mathrm{id} - dx^j \circ i_a(\frac{\partial}{\partial x^i})) \\ &= \mathrm{id} \otimes dx^i \circ i_a(\frac{\partial}{\partial x^i}) + dx^i \circ i_s(\frac{\partial}{\partial x^i}) \otimes \mathrm{id} + 0 \\ &= \deg_s + \deg_a,\end{aligned}$$

where the last equality follows from lemma 4.1.6. □

If we define a new map δ' such that on homogeneous elements it is given by

$$\delta'(x) = 0 \text{ if } |x|_s + |x|_a = 0$$
$$\delta'(x) = \frac{1}{|x|_s + |x|_a}\delta^* \text{ if } |x|_s + |x|_a \neq 0,$$

then lemma 4.1.8 says that $\delta\delta' + \delta'\delta = \mathrm{id}$, for $x \in \mathcal{M}(M)$ such that $|x|_s + |x|_a \neq 0$. For the general case, let pr: $\mathcal{M}(M) \to \mathcal{C}^\infty(M)$ denote the projection map, then we have

$$\delta\delta' + \delta'\delta = \mathrm{id} - pr.$$

If $\mathcal{M}(M)$ is equipped with the antisymmetric grading, then the map pr induces a map $\{pr_k\}_{k\in\mathbb{N}}$: $\mathcal{M}(M) \to \mathcal{M}(M)$ with

$$pr_k := pr, \text{ if } k = 0$$
$$pr_k := 0, \text{ if } k \neq 0$$

This new map is a chain map of complexes $\mathcal{M}(M)$, i.e. $\delta_n \circ pr_{n-1} = pr_n \circ \delta$. One notes that the degree of this complex is given by the antisymmetric degree. We have seen that δ' has antisymmetric degree $+1$, therefore we can interpret δ' as a homotopy between two chain maps, id and pr. In this case equation $\delta\delta' + \delta'\delta = \mathrm{id} - pr$ says that they are homotopic. Since homotopic maps are in particular induce the same homomorphism in homology, we have

$$H^n(\mathcal{M}(M)) \cong \mathcal{C}^\infty(M), \text{ if } n = 0$$
$$H^n(\mathcal{M}(M)) \cong 0, \text{ if } n \geq 1.$$

Remark 4.1.9. If $\mathcal{M}(M)[[t]] = \mathcal{M}(M) \otimes \mathbb{C}[[t]]$ denotes the deformation of the mixed algebra defined in 4.1.3, then derivations such as δ, δ^* and δ' and degree maps such as $\deg_s$ and $\deg_a$ can be extened to the algebra $\mathcal{M}(M)[[t]]$ in a canonical way and we will keep these notations.

4.2 Useful Derivations on the Mixed Algebra

Definition 4.2.1. Let (M, ω) be a symmetric manifold and let π be the associated Poisson structure. A formal deformation $*$: $\mathcal{M}(M)[[t]] \otimes \mathcal{M}(M)[[t]] \to \mathcal{M}(M)[[t]]$ of the multiplication map μ is defined as follows:

$$x * y := \mu \circ e^{\frac{it}{2}\pi^{ij} i_s(\frac{\partial}{\partial x^i}) \otimes_{\mathcal{C}^\infty(M)} i_s(\frac{\partial}{\partial x^j})}(x, y),$$

where $x, y \in \mathcal{M}(M)[[t]]$. This multiplication map $*$ is called *Weyl-Moyal star product.*

Indeed, by theorem 3.2.7, this star product is an associative formal deformation.

The algebra $\mathcal{M}(M)[[t]]$ comes with the canonical degree map $|-|_t$: $\mathcal{M}(M)[[t]] \to \mathbb{N}$, therefore, we can define a map $\deg_t$ to be

$$\deg_t(x) = |x|_t x$$

for a homogeneous element $x \in \mathcal{M}(M)[[t]]$. Obviously, $\deg_t$ is a derivation on $(\mathcal{M}(M)[[t]], *)$.

The Weyl-Moyal product really defines an associative deformation of the mixed algebra $\mathcal{M}(M)$, because the insertion operators $i_s(\frac{\partial}{\partial x^i})$ commute with each other. Moreover, since the Weyl-Moyal star product is identical with the multiplication μ on the antisymmetric part of $\mathcal{M}(M)[[t]]$, the mixed algebra inherits the antisymmetric grading, $|-|_a$ is its degree map and $\deg_a$ is still a derivation with respect to $*$.

For each degree in t, every insertion operator in the definition of the Weyl-Moyal product reduces the symmetric degree by 2. Therefore, $\deg_s$ cannot be a derivation on $(\mathcal{M}(M)[[t]], *)$ anymore. In order to repair this flaw, we define a new degree map:

Definition 4.2.2. The total degree map Deg on $(\mathcal{M}(M)[[t]], *)$ is defined as

$$\mathrm{Deg} := \deg_s + 2\deg_t .$$

One easily verifies that Deg is a derivation on $(\mathcal{M}(M)[[t]], *)$. The mixed algebra equipped with the Weyl-Moyal star product is $(\mathcal{M}(M)[[t]], *)$ now a graded algebra with respect to Deg. To be more precise, we have

$$\mathcal{M}(M)[[t]] = \prod_{k=0}^{\infty} (\mathcal{M}(M)[[t]])_k$$

where

$$(\mathcal{M}(M)[[t]])_k = \{x \in \mathcal{M}(M)[[t]] | \, \mathrm{Deg}(x) = kx\}.$$

For a symplectic manifold (M, ω), the symplectic form ω can be written as $\omega_a = 1 \otimes \omega \in \mathcal{M}(M)[[t]]$ and it has symmetric degree 0 and antisymmetric degree 2. It is also possible to write ω in the form $\omega_{sa} = \omega_{kl} dx^k \otimes dx^l \in \mathcal{M}(M)[[t]]$, hence it has symmetric degree 1 and antisymmetric degree 1. The three different degree maps are invariant on ω_{sa}, i.e. $\deg_s(\omega_{sa}) = \deg_a(\omega_{sa}) = \mathrm{Deg}(\omega_{sa}) = \omega_{sa}$. The lemma below shows that δ can be defined in terms of ω_{sa} and it provides an alternative proof for the derivation property of δ.

Lemma 4.2.3. *For $\omega_{sa} = \omega_{kl} dx^k \otimes dx^l \in \mathcal{M}(M)[[t]]$, it holds:*

$$\delta = -\frac{i}{t} ad(\omega_{sa}).$$

Proof. Since $*$ is a deformation of μ and μ is graded commutative, in the zero-th level $ad(\omega_{sa}) = 0$. For an $x \in \mathcal{M}(M)[[t]]$, we write $\omega_{sa} * x$ in a sum of elements in $\mathcal{M}(M)$ which are homogeneous in t. The elements of higher t-degree than 1 must vanish, because $\omega_{sa} = \omega_{kl} dx^k \otimes dx^l$ and $i_s(\frac{\partial}{\partial x^i}) \circ i_s(\frac{\partial}{\partial x^j})(\omega_{kl} dx^k \otimes dx^l) = 0$. Therefore, $ad(\omega_{sa})x$ only consists of elements of degree 1 in t.

A straightforward calculation using the equality $\pi^{kl}\omega_{lm} = -\delta^k_m$ now proves the claim. For an

$x \in \mathcal{M}(M)[[t]]$ with $|x|_a = i$, we have:

$$\begin{aligned}
-\frac{i}{t}ad(\omega_{sa})x &= -\frac{i}{t}\frac{it}{2}\pi^{kl}(i_s(\frac{\partial}{\partial x^k})(\omega_{nm}dx^n \otimes dx^m)i_s(\frac{\partial}{\partial x^l})(x) \\
&\quad - (-1)^i i_s(\frac{\partial}{\partial x^k})(x)i_s(\frac{\partial}{\partial x^l})(\omega_{nm}dx^n \otimes dx^m)) \\
&= \frac{1}{2}\pi^{kl}(i_s(\frac{\partial}{\partial x^k})(\omega_{nm}dx^n \otimes dx^m)i_s(\frac{\partial}{\partial x^l})x - i_s(\frac{\partial}{\partial x^l})(\omega_{nm}dx^n \otimes dx^m)i_s(\frac{\partial}{\partial x^k})x) \\
&= \frac{1}{2}\pi^{kl}(\omega_{km}(1 \otimes dx^m)i_s(\frac{\partial}{\partial x^l})x - \omega_{lm}(1 \otimes dx^m)i_s(\frac{\partial}{\partial x^k})x) \\
&= \frac{1}{2}((\mathrm{id} \otimes dx^m) \circ i_s(\frac{\partial}{\partial x^m})x + (\mathrm{id} \otimes dx^m) \circ i_s(\frac{\partial}{\partial x^m})x) = \delta x.
\end{aligned}$$

□

Compared with μ, its deformation $*$ is no longer graded commutative and we can define a nontrivial inner derivation by using $| - |_a$. For $x, y \in \mathcal{M}(M)[[t]]$,

$$ad(x)y = [x, y] = x * y - (-1)^{|x|_a|y|_a} y * x.$$

Since $*$ is a deformation of μ, these two multiplication maps are equal and $[-,-]$ vanishes at the zero-th level. Now we want to determine the center of our algebra $(\mathcal{M}(M)[[t]], *)$.

Lemma 4.2.4. *An element x is in the center of $(\mathcal{M}(M)[[t]], *)$, i.e. x lies in the kernel of ad, if and only if x is in the kernel of $| - |_s$.*

Proof. If x lies in the kernel of ad, then $ad(x) = 0$. This implies, for $y = y_i dx^i \otimes 1$ we have $0 = ad(x)y = 0 + \frac{it}{2}\pi^{kl}(i_s(\frac{\partial}{\partial x^k})x i_s(\frac{\partial}{\partial x^l})y - i_s(\frac{\partial}{\partial x^k})y i_s(\frac{\partial}{\partial x^l})x)) = it\pi^{kl}y_l i_s(\frac{\partial}{\partial x^k})x$. We assumed that manifold M is symplectic, hence the associated Poisson structure π is nondegenerated. It follows that $i_s(\frac{\partial}{\partial x^k})x = 0$ and by the linearity of insertion operatiors, we can assume that x is homogeneous. Then it is clear, that $|x|_s$ must be 0 and x lies in the kernel of $| - |_s$.

Conversely, if x is in the kernel of $| - |_s = 0$, then the Weyl-Moyal product $*$ equals the multiplication μ which is graded commutative. □

It follows from the general theory of symplectic manifold that there always exists a torsionfree covariant derivative ∇ such that $\nabla\omega = 0$, where ω denotes the symplectic form. See also [Hes81] for a proof. One also easily verifies that for every torsionfree covariant derivative and for every k-form α it holds: $d\alpha = dx^i \wedge \nabla_{\frac{\partial}{\partial x^i}}$.

Definition 4.2.5. Let (M, ω) be a symplectic manifold. The antisymmetric degree 1 map D: $\mathcal{M}(M)[[t]] \to \mathcal{M}(M)[[t]]$ is defined locally by

$$D := (\mathrm{id} \otimes dx^i) \circ \nabla_{\frac{\partial}{\partial x^i}}$$

This means, for an object $a \otimes \alpha \in \mathcal{M}(M)$

$$D(a \otimes \alpha) = \nabla_{\frac{\partial}{\partial x^i}} a \otimes dx^i \wedge \alpha + a \otimes dx^i \wedge \nabla_{\frac{\partial}{\partial x^i}}\alpha = \nabla_{\frac{\partial}{\partial x^i}} a \otimes dx^i \wedge \alpha + a \otimes d\alpha.$$

It follows directly from the definition that D has symmetric degree 0 and antisymmetric degree 1.

