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Abstract

Space and numbers are two important concepts which philosophers created for the description of natural phenamena. From the beginnig, it seemed that there is a kind of equivalence (duality) between the space (geometry) and numbers (algebra).
In modern mathematics, this equivalence can be expressed by category theory. An important example is duality between commutative C^*–algebras and locally compact topological Hausdorff spaces. But there is a wide array of algebras which have noncommutative product between the members. Noncommutative geometry is the study of topological spaces whose commutative C^*–algebra of functions is replaced by a noncommutative algebra.
There are some physical evidences that tell us space time has a noncommutative structure in very small lenght scale.
The aim of my Thesis is to introduce deformation quantization which arises from the noncommutativity of space time. In this view point, deformation quantization appears as a product on the deformed algebra of smooth functions on space time manifold. This product is an associative and noncommutative product and called star product.
Kontsevich has shown that star product exists as a general formula of quantization of an arbitrary Poisson manifold. Physical interpretation of star product appears in Poisson sigma models which are topological quantum field theories and provide a unification for some class of field theories. It seems that maybe there is a grand unified field theory behind noncommutativity properties of space time.

Zusammenfassung

Räume und Zahlen sind zwei wichtige philosophische Konzepte für die Beschreibung natürlichen Phänomene. Von Anfang an schien es, dass eine Art von Äquivalenz (Dualität) zwischen geometrischen Räumen (also der Geometrie) und Zahlen (also Algebren) existiert.
In der modernen Mathematik kann diese Äquivalenz z.B. mittels der Kategorientheorie ausgedrückt werden. Ein wichtiges Beispiel ist dabei die Dualität zwischen kommutativen C^*-Algebren und lokal kompakten topologischen Hausdorff Räumen. Neben diesen kommutativen Algebren gibt es aber auch eine breite Palette von nichtkommutativen Algebren, d.h. Algebren deren Multiplikation von der Reihenfolge der Faktoren abhängt. Nichtkommutative Geometrie ist demnach das Studium von topologischen Räumen, wobei jeweils eine nichtkommutative Funktionenalgebra (C^*-Algebra) das duale Gegenstück bildet.
Es gibt einige physikalische Hinweise auf eine nichtkommutative Struktur der Raumzeit für sehr kleine Läengenskalen.
Das Ziel meiner Arbeit ist es, ausgehend von einer nichtkommutativer Raumzeit Deformationsquantisierungen von Funktionenalgebren zu diskutieren. In dieser Hinsicht erscheint eine Deformationsquantisierung als nichtkommutatives Produkt auf der Algebra der glatten Funktionen auf der Raumzeit-Mannigfaltigkeit. Dieses assoziative und nichtkommutative Produkt wird als Sternprodukt (star product) bezeichnet.
Kontsevich konnte zeigen, dass für jede Mannigfaltigkeit mit einer sogenannten Poisson Struktur ein Sternprodukt definiert werden kann und konnte eine explizite Formel dafür angeben. Allerdings ist diese Zuweisung nicht eindeutig möglich, es müssen Äquivalenzklassen von Sternprodukten betrachtet werden. Eine physikalische Interpretation von Sternprodukten wird im Rahmen von Poisson Sigmamodelle klar. Poisson Sigmamodelle sind topologische Quantenfeldtheorien und eine Verallgemeinerung einer großen Klasse von Feldtheorien.
Es deutet sehr vieles darauf hin, dass die Nichtkommutativität von Raum und Zeit wertvolle Hinweise für die Struktur einer möglichen großen vereinheitlichten Theorie, also einer *grand unified field theory*, geben könnte.

Chapter 1

Some historical introduction

1.1 Foundamental Duality

Space and numbers are two important concepts which philosophers created for the description of natural phenamena. From the beginning, it seemed that there is a kind of duality between space (geometry) and numbers (algebra). The first equivalence was found by the Greek mathematician *Menaechmus* $(380 - 320BC)$. He solved problems and theorems by using a method that has a strong resemblance to the use of coordinates. Actually he introduced analytic geometry.
The next evidence was discovered by the Persian mathematician *Khayyam* in the eleventh century. He saw a strong relationship between geometry and algebra, when he wanted to solve the cubic equation with geometrical methods.
At the end, analytic geometry has traditionally been attributed to *Rene Descartes*. He has shown that the geometrical point can be represented by a set of numbers. [1]
A revolution in geometry happened in the 19th century by *Gauss, Riemann, Hilbert, Poincare and Klein* which is based on the *Cantor* set theory and a revolution in algebra by *Galois, Dedekind, Kronecher,....* In this modern perspective, space is a set with an inner structure (a finite number of axioms). For example:Topology, smooth structures, measures. An algebra is a set with some operations between the members. The duality between geometry and algebra can be expressed by category theory (in other words, we can express the geometrical information of a space by the algebra of functions defined on that space).
The two famous examples for dualities are:
1) Hilberts Nullstellen theorem: Duality between affine algebraic varieties and finitely generated commutative algebras.
2) The Gelfand-Naimark theorem: Duality between commutative C*-algebras and locally compact topological Hausdorff spaces.
But there is a wide array of algebras which have noncommutative product between the members. The most important of them are noncommutative C*-algebras which naturally appear in mathematics and theoretical physics, for examle in Harmonic analysis, differential geometry and quantum mechanics.
This led us to the theory of *Von Neumann* algebras and was the birth of noncommutative geometry. Actually noncommutative geometry is the study of topological spaces whose commutative C^*–algebra of functions is replaced by a noncommutative algebra.
The idea of noncommutative geometry was revived by *Alain Connes*. [2]

So we can replace the old concept of space (as a set of points) with the new concept of space (as a noncommutative algebra of functions).

1.2 Physical evidence behind noncommutative space time

Noncommutative structure of phase space was suggested by the creators of quantum mechanics: Phase space in quantum mechanics is defined by replacing the classical momentum p_i and position q_i by Hermitian operators $\hat{p_i}$ and $\hat{q_i}$ which satisfy the Heisenberg commutation relation and therefore the uncertainty principle. It means that the notion of a point in quantum phase space is meaningless.
We can reformulate quantum field theory with a noncommutative structure for space time coordinates at Planck scale in order to introduce an effective ultraviolet cutoff. Ultraviolet divergences arise when we try to measure the amplitude of field oscillations at a point in space time. We can assume the structure of space time in the small length scale should be noncommutative. This means that it is impossible to measure exactly the position of a particle since the space coordinates could not be simultaneously diagonalized (quantization of phase space).
Thus quantization of a classical space is defined by replacing space time coordinates x_i by the Hermitian generators $\hat{x_i}$ of a noncommutative C^*–algebra of functions on space time manifold and satisfying the commutation relations:

$$[\hat{x_i}, \hat{x_j}] = i\theta_{ij}(\hat{x}). \tag{1.1}$$

.

The simplest case is where θ_{ij} is a constant, real valued antisymmetric $M \times M$ matrix (M is the dimention of space time).
So, we can write uncertainty relation for space time coordinates:

$$\Delta x_i.\Delta x_j \geq (\frac{1}{2})\theta_{ij}. \tag{1.2}$$

.

It means that a point in space time is replaced by a Planck cell .
Quantized space time was suggested by *Snyder* in 1947 [5]. His idea was that if we could create a structure of space time that is pointless on very small length scales, then the ultraviolet divergence of quantum field theory might disappear. One consequence of this idea is that the classical general relativity would break down at this scale because space time would not be described by a differential manifold any more.
Now, we study some physical evidence for noncommutative space time:

1) Landau problem:
Consider a homogeneous and constant magnetic field B and a particle with charge e moving in B. The action is:

$$S = \int dt(\frac{1}{2}m\dot{x_i}\dot{x^i} - \frac{e}{c}B_{\mu\nu}x^i\dot{x^i}). \tag{1.3}$$

Where $B_{\mu\nu} = -B_{\nu\mu}$ and $A_\nu = B_{\mu\nu}x^\mu$ and the canonical momentum is:

$$\pi_i = \frac{\partial L}{\partial \dot{x_i}} = m\dot{x_i} + (\frac{e}{c})B_{ij}x_j. \tag{1.4}$$

The poisson bracket between coordinate and momentum is:

$$\left\{\pi_i, x^j\right\} = \delta_i^j. \tag{1.5}$$

From (1.4) and (1.5) we can write:

$$\begin{aligned}
&\left\{\dot{x_i}, x^j\right\} + (\frac{e}{mc})B_{ik}\left\{x^k, x^j\right\} = (\frac{1}{m})\delta_i^j \\
&\Rightarrow \left\{x^k, x^j\right\} = (\frac{c(B^{-1})^{kj}}{e})\delta_i^j - (\frac{mc(B^{-1})^{ki}}{e})\left\{\dot{x_i}, x^j\right\}.
\end{aligned} \tag{1.6}$$

We assume strong magnetic field and small mass. In this limit, the last equation simplifies and we have:

$$\left\{x^k, x^j\right\} = (\frac{c}{e})(B^{-1})^{kj}. \tag{1.7}$$

It means that the coordinates do not commute.

2) Divergencies in quantum field theory:
In quantum field theory, loop contributions to the transition amplitudes diverge. For example in Euclidian space, the real scalar field Φ^4 theory which is described by the Lagrangian [15]:

$$L = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}m^2\Phi^2 + \frac{g}{4!}\Phi^4. \tag{1.8}$$

The generating functional of correlation functions is:

$$Z[J] = \int [d\Phi]exp(-iS), \tag{1.9}$$

$$\begin{aligned}
&Z[J] = \{1 + \frac{g}{4}\Delta_F(0)\int d^4x \int d^4y \int d^4z J(y)\Delta_F(y-z)J(x)\Delta_F(x-z) + ...\} \\
&\times exp\{-\frac{i}{2}\int d^4x \int d^4y J(x)\Delta_F(x-y)J(y)\}.
\end{aligned} \tag{1.10}$$

The loop contribution in the formula appears in term $\Delta_F(0) = \int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2-m^2}$, thus the result is divergent. There may be other divergences and the renormalisation procedure may remove some of them. If all divergencies can be removed with a finite number of counter terms, the theory is

called renormalisable. The Φ^4 theory in 4 dimensions is renormalisable.
The quantum theory of gravity is not renormalisable. The hope to have a renormalisable theory of quantum gravity is a motivation for noncommutative coordinates.

3) Quantization of gravity:
In the way of quantization of gravity, it clarified that the space time structure might have a non-commutative structure. For measuring the distance l between two points in space time, energy E has to be deposited in that region of space time:

$$\lambda \approx l : E = (hc)/\lambda \approx (hc)/l. \tag{1.11}$$

It means, as the distance l decreases the energy E increases. At the Planck scale we have:

$$\begin{aligned} & l \approx l_{pl} = \sqrt{(\hbar G)/c^3} \\ \Rightarrow \quad & E = (hc)\sqrt{c^3/(\hbar G)}. \end{aligned} \tag{1.12}$$

The Schwarzshild radius is:

$$r_s = 2GM/c^2 = 2\sqrt{\hbar G/c^3} = 2l_{pl}. \tag{1.13}$$

It means that a black hole is generated during the measurement. In order to avoid the appearence of a black hole, we need a minimal length which can be implemented by commutation relations [16].

1.3 Deformation and quantization

The 20th century started with two revolutions in theoretical physics:
1) Relativity,
2) Quantum mechanics.
The aim of both classical and quantum mechanics are to study the evolution of observable quantities. An observable in classical mechanics is a smooth function $f \in C^\infty(M)$ on a Poisson manifold M, called the phase space. The time evolution of an observable f is given by:

$$\frac{df}{dt} = \{H, f\}. \tag{1.14}$$

Where $\{,\}$ is the Poisson bracket on $C^\infty(M)$ and H is the Hamiltonian of the system. The Poisson bracket is completley determined by its action on coordinate functions $(p_i, q_i)_{i=1}^n$:

$$\{p_i, q_j\} = -\delta_{ij}, \tag{1.15}$$

where p_i and q_j are momentum and position, local coordinates on the $2n$ dimensional Poisson manifold M.

An observable in quantum mechanics is a self adjoint operator $\hat{f} \in B(H)$ on a complex projective Hilbert space H, called state space. The time evolution of an observable (Heisenberg picture) is given by:

$$\frac{d\hat{f}}{dt} = -i\hbar \left[\hat{H}, \hat{F}\right], \tag{1.16}$$

where $[,]$ is a Lie algebra structure on $B(H)$. It satisfies the canonical commutaion relation:

$$[\hat{p}_i, \hat{q}_j] = -i\hbar\delta_{ij}. \tag{1.17}$$

Therefore the following question naturally arises:
Is there any way to construct quantum mechanics from classical mechanics? The answer is yes! This method is called quantization.

Deformation quantization:
As we can see in the last part of my discussion, the word *Quantization* played a very important role in the revolution of physics. From the beginning of quantum mechanics, it seemed that quantum mechanics is some kind of deformed classical mechanics. The method we can use in order to construct quantum mechanics from classical mechanics is called *Deformation quantization*. Deformation quantization was discussed by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [18]. The observables in classical mechanics ($f \in C^{\infty}(M)$) have a commutative C^*–algebra structure and in quantum mechanics ($\hat{f} \in B(H)$) a noncommutative C^*–algebra structure. Deformation quantization is a way to construct a noncommutative C^*–algebra of quantum observables from a commutative C^*–algebra of classical observables. See [25] for a general definition of quantization.
The starting point is the phase space (a Poisson manifold) with the commutative algebra of observables. Then one tries to construct a noncommutative algebra of quantum observables by defining a new noncommutative product (star product) which introduces a noncommutative structure to the algebra of functions. As we will see, deformation quantization is based on algebraic structures. We can also extend our discussion in the geometrical language (by the duality between algebra and geometry). Thus we can explain it with the geometrical perspective.
We can illustrate the quantization method by the following diagram [20]:

$$\begin{aligned} \text{classical mechanics} &\longleftrightarrow \text{quantum mechanics} \\ \text{algebra of functions on manifold} &\longleftrightarrow \text{algebra of observables} \\ \text{manifold phase space} &\longleftrightarrow \text{Hilbert space} \end{aligned}$$

Generally, a deformation of mathematical objects is a family of some kind of objects depending on some parameters [19]. We are concerned with deformation of algebras.
Deformation of an algebra is encoded as a product, so called *Star product*. Let $\hat{A}$ be a noncommutative algebra of functions on a noncommutative space:

$$\hat{A} = \frac{\mathbb{C}\langle\langle \hat{x^1}, ..., \hat{x^n} \rangle\rangle}{I}, \tag{1.18}$$

where I is the ideal generated by the commutation relation of the coordinate functions, and the commutative algebra of functions:

$$A = \frac{\mathbb{C}\langle\langle x^1, ..., x^n \rangle\rangle}{[x^i, x^j]}, \tag{1.19}$$

i.e., $\left[x^i, x^j\right] = 0$. Our aim is to relate these algebras by an isomorphism $W : A \to \hat{A}$. Let us first consider the vector space structure of the algebras. In order to construct a vector space isomorphism, a basis has to be choosen in both algebras and then they have to be satisfied by Poincare-Birkhoff-Witt property, then W is an isomorphism of vector spaces [16].
In order to extend W to an algebra isomorphism, we have to introduce a new noncommutative multipication $*$ in A. This star product in defined by:

$$W(f * g) = W(f) \cdot W(g) = \hat{f} \cdot \hat{g}, \tag{1.20}$$

where $f, g \in A$ and $\hat{f}, \hat{g} \in \hat{A}$. W is an algebra isomorphism:

$$(A, *) \cong (\hat{A}, \cdot) \tag{1.21}$$

The information on the noncommutativity of $\hat{A}$ is encoded in the star product.

