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„Deformed $\mathbb{R}^3$ as a physical framework for quantum mechanical problems"

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Introduction

After E. Schrödinger and W. Heisenberg laid the foundations of non-relativistic quantum mechanics around 1925, P.A.M. Dirac [6] created the subject of quantum field theory (QFT) in 1927 by first describing the electromagnetic field and charged matter in the framework of quantum mechanics. A year later he found an equation describing relativistic spin-1/2 particles, which was named Dirac equation in his honor. It was apparent that this equation as well as QFT incorporated special relativity, thus allowing for a relativistic treatment of many-particle systems. However, at small distances around the Planck length $l_P = \sqrt{\frac{\hbar G}{c^3}}$, QFT produced divergent quantities, so-called ultraviolet divergences, rendering the results of the theory in this regime useless. The appearance of these divergences in the calculations indicated that at small distances space-time might not exhibit classical behavior; for example, Heisenberg [19] reasoned in 1938 that there exists a fundamental length scale beyond which quantum mechanics in its then common interpretation is not applicable. Schrödinger [28] made more general remarks about the measure process and geometric notions at small scales in 1934.

There is a simple, heuristic argument for a possible 'quantum' nature of space-time at the Planck scale: Suppose we have an object of length-dimension Δx. According to Heisenberg's uncertainty principle $\Delta x \Delta p \geq \frac{\hbar}{2}$ there is an associated momentum uncertainty of $\Delta p \geq \frac{\hbar}{2\Delta x}$. Since the relativistic energy is given by $E = \sqrt{p^2c^2 + m^2c^4}$, we certainly have a lower bound for the energy (resp. the rest mass) given by $E = mc^2 \geq \Delta pc$. General relativity allows us to associate a Schwarzschild radius of $R_{SS} \simeq \frac{2GE}{c^4}$ to the energy of this object. A reasonable description of the object beyond the Schwarzschild radius is not possible; hence, we infer by the above reasoning that

$$\Delta x \geq R_{SS} \simeq \frac{2GE}{c^4} \geq \frac{2G\Delta p}{c^3} \geq \frac{G\hbar}{\Delta x c^3}$$

and hence

$$(\Delta x)^2 \geq l_p^2. \tag{0.1}$$

The above argument is obviously rather hand-waving; however, in 1995 S. Doplicher,

K. Fredenhagen and J.E. Roberts [7] provided more detailed arguments in favor of the idea of the quantum nature of space-time. Note that during the reasoning leading to (0.1) we used arguments from both quantum mechanics and general relativity; we will return to this thought later on.

Recall that Heisenberg's uncertainty principle $\Delta x \Delta p \geq \frac{\hbar}{2}$ stems from the commutation relation $[\hat{X}^i, \hat{P}^j] = i\hbar\delta^{ij}$; hence, we can interpret (0.1) as the result of prescribing a similar commutation relation for the coordinates (or better: coordinate operators) $\hat{X}^i$. This was first formalized by H.S. Snyder [30], who used a commutation relation in the form of

$$[\hat{X}^i, \hat{X}^j] = \theta^{ij}\,\mathrm{id}$$

where θ^{ij} is an anti-symmetric tensor. Although Snyder's approach to quantizing space-time seemed promising, it failed to receive proper resonance in the scientific community after R. Feynman, J. Schwinger, S. Tomonaga and F. Dyson developed renormalization and successfully applied it to electromagnetic QFT, thus founding quantum electrodynamics (QED). Due to the overwhelming accuracy of QED in the prediction of electromagnetic quantities such as the anomalous magnetic moment of the electron, Snyder's idea of quantizing space-time was discarded in favor of the newly developed renormalization techniques. However, theoretical physicists were very well aware of the shortcomings of renormalization; Dirac for example was never really content with its lack of mathematical rigor.

The fact that the Physics community turned away from quantizing space-time through non-commuting position operators did not prevent mathematicians from picking up the idea and formalizing the concepts. The de-facto standard and highlight in this undertaking is undoubtedly A. Connes' seminal publication [5] in which he developed the theory of non-commutative geometry by using methods from K-theory and cyclic cohomology. Connes' work drew the attention of many fellow mathematicians such as M. Rieffel and J. Lott, as well as revitalizing the interest of physicists in non-commutative geometry. In 1992 J. Madore [20] gave a description of the non-commutative fuzzy sphere S_F^2 in terms of an algebra of matrices; together with H. Grosse [15] he applied this as a regularization to the Schwinger model. A year later Grosse and P. Prešnajder [16] described a method of constructing non-commutative manifolds using coherent states, which they applied to Madore's fuzzy sphere. A non-commutative differential calculus based on the Frolicher-Nijenhuis bracket and derivations was developed by M. Dubois-Violette [9] in 1988, also published co-authored with P. Michor [10] in 1994. Furthermore, Grosse, C. Klimčik and Presnajder discussed field theories on non-commutative manifolds in a series of papers, including [13] and [14]. In 2004 Grosse and R. Wulkenhaar [17] showed that the four-dimensional non-commutative ϕ^4-model is renormalizable to all orders by reformulating it as a dynamical matrix model.

The year 1999 saw important developments in non-commutative geometry. S. Minwalla, M. van Raamsdonk and N. Seiberg investigated the perturbative dynamics of non-commutative field theories on $\mathbb{R}^d$, discovering an IR/UV mixing in the theory. Furthermore, a connection between non-commutative geometry and string theory was established when A. Alekseev, A. Recknagel, and V. Schomerus [2] on the one hand and Seiberg and E. Witten [29] on the other hand discovered the appearance of non-commutative geometries in string theory in presence of a non-vanishing magnetic field. These developments not only spurred the interest of string theorists in non-commutative geometry, but also substantiated the stand-alone role of the field. This is based on the fact that non-commutative geometry adds to the existing candidates for a theory of quantum gravity, the most famous contestants being string theory and loop quantum gravity. In certain matrix models in non-commutative geometry, gravity enters the stage through an effect called 'emergent gravity'.

The thesis at hand provides an introduction to non-commutative geometry through the investigation of deformed $\mathbb{R}^3$, denoted by $\mathbb{R}^3_\lambda$. The parameter λ governs the non-commutativity of the space in a way which will become clear in the course of the discussion. This thesis is divided into three chapters.

In the first chapter we acquaint the reader with the field of non-commutative geometry by first introducing the Moyal plane, in a sense the simplest and most comprehensible example of a non-commutative space. This is achieved by defining the canonical non-commutative Groenewold-Moyal star product on the algebra of functions on $\mathbb{R}^4$ through the use of Weyl operators. We also give a definition of a different star product using the coherent state method. Not only does this alternative approach demonstrate that there are different star products and hence different non-commutative geometries on a given base space; this method is also convenient to define a star product on $\mathbb{R}^3_\lambda$, the central object of this thesis. We also point out the relationship between $\mathbb{R}^3_\lambda$ and the fuzzy sphere $S^2_{\lambda,N/2}$ by recognizing $\mathbb{R}^3_\lambda$ as a direct sum of fuzzy spheres of increasing radii. It is further demonstrated that the star product on $\mathbb{R}^3_\lambda$ defined by coherent states can be easily reduced to a fuzzy sphere of given radius.

The second chapter deals with differential calculi on non-commutative spaces. After a short recapitulation of the de Rham-calculus on commutative manifolds, we first discuss the universal calculus on arbitrary algebras. An important theorem states that every given differential calculus on an algebra can be obtained as a quotient object of the universal calculus. We then take a short detour and introduce quantum groups (or Hopf algebras), a special class of bialgebras whose additional structure permits an explicit construction of a differential calculus. The identification of $\mathbb{R}^3_\lambda$ with the universal enveloping algebra $\mathcal{U}(\mathfrak{su}(2))$ reveals its quantum group structure and allows us to present a concrete example of a four-dimensional calculus on $\mathbb{R}^3_\lambda$. In the commutative limit this calculus reduces to the ordinary three-dimensional calculus on $\mathbb{R}^3$, a fact which we support by evaluating the exterior derivative acting on plane waves and showing that we can reproduce the known result of the commutative calculus. Since the subject of quantum groups would deserve a thesis of its own, the treatment is kept rather short and only intends to provide the most basic notions of quantum groups. However, references to the literature are included for the interested reader.

After the general considerations in the first two chapters, the last chapter focuses on a concrete quantum mechanical problem: the Coulomb problem formulated on $\mathbb{R}^3_\lambda$. We repeat the realization of $\mathbb{R}^3_\lambda$ via the Hopf fibration and bosonic creation and annihilation operators from Chapter 1 and discuss the most important properties of the coordinate operators $\hat{x}^i, i = 1, 2, 3$. After defining the angular momentum operators $\hat{L}^i, i = 1, 2, 3$ and identifying the eigenstates of $\mathcal{L}^2 := \sum_i (\hat{L}^i)^2$ and $\hat{L}^3$, we investigate the Hilbert space generated by these eigenstates. In order to formulate the Coulomb problem in this space, we define a Laplace operator and analyze its action on the eigenstates of the angular momentum operators. Together with a potential term the Laplace operator forms the Hamiltonian of the Coulomb problem. We compute the spectrum of the Hamiltonian and learn that the energy values are essentially the solutions from the commutative problem modified by certain correction terms due to the non-commutativity of $\mathbb{R}^3_\lambda$.

Since this thesis is intended as an introduction to non-commutative geometry on a graduate level, the reader is not assumed to be familiar with the subject. However, a sound knowledge of algebra and differential geometry surely aids in following the main arguments. Detailed computations of the results are given wherever possible. Proofs of the mathematical theorems (especially in the sections about the universal calculus and quantum groups) are mainly omitted in order to focus on the explicit aspects of the non-commutative geometry of $\mathbb{R}^3_\lambda$. The set of natural numbers, integers, real numbers and complex numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ respectively. By convention, $\mathbb{N}$ does not contain 0.

1 Examples of non-commutative spaces

The most general approach to defining the concept of non-commutative geometry, as introduced by Connes in [5], is the following: Consider a general manifold M and the $\mathbb{R}$-algebra $C^\infty(M)$ of smooth, real-valued functions on M. Addition and multiplication of functions in $C^\infty(M)$ are defined pointwise, turning it into a commutative algebra. It is well-known that this algebra encodes much information about the topology and differential structure of M in an algebraic sense. Hence, the algebra of functions provides a good starting point for generalizing the concept of manifolds to non-commutative spaces. The crucial idea is to replace $C^\infty(M)$ by a non-commutative $\mathbb{R}$-algebra $\hat{\mathcal{A}}$ and construct a differential calculus in resemblance to the concepts in commutative differential geometry (cf. Section 2). Furthermore, one often chooses an isomorphism between $\hat{\mathcal{A}}$ and $(C^\infty(M), +, \star)$ where $\star$ is an associative, non-commutative product for ordinary functions in $C^\infty(M)$. We will discuss two possible approaches to defining this star product in Sections 1.1.2 and 1.1.3 respectively.

However, there is a more straight-forward way of defining non-commutative spaces. We simply prescribe commutation relations for the coordinates x^i, $i = 1, \dots, D$ of a manifold M with $D = \dim(M)$, hence turning them into non-commutative operators. A reasonable ansatz is a relation of the form

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij} \tag{1.1}$$

where θ^{ij} is an anti-symmetric tensor of type $(2, 0)$ and rank 4 (thus invertible), the entries having dimension $(\text{length})^2$. The hats over the coordinates $\hat{x}^i$ indicate their operator nature.

In the following sections we are going to discuss two special cases of (1.1). The treatment at hand is based on [31, Ch. 2] and [18, Ch. 2 and 3], but more detailed and including proofs and explicit calculations whenever it does not impair readability.

1.1 Moyal plane

1.1.1 Definition of the Moyal plane

The simplest case of (1.1) is to consider a constant matrix θ^{ij}. It follows from the theory of anti-symmetric matrices that after a suitable basis transformation one can achieve the following block form of the matrix θ^{ij}:

$$(\theta^{ij}) = \begin{pmatrix} \begin{matrix} 0 & \theta_1 \\ -\theta_1 & 0 \end{matrix} & 0 & \cdots & & & \\ 0 & \begin{matrix} 0 & \theta_2 \\ -\theta_2 & 0 \end{matrix} & & & & \\ \vdots & & \ddots & & & \\ & & & \begin{matrix} 0 & \theta_n \\ -\theta_n & 0 \end{matrix} & & \\ & & & & 0 & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix} \tag{1.2}$$

where $\theta_1, \dots, \theta_n \in \mathbb{R}$ and $\pm i\theta_1, \dots, \pm i\theta_n, 0$ are the eigenvalues of (θ^{ij}). Setting $D = 4$ and assuming non-degeneracy of θ^{ij} we obtain the Moyal plane $\mathbb{R}^4_\theta$:

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij} \quad \text{with} \quad (\theta^{ij}) = \begin{pmatrix} \begin{matrix} 0 & \theta \\ -\theta & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & \theta \\ -\theta & 0 \end{matrix} \end{pmatrix}. \tag{1.3}$$

Note that we introduced another basis transformation (which is effectively a rescaling of the coordinates $\hat{x}^i$) to achieve $\theta_1 = \theta_2 = \theta$ in (1.2). The real parameter θ governs the non-commutativity of $\mathbb{R}^4_\theta$; in the commutative limes $\theta \to 0$ we obtain $\mathbb{R}^4_\theta \to \mathbb{R}^4$ according to (1.1). We denote by $\hat{\mathcal{A}}$ the non-commutative algebra generated by the coordinate operators $\hat{x}^i$.

1.1.2 Weyl operators and Moyal star product

The prescription of commutation relations for the coordinates $\hat{x}^i$ of the manifold M also turns fields (which are essentially functions of the coordinates) into operators. Therefore, studying field theory requires special care when dealing with these operator due to the need to specify an ordering prescription. One way of facilitating this task is to relate the algebra $\hat{\mathcal{A}}$ generated by the operators $\hat{x}^i$ to ordinary functions in $C^\infty(M)$. The non-commutativity of $\hat{\mathcal{A}}$ is incorporated by defining a non-commutative, associative product on $C^\infty(M)$, the so-called star product. In this section we discuss the Moyal star product on the Moyal plane $\mathbb{R}^4_\theta$.

In order to ensure that Fourier transformation is well defined, we first restrict ourselves to the Schwartz space

$$\mathcal{S}(\mathbb{R}^4) = \left\{ f \in C^\infty(\mathbb{R}^4) \mid \sup_{x\in\mathbb{R}^4} |x^\alpha D^\beta f(x)| < \infty \; \forall \alpha, \beta \right\}$$

where α and β are multi-indices. More precisely, the Fourier transform $\mathcal{F}$ is a linear automorphism of $\mathcal{S}(\mathbb{R}^4)$. Furthermore, for vectors $v, w \in \mathbb{R}^4$ we denote by vw the standard Euclidean scalar product $\delta_{ij} v^i w^j$. Similarly, $k\hat{x} \equiv k_i \hat{x}^i$ where $k \in \mathbb{R}^4$ and $\hat{x}^i$, $i = 1, \ldots, 4$ are the coordinate operators.

In order to define the star product, we need to introduce the notion of Weyl operators. Given a function $f(x) \in \mathcal{S}(\mathbb{R}^4)$ and the corresponding Fourier transform

$$\tilde{f}(k) = \frac{1}{(2\pi)^2} \int d^4x \, e^{-ikx} f(x),$$

the Weyl operator $\hat{\mathcal{W}}[f]$ is defined by

$$\hat{\mathcal{W}}[f] = \frac{1}{(2\pi)^2} \int d^4k \, \tilde{f}(k) e^{ik\hat{x}} \tag{1.4}$$

where $\hat{x}^i$ are the coordinate operators satisfying (1.3). Since $e^{ik\hat{x}}$ is defined by the Taylor expansion of the exponential function, we have to fix an ordering prescription in (1.4). A natural choice is the symmetric Weyl ordering, defined by requiring $\hat{\mathcal{W}}[e^{ikx}] = e^{ik\hat{x}}$. Denoting the mapping of a function $f(x)$ to its Weyl operator $\hat{\mathcal{W}}[f]$ by

$$\begin{aligned} S : \mathcal{S}(\mathbb{R}^4) &\longrightarrow \hat{\mathcal{A}} \\ f(x) &\longmapsto \hat{\mathcal{W}}[f], \end{aligned}$$

we observe that we may define an inverse mapping S^{-1} by

$$S^{-1}(\hat{\mathcal{W}}[\hat{f}])(k) = \mathrm{tr}(e^{-ik\hat{x}} \hat{f}). \tag{1.5}$$

Note however that S^{-1} gives a function in momentum space, i.e., we have to perform a Fourier transform of $S^{-1}(\hat{\mathcal{W}}[\hat{f}])(k)$ in order to obtain $f(x)$ in position space.

Formally, the star product $(f \star g)(x)$ of two functions $f, g \in \mathcal{S}(\mathbb{R}^4)$ is defined as

$$(f \star g)(x) := \frac{1}{(2\pi)^2} \int d^4k \, e^{ikx} S^{-1}(\hat{\mathcal{W}}[f]\hat{\mathcal{W}}[g]). \tag{1.6}$$

We would like to obtain an explicit formula for the star product in (1.6). To this end, we observe using (1.4) that

$$\hat{\mathcal{W}}[f]\hat{\mathcal{W}}[g] = \frac{1}{(2\pi)^4} \int d^4k d^4l \, e^{ik\hat{x}} e^{il\hat{x}} \tilde{f}(k)\tilde{g}(l). \tag{1.7}$$

Employing the Baker-Campbell-Hausdorff formula

$$\begin{aligned} \log(e^A e^B) = A + B + \frac{1}{2}[A, B] + \frac{1}{12}[A, [A, B]] \\ - \frac{1}{12}[B, [A, B]] - \frac{1}{24}[B, [A, [A, B]]] \pm \ldots, \end{aligned} \tag{1.8}$$

which continues with terms involving an increasing number of commutators, the product $e^{ik\hat{x}} e^{il\hat{x}}$ is evaluated as

$$\begin{aligned} e^{ik\hat{x}} e^{il\hat{x}} &= \exp(i(k + l)\hat{x} - \frac{1}{2} k_i l_j \underbrace{[\hat{x}^i, \hat{x}^j]}_{=i\theta^{ij}}) \\ &= e^{i(k+l)\hat{x}} e^{-\frac{1}{2} k_i l_j \theta^{ij}}. \end{aligned} \tag{1.9}$$

The key observation here is that the commutation relation (1.1) is constant; therefore, terms involving two or more commutators drop out. By inserting (1.9) into (1.7) and comparing it with (1.4), we infer the explicit form of the Moyal star product in position space as

$$(f \star g)(x) := f(x) \exp(\overleftarrow{\partial_i} \frac{i}{2} \theta^{ij} \overrightarrow{\partial_j}) g(y) \Big|_{x=y}. \tag{1.10}$$

It is important to note that different ordering prescriptions in (1.4) lead to different star products. The form given in (1.10) corresponds to the symmetric Weyl ordering $\hat{\mathcal{W}}[e^{ikx}] = e^{ik\hat{x}}$. We also observe that in the commutative limit $\theta \to 0$ the star product (1.10) reduces to the usual commutative product of functions in analogy to $\mathbb{R}^4_\theta \to \mathbb{R}^4$.

Let us check the consistency of (1.10) with the commutation relation $[\hat{x}^i, \hat{x}^j] = i\theta^{ij}$:

$$x^i \star x^j = x^i \exp(\overleftarrow{\partial_i} \frac{i}{2} \theta^{ij} \overrightarrow{\partial_j}) y^j \Big|_{x=y}$$

$$= x^i(1 + \overleftarrow{\partial}_i \frac{i}{2}\theta^{ij}\overrightarrow{\partial}_j + \frac{1}{2}\overleftarrow{\partial}_i \frac{i}{2}\theta^{ij}\overrightarrow{\partial}_j \overleftarrow{\partial}_m \frac{i}{2}\theta^{mn}\overrightarrow{\partial}_n + \dots)y^j\Big|_{x=y}$$
$$= x^i x^j + \frac{i}{2}\theta^{ij}$$

Hence, $\{x^i, x^j\}_\star := x^i \star x^j - x^j \star x^i = i\theta^{ij}$, implying that $(\mathcal{S}(\mathbb{R}^4), \star) \cong \hat{\mathcal{A}}$, that is, they are isomorphic as non-commutative rings. Note that the above reasoning readily generalizes to higher (even) dimensions D. The star product has the following important property:

Lemma 1. $\int d^4x\, (f \star g)(x) = \int d^4x\, f(x)g(x)$

Proof. We first compute the Fourier transform of the star product (1.10):

$$(f \star g)(x) = f(x)\exp(\overleftarrow{\partial}_i \frac{i}{2}\theta^{ij}\overrightarrow{\partial}_j)g(y)\Big|_{x=y}$$
$$= \int d^4p\, d^4q\, e^{ipx}\tilde{f}(p)e^{iqx}\tilde{g}(q)\exp\left(\frac{i}{2}\theta^{ij}p_i q_j\right)$$

Hence,

$$\int d^4x\, (f \star g)(x) = \int d^4p\, d^4q \underbrace{\int d^4x\, e^{i(p+q)x}}_{=\delta(p+q)} \tilde{f}(p)\tilde{g}(q)\exp\left(\frac{i}{2}\theta^{ij}p_i q_j\right)$$
$$= \int d^4p\, \tilde{f}(p)\tilde{g}(-p)\exp\Big(-\frac{i}{2}\underbrace{\theta^{ij}p_i p_j}_{=0}\Big)$$
$$= \int d^4p\, \tilde{f}(p)\tilde{g}^*(p)$$
$$= \int d^4x\, f(x)g(x);$$

the last equality is Parseval's theorem. Note that $\theta^{ij}p_i p_j = 0$, since it is the contraction of an anti-symmetric and a symmetric quantity. □

Corollary 2. $\int d^4x\, (f_1 \star f_2 \star \dots \star f_n)(x) = \int d^4x\, (f_{\sigma(1)} \star f_{\sigma(2)} \star \dots \star f_{\sigma(n)})$ *where* σ *is any power of the cyclic permutation* $(1\,2\,\dots\,n)$.

Lemma 1 has important applications in field theory. For example, a quadratic term $\int d^4x\, \phi \star \phi$ in the action can be replaced by the integral over the ordinary product $\int d^4x\, \phi^2$.

1.1.3 Star product using coherent states

After having defined the star product on $\mathbb{R}^4_\theta$ via Weyl operators of functions, we want to investigate a different approach using coherent states on a suitable Fock space.

Let us first identify $\mathbb{R}^4$ with $\mathbb{C}^2$ via

$$z_1 = x_1 + ix_2 \qquad\qquad z_2 = x_3 + ix_4. \tag{1.11}$$

If we deform $\mathbb{R}^4$ to the Moyal plane by imposing the commutation relation (1.3), the complex coordinates z_α also become operators, which we denote by $\hat{a}_\alpha$. Upon setting[1] $\theta \equiv \frac{1}{2}$, their commutation relation is immediate from (1.11):

$$[\hat{a}_\alpha, \hat{a}^\dagger_\beta] = \delta_{\alpha\beta} \tag{1.12a}$$

$$[\hat{a}_\alpha, \hat{a}_\beta] = 0 = [\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\beta] \quad \text{for} \quad \alpha, \beta = 1, 2 \tag{1.12b}$$

Hence, $\hat{a}_\alpha$ and $\hat{a}^\dagger_\alpha$ are the creation and annihilation operators of a two-dimensional harmonic oscillator acting on the Fock space $\mathcal{F}$ defined by

$$\mathcal{F} := \text{span}(|n_1, n_2\rangle \;|\; n_1, n_2 \in \mathbb{N}) \tag{1.13a}$$

$$|n_1, n_2\rangle := \frac{(\hat{a}^\dagger_1)^{n_1}(\hat{a}^\dagger_2)^{n_2}}{\sqrt{n_1! n_2!}}|0\rangle \tag{1.13b}$$

where $|0\rangle := |0, 0\rangle$ is the normalized vacuum state with $\hat{a}_1|0\rangle = \hat{a}_2|0\rangle = 0$. Given a vector $z \in \mathbb{C}^2$ we can now define the coherent state

$$|z\rangle := e^{z^* z/2} e^{z_\alpha \hat{a}^\dagger_\alpha}|0\rangle \tag{1.14}$$

where $z^* z$ is understood as $z^*_\alpha z_\alpha$. We can bring (1.14) into a more practical form using (1.13b):

$$\begin{aligned} |z\rangle &= e^{-z^* z/2} e^{z_1 \hat{a}^\dagger_1 + z_2 \hat{a}^\dagger_2}|0\rangle \\ &= e^{-z^* z/2} \sum_{n_1, n_2} \frac{z_1^{n_1} z_2^{n_2}}{n_1! n_2!} (\hat{a}^\dagger_1)^{n_1} (\hat{a}^\dagger_2)^{n_2}|0\rangle \\ &= e^{-z^* z/2} \sum_{n_1, n_2} \frac{z_1^{n_1} z_2^{n_2}}{\sqrt{n_1! n_2!}}|n_1, n_2\rangle \end{aligned} \tag{1.15}$$

[1] The reason for this is to retain the standard form of (1.12). We will reinsert the non-commutativity parameter θ when discussing the commutative limit $\theta \to 0$.

Let us record a few properties of these coherent states in the following

Lemma 3. *For $w, z \in \mathbb{C}^2$ coherent states*

(i) *are normalized:* $\langle z|z\rangle = 1$

(ii) *are eigenstates of the annihilation operators* $\hat{a}_\alpha$*:* $\hat{a}_\alpha|z\rangle = z_\alpha|z\rangle$

(iii) *are not orthogonal:* $\langle w|z\rangle = e^{(-w^*w-z^*z)/2+w^*z}$

(iv) *satisfy the completeness relation*

$$\int d\mu(z^*, z)\,|z\rangle\langle z| = 1$$

where $d\mu(z^*, z) = \frac{1}{\pi^2}dz_1^*dz_1dz_2^*dz_2$ *is the canonical measure on the complex plane* $\mathbb{C}^2$*.*

Proof. Setting $w = z$ in (iii) yields (i). To show (iii), we use (1.15) to observe:

$$\begin{aligned}\langle w|z\rangle &= e^{(-w^*w-z^*z)/2}\sum_{\substack{n_1,n_2\\ m_1,m_2}}\frac{(w_1^*)^{m_1}(w_2^*)^{m_2}z_1^{n_1}z_2^{n_2}}{\sqrt{n_1!n_2!m_1!m_2}}\underbrace{\langle m_1, m_2|n_1, n_2\rangle}_{=\delta_{m_1n_1}\delta_{m_2n_2}}\\ &= e^{(-w^*w-z^*z)/2}\underbrace{\sum_{n_1}\frac{(w_1^*)^{n_1}z_1^{n_1}}{n_1!}\sum_{n_2}\frac{(w_2^*)^{n_2}z_2^{n_2}}{n_2!}}_{=\exp(w_1^*z_1+w_2^*z_2)}\\ &= e^{(-w^*w-z^*z)/2+w^*z}\end{aligned}$$

(ii) First, consider $\hat{a}_1$: We set

$$f_{z_1} = e^{-z_\alpha\hat{a}_\alpha^\dagger}\hat{a}_1e^{z_\alpha\hat{a}_\alpha^\dagger}$$

and differentiate f_{z_1} with respect to z_1:

$$\begin{aligned}\frac{\partial f_{z_1}}{\partial z_1} &= e^{-z_\alpha\hat{a}_\alpha^\dagger}(-\hat{a}_1^\dagger\hat{a}_1)e^{z_\alpha\hat{a}_\alpha^\dagger} + e^{-z_\alpha\hat{a}_\alpha^\dagger}\hat{a}_1\hat{a}_1^\dagger e^{z_\alpha\hat{a}_\alpha^\dagger}\\ &= e^{-z_\alpha\hat{a}_\alpha^\dagger}[\hat{a}_1, \hat{a}_1^\dagger]e^{z_\alpha\hat{a}_\alpha^\dagger}\\ &= 1,\end{aligned}$$

where the 1 is understood as the identity operator. Thus, we have the ordinary differential equation

$$\frac{\partial f_{z_1}}{\partial z_1} = 1$$

$$f_0 = \hat{a}_1,$$

which has the solution $f_{z_1} = z_1 + \hat{a}_1$. Similarly, $f_{z_2} = z_2 + \hat{a}_2$, giving

$$\hat{a}_1 e^{z_\alpha \hat{a}_\alpha^\dagger} = z_1 e^{z_\alpha \hat{a}_\alpha^\dagger} + e^{z_\alpha \hat{a}_\alpha^\dagger} \hat{a}_1$$
$$\hat{a}_2 e^{z_\alpha \hat{a}_\alpha^\dagger} = z_2 e^{z_\alpha \hat{a}_\alpha^\dagger} + e^{z_\alpha \hat{a}_\alpha^\dagger} \hat{a}_2.$$

Applying these operators to the vacuum state $|0\rangle$ and recalling that $|0\rangle$ is annihilated by the term $e^{z_\alpha \hat{a}_\alpha^\dagger} \hat{a}_\beta$, $\beta = 1, 2$, yields the desired identity $\hat{a}_\alpha |z\rangle = z_a |z\rangle$.