We calculate for two homogeneous elements x and y in $(\mathcal{M}(M)[[t]],\mu)$:

$$\begin{aligned}
&(\mathrm{id}\otimes dx^i)\circ\nabla_{\frac{\partial}{\partial x^i}}\circ\mu(x,y)\\
&=(\mathrm{id}\otimes dx^i)\circ\mu\circ(\nabla_{\frac{\partial}{\partial x^i}}\times\mathrm{id}+\mathrm{id}\times\nabla_{\frac{\partial}{\partial x^i}})(x,y)\\
&=(\mu\circ((\mathrm{id}\otimes dx^i)\times\mathrm{id})\circ(\nabla_{\frac{\partial}{\partial x^i}}\times\mathrm{id})+\mu\circ((\mathrm{id}\otimes dx^i)\times\mathrm{id})\circ(\mathrm{id}\times\nabla_{\frac{\partial}{\partial x^i}}))(x,y)\\
&=\mu\circ(D\times\mathrm{id}+(-1)^{|x|_a}\,\mathrm{id}\times D)(x,y)
\end{aligned}$$

The first equality follows from the fact that ∇ is a derivation with respect to tensor product, therefore, it is also a derivation with respect to $\vee$ and $\wedge$. For the second and the third equality one observes that

$$\begin{aligned}
(\mathrm{id}\otimes dx^i)\circ\mu(a\otimes\alpha,b\otimes\beta)&=a\vee b\otimes dx^i\wedge\alpha\wedge\beta\\
&=\mu\circ((\mathrm{id}\otimes dx^i)\times\mathrm{id})(a\otimes\alpha,b\otimes\beta)\\
&=a\vee b\otimes(-1)^{|a\otimes\alpha|_a}\alpha\wedge dx^i\wedge\beta\\
&=(-1)^{|a\otimes\alpha|_a}\mu\circ(\mathrm{id}\times(\mathrm{id}\otimes dx^i))(a\otimes\alpha,b\otimes\beta).
\end{aligned}$$

By linearity we conclude that D is a graded derivation on $(\mathcal{M}(M)[[t]],\mu)$. The following proposition states that it remains true for the Weyl-Moyal product.

Proposition 4.2.6. *If D is defined as in 4.2.5, then it is also a graded derivation on $(\mathcal{M}(M)[[t]],*)$. For homogeneous elements x and y we have*

$$D\circ *(x,y)=*\circ(D\otimes\mathrm{id}+(-1)^{|x|_a}\,\mathrm{id}\otimes D)(x,y).$$

Proof. As above, we first verify that $\nabla_{\frac{\partial}{\partial x^i}}$ is a derivation with respect to the Weyl-Moyal product. Recall that Weyl-Moyal product is defined by $*:=\mu\circ e^{\frac{it}{2}\pi^{kl}i_s(\frac{\partial}{\partial x^k}\otimes\frac{\partial}{\partial x^l})}$, therefore, it is necessary that $\nabla_{\frac{\partial}{\partial x^i}}\otimes\mathrm{id}+\mathrm{id}\otimes\nabla_{\frac{\partial}{\partial x^i}}$ commutes with $\pi^{kl}i_s(\frac{\partial}{\partial x^k})\otimes i_s(\frac{\partial}{\partial x^l})$. If we use the equality $[\nabla_{\frac{\partial}{\partial x^m}},i_s(\frac{\partial}{\partial x^n})]=i_s(\nabla_{\frac{\partial}{\partial x^m}}\frac{\partial}{\partial x^n})$, we have

$$\begin{aligned}
&[\nabla_{\frac{\partial}{\partial x^i}}\otimes\mathrm{id}+\mathrm{id}\otimes\nabla_{\frac{\partial}{\partial x^i}},\pi^{kl}i_s(\frac{\partial}{\partial x^k})\otimes i_s(\frac{\partial}{\partial x^l})]\\
&=[\nabla_{\frac{\partial}{\partial x^i}},\pi^{kl}i_s(\frac{\partial}{\partial x^k})]\otimes i_s(\frac{\partial}{\partial x^l})+\pi^{kl}i_s(\frac{\partial}{\partial x^k})\otimes[\nabla_{\frac{\partial}{\partial x^i}},i_s(\frac{\partial}{\partial x^l})]\\
&=\frac{\partial\pi^{kl}}{\partial x^i}i_s(\frac{\partial}{\partial x^k})\otimes i_s(\frac{\partial}{\partial x^l})+\pi^{kl}i_s(\nabla_{\frac{\partial}{\partial x^i}})\frac{\partial}{\partial x^k}\otimes i_s(\frac{\partial}{\partial x^l})+\pi^{kl}i_s(\frac{\partial}{\partial x^k})\otimes i_s(\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^l})\\
&=(\frac{\partial\pi^{kl}}{\partial x^i}+\pi^{ml}\Gamma^k_{im}+\pi^{km}\Gamma^l_{im})i_s(\frac{\partial}{\partial x^k})\otimes i_s(\frac{\partial}{\partial x^l}).
\end{aligned}$$

Since ∇ was assumed to be a symplectic covariant derivative, hence, $\nabla\pi=0$ and in coordinates it reads $\frac{\partial \pi^{kl}}{\partial x^i} + \pi^{ml}\Gamma^k_{im} + \pi^{km}\Gamma^l_{im} = 0$. Thus, we proved our claim and it implies:

$$\begin{aligned}\nabla_{\frac{\partial}{\partial x^i}} \circ * &= \nabla_{\frac{\partial}{\partial x^i}} \circ \mu \circ e^{\frac{it}{2}\pi^{kl} i_s(\frac{\partial}{\partial x^k} \otimes \frac{\partial}{\partial x^l})} \\ &= \mu \circ (\nabla_{\frac{\partial}{\partial x^i}} \otimes \mathrm{id} + \mathrm{id} \otimes \nabla_{\frac{\partial}{\partial x^i}}) \circ e^{\frac{it}{2}\pi^{kl} i_s(\frac{\partial}{\partial x^k} \otimes \frac{\partial}{\partial x^l})} \\ &= \mu \circ e^{\frac{it}{2}\pi^{kl} i_s(\frac{\partial}{\partial x^k} \otimes \frac{\partial}{\partial x^l})} \circ (\nabla_{\frac{\partial}{\partial x^i}} \otimes \mathrm{id} + \mathrm{id} \otimes \nabla_{\frac{\partial}{\partial x^i}}) \\ &= * \circ (\nabla_{\frac{\partial}{\partial x^i}} \otimes \mathrm{id} + \mathrm{id} \otimes \nabla_{\frac{\partial}{\partial x^i}}).\end{aligned}$$

The verification of the equality $(\mathrm{id} \otimes dx^i) \circ *(a \otimes \alpha, b \otimes \beta) = (-1)^{|a \otimes \alpha|_a} * \circ (\mathrm{id} \times (\mathrm{id} \otimes dx^i))(a \otimes \alpha, b \otimes \beta)$ can be proved as in the case of μ, because $*$ equals μ on the antisymmetric part of $\mathcal{M}(M)$. This observation completes the proof. $\square$

Lemma 4.2.7. *For maps D and δ as defined in 4.2.5 and 4.1.7 we have*

$$[D, \delta] = 0.$$

Proof. We first prove that for every graded algebra A, $x \in A$ and a derivation D on A, we have

$$[D, ad(x)] = ad(Dx).$$

Once more, by linearity one only needs to show the identity for homogeneous elements. Also note that $|ad(x)| = |x|$ and $|Dx| = |D| + |x|$.

$$\begin{aligned}[D, ad(x)]y &= D \circ ad(x)(y) - (-1)^{|D||x|} ad(x) Dy \\ &= D(xy - (-1)^{|x||y|} yx) - (-1)^{|D||x|} x(Dy) - (-1)^{|x||Dy| + |x||D|}(Dy)x \\ &= (Dx)y + (-1)^{|D||x|} x(Dy) - (-1)^{|x||y|}(Dy)x + (-1)^{|x||y| + |D||y|} y(Dx) \\ &\quad - (-1)^{|D||x|} x(Dy) + (-1)^{|x||y|}(Dy)x \\ &= (Dx)y + (-1)^{|Dx||y|} y(Dx) \\ &= ad(Dx)y.\end{aligned}$$

This implies that $[D, \delta] = [D, -\frac{i}{t} ad(\omega_{sa})] = -\frac{i}{t} ad(D\omega_{sa})$, but this is zero because $D = (\mathrm{id} \otimes dx^i)\nabla_{\frac{\partial}{\partial x^i}}$ and $\nabla\omega = 0$. $\square$

Definition 4.2.8. Let (M, ω) be a symplectic manifold and let ∇ be a symplectic covariant derivative. If R denotes the associated curvature tensor, i.e. for vector fields X, Y and Z $R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, then $\tilde{R}$ is defined by

$$\tilde{R}(X, Y, U, V) := \omega(X, R(U, V)Y),$$

where X, Y, U, V are vector fields.

In local coordinates, we have

$$\tilde{R} = \frac{1}{4} \omega_{in} R^n_{jkl} dx^i \vee dx^j \otimes dx^k \wedge dx^l$$

and we see that $\tilde{R}$ is an element in $\Gamma(\bigvee^2 T^*M) \otimes \Lambda^2(M) \subset \mathcal{M}(M)$.

Proposition 4.2.9. *If D is the graded derivation as defined in 4.2.5, then it holds*

$$[D, D] = \frac{2i}{t} ad(\tilde{R}),$$

where $\tilde{R}$ is defined as above.

Proof. We have $[D, D] = D^2 + D^2 = 2D^2$ and D^2 is a derivation of antisymmetric degree 2. For $a \otimes \alpha \in \mathcal{M}(M)$, by using torsionfreeness of symplectic covariant derivative we have

$$\begin{aligned}
D^2(a \otimes \alpha) &= D(\nabla_{\frac{\partial}{\partial x^l}} a \otimes dx^k \wedge \alpha + a \otimes d\alpha) \\
&= \nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^l}} a \otimes dx^k \wedge dx^l \wedge \alpha + \nabla_{\frac{\partial}{\partial x^k}} a \otimes dx^k \wedge d\alpha + \nabla_{\frac{\partial}{\partial x^l}} a \otimes d(dx^l \wedge \alpha) + 0 \\
&= \frac{1}{2}(\nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^l}} - \nabla_{\frac{\partial}{\partial x^l}} \nabla_{\frac{\partial}{\partial x^k}}) a \otimes dx^k \wedge dx^l \wedge \alpha.
\end{aligned}$$

Here, in the last equality, we used $d(dx^l \wedge \alpha) = -dx^l \wedge d\alpha$.

Once again, by torsionfreeness $\nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^l}} - \nabla_{\frac{\partial}{\partial x^l}} \nabla_{\frac{\partial}{\partial x^k}} = [\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}] = R(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l})$. Since ∇_X is a derivation with respect to tensor product, it is a derivation with respect to $\vee$. It follows that the commutator $[\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}]$ is also a derivation with respect to $\vee$. On functions it is zero, therefore, it is already determined by evaluation on one-forms. If a is a one-form and X a vector field, then usual formula provides

$$\begin{aligned}
(R(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l})a)(X) &= R(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l})(a(X)) - a(R(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l})X) \\
&= 0 - a(R(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l})X) \\
&= -(dx^j \vee i_s(R(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l})\frac{\partial}{\partial x^j})a)(X) \\
&= -(R^i_{jkl} dx^j \vee i_s(\frac{\partial}{\partial x^i})a)(X)
\end{aligned}$$

Hence, we have

$$D^2(a \otimes \alpha) = -\frac{1}{2}(R^i_{jkl} dx^j \otimes dx^k \wedge dx^l)(i_s(\frac{\partial}{\partial x^i})(a \otimes \alpha))$$

and we want to compare this expression with $\frac{i}{t} ad(\tilde{R})$.

