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Verfasser:	Sebastian Woblistin
Studienkennzahl:	A 405
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Betreuer:	Bernhard Lamel

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Chapter 1
Topological Preliminaries

In this chapter we will prepare the necessary topological tools and terminology for the study of non-linear maps between non-metrizable locally convex spaces. The first part introduces different types of spaces, in which sequences suffice to describe certain topological features such as continuity or the closure of a set. It turns out that for locally convex spaces which most commonly appear in applications the property of being a Fréchet-Uryson space is – like first-countability – equivalent to metrizability (Theorem 2.5.16), while many important non-metrizable spaces are (k)- and sequential spaces. In the second part we briefly discuss the Arzelà-Ascoli-theorem and the compact-open topology.

1.1 Sequential, (k)- and Fréchet-Uryson-spaces

Definition 1.1.1. Let X be a set, $U \subset X$ and let $(x_i)_{i\in I}$ be a net in X. We say that $(x_i)_{i\in I}$ is *finally contained* in U, if there exists an index $i_0 \in I$ so that $\{x_i \mid i \geq i_0\} \subset U$. If X is a topological space and U an open subset of X, then every net in X which converges to an element of U is finally contained in U.

Definition 1.1.2. Let (X, τ) be a topological space. A subset U of X is called *sequentially open*, if every *sequence* which converges to an element of U is finally contained in U. For a subset A of X, we set $[A]_{\mathrm{seq}} := \Big\{x \in X \mid \exists\, (a_n)_{n\in\mathbb{N}} \subset A\colon a_n \to x\Big\}$, which we call the *sequential adherence* of A. If $A = [A]_{\mathrm{seq}}$, then A is called *sequentially closed*; i.e. a subset is sequentially closed iff all sequences of A which converge in X have all their limit points in A. The family of sequentially open sets defines a topology τ_{s} on X and the closed sets of (X, τ_{s}) are exactly the sequentially closed ones. A topological space (X, τ) is called *sequential* if $\tau = \tau_{\mathrm{s}}$, i.e. if every sequentially open set is τ-open. We note that in an arbitrary topological space (X, τ) every τ-open/closed set is also τ_{s}-open/closed, but in general the converse does not hold. However, τ_{s} has the same convergent sequences as τ: If $x_n \xrightarrow{\tau} x$ and if U is a τ_{s}-open-neighborhood of x, then there exists an index n_0 so that $(x_n)_{n>n_0} \subset U$, hence $x_n \xrightarrow{\tau_{\mathrm{s}}} x$. Conversely, every τ_{s}-convergent sequence is τ-convergent since $\tau \prec \tau_{\mathrm{s}}$. In fact, τ_{s} is the finest topology on X which has the same convergent sequences as τ (see Lemma 1.1.4). Note that the operation of forming the sequential topology τ_s is idempotent, i.e. $(\tau_s)_s = \tau_s$.

Lemma 1.1.3 ([Eng89, p.53])**.** *Let (X, τ) be a topological space. TFAE:*

(1) *(X, τ) is sequential.*

(2) *For every topological space (Y, σ), a function $f\colon X \to Y$ is continuous if it is sequentially continuous.*

Proof: (1)⇒ (2) Let (Y, σ) be an arbitrary topological space and $f\colon X \to Y$ a sequentially continuous map. Suppose that f is not continuous. Then there exists a (non-empty) open set $U \subset Y$ so that $f^{-1}(U)$ is not open. Hence there exists a sequence $(x_n)_{n\in\mathbb{N}}$ so that $x_n \to x \in f^{-1}(U)$ but which is not finally contained in $f^{-1}(U)$. So we can extract a subsequence $(x_{n_k})_{k\in\mathbb{N}} \subset X \setminus f^{-1}(U)$. Thus $x_{n_k} \to x$ but $f(x_{n_k}) \not\to f(x)$, which contradicts the sequential continuity of f. (2)⇒(1) The identity map $\mathrm{id}\colon (X,\tau) \to (X, \tau_S)$ is sequentially continuous and hence continuous, which implies $\tau = \tau_S$. □

Lemma 1.1.4. *Let (X, τ) be a topological space.*

(1) *τ_S is the finest topology on X which has the same convergent sequences as τ.*

(2) [KM97, p.37], [Gor] *τ_S is the final topology with respect to all convergent sequences; i.e. the final topology with respect to all continuous maps $f\colon \mathbb{N}_\infty \to (X, \tau)$, where $\mathbb{N}_\infty$ denotes the Alexandroff-compactification of $\mathbb{N}$.*

Proof: (1) Let σ another topology on X which has the same convergent sequences as τ. Then $\mathrm{id}\colon (X, \tau_S) \to (X, \sigma)$ is sequentially continuous, and thus by the above lemma continuous, which means that τ_{s} is finer than σ.
(2) Let $U \in \tau_S$, let $s\colon \mathbb{N}_\infty \to X$ be continuous and let $s(\infty) =: x$. If $x \notin U$, then $\infty \notin s^{-1}(U)$ and hence $s^{-1}(U)$ is open in $\mathbb{N}_\infty$. If $x \in U$, then there exists a number n_0 so that $(s_n)_{n\geq n_0} \subset U$ and hence $\{n \geq n_0\} \subset s^{-1}(U)$, so the preimage of U under s is open. Since s was arbitrary, it follows that U is open in the final topology of all convergent sequences. Conversely, suppose that U is open in the final topology. Let $x_n \xrightarrow{\tau} x \in U$ and let s be the continuous extension of the sequence to $\mathbb{N}_\infty$. Since U is open and since $\infty \in s^{-1}(U)$, there is an n_0 so that $\{n \geq n_0\} \subset s^{-1}(U)$, and thus $x_n = s(n) \in U$ for $n \geq n_0$. Hence U is sequentially open. □

Example 1.1.5. Every topological space which satisfies the first axiom of countability (AA1) is a sequential space. For separated topological vector spaces the properties metrizable and (AA1) are equivalent. Nevertheless, there are important classes of non-metrizable sequential LCVS: A (DFM)-*space* is a locally convex space which is the strong dual of a Fréchet-Montel space (see Definition 2.5.12). Webb ([Web68]) showed that every (DFM)-space is sequential.

Theorem 1.1.6 ([Eng89, p.54, Appendix], [Fra67])**.**
Let X be a sequential space and let $A \subset X$.

(1) *Sequentiality is passed over to open and closed subspaces of X, while arbitrary subspaces of X need not to be sequential.*

(2) *In general $\overline{A} = [A]_{\mathrm{seq}}$ does not hold.*

Definition 1.1.7. A topological space X is called a *Fréchet-Urysohn*-space (shortly (FU)-space) if $\overline{A} = [A]_{\mathrm{seq}}$ holds for every subset A of X.

Remark. It is easy to see that every (FU)-space is sequential, while the converse is false ([Eng89, p.54]). (FU)-spaces can be characterized as hereditary sequential spaces: A topological space (X, τ) is an (FU)-space if and only if every subspace of X is sequential. ([Fra67, Proposition 7.2])

Definition 1.1.8. A topological space (X, τ) is called a *(k)-space* or *compactly generated space* if a subset U of X is open if and only if $U \cap K$ is open in K for all compact subsets K of X. In other words a topological space is a (k)-space if its topology coincides with τ_{K} - the final topology with respect to all pairs (K, ι_K) where K is a compact subset of X and where ι_K denotes the inclusion map $K \to X$. If X is Hausdorff then τ_{K} can also be described as the final topology with respect to all pairs (K, f) where K is a compact topological space and $f\colon K \to X$ a τ- continuous map. We note that a topological space is a (k)-space if and only if a function $f\colon X \to Y$ to an arbitrary topological space is continuous iff all its restrictions to compact subsets of X are continuous.

Remark. It is easy to see that every Hausdorff sequential space is a (k)-space.

Definition 1.1.9. Let X, Y be topological spaces and $\mathcal{F} \subset \mathscr{C}(X, Y)$. A topology on $\mathcal{F}$ is called *jointly continuous* or *admissible* if the evaluation $\mathrm{ev}\colon \mathcal{F} \times X \to Y$ is continuous.

Theorem 1.1.10 ([Wil04, p.288])**.** *Let (X, τ) be Hausdorff and a (k)-space, let Y be a topological space and $\mathcal{F} \subset \mathscr{C}(X, Y)$ a family of continuous functions. The compact-open topology on $\mathcal{F}$ is jointly continuous and it is the coarsest topology with this property.*

Definition 1.1.11. A topological space (X, τ) is called a *Lindelöf* space if every open cover of X possesses a countable subcover. We say that X is *strongly* or *hereditarily Lindelöf* if every open subspace of X is again a Lindelöf space.

Theorem 1.1.12 ([Eng89, p.256])**.** *A metrizable space is a Lindelöf space iff it is second-countable.*

Definition 1.1.13. X is called *hemicompact* if there exists a sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of X so that every compact subset K of X is contained in some K_{N_0}. We say that X is *hereditarily hemicompact* if every open subspace of X is hemicompact.

Definition 1.1.14. A Hausdorff topological space is a *Tychonoff* space or a *completely regular* space if for every closed set $A \subset X$ and every $x_0 \in X \setminus A$ there is a continuous function $f\colon X \to [0,1]$ so that $f|_A = 0$ and $f(x_0) = 1$. Every Hausdorff topological vector space is a Tychonoff space (see for example [Sch71]).

Theorem 1.1.15 ([Wil04, p.289] [Eng89, p.165])**.**
Let X be a Hausdorff topological space.
(1) *If X is hemicompact, then $(\mathscr{C}(X, \mathbb{R}), \tau_{\mathrm{co}})$ is metrizable.*
(2) *Suppose that X is Tychonoff space. Then $(\mathscr{C}(X, \mathbb{R}), \tau_{\mathrm{co}})$ is metrizable if and only if X is hemicompact.*
(3) *If X is first-countable and hemicompact, then it is locally compact.*

Corollary 1.1.16. *A metrizable topological vector space (over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) is hemicompact iff it is finite-dimensional.*

Proof: Recall that a Hausdorff topological vector space is locally compact iff it is finite-dimensional, in which case it is isomorphic to some $\mathbb{K}^N$ (see for example [Sch71]). Then the claim is an immediate consequence of part (3) of the above theorem. □

1.2 Function spaces

Definition 1.2.1. Let X be a topological space and let Y be a uniform space. A family $\mathcal{F}$ of functions $X \to Y$ is said to be *equicontinuous on a subset A of X* if the family of restriction $\mathcal{F}|_A$ is equicontinuous as a family of functions $A \to Y$. In general, this does not imply that $\mathcal{F}$ is equicontinuous at a point $x \in A$, this (in general) holds true only if $x \in A^\circ$.

Theorem 1.2.2 (Ascoli [Kel75, p.234])**.** *Let X be a (k)-space which is either Hausdorff or regular, let Y be a Hausdorff uniform space. A subfamiliy $\mathcal{F}$ of $\mathscr{C}(X,Y)$ is compact in $(\mathscr{C}(X,Y), \tau_{co})$ if and only if satisfies the following:*
(i) $\mathcal{F}$ is closed in $(\mathscr{C}(X,Y), \tau_{co})$.
(ii) $\mathcal{F}(x)$ is relatively compact for each $x \in X$.
(iii) $\mathcal{F}$ is equicontinuous on every compact subset of X.

Definition 1.2.3. Let X and Y be topological spaces. A family $\mathcal{F}$ of functions $X \to Y$ is called *evenly continuous* if for every $x \in X$, every $y \in Y$ and each neighborhood U of y there is a neighborhood V of x and a neighborhood W of y so that $f(x) \in W$ implies that $f(V) \subset U$.

Theorem 1.2.4 ([Kel75, p.236])**.** *Let X, Y be topological spaces and let Y be a regular Hausdorff space. If a family $\mathcal{F} \subset \mathscr{C}(X,Y)$ is compact with respect to a jointly continuous topology, then $\mathcal{F}$ is evenly continuous.*

Theorem 1.2.5 ([Kel75, p.237])**.** *Let X be a topological space, let Y be a uniform space and $\mathcal{F} \subset \mathscr{C}(X,Y)$.*
(1) *If $\mathcal{F}$ is equicontinuous, then it is evenly continuous.*
(2) *If $\mathcal{F}$ is evenly continuous and $x \in X$ is a point of X so that $\mathcal{F}(x)$ is relatively compact in Y, then $\mathcal{F}$ is equicontinuous at x.*

Proposition 1.2.6. *Let X be a (k)-space and let Y be a regular Hausdorff uniform space. A family $\mathcal{F} \subset \mathscr{C}(X,Y)$ is compact in $(\mathscr{C}(X,Y), \tau_{co})$ if and only if*
(i) $\mathcal{F}$ is closed in $(\mathscr{C}(X,Y), \tau_{co})$
(ii) $\mathcal{F}(x)$ is relatively compact for all $x \in X$.
(iii) $\mathcal{F}$ is equicontinuous.

Proof: Sufficiency follows from Theorem 1.2.2. Suppose that $\mathcal{F}$ is compact. Then (1) and (2) follow again from Theorem 1.2.2. The compact-open topology is jointly continuous on (k)-spaces ([Wil04, p.288]) so Theorem 1.2.4 can be applied to conclude that $\mathcal{F}$ is evenly continuous. Because of (2) we an apply Theorem 1.2.5 to conclude that $\mathcal{F}$ is equicontinuous. □

This form of the Arzelà-Ascoli-theorem is particularly useful in the setting of locally convex spaces: They are always regular, and the class of locally convex spaces which are (k)-spaces covers metrizable as well as (DFM)-spaces (see Proposition 2.5.15).

Theorem 1.2.7 ([Din99, p.166]). *Let X be a topological space and let F be a locally convex space whose topology is generated by the set of semi-norms $\mathcal{P}$. Then $(\mathscr{C}(X,F),\tau_{\text{co}})$ is a locally convex space and τ_{co} is described by the system of semi-norms $\left\{\| \ \|_{K,\phi} \ : \ K \overset{\text{co}}{\subset} X, \ \phi \in \mathcal{P}\right\}$, where $\|F\|_{K,\phi} := \sup_{a\in K} \phi\left(F(a)\right)$.*

Chapter 2
Functional Analysis

After recalling basic definitions and properties of locally convex spaces we give a broader survey of projective and especially inductive limits of locally convex spaces, which are in applications the most important spaces beyond Banach or Fréchet spaces. Even the inductive limit of a sequence of normed spaces may have pathological topological properties, hence additional regularity assumptions are needed to ensure that the limit topology enjoys good features and relates to the generating sequence. Particular strong results hold for (DFM)-spaces (a limit of normed spaces which has the Heine-Borel property), which turn out to be sequential spaces, and the subclass of (DFS)-spaces (a limit of a sequence of normed spaces with compact linking maps), which additionally satisfy that the convergent sequences are exactly those of the "steps" of a generating sequence. In the last part we discuss Mackey-convergence and local completeness, which play an important role in infinite-dimensional calculus.

Basic definitions. We call a topological vector space (TVS) over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ *locally convex space* (LCVS) if it is *Hausdorff* (instead of Hausdorff we will also use the term *separated*) and if its topology admits a zero-neighborhood-base of absolutely convex sets. A *topology* τ on a vector space E is called *locally convex* if (E, τ) is a topological vector space (over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) and if the topology admits a zero-neighborhood-base of absolutely convex sets. A locally convex topology may not be Hausdorff. By $\operatorname{cs}((E, \tau))$ we denote the set of all continuous semi-norms on a vector space E carrying a locally convex topology τ. We say that a system of semi-norms $\mathcal{P}$ on E describes the topology τ if a net $(x_\alpha)_{\alpha \in \mathcal{A}}$ converges to x in (E, τ) if and only if $p(x - x_\alpha) \to 0$ for all $p \in \mathcal{P}$. By $\mathcal{U}_0^E$ we denote the system of zero neighborhoods of E. A subset B of a topological vector space is called *bounded* if for every $U \in \mathcal{U}_0^E$ there is a $\lambda > 0$ so that $B \subset \lambda U$. If E is equipped with a locally convex topology which is generated by the system of semi-norms $\mathcal{P}$ then a subset B of E is bounded iff $\sup\{p(x) \mid x \in B\} < \infty$ for every $p \in \mathcal{P}$. As a consequence, the absolutely convex hull of a bounded set in a TVS with a locally convex topology is again bounded. By $\mathcal{B}(E)$ we denote the family of bounded subsets of E. A set $M \subset E$ is called *bornivorous* if for every $B \in \mathcal{B}(E)$ there exists a $\lambda > 0$ so that $B \subset \lambda M$ and a locally convex space in which every absolutely convex bornivorous set is a neighborhood of zero is called *bornological*. A bounded and absolutely convex subset B of a LCVS (E, τ) is called a *disc*. We write $\mathbb{D}(E)$ for the family of all discs in E. If B is a disc in E, then E_B denotes the linear span of B endowed with the topology defined by the Minkowski functional ρ_B, which is a norm on E_B. The gauge-topology on E_B is finer than the subspace-topology of E_B as the inclusion map $\iota \colon E_B \to E$ is continuous since it is bornological. A disc B is called a *Banach disc* if E_B is complete. We note that every compact disc is a Banach disc (see [PCB87, p.83]). E' denotes the dual space of a LCVS E and E' equipped with the topology of uniform convergence on bounded subsets of E is called the strong *strong dual* of E which we will denote by E'_b.

2.1 Inductive Limits of locally convex spaces

Definition 2.1.1. A partially ordered set $(\mathcal{A}, \preceq)$ is called a *directed set* if for any $\alpha, \beta \in \mathcal{A}$ there exists a $\gamma \in \mathcal{A}$ so that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$. A subset $\mathcal{J}$ of $\mathcal{A}$ is *cofinal* in $\mathcal{A}$ if for any $\alpha \in \mathcal{A}$ there exists $\gamma \in \mathcal{J}$ so that $\alpha \preceq \gamma$.

Definition 2.1.2. Let $\mathcal{S} = (E_i, \tau_i)_{i \in \mathcal{I}}$ be a system of LCVS, let E be a vector space and $\mathcal{F} = (f_i)_{i \in \mathcal{I}}$ be a system of linear maps, where $f_i \colon E_i \to E$. The *inductive topology* on E with respect to $(\mathcal{S}, \mathcal{F})$ is the finest *locally convex* topology on E for which all maps f_i are continuous. Note that - in general - it is *not* Hausdorff and that it is strictly coarser than the final topology on E with respect to $(\mathcal{S}, \mathcal{F})$ (i.e. the finest topology on E for which all f_i are continuous). The inductive topology is generated by the set of semi-norms $\{q \text{ seminorm on E} \mid q \circ f_i \text{ continuous } \forall i \in \mathcal{I}\}$. Let (F, σ) be a topological vector space carrying a locally convex topology. Then a linear map $T \colon E \to F$ is continuous if and only if all compositions $T \circ f_i$ are continuous.

Definition 2.1.3. An *inductive spectrum* is a system $\mathcal{S} = (E_\alpha, \tau_\alpha)_{\alpha \in \mathcal{A}}$ of locally convex spaces which is indexed by a directed set $\mathcal{A}$ together with a system $\Pi = (\pi_{\alpha,\beta})_{\alpha \preceq \beta \in \mathcal{A}}$ of continuous linear maps where $\pi_{\alpha,\beta} \colon E_\alpha \to E_\beta$, which satisfies that $\pi_{\alpha,\alpha} = \mathrm{id}_\alpha$ and that $\pi_{\beta,\gamma} \circ \pi_{\alpha,\beta} = \pi_{\alpha,\gamma}$. A topological vector space (E, τ) carrying a locally convex topology τ is called *inductive limit* of an inductive spectrum $(\mathcal{S}, \Pi)$, if there exists a system of continuous linear maps $\mathcal{E} = (\epsilon_\alpha)_{\alpha \in \mathcal{A}}$ (which we call a *universal inductive cone*), where $\epsilon_\alpha \colon E_\alpha \to E$, which satisfies:

(i) $\epsilon_\beta \circ \pi_{\alpha,\beta} = \epsilon_\alpha$

(ii) For any locally convex space (H, σ) and a system of continuous linear maps $f_\alpha \colon E_\alpha \to H$ satisfying $f_\beta \circ \pi_{\alpha,\beta} = f_\alpha$ (any such system is called an *inductive cone*), there exists a unique continuous linear map $F \colon E \to H$ with $F \circ \epsilon_\alpha = f_\alpha$.

The inductive limit exists for every inductive spectrum and is - up to isomorphism in the category of topological vector spaces - uniquely determined. We denote it by $\varinjlim(\mathcal{S}, \Pi)$. The topology of the inductive limit is the inductive topology with respect to $(\mathcal{S}, \mathcal{E})$. It might fail to be Hausdorff, even if all ϵ_α are injective and all steps E_α are Hausdorff (see Example 2.1.7).

Theorem 2.1.4 (Construction of the inductive limit)**.** *Let* $(\mathcal{S}, \Pi)$ *be an inductive spectrum indexed by the directed set* $\mathcal{A}$*. Let* $\iota_\alpha \colon E_\alpha \to \bigoplus_{\alpha \in \mathcal{A}} E_\alpha$ *be the canonical injection and let* $M := \operatorname{span} \bigcup_{\alpha \preceq \beta} \operatorname{Im}(\iota_\alpha - \iota_\beta \circ \pi_{\alpha,\beta})$*. Set* $E := \bigoplus_{\alpha \in \mathcal{A}} E_\alpha \big/ M$*, and let* $\Phi \colon \bigoplus_{\alpha \in \mathcal{A}} E_\alpha \to E$ *be the canonical quotient map. Furthermore, let* $\epsilon_\alpha := \Phi \circ \iota_\alpha$ *and let* $\mathcal{E} = (\epsilon_\alpha)_{\alpha \in \mathcal{A}}$*. Then* E *equipped with the inductive topology with respect to* $(\mathcal{S}, \mathcal{E})$ *is an inductive limit of* $(\mathcal{S}, \Pi)$ *and is unique up to isomorphism.*

Proof: For $\alpha \preceq \beta$, observe that $\epsilon_\alpha - \epsilon_\beta \circ \pi_{\alpha,\beta} = \Phi \circ (\iota_\alpha - \iota_\beta \circ \pi_{\alpha,\beta}) = 0$, so $\epsilon_\alpha = \epsilon_\beta \circ \pi_{\alpha,\beta}$. Let (H, σ) be a topological vector space and let $\mathcal{F} = (f_\alpha)_{\alpha \in \mathcal{A}}$ be a system of continuous linear maps such that $f_\alpha \colon E_\alpha \to H$ and $f_\alpha = f_\beta \circ \pi_{\alpha,\beta}$ for $\alpha \preceq \beta$. Let $T \colon \bigoplus_{\alpha \in \mathcal{A}} E_\alpha \to H,\ T(x) := \sum f_\alpha (p_\alpha (x))$, where $p_\alpha \colon \bigoplus_{\alpha \in \mathcal{A}} E_\alpha \to E_\alpha$ is the canonical projection. T is linear by definition and we observe that $T \circ (\iota_\alpha - \iota_\beta \circ \pi_{\alpha,\beta}) = f_\alpha - f_\beta \circ \pi_{\alpha,\beta} = f_\alpha - f_\alpha = 0$. Hence $M \subset \ker(T)$. Let F be the unique linear map $F \colon E \to H$ which satisfies that $F \circ \Phi = T$. Since $F \circ \epsilon_\alpha = f_\alpha$ we can conclude that F is continuous. Also, F is uniquely determined by the condition $F \circ \epsilon_\alpha = f_\alpha$ since $\operatorname{span} \bigcup_{\alpha \in \mathcal{A}} \epsilon_\alpha (E_\alpha) = E$. So far we have shown that (E, τ) is an inductive limit of (S, Π). Let (E_2, τ_2) be another inductive limit with lifting maps $\delta = (\delta_\alpha)_{\alpha \in \mathcal{A}}$, where $\delta_\alpha \colon E_\alpha \to E_2$. Then there exist continuous linear maps $T \colon E_1 \to E_2$ and $U \colon E_2 \to E_1$ so that $T \circ E_\alpha = \delta_\alpha$ and $U \circ \delta_\alpha = \epsilon_\alpha$. Hence $U \circ T \circ \epsilon_\alpha = \epsilon_\alpha$. This means that $U \circ T$ lifts the system of maps $\mathcal{E} \colon \mathcal{S} \to (E, \tau)$. However, id_E has the same property and by uniqueness we conclude that $\mathrm{id}_E = U \circ T$. By the same arguments we can also conclude that $\mathrm{id}_{E_2} = T \circ U$. Hence T is a continuous linear bijection between E and E_1 with continuous inverse. □

Definition 2.1.5. An *inductive net* is a family of locally convex spaces $(E_\alpha, \tau_\alpha)_{\alpha \in \mathcal{A}}$ which is indexed by a directed set $\mathcal{A}$ and which satisfies that $E_\alpha \subset E_\beta$ and that the inclusion map $\iota_{\alpha,\beta} \colon E_\alpha \to E_\beta$ is continuous (which we are going to denote by $E_\alpha \hookrightarrow E_\beta$), for $\alpha < \beta$. If the index set is $\mathbb{N}$ with the usual ordering, the inductive net is called an *inductive sequence*. The *inductive limit* of an inductive net $(E_\alpha, \tau_\alpha)_{\alpha \in \mathcal{A}}$ is the vector space $E := \bigcup_{\alpha \in \mathcal{A}} E_\alpha$ equipped with the inductive topology with respect to all inclusion $\iota_\alpha \colon E_\alpha \to E$. We say that a given topological vector space with a locally convex topology (E, τ) is generated by an inductive net $(E_\alpha, \tau_\alpha)_{\alpha \in \mathcal{A}}$ if $\varinjlim E_\alpha = (E, \tau)$. We call an inductive net *proper* if $E_\alpha \subsetneq E_\beta$ for $\alpha \precneqq \beta$.