(iv) In the following, $d\mu \equiv d\mu(z^*, z)$:

$$\int d\mu \, |z\rangle\langle z| = \sum_{\substack{n_1, n_2 \\ m_1, m_2}} \frac{1}{\sqrt{n_1! n_2! m_1! m_2!}} |n_1, n_2\rangle\langle m_1, m_2| \times \int d\mu \, e^{-z_1^* z_1 - z_2^* z_2} z_1^{n_1} (z_1^*)^{m_1} z_2^{n_2} (z_2^*)^{m_2}$$

We take a closer look at the remaining integral:

$$\int d\mu \, e^{-z_1^* z_1 - z_2^* z_2} z_1^{n_1} (z_1^*)^{m_1} z_2^{n_2} (z_2^*)^{m_2} = \left(\frac{1}{\pi} \int dz_1^* dz_1 \, e^{-z_1^* z_1} z_1^{n_1} (z_1^*)^{m_1} \right) \left(\frac{1}{\pi} \int dz_2^* dz_2 \, e^{-z_2^* z_2} z_2^{n_2} (z_2^*)^{m_2} \right)$$

These integrals are best solved using polar coordinates, setting $z_1 = re^{i\varphi}$:

$$\frac{1}{\pi} \int dz_1^* dz_1 \, e^{-z_1^* z_1} z_1^{n_1} (z_1^*)^{m_1} = \frac{1}{\pi} \int_0^\infty dr \, r e^{-r^2} r^{n_1 + m_1} \underbrace{\int_0^{2\pi} d\varphi \, e^{i(n_1 - m_1)\varphi}}_{= 2\pi \delta_{n_1 m_1}}$$
$$= 2 \int_0^\infty dr \, r e^{-r^2} r^{2n_1}$$

and changing variables to $t = r^2, dt = 2rdr$ gives

$$= \int_0^\infty e^{-t} t^{n_1} = \Gamma(n_1 + 1) = n_1!.$$

Analogously,

$$\frac{1}{\pi} \int dz_2^* dz_2 \, e^{-z_2^* z_2} z_2^{n_2} (z_2^*)^{m_2} = n_2!,$$

resulting in

$$\int d\mu\, |z\rangle\langle z| = \sum_{n_1,n_2} \frac{n_1!n_2!}{\sqrt{n_1!n_2!n_1!n_2!}} |n_1,n_2\rangle\langle n_1,n_2| = 1. \qquad \square$$

The coherent states $|z\rangle$ can be used to associate to an operator $\hat{f} \in \hat{\mathcal{A}}$ a function $f(z^*, z)$ on the complex plane in the following way:

$$f(z^*, z) := \langle z|\hat{f}|z\rangle \tag{1.16}$$

From this we immediately obtain a definition for the star product of two functions:

$$(f \star g)(z^*, z) = \langle z|\hat{f}\hat{g}|z\rangle = \int d\mu(w^*, w)\, \langle z|\hat{f}|w\rangle\langle w|\hat{g}|z\rangle \tag{1.17}$$

where we implicitly used Lemma 3(iv). An explicit formula for (1.17) is stated in the following

Proposition 4.

$$(f \star g)(z^*, z) = f(z^*, z) \exp\left(\frac{\overleftarrow{\partial}}{\partial z_\alpha} \frac{\overrightarrow{\partial}}{\partial z_\alpha^*}\right) g(z^*, z) \quad \textit{for} \quad f, g \in C^\infty(\mathbb{C}^2) \tag{1.18}$$

Proof. We first introduce the translation operator $\exp(w\frac{\partial}{\partial z})$, whose action on $f(z^*, z)$ is given by

$$e^{w_\alpha \frac{\partial}{\partial z_\alpha}} f(z^*, z) = \frac{\langle z|\hat{f}|z+w\rangle}{\langle z|z+w\rangle}.$$

Hence,

$$e^{-z_\alpha \frac{\partial}{\partial w_\alpha} + w_\alpha \frac{\partial}{\partial z_\alpha}} f(z^*, z) = \frac{\langle z|\hat{f}|w\rangle}{\langle z|w\rangle} =: e^{(w_\alpha - z_\alpha)\frac{\overrightarrow{\partial}}{\partial z_\alpha}} : f(z^*, z) \tag{1.19}$$

where we define the ordering : : by moving every derivative in the Taylor expansion of $e^{(w_\alpha - z_\alpha)\frac{\overrightarrow{\partial}}{\partial z_\alpha}}$ to the right. Similarly, in : $e^{\frac{\overleftarrow{\partial}}{\partial z_\alpha}(w_\alpha - z_\alpha)}$: every derivative in the Taylor expansion is moved to the left. Plugging (1.19) into (1.17) gives

$$(f \star g)(z^*, z) = \int d\mu(w^*, w)\, \langle z|\hat{f}|w\rangle\langle w|\hat{g}|z\rangle$$

$$= f(z^*, z) \left(\int d\mu(w^*, w) \; : e^{\frac{\overleftarrow{\partial}}{\partial z_\alpha}(w_\alpha - z_\alpha)} : |\langle z|w\rangle|^2 : e^{(w^*_\alpha - z^*_\alpha)\frac{\overrightarrow{\partial}}{\partial z^*_\alpha}} : \right) g(z^*, z).$$

Lemma 3(iii) implies that $|\langle z|w\rangle|^2 = e^{-|w_1 - z_1|^2 - |w_2 - z_2|^2}$. Hence, we see that similar to the proof of Lemma 3(iv) the integral factorizes into integrals over the two complex coordinates w_1 and w_2:

$$\overbrace{\frac{1}{\pi}\int dw_1^* dw_1 \; : e^{\frac{\overleftarrow{\partial}}{\partial z_1}(w_1 - z_1)} : e^{-|w_1 - z_1|^2} : e^{(w_1^* - z_1^*)\frac{\overrightarrow{\partial}}{\partial z_1^*}} :}^{(*)}$$

$$\times \underbrace{\frac{1}{\pi}\int dw_2^* dw_2 \; : e^{\frac{\overleftarrow{\partial}}{\partial z_2}(w_2 - z_2)} : e^{-|w_2 - z_2|^2} : e^{(w_2^* - z_2^*)\frac{\overrightarrow{\partial}}{\partial z_2^*}} :}_{(**)}$$

Using the variable transformation $u = w_1 - z_1$, abbreviating $a \equiv \frac{\partial}{\partial z_1}$ $(a^* \equiv \frac{\partial}{\partial z_1^*})$ and using polar coordinates $u = re^{i\varphi}$, the integral $(*)$ can be expressed as:

$$\begin{aligned} (*) &= \frac{1}{\pi}\int_0^\infty dr\, re^{-r^2} \int_0^{2\pi} d\varphi \, \exp(are^{i\varphi} + a^* re^{-i\varphi}) \\ &= \frac{1}{\pi}\int_0^\infty dr\, re^{-r^2} \\ &\times \int_0^{2\pi} d\varphi \,(1 + are^{i\varphi} + a^* re^{-i\varphi} + \frac{1}{2}(a^2 r^2 e^{2i\varphi} + 2aa^* r^2 + (a^*)^2 r^2 e^{-2i\varphi}) + \dots) \end{aligned}$$

Every term in the Taylor expansion of $\exp(are^{i\varphi} + a^* re^{-i\varphi})$ containing a factor $e^{ik\varphi}$ for some non-zero $k \in \mathbb{Z}$ is integrated to zero; thus, only the terms constant with respect to φ give non-zero contributions:[2]

$$\begin{aligned} (*) &= 2\int_0^\infty dr \left(re^{-r^2} + \frac{1}{2} 2aa^* r^3 e^{-r^2} + \frac{1}{4!}\binom{4}{2}(aa^*)^2 r^5 e^{-r^2} + \dots \right) \\ &= 1 + aa^* + \frac{1}{2}(aa^*)^2 + \frac{1}{3!}(aa^*)^3 + \dots \\ &= \exp(\frac{\overleftarrow{\partial}}{\partial z_1}\frac{\overrightarrow{\partial}}{\partial z_1^*}) \end{aligned}$$

[2]The occurring integrals can be evaluated with any decent CAS.

Similarly,

$$(**) = \exp(\frac{\overleftarrow{\partial}}{\partial z_2}\frac{\overrightarrow{\partial}}{\partial z_2^*}).$$

Combining these results finally gives the explicit formula for $(f \star g)(z^*, z)$. □

Corollary 5. *We have*

$$\int d\mu(z^*, z)\,(f \star g)(z^*, z) = \int d\mu(z^*, z)\,(g \star f)(z^*, z)$$

where $d\mu(z^*, z) = \frac{1}{\pi^2} dz_1^* dz_1 dz_2^* dz_2$ *is the standard four-dimensional measure on* $\mathbb{C}^2$*.*

Proof. This is easily verified by employing a Fourier transformation. Since

$$\exp\left(\frac{\overleftarrow{\partial}}{\partial z_\alpha}\frac{\overrightarrow{\partial}}{\partial z_\alpha^*}\right) \xrightarrow{\mathcal{F}} \exp(p_\alpha p_\alpha^*),$$

the result is obtained by a simple change of ordering. □

Remark. The exponential does not vanish after Fourier transforming as it did in Lemma 1. Thus, we stress that in general

$$\int f \star g \neq \int fg$$

with this particular star product, in contrast to the Moyal star product (1.10). ◇

Recall that we set the non-commutativity parameter $\theta \equiv \frac{1}{2}$ at the beginning of this section in order to retain the well-known form of the commutator relations (1.12). Reintroducing θ in the formula for the star-product of Proposition 4 yields

$$(f \star g)(z^*, z) = f(z^*, z) \exp\left(\frac{\overleftarrow{\partial}}{\partial z_\alpha}\frac{\theta}{2}\frac{\overrightarrow{\partial}}{\partial z_\alpha^*}\right) g(z^*, z). \tag{1.20}$$

It is easy to see that (1.20) reduces to the usual commutative product $f(z^*, z)g(z^*, z)$ in the commutative limit $\theta \to 0$.

Further, we observe that the star product (1.20) (denoted by $\star_C$) is different from the star product (1.10) (denoted by $\star_M$) in Section 1.1.2, as can be seen in Corollary 5 and the following remark. However, as both $(C^\infty(\mathbb{R}^4), \star_M)$ and $(C^\infty(\mathbb{C}^2), \star_C)$ are isomorphic to the non-commutative algebra $\hat{\mathcal{A}}$, they are also isomorphic to each other. For details of this isomorphism we refer the reader to [1, p. 14, Sec. 3.3.3].

1.2 Deformed $\mathbb{R}^3$ and the fuzzy sphere

In Section 1.1 we discussed the commutation relation (1.1) with $D = 4$ and a constant anti-symmetric matrix (θ^{ij}). These assumptions resulted in the four-dimensional Moyal plane $\mathbb{R}^4_\theta$, which we equipped with the star product $\star_M$ in (1.10) respectively $\star_C$ in (1.18). In the following section, we pass from a constant commutation relation to a more general setting by assuming a coordinate dependence in (1.1). We then define the star product on $\mathbb{R}^3_\lambda$ by restricting the star product $\star_C$ of Section 1.1.3 in a suitable way. To this end we employ the Hopf fibration $S^3 \to S^2$ and anticipate the Jordan-Schwinger construction for the coordinate operators $\hat{x}^i$ from 3.1.1 in a slightly different way, thus enabling us to work out an explicit formula for the star product on $\mathbb{R}^3_\lambda$.

1.2.1 Definition and relation between the two spaces

We set $D = 3$ and consider the commutation relation

$$[\hat{x}^i, \hat{x}^j] = i\lambda \varepsilon^{ij}{}_k \hat{x}^k \quad \text{for} \quad i, j, k \in \{1, 2, 3\} \tag{1.21}$$

where λ is a non-commutative parameter playing the same role as θ in Section 1.1.1. Relation (1.21) is immediately recognized as the defining commutation relation of the Lie algebra $\mathfrak{su}(2)$, that is, $\mathbb{R}^3_\lambda$ comprises a reducible $SU(2)$-representation on a suitable Fock space

$$\mathcal{F} = \bigoplus_{N=1}^{\infty} \mathcal{F}_N .$$

This Fock space $\mathcal{F}$ is defined in Sections 1.1.3 resp. 3.1.1; in the latter we show explicitly (using the Jordan-Schwinger construction for the coordinate operators $\hat{x}^i$) that (1.21) defines an irreducible spin-$N/2$ representation of $SU(2)$ on the subspace $\mathcal{F}_N$:

$$\sum_{i=1}^{3} (\hat{x}^i)^2\big|_{\mathcal{F}_N} = \lambda^2 \frac{N}{2}\left(\frac{N}{2} + 1\right) \mathrm{id}_{\mathcal{F}_N} \tag{1.22}$$

This is the defining relation of the fuzzy sphere $S^2_{\lambda,N/2}$ with radius $\lambda\sqrt{\frac{N}{2}\left(\frac{N}{2}+1\right)}$. Hence, we can view $\mathbb{R}^3_\lambda$ as a direct sum of infinitely many fuzzy spheres $S^2_{\lambda,N/2}$:

$$\mathbb{R}^3_\lambda = \bigoplus_{N=1}^{\infty} S^2_{\lambda,N/2} \tag{1.23}$$

The fuzzy sphere $S^2_{\lambda,N/2}$ can also be obtained using the orbit quantization method; see for example [4], where an orbit quantization of the projective complex space $\mathbb{C}P^2$ is carried out explicitly. Since $S^2 \cong \mathbb{C}P^1$, the method can be readily applied to the fuzzy sphere.

1.2.2 Star product on $\mathbb{R}^3_\lambda$

In the following section we solely concentrate on the star product $\star_C$ coming from the coherent states; therefore, we simply write $\star$.

Quantizing the Hopf fibration

The two-sphere S^2 can be regarded as the image under the Hopf fibration $S^3 \to S^2$ in the following way: first, we embed the 3-sphere S^3 in $\mathbb{C}^2$ via

$$\begin{aligned} \iota : S^3 &\longrightarrow \mathbb{C}^2 \\ (x_1, x_2, x_3, x_4) &\longmapsto (z_1 = x_1 + x_2 i, z_2 = x_3 + x_4 i) \end{aligned}$$

where $x_i \in \mathbb{R}$ and $\sum_i x_i^2 = 1$. Second, we declare two points on S^3 equivalent if both lie on a complex line through the origin:

$$x \sim_{S^3} y :\Leftrightarrow x = \lambda y \text{ for a } \lambda \in \mathbb{C} \text{ with } |\lambda| = 1 \tag{1.24}$$

Since $\mathbb{C}P^1$ is the space of complex lines through the origin, this equivalence map gives rise to the quotient map $\pi : S^3 \to \mathbb{C}P^1$ of $\sim_{S^3}$, assigning to each $x \in S^3$ the corresponding complex line $[x] \in \mathbb{C}P^1$. The pre-image $\pi^{-1}([x]) = \{e^{i\theta}x \mid \theta \in [0, 2\pi)\}$, also called the fiber of $[x]$, is isomorphic to S^1. Employing the isomorphism $S_2 \cong \mathbb{C}P^1$ finally gives the Hopf fibration $S^3 \to S^2$. The corresponding Hopf map can be written compactly as

$$x^i(z) = \frac{1}{2} z^*_\alpha \sigma^i_{\alpha\beta} z_\beta \tag{1.25}$$

where $z = (z_1, z_2) \in \mathbb{C}^2$ with $|z_1|^2 + |z_2|^2 = 1$ and σ^i, $i = 1, 2, 3$, are the usual Pauli matrices.

To quantize the Hopf map we simply replace the complex coordinates χ_α with bosonic creation and annihilation operators $\hat{a}_\alpha$ of a 2-dimensional harmonic oscillator satisfying the canonical commutation relations (1.12). One easily checks (cf. Section 3.1.1) that the coordinates

$$\hat{x}^i := \frac{1}{2}\hat{a}^\dagger_\alpha \sigma^i_{\alpha\beta}\hat{a}_\beta \tag{1.26}$$

satisfy the defining commutation relation (1.21) for $\lambda = 1$.

Reducing the algebra of the Moyal plane

Recall that we defined $\hat{\mathcal{A}}$ as the non-commutative algebra defined by the coordinate operators $\hat{x}^i_M$, $i = 1, \dots, 4$ of the Moyal plane $\mathbb{R}^4_\theta$. Consider now the sub-algebra $\hat{\mathcal{A}}_3 \subset \hat{\mathcal{A}}$ generated by the coordinate operators $\hat{x}^i = \frac{1}{2}\hat{a}^\dagger_\alpha \sigma^i_{\alpha\beta}\hat{a}_\beta$. Since $(\hat{x}^0)^2 := \sum_{i=1}^3 (\hat{x}^i)^2$ commutes with $\hat{x}^i, i = 1, 2, 3$ by (1.22), we see that $\hat{\mathcal{A}}_3$ is the algebra of elements of $\hat{\mathcal{A}}$ commuting with $\hat{x}^0$. The function x^0 defined by $(x^0)^2 = \sum_i (x^i)^2$ is associated to the operator $\hat{x}^0$. To explicitly compute x^0, we first write out the Hopf coordinates x^i:

$$\begin{aligned}
x^1 &= \frac{1}{2}(z_1^* z_2 + z_1 z_2^*) \\
x^2 &= \frac{i}{2}(z_1 z_2^* - z_1^* z_2) \\
x^3 &= \frac{1}{2}(z_1 z_1^* - z_2 z_2^*)
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_i (x^i)^2 &= \frac{1}{4}((z_1^*)^2 z_2^2 + 2 z_1^* z_1 z_2^* z_2 + z_1^2 (z_2^*)^2 - z_1^2 (z_2^*)^2 + 2 z_1^* z_1 z_2^* z_2 - (z_1^*)^2 z_2^2 \\
&\quad + (z_1^* z_1)^2 - 2 z_1^* z_1 z_2^* z_2 + (z_2^* z_2)^2 \\
&= \frac{1}{4}(z_1^* z_1 + z_2^* z_2)^2,
\end{aligned}$$

implying $x^0 = \frac{1}{2} z_\alpha^* z_\alpha$. Employing the identification

$$i[\hat{x}^0, \hat{f}] \longleftrightarrow i[x^0, f]_\star := i(x^0 \star f(z^*, z) - f(z^*, z) \star x^0)$$

we compute the equivalent of the commutation relation $i[\hat{x}^0, \hat{f}]$ in $(C^\infty(\mathbb{C}^2), \star)$:

$$\begin{aligned}
x^0 \star f(z^*, z) &= \frac{1}{2} z_\alpha^* z_\alpha \exp\left(\frac{\overleftarrow{\partial}}{\partial z_\alpha}\frac{\overrightarrow{\partial}}{\partial z_\alpha^*}\right) f(z^*, z) \\
&= \frac{1}{2}(z_\alpha^* z_\alpha (1 + \frac{\overleftarrow{\partial}}{\partial z_\alpha}\frac{\overrightarrow{\partial}}{\partial z_\alpha^*} + \dots) f(z^*, z))
\end{aligned}$$

$$= \frac{1}{2}(z^*_\alpha z_\alpha f(z^*, z) + z^*_\alpha \frac{\partial}{\partial z^*_\alpha} f(z^*, z)),$$

since terms with higher-order derivatives annihilate $z^*_\alpha z_\alpha$. Analogously,

$$f(z^*, z) \star x^0 = \frac{1}{2}(z^*_\alpha z_\alpha f(z^*, z) + z_\alpha \frac{\partial}{\partial z_\alpha} f(z^*, z)),$$

resulting in

$$i[x^0, f]_\star = \frac{i}{2}\left(z^*_\alpha \frac{\partial}{\partial z^*_\alpha} - z_\alpha \frac{\partial}{\partial z_\alpha}\right) f(z^*, z) =: \mathcal{L}_0 f.$$

The map $\mathcal{L}_0$ is actually a derivation with respect to $\star$:

Lemma 6. $\mathcal{L}_0 \in \mathrm{Der}((C^\infty(\mathbb{C}^2), \star))$, *that is,* $\mathcal{L}_0(f \star g) = \mathcal{L}_0(f) \star g + f \star \mathcal{L}_0 g$.

Proof. We compute:

$$\begin{aligned}
\mathcal{L}_0(f \star g) &= \mathcal{L}_0(f(z^*, z) \exp\left(\frac{\overleftarrow{\partial}}{\partial z_\alpha}\frac{\overrightarrow{\partial}}{\partial z^*_\alpha}\right) g(z^*, z)) \\
&= \frac{i}{2}\left(z^*_\alpha \frac{\partial}{\partial z^*_\alpha} - z_\alpha \frac{\partial}{\partial z_\alpha}\right)(f(z^*, z) \exp\left(\frac{\overleftarrow{\partial}}{\partial z_\alpha}\frac{\overrightarrow{\partial}}{\partial z^*_\alpha}\right) g(z^*, z)) \\
&= \frac{i}{2}\left(z^*_\alpha \frac{\partial}{\partial z^*_\alpha} f(z^*, z)\right) \exp\left(\frac{\overleftarrow{\partial}}{\partial z_\alpha}\frac{\overrightarrow{\partial}}{\partial z^*_\alpha}\right) g(z^*, z) \\
&\quad + \frac{i}{2} f(z^*, z) \exp\left(\frac{\overleftarrow{\partial}}{\partial z_\alpha}\frac{\overrightarrow{\partial}}{\partial z^*_\alpha}\right) z^*_\alpha \frac{\partial}{\partial z^*_\alpha} g(z^*, z) \\
&\quad - \frac{i}{2}\left(z_\alpha \frac{\partial}{\partial z_\alpha} f(z^*, z)\right) \exp\left(\frac{\overleftarrow{\partial}}{\partial z_\alpha}\frac{\overrightarrow{\partial}}{\partial z^*_\alpha}\right) g(z^*, z) \\
&\quad - \frac{i}{2} f(z^*, z) \exp\left(\frac{\overleftarrow{\partial}}{\partial z_\alpha}\frac{\overrightarrow{\partial}}{\partial z_\alpha}\right) z_\alpha \frac{\partial}{\partial z^*_\alpha} g(z^*, z) \\
&= \frac{i}{2}\left(\left(z^*_\alpha \frac{\partial}{\partial z^*_\alpha} - z_\alpha \frac{\partial}{\partial z_\alpha}\right) f(z^*, z)\right) \exp\left(\frac{\overleftarrow{\partial}}{\partial z_\alpha}\frac{\overrightarrow{\partial}}{\partial z^*_\alpha}\right) g(z^*, z) \\
&\quad + \frac{i}{2} f(z^*, z) \exp\left(\frac{\overleftarrow{\partial}}{\partial z_\alpha}\frac{\overrightarrow{\partial}}{\partial z_\alpha}\right)\left(z^*_\alpha \frac{\partial}{\partial z^*_\alpha} - z_\alpha \frac{\partial}{\partial z_\alpha}\right) g(z^*, z) \\
&= (\mathcal{L}_0 f) g + f \mathcal{L}_0 g
\end{aligned}$$

□

Since we have the correspondence

$$i[\hat{x}^0, \hat{f}] = 0 \text{ in } \hat{\mathcal{A}} \quad \longleftrightarrow \quad \mathcal{L}_0 f = 0 \text{ in } (C^\infty(\mathbb{C}^2), \star),$$

Lemma 6 implies that the function algebra $\mathcal{A}_3$ corresponding to the operator sub-algebra $\hat{\mathcal{A}}_3 \subset \hat{\mathcal{A}}$ is closed under the star product $\star$. We can thus simply restrict the star product (1.17) to $\mathbb{R}^3_\lambda$ and denote the corresponding function algebra by $\mathcal{A}_3$. Let us find an explicit formula for $\star$ of (1.18) on $\mathcal{A}_3$:

Proposition 7.

$$(f \star g)(x) = f(u) \exp\left(\frac{\overleftarrow{\partial}}{\partial u^i} \frac{1}{2} (\delta^{ij} x^0 + i\varepsilon^{ij}{}_k x^k) \frac{\overrightarrow{\partial}}{\partial v^j} \right) g(v) \bigg|_{u=v=x} \tag{1.27}$$

for $f, g \in \mathcal{A}_3$.

Proof. First, we have to change the differentials $\frac{\partial}{\partial z_\alpha}|_{\mathcal{A}_3}$ and $\frac{\partial}{\partial z^*_\alpha}|_{\mathcal{A}_3}$ to the new coordinates $x^i = \frac{1}{2} z^*_\alpha \sigma^i_{\alpha\beta} z_\beta$:

$$\begin{aligned} \frac{\partial}{\partial z_\alpha}\bigg|_{\mathcal{A}_3} &= \frac{dx^i}{dz_\alpha} \frac{\partial}{\partial x^i} = \frac{1}{2} z^*_\beta \sigma^i_{\beta\alpha} \frac{\partial}{\partial x^i} \\ \frac{\partial}{\partial z^*_\alpha}\bigg|_{\mathcal{A}_3} &= \frac{dx^i}{dz^*_\alpha} \frac{\partial}{\partial x^i} = \frac{1}{2} \frac{\partial}{\partial x^i} \sigma^i_{\alpha\beta} z_\beta \end{aligned}$$

Plugging this into the exponential in (1.18) gives

$$\exp\left(\frac{\overleftarrow{\partial}}{\partial z_\alpha} \frac{\overrightarrow{\partial}}{\partial z^*_\alpha} \right) = \exp\left(\frac{\overleftarrow{\partial}}{\partial x^i} \frac{1}{4} \underbrace{z^*_\beta \sigma^i_{\beta\alpha} \sigma^j_{\alpha\gamma} z_\gamma}_{(*)} \frac{\overrightarrow{\partial}}{\partial x^j} \right)$$

where we take a closer look at the term $(*)$:

$$\begin{aligned} (*) &= z^*_\beta (\sigma^i \sigma^j)_{\beta\gamma} z_\gamma \\ &= z^*_\beta (\delta^{ij} \delta_{\alpha\beta} + i\varepsilon^{ij}{}_k \sigma^k_{\beta\alpha}) z_\alpha \\ &= 2(\delta^{ij} x^0 + i\varepsilon^{ij}{}_k x^k) \end{aligned}$$

Putting everything together yields the desired result. □

Let us check that we can recover the commutator relation (1.21) from the star product

(1.27):

$$\begin{aligned} x^i \star x^j &= x^i \exp\left(\frac{\overleftarrow{\partial}}{\partial x^k}\frac{1}{2}(\delta^{kl}x^0 + i\varepsilon^{kl}{}_m x^m)\frac{\overrightarrow{\partial}}{\partial x^l}\right) x^j \\ &= x^i(1 + \frac{\overleftarrow{\partial}}{\partial x^k}\frac{1}{2}(\delta^{kl}x^0 + i\varepsilon^{kl}{}_m x^m)\frac{\overrightarrow{\partial}}{\partial x^l} + \dots)x^j \\ &= x^i x^j + \frac{1}{2}(\delta^{ij}x^0 + i\varepsilon^{ij}{}_k x^k) \end{aligned}$$

Therefore,

$$\begin{aligned} [x^i, x^j]_\star &= x^i \star x^j - x^j \star x^i \\ &= \frac{i}{2}(\varepsilon^{ij}{}_k x^k - \varepsilon^{ji}{}_k x^k) \\ &= i\varepsilon^{ij}{}_k x^k, \end{aligned}$$

establishing the isomorphism between $\hat{\mathcal{A}}_3$ and $\mathcal{A}_3$.

Suitable measure on $\mathbb{R}^3_\lambda$

In order to formulate field theories on $\mathbb{R}^3_\lambda$, one needs to specify an integration measure on the algebra $\mathcal{A}_3$ corresponding to $\mathbb{R}^3_\lambda$. In the spirit of Section 1.2.2, we take the four-dimensional measure

$$d\mu = \frac{1}{\pi^2} dz_1^* dz_1 dz_2^* dz_2 \tag{1.28}$$

on $\mathbb{C}^2$ as in Corollary 5 and try to extract a suitable measure for $\mathcal{A}_3$.

Our ansatz is the following coordinate transformation:

$$\begin{aligned} z_1 &= R\cos\theta' e^{i\varphi_1} \qquad \text{with } 0 \leq \theta' \leq \frac{\pi}{2}, 0 \leq \varphi_\alpha \leq \pi \\ z_2 &= R\sin\theta' e^{i\varphi_2} \end{aligned}$$

We need to express the coordinate differentials dz_α and dz_α^* in the new coordinates:

$$\begin{aligned} dz_1 &= \cos\theta' e^{i\varphi_1} dR - R\sin\theta' e^{i\varphi_1} d\theta' + iR\cos\theta' e^{i\varphi_1} d\varphi_1 \\ dz_1^* &= \cos\theta' e^{-i\varphi_1} dR - R\sin\theta' e^{-i\varphi_1} d\theta' - iR\cos\theta' e^{-i\varphi_1} d\varphi_1 \\ dz_2 &= \sin\theta' e^{i\varphi_2} dR + R\cos\theta' e^{i\varphi_2} d\theta' + iR\sin\theta' e^{i\varphi_2} d\varphi_2 \\ dz_2^* &= \sin\theta' e^{-i\varphi_2} dR + R\cos\theta' e^{-i\varphi_2} d\theta' - iR\sin\theta' e^{-i\varphi_2} d\varphi_2 \end{aligned}$$

Further, we compute the exterior product of these differentials:

$$
\begin{aligned}
dz_1^* \wedge dz_1 &= -R\sin\theta'\cos\theta' dR\wedge d\theta' + iR\cos^2\theta' dR\wedge d\varphi_1 \\
&\quad - R\sin\theta'\cos\theta' d\theta'\wedge dR - iR\sin\theta'\cos\theta' d\theta'\wedge d\varphi_1 \\
&\quad - iR\cos^2\theta' d\varphi_1\wedge dR + iR^2\sin\theta'\cos\theta' d\varphi_1\wedge d\theta' \\
&= 2iR\cos^2\theta' dR\wedge d\varphi_1 - 2iR^2\sin\theta'\cos\theta' d\theta'\wedge d\varphi_1
\end{aligned}
$$

Analogously,

$$
dz_2^* \wedge dz_2 = 2iR^2\sin^2\theta dR\wedge d\varphi_2 + 2iR^2\sin\theta'\cos\theta' d\theta'\wedge d\varphi_2.
$$

This finally gives

$$
\begin{aligned}
d\mu(z^*, z) &= \frac{1}{\pi^2} dz_1^*\wedge dz_1\wedge dz_2^*\wedge dz_2 \\
&= \frac{1}{\pi^2} R^3\sin(2\theta') dR d(2\theta') d\varphi_1 d\varphi_2.
\end{aligned}
$$

Let us also express the coordinates x^i in the new basis $(R, \theta', \varphi_1, \varphi_2)$:

$$
x^0 = \frac{1}{2} z_\alpha^* z_\alpha = \frac{1}{2} R^2(cos^2(\theta') + \sin^2(\theta')) = \frac{1}{2} R^2 \tag{1.29a}
$$

$$
\begin{aligned}
x^1 &= \frac{1}{2} z_\alpha^* \sigma^1_{\alpha\beta} z_\beta = \frac{1}{2}(z_1^* z_2 + z_2^* z_1) \\
&= \frac{1}{2} R^2\sin\theta'\cos\theta'(e^{i(\varphi_2-\varphi_1)} + e^{i(\varphi_1-\varphi_2)}) \\
&= \frac{1}{4} R^2\sin(2\theta')(e^{i(\varphi_2-\varphi_1)} + e^{-i(\varphi_2-\varphi_1)})
\end{aligned} \tag{1.29b}
$$

Similarly,

$$
x^2 = \frac{i}{4} R^2\sin(2\theta')(e^{-i(\varphi_2-\varphi_1)} - e^{i(\varphi_2-\varphi_1)}) \tag{1.29c}
$$

$$
x^3 = \frac{1}{4} R^2\cos(2\theta'). \tag{1.29d}
$$

We see in (1.29) that the x^i only depend on the variables $(R, \theta := 2\theta', \varphi := \varphi_2 - \varphi_1)$. This is not surprising, given that $\mathbb{R}^3_\lambda$ is a three-dimensional space. We change to this new coordinate system (R, θ, φ) and observe that the coordinates x^i are now of the form

$$
x^1 = \frac{R^2}{2}\sin\theta\cos\varphi
$$

$$x^2 = \frac{R^2}{2}\sin\theta\sin\varphi$$
$$x^3 = \frac{R^2}{2}\cos\theta,$$

that is, the usual spherical coordinates with radius $x^0 = \frac{R^2}{2}$. To express the measure (1.28) in the basis (R, θ, φ), we also note the following:

$$R^2 = 2x^0 \quad \Rightarrow \begin{cases} RdR = dx^0 \\ R^3 dR = 2x^0 dx^0 \end{cases}$$
$$\left.\begin{array}{r} \varphi = \varphi_2 - \varphi_1 \\ \varphi' = \varphi_2 + \varphi_1 \end{array}\right\} \Rightarrow \begin{cases} \varphi_1 = \frac{1}{2}(\varphi' - \varphi) \\ \varphi_2 = \frac{1}{2}(\varphi' + \varphi) \end{cases}$$
$$d\varphi_1 d\varphi_2 = \frac{1}{4}(d\varphi' d\varphi - d\varphi d\varphi')$$
$$= \frac{1}{2} d\varphi' d\varphi$$

Thus, the measure $d\mu(z^*, z)$ takes the form

$$d\mu(z^*, z) = \frac{1}{4\pi^2} x^0 \sin\theta dx^0 d\theta d\varphi' d\varphi$$

and the integral of an arbitrary function $f(x) \in \mathcal{A}_3$ can be written as

$$\int d\mu(z^*, z)\, f(x) = \frac{1}{2\pi}\int_0^\infty x^0 dx^0 \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi\, f(x)$$
$$= \frac{1}{2\pi}\int \frac{d^3x}{x^0}\, f(x)$$

where we already carried out the trivial integration over φ'.