As in the proof of 4.2.3 by using the fact that $*$ is a deformation of a graded commutation multiplication, $\frac{i}{t} ad(\tilde{R})$ is zero in the zero-th level. By applying the explicit formula of the Weyl-Moyal product, we see that in the k-level we have

$$\mu \circ (\pi^{ij} i_s(\frac{\partial}{\partial x^i} \otimes_{\mathcal{C}^\infty(M)} \frac{\partial}{\partial x^j}))^k (x \otimes y) = (-1)^{|x|_a |y|_a + k} \mu \circ (\pi^{ij} i_s(\frac{\partial}{\partial x^i} \otimes_{\mathcal{C}^\infty(M)} \frac{\partial}{\partial x^j}))^k (y \otimes x),$$

for antisymmetric homogeneous objects $x, y \in \mathcal{M}(M)[[t]]$. In particular, we see that objects graded commutate with each other if k is even, it follows that $\frac{i}{t} ad(\tilde{R})$ is zero in the second order of t. Higher orders of t do not exist anyway, because R has symmetric degree 2 and the operator $\pi^{ij} i_s(\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$ decreases the symmetric degree by 1 each time.

Therefore, only the term of t-order 1 remains:

$$
\begin{aligned}
\frac{i}{t}ad(\tilde{R})(x) &= \frac{i}{t}(\tilde{R} * x - x * \tilde{R}) \\
&= \frac{i}{t}\frac{it}{2}(\pi^{ij} i_s \frac{\partial}{\partial x^i}\tilde{R}\frac{\partial}{\partial x^j}x - \pi^{ij} i_s \frac{\partial}{\partial x^i}\tilde{x}\frac{\partial}{\partial x^j}\tilde{R}) \\
&= -\pi^{ij} i_s \frac{\partial}{\partial x^i}\tilde{R}\frac{\partial}{\partial x^j}x \\
&= -\frac{1}{4}\pi^{ij} i_s \frac{\partial}{\partial x^i}\omega_{mn}R^n_{rkl}dx^m \vee dx^r \otimes dx^k \wedge dx^l \frac{\partial}{\partial x^j}x \\
&= -\frac{1}{2}\pi^{ij}\omega_{in}R^n_{rkl}dx^r \otimes dx^k \wedge dx^l \frac{\partial}{\partial x^j}x \\
&= -\frac{1}{2}\delta^j_n R^n_{rkl}dx^r \otimes dx^k \wedge dx^l \frac{\partial}{\partial x^j}x \\
&= D^2(x) = \frac{1}{2}[D, D](x).
\end{aligned}
$$

□

Lemma 4.2.10. *For $\tilde{R}$ as defined in 4.2.8 it holds:*

$$\delta\tilde{R} = 0 \text{ and } D\tilde{R} = 0.$$

Proof. It is possible to prove these identities by a straightforward calculation. One will realize that $\delta\tilde{R}$ follows from the first Bianchi-identity, while $D\tilde{R} = 0$ from the second Bianchi-identity. Here, we will present another proof which uses the results from above.

First, one observes that for a graded derivation of antisymmetric degree 1, such as D, the graded Jacobi rule implies that $[D, [D, D]] = 0$. According to proposition 4.2.9 we have $[D, D] = \frac{2i}{t}ad(\tilde{R})$. Thus,

$$0 = [D, \frac{2i}{t}ad(\tilde{R})] = \frac{2i}{t}ad(D\tilde{R})$$

and $D\tilde{R}$ lies in the center of the algebra. Since $|D|_s = 0$ and $|\tilde{R}|_s = 2$, it holds $\deg_s(D\tilde{R}) = 2D\tilde{R}$. By lemma 4.2.4, $D\tilde{R} = 0$.

In the second case, by lemma 4.2.7 we have $[D, \delta] = 0$. This implies $[\delta, [D, D]] = [[\delta, D], D] - [D, [\delta, D]] = 0$. In the same way one realizes that $ad(\delta\tilde{R})$ lies in the center again. The operator δ decreases the symmetric degree by 1, hence $\deg_s(\delta\tilde{R}) = \delta\tilde{R}$ and $\delta\tilde{R}$ must be 0. □

4.3 Fedosov Derivation

This section is the heart of the Fedosov construction.

Up to now, the reason for the introduction of the algebra $\mathcal{M}(M)[[t]]$ of a symplectic manifold M stays unclear. Although there exists a star product on the algebra $\mathcal{M}(M)[[t]]$, but there is no hint how to pull it back to the algebra of interest, namely $\mathcal{C}^\infty(M)[[t]]$. The idea of Fedosov is to define a degree 1 graded derivation $\mathcal{D}$: $\mathcal{M}(M)[[t]] \to \mathcal{M}(M)[[t]]$ such that the projection map pr: $\mathcal{M}(M)[[t]] \to \mathcal{C}^\infty(M)[[t]]$ is bijective on the intersection of the kernel of both maps $\mathcal{D}$ and $\deg_a$. If one can prove that for any symplectic manifold there is a graded derivation as described, then, of course, the induced bijection provides a star product on $\mathcal{C}^\infty(M)[[t]]$. Maybe, the first derivation that comes to one's mind is the map δ, but after some calculation one will realize that this cannot be the right choice. One can make a more general ansatz and write $\mathcal{D}$ as a power series of derivations

$$\mathcal{D} := -\delta + D + \sum_{k=2}^{\infty} D_k,$$

where $\mathrm{Deg}(D_k) = k$. Since δ and D have total degree -1 and 0 respectively, the infinite sum above is a power series with respect to the total degree. We can require $\mathcal{D}$ to have antisymmetric degree 1, hence, in particular, $\deg_a(D_k) = 1$. Fedosov realized that D_k can be defined to be inner derivations. He made the following ansatz. $\mathcal{D} := -\delta + D + \sum_{k=1}^{\infty} \frac{i}{t} ad(x_k)$, where $x_k \in \mathcal{M}(M)$. One notes that the condition $\deg_a(D_k) = 1$ is equivalent to requiring $\deg_a(x_k) = \deg_a(D_k) = 1$. Moreover, since $\frac{i}{t} ad(x_k)$ must be a derivation of total degree $k-2$, x_k must have total degree k. Since both derivations δ and D are graded derivations of antisymmetric degree 1, then for

$$x := \sum_{k=2}^{\infty} x_k$$

we have a derivation of antisymmetric degree 1

$$\mathcal{D} = -\delta + D + \frac{i}{t} ad(x).$$

4.4 Fedosov Derivation is a Differential

Theorem 4.4.1. *There exist elements $x_k \in \mathcal{M}(M)$ with $\deg_a(x_k) = x_k$ and $\mathrm{Deg}(x_k) = kx_k$ such that for $x := \sum_{k=1}^{\infty} x_k$*

$$\mathcal{D} = -\delta + D + \frac{i}{t} ad(x)$$

is a differential, i.e. $\mathcal{D}^2 = 0$.

The elements x_k are unique if we require in addition that $\delta' x_k = 0$ for all k.

Before we can prove this theorem in proposition 4.4.4 we need some auxiliary results.

Lemma 4.4.2. *For an element x in $\mathcal{M}(M)$, if $\mathcal{D} = -\delta + D + \frac{i}{t} ad(x)$, then*

$$\mathcal{D}^2 = \frac{i}{t} ad(-\delta x + \tilde{R} + Dx + \frac{i}{2t}[x, x]).$$

Proof. Since $|\mathcal{D}|_a = 1$, it holds $\mathcal{D}^2 = \mathcal{D} \circ \mathcal{D} - (-1)^1 \mathcal{D} \circ \mathcal{D} = \frac{1}{2}[\mathcal{D}, \mathcal{D}]$. Therefore, we have

$$\begin{aligned}
\mathcal{D}^2 &= \frac{1}{2}[-\delta + D + \frac{i}{t} ad(x), -\delta + D + \frac{i}{t} ad(x)] \\
&= \frac{1}{2}([\delta, \delta] - [\delta, D] - [\delta, \frac{i}{t} ad(x)] - [D, \delta] + [D, D] \\
&\quad + [D, \frac{i}{t} ad(x)] - [\frac{i}{t} ad(x), \delta] + [\frac{i}{t} ad(x), D] + [\frac{i}{t} ad(x), \frac{i}{t} ad(x)])
\end{aligned}$$

By lemma 4.1.8 the first term vanishes and by lemma 4.2.7 the second and the fourth term also vanish. Lemma 4.2.9 says $[D, D] = \frac{2i}{t} ad(\tilde{R})$ and $[\frac{i}{t} ad(x), \frac{i}{t} ad(x)] = (\frac{i}{t})^2 ad([x, x])$. Since D, δ and x have antisymmetric degree 1, we have

$$\mathcal{D}^2 = \frac{i}{t} ad(-\delta x + \tilde{R} + Dx + \frac{i}{2t}[x, x]).$$

□

Therefore, requiring that $\mathcal{D}$ is a differential is equivalent to requiring the existence of an x such that $-\delta x + \tilde{R} + Dx + \frac{i}{t}[x, x]$ lies in the center of the algebra $\mathcal{M}(M)$. We even claim more, we will show that we can choose x such that the term in question vanishes, see 4.4.4.

Lemma 4.4.3. *For an element x in $\mathcal{M}(M)$ with $|x|_a = 1$, if $\mathcal{D} = -\delta + D + \frac{i}{t} ad(x)$, then*

$$\mathcal{D}(-\delta x + \tilde{R} + Dx + \frac{i}{2t}[x, x]) = 0.$$

Proof. By using the results from previous sections, a straightforward calculation will prove the claim:

$$\begin{aligned}
\mathcal{D}(-\delta x + \tilde{R} + Dx + \frac{i}{2t}[x, x]) &= (-\delta + D + \frac{i}{t} ad(x))(-\delta x + \tilde{R} + Dx + \frac{i}{2t}[x, x]) \\
&= \delta^2 x - \delta\tilde{R} - \delta Dx - \frac{i}{2t}\delta[x, x] - D\delta x + D\tilde{R} + D^2 x + \frac{i}{2t} D[x, x] \\
&- \frac{i}{t} ad(x)\delta x + \frac{i}{t} ad(x)\tilde{R} + \frac{i}{t} ad(x) Dx - \frac{1}{2t^2} ad(x)[x, x] \\
&= -[\delta, D]x - \frac{i}{t}(\frac{1}{2}\delta[x, x] + ad(x)\delta x) + \frac{i}{t}(ad(\tilde{R})x \\
&+ ad(x)\tilde{R}) + \frac{i}{t}(\frac{1}{2} D[x, x] + ad(x) Dx) - \frac{1}{2t^2}[x, [x, x]] = 0
\end{aligned}$$

The third equality follows from $\delta^2 = \delta\tilde{R} = D\tilde{R} = 0$, see lemma 4.1.8 and lemma 4.2.10. Since x, δ and D have antisymmetric degree 1, $ad(x)\delta x = ad(\delta x)x$ and $ad(x)Dx = adxDx$. Then, the last equality is justified by lemma 4.2.7 and by the graded Jacobi identity. □

Now, we are able to prove the existence of an element x such that the Fedosov derivation $\mathcal{D}$ is a differential.