Lemma 2.1.6. *Let $(\mathcal{S}, \Pi)$ be an inductive spectrum indexed by $\mathcal{A}$.*

(1) [Mor93, p.246] *Let $\mathcal{J}$ be a cofinal subset of $\mathcal{A}$, let $\mathcal{S}_{|\mathcal{J}} := (E_j, \tau_j)_{j \in \mathcal{J}}$ and $\Pi_{|\mathcal{J}} := \{\pi_{\alpha,\beta} \ : \ \alpha \preceq \beta,\ \alpha, \beta \in \mathcal{J}\}$. Then $\varinjlim (\mathcal{S}, \Pi) = \varinjlim \left(\mathcal{S}_{|\mathcal{J}}, \Pi_{|\mathcal{J}}\right)$.*

(2) [FW68, p.120] *Suppose that all $\pi_{\alpha,\beta}$ are injective. Then all ϵ_α are injective and we have that $\epsilon_\alpha (E_\alpha) \subset \epsilon_\beta (E_\beta)$. Let $F_\alpha = \epsilon_\alpha (E_\alpha)$ and let σ_α be the topology induced by ϵ_α on F_α so that $(E_\alpha, \tau_\alpha) \cong (F_\alpha, \sigma_\alpha)$. Then $(F_\alpha, \sigma_\alpha)_{\alpha \in \mathcal{A}}$ is an inductive net which produces the same inductive limit as $(\mathcal{S}, \Pi)$. Furthermore, all properties of the linkinkg maps $\pi_{\alpha,\beta}$ which are invariant under composition with isomorphisms (of LCVS) carry over to the injections $\iota_{\alpha,\beta} \colon F_\alpha \hookrightarrow F_\beta$.*

Remark. From now on we will not distinguish between inductive nets and inductive spectra with *injective* linking maps, which is justified by the above lemma. We will intrinsically assume that we pass over to the representation as an inductive net if we deal with an inductive spectrum with injective linking maps.

Remark. One of the consequences of the first part of the lemma is that we can enlarge diagrams as long as the linking maps in the new diagram lead back to the old one (in the fashion as required in the lemma) and that in applications it is usually enough to consider enumerable spectra: For example, we consider the *inverse* half-ordering $\preceq_{inv}$ on $\mathbb{R}^d_+$, where $r \preceq_{inv} s$ iff $r_i \geq s_i$ for all $1 \leq i \leq d$. If $(\mathcal{S}, \Pi)$ is an inductive spectrum indexed by $\left(\mathbb{R}^d_+, \preceq_{inv}\right)$ and $(r_n)_{n\in\mathbb{N}}$ is an arbitrary null sequence in $\mathbb{R}^d_+$, then $\varinjlim (\mathcal{S}, \Pi) = \varinjlim \left(\mathcal{S}|_{(r_n)_{n\in\mathbb{N}}}, \Pi|_{(r_n)_{n\in\mathbb{N}}}\right)$. Of course the same holds whenever the index set has a countable cofinal subset. Suppose now that in addition all linking maps $\pi_{\alpha,\beta}$ are injective. Then $\left(\mathcal{S}|_{(r_n)_{n\in\mathbb{N}}}, \Pi|_{(r_n)_{n\in\mathbb{N}}}\right)$ is equivalent to an inductive sequence. From now on, we will only regard inductive nets, which are in applications the most common form of inductive spectra. Still, all problems (concerning the topology) of the general concept for inductive spectra appear when dealing with inductive nets:

Example 2.1.7 ([Flo80, p.207])**.** The inductive topology might decay to the chaotic topology, even if all steps are normed spaces: Let P_0 be the space of all real polynomials vanishing at 0 and let $E_n := (P_0, \|\ \|_n)$, where $\|f\|_n := \max\left\{|f(x)| : x \in \left[0, \frac{1}{n}\right]\right\}$. Then $E = \varinjlim E_n$ carries the chaotic topology (= indiscrete topology).

Definition 2.1.8. We are going to introduce different types of inductive limits. A LCVS (E, τ) is called an *(LF)/(LB)/(LNORM)/(LM)-space*, if there is an inductive net of *Fréchet/Banach/Normed/Pre-Fréchet spaces* $(E_n)_{n\in\mathbb{N}}$ with $\varinjlim (E_n)_{n\in\mathbb{N}} = (E, \tau)$. The terminology is not unified in the literature - for example Schaefer reserves the term (LF)-space to limits of *strict* inductive sequences of Fréchet spaces - but in more recent publications the terms (LB) and (LF) are used according to the above definitions.

Despite the above example the most important spaces carrying an inductive limit structure (e.g. the space of bump functions or the space of germs of holomorphic functions) have good properties in the sense that their (linear-)topological features are strongly related with those of the spaces of a defining inductive net of locally convex spaces (which has the space under consideration as inductive limit). Indeed these properties may depend on the choice of the defining net, but usually the regularity properties are passed over to equivalent inductive nets.

Definition 2.1.9. Two inductive sequences $(E_n)_{n\in\mathbb{N}}$, $(F_n)_{n\in\mathbb{N}}$ are called equivalent if for every $n \in \mathbb{N}$ there exists an $m \in \mathbb{N}$ so that $E_n \hookrightarrow F_m$ and if vice versa for all $m \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ so that $F_m \hookrightarrow E_n$. Equivalent sequences have the same inductive limit, the converse is true if the limit is Hausdorff and if all sequence members are Fréchet spaces (Theorem 2.1.11).

Theorem 2.1.10 ([Flo71, p.161])**.** *Let F and $(E_n)_{n\in\mathbb{N}}$ be Fréchet-spaces and suppose that $E = \lim E_n$ is separated. For every continuous linear map $T\colon F \to E$ there is an index n_0 so that $T(F) \subset E_{n_0}$ and $T\colon F \to (E_{n_0}, \tau_{n_0})$ is continuous.*

Theorem 2.1.11 ([Flo80, p.209])**.** *Let E be a Hausdorff-(LF)-space and let $(E_n)_{n\in\mathbb{N}}$ and $(F_n)_{n\in\mathbb{N}}$ be inductive sequences of Fréchet spaces which generate E, i.e. $E = \varinjlim E_n = \varinjlim F_n$. Then $(E_n)_{n\in\mathbb{N}}$ and $(F_n)_{n\in\mathbb{N}}$ are equivalent.*

As a consequence the usual regularity properties of separated (LF)-spaces which relate to the steps of a generating sequence (of Fréchet-spaces) are independent of the particular choice of the generating sequence, which is quite useful as one can be sure that the lack of good properties of a sequence of Fréchet spaces is not due to a bad choice of the generating sequence but a defect of the limit space itself. However, there are (LM)-spaces with nonequivalent generating sequences (Example 2.2.9).

2.2 Regularity concepts for inductive limits

Definition 2.2.1. Let $(E_\alpha, \tau_\alpha)_{\alpha\in\mathcal{A}}$ be an inductive net of LCVS.

(1) We call $(E_\alpha, \tau_\alpha)_{\alpha\in\mathcal{A}}$ *regular* if for every bounded set $B \subset \varinjlim E_\alpha$ there exists a step E_{α_0} so that B is bounded in $(E_{\alpha_0}, \tau_{\alpha_0})$.

(2) $(E_\alpha, \tau_\alpha)_{\alpha\in\mathcal{A}}$ is *sequentially retractive* if for every convergent *sequence* $(x_n)_{n\in\mathbb{N}}$ in $\varinjlim E_\alpha$ there is a step E_{α_0} so that $(x_n)_{n\in\mathbb{N}} \subset E_{\alpha_0}$ and x_n converges in $(E_{\alpha_0}, \tau_{\alpha_0})$.

(3) Let $\mathcal{A} = \mathbb{N}$ with the usual ordering. We say that $(E_n, \tau_n)_{n\in\mathbb{N}}$ has property (M) if there exists a sequence (U_n) of 0-neighborhoods U_n in (E_n, τ_n) so that $U_n \subset U_{n+1}$ and which satisfies

$$\forall n\ \exists j > n\ \forall k > j : (U_n, \tau_j) = (U_n, \tau_k)$$

We will call property (M) also *Retakh's condition.*

Lemma 2.2.2. *If $(E_n)_{n\in\mathbb{N}}$ is regular, then $\varinjlim E_n$ is Hausdorff.*

Proof: $E := \varinjlim E_n$ is Hausdorff if and only if $N := \bigcap_{U\in\mathcal{U}_0^E} U = \{0\}$. N is the closure of $\{0\}$ in E and hence a subspace of E, which is bounded and therefore contained and bounded in some step E_{n_0}. For $U \in \mathcal{U}_0^{E_{n_0}}$ there is a $\lambda > 0$ so that $N \subset \lambda U$, and hence $N \subset U$. This implies that $N \subset \bigcap_{U\in U_0^{E_{n_0}}} U$ and thus $N = \{0\}$ as E_{n_0} is Hausdorff. □

Theorem 2.2.3 ([Flo80, p.214])**.** *Let $(E_n)_{n\in\mathbb{N}}$ be an inductive sequence of normed spaces, and let K_n be the closed unit ball in E_n. If the set $\sum_{i=1}^m \epsilon_i K_i$ is closed in E_{m+1} for all $m \in \mathbb{N}$ and for all $\epsilon_1, \dots, \epsilon_m > 0$, then $(E_n)_{n\in\mathbb{N}}$ is regular.*

Lemma 2.2.4 ([Flo73], [Kuc01])**.** *For an inductive sequence $(E_n)_{n\in\mathbb{N}}$ of Fréchet-spaces, TFAE:*

(1) $\varinjlim E_n$ *is regular.*

(2) $\varinjlim E_n$ *is Hausdorff and sequentially complete.*

(3) $\varinjlim E_n$ *is Hausdorff and Mackey-complete (see Definition 2.9.1).*

The following theorem is a generalization of Grothendieck's factorization theorem due to K. Floret, which we present in a slightly more general way as we make no local convexity assumptions about the source space.

Theorem 2.2.5 ([Flo73, p.69])**.** *Let (F, τ) be a metrizable topological vector space, and let $((E_n, \tau_n))_{n\in\mathbb{N}}$ be a sequentially retractive sequence. For every continuous linear map $T\colon F \to \varinjlim E_n$ there is a number $n_0 \in \mathbb{N}$ so that T factorizes continuously over some E_{n_0}, i.e. there is an $n_0 \in \mathbb{N}$ so that $T \in \mathcal{L}(F, (E_{n_0}, \tau_{n_0}))$.*

Proof: Let $(V_n)_{n\in\mathbb{N}}$ be a countable zero-neighborhood-base of F with $V_{n+1} \subset V_n$. First we show that $T(F) \subset E_{n_0}$ for some $n_0 \in \mathbb{N}$. Suppose that this does not hold. Then for every $n \in \mathbb{N}$ there is a $x_n \in F$ so that $T(x_n) \in E \setminus E_n$ and we may choose x_n so that $x_n \in V_n$. By construction, $x_n \to 0$ but $(T(x_n))$ cannot be a null sequence in $\varinjlim E_n$ as it is not contained in any step E_n, which contradicts the continuity of T and thus $T(F)$ has to be contained in some E_{n_0}. It remains to show that $T\colon F \to (E_{n_1}, \tau_{n_1})$ is continuous for some $n_1 \geq n_0$. WLOG we may suppose $n_0 = 1$. Suppose $T\colon F \to (E_k, \tau_k)$ is discontinuous for every $k \in \mathbb{N}$. Then for every $k \in \mathbb{N}$ there is a null-sequence $\left(x_n^k\right)_{n\in\mathbb{N}}$, where $x_n^k \in V_n$, for which $\left(T(x_n^k)\right)_{n\in\mathbb{N}}$ does not converge to 0 with respect to τ_k. Then the sequence $x_1^1, x_2^1, x_2^2, x_3^1, x_3^2, x_3^3, x_4^1, \ldots$ is a null-sequence in F, but the image sequence cannot converge to 0 in any step E_n, which again contradicts the continuity of T because of the supposed sequential retractivity of $(E_n)_{n\in\mathbb{N}}$, which proves the claim. □

Lemma 2.2.6 ([Flo73, p.67/68])**.** *For a sequentially retractive sequence of LCVS $(E_n)_{n\in\mathbb{N}}$, the following holds:*

(1) *$(E_n)_{n\in\mathbb{N}}$ is regular.*
(2) *$\varinjlim E_n$ is separated.*
(3) *$x_n \to x$ in $\varinjlim E_n$ implies that $x_n \to x$ in some (E_{n_0}, τ_0).*
(4) *If all E_n are Fréchet-spaces, then any basis of $\varinjlim E_n$ is a Schauder-basis.*
(5) *Any equivalent sequence to $(E_n)_{n\in\mathbb{N}}$ is also sequentially retractive.*

Proof: (1) Let B be bounded in $\varinjlim E_n$. WLOG we may suppose B to be absolutely convex and we define E_B as the linear span of B equipped with the gauge-topology associated to B. The space E_B is metrizable and the injection $\iota\colon E_B \to \varinjlim E_n$ is continuous, thus (by Theorem 2.2.5) ι has to continuously factorize over some step E_{n_0}, which implies that B has to be bounded in E_{n_0}. (2) and (3) follow from Lemma 2.2.2. (4) see [Flo70] . (5) follows from the definition. □

As an immediate consequence of the preceding lemma and Theorem 2.1.11 we obtain the following result:

Lemma 2.2.7. *If an (LF)-space E possesses a sequentially retractive generating sequence of Fréchet spaces, then every other generating sequence of Fréchet spaces for E is also sequentially retractive.*

Definition 2.2.8. An (LF)-space which can be generated by a sequentially retractive sequence of Fréchet spaces is called a sequentially retractive (LF)-space. By the preceding lemma this property is independent of the particular choice of the sequence. However, this is not true if the sequences consist of Pre-Fréchet spaces:

Example 2.2.9 ([Flo73, p.69/4.1 and p.70])**.** Let (H, τ) be a Banach-space and let $(H_n)_{n\in\mathbb{N}}$ be a sequence of proper and dense subspaces with $H = \bigcup_{n\in\mathbb{N}} H_n$. Then $(H, \tau) = \varinjlim H_n$ and $(H_n)_{n\in\mathbb{N}}$ is a generating sequence of (H, τ) which is not sequentially retractive. Now set $(F_n, \tau_n) = (H, \tau)$. Then also $(F_n)_{n\in\mathbb{N}}$ is a generating sequence of (H, τ), but which is of course sequentially retractive.

There are other strong regularity concepts which we only mention briefly (see for example [Wen96]).

Definition 2.2.10. An inductive sequence $(E_n, \tau_n)_{n\in\mathbb{N}}$ with inductive limit (E, τ) is called *boundedly retractive* if for every bounded set B in E there is $n \in \mathbb{N}$ such that $B \subset E_n$ and the topologies τ and τ_n coincide on B. The sequence $(E_n, \tau_n)_{n\in\mathbb{N}}$ is called *(sequentially) compactly regular* if every (sequentially) compact subset of the inductive limit is (sequentially) compact in some step.

Theorem 2.2.11 ([Wen96, Theorem 2.7]).
For an inductive sequence $(E_n, \tau_n)_{n\in\mathbb{N}}$ of Fréchet-spaces, TFAE:
(1) $(E_n, \tau_n)_{n\in\mathbb{N}}$ *is sequentially retractive.*
(2) $(E_n, \tau_n)_{n\in\mathbb{N}}$ *is boundedly retractive.*
(3) $(E_n, \tau_n)_{n\in\mathbb{N}}$ *is sequentially compactly regular.*
(4) $(E_n, \tau_n)_{n\in\mathbb{N}}$ *is compactly regular.*
(5) $(E_n, \tau_n)_{n\in\mathbb{N}}$ *satisfies property (M).*

Theorem 2.2.12 ([Wen96, Corollary 2.8]). *Let E be an (LF)-space. If E is sequentially retractive, then E is complete.*

2.3 Projective Limits of locally convex spaces

The dual concept to inductive spectra is that of projective spectra:

Definition 2.3.1. Let $S = (E_\alpha, \tau_\alpha)_{\alpha\in\mathcal{A}}$ be a system of LCVS. Let E be a vector space and $\mathcal{E} = (\epsilon_\alpha)_{\alpha\in\mathcal{A}}$ be a system of linear maps, where $\epsilon_\alpha \colon E \to E_\alpha$. The *projective topology τ_{proj}* on E with respect to $(\mathcal{S}, \mathcal{E})$ is the coarsest topology on E for which all maps ϵ_α are continuous (it coincides with the initial topology from general topology). τ_{proj} is locally convex and $\{q \circ f_\alpha \,|q \in \mathrm{cs}\,(E_\alpha, \tau_\alpha)\,,\ \alpha \in \mathcal{A}\}$ describes τ_{proj}. If $\mathcal{E}$ *separates points* (i.e. $\forall x \in E\ \exists f \in \mathcal{E} \colon f(x) \neq 0$), then τ_{proj} is Hausdorff. Let (X, τ) be an arbitrary topological space and $f \colon X \to (E, \tau_{\mathrm{proj}})$. Then f is continuous if and only if $\epsilon_\alpha \circ f$ is continuous for every $\alpha \in \mathcal{A}$.

Definition 2.3.2. A *projective spectrum* is a system of LCVS $\mathcal{S} = (E_\alpha, \tau_\alpha)_{\alpha\in\mathcal{A}}$ which is indexed by a directed set $\mathcal{A}$ together with a system $\Pi = (\pi_{\alpha,\beta})_{\alpha\preceq\beta\in\mathcal{A}}$ of continuous linear maps where $\pi_{\alpha,\beta} \colon E_\beta \to E_\alpha$, which satisfies that $\pi_{\alpha,\beta} \circ \pi_{\beta,\gamma} = \pi_{\alpha,\gamma}$ and that $\pi_{\alpha,\alpha} = \mathrm{id}_\alpha$. We say that a projective spectrum is *enumerable* if its index set is enumerable. To a given projective spectrum $(\mathcal{S}, \Pi)$, a topological vector space (E, τ) carrying a locally convex topology τ is called its *projective limit*, if there exists a system of continuous linear maps $\mathcal{E} = (\epsilon_\alpha)_{\alpha\in\mathcal{A}}$ (which we call a *universal projective cone*), where $\epsilon_\alpha \colon E \to E_\alpha$, which satisfies:

(i) $\pi_{\alpha,\beta} \circ \epsilon_\beta = \epsilon_\alpha$

(ii) For any locally convex space (H, σ) and any system of continuous linear maps $f_\alpha \colon H \to E_\alpha$ satisfying $\pi_{\alpha,\beta} \circ f_\beta = f_\alpha$ (any such system is called a *projective cone*), there exists a unique continuous linear map $F \colon H \to E$ with $\epsilon_\alpha \circ F = f_\alpha$.

The projective limit exists for every projective spectrum and is - up to isomorphism in the category of topological vector spaces - uniquely determined. We denote it by $\varprojlim(\mathcal{S}, \Pi)$. The topology of the projective limit is the projective topology with respect to $(\mathcal{S}, \mathcal{E})$.

Theorem 2.3.3 (Construction and properties of the projective limit)**.** *Let* (S, Π) *be a projective spectrum indexed by* $\mathcal{A}$ *and let* $\epsilon_\alpha \colon \prod_{\beta\in\mathcal{A}} E_\beta \to E_\alpha$ *be the projection onto* E_α. *Set*

$$E := \left\{ x \in \prod_{\alpha\in\mathcal{A}} E_\alpha \,\middle|\, \epsilon_\alpha(x) = \pi_{\alpha,\beta} \circ \epsilon_\beta(x) \ \forall \alpha \le \beta \in \mathcal{A} \right\}, \ \mathcal{E} := (\epsilon_\alpha)_{\alpha\in\mathcal{A}}$$

(1) *E equipped with the projective topology with respect to $(\mathcal{S}, \mathcal{E})$ is a projective limit of $(\mathcal{S}, \Pi)$ with universal projective cone $\mathcal{E}$ and is unique up to isomorphism.*
(2) *E is a closed subspace of $\prod_{\alpha\in\mathcal{A}} E_\alpha$ and the projective topology on E is the subspace-topology induced by $\prod_{\alpha\in\mathcal{A}} E_\alpha$.*
(3) *E is always Hausdorff and is complete if all (E_α, τ_α) are complete.*
(4) *E is metrizable if $\mathcal{A}$ is countable and if all (E_α, τ_α) are metrizable.*
(5) *Every complete LCVS can be represented as the limit of a projective spectrum of Banach spaces.*
(6) *Let $\mathcal{J}$ be a cofinal subset of $\mathcal{A}$. Then* $\varprojlim (\mathcal{S}, \Pi) = \varprojlim \left(\mathcal{S}_{|\mathcal{J}}, \Pi_{|\mathcal{J}} \right)$
(7) *If $E = \lim(\mathcal{S}, \Pi)$ is complete, then a subset K of E is relatively compact if and only if $\epsilon_\alpha(K)$ is relatively compact in E_α for all $\alpha \in \mathcal{A}$.*

The proof of (1) is simple and similar to that of Theorem 2.1.4, proofs to (2)-(5) can be found in [FW68, §6 and §12] and for (6) we refer to [Mor93, p.294]. (7) is a direct consequence of Tychonoff's Theorem (see [FW68, p.75]).

Definition 2.3.4. A projective spectrum $(\mathcal{S}, \Pi)$ indexed by $\mathcal{A}$ with projective limit E and a universal cone $\mathcal{E} = (\epsilon_\alpha)_{\alpha\in\mathcal{A}}$ is called *reduced* if $\epsilon_\alpha(E)$ is dense in E_α for every $\alpha \in \mathcal{A}$ and we say that $(\mathcal{S}, \Pi)$ is *strict* if $\pi_{\alpha,\beta}(E_\beta)$ is dense in E_α. By construction we have $\pi_{\alpha,\beta}(\epsilon_\beta(E_\beta)) = \epsilon_\alpha(E)$, which shows that every reduced spectrum is strict.

Definition 2.3.5. A *projective net* is a family $(E_\alpha)_{\alpha\in\mathcal{A}}$ of LCVS indexed by a directed set which satisfies that $E_\beta \hookrightarrow E_\alpha$ and that the inclusion is continuous for $\alpha \le \beta$. A *projective sequence* is a projective net whose index set is $\mathbb{N}$ with the usual ordering. The limit of a projective net is $\bigcap\limits_{\alpha\in\mathcal{A}} E_\alpha$.