The main goal of my Thesis is to introduce star products and study some physical applications which appear in Poisson sigma models. The main reasons which motivate us to consider Poisson sigma models is the relation of this model with quantization theory which appear in the *Kontsevich* quantization theorem. In this theorem, Kontsevich gives us a universal formula for deformation quantization expressed in terms of some graphs. This formula imitates the Feynman diagrams expansion in quantum field theory. The Poisson sigma model provides a unification for some class of field theories.

In chapter 2, I will strudy some applicatoins of commutative C^*–algebras and some topological concepts which are related to the Gelfand-Naimark theorem. In chapter 3, I will study noncommutative space time, some examples and Hopf algebras.
I will study the mathematics of star products (deformation quantization) in chapter 4 and introduce a very important theorem which was proved by Kontsevich.
In chapter 5, I will introduce Poisson sigma models and the physical interpretation of star products.

Chapter 2

C^*–algebras and their geometry

2.1 Introduction to C^*–algebras

The aim of this chapter is to introduce the Gelfand-Naimark theorem. I will start with the C^*–algebra and continue with the fundamental theorems which lead us to the Gelfand-Naimark theorem. The C^*–algebras are based on Banach algebras, and Banach algebras are a special case of algebras which have a normed space structure. There is also an interesting relation between normed spaces and topological vectore spaces [21], [22], [23], [24].
For constructing a Banach algebra, in a first step, we need to define a norm on an arbitrary vector space X and make a Banach space from X.

Definition 2.1.1 (seminorm)**.** *Let X be a vector space on a field F. A map $X \longrightarrow \mathbb{R}$ is called a seminorm on X if it satisfies the following axioms:*

$$\begin{aligned} &1) \quad \forall x, y \in X : \|x + y| \leq \|x\| + \|y\|\,, \\ &2) \quad \forall x \in X, \alpha \in F : \|\alpha x\| \leq |\alpha|\,.\,\|x\|\,. \end{aligned} \tag{2.1}$$

Definition 2.1.2. *Let X be a F-vector space with a seminorm. X is called a normed space if :*

$$\forall x \in X : \|x\| = 0 \Longrightarrow x = 0. \tag{2.2}$$

Then the pair $(X, \|.\|)$ is called a normed space.

Definition 2.1.3. *A normed space X is called a Banach space if every Cauchy sequence (with respect to the metric which defines the norm) is convergent.*

Definition 2.1.4. *Let A be an algebra over a ring R. It is called a normed algebra if it satisfies the following axioms:*

$$1) \quad A \text{ is a normed space}, \tag{2.3}$$

$$2) \quad \forall x, y \in A : \|xy\| \leq \|x\|\,.\,\|y\|\,. \tag{2.4}$$

Definition 2.1.5. *A normed algebra A is called a Banach algebra if A is a Banach space.*

We have introduced a Banach algebra structure. The next step is to construct a $*$-algebra from a normed algebra A. Thus we need an additional structure on the algebra A.

Definition 2.1.6. *Let A be a (complex) normed algebra. An involution is a map $A \to A(a \mapsto a^*)$ such that the following axioms hold:*
$\forall a, b \in A, \alpha \in F$:

$$\begin{aligned} &1) \quad (a^*)^* = a, &(2.5)\\ &2) \quad (a+b)^* = a^* + b^*, &(2.6)\\ &3) \quad (ab)^* = b^* a^*, &(2.7)\\ &4) \quad (\alpha a)^* = \bar{\alpha} a^*. &(2.8)\end{aligned}$$

Definition 2.1.7. *A normed algebra A with an involution $* : A \to A$ is called $*$-algebra and denoted by $(A, *)$.*

Now we are ready to define a C^*-algebra:

Definition 2.1.8. *A star Banach algebra is called C^*–algebra if:*

$$\forall x \in A : \|x^* x\| = \|x\|^2 . \tag{2.9}$$

Definition 2.1.9. *Let A be a C^*–algebra and $x \in A$. Then we have the following definitions:*

$$\begin{aligned} &1) \quad x \quad \textit{is Hermitian if} \quad x = x^*, &(2.10)\\ &2) \quad x \quad \textit{is normal if} \quad xx^* = x^* x, &(2.11)\\ &3) \quad x \quad \textit{is unitary if} \quad xx^* = x^* x = 1, &(2.12)\\ &4) \quad x \quad \textit{is a projection if} \quad x^2 = x^* = x &(2.13)\\ &5) \quad x \quad \textit{is a partial isometry if} \quad xx^* \textit{ or } x^* x \quad \textit{is a projection.} &(2.14)\end{aligned}$$

The C^*-algebras have some important properties. One of them will be described in the next theorem. These properties arise from the involution map defined in definition (2.1.7).

Theorem 2.1.10. *Let A be a C^*–algebra, then:*

$$\begin{aligned} &1) \quad \forall x \in A : \|x^*\| = \|x\| , &(2.15)\\ &2) \quad \forall a \in A : \|a\| = sup \{\|ax\| ; x \in A, \|x\| \le 1\} . &(2.16)\end{aligned}$$

Proof. 1) By the definition of norm on A, we have:

$$\forall x \in A : \|x\|^2 = \|x^* x\| \le \|x^*\| . \|x\| \Rightarrow \|x\|^2 \le \|x^*\| . \|x\| \Rightarrow \|x\| \le \|x^*\| . \tag{2.17}$$

On the other side:

$$(x^*)^* = x \Rightarrow \|x^*\|^2 = \|x^{**} x^*\| = \|xx^*\| \le \|x\| . \|x^*\| \Rightarrow \|x^*\| \le \|x\| . \tag{2.18}$$

Thus by the last two equations we have:

$$\|x\| = \|x^*\| . \tag{2.19}$$

2) By definition (2.1.1) we have

$$\|ax\| \le \|a\| . \|x\| , \tag{2.20}$$

since $\|x\| \leq 1$ we have

$$\|ax\| \leq \|a\|, \tag{2.21}$$

so by the definition of supremum of a set, we have:

$$\|a\| = sup\{\|ax\|; x \in A, \|x\| \leq 1\}. \tag{2.22}$$

$\square$

Proposition 2.1.11. *Let A be a C^*–algebra. If $x \in A$ is an invertible element, then x^* is also invertible and*

$$(x^{-1})^* = (x^*)^{-1}. \tag{2.23}$$

Example The complex field is a unital C^*–algebra with involution given by complex conjugation:

$$\alpha \in \mathbb{C} : \alpha \to \bar{\alpha}. \tag{2.24}$$

2.2 Algebra of continuous functions

In this section we study a special case of C^*–algebras, the algebra of continuous functions over the topological spaces which we denoted by $C(X)$ (X is a topological space). This algebra plays a very important role in our goal.
I start with the definition of a topology on a set:

Definition 2.2.1. *Let X be an arbitrary set and T be a set of subsets of X. T is called a topology on X, if satisfies the following axioms:*

$$1) \quad X \in T, \varnothing \in T, \tag{2.25}$$

$$2) \quad \forall i \in I, G_i \in T \Rightarrow \bigcup_{i \in I} G_i \in T, \tag{2.26}$$

$$3) \quad \forall i \in I, G_i \in T \Rightarrow \bigcap_{i=1}^{n} G_i \in T. \tag{2.27}$$

I is an index set.

Definition 2.2.2. *The set $G_i \in T$ in the last definition is called open set in X.*

Definition 2.2.3. *Let X be a topological space. An open cover of X is a set $\{G_i\}_{i \in I}$ of open subset of X such that:*

$$X \subseteq \cup_{i \in I} G_i. \tag{2.28}$$

Now, I introduce some important examples of topological spaces.

Definition 2.2.4. *Let X be a topological space. Then we have the following definitions:*
1) X is called compact if each open cover of X has a finite sub cover.
2) X is called locally compact if every point of X has a compact neighbourhood.
3) X is called Hausdorff if two distinct points have disjoint neighbourhoods.

Example Let X be a compact topological Hausdorff space. $C(X)$ is the set of all complex continuous functions $X \to \mathbb{C}$. Then the set $C(X)$ is a Banach algebra with the norm:

$$\forall f \in C(X) : \|f\| = sup\{\|f(x)\|\,; x \in X, \|x\| \leq 1\} \tag{2.29}$$

and it is a unital commutative C^*-algebra with the following involution $f \to f^*$:

$$\forall x \in X : f^*(x) = f(\bar{x}) \to \|f^* f\| = \|f\|^2 , \tag{2.30}$$

with unital element: $\forall x \in X : I(x) = 1$.

Example Let X be a locally compact topological Hausdorff space, then

$$C_0(X) = \{f : X \to \mathbb{C}; f(\infty) = 0\} \tag{2.31}$$

is a commutative C^*–algebra without unit element,

$$\forall f \in C_0(X) : \|f\|_\infty = \|f\| . \tag{2.32}$$

Proposition 2.2.5. *The commutative C^*–algebra $C_0(X)$ is a unital algebra if and only if X is a compact space.*

Example Let H be a Hibert space and $B(H)$ the set of all bounded linear operators on H. Then $B(H)$ is a Banach algebra under the uniform norm:

$$\forall o \in B(H) : \|o\| := sup\{\|o(x)\|\,; x \in H, \|x\| \leq 1\} , \tag{2.33}$$

and it is a noncommutative C^*–algebra with an involution:

$$o \to o^* : \forall x, y \in H : \langle o(x), y\rangle = \langle x, o^*(y)\rangle . \tag{2.34}$$

Remark Let A be an algebra over a ring R. A subset $I \subset A$ is called a left ideal of A, if it satisfies the following axioms:
1) I is a submodule of A,
2)$\forall x \in A, i \in I : ix \in I$.

Example Let A be a $*$–algebra. Let I be a self adjiont ideal of A ($I = I^*$). Then the quotient algebra A/I is a $*$–algebra with the involution given by: $\forall x \in A$:$(x + I)^* = x^* + I$.

2.3 Spectral theory

The spectral theory was introduced by David Hilbert in his formulation of Hilbert space theory and generally in functional analysis, bounded operators. The important point is that the spectrum of a point of a unital algebra is a non-empty set (Gelfand theorem). The spectrum of a bounded operator is always a closed, bounded and non-empty subset of the complex plane.

Definition 2.3.1. *Let A be an algebra (over a complex ring). The spectrum of $x \in A$ is defined by:*

$$Sp_A(x) = \{\alpha \in \mathbb{C}; (x - \alpha I) \quad \textit{is not invertible}\} \tag{2.35}$$

Theorem 2.3.2. *Let A be a unital algebra. Then:*

$$\forall x, y \in A : Sp_A(xy) \cup \{0\} = Sp_A(yx) \cup \{0\} \tag{2.36}$$

Proof. Let $\alpha \notin Sp_A(xy) \cup \{0\}$, then there is an inverse of $(xy - \alpha I)$. We denote by u. If $u = (xy - \alpha I)^{-1}$, thus:

$$\begin{aligned} & u(xy - \alpha I) = uxy - \alpha u = I \\ & (xy - \alpha I)u = xyu - \alpha u = I \\ \Rightarrow \quad & xyu - \alpha u = I \\ \Rightarrow \quad & xyu = I + \alpha u \\ \Rightarrow \quad & y(xyu)x = yx + \alpha yux + \alpha - \alpha \\ \Rightarrow \quad & (yx - \alpha)(yux - I) = \alpha \quad \text{and} \quad (yux - I)(yx - \alpha) = \alpha \end{aligned} \tag{2.37}$$

Thus $(yx - \alpha I)$ is invertible so $\alpha \notin Sp_A(yx) \cup \{0\}$, thus $Sp_A(xy) \cup 0 \subseteq Sp_A(yx) \cup \{0\}$. Similary we can show that $Sp_A(yx) \cup \{0\} \subseteq Sp_A(xy) \cup \{0\}$, thus $Sp_A(xy) \cup \{0\} = Sp_A(yx) \cup \{0\}$. □

Now we start to review some important theorems which are the basis of commutative spaces.

Theorem 2.3.3 (Gelfand Theorem)**.** *Let A be a unital Banach algebra. If $x \in A$, then the spectrum of x is not empty.*

$$Sp_A(x) \neq \varnothing. \tag{2.38}$$

Theorem 2.3.4 (Gelfand-Mazur Theorem)**.** *A unital Banach algebra in which every nonzero element is invertible, is an isometry-isomorphic to the complex field.*

It means that every complex unital Banach algebra which is a division algebra is isomorphic to the complex numbers.

Definition 2.3.5. *Let A be an algebra. The spectral radius of $x \in A$ is defined by:*

$$r(x) = sup\left\{|\alpha|, \alpha \in Sp_A(x)\right\}. \tag{2.39}$$

Theorem 2.3.6 (Beurling Theorem)**.** *Let A be a unital Banach algebra and $x \in A$. Then:*

$$r(x) = inf_{n \geq 1} \|x^n\|^{1/n} = lim_{n \to \infty} \|x^n\|^{1/n}. \tag{2.40}$$

Proposition 2.3.7. *If x is a self adjoint element of a C^*–algebra A, then:*

$$r(x) = \|x\|. \tag{2.41}$$

Theorem 2.3.8. *Let A be a C^*–algebra and $x \in A$. Then $Sp_A(x)$ is a compact subset of the complex disk of radius $r(x) = \|x\|$ and so:*

$$Sp_A(x^*) = \{\bar{\alpha}; \alpha \in Sp_A(x)\}, \tag{2.42}$$

$$Sp_A(x^2) = \left\{\alpha^2; \alpha \in Sp_A(x)\right\}. \tag{2.43}$$

2.4 Gelfand-Naimark theorem

We are ready to study the Gelfand-Naimark theorem. In the last section, we studied a special class of algebras (C^*–algebras). Now we introduce some geometrical structures on the algebras, it means that we should define a topology which arises from continuous functions.