Proposition 4.4.4. *One can construct elements $x_k \in \mathcal{M}(M)$ of antisymmetric degree 1 and total degree $k + 2$ such that*

$$-\delta x + \tilde{R} + Dx + \frac{i}{2t}[x, x] = 0,$$

for $x = \sum_{k=2}^{\infty} x_k$. The construction only depends on D, hence only the symplectic covariant derivative ∇ of the manifold M, if one requires in addition that

$$\delta' x_k = 0$$

holds for every k.

Proof. Since $[x,x] = 2x * x$, we have $-\delta x + \tilde{R} + Dx + \frac{i}{2t}[x,x] = -\delta x + \tilde{R} + Dx + \frac{i}{t}x * x := \chi$. For $x = \sum_{k=2}^{\infty} x_k$, we have

$$\chi = -\delta x_2 - \delta x_3 + \tilde{R} + Dx_2 + \sum_{k=4}^{\infty}(-\delta x_k + Dx_{k-1} + \frac{i}{t}\sum_{l=2}^{k-1} x_l * x_{k+1-l}).$$

We see, if we choose

$$x_2 = 0,$$

then χ is zero in degree 1, because δ decreases the total degree by 1. Furthermore, the choice x_2 and the condition that χ vanishes in degree 2 forces that $\delta x_3 = \tilde{R}$, because D has total degree 0. Since $|\tilde{R}|_a = 2 = |\tilde{R}|_s$, according to the discussion after lemma 4.1.8 we have $\delta\delta'\tilde{R} + \delta'\delta\tilde{R} = \tilde{R}$. By lemma 4.2.10 $\tilde{R} = \delta\delta'\tilde{R}$ and choosing

$$x_3 = \delta'\tilde{R}$$

implies that χ vanishes in degree 2.

The derivation δ is even a differential by lemma 4.1.8, hence, for every $y_3 \in \mathcal{M}(M)$, the object $x_3 = \delta'\tilde{R} + \delta y_3$ also implies that $\chi_2 = 0$. The requirement that x_3 has antisymmetric degree 1 forces that the antisymmetric degree of an nontrivial y_3 to be 0 and $|y_3|_s \geq 1$. Therefore, it must hold

$$\delta' x_3 = \delta'\delta y_3 = -\delta\delta' y_3 + y_3 = y_3,$$

where the last equality follows from the fact that δ' annihilates every element of antisymmetric degree 0. This shows that requiring $\delta' x_3 = 0$ is equivent to requiring $y_3 = 0$.

We saw by the explicit formula of χ that if we choose $x_2 = 0$ and for every $k \geq 1$ an x_k such that $\delta x_{k+3} = Dx_{k+2} + \frac{i}{t}\sum_{l=2}^{k+2} x_l * x_{k+4-l} = Dx_{k+2} + \frac{i}{t}\sum_{l=1}^{k-1} x_{l+2} * x_{k+2-l}$, then χ_{k+2} vanishes. Assuming we have choosen for $k \geq 1$ objects $x_3, ..., x_{k+2}$ in $\mathcal{M}(M)$ such that $\chi_2 = ... = \chi_{k+1} = 0$, then lemma 4.4.3 tells us that $\mathcal{D}\chi = 0$ and in particular we have in $(k+1)$-th degree

$$\mathcal{D}_{-1}\chi_{k+2} + ... + \mathcal{D}_{k-1}\chi_2 = \mathcal{D}_{-1}\chi_{k+2} = 0.$$

Since $\mathcal{D}_{-1} = \delta$, we have $\delta\chi_{k+2} = 0$ and for χ_{k+2} it holds $\chi_{k+2} = -\delta x_{k+3} + Dx_{k+2} + \frac{i}{t}\sum_{l=1}^{k-1} x_{l+2} * x_{k+2-l}$. Hence, it shows that

$$\delta(Dx_{k+2} + \frac{i}{t}\sum_{l=1}^{k-1} x_{l+2} * x_{k+2-l}) = 0.$$

The object $Dx_{k+2} + \frac{i}{t}\sum_{l=1}^{k-1} x_{l+2} * x_{k+2-l}$ in $\mathcal{M}(M)$ has antisymmetric degree 2, therefore the equation above and the identity $\delta\delta' + \delta'\delta = \mathrm{id}$, which holds for $z \in \mathcal{M}(M)$ with $|z|_s + |z|_a \neq 0$, show that

$$\delta\delta'(Dx_{k+2} + \frac{i}{t}\sum_{l=1}^{k-1} x_{l+2} * x_{k+2-l}) = Dx_{k+2} + \frac{i}{t}\sum_{l=1}^{k-1} x_{l+2} * x_{k+2-l}.$$

This means, by choosing

$$x_{k+3} = \delta'(Dx_{k+2} + \frac{i}{t}\sum_{l=1}^{k-1} x_{l+2} * x_{k+2-l})$$

χ_{k+2} vanishes.

As in the case of $k = 3$, we can add an object y_{k+3} to x_{k+3}. For the same reason the symmetric degree of y_{k+3} has to be greater than zero and

$$\delta' x_{k+3} = y_{k+3}.$$

Hence, the condition $\delta' x_k = 0$ for all k guarantees the uniqueness of the elements x_k. □

Remark 4.4.5. The given proof is a constructive proof. For the reader's convenience, we summarize here the recursion formula from above.

1. Starting values of the recursion: $x_2 = 0$ and $x_3 = \delta'\tilde{R}$.
2. The recursion formula: For $k \geq 1$, $x_{k+3} = \delta'(Dx_{k+2} + \frac{i}{t}\sum_{l=1}^{k-1} x_{l+2} * x_{k+2-l})$.

Remark 4.4.6. If the symplectic manifold M is flat, i.e. $R = 0$, then $\tilde{R} = 0$ and by the recursion formula from above $x_k = 0$ for all k. In this case the Fedosov derivation $\mathcal{D}$ is given by

$$\mathcal{D} = -\delta + D.$$

4.5 A Non-constructive Proof of $\mathcal{D}^2 = 0$

Above, we gave the order of constructing an element x such the term $-\delta x + \tilde{R} + Dx + \frac{i}{t}x * x$ vanishes. In particular, this implies that the Fedosov derivation $\mathcal{D}$ is a differential. In the following, we investigate the case, where the two-form $-\delta x + \tilde{R} + Dx + \frac{i}{t}x * x$ does not vanish, but lies in the center of the algebra $\mathcal{M}(M)[[t]]$. We will see that the existence and the uniquess of the element x is depending on two initial data Θ and z. The proof will use the *Banach fixed point theorem*. We decided not to start with this more abstact version of the proof, because the Banach fixed point theorem is an existence proof and it does not give any hint of a construction. Nevertheless, we have to discuss the basic idea of this method, because it is often used in deformation quantization in order to proof the existence of certain objects and we will use the results in the discussion of equivalence of Fedosov star products in the next section.

Definition 4.5.1. Let the grading on $\mathcal{M}(M)[[t]]$ be the total degree and let the map

$$o\colon \mathcal{M}(M)[[t]] \to \mathbb{N} \cup \{\infty\}$$

be defined by

1. $o(x) = \min\{k | x_k \neq 0 \text{ for } x = \sum_{k=0}^{\infty} t^k x_k\}$, if $x \neq 0$.
2. $o(0) = \infty$.

Definition 4.5.2. The *t-adic metric* on $\mathcal{M}(M)[[t]]$ is given by the distance function d: $\mathcal{M}(M)[[t]] \times \mathcal{M}(M)[[t]] \to \mathbb{R} \cup \infty$ defined by

$$d(x,y) = 2^{-o(x-y)}.$$

In the following proposition we recall some basic facts about the distance function d and the t-adic topology on $\mathcal{M}(M)[[t]]$, a proof can be found in [Wal07].

Proposition 4.5.3. *The distance function d as just defined induces an* ultrametric *on $\mathcal{M}(M)[[t]]$, i.e. the following hold for $x, y, z \in \mathcal{M}(M)[[t]]$:*

1. $d(x,y) \geq 0$.
2. $d(x,y) = 0$ *if and only if* $x = y$.
3. $d(x,y) = d(y,x)$
4. $d(x,z) \leq \max(d(x,y), d(y,z))$.

The topological space $(\mathcal{M}(M)[[t]], d)$ is a complete space.

This proposition allows us to apply the Banach fixed point theorem on $\mathcal{M}(M)[[t]]$.

Theorem 4.5.4 (Banach fixed point theorem)**.** *Let M be a complete metric space and let C: $M \to M$ be a contraction, i.e. there exists a $0 \leq q < 1$ such that $d(Cx, Cy) \leq qd(v,w)$ for all $v, w \in M$. There exist a unique fixed point x_∞ of C and for every $x_0 \in M$ and it holds $x_\infty = \lim_{n\to\infty} C^n(x_0)$.*

One easily verifies the lemma below.

Lemma 4.5.5. *A map C: $\mathcal{M}(M)[[t]] \to \mathcal{M}(M)[[t]]$ is a contraction if and only if C increases the degree, i.e. there exists $k \in \mathbb{N} \cup \{\infty\}, k \geq 1$ such that*

$$o(Cv - Cw) \geq k + o(v - w).$$

The main observation now is that one can define a contraction C: $\mathcal{M}(M)[[t]] \to \mathcal{M}(M)[[t]]$ which guarantees the existence of a fixed point x such that $\chi = -\delta x + \tilde{R} + Dx + \frac{i}{t} x * x$ lies in the center of $\mathcal{M}(M)[[t]]$. Moreover, C should preserve the antisymmetric degree of objects if its antisymmetric degree is 1, because $|x|_a$ was assumed to be 1. Once one defined the contraction, the Banach fixed point theorem states that there exists a unique fixed point x, i.e. $C(x) = x$. In lemma 4.4.2 we saw that for $\mathcal{D}$ being a differential the element χ must lie in the center of $\mathcal{M}(M)[[t]]$ and by lemma 4.2.4 this is equivalent to requiring $|\chi|_s = 0$. If it is possible to find an object x for every $\Theta \in \mathcal{M}(M)[[t]]$ with $|\Theta|_s = 0$ such that

$$\Theta = \chi$$

then we are done. As we will see, the uniquess of the element x is provided by two conditions:

$$\delta' x = z \text{ and } pr(z) = 0$$

for element $z \in \mathcal{M}(M)[[t]]$ of total degree 3 and antisymmetric degree 0. For the particular case $\Theta = 0$, we recover the statement of the constructive proof of proposition 4.4.4.

For $\chi = -\delta x + \tilde{R} + Dx + \frac{i}{t} x * x$, we observe that x has total degree greater than 2, hence, δx has total degree greater than 1. Since the other terms in χ all have total degree greater than 2, the total degree of χ is greater than 1. If χ lies in the center, its symmetric degree must vanish, therefore, its total degree is even and it is two times its t-degree. Therefore, we have for the two-form Θ:

$$\Theta = \sum_{k=1}^{\infty} t^k \Theta_k.$$

Lemma 4.5.6. *Suppose there is an element x of total degree 2 and antisymmetric degree 1, such that $\chi = -\delta x + \tilde{R} + Dx + \frac{i}{t} x * x$ lies in the center, then χ is a closed form.*

Proof. Since χ lies in the center, by lemma 4.2.4, it has vanishing symmetric degree. It follows that $\mathcal{D}\chi = D\chi$, because δ decreases the symmetric degree and $\frac{i}{t} ad(x)$ is an inner derivation. By the definition of the derivation D, we have $D\chi = d\chi$. The claim of the lemma now follows from lemma 4.4.3. $\square$

This lemma implies in particular that if $\Theta = \sum_{k=1}^{\infty} t^k \Theta_k$, then every Θ_k is a closed two-form.