2.4 Duality of Projective and Inductive spectra

Definition 2.4.1. For $T \in \mathcal{L}_B(E, F)$ the *dual* or *adjoint* map $T^* \colon F'_b \to E'_b$ is the continuous linear mapping $f \mapsto f \circ T$. Let $(\mathcal{S}, \Pi)$ be a projective/ inductive spectrum with index set $\mathcal{A}$. Set $\mathcal{S}^* := \left\{ (S_\alpha)'_b \mid \alpha \in \mathcal{A} \right\}$ and $\Pi^* := \left\{ \pi^*_{\alpha,\beta} \,\middle|\, \pi_{\alpha,\beta} \in \Pi \right\}$. Then $(\mathcal{S}^*, \Pi^*)$ is an inductive/projective spectrum which we call the *dual inductive/ projective spectrum* of $(\mathcal{S}, \Pi)$.

Definition 2.4.2. If $\iota \colon E \hookrightarrow F$ is the inclusion map then we write ρ for its dual map ι^*. It is easy to see that $\rho \colon F'_b \to E'_b$ is the restriction mapping $f \mapsto f|_E$. The dual spectrum to an inductive sequence $(E_n)_{n\in\mathbb{N}}$ will be written as $((E_n)'_b, \rho_n)_{n\in\mathbb{N}}$, where $\rho_n \colon (E_{n+1})'_b \to (E_n)'_b$ denotes the restriction mapping.

Lemma 2.4.3 ([Flo71, p.158])**.**

(1) *Let E, F be locally convex spaces and let $T \in \mathcal{L}_B(E, F)$. T has dense range if and only if the dual map T^* is injective.*

(2) *Let $(\mathcal{S}, \Pi)$ be a projective spectrum. $(\mathcal{S}, \Pi)$ is strict if and only if all dual maps $\pi^*_{\alpha,\beta}\colon (E_\alpha)'_b \to (E_\beta)'_b$ are injective.*

Theorem 2.4.4 ([FW68, p.143])**.**

(1) *Let $(\mathcal{S}, \Pi)$ be a* reduced *projective spectrum indexed by $\mathcal{A}$ and let $\mathcal{E} = (\epsilon_\alpha)_{\alpha\in\mathcal{A}}$ be a universal projective cone. Let $\mathcal{E}^* := (\epsilon^*_\alpha)_{\alpha\in\mathcal{A}}$ be the system of dual maps, where $\epsilon^*_\alpha\colon (E_\alpha)'_b \to \left(\varprojlim (\mathcal{S}, \Pi)\right)'_b$. The family $\mathcal{E}$ is an inductive cone and can thus be lifted to a linear map*

$$\iota\colon \varinjlim (\mathcal{S}^*, \Pi^*) \to \left(\varprojlim (\mathcal{S}, \Pi)\right)'_b$$

which is bijective and continuous.

(2) *Let $(\mathcal{S}, \Pi)$ be an inductive spectrum indexed by $\mathcal{A}$ and let $\mathcal{E} = (\epsilon_\alpha)_{\alpha\in\mathcal{A}}$ be a universal inductive cone, where $\epsilon_\alpha\colon E_\alpha \to \varinjlim(\mathcal{S}, \Pi)$. Let $\mathcal{E}^* := (\epsilon^*_\alpha)_{\alpha\in\mathcal{A}}$ be the system of dual maps, where $\epsilon^*_\alpha\colon \left(\varinjlim (\mathcal{S}, \Pi)\right)'_b \to (E_\alpha)'_b$. The family $\mathcal{E}$ is a projective cone and can thus be lifted to a linear map*

$$\iota\colon \left(\varinjlim (\mathcal{S}, \Pi)\right)'_b \to \varprojlim (\mathcal{S}^*, \Pi^*)$$

which is bijective and continuous.

In the next theorems we are going to state conditions on the inductive and projective spectrum, respectively, which assure that the dual spectrum generates the dual space of the limit of the spectrum under consideration. For this we need the following generalization of regularity:

Definition 2.4.5. Let $(\mathcal{S}, \Pi)$ be an inductive spectrum, $E = \varinjlim (\mathcal{S}, \Pi)$ and let $\mathcal{E} = (\epsilon_\alpha)_{\alpha\in\mathcal{A}}$ be a universal inductive cone. $(\mathcal{S}, \Pi)$ is called *regular*, if for every $B \in \mathcal{B}(E)$ there exists an $\alpha_0 \in \mathcal{A}$ and a $\tilde{B} \in \mathcal{B}(E_{\alpha_0})$ so that $\epsilon_{\alpha_0}(\tilde{B}) = B$.

Theorem 2.4.6 ([FW68, p.145])**.** *If $(\mathcal{S}, \Pi)$ is a regular inductive spectrum, then*

$$\left(\varinjlim (\mathcal{S}, \Pi)\right)'_b \cong \varprojlim (\mathcal{S}^*, \Pi^*)$$

Theorem 2.4.7. *Let $(E_n)_{n\in\mathbb{N}}$ be a reduced projective sequence of normed spaces and suppose that $\left(\varprojlim E_n\right)'_b$ is bornological. Then $((E_n)'_b)_{n\in\mathbb{N}}$ is an inductive sequence of Banach spaces and*

$$\varinjlim (E_n)'_b \cong \left(\varprojlim E_n\right)'_b$$

Proof: Since $E = \varprojlim E_n$ is a metrizable LCVS and thus bornological, its strong dual is complete and as E'_b is bornological this implies that it is already ultrabornological [MV92, p.283]. Since $(E_n)_{n\in\mathbb{N}}$ is reduced the dual maps $\iota^*_n\colon (E_n)'_b \to (E_{n+1})'_b$ are injective (by Lemma 2.4.3) and thus $\left((E_n)'_b\right)_{n\in\mathbb{N}}$ is an inductive sequence. The limit of an inductive sequence of normed spaces is webbed (see [Jar81, p.92]) and so we can apply Webb's open mapping theorem to conclude that the continuous linear bijection $\iota\colon \varinjlim \left((E_n)'_b\right)_{n\in\mathbb{N}} \to E'_b$ (see Theorem 2.4.4) is an isomorphism. □

2.5 Some classes of locally convex spaces

Definition 2.5.1. A sequence of bounded sets $(B_n)_{n\in\mathbb{N}}$ in a locally convex space E is called a *fundamental sequence of bounded sets* if for every $B \in \mathcal{B}(E)$ there exists a $\lambda > 0$ and an $n_0 \in \mathbb{N}$ so that $B \subset \lambda \cdot B_{n_0}$. A locally convex space E is called a *(DF)-space* if it possesses a fundamental sequence of bounded sets and if for every sequence of absolutely convex 0-neighborhoods the set $\bigcap_{n\in\mathbb{N}} V_n$ is a 0-neighborhood in E whenever $\bigcap_{n\in\mathbb{N}} V_n$ is bornivorous.

Remark. The letters (DF) stand for Dual-Fréchet. Indeed the dual of a Fréchet-space is a (DF)-space (see [MV92, p.298]) and the strong dual of a (DF)-space is a Fréchet-space. Note that a (DF)-space is metrizable if and only if it is normable ([Jar81, p.259]).

Another important example of (DF)-spaces are *separated* (LNorm)-spaces. In general, a generating sequence for such a space need *not* to be regular [Flo71, p.163]. Still the family of closures of bounded sets of a generating sequence forms a fundamental system of bounded sets and we will see in Theorem 2.5.4 that every separated (LNorm)-space possesses a regular generating sequence.

Theorem 2.5.2 (Grothendieck [Flo71, p.163])**.** *Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of normed spaces and suppose that $E = \varinjlim E_n$ is separated and let B_n denote the unit ball of E_n. If B is a bounded subset of E, then there exists an $n_0 \in \mathbb{N}$ such that $B \subset \lambda\overline{B_{n_0}}^{E}$. In particular, $\varinjlim E_n$ is a bornological (DF)-space.*

Theorem 2.5.3. *For a locally convex space E, TFAE:*
1. *E has a fundamental sequence of bounded sets and is bornological.*
2. *E is a bornological (DF)-space.*
3. *E is the limit of a regular inductive sequence of normed spaces.*

Proof: (3) ⇒ (1) Let $(E_n)_{n\in\mathbb{N}}$ be a regular inductive sequence of normed spaces which generates E and let B_n be the unit ball in E_n. As every bounded set of E is contained in some step E_{n_0} and hence in some λB_{n_0}, it follows that $(B_n)_{n\in\mathbb{N}}$ forms a fundamental sequence of bounded sets. Normed spaces are bornological and the inductive limit of a family of bornological spaces is again bornological.
(1) ⇒ (2) If $(V_n)_{n\in\mathbb{N}}$ is a sequence of absolutely convex zero-neighborhoods then $V = \bigcap_{n\in\mathbb{N}} V_n$ is absolutely convex and hence a zero-neighborhood if it is bornivorous, since E is bornological by assumption.
(2) ⇒ (3) Suppose that E is a (DF)-space with fundamental sequence $(B_n)_{n\in\mathbb{N}}$. As the closed absolutely convex hull of a bounded set is again bounded, we may WLOG assume that all members of the fundamental sequence are bounded discs.

Let $E_n := E_{B_n}$. Then $E = \bigcup_{n\in\mathbb{N}} E_n$ and even $E \cong \varinjlim E_n$ holds: As every E_n injects continuously into E we have that $(\varinjlim E_n) \hookrightarrow E$. Since E is bornological, the map $\mathrm{id}\colon E \to \varinjlim E_n$ is continuous if it is bornological. Let $B \in \mathcal{B}(E)$. Then there is an $n_0 \in \mathbb{N}$ and a $\lambda > 0$ so that $B \subset \lambda B_{n_0}$. Hence B is bounded in E_{n_0} and hence bounded in $\varinjlim E_n$, which shows that $E \cong \varinjlim E_n$. As every bounded subset of E is contained in some step E_n, the sequence is regular. □

As an immediate consequence of Theorem 2.5.2 and Theorem 2.5.3 we obtain:

Corollary 2.5.4. *An (LNorm)-space admits a regular generating sequence iff it is separated.*

Definition 2.5.5. A locally convex space E is called *semi-Montel* if it satisfies the Heine-Borel-property - i.e. if every bounded subset of E is relatively compact in E. We say that E is a *Montel space* if it is semi-Montel *and* barrelled.

Definition 2.5.6. Let (E, τ) be a locally convex space and let ι denote the natural injection $E \to (E'_b)'$ into the bidual space of E, where $(\iota(x))\colon f \mapsto f(x)$ is the evaluation map, for $x \in E,\ f \in E'$. The mapping $\iota\colon E \to (E'_b)'_b$ is injective and continuous and we say that E is *semi-reflexive* if it is surjective and E is called *reflexive* if ι is an isomorphism of LCVS.

Theorem 2.5.7 ([Mor93, p.242])**.** *A semi-reflexive space is reflexive if and only if it is barrelled.*

Theorem 2.5.8 ([Mor93, p.242], [FW68, p.108], [KC73], [AK68])**.**
(1) *Semi-Montel spaces are semi-reflexive and Montel spaces are reflexive.*
(2) *The strong dual of a Montel space is again a Montel space.*
(3) *Montel spaces are quasi-complete (every* bounded *Cauchy net converges).*
(4) *There are Montel spaces which fail to be complete.*

Theorem 2.5.9 ([FW68, p.108])**.** *Let E be a Montel space and $B \subset E'$ be a bounded subset. Then the strong topology coincides with the weak topology on B. In particular, a* sequence *in E converges strongly iff it converges weakly.*

Remark. Note that in general the topology on a Montel space is different from its weak topology, even though both topologies have the same convergent sequences. For example (DFM)-spaces (defined below) carry the weak topology iff they are finite-dimensional (see [KŜ92]).

Example 2.5.10. For any open subset Ω of $\mathbb{C}^d$, the space of $\mathbb{C}$-valued holomorphic functions $\mathcal{H}(\Omega)$ equipped with the compact-open topology is a Montel space.

Example 2.5.11. By the Riesz lemma a normed space is semi-Montel iff it is finite-dimensional: Suppose that $(E, \|\ \|)$ is a normed semi-Montel-space. Then its closed unit ball is by assumption compact, so E has to be finite-dimensional by Riesz' lemma.

Definition 2.5.12. An (FM)-space (*Fréchet-Montel space*) is a Fréchet space which is also a Montel space. A (DFM)-space is a (DF)-space which is also a Montel space.

Theorem 2.5.13 (Duality of (FM)- and (DFM)-spaces)**.**
(1) *The strong dual of an (FM)-space is a (DFM)-space.*
(2) *The strong dual of a (DFM)-space is an (FM)-space.*

Proof: (1) If F is a Fréchet space then its strong dual is a (DF)-space and the strong dual of a Montel space is a Montel space. (2) The dual of a (DF)-space is a Fréchet space and again the property of being Montel transfers to the strong dual. □

Corollary 2.5.14. *(DFM)-spaces are complete.*

Proof: Every (DFM)-space is the strong dual of a metrizable (hence bornological) LCVS, and hence complete, since the space of continuous linear operators $\mathcal{L}_b(E, F)$ equipped with the topology of uniform convergence on bounded subsets of E is complete if E is bornological and F is complete (see [Sch71, p.117]). □

Proposition 2.5.15 ([KS02, p.397], [CO86])**.** *For an (LM)-space E, TFAE:*
(1) *E is sequential.*
(2) *E is a (k)-space.*
(3) *E is metrizable or a (DFM)-space.*

Theorem 2.5.16 ([KS02, p.2])**.** *Let E be either an (LM)- or a (DF)-space. TFAE:*
(1) *E is a Fréchet-Urysohn-space.*
(2) *E is metrizable.*

Corollary 2.5.17. *A (DFM)-space is a Fréchet-Urysohn-space if and only if it is finite-dimensional.*

Proof: Let E be a (DFM)- and (FU)-space. E is metrizable by the above theorem and even normable, as it is a (DF)-space ([Jar81, p.259]). A normed space is a Montel space if and only if it is finite-dimensional, as the closed unit ball of a normed space is compact if and only if its dimension is finite (Riesz' lemma). The converse of the theorem holds as every finite-dimensional Hausdorff topological vector space is isomorphic to some $\mathbb{K}^N$ (see [Sch71]). □

Despite sequentiallity, (DFM)-spaces satisfy other strong topological properties:

Theorem 2.5.18 ([Din75, p.462, 465])**.** *(DFM)-spaces are strongly Lindelöf and hereditarily hemicompact.*

2.6 Special classes of inductive and projective limits

Definition 2.6.1. A linear map $T : E \to F$ between locally convex spaces E, F is said to be *compact* if there exists a 0-neighborhood U in E so that $T(U)$ is relatively compact in F. If $T(U)$ is already compact in F, then T is called *bicompact.*

Remark. As the absolutely convex zero-neighborhoods form a zero-neighborhood base, one can always choose an absolutely convex zero-neighborhood V so that $T(V)$ is relatively compact, if T is a compact linear map. It is easy to see that a compact map is continuous and maps bounded sets to relatively compact sets. The second property is equivalent to the compactness of an operator if E and F are both normed spaces.

Theorem 2.6.2 (Bicompact factorization of compact maps [FW68, p.88])**.**
Let (E,τ), (F,σ) be LCVS, and let $T\colon E \to F$ be compact. Let U be an absolutely convex zero-neighborhood so that $K := \overline{T(U)}$ is compact.
Then $T \in \mathcal{L}_B(E, F_K)$, the inclusion map $\iota\colon F_K \to F$ is bicompact and the closed unit ball of F_K is compact in F.

$$\begin{tikzcd}
E \arrow[r, "T"] \arrow[dr, "T"'] & F \\
& F_K \arrow[u, "\iota"']
\end{tikzcd}$$

Proof: F_K is a Banach space since K is a compact disc in F. Note that K is the closed unit ball of F_K (i.e. $K = \{x \in F_K \mid \rho_K(x) \le 1\}$) as it is closed in F, which shows that $\iota\colon F_K \to F$ is bicompact. Since $T(U) \subset F_K$ and hence $T(E) \subset F_K$ it remains to show that $T\colon F \to F_K$ is continuous. For any $\epsilon > 0$ we have that $\epsilon \cdot U \subset T^{-1}(\epsilon K)$ and consequently $T^{-1}(\epsilon K) \in \mathcal{U}_0^E$ which means that T is continuous at 0 and hence $T\colon E \to F_K$ is continuous. □

Definition 2.6.3. A projective spectrum $(\mathcal{S}, \Pi)$ (index by a directed set $\mathcal{A}$) is called *compact* if for every $\alpha \in \mathcal{A}$ there exists an index $\beta \ge \alpha \in \mathcal{A}$ so that the linking map $\pi_{\alpha,\beta}\colon E_\beta \to E_\alpha$ is compact. Likewise, an inductive spectrum $(\mathcal{S}, \Pi)$ (index by a directed set $\mathcal{A}$) is called *compact* if for every $\alpha \in \mathcal{A}$ there exists an index $\beta \ge \alpha \in \mathcal{A}$ so that the linking map $\pi_{\alpha,\beta}\colon E_\alpha \to E_\beta$ is compact.

Corollary 2.6.4. *Any compact inductive sequence $(E_n)_{n\in\mathbb{N}}$ of LCVS is equivalent to a bicompact inductive sequence of Banach spaces $(G_l)_{l\in\mathbb{N}}$, such that the closed unit ball of every G_l is compact in G_{l+1}.*

Proof: Let $(E_{n_l})_{l\in\mathbb{N}}$ be a subsequence so that $\iota\colon E_{n_l} \to E_{n_{l+1}}$ is compact. Theorem 2.6.2 shows that for any $l \in \mathbb{N}$ there is a Banach-space $(G_l, \|\ \|)$ so that E_{n_l} includes continuously into G_l; furthermore $\iota\colon G_l \to E_{n_{l+1}}$ is bicompact and the closed unit ball B_l of G_l is compact in $E_{n_{l+1}}$. Consequently B_l is compact in G_{l+1}, and $(E_n)_{n\in\mathbb{N}}$ is equivalent to $(G_l)_{l\in\mathbb{N}}$, which is illustrated by the following commutative diagram (the arrows being the (continuous) inclusion maps):

$$\begin{tikzcd}
\cdots \arrow[r] & E_{n_l} \arrow[r] \arrow[dr] & E_{n_{l+1}} \arrow[r] \arrow[dr] & E_{n_{l+2}} \arrow[r] \arrow[dr] & \cdots \\
& \cdots \arrow[r] & G_l \arrow[r] \arrow[u] & G_{l+1} \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

□

Theorem 2.6.5 (Schauder [Sch71, p.111])**.** *Let E and F be Banach spaces and let $T\colon E \to F$ be continuous. T is compact if and only if the dual operator $T^*\colon F_b' \to E_b'$ is compact.*

Definition 2.6.6. A linear map $T\colon E \to F$ between topological vector spaces is called *nuclear*, if it satisfies the following: There exist

(1) an equicontinuous sequence $(f_n)_{n\in\mathbb{N}} \subset E'$
(2) a sequence $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ with $\sum |\lambda_n| < \infty$
(3) a bounded sequence $(y_n)_{n\in\mathbb{N}} \subset F$

so that

$$T(x) = \sum_{n=1}^{\infty} \lambda_n y_n f_n(x)$$

Lemma 2.6.7 ([FW68, p.101])**.** *Let E, F be locally convex spaces, and let $T\colon E \to F$ be nuclear.*

(1) *If F is complete, then T is compact.*
(2) *The dual map $T'\colon F' \to E'$ is nuclear.*

Definition 2.6.8. A projective spectrum $(\mathcal{S}, \Pi)$ (index by a directed set $\mathcal{A}$) is called *nuclear* if for every $\alpha \in \mathcal{A}$ there exists an index $\beta \geq \alpha \in \mathcal{A}$ so that the linking map $\pi_{\alpha,\beta}\colon E_\beta \to E_\alpha$ is nuclear. Likewise, an inductive spectrum $(\mathcal{S}, \Pi)$ (index by a directed set $\mathcal{A}$) is called *nuclear* if for every $\alpha \in \mathcal{A}$ there exists an index $\beta \geq \alpha \in \mathcal{A}$ so that the linking map $\pi_{\alpha,\beta}\colon E_\alpha \to E_\beta$ is nuclear.

Definition 2.6.9. A LCVS E is called *nuclear*, if its *completion* $\tilde{E}$ can be represented as the projective limit of a nuclear projective spectrum of sequentially complete LCVS.

Theorem 2.6.10. *Let E be a locally convex space. The following are equivalent:*

(1) *E is nuclear.*
(2) *Every continuous linear map from E to a Banach space is already nuclear.*
(3) *The completion $\tilde{E}$ of E is the limit of a nuclear projective spectrum of Hilbert spaces.*

For Fréchet spaces it is possible to give another characterization of nuclearity in terms of absolute summability:

Definition 2.6.11. A subset $\mathcal{S}$ of a locally convex space E is called *summable* to x in E if for every $\epsilon > 0$ and each $q \in \mathrm{cs}(E)$ there is a finite subset $\mathcal{F}$ of S so that $q(x - \sum_{s\in\mathcal{A}} s) < \epsilon$ holds for any finite subset $\mathcal{A}$ of $\mathcal{S}$ which contains $\mathcal{F}$. We write $\sum_{s\in\mathcal{S}} s = x$ if $\mathcal{S}$ is summable to x. Call a sequence $(x_n)_{n\in\mathbb{N}}$ *absolutely summable* if it is summable and if for every $q \in \mathrm{cs}(E)$ the sequence $(q(x_n))_{n\in\mathbb{N}}$ is summable in $\mathbb{R}$. $(x_n)_{n\in\mathbb{N}}$ is absolutely summable if and only if $\lim_{N\to\infty} \sum_{k=0}^{N} x_k$ exists in E and if $\sum_{n=0}^{\infty} q(x_n) < \infty$ for all $q \in \mathrm{cs}(E)$. If E is complete, then a sequence $(x_n)_{n\in\mathbb{N}}$ is absolutely summable if and only if for each $q \in \mathrm{cs}(E)$ the sequence $(q(x_n))_{n\in\mathbb{N}}$ is summable in $\mathbb{R}$ (see [Jar81][p.305]).

Theorem 2.6.12 ([Sch71, p.184]).
(1) *A Fréchet space is nuclear if and only if each summable sequence is also absolutely summable.*
(2) *A Banach space is nuclear if and only if it is finite-dimensional.*

Theorem 2.6.13 (Permanence Properties of nuclear spaces [FW68, pp155]).
The class of nuclear spaces is closed under forming subspaces, projective limits, products and quotients by closed subspaces.

2.7 Fréchet-Schwartz and (DFS)-spaces

Definition 2.7.1. We call a locally convex space $(\overline{S})$*-space* if it is the projective limit of a compact spectrum of locally convex spaces. A *Schwartz* space is a LCVS whose completion is an $(\overline{S})$-space. A *Fréchet-Schwartz* (shortly (FS)-space) is a LCVS which is the projective limit of an *enumerable compact spectrum* of locally convex spaces. The factorization theorem for compact maps between LCVS (Theorem 2.6.2) yields that every (FS)-space can be generated by a compact projective sequence of Banach spaces. Dually, the inductive limit of a compact *sequence* of locally convex spaces is called (DFS)*-space*. Indeed (FS)-spaces and (DFS)-spaces are dual (as locally convex spaces) to each other.

Remark. Note that the class of limits of compact inductive *nets* is not designated with its own name - the reason for this lies in the fact that every ultrabornological space is the limit of a compact net of Banach spaces (which was proven by Raikov - see [Flo71, p.168]). So according to Floret [Flo71, p.168] "there is little hope that special properties could be deduced from the compactness of the linking maps". Note that every Banach space is an ultrabornological space and hence limits of compact inductive nets are in general not semi-Montel spaces - unlike projective or inductive sequences with compact linking maps.