We observe that the measure on $\mathbb{R}^3_\lambda$ differs from the ordinary measure on $\mathbb{R}^3$ by a factor $\frac{1}{x^0}$. This stems from the radial part $x^0 dx^0$ of the integration measure on $\mathbb{R}^3_\lambda$; since the usual radial part of the integration measure on $\mathbb{R}^3$ is $r^2 dr$, we need the extra factor $\frac{1}{x^0}$. This is tied to the fact that the radial direction in the deformed space $\mathbb{R}^3_\lambda$ plays a special role, since it encodes all different fuzzy spheres with radius $\lambda\sqrt{\frac{N}{2}\left(\frac{N}{2}+1\right)}$, expressed in (1.23).

Remark. Clearly, Corollary 5 and the remark thereafter also hold for the star product

(1.27) on $\mathbb{R}^3_\lambda$. We therefore have

$$\int \frac{d^3x}{x^0}(f \star g)(x) = \int \frac{d^3x}{x^0}(g \star f)(x),$$

but $\int f \star g \neq \int fg$ in general. ◇

1.2.3 Projection operator $\hat{P}_J$

In Section 1.2.2 we defined the star product on $\mathbb{R}^3_\lambda$ as the restriction of the star product $\star_C$ on the Moyal plane $\mathbb{R}^4_\theta$. Furthermore, in Section 1.2.1 we saw that $\mathbb{R}^3_\lambda$ can be regarded as the direct sum over the radii of infinitely many fuzzy spheres. We now employ this connection to define the star product on a fuzzy sphere of given radius by means of a projection operator $\hat{P}_J$.

In order to define $\hat{P}_J$ we first switch to a more suitable basis of the Fock space $\mathcal{F}$. Remember that we initially defined $\mathcal{F}$ as the span of the vectors

$$|n_1, n_2\rangle = \frac{(\hat{a}_1^\dagger)^{n_1}(\hat{a}_2^\dagger)^{n_2}}{\sqrt{(n_1!n_2!)}}|0\rangle,$$

that is, eigenstates of the number operator $\hat{N} = \hat{a}_\alpha^\dagger \hat{a}_\alpha$ (as shown in Lemma 28 in Section 3.1.2). However, for our purpose it is more convenient to work in the so-called Schwinger basis:

$$|j, m\rangle = \frac{(\hat{a}_1^\dagger)^{j+m}(\hat{a}_2^\dagger)^{j-m}}{\sqrt{(j+m)!(j-m)!}}|0\rangle \tag{1.30}$$

where $j \in \frac{1}{2}\mathbb{N}$ and $m \in \{-j, -j+1, \ldots, j-1, j\}$. To express the coherent state $|z\rangle$ in the basis (1.30), we use the expression (1.15) for $|z\rangle$ to compute:

$$\begin{aligned}\langle j, m|z\rangle &= e^{-z^*z/2}\frac{1}{\sqrt{(j+m)!(j-m)!}}\sum_{n_1,n_2}\frac{z_1^{n_1}z_2^{n_2}}{\sqrt{n_1!n_2!}}\underbrace{\langle 0|\hat{a}_1^{j+m}\hat{a}_2^{j-m}|n_1, n_2\rangle}_{=\delta_{n_1,j+m}\delta_{n_2,j-m}} \\ &= e^{-z^*z/2}\frac{z_1^{j+m}z_2^{j-m}}{(j+m)!(j-m)!}\end{aligned}$$

Hence, the coherent state in the Schwinger basis reads

$$|z\rangle = e^{-z^*z/2}\sum_{j\in\frac{1}{2}\mathbb{N}}\sum_{m=-j}^{j}\frac{z_1^{j+m}z_2^{j-m}}{(j+m)!(j-m)!}|j, m\rangle. \tag{1.31}$$

Setting $J \equiv \frac{N}{2}$ (i.e., J is a specific value of j in the Schwinger expansion $|j,m\rangle$), we denote by $\hat{\mathcal{A}}_J$ the algebra of operators on the fuzzy sphere $S^2_{\lambda,J}$ defined by

$$[\hat{x}^i, \hat{x}^j] = i\lambda\varepsilon^{ij}{}_k\hat{x}^k \qquad\qquad \sum_{i=1}^{3}(\hat{x}^i)^2 = \lambda^2 J(J+1). \tag{1.32}$$

An operator $\hat{f}_J \in \hat{\mathcal{A}}_J$ can be written with respect to the Schwinger basis (1.30) as

$$\hat{f}_J = \sum_{m,m'=-J}^{J} f^J_{m,m'}|J,m\rangle\langle J,m'| \quad \text{with} \quad f^J_{m,m'} \in \mathbb{C}. \tag{1.33}$$

Recalling that $\mathbb{R}^3_\lambda = \bigoplus_{J\in\frac{1}{2}\mathbb{N}} S^2_{\lambda,J}$, we can certainly decompose any operator $\hat{f} \in \hat{\mathcal{A}}_3$ as

$$\hat{f} = \sum_{J\in\frac{1}{2}\mathbb{N}} \hat{f}_J, \tag{1.34}$$

and it is obvious that (1.33) and (1.34) are related by the projection operator

$$\hat{P}_J := \sum_{m=-J}^{J} |J,m\rangle\langle J,m|. \tag{1.35}$$

More precisely, we have

$$\hat{f}_J = \hat{P}_J^\dagger \hat{f} \hat{P}_J, \tag{1.36}$$

which is obvious from the definitions of $\hat{f}_J$ and $\hat{P}_J$. It is easy to see that $\hat{P}_J$ is indeed a projection operator:

Lemma 8. *$\hat{P}_J$ is a projection operator of rank $2J+1$, that is:*

(i) $\hat{P}_J^2 = \hat{P}_J$

(ii) $\hat{P}_J^\dagger = \hat{P}_J$

(iii) $\dim(\hat{P}_J(\mathbb{R}^3_\lambda)) = 2J+1$ *and* $\sum_{J\in\frac{1}{2}\mathbb{N}} \hat{P}_J = \mathrm{id}_{\mathbb{R}^3_\lambda}$

Proof. Prove this by using the definition of $\hat{P}_J$ and the decomposition $\mathbb{R}^3_\lambda = \bigoplus_{J\in\mathbb{N}} S^2_{\lambda,J}$. □

The projection operator $\hat{P}_J$ can be used to define a star product on the fuzzy sphere $S^2_{\lambda,J}$ related to the star product $\star_C$ on $\mathbb{R}^3_\lambda$. Since we used coherent states to define $\star_C$

we first need to investigate the action of $\hat{P}_J$ on $\hat{\mathcal{A}}_3$. The following Lemma proves to be useful:

Lemma 9.

(i) $\hat{a}_\alpha \hat{P}_J = \hat{P}_{J-\frac{1}{2}} \hat{a}_\alpha$ *and* $\hat{a}^\dagger_\alpha \hat{P}_J = \hat{P}_{J+\frac{1}{2}} \hat{a}^\dagger_\alpha$

(ii) $[\hat{x}^i, \hat{P}_J] = 0$ *for* $i = 1, 2, 3$

Proof. (i) First we calculate the action of $\hat{a}_\alpha$ on a Schwinger basis vector $|j, m\rangle$:

$$\begin{aligned}
\hat{a}_1|j,m\rangle &= \frac{(\hat{a}^\dagger_1)^{j+m}(\hat{a}^\dagger_2)^{j-m}}{\sqrt{(j+m)!(j-m)!}}|0\rangle \\
&= \frac{(j+m)(\hat{a}^\dagger_1)^{j+m-1}(\hat{a}^\dagger_2)^{j-m}}{\sqrt{(j+m)!(j-m)!}}|0\rangle \\
&= \sqrt{j+m}\frac{(\hat{a}^\dagger_1)^{(j-\frac{1}{2})+(m-\frac{1}{2})}(\hat{a}^\dagger_2)^{(j-\frac{1}{2})-(m-\frac{1}{2})}}{\sqrt{((j-\frac{1}{2})+(m-\frac{1}{2}))!((j-\frac{1}{2})-(m-\frac{1}{2}))!}}|0\rangle \\
&= \sqrt{j+m}|j-\tfrac{1}{2}, m-\tfrac{1}{2}\rangle
\end{aligned}$$

Similarly,

$$\hat{a}_2|j,m\rangle = \sqrt{j-m}|j-\tfrac{1}{2}, m-\tfrac{1}{2}\rangle.$$

Therefore:

$$\begin{aligned}
\hat{a}_1\hat{P}_J &= \sum_{m=-J}^{J} \hat{a}_1|j,m\rangle\langle j,m| \\
&= \sum_{m=-J}^{J} \sqrt{j+m}|j-\tfrac{1}{2}, m-\tfrac{1}{2}\rangle\langle j,m| \\
&= \sum_{m=-J}^{J} |j-\tfrac{1}{2}, m-\tfrac{1}{2}\rangle\langle 0|\hat{a}_1^{j+m}\hat{a}_2^{j-m}\frac{\sqrt{j+m}}{\sqrt{(j+m)!(j-m)!}} \\
&= \sum_{m=-J}^{J} |j-\tfrac{1}{2}, m-\tfrac{1}{2}\rangle\langle 0|\frac{\hat{a}_1^{j+m-1}\hat{a}_2^{j-m}}{\sqrt{(j+m-1)!(j-m)!}}\hat{a}_1 \\
&= \sum_{m=-J}^{J} |j-\tfrac{1}{2}, m-\tfrac{1}{2}\rangle\langle j-\tfrac{1}{2}, m-\tfrac{1}{2}|\hat{a}_1 \\
&= \hat{P}_{J-\frac{1}{2}}\hat{a}_1
\end{aligned}$$

upon setting $m' = m - \frac{1}{2}$ and $J' = J - \frac{1}{2}$. Analogously,

$$\hat{a}_2 \hat{P}_J = \hat{P}_{J-\frac{1}{2}} \hat{a}_2.$$

From $\hat{a}_\alpha \hat{P}_J = \hat{P}_{J-\frac{1}{2}} \hat{a}_\alpha$ we immediately get $\hat{a}^\dagger_\alpha \hat{P}_J = \hat{P}_{J+\frac{1}{2}} \hat{a}^\dagger_\alpha$ by Hermitian conjugation.

(ii) Using (i), we calculate:

$$\begin{aligned}
[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{P}_J] &= [\hat{a}^\dagger_\alpha, \hat{P}_J]\hat{a}_\beta + \hat{a}^\dagger_\alpha [\hat{a}_\beta, \hat{P}_J] \\
&= \hat{a}^\dagger_\alpha \hat{P}_J \hat{a}_\beta - \hat{P}_J \hat{a}^\dagger_\alpha \hat{a}_\beta + \hat{a}^\dagger_\alpha \hat{a}_\beta \hat{P}_J - \hat{a}^\dagger_\alpha \hat{P}_J \hat{a}_\beta \\
&= \hat{P}_{J+\frac{1}{2}} \hat{a}^\dagger_\alpha \hat{a}_\beta - \hat{P}_J \hat{a}^\dagger_\alpha \hat{a}_\beta + \underbrace{\hat{a}^\dagger_\alpha \hat{P}_{J-\frac{1}{2}} \hat{a}_\beta}_{=\hat{P}_J \hat{a}^\dagger_\alpha \hat{a}_\beta} - \hat{P}_{J+\frac{1}{2}} \hat{a}^\dagger_\alpha \hat{a}_\beta = 0,
\end{aligned}$$

yielding

$$[\hat{x}^i, \hat{P}_J] = \frac{1}{2} \sigma^i_{\alpha\beta} [\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{P}_J] = 0. \qquad \square$$

Lemma 9 implies that any operator in $\hat{\mathcal{A}}_3 = \mathrm{span}(\hat{x}^i \mid i = 1, 2, 3)$ commutes with the projection operator $\hat{P}_J$. Hence, (1.36) can be refined to

$$\hat{f}_J = \hat{P}^\dagger_J \hat{f} \hat{P}_J = \hat{f} \hat{P}_J \tag{1.37}$$

and therefore also $\hat{\mathcal{A}}_J = \hat{P}_J \hat{\mathcal{A}}_3 = \hat{\mathcal{A}}_3 \hat{P}_J = \hat{P}_J \hat{\mathcal{A}}_4 \hat{P}_J$. In order to relate $\hat{P}_J$ to the star product on $\hat{\mathcal{A}}_J$ we need to compute its action on a coherent state $|z\rangle$. To this end, we rewrite (1.31) as

$$|z\rangle = \sum_{j \in \frac{1}{2}\mathbb{N}} |z\rangle_j \quad \text{with} \quad |z\rangle_j = e^{-z^* z/2} \sum_{m=-j}^{j} \frac{z_1^{j+m} z_2^{j-m}}{(j+m)!(j-m)!} |j, m\rangle \tag{1.38}$$

which immediately implies $\hat{P}_J |z\rangle = |z\rangle_J$. Further, the coordinate representation $P_J \in \mathcal{A}_3$ of $\hat{P}_J$ is given by

$$P_J = \langle z | \hat{P}_J | z \rangle = \langle z | z \rangle_J = \frac{1}{(2J)!} e^{-z^* z} (z^* z)^{2J} = \frac{1}{(2J)!} e^{-2x^0} (2x^0)^{2J}. \tag{1.39}$$

Note that $P_J = P_J(x^0)$ as expected. We now define

$$f_j(z^*, z) =_j \langle z | \hat{f} | z \rangle_j = \langle z | \hat{P}_j \hat{f} \hat{P}_j | z \rangle = P_j \star f \star P_j \tag{1.40}$$

for $f \in \mathcal{A}_4$ using the star product in (1.17); the star product on $\mathcal{A}_J$ for $f, g \in \mathcal{A}_4$ is then given by $(f \star g)_j := f_j \star g_j$. Introducing the angular coordinates $\tilde{x}^i := \frac{x^i}{x^0}$ it is straight-forward to prove that $(\tilde{x}^i \star x^0)_j = \tilde{x}^i x^0$ and hence $(f(\tilde{x}^i) \star x^0)_j = (x^0 \star f(\tilde{x}^i))_j$.

2 Differential calculi

The purpose of this chapter is to define a differential calculus on $\mathbb{R}^3_\lambda$. After a short repetition of the de Rham-calculus on commutative manifolds, we first discuss the universal calculus on an arbitrary unital algebra, from which every differential calculus can be obtained as a quotient object. Turning to a certain class of algebras called quantum groups or Hopf algebras, we describe an explicit method of defining differential calculi on them. Since $\mathbb{R}^3_\lambda$ can be regarded as a quantum group, we can use the techniques from the previous sections to introduce an example of a four-dimensional calculus on $\mathbb{R}^3_\lambda$. The results are compared to the commutative setting.

The first section about the de Rham-calculus is part of every standard textbook on differential geometry. The discussion of the universal calculus is inspired by [21, Sec. 6.1], the sections on quantum groups draw from [21, Sec. 4.4] and [23, Sec. 1 and 24]. The explicit construction of the four-dimensional calculus on $\mathbb{R}^3_\lambda$ is taken from [3, Sec. 4 and 5], extended by explicit calculations.

2.1 Commutative manifolds

In this section we briefly recapitulate the construction of the de Rham differential calculus on a general, commutative manifold M. We set $\mathcal{C}(M) := C^\infty(M)$.

2.1.1 Vector fields

Let M be a smooth manifold of dimension n. A smooth vector field $X : M \longrightarrow TM$ is a smooth map assigning to each point $p \in M$ a tangent vector in T_pM. If we denote by $\pi : TM \longrightarrow M$ the projection of the tangent bundle, we have $\pi \circ X = \mathrm{id}_M$; hence, a vector field X is a smooth section of the tangent bundle $TM \longrightarrow M$. The space $\mathfrak{X}(M)$ of smooth vector fields on M is a left $\mathcal{C}(M)$-module via $(fX)(p) := f(p)X(p)$ for $f \in \mathcal{C}(M)$ and $X \in \mathfrak{X}(M)$. In the special case of the manifold $\mathbb{R}^n$ the vector fields form the free module $\mathcal{C}(\mathbb{R}^n)^n$, i.e., there is a global basis $(\partial_1, \dots, \partial_n)$ for $\mathfrak{X}(\mathbb{R}^n)$ and every vector field X on $\mathbb{R}^n$ can be uniquely written as $X = X^i\partial_i$ with $X^i \in \mathcal{C}(\mathbb{R}^n)$ (Einstein summation convention implied). In general, the module $\mathfrak{X}(M)$ is not free, i.e., such a (global) basis does not always exist. However, we can always find a $\mathcal{C}(M)$-module N such that $\mathfrak{X}(M) \oplus N$ is free. This is equivalent to saying that $\mathfrak{X}(M)$ is a projective module. By the observation above, $\mathfrak{X}(M)$ is free if and only if M is parallelizable.

An important result states that vector fields can be identified with derivations on $\mathcal{C}(M)$:

$$\mathfrak{X}(M) \cong \mathrm{Der}(\mathcal{C}(M))$$

We write $X(f)$ for $X \in \mathfrak{X}(M)$ and $f \in \mathcal{C}(M)$ to indicate that X acts on f as a derivation. The relationship between vector fields and derivations is supported by the representation of vectors as $X = X^i \partial_i$ where ∂_i are the ordinary partial derivatives with respect to some local coordinate system in $p \in M$. The isomorphism $\mathfrak{X}(M) \cong \mathrm{Der}(\mathcal{C}(M))$ however is not trivial.

2.1.2 De Rham-calculus

A differential calculus on a manifold M consists of a space $\Omega^*(M)$ of differential forms and an exterior derivative d defined on these forms. We familiarize ourselves with the concept by considering the de Rham-calculus.

A p-form α is a smooth section of $\Lambda^p(T^*M)$, the p-th exterior power of the cotangent bundle. Explicitly, for every $x \in M$ the smooth map $x \mapsto \alpha_x$ defines a p-linear, alternating map $\alpha(x) = \alpha_x : T_xM \times \cdots \times T_xM \longrightarrow \mathbb{R}$. The set $\Omega^p(M)$ of p-forms on M is turned into a $\mathcal{C}(M)$-bimodule by means of

$$(f\alpha)(x) = (\alpha f)(x) = f(x)\alpha(x)$$

for $f \in \mathcal{C}(M)$, $\alpha \in \Omega^p(M)$ and $x \in M$. For a p-form $\alpha \in \Omega^p(M)$ and a q-form $\beta \in \Omega^q(M)$ we define the exterior or wedge product $\alpha \wedge \beta \in \Omega^{p+q}(M)$ as

$$(\alpha \wedge \beta)(X_1, \ldots, X_{p+q}) := \frac{1}{p!q!} \sum_{\sigma \in \mathfrak{S}_{p+q}} \mathrm{sgn}(\sigma)\alpha(X_{\sigma(1)}, \ldots, X_{\sigma(p)})\beta(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)})$$

where $\mathfrak{S}_n$ is the symmetric group on $\{1, \ldots, n\}$. Note that $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$, i.e., the exterior product is graded commutative. Setting $\Omega^0(M) := \mathcal{C}(M)$ we define the space $\Omega^*(M)$ of differential forms as

$$\Omega^*(M) = \bigoplus_{p=0}^{\infty} \Omega^p(M),$$

which by the above is turned into a graded algebra $(\Omega^*(M), \wedge)$ by noting that

$$\Omega^p(M) \wedge \Omega^q(M) \subset \Omega^{p+q}(M).$$

The second ingredient in a differential calculus is an exterior derivative d, i.e., an $\mathbb{R}$-linear map satisfying $d^2 = 0$ and a graded Leibniz rule. For $\alpha \in \Omega^p(M)$ we define

$$\begin{aligned} d\alpha(X_0, \ldots, X_p) := &\sum_{i=0}^{p} (-1)^i X_i(\alpha(X_0, \ldots, \underbrace{X_i}, \ldots, X_p)) \\ &+ \sum_{0 \le i < j \le p} (-1)^{i+j} \alpha([X_i, X_j], X_0, \ldots, \underbrace{X_i}, \ldots, \underbrace{X_j}, \ldots, X_p) \end{aligned} \tag{2.1}$$

where a square bracket means that this vector is left out and $[X_i, X_j]$ is the Lie bracket of the vector fields X_i and X_j. Observe that $d\alpha \in \Omega^{p+1}(M)$. Given $f \in \mathcal{C}(M) = \Omega^0(M)$, definition (2.1) reduces to $df(X) = X(f)$, i.e., this is just the derivation X acting on f. The exterior derivative is a graded derivation with respect to the exterior product:

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$$

Another important observation is $d^2 = 0$, which follows directly from (2.1). A p-form α is called exact if there is a form $\beta \in \Omega^{p-1}(M)$ with $\alpha = d\beta$, and closed if $d\alpha = 0$. Since $d^2 = 0$, every exact form is automatically closed; one can then start with a particular space $\Omega^p(M)$ and investigate the quotient space (closed forms)/(exact forms) to obtain topological information about the manifold M. This is the subject of the field of cohomology. The pair $(\Omega^*(M), d)$ is called the de Rham-calculus on a commutative manifold M.

Let us also introduce two additional operations on differential forms: the interior product ι_X and the Lie derivative $\mathcal{L}_X$. Given a vector field X and a p-form α we define

$$(\iota_X \alpha)(X_1, \ldots, X_{p-1}) := p\alpha(X, X_1, \ldots, \ldots, X_{p-1}),$$

which is just the contraction of α with the vector field X, producing the form $\iota_X \alpha \in \Omega^{p-1}(M)$. The Lie derivative $\mathcal{L}_X$ is now defined as

$$\mathcal{L}_X := \iota_X d + d\iota_X.$$

Hence, $\mathcal{L}_X$ is a map $\Omega^p(M) \longrightarrow \Omega^p(M)$. Note that the notion of the Lie derivative can be extended to arbitrary tensor fields. In the special case of vector fields, the Lie derivative of a vector field Y with respect to a given vector field X is just the Lie bracket of X and Y:

$$\mathcal{L}_X Y = [X, Y]$$

2.1.3 Generalized construction

In section 2.1.2 we outlined the explicit construction of the de Rham-calculus on a commutative manifold M. However, this procedure is not suitable for generalizing the concept of differential calculi to non-commutative spaces. Therefore, we consider an equivalent procedure, which is easily adopted to the non-commutative setting:

(1) Identify the algebra of functions $\mathcal{C}(M)$ and the vector fields $\mathfrak{X}(M)$ on M.

(2) Set $\Omega^0(M) := \mathcal{C}(M)$.

(3) For $f \in \mathcal{C}(M)$ and $X \in \mathfrak{X}(M)$ define $df(X) = X(f)$.

(4) Let $\mathcal{D}$ be the $\mathcal{C}(M)$-module generated by the elements df for $f \in \mathcal{C}(M)$ and let $\mathcal{I} \subset \mathcal{D}$ be the submodule generated by elements of the form $f dg - (dg) f$ for $f, g \in \mathcal{C}(M)$. Define $\Omega^1(M) := \mathcal{D}/\mathcal{I}$.

(5) Define the space $\Omega^p(M)$ of p-forms as the $\mathcal{C}(M)$-module generated by p-fold exterior products of elements in $\Omega^1(M)$.

Note that in the commutative case step (1) amounts to setting $\mathcal{C}(M) = C^\infty(M)$ and $\mathfrak{X}(M) = \mathrm{Der}(C^\infty(M))$. There are more possibilities in the non-commutative case and obtaining a differential calculus requires reasonable choices. Recall also the different choices for non-commutative algebras shown in Chapter 1.

2.2 Non-commutative spaces

In Section (2.1.2) we used standard objects and techniques from differential geometry to construct the de Rham-calculus on an arbitrary commutative manifold M. However, a differential calculus can be defined in a purely algebraic way on the algebra $C^\infty(M)$ of smooth functions on M; this was already indicated in Section 2.1.3. One can even go one step further and start with an arbitrary, possibly non-commutative associative algebra. We learned in Chapter 1 that non-commutative spaces are defined by replacing the commutative algebra of functions on a manifold by a non-commutative algebra; hence, we need to be able to construct differential calculi on non-commutative algebras in order to have geometric tools on non-commutative spaces at our disposal. The most important observation is that, given an associative algebra, there is always a 'minimal' choice of such a differential calculus, called the universal calculus. It is minimal in the sense that every other differential calculus can be expressed as a suitable quotient of the universal calculus.

In this section we will first define the universal calculus and show a few properties. Secondly, we take a closer look at quantum groups and outline how to obtain a non-universal differential calculus on them by taking a quotient of the universal calculus. Finally, these results are used to explicitly construct a four-dimensional calculus on $\mathbb{R}^3_\lambda$ and the fuzzy sphere.

2.2.1 Universal calculus

Given a unital associative algebra $\mathcal{A}$ we define $C_p := \mathcal{A}^{\otimes(p+1)}$ as the set of p-chains (e.g., 0-chains are elements of $\mathcal{A}$, 1-chains are elements in $\mathcal{A}\otimes\mathcal{A}$, and so forth) and abbreviate $(a_0, \ldots, a_p) := a_0 \otimes \cdots \otimes a_p \in C_p$ where $a_i \in \mathcal{A}, i = 0, \ldots, p$. We now define the central object of a differential calculus, the differential map[1] $d_u : C_p \longrightarrow C_{p+1}$ by:

$$\begin{aligned} d_u(a_0, \ldots, a_p) = (1, a_0, \ldots, a_p) \\ + \sum_{i=1}^{p} (-1)^p (a_0, \ldots, a_{i-1}, 1, a_i, \ldots, a_p) \\ + (-1)^{p+1} (a_0, \ldots, a_p, 1) \end{aligned} \tag{2.2}$$

For example, $d_u a = 1 \otimes a - a \otimes 1$ and $d_u(a, b) = (1, a, b) - (a, 1, b) + (a, b, 1)$. We need to ensure that $d_u^2 = 0$, as we would expect from a differential map. Furthermore, d_u should satisfy the Leibniz rule on 0-chains $a \in \mathcal{A}$ (i.e., 'functions') and a graded Leibniz rule on higher forms. The former is shown in the following proposition, the latter will be shown in Proposition 11.

Proposition 10.

(i) $d_u^2 = 0$

(ii) $d_u(ab) = (d_u a)b + a d_u b$ *for* $a, b \in \mathcal{A}$

Proof. Throughout the proof we abbreviate $d \equiv d_u$. Further, for $a = (a_0, \ldots, a_p) \in C_p$ we define an insertion operator ι_i by $\iota_i(a_0, \ldots, a_p) := (a_0, \ldots, a_{i-1}, 1, a_i, \ldots, a_p)$ for $i =$

[1] In the language of homological algebra, the differential d_u is called a coboundary.

$1, \dots, p$, that is, it inserts the unit 1 in the $(i+1)$-th slot of $a \in C_p$. Note that this is just in order to ensure a clean notation in the proof.