Theorem 4.5.7. *Let Θ be a closed two-form in $\mathcal{M}(M)[[t]]$ which lies in the center. Let z be an element in $\mathcal{M}(M)[[t]]$ of total degree 3 and $|z|_a = 0$. There exists $x \in \mathcal{M}(M)[[t]]$ of total degree 2 and $|x|_a = 1$ such that*

$$\Theta = -\delta x + \tilde{R} + Dr + \frac{i}{t} x * x.$$

Moreover, x is unique, if we also assume

$$\delta' x = z.$$

Proof. If there exists an x satisfying the condition in the theorem, then by applying δ' to the equation $\Theta = -\delta x + \tilde{R} + Dx + \frac{i}{t} x * x$ from the left, we have $\delta'\Theta = -\delta'\delta x + \delta'(\tilde{R} + Dx + \frac{i}{t} x * x)$. For objects which do not lie in $\mathcal{C}^\infty(M)[t]] \subset \mathcal{M}(M)[[t]]$, the identity $\delta\delta' + \delta'\delta = \mathrm{id}$ and the assumption $\delta' x = z$ imply that

$$x = \delta'(\tilde{R} + Dx + \frac{i}{t} x * x - \Theta) + \delta z.$$

This is a necessary condition for the existence of such an object x.

If we define a map C by

$$C(v) = \delta'(\tilde{R} + Dv + \frac{i}{t} v * v - \Theta) + \delta z,$$

$v \in \mathcal{M}(M)[[t]]$, and if we can verify that C is a contraction, then the Banach fixed point theorem guarantees the existence of such a fixed point x which is even unique.

For the contraction condition, we first check that $|Cv|_a = 1$, if $|v|_a = 1$. Since $|Dv|_a = |\tilde{R}|_a = |\Theta|_a = |\frac{i}{t} x * x|_a = 2$, all the elements $\delta' Dv$, $\delta'\tilde{R}$, $\delta'\Theta$ and $\delta' \frac{i}{t} x * x$ have antisymmetric degree 1, $|Cv|_a = 1$. For two objects v and v' with $|v|_a = 1 = |v'|_a = |v - v'|_a$, if $v - v'$ have total degree greater than n, then we want to show that after applying the map C the element $C(v) - C(v')$

has a total degree which is greater than $n+1$. We have

$$Cv - Cv' = \delta' D(v - v') + \frac{i}{t}\delta'(v * v - v' * v'),$$

where the first term has total degree $n+1$, because the map $\delta' D$ increases the symmetric degree by 1. The object $v - v'$ was assumed to have total degree greater than n, therefore $(v - v') * v'$ and $v * (v - v')$ have total degree greater than $n+2$. Since the map $\frac{i}{t}\delta'$ decreases the total degree by 1, the object

$$\frac{i}{t}\delta'(v * v - v' * v') = \frac{i}{t}\delta'((v - v') * v' + v * (v - v'))$$

has total degree greater than $n+1$. By lemma 4.5.5 this observation shows that C is indeed a contraction and there exists a unique x such that $Cx = x$. Since $|z|_a = 0$, we have $\delta' z = 0$ and $\delta' x = \delta' \delta z = z$. Therefore, the fixed point of C already satisfies the equation $\delta' x = z$.

Nevertheless, we still have to show that $x = \delta'(\tilde{R} + Dx + \frac{i}{t}x * x - \Theta) + \delta z$ is not only a necessary but also a sufficient condition. This means, we have to verify that the fixed point of C also satisfies $\Theta = -\delta x + \tilde{R} + Dx + \frac{i}{t}x * x$.

If we define

$$a := \delta x - \tilde{R} - Dx - \frac{i}{t}x * x + \Theta,$$

then we have

$$\begin{aligned}
\delta a &= -\delta D x - \frac{i}{t}(\delta x * x - x * \delta x) - \delta\Theta = D\delta x + \frac{i}{t}[x, \delta x] \\
&= D(a + \tilde{R} + Dx + \frac{i}{t}x * x - \Theta) + \frac{i}{t}ad(x)(a + \tilde{R} + Dx + \frac{i}{t}x * x - \Theta) \\
&= Da + D\tilde{R} + \frac{i}{t}[\tilde{R}, x] + \frac{i}{t}D(x * x) + \frac{i}{t}[x, \tilde{R}] + \frac{i}{t}[x, Dx] - \frac{1}{t^2}[x, [x, x]] \\
&= Da + \frac{i}{t}[x, a],
\end{aligned}$$

where the first and the second equalities follows from $\delta\tilde{R} = 0$ and the equation $\delta D + D\delta = 0$ (see lemma 4.2.10 and lemma 4.2.7). The third equality is justified by $D\tilde{R} = 0$, $D^2 x = \frac{i}{t}ad\tilde{R}x$, $D\Theta = d\Theta = 0$ (see lemma 4.2.10, 4.4.2 and 4.5.6) and the Jacobi-identity applied on elements of antisymmetric degree 1. For the last equality, one notes that $[\tilde{R}, x] = -[x, \tilde{R}]$ and $D(x * x) = -[x, Dx]$, because $|\tilde{R}|_a = 1 = |Dx|_a$ and $x * x = \frac{1}{2}[x, x]$.

On the other side, since x is the fixed point of contraction C, we have

$$\begin{aligned}
\delta' a &= \delta'(\delta x - \tilde{R} - Dx - \frac{i}{t}x * x + \Theta) \\
&= \delta'\delta x - x + \delta z = \delta z - \delta z = 0.
\end{aligned}$$

The object a has antisymmetric degree 2, therefore, we can apply the equation $\delta\delta' + \delta'\delta = \mathrm{id}$ to a. By using the previous results, we have

$$a = \delta'(DA + \frac{i}{t}[x, a]).$$

We can interpret this result by saying that a is a fixed point of the map

$$C' := \delta'(D + \frac{i}{t}ad(x)).$$

Since $D+\frac{i}{t}ad(x)$ does not decrease the total degree, while δ' increases it by 1, by applying lemma 4.5.5, we see that C' is a contraction too. Obviously, 0 is a fixed point, therefore, by uniqueness $a = 0$ and a fixed point of C already satisfies the claim. □

Notation 4.5.8. Sometimes one uses the notation $*_{(\nabla,\Theta,z)}$, in order to emphasize the dependence of a Fedosov star product on its inital data (∇, Θ, z). Here, ∇, Θ and z denote a symplectic covariant derivative, a closed symmetric two-form and an object in $\mathcal{M}(M)[[t]]$ of total degree 3 and antisymmetric degree 0, respectively, and they satisfy the conditions as stated above.

4.6 Fedosov Taylor Series

In this section we want to examine the elements in the kernel of the Fedosov derivation which have antisymmetric degree 0. Obviously, the these objects form a subalgebra in $\mathcal{M}(M)[[t]]$ which will denoted by $\ker_0$. We will introduce a linear map T: $\mathcal{C}^\infty(M)[[t]] \to \mathcal{M}(M)[[t]]$ which will provide a bijection between $\mathcal{C}^\infty(M)[[t]] \subset \mathcal{M}(M)[[t]]$ and $\ker_0$. The evaluation of T on a function f is called the *Fedosov Taylor series* of f. This map is the most important map in the Fedosov construction, because, as we will see in the following section, T allows one to define a star product on $\mathcal{C}^\infty(M)[[t]]$, called the Fedosov star product.

The motivation behind all the nontrivial definitions given below is the idea to "deform" the equation $\delta\delta' + \delta'\delta = \mathrm{id} - pr$. By definition, the Fedosov derivation $\mathcal{D} = -\delta + D + \frac{i}{t}ad(x)$, hence it is a deformation of δ because only δ has total degree 0. The theorem 4.4.1 guarantees that it is also a differential. Therefore, in order to deform $\delta\delta' + \delta'\delta = \mathrm{id} - pr$ we have to find the right maps replacing δ' and pr.

Proposition 4.6.1. *Let the object x and the differential $\mathcal{D}$ be as defined in theorem 4.4.1. If the map γ denotes $[\delta', D + \frac{i}{t}ad(x)]$, then the identity is homotopic to $(\mathrm{id} - \gamma)^{-1}pr$, i.e. it holds*

$$\mathcal{D}\mathcal{D}' + \mathcal{D}'\mathcal{D} = \mathrm{id} - (\mathrm{id} - \gamma)^{-1}pr.$$

The map $\mathcal{D}'$: $\mathcal{M}(M)[[t]] \to \mathcal{M}(M)[[t]]$ has total degree -1 and it is defined as follows

$$\mathcal{D}' := -\delta'(\mathrm{id} - \gamma)^{-1}.$$

Proof. We will prove the proposition in two steps. First we show that δ' and $\mathcal{D}'$ commute with each other and then we will use this fact to prove the homotopy property.

Since δ' has total degree 1 and $D+\frac{i}{t}ad(x)$ does not decrease it, the map γ increases the total degree at least by 1. Therefore, for every $z \in \mathcal{M}(M)[[t]]$, the series $\sum_{k=0}^{\infty} \gamma^k(z)$ is convergent in the complete metric space $\mathcal{M}(M)[[t]]$. This shows that $\sum_{k=0}^{\infty} \gamma^k$ is the inverse map of $\mathrm{id} - \gamma$ and the definition of $\mathcal{D}'$ makes sense. We calculate

$$\begin{aligned} pr - \mathcal{D}\delta' - \delta'\mathcal{D} &= pr + \delta\delta' - (D + \frac{i}{t}ad(x)) \circ \delta' + \delta'\delta - \delta' \circ (D + \frac{i}{t}ad(x)) \\ &= \mathrm{id} - [\delta', D + \frac{i}{t}ad(x)] = \mathrm{id} - \gamma. \end{aligned}$$

Since δ' has symmetric degree 1 and antisymmetric degree -1, it holds $\delta' pr = 0$. Together with $\delta'^2 = 0$ (see lemma 4.1.8) this identities imply that by multiplying δ' to the equation above from the left, we get

$$-\delta'\mathcal{D}\delta' = \delta' - \delta'\gamma.$$

Similarly, by multiplying δ' from the right, we get

$$-\delta'\mathcal{D}\delta' = \delta' - \gamma\delta'.$$

This shows that δ commutes with γ and therefore also with $\sum_{k=0}^{\infty} \gamma^k = (\mathrm{id} - \gamma)^{-1}$.