Definition 2.4.1. *Let A be a commutative Banach algebra and $\Omega(A)$ the set of all non-zero complex linear functionals on A. We define a map $\hat{x} : \Omega(A) \to \mathbb{C}$ as follows:*

$$\forall f \in \Omega(A) : \hat{x}(f) = f(x). \tag{2.44}$$

x is a member of A.

In the next definition, we make a topological structure on $\Omega(A)$.

Definition 2.4.2. *The topology defined by $\{\hat{x}\}_{x\in A}$ on $\Omega(A)$ (weak topology on $\Omega(A)$ with subbase $\left\{x^{-1}(O_i)\right\}_{i\in I}$ for open set O_i in $\mathbb{C}$) is called Gelfand topology. Therefore $\{\hat{x}\}_{x\in A}$ are continuous with respect to the Gelfand topology.*

Definition 2.4.3. *Let I be a left (right) Ideal in the ring R. I is called a maximal Ideal on$R$ if $I \neq R$ and for every left (right) Ideal J of R, $I \subset J \subset R$ implies $J = I$ or $J = R$.*

Definition 2.4.4. *There is a bijection between the maximal ideal of A and $\Omega(A)$. The set $\Omega(A)$ with the Gelfand topology is called the space of maximal ideal of A*

Remark (First isomorphism theorem in ring theory).
Let R and R' be two rings and $\varphi : R \longrightarrow R'$ be a ring homomorphism with kernel K. Then we have $R/Ker(\varphi) \cong Im(\varphi)$.

Proposition 2.4.5. *Let I be a maximal ideal of a Banach algebra A, then there is a $f \in \Omega(A)$ such that $I = ker f$, so by the first isomorphism theorem we have:*

$$A/I \approx \mathbb{C}. \tag{2.45}$$

Definition 2.4.6. *Let A be a commutative Banach algebra. Then the Gelfand transformation is defined as a map $G : A \to C(\Omega(A))$ satisfying:*

$$\forall x \in A : G(x) = \hat{x}. \tag{2.46}$$

Proposition 2.4.7. *The Gelfand transformation is an algebraic homomorphism. It is an isomorphism if and only if A is a semisimple algebra.*

Theorem 2.4.8. *Let A be a commutative Banach algebra. Then $\Omega(A)$ is a locally compact topological Hausdorff space with respect to the Gelfand topology.*

Definition 2.4.9. *Let A and B be two $*$-algebras. A $*$-homomorphism between A and B is a complex linear map $\pi : A \to B$ such that the following axioms are satisfied:*

$$1) \quad \forall x, y \in A : \pi(xy) = \pi(x)\pi(y), \tag{2.47}$$

$$2) \quad \forall x \in A : \pi(x^*) = \pi(x)^*. \tag{2.48}$$

Proposition 2.4.10. *By the last definition the map π is a complex-module homomorphism and also is a complex-algebra homomorphism.*

Proposition 2.4.11. *Let A be a C^*–algebra, then the Gelfand transformation $G : A \to C(\Omega(A))$ can be written as follows:*

$$\forall f \in \Omega(A), x \in A : \hat{x}(f) = f(x) \Rightarrow Sp_A(x) = \{f(x); x \in A\} = \hat{x}(\Omega(A)) \tag{2.49}$$

In the commutative case, G is a bijection.

Now, we are ready for the Gelfand–Naimark theorem.

Theorem 2.4.12. *Let A be a commutative C^*–algebra, then the Gelfand transformation is a C^*–algebra isomorphism:*
1) If A is a unital algebra, there exist a unique compact topological Hausdorff space $X = \Omega(A)$ such that:$A \approx C(X)$.
2) If A is not unital, there exist a unique locally compact topological Hausdorff space $X = \Omega(A)$ but noncompact such that: $A \approx C_0(X)$.

So, the Gelfand-Naimark theorem shows that there is an equivalence between a locally compact topological Hausdorff space and a commutative C^*–algebra. In the other words, all information about a compact topological Hausdorff space X is encoded in the commutative C^*–algebra of complex continuous functions on X. Thus, a noncommutative C^*–algebras might be interpreted by the algebra of continuous functions of a noncommutative space. This is the substance of noncommutative geometry.

Chapter 3

Noncommutative space time

3.1 Commutative spaces

Space time in classical physics has the structure of a differentiable manifold. Let M be a smooth, compact, oriented and real or complex manifold. Let $C^\infty(M)$ be the set of all infinitely differentiable (real or complex) functions on M then $C^\infty(M)$ is a commutative, associative algebra with pointwise addition and multipilication:

$$1) \quad \forall f, g \in C^\infty(M), x \in M : (f+g)(x) = f(x) + g(x), \tag{3.1}$$

$$2) \quad \forall f, g \in C^\infty(M), x \in M : (f.g)(x) = f(x).g(x). \tag{3.2}$$

Now, we review two important theorems:
1) Banach-Stone theorem.
2) Nash theorem.

Theorem 3.1.1 (Banach-Stone theorem)**.** *Let X and Y be compact topological Hausdorff spaces and $T : C(X) \to C(Y)$ be a surjective linear isometry, then there exists a homomorphism $\varphi : Y \to X$ with:*

$$1)\forall y \in Y; \exists g \in C(Y) : |g(y)| = 1; \tag{3.3}$$

$$2)\forall y \in Y; \forall f \in C(X) : (Tf)(y) = g(y)f(\varphi(y)). \tag{3.4}$$

This theorem implies the next proposition:

Proposition 3.1.2. *Let M and N be differentiable manifolds. They are diffeomorphic if and only if the commutative algebras of smooth functions $C^\infty(M)$ and $C^\infty(N)$ are isomorphic.*

Thus, the informations on a manifold M can be expressed as a commutative algebra $C^\infty(M)$. For example, every point of manifold M can be identified with pure states of $C^\infty(M)$ or geometrical objects on a manifold such as a differential forms, tensor fields, vector bundles, etc can all be defined algebraically in term of $C^\infty(M)$.

Example Let $\Gamma(M)$ be a set of all smooth vector fields X on a manifold M, then $\Gamma(M)$ is a left $C^{\infty}(M)$ -module. An important property of X is the fact that it can be defined as a derivation of the commutative algebra $C^{\infty}(M)$:

$$\begin{aligned} &X : C^{\infty}(M) \to C^{\infty}(M) \\ &\forall f, g \in C^{\infty}(M) : \\ &X(fg) = X(f)g + fXg \end{aligned} \tag{3.5}$$

It leads us to identify $\Gamma(M)$ with the derivations of the algebra $C^{\infty}(M)$:

$$\Gamma(M) \equiv Der(C^{\infty}(M)). \tag{3.6}$$

Generally, we can construct differential calculus in purley algebraic term over commutative algebras as follow:
1) Topological information of a smooth manifold M is encoded in smooth function $A = C^{\infty}(M)$ as in Banach-Stone theorem.
2) Vector bundles over M correspond to projective finitely generated modules over A.
3) Vector fields on M are identified with derivations of A.
4) In general, a linear differential operator of order $p+1$, which images section of a vector bundle $E \to M$ to section of another bundle $F \to M$ is seem to be an $\mathbb{R}$-linear map $\Delta : \Gamma(E) \to \Gamma(F)$ such that for any $p+1$ element $f_1, ...f_{p+1} \in A$:

$$[f_{p+1}, [f_p, ... [f_1, \Delta] ...]] = 0. \tag{3.7}$$

Where the bracket $[f, \Delta] : \Gamma(E) \to \Gamma(F)$ is defined:

$$\forall s \in \Gamma(E) : [f, \Delta](s) = \Delta(f \cdot s) - f \cdot \Delta(s) \tag{3.8}$$

For more details see [30].

To define a differential calculus on an arbitrary algebra A, we should pay attention to the partial derivations and the observations that they fulfill two important properties:
1) It is a linear operator.
2) It satisfies the Leibniz rule.
Thus our aim is to find the operators acting on the algebra A that satisfy the last two conditions. One type of such operators are given by the *adjiont action* of an element of a Lie algebra L on L itself. Generally we have the definition:

Definition 3.1.3. *The adjoint endomorphism or adjiont action is an endomorphism of a Lie algebra L such that for every $x \in L$:*

$$\begin{aligned} &ad_x : L \to L, \\ &\forall y \in L : ad_x(y) = [x, y]. \end{aligned} \tag{3.9}$$

This definition satisfies the linearity condition and the Leibniz rule (by using the Jacobi identity).

Definition 3.1.4. *The mapping $ad : L \to End(L)$ given by $x \mapsto ad_x$ is a representation of a Lie algebra and is called the adjiont representation of the algebra.*

Theorem 3.1.5. *Let L be a semisimple Lie algebra, then $adL = Der(L)$.*

And the second important theorem in my discussion is *Nash* theorem:

Theorem 3.1.6 (Nash). *Let M be a compact manifold of dimansion m. For every positive and integer number $n \succ m$, M can be embedded in $\mathbb{R}^n$.*

The coordinates of the embedding space M are generators of an algebra of polynomials which is dense in $C(\mathbb{R}^n)$ (Stone-Weierstrass theorem) and the equations which define the manifold M are relations in $C(\mathbb{R}^n)$. The ideal I generated by the relations makes a quotient algebra structure $C(\mathbb{R}^n)/I$ and the quotient of $C(\mathbb{R}^n)$ by the ideal I is equal to $C^\infty(M)$ (first isomorphism theorem).

Example Sphere S^2 is embedded in $\mathbb{R}^3$ and the equation which defines S^2 is:

$$g_{ab}x^a x^b = R^2, \tag{3.10}$$

Where x^a are coordinates in $\mathbb{R}^3$. Consider the algebra *Pol* of polynomials in the x^a and let I be the ideal generated by the relation $g_{ab}x^a x^b = r^2$ so by the definition of an ideal, I consists of elements of *Pol* with $g_{ab}x^a x^b - r^2$ as factor. Then the quotient algebra $A = Pol/I$ is dense in the algebra $C(S^2)$. Any element of A can be represented as a finite multipole expansion of the form:

$$f(x^a) = f_0 + f_a x^a + \frac{1}{2} f_{ab} x^a x^b + ... \tag{3.11}$$

Later we will discuss more about the vector space of $f_{a_1 a_2 \ldots}$ and define the product of the x^a in order to introduce a multipicative structure and promote the vector space to an algebra.
To construct the derivations of A we use the Lie algebra of $su(2)$ ($S^2 \simeq SU(2)$). Let λ_a be a base of $su(2)$. The adjoint action of $su(2)$, $\lambda_a \mapsto g^{-1}\lambda_a g = \Lambda_a^b \lambda_b$ maps the element $g \in SU(2)$ onto the element $\Lambda_a^b \in SO(3)$. The base of $su(2)$ can be considered as a set e_a of vector fields on S^2. For more detail see [3].

So we have a commutative differential geometry and we present the following diagram:

$$\text{Manifold M} \longrightarrow \text{commutative algebra } A = C^\infty(M) \xrightarrow[\text{theorem}]{\text{G.N}} \text{geometrical object, induced by A}$$

Figure 3.1: Duality in commutative case

This leads us to the suggestion to generalize this diagram to noncommutative manifolds with noncommutative algebras. But we should not forget that over an arbitrary algebra there are many differential calculi.

3.2 Noncommutative spaces

The aim of noncommutative geometry is the study of topological spaces whose commutative algebra of functions is replaced by a noncommutative algebra. We should know that there are many sources of noncommutative spaces. We can identify at least four methods to construct noncommutative spaces [12]:
1) noncommutative quotient;
2) algebraic and C^*-algebraic deformations;
3) Hopf algebras and quantum groups;
4) cohomological constructions.

The main notion in noncommutative geometry is that *noncommutative geometry is a pointless geometry*. It means that we should swich to the algebraic perspective of geometrical structures. Since one of the important concepts that we need in physics is differentibale structures and also differential calculus on a space, then we focus on these structures.
The algebraic viewpoint of differential calculus on commutative spaces can motivate us to define differential calculus on noncommutative spaces. As we have seen in the last part, the differential geometry of a manifold M can be described as an algebra of functions defined on M. The coordinates of M are generators of the algebra and vector fields are the derivations, thus differential geometry on M can be described as operations on an algebra of functions. It leads us to develop a noncommutative version of differential geometry by replacing the algebra of functions $C^\infty(M)$ by an abstract associative and noncommutative algebra A. One possibility is the algebra M_n of $n \times n$ complex matrices. Since M_n is finite dimensional we have a finite noncommutative geometry.
Thus our aim is to find the operators acting on M_n. By equation (3.9) and theorem (3.1.5) we can construct a differential calculus on the matrix algebras as follow:

$$\forall A, B \in M_n : ad_A(B) := [A, B] \,. \tag{3.12}$$

The set of all derivations on the matrix algebra M_n, one interest are differentials d of an element $A \in M_n$ defined by:

$$dA := [F, A] \,. \tag{3.13}$$

where F is any fix self adjoint matrix of square one. Thus $F^2 = 1$ and it implies $d^2 = 0$.
This method opens the door to make a differential calculus on matrix algebra and also to formulate noncommutative structures of many physical theories.