Remark. In [FW68] Wloka and Floret study compact inductive spectra with linking maps which are not necessarily injective. Limits of such spectra are called (LS)-spaces. Recently also the class of projective limits of sequences of (LS)-spaces received some attention, which are called (PLS)-spaces. Examples for (PLS)-spaces are the space $\mathcal{L}_B(E,F)$ if both E and F are (FS)-spaces ([DL08, p.17]) and the space $\mathcal{A}(\Omega)$ of real-analytic functions on a non-compact real-analytic manifold Ω (see [Dom12]).

Theorem 2.7.2 ([FW68]).
(1) *Every* $(\overline{S})$*-space is a complete semi-Montel space.*
(2) *Every (FS)-space is a Fréchet-Montel space.*

Proof: (1) Let $(\mathcal{S},\Pi)$ be a compact projective spectrum indexed by $\mathcal{A}$ and let $E = \varprojlim(\mathcal{S},\Pi)$. The compact factorization theorem implies that there exists a spectrum of Banach space $(\tilde{\mathcal{S}},\tilde{\Pi})$ indexed by a cofinal subset $\mathcal{J}$ of $\mathcal{A}$ so that any mapping $\tilde{\pi}_{\alpha,\beta} \in \tilde{\Pi}$ is bicompact and which is equivalent to $(\mathcal{S},\Pi)$. Hence $E = \varprojlim(\tilde{\mathcal{S}},\tilde{\Pi})$ is complete (see Theorem 2.3.3). Let $\mathcal{E} = (\epsilon_j)_{j\in\mathcal{J}}$ be a universal cone of $(\tilde{\mathcal{S}},\tilde{\Pi})$. As $\tilde{\Pi}$

consists of bicompact maps it follows that all ϵ_j are compact maps. If $B \in \mathcal{B}(E)$, then $\epsilon_j(B)$ is relatively compact for all $j \in \mathcal{J}$. By Theorem 2.3.3 this implies that B is relatively compact in E which is thus a semi-Montel space. (2) Again the compact factorization theorem yields that an (FS)-space can be generated by a bicompact enumerable spectrum of Banach spaces and is hence metrizable (Theorem 2.3.3(7)). □

Theorem 2.7.3 (Duality between (FS)- and (DFS)-spaces)**.**

(1) *Let $(E_n)_{n\in\mathbb{N}}$ be a compact inductive sequence of Banach spaces which generates the (DFS)-space E. Then E'_b is an (FS)-space which is generated by the compact projective spectrum $\left(((E_n)'_b)_{n\in\mathbb{N}}, (\rho_n)_{n\in\mathbb{N}}\right)$, where $\rho_n\colon (E_{n+1})'_b \to (E_n)'_b$ denotes the restriction $f \mapsto f|_{E_n}$.*

(2) *Let $((F_n)_{n\in\mathbb{N}}, (\rho_n)_{n\in\mathbb{N}})$ be a compact projective sequence of Banach spaces which generates the (FS)-space F. Then F'_b is a (DFS)-space and if $(F_n)_{n\in\mathbb{N}}$ is reduced then the compact inductive sequence $((F_n)'_b)_{n\in\mathbb{N}}$ generates F'_b.*

Proof: (1) Let $(\mathcal{S}^*, \Pi^*) := \left(((E_n)'_b)_{n\in\mathbb{N}}, (\rho_n)_{n\in\mathbb{N}}\right)$. By Theorem 2.4.6 we have that $E'_b \cong \varprojlim(\mathcal{S}^*, \Pi^*)$. The compactness of $\iota_{n,n+k}\colon E_n \to E_{n+k}$ carries over to the dual map $\iota^*_{n,n+k} = \rho_{n,n+k}\colon (E_{n+k})'_b \to (E_n)'_b$. Hence $(\mathcal{S}^*, \Pi^*)$ is a compact spectrum of Banach spaces and E'_b is an (FS)-space. (2) As before the compactness of $\rho^*_{n,n+k}\colon (E_n)'_b \to (E_{n+k})'_b$ is inherited from the compactness of $\rho_{n,n+k}\colon E_{n+k} \to E_n$ and $\rho^*_{n,n+k}$ is injective as $\rho_n(E_{n+1})$ is dense in E_n. Because of Theorem 2.4.4 we have $\varinjlim ((F_n)'_b)_{n\in\mathbb{N}} = E'_b$. □

In the next theorem we will see that (DFS)-spaces have all desirable properties a locally convex space could have. Even more, the fact that (DFS)-spaces are Montel spaces, sequential and that their convergent sequences are exactly the convergent sequences of the steps of a generating sequence of Banach spaces, makes them the optimal setting if one studies non-linear maps on non-metrizable locally convex spaces.

Proposition 2.7.4. *Let $(E_n)_{n\in\mathbb{N}} = (E_n, \tau_n)_{n\in\mathbb{N}}$ be a compact sequence of Banach-spaces which generates E. Then:*

(1) *E is Hausdorff.*
(2) *E is a (DF)-space.*
(3) *E is a Montel space.*
(4) *E is a sequential space.*
(5) *E is complete.*
(6) *E is webbed and ultrabornological.*
(7) *$(E_n)_{n\in\mathbb{N}}$ is compactly retractive.*
(8) *Every generating sequence of E consisting of Fréchet spaces is compact.*
(9) *Every generating sequence of E consisting of Fréchet spaces is sequentially retractive.*
(10) *Every basis of E is a Schauder basis.* [Flo73]
(11) *E carries the final topology with respect to $\left((E_n)_{n\in\mathbb{N}}, \mathcal{E}\right)$ where $\mathcal{E}$ denotes the family of all inclusion mapping $E_n \hookrightarrow E$. Consequently a map $f : E \to X$ to an arbitrary topological space is continuous if and only if all restrictions $f|_{(E_n,\tau_n)}$ are continuous.* [FW68, p.135].

Proof: We show that $(E_n)_{n\in\mathbb{N}}$ satisfies Retakh's condition (see Definition 2.2.1(3)) that E is semi-Montel. Then the rest of the properties (1)-(10) follow directly from the theorems stated about sequentially retractive (LF) spaces and (DF) Montel spaces. For more direct proofs we refer to [FW68]. WLOG we may assume that all $\iota_n\colon E_n \to E_{n+1}$ are compact since there exists in any case a subsequence $(E_{n_k})_{k\in\mathbb{N}}$ with this property. Let B_n denote the closed unit ball in E_n and choose $\lambda_n \in \mathbb{R}_+$ so that $\lambda_n B_n \subset \lambda_{n+1} B_{n+1}$ for all $n \in \mathbb{N}$. As $\lambda_n B_n$ is relatively compact in E_{n+1} the topology induced on B_n by E_{n+k} (for any $k \geq 1$) coincides with that induced by E_{n+1} which shows that $(E_n)_{n\in\mathbb{N}}$ satisfies Retakh's condition (M). Hence (by Theorem 2.2.11) $(E_n)_{n\in\mathbb{N}}$ is sequentially retractive, regular and consequently Hausdorff. Thus, if $B \in \mathcal{B}(E)$, there is an $n_0 \in \mathbb{N}$ so that $B \in \mathcal{B}(E_{n_0})$ and hence B is relatively compact in E_{n_0+1} (because of the compactness of the inclusion ι_{n_0}) and hence in E. Now we show (11). Let X be a topological space, let $f\colon E \to X$ be a function and let ϵ_n denote the inclusion map $E_n \hookrightarrow E$. If f is continuous, then all $f_n := f \circ \epsilon_n\colon (E_n, \tau_n) \to X$ are continuous since all $\epsilon_n\colon (E_n, \tau_n) \to (E, \tau)$ are continuous. Now suppose that all f_n are continuous. We want to conclude that f is continuous and since (E, τ) is a (k)-space it suffices to show that all restrictions of f to compact sets are continuous. Let K be a compact subset of E. Then there exists a step E_n so that K is contained in E_n and compact with respect to τ_n and consequently $(K, \tau_n) \cong (K, \tau)$ (since $(K, \tau_n) \hookrightarrow (K, \tau)$). As f_n is continuous, the restriction $f_n|_{(K,\tau_n)}$ is continuous and hence $f|_{(K,\tau)}$ is continuous. This shows that a function $f\colon E \to X$ is continuous if and only if all compositions $f \circ \epsilon_n$ are continuous and hence E carries the final topology with respect to $\Big((E_n)_{n\in\mathbb{N}}, (\epsilon_n)_{n\in\mathbb{N}}\Big)$. ☐

Lemma 2.7.5. *Let $(E_n)_{n\in\mathbb{N}}$ be an inductive sequence of Banach spaces and let $E = \varinjlim E_n$. Then E is a (DFS)-space if and only if E is semi-Montel and $(E_n)_{n\in\mathbb{N}}$ is sequentially retractive.*

Proof: If E is a (DFS)-space then it is a Montel space. A compact sequence of Banach spaces is sequentially retractive and since all generating sequences of an (LF)-space are equivalent (Theorem 2.1.11) and sequential retractivity is passed over to equivalent sequences it follows that $(E_n)_{n\in\mathbb{N}}$ is sequentially retractive. Now suppose that E is semi-Montel and that $(E_n)_{n\in\mathbb{N}}$ is sequentially retractive. Let B_n be the unit ball in E_n. For any $n_0 \in \mathbb{N}$, we have that $K = \overline{B_{n_0}}^E$ is compact in E and since $(E_n)_{n\in\mathbb{N}}$ is compactly retractive (by Theorem 2.2.11) there exists an $n_1 > n_0$ so that K is compact in E_{n_1}. Hence B_{n_0} is relatively compact in E_{n_1}, which shows that $(E_n)_{n\in\mathbb{N}}$ is a compact sequence and consequently E is a (DFS)-space. ☐

Theorem 2.7.6 (Permanence properties of (DFS)/(FS)-spaces [Flo71, p.182]).

(1) *The class of (FS)-spaces is closed under taking closed subspaces, quotients with respect to closed subspaces and finite products.*
(2) *The class of (DFS)-spaces is closed under taking closed subspaces, quotients with respect to closed subspaces and finite products.*
(3) *Let $(E_n)_{n\in N}$ be a compact sequence of Banach spaces, let $E = \varinjlim E_n$ and let F be a closed subspace of E. Then F is also a (DFS)-space and is generated by the compact sequence $(F \cap E_n)_{n\in\mathbb{N}}$.*

Next we study nuclear inductive sequences of Fréchet spaces. Since a nuclear map $T\colon E \to F$ is compact if F is complete, these are special cases of compact sequences and the limit spaces obtained (i.e. (DFN)-spaces are (DFS)-spaces) and it turns out that nuclear (DFS)-spaces are exactly the (DFN)-spaces.

Definition 2.7.7. We call a locally convex space E (DFN)-*space* if it is the inductive limit of a nuclear sequence of Frechet spaces. Dually, the projective limit of a nuclear sequence of Fréchet spaces is called an (FN)-*space*.

Theorem 2.7.8 ([FW68, p.160-163] [Flo71, p.175])**.**

(1) *The classes of (FN)- and (DFN)-spaces are dual to each other: The strong dual of a (DFN)-space is an (FN)-space and the strong dual of an (FN)-space is a (DFN)-space.*

(2) *Every (DFN)-space is a (DFS)-space and every (FN)-space is an (FS)-space.*

(3) *A (DFS)-space is nuclear if and only if it is a (DFN)-space.*

(4) *An (FS)-space is nuclear if and only if it is an (FN)-space.*

(5) *The inductive limit of a nuclear sequence of sequentially complete LCVS is a (DFN)-space.*

As a consequence of the stability properties of (DFS)-, (FS)- and nuclear spaces and the characterization of (DFN)/(FN)-spaces as nuclear (DFS)/(FS)-spaces we immediately obtain the following corollary:

Corollary 2.7.9 (Stability properties of (FN)- and (DFN)- spaces)**.** *The classes of (FN)-spaces and (DFN)-spaces satisfy the same stability properties as (FS)- and (DFS)-spaces, respectively.*

2.8 Non-linear maps between LCVS

Definition 2.8.1. Let E, F be LCVS. A map $T : E \to F$ is called *bornological* if the image of every bounded set under T is again bounded and is called *sequentially bornological* if $(T(a_n))_{n\in\mathbb{N}}$ is bounded whenever $(a_n)_{n\in\mathbb{N}}$ is bounded.

Theorem 2.8.2. *Let E be a sequential semi-Montel space and let F be a semi-Montel space which satisfies that all its compact subsets are metrizable (in the topology induced by F) and which possesses a Schauder basis $\{x_n\}_{n\in\mathbb{N}}$. Let $T\colon E \to F$ be a (possibly non-linear) map, let $T_n\colon E \to \mathbb{C}$ denote the n^{th} coordinate function of T. TFAE:*

(1) *T is continuous.*

(2) *T is bornological and all T_n are continuous.*

(3) *T is sequentially bornological and all T_n are continuous.*

Remark. Despite the strong requirements on the spaces E, F there are huge classes of locally convex spaces to which the theorem applies:

(1) Every (DFM)-space is (by definition) a Montel space and sequential by a result of Webb ([Web68]).

(2) In every (LM)-space all pre-compact subsets are metrizable by a result of Cascales and Orihuela (see [CO86]).

(3) In particular the result applies to the case where E is a (DFM) and F a (DFS)-space with a basis: Floret showed in ([Flo70]) that in a sequentially retrative (LF)-space (and thus in every (DFS)- space) every basis is already a Schauder basis.

Proof: (1) $\Rightarrow$ (2) Let B be a bounded set in E. Then B is relatively compact, hence $T(B)$ is relatively compact and thus bounded in F. The coordinate projections are continuous, and thus all T_n are so. (2) $\Rightarrow$ (3) is trivial. (3) $\Rightarrow$ (1) Let $a_n \to a$ in E. Then $A := \{a_n\}_{n\in\mathbb{N}} \cup \{a\}$ is bounded in E, hence $T(A)$ is bounded and thus $\overline{T(A)}$ is compact and by assumption metrizable. Let $p_m \colon F \to \mathbb{C}$ be the m^{th} coordinate projection. By compactness we can extract a subsequence $(T(a_{n_k}))_{k\in\mathbb{N}}$ which converges to an element $y \in F$. Then for all $m \in \mathbb{N}$ we have $p_m(T(a_{n_k})) \to p_m(y)$. By continuity of the coefficient functions $\lim p_m(T(a_{n_k})) = \lim p_m(T(a_n)) = T_m(a)$, which means that $p_m(y) = p_m(T(a))$ and we conclude that $y = T(a)$. We have shown that any convergent subsequence of $T(a_n)$ converges to $T(a)$, and since $\overline{T(A)}$ is a compact metrizable space it follows that $T(a_n) \to T(a)$ in F. $\square$

We can drop the requirement that F has a basis and substitute the condition that the coefficient functions are continuous for example by demanding that T is continuous if F is equipped with the weak topology:

Theorem 2.8.3. *Let E be a sequential semi-Montel space and let F be a semi-Montel space which satisfies that all its compact subsets are metrizable (in the topology induced by F). Let $T\colon E \to F$ be a (possibly non-linear) map. Let $\mathcal{F} \subset F'$ be any family of functionals which separates points. TFAE:*

(1) *T is continuous.*
(2) *T is sequentially bornological and $f \circ T$ is continuous for all $f \in F'$.*
(3) *T is sequentially bornological and $f \circ T$ is continuous for all $f \in \mathcal{F}$.*

Proof: The fact that bounded sets are relatively compact yields (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) is clear. (3) $\Rightarrow$ (1) Let $a_n \to a$ in E. Then $\overline{\{T(a_n)\}_{n\in\mathbb{N}}}$ is compact and by assumption metrizable. By compactness we can extract a subsequence $(T(a_{n_k}))_{k\in\mathbb{N}}$ which converges to an element $y \in F$. Then for all $f \in \mathcal{F}$ we have $f(T(a_{n_k})) \to f(y)$. Since $f \circ T$ is continuous, $\lim f(T(a_{n_k})) = f(T(a))$, which means that $f(y) = f(T(a))$. This holds for all $f \in \mathcal{F}$ and as the family $\mathcal{F}$ separates points we have $y = T(a)$. We have shown that any convergent subsequence of $T(a_n)$ converges to $T(a)$, and since $\overline{T(A)}$ is a compact metrizable space it follows that $T(a_n) \to T(a)$ in F. $\square$

Theorem 2.8.4. *Suppose that (X, τ) is a regular Lindelöf space and let (E, σ) be a LCVS. Let $\mathcal{F} \subset \mathscr{C}(X, E)$ be a vector space which satisfies the following:*

(i) $\mathcal{F}$ is closed in the compact-open-topology τ_{co}.
(ii) For every $x \in X$, there is a neighborhood U, so that that the restriction map $\rho\colon (\mathcal{F}, \tau_{co}) \to (\mathscr{C}(U, E), \tau_{co})$ is compact.
(iii) $(\mathcal{F}, \tau_{co})$ is metrizable.

Then (F, τ_{co}) is a semi-Montel space.

Note that we don't require the restriction map to be injective. We need a small lemma concerning coverings by compact sets in regular topological spaces before we can prove the theorem stated above. We recall that a *regular* topological space is a topological space in which points and closed sets can be separated by open sets. In a regular space every point possesses a neighborhood-base of closed sets. Important examples of regular topological spaces are Hausdorff topological vector spaces.

Lemma 2.8.5. *Let (X, τ) be a regular Hausdorff topological space. Let K be compact in X and let $U_1, \dots, U_n$ be a covering by open sets of K. Then there exist compact sets $K_1, \dots, K_n$ with $K_i \subset U_i$ so that $K \subset K_1 \cup \dots \cup K_n$.*

Proof: Let $R_1 := \delta U_1 \cap K$. $R_1 \subset U_1^c$, hence $R_1 \subset U_2 \cup \dots \cup U_n$. For $x \in R_1$ there exists a closed neighborhood V_x of x which is contained in $U_2 \cup \dots \cup U_n$. Since R_1 is compact, there exist $x_1, \dots, x_N$ so that $R_1 \subset V := \bigcup_{i=1}^N V_{x_i} \subset \bigcup_{j=2}^N U_j$. Let $K_1 := \overline{K \cap U_1 \setminus V}$ and let $K_2 := \overline{(K \setminus K_1)} \cup (V \cap K)$. Then $K = K_1 \cup K_2$, $K_2 \subset \bigcup_{j=2}^n U_j$ and K_1 is a compact subset of U_1. Now we show that $\overline{(K \setminus K_1)}$ is compact:
$(K \setminus K_1) = (K \setminus (K \cap (U_1 \setminus V)))^\circ = ((K \cap U_1^c) \cup K \cap V)^\circ = (R_1 \cup (K \cap V))^\circ$.
Hence $\overline{K \setminus K_1} = \overline{(R_1 \cup (K \cap V))}$, which is a compact set. So K_2 is also a compact set. We have shown that there exist compact sets K_1, K_2 so that $K = K_1 \cup K_2$ with $K_1 \subset U_1$ and $K_2 \subset U_2 \cup \dots U_n$. Proceeding in this manner, we can find compact sets $K_1, \dots, K_n$ so that $K_i \subset U_i$ and $K = K_1 \cup \dots \cup K_n$. □

Proof: (of the theorem): Let B be a bounded subset of $\mathcal{F}$. Since $(\mathcal{F}, \tau_{co})$ is metrizable, it is enough to show that every sequence $(f_n)_{n \in \mathbb{N}}$ possesses a subsequence which converges in $\mathcal{F}$ to prove that B is relatively compact in $\mathcal{F}$. For $x \in X$ there exists a neighborhood U_x of x so that $B|_{U_x}$ is relatively compact in $(\mathscr{C}(U, E), \tau_{co})$. By the Lindelöf property of X there exists $(x_i)_{i \in \mathbb{N}}$ so that $X = \bigcup_{i \in \mathbb{N}} U_{x_i}$. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in B. For any $i \in \mathbb{N}$, there exists a subsequence $(f_{n_i})_{i \in \mathbb{N}}$ which converges in $(\mathscr{C}(U_{x_i}), \tau_{co})$. Using the diagonal argument, we can thus extract a subsequence $(f_{n_l})_{l \in \mathbb{N}}$ which converges in $(\mathscr{C}(U_{x_i}), \tau_{co})$ for all $i \in \mathbb{N}$. Let $f(x) = \lim_{l \to \infty} f_{n_l}(x)$. Let K be a compact subset of X. Then there exists a finite subset $A = \{y_1, \dots, y_n\}$ of $(x_i)_{i \in \mathbb{N}}$ so that $K \subset \bigcup_{i=1}^n U_{y_i}$. Let $\{K_j\}_{j=1}^n$ be compact subsets of X with $K_j \subset U_{y_j}$ so that $K \subset \bigcup_{j=1}^n K_j$ Since $(f_{n_l})_{l \in \mathbb{N}}$ converges uniformly on compact subsets of the sets U_{y_i}, it follows that $(f_{n_l})_{l \in \mathbb{N}}$ converges uniformly on K. Hence $(f_{n_l})_{l \in \mathbb{N}}$ converges in the compact-open topology to a function $f \in \mathscr{C}(X, \tau_{co})$, and since $\mathcal{F}$ is closed, $f \in \mathcal{F}$ and thus B is relatively compact in $\mathcal{F}$. □

2.9 Local convergence and local completeness

Definition 2.9.1. A net $(x_i)_{i \in I}$ is *Mackey-* (or *locally-*) *convergent* to x, if there exists a disc B, so that x_i converges to x in E_B. We write $x_i \xrightarrow{M} x$ if $(x_i)_{i \in I}$ converges locally to x and $x_i \xrightarrow{E_B} x$ if it converges in E_B to x. A LCVS (E, τ) satisfies the *Mackey condition* (Mc) if every τ-convergent *sequence* is Mackey-convergent. We say that E satisfies the *strict Mackey condition* (sMc) if for every bounded set B in E there is a disc D in E so that the relative topologies on B with respect to E_D and (E, τ) coincide. E is called *Mackey-* (or *locally-*) *complete* or *convenient* if E_B is complete for any closed disc B in E – i.e., if every closed disc is already a Banach disc ([PCB87, p.83]). We note that every semi-Montel space is locally complete as every compact disc is a Banach disc. We have the implications complete $\Rightarrow$ sequentially complete $\Rightarrow$ locally complete ([KM97, p.15]).

Theorem 2.9.2 ([Val82, p.167] and [Jar81, p.265/266])**.**
(1) *(sMc) implies (Mc).*
(2) *If E is an (LNorm)-space, then E satisfies (Mc) if and only if it satisfies (sMc).*
(3) *A (DFM)-space is a (DFS)-space if and only if it satisfies (Mc).*

Proof: (1) Let E be a LCVS with property (sMc) and let $x_n \to x$. The set $B := \{x_n\}_{n\in\mathbb{N}} \cup \{x\}$ is bounded, hence there exists a $D \in \mathbb{D}(E)$ so that the topologies induced by E and E_D on B coincide. As a consequence $x_n \xrightarrow{E_D} x$. (2) See [Val82, p.167] (3) A (DFM)-space E admits a regular generating sequence $(E_n)_{n\in\mathbb{N}}$ which consists of normed spaces. Let B_n be the closed unit ball of E_n. Then $(B_n)_{n\in\mathbb{N}}$ is a fundamental sequence of bounded sets. Let $n_0 \in \mathbb{N}$. Then $K := \overline{B_{n_0}}^E$ is compact in E and since E satisfies (sMc) there is an $n_1 > n_0$ so that the topologies induced on K by E_{n_1} and E coincide. Consequently B_{n_0} is relatively compact in E_{n_1} and hence $(E_n)_{n\in\mathbb{N}}$ is a compact spectrum, which shows that E is a (DFS)-space. $\square$

Corollary 2.9.3. *A bornological (DF)-space satisfies (Mc) if and only if it is the limit of a sequentially retractive sequence of normed spaces.*

Lemma 2.9.4 ([KM97, p.12])**.** *Let B be a bounded and absolutely convex subset of E and let $(x_\gamma)_{\gamma\in\Gamma}$ be a net in E_B. TFAE:*
(1) $x_\gamma \xrightarrow{E_B} 0$
(2) *There exists a net $\mu_\gamma \to 0$ in $\mathbb{R}_+$, such that $x_\gamma \in \mu_\gamma B$.*

Proof: (2) $\Rightarrow$ (1) Note that $\rho_B(x_\gamma) \leq |\mu_\gamma|$ and hence $\rho_B(x_\gamma) \to 0$. (1) $\Rightarrow$ (2) Recall that $\rho_B(y) = \inf\{r \geq 0 \mid y \in rB\}$. Hence for any $\epsilon > 0$: $y \in (\rho_B(y) + \epsilon) B$. Set $\mu_\gamma := \rho_B(x_\gamma) + \exp\left(\frac{-1}{\rho_B(x_\gamma)}\right) > \rho_B(x_\gamma)$. Then $\mu_\gamma \to 0$ and $x_\gamma \in \mu_\gamma B$. $\square$

Example 2.9.5. Every metrizable LCVS (E, τ) satisfies the Mackey-condition: E possesses an enumerable and absolutely convex 0-neighborhood-base $(U_n)_{n\in\mathbb{N}}$ satisfying $U_{n+1} \subset U_n$. Let $x_n \xrightarrow{\tau} x$. We are going to construct a sequence $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ with $\lambda_n \to \infty$ and $\lambda_n(x_n - x) \to 0$. WLOG we may assume that all x_n are contained in U_1. Let $n_1 = 1$ and let $n_{k+1} = \min\left\{N > n_k \,\middle|\, x_n - x \in \frac{1}{k+1}U_{k+1}\ \forall n \geq N\right\}$. Set $\lambda_1 = 1$ and $\lambda_j = k$ for $n_k \leq j < n_{k+1}$. Hence $\lambda_j \to \infty$ and $\lambda_j(x_j - x) \to 0$ by construction. Let B be the absolute convex hull of $(\lambda_j(x_j - x))_{j\in\mathbb{N}}$. We have that $(x_j - x) \in \frac{1}{\lambda_j}B$, so by the above lemma the sequence x_j is Mackey-convergent.