Now, we use the same trick once more, but with $\mathcal{D}$ instead of δ'. A multiplication from the left gives this time the equation

$$-\mathcal{D}\delta'\mathcal{D} + \mathcal{D}pr = \mathcal{D} - \mathcal{D}\gamma,$$

while a multiplication from the right gives

$$-\mathcal{D}\delta'\mathcal{D} = \mathcal{D} - \gamma\mathcal{D}.$$

Note that in the calculation we used the fact that $\mathcal{D}$ increases the antisymmetric degree by 1 and that it is a differential, see theorem 4.4.1. The last two equations imply that $\mathcal{D}(\mathrm{id} - \gamma) = \mathcal{D}pr + (\mathrm{id} - \gamma)\mathcal{D}$ and therefore, by $pr\delta' = 0$, it holds $\mathcal{D}(\mathrm{id} - \gamma)\delta' = (\mathrm{id} - \gamma)\mathcal{D}\delta'$. By $[\gamma, \delta'] = 0$, it follows that

$$(\mathrm{id} - \gamma)^{-1}\mathcal{D}\delta' = \mathcal{D}(\mathrm{id} - \gamma)^{-1}\delta'.$$

Moreover, we have

$$\begin{aligned} \mathrm{id} &= (\mathrm{id} - \gamma)^{-1}(pr - \mathcal{D}\delta' - \delta'\mathcal{D}) \\ &= (\mathrm{id} - \gamma)^{-1}pr - \mathcal{D}(\mathrm{id} - \gamma)^{-1}\delta' - (\mathrm{id} - \gamma)^{-1}\delta'\mathcal{D} \\ &= \mathcal{D}\mathcal{D}' + \mathcal{D}'\mathcal{D} + (\mathrm{id} - \gamma)^{-1}pr. \end{aligned}$$

□

Lemma 4.6.2. *Let γ denote $[\delta', D + \frac{i}{t}ad(x)]$. If x is an element in $\mathcal{M}(M)[[t]]$ of antisymmetric degree 0, then $\mathcal{D}x = 0$ if and only if*

$$x = (\mathrm{id} - \gamma)^{-1}pr(x).$$

Proof. If $\mathcal{D}x = 0$, then, by proposition 4.6.1

$$x = \mathcal{D}\mathcal{D}'x + (\mathrm{id} - \gamma)^{-1}pr(x) = (\mathrm{id} - \gamma)^{-1}pr(x).$$

The last equality follows from $\mathcal{D}'(x) = -\delta'(\mathrm{id} - \gamma)^{-1}(x) = -(\mathrm{id} - \gamma)^{-1}\delta'(x)$ and $\delta'(x) = 0$.

If $x = (\mathrm{id} - \gamma)^{-1} pr(x)$, then by the theorem above and by $\delta' x = 0$, we have $\mathcal{D}\mathcal{D}'x + \mathcal{D}'\mathcal{D}x = \mathcal{D}'\mathcal{D}x = 0$. Since $|\mathcal{D}x|_a \geq 1$, it holds $\mathcal{D}x = \mathcal{D}\mathcal{D}'\mathcal{D}x$ which is 0, because $\mathcal{D}'\mathcal{D}x = 0$. □

As we will see shortly later, this lemma provides that the map $(\mathrm{id} - \gamma)^{-1}$ induces an isomomorphism between the subalgebra $\ker_0$ and the algebra $\mathcal{C}^\infty(M)[[t]]$. In particular, the objects in $\ker_0$ are totally determined by the evaluation of the projection map pr. Because of the importance of the map $(\mathrm{id} - \gamma)^{-1}$, it deserves a name.

Definition 4.6.3. For $f \in \mathcal{C}^\infty(M)[[t]]$, the *Fedosov Taylor series* of f is defined by

$$T(f) := (\mathrm{id} - [\delta', D + \frac{i}{t} ad(x)])^{-1}(f).$$

The notation T should suggest the relationship with the ordering map S and we will see in definition 4.7.1 that T really plays the role of S in definition 3.6.1.

It is clear that for an f in $\mathcal{C}^\infty(M)[[t]]$, the Fedosov Taylor series is given by

$$T(f) = \sum_{k=0}^{\infty} [\delta', D + \frac{i}{t} ad(x)]^k (f).$$

Since $[\delta', D + \frac{i}{t} ad(x)]$ increases the symmetric degree at least by 1, we have $pr \circ [\delta', D + \frac{i}{t} ad(x)] = 0$ and on the algebra $\mathcal{C}^\infty(M)[[t]]$,

$$pr \circ T = \mathrm{id}\,.$$

By lemma 4.6.2, we already know that $T \circ pr = id$ on $\ker_0$. Hence, T: $\mathcal{C}^\infty(M)[[t]] \to \ker_0$ is an algebra isomomorphism with the inverse map pr.

Lemma 4.6.4. *For $f \in \mathcal{C}^\infty(M)[[t]]$, it holds*

1. $T(f)_0 = f$
2. $T(f)_1 = df$.

Proof. The total degree of $\frac{i}{t} ad(x)$ and δ' is 1, therefore, the map $[\delta', \frac{i}{t} ad(x)]$ increases the total degree at least by 2. Moreover, it holds

$$\begin{aligned}
[\delta', D](f) &= \delta' D(f) - D\delta'(f) = \delta' D(f) \\
&= (dx^i \otimes \mathrm{id}) \circ i_a(\frac{\partial}{\partial x^i}) \circ D(f) \\
&= (dx^i \otimes \mathrm{id}) \circ i_a(\frac{\partial}{\partial x^i})(\nabla_{\frac{\partial}{\partial x^i}} f \otimes dx^i) \\
&= \frac{\partial f}{\partial x^i} dx^i \otimes 1 = df \otimes 1 = df.
\end{aligned}$$

Altogether, we have $T(f) = (\mathrm{id} + [\delta', D] + ...)(f) = f + df + ...$ □

4.7 Fedosov Star Products

By lemma 4.6.2

$$T\colon \mathcal{C}^\infty(M)[[t]] \to \ker_0$$

is a bijection. Using this bijection we can define a star product.

Definition 4.7.1. Let T be defined as in definition 4.6.3 and let $*$ denotes the Weyl-Moyal star product on $\mathcal{M}(M)[[t]]$, then the *Fedosov star product* $*$ on $\mathcal{C}^\infty(M)[[t]]$ is defined by

$$f *_{fed} g := pr(T(f) * T(g)).$$

Proposition 4.7.2. *Let $*_{fed}$ denote the Fedosov star product, then it is a star product in the sense of definition 3.2.1. In particular, for f and g it holds*

$$f *_{fed} g = fg + \frac{it}{2}\{f, g\} + ...$$

Moreover, the Fedosov star product is a differential star product (see definition 3.2.3).

Proof. First, the Fedosov star product is associative, because

$$\begin{aligned}(f *_{fed} g) *_{fed} h &= pr(T(pr(Tf * Tg)) * Th) = pr((Tf * Tg) * Th)\\ &= pr(Tf * (Tg * Th)) = pr(Tf * T(pr(Tg * Th)))\\ &= f *_{fed} (g *_{fed} h).\end{aligned}$$

The second and the fourth equality follow from $T \circ pr = \mathrm{id}$ (see lemma 4.6.2).

Since 1 is the unit with respect to the Weyl-Moyal product and since $T(1) = 1$, 1 is also the unit with respect to the Fedosov star product, i.e.

$$f *_{fed} 1 = 1 *_{fed} f.$$

For the lowest degree of the product $f *_{fed} g$ we calculate

$$\begin{aligned}f *_{fed} g &= pr(Tf * Tg) = pr(\mu \circ e^{\frac{it}{2}\pi^{ij} i_s(\frac{\partial}{\partial x^i}) \otimes_{\mathcal{C}^\infty(M)} i_s(\frac{\partial}{\partial x^j})}(Tf \otimes Tg))\\ &= pr(\mu + \mu \circ \frac{it}{2}\pi^{ij} i_s(\frac{\partial}{\partial x^i}) \otimes i_s(\frac{\partial}{\partial x^j}) + ...)(f + df + ... \otimes g + dg + ...)\\ &= fg + \frac{it}{2}\{f, g\} + ...\end{aligned}$$

The last equality follows from the fact that pr is trivial on elements with either nonzero symmetric degree or nonzero antisymmetric degree.

The last claim is verified by the observation that $*$ is a sum of compositions of insertion operators while T is a power series of differential operators, hence as a composition of these maps $*_{fed}$ is a differential star product. □

In remark 4.5.7 we have seen that for an element Θ in $\mathcal{M}(M)[[t]]$ of total degree greater than

1 and symmetric degree 0, one can always find an $x \in \mathcal{M}(M)[[t]]$ such that

$$\delta x = \tilde{R} + Dx + \frac{i}{t} x * x + \Theta.$$

In order to guarantee the uniqueness, we further assumed that

$$\delta' x = z,$$

for a z with total degree 3 and antisymmetric degree 0. Although the Fedosov star product does not need to be hermitian in general, it is hermitian, if we choose these two initial values properly.

Proposition 4.7.3. *If the objects Θ and z defined above are real, then it holds*

1. $\overline{(-)} \circ \mathcal{D} = \mathcal{D} \circ \overline{(-)}$ *and* $\overline{(-)} \circ \mathcal{D}' = \mathcal{D}' \circ \overline{(-)}$
2. $\overline{(-)} \circ T = T \circ \overline{(-)}$
3. $\overline{(-)} \circ *_{fed} = *_{fed} \circ \overline{(-)} \otimes \overline{(-)} \circ \tau$

Here, $\overline{(-)}$ denotes the complex conjugation map and τ denotes the flipping map in definition 1.4.1.

Proof. For objects y with $|y|_a = i$ and y' with $|y'|_a = j$, the equations below

$$\begin{aligned}\overline{(-)} \circ *(y \otimes y') &= \overline{(-)} \circ \mu \circ e^{\frac{it}{2}\pi^{ij} i_s(\frac{\partial}{\partial x^i}) \otimes_{\mathcal{C}^\infty(M)} i_s(\frac{\partial}{\partial x^j})}(y \otimes y') \\ &= (-1)^{ij} \mu \circ e^{\frac{it}{2}\pi^{ij} i_s(\frac{\partial}{\partial x^i}) \otimes_{\mathcal{C}^\infty(M)} i_s(\frac{\partial}{\partial x^j})} \circ \tau \circ \overline{(-)} \otimes \overline{(-)}(y \otimes y')\end{aligned}$$

show that $\overline{y * y'} = (-1)^{ij} \overline{y'} * \overline{y}$. Hence, we have $\overline{\frac{i}{2t}[y, y']} = -\frac{i}{2t}(\overline{y * y'} - (-1)^{ij}\overline{y' * y}) = \frac{i}{2t}(\overline{y} * \overline{y'} - (-1)^{ij}\overline{y'} * \overline{y}) = \frac{i}{2t}[\overline{y}, \overline{y'}]$. Moreover, we see that for elements with vanishing antisymmetric degree we have the following identity $\overline{(-)} \circ * = * \circ (\overline{(-)} \otimes \overline{(-)}) \circ \tau$.

Since δ, δ', D and pr are real maps and $\frac{1}{2}[x, x] = x * x$, we have

$$\delta\overline{x} = \overline{\tilde{R} + Dx + \frac{i}{t} x * x + \Theta} = \overline{\tilde{R}} + D\overline{x} + \frac{i}{t}\overline{x} * \overline{x} + \overline{\Theta} = \tilde{R} + D\overline{x} + \frac{i}{t}\overline{x} * \overline{x} + \Theta.$$

The last equality follows from the observation $\tilde{R} = \overline{\tilde{R}}$ and the assumption $\Theta = \overline{\Theta}$. Together with

$$\delta'\overline{x} = \overline{\delta' x} = \overline{s} = s = \delta' x$$

we realize that both elements x and $\overline{x}$ satisfy the same initial conditions. By the uniquess, this implies that $x = \overline{x}$.