Example The fuzzy sphere is a simple example for a noncommutative space. The algebra of fuzzy sphere can be constructed by using Berezin's quantization [7], [8] for the function algebra over the sphere and the result is the matrix algebra of finite dimension. But we should keep in mind that the way to introduce a differential calculus is not unique. For example see [9].
Let us consider commutative functions on the sphere S^2. As I said in the last part, smooth functions on S^2 can be written as a polynomial in x^a:

$$f(x^a) = f_0 + f_{a_1} x^{a_1} + \frac{1}{2} f_{a_1 a_2} x^{a_1} x^{a_2} + \ldots \tag{3.14}$$

We shall now construct a sequence of noncommutative approximation to $C(S^2)$. We start with truncating the expansion of all functions of $C(S^2)$ Eq(3.18) to the constant term. This reduces the algebra $C(S^2)$ to the algebra of complex numbers $A_1 = \mathbb{C}$, so the geometry of S^2 reduces to that of a point. Now we assume that we keep the expansion up to the term linear in x^a:

$$f(x^a) = f_0 + f_a x^a, \tag{3.15}$$

and the set A_2 of all these functions is a four dimention vector space. Our aim is to introduce a new product which establishes A_2 as an algebra. One possibility is the algebra M_2 of complex 2×2 matrices. We replace:

$$x^a \mapsto \tilde{x}^a = k\sigma^a, \tag{3.16}$$

where σ^a are the Pauli matrices $\left[\sigma^a, \sigma^b\right] = 2i\epsilon_{abc}\sigma^c$ and the parameter k must be related to the radius R:

$$x^a x^a = \tilde{x}^a \tilde{x}^a = R^2 \Rightarrow 3k^2 = R^2. \tag{3.17}$$

Suppose next that we also keep the quadratic term of expansion (3.14) in x^a:

$$f(x^a) = f_0 + f_{a_1} x^{a_1} + \frac{1}{2} f_{a_1 a_2} x^{a_1} x^{a_2}. \tag{3.18}$$

This is we consider the set of functions $A_3$of this form. This set is a nine dimentional vector space. As in the case of A_2, we introduce the algebra M_3 of 3×3 matrices. We replace:

$$x^a \mapsto \tilde{x}^a = kJ^a, \tag{3.19}$$

where J^a are the three dimensional irreducible representation of $SU(2)$ and k must be $8k^2 = R^2$. In general, suppose we keep the terms up to n-th order in the x^a and denote the resulting vector space by A_n. Because of the constraint (3.10), there are $(2r+1)$ independent monomials to n-th order in the expansion (3.11) and $\sum_{r=0}^{n-1}(2r+1) = n^2$ is the dimension of vector space A_n. As before we can replace n-dimensional irreducible representation of the Lie algebra $su(2)$:

$$\left[J^a, J^b\right] = 2i\varepsilon_{abc} J^c, \tag{3.20}$$

and also parameter k is related to R by the equation:

$$R^2 = (n^2 - 1)k^2. \tag{3.21}$$

For large n we have $k = \frac{r}{n}$ and when $k \to 0$ then $n \to \infty$.
We introduce a new constant $K = 4\pi kR$, It has a dimension of $(Length)^2$ and plays a rule like Planck's constant in quantum mechanics. So we have:

$$\bar{K} = \frac{K}{2\pi} = 2kR. \tag{3.22}$$

By the (3.19), (3.20) and (3.22) we can write:

$$\begin{aligned}\frac{1}{k^2}\left[\tilde{x}^a, \tilde{x}^b\right] &= 2i\varepsilon_{abc}\frac{1}{k}\tilde{x}^c \Rightarrow \left[\tilde{x}^a, \tilde{x}^b\right] = 2i\varepsilon_{abc} k\tilde{x}^c \\ &= 2i\varepsilon_{abc}\frac{\bar{K}}{2R}x^c \Rightarrow \left[\tilde{x}^a, \tilde{x}^b\right] = i\varepsilon_{abc}\bar{K}R^{-1}\tilde{x}^c,\end{aligned} \tag{3.23}$$

and we introduce $C_{abc} := R^{-1}\varepsilon_{abc}$, then we have:

$$\left[\tilde{x}^a, \tilde{x}^b\right] = iC_{abc}\bar{K}\tilde{x}^c. \tag{3.24}$$

So in the limit $\bar{K} \to 0$ we have commutativity and all of points of the sphere can be distinguished. We shall be more interested in differential calculus on the fuzzy sphere. Since we have a matrix algebra then we can use of the results for the differential calculus on matrix algebras. For more details see [3], [4].

Now we have a noncommutative differential geometry and we can show the following diagrams:

$$\begin{tikzcd}
\text{Noncommutative manifold} \arrow[r] & \text{Noncommutative algebra } \hat{A} \arrow[r] & \text{geometrical objects induced by } \hat{A}
\end{tikzcd}$$

Figure 3.2: Duality in noncommutative case

$$\begin{tikzcd}
\text{Manifold M} \arrow[r] \arrow[d] & \text{Commutative algebra } A = C^{\infty}(M) \arrow[r] \arrow[d, "\text{Deformation}"'] & \text{geometrical objects induced by } A \arrow[d, "\text{Deformation}"'] \\
\text{Noncommutative manifold } M \arrow[r] & \text{Noncommutative algebra } \hat{A} \arrow[r] & \text{geometrical objects induced by } \hat{A}
\end{tikzcd}$$

Figure 3.3: Duality in general form

By the last two figures, we can illustrate the following diagram:

$$\begin{tikzcd}
\text{Phase space } M \arrow[r] \arrow[d] & \text{Physical observable } A = C^{\infty}(M) \arrow[r] \arrow[d, "\text{Deformation}"'] & \text{Classical mechanics} \arrow[d, "\text{Deformation}"'] \\
\text{Noncommutative space time} \arrow[r] & \text{Physical observable } \hat{f} \in \hat{A} \arrow[r] & \text{Quantum mechanics}
\end{tikzcd}$$

Figure 3.4: Physical interpretation

Remark Since the product of Hermitian operators is not Hermitian (unless they commute), then the set of all Hermitian operators does not form an algebra. Therfore $\hat{A}$ should be a complex algebra of all operators which include the Hermitian operators.

3.3 Symmetry and Hopf algebra

In the commutative space time, symmetries are described by Lie groups and their Lie algebras, but in the noncommutative space time, this is no longer true.
Let us consider the noncommutative relation between coordinate functions $\left[\hat{x^\mu}, \hat{x^\nu}\right] = i\theta^{\mu\nu}$. Now we want to check Poincare symmetry in the noncommutative case. The derivative act on coordinate in classical case: $\left[\hat{\partial}_\nu, \hat{x^\mu}\right] = \delta^\mu_\nu$.
There are tow consistent ways to define commutation relation of derivatives. Simplest choice is:

$$\left[\hat{\partial}_\nu, \hat{\partial}_\mu\right] = 0. \tag{3.25}$$

Another possibility compatible with the coordinate algebra is obtained by observing that:

$$\hat{x^\mu} - i\theta^{\mu\nu}\hat{\partial}_\nu \tag{3.26}$$

commutes with all coordinates $\hat{x^\mu}$ and all derivatives $\hat{\partial}_\nu$.It is possible when we assume that this expression equals some constant, 0 say [17]. Then we have:

$$\hat{x^\mu} - i\theta^{\mu\nu}\hat{\partial}_\nu = 0 \Rightarrow \hat{\partial}_\nu = -i\theta^{-1}_{\mu\nu}\hat{x^\mu} \Rightarrow \left[\hat{\partial}_\mu, \hat{\partial}_\nu\right] = -i\theta^{-1}_{\mu\nu}. \tag{3.27}$$

.

So the Poincare Lie algebra (symmetry) is broken by the existence of the noncommutative structure $\theta^{\mu\nu}$. Thus we need a new structure such that the deformed space is invariant under that. This structure is called a Hopf algebra. Hopf algebras occur naturally is in algebraic topology [31], [32], [33].
We start to define some structures which used in the definition of Hopf algebra. These structures are coalgebras and bialgebras.

Definition 3.3.1. *A triple (H, π, η) is called a unital associative algebra over a ring R if it satisfies the following axioms:*
1) H is a R-module.
2) The map $\pi : H \otimes H \to H$ such that:

$$\forall h_i, h_j \in H : \pi(h_i \otimes h_j) = h_i h_j.$$

is a module homomorphism.
3) The map $\eta : R \to H$ is a module homomorphism.
4) The following diagrams are commutative:

$$\begin{tikzcd}
H \otimes H \otimes H \arrow[r, "\pi \otimes \mathbf{1}"] \arrow[d, "\mathbf{1} \otimes \pi"] & H \otimes H \arrow[d, "\pi"] \\
H \otimes H \arrow[r, "\pi"] & H
\end{tikzcd} \qquad \widehat{=} \qquad (h_i h_j)h_k = h_i(h_j h_k)$$

Figure 3.5: Associativity

$$\begin{tikzcd}
R \otimes H \arrow[r, "\eta \otimes \mathbf{1}"] \arrow[dr, "\cong"'] & H \otimes H \arrow[d, "\nabla"] & H \otimes R \arrow[l, "\mathbf{1} \otimes \eta"'] \arrow[dl, "\cong"] \\
& H &
\end{tikzcd} \quad \widehat{=} \quad 1_H \cdot h = h \cdot 1_H = h$$

Figure 3.6: Unity

Definition 3.3.2. *A triple $(H, \Delta, \epsilon)$is called a coalgebra over a ring R if it satisfies the following axioms:*
1) H is a R-module.
2) The map $\Delta : H \to H \otimes H$ such that:

$$\forall h_i, h_j \in H : \Delta(h_i h_j) = h_i \otimes h_j. \tag{3.28}$$

is a module homomorphism.
3) The map $\epsilon : H \to R$ in a module homomorphism.
4) The following diagrams are commutative:

$$\begin{tikzcd}
H \otimes H \otimes H & H \otimes H \arrow[l, "\Delta \otimes \mathbf{1}"'] \\
H \otimes H \arrow[u, "\mathbf{1} \otimes \Delta"'] & H \arrow[l, "\Delta"'] \arrow[u, "\Delta"']
\end{tikzcd}$$

Figure 3.7: co-associativity

$$\begin{tikzcd}
H \otimes R & H \otimes H \arrow[l, "\epsilon \otimes \mathbf{1}"'] \arrow[r, "\mathbf{1} \otimes \epsilon"] & R \otimes H \arrow[dl, "\cong"] \\
& H \arrow[u, "\Delta"'] &
\arrow[from=1-1, to=2-2, "\cong"']
\end{tikzcd}$$

Figure 3.8: co-unity axiom

Remark Fig. (3.7) and (3.8) obtained from (3.5) and (3.6) by inversing arrows (dual structures).

Definition 3.3.3. *A quintuple $(H, \pi, \Delta, \eta, \epsilon)$ is called a bialgebra over a ring R if satisfies the following axioms:*
1) The triple (H, π, η) is a unital associative algebra.
2) The triple (H, Δ, ϵ) is a coalgebra.
3) The following diagram are commutative:

$$a) \quad \Delta \circ \pi = (\pi \otimes \pi) \circ (1_H \otimes \tau \otimes 1_H) \circ (\Delta \otimes \Delta). \tag{3.29}$$

$$b) \quad \Delta \circ \eta = \eta \otimes \eta. \tag{3.30}$$

$$c) \quad \epsilon \circ \pi = \epsilon \otimes \epsilon. \tag{3.31}$$

$$d) \quad \epsilon \circ \eta = \mathbf{1}_R. \tag{3.32}$$

τ is a module homomorphism:

$$\tau : H \otimes H \to H \otimes H \tag{3.33}$$
$$\tau(h_1 \otimes h_2) = h_2 \otimes h_1. \tag{3.34}$$

.

Definition 3.3.4. *$(H, \pi, \Delta, \eta, \epsilon, S)$ is called a Hopf algebra if satisfies the following axioms:*
1) The quintuple $(H, \pi, \Delta, \eta, \epsilon)$ is a bialgebra over a ring R.
2) The map $S : H \to H$ is an algebra anti homomorphism such that:

$$\pi \circ (\mathbf{1} \otimes S) \circ \Delta = \pi \circ (S \otimes \mathbf{1}) \circ \Delta = \eta \circ \epsilon \tag{3.35}$$

Definition 3.3.5. *The algebra anti homomorphism $S : H \to H$ in the last definition is called the antipode.*

Definition 3.3.6. *An invertible element $F \in H \otimes H$ is called a twist if it satisfies:*

$$(F \otimes 1_H)(\Delta \otimes \mathbf{1})F = (1_H \otimes F)(\mathbf{1} \otimes \Delta)F, \tag{3.36}$$
$$(\epsilon \otimes \mathbf{1})F = 1_{H\otimes H} = (\mathbf{1} \otimes \epsilon)F. \tag{3.37}$$

Using a element F one can define a new Hopf algebra structure on H with a new coproduct Δ_F:

$$\Delta_F(h) = F\Delta\,(h)\,F^{-1}. \tag{3.38}$$

This called twisted coproduct.

Definition 3.3.7. *Let H be a Hopf algebra and A be a unital associative algebra. We say H (left) act on A if there is bilinear map $H \otimes A \to A$ such that:*

$$h \otimes a \mapsto h \triangleright a. \tag{3.39}$$

compatible with the algebra structure on H:

$$(hh') \triangleright a = h \triangleright (h' \triangleright a), \tag{3.40}$$
$$1_H \triangleright f = f. \tag{3.41}$$

And also with the coalgebra structure on H:

$$h \triangleright \pi(a \otimes a') = \pi \circ \Delta(h) \triangleright (a \otimes a'), \tag{3.42}$$
$$h \triangleright 1_A = 1_A \circ \epsilon(h). \tag{3.43}$$

Definition 3.3.8. *A Hopf algebra is called cocommutative if:*

$$\tau \circ \Delta\,(x) = \Delta\,(x)\,. \tag{3.44}$$

Example A simple example of a Hopf algebra is the universal envelope of a Lie algebra. Let L be a Lie algebra with generators T^i satisfying:

$$\left[T^i, T^j\right] = i\epsilon_k^{ij} T^k. \tag{3.45}$$

The universal enveloping algebra of L is defined as the quotient algebra $u(L) = \frac{T(L)}{I}$ such that $T(L)$ is tensor algebra of generators and I is an Ideal generated by $T^i \otimes T^j - T^j \otimes T^i - [T^i, T^j]$. Then $u(g)$ is a Hopf algebra with the following structure maps:

$$\begin{aligned} 1) \quad & \Delta : u(L) \to u(L) \otimes u(L), & (3.46)\\ & \Delta(x) = (x \otimes 1) + (1 \otimes x) \\ 2) \quad & \epsilon : u(g) \to \mathbb{C}, & (3.47)\\ & \epsilon(x) = 0 \\ 3) \quad & S : u(L) \to u(L), & (3.48)\\ & s(x) = -x. \end{aligned}$$

The universal enveloping algebra is cocommutative, which means that:

$$\tau \circ \Delta(x) = \Delta(x) \tag{3.49}$$

with the permutation $\tau(x \otimes y) = y \otimes x$.