Theorem 2.9.6 ([PCB87, p.157/Thm 5.1.27])**.** *Every metrizable LCVS satisfies the strict Mackey-condition.*

Definition 2.9.7. Let $(\mu_n)_{n\in\mathbb{N}} \to \infty$ be a real-valued sequence. A sequence $(x_n)_{n\in\mathbb{N}} \subset E$ converges to x with quality $(\mu_n)_{n\in\mathbb{N}}$ if $(x_n - x)\,\mu_n$ is bounded and is said to *converge fast* to x if $(x_n)_{n\in\mathbb{N}}$ converges to x with quality $\left(n^k\right)_{n\in\mathbb{N}}$ for all $k \in \mathbb{N}$.

Lemma 2.9.8. *Let E be an LCVS and let $(\mu_n)_{n\in\mathbb{N}}$ be a real-valued sequence with $(\mu_n)_{n\in\mathbb{N}} \to \infty$.*

(1) *Any locally convergent sequence in E possesses a subsequence which converges with quality $(\mu_n)_{n\in\mathbb{N}}$.*
(2) *Any locally convergent sequence in E possesses a fast convergent subsequence.*

Proof: (1) WLOG we can assume that $|\mu_n| \neq 0$ for all $n \in \mathbb{N}$. Let $x_n \xrightarrow{E_B} x$ and $\|y\| := \rho_B(y)$. Let $n_1 := \min\left\{N \in \mathbb{N} \;\middle|\; \forall n \geq N : \|x_n - x\| \leq \frac{1}{|\mu_1|}\right\}$. For $n_1, \ldots, n_k$ already chosen, we set $n_{k+1} := \min\left\{N \in \mathbb{N} \;\middle|\; N > n_k,\ \forall n \geq N : \|x_n - x\| \leq \frac{1}{(k+1)|\mu_{k+1}|}\right\}$. Then we have $\|(x_{n_k} - x)\mu_k\| \leq \frac{1}{k}$, hence $(x_{n_k} - x)\,\mu_k \xrightarrow{k\to\infty} 0$. (2) Let $y_n := x_n - x$ and let $(y_{n_1})_{n_1\in\mathbb{N}}$ be a subsequence of $(y_n)_{n\in\mathbb{N}}$ which converges with quality $(n^1)_{n\in\mathbb{N}}$ to 0. Again we can extract a subsequence of y_{n_1} which converges with quality $(n^2)_{n\in\mathbb{N}}$ to 0 and so forth. The diagonal sequence of this family of subsequences then converges fast to 0. □

Definition 2.9.9 (Mackey-closure-topology)**.** A subset A of a LCVS E is called *locally*- or *Mackey*-closed if $x_n \xrightarrow{M} x$ implies that $x \in A$ for any sequence $(x_n)_{n\in\mathbb{N}}$ in A. It is easy to check that the family of Mackey-closed sets satisfies the axioms of a family of closed sets of a topology, which is called the *Mackey-closure-topology* and which we will denote by τ_M. Every τ-closed set is Mackey-closed and hence $\tau \preceq \tau_M$. Note however that τ_M is in general strictly finer than τ and that τ_M in general does *not* define a linear topology - i.e. (E, τ_M) is in general not a topological vector space. Also note that the Mackey-closure topology is not the Mackey-topology from duality theory.

Example 2.9.10. Every locally convex space (E, τ) which is sequential and satisfies the Mackey-convergence-condition carries the Mackey-closure topology τ_M: By construction, every Mackey-convergent sequence converges with respect to τ_M, which shows that $\mathrm{id}\colon (E,\tau) \to (E, \tau_M)$ is sequentially continuous and hence continuous as E was assumed to be sequential. Hence every (DFS)- space and every metrizable LCVS carries the Mackey-closure topology.

Lemma 2.9.11. *For $B \in \mathbb{D}(E)$ let $\iota_B : E_B \to E$ denote the inclusion map. The Mackey-closure topology is the final topology with respect to $\left((E_B, \rho_B), \iota_B\right)_{B\in\mathbb{D}(E)}$.*

Proof:

$$
\begin{aligned}
A \subset E \text{ is Mackey-closed} &\Leftrightarrow \forall\,(x_n)_{n\in\mathbb{N}} \subset A : x_n \xrightarrow{M} x \text{ implies } x \in A \\
&\Leftrightarrow\ \forall B \in \mathcal{B}(E)\ \ \forall\,(x_n)_{n\in\mathbb{N}} \subset A :\ x_n \xrightarrow{E_B} x \text{ implies } x \in A \\
&\Leftrightarrow\ \forall B \in \mathcal{B}(E) :\ A \cap E_B \text{ is closed in } E_B \\
&\Leftrightarrow\ A \text{ is closed in the final topology of } \left((E_B, \rho_B), \iota_B\right)_{B\in\mathcal{B}(E)}\,. \square
\end{aligned}
$$

Chapter 3
Infinite-dimensional Calculus

The aim of this chapter is to give a short survey of two of the most common concepts of holomorphicity in the setting of locally convex spaces. From now on we will assume that any locally convex space is a vector space over $\mathbb{C}$. Note that there exist 25 non-equivalent definitions for the differentiability of a function between topological vector spaces (cite [KM97, p.76],[AS68]). We will begin by giving a brief overview of the convenient calculus as proposed by Kriegl and Michor (see [KM97]), which starts by introducing the notion of smooth curves and then defines smooth mappings $f \colon E \to F$ between locally convex spaces E and F as those which are smooth along smooth curves. In an infinite-dimensional setting, smoothness may not imply the continuity of a function, but in the setting of convenient vector spaces, it is always continuous with respect to the c^∞-topologies associated to of E and F. The c^∞-topology turns out to be the Mackey-closure-topology, as introduced in the last chapter. In a similar fashion the concept of a curve-holomorphic mapping is introduced. Another definition of holomorphy is used for example by S.Dineen: A map $f \colon E \to F$ between locally convex spaces is holomorphic if for all $\phi \in F'$ the composition $\phi \circ f$ is holomorphic if restricted to any finite-dimensional subspace (it is G-holomorphic) and if f is continuous. Curve-holomorphic mappings can be characterized as G-holomorphic mappings which are continuous with respect to the c^∞-topologies. So the two concepts coincide whenever the spaces under consideration carry the respective Mackey-closure-topologies, which is the case for example if the spaces are metrizable or (DFS)-spaces.

3.1 Curves and convenient calculus

Definition 3.1.1. Let E and F be LCVS. A curve $c : \mathbb{R} \to E$ is called *differentiable* if the derivative $c'(t) := \lim\limits_{h\to 0} \frac{c(t+h)-c(t)}{h}$ exists for all $t \in \mathbb{R}$. It is called *smooth* if all iterated derivatives exist and is called C^n if all derivatives up to order n exist. A curve c is called *locally Lipschitz* if its difference quotient is bounded on every bounded interval, i.e. if for every bounded interval I the set $\left\{ \frac{c(t)-c(s)}{t-s} \mid t \neq s; t, s \in I \right\}$ is bounded. Note that differentiable curves are locally Lipschitzian and hence continuous. A LCVS is called *convenient* if it is Mackey complete. Among others [KM97, p.20], this is equivalent to the property that a curve c into E is smooth if and only if for any continuous functional ϕ the composition $\phi \circ c$ is a smooth mapping.

Lemma 3.1.2 (Mean value theorem, [KM97, p.10])**.** *Let $c \colon [a, b] \to E$ be a differentiable curve into a LCVS E. Then $c(b) - c(a) \in \overline{co}\,\{c'(t)\,(b-a) \mid t \in (a, b)\}$.*

Proof: Suppose that $c(b) - c(a) \notin \overline{\mathrm{co}} \{c'(t) \mid t \in (a,b)\} (b-a)$. Then by the Hahn-Banach theorem there exists a continuous linear functional ϕ so that

$$\phi(c(b) - c(a)) \notin \phi\left(\overline{\mathrm{co}} \{c'(t)\,(b-a) \mid t \in (a,b)\}\right).$$

Note that $\phi \circ c \colon [a,b] \to \mathbb{R}$ is differentiable on (a,b) and that $(\phi \circ c)'(t) = \phi\left(c'(t)\right)$. We conclude that $(\phi \circ c)(b) - (\phi \circ c)(a) \notin \left\{(\phi \circ c)'(t)(b-a) \mid t \in (a,b)\right\}$ - a contradiction to the classical mean value theorem. $\square$

Lemma 3.1.3 (Special curve lemma, [KM97, p.18])**.** *Let E be a LCVS. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence which converges fast to x. Then there exists a smooth curve c so that $c\left(\frac{1}{n}\right) = x_n$ and $c(0) = x$.*

Definition 3.1.4. The c^∞ topology on a LCVS (E, τ) is defined as the final topology with respect to all smooth curves. It coincides with the Mackey-closure topology τ_M, but in general not with the given topology τ. Let U be a c^∞-open subset of E. A map $f \colon U \to F$ is called smooth if it maps smooth curves to smooth curves, i.e. if for any smooth curve $c \colon \mathbb{R} \to E$ the composition $f \circ c$ is a smooth curve to F. Smooth mappings are c^∞-continuous but may fail to be continuous with respect to τ.

Theorem 3.1.5 ([KM97, p.19, Thm 2.13.])**.** *The c^∞-topology coincides with the Mackey-closure-topology τ_M.*

Proof: Let $U \subset E$ be c^∞-open. Let $x \in U$ and $x_n \xrightarrow{M} x$. Suppose that there is a subsequence $(x_{n_k})_{k\in\mathbb{N}} \subset E \setminus U$. We can extract a subsequence $\left(x_{n_{k_l}}\right)_{l\in\mathbb{N}}$ which is fast convergent to x and by the special curve lemma there is a smooth curve $c : \mathbb{R} \to E$ so that $c\left(\frac{1}{l}\right) = x_{n_{k_l}}$ and $c(0) = x$. But this (by the continuity of c) means that $x_{n_{k_l}} \xrightarrow{l\to\infty} x$ with respect to c^∞ - a contradiction to $(x_{n_k}) \subset E \setminus U$, and hence $(x_n)_{n\in\mathbb{N}}$ must be finally contained in U. So U is open in τ_M, and $c^\infty \prec \tau_M$. Now consider the identity map $\mathrm{id} \colon (E, c^\infty) \to (E, \tau_M)$. We want to show that id is continuous, hence we need to show that every smooth curve into E is continuous into (E, τ_M). Let c be a smooth curve and $I \subset \mathbb{R}$ be a compact interval. Since differentiable curves are locally Lipschitzian, there exists a disc B so that for all $t, s \in I : c(t) - c(s) \in (t-s)B$. Hence $c_{|_J} : J \to E_B$ and c is continuous since $\rho_B\left(c(t) - c(s)\right) \leq |t-s|$. $\square$

3.1.1 Curve-holomorphic mappings

Definition 3.1.6. Let $\mathbb{D}$ be the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and let E be a LCVS. A map $c \colon \mathbb{D} \to E$ is called *holomorphic curve* if $c'(z) := \lim\limits_{w\to 0} \frac{c(z+w)-c(z)}{w}$ exists for all $z \in \mathbb{D}$. We say that a family of functions $f_n(z) : \mathbb{D} \to E$ converges *Mackey-uniformly* to a function $f \colon \mathbb{D} \to E$, if for any compact subset K of $\mathbb{D}$ there exists a $B \in \mathbb{D}(E)$ so that $f_{n_{|_K}}$ converges uniformly to $f_{|_K}$ as functions $K \to E_B$.

Lemma 3.1.7. *Holomorphic curves are continuous.*

Proof: Let E be an arbitrary locally convex space, let $c\colon \mathbb{D} \to E$ be a holomorphic curve and let $z \in \mathbb{D}$. Let $U \in U_0^E$ and let $V \in U_0^E$ be absolutely convex and such that $V + V \subset U$. Then there exists an open disc $\mathbb{D}_r$ with $\mathbb{D}_r + z \subset \mathbb{D}$ and so that $c'(z) - \frac{c(z+w)-c(z)}{w} \in V$ for all $w \in \mathbb{D}_r$. Hence $c(z+w) - c(z) \in wc'(z) + V$ for all $w \in \mathbb{D}_r$. Now choose $0 < r' < r$ so that $\mathbb{D}_{r'} \cdot c'(z) \subset V$. Then for all $w \in \mathbb{D}_{r'}$ we have $c(z+w) - c(z) \in V + V \subset U$, which shows that c is continuous. □

Theorem 3.1.8 ([KM97, p.81-82]). *Let E be a convenient vector space and let $c\colon \mathbb{D} \to E$ be a mapping. TFAE:*

(1) *c is a holomorphic curve.*
(2) *$\phi \circ c\colon \mathbb{D} \to \mathbb{C}$ is holomorphic for all $\phi \in E'$.*
(3) *c factors locally to a holomorphic curve to some E_B, i.e. for any open and relatively-compact subset U of $\mathbb{D}$ there exists a disc B in E so that $c|_U \to E_B$ is a holomorphic curve.*
(4) *All complex higher derivatives $c^{(n)}(0)$ exist and $c(z) = \sum\limits_{n=0}^{\infty} \frac{z^n}{n!} c^{(n)}(0)$ converges Mackey-uniformly.*

Definition 3.1.9. Let $U \subset E$ be a c^∞-open subset of U and F a LCVS. A map $f\colon E \to F$ is called *curve-holomorphic* if it maps holomorphic curves to holomorphic curves.

Theorem 3.1.10 ([KM97, Theorem 7.19]). *Let E, F be convenient vector spaces, let U be a $c^\infty -$ open subset of E, and let $f\colon U \to F$ be a mapping. TFAE:*

(1) *f is curve-holomorphic.*
(2) *For all $\phi \in F'$ and for all $B \in \mathbb{D}(E)$ the mapping $\phi \circ f\colon E_B \to \mathbb{C}$ is curve-holomorphic.*
(3) *For all $\phi \in F'$ and for any holomorphic curve $c\colon \mathbb{D} \to E$ the mapping $\phi \circ f \circ c$ is curve-holomorphic.*
(4) *f is holomorphic along all affine (complex) lines (*G-holomorphic - *see below) and is c^∞-continuous.*

3.2 Holomorphic maps between LVCS

Definition 3.2.1. Let E, F be LCVS. We say that a subset U of E is finitely open if $U \cap M$ is open in M for every finite-dimensional subspace M of E. Let U be a finitely open subset. A function $f\colon U \to F$ is called *Gâteaux-* or *G-holomorphic* if for any $\zeta \in U,\ \omega \in E,$ and $\phi \in F'$ the $\mathbb{C}$-valued function of one complex variable

$$H_{\zeta,\omega,\phi}\colon z \mapsto \phi \circ f\left(\zeta + z \cdot \omega\right)$$

is holomorphic in a neighborhood of $0 \in \mathbb{C}$. We say that a function $f\colon U \to F$ is Gâteaux-differentiable if for any $\zeta \in U,\ \omega \in E$

$$\lim_{\substack{\lambda \to 0 \\ \lambda \in \mathbb{C}}} \frac{f(\zeta + \lambda \cdot \omega) - f(\zeta)}{\lambda}$$

exists in the completion $\widehat{F}$ of F. A function is Gâteaux-differentiable if and only it is G-holomorphic ([Din99, p.149]).

We call a function $f\colon U \to \mathbb{C}$ *holomorphic* if it is *G-holomorphic* and continuous with respect to the locally-convex topologies of E and F. By $\mathcal{H}_G(U,F)$ we denote the family of G-holomorphic functions $U \to F$ and by $\mathcal{H}(U,F)$ the family of holomorphic functions $U \to F$. Instead of $\mathcal{H}(U,\mathbb{C})$ we also write $\mathcal{H}(U)$.

Lemma 3.2.2. *Let E be a convenient vector space and let $c\colon \mathbb{D} \to E$. c is a holomorphic curve (in the sense of Definition 3.1.6) if and only if it is a holomorphic mapping (in the sense of Definition 3.2.1).*

Proof: By Theorem 3.1.8, c is a holomorphic curve if and only if $\phi \circ c$ is a holomorphic function $\mathbb{D} \to \mathbb{C}$ for all $\phi \in E'$, which means that c is a holomorphic curve if and only if c is G-holomorphic. By Lemma 3.1.7 holomorphic curves are continuous, which proves the claim. □

Theorem 3.2.3. *Let (E,τ_E) and (F,τ_F) be convenient vector spaces and let U be an open subset of (E,τ). Suppose that $c^\infty_E = \tau_E$ and that $c^\infty_F = \tau_F$. Then a function $f\colon U \to F$ is holomorphic if and only if it is curve-holomorphic.*

Proof: By Theorem 3.1.10 f is curve-holomorphic if and only if it is G-holomorphic and c^∞-continuous. As E and F carry the respective c^∞-topologies a function $f\colon E \to F$ is c^∞-continuous iff it is continuous. □

Definition 3.2.4. Let E, F be LCVS, let $A \subset E$ and let $\mathcal{F}$ be a family of functions $A \to F$. $\mathcal{F}$ is called *locally bounded* at $x \in A^\circ$ if there exists a neighborhood $U \subset A$ of x so that $\bigcup_{f\in\mathcal{F}} f(U)$ is bounded in F and if V is an open subset of E we say that a family $\mathcal{F} \subset \mathcal{H}_G(V,F)$ is locally bounded if it is locally bounded at every point $x \in V$.

Proposition 3.2.5 ([Din99, p.153])**.** *If U is an open subset of a LCVS E and F is a normed linear space then $f \in \mathcal{H}_G(U,F)$ is holomorphic if and only if it is locally bounded.*

Corollary 3.2.6. *Let U be an open subset of a (DFS)-space E and let F be a normed space. Then a function $f\colon U \to F$ is holomorphic iff it is G-holomorphic and if $f(K)$ is bounded in F for every compact subset K of U.*

Proof: ($\Rightarrow$) T is G-holomorphic and continuous by definition, hence $f(K)$ is compact whenever K is a compact subset of U. ($\Leftarrow$) Let $(E_n)_{n\in\mathbb{N}}$ be a bicompact generating sequence of Banach spaces for E so that the closed unit ball B_n of E_n is compact in E_{n+1} (see Theorem 2.6.4). Then T is continuous iff $T|_{U_n}$ is continuous for all $n \in \mathbb{N}$, where $U_n = E_n \cap U$ (see Proposition 2.7.4 (11)). Let $x \in U_n$ and let $\epsilon > 0$ so that $x + \epsilon \cdot B_n \subset U_n$. Hence $x + \epsilon \cdot B_n$ is compact in U_{n+1} and thus in U. By assumption $T(x + \epsilon \cdot B_n)$ is bounded, which means that $T|_{U_n}$ is locally bounded and hence holomorphic by Proposition 3.2.5. Consequently $T|_{U_n}$ is continuous for all $n \in \mathbb{N}$, yielding the continuity of T, which is thus holomorphic. □

Theorem 3.2.7. *Let E and F be LCVS and let U be an open subset of E. For a family $\mathcal{F} \subset \mathcal{H}(U, F)$, the following holds:*

(1) *If $\mathcal{F}$ is locally bounded, then $\mathcal{F}$ is bounded in the compact-open topology.*

(2) *If E is a (DFM)-space, and F a Banach space, then TFAE:*

  (i) $\mathcal{F}$ is locally bounded.

  (ii) $\mathcal{F}$ is point-wise bounded and equicontinuous.

  (iii) $\mathcal{F}$ is bounded in the compact-open topology.

Proof: If $\mathcal{F}$ is locally bounded, then every compact subset of U can be covered by a finite number of open subsets of U on which $\mathcal{F}$ is bounded, hence $\mathcal{F}$ is bounded on every compact subset and is thus bounded in τ_{co}, which shows (1) and $(i) \Rightarrow (iii)$. Now suppose that E is a (DFM)-space and that F is a Banach space. For $(iii) \Rightarrow (ii)$ see [Din99, p.157]. Suppose that $\mathcal{F}$ is equicontinuous and pointwise bounded and let $x \in U$. Then there exists a neighborhood W of x so that $\|F(y) - F(x)\| \leq 1$ for all $F \in \mathcal{F}$ and all $y \in W$. Further there is an $M > 0$ so that $\mathcal{F}(x) \subset \{\|z\| \leq M\}$ and so $\|F(y)\| \leq 1 + M$ for all $y \in W$, which means that $\mathcal{F}$ is locally bounded, which shows $(ii) \Rightarrow (i)$. □

Corollary 3.2.8. *Let U be an open subset of a (DFM)-space E and let F be a Banach space.*

(1) [Din99, p.172] *$(\mathcal{H}(U, F), \tau_{\text{co}})$ is a Fréchet space.*

(2) *If $F = \mathbb{C}^d$, then $(\mathcal{H}(U, F), \tau_{\text{co}})$ is a Fréchet-Montel space.*

Proof: (1) Any open subset of a (DFM)-space has a countable fundamental system of compact sets, hence $\mathcal{H}(U, F)$ is metrizable. By [Din99, p.157] a function $f\colon U \to F$ is holomorphic if it is G-holomorphic and bounded on the compact subsets of U. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy-sequence in $\mathcal{H}(U, F)$ and let $f\colon U \to F$ be its pointwise limit. Let $\phi \in F'$, let $x \in U$ and W be an absolutely convex 0-neighborhood in E so that $x + W \subset U$. For $\nu \in E$ set $g_n\colon \mathbb{D} \to \mathbb{C},\ g_n(z) := \phi \circ f_n(x + \nu \cdot z)$. The sequence $(g_n)_{n\in\mathbb{N}}$ converges uniformly on compact subsets to $g(z) := \phi \circ f(x + \nu \cdot z)$ which implies that g is holomorphic and that f is G-holomorphic. Obviously f is bounded on compact subsets of U and is hence holomorphic. (2) By Theorem 3.2.7, every bounded subset of $\mathcal{H}(U, \mathbb{C}^d)$ is pointwise bounded and equicontinuous, and hence relatively compact by the theorem of Arzelà-Ascoli.