(i) Clearly, by definition of d we have $d(a \pm b) = da \pm db$ where $a = (a_0, \dots, a_p) \in C_p, b = (b_0, \dots, b_p) \in C_p$ and $a \pm b := (a_0 \pm b_0, \dots, a_p \pm b_p)$. Hence, we compute:

$$\begin{aligned}
d^2(a_0, \dots, a_p) &= d(1, a_0, \dots, a_p) + \sum_{i=1}^{p} (-1)^p d(a_0, \dots, a_{i-1}, 1, a_i, \dots, a_p) \\
&\quad + (-1)^{p+1} d(a_0, \dots, a_p, 1) \\
&= \underbrace{(1, 1, a_0, \dots, a_p)}_{(\clubsuit)} + \underbrace{\sum_{i=1}^{p+1} (-1)^i \iota_i (1, a_0, \dots, a_p)}_{(\clubsuit)} + \underbrace{(-1)^{p+2}(1, a_0, \dots, a_p, 1)}_{(\heartsuit)} \\
&\quad + \underbrace{\sum_{i=1}^{p} (-1)^i (1, a_0, \dots, a_{i-1}, 1, a_i, \dots, a_p)}_{(\clubsuit)} \\
&\quad + \underbrace{\sum_{i=1}^{p} \sum_{j=1}^{p+1} (-1)^{i+j} \iota_j (a_0, \dots, a_{i-1}, 1, a_i, \dots, a_p)}_{(\diamondsuit)} \\
&\quad + \underbrace{\sum_{i=1}^{p} (-1)^{i+p+1} (a_0, \dots, a_{i-1}, 1, a_i, \dots, a_p, 1)}_{(\spadesuit)} + \underbrace{(-1)^{p+1}(1, a_0, \dots, a_p, 1)}_{(\heartsuit)} \\
&\quad + \underbrace{\sum_{i=1}^{p+1} (-1)^{i+p+1} \iota_i (a_0, \dots, a_p, 1)}_{(\spadesuit)} + \underbrace{(-1)^{2p+3}(a_0, \dots, a_p, 1, 1)}_{(\spadesuit)}
\end{aligned}$$

The terms $(\heartsuit), (\spadesuit)$ and $(\clubsuit)$ cancel. It remains to check that the term $(\diamondsuit)$ vanishes. If we write

$$(\diamondsuit) = \sum_{i=1}^{p} \sum_{j=1}^{p+1} (-1)^{i+j} \iota_j (\iota_i (a_0, \dots, a_p)),$$

we see that the pairs (i, j) and $(j, i+1)$ cancel each other out. Since there is an even number of terms, $p(p+1)$, the term $(\diamondsuit)$ vanishes altogether, and we obtain

$$d^2(a_0, \dots, a_p) = 0$$

for all p. Note that the whole proof is actually straightforward and merely a problem of writing everything out.

(ii) In order to verify the Leibniz rule, we first need to specify what we mean by the expressions $(a, b)c$ and $c(a, b)$ for $a, b, c \in \mathcal{A}$:

$$\begin{aligned} (a, b)c &:= (a, bc) = (ac, b) \\ c(a, b) &:= (ca, b) = (a, cb) \end{aligned} \tag{2.3}$$

Hence, we can choose either the embedding $\mathcal{A} \hookrightarrow \mathcal{A} \otimes \mathcal{A}, c \mapsto c \otimes 1$ or $\mathcal{A} \hookrightarrow \mathcal{A} \otimes \mathcal{A}, c \mapsto 1 \otimes c$; the non-commutativity of the algebra $\mathcal{A}$ however forces us to be careful with the order of multiplication. The Leibniz rule is now easily checked:

$$\begin{aligned} (da)b + adb &= ((1, a) - (a, 1))b + a((1, b) - (b, 1)) \\ &= (1, ab) - (a, b) + (a, b) - (ab, 1) \\ &= (1, ab) - (ab, 1) \\ &= d(ab) \end{aligned}$$

□

Remark. The choice in (2.3) fixes the left and right $\mathcal{A}$-module structure of $C_2 = \mathcal{A} \otimes \mathcal{A}$. We will shortly see that the space $\Omega^1_u(\mathcal{A})$ of 1-forms in the universal calculus is a submodule of C_2 with this module structure. ◇

The previous remark has already indicated the nature of the space $\Omega^1_u(\mathcal{A})$ of 1-forms. In order to construct the space $\Omega^p_u(\mathcal{A})$ of p-forms with $p > 1$, we have to define a multiplication for elements in C_p. For $\alpha = (a_0, \dots, a_p) \in C_p$ and $\beta = (b_0, \dots, b_q) \in C_q$ we set

$$\alpha * \beta := (a_0, \dots, a_{p-1}, a_p b_0, b_1, \dots, b_q).$$

Since $\mathcal{A}$ is an associative algebra, so is the product $*$; furthermore, by definition we clearly have $C_p * C_q \subset C_{p+q}$. Thus, $C^* := \bigcup_{p \in \mathbb{N}} C_p$ is a graded associative algebra, on which the differential d_u satisfies a graded Leibniz rule:

Proposition 11.

$$d_u(\alpha * \beta) = (d_u \alpha) * \beta + (-1)^p \alpha * d_u \beta \quad \textit{for } \alpha \in C_p, \beta \in C_q$$

Proof. We set $\alpha = (a_0, \dots, a_p), \beta = (b_0, \dots, b_q)$ and calculate the left-hand side of the equation:

$$\begin{aligned} d_u(\alpha * \beta) &= d_u(a_0, \dots, a_p b_0, \dots, b_q) \\ &= (1, a_0, \dots, a_p b_0, \dots, b_q) + \sum_{i=1}^{p+q} (-1)^i \iota_i(a_0, \dots, a_p b_0, \dots, b_q) \end{aligned}$$

$$+ (-1)^{p+q+1}(a_0, \ldots, a_p b_0, \ldots, b_q, 1),$$

where ι_i is the insertion operator from the proof of Proposition 10. The right-hand side amounts to

$$\begin{aligned}(d_u\alpha) * \beta + (-1)^p \alpha * d_u\beta = (1, a_0, \ldots, a_p b_0, \ldots, b_q) + \underbrace{\sum_{i=1}^{p}(-1)^i \iota_i(a_0, \ldots, a_p b_0, \ldots, b_q)}_{(*)} \\ + \underbrace{(-1)^{p+1}(a_0, \ldots, a_p, b_0, \ldots, b_q)}_{(**)} \\ + \underbrace{(-1)^p(a_0, \ldots, a_p, b_0, \ldots, b_q)}_{(**)} \\ + \underbrace{(-1)^p \sum_{i=p+1}^{p+q} (-1)^{p+i} \iota_{p+1}(a_0, \ldots, a_p b_0, \ldots, b_q)}_{(*)} \\ + (-1)^{p+q+1}(a_0, \ldots, a_p b_0, \ldots, b_q, 1)\end{aligned}$$

The terms $(**)$ cancel. Since $(-1)^{2p+i} = (-1)^i$, combining the terms $(*)$ gives exactly $\sum_{i=1}^{p+q}(-1)^i \iota_i(a_0, \ldots, a_p b_0, \ldots, b_q)$; hence, the two sides of the equation are equal, proving the claim. □

Finally, we are ready to give the definition of the universal calculus over $\mathcal{A}$:

Definition 12 (Universal calculus)**.** Let $\mathcal{A}$ be a unital associative algebra. Set $\Omega_u^0(\mathcal{A}) = \mathcal{A}$ and let $\Omega_u^1(\mathcal{A}) \subset C_1 = \mathcal{A} \otimes \mathcal{A}$ be the $\mathcal{A}$-bimodule generated by the set $\{d_u a \mid a \in \mathcal{A}\}$. The left and right module structure are given by

$$\begin{aligned} c(a_1, a_2) &= (ca_1, a_2) = (a_1, ca_2) \\ (a_1, a_2)c &= (a_1, a_2 c) = (a_1 c, a_2) \end{aligned}$$

for $a_1, a_2, c \in \mathcal{A}$. For $p \geq 2$, $\Omega_u^p(\mathcal{A})$ is the $\mathcal{A}$-bimodule generated by the set

$$\{d_u a_1 * \cdots * d_u a_p \mid a_1, \ldots, a_p \in \mathcal{A}\}.$$

The left and right module structure are given by

$$\begin{aligned} c(a_1, \ldots, a_p) &= (ca_1, \ldots, a_p) = (a_1, \ldots, ca_i, \ldots, a_p) = (a_1, \ldots, ca_p) \\ (a_1, \ldots, a_p)c &= (a_1, \ldots, a_p c) = (a_1, \ldots, a_i c, \ldots, a_p) = (a_1, \ldots, a_p c) \end{aligned}$$

for $a_1, \ldots, a_p, c \in \mathcal{A}$. With $\Omega_u^*(\mathcal{A}) = \bigcup_{p\in\mathbb{N}} \Omega_u^p(\mathcal{A})$, the graded algebra $(\Omega_u^*, *, d)$ is called the universal calculus of $\mathcal{A}$. ◇

Note that $\Omega_u^p(\mathcal{A}) * \Omega_u^q(\mathcal{A}) \subset \Omega_u^{p+q}(\mathcal{A})$ by definition. Although the $*$-product is the analogue of the wedge product in the de Rham-calculus of commutative manifolds, we have in general

$$d_u a * d_u b \neq -d_u b * d_u a$$

for $a, b \in \mathcal{A}$ due to the module structure of $\Omega_u^1(\mathcal{A})$ (which is in turn a consequence of the non-commutativity of $\mathcal{A}$). The universal calculus is characterized by the following universal property:

Theorem 13 (Universal property of the universal calculus)**.** *Let $\mathcal{A}$ be a unital associative algebra, $(\Omega_u^*(\mathcal{A}), d_u)$ the universal calculus over $\mathcal{A}$ and $(\Omega^*(\mathcal{A}), *, d)$ some other calculus over $\mathcal{A}$. Then there is a unique surjective algebra homomorphism*

$$\phi : \Omega_u^*(\mathcal{A}) \longrightarrow \Omega^*(\mathcal{A})$$

with $\phi(d_u\xi) = d\phi(\xi)$ for $\xi \in \Omega_u^*(\mathcal{A})$.

Proof. See [23, p. 150, Ex. 23.7]. □

Remark. The theorem states that every calculus $\Omega^*(\mathcal{A})$ on $\mathcal{A}$ can be obtained as a suitable quotient of the universal calculus $\Omega_u^*(\mathcal{A})$. ◇

2.2.2 Quantum groups

The universal calculus of Section 2.2.1 can be defined for arbitrary unital associative algebras. The focus of this thesis however lies on the particular algebra $\mathbb{R}^3_\lambda$ introduced in Section 1.2, which can be regarded as a so-called quantum group or Hopf algebra.

The following section contains a very short introduction to quantum groups as well as an outline of the construction of a differential calculus on quantum groups using the quotient method of Theorem 13. Our goal is an explicit formula for the differential d on $\mathbb{R}^3_\lambda$, enabling us to work out the computations in Section 2.2.4. Since a thorough treatment of quantum groups would go beyond the scope of this work, we will be short and only state the necessary theorems without proof; the interested reader is referred to the literature, especially the textbooks [21] and [23] by Shahn Majid.

In order to define quantum groups, which are a certain type of a bialgebra, we need to consider coalgebras, the dual objects of algebras. To emphasize this duality, we restate the already familiar definition of an algebra. Throughout this section k denotes an arbitrary field.

Definition 14 (Algebra)**.** An algebra $\mathcal{A}$ over k is a k-vector space together with an associative multiplication map $m : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ and a unit $i : k \longrightarrow \mathcal{A}$ satisfying $i(1_k) = 1_\mathcal{A}$. The associativity condition can be expressed by defining the maps

$$m \otimes \mathrm{id} : (\mathcal{A} \otimes \mathcal{A}) \otimes \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A} \qquad \mathrm{id} \otimes m : \mathcal{A} \otimes (\mathcal{A} \otimes \mathcal{A}) \longrightarrow \mathcal{A} \otimes \mathcal{A}$$

and requiring $m \circ (m \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes m)$. We frequently abbreviate $ab \equiv m(a \otimes b)$ for $a, b \in \mathcal{A}$. The unit i has to satisfy $m \circ (\mathrm{id} \otimes i) = m \circ (i \otimes \mathrm{id}) = \mathrm{id}$, which just states that $a 1_\mathcal{A} = 1_\mathcal{A} a = a$ for $a \in \mathcal{A}$. ◇

If we reverse the direction of the arrows of the maps m and i in Definition 14, we obtain the dual object to an algebra, a coalgebra:

Definition 15 (Coalgebra)**.** A coalgebra $\mathcal{A}$ over k is a k-vector space together with a coassociative coproduct $\Delta : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$ and a counit $\epsilon : \mathcal{A} \longrightarrow k$ satisfying

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$$
$$(\mathrm{id} \otimes \epsilon) \circ \Delta = (\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id};$$

the first condition is the coassociativity of the coproduct Δ. ◇

We see that Definition 15 is, after reversing all arrows, entirely analogous to Definition 14. Now, if $\mathcal{A}(m, i, \Delta, \epsilon)$ has the structure of both an algebra and a coalgebra and satisfies certain compatibility conditions, it is called a bialgebra. In order to give an exact definition, we need the following map:

$$\tau : \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$$
$$a \otimes b \otimes c \otimes d \longmapsto a \otimes c \otimes b \otimes d$$

This helps us in stating the following

Definition 16 (Bialgebra)**.** A bialgebra $\mathcal{A}(m, i, \Delta, \epsilon)$ is a k-vector space having the structure of an associative algebra and a coassociative coalgebra satisfying the following compatibility conditions:

$$\Delta \circ m = (m \otimes m) \circ \tau \circ (\Delta \otimes \Delta) \qquad \epsilon \circ m = m \circ (\epsilon \otimes \epsilon)$$

$$\Delta \circ i = i \otimes i \qquad\qquad \epsilon \circ i = \text{id}$$

For example, the second condition $\epsilon \circ m = m \circ (\epsilon \otimes \epsilon)$ requires that the counit ϵ is an algebra homomorphism: $\epsilon(ab) = \epsilon(a)\epsilon(b)$. ◇

Example (Smooth functions on a group as bialgebra). Consider the set $C^\infty(G)$ of smooth complex-valued functions on a group G with identity e_G. For $f, g \in C^\infty(G)$ and $x, y \in G$ define

$$\begin{aligned}(m(f \otimes g))(x) &:= f(x)g(x)\\ (\Delta f)(x, y) &:= f(xy)\\ \epsilon(f)(x) &:= f(e_G);\end{aligned}$$

the unit i is fixed by the requirement $\epsilon \circ i = \text{id}$. Then it is straightforward to check the compatibility conditions such that $C^\infty(G)(m, i, \Delta, \epsilon)$ becomes a bialgebra. ◇

A Hopf algebra is a certain type of bialgebra. The term quantum group was coined by Drinfeld and Jimbo, who originally used a special q-deformed version of Hopf algebras in physics, q being a deformation parameter. However, up to this point there is no universally accepted definition of a quantum group; some authors use the terms quantum group and Hopf algebra interchangeably, and we will adhere to this convention.

Definition 17 (Quantum group or Hopf algebra)**.** A Hopf algebra $\mathcal{A}$ is a bialgebra $\mathcal{A}(m, i, \Delta, \epsilon)$ together with a linear map $S : \mathcal{A} \longrightarrow \mathcal{A}$ called the antipode, which satisfies the compatibility condition $m \circ (\text{id} \otimes S) \circ \Delta = m \circ (S \otimes \text{id}) \circ \Delta = i \circ \epsilon$. ◇

Example (Group algebra of a finite group). Let G be a finite group, k an arbitrary field and kG the group algebra of G. Defining the coproduct, counit and antipode on elements $g \in G$ as

$$\Delta g := g \otimes g \qquad\qquad \epsilon(g) := 1_k \qquad\qquad S(g) := g^{-1}$$

and extending them by linearity to all of kG turns $(kG, \Delta, \epsilon, S)$ into a Hopf algebra. ◇

2.2.3 Non-universal differential calculi for quantum groups

We have seen in the previous discussion of differential calculi that a key ingredient in defining a differential calculus $\Omega^*(\mathcal{A})$ on an algebra $\mathcal{A}$ is the space $\Omega^1(\mathcal{A})$ of 1-forms. For quantum groups H one is usually interested in so-called bicovariant modules. This essentially means that the H-module $\Omega^1 \equiv \Omega^1(H)$ admits a left coaction $\Omega^1 \longrightarrow H \otimes \Omega^1$ and right coaction $\Omega^1 \longrightarrow \Omega^1 \otimes H$, both being bimodule homomorphisms. The axioms for a coaction are the same as the usual axioms for actions, except with all arrows in the defining maps reversed. We have the following

Theorem 18 (Classification of bicovariant differential calculi on quantum groups)**.** *For a quantum group H the bicovariant differential calculi $\Omega^1(H)$ on H are in 1:1-correspondence with quotients Λ^1 of $\ker\epsilon$, that is, there is a two-sided ideal $\mathcal{I} \subset \ker\epsilon$ such that $\Lambda^1 = \ker\epsilon/\mathcal{I}$. The first-order calculus $(\Omega^1(H), d)$ is then given by*

$$\begin{aligned}\Omega^1(H) &= \Lambda^1 \otimes H \\ dh &= (\pi \otimes \mathrm{id})(\Delta h - 1 \otimes h) \quad \textit{for} \quad h \in H,\end{aligned}$$

where $\pi : \ker\epsilon \longrightarrow \Lambda^1 = \ker\epsilon/\mathcal{I}$ is the natural projection.

Proof. See [23, p. 156f., Lem. 24.6 and Thm. 24.7]. □

This result needs some explanations. Before we start, we introduce the Sweedler notation. For $\omega \in H \otimes H$ we can always write $\omega = \sum_i x_{(1),i} \otimes x_{(2),i}$ for suitable $x_{(1),i}, x_{(2),i} \in H$. Note that the indices (1) and (2) only keep track of the corresponding factor in $H \otimes H$. Sweedler's notation abbreviates this expression to

$$\omega = \sum x_{(1)} \otimes x_{(2)} \equiv x_{(1)} \otimes x_{(2)},$$

that is, summation is always implied. An advantage of this notation is that the coproduct Δh for $h \in H$ can be written in compact form as

$$\Delta h = h_{(1)} \otimes h_{(2)}.$$

Let us now turn to Theorem 18. By [23, p. 156, Lem. 24.6(i)] we have an isomorphism

$$\begin{aligned}\psi : \Omega^1_u(H) &\xrightarrow{\sim} \ker\epsilon \otimes H \\ h \otimes g &\longmapsto h_{(1)} \otimes h_{(2)} g\end{aligned} \tag{2.4}$$

where $h_{(1)}, h_{(2)} \in H$ are given by $\Delta h = h_{(1)} \otimes h_{(2)}$, that is, the isomorphism 'twists' the coproduct by g in the second factor. The image of the universal differential[2] $d_u h =$

[2]With the sign convention of Section 2.2.1 this is actually $-d$; clearly, both definitions are equivalent.

$h \otimes 1 - 1 \otimes h$ under the isomorphism (2.4) is

$$\begin{aligned}\psi(dh) &= \psi(h \otimes 1 - 1 \otimes h) \\ &= h_{(1)} \otimes h_{(2)} - 1 \otimes h \qquad (*) \\ &= \Delta h - 1 \otimes h.\end{aligned}$$

In $(*)$ we assumed that $1_{(1)} = 1_{(2)} = 1$. It remains to map the first factor of the differential $dh \in \Omega^1_u(H) \cong \ker \epsilon \otimes H$ to $\Lambda^1(H)$ via the projection $\pi : \ker \epsilon \to \Lambda^1$; this gives exactly $dh = (\pi \otimes \mathrm{id})(\Delta h - 1 \otimes h)$, as stated in the theorem. In summary, we used the following maps:

$$\Omega^1_u(H) \xrightarrow{\psi} \ker \epsilon \otimes H \xrightarrow{\pi \otimes \mathrm{id}} \Lambda^1 \otimes H = \Omega^1(H)$$

Theorem 18 states that the choice of a first-order calculus for a quantum group H amounts to finding a two-sided ideal $\mathcal{I}$ of $\ker \epsilon$. A practical method is to find a surjective representation of $\ker \epsilon \subset H$ on $\mathbb{C}^n$, that is, a surjective algebra homomorphism $\rho : \ker \epsilon \longrightarrow M_n(\mathbb{C})$ such that $\Lambda^1 \cong \ker \epsilon / \ker \rho$. We will use this ansatz in Section 2.2.4 to define a differential calculus on $\mathbb{R}^3_\lambda$.

However, note that Theorem 18 only defines a first-order differential calculus, that is, the 1-forms $\Omega^1(H)$, along with an exterior derivative defined on the 0-forms $\Omega^0(H) = H$. The following part deals with the construction of the spaces $\Omega^p(H)$ of p-forms for $p \geq 2$.

Constructing $\Omega^p(H)$ for $p \geq 2$

In Theorem 13 we stated that every differential calculus $\Omega^*(\mathcal{A})$ for an arbitrary algebra $\mathcal{A}$ can be obtained as the image of the universal calculus $\Omega_u(\mathcal{A})$ under a surjective algebra homomorphism φ. This homomorphism can be used to construct $\Omega^p(H)$ for $p \geq 2$. We will discuss the case $p = 2$, from which the general procedure can be inferred.

To this end, let H be a Hopf algebra and $\Omega^1(H)$ be a first-order calculus for H as in Theorem 18. Further, denote by $\varphi_1 = (\pi \otimes \mathrm{id}) \circ \psi$ the projection from $\Omega^1_u(H)$ onto $\Omega^1(H)$ (after having defined $\Omega^*(H)$, the projection φ_1 will be equal to the restriction of φ from Theorem 13 to $\Omega^1_u(H)$). Our goal is to find a map φ_2 playing the role of φ_1, that is, the restriction of φ to $\Omega^2_u(H)$; the 2-forms $\Omega^2(H)$ are then defined as the image of $\Omega^2_u(H)$ under φ_2. Further, we need to extend the definition of the exterior derivative d by a map $d_1 : \Omega^1(H) \longrightarrow \Omega^2(H)$, i.e., $d_1 = d|_{\Omega^1(H)}$. We collect these facts in the following diagram:

$$\begin{array}{ccc} \Omega^1_u(H) & \xrightarrow{d_u} & \Omega^2_u(H) \\ {\scriptstyle (\pi\otimes\mathrm{id})\circ\psi=\varphi_1}\downarrow & & \vdots\, {\scriptstyle \varphi_2} \\ \Omega^1(H) & \overset{d_1}{\cdots\cdots>} & \Omega^2(H) \end{array}$$

We make the ansatz

$$\Omega^2(H) := \Omega^2_u(H)/\mathcal{N}$$

where $\mathcal{N} \leq \Omega^2_u(H)$ is the submodule generated by $d_u \ker \varphi_1$. The map φ_2 is defined as the natural projection of this quotient. Furthermore, for $a \in \Omega^1(H)$ we set

$$d_1 a := \varphi_2(d_u a_u)$$

where $a_u \in \Omega^1_u(H)$ such that $a = \varphi_1(a_u)$ (remember that φ_1 is surjective, hence such an element always exists). Primarily, the choice for $\Omega^2(H)$ just ensures the natural requirement that $\varphi_2(d_u a) = 0$ for $a \in \ker \varphi_1$, that is, $d(\varphi_1(a)) = d0 = 0$. It turns out that this 'minimal' choice already suffices, as the derivative d_1 satisfies the Leibniz rule and ensures that $d^2|_H = 0$:

Proposition 19.

(i) $d_1(ab) = (d_1 a)b + a d_1 b$ *for* $a, b \in \Omega^1(H)$.

(ii) $d_1 \circ d = 0$.

Proof. (i) Let $a, b \in \Omega^1(H)$ and a_u, b_u such that $a = \varphi_1(a_u)$ and $b = \varphi_1(b_u)$. Since φ_2 is an algebra homomorphism with respect to the multiplication $*$ as in Definition 12, we also have $ab = \varphi_2(a_u b_u)$. We compute:

$$\begin{aligned} d_1(ab) &= \varphi_2(d_u(a_u b_u)) \\ &= \varphi_2((d_u a_u)b_u + a_u d_u b_u) \\ &= \varphi_2((d_u a_u)b_u) + \varphi_2(a_u d_u b_u) \\ &= \varphi_2(d_u a_u)b + a\varphi_2(d_u b_u) \\ &= (d_1 a)b + a d_1 b \end{aligned}$$

(ii) Let $h \in H$. By Theorem 13 and 18 we have $\varphi_1(d_u h) = dh$ and therefore

$$d_1(dh) = \varphi_2(d_u(d_u h)) = \varphi_2(0) = 0.$$ □

Using this procedure one can inductively define $\Omega^p(H)$ for $p \geq 2$ to arrive at the full differential calculus $\Omega^*(H)$.

$\mathbb{R}^3_\lambda$ as a quantum group

In order to apply the previous considerations to $\mathbb{R}^3_\lambda$ we need to establish its structure as a quantum group. To this end, we first introduce the universal enveloping algebra of a finite-dimensional Lie algebra:

Definition 20 (Universal enveloping algebra of a Lie algebra)**.** Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over the field k and consider the tensor algebra

$$\mathcal{T}(\mathfrak{g}) := \bigoplus_{n\in\mathbb{N}} \mathfrak{g}^{\otimes n} = k \oplus \mathfrak{g} \oplus (\mathfrak{g}\otimes\mathfrak{g}) \oplus (\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}) \oplus \ldots .$$

Further, let $\mathcal{J}$ be the two-sided ideal generated by the terms $X \otimes Y - Y \otimes X - [X, Y]$ for $X, Y \in \mathfrak{g}$. The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is defined as the quotient

$$\mathcal{U}(\mathfrak{g}) := \mathcal{T}(\mathfrak{g})/\mathcal{J}. \qquad \diamond$$

Remark. There is an obvious embedding $\iota : \mathfrak{g} \hookrightarrow \mathcal{U}(\mathfrak{g})$. Furthermore, the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is characterized by a universal property: for every algebra homomorphism $\varphi : \mathfrak{g} \longrightarrow \mathcal{A}$ where $\mathcal{A}$ is a unital associative algebra, there is a unique homomorphism $\tilde{\varphi} : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{A}$ such that $\varphi = \tilde{\varphi} \circ \iota$. $\diamond$

An important observation is that every universal enveloping algebra of a Lie algebra has the structure of a quantum group:

Proposition 21. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ can be regarded as a Hopf algebra by setting*

$$\begin{aligned}
\Delta(\xi) &= \xi \otimes 1 + 1 \otimes \xi \\
\epsilon(\xi) &= 0 \\
S(\xi) &= -\xi
\end{aligned}$$

for $\xi \in \mathfrak{g}$ and extending Δ, ϵ as algebra homomorphisms and S as an anti-algebra homomorphism to all of $\mathcal{U}(\mathfrak{g})$.

Proof. This is simply checked by evaluating the maps on products $\xi\eta$ and $\eta\xi$ for $\xi, \eta \in \mathfrak{g}$ and using the defining relations of $\mathcal{U}(\mathfrak{g})$. $\square$

Remark. An anti-algebra homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ between associative k-algebras $\mathcal{A}$ and $\mathcal{B}$ is a k-linear map $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ satisfying $\varphi(xy) = \varphi(y)\varphi(x)$ for $x, y \in \mathcal{A}$. ◇

Due to the commutation relations (1.21) for the generators $\hat{x}^i$, $i = 1, 2, 3$ of $\mathbb{R}^3_\lambda$, we immediately recognize $\mathbb{R}^3_\lambda$ as the universal enveloping algebra of the Lie algebra $\mathfrak{su}(2)$:

$$\mathbb{R}^3_\lambda \cong \mathcal{U}(\mathfrak{su}(2)) \tag{2.5}$$

This establishes the structure of $\mathbb{R}^3_\lambda$ as a quantum group, enabling us to apply Theorem 18. Let us compute the differential $d\xi = (\pi \otimes \mathrm{id})(\Delta\xi - 1 \otimes \xi)$ for $\xi \in \mathfrak{su}(2)$ using the definitions from Proposition 21:

$$\begin{aligned} d\xi &= (\pi \otimes \mathrm{id})(\Delta\xi - 1 \otimes \xi) \\ &= (\pi \otimes \mathrm{id})(\xi \otimes 1 + 1 \otimes \xi - 1 \otimes \xi) \\ &= \pi(\xi) \otimes 1 \end{aligned}$$

Recall that given a surjective representation $\rho : \ker \epsilon \longrightarrow M_n(\mathbb{C})$ the map π is just the canonical projection $\ker \epsilon \longrightarrow \Lambda^1 = \ker \epsilon / \ker \rho$. We have $\mathfrak{su}(2) \subset \ker \epsilon$ by the definition of ϵ in Proposition 21; furthermore, $\mathfrak{su}(2) \cap \ker \rho = \emptyset$, since ρ is surjective and in particular non-zero on basis elements of $\mathfrak{su}(2)$. Therefore, we can make the identifications $\pi(\xi) \equiv \rho(\xi)$ for $\xi \in \mathfrak{su}(2)$ and $\rho(\xi) \otimes 1 \equiv \rho(\xi)$ to obtain

$$d\xi = \lambda^{-1}\rho(\xi). \tag{2.6}$$

The factor λ^{-1} is introduced as a length scaling.

Consider further a 'group-like' element $\exp(i\xi)$ where $\xi \in \mathfrak{su}(2)$ and $\exp : \mathfrak{su}(2) \to SU(2)$ is the exponential map of the Lie algebra $\mathfrak{su}(2)$. The coproduct Δ on group-like elements $g \in \mathcal{U}(\mathfrak{su}(2))$ is defined as $\Delta g = g \otimes g$ (cf. [3, p. 12] and the example after Definition 17). Hence, the exterior derivative on group-like elements g is as follows (already including the length scaling λ^{-1}):

$$\begin{aligned} dg &= \lambda^{-1}(\pi \otimes \mathrm{id})(\Delta g - 1 \otimes g) \\ &= \lambda^{-1}(\pi \otimes \mathrm{id})(g \otimes g - 1 \otimes g) \\ &= \lambda^{-1}(\pi \otimes \mathrm{id})((g - 1) \otimes g) \\ &= \lambda^{-1}\pi(g - 1) \otimes g \end{aligned}$$

According to the reasoning above we can write

$$\pi(g-1)\otimes g \equiv \rho(g-1)\otimes g = (\rho(g)-\theta)\otimes g \equiv (\rho(g)-\theta)g$$

and arrive at

$$dg = \lambda^{-1}(\rho(g)-\theta)g. \tag{2.7}$$

2.2.4 A four-dimensional calculus on $\mathbb{R}^3_\lambda$

As we saw in Section 2.2.2, the construction of a differential calculus on $\mathbb{R}^3_\lambda$ amounts to choosing an ideal $\mathcal{I} \subset \ker\epsilon$ where $\epsilon : \mathbb{R}^3_\lambda \longrightarrow \mathbb{C}$ is the co-unit of $\mathbb{R}^3_\lambda$. To this end we fix a certain irreducible representation $\rho : \mathbb{R}^3_\lambda \longrightarrow \mathrm{End}(\mathbb{C}^2)$ whose restriction to $\ker\epsilon$ gives a surjective map onto $\mathrm{End}(\mathbb{C}^2) = M_2(\mathbb{C})$. The fundamental theorem on homomorphisms then guarantees that $M_2(\mathbb{C}) = \ker\epsilon/\ker\rho$, allowing us to identify the space of 1-forms with complex-valued 2×2 matrices. Finally, we set $\Omega^1(\mathbb{R}^3_\lambda) = M_2(\mathbb{C})\otimes\mathbb{R}^3_\lambda$. In the following we work out the details.