Now we calculate for an object $y \in \mathcal{M}(M)[[t]]$:

$$\overline{\mathcal{D}y} = -\delta\overline{y} + D\overline{y} + \frac{i}{t}[x, \overline{y}] = \mathcal{D}\overline{y}.$$

Similar, one proves that the complex conjugation commutes with $\mathcal{D}'$. For $f \in \mathcal{C}^\infty(M)[[t]]$ and for every k, it holds $\overline{[\delta', D + \frac{i}{t}ad(x)]^k(f)} = [\delta', D + \frac{i}{t}ad(x)]^k(\overline{f})$. Therefore, it is clear that T

commutes with the complex conjugation. Finally we have:

$$\overline{(-)} \circ *_{fed} = \overline{(-)} \circ pr \circ * \circ (T \otimes T) = pr \circ * \circ (T \otimes T) \circ (\overline{(-)} \otimes \overline{(-)}) \circ \tau = *_{fed} \circ \overline{(-)} \otimes \overline{(-)} \circ \tau.$$

□

4.8 Equivalences of Fedosov Star Products

The existence of a star product on a symplectic manifold was proven in the last section by constructing a Fedosov star product explicitly. The more general theorem discussed in remark 4.5.7 implies that there exists a Fedosov star product $*_{\nabla,\Theta,z}$ for every choosen triple (∇, Θ, z). If we are thinking about the function which maps every Fedosov star product $*_{\nabla,\Theta,z}$ to its equivalence class of star products, then two questions immediately arise.

1. Is this map injective?
2. Is this map even surjective, i.e. is every star product equivalent to a Fedosov star product?

Concerning the first question, Fedosov gave an answer in [Fed94] for the case of trivial s, and general case is treated by Neumaier in [Neu01] and [Neu02]. It will reveal that the objects Θ in the Fedosov construction plays an essential role in the formal deformation theory of symplectic manifolds. We will not prove the theorem and refer the reader to the sources mentioned above.

Theorem 4.8.1. *The function which maps each Fedosov star product $*_{\nabla,\Theta,z}$ to its equivalence class of star product is not injective. Two Fedosov star products $*_{\nabla,\Theta,z}$ and $*_{\nabla',\Theta',z'}$ are equivalent if and only if $[\Theta] = [\Theta']$ in $tH^2_{dR}(M, \mathbb{C})[[t]]$.*

This surprising result states in particular that even if a Fedosov star product $*_{\nabla,\Theta,z}$ depends on the initial values (∇, Θ, z), its equivalence class only depends on the de Rham cohomology class of the closed two form Θ.

Since we will use the Jacobi identity in the proof of the lemma below, we want to introduce a new notation.

Notation 4.8.2. Let $a \in C^2_{hoch}(\mathcal{C}^\infty(M))$, then $\mathcal{S}(a(f,g,h))$ denotes $a(f,g,h)+a(g,h,f)+a(h,f,g)$, for $f, g, h \in \mathcal{C}^\infty(M)$.

In particular, if $\mathfrak{g}$ is a Lie algebra and $[-,-]$ is its Lie bracket, then, for $a, b, c \in \mathfrak{g}$, $\mathcal{S}([a,[b,c]]) = [a,[b,c]] + [b,[c,a]] + [c,[a,b]]$ and requiring $[-,-]$ satisfies the Jacobi identity is equivalent to requiring $\mathcal{S}([a,[b,c]]) = 0$.

Lemma 4.8.3. *If two local star products $*$ and $*'$ on a symplectic manifold (M, ω) are equal up to degree n, then for degree $n+1$ there is $a_{n+1} \in C^1_{loc}(\mathcal{C}^\infty(M))$ and a closed two-form α_{n+1} such that*

$$F_{n+1}(f,g) - F'_{n+1}(f,g) = (\delta a_{n+1})(f,g) + \alpha_{n+1}(X_f, X_g),$$

where X_f and X_g denote the Hamilton vector fields associated with the functions f and g.

Proof. According to the discussion after proposition 2.4.3 it holds $\delta_H(F_{n+1} - F'_{n+1}) = 0$. Theorem 2.5.9 provides the existence of $a \in C^1_{loc}(\mathcal{C}^\infty(M))$ and a bivector field X such that $F_{n+1} - F'_{n+1} = \delta a_{n+1} + \mathscr{F}_1(X)$. For two functions f and g, by the definition of the Hochschild-Kostant-Rosenberg map $\mathscr{F}_1$, $\mathscr{F}_1 X(f,g) = \frac{1}{2}X(df,dg)$. Since the symplectic form ω is nondegenerated, there exists a two-form α such that $X(df,dg) = \alpha_{n+1}(X_f, X_g)$. Therefore, we have

$$F_{n+1}(f,g) - F'_{n+1}(f,g) = \delta a_{n+1}(f,g) + \alpha_{n+1}(X_f, X_g).$$

If we can show the closeness, the proof is done.

If $[-,-]_*$ denotes the commutator with respect to the star product $*$, then, for two functions f and g, we have

$$[f,g]_* = f*g - g*f = \sum_{i=1}^{\infty} t^i C_i(f,g),$$

for some $C_i \in C^2_{loc}(\mathcal{C}^\infty(M))$. Note that the sum starts by 1, because μ is commutative. Moreover the commutator is a Lie bracket, because $*$ is associative. Since $*$ as well as $*'$ is associative, they both satisfy the Jacobi identity. If we use the notation of 4.8.2, we have for $f,g,h \in \mathcal{C}^\infty(M)$

$$\begin{aligned}
0 &= \mathcal{S}([f,[g,h]_*]_*) - \mathcal{S}([f,[g,h]_{*'}]_{*'}) \\
&= \mathcal{S}C_1(f, C_{n+1}(g,h)) + \mathcal{S}C_{n+1}(f, C_1(g,h)) - (\mathcal{S}C'_1(f, C'_{n+1}(g,h)) + \mathcal{S}C'_{n+1}(f,(C'_1(g,h)))) \\
&= \mathcal{S}\{f, C_{n+1}(g,h) - C'_{n+1}(g,h)\} + \mathcal{S}((C_{n+1} - C'_{n+1})(f,\{g,h\})).
\end{aligned}$$

The second equality follows from the assumption that $*$ and $*'$ are equal up to degree n, while we used the star product property $C_1(f,g) = i\{f,g\} = C'_1(f,g)$ in the third equality. Furthermore, by the symmetry of δa_{n+1} and by the antisymmetry of α_{n+1}, we have the following auxiliary step

$$\begin{aligned}
(C_{n+1} - C'_{n+1})(f,g) &= F_{n+1}(f,g) - F'_{n+1}(f,g) - (F_{n+1}(g,f) - F'_{n+1}(g,f)) \\
&= \delta a_{n+1}(f,g) + \alpha_{n+1}(X_f,X_g) - \delta a_{n+1}(g,f) - \alpha_{n+1}(X_g,X_f) \\
&= 2\alpha_{n+1}(X_f,X_g).
\end{aligned}$$

Therefore, the calculation from above goes on with

$$\begin{aligned}
0 &= -\mathcal{S}(X_f(2\alpha_{n+1}(X_g,X_h))) + 2\mathcal{S}(\alpha_{n+1}(X_f, X_{g,h})) \\
&= -2\mathcal{S}(X_f(\alpha_{n+1}(X_g,X_h)) + \alpha_{n+1}(X_f,[X_g,X_h])) \\
&= -2d\alpha_{n+1}(X_f,X_g,X_h).
\end{aligned}$$

Since $d\alpha_{n+1}(X_f,X_g,X_h) = 0$ for every Hamilton vector fields X_f, X_g, X_h and since Hamilton vector fields generate the tangent space, the calculation implies $d\alpha_{n+1} = 0$ and α_{n+1} is a closed form. □

Lemma 4.8.4. *For initial data, ∇, $X \in t\Gamma^\infty(\Lambda^2 T^*M)[[t]]$ and $z = 0$, let the element x be constructed by the method discussed in remark 4.5.7. Let $*$ denotes the Fedosov star product associated to the Fedosov derivation $\mathcal{D} = -\delta + D + \frac{i}{t}ad(x)$. The following hold*

1. *For the construction of x, Θ_n is only needed for elements of higher* Deg*-degree than $2n+1$ and we have*
$$x^{2n+1} = t^n \delta' \Theta_n + \textit{ terms without } \Theta_n.$$

2. *Let $f \in \mathcal{C}^\infty(M)$. For the construction of $T(f)$, Θ_n is only needed for elements of higher* Deg*-degree than $2n+1$ and we have*

$$T^{2n+1}(f) = -\frac{t^n}{2}\delta' i_a(X_f)\Theta_n + \text{ terms without } \Theta_n.$$

3. *Let $f, g \in \mathcal{C}^\infty(M)$. For the construction of $f * g$, Θ_n is only needed for elements of higher t-degree than $n+1$ and we have*

$$C_{n+1}(f,g) = -\frac{i}{2}\Theta_n(X_f, X_g) + \text{ terms without } \Theta_n.$$

Proof. 1. Since $t^n\delta'\Theta_n$ has total degree $2n+1$, the first assertion follows from the recursion formula for $z = 0$ $x = \delta'(R + Dx + \frac{i}{t}x * x + \Theta)$.

2. In order to prove the second assertion, we calculate the leading terms of $T(f)$.

$$T(f) = \sum_{n=0}^{\infty}[\delta', D + \frac{i}{t}ad(x)]^n(f) = f + df + \delta'(D + \frac{i}{t}ad(x))df + ...$$

The first two terms follow from lemma 4.6.4. The equality $\delta' f = 0 = \frac{i}{t}ad(x)f$ implies that the third term is $[\delta', D + \frac{i}{t}ad(x)]df = \delta'(D + \frac{i}{t}ad(x))df$. Since the element of lowest total degree which also contains Θ_n is $\delta'\frac{i}{t}ad(t^n\delta'\Theta_n)df$, the equation $ad(\delta'\Theta_n)df = \frac{it}{2}i_a(X_f)\Theta_n$ proves the second assertion.

3. The map pr: $\mathcal{M}(M)[[t]] \to \mathcal{C}^\infty(M)[[t]]$ does not change the total degree, therefore, by the second assertion, the first term in $f * g = pr(Tf * Tg)$, which contains Θ_n, only concerns the total degree $2n+1$ elements in Tf and Tg. For pairs Tf_{2n+1}, Tg_1 and Tf_1, Tg_{2n+1}, by the calculation above, we have $(Tf * Tg)_{2n+2} = -\frac{it^{n+1}}{2}\Theta_n(X_f, X_g) +$ terms without Θ_n. □

Now, we are able to prove the second main theorem of this section.

Theorem 4.8.5. *Let (M, ω) be a symplectic manifold and let $*$ denotes a differential and local star product on M. There exists a closed two-form $\Theta \in t\Gamma^\infty(\Lambda^2 T^* M)[[t]]$, such that the Fedosov star product $*'$ corresponding to Θ is equivalent to $*$.*

Proof. If the two star products $*$ and $*'$ are equivalent up to degree n, then, by lemma 2.2.8, we can assume that they are even equal up to degree n. According to lemma 4.8.3, we can write

$$F_{n+1}(f,g) - F'_{n+1}(f,g) = \delta a_{n+1}(f,g) + \alpha_{n+1}(X_f, X_g),$$

for some appropriate $a_{n+1} \in C^1_{loc}(\mathcal{C}^\infty(M))$ and a closed two-form α_{n+1}. If we define $\Theta'' := \Theta + 2i\alpha_{n+1}$, then Θ'' is still a closed two-form. If $*''$ denotes Fedosov star product associated to Θ'', then, by lemma above,

$$\begin{aligned} F_{n+1}(f,g) - F''_{n+1}(f,g) &= F_{n+1}(f,g) + \frac{i}{2}\Theta(X_f, X_g) - \alpha_{n+1} + \text{ terms without } \Theta_n \\ &= F_{n+1}(f,g) - F'_{n+1}(f,g) - \alpha_{n+1} = \delta a_{n+1}(f,g). \end{aligned}$$

Lemma 2.2.6 implies that star product $*$ is $n+1$-equivalent to the Fedosov star product $*''$ constructed by Θ''. By an induction over n, we see that every star product is equivalent to a Fedosov star product. $\square$

Theorems 4.8.1 and 4.8.5 states that if one is interested in equivalence classes of star products, then one only has to look at the complex of de Rham cohomology classes in $tH^2_{dR}(M,\mathbb{C})[[t]]$, because the equivalence class is fully determined by objects $[\Theta]$. One way to interpret the importance of the objects in $tH^2_{dR}(M,\mathbb{C})[[t]]$ is to view $[\Theta]$ as a deformation of $[\omega]$. In particular, for a symplectic manifold (M,ω) the object $[\omega]+[\Theta]$ can be interpreted as an object in the affine vector space $[\omega]+tH^2_{dR}(M,\mathbb{C})[[t]]$, where $[\omega]$ is the origin. This leads us to the definition of a Fedosov class.