□

Chapter 4

Rings of convergent power series

After establishing the necessary notational framework we will investigate the topological features of the ring of convergent power series $\mathcal{O}_d$ equipped with its natural inductive topology, which turns it into a (DFN)-space. Then we are going to examine holomorphic maps $\mathcal{O}_d \to \mathbb{C}$. These maps appear rather naturally also in the study of finite-dimensional holomorphy, for example if one is interested in describing the coefficients of substitution maps $\phi \mapsto F(x, \phi(x))$. Another example is the map $(a_n)_{n\in\mathbb{N}} \mapsto \prod \frac{1}{1-a_n}$, which locally in $\mathcal{O}_1$ defines a holomorphic map. We will see that holomorphic maps on $\mathcal{O}_d$ can be expanded again into convergent power series $\sum c_\gamma x^\gamma$, which enables concrete computations. We will then turn our attention to holomorphic functions $\mathcal{O}_d \to \mathcal{O}_d$. After investigating the topological properties of $\mathcal{H}(\mathcal{O}_d, \mathcal{O}_d)$ we will establish the monomial series expansion for maps $\mathcal{O}_d \to \mathcal{O}_d$.

Basic definitions and Notation. By $\mathcal{P}_d(R)$ we will denote the ring of formal power series in d variables over the commutative ring R and we shortly write $\mathcal{P}_d$ for $\mathcal{P}_d(\mathbb{C})$. The subring of $\mathcal{P}_d(\mathbb{C})$ of convergent power series will be denoted by $\mathcal{O}_d$. We will write elements of $\mathcal{P}_d^p$ in the form $\left(\sum_\alpha c_{\alpha,k} x^\alpha\right)_{1\le k\le p} := \left(\sum_{\alpha\in\mathbb{N}^d} c_{\alpha,1}, \ldots, \sum_{\alpha\in\mathbb{N}^d} c_{\alpha,p} x^\alpha\right)$. For $\phi = \left(\sum c_{\alpha,k} x^\alpha\right)_{1\le k\le p} \in \mathcal{P}_d^p$, let $|\phi| := \left(\sum |c_{\alpha,k}|\, x^\alpha\right)_{1\le k\le p} \in \mathcal{P}_d^p$ and for $\epsilon \in \mathbb{N}^d$, $1 \le j \le p$, let $\phi[\epsilon, j] := c_{\epsilon,j}$. The ϵ-jet of ϕ is $j_\epsilon(\phi) := \sum_{\alpha\le\epsilon} c_{\alpha,k} x^\alpha$, where for $\alpha, \beta \in \mathbb{N}^d$ we write $\alpha \le \beta$ if $\alpha_k \le \beta_k$ holds for all k. We set $|\alpha| := \alpha_1 + \cdots + \alpha_d$. For $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$, $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, let $z^\alpha := z_1^{\alpha_1} \cdots z_d^{\alpha_d}$, and for $S = (S_1, \ldots, S_p) \in (\mathbb{C}^d)^p$ and $V = (V_1, \ldots, V_p) \in M_{d,p}(\mathbb{N})$ let $S^V := S_1^{V_1} \cdots S_p^{V_p}$. For $T \in \mathbb{R}_+^d$ and $F = \sum_{\alpha\in\mathbb{N}^d} c_\alpha x^\alpha \in \mathcal{P}_d$, let $\|F\|_T^q := \left(\sum_{\alpha\in\mathbb{N}^d} |c_\alpha T^\alpha|^q\right)^{1/q}$ for $q \in [1, \infty)$ and let $\|F\|_T^\infty = \sup_{\alpha\in\mathbb{N}^d} |c_\alpha|\, T^\alpha$. For $d, p \in \mathbb{N}, q \in [1, \infty],\ M = (M_1, \ldots, M_p) \in \mathbb{R}_+^p$, $S = (S_1, \ldots, S_p) \in \left(\mathbb{R}_+^d\right)^p$ we set

$$H_{S,M} := \left\{F = (F_1, \ldots, F_p) \in \mathcal{P}_d^p \;:\; \|F_k\|_{S_k}^\infty \le M_k \text{ for } k = 1, \ldots, p\right\}$$

which will play the role of an infinite-dimensional polydisc. Furthermore, let

$$\ell^q(S) := \left\{F = (F_1, \ldots, F_p) \in \mathcal{P}_d^p \;:\; \|F_k\|_{S_k}^q < \infty \text{ for } k = 1, \ldots, p\right\}$$

We equip $\ell^q(S)$ with the norm $\|\ \|_S^q$, where $\|(F_1, \ldots, F_p)\|_S^q := \max_{1\le k\le p} \|F_k\|_{S_k}^q$ and note that $\ell^q(S)$ is a Banach space. For $S = (S_1, \ldots, S_d) \in \mathbb{R}_+^d$ and

$T = (T_1, \ldots, T_d) \in \mathbb{R}_+^d$, we write $S < T$ if $S_i < T_i$ for $1 \leq i \leq d$ and we set

$$\frac{S}{T} = \left(\frac{S_1}{T_1}, \ldots, \frac{S_d}{T_d}\right) \in \mathbb{R}_+^d$$

Likewise, for $S = (S_1, \ldots, S_p) \in \left(\mathbb{R}_+^d\right)^p$, $T = (T_1, \ldots, T_p) \in \left(\mathbb{R}_+^d\right)^p$, we set

$$\frac{S}{T} := \left(\frac{S_1}{T_1}, \ldots, \frac{S_p}{T_p}\right) \in \left(\mathbb{R}_+^d\right)^p$$

For $n \in \mathbb{Z}$ and $S = (S_1, \ldots, S_d) \in \mathbb{R}_+^d$ let

$$S^n := (S_1^n, \ldots, S_d^n)$$

Definition 4.0.9. The ring of convergent power series in d variables $\mathcal{O}_d$ is the set of power series which converge locally at 0, equipped with the usual operations (Cauchy-product multiplication plus coefficient-wise scalar multiplication and addition) inherited from the ring of formal power series $\mathcal{P}_d$ in d variables. For $p \in \mathbb{N}$ we set $\mathcal{O}_d^p = \mathcal{O}_d \times \cdots \times \mathcal{O}_d$. A power series $\sum_{\alpha \in \mathbb{N}^d} c_\alpha x^\alpha \in \mathcal{P}_d$ is locally convergent iff there is a $T \in \mathbb{R}_+^d$ with $\sup_{\alpha \in \mathbb{N}^d} |c_\alpha| T^\alpha < \infty$. Hence $\mathcal{O}_d = \bigcup_{T \in \mathbb{R}_+^d} \ell^\infty(T)$. We equip $\mathcal{O}_d$ with the locally convex topology with which it becomes the inductive limit of the inductive net of Banach spaces $(\ell_d^\infty(T))_{T \in \mathbb{R}_+^d}$ (indexed by $\left(\mathbb{R}_+^d, \preceq_{inv}\right)$). For any null sequence $(T_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+^d$ the inductive sequence $(\ell^\infty(T_n))_{n \in \mathbb{N}}$ is equivalent to the net $(\ell^\infty(T))_{T \in \mathbb{R}_+^d}$, and thus induces the same inductive limit. Note that for $1 \leq p \leq q \leq \infty$ we have that $\ell^q(T) \hookrightarrow \ell^p(S)$ for $T > S$. The inclusions are continuous, and so the inductive nets $(\ell^q(T))_{T \in \mathbb{R}_+^d}$ and $(\ell^p(T))_{T \in \mathbb{R}_+^d}$ are equivalent for any $1 \leq p \leq q \leq \infty$.

4.1 The topology of $\mathcal{O}_d^p$

Theorem 4.1.1.

(1) *$\mathcal{O}_d^p$ is a (DFN)-space.*

(2) *The set of monomials forms a Schauder-basis for $\mathcal{O}_d$.*

(3) *The set of polydisc $H_{S,M}$ forms a fundamental system of compact sets of $\mathcal{O}_d^p$ and the compact subsets of $\mathcal{O}_d^p$ are metrizable.*

Proof: (1) Let p =1 and let $R > S \in \mathbb{R}_+^d$. We are going to show that the inclusion map $T\colon \ell^\infty(R) \to \ell^\infty(S)$ is nuclear. Let $f_\alpha = R^\alpha p_\alpha \in (\ell^\infty(R))'$, let $\lambda_\alpha = \left(\frac{S}{R}\right)^\alpha$ and $y_\alpha = \frac{x^\alpha}{S^\alpha} \in \ell^\infty(S)$, and let $a = \sum_{\alpha \in \mathbb{N}^d} c_\alpha x^\alpha \in \ell^\infty(R)$. Then

$$\sum_{\alpha \in \mathbb{N}^d} \lambda_\alpha f_\alpha(a) y_\alpha = \sum_{\alpha \in \mathbb{N}^d} \frac{S^\alpha}{R^\alpha} c_\alpha R^\alpha \frac{x^\alpha}{S^\alpha} = \sum_{\alpha \in \mathbb{N}^d} c_\alpha x^\alpha = T(a)$$

The sequence of functionals $(f_\alpha)_{\alpha \in \mathbb{N}^d}$ is uniformly bounded on the unit ball of $\ell^\infty(R)$ and hence equicontinuous, the sequence $(\lambda_\alpha)_{\alpha \in \mathbb{N}^d}$ is absolutely summable and $(y_\alpha)_{\alpha \in \mathbb{N}^d}$

is bounded in $\ell^\infty(S)$. By Definition 2.6.6 this means that T is nuclear and hence any generating sequence $(\ell^\infty(R_n))_{n\in\mathbb{N}}$ (where $R_n \to 0$) is nuclear, which shows by Theorem 2.7.8 that $\mathcal{O}_d$ is a (DFN)-space. And since both the class of nuclear and (DFS)-spaces are closed with respect to taking finite products we conclude that $\mathcal{O}_d^p$ is a nuclear (DFS)-space and hence a (DFN)-space. (Theorem 2.7.8).
(2) Let $a = \sum_{\alpha\in\mathbb{N}^d} c_\alpha x^\alpha \in \mathcal{O}_d$. Then there is an $R \in \mathbb{R}_+^d$ so that $a \in \ell^\infty(R)$. For any $S < R$ we have that $\left\|\sum_{|\alpha|\leq n} c_\alpha x^\alpha - a\right\|_S^\infty \xrightarrow{n\to\infty} 0$ and hence $\sum_{|\alpha|\leq n} c_\alpha x^\alpha \to a$ in $\mathcal{O}_d$. As $\mathcal{O}_d$ is sequentially retractive, it follows that any basis of $\mathcal{O}_d$ is Schauder basis by a result of Floret ([Flo73, p.67/68]).
(3) Let $(R_n)_{n\in\mathbb{N}}$ be a strictly monotonous null sequence in $(\mathbb{R}_+^d)^p$ and let K be a compact subset of $\mathcal{O}_d^p$. By Theorem 2.2.11 we know that $\ell^\infty(R_n)$ is compactly regular, hence K is a compact subset of some $\ell^\infty(R_{N_0})$. Consequently there exists an $M \in \mathbb{R}_+^p$ so that $K \subset H_{R_{N_0},M}$. As the embedding $\ell^\infty(R_{N_0}) \to \ell^\infty(R_{N_0+1})$ is compact (by (1)) we conclude that $H_{R_{N_0},M}$ is relatively compact in $\ell^\infty(R_{N_0+1})$. In fact, $H_{R_{N_0},M}$ is even compact as it is closed in $\ell^\infty(R_{N_0+1})$ and is thus compact in $\mathcal{O}_d^p$, which shows that the family of polydiscs forms a fundamental system of compact discs. Since $H_{R_0,M}$ is compact in $\ell^\infty(R_{N_0+1})$, the subspace topologies induced by $\ell^\infty(R_{N_0+1})$ and $\mathcal{O}_d^p$ coincide, which shows that compact subsets of $\mathcal{O}_d^p$ are metrizable. □

Corollary 4.1.2 (Montel's Theorem). *Let Ω be an open subset of $\mathbb{C}^{d_1}$ and let $\mathcal{B} \subset \mathcal{H}\left(\Omega, \mathbb{C}^{d_2}\right)$. The family $\mathcal{B}$ is relatively compact if and only if it is locally bounded.*

Proof: If $\mathcal{B}$ is locally bounded, then it is bounded on compact subsets of Ω and is hence a bounded subset of $\left(\mathcal{H}\left(\Omega, \mathbb{C}^{d_2}\right), \tau_{co}\right)$, which is a semi-Montel space by Theorem 2.8.4 and hence $\mathcal{B}$ is relatively compact. Conversely, suppose that $\mathcal{B}$ is relatively compact and let $\overline{P_r(z_0)} \subset \Omega$. The evaluation is continuous on $\left(\mathcal{H}\left(\Omega, \mathbb{C}^{d_2}\right), \tau_{co}\right) \times \Omega$, hence $\mathcal{F}\left(\overline{P_r(z_0)}\right)$ is relatively compact in $\mathbb{C}^d$ and so $\mathcal{B}$ is locally bounded. □

Now we show that the strong dual of $\mathcal{O}_d$ is isomorphic to the space of entire functions $\mathcal{H}(\mathbb{C}^d)$ equipped with the compact open topology.

Theorem 4.1.3. *Let $d, p \in \mathbb{N}$. For $\phi = \sum_{\alpha\in\mathbb{N}^d} \phi_\alpha x^\alpha \in \mathcal{H}\left(\mathbb{C}^d\right)$ and $a = \sum_{\alpha\in\mathbb{N}^d} c_\alpha x^\alpha$ set $\phi^*(a) := \sum_{\alpha\in\mathbb{N}^d} c_\alpha \phi_\alpha$. The map $*\colon \left(\mathcal{H}\left(\mathbb{C}^d\right), \tau_{\mathrm{co}}\right) \to (\mathcal{O}_d)'_\beta$, $\phi \mapsto \phi^*$ is an isomorphism of locally convex spaces.*

Proof: As $\mathcal{O}_d$ is a (DF)-space, its strong dual is a Fréchet space and thus it suffices to show that the map $*$ is a continuous linear bijection as the open mapping theorem holds for Fréchet spaces. Let $\phi = \sum_\alpha \phi_\alpha x^\alpha \in \mathcal{H}\left(\mathbb{C}^d\right)$ and $a = \sum_{\alpha\in\mathbb{N}^d} c_\alpha x^\alpha \in \mathcal{O}_d$. There exist $M \in \mathbb{R}_+$ and $S \in \mathbb{R}_+^d$ so that $\sup |c_\alpha| S^\alpha \leq M$ and since $\sum_{\alpha\in\mathbb{N}^d} \phi_\alpha z^\alpha$ converges absolutely for all $z \in \mathbb{C}^d$ we conclude that ϕ^* is well-defined. Next we show that $*$ is surjective. Let $f \in \mathcal{O}_d'$ and $a = \sum_{\alpha\in\mathbb{N}^d} c_\alpha x^\alpha \in \mathcal{O}_d$. Then $f(a) = \sum_\alpha c_\alpha f(x^\alpha) = \sum_{\alpha\in\mathbb{N}^d} c_\alpha \phi_\alpha = \phi^*(a)$, where we set $\phi_\alpha = f(x^\alpha)$ and $\phi = \sum_\alpha \phi_\alpha x^\alpha \in \mathcal{P}_d$. As $a(z) := \sum_\alpha z^\alpha x^\alpha \in \mathcal{O}_d$ and since $\phi(z) = \sum_{\alpha\in\mathbb{N}^d} \phi_\alpha z^\alpha = f(a(z)) < \infty$ for all $z \in \mathbb{C}^d$ we have that $\phi \in \mathcal{H}\left(\mathbb{C}^d\right)$ and it follows that $*$ is surjective. The linearity and injectivity of $*$ are obvious, so it remains to show that $*$ is continuous. Let

$\phi_i = \sum_{\alpha\in\mathbb{N}^d} \phi^i_\alpha x^\alpha$ be a sequence in $\mathcal{H}\left(\mathbb{C}^d\right)$ which converges in the compact-topology to ϕ. Let B be a bounded set in $\mathcal{O}_d$ and $\epsilon > 0$. Then there are $S \in \mathbb{R}^d_+, M \in \mathbb{R}_+$ so that $B \subset B_{S,M}$. Let $i_0 \in \mathbb{N}$ so that $\|\phi^i - \phi\|_{\Delta_{2\dot{S}^{-1}}} \leq \epsilon$ holds for $i \geq i_0$. Let $a \in B_{S,M}$. We have

$$|(\phi_i^* - \phi^*)(a)| \leq \sum_{\alpha\in\mathbb{N}^d} |\phi^i_\alpha - \phi_\alpha||c_\alpha| \leq \sum_{\alpha\in\mathbb{N}^d} \left(\frac{\epsilon S^\alpha}{2^{|\alpha|}}\right)\left(\frac{M}{S^\alpha}\right) \leq$$

$$\leq \epsilon M \sum_{\alpha\in\mathbb{N}^d} \left(\frac{1}{2}\right)^{|\alpha|} \leq \epsilon M 2^d$$

We conclude that $(\phi^i)^*$ converges to ϕ^* and hence $*$ is continuous, which gives the desired result.

$\square$

Definition 4.1.4. For $\phi = (\phi_1, \ldots, \phi_p) \in \mathcal{H}\left(\mathbb{C}^d\right)^p$, where $\phi_k = \sum\limits_{\alpha\in\mathbb{N}^d} \phi_{\alpha,k} x^\alpha$, we set $\phi^* : \mathcal{O}^p_d \to \mathbb{C},\ (\sum c_{\alpha,1}x^\alpha, \ldots, \sum c_{\alpha,p}x^\alpha) \mapsto \sum\limits_{k=1}^{p} \sum\limits_{\alpha\in\mathbb{N}^d} \phi_{\alpha,k} c_{\alpha,k}$

Corollary 4.1.5. *The map $\phi \mapsto \phi^*$ is an isomorphism $\left(\mathcal{H}\left(\mathbb{C}^d\right), \tau_{\mathrm{co}}\right)^p \to (\mathcal{O}^p_d)'_\beta$.*

Theorem 4.1.6. *Let $T : \mathcal{O}^p_{d_1} \to \mathcal{O}_{d_2}$ be a linear map.*
Then T is continuous iff there exists a sequence $(\phi_{\delta,k})_{\substack{\delta\in\mathbb{N}^{d_2}\\ 1\leq k\leq p}} \subset \left(\mathcal{H}\left(\mathbb{C}^d\right)\right)^p$ which satisfies

$$\forall S \in (\mathbb{R}^d_+)^p\ \forall M \in \mathbb{R}^p_+\ \exists R \in \mathbb{R}^{d_2}_+ : \sup_{\delta\in\mathbb{N}^{d_2},\ 1\leq k\leq p} \|\phi_{\delta,k}\|_{\Delta_{S,M}} R^\delta < \infty$$

so that $T(a) = \sum_{\substack{\delta\in\mathbb{N}^d\\ 1\leq k\leq p}} (\phi_{\delta,k})^*(a)\, x^\delta$.

Proof: Suppose that T is continuous. Then all coordinates T_δ are continuous, hence there exist $(\phi_\delta)_{\delta\in\mathbb{N}^{d_2}} \subset \mathcal{H}\left(\mathbb{C}^d\right)^p$ such that $\phi^*_\delta = T_\delta$. Since T is continuous and hence bornological, we have that for every $S \in (\mathbb{R}^d_+)^p$ and $M \in \mathbb{R}^p_+$ there exists an $R \in \mathbb{R}^{d_2}_+$ and a $K \in \mathbb{R}^p_+$ so that $T\left(H_{S,M}\right) \subset H_{R,K}$. Hence we obtain the estimate $\|\phi_{\delta,k}\|_{\Delta_{S,M}} \leq \frac{K}{R^\delta}$, which holds $\forall 1 \leq k \leq p\ \forall \delta \in \mathbb{N}^{d_2}$. Conversely, suppose that the sequence $(\phi_{\delta,k})_{\substack{\delta\in\mathbb{N}^{d_2}\\ 1\leq k\leq p}} \subset \left(\mathcal{H}\left(\mathbb{C}^d\right)\right)^p$ satisfies the hypothesis of the theorem, and let $T(a) := \sum_{\substack{\delta\in\mathbb{N}^d\\ 1\leq k\leq p}} (\phi_{\delta,k})^*(a)\, x^\delta$. As in the proof of Theorem 4.1.3, one sees that $|\phi^*(a)| \leq\ M\, \|\phi\|_{\Delta_{S^{-1}}}$ for any $\phi \in \mathcal{H}\left(\mathbb{C}^d\right)$, and $a \in H_{S,M}$. Hence for any $S, M > 0$ there exist R, K so that $\left|\phi^*_{\delta,k}(a)\right| \leq \frac{K}{T^\delta}$ holds $\forall 1 \leq k \leq p,\ \forall \delta \in \mathbb{N}^{d_2}$ and $\forall a \in H_{S,M}$, which means that $T\colon H_{S,M} \to H_{R,p\cdot K}$. Consequently T is bornological and hence continuous. $\square$

Lemma 4.1.7 ([MV92, p.332]).

(1) *A sequence f_k converges to f in $\mathcal{O}_d$ iff it converges weakly to f.*

(2) *For $\phi = \sum \phi_\alpha x^\alpha \in \mathcal{H}\left(\mathbb{C}^d\right)$, set $p_\phi(\sum c_\alpha x^\alpha) := \sum |c_\alpha \phi_\alpha|$. The system of semi-norms $(p_\phi)_{\phi\in\mathcal{H}\left(\mathbb{C}^d\right)}$ generates the inductive topology of $\mathcal{O}_d$.*

(3) *For $\phi = \sum \phi_\alpha x^\alpha \in \mathcal{H}\left(\mathbb{C}^d\right)$, set $p_\phi^\infty(\sum c_\alpha x^\alpha) := \sup |\phi_\alpha c_\alpha|$. The system of semi-norms $(p_\phi^\infty)_{\phi\in\mathcal{H}\left(\mathbb{C}^d\right)}$ generates the inductive topology of $\mathcal{O}_d$.*

(4) *A formal power series $\sum_{\alpha\in\mathbb{N}^d} c_\alpha x^\alpha$ is convergent iff $\sum_{\alpha\in\mathbb{N}^d} c_\alpha \phi_\alpha < \infty$ for all $\sum \phi_\alpha x^\alpha \in \mathcal{H}\left(\mathbb{C}^d\right)$*

Proof: (1) If f_k converges weakly to f, then the set $B := \{f_k\}_{k\in\mathbb{N}} \cup \{f\}$ is weakly bounded and weakly closed, and hence bounded and closed in $\mathcal{O}_d^p$, which means that B is compact. Compact subspaces of $\mathcal{O}_d^p$ are metrizable, hence f_k converges to f in the topology of $\mathcal{O}_d^p$ if and only if all convergent subsequences of f_k converge to f, which follows from the weak convergence.
(2) Recall that the topology of a locally convex space is described by its set of continuous semi-norm. A semi-norm $q\colon \mathcal{O}_d \to [0,\infty)$ is continuous if and only if $q|_{\ell^\infty(R)}$ is continuous for all $R \in \mathbb{R}_+^d$. Let $q \in cs(\mathcal{O}_d)$ and set $\phi_\alpha := q(x^\alpha)$. Since $q|$ is a continuous seminorm on $\ell^\infty(R)$ there is a $C_R \geq 0$ so that $q(a) \leq C_R \cdot \|a\|_R^\infty$ for all $a \in \ell_R^\infty$, which implies that $\phi_\alpha \leq C_R R^\alpha$ for all $\alpha \in \mathbb{N}^d$. We conclude that $\phi = \sum_{\alpha\in\mathbb{N}^d} \phi_\alpha x^\alpha \in \mathcal{H}\left(\mathbb{C}^d\right)$ and since $q(a) \leq p_\phi(a)$ for all $a \in \mathcal{O}_d$ the claim follows.
(3) Let $\phi = \sum \phi_\alpha x^\alpha \in \mathcal{H}\left(\mathbb{C}^d\right)$. Set $\psi = \sum 2^d \phi_\alpha 2^{|\alpha|} x^\alpha \in \mathcal{H}\left(\mathbb{C}^d\right)$ and let $a = \sum c_\alpha x^\alpha \in \mathcal{O}_d$. Then

$$p_\phi(a) = \sum |\phi_\alpha c_\alpha| = \sum \left|\frac{\phi_\alpha}{2^{|\alpha|}} c_\alpha\right| \left(\frac{1}{2}\right)^{|\alpha|} \leq 2^d \cdot \sup_{\alpha\in\mathbb{N}^d} \left|\frac{\phi_\alpha}{2^{|\alpha|}} c_\alpha\right| = p_\psi^\infty(a)$$

It is easy to see the p_ψ^∞ is indeed a semi-norm. The above estimate and (3) shows that the system $\{p_\psi^\infty \mid \psi \in \mathcal{H}\left(\mathbb{C}^d\right)\}$ generates the inductive topology.
(4) is a consequence of the fact that $\left(\mathcal{H}\left(\mathbb{C}^d\right), \tau_{\text{co}}\right)$ can be represented as a Köthe sequence space and the canonical representation of its dual, as well as of the reflexivity of $\mathcal{O}_d$ - see [MV92]. □

Remark. Considering (1) and (2) of the theorem above, one might suppose that $\mathcal{O}_d$ carries the weak topology. However, this is wrong, which was pointed out to us by user "jbc" in [staa]. In fact, if E is a Fréchet-space, then the strong topology coincides with the weak topology on E' if and only if E is finite-dimensional (see [KṠ92]).