We define the representation ρ via its images of the generators $\hat{x}^i$ of $\mathbb{R}^3_\lambda$:

$$\rho(\hat{x}^1) = \frac{\lambda}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \rho(\hat{x}^2) = \frac{\lambda}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \qquad \rho(\hat{x}^3) = \frac{\lambda}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{2.8}$$

Note that these are essentially the Pauli matrices σ^i for $i = 1,2,3$. By (2.6) the coordinate differentials are given as

$$\begin{aligned} d\hat{x}^i &= \frac{1}{2}\sigma^i \\ \theta &= \mathrm{id}\,. \end{aligned} \tag{2.9}$$

The fourth differential θ serves to constitute a basis for $M_2(\mathbb{C})$ ($\dim M_2(\mathbb{C}) = 4$ as a vector space). For $\xi \in \mathfrak{su}(2)$ and $v \in \Omega^1(\mathbb{R}^3_\lambda)$ the commutation relations are given in [24, p. 148, Prop. 4.5 and proof] as

$$[\xi, v] = \rho(\xi)v. \tag{2.10}$$

The product on the right-hand side of the equation is the ordinary matrix product in $M_2(\mathbb{C})$; however, the entries in v might be elements in $\mathbb{R}^3_\lambda$, since $\Omega^1(\mathbb{R}^3_\lambda) \cong M_2(\mathbb{C})\otimes\mathbb{R}^3_\lambda$. We need to check that (2.10) is consistent with the definition (2.6) of the differential on

$\mathfrak{su}(2)$. To this end, let $\xi, \eta \in \mathfrak{su}(2)$:

$$\begin{aligned} d(\xi\eta) &= (d\xi)\eta + \xi d\eta \\ &= \lambda\rho(\xi)\eta + \lambda\xi\rho(\eta) \\ &= \lambda\rho(\xi)\eta + \lambda\rho(\eta)\xi + \lambda \underbrace{\rho(\xi)\rho(\eta)}_{=\rho(\xi\eta)} && \text{using (2.10)} \\ d(\eta\xi) &= \lambda\rho(\eta)\xi + \lambda\rho(\xi)\eta + \lambda\rho(\eta\xi) \end{aligned}$$

and hence,

$$\begin{aligned} d(\xi\eta - \eta\xi) &= \lambda(\rho(\xi\eta) - \rho(\eta\xi)) \\ &= \lambda\rho(\xi\eta - \eta\xi) \\ &= \lambda\rho([\xi, \eta]) \\ &= d([\xi, \eta]). \end{aligned}$$

Note that we used the fact that ρ is an algebra homomorphism twice in this computation. If (2.10) is written as $\xi v = v\xi + \rho(\xi)v$, it can also be interpreted as the definition of a left $\mathbb{R}^3_\lambda$-module structure of $\Omega^1(\mathbb{R}^3_\lambda)$; the term $v\xi$ is determined by the natural right $\mathbb{R}^3_\lambda$-module structure of $\Omega^1(\mathbb{R}^3_\lambda) \cong M_2(\mathbb{C}) \otimes \mathbb{R}^3_\lambda$ and $\rho(\xi)v$ is again the usual matrix product in $M_2(\mathbb{C})$.

The above discussion allows us to compute the commutation relations of the coordinates $\hat{x}^i$ and the coordinate differentials $\{d\hat{x}^i, \theta\}$:

$$\begin{aligned} [\hat{x}^i, d\hat{x}^j] &= \frac{\lambda}{4}\sigma^i\sigma^j \\ &= \frac{\lambda}{4}(\delta^{ij}\,\mathrm{id} + i\varepsilon^{ij}{}_k\sigma^k) \\ &= \frac{\lambda}{4}\delta^{ij} + \frac{\lambda}{2}i\varepsilon^{ij}{}_k d\hat{x}^k && \text{(2.11a)} \\ [\hat{x}^i, \theta] &= \frac{\lambda}{2}\sigma^i \\ &= \lambda d\hat{x}^i && \text{(2.11b)} \end{aligned}$$

After having defined $\Omega^1(\mathbb{R}^3_\lambda)$, the full calculus $\Omega^*(\mathbb{R}^3_\lambda)$ is now constructed along the lines of Section 2.2.3. We denote the exterior product by $\wedge$ as in the commutative case, since it can be shown (cf. [3, p. 16]) that on $\Omega^1(\mathbb{R}^3_\lambda)$ the exterior product $\wedge$ is in fact antisymmetric, that is, $a \wedge b = -b \wedge a$ for $a, b \in \Omega^1(\mathbb{R}^3_\lambda)$.

Using the commutation relations (2.11) we can compute the differential of the Casimir operator $\mathcal{C} := \sum_{i=1}^{3}(\hat{x}^i)^2$:

$$\begin{aligned} d\mathcal{C} &= \sum_{i=1}^{3}((d\hat{x}^i)\hat{x}^i + \hat{x}^i d\hat{x}^i) \\ &= 2\sum_{i=1}^{3}(d\hat{x}^i)\hat{x}^i + \frac{3\lambda}{4}\theta \end{aligned} \tag{2.12}$$

Thus, the differential of the Casimir operator becomes $d\mathcal{C} = 2\sum_{i=1}^{3}(dx^i)x^i$ in the commutative limit $\lambda \to 0$; the fourth basis differential θ, which constitutes the additional dimension in the four-dimensional calculus on $\mathbb{R}^3_\lambda$, disappears as $\lambda \to 0$.

An immediate application of the preceding discussion is the calculation of the derivative $d(\exp(ik\hat{x}))$ of plane waves. The result is stated in the following

Proposition 22.

$$d(e^{ik\hat{x}}) = \left(\frac{\theta}{\lambda}\left(\cos\left(\frac{\lambda|k|}{2}\right) - 1\right) + \frac{2i}{\lambda|k|}\sin\left(\frac{\lambda|k|}{2}\right) kd\hat{x}\right) e^{ik\hat{x}}$$

where $k\hat{x} \equiv k_i\hat{x}^i$ and $kd\hat{x} \equiv k_i d\hat{x}^i$.

Proof. We will employ the identity

$$e^{ia_i\sigma^i} = I_2\cos|a| + \frac{ia_i\sigma^i}{|a|}\sin|a| \tag{$*$}$$

where I_2 is the 2×2-identity matrix, $a \in \mathbb{R}^3$ and $|a| = \sqrt{a_i a^i}$. Since $e^{ik\hat{x}}$ is a group-like element in $\mathcal{U}(\mathfrak{su}(2))$, the formula (2.7) for the exterior derivative applies here. Further using (2.8) and the coordinate differentials from (2.9) we compute:

$$\begin{aligned} d(e^{ik\hat{x}}) &= \lambda^{-1}(\rho(e^{ik\hat{x}}) - \theta)e^{ik\hat{x}} \\ &= \lambda^{-1}(e^{ik_i\rho(\hat{x}^i)} - \theta)e^{ik\hat{x}} \\ &= \lambda^{-1}(e^{i\lambda k_i\sigma^i/2} - \theta)e^{ik\hat{x}} \\ &= \lambda^{-1}\left(\theta\cos\left(\frac{\lambda|k|}{2}\right) + \frac{i\lambda k_i\sigma^i}{2}\frac{2}{\lambda|k|}\sin\left(\frac{\lambda|k|}{2}\right) - \theta\right)e^{ik\hat{x}} \qquad \text{using } (*) \\ &= \left(\frac{\theta}{\lambda}\left(\cos\left(\frac{\lambda|k|}{2}\right) - 1\right) + \frac{2i}{\lambda|k|}\sin\left(\frac{\lambda|k|}{2}\right) kd\hat{x}\right) e^{ik\hat{x}} \qquad \square \end{aligned}$$

In order to obtain the commutative limit $\lambda \to 0$, we compute the following terms using l'Hôpital's rule:

$$\lim_{\lambda\to 0} \frac{\cos\left(\frac{\lambda|k|}{2}\right) - 1}{\frac{\lambda|k|}{2}} = -\lim_{\lambda\to 0} \sin\left(\frac{\lambda|k|}{2}\right) = 0$$
$$\lim_{\lambda\to 0} \frac{\sin\left(\frac{\lambda|k|}{2}\right)}{\frac{\lambda|k|}{2}} = \lim_{\lambda\to 0} \cos\left(\frac{\lambda|k|}{2}\right) = 1$$

Hence, $\lim_{\lambda\to 0} de^{ik\hat{x}} = ik(dx)e^{ikx}$, that is, we recover the well-known result of the ordinary three-dimensional differential calculus on commutative $\mathbb{R}^3$.

Hodge $*$-operator

In analogy to ordinary differential geometry one can define a Hodge $*$-operator for an n-dimensional differential calculus on a non-commutative manifold M by declaring its image on basis elements $e^{i_1} \wedge \cdots \wedge e^{i_k} \in \Omega^k(M)$:

$$*(e^{i_1} \wedge \cdots \wedge e^{i_k}) := \frac{1}{(n-k)!} \varepsilon^{i_1 \dots i_k i_{k+1} \dots i_n} \eta_{i_{k+1} j_1} \dots \eta_{i_n j_{n-k}} e^{j_1} \wedge \cdots \wedge e^{j_{n-k}}$$

where η is a non-degenerate metric on M with $\det(\eta) = 1$. In our case we choose[3] the Minkowskian metric

$$\eta = \sum_{i=1}^{3} d\hat{x}^i \otimes d\hat{x}^i - \theta \otimes \theta,$$

leading to

$$** |_{\Omega^k(M)} = (-1)^{1+k(4-k)}$$

as in the commutative setting.

The Hodge $*$-operator allows to form the coderivative $\delta = *d*$ and the Laplacian $\Delta = \delta d + d\delta$. However, we are more interested in the 'box' operator $\Box = \delta d = *d*d$ (as introduced in [3]) and its action on plane waves $e^{ik\hat{x}}$. To compute this action, we need to evaluate the hodge dual of the coordinate differentials $d\hat{x}^i$:

$$*d\hat{x}^i = \frac{1}{3!}(\varepsilon^{1abc} \eta_{aa'} \eta_{bb'} \eta_{cc'} d\hat{x}^{a'} \wedge d\hat{x}^{b'} \wedge d\hat{x}^{c'})$$

Let us start with $d\hat{x}^1$. Since one of the η factors in each summand is $\eta_{\theta\theta} = -1$, we get

[3] Another option is the Euclidean metric $\eta = \sum_{i=1}^{3} d\hat{x}^i \otimes d\hat{x}^i + \theta \otimes \theta$.

an overall minus sign:

$$\begin{aligned}*d\hat{x}^1 = -\frac{1}{6}(&d\hat{x}^2 \wedge d\hat{x}^3 \wedge \theta - d\hat{x}^3 \wedge d\hat{x}^2 \wedge \theta - d\hat{x}^2 \wedge \theta \wedge d\hat{x}^3 \\ &+ d\hat{x}^3 \wedge \theta \wedge d\hat{x}^2 - \theta \wedge d\hat{x}^3 \wedge d\hat{x}^2 + \theta \wedge d\hat{x}^2 \wedge d\hat{x}^3)\end{aligned}$$

Therefore,

$$*d\hat{x}^1 = -d\hat{x}^2 \wedge d\hat{x}^3 \wedge \theta \tag{2.13a}$$

after reordering the wedge-products while keeping track of the signs. In analogy,

$$*d\hat{x}^2 = d\hat{x}^1 \wedge d\hat{x}^3 \wedge \theta \tag{2.13b}$$

$$*d\hat{x}^3 = -d\hat{x}^1 \wedge d\hat{x}^2 \wedge \theta \tag{2.13c}$$

$$*\theta = -d\hat{x}^1 \wedge d\hat{x}^2 \wedge d\hat{x}^3. \tag{2.13d}$$

Furthermore, we trivially have

$$*(d\hat{x}^1 \wedge d\hat{x}^2 \wedge d\hat{x}^3 \wedge \theta) = 1$$

We are now able to compute $\Box e^{ik\hat{x}}$:

Proposition 23.

$$\Box e^{ik\hat{x}} = -\frac{1}{\lambda^2}\left(4\sin^2\left(\frac{\lambda|k|}{2}\right) + \left(\cos\left(\frac{\lambda|k|}{2}\right) - 1\right)^2\right)e^{ik\hat{x}}$$

Proof. We simply use the definition $\Box := *d * d$ of the box operator together with the result for $de^{ik\hat{x}}$ from Proposition 22:

$$\begin{aligned}\Box e^{ik\hat{x}} &= *d * de^{ik\hat{x}} \\ &= *d * \left(\frac{\theta}{\lambda}\left(\cos\left(\frac{\lambda|k|}{2}\right) - 1\right) + \frac{2i}{\lambda|k|}\sin\left(\frac{\lambda|k|}{2}\right) kd\hat{x}\right)e^{ik\hat{x}} && \text{by Prop. 22} \\ &= *d\Bigg[-\underbrace{\frac{d\hat{x}^1 \wedge d\hat{x}^2 \wedge d\hat{x}^3}{\lambda}}_{(*)}\left(\cos\left(\frac{\lambda|k|}{2}\right) - 1\right) + \frac{2i}{\lambda|k|}\sin\left(\frac{\lambda|k|}{2}\right) \\ &\quad \times (-k_1 d\hat{x}^2 \wedge d\hat{x}^3 \wedge \theta + k_2 d\hat{x}^1 \wedge d\hat{x}^3 \wedge \theta - k_3 d\hat{x}^1 \wedge d\hat{x}^2 \wedge \theta)\Bigg]e^{ik\hat{x}} && \text{by (2.13)}\end{aligned}$$

When applying d to the bracket, the term $(*)$ vanishes because of $d^2 = 0$. Furthermore,

$d\theta = 0$, that is, θ is closed. Thus, d only acts on $e^{ik\hat{x}}$:

$$
\begin{aligned}
\Box e^{ik\hat{x}} &= *\Bigg(\Bigg[-\frac{d\hat{x}^1\wedge d\hat{x}^2\wedge d\hat{x}^3}{\lambda}\left(\cos\left(\frac{\lambda|k|}{2}\right)-1\right)+\frac{2i}{\lambda|k|}\sin\left(\frac{\lambda|k|}{2}\right) \\
&\quad\times\left(-k_1 d\hat{x}^2\wedge d\hat{x}^3\wedge\theta + k_2 d\hat{x}^1\wedge d\hat{x}^3\wedge\theta - k_3 d\hat{x}^1\wedge d\hat{x}^2\wedge\theta\right)\Bigg] \\
&\quad\wedge\left[\frac{\theta}{\lambda}\left(\cos\left(\frac{\lambda|k|}{2}\right)-1\right)+\frac{2i}{\lambda|k|}\sin\left(\frac{\lambda|k|}{2}\right)kd\hat{x}\right]\Bigg)e^{ik\hat{x}}
\end{aligned}
$$

To evaluate the wedge product between the square brackets, we observe that $a\wedge a = 0$, so that we only have to keep track of terms of the form $d\hat{x}^1\wedge d\hat{x}^2\wedge d\hat{x}^3\wedge\theta$:

$$
\begin{aligned}
&= *\Bigg(-\frac{1}{\lambda^2}d\hat{x}^1\wedge d\hat{x}^2\wedge d\hat{x}^3\wedge\theta\left(\cos\left(\frac{\lambda|k|\lambda}{2}\right)-1\right)^2-\frac{4}{\lambda^2|k|^2}\sin^2\left(\frac{\lambda|k|}{2}\right) \\
&\quad\times\Big(k_1^2\underbrace{(-1)d\hat{x}^2\wedge d\hat{x}^3\wedge\theta\wedge d\hat{x}^1}_{(\heartsuit)}+k_2^2\underbrace{d\hat{x}^1\wedge d\hat{x}^3\wedge\theta\wedge d\hat{x}^2}_{(\clubsuit)} \\
&\quad+k_3^2\underbrace{(-1)d\hat{x}^1\wedge d\hat{x}^2\wedge\theta\wedge d\hat{x}^3}_{(\spadesuit)}\Big)\Bigg)e^{ikx}
\end{aligned}
$$

The terms ($\heartsuit$), ($\clubsuit$) and ($\spadesuit$) are all equal to $d\hat{x}^1\wedge d\hat{x}^2\wedge d\hat{x}^3\wedge\theta$ after changing order and keeping track of the correct sign. Applying the last Hodge $*$-operator via (2.13d) and rearranging the terms, we get the final result:

$$
\Box e^{ik\hat{x}} = -\frac{1}{\lambda^2}\left(4\sin^2\left(\frac{\lambda|k|}{2}\right)+\left(\cos\left(\frac{\lambda|k|}{2}\right)-1\right)^2\right)e^{ik\hat{x}} \qquad \Box
$$

Corollary 24. *In the commutative limit $\lambda\to 0$ the box operator $\Box$ reduces to the ordinary three-dimensional Laplace operator.*

Proof. This is checked by computing the commutative limit of the eigenvalues

$$
-\frac{1}{\lambda^2}\left(4\sin^2\left(\frac{\lambda|k|}{2}\right)+\left(\cos\left(\frac{\lambda|k|}{2}\right)-1\right)^2\right)
$$

of $\Box$ acting on plane waves $e^{ik\hat{x}}$. We start with the first term, using l'Hôpital's rule twice:

$$
\begin{aligned}
\lim_{\lambda\to 0} -\frac{4\sin^2\left(\frac{\lambda|k|}{2}\right)}{\lambda^2} &= \lim_{\lambda\to 0} -\frac{8\sin\left(\frac{\lambda|k|}{2}\right)\cos\left(\frac{\lambda|k|}{2}\right)\frac{|k|}{2}}{2\lambda} \\
&= \lim_{\lambda\to 0} -\frac{\sin(\lambda|k|)|k|}{\lambda}
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{\lambda\to 0} -\frac{\cos(\lambda|k|)|k|^2}{1} \\
&= -|k|^2
\end{aligned}
$$

Similarly for the second term,

$$
\begin{aligned}
\lim_{\lambda\to 0} -\frac{\left(\cos\left(\frac{\lambda|k|}{2}\right)-1\right)^2}{\lambda^2} &= \lim_{\lambda\to 0} -\frac{2\left(\cos\left(\frac{\lambda|k|}{2}\right)-1\right)\left(-\sin\left(\frac{\lambda|k|}{2}\right)\right)\frac{|k|}{2}}{2\lambda} \\
&= \lim_{\lambda\to 0} \frac{\frac{1}{2}\sin(\lambda|k|)-\sin\left(\frac{\lambda|k|}{2}\right)}{2\lambda}|k| \\
&= \lim_{\lambda\to 0} \frac{\frac{1}{2}\cos(\lambda|k|)-\cos\left(\frac{\lambda|k|}{2}\right)\frac{1}{2}}{2}|k|^2 \\
&= 0.
\end{aligned}
$$

In summary, we have

$$
\begin{aligned}
\lim_{\lambda\to 0} \Box e^{ik\hat{x}} &= \lim_{\lambda\to 0} -\frac{1}{\lambda^2}\left(4\sin^2\left(\frac{\lambda|k|}{2}\right)+\left(\cos\left(\frac{\lambda|k|}{2}\right)-1\right)^2\right)e^{ik\hat{x}} \\
&= -|k|^2 e^{ikx} \\
&= \Delta e^{ikx},
\end{aligned}
$$

which is the eigenvalue of the three-dimensional Laplace operator in commutative $\mathbb{R}^3$. □

Reduction to the fuzzy sphere

The four-dimensional calculus introduced on $\mathbb{R}^3_\lambda$ in the previous section can be reduced to the fuzzy sphere. We will see that we lose one direction along the way, making the reduced differential calculus on the fuzzy sphere three-dimensional.

Let us start by recalling the defining relation for the fuzzy sphere from (1.22) or (1.32):

$$
\mathcal{C} = \sum (\hat{x}^i)^2 = const.
$$

Using (2.12) for the differential of the Casimir we obtain

$$
d\mathcal{C} = 2\sum (d\hat{x}^a)\hat{x}^a + \frac{3}{4}\lambda\theta = 0.
$$

This means that the four differentials $d\hat{x}^i, i = 1,2,3$ and θ have become linearly dependent. For instance,

$$\theta = -\frac{8}{3}\lambda \sum (d\hat{x}^a)\hat{x}^a, \tag{2.14}$$

turning the obtained calculus into a three-dimensional one. The commutation relations (2.11) can be rewritten using (2.14):

$$[\hat{x}^i, d\hat{x}^j] = \frac{i}{2}\lambda \varepsilon^{ij}{}_k d\hat{x}^k - \frac{2}{3}\delta^{ij} \sum_k (d\hat{x}^k)\hat{x}^k \tag{2.15}$$

Since

$$\lim_{\lambda \to 0} d\mathcal{C} = 2 \sum (d\hat{x}^a)\hat{x}^a = 0,$$

we observe that in the commutative limit the three-dimensional calculus $d\hat{x}^i, i = 1, 2, 3$ reduces to the ordinary two-dimensional calculus on the sphere.

3 The non-commutative Coulomb problem on $\mathbb{R}^3_\lambda$

The previous chapters introduced the non-commutative space $\mathbb{R}^3_\lambda$ and illustrated the mathematical concepts behind it. In this chapter we turn to a prototypical quantum mechanical problem, the Coulomb problem or H-atom, which we want to describe on deformed $\mathbb{R}^3$.

The first section describes an explicit realization of the space $\mathbb{R}^3_\lambda$ via bosonic creation and annihilation operators, which is essentially identical to the approach in Section 1.2.1. It comprises a detailed discussion of the coordinate operators $\hat{x}^i$ and the angular momentum operators $\hat{L}^i$, including the identification of the eigenfunctions of the operators $\hat{L}^i$. They generate a Hilbert space in which a suitable Laplace operator can be defined; this is the subject of the next section. Finally, the Laplace operator and a certain potential operator are used to form the Hamiltonian of the Coulomb problem in $\mathbb{R}^3_\lambda$. We investigate the spectrum of the Hamiltonian and compare it to the energy levels of the commutative problem.

The discussion follows [11], however using a different method to find the eigenvalues of the Hamiltonian. Furthermore, detailed calculations are always included in order to accommodate readers who are not familiar with the subject.

3.1 Realization of $\mathbb{R}^3_\lambda$

3.1.1 Coordinate operators

The non-commutative coordinates $\hat{x}^i$, $i = 1, 2, 3$, of the deformed space $\mathbb{R}^3_\lambda$ are defined via a set of two bosonic creation and annihilation operators $\hat{a}_\alpha$, $\alpha = 1, 2$ satisfying the canonical commutation relations (1.12); as in Section 1.1.3 they act in the Fock space (1.13). The coordinate operators $\hat{x}^i$ are realized by setting

$$\hat{x}^i := \lambda \sigma^i_{\alpha\beta} \hat{a}^\dagger_\alpha \hat{a}_\beta \equiv \lambda \hat{a}^\dagger \sigma^i \hat{a} \quad \text{for} \quad i = 1, 2, 3. \tag{3.1}$$

Here, σ^i are the usual Pauli matrices. The parameter λ has dimension length and measures the non-commutativity of the space as we will see in the course of this section. From (1.12) we derive commutation relations for the coordinates $\hat{x}^i$:

$$\begin{aligned}
\frac{1}{\lambda^2}[\hat{x}^i, \hat{x}^j] &= [\sigma^i_{\alpha\beta}\hat{a}^\dagger_\alpha \hat{a}_\beta, \sigma^j_{\gamma\delta}\hat{a}^\dagger_\gamma \hat{a}_\delta] \\
&= \sigma^i_{\alpha\beta}\sigma^j_{\gamma\delta}[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma \hat{a}_\delta] \\
&= \sigma^i_{\alpha\beta}\sigma^j_{\gamma\delta}(\hat{a}^\dagger_\alpha \underbrace{[\hat{a}_\beta, \hat{a}^\dagger_\gamma]}_{=\delta_{\beta\gamma}} \hat{a}_\delta + \hat{a}^\dagger_\gamma \underbrace{[\hat{a}^\dagger_\alpha, \hat{a}_\delta]}_{=-\delta_{\alpha\delta}} \hat{a}_\beta) && \text{by (3.2) and (1.12)} \\
&= \sigma^i_{\alpha\beta}\sigma^j_{\beta\delta}\hat{a}^\dagger_\alpha \hat{a}_\delta - \sigma^i_{\alpha\beta}\sigma^j_{\gamma\alpha}\hat{a}^\dagger_\gamma \hat{a}_\beta \\
&= (\sigma^i\sigma^j - \sigma^j\sigma^i)_{\alpha\delta}\hat{a}^\dagger_\alpha \hat{a}_\delta \\
&= [\sigma^i, \sigma^j]_{\alpha\delta}\hat{a}^\dagger_\alpha \hat{a}_\delta \\
&= 2i\varepsilon^{ij}{}_k \sigma^k_{\alpha\delta}\hat{a}^\dagger_\alpha \hat{a}_\delta && \text{by (3.3)} \\
&= \frac{2i}{\lambda}\varepsilon^{ij}{}_k \hat{x}^k
\end{aligned}$$

In the derivation we used the commutator identity

$$[AB, CD] = A[B, C]D + AC[B, D] + [A, C]DB + C[A, D]B \tag{3.2}$$

and the well-known commutation relation for the Pauli matrices

$$[\sigma^i, \sigma^j] = 2i\varepsilon^{ij}{}_k \sigma^k. \tag{3.3}$$

In summary, the commutation relation for the coordinates $\hat{x}^i$ reads

$$[\hat{x}^i, \hat{x}^j] = 2i\lambda\varepsilon^{ij}{}_k \hat{x}^k. \tag{3.4}$$

Relation (3.4) suggests that the coordinates $\hat{x}^i$ define an irreducible $SU(2)$-representation on the Fock space $\mathcal{F}_N$. Let us verify this by computing the action of the Casimir operator $\mathcal{X}^2 := \sum_i (\hat{x}^i)^2$ on $\mathcal{F}_N$. For this purpose it is advisable to write out the coordinates $\hat{x}^i$ explicitly using the standard two-dimensional representation of the Pauli matrices:

$$\hat{x}^1 = \lambda(\hat{a}^\dagger_1 \hat{a}_2 + \hat{a}^\dagger_2 \hat{a}_1) \tag{3.5a}$$

$$\hat{x}^2 = i\lambda(\hat{a}^\dagger_2 \hat{a}_1 - \hat{a}^\dagger_1 \hat{a}_2) \tag{3.5b}$$

$$\hat{x}^3 = \lambda(\hat{a}^\dagger_1 \hat{a}_1 - \hat{a}^\dagger_2 \hat{a}_2) \tag{3.5c}$$

Further, we define the the number operator[1]

$$\hat{N} := \hat{a}^\dagger_\alpha \hat{a}_\alpha.$$

Now we are ready to compute the action of $\mathcal{X}^2$ on $\mathcal{F}_N$. Note that the commutation relations (1.12) are deployed numerous times and their use is not stated explicitly in order to retain a compact computation.

$$\begin{aligned}
\frac{1}{\lambda^2}\mathcal{X}^2 &= (\hat{a}^\dagger_1\hat{a}_2 + \hat{a}^\dagger_2\hat{a}_1)^2 - (\hat{a}^\dagger_2\hat{a}_1 - \hat{a}^\dagger_1\hat{a}_2)^2 + (\hat{a}^\dagger_1\hat{a}_1 - \hat{a}^\dagger_2\hat{a}_2)^2 \\
&= \hat{a}^\dagger_1\hat{a}_2\hat{a}^\dagger_1\hat{a}_2 + \hat{a}^\dagger_1\hat{a}_2\hat{a}^\dagger_2\hat{a}_1 + \hat{a}^\dagger_2\hat{a}_1\hat{a}^\dagger_1\hat{a}_2 + \hat{a}^\dagger_2\hat{a}_1\hat{a}^\dagger_2\hat{a}_1 - \hat{a}^\dagger_2\hat{a}_1\hat{a}^\dagger_2\hat{a}_1 + \hat{a}^\dagger_2\hat{a}_1\hat{a}^\dagger_1\hat{a}_2 \\
&\quad + \hat{a}^\dagger_1\hat{a}_2\hat{a}^\dagger_2\hat{a}_1 - \hat{a}^\dagger_1\hat{a}_2\hat{a}^\dagger_1\hat{a}_2 + \hat{a}^\dagger_1\hat{a}_1\hat{a}^\dagger_1\hat{a}_1 - \hat{a}^\dagger_1\hat{a}_1\hat{a}^\dagger_2\hat{a}_2 - \hat{a}^\dagger_2\hat{a}_2\hat{a}^\dagger_1\hat{a}_1 + \hat{a}^\dagger_2\hat{a}_2\hat{a}^\dagger_2\hat{a}_2 \\
&= 2\hat{a}^\dagger_1\hat{a}_2\hat{a}^\dagger_2\hat{a}_1 + 2\hat{a}^\dagger_2\hat{a}_1\hat{a}^\dagger_1\hat{a}_2 + \hat{a}^\dagger_1\hat{a}_1\hat{a}^\dagger_1\hat{a}_1 + \hat{a}^\dagger_2\hat{a}_2\hat{a}^\dagger_2\hat{a}_2 - \hat{a}^\dagger_1\hat{a}_1\hat{a}^\dagger_2\hat{a}_2 - \hat{a}^\dagger_2\hat{a}_2\hat{a}^\dagger_1\hat{a}_1 \\
&= 2\hat{a}^\dagger_1\hat{a}_2\hat{a}^\dagger_2\hat{a}_1 + 2\hat{a}^\dagger_2\hat{a}_1\hat{a}^\dagger_1\hat{a}_2 + \hat{a}^\dagger_1\hat{a}_1\hat{a}^\dagger_1\hat{a}_1 + \hat{a}^\dagger_2\hat{a}_2\hat{a}^\dagger_2\hat{a}_2 \\
&\quad + \hat{a}^\dagger_1\hat{a}_1 - \hat{a}^\dagger_1\hat{a}_1\hat{a}_2\hat{a}^\dagger_2 + \hat{a}^\dagger_2\hat{a}_2 - \hat{a}^\dagger_2\hat{a}_2\hat{a}_1\hat{a}^\dagger_1 \\
&= \hat{a}^\dagger_1\hat{a}_1\hat{a}^\dagger_1\hat{a}_1 + \hat{a}^\dagger_2\hat{a}_2\hat{a}^\dagger_2\hat{a}_2 + \underbrace{\hat{a}^\dagger_1\hat{a}_1 + \hat{a}^\dagger_2\hat{a}_2}_{=\hat{N}} + \hat{a}^\dagger_1\hat{a}_2\hat{a}^\dagger_2\hat{a}_1 + \hat{a}^\dagger_2\hat{a}_2\hat{a}_1\hat{a}^\dagger_1 \\
&= \hat{a}^\dagger_1\hat{a}_1(\hat{a}^\dagger_1\hat{a}_1 + \hat{a}_2\hat{a}^\dagger_2) + \hat{a}^\dagger_2\hat{a}_2(\hat{a}^\dagger_2\hat{a}_2 + \hat{a}_1\hat{a}^\dagger_1) + \hat{N} \\
&= \hat{a}^\dagger_1\hat{a}_1(\underbrace{\hat{a}^\dagger_1\hat{a}_1 + \hat{a}^\dagger_2\hat{a}_2}_{=\hat{N}}) + \hat{a}^\dagger_1\hat{a}_1 + \hat{a}^\dagger_2\hat{a}_2(\underbrace{\hat{a}^\dagger_2\hat{a}_2 + \hat{a}^\dagger_1\hat{a}_1}_{=\hat{N}}) + \hat{a}^\dagger_2\hat{a}_2 + \hat{N} \\
&= \hat{N}^2 + 2\hat{N}
\end{aligned}$$

Hence,

$$\mathcal{X}^2 \,|_{\mathcal{F}_N} = \lambda^2(N^2 + 2N)\,\mathrm{id}_{\mathcal{F}_N}, \tag{3.6}$$

i.e., the modified coordinates $\hat{y}^i := \frac{\hat{x}^i}{2\lambda}$ define a spin-$N/2$ representation of $SU(2)$ on $\mathcal{F}_N$.