Example Let G be an arbitrary finite group and $F(G) = \{f : G \to \mathbb{C}\}$ be the set of all complex functions on G, then $F(G)$ is a Hopf algebra:
1) The algebra structure with the following maps:

$$\begin{aligned} a) \quad & \pi : F(G) \otimes F(G) \to F(G), & (3.50)\\ & \pi(f_1 \otimes f_2)(g) = f_1(g) f_2(g), \\ b) \quad & \eta : \mathbb{C} \to F(G), & (3.51)\\ & \eta(k) = k 1_{F(G)}, \end{aligned}$$

where $f_1, f_2 \in F(G)$ and $g \in G$.
2) The coalgebra structure with the following maps:

$$\begin{aligned} a) \quad & \Delta : F(G) \to F(G) \otimes F(G), & (3.52)\\ & \Delta(f)(g_1 \otimes g_2) = f(g_1 . g_2), \\ b) \quad & \epsilon : F(G) \to \mathbb{C}, & (3.53)\\ & \epsilon(f) = f(e), \end{aligned}$$

where $f \in F(G)$ and $g_1, g_2 \in G$ and e is the unit element of G. The antipode $S : F(G) \to F(G)$ is also given by:

$$S(f)(g) = f(g^{-1}). \tag{3.54}$$

So $F(G)$ is a Hopf algebra and it is a commutative Hopf algebra because the algebra of functions is commutative.

Example (Group Hopf algebra) Let G be a matrix group of complex $n \times n$ matrices. The $\mathbb{C}G$ is a $\mathbb{C}$-algebra and is called group algebra of G over $\mathbb{C}$. An element of $\mathbb{C}G$ is given by :

$$\mathbb{C}G \ni x = \Sigma_{g \in G} a_g g, \tag{3.55}$$

such that $a_g \in \mathbb{C}$. The generators t^i_j form a basis of $\mathbb{C}G$; $i, j = 1, ..., n$ and the generators have 1 in the i^{th} row and j^{th}column, and 0 everywhere else. It is enough to specify the structure maps for generators because the generators generate $\mathbb{C}G$. Thus we have:

$$\Delta(t^i_j) = \Sigma_k t^i_k \otimes t^k_j, \tag{3.56}$$
$$\epsilon(t^i_j) = 1, \tag{3.57}$$
$$S(t^i_j) = (t^{-1})^i_j. \tag{3.58}$$

This Hopf algebra is commutative because:

$$t^i_j t^m_n = t^m_n t^i_j. \tag{3.59}$$

Definition 3.3.9. *A quantum group is a pair (Q, R) such that Q is a Hopf algebra and R is an additional structure, which we will describe with following:*

Let us consider the group Hopf algebra $\mathbb{C}G$ of some matrix group G. The additional structure is defined as a R-Matrix:

$$R : \mathbb{C}G \otimes \mathbb{C}G \to \mathbb{C},$$
$$R(t^i_k \otimes t^j_l) \equiv R^{ij}_{kl}. \tag{3.60}$$

The R-matrix is not an arbitrary matrix. It has to be a solution of the *Quantum-Yang-Baxter-Equation* [17]

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{3.61}$$

where $(R_{13})^{ijk}_{lmn} = \delta^j_m R^{ik}_{ln}$, R_{12} and R_{23} are defined accordingly.

As we have seen in the last example, the group Hopf algebra $\mathbb{C}G$ is a commutative Hopf algebra. The R-matrix introduces a deformation to a noncommutative Hopf algebra (because generally tensor product is not commutative). Thus the quantum group is a noncommutative Hopf algebra. We denote it by $\mathbb{C}G_q$.

Now we come back to the Poincare symmetry which we discussed before. The starting point is the universal enveloping algebra of Poincare algebra, $u(p)$. This Hopf algebra has the coproduct:

$$\forall x \in u(p) : \Delta(x) = x \otimes 1 + 1 \otimes x. \tag{3.62}$$

We introduce a twist $F \in u(g) \otimes u(g)$, which satisfies the cocycle condition:

$$(F \otimes 1)(\Delta \otimes 1)F = (1 \otimes F)(1 \otimes \Delta)F. \tag{3.63}$$

Thus we can define a coproduct $\Delta_F(x)$ by the twist F:

$$\Delta_F(x) = F\Delta(x)F^{-1}. \tag{3.64}$$

In the case of canonical deformation, the twist is:

$$F = exp\left(\frac{i}{2}\theta^{\mu\nu}P_\mu \otimes P_\nu\right), \tag{3.65}$$

where P_μ are the translation generators. Therefore, we obtain:

$$\Delta_F P_\mu = F\Delta P_\mu F^{-1} = exp\left(\frac{i}{2}\theta^{\mu\nu}P_\mu \otimes P_\nu\right)(P_\mu \otimes 1 + 1 \otimes P_\mu)\, exp\left(\frac{-i}{2}\theta^{\mu\nu}P_\mu \otimes P_\nu\right)$$
$$\Rightarrow \Delta_F P_\mu = P_\mu \otimes 1 + 1 \otimes P_\mu, \tag{3.66}$$

since the translation generators commute.
And for the Lorentz generator $M_{\mu\nu}$, we have:

$$\Delta_F M_{\mu\nu} = exp\left(\frac{i}{2}\theta^{\mu\nu}P_\mu \otimes P_\nu\right)(M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu})\, exp\left(\frac{-i}{2}\theta^{\mu\nu}P_\mu \otimes P_\nu\right)$$
$$\Rightarrow \Delta_F M_{\mu\nu} = (M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu}) + \frac{i}{2}\theta^{\alpha\beta}\Big(((\eta_{\alpha\mu}P_\nu - \eta_{\alpha\nu}P_\mu) \otimes P_\beta$$
$$+ P_\alpha \otimes (\eta_{\beta\mu}P_\nu - \eta_{\beta\nu}P_\mu)\Big) \tag{3.67}$$

And also the action of generators on coordinates are:

$$(P_\mu \hat{x}^\alpha) = i\delta^\alpha_\mu, \tag{3.68}$$
$$(M_{\mu\nu}\hat{x}^\alpha) = i\left(\hat{x_\mu}\partial_\nu - \hat{x_\nu}\partial_\mu\right)\hat{x^\alpha} = i\left(\hat{x_\mu}\delta^\alpha_\nu - \hat{x_\nu}\delta^\alpha_\mu\right). \tag{3.69}$$

Using the definition (3.3.7), due to the deformed coproduct of the generators above and their action on coordinates, we have:

$$P_\mu \triangleright \pi\left(\hat{x}^\alpha \otimes \hat{x}^\beta - \hat{x}^\beta \otimes \hat{x}^\alpha\right) = P_\mu \triangleright \left[\hat{x}^\alpha, \hat{x}^\beta\right] := \Delta_F P_\mu \left(\hat{x}^\alpha \otimes \hat{x}^\beta - \hat{x}^\beta \otimes \hat{x}^\alpha\right) \tag{3.70}$$
$$\Rightarrow P_\mu \triangleright \left[\hat{x}^\alpha, \hat{x}^\beta\right] = (P_\mu \otimes 1 + 1 \otimes P_\mu)\left(\hat{x}^\alpha \otimes \hat{x}^\beta - \hat{x}^\beta \otimes \hat{x}^\alpha\right) = 0. \tag{3.71}$$

And acting on the right hand side of the (1.1) yields the same result:

$$P_\mu \triangleright i\theta^{\alpha\beta} = 0. \tag{3.72}$$

Next consider the Lorentz generator $M_{\mu\nu}$:

$$M_{\mu\nu} \triangleright \pi\left(\hat{x}^\alpha \otimes \hat{x}^\beta - \hat{x}^\beta \otimes \hat{x}^\alpha\right) = M_{\mu\nu} \triangleright \left[\hat{x}^\alpha, \hat{x}^\beta\right] := \Delta_F M_{\mu\nu}\left(\hat{x}^\alpha \otimes \hat{x}^\beta - \hat{x}^\beta \otimes \hat{x}^\alpha\right). \tag{3.73}$$

As before, the result is equal to zero, acting on the right hand side of (1.1) yields:

$$M_{\mu\nu} \triangleright i\theta^{\alpha\beta} = 0. \tag{3.74}$$

Therefore, the noncommutative relations are consistent with the twisted Poincare Hopf algebra. It means that the symmetry of the noncommutative space is expressed by the Hopf algebra.

Chapter 4

Star Product

4.1 Deformation

Now we are ready to study deformation quantization. Let us start with the general definition of deformation in pure mathematics, and then we come back to our case, namely Poisson manifolds.

Definition 4.1.1. *Lex X be a module over a ring R and $\alpha \in R$. The set $X\left[[\alpha]\right]$ of formal power series in the formal parameter α with cofficients in X is given by the Cartesian product $X\left[[\alpha]\right] = \prod_{r=0}^{\infty}(X_r)$ such that $X_r = X$ for all r. Each sequence $(x_r) = (x_0, x_1, ...)$ in X is denoted as a formal power series:*

$$x = \sum_{r=0}^{\infty}(\alpha^r x_r). \tag{4.1}$$

Remark . $X\left[[\alpha]\right]$ is also a module over the ring R.

Remark . Let X, Y, Z be R-modules. Then an operation $\bullet : X \times Y \longrightarrow Z$ induces a corresponding operation $\circ : X\left[[\alpha]\right] \times Y\left[[\alpha]\right] \longrightarrow Z\left[[\alpha]\right]$ such that:

$$(\sum_{r=0}^{\infty}(\alpha^r x_r)) \circ (\sum_{s=0}^{\infty}(\alpha^s x_s)) = \sum_{r=0}^{\infty}(\alpha^r)\sum_{s=0}^{\infty}(x_s \bullet y_{r-s}) \tag{4.2}$$

.

Remark Let X and Y be R-modules and $Hom_R(X, Y)$ is a set of all R-module homomorphisms. Then:

$$Hom_R(X, Y)\left[[\alpha]\right] \cong Hom_{R[[\alpha]]}(X\left[[\alpha]\right], Y\left[[\alpha]\right]). \tag{4.3}$$

We can extent all algebraic structures to formal power series.

Definition 4.1.2. *Let A be an algebra over a field F (of characteristic zero). A formal deformation of A is a $F[[\alpha]]$-bilinear multipication law $\mu_\alpha : A[[\alpha]] \otimes A[[\alpha]] \to A[[\alpha]]$ on the algebra $A[[\alpha]]$ of formal power series in a variable α with coeffcients in A, satisfying the following axioms:*

$$1)\forall a, b \in A : \mu_\alpha(a \otimes b) = a \cdot b + \mu_1(a \otimes b)\,\alpha + \mu_2(a \otimes b)\,\alpha^2 + \ldots \tag{4.4}$$

where a.b is the pointwise multipication.

$$2)\forall a, b, c \in A : \mu_\alpha(\mu_\alpha(a \otimes b) \otimes c) = \mu_\alpha(a \otimes \mu_\alpha(b \otimes c)), \qquad \textit{Associativity}.$$

Since A is an algebra, then there is a multipication map [26], [27]:

$$\pi : A \otimes A \to A \tag{4.5}$$

$$\forall a, b \in A : \pi(a \otimes b) = a \cdot b. \tag{4.6}$$

And since $A[[\alpha]]$ is a formal power series of A, then there is a multipication map:

$$\pi_\alpha : A[[\alpha]] \otimes A[[\alpha]] \to A[[\alpha]] \tag{4.7}$$

$$\forall a, b \in A[[\alpha]] : \pi_\alpha(a \otimes b) =: a \bullet b. \tag{4.8}$$

Thus, by the definition (4.1.2), if $A[[\alpha]]$ be a formal deformation of A, then we can write:

$$a \bullet b = a \cdot b + \Sigma_{r=0}^{\infty} \alpha^r \pi_r(a \otimes b). \tag{4.9}$$

In physical terms, A has a Poisson structure, such that A is the quasi-classical limit of the associative algebra $A[[\alpha]]$, and the algebra $A[[\alpha]]$ is called deformation quantization of the Poisson algebra A. Also α can be regarded as a quantum parameter such as the Planck constant [28]. Now, we come back and limit ourselves to Poisson manifolds.

4.2 Poisson manifold

Definition 4.2.1. *Let A be a vector space over a field F. A Poisson bracket on A is a bilinear function $\{,\} : A \times A \longrightarrow A$ such that the following axioms are satisfied:*

$$1)\quad \forall x_1, x_2 \in A : \{x_1, x_2\} = -\{x_2, x_1\}\,(\textit{skew symmetry}), \tag{4.10}$$

$$2)\quad \forall x_1, x_2, x_3 \in A : \{\{x_1, x_2\}, x_3\} + \{\{x_2, x_3\}, x_1\} + \{\{x_3, x_1\}, x_2\} = 0(\textit{Jacobi identity}). \tag{4.11}$$

Definition 4.2.2. *Let A be a F-vector space with a Poisson bracket. A is called a Poisson algebra if there is a commutative product on A, $A \times A \longrightarrow A$ such that satisfies the follow axiom:*

$$\forall x_1, x_2, x_3 \in A : \{x_1, x_2 x_3\} = x_2\{x_1 x_3\} + \{x_1, x_2\} x_3 \ \textit{(Leibniz rule)}. \tag{4.12}$$

Definition 4.2.3. *A manifold M with a Poisson structure is called Poisson manifold.*

Remark . Let M be a Poisson manifold and $g \in C^\infty(M)$. Let the map $f \mapsto \{f, g\}$ be a derivation on $C^\infty(M)$. This implies the existence of a vector field X_g (Hamiltonian vector field) on M such that:

$$\forall f \in C^\infty(M) : X_g(f) = \{f, g\}. \tag{4.13}$$

Thus the Poisson bracket depends only on the derivative of $f \in C^{\infty}(M)$.

Remark . Let M be a Poisson manifold. The map between the tangent and cotangent bundle on M implies the existence of a bivector field α on M such that:

$$\forall f, g \in C^{\infty}(M) : \{f, g\} = \langle df \otimes dg, \alpha \rangle \,, \tag{4.14}$$

where $\langle , \rangle$ is the usual pairing between the tangent and cotangent bundle.

Remark . Locally we can write the Poisson bracket with an antisymmetric tensor $\alpha^{ij} = -\alpha^{ji}$:

$$\forall f, g \in C^{\infty}(M) : \{f, g\} = \alpha^{ij}(X)\partial_i(f)\partial_j(g), \tag{4.15}$$

and by the Jacobi identity we have:

$$\alpha^{ij}\partial_j\alpha^{kl} + \alpha^{kj}\partial_j\alpha^{li} + \alpha^{lj}\partial_j\alpha^{ik} = 0. \tag{4.16}$$

Now we can define a deformation on the algebra of smooth complex functions on a Poisson manifold. This deformation is called star product.