4.2 The space of entire functions

As a direct consequence of the duality between (FN)- and (DFN)- spaces (Theorem 2.7.8) we obtain the following result:

Lemma 4.2.1. $\left(\mathcal{H}(\mathbb{C}^d), \tau_{\text{co}}\right)$ *is a nuclear Fréchet space.*

We note that the topology induced by $\mathcal{O}_d$ on $\mathcal{H}\left(\mathbb{C}^d\right)$ is strictly weaker than the compact-open topology: the sequence $(x^\alpha)_{\alpha\in\mathbb{N}^d}$ is a null-sequence in $\mathcal{O}_d$ but is unbounded in $\left(\mathcal{H}\left(\mathbb{C}^d\right), \tau_{\mathrm{co}}\right)$.

Next we are going to discuss *point-wise* convergence of entire functions.

Definition 4.2.2. The topology of *point-wise convergence* on $\mathcal{H}\left(\mathbb{C}^d\right)$ is the topology initiated by all semi-norms $\left\{\hat{\epsilon}_x \mid x \in \mathbb{C}^d\right\}$ where $\hat{\epsilon}_x(f) = |f(x)|$. A net $(\phi_j)_{j\in J}$ converges *point-wisely* to ϕ if $\phi_j(x) \to \phi(x)$ for all $x \in \mathbb{C}^d$.

The pointwise limit of a sequence of entire functions need not to be holomorphic, but still there is an open and dense subset of $\mathbb{C}^d$ on which the convergence is locally uniform and on which the limit function is holomorphic.

Theorem 4.2.3 (Osgood [Kra]). *Let $(f_j)_{j\in\mathbb{N}}$ be a sequence of holomorphic functions on a domain $\Omega \subset \mathbb{C}$. Assume that $(f_j)_{j\in\mathbb{N}}$ converges pointwise to a limit function f on Ω. Then f is holomorphic on a dense, open subset of Ω. The convergence is uniform on compact subsets of the dense, open set.*

In the light of Osgood's theorem it makes sense to ask whether a sequence of holomorphic functions which converges pointwise to a function f converges uniformly on all compact subsets, if we further suppose that the limit function f is *holomorphic* on Ω - i.e. whether pointwise and uniform convergence coincide on $\mathcal{H}(\Omega)$. However, it turns out that the answer is negative, even if all sequence members are entire functions:

Lemma 4.2.4 ([stab]). *The topology of pointwise convergence on $\mathcal{H}\left(\mathbb{C}^d\right)$ is strictly weaker than the compact-open topology on $\mathcal{H}\left(\mathbb{C}^d\right)$.*

Proof: Let d=1. Using Runge's theorem we are going to construct a sequence of polynomials which converges to zero pointwise but fails to converge uniformly to zero in any neighborhood of $0 \in \mathbb{C}$. Let $K_n := \left\{re^{i\theta} \mid r \in [0,1],\ \frac{2}{n} \leq \theta \leq 2\pi\right\}$, let $A_n := n \cdot K_n$, let $z_n := \frac{e^{i\frac{1}{n}}}{n}$ and $L_n := A_n \cup \{z_n\}$. Let V_n^1 and V_n^2 be open and disjoint neighborhoods of z_{n_0} and A_n, respectively, and let $V_n = V_n^1 \cup V_n^2$. Set

$$f_n \colon V \to \mathbb{C},\ f_n(z) := \begin{cases} 0 & z \in V_n^2 \\ \frac{z}{z_n} & z \in V_n^1 \end{cases}$$

By Runge's Theorem we can find a sequence of polynomials p_n so that $\|f_n - p_n\|_{V_n} < \frac{1}{n}$. Since $A_n \subset A_{n+1}$ and $\bigcup\limits_{n\in\mathbb{N}} A_n = \mathbb{C}$, we have that $(p_n)_{n\in\mathbb{N}}$ converges pointwise to zero. But $(p_n)_{n\in\mathbb{N}}$ does not converge in any neighborhood of 0: Note that $z_n \to 0$. Given $\epsilon > 0$, let $N_0 \in \mathbb{N}$ so that $z_n \in \mathbb{D}_\epsilon$ for $n > N_0$. So $\|p_n\|_{\mathbb{D}_\epsilon} \geq 1 - \frac{1}{n}$ for $n \geq N_0$. The same counterexample applies also for d>1. □

A positive result for the question when pointwise convergence coincides with locally uniform convergence can be achieved for sequences of univalent functions. Let $\Omega \subset \mathbb{C}$ be a domain. A function $f \colon \Omega \to \mathbb{C}$ is called *univalent* if f is holomorphic and injective.

Theorem 4.2.5 ([BM03]). *Let $\Omega \subset \mathbb{C}$ be a domain, $f_n \in \mathcal{H}(\Omega)$ be a family of univalent functions which converges pointwise to some function $f\colon \Omega \to \mathbb{C}$. Then f is analytic and f_n converges locally uniformly to f. If Ω is connected, then f is either an univalent function or a constant function.*

The proof is done by using that fact that around an arbitrary point $z_0 \in \Omega$ one can find a disc centered at a point at which the sequence converges locally uniform (which follows from Osgoods theorem). Then the growth theorem for *schlicht* functions on the unit disc can be applied to show that the sequence is bounded on compact subsets of the disc containing z_0.

Other topologies which naturally appear in the study of power series spaces are the simple topology and the Krull topology inherited by $\mathcal{P}_d$. The following results are taken from [BZ79] and we state them for the sake of completeness of our study of topological properties of $\mathcal{O}_d$ without using them later.

Definition 4.2.6. The *simple* topology τ_{sim} of $\mathcal{O}_d$ is the metrizable locally-convex topology described by the set of seminorms $\{q_\beta\}_{\beta\in\mathbb{N}^d}$, where
$q_\beta(\sum_{\alpha\in\mathbb{N}^d} c_\alpha x^\alpha) := |c_\beta|$. A net $f_i = \sum_{\alpha\in\mathbb{N}^d} c_\alpha^i x^\alpha$ converges to $f = \sum_{\alpha\in\mathbb{N}^d} c_\alpha x^\alpha$ in $(\mathcal{O}_d, \tau_s)$ iff $c_\alpha^i \xrightarrow{i\in I} c_\alpha$ for all $\alpha \in \mathbb{N}^d$. $(\mathcal{O}_d, \tau_s)$ is a Fréchet-algebra – a complete commutative metrizable topological algebra.

Lemma 4.2.7 ([BZ79, p.37/38/39]). *Let J be an ideal in $\mathcal{O}_d$.*
(1) *J is closed in the simple topology.*
(2) *$\mathcal{O}_d \big/ J$ has a unique Fréchet-topology.*
(3) *The Krull topology and the simple topology admit the same continuous linear functionals.*

Note that this also implies that $\mathcal{O}_d$ has a unique topology which turns it into a Fréchet-algebra.

4.3 Holomorphic maps between rings of convergent power series

In the first part of this section we discuss different characterizations of holomorphy for maps between rings of convergent power series. In the second part we establish the monomial series expansion for holomorphic functions $U \to \mathbb{C}$ for suitable open subsets U of $\mathcal{O}_d^p$ - a result which is due to Boland and Dineen ([BD78]). We are going to show that a map $\mathcal{O}_d^p \to \mathcal{O}_d^q$ is holomorphic iff it is bornological and if its coefficients are holomorphic, which allows us to characterize holomorphic maps in this setting as sequences of power series whose coefficients satisfy certain Cauchy-type estimates, which results in a projective-inductive description of $\mathcal{H}(\mathcal{O}_d^p, \mathcal{O}_d^q)$ (see Definition 4.4.15). Then we will investigate the compact-open topology on $\mathcal{H}(\mathcal{O}_d^p, \mathcal{O}_d)$ and show that $(\mathcal{H}(\mathcal{O}_d^p, \mathcal{O}_d^q), \tau_{\text{co}})$ has a basis.

Theorem 4.3.1. *Let U be an open subset of $\mathcal{O}^p_{d_1}$. Let $F : U \to \mathcal{O}^q_{d_2}$. TFAE:*
(1) *F is holomorphic.*
(2) *F is continuous and has holomorphic coefficient functions.*
(3) *F has holomorphic coefficient functions and $F(K)$ is bounded for all $K \underset{\mathrm{co}}{\subseteq} U$.*
(4) *For any holomorphic curve $c\colon \mathbb{D} \to \mathcal{O}^p_{d_1}$ the composition $F \circ c$ is bounded on compact subsets of the open unit disc $\mathbb{D}$ and F has holomorphic coefficient functions.*
(5) *F is curve-holomorphic.*
(6) *F is G-holomorphic and $F(K)$ is bounded for all $K \underset{\mathrm{co}}{\subseteq} U$.*

Proof: (1) $\Rightarrow$ (2) Let $F_{\alpha,k} = p_{\alpha,k} \circ F$, where $p_{\alpha,k}$ denotes the continuous coefficient projection $(\sum_{\beta\in\mathbb{N}^{d_2}} c_{\beta,1}x^\beta, \dots, \sum_{\beta\in\mathbb{N}^{d_2}} c_{\beta,q}x^\beta) \mapsto c_{\alpha,k}$. Note that continuous linear maps between locally convex spaces (over $\mathbb{C}$) are holomorphic and that the composition of holomorphic functions is again holomorphic [Din99, p.219].
(2) $\Rightarrow$ (3) F is continuous, therefore $F(K)$ is compact and thus bounded, for $K \underset{\mathrm{co}}{\subseteq} U$.
(3) $\Rightarrow$ (4) The image of a compact subset of $\mathbb{D}$ under a holomorphic curve is compact since holomorphic curves are continuous (see Lemma 3.1.7), therefore $F \circ c$ is bounded on compact subsets of $\mathbb{D}$.
(4) $\Rightarrow$ (5) We need to show that for any holomorphic curve $c\colon \mathbb{D} \to U$ and any $\psi \in (\mathcal{O}^q_{d_2})'$ the composition $\psi \circ F \circ c$ is a holomorphic function $\mathbb{D} \to \mathbb{C}$. Recall that ${}^*\colon \left(\mathcal{H}\left(\mathbb{C}^{d_2}\right)\right)^q \to \left(\mathcal{O}^q_{d_2}\right)'$ is an isomorphism (see Theorem 4.1.5). Let $c\colon \mathbb{D} \to U$, set $f_{\alpha,k} = p_{\alpha,k} \circ F \circ c$ and let $\phi = (\phi_1, \dots, \phi_q) \in \left(\mathcal{H}\left(\mathbb{C}^{d_2}\right)\right)^q$, $\phi_k = \sum_{\alpha\in\mathbb{N}^d} c_{\alpha,k}x^\alpha$. By Lemma 3.2.2 holomorphic curves into $(\mathcal{O}_d)^p$ are holomorphic functions and as the composition of holomorphic functions is holomorphic ([Din99, p.219]) we have that all $f_{\alpha,k}\colon \mathbb{D} \to \mathbb{C}$ are holomorphic. For $r \in (0,1)$, there exist $S \in (\mathbb{R}^{d_2}_+)^q$ and $M \in \mathbb{R}^q_+$ so that $(F \circ c)(\overline{\Delta_r}) \subset H_{S,M}$, which implies that $\|f_{\alpha,k}\|_{\overline{\mathbb{D}_r}} \le \frac{M_k}{S^\alpha_k}$. For $z \in \overline{\Delta_r}$ we have

$$\begin{aligned} |(\phi^* \circ F \circ c)(z)| = \left| \sum_{1\le k\le q,\ \alpha\in\mathbb{N}^{d_2}} \phi_{\alpha,k} f_{\alpha,k}(z) \right| &\le \sum_{1\le k\le q,\ \alpha\in\mathbb{N}^{d_2}} |\phi_{\alpha,k}| \, \|f_{\alpha,k}\|_{\overline{\mathbb{D}_r}} \le \\ &\le \sum_{1\le k\le q,\ \alpha\in\mathbb{N}^{d_2}} |\phi_{\alpha,k}| \frac{M_k}{S^\alpha_k} < \infty \end{aligned}$$

This shows that the series $\sum_{1\le k\le q,\alpha\in\mathbb{N}^{d_2}} \phi_{\alpha,k} f_{\alpha,k} = \phi^* \circ F \circ c$ converges locally uniformly on $\mathbb{D}$ and as the coefficient functions $f_{\alpha,k}$ are holomorphic we conclude that $\phi^* \circ F \circ c$ is a holomorphic function $\mathbb{D} \to \mathbb{C}$. Hence F is curve-holomorphic.
For (1) $\Leftrightarrow$ (5) see Theorem 3.2.3 and Theorem 3.1.10, and recall that the (inductive) topology on $\mathcal{O}^p_d$ coincides with its c^∞-topology as it is a (DFS)-space and therefore also a convenient space as it is complete and hence locally complete ($\Leftrightarrow$ convenient).
(1) $\Rightarrow$ (6) Holomorphic maps are G-holomorphic and continuous.
(6) $\Rightarrow$ (3) All coordinates of F are G-holomorphic and bounded on compact subsets. Theorem 3.2.6 then yields that all coordinates of F are holomorphic. $\square$

Since bounded subsets of $\mathcal{O}^p_d$ are relatively compact, we obtain the following characterization of entire functions:

Theorem 4.3.2. *For a map $F : \mathcal{O}_{d_1}^p \to \mathcal{O}_{d_2}^q$, the following are equivalent:*

(1) *F is holomorphic.*
(2) *F is continuous and has holomorphic coefficient functions.*
(3) *F is bornological and has holomorphic coefficient functions.*
(4) *For any holomorphic curve $c\colon \mathbb{D} \to \mathcal{O}_{d_1}^p$ the composition $F \circ c$ is bounded on compact subsets of the open unit disc $\mathbb{D}$ and F has holomorphic coefficient functions.*
(5) *F is curve-holomorphic.*
(6) *F is bornological and G-holomorphic.*

Definition 4.3.3. A sequence $(x_n)_{n\in\mathbb{N}}$ of elements in a locally convex space E is called a *basis* for E if for every element $a \in E$ there exists a *unique* sequence $(a_k)_{k\in\mathbb{N}} \subset \mathbb{C}$ so that $a = \sum_{k=0}^\infty a_k x_k := \lim_{n\to\infty} \sum_{k=0}^n a_k x_k$. A basis is called a *Schauder-basis* if all projections $p_k\colon E \to \mathbb{C}$, $p_m\left(\sum_{k=0}^\infty a_k x^k\right) := a_m$ are continuous. For $n \in \mathbb{N}$ we set $s_n\left(\sum_{k=0}^\infty a_k x_k\right) := \sum_{k=0}^n a_k x_k$. An *equi-Schauder-basis* is a basis which satisfies that the set of sum-operator $\{s_n\}_{n=0}^\infty$ is equicontinuous.

Lemma 4.3.4. *Let E be a locally convex space.*

(1) *If E is the inductive limit of a sequentially retractive sequence of locally convex spaces, then every basis in E is already a Schauder-basis.* [Flo73]
(2) *If E is barrelled, then every Schauder-basis is already an equi-Schauder-basis.* [Din99, p.188]

Definition 4.3.5. We say that a series $\sum_{n=0}^\infty y_n$ converges *unconditionally* if $\sum_{n=0}^\infty y_n = \sum_{k=0}^\infty y_{\pi(k)}$ for any permutation π of the natural numbers. A series $\sum_{n=0}^\infty y_n$ converges unconditionally to S if and only if the set $\{y_n \mid n \in \mathbb{N}\}$ is summable to S (see Definition 2.6.11). A basis $(x_n)_{n\in\mathbb{N}}$ for a space E is called *unconditional* if $\sum_{k=0}^\infty a_k x_k$ converges unconditionally for every $\sum_{k=0}^\infty a_k x_k \in E$. A basis $(x_n)_{n\in\mathbb{N}}$ is called *absolute*, if $\rho_q(\sum_{n\in\mathbb{N}} c_n x_n) := \sum_{n\in\mathbb{N}} |c_n| q(x_n)$ defines a continuous semi-norm for every $q \in \mathrm{cs}(E)$. If $\{x_k\}_{k\in\mathbb{N}}$ is an absolute basis, then it is an equi-Schauder-basis and $\sum_{k\in\mathbb{N}} a_k x_k$ converges absolutely (and hence unconditionally) whenever it converges (see [Din99, p.189]).

Lemma 4.3.6. *The monomials form an absolute basis for $\mathcal{O}_d^p$.*

Proof: WLOG p=1. A semi-norm $q\colon \mathcal{O}_d \to \mathbb{R}_+$ is continuous iff there exists a $C > 0$ and a $\phi = \sum \phi_\alpha x^\alpha \in \mathcal{H}(\mathbb{C}^d)$ so that $q(a) \le C \cdot p_\phi(a)$ for all $a \in \mathcal{O}_d$. Set $q_\alpha = q(x^\alpha)$. As $q_\alpha < C|\phi_\alpha|$ we see that $\rho_q(\sum c_\alpha x^\alpha) = \sum |c_\alpha||q(x^\alpha)| \le C p_\phi(a)$, which shows both that ρ_q is well-defined and continuous. $\square$

Definition 4.3.7. We define

$$\mathcal{M}_d := \{f\colon \mathbb{N}^d \to \mathbb{N} \mid f(\alpha) \neq 0 \text{ for only finitely many } \alpha \in \mathbb{N}^d\}$$

For $\gamma \in \mathcal{M}_d$ and $\phi = \sum_{\alpha\in\mathbb{N}^d} c_\alpha x^\alpha \in \mathcal{P}_d(R)$, we set $\phi^\gamma := \prod_{\alpha\in\mathbb{N}^d} c_\alpha^{\gamma(\alpha)}$, where we use the convention $0^0 = 1$. Analogously, for $\gamma = (\gamma_1, \dots, \gamma_p) \in \mathcal{M}_d^p$ and

$\phi = (\phi_1, \dots, \phi_p) \in \mathcal{P}_d^p(R)$ let $\phi^\gamma := \phi_1^{\gamma_1} \dots \phi_p^{\gamma_p}$. By e_k we denote the k^{th} standard unit vector of $\mathbb{N}^p$. For $\gamma \in \mathcal{M}_d^p$, we set

$$\begin{aligned} \mathrm{sh}(\gamma) &:= \sum_{k=1}^{p} \sum_{\alpha \in \mathbb{N}^d} \gamma_k(\alpha) \cdot e_k \in \mathbb{N}^p \\ \mathrm{wt}(\gamma) &:= \sum_{k=1}^{p} \sum_{\alpha \in \mathbb{N}^d} \gamma_k(\alpha) \cdot \alpha \in \mathbb{N}^d \\ \mathrm{wt_v}(\gamma) &:= (\mathrm{wt}(\gamma_1), \dots, \mathrm{wt}(\gamma_p)) \in M_{d\times p}(\mathbb{N}) \end{aligned}$$

We note that $\mathrm{sh}(\gamma) = \sum_{k=1}^p \mathrm{sh}(\gamma_k)e_k$ and that $\mathrm{wt}(\gamma) = \sum_{k=1}^p \mathrm{wt}(\gamma_k)$. The support $\mathrm{supp}(\gamma)$ of $\gamma \in \mathcal{M}_d^p$ is defined as the set $\{\alpha \in \mathbb{N}^d \mid \exists k : \ \gamma_k(\alpha) \neq 0\}$. We set

$$\begin{aligned} I_{d,n} &:= \left\{\alpha \in \mathbb{N}^d \middle| \, \alpha_k \leq n\right\} \\ A_{d,n}^p &:= \left\{\gamma \in \mathcal{M}_d^p | \, \mathrm{supp}(\gamma) \subset I_{d,n} \text{ and } \max_{1\leq k\leq p} \max_{\alpha\in\mathbb{N}^d} \gamma_k(\alpha) \leq n\right\}. \end{aligned}$$

For $S \in (\mathbb{R}_+^d)^p, \ M \in \mathbb{R}_+^p$, let $\mathfrak{g}_{S,M} = \left(\sum_{\alpha\in\mathbb{N}^d} \frac{M_1 \ x^\alpha}{S_1^\alpha}, \dots, \sum_{\alpha\in\mathbb{N}^d} \frac{M_p \ x^\alpha}{S_p^\alpha}\right) \in \mathcal{O}_d^p$.