Introducing the radial operator $\hat{\rho} := \lambda\hat{N}$, we compute

$$\begin{aligned}
\frac{1}{\lambda^2}[\hat{x}^i, \rho] &= [\sigma^i_{\alpha\beta}\hat{a}^\dagger_\alpha\hat{a}_\beta, \hat{a}^\dagger_\gamma\hat{a}_\gamma] \\
&= \sigma^i_{\alpha\beta}[\hat{a}^\dagger_\alpha\hat{a}_\beta, \hat{a}^\dagger_\gamma\hat{a}_\gamma] \\
&= \sigma^i_{\alpha\beta}(\hat{a}^\dagger_\alpha[\hat{a}_\beta, \hat{a}^\dagger_\gamma]\hat{a}_\gamma + \hat{a}^\dagger_\gamma[\hat{a}^\dagger_\alpha, \hat{a}_\gamma]\hat{a}_\beta) && \text{by (3.2)} \\
&= \sigma^i_{\alpha\beta}(\hat{a}^\dagger_\alpha\hat{a}_\beta - \hat{a}^\dagger_\alpha\hat{a}_\beta) && \text{by (1.12)}
\end{aligned}$$

[1] As the name suggests, the number operator satisfies $\hat{N}|n_1, n_2\rangle = (n_1 + n_2)|n_1, n_2\rangle$, which is shown in the proof of Lemma 28.

$$= 0.$$

Hence, the coordinates $\hat{x}^i$ commute with the radial variable $\hat{\rho}$:

$$[\hat{x}^i, \hat{\rho}] = 0 \tag{3.7}$$

However, $\hat{\rho}$ should not directly be interpreted as a radius in the non-commutative space. This can be seen by computing the quantity $\hat{\rho}^2 - \mathcal{X}^2$, which should have dimension (length)^2, i.e., λ^2 according to the choice of our coordinates in (3.1). Using the result of the computation which led to (3.6) we find

$$\hat{\rho}^2 - \mathcal{X}^2 = \lambda^2(\hat{N}^2 - \hat{N}^2 - 2\hat{N}) = -2\lambda^2\hat{N}. \tag{3.8}$$

In order to get rid of the number operator $\hat{N}$ in expression (3.8) we introduce a modified radial operator $\hat{r}$ as follows (here, $1 \equiv \mathrm{id}_{\mathcal{F}}$, the identity operator in $\mathcal{F}$):

$$\hat{r} := \lambda(\hat{N} + 1) \tag{3.9}$$

Revisiting (3.8) we arrive at

$$\hat{r}^2 - \mathcal{X}^2 = \lambda^2(\hat{N}^2 + 2\hat{N} + 1 - \hat{N}^2 - 2\hat{N}) = \lambda^2, \tag{3.10}$$

having successfully discarded the number operator $\hat{N}$ and ensuring the correct dimensionality of the expression $\hat{r}^2 - \mathcal{X}^2$. Furthermore, the operator $\hat{r}$ will play a crucial role in defining the Laplace operator in the next section.

3.1.2 Angular momentum operators

The angular momentum operators $\hat{L}^i$, $i = 1, 2, 3$, serving as the generators of rotations (i.e., generators of the Lie algebra $\mathfrak{su}(2)$) are defined by their action on the vector space $\mathcal{H}_j$ of normal ordered polynomials in $\hat{a}_\alpha$ and $\hat{a}^\dagger_\beta$ having the same number of creation and annihilation operators:

$$\mathcal{H}_j := \mathrm{span}\left((\hat{a}_1^\dagger)^{m_1}(\hat{a}_2^\dagger)^{m_2}(\hat{a}_1)^{n_1}(\hat{a}_2)^{n_2} \;\middle|\; m_1 + m_2 = j = n_1 + n_2\right) \tag{3.11}$$

Note that $\hat{\Psi} \in \mathcal{H}_j$ leaves the particle number in the Fock space $\mathcal{F}$ invariant, i.e., $\hat{\Psi}(\mathcal{F}_N) \subset \mathcal{F}_N$ where $\mathcal{F}_N := \mathrm{span}(|n_1, n_2\rangle \mid n_1 + n_2 = N)$ and $j \leq N$. Furthermore, we have $\hat{\Psi}\,|_{\mathcal{F}_N} = 0$ for $\hat{\Psi} \in \mathcal{H}_j$ if $j > N$.

We can now define the angular momentum operators L^i:

$$\hat{L}^i\hat{\Psi} := \frac{1}{2}[\sigma^i_{\alpha\beta}\hat{a}^\dagger_\alpha\hat{a}_\beta, \hat{\Psi}] = \frac{1}{2\lambda}[\hat{x}^i, \hat{\Psi}] \qquad \text{for } i = 1,2,3 \text{ and } \hat{\Psi} \in \mathcal{H}_j \tag{3.12}$$

Computing their commutation relation shows that the L^i can be identified with generators of rotations. Let us first evaluate $\hat{L}^i(\hat{L}^j\hat{\Psi})$ for $\hat{\Psi} \in \mathcal{H}_j$:

$$\begin{aligned}
4\lambda^2\hat{L}^i(\hat{L}^j\hat{\Psi}) &= 4\lambda^2\hat{L}^i\left(\frac{1}{2\lambda}[\hat{x}^j, \hat{\Psi}]\right) \\
&= [\hat{x}^i, [\hat{x}^j, \hat{\Psi}]] \\
&= -[\hat{x}^j, [\hat{\Psi}, \hat{x}^i]] - [\hat{\Psi}, [\hat{x}^i, \hat{x}^j]] && \text{by (3.13)} \\
&= [\hat{x}^j, [\hat{x}^i, \hat{\Psi}]] - 2i\lambda\varepsilon^{ij}{}_k[\hat{\Psi}, \hat{x}^k] \\
&= [\hat{x}^j, [\hat{x}^i, \hat{\Psi}]] + 2i\lambda\varepsilon^{ij}{}_k[\hat{x}^k, \hat{\Psi}] \\
&= 2\lambda[\hat{x}^j, \hat{L}^i\hat{\Psi}] + 4i\lambda^2\varepsilon^{ij}{}_k\hat{L}^k\hat{\Psi} && \text{by (3.12)} \\
&= 4\lambda^2\hat{L}^j(\hat{L}^i\hat{\Psi}) + 4i\lambda^2\varepsilon^{ij}{}_k\hat{L}^k\hat{\Psi} && \text{by (3.12)}
\end{aligned}$$

Note that we used Jacobi's identity

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \tag{3.13}$$

in the derivation. Hence, the commutation relation reads

$$[\hat{L}^i, \hat{L}^j] = i\varepsilon^{ij}{}_k\hat{L}^k. \tag{3.14}$$

Let us also record the transformation properties of the ladder operators and the coordinates under the $\mathfrak{su}(2)$-rotations $\hat{L}^i$:

$$\begin{aligned}
\hat{L}^i\hat{a}_\alpha &= \frac{1}{2}[\sigma^i_{\beta\gamma}\hat{a}^\dagger_\beta\hat{a}_\gamma, \hat{a}_\alpha] \\
&= \frac{1}{2}\sigma^i_{\beta\gamma}[\hat{a}^\dagger_\beta, \hat{a}_\alpha]\hat{a}_\gamma \\
&= -\frac{1}{2}\sigma^i_{\beta\gamma}\delta_{\beta\alpha}\hat{a}_\gamma && \Rightarrow \hat{L}^i\hat{a}_\alpha = -\frac{1}{2}\sigma^i_{\alpha\gamma}\hat{a}_\gamma && \text{(3.15a)} \\
\hat{L}^i\hat{a}^\dagger_\alpha &= \frac{1}{2}[\sigma^i_{\beta\gamma}\hat{a}^\dagger_\beta\hat{a}_\gamma, \hat{a}^\dagger_\alpha] \\
&= \frac{1}{2}\sigma^i_{\beta\gamma}\hat{a}^\dagger_\beta\delta_{\gamma\alpha} && \Rightarrow \hat{L}^i\hat{a}^\dagger_\alpha = \frac{1}{2}\sigma^i_{\beta\alpha}\hat{a}^\dagger_\beta && \text{(3.15b)} \\
\hat{L}^i\hat{x}^j &= \frac{1}{2\lambda}[\hat{x}^i, \hat{x}^j] && \Rightarrow \hat{L}^i\hat{x}^j = i\varepsilon^{ij}{}_k\hat{x}^k && \text{(3.15c)}
\end{aligned}$$

In analogy to quantum mechanics, we investigate the spectrum of the commuting operators $\mathcal{L}^2 := \sum_i (\hat{L}^i)^2$ and $\hat{L}^3$. Their eigenfunctions[2] for $j \in \mathbb{N}$ and $m = -j, \dots, j$ are given as

$$\hat{\Psi}_{jm} = \lambda^j \sum_{m_1,m_2,n_1,n_2} \frac{(\hat{a}_1^\dagger)^{m_1}(\hat{a}_2^\dagger)^{m_2}}{m_1!m_2!} : R_j(\hat{\rho}) : \frac{(\hat{a}_1)^{n_1}(-\hat{a}_2)^{n_2}}{n_1!n_2!} \tag{3.16}$$

where the range of the integers m_1, m_2, n_1 and n_2 is restricted by the conditions

$$m_1 + m_2 = n_1 + n_2 = j \qquad\qquad m_1 - m_2 - n_1 + n_2 = 2m \tag{3.17}$$

and R_j is an analytic function in $\hat{\rho}$. As stated before, $\hat{\Psi}_{jm}\mid_{\mathcal{F}_N} = 0$ if $j > N$. We need to verify that the $\hat{\Psi}_{jm}$ are indeed eigenfunctions of $\mathcal{L}^2$ and $\hat{L}^3$. To this end the following two Lemmata are useful:

Lemma 25. $\hat{L}^3 : R_j(\hat{\rho}) := 0$ *for analytic functions* R_j.

Proof. Let $R_j(\hat{\rho}) = \sum_{k=0}^{\infty} c_k \hat{\rho}^k$. We prove the claim separately for each term $: \hat{\rho}^k :$ and use induction over k. Relation (3.7) implies that $\hat{L}^3\hat{\rho} = 0$ (note that $\hat{\rho}$ is already normal ordered), so suppose the claim is true for $k-1$. Since the $\hat{L}^i$ are defined via the commutators $[\hat{x}^i, .]$, we have to use the Leibniz rule when applying $\hat{L}^3$ to products of operators.

$$\begin{aligned}
\hat{L}^3 : \hat{\rho}^k : &= \lambda^k \hat{L}^3 \Big(\hat{a}^\dagger_{\alpha_1} \dots \hat{a}^\dagger_{\alpha_k} \underbrace{\hat{a}_{\alpha_1} \dots \hat{a}_{\alpha_k}}_{=\hat{a}_{\alpha_2}\dots\hat{a}_{\alpha_k}\hat{a}_{\alpha_1}} \Big) \\
&= \lambda^k \Big(\hat{L}^3(\hat{a}^\dagger_{\alpha_1}) \hat{a}^\dagger_{\alpha_2} \dots \hat{a}^\dagger_{\alpha_k} \hat{a}_{\alpha_2} \dots \hat{a}_{\alpha_k} + \hat{a}^\dagger_{\alpha_1} \underbrace{\hat{L}^3 \Big(\hat{a}^\dagger_{\alpha_2} \dots \hat{a}^\dagger_{\alpha_k} \hat{a}_{\alpha_2} \dots \hat{a}_{\alpha_k} \Big)}_{=\hat{L}^3 : \hat{\rho}^{k-1} := 0} \hat{a}_{\alpha_1} \\
&\quad + \hat{a}^\dagger_{\alpha_1} \hat{a}^\dagger_{\alpha_2} \dots \hat{a}^\dagger_{\alpha_k} \hat{a}_{\alpha_2} \dots \hat{a}_{\alpha_k} \hat{L}^3 (\hat{a}_{\alpha_1}) \Big) \\
&= \lambda^k \Big(\hat{L}^3(\hat{a}^\dagger_{\alpha_1}) : \hat{\rho}^{k-1} : \hat{a}_{\alpha_1} + \hat{a}^\dagger_{\alpha_1} : \hat{\rho}^{k-1} : \hat{L}^3 \hat{a}_{\alpha_1} \Big)
\end{aligned}$$

We rewrite (3.15a) and (3.15b) as

$$\begin{aligned}
\hat{L}^3 \hat{a}^\dagger_{\alpha_1} &= \frac{1}{2} \sigma^3_{\beta\alpha_1} \hat{a}^\dagger_\beta = \frac{1}{2}(-1)^{\alpha_1+1} \hat{a}^\dagger_{\alpha_1} \\
\hat{L}^3 \hat{a}_{\alpha_1} &= -\frac{1}{2} \sigma^3_{\beta\alpha_1} \hat{a}_\beta = -\frac{1}{2}(-1)^{\alpha_1+1} \hat{a}_{\alpha_1}
\end{aligned}$$

[2] Note that we actually mean operators acting on $\mathcal{H}_j$ here. However, we will continue to use the term eigenfunction.

to get

$$\hat{L}^3 : \hat{\rho}^k := \frac{(-1)^{\alpha_1+1}}{2}\lambda^k \left(\hat{a}^\dagger_{\alpha_1} : \hat{\rho}^{k-1} : \hat{a}_{\alpha_1} - \hat{a}^\dagger_{\alpha_1} : \hat{\rho}^{k-1} : \hat{a}_{\alpha_1}\right) = 0. \qquad \square$$

Lemma 26.

(i) *The ladder operators 'differentiate' powers of themselves:*

$$[\hat{a}_\alpha, (\hat{a}^\dagger_\beta)^n] = n\delta_{\alpha\beta}(\hat{a}^\dagger_\beta)^{n-1} \qquad [\hat{a}^\dagger_\alpha, (\hat{a}_\beta)^n] = -n\delta_{\alpha\beta}(\hat{a}_\beta)^{n-1}$$

(ii) *Let* $R_j(\hat{\rho}) = \sum_{k=0}^\infty c_k\hat{\rho}^k$ *be an analytic function and* $\partial_{\hat{N}} R_j(\hat{\rho}) := \sum_{k=1}^\infty kc_k\lambda^k\hat{N}^{k-1}$ *be its formal derivative with respect to* $\hat{N} = \frac{\hat{\rho}}{\lambda}$*, then*

$$[\hat{a}_\alpha, : R_j(\hat{\rho}) :] =: \partial_{\hat{N}} R_j(\hat{\rho}) : \hat{a}_\alpha \qquad [\hat{a}^\dagger_\alpha, : R_j(\hat{\rho}) :] = -\hat{a}^\dagger_\alpha : \partial_{\hat{N}} R_j(\hat{\rho}) : .$$

Proof. (i)

$$\begin{aligned}
\hat{a}_\alpha(a^\dagger_\beta)^n &= \hat{a}^\dagger_1\hat{a}_\alpha(\hat{a}^\dagger_\beta)^{n-1} + \delta_{\alpha\beta}(\hat{a}^\dagger_\beta)^{n-1} \\
&= (\hat{a}^\dagger_\beta)^2\hat{a}_\alpha(\hat{a}^\dagger_\beta)^{n-2} + \delta_{\alpha\beta}(\hat{a}^\dagger_\beta)^{n-1} + \delta_{\alpha\beta}(\hat{a}^\dagger_\beta)^{n-1} \\
&\vdots \\
&= (\hat{a}^\dagger_\beta)^n\hat{a}_\alpha + n\delta_{\alpha\beta}(\hat{a}^\dagger_\beta)^{n-1}
\end{aligned}$$

and hence, $[\hat{a}_\alpha, (\hat{a}^\dagger_\beta)^n] = n\delta_{\alpha\beta}(\hat{a}^\dagger_\beta)^{n-1}$. The second commutator can be obtained by an entirely analogous calculation.

(ii) A square bracket under an operator means that it is dropped in the summation.

$$\begin{aligned}
[\hat{a}^\dagger_\alpha, : \hat{N}^k :] &= [\hat{a}^\dagger_\alpha, \hat{a}^\dagger_{\alpha_1} \dots \hat{a}^\dagger_{\alpha_k}\hat{a}_{\alpha_1} \dots \hat{a}_{\alpha_k}] \\
&= \hat{a}^\dagger_{\alpha_1} \dots \hat{a}^\dagger_{\alpha_k}[\hat{a}^\dagger_\alpha, \hat{a}_{\alpha_1} \dots \hat{a}_{\alpha_k}] \\
&= -\hat{a}^\dagger_{\alpha_1} \dots \hat{a}^\dagger_{\alpha_k}\sum_{j=1}^k \delta_{\alpha\alpha_j}\hat{a}_{\alpha_1} \dots \underbrace{\hat{a}_{\alpha_j}} \dots \hat{a}_{\alpha_k} \\
&= -k\hat{a}^\dagger_\alpha : \hat{N}^{k-1} :
\end{aligned}$$

and similarly

$$[\hat{a}_\alpha, : \hat{N}^k :] = k : \hat{N}^{k-1} : \hat{a}_\alpha.$$

Using $R_j(\hat{\rho}) = \sum_{k=0}^\infty c_k\hat{\rho}^k$ and $\partial_{\hat{N}} R_j(\hat{\rho}) := \sum_{k=1}^\infty kc_k\lambda^k\hat{N}^{k-1}$ now proves the claim. $\square$

These results enable us to prove the following

Proposition 27. *Given $\hat{\Psi}_{jm}$ as in* (3.16)*, we have for $j \in \mathbb{N}$ and $m = -j, \dots, j$:*

(i) $\hat{L}^3 \hat{\Psi}_{jm} = m \hat{\Psi}_{jm}$

(ii) $\mathcal{L}^2 \hat{\Psi}_{jm} = j(j+1) \hat{\Psi}_{jm}$

Proof. (i)

$$\begin{aligned}
\hat{L}^3 \Psi_{jm} &= \lambda^j \sum \hat{L}^3 \left(\frac{(\hat{a}_1^\dagger)^{m_1} (\hat{a}_2^\dagger)^{m_2}}{m_1! m_2!} : R_j(\hat{\rho}) : \frac{(\hat{a}_1)^{n_1} (-\hat{a}_2)^{n_2}}{n_1! n_2!} \right) \\
&= \lambda^j \sum \underbrace{\hat{L}^3 \left(\frac{(\hat{a}_1^\dagger)^{m_1} (\hat{a}_2^\dagger)^{m_2}}{m_1! m_2!} \right)}_{(*)} : R_j(\hat{\rho}) : \frac{(\hat{a}_1)^{n_1} (-\hat{a}_2)^{n_2}}{n_1! n_2!} \\
&\qquad + \frac{(\hat{a}_1^\dagger)^{m_1} (\hat{a}_2^\dagger)^{m_2}}{m_1! m_2!} : R_j(\hat{\rho}) : \underbrace{\hat{L}^3 \left(\frac{(\hat{a}_1)^{n_1} (-\hat{a}_2)^{n_2}}{n_1! n_2!} \right)}_{(**)} \qquad \text{by Lemma 25}
\end{aligned}$$

Keeping in my mind that by (3.15a) we have $\hat{L}^3 \hat{a}_1^\dagger = \frac{1}{2} \hat{a}_1^\dagger$ and $\hat{L}^3 \hat{a}_2^\dagger = -\frac{1}{2} \hat{a}_2^\dagger$, we get for the term $(*)$:

$$\begin{aligned}
\hat{L}^3 \left((\hat{a}_1^\dagger)^{m_1} (\hat{a}_2^\dagger)^{m_2} \right) &= \hat{L}^3 \left((\hat{a}_1^\dagger)^{m_1} \right) (\hat{a}_2^\dagger)^{m_2} + (\hat{a}_1^\dagger)^{m_1} \hat{L}^3 \left((\hat{a}_2^\dagger)^{m_2} \right) \\
&= m_1 (\hat{a}_1^\dagger)^{m_1 - 1} \left(\hat{L}^3 \hat{a}_1^\dagger \right) (\hat{a}_2^\dagger)^{m_2} + m_2 (\hat{a}_1^\dagger)^{m_1} (\hat{a}_2^\dagger)^{m_2 - 1} \hat{L}^3 \hat{a}_2^\dagger \\
&= \frac{1}{2} \left(m_1 (\hat{a}_1^\dagger)^{m_1} (\hat{a}_2^\dagger)^{m_2} - m_2 (\hat{a}_1^\dagger)^{m_1} (\hat{a}_2^\dagger)^{m_2} \right)
\end{aligned}$$

For the term $(**)$ we note that (3.15b) implies $\hat{L}^3 \hat{a}_1 = -\frac{1}{2} \hat{a}_1$ and $\hat{L}^3 \hat{a}_2 = \frac{1}{2} \hat{a}_2$. A similar calculation then leads to

$$\hat{L}^3 \Big((\hat{a}_1)^{n_1} (-\hat{a}_2)^{n_2} \Big) = \frac{1}{2} \Big(- n_1 (\hat{a}_1)^{n_1} (-\hat{a}_2)^{n_2} + n_2 (\hat{a}_1)^{n_1} (-\hat{a}_2)^{n_2} \Big).$$

Hence

$$\begin{aligned}
\hat{L}^3 \hat{\Psi}_{jm} &= \lambda^j \sum \underbrace{\frac{m_1 - m_2 - n_1 + n_2}{2}}_{=m} \frac{(\hat{a}_1^\dagger)^{m_1} (\hat{a}_2^\dagger)^{m_2}}{m_1! m_2!} : R_j(\hat{\rho}) : \frac{(\hat{a}_1)^{n_1} (-\hat{a}_2)^{n_2}}{n_1! n_2!} \\
&= m \hat{\Psi}_{jm},
\end{aligned}$$

which is the desired result.

(ii) To compute the eigenvalue of $\mathcal{L}^2$ we build the operators

$$\hat{L}_\pm := \hat{L}_1 \pm i\hat{L}_2$$

and observe that

$$\begin{aligned}\hat{L}_-\hat{L}_+ &= (\hat{L}^1 - i\hat{L}^2)(\hat{L}^1 + i\hat{L}^2)\\ &= (\hat{L}^1)^2 - i\hat{L}^2\hat{L}^1 + i\hat{L}^1\hat{L}^2 + (\hat{L}^2)^2\\ &= (\hat{L}^1)^2 + (\hat{L}^2)^2 + i\underbrace{[\hat{L}^1, \hat{L}^2]}_{=i\hat{L}^3}\\ &= (\hat{L}^1)^2 + (\hat{L}^2)^2 - \hat{L}^3\end{aligned}$$

and hence,

$$\mathcal{L}^2 = \hat{L}_-\hat{L}_+ + \hat{L}^3 + (\hat{L}^3)^2.$$

We need to compute the action of $\hat{L}_-\hat{L}_+$ on $\hat{\Psi}_{jm}$. To this end, we note that we can write

$$\begin{aligned}\hat{L}_+\hat{\Psi}_{jm} &= (\hat{L}^1 + i\hat{L}^2)\hat{\Psi}_{jm}\\ &= \frac{1}{2\lambda}[\hat{x}^1 + i\hat{x}^2, \hat{\Psi}_{jm}] && \text{using (3.12)}\\ &= [\hat{a}_1^\dagger\hat{a}_2, \hat{\Psi}_{jm}] && \text{using (3.5)}\end{aligned}$$

and similarly

$$\hat{L}_-\hat{\Psi}_{jm} = [\hat{a}_2^\dagger\hat{a}_1, \hat{\Psi}_{jm}].$$

Thus,

$$\hat{L}_-\hat{L}_+\hat{\Psi}_{jm} = [\hat{a}_2^\dagger\hat{a}_1, \underbrace{[\hat{a}_1^\dagger\hat{a}_2, \hat{\Psi}_{jm}]}_{(*)}]$$

Let us first deal with the term $(*)$. We will use the results from Lemma 26 in the computation. Note that we only need to keep commutators acting on $: R_j(\hat{\rho}) :$ and commutators of $\hat{a}_\alpha$ acting on $(\hat{a}_\beta^\dagger)^n$ and vice versa.

$$[\hat{a}_1^\dagger\hat{a}_2, \hat{\Psi}_{jm}] = \hat{a}_1^\dagger[\hat{a}_2, \hat{\Psi}_{jm}] + [\hat{a}_1^\dagger, \hat{\Psi}_{jm}]\hat{a}_2$$

$$
\begin{aligned}
&= \lambda^j \sum \hat{a}_1^\dagger \left[\hat{a}_2, \frac{(\hat{a}_1^\dagger)^{m_1}(\hat{a}_2^\dagger)^{m_2}}{m_1!m_2!}\right] : R_j(\hat{\rho}) : \frac{(\hat{a}_1)^{n_1}(-\hat{a}_2)^{n_2}}{n_1!n_2!} \\
&\quad + \lambda^j \sum \frac{(\hat{a}_1^\dagger)^{m_1+1}(\hat{a}_2^\dagger)^{m_2}}{m_1!m_2!} [\hat{a}_2, : R_j(\hat{\rho}) :] \frac{(\hat{a}_1)^{n_1}(-\hat{a}_2)^{n_2}}{n_1!n_2!} \\
&\quad - \lambda^j \sum \frac{(\hat{a}_1^\dagger)^{m_1}(\hat{a}_2^\dagger)^{m_2}}{m_1!m_2!} \left[\hat{a}_1^\dagger, : R_j(\hat{\rho}) :\right] \frac{(\hat{a}_1)^{n_1}(-\hat{a}_2)^{n_2+1}}{n_1!n_2!} \\
&\quad + \lambda^j \sum \frac{(\hat{a}_1^\dagger)^{m_1}(\hat{a}_2^\dagger)^{m_2}}{m_1!m_2!} : R_j(\hat{\rho}) : \left[\hat{a}_1^\dagger, \frac{(\hat{a}_1)^{n_1}(-\hat{a}_2)^{n_2+1}}{n_1!n_2!}\right] \hat{a}_2 \\
&= \lambda^j \sum \frac{m_2(\hat{a}_1^\dagger)^{m_1+1}(\hat{a}_2^\dagger)^{m_2-1}}{m_1!m_2!} : R_j(\hat{\rho}) : \frac{(\hat{a}_1)^{n_1}(-\hat{a}_2)^{n_2}}{n_1!n_2!} \\
&\quad - \lambda^j \sum \frac{(\hat{a}_1^\dagger)^{m_1+1}(\hat{a}_2^\dagger)^{m_2}}{m_1!m_2!} : \partial_{\hat{N}} R_j(\hat{\rho}) : \frac{(\hat{a}_1)^{n_1}(-\hat{a}_2)^{n_2+1}}{n_1!n_2!} \\
&\quad + \lambda^j \sum \frac{(\hat{a}_1^\dagger)^{m_1+1}(\hat{a}_2^\dagger)^{m_2}}{m_1!m_2!} : \partial_{\hat{N}} R_j(\hat{\rho}) : \frac{(\hat{a}_1)^{n_1}(-\hat{a}_2)^{n_2+1}}{n_1!n_2!} \\
&\quad + \lambda^j \sum \frac{m_2(\hat{a}_1^\dagger)^{m_1}(\hat{a}_2^\dagger)^{m_2}}{m_1!m_2!} : R_j(\hat{\rho}) : \frac{(\hat{a}_1)^{n_1-1}(-\hat{a}_2)^{n_2+1}}{n_1!n_2!}
\end{aligned}
$$