Definition 4.2.4 (Star product)**.** *Let M be a Poisson manifold and $C^{\infty}(M)$ be a set of all smooth complex functions on M. The formal star product on M is a $\mathbb{C}[[\alpha]]$-bilinear product $* : C^{\infty}(M)[[\alpha]] \times C^{\infty}(M)[[\alpha]] \longrightarrow C^{\infty}(M)[[\alpha]]$:*

$$\forall f, g \in C^{\infty}(M) : f * g = \sum_{r=0}^{\infty}(\alpha^r C_r(f, g)), \tag{4.17}$$

Such that $C_r : C^{\infty}(M) \times C^{\infty}(M) \longrightarrow C^{\infty}(M)$ are C- bilinear function, satisfing the following axioms:

$$\begin{aligned}
&1) \quad * \text{ is an associative product:} \forall f, g, h \in C^{\infty}(M) : (f * g) * h = f * (g * h),\\
&2) \quad C_0(f, g) = f.g,\\
&3) \quad 1 * f = f * 1 = f \Longrightarrow C_r(1, f) = C_r(f, 1) = 0, \forall r \succ 0\\
&4) \quad \forall f, g \in C^{\infty}(M) : i\alpha\{f, g\} = (C_1(f, g) - C_1(g, f)).
\end{aligned} \tag{4.18}$$

Definition 4.2.5. *Let M be a Poisson manifold and $C^{\infty}(M)$ is the set of all smooth complex functions on M. The pair $(C^{\infty}(M)[[\alpha]], *)$ such that $*$ is a star product on M is called deformation quantization on M.*

Example Let $M = \mathbb{R}^n$ and $\alpha \in \Gamma(\wedge^2 TM)$ be a Poisson structure with constant coefficient $\alpha = \alpha^{ij}\partial_i \otimes \partial_j$ such that $\alpha^{ij} = -\alpha^{ji} \in \mathbb{R}$.
The Moyal product is defined as a formal exponential of α:

$$\forall f, g \in C^{\infty}(M) : \tag{4.19}$$

$$f * g = \pi \circ exp\,(i\alpha\hbar)\,(f \otimes g)\,, \tag{4.20}$$

where $\pi : C^{\infty}(M) \otimes C^{\infty}(M) \to C^{\infty}(M)$ is the ordinary product on $C^{\infty}(M)$. Then by definition of Moyal product we have:

$$f * g = \pi\,(f \otimes g) + \left(i\alpha^{ij}\hbar\right)\pi\,(\partial_i \otimes \partial_j)\,(f \otimes g) + O\left(\hbar^2\right) \Rightarrow \tag{4.21}$$

$$f * g = f \cdot g + \left(i\alpha^{ij}\hbar\right)\pi\,(\partial_i f \otimes \partial_j g) + O\left(\hbar^2\right) \Rightarrow f * g = f \cdot g + \left(i\alpha^{ij}\hbar\right)\partial_i f \partial_j g + O\left(\hbar^2\right) \tag{4.22}$$

$$\Rightarrow \forall x \in M : f * g\,(x) = exp\left\{i\hbar\alpha^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}\right\} f\,(x)\,g\,(y)\,|_{x=y}\,. \tag{4.23}$$

Now we check the axioms of the definition (4.2.4):
The second, third and fourth axioms are simply satisfied. For the associativity, we have:

$$\begin{aligned}
&((f*g)*h)(x) = exp\left\{i\hbar\alpha^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial z^j}\right\}(f*g)(x)\,h(z)\,|_{x=z} \\
&= \left(exp\left\{i\hbar\alpha^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial z^j}\right\}\left(exp\left\{i\hbar\alpha^{lk}\frac{\partial}{\partial x^l}\frac{\partial}{\partial y^k}\right\}f(x)g(y)\right)_{x=y}h(z)\right)_{x=z} \\
&= \left(exp\left\{i\hbar\alpha^{ij}\left(\frac{\partial}{\partial x^i}+\frac{\partial}{\partial y^i}\right)\frac{\partial}{\partial z^j}\right\}exp\left\{i\hbar\alpha^{lk}\frac{\partial}{\partial x^l}\frac{\partial}{\partial y^k}\right\}f(x)g(y)h(z)\right)_{x=y=z} \\
&= \left(exp\left\{i\hbar\alpha^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial z^j}\right\}exp\left\{i\hbar\alpha^{ij}\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}\right\}exp\left\{i\hbar\alpha^{lk}\frac{\partial}{\partial x^l}\frac{\partial}{\partial y^k}\right\}f(x)g(y)h(z)\right)_{x=y=z} \\
&= \left(exp\left\{i\hbar\alpha^{ij}\frac{\partial}{\partial x^i}\left(\frac{\partial}{\partial z^j}+\frac{\partial}{\partial y^j}\right)\right\}f(x)(g*h)(y)\right)_{x=y} \\
&= \left(exp\left\{i\hbar\alpha^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}\right\}f(x)(g*h)(y)\right)_{x=y} = (f*(g*h))(x).
\end{aligned} \tag{4.24}$$

Example (quantum plane) [39]. Let us consider two vector fields $x\frac{\partial}{\partial x}$ and $x\frac{\partial}{\partial y}$ on the two dimensional real space $\mathbb{R}^2$. A star product is defined on the functions $f, g \in C^\infty(M)$ as follows:

$$\begin{aligned}
f*g &:= \pi\circ exp\left(i\hbar x\frac{\partial}{\partial x}\otimes y\frac{\partial}{\partial y}\right)(f\otimes g) = \pi(f\otimes g) + i\hbar\pi\left(x\frac{\partial}{\partial x}\otimes y\frac{\partial}{\partial y}\right)(f\otimes g) + O\left(\hbar^2\right) \\
&\Rightarrow f*g = f\cdot g + i\hbar\pi\left(x\frac{\partial f}{\partial x}\otimes y\frac{\partial g}{\partial y}\right) + O\left(\hbar^2\right) \Rightarrow f*g = f\cdot g + i\hbar\left(xy\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}\right) + O\left(\hbar^2\right).
\end{aligned} \tag{4.25}$$

And the Poisson bracket is :

$$\{f,g\} = \frac{1}{i\hbar}xy\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right). \tag{4.26}$$

By replacing f, g with the coordinate functions, we have:

$$x*y = x\cdot y + i\hbar(x\cdot y) + O\left(\hbar^2\right) \Rightarrow x*y = \Sigma_{n=0}^{\infty}\left(\frac{i\hbar}{n!}\right)x\cdot y \Rightarrow x*y = e^{i\hbar}(x\cdot y). \tag{4.27}$$

And simply we can calculate:

$$y*x = x\cdot y. \tag{4.28}$$

Thus by the last two equations we obtain:

$$x*y = e^{i\hbar}(y*x). \tag{4.29}$$

Example [16] In Quantum mechanics, we have the Heisenberg commutation relations between momenta and position operators,

$$\left[Q^i, P_j\right] = i\hbar\delta^i_j. \tag{4.30}$$

In normal ordering, where all momenta are on the right and all coordinates on the left, thus we obtain for the $*$-product:

$$f *_N g(Q,P) = \pi\circ exp(-i\hbar\partial_{P_i}\otimes\partial_{Q^i})f(Q,P)\otimes g(Q,P) \tag{4.31}$$

For symmetrical ordering we have:

$$f *_S g(P,Q) = \pi \circ exp(\frac{i\hbar}{2}(\partial_{Q^i} \otimes \partial_{P_i} - \partial_{P_i} \otimes \partial_{Q^i}))f(Q,P) \otimes g(Q,P). \tag{4.32}$$

Definition 4.2.6. *Let M be a smooth manifold and $*$ and $\tilde{*}$ be two star products on M. This two products are equivalent if there is a formal power series $\mu = I_{C^\infty(M)} + \sum_{r=1}^{\infty}(\alpha^r \mu_r)$ of linear maps $\mu_r : C^\infty(M) \longrightarrow C^\infty(M)$ such that:*

$$\begin{aligned} &1) \quad \mu(I) = I, \\ &2) \quad \forall f, g \in C^\infty(M) : \mu(f * g) = \mu(f)\tilde{*}\mu(g). \end{aligned}$$

Definition 4.2.7. *A star product is called differentiable if C_r are differentiable operators.*

Example Let us consider the last example. There $*_N$ and $*_S$ are equivalent $*$-product. Using matrices in the exponent, these formulae can be written as follow:

$$f *_N g(q,p) = \pi \circ exp(\frac{i\hbar}{2}\widetilde{\alpha}^{ij}\partial_i \otimes \partial_j)f(q,p) \otimes g(q,p), \tag{4.33}$$

$$f *_S g(q,p) = \pi \circ exp(\frac{i\hbar}{2}\alpha^{ij}\partial_i \otimes \partial_j)f(q,p) \otimes g(q,p), \tag{4.34}$$

where

$$(\widetilde{\alpha}^{\mathbf{ij}})_{\mathbf{i,j=q,p}} = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}$$

and

$$(\alpha^{\mathbf{ij}})_{\mathbf{i,j=q,p}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

α is the antisymmetric part of $\widetilde{\alpha}$ (remember that every matrix can be written as a symmetric and antisymmetric parts: $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$).
These $*$-products are connected by the transformation μ,

$$\mu = exp(\frac{i\hbar}{4}\theta_S^{ij}\partial_i\partial_j) = exp(-\frac{i\hbar}{2}\partial_q\partial_p), \tag{4.35}$$

where θ_S is the symmetric part of $\widetilde{\alpha}$, $\widetilde{\alpha}^{ij} = \alpha^{ij} + \theta_S^{ij}$:

$$(\theta^{\mathbf{ij}})_{\mathbf{S}} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Then we have:

$$f *_S g = \mu^{-1}(\mu f *_N \mu g). \tag{4.36}$$

4.3 Kontsevich theorem

The natural subsequent step is to look for the existence of a star product on an arbitrary Poisson manifold (i.e. existence of deformation quantization on an arbitrary Poisson manifold).

The existence of a star product on a sympletic phase space at first was shown by *DeWilde* and *Lecomte* in 1983 [34]. The existence of star product on a general Poisson manifold was solved by *Kontsevich* in 1997 [35].

Theorem 4.3.1 (Kontsevich)**.** *For any Poisson bivector field α in some domain of R^n, there exists a star product $*_k$. It is given by the following formula:*

$$\forall f, g \in C^\infty(R^n):$$
$$f *_k g = \sum_{\infty}^{n=0} h^n \sum_{\Gamma \in G_n} \omega_\Gamma B_{\Gamma,\alpha}(f,g) \tag{4.37}$$

Let us explain the symbols which appear in the Kontsevich formula:

1. h is the formal deformation parameter.
2. The index n in G_n $(n \geq 0)$ is the order in the formal deformation parameter.
3. $\Gamma \in G_n$ is an admissible graph of order n, which satisfies the following properties:
   a) Γ has $(n+2)$ vertices labeled by $\{1, 2, ...n, L, R\}$. L and R are just symbols for left and right.
   b) Γ has $2n$ edges, labeled by $i_1, j_1, ..., i_n, j_n$.
   c) The edges i_k and j_k start at the vertex k and end there.
   d) The edges i_k and j_k have to end at different vertices.
4. ω_Γ is called the weight for the graphs Γ and is more complicated [19]
5. For every $\Gamma \in G_n$, there is an associated bi-differentiable operator $B_{\Gamma,\alpha}$ and a weight ω_Γ.
6. The edges correspond to derivatives and the vertices correspond to the Poisson structure α. If an edge ends at a vertex, it means that the derivative of the corresponding Poisson structure acts on the corresponding Poisson structure.
   If it ends on L or R, it means that the derivative acts on the function f or g, respectively.

Now, we review some examples of Kontsevich graphs.
Examples:
1) Graph to n^{th} order for the Moyal product. The Poisson structure is constant and derivatives can only act on the functions f and g:

$$f * g = \sum_{r=0}^{\infty} \frac{\hbar}{r!} (\alpha^{i_1 j_1} ... \alpha^{i_r j_r})(\partial_{i_1} ... \partial_{i_r} f)(\partial_{j_1} ... \partial_{j_r} g) \tag{4.38}$$

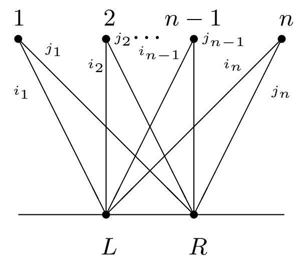

Figure 4.1: Kontsevich diagram, α is constant.

2) Third order, nonconstant Poisson structure:
In this case we have:

$$B_\Gamma(f,g) = \alpha^{i_1 j_1}(\partial_{j_1}\partial_{i_3}\alpha^{i_2 j_2})\alpha^{i_3 j_3}(\partial_{i_1}\partial_{i_2} f)(\partial_{j_2}\partial_{j_3} g) \tag{4.39}$$

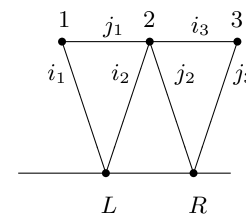

Figure 4.2: nonconstant Poisson structure

The physical interpretation of the Kontsevich construction was given by *Cattaneo* and *Felder* [50]. It is a topological quantum field theory in terms of the Poisson sigma model, which we discuss in the next chapter.

Chapter 5

Poisson sigma model

5.1 Introduction

The Poisson sigma models were defined for the first time in 1993 by *Ikeda* [42] and independly by *Schaller* and *Strobl* [43], [44]. This model is a topological field theory where the target space is a Poisson manifold and the world sheet is a 2- dimensional surface with boundary (like a disk). It provides a unification for some class of field theories like: 2-dimentional gravity theory and Yang-Mills field theory [46]. In this chapter we will survey some aspects of Poisson sigma models. After that, we will study the relation between Poisson sigma models on surfaces with boundary and deformation quantization.