Definition 4.8.6 (Fedosov class)**.** Let (M,ω) be a symplectic manifold. For a star product $*$ on $\mathcal{C}^\infty(M)[[t]]$, the *Fedosov class* $\mathrm{Fed}(*)$ is defined by

$$\mathrm{Fed}(*) := [\omega]+[\Theta] \in [\omega]+tH^2_{dR}(M,\mathbb{C})[[t]]$$

such that the Fedosov star product associated to Θ is equivalent to $*$.

By using this notation, theorems 4.8.1 and 4.8.5 states that Fed is bijective map from the set of equivalence classes of star products on $\mathcal{C}^\infty(M)[[t]]$ to the affine vector space $[\omega]+tH^2_{dR}(M,\mathbb{C})[[t]]$.

4.9 Kontsevich's formality theorem

We have seen that there exists a star product on every symplectic manifold and we mentioned that this is also true for a Poisson manifold. Unfortunately, we cannot discuss Kontsevich's proof exhaustively, because it would be far beyond the scope of this diploma thesis. Nevertheless, we want to give some basic ideas and we refer the readers to [Kon97] and [Kon03] for details.

We alredy know that we have to find functions F_n: $\mathcal{C}^\infty(M)\times\mathcal{C}^\infty(M)\to\mathcal{C}^\infty(M)$ such that $*$: $(f,g)\mapsto f*g = fg+\hbar F_1(f,g)+\hbar^2F_2(f,g)+\ldots\in\mathcal{C}^\infty(M)[[\hbar]]$ defines a star product. Kontsevich's idea is to solve this problem in two steps. In the first step one uses graph theoretical considerations in order to enumerate bidifferential operators. More precisely, for each $n\in\mathbb{N}$ and for any given pair of smooth functions $f,g\in\mathcal{C}^\infty(M)$, there is graph Γ which is associated to a bidifferential operator D_Γ. In the second step, one defines a weight ω_Γ for each D_Γ such that the functions F_k are given by a sum of weighted bidifferential operators $\omega_\Gamma D_\Gamma$.

Definition 4.9.1. An *oriented graph* Γ is an ordered pair (V,E) of two sets such that E is a subset of $V\times V$. The elements of V and E are called *vertices* and *edges*, respectively. A graph Γ is called *finite*, if the sets V and E are finite. A graph Γ is a *weighted graph*, if every edge has an associated real number.

We are interested in a special class of graphs, denoted by G_n.

Definition 4.9.2. The class of graphs G_n consists of orientated graphs Γ satisfying the following properties:

1. Γ has $n+2$ vertices and $2n$ edges.

2. The set of vertices V consists of $\{1,...,n\} \sqcup \{f,g\}$. We use the notations f and g in order to indicate that f and g are those functions whose product we want to deform.

3. The $2n$ edges in E are labeled by $e_1^1, e_1^2, e_2^1, e_2^2, ..., e_n^1, e_n^2$, where e_k^f and e_k^g are edges which start at the vertex k and goes to different vertices.

4. For $v \in V$ the ordered pair (v,v) is not an element of E.

Given a Poisson manifold (M,π), one can now assign a bidifferential operator

$$D_\Gamma \colon \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$$

to each graph Γ in G_n. If I denotes a map which enumerate the edges in Γ, then D_Γ is defined by

$$D_\Gamma(f,g) := \sum_I (\prod_{k=1}^n \prod_{e\in E, e=_k^*} \partial_{I(e)} \pi^{I(e_k^f)I(e_k^g)}) \times (\prod_{e\in E, e=e_*^f} \partial_{I(e)}) f \times (\prod_{e\in E, e=e_*^g} \partial_{I(e)}) g.$$

By definition, the set G_0 has only one element. Using combinatorial consideration, a close examination reveals that the set G_n has $(n(n+1))^n$ elements for $n \geq 1$. In the second step, we associate a weight $w_\Gamma \in \mathbb{R}$ to each of the graphs in G_n. We embed a graph Γ into the union of the upper half-plane $H := \{z = x + iy \in \mathbb{C} | y > 0\}$ and the real line $\mathbb{R}$ such that the vertex f and g is mapped to 0 and 1, respectively. The topological space H admits a metric called the Lobachevsky metric which is given by

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

For two points $a, b \in H, a \neq b$, let $s(a,b)$ and $s(a,\infty)$ denote the geodesics from a to b and from a to ∞, respectively. The angle $\phi(a,b)$ between these two lines is measured counterclockwise and it is given by

$$\phi(a,b) = \frac{1}{2i} \log \frac{(a-b)(a-\bar{b})}{(\bar{a}-b)(\bar{a}-\bar{b})}.$$

For an edge e_k^f of the graph Γ, we write $\phi(e_k^f)$ instead of $\phi(k,f)$.

If the domain of integration $C_n(H)$ is defined by $C_n(H) := \{(a_1,...,a_n) | a_k \in H, a_k \neq a_l \text{ for } k \neq l\}$, then the weight of a graph Γ is defined by

$$w_\Gamma := \frac{1}{n!(2\pi)^{2n}} \int_{C_n(H)} \bigvee_{k=1}^n (d\phi(e_k^f) \wedge d\phi(e_k^g)).$$

This term is absolutely convergent, as Kontsevich proved in [Kon97].

The functions F_n are now defined by the bidifferential operators D_Γ and their weights w_Γ.

Theorem 4.9.3. *[Kon97] Let $(\mathbb{R}^n, \pi)$ be a Poisson manifold. The bilinear map $*$ given by*

$$f * g := \sum_{n=0}^\infty \hbar^n \sum_{\Gamma \in G_n} w_\Gamma D_\Gamma(f,g)$$

defines a star product on $(\mathbb{R}^n, \pi)$.

The method given above is a local description of constructing a star product which can then be extended to the global case, which shows that every Poisson manifold admits a star product.

Summary

This diploma thesis consists of four chapters. In the first chapter we discuss methods and difficulties of quantization. As a motivation, we first compare classical mechanics with quantum mechanics where we put special emphasis on the algebra of observables, which is the central object in classical physics. After clarifying the meaning of quantization, we give reasonable properties a quantization has to satisfy. Then, we introduce the concept of deformation quantization and we discuss the possibility of quantizing a physical system in a canoncial way. However, due to Groenewold-van Hove theorem, this is impossible, as we see at the end of the chapter.

The second chapter is devoted to the mathematical theory behind the deformation quantization. After a short algebraic preliminary we introduce the concept of deformations of algebras. Then, we discuss related topics such as equivalences and extension of deformations. Important terminologies, like the Hochschild complex and the Gerstenhaber algebra, are defined and their properties are studied in this chapter. Since we will need the results of the Hochschild-Kostant-Rosenberg theorem in the last chapter, we present its proof here and discuss some of its consequences.

In the third chapter, after recalling some definitions and results of differential geometry, we use the techniques developed in the previous chapters to define a star product. The ambiguity of star products comes from the different orderings. Therefore, the main focus of this chapter lies on discussing t-orderings and $\tilde{t}$-orderings and their associated star products. As we will see in this chapter, many well-known star products such as the Weyl-Moyal star product and the Wick star product are special cases of t-ordered and $\tilde{t}$-ordered star products.

The last chapter is devoted to the Fedosov construction. As a preparation, we first introduce the so-called mixed algebra and some derivation maps operating on this algebra. Then, we define the Fedosov derivation and show that it is a differential. At first, we give a constructive proof which can be useful for calculations. Later, we present a more general, non-constructive proof which is based on homotopical arguments. These result will lead us to the definition of Fedosov Taylor series which we use to introduce the Fedosov star products. Properties and equivalences of Fedosov star products are also discussed. We conclude this diploma thesis by outlining some basic ideas of the proof of Kontsevich's formality theorem.

Zusammenfassung

Diese Diplomarbeit besteht aus vier Kapiteln. Im ersten Kapitel diskutieren wir die Methoden und Schwierigkeiten der Quantisierung. Zur Motivation werden wir zuerst die klassische Mechanik mit der Quantenmechanik vergleichen. Dabei stellt sich heraus, dass die Observablenalgebra das zentrale Objekt in der klassichen Physik ist. Nachdem wir den Begriff "Quantisierung" präzisiert haben, geben wir die Eigenschaften an, die eine vernünftige Quantisierung erfüllen sollte. Als Nächstes führen wir das Konzept der Deformationsquantisierung ein und diskutieren über die Möglichkeit ein physikalisches System auf kanonische Weise zu quantisieren. Wie wir am Ende dieses Kapitels sehen werden, ist dies jedoch durch das Groenewold-van Hove Theorem ausgeschloßen.

Im zweiten Kapitel widmen wir uns der mathematischen Grundlage der Deformationsquantisierung. Nach einer kurzen Erläuterung einiger algebraischer Begriffe, führen wir das Konzept der Algebradeformation ein und besprechen damit verwandte Begriffe, wie etwa Äquivalenzen und Erweiterungen von Deformationen. Weiters werden Hochschild-Komplexe sowie Gerstenhaber-Algebren definiert und deren Eigenschaften studiert. Da wir die Resultate des Hochschild-Kostant-Rosenberg Theorems im letzten Kapitel benötigen werden, wird dessen Beweis schon in diesem Kapitel präsentiert.

Im dritten Kapitel werden wir zuerst einige Definitionen und Resultate aus der Differentialgeometrie kurz wiederholen und dann das Sternprodukt definieren. Da die Vielzahl der Ordnungen der Grund für die fehlende Eindeutigkeit des Sternprodukts ist, liegt der Fokus dieses Kapitels auf der Beschreibung der t-Ordnung und der $\tilde{t}$-Ordnung. Außerdem werden wir in dem Kapitel sehen, dass die bekannten Sternprodukte, wie das Weyl-Moyal Sternprodukt und das Wick Sternprodukt, Spezialfälle von t- bzw. $\tilde{t}$-Ordnungen sind.

Im letzten Kapitel beschäftigen wir uns mit der Fedosov-Konstruktion. Als Vorbereitung führen wir die sogenannte gemischte Algebra ein, danach definieren wir die Fedosov-Derivation. Anschließend werden wir auf zwei verschiedenen Arten zeigen, dass diese sogar ein Differential ist. Der erste Beweis ist konstruktiv und kann für etwaige Berechnungen herangezogen werden. Danach geben wir noch einen allgemeineren, nichtkonstruktiven Beweis an, der auf homotopischen Überlegungen basiert. Diese Resultate führen uns zur Definition der Fedosov Taylorreihe, mit dessen Hilfe das Fedosov-Sternprodukt definiert wird. Eigenschaften und Äquivalenzen der Fedosov-Sternprodukte werden ebenfalls diskutiert. Am Ende der Diplomarbeit präsentieren wir die grundlegenden Beweisideen des Formalitätstheorem von Kontsevich.

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