The two middle terms cancel, giving

$$
\begin{aligned}
[\hat{a}_1^\dagger \hat{a}_2, \hat{\Psi}_{jm}] &= \lambda^j \sum \frac{m_2(\hat{a}_1^\dagger)^{m_1+1}(\hat{a}_2^\dagger)^{m_2-1}}{m_1!m_2!} : R_j(\hat{\rho}) : \frac{(\hat{a}_1)^{n_1}(-\hat{a}_2)^{n_2}}{n_1!n_2!} \\
&\quad + \lambda^j \sum \frac{m_2(\hat{a}_1^\dagger)^{m_1}(\hat{a}_2^\dagger)^{m_2}}{m_1!m_2!} : R_j(\hat{\rho}) : \frac{(\hat{a}_1)^{n_1-1}(-\hat{a}_2)^{n_2+1}}{n_1!n_2!}
\end{aligned}
$$

We proceed with the calculation. In the following, we abbreviate $\hat{\phi} := [\hat{a}_1^\dagger \hat{a}_2, \hat{\Psi}_{jm}]$ and leave out intermediate steps in order to shorten the proof.

$$
\begin{aligned}
[\hat{a}_2^\dagger \hat{a}_1, \hat{\phi}] &= [\hat{a}_2^\dagger, \hat{\phi}]\hat{a}_1 + \hat{a}_2^\dagger[\hat{a}_1, \hat{\phi}] \\
&= -\lambda^j \sum \frac{m_2(\hat{a}_1^\dagger)^{m_1+1}(\hat{a}_2^\dagger)^{m_2}}{m_1!m_2!} : \partial_{\hat{N}} R_j(\hat{\rho}) : \frac{(\hat{a}_1)^{n_1+1}(-\hat{a}_2)^{n_2}}{n_1!n_2!} \\
&\quad + \lambda^j \sum \frac{m_2(\hat{a}_1^\dagger)^{m_1+1}(\hat{a}_2^\dagger)^{m_2-1}}{m_1!m_2!} : R_j(\hat{\rho}) : \frac{n_2(\hat{a}_1)^{n_1+1}(-\hat{a}_2)^{n_2-1}}{n_1!n_2!} \\
&\quad - \lambda^j \sum \frac{(\hat{a}_1^\dagger)^{m_1}(\hat{a}_2^\dagger)^{m_2+1}}{m_1!m_2!} \partial_{\hat{N}} : R_j(\hat{\rho}) : \frac{n_1(\hat{a}_1)^{n_1}(-\hat{a}_2)^{n_2+1}}{n_1!n_2!} \\
&\quad + \lambda^j \sum \frac{(\hat{a}_1^\dagger)^{m_1}(\hat{a}_2^\dagger)^{m_2}}{m_1!m_2!} : R_j(\hat{\rho}) : \frac{n_1(n_2+1)(\hat{a}_1)^{n_1}(-\hat{a}_2)^{n_2}}{n_1!n_2!} \\
&\quad + \lambda^j \sum \frac{(m_1+1)m_2(\hat{a}_1^\dagger)^{m_1}(\hat{a}_2^\dagger)^{m_2}}{m_1!m_2!} : R_j(\hat{\rho}) : \frac{(\hat{a}_1)^{n_1}(-\hat{a}_2)^{n_2}}{n_1!n_2!}
\end{aligned}
$$

$$
\begin{aligned}
&+ \lambda^j \sum \frac{m_2(\hat{a}_1^\dagger)^{m_1+1}(\hat{a}_2^\dagger)^{m_2}}{m_1!m_2!} : \partial_{\hat{N}} R_j(\hat{\rho}) : \frac{(\hat{a}_1)^{n_1+1}(-\hat{a}_2)^{n_2}}{n_1!n_2!} \\
&+ \lambda^j \sum \frac{m_1(\hat{a}_1^\dagger)^{m_1-1}(\hat{a}_2^\dagger)^{m_2+1}}{m_1!m_2!} : R_j(\hat{\rho}) : \frac{n_1(\hat{a}_1)^{n_1-1}(-\hat{a}_2)^{n_2+1}}{n_1!n_2!} \\
&+ \lambda^j \sum \frac{(\hat{a}_1^\dagger)^{m_1}(\hat{a}_2^\dagger)^{m_2+1}}{m_1!m_2!} : R_j(\hat{\rho}) : \frac{n_1(\hat{a}_1)^{n_1}(-\hat{a}_2)^{n_2+1}}{n_1!n_2!}
\end{aligned}
$$

We observe that the first term cancels with the sixth and the third term cancels with the eighth. The remaining four terms are:

$$
\begin{aligned}
[\hat{a}_2^\dagger \hat{a}_1, \hat{\phi}] &= \lambda^j \sum ((m_1+1)m_2 + n_1(n_2+1)) \frac{(\hat{a}_1^\dagger)^{m_1}(\hat{a}_2^\dagger)^{m_2}}{m_1!m_2!} : R_j(\hat{\rho}) : \frac{(\hat{a}_1)^{n_1}(-\hat{a}_2)^{n_2}}{n_1!n_2!} \\
&\quad + \lambda^j \sum m_1 n_1 \frac{(\hat{a}_1^\dagger)^{m_1-1}(\hat{a}_2^\dagger)^{m_2+1}}{m_1!m_2!} : R_j(\hat{\rho}) : \frac{(\hat{a}_1)^{n_1-1}(-\hat{a}_2)^{n_2+1}}{n_1!n_2!} \\
&\quad + \lambda^j \sum m_2 n_2 \frac{(\hat{a}_1^\dagger)^{m_1+1}(\hat{a}_2^\dagger)^{m_2-1}}{m_1!m_2!} : R_j(\hat{\rho}) : \frac{(\hat{a}_1)^{n_1+1}(-\hat{a}_2)^{n_2-1}}{n_1!n_2!} \\
&= \lambda^j \sum ((m_1+1)m_2 + n_1(n_2+1)) \frac{(\hat{a}_1^\dagger)^{m_1}(\hat{a}_2^\dagger)^{m_2}}{m_1!m_2!} : R_j(\hat{\rho}) : \frac{(\hat{a}_1)^{n_1}(-\hat{a}_2)^{n_2}}{n_1!n_2!} \\
&\quad + \underbrace{\lambda^j \sum (m_2+1)(n_2+1) \frac{(\hat{a}_1^\dagger)^{m_1-1}(\hat{a}_2^\dagger)^{m_2+1}}{(m_1-1)!(m_2+1)!} : R_j(\hat{\rho}) : \frac{(\hat{a}_1)^{n_1-1}(-\hat{a}_2)^{n_2+1}}{(n_1-1)!(n_2+1)!}}_{(\spadesuit)} \\
&\quad + \underbrace{\lambda^j \sum (m_1+1)(n_1+1) \frac{(\hat{a}_1^\dagger)^{m_1+1}(\hat{a}_2^\dagger)^{m_2-1}}{(m_1+1)!(m_2-1)!} : R_j(\hat{\rho}) : \frac{(\hat{a}_1)^{n_1+1}(-\hat{a}_2)^{n_2-1}}{(n_1+1)!(n_2-1)!}}_{(\clubsuit)}
\end{aligned}
$$

Let us change indices in the term ($\spadesuit$) to

$$
\begin{aligned}
s_1 &:= m_1 - 1 & t_1 &:= n_1 - 1 \\
s_2 &:= m_2 + 1 & t_2 &:= n_2 + 1.
\end{aligned}
$$

Remember that the summation runs over $j = m_1 + m_2 = n_1 + n_2$ and $m = \frac{1}{2}(m_1 - m_2 - n_1 + n_2)$; since $s_1 + s_2 = t_1 + t_2 = j$ and $s_1 - s_2 - t_1 + t_2 = 2m$, the summation remains unchanged. The factor $(m_2+1)(n_2+1)$ simply becomes $s_2 t_2$. We do the analogous index change in the term ($\clubsuit$), so that we can finally write (by changing the index names back to m_α and n_α)

$$
[\hat{a}_2^\dagger \hat{a}_1, \hat{\phi}] = \lambda^j \sum ((m_1+1)m_2 + n_1(n_2+1) + m_2 n_2 + m_1 n_1)
$$

$$\times \frac{(\hat{a}_1^\dagger)^{m_1}(\hat{a}_2^\dagger)^{m_2}}{m_1!m_2!} : R_j(\hat{\rho}) : \frac{(\hat{a}_1)^{n_1}(-\hat{a}_2)^{n_2}}{n_1!n_2!}.$$

Now we are almost done. By using $j = m_1 + m_2 = n_1 + n_2$ and $2m = m_1 - m_2 - n_1 + n_2$ we observe that we have the identities

$$m_2 + n_1 = j - m \qquad\qquad m_1 + n_2 = j + m,$$

leading to

$$\begin{aligned}(m_1+1)m_2 + n_1(n_2+1) + m_2 n_2 + m_1 n_1 &= m_1 m_2 + m_2 + n_1 n_2 + n_1 + m_2 n_2 + m_1 n_1 \\ &= (m_1+n_2)(m_2+n_1) + n_1 + m_2 \\ &= (j+m)(j-m) + j - m \\ &= j(j+1) - m^2 - m.\end{aligned}$$

Thus, we have in summary:

$$\hat{L}_-\hat{L}_+\hat{\Psi}_{jm} = (j(j+1) - m^2 - m)\hat{\Psi}_{jm}$$

By recalling that $\mathcal{L}^2 = \hat{L}_-\hat{L}_+ + \hat{L}^3 + (\hat{L}^3)^2$ and using (i) we have finally shown that $\mathcal{L}^2\hat{\Psi}_{jm} = j(j+1)\hat{\Psi}_{jm}$. □

To conclude this section we derive a different expression for the analytic function $: R_j(\hat{\rho}) :$ in (3.16). We first express $R_j(\hat{\rho})$ as a power series:

$$: R_j(\hat{\rho}) : = \sum_k c_k : \hat{\rho}^k := \sum_k c_k \lambda^k : \hat{N}^k :$$

Furthermore, we need the following

Lemma 28. $: \hat{N}^k : |n_1, n_2\rangle = \frac{N!}{(N-k)!}|n_1, n_2\rangle$ *for* $N = n_1 + n_2$.

Proof. As before, we use induction over k. Let $k = 1$. We have

$$\hat{N}|n_1, n_2\rangle = \hat{a}_\alpha^\dagger \hat{a}_\alpha |n_1, n_2\rangle = \hat{a}_\alpha^\dagger \hat{a}_\alpha \frac{(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}}{\sqrt{n_1!n_2!}}|0\rangle.$$

By Lemma 26(i) we also have

$$\hat{a}_\alpha (a_1^\dagger)^{n_1} = \underbrace{(\hat{a}_1^\dagger)^{n_1}\hat{a}_\alpha}_{(*)} + n_1 \delta_{\alpha 1}(\hat{a}_1^\dagger)^{n_1-1}$$

$$\hat{a}_\alpha(\hat{a}_2^\dagger)^{n_2} = \underbrace{(\hat{a}_2^\dagger)^{n_2}\hat{a}_\alpha}_{(**)} + n_2\delta_{\alpha 2}(\hat{a}_2^\dagger)^{n_2-1}$$

Note that the terms $(*)$ and $(**)$, give no contribution when acting on $|0\rangle$, since $\hat{a}_\alpha|0\rangle = 0$. Hence, we get:

$$\begin{aligned}
\hat{a}_\alpha^\dagger\hat{a}_\alpha|n_1,n_2\rangle &= n_1\underbrace{\delta_{\alpha 1}\hat{a}_\alpha^\dagger\frac{(a_1^\dagger)^{n_1-1}(a_2^\dagger)^{n_2}}{\sqrt{n_1!n_2!}}|0\rangle}_{=|n_1,n_2\rangle} + n_2\underbrace{\delta_{\alpha 2}\hat{a}_\alpha^\dagger\frac{(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2-2}}{\sqrt{n_1!n_2!}}|0\rangle}_{=|n_1,n_2\rangle} \\
&= (n_1+n_2)|n_1,n_2\rangle \\
&= \frac{N!}{(N-1)!}|n_1,n_2\rangle
\end{aligned}$$

Assume now that the claim is true for $k-1$. We want to compute

$$:\hat{N}^k:|n_1,n_2\rangle = \hat{a}_{\alpha_1}^\dagger\ldots\hat{a}_{\alpha_k}^\dagger\hat{a}_{\alpha_1}\ldots\hat{a}_{\alpha_k}|n_1,n_2\rangle.$$

In a similar manner to before, we compute the right-hand side step by step (the square bracket under an operator means that this operator is omitted):

$$\begin{aligned}
\hat{a}_{\alpha_1}^\dagger\ldots\hat{a}_{\alpha_k}^\dagger\hat{a}_{\alpha_1} &= \hat{a}_{\alpha_1}^\dagger\ldots\hat{a}_{\alpha_{k-1}}^\dagger\hat{a}_{\alpha_1}\hat{a}_{\alpha_k}^\dagger - \delta_{\alpha_1\alpha_k}\hat{a}_{\alpha_1}^\dagger\ldots\hat{a}_{\alpha_{k-1}}^\dagger \\
&= \hat{a}_{\alpha_1}^\dagger\ldots\hat{a}_{\alpha_{k-2}}^\dagger\hat{a}_{\alpha_1}\hat{a}_{\alpha_{k-1}}^\dagger\hat{a}_{\alpha_k}^\dagger - \delta_{\alpha_1\alpha_{k-1}}\hat{a}_{\alpha_1}^\dagger\ldots\hat{a}_{\alpha_{k-2}}^\dagger\hat{a}_{\alpha_k}^\dagger \\
&\quad - \delta_{\alpha_1\alpha_k}\hat{a}_{\alpha_1}^\dagger\ldots\hat{a}_{\alpha_{k-1}}^\dagger \\
&\ \vdots \\
&= \hat{a}_{\alpha_1}\hat{a}_{\alpha_1}^\dagger\ldots\hat{a}_{\alpha_k}^\dagger - \sum_{j=1}^{k}\delta_{\alpha_1\alpha_j}\hat{a}_{\alpha_1}^\dagger\ldots\underbracket{\hat{a}_{\alpha_j}^\dagger}\ldots\hat{a}_{\alpha_k}^\dagger
\end{aligned}$$

This allows us to deploy the induction hypothesis twice:

$$\begin{aligned}
:\hat{N}^k:|n_1,n_2\rangle &= \hat{a}_{\alpha_1}\hat{a}_{\alpha_1}^\dagger\hat{a}_{\alpha_2}^\dagger\ldots\hat{a}_{\alpha_k}^\dagger\hat{a}_{\alpha_2}\ldots\hat{a}_{\alpha_k}|n_1,n_2\rangle \\
&\quad - \sum_{j=1}^{k}\delta_{\alpha_1\alpha_j}\hat{a}_{\alpha_1}^\dagger\ldots\underbracket{\hat{a}_{\alpha_j}^\dagger}\ldots\hat{a}_{\alpha_k}^\dagger\hat{a}_{\alpha_2}\ldots\hat{a}_{\alpha_k}|n_1,n_2\rangle \\
&= \hat{a}_{\alpha_1}\hat{a}_{\alpha_1}^\dagger\frac{N!}{(N-k+1)!}|n_1,n_2\rangle - \sum_{j=1}^{k}\frac{N!}{(N-k+1)!}|n_1,n_2\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \underbrace{\hat{a}^\dagger_{\alpha_1}\hat{a}_{\alpha_1}\frac{N!}{(N-k+1)!}|n_1,n_2\rangle}_{=N\frac{N!}{(N-k+1)!}|n_1,n_2\rangle} + \frac{N!}{(N-k+1)!}|n_1,n_2\rangle \\
&\quad - k\frac{N!}{(N-k+1)!}|n_1,n_2\rangle \\
&= (N-k+1)\frac{N!}{(N-k+1)!}|n_1,n_2\rangle \\
&= \frac{N!}{(N-k)!}|n_1,n_2\rangle \qquad \square
\end{aligned}
$$

Using Lemma 28 we can rewrite the analytic function $:R_j(\hat\rho):$ as

$$
\begin{aligned}
:R_j(\hat\rho): &= \sum_k c_k :\hat\rho^k: \\
&= \sum_k c_k \lambda^k \frac{\hat N!}{(\hat N - k)!}
\end{aligned} \tag{3.18}
$$

when restricting $:R_j(\hat\rho):$ to the subspace $\mathcal{F}_N$. The operator $\hat N!$ is defined on $\mathcal{F}_N$ by the eigenvalue equation $\hat N!|n_1,n_2\rangle = N!|n_1,n_2\rangle$ with $N = n_1 + n_2$.

3.2 The Laplace operator

3.2.1 Identifying the Hilbert space

Let $\hat{\mathcal{H}}$ be the Hilbert space generated by the operators $\hat\Psi_{jm}$, $j \in \mathbb{N}$, $m = -j, \dots, j$. The scalar product on $\hat{\mathcal{H}}$ is given by

$$
\langle\hat\Psi|\hat\Phi\rangle = \mathrm{tr}\left(w(\hat r)\hat\Psi^\dagger\hat\Phi\right) \quad \text{for } \hat\Psi, \hat\Phi \in \hat{\mathcal{H}} \tag{3.19}
$$

where $w(\hat r)$ is an (a priori) arbitrary, rotationally invariant weight function. With respect to the scalar product in (3.19), the generators of rotations $\hat L^i$ are Hermitian (for simplicity, we set $w(\hat r) = 1$):

$$
\begin{aligned}
\langle\hat\Phi|\hat L^i\hat\Psi\rangle &= \mathrm{tr}\left(\hat\Phi^\dagger\hat L^i\hat\Psi\right) = \frac{1}{2\lambda}\,\mathrm{tr}\left(\hat\Phi^\dagger[\hat x^i,\hat\Psi]\right) \\
&= \frac{1}{2\lambda}\left(\mathrm{tr}\left(\hat\Phi^\dagger\hat x^i\hat\Psi\right) - \mathrm{tr}\left(\hat\Phi^\dagger\hat\Psi\hat x^i\right)\right) \\
&= \frac{1}{2\lambda}\left(\mathrm{tr}\left((\hat x^i\hat\Phi)^\dagger\hat\Psi\right) - \mathrm{tr}\left(\hat x^i\hat\Phi^\dagger\hat\Psi\right)\right) \\
&= \frac{1}{2\lambda}\left(\mathrm{tr}\left((\hat x^i\hat\Phi)^\dagger\hat\Psi\right) - \mathrm{tr}\left((\hat\Phi\hat x_i)^\dagger\hat\Psi\right)\right)
\end{aligned}
$$

$$= \frac{1}{2\lambda} \operatorname{tr}\left([\hat{x}^i, \hat{\Phi}]^\dagger \hat{\Psi}\right)$$
$$= \langle \hat{L}^i \hat{\Phi} | \hat{\Psi} \rangle$$

Therefore, the operators $\hat{\Psi}_{jm} = \sum \ldots : R_j(\hat{\rho}) : \ldots$ and $\hat{\Psi}_{j'm'} = \sum \ldots : S_{j'}(\hat{\rho}) : \ldots$ with arbitrary analytic functions R_j and $S_{j'}$ are orthogonal with respect to (3.19) as eigenfunctions of a Hermitian operator.

3.2.2 Definition of the Laplace operator

In $\hat{\mathcal{H}}$ we define the Laplace operator as

$$\Delta_\lambda := -\frac{1}{\lambda \hat{r}} [\hat{a}^\dagger_\alpha, [\hat{a}_\alpha, .]] \tag{3.20}$$

Its action on the eigenfunctions $\hat{\Psi}_{jm}$ of $\mathcal{L}^2$ and $\hat{L}^3$ (neglecting for now the factor $\frac{1}{\hat{r}}$, cf. section 3.3) is recorded in the following

Proposition 29. *Let $\hat{\Psi}_{jm}$ be an eigenfunction of $\mathcal{L}^2$ and $\hat{L}^3$ as given in* (3.16)*. Then*

$$\hat{r}\,\Delta_\lambda\,\hat{\Psi}_{jm} = \lambda^j \sum \frac{(\hat{a}^\dagger_1)^{m_1}(\hat{a}^\dagger_2)^{m_2}}{m_1! m_2!} : \hat{\rho} R''_j(\hat{\rho}) + 2(j+1) R'_j(\hat{\rho}) : \frac{\hat{a}_1^{n_1}(-\hat{a}_2)^{n_2}}{n_1! n_2!}$$

where $R'_j(\hat{\rho}) := \sum_{k=1}^\infty k c_k \hat{\rho}^{k-1}$ and $R''_j(\hat{\rho}) := \sum_{k=2}^\infty k(k-1) c_k \hat{\rho}^{k-2}$.

Proof. We start with the computation:

$$[\hat{a}^\dagger_\alpha, [\hat{a}_\alpha, \hat{\Psi}_{jm}] = \lambda^j \left[\hat{a}^\dagger_\alpha, \left[\hat{a}_\alpha, \sum \frac{(\hat{a}^\dagger_1)^{m_1}(\hat{a}^\dagger_2)^{m_2}}{m_1! m_2!} : R_j(\hat{\rho}) : \frac{\hat{a}_1^{n_1}(-\hat{a}_2)^{n_2}}{n_1! n_2!}\right]\right]$$

We only need to keep the non-vanishing terms, i.e., commutators acting on $: R_j(\hat{\rho}) :$ and commutators of $\hat{a}_\alpha$ acting on $\hat{a}^\dagger$-terms and vice versa:

$$\begin{aligned}
[\hat{a}^\dagger_\alpha, [\hat{a}_\alpha, \hat{\Psi}_{jm}] &= \lambda^j \sum \left[\hat{a}_\alpha, \frac{(\hat{a}^\dagger_1)^{m_1}(\hat{a}^\dagger_2)^{m_2}}{m_1! m_2!}\right] [\hat{a}^\dagger_\alpha, : R_j(\hat{\rho})] : \frac{\hat{a}_1^{n_1}(-\hat{a}_2)^{n_2}}{n_1! n_2!} && (\spadesuit)\\
&+ \lambda^j \sum \left[\hat{a}_\alpha, \frac{(\hat{a}^\dagger_1)^{m_1}(\hat{a}^\dagger_2)^{m_2}}{m_1! m_2!}\right] : R_j(\hat{\rho}) : \left[\hat{a}^\dagger_\alpha, \frac{\hat{a}_1^{n_1}(-\hat{a}_2)^{n_2}}{n_1! n_2!}\right] && (\clubsuit)\\
&+ \lambda^j \frac{(\hat{a}^\dagger_1)^{m_1}(\hat{a}^\dagger_2)^{m_2}}{m_1! m_2!} [\hat{a}^\dagger_\alpha, [\hat{a}_\alpha, : R_j(\hat{\rho})]] \frac{\hat{a}_1^{n_1}(-\hat{a}_2)^{n_2}}{n_1! n_2!} && (\heartsuit)\\
&+ \lambda^j \frac{(\hat{a}^\dagger_1)^{m_1}(\hat{a}^\dagger_2)^{m_2}}{m_1! m_2!} [\hat{a}_\alpha, : R_j(\hat{\rho}) :] \left[\hat{a}^\dagger_\alpha, \frac{\hat{a}_1^{n_1}(-\hat{a}_2)^{n_2}}{n_1! n_2!}\right] && (\diamondsuit)
\end{aligned}$$

The commutator appearing first is evaluated as:

$$\begin{aligned}[\hat{a}_\alpha, (\hat{a}_1^\dagger)^{m_1}(\hat{a}_2^\dagger)^{m_2}] &= (\hat{a}_1^\dagger)^{m_1}[\hat{a}_\alpha, (\hat{a}_2^\dagger)^{m_2}] + [\hat{a}_\alpha, (\hat{a}_1^\dagger)^{m_1}](\hat{a}_2^\dagger)^{m_2} \\ &= m_2\delta_{\alpha 2}(\hat{a}_1^\dagger)^{m_1}(\hat{a}_2^\dagger)^{m_2-1} + m_1\delta_{\alpha 1}(\hat{a}_1^\dagger)^{m_1-1}(\hat{a}_2^\dagger)^{m_2} \qquad (*)\end{aligned}$$

Observe further that

$$\begin{aligned}[\hat{a}_2^\dagger, \hat{a}_1^{n_1}(-\hat{a}_2)^{n_2}] &= n_2\hat{a}_1^{n_1}(-\hat{a}_2)^{n_2-1} \\ [\hat{a}_1^\dagger, \hat{a}_1^{n_1}(-\hat{a}_2)^{n_2}] &= -n_1\hat{a}_1^{n_1-1}(-\hat{a}_2)^{n_2}.\end{aligned}$$

Hence,

$$\begin{aligned}(\clubsuit) = \quad & \lambda^j \sum \frac{m_2(\hat{a}_1^\dagger)^{m_1}(\hat{a}_2^\dagger)^{m_2-1}}{m_1!m_2!} : R_j(\hat{\rho}) : \frac{n_2\hat{a}_1^{n_1}(-\hat{a}_2)^{n_2-1}}{n_1!n_2!} \\ & - \lambda^j \sum \frac{m_1(\hat{a}_1^\dagger)^{m_1-1}(\hat{a}_2^\dagger)^{m_2}}{m_1!m_2!} : R_j(\hat{\rho}) : \frac{n_1\hat{a}_1^{n_1-1}(-\hat{a}_2)^{n_2}}{n_1!n_2!} \\ = & \lambda^j \sum \frac{(\hat{a}_1^\dagger)^{m_1}(\hat{a}_2^\dagger)^{m_2-1}}{m_1!(m_2-1)!} : R_j(\hat{\rho}) : \frac{\hat{a}_1^{n_1}(-\hat{a}_2)^{n_2-1}}{n_1!(n_2-1)!} \\ & - \lambda^j \sum \frac{(\hat{a}_1^\dagger)^{m_1-1}(\hat{a}_2^\dagger)^{m_2}}{(m_1-1)!m_2!} : R_j(\hat{\rho}) : \frac{\hat{a}_1^{n_1-1}(-\hat{a}_2)^{n_2}}{(n_1-1)!n_2!} \\ = & 0\end{aligned}$$

since every summand in the left term also appears in the right term. To compute the commutators acting on $: R_j(\hat{\rho}) :$ we recall Lemma 26(ii), which states:

$$\begin{aligned}[\hat{a}_\alpha^\dagger, : R_j(\hat{\rho}) :] &= -\hat{a}_\alpha^\dagger : \partial_{\hat{N}} R_j(\hat{\rho}) : \\ [\hat{a}_\alpha, : R_j(\hat{\rho}) :] &= : \partial_{\hat{N}} R_j(\hat{\rho}) : \hat{a}_\alpha\end{aligned}$$

Using this and $(*)$ we obtain

$$\begin{aligned}(\spadesuit) &= \lambda^j \sum \left[\hat{a}_\alpha, \frac{(\hat{a}_1^\dagger)^{m_1}(\hat{a}_2^\dagger)^{m_2}}{m_1!m_2!}\right] [\hat{a}_\alpha^\dagger, : R_j(\hat{\rho})] : \frac{\hat{a}_1^{n_1}(-\hat{a}_2)^{n_2}}{n_1!n_2!} \\ &= \lambda^j \sum \frac{1}{m_1!m_2!}(m_2\delta_{\alpha 2}(\hat{a}_1^\dagger)^{m_1}(\hat{a}_2^\dagger)^{m_2-1} + m_1\delta_{\alpha 1}(\hat{a}_1^\dagger)^{m_1-1}(\hat{a}_2^\dagger)^{m_2}) \\ &\quad \times (-\hat{a}_\alpha^\dagger : \partial_{\hat{N}} R_j(\hat{\rho}) :)\frac{\hat{a}_1^{n_1}(-\hat{a}_2)^{n_2}}{n_1!n_2!} \\ &= \lambda^j \sum \frac{(\hat{a}_1^\dagger)^{m_1}(\hat{a}_2^\dagger)^{m_2}}{m_1!m_2!}(-j : \partial_{\hat{N}} R_j(\hat{\rho}) :)\frac{\hat{a}_1^{n_1}(-\hat{a}_2)^{n_2}}{n_1!n_2!}.\end{aligned}$$

It is easy to see that ($\diamondsuit$) gives the same contribution. Thus, the last thing we need to calculate is line ($\heartsuit$). To this end, we evaluate the double commutator acting on $: R_j(\hat{\rho}) :$, which is given as:

$$\begin{aligned}[\hat{a}^\dagger_\alpha, [\hat{a}_\alpha, : R_j(\hat{\rho}) :]] &= [\hat{a}^\dagger_\alpha, : \partial_{\hat{N}} R_j(\hat{\rho}) : \hat{a}_\alpha] \\ &= -\hat{a}^\dagger_\alpha : \partial^2_{\hat{N}} R_j(\hat{\rho}) : \hat{a}_\alpha - 2 : \partial_{\hat{N}} R_j(\hat{\rho}) : \\ &= - : \hat{N}\partial^2_{\hat{N}} R_j(\hat{\rho}) : -2 : \partial_{\hat{N}} R_j(\hat{\rho}) :\end{aligned}$$

In summary we have:

$$-\frac{1}{\lambda}[\hat{a}^\dagger_\alpha, [\hat{a}_\alpha, \hat{\Psi}_{jm}]] = \lambda^j \sum \frac{(\hat{a}^\dagger_1)^{m_1}(\hat{a}^\dagger_2)^{m_2}}{m_1!m_2!} : \hat{N}\partial^2_{\hat{N}} R_j(\hat{\rho}) + 2(j+1)\partial_{\hat{N}} R'_j(\hat{\rho}) : \frac{\hat{a}_1^{n_1}(-\hat{a}_2)^{n_2}}{n_1!n_2!}$$

Remembering that $\hat{\rho} = \lambda\hat{N}$, we can switch 'derivatives' from $\partial_{\hat{N}}$ to $\partial_{\hat{\rho}}$. More precisely, we have $\hat{N}\partial^2_{\hat{N}} = \hat{N}\partial_{\hat{N}}\lambda\partial_{\hat{\rho}} = \lambda\hat{\rho}\partial^2_{\hat{\rho}}$ and $\partial_{\hat{N}} = \lambda\partial_\rho$. Setting $()' \equiv \partial_\rho$ we obtain the result. □

It is important to note that the 'differential operators' $\partial_{\hat{N}}$ and $\partial_{\hat{\rho}}$ are only defined via their action on the power series expansion of analytic functions. They cannot (yet) be regarded as ordinary differential operators, since $\hat{N}$ respectively $\hat{\rho}$ are discrete 'variables', that is, operators acting on the Fock subspace $\mathcal{F}$.