Classical field theory :
Let (M, Σ, π, F) be a fiber bundle with fiber F. Σ, M and F are smooth manifolds and $\pi : M \longrightarrow \Sigma$ is a smooth function. M is called target space and Σ is called base space.
Let Σ be an affine Minkowski space time. Fields on Σ are smooth sections of $\pi : M \longrightarrow \Sigma$ (π usually called fiber bundle) denoted by ϕ. Thus we have: $\phi : \Sigma \longrightarrow M$.
Classical field theory starts with a smooth manifold Σ, which represents space time and a bundle π on Σ. The observables (Physical quantities) constructed from $\Gamma^{\infty}(\Sigma)$, ($\Gamma^{\infty}(\Sigma)$ is the set of sections on Σ) and have an algebraic form, usually formal polynomials in the fields and their derivatives [19].
An action is defined as a functional consisting of an integral over Σ as follows:

$$S : \Gamma^{\infty}(\Sigma) \rightarrow \mathbb{R} \tag{5.1}$$

$$S[\phi] = \int_{\Sigma} L[\phi]. \tag{5.2}$$

In local coordinates the action $L[\phi]$ can be expressed in terms of polynomials in the field and their derivatives. The state of the system is a field (in the language of C^*-algebra, state is a map from the algebra of observables to the complex numbers [45]), which minimizes the action $S[\phi]$. We call it classical solution of the system. The Euler-Lagrange equations arises from this minimalization.
An observable A is a function of the field $\Gamma^{\infty}(\Sigma)$, namley a polynomial in them and their deriva-

tives. Let us write $A = F_A(\phi)$. We declare that:

$$\langle A \rangle = \frac{\int [D\phi]\, e^{-\frac{i}{\hbar}S[\phi]} F_A(\phi)}{\int [D\phi]\, e^{-\frac{i}{\hbar}S[\phi]}}. \tag{5.3}$$

$[D\phi]$ is a measure on the space of the fields. This formula is called Feynman path integral [15]. In the laboratory, we can study scattering processes by use of cross sections. These cross sections can be expressed in terms of $\langle A \rangle$, see e.g. [47], [48].

Sigma model:
In this model, the field operator X is a map from the space time manifold Σ (with boundary), called *world sheet* to an auxiliary space M, usually a Riemannian manifold.(In the two dimentional case, it can be illustrated as a surface which is swept out in space time by a string). A Lagrangian is a functional of the field: $L(X, \dot{X})$.
There are also sigma models coupled to a gauge theory. In this model there is a principle G-bundle over space time, denoted by P. G is a compact Lie group which acts by isometries on M. The Lagrangian is written in term of two elementary fields:
1) A section X of the fiber bundle $M^P = P \times_G M$ with fiber M,
2) Connection A on P.
The sigma-model where M is a Poisson manifold, is called Poisson sigma-model.

Topological quantum field theory:
A topological quantum field theory is a field theory such that the partition function, depends only on the topological structure of the base manifold (worldsheet). For example, the partition function does not depend on the metric of space time. In the mathematical view point, a topological field theory can be defined as a functor from the category of topological manifolds with boundary to the category of vector spaces [49].

5.2 Lagrangian and field equations for Poisson sigma models

Let M (target space) be a smooth manifold endowed with a Poisson structute α. The field is defined as a pair (A, X) where X is a section of bundle: $X : \Sigma \to M$ and takes values in the Poisson manifold M and A is a connection on the budle, a one-form on Σ and takes values in the pull-back of the cotangent bundle $X^*(T^*M)$. The action is defined as:

$$S(X, A) = \int_\Sigma \langle A, dX \rangle + \frac{1}{2} \langle A, \alpha(X) A \rangle . \tag{5.4}$$

Let (x^μ) be local coordiante functions on Σ and (X^i) be coordinate functions on M, then we can write:

$$\begin{aligned} & v \in T_\sigma \Sigma \Rightarrow A(v) = A_{\mu\nu} dx^\mu(v) dX^i \Rightarrow A(v) = \widetilde{A}_i(v) dX^i \in T^*M \\ & \Rightarrow A : T\Sigma \to T^*M. \end{aligned} \tag{5.5}$$

So, the connection A is understood locally as a map from $T\Sigma$ to T^*M. And also:

$$X:\Sigma\to M\Rightarrow\forall\sigma\in\Sigma: X(\sigma)=(X^1(\sigma),...,X^n(\sigma))\Rightarrow X(\sigma)=X^i(\sigma)\partial_i, \tag{5.6}$$
$$dX=d(X^i\partial_i)=dX^i\partial_i. \tag{5.7}$$

Thus we can write the action more clearly in a local form. For the first term in the action we have:

$$\begin{aligned}&\langle A,dX\rangle=\left\langle A_{\mu i}dx^\mu\wedge dX^i,dX^j(\partial_j\right\rangle=A_{\mu i}dx^\mu\wedge dX^j\left\langle dX^i,\partial_j\right\rangle=\\&=A_{\mu i}dx^\mu\wedge dX^j\delta^i_j\Rightarrow\langle A,dX\rangle=A_{\mu j}dx^\mu\wedge dX^j=A_j\wedge dX^j.\\&\Rightarrow\langle A,dX\rangle=A_j\wedge dX^j.\end{aligned} \tag{5.8}$$

For second term, we can write:

$$\begin{aligned}&\langle A,\alpha(X)A\rangle=\left\langle A_{\mu l}dx^\mu\wedge dX^l,(\alpha^{ij}(X)\partial_i\wedge\partial_j)(A_{\nu k}dx^\nu\wedge dX^k)\right\rangle=\\&=A_{\mu l}A_{\nu k}\alpha^{ij}(X)dx^\mu\wedge dx^\nu(\partial_i\wedge\partial j)(dX^l\wedge dX^k)\\&\Rightarrow\langle A,\alpha(X)A\rangle=A_{\mu l}A_{\nu k}\alpha^{ij}(X)dx^\mu\wedge dx^\nu(\partial_i(dX^l)\wedge\partial_j(dX^k))=\\&=A_{\mu l}A_{\nu k}\alpha^{ij}(X)dx^\mu\wedge dx^\nu\delta^l_i\delta^k_j\\&\Rightarrow\langle A,\alpha(X)A\rangle=A_{\mu i}A_{\nu j}\alpha^{ij}(X)dx^\mu\wedge dx^\nu=\alpha^{ij}(X)A_i\wedge A_j\\&\Rightarrow\langle A,\alpha(X)A\rangle=\alpha^{ij}(X)A_i\wedge A_j.\end{aligned} \tag{5.9}$$

So we can write the action in the local form as:

$$S(X,A)=\int A_i\wedge dX^i+\frac{1}{2}\alpha^{ij}(X)A_i\wedge A_j. \tag{5.10}$$

By the Euler-Lagrange equation we can calculate the field equations as follows:

$$\partial_k(\frac{L}{\partial(\partial_k A_i)})-\frac{\partial L}{\partial A_i}=0\Rightarrow\frac{\partial L}{\partial A_i}=dX^i+\alpha^{ij}(X)A_j=0, \tag{5.11}$$
$$\partial_k(\frac{\partial L}{\partial(\partial_k X^i)})-\frac{\partial L}{\partial X^i}=0\Rightarrow\frac{\partial L}{\partial X^i}=dA_i+\frac{1}{2}\frac{\partial\alpha^{jk}(X)}{\partial X^i}A_j\wedge A_k=0. \tag{5.12}$$

The temrs $A_i\wedge dX^i$ and $\frac{1}{2}\alpha^{ij}(X)A_i\wedge A_j$ are 2-forms on Σ. It means that the action only depends on the topology of Σ.
Now, one question may be asked: Which transformations preserve the action? The action is invariant under the following local gauge transformations:

$$\delta_\epsilon X^i=\alpha^{ij}(X)\epsilon_j, \tag{5.13}$$
$$\delta_\epsilon A_i=-d\epsilon_i-\frac{\partial\alpha^{ij}}{\partial X^i}A_j\epsilon_k, \tag{5.14}$$

where ϵ_i are the infinitesmal parameters and functions on Σ. Now we show that the action is invariant under the gauge transformations:

$$\begin{aligned}&\delta_\epsilon\left(A_i\wedge dX^i\right)=\delta_\epsilon A_i\wedge dX^i+A_i\wedge d\left(\delta_\epsilon X^i\right)=\left(-d\epsilon_i-\frac{\partial\alpha^{ij}}{\partial X^i}A_j\epsilon_k\right)\wedge dX^i+\\&+A_i\wedge\left(d\alpha^{ij}(X)\epsilon_j+\alpha^{ij}(X)d\epsilon_j(X)\right)\\&\Rightarrow\delta_\epsilon\left(A_i\wedge dX^i\right)=-d\epsilon_i\wedge dX^i-\frac{\partial\alpha^{jk}}{\partial X^i}\epsilon_kA_j\wedge dX^i+A_i\wedge\frac{\partial\alpha^{ij}}{\partial X^k}dX^k\epsilon_j+A_i\wedge\alpha^{ij}(X)d\epsilon_j.\end{aligned} \tag{5.15}$$

We used: $d\alpha^{ij}(X) = \frac{\partial \alpha^{ij}}{\partial X^k} dX^k$. And for the second term in the action, we have:

$$
\begin{aligned}
&\delta_\epsilon \left(\alpha^{ij}(X) A_i \wedge A_j\right) = \delta_\epsilon \alpha^{ij}(X) A_i \wedge A_j + \alpha^{ij}(X) \delta_\epsilon A_i \wedge A_j + \alpha^{ij}(X) A_i \wedge \delta_\epsilon A_j \\
&\Rightarrow \delta_\epsilon \left(\alpha^{ij}(X) A_i \wedge A_j\right) = \frac{\partial \alpha^{ij}}{\partial X^k} \alpha^{kl}(X) \epsilon_l A_i \wedge A_j - \alpha^{ij}(X) d\epsilon_i \wedge A_j - \frac{\partial \alpha^{lk}}{\partial X^i} \alpha^{ij}(X) \epsilon_k A_l \wedge A_j - \\
&- \frac{\partial \alpha^{lk}}{\partial X^j} \alpha^{ij}(X) \epsilon_k A_i \wedge A_l - \alpha^{ij}(X) A_i \wedge d\epsilon_j. \qquad (5.16)
\end{aligned}
$$

By this calculation we can write the variation of the action as follows:

$$
\begin{aligned}
\delta_\epsilon S = &- \int d\epsilon_i \wedge dX^i + \int \left(- \frac{\partial \alpha^{jk}}{\partial X^i} \epsilon_k A_j \wedge dX^i + \frac{\partial \alpha^{ij}}{\partial X^k} \epsilon_j A_i \wedge dX^k \right) + \\
&+ \int \left(\alpha^{ij} A_i \wedge d\epsilon_j - \frac{1}{2} \alpha^{ij} A_i \wedge d\epsilon_j - \frac{1}{2} \alpha^{ij} A_i \wedge d\epsilon_j \right) + \int \frac{1}{2} \left(\frac{\partial \alpha^{ij}}{\partial X^k} \alpha^{kl} \epsilon_l A_i \wedge A_j - \right. \\
&\left. - \frac{\partial \alpha^{lk}}{\partial X^i} \alpha^{ij} \epsilon_k A_l \wedge A_j - \frac{\partial \alpha^{lk}}{\partial X^j} \alpha^{ij} \epsilon_k A_i \wedge A_l \right). \qquad (5.17)
\end{aligned}
$$

We can rewrite the term $-\int d\epsilon_i \wedge dX^i$ as follows:

$$
- \int d\epsilon_i \wedge dX^i = - \int d \wedge \left(\epsilon_i \wedge dX^i\right). \qquad (5.18)
$$

Since the integration is over a closed surface, by Stokes' theorem the first integration is equal to zero. The second and third integrations are also canceled, and thus remains just the last integration:

$$
\delta_\epsilon S = \int \frac{1}{2} \left(\frac{\partial \alpha^{ij}}{\partial X^k} \alpha^{kl} \epsilon_l A_i \wedge A_j - \frac{\partial \alpha^{lk}}{\partial X^i} \alpha^{ij} \epsilon_k A_l \wedge A_j - \frac{\partial \alpha^{lk}}{\partial X^j} \alpha^{ij} \epsilon_k A_i \wedge A_l \right). \qquad (5.19)
$$

By changing the indices and use $\alpha^{ij} = -\alpha^{ji}$, we have:

$$
\delta_\epsilon S = \int \frac{1}{2} \epsilon_k \left(\alpha^{kl} \frac{\partial \alpha^{ji}}{\partial X^l} + \alpha^{jl} \frac{\partial \alpha^{ik}}{\partial X^l} + \alpha^{il} \frac{\partial \alpha^{kj}}{\partial X^l} \right) A_i \wedge A_l. \qquad (5.20)
$$

This is just the Jacoby Identity and so is zero. Thus the variation of the action with respect to the infinitesimal transformations (5.13) and (5.14) is zero. This symmetry is an extension of more familiar gauge symmetries. [50].

5.3 Path integral approach to the Kontsevich formula

In this section, we will show that the star product between two functions $f, g \in C^\infty(M)$ is given by the semiclassical expansion of the path integral over all (X, A) of the Poisson sigma model on a disk Σ.

We claim that the formula of the star product is given by:

$$
f * g(x) = \int_{X(\infty)=x} f(X(1))\, g(X(0))\, e^{\frac{i}{\hbar} S[X,A]} dX dA. \qquad (5.21)
$$

Cattaneo and Felder suggested this formula using the insights and ideas of quantization theory in physics [50].

In this formula, the three points $\{0, 1, \infty\}$ are distinct and in cyclic order on the unit circle. Cyclic

means that if we start from 0 and move counterclockwise on the circle, we first meet 1 and then ∞. Semiclassical expansion means an expansion around the classical solution $X(u) = x$ and $A(u) = 0$ for every point belonging to the boundary: $u \in \partial\Sigma$.
Now, we should check the three axioms of star products in the definition (4.2.4):

$$\begin{aligned}
&1) f * \mathbf{1} = \mathbf{1} * f = f, \\
&2) f * (g * h) = (f * g) * h, \\
&3) C_0\left(f, g\right) = f \cdot g \\
&4) \left\{f, g\right\} = \frac{1}{i\hbar}(C_1(f,g) - C_1(g,f))
\end{aligned}$$

For the first axiom we have:

$$f * \mathbf{1}\left(x\right) = \int_{X(\infty)=x} f\left(X\left(1\right)\right) \mathbf{1}\left(X\left(0\right)\right) e^{\frac{i}{\hbar}S[X,A]} dX dA =$$
$$\int_{X(\infty)=x} f\left(X\left(1\right)\right) e^{\frac{i}{\hbar}S[X,A]} dX dA = f(x) \tag{5.22}$$

Similary, we have $\mathbf{1} * f = f$, thus the first axiom is satisfied.
The second axiom:

$$(f * g) * h\left(x\right) = \int_{X(\infty)=x} \left(f * g\right)\left(X\left(1\right)\right) h\left(X\left(0\right)\right) e^{\frac{i}{\hbar}S[X,A]} dX dA. \tag{5.23}$$

The topological nature of the model allows us to move the points on the boundary and since the algebra $C^\infty(M)$ is an associative algebra, then:

$$\begin{aligned}
(f * g) * h\left(x\right) &= \int_{X(\infty)=x} \int_{X'(\infty)=X(1)} \left\{f(X'(1)) g(X'(0))\right\} h(X(0)) e^{\frac{i}{\hbar}S[X,A]} e^{\frac{i}{\hbar}S\left[X',A'\right]} dX dA dX' dA' \\
&= \int_{X(\infty)=x} \int_{X'(\infty)=X(1)} f(X'(1)) \left\{g(X(\infty)) h(X(0))\right\} e^{\frac{i}{\hbar}S[X,A]} e^{\frac{i}{\hbar}S\left[X',A'\right]} dX dA dX' dA' \\
&= \int_{X(\infty)=x=X'(0)} f(X'(1)) (g * h)(X'(\infty)) e^{\frac{i}{\hbar}S\left[X',A'\right]} dX' dA' = f * (g * h)(x).
\end{aligned} \tag{5.24}$$

Let us turn to the third axiom. We expand the exponential in the integration:

$$\begin{aligned}
f * g\left(x\right) &= \int_{X(\infty)=x} f\left(X\left(1\right)\right) g\left(X\left(0\right)\right) e^{\frac{i}{\hbar}S[X,A]} dX dA \\
&= \int_{X(\infty)=x} f\left(X\left(1\right)\right) g\left(X\left(0\right)\right) dX dA + O\left(\hbar\right).
\end{aligned} \tag{5.25}$$

By the integration of the first term in the right hand side, we have:

$$\int_{X(\infty)=x} f\left(X\left(1\right)\right) g\left(X\left(0\right)\right) dX dA = f\left(x\right) g\left(x\right) |_{X(\infty)=x} = \left(f \cdot g\right)\left(x\right) |_{X(\infty)=x} . \tag{5.26}$$

So the zeroth term is satisfied. To prove the higher order terms, there are some quantum field theory methods [51]

Remark We assumed that Σ is a disk. It means we considerd the simplest topology with boundary. In general case, when Σ is an arbitrary topological space with boundary, it is not easy to check the associativity condition. It means, associativity depends on the toplogical structure of the word sheet. In the sympletic case the above formula essentially reduces to the original Feynman path integral formula for quantum mechanics [50] .

5.4 Conclusion

The dream of physicists is to unify all four forces in one fundamental force. This method is called unified field theory. The main problem is to unify quantum physics and Einstein's general relativity. Actually we try to reconcile them in order to treat them as a single theory. There are three famous condidates for unification:
Loop quantum gravity, string theory and noncommutative geometry, which are failure to unify quantum physics and general relativity unitl now.
As we studied in this thesis, the quantization method which arises from the noncommutativity idea of space time, implies the Kontsevich theorem. This theorem gives us a general formula of quantization of an arbitrary Poisson manifold. The physical interpretation of Kontsevich theorem appears in Poisson sigma models which are topological quantum field theories and provide a unification for some class of field theories. It seems that maybe there is a grand unified field theory behind noncommutativity properties of space time.
Let us come back to the word *unification*. This word has a deep meaning in philosophy, physics and also mathematics. I would like to start with some kind of unification in physics and mathematics and then compare them with the unified field theory which physicists try to construct.

Newtonian mechanics as a unifying theory:
From the Greek philosophy until Newtonian mechanics, there was a philosophical framework of physics, it contained two different subjects, physics of the cosmos which described the moving of planets and physics of the Earth, which described the phenomenas on earth. This framework culminated when Kepler described the cosmology with three laws and Galileo postulated mechanics and calculated projectile motion with some geometrical methods (Galileo dynamics).
"My dream is to introduce all natural phenomenas from the mechanical principles"(Isaac Newton).
Newtonian mechanics is a set of postulates such that Kepler's laws and Galileo's principles follow from it as a consequence, so Newtonian mechanics behaves as a unifying theory:
Unification of cosmology and Galileo dynamics.
The important note is, that Newton used Kepler's and Galileo's principles as a motivation and postulated classical mechanics.

Special theory of relativity as a unifying theory:
This theory was born because of some inconsistencies. One of them was in electrodynamics. The Maxwell equations are not invariant under the Galileo transformations. Special relativity actually is a theory of space and time unifying the magnetic and electric forces. These are two succesfull examples for unification of two different subjects.

Why is the unification of gravitaion and quantum physics unsuccessfull? Some mathematicians believe that the problem is behind the mathematical structure of these theories. We just mix the theories whitout attention to the contradictions which arise from their mathematical concepts. Maybe we need to have some kind of unification in the mathematics and from there, we should construct a unification in physics!

Category theory as a unifying theory in mathematics:
Category theory was developed in the second half of the 20th century. Categories were first introduced by *Saunders Mac Lane* and *Samuel Eilenberg* during 1942-1945 in algebraic topology and then developed independently as a branch of mathematics. This theory starts with the obser-

vation that all mathematical structures (algebras, topology, geometry,...) can be unified within framework. We can simplify mathematical structures by introducing a family of objects (like sets) and arrows (like morphisms between sets). A key goal is to represent any physical quantity A with an arrow between two special objects (the state-object and quantity-value object) in the appropriate a *Topos*.

Topos is a special type of category. The construction of a theory in physics is equivalent to finding a representation in a topos. For example classical physics uses the topos of sets and other theories introduce a different topos. This fact about pure mathematics motivates us to develop fundamentally new ways of constructing theories in physics [54].

If there is a unification of theories, noncommutativity of space time may help us to understand, how the final theory should look like.

Bibliography

[1] Thomas Little Heath, *A history of Greek mathematics* , Volume 1, 2, (1981).

[2] Alain Connes, *Noncommutative geometry* , Academic Press, (1994).

[3] J. Madore, *An introduction to noncommutative differential geometry and its physical applications* , London mathematical society, Lecture Note 257, second edition (1999).

[4] J. Madore, *The fuzzy sphere* , Laboratoire de Physique Theorique et Hautes Energies, Universite de Paris-Sud, Bat.

[5] H. S. Snyder, *Quantized space-time* , Phys.Rev. 71 (1947)

[6] Jose M.Bondia, Joseph C.Varilly, Hector Figueroa, *Elements of noncommutative geometry* , Birkhaeuser Advanced texts (2000).

[7] F. A. Berezin, *Quantization* , Math, USSR, Izvestija 8, 1109-1165, (1974).

[8] F. A. Berezin, *General cocept of quantization* , Commun. Math. Phys. 40 153-174 (1975).

[9] U. C. Watamura, S. Watamura *Differenrial calculus on fuzzy sphere and scalar field* arXiv: q-alg/0710034v1.

[10] Richard J. Szabo, *Quantum Field Theory on Noncommutative Spaces* , arXiv: hep-th/0109162v4.

[11] Giovanni Landi, *An Introduction to Noncommutative Spaces and their Geometry* , arxiv: hep-th/9701078v1

[12] Masoud Khalkhali, *Lectures on noncommutative geometry* , arXiv: math/0702140v2.

[13] Masoud Khalkhali, *Very basic noncommutative geometry* , arXiv: math/0408416v1.

[14] James E. Humphreys, *Introduction to Lie algebras and representation theory* , Springer (1972)

[15] Peskin, Schroeder, *Introduction to quantum field theory* , Westview, (1995).

[16] Michael Wohlgenannt, *Field theoretical Models on Non-Commutative Spaces* , Ph.D Thesis, Institute for theoretical Physics, Ludwig-Maximilians-University of Muenchen, (2003).

[17] Michael Wohlgenannt, *Introduction to Noncommutative QFT* , Lecture given at the Vienna University of Technology, Summer Term 2010.

[18] F. Bayen, M. Flato, C.Fronsdal, A.Lichnerwicz, D. Sternheimer, *Deformation theory and quantization I, II* ; Ann. Phys. 111 (1978), 61-110, 111-151.

[19] Alfonso Gracia-Saz, *On the Kontsevich formula for deformation quantization* . http://math.berkeley.edu/ alanw/242papers02/graciasaz.pdf

[20] Maxim Kontsevich, *"Operads and motives in deformation quantization"*, arXiv: math/9904055v1 [math.QA], Lett.Math.Phys. 48 (1999) 35-72.

[21] Larsen, *Functional Analysis* .

[22] John B.Conway, *A course in Functional Analysis* , Springer, (2003).

[23] Walter Rudin, *Functional Analysis* , Mc Grawhill, (1973).

[24] Schaefer, Helmut H, *Topological vectore spaces* , Springer, (1999).

[25] F. A. Berezin, *General concepts of quantization* , Comm. Math. Phys. 40 (1975), 153-174.

[26] Hungerford, *Algebra* , Springer, (1982)

[27] Rotman, *Advance algebra* , Springer, (20023)

[28] Alexander A.Voronov, *Deformation theory of associative algebras* , Lecture 16, seminar course, November 16, 19 and 21, 2001.

[29] Ana Cannas da Silva, Alan Weinstein, *Geometric models for noncommutative algebras* , Lecture not, University of Berkeley; (1998).

[30] Joseph Krasilshchik, Barbara Prinari, *Lectures on linear differential operators over commutative algebras*, The 1st Diffiety school, Forino, Italy-July, 1998.

[31] Eiichi Abe, *Hopf algebras* , Cambridge university press (1977).

[32] Mosse E. Sweedler, *Hopf algebras* , Cornell university (1969).

[33] Allen Hatcher, *Algebraic topology* , Cambridge university press (2010).

[34] DeWilde, M.,Lecomte, P.B.A, *Existence of star product and of formal Deformations of the Poisson Lie algebra of Arbitrary Sympletic Manifolds* , Lett.Math. Phys. 7 (1983), 487-496.

[35] Kontsevich. M, *Deformation quantization of Poisson Manifolds, I* ; Lett.Math.Phys. 66 (2003) 157-216, arXiv:q-alg/9709040v1, .

[36] Pierre Deligne, Pavel Etingof, Daniel S. Freed, Lisa C. Jeffrey, David Kazhdan, john W. Morgan, David R. Morrison, Edward Witten, *Quantum fields and strings:A course for mathematicians I, II* , American Mathematical Society (1999).

[37] Xavier Bakaert, *Universal enveloping algebras and some applications in Physics* , Lecture given at the first Modave Summer School in Mathematical Physics (Belgium, June 2005).

[38] Arturo Echeverria-Enriquez, Miguel C. Munoz-Lecanda, Narciso Roman-Roy, Carles Victoria-Monge, *Mathematical foundations of geometric quantization* , arXiv:math-ph/9904008v1, Extracta Math. 13 (1998) 135-238.

[39] Stefan Waldmann, *Deformation quantization: Observable algebras, states and representation theory* , Lectures not for the summer school in Kopaonik, (2002).

[40] Stefan Waldmann, *Poisson-geometrie und Deformations quantisierung* , Springer (2007).

[41] Schaller. P, Strobl. T, *Poisson structure induced(topological) field theories* , Modern Phys. Lett.A 9 (1994).

[42] Ikeda, N, *Two dimentional gravity and nonlinear gauge theory* , Ann. Phya. 235, (1994) 435-464.

[43] Schaller. P, Thomas Strobl, *Poisson sigma models:A generalization of 2d Gravity-Yang-Mills systems* , Dubna 1994, Finite dimensional integrable systems* 181-190 TUW-94-21, PITHA-94-49, C94-07-18.7, arXiv: hep-th/9411163v1.

[44] Schaller. p, Thomas Strobl, *A brief introduction to Poisson sigma model* , Schladming 1995, Low dimensional models in statistical physics and quantum field theory* 321-333 TUW-95-15, PITHA-95-15, C95-03-04 , C95-03-04, arXiv: hep-th/9507020v1.

[45] Shoichiro Sakai, *C^*-algebras and W^*-algebras* , Springer (1971).

[46] Do Duc Hanh, *Some aspects of Piosson sigma models* , Department of Mathematics, University of Berkeley, (2005).

[47] Walter Greiner, Joachim Reinhardt, *Field quantization* , Springer (1992).

[48] Walter Greiner, Joachim Reinhardt, *Quantum electrodynamics* , Springer (1992).

[49] Joachim Koch, *Frobenius algebras and 2D topological quantum field theory* , Cambridge university press, (2004).

[50] Cattaneo, A., Felder, G, *A Path Integral approach to the Kontsevich quantization formula* , Math.Phys. 212 (2000) 591-611, arXiv: math/9902090v3.

[51] A. S. Cattaneo, G. Felder, *Poisson sigma models and deformation quantization*, Mod.Phys.Lett. A16 (2001) 179-190, arXiv: hep-th/0102208v1.

[52] Ivan Calbo Rubio, *Poisson sigma models on surfaces with boundary:Classical and quantum aspects* , arXiv: aunt-ph/0708.2861v1.

[53] Mac Lane, Saunders, *Categories theory for working mathematicians* , Graduate text in mathematics, Springer (1998).

[54] C.J. Isham; A.Doering, *A topos foundation of physics: Formal lanquages for physics* , J.Math.Phys. 49 (2008) 053515, arXiv: quant-ph/0703060v1.

[55] C.J. Isham; A. Doering, *A topos foundation of physics: Daseinisation and the liberation of quantum theory* , J.Math.Phys. 49 (2008) 053516, arXiv: quant-ph/0703062v1.

[56] C.J. Isham, A. Doering, *A topos foundation of physics: The representation of physical quantities with arrows* , J.Math.Phys. 49 (2008) 053517, arXiv: quant-ph/0703064v1.

[57] C.J. Isham; A. Doering, *A topos foundation of physics: Categories of systems* , J.Math.Phys. 49 (2008) 053518, arXiv: quant-ph/0703066v1. .

[58] C.J. Isham, A. Doering, *What is a thing?: Topos theory in the foundations of physics* , arXiv: 0803.0417v1 [quant-ph].

[59] C.J. Isham, *Topos method in the foundations of physics* , arXiv: 1004.3564v1 [quant-ph].

[60] M. Barr; C. Wells, *Toposes, Triples and Theories* , Springer (1985).

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