3.3 The Hamiltonian and its eigenvalues

3.3.1 The potential term

The Laplace operator Δ_λ from the previous section forms the kinetic part of the Hamiltonian H for the Coulomb problem in $\mathbb{R}^3_\lambda$ (setting $\hbar = 1$ and scaling the mass m such that $\frac{1}{2m} = 1$). The total Hamiltonian H is given as

$$\begin{aligned} H &= -\Delta_\lambda + V(\hat{r}) \\ &= \frac{1}{\lambda\hat{r}}[\hat{a}^\dagger_\alpha, [\hat{a}_\alpha, .]] + V(\hat{r}) \end{aligned} \tag{3.21}$$

where $V(\hat{r})$ is a rotationally invariant potential term yet to be determined. In the commutative Coulomb problem V is the fundamental solution of the Laplace equation $\Delta u = 0$. It turns out that the potential retains the same form in the non-commutative case as the fundamental solution of $\Delta u = 0$:

Lemma 30. *Let $V(\hat{r})$ be a solution of $\Delta_\lambda \hat{u} = 0$. Then $V(\hat{r})$ is of the form*

$$V(\hat{r}) = -\frac{q}{\hat{r}} + q_0$$

where q and q_0 are constants.

Proof. We need to solve $[\hat{a}^\dagger_\alpha, [\hat{a}_\alpha, V(\hat{N})]] = 0$. To this end, we restrict this equation to $\mathcal{F}_N$ for $n_1 + n_2 = N$:

$$[\hat{a}^\dagger_\alpha, [\hat{a}_\alpha, V(\hat{N})]]|n_1, n_2\rangle = (\hat{a}^\dagger_\alpha \hat{a}_\alpha V(\hat{N}) - \hat{a}^\dagger_\alpha V(\hat{N})\hat{a}_\alpha - \hat{a}_\alpha V(\hat{N})\hat{a}^\dagger_\alpha + V(\hat{N}) \underbrace{\hat{a}_\alpha \hat{a}^\dagger_\alpha}_{=\hat{N}+2})|n_1, n_2\rangle$$

The action of $\hat{a}_\alpha$ and $\hat{a}^\dagger_\alpha$ on $|n_1, n_2\rangle$ is:

$$\begin{aligned} \hat{a}_1|n_1, n_2\rangle &= \sqrt{n_1}|n_1 - 1, n_2\rangle & \hat{a}^\dagger_1|n_1, n_2\rangle &= \sqrt{n_1 + 1}|n_1 + 1, n_2\rangle \\ \hat{a}_2|n_1, n_2\rangle &= \sqrt{n_2}|n_1, n_2 - 1\rangle & \hat{a}^\dagger_2|n_1, n_2\rangle &= \sqrt{n_2 + 1}|n_1, n_2 + 1\rangle \end{aligned}$$

Furthermore, if V is analytic then $V(\hat{N})|n_1, n_2\rangle = V(N)|n_1, n_2\rangle$. Therefore,

$$\begin{aligned} \hat{a}^\dagger_1 V(\hat{N})\hat{a}_1|n_1, n_2\rangle &= \hat{a}^\dagger_1 V(\hat{N})\sqrt{n_1}|n_1 - 1, n_2\rangle + \\ &= \sqrt{n_1}\hat{a}^\dagger_1 V(N - 1)|n_1 - 1, n_2\rangle \\ &= n_1 V(N - 1)|n_1, n_2\rangle \\ \hat{a}^\dagger_2 V(\hat{N})\hat{a}_2|n_1, n_2\rangle &= n_2 V(N - 1)|n_1, n_2\rangle \\ \hat{a}_1 V(\hat{N})\hat{a}^\dagger_1|n_1, n_2\rangle &= \hat{a}_1 V(\hat{N})\sqrt{n_1 + 1}|n_1 + 1, n_2\rangle \\ &= \sqrt{n_1 + 1}\hat{a}_1 V(N + 1)|n_1 + 1, n_2\rangle \\ &= (n_1 + 1)V(N + 1)|n_1, n_2\rangle \\ \hat{a}_2 V(\hat{N})\hat{a}^\dagger_2|n_1, n_2\rangle &= (n_2 + 1)V(N + 1)|n_1, n_2\rangle, \end{aligned}$$

and putting everything together we obtain

$$\begin{aligned} 0 &= [\hat{a}^\dagger_\alpha, [\hat{a}_\alpha, V(\hat{N})]]|n_1, n_2\rangle \\ &= (NV(N) - (n_1 + n_2)V(N - 1) - (n_1 + n_2 + 2)V(N + 1) \\ &\quad + V(N)N + 2V(N))|n_1, n_2\rangle. \end{aligned}$$

Hence, the operator in the last line is zero,

$$2(\hat{N} + 1)V(\hat{N}) - \hat{N}V(\hat{N} - 1) - (\hat{N} + 2)V(\hat{N} + 1) = 0,$$

and we rewrite this as a recurrence relation:

$$(\hat{N}+2)V(\hat{N}+1) - (\hat{N}+1)V(\hat{N}) = (\hat{N}+1)V(\hat{N}) - \hat{N}V(\hat{N}-1) \tag{3.22}$$

For $M \in \mathbb{N}$ equation (3.22) implies that the operator $(\hat{M}+1)V(\hat{M}) - \hat{M}V(\hat{M}-1)$ is constant and therefore a multiple of the identity operator, say, q_0. Set $V(0) = q_0 - \frac{q}{\lambda}$ and compute (the summation over $\hat{M}$ is symbolic and understood as summing over the eigenvalues when acting on $\mathcal{F}$):

$$\sum_{\hat{M}=1}^{\hat{N}} (\hat{M}+1)V(\hat{M}) - \hat{M}V(\hat{M}-1) = \hat{N}q_0$$
$$(\hat{N}+1)V(\hat{N}) - V(0) = \hat{N}q_0$$

which gives the final result:

$$V(\hat{N}) = -\frac{q}{\lambda(\hat{N}+1)} + q_0 = -\frac{q}{\hat{r}} + q_0 \qquad \square$$

Therefore, the Hamiltonian of the Coulomb problem in $\mathbb{R}^3_\lambda$ is

$$H = \frac{1}{\lambda\hat{r}}[\hat{a}^\dagger_\alpha, [\hat{a}_\alpha, .]] - \frac{q}{\hat{r}}, \tag{3.23}$$

where we have set $q_0 = 0$.

3.3.2 Solving the Schrödinger equation

As in the commutative case, the constant q is obviously proportional to the square of the unit charge e. With (3.23) the stationary Schrödinger equation is given as:

$$\frac{1}{\lambda\hat{r}}[\hat{a}^\dagger_\alpha, [\hat{a}_\alpha, \hat{\Psi}]] - \frac{q}{\hat{r}}\hat{\Psi} = E\hat{\Psi}$$
$$\Longleftrightarrow \quad \frac{1}{\lambda}[\hat{a}^\dagger_\alpha, [\hat{a}_\alpha, \hat{\Psi}]] - q\hat{\Psi} = -\kappa^2\hat{r}\hat{\Psi} \qquad \text{with } \kappa := \sqrt{-E} \tag{3.24}$$

Similarly to the commutative case, we look for solutions among the eigenfunctions $\hat{\Psi}_{jm}$ of $\mathcal{L}^2$ and $\hat{L}^3$. To this end, we need to calculate the action of $\hat{r}$ on $\hat{\Psi}$:

Lemma 31. *Let $\hat{\Psi}_{jm}$ be an eigenfunction of $\mathcal{L}^2$ and $\hat{L}^3$ as given in* (3.16)*. Then*

$$\hat{r}\hat{\Psi}_{jm} = \lambda^j \sum \frac{(\hat{a}^\dagger_1)^{m_1}(\hat{a}^\dagger_2)^{m_2}}{m_1!m_2!} : (\hat{\rho} + \lambda j + \lambda)R_j(\hat{\rho}) + \lambda\hat{\rho}R'_j(\hat{\rho}) : \frac{\hat{a}_1^{n_1}(-\hat{a}_2)^{n_2}}{n_1!n_2!}.$$

Proof. We first calculate:

$$\begin{aligned}
\hat{N} : \hat{N}^k : &= \hat{a}^\dagger_\alpha \underbrace{\hat{a}_\alpha \hat{a}^\dagger_{\alpha_1}}_{=\hat{a}^\dagger_{\alpha_1}\hat{a}_\alpha + \delta_{\alpha\alpha_1}} \dots \hat{a}^\dagger_{\alpha_k} \hat{a}_{\alpha_1} \dots \hat{a}_{\alpha_k} \\
&=: \hat{N}^k : + \hat{a}^\dagger_\alpha \hat{a}^\dagger_{\alpha_1} \hat{a}_\alpha \hat{a}^\dagger_{\alpha_2} \dots \hat{a}^\dagger_{\alpha_k} \hat{a}_{\alpha_1} \dots \hat{a}_{\alpha_k} \\
&= 2 : \hat{N}^k : + \hat{a}^\dagger_\alpha \hat{a}^\dagger_{\alpha_1} \hat{a}^\dagger_{\alpha_2} \hat{a}_\alpha \hat{a}^\dagger_{\alpha_3} \dots \hat{a}^\dagger_{\alpha_k} \hat{a}_{\alpha_1} \dots \hat{a}_{\alpha_k} \\
&\vdots \\
&= k : \hat{N}^k : + : \hat{N}^{k+1} :
\end{aligned}$$

Hence,

$$\hat{N} : R_j(\hat{\rho}) : =: \hat{N} R_j(\hat{\rho}) : + : \hat{N} \partial_{\hat{N}} R_j(\hat{\rho}) : . \tag{$*$}$$

Furthermore, we have

$$\begin{aligned}
[\hat{N}, \hat{a}^\dagger_\alpha] &= [\hat{a}^\dagger_\beta \hat{a}_\beta, \hat{a}^\dagger_\alpha] \\
&= \hat{a}^\dagger_\beta [\hat{a}_\beta, \hat{a}^\dagger_\alpha] \\
&= \hat{a}^\dagger_\alpha,
\end{aligned}$$

resulting in

$$\begin{aligned}
\hat{N} (\hat{a}^\dagger_1)^{m_1} (\hat{a}^\dagger_2)^{m_2} &= m_1 (\hat{a}^\dagger_1)^{m_1} (\hat{a}^\dagger_2)^{m_2} + (ac^\dagger_1)^{m_1} \hat{N} (\hat{a}^\dagger_2)^{m_2} \\
&= \underbrace{(m_1 + m_2)}_{=j} (\hat{a}^\dagger_1)^{m_1} (\hat{a}^\dagger_2)^{m_2} \hat{N}.
\end{aligned} \tag{$**$}$$

Using $(*)$ and $(**)$ we arrive at

$$\begin{aligned}
\hat{r} \hat{\Psi}_{jm} &= \lambda^j \sum \lambda (\hat{N} + 1) \frac{(\hat{a}^\dagger_1)^{m_1} (\hat{a}^\dagger_2)^{m_2}}{m_1! m_2!} : R_j(\hat{\rho}) : \frac{\hat{a}_1^{n_1} (-\hat{a}_2)^{n_2}}{n_1! n_2!} \\
&= \lambda^j \sum \frac{(\hat{a}^\dagger_1)^{m_1} (\hat{a}^\dagger_2)^{m_2}}{m_1! m_2!} \lambda (\hat{N} + j + 1) : R_j(\hat{\rho}) : \frac{\hat{a}_1^{n_1} (-\hat{a}_2)^{n_2}}{n_1! n_2!} \\
&= \lambda^j \sum \frac{(\hat{a}^\dagger_1)^{m_1} (\hat{a}^\dagger_2)^{m_2}}{m_1! m_2!} : (\hat{\rho} + \lambda j + \lambda) R_j(\hat{\rho}) + \lambda \hat{N} \partial_{\hat{N}} R_j(\hat{\rho}) : \frac{\hat{a}_1^{n_1} (-\hat{a}_2)^{n_2}}{n_1! n_2!},
\end{aligned}$$

which is the desired result if we again change the 'derivatives' from $\partial_{\hat{N}}$ to $\partial_{\hat{\rho}}$ as in the proof of Proposition 29. □

Hence, the Schrödinger equation

$$\frac{1}{\lambda}[\hat{a}_\alpha^\dagger, [\hat{a}_\alpha, \hat{\Psi}_{jm}]] - q\hat{\Psi}_{jm} = -\kappa^2 \hat{r}\hat{\Psi}_{jm} \tag{3.25}$$

can be translated into a 'differential' equation for the radial part of $\hat{\Psi}_{jm}$ using Proposition 29 and Lemma 31:

$$:\hat{\rho}R_j''(\hat{\rho}) + 2(j+1)R_j'(\hat{\rho}) + qR_j(\hat{\rho}) : = \kappa^2 : \hat{\rho}R_j(\hat{\rho}) + \lambda((j+1)R_j(\hat{\rho}) + \hat{\rho}R_j'(\hat{\rho}) : \tag{3.26}$$

However, the use of derivatives in (3.26) should not be misunderstood. As $\hat{\rho}$ is a discrete 'variable', i.e., an operator with discrete spectrum $\mathrm{spec}(\hat{\rho}) = \{\lambda N \mid N \in \mathbb{N}\}$ acting on the Fock space $\mathcal{F}$, the 'derivative' $R_j'(\hat{\rho})$ is solely defined algebraically using the representation of $R_j(\hat{\rho})$ as a power series:

$$R_j'(\hat{\rho}) := \sum_{k=1}^{\infty} c_k k \hat{\rho}^{k-1} \quad \text{for} \quad R_j = \sum_{k=0}^{\infty} c_k \hat{\rho}^k$$

However, in either the commutative limit $\lambda \to 0$ or the quasi-classical limit $N \to \infty$ for fixed λ, we can reinterpret $R_j'(\hat{\rho})$ as the analytic derivative of $R_j(\hat{\rho})$ with respect to the now continuous variable $\hat{\rho}$.

Let us work in the quasi-classical limit in order to investigate any possible non-commutative effects visible in the spectrum of H. To this end, we change from the operator $\hat{\rho}$ to the continuous variable ρ and associate the following ordinary differential equation to (3.26):

$$\rho R_j'' + 2(j+1)R_j' + qR_j = \kappa^2(\rho R_j + \lambda((j+1)R_j + \rho R_j')) \tag{3.27}$$

Since we are only interested in the energy levels of the solutions of (3.27), we are not going to solve the ordinary differential equation explicitly. Rather, our strategy is to bring it into a form resembling the radial Schrödinger equation for the commutative Coulomb problem (cf. [25, p. 412, eq. (XI.4)]):

$$-R_j'' + \left(\frac{j(j+1)}{\rho^2} - \frac{q}{\rho}\right) R_j = -\kappa^2 R_j \tag{3.28}$$

As we will see in the course of the computation, the right-hand side of (3.28) is going to receive a correction term accounting for the effects stemming from the non-commutative setting. This is achieved by first introducing new variables:

$$R =: \chi S \quad \Rightarrow \quad \begin{aligned} R' &= \chi' S + \chi S' \\ R'' &= \chi'' S + 2\chi' S' + \chi S'' \end{aligned}$$

Plugging this into (3.27) gives

$$\rho\chi'' S + 2\rho\chi' S' + \rho\chi S'' + (2(j+1) - \kappa^2\lambda\rho)\chi' S \\ + (2(j+1) - \kappa^2\lambda\rho)\chi S' + (q - \kappa^2\rho - \kappa^2\lambda(j+1))\chi S = 0. \quad (3.29)$$

Since we want to get rid of the first derivative in (3.27), we set the coefficient of S' to zero:

$$2\rho\chi' + (2(j+1) - \kappa^2\lambda\rho)\chi = 0 \quad \Leftrightarrow \quad \frac{\chi'}{\chi} = -\frac{j+1}{\rho} + \frac{\kappa^2\lambda}{2}$$

Solving this differential equation for χ results in

$$\chi = \rho^{-(j+1)} e^{\kappa^2\lambda\rho/2} \quad (3.30a)$$

$$\chi' = \left(-\frac{j+1}{\rho} + \frac{\kappa^2\lambda}{2}\right)\chi \quad (3.30b)$$

$$\chi'' = \left(\frac{j+1}{\rho^2} + \left(-\frac{j+1}{\rho} + \frac{\kappa^2\lambda}{2}\right)^2\right)\chi, \quad (3.30c)$$

and inserting (3.30) into (3.29) we obtain

$$\rho S'' + \left(\rho\left(\frac{j+1}{\rho^2} + \left(-\frac{j+1}{\rho} + \frac{\kappa^2\lambda}{2}\right)^2\right)\right. \\ \left. + (2(j+1) - \kappa^2\lambda\rho)\left(-\frac{j+1}{\rho} + \frac{\kappa^2\lambda}{2}\right) + q - \kappa^2\rho - \kappa^2\lambda(j+1)\right) S = 0. \quad (3.31)$$

Carrying out the multiplications and dividing by ρ, we see that the majority of terms in (3.31) cancel each other out; this results in

$$S'' + \left(\frac{-j^2 - j}{\rho^2} - \frac{\lambda^2}{4}\kappa^4 + \frac{q}{\rho} - \kappa^2\right) S = 0,$$

which can be regrouped to give the final result:

$$-S'' + \left(\frac{j(j+1)}{\rho^2} - \frac{q}{\rho}\right) S = \left(-\kappa^2 - \frac{\lambda^2}{4}\kappa^4\right) S \qquad (3.32)$$

3.3.3 Interpretation of the results

Comparing (3.32) with (3.28) we observe that the energy $-\kappa^2$ in the commutative Coulomb problem is replaced by the term $-\kappa^2 - \frac{\lambda^2}{4}\kappa^4$ in the non-commutative Coulomb problem. We set $-\kappa^2 \equiv E$ and investigate the condition

$$E - \frac{\lambda^2}{4}E^2 = -\frac{C}{M^2} \qquad \text{where } M \in \mathbb{N} \text{ and } C = const. \qquad (3.33)$$

corresponding to bound states in the Coulomb problem. Solving for E gives the result

$$\begin{aligned} E &= \frac{2}{\lambda^2}\left(1 \pm \sqrt{1 + \frac{\lambda^2 C}{M^2}}\right) \\ &= -\frac{C}{M^2} \times \frac{2}{1 \mp \sqrt{1 + \frac{\lambda^2 C}{M^2}}}. \end{aligned} \qquad (3.34)$$

Hence, the energy E in the non-commutative Coulomb problem consists of the well-known energy levels $-C/M^2$ for $M \in \mathbb{N}$ of the commutative problem (cf. [25, p. 417, eq. (XI.17)]) and a factor $2(1 \mp \sqrt{1 + \lambda^2 C/M^2})^{-1}$ containing the non-commutative corrections, see Figure 3.1. Observe that the correctional factor vanishes in the commutative limit $\lambda \to 0$.

Figure 3.2 shows a plot of the energy function $f(E) = E - \frac{\lambda^2 E^2}{4}$ and the first few energy levels $-C/M^2$ of the bound states. The spectrum is symmetric with respect to the vertex at $E = 2/\lambda^2$, the two branches resulting from the different signs in (3.34). The region where $f(E) > 0$ corresponds to scattering states, which we do not consider here. Furthermore, in the commutative limit $\lambda \to 0$ the distance $4/\lambda^2$ between the two energy bounds at $E = 0$ and $E = 4/\lambda^2$ goes to infinity. Thus, the second branch of $f(E)$ disappears, giving back the well-known energy levels in the (commutative) Coulomb problem. Of course, this corresponds to the simple fact that $f(E) \to E$ for $\lambda \to 0$.

Reintroducing $\hbar$, the electron mass m_e and the electron charge e into the equations, the constant C can be computed. According to [11, Sec. 4], the value of C is (in Gaussian

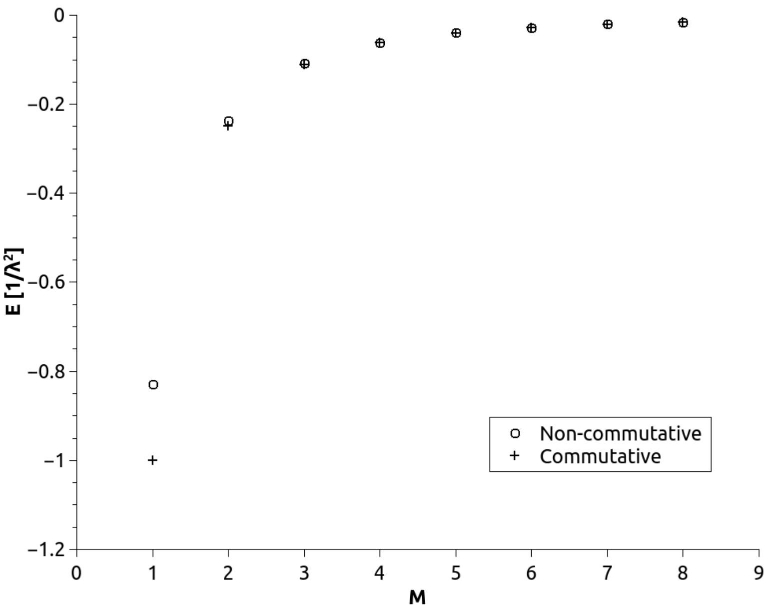

Figure 3.1: Comparison of the energy levels $-C/M^2$ in the commutative case and $-C/M^2 \times 2(1 \mp \sqrt{1+\lambda^2 C/M^2})^{-1}$ in the non-commutative case.

units)

$$C = \frac{m_e e^4}{2\hbar^2}. \tag{3.35}$$

This can be used to find an estimate for the non-commutativity parameter λ. After including the physical constants into the equations, the energy levels (3.34) read

$$E = -\frac{C}{M^2} \times \frac{2}{1 \mp \sqrt{1 + \frac{\lambda^2}{a_0^2 M^2}}} \tag{3.36}$$

with C as in (3.35) and the Bohr radius $a_0 = \frac{\hbar^2}{m_e e^2}$ in Gaussian units. The energy level $M = 1$ corresponds to the ionisation energy of hydrogen, given by the Rydberg unit

$$1\, Ry = 13.605\,692\,53(30)\, eV.$$

Let us assume that the non-commutative correction factor $2\left(1 + \sqrt{1 + \frac{\lambda^2}{a_0^2}}\right)^{-1}$ is in the order of magnitude of the uncertainty of the Rydberg unit. This is achieved by setting

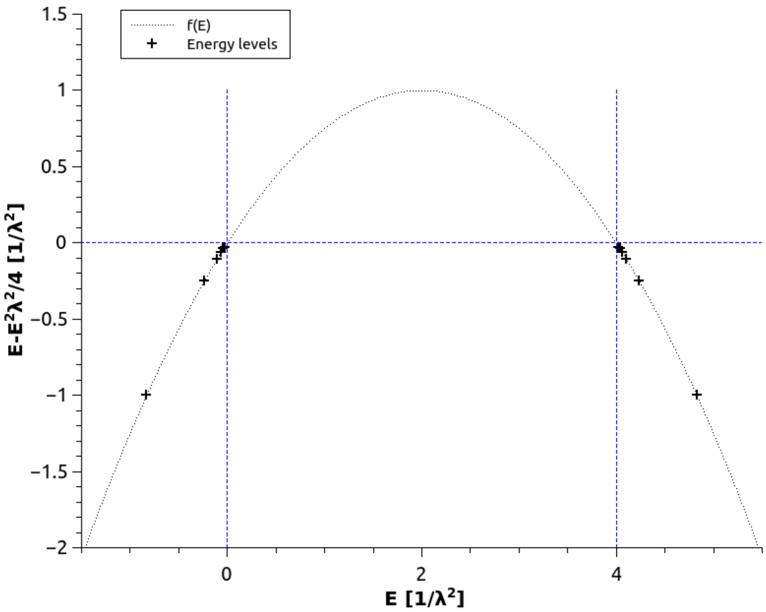

Figure 3.2: Plot of the energy function $f(E)$ and the energy levels $-C/M^2$.

$E = 13.605\,692\,83\,eV$. Solving (3.36) with $M = 1$ for λ then leads to

$$\begin{aligned}\lambda &= a_0\sqrt{\left(\frac{2C}{E}+1\right)^2-1}\\ &\approx 4.67\times 10^{-18}\,m \end{aligned} \tag{3.37}$$

for the non-commutativity parameter λ. Let us compare this to the available amount of energy in current particle accelerators. The LHC at CERN operates its two beams at an energy of $4\,TeV$ each, resulting in $E = 8\,TeV$ being released in the collision of the beams. Employing $E = h\nu$ and $c = r\nu$ where ν is the frequency corresponding to the operating energy E and r is the associated wavelength, we find a rough estimate for the resolution of the LHC at

$$r = \frac{hc}{E} \approx 10^{-18}\,m.$$

This means that the order of magnitude of the non-commutativity parameter λ in (3.37) should in principle be accessible to modern particle accelerators. However, at this scale relativistic and quantum field theoretic effects have to be taken into account, which would certainly modify the above reasoning and estimation.

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Abstract (German)

Die vorliegende Diplomarbeit behandelt den nichtkommutativen Raum $\mathbb{R}^3_\lambda$ als physikalischen Rahmen für quantenmechanische Problemstellungen. Zunächst wird die nichtkommutative Struktur dieses Raumes untersucht und ein Differentialkalkül konstruiert. Dann wird das Coulomb-Problem auf $\mathbb{R}^3_\lambda$ formuliert und dessen Energieniveaus werden mit dem kommutativen Fall verglichen.

Das erste Kapitel enthält eine kurze Einführung in nichtkommutative Räume anhand der Moyalebene. Dazu werden auf diesem Raum sowohl das kanonische Moyal-Sternprodukt als auch ein Sternprodukt basierend auf kohärenten Zuständen eingeführt. Mit Hilfe der zweiten Methode definieren wir auch auf dem deformierten $\mathbb{R}^3$ ein Sternprodukt und erläutern den Zusammenhang mit der Fuzzysphäre.

Im zweiten Kapitel werden Differentialkalküle auf nicht-kommutativen Räumen behandelt, wobei zunächst ein kurzer Überblick über den de Rham-Kalkül auf kommutativen Mannigfaltigkeiten gegeben wird. Dann führen wir den Universalkalkül auf allgemeinen assoziativen Algebren mit 1 ein und geben außerdem eine kurze Einführung in Quantengruppen. Wir entwickeln eine Methode einen Differentialkalkül auf Quantengruppen zu konstruieren. Da der Raum $\mathbb{R}^3_\lambda$ eine Quantengruppenstruktur besitzt, kann diese Methode dazu verwendet werden, ein explizites Beispiel eines vierdimensionalen Kalküls auf $\mathbb{R}^3_\lambda$ anzugeben. Wir berechnen in diesem Kalkül die äußere Ableitung von ebenen Wellen und vergleichen das Resultat mit dem kommutativen Fall.

Das dritte Kapitel beschäftigt sich mit dem Coulomb-Problem auf $\mathbb{R}^3_\lambda$. Dazu werden die Drehimpulsoperatoren $\hat{L}^i, i = 1, 2, 3$ definiert und analog zur kommutativen Quantenmechanik die Eigenfunktionen $\hat{\Psi}_{jm}$ der Operatoren $\mathcal{L}^2 := \sum_i (\hat{L}^i)^2$ and $\hat{L}^3$ berechnet. Diese erzeugen einen Hilbertraum, auf dem wir einen Laplace-Operator und einen Potentialoperator definieren und somit den Hamiltonoperator des Coulomb-Problems bestimmen können. Die Berechnung des Spektrums des Hamiltonoperators führt auf eine gewöhnliche Differentialgleichung zweiter Ordnung. Deren Lösungen bestehen aus den bekannten Energieniveaus des kommutativen Problems sowie einem Korrekturterm, der im kommutativen Limes $\lambda \to 0$ verschwindet. Weiters wird eine Abschätzung für den Nichtkommutativitätsparameter λ angegeben.

Abstract (English)

The thesis at hand discusses the non-commutative space $\mathbb{R}^3_\lambda$ as a physical framework for quantum mechanical problems. After investigating its non-commutative structure and constructing a differential calculus, we formulate the Coulomb problem on $\mathbb{R}^3_\lambda$ and compare its energy levels to the commutative case.

In the first chapter we give a short introduction to non-commutative spaces on the basis of the Moyal plane, defining both the canonical Moyal star product and a star product based on coherent states. We then specialize to deformed $\mathbb{R}^3$ and adapt the previous method of using coherent states to define a star product on this space. Furthermore, the connection between $\mathbb{R}^3_\lambda$ and the fuzzy sphere is made clear.

The second chapter deals with differential calculi on non-commutative spaces, starting with a short overview of the de Rham-calculus on commutative manifolds. A treatment of the universal calculus on arbitrary associative unital algebras is followed by an introduction to quantum groups. We develop a method to construct a differential calculus on quantum groups and apply it to $\mathbb{R}^3_\lambda$ after identifying its quantum group structure. The chapter concludes with the definition of an explicit four-dimensional differential calculus on $\mathbb{R}^3_\lambda$. We calculate the action of the exterior derivative on plane waves and learn that the results from the commutative de Rham-calculus are recovered in the commutative limit $\lambda \to 0$.

In the third chapter we discuss the Coulomb problem formulated on $\mathbb{R}^3_\lambda$. To this end, we define the angular momentum operators $\hat{L}^i, i = 1, 2, 3$ and identify the eigenfunctions $\hat{\Psi}_{jm}$ of $\mathcal{L}^2 := \sum_i (\hat{L}^i)^2$ and $\hat{L}^3$, in analogy to commutative quantum mechanics. The eigenfunctions $\hat{\Psi}_{jm}$ span a Hilbert space, on which we define a Laplace operator and a potential operator, leading to the Hamiltonian of the non-commutative Coulomb problem. The computation of the spectrum of the Hamiltonian amounts to solving an ordinary second-order differential equation. Its solutions consist of the usual energy levels of the commutative problem multiplied by a non-commutative correction term, which vanishes in the commutative limit. We also provide an estimate of the non-commutativity parameter λ.

Curriculum Vitae

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