Theorem 4.3.8. *Let $d, p \in \mathbb{N}$. The infinite-dimensional geometric series*

$$g(a) = \sum_{\gamma\in\mathcal{M}_d} a^\gamma$$

converges on the infinite-dimensional polydisc $H_{\frac{1}{S},Q} = \{\sum_{\alpha\in\mathbb{N}^d} c_\alpha x^\alpha \ : \ |c_\alpha| < QS^\alpha\}$ for $0 < Q < 1, \ S < (1, \dots, 1), S \in \mathbb{R}_+^d$. Furthermore, for $a \in H_{\frac{1}{S},Q}$ we have the following estimate:

$$|g(a)| \leq g(\mathfrak{g}_{S^{-1},Q}) = \sum_{\gamma\in\mathcal{M}_d} Q^{\mathrm{sh}(\gamma)} S^{\mathrm{wt_v}(\gamma)} = \prod_{\alpha\in\mathbb{N}^d} \frac{1}{1 - QS^\alpha} \tag{4.1}$$

Analogously, let $S = (S_1, \dots, S_p) \in (\mathbb{R}_+^d)^p$ such that $S_k < (1, \dots, 1)$ for all $k \leq p$, and let $Q = (Q_1, \dots, Q_p) \in \mathbb{R}_+^p$ satisfying $Q_k < 1$ for all $k \leq p$. Then $g := \sum_{\gamma\in\mathcal{M}_d^p} x^\gamma$ converges on $H_{\frac{1}{S},Q}$, and for $a \in H_{\frac{1}{S},Q}$

$$|g(a)| \leq g(\mathfrak{g}_{S^{-1},Q}) = \sum_{\gamma\in\mathcal{M}_d^p} Q^{\mathrm{sh}(\gamma)} S^{\mathrm{wt_v}(\gamma)} = \prod_{j=1}^{p} \prod_{\alpha\in\mathbb{N}^d} \frac{1}{1 - Q_j S_j^\alpha} \tag{4.2}$$

Proof: First, we show the case p=1. Let $(\alpha_k)_{k=1}^N$ be an enumeration of $I_{d,n}$ and set $\gamma_k = \gamma(\alpha_k)$ for $\gamma \in \mathcal{M}_d$.

$$\begin{aligned} g(\mathfrak{g}_{S^{-1},Q}) &= \sum_{\gamma\in\mathcal{M}_d^p} Q^{\mathrm{sh}(\gamma)} S^{\mathrm{wt_v}(\gamma)} = \lim_{n\to\infty} \sum_{\gamma\in A_{d,n}} Q^{\mathrm{sh}(\gamma)} S^{\mathrm{wt_v}(\gamma)} = \\ &= \lim_{n\to\infty} \sum_{\gamma\in A_{d,n}} Q^{\gamma_1+\dots+\gamma_N} (S^{\alpha_1})^{\gamma_1} \cdots (S^{\alpha_N})^{\gamma_N} = \end{aligned}$$

$$= \lim_{n\to\infty} \sum_{0\le\gamma_n\le N} Q^{\gamma_1+\dots+\gamma_N} (S^{\alpha_1})^{\gamma_1} \dots (S^{\alpha_N})^{\gamma_N} =$$

$$= \lim_{n\to\infty} \left(\sum_{\gamma_1=0}^{n} Q^{\gamma_1} (S^{\alpha_1})^{\gamma_1} \right) \cdots \left(\sum_{\gamma_N=0}^{n} Q^{\gamma_N} (S^{\alpha_N})^{\gamma_N} \right)$$

$$= \lim_{n\to\infty} \prod_{\alpha\in I_{d,n}} \frac{1-(QS^\alpha)^{n+1}}{1-QS^\alpha}$$

If we set

$$T = \prod_{\alpha\in\mathbb{N}^d} \frac{1}{1-QS^\alpha} \text{ and } T_n = \prod_{\alpha\in I_{d,n}} \frac{1-(QS^\alpha)^{n+1}}{1-QS^\alpha},$$

then

$$T - T_n = \prod_{\alpha\in I_{d,n}} \frac{1}{1-QS^\alpha} \Bigg(\underbrace{\prod_{\alpha\notin I_{d,n}} \frac{1}{1-QS^\alpha}}_{u_n} - \underbrace{\prod_{\alpha\in I_{d,n}} \left(1-(QS^\alpha)^{n+1}\right)}_{v_n} \Bigg).$$

We are going to show that $\prod_{\alpha\in\mathbb{N}^d} \frac{1}{1-QS^\alpha}$ is convergent (which implies that $u_n \xrightarrow{n\to\infty} 1$) and that $v_n \xrightarrow{n\to\infty} 1$. Set $f_n(x,y) := \prod_{\alpha\in I_{d,n}} (1-(yx^\alpha))$ and $S^n := (S_1^n, \dots, S_d^n)$ and observe that $v_n = f_n(S^{n+1}, Q^{n+1})$. Next we show that $f_n(x,y)$ converges uniformly to $f(x,y) := \prod_{\alpha\in\mathbb{N}^d} (1-(yx^\alpha))$ on $\Delta_r \times \Delta_t$ for $r \in \mathbb{R}_+^d$ with $r < (1,\dots,1)$ and $0 < t < 1$. Let log denote the principal branch of the complex logarithm. The series $\sum_{k=1}^\infty \frac{z^k}{k}$ converges locally uniformly to $-\log(1-z)$ on the open unit disc in the complex plane, which yields the inequality $|\log(1-z)| \le \frac{|z|}{1-|z|}$ for $z \in \mathbb{C}$, $|z| < 1$. For $q > 0$ the sequence $\frac{1}{1-q^n}$ is bounded and so there is a constant $C > 0$ depending only on r, t such that $\frac{1}{1-|yx^\alpha|} \le C$ for all $\alpha \in \mathbb{N}^d$ and $(x,y) \in \Delta_r \times \Delta_t$.

$$\left| \sum_{\alpha\in\mathbb{N}^d} \log(1-(yx^\alpha)) \right| \le \sum_{\alpha\in\mathbb{N}^d} \frac{|yx^\alpha|}{1-|yx^\alpha|} \le C \sum_{\alpha\in\mathbb{N}^d} |yx^\alpha| \tag{4.3}$$

The inequality shows both that $\prod_{\alpha\in\mathbb{N}^d} \frac{1}{1-QS^\alpha}$ converges and that f_n converges locally uniformly to f and it follows readily that

$$\lim_{n\to\infty} v_n = = \lim_{n\to\infty} f_n(S^{n+1}, Q^{n+1}) = f(0,0) = 1.$$

Therefore T_n converges to T, which completes the case $p = 1$. For $p > 1$, the validity of our claim can now be deduced easily:

$$g(\mathfrak{g}_{S,Q}) = \sum_{\gamma\in\mathcal{M}_d^p} \mathfrak{g}_{S,Q}^\gamma = \sum_{\gamma\in\mathcal{M}_d^p} Q^{\mathrm{sh}(\gamma)} S^{\mathrm{wt_v}(\gamma)} = \lim_{n\to\infty} \sum_{\gamma\in A_{d,n}^p} Q^{\mathrm{sh}(\gamma)} S^{\mathrm{wt_v}(\gamma)} =$$

$$= \lim_{n\to\infty} \left(\sum_{\gamma_1\in\mathcal{A}_{d,n}} Q_1^{\mathrm{sh}(\gamma_1)} S_1^{\mathrm{wt}(\gamma_1)} \right) \cdots \left(\sum_{\gamma_p\in\mathcal{A}_{d,n}} Q_p^{\mathrm{sh}(\gamma_p)} S_p^{\mathrm{wt}(\gamma_p)} \right) =$$

$$= \prod_{j=1}^{p} \prod_{\alpha\in\mathbb{N}^d} \frac{1}{1-Q_j S_j^\alpha}$$

□

Corollary 4.3.9. *Let $0 < S < 1$ and $0 < Q < 1$, let $a = \sum_{n\in\mathbb{N}} a_n x^n \in H_{S^{-1},Q}$ and $g(x) = \sum_{\gamma\in\mathcal{M}_1} x^\gamma$. Then*

$$g(a) = \prod_{n=0}^{\infty} \frac{1}{1-a_n}$$

Proof: $g(a) = \sum_{\gamma\in\mathcal{M}_1^1} a^\gamma = \lim_{n\to\infty} \prod_{k=0}^{n} \frac{1-a_k^{n+1}}{1-a_k}$. We have already seen that $\prod_{n\geq 0} \frac{1}{1-a_n}$ converges. As in the proof above we establish the estimate

$$\left|\log(1-a_k^{n+1})\right| \leq \frac{\left|a_k^{n+1}\right|}{1-\left|a_k^{n+1}\right|} \leq \frac{(MS^k)^{n+1}}{1-(MS^k)^{n+1}}$$

And hence

$$\begin{aligned}\left|\sum_{k=0}^{n} \log(1-a_k^{n+1})\right| \leq \sum_{k=0}^{n} \frac{(MS^k)^{n+1}}{1-(MS^k)^{n+1}} &\leq \frac{M^{n+1}}{1-MS}\sum_{k=0}^{n}(s^k)^{n+1} \leq \\ &\leq M^{n+1}\frac{1}{1-MS}\frac{1}{1-S}\end{aligned}$$

which yields $\lim_{n\to\infty}\sum_{k=0}^{n}\log(1-a_k^{n+1}) = 0$ and $\lim_{n\to\infty}\prod_{k=0}^{n}(1-a_k^{n+1}) = 1$. Hence

$$\lim_{n\to\infty}\prod_{k=0}^{n}\frac{1-a_k^{n+1}}{1-a_k} = \lim_{n\to\infty}\prod_{k=0}^{n}\left(\frac{1}{1-a_k}\right)\prod_{k=0}^{n}(1-a_k^{n+1}) = \prod_{k=0}^{\infty}\frac{1}{1-a_k}$$

. □

Example 4.3.10. The following representation of the Dirichlet series as the value of an infinite dimensional holomorphic function is a result of H. Bohr (see [Din99, p.231]). Let p_n denote the n^{th} prime number ($p_0 = 2, p_1 = 3, \dots$) and for $z \in \mathbb{C}$ let $P(z) = \sum_{n\geq 0} p_n^z x^n$. For $n \in \mathbb{N}_+$ there is a unique $\gamma \in \mathcal{M}_1^1$ so that $n = P(1)^\gamma$.
Let $\Re(z) > 1, 0 < S < 1, 0 < M < 1$ and $\sum_{n=0} a_n x^n \in H_{S^{-1},M}$. Set $a_\gamma = a_n$ for $n = P(1)^\gamma \in \mathbb{N}_+$. Then $\sum_{n\geq 1}\frac{a_n}{n^z} = \sum_{\gamma\in\mathcal{M}_1} a_\gamma P(-z)^\gamma$.

Definition 4.3.11. Let $S \in (\mathbb{R}_+^d)^p, M \in \mathbb{R}_+^p$ and $a \in \mathcal{O}_d^p$. The set $a + H_{S,M}$ is called a *compact polydisc* with *center* a. For $S \in (\mathbb{R}_+^d)^p, M \in \mathbb{R}_+^p, R = (R_{\alpha,k})_{\alpha\in I_{d,n},1\leq k\leq p}$, such that all $R_{\alpha,k} \in \mathbb{R}_+$, we set

$$H_{R,S,M} := \left\{\left(\sum c_{\alpha,k}x^\alpha\right)_{1\leq k\leq p} \;:\; |c_{\alpha,k}| \leq R_{\alpha,k} \text{ if } \alpha\in I_{d,n},\ |c_{\alpha,k}| \leq \frac{M_k}{S_k^\alpha} \text{ if } \alpha\notin I_{d,n}\right\}$$

and $H_{R,S,M}(a) := a + H_{R,S,M}$. These sets will be called *quasi-polydiscs with center a*. For $\phi = (\phi_1,\dots,\phi_p) \in (\mathcal{H}\left(\mathbb{C}^d\right))^p$ and $\epsilon > 0$ the set

$$P_{\phi,\epsilon}(a) = \left\{f \in \mathcal{O}_d^p \,\middle|\, \max_{1\leq k\leq p} p_{\phi_k}^\infty(a-f) < \epsilon\right\}$$

is called an *open polydisc* with *center* a. Since the semi-norms p_ϕ^∞ are continuous (Lemma 4.1.7), the open polydiscs are indeed open subsets of $\mathcal{O}_d^p$.

Lemma 4.3.12. *Let U be an open subset of $\mathcal{O}_d^p$.*

(1) *The family of finite unions of compact polydiscs contained in U forms a fundamental system of compact sets for U.*

(2) *Suppose that U is an open polydisc $P_{\phi,\epsilon}$. Then there exists a fundamental system of compact sets consisting of quasi-polydiscs centered at a for U.*

Proof: (1) Let K be a compact subset of U. Then there is an $S \in \left(\mathbb{R}_+^d\right)^p$ so that K is a compact subset of $\ell^\infty(S)$. Since $U \cap \ell^\infty(S)$ is open, we can cover K with a finite number of translated polydiscs of radius S which are contained in U.
(2) WLOG $a = 0$. Let K be a compact subset of U. Since the compact polydiscs form a fundamental system of compact sets for $\mathcal{O}_d^p$, there are $S, M > 0$ so that $K \subset H_{S,M}$. There exists an $n_0 \in \mathbb{N}$ so that $|\phi_{\alpha,k}|\frac{M_k}{S_k^\alpha} < \epsilon/2$, for all $\alpha \geq (n_0, \ldots, n_0)$. For $\alpha \in I_{d,n_0}$, let $R_{\alpha,j} := \max_{a\in K} |a[\alpha, j]| < \frac{\epsilon}{|\phi_{\alpha,j}|}$ and $R = (R_{\alpha,k})_{\alpha\in I_{d,n}} \in \mathbb{R}_+^{N_0}$. By construction, $K \subset H_{R,S,M} \subset U$ and it is easy to see that $H_{R,S,M}$ is compact. $\square$

Theorem 4.3.13. *Let $U = P_{\phi,\epsilon}(a)$ be an open polydisc and let $H_{R,S,M}(a) \subset U$. There exist $K > M,\ 0 < T < S$ and $\tilde{R} > R$ so that $H_{R,S,M} \subset H_{\tilde{R},T,K} \subset U$.*

Proof: WLOG a=0. For $n \in \mathbb{N},\ R = (R_{\alpha,k})_{\alpha\in I_{d,n},1\leq k\leq p},\ S \in (\mathbb{R}_+^d)^p,\ M \in \mathbb{R}_+^p$ let $h_{R,S,M} = (\sum_\alpha h_{\alpha,k}x^\alpha)_{1\leq k\leq p} \in \mathcal{O}_d^p$, where

$$h_{\alpha,k} = \begin{cases} R_{\alpha,k} & \text{if } \alpha \in I_{d,n} \\ \frac{M_k}{S_k^\alpha} & \text{else} \end{cases}.$$

Let $c\colon \mathbb{C} \setminus \{0\} \to \mathcal{O}_d^p,\ z \mapsto h_{z\cdot R,z\cdot S,z\cdot M}$. The coefficients of c are polynomials and c is bornological, hence c is a holomorphic function. Consequently $v(t) := p_\phi^\infty(h(t))$ is a continuous function $\left[\frac{1}{2}, \frac{3}{2}\right] \to \mathbb{R}_+$, with $v(1) < \epsilon$. Hence there is a $\lambda > 1$ so that $p_\phi(\lambda) < \epsilon$. This means that $H_{\lambda R,\frac{S}{\lambda},M\lambda} \subset U$: For $b \in H_{\lambda R,\frac{S}{\lambda},M\lambda}$ we have $p_\phi^\infty(b) \leq p_\phi^\infty(h_{\lambda R,\frac{S}{\lambda},M\lambda}) = v(\lambda) < \epsilon$, so $b \in U_{\phi,\epsilon}$. $\square$

Theorem 4.3.14. *Let $S = (S_1, \ldots, S_p) \in (\mathbb{R}_+^d)^p$ satisfy $S_k < (1, \ldots, 1)$ for all $k \leq p$, and assume that $Q = (Q_1, \ldots, Q_p) \in \mathbb{R}_+^p$ satisfies $Q_k < 1$ for all $k \leq p$. Let $n \in \mathbb{N}$, let $0 < R_{\alpha,k} < 1$ for all $\alpha \in I_{d,n}$, $1 \leq k \leq p$ and let $R = (R_{\alpha,k})_{\alpha\in I_{d,n},1\leq k\leq p}$. For $\gamma \in \mathcal{M}_d^p$, $n \in \mathbb{N}$ set $\rho_n(\gamma) := \gamma|_{I_{d,n}^p}$ and $R^{\rho_n(\gamma)} := \prod_{\alpha\in I_{d,n},1\leq k\leq p} R_{\alpha,k}^{\gamma_k(\alpha)}$. Then $g := \sum_{\gamma\in\mathcal{M}_d^p} x^\gamma$ converges on $H_{R,\frac{1}{S},Q}$, and for $a \in H_{R,S,M}$ we obtain the estimate*

$$|g(a)| \leq \sum_{\gamma\in\mathcal{M}_d^p} R^{\rho_n(\gamma)} Q^{\mathrm{sh}(\gamma-\rho_n(\gamma))} S^{\mathrm{wt_v}(\gamma-\rho_n(\gamma))} = \prod_{k=1}^p \left(\prod_{\alpha\in\mathbb{N}^d\setminus I_{d,n}} \frac{1}{1 - Q_k S_k^\alpha} \prod_{\alpha\in I_{d,n}} \frac{1}{1 - R_{\alpha,k}} \right)$$

Proof:

$$
\begin{aligned}
|g(a)| &\leq \sum_{\gamma\in\mathcal{M}_d^p} |a|^\gamma \leq \sum_{\gamma\in\mathcal{M}_d^p} R^{\rho_n(\gamma)} Q^{\mathrm{sh}(\gamma-\rho_n(\gamma))} S^{\mathrm{wt_v}(\gamma-\rho_n(\gamma))} = \\
&= \lim_{M\to\infty} \sum_{\gamma\in A_{d,M}^p} R^{\rho_n(\gamma)} Q^{\mathrm{sh}(\gamma-\rho_n(\gamma))} S^{\mathrm{wt_v}(\gamma-\rho_n(\gamma))} = \\
&= \lim_{M\to\infty} \prod_{k=1}^p \left(\left(\prod_{\alpha\in I_{d,M}\cap I_{d,n}} \sum_{j_{\alpha,k}=1}^M R_{\alpha,k}^{j_{\alpha,k}} \right) \cdot \left(\prod_{\alpha\in I_{d,M}\setminus I_{d,n}} \sum_{j_{\alpha,k}=1}^M (Q_k S_k^\alpha)^{j_{\alpha,k}} \right) \right) = \\
&= \lim_{M\to\infty} \prod_{k=1}^p \left(\prod_{\alpha\in I_{d,M}} \frac{R_{\alpha,k}^{M+1}-1}{R_{\alpha,k}-1} \prod_{\alpha\in I_{d,M}\setminus I_{d,n}} \frac{(Q_k S_k^\alpha)^{M+1}-1}{(Q_k S_k^\alpha)-1} \right) = \\
&= \lim_{M\to\infty} \prod_{k=1}^p \left(\prod_{\alpha\in I_{d,M}} \frac{R_{\alpha,k}^{M+1}-1}{R_{\alpha,k}-1} \prod_{\alpha\in I_{d,M}} \frac{(Q_k S_k^\alpha)^{M+1}-1}{(Q_k S_k^\alpha)-1} \left(\prod_{\alpha\in I_{d,n}} \frac{(Q_k S_k^\alpha)^{M+1}-1}{(Q_k S_k^\alpha)-1} \right)^{-1} \right) = \\
&= \prod_{k=1}^p \left(\prod_{\alpha\in I_{d,n}} \frac{1}{R_{\alpha,k}-1} \prod_{\alpha\in\mathbb{N}^d} \frac{1}{(Q_k S_k^\alpha)-1} \left(\prod_{\alpha\in I_{d,n}} \frac{1}{(Q_k S_k^\alpha)-1} \right)^{-1} \right)
\end{aligned}
$$

$\square$

Proposition 4.3.15 ([Din99, p.205,208,172])**.**
Let $U = U_{\phi,\epsilon}$ be an open polydisc in $\mathcal{O}_d^p$ and let $f \in \mathcal{H}(U,\mathbb{C})$.

(1) *There exists a unique sequence of coefficients $(c_\gamma)_{\gamma\in\mathcal{M}_d^p}$ so that*

$$f(x) = \sum_{\gamma\in\mathcal{M}_d^p} c_\gamma\,(x-a)^\gamma \tag{4.4}$$

for all $x \in U$. The series converges absolutely and uniformly on the compact subsets of U.

(2) *(Cauchy-estimates) Let $H_{R,S,M}(a) \subset U$, where $R = (R_{\alpha,k})_{\alpha\in I_{d,n},1\leq k\leq p}$. Then*

$$|c_\gamma| \leq \|f\|_{H_{R,S,M}(a)} \frac{S^{\mathrm{wt_v}(\gamma-\rho_n(\gamma))}}{M^{\mathrm{sh}(\gamma-\rho_n(\gamma))}} \frac{1}{R^{\rho_n(\gamma)}}$$

Especially, if $H_{S,M}(a) \subset U$, then

$$|c_\gamma| \leq \|f\|_{H_{S,M}(a)} \frac{S^{\mathrm{wt_v}(\gamma)}}{M^{\mathrm{sh}(\gamma)}}$$

(3) *The monomials form an absolute basis for $(\mathcal{H}(U),\tau_{co})$.*

(4) *$(\mathcal{H}(U),\tau_{co})$ is a nuclear Fréchet space.*

Proof: (1),(2),(3) WLOG let a=0. Let $H_{R,S,M} \subset U$. By Theorem 4.3.13 there exist $0 < T < S$, $\tilde{R} > R$ and $K > M$ such that $H_{R,S,M} \subset H_{\tilde{R},T,K} \subset U$. Set
$E_N := \{(f_1,\ldots,f_p) \in \mathcal{O}_d^p \mid f_k[\alpha] = 0 \text{ for } \alpha \notin I_{d,N}\}$, $\Delta_{R,S,M}(N) := E_N \cap H_{R,S,M}$, $M_N := \{\gamma \in \mathcal{M}_d^p \mid \mathrm{supp}(\gamma) \subset E_N\}$ and $d(N) = \dim E_N$. Since the restriction of f to any finite-dimensional open subset of U is holomorphic we can expand $f_N := f|_{\Delta_{R,S,M}(N)}$

into a locally uniformly convergent Taylor series $\sum_{\delta \in \mathbb{N}^{d(n)}} c_\delta z^\delta$. Let $\gamma \in \mathcal{M}_d^p$ and let N be large enough so that $\gamma \in M_N$. Then the monomial x^γ can be identified with its restriction to E_n and thus there exists a unique $\delta(\gamma) \in \mathbb{N}^{d(n)}$ such that $x^\gamma|_{E_N} = z^{\delta(\gamma)}$ and we set $c_\gamma = c_{\delta(\gamma)}$. Note that c_γ is independent of N as $f_{N+1}|_{\Delta_{R,S,M}(N)} = f_N$. The Cauchy-estimates yield $|c_\gamma| \leq \|f\|_{H_{R,S,M}} \frac{T^{\mathrm{wt_v}(\gamma)}}{K^{\mathrm{sh}(\gamma-\rho_n(\gamma))}} \frac{1}{R^{\gamma_n}(\gamma)}$. Set $g(x) = \sum_{\mathcal{M}_d^p} c_\gamma x^\gamma$. For $b \in H_{R,S,M}$ we have

$$|g(b)| \leq \sum_{\gamma \in \mathcal{M}_d^p} |c_\gamma| \, |b|^\gamma \leq \sum_{\gamma \in \mathcal{M}_d^p} \left(\|f\|_{H_{\tilde{R},T,K}} \frac{1}{\tilde{R}^{\rho_n(\gamma)}} \frac{T^{\mathrm{wt_v}(\gamma-\rho_n(\gamma))}}{K^{\mathrm{sh}(\gamma-\rho_n(\gamma))}} \right) \left(\frac{M^{\mathrm{sh}(\gamma-\rho_n(\gamma))}}{S^{\mathrm{wt_v}(\gamma-\rho_n(\gamma))}} R^{\rho_n(\gamma)} \right) \leq$$

$$\leq \|f\|_{H_{\tilde{R},T,K}} \sum_{\gamma \in \mathcal{M}_d^p} \left(\frac{M}{K}\right)^{\mathrm{sh}(\gamma-\rho_n(\gamma))} \left(\frac{T}{S}\right)^{\mathrm{wt_v}(\gamma-\rho_n(\gamma))} \left(\frac{R}{\tilde{R}}\right)^{\rho_n(\gamma)} < \infty$$

The convergence of the geometric series is the content of Theorem 4.3.14 and the above estimate implies that $\sum_{\gamma \in \mathcal{M}_d^p} c_\gamma x^\gamma$ converges absolutely and uniformly on $H_{R,S,M}$, thus $g\colon H_{R,S,M} \to \mathbb{C}$ is continuous. By construction, $g\left(j_\epsilon(a)\right) = f\left(j_\epsilon(a)\right)$ for all $\epsilon \in \mathbb{N}^d$. The continuity of $f-g$ yields $(f-g)(a) = \lim_{|\alpha|\to\infty}(f-g)(j_\alpha(a)) = 0$, which together with the uniqueness of the Taylor-expansion on finite-dimensional subspaces of E implies that the monomials form a basis for $(\mathcal{H}(U,\mathbb{C}), \tau_{\mathrm{co}})$. Again the Cauchy-estimates imply that the monomials even form an absolute basis.
For the proof of (4) we refer the reader to [Din99, p.208] and [Din99, p.172].

□

Corollary 4.3.16. *A function $F\colon \mathcal{O}_d^p \to \mathbb{C}$ is holomorphic if and only if F is a power series $\sum_{\gamma \in \mathcal{M}_d^p} c_\gamma x^\gamma$ which converges uniformly and unconditionally on the compact subsets of $\mathcal{O}_d^p$.*

Proof: ($\Rightarrow$) $\mathcal{O}_d^p$ is an open polydisc since $\mathcal{O}_d^p = P_{\phi,1}$ with $\phi = 0$.
($\Leftarrow$) Clearly every monomial and hence every linear combination of monomials is a holomorphic function. As $\left(\mathcal{H}\left(\mathcal{O}_d, \mathbb{C}\right), \tau_{co}\right)$ is complete it follows that $\sum_{\gamma \in \mathcal{M}_d^p} c_\gamma x^\gamma$ is holomorphic. □

Lemma 4.3.17 (Lifting of finite-dimensional holomorphic maps)**.** *Let $f_k\colon \mathbb{C}^k \to \mathbb{C}$ be a sequence of holomorphic functions such that $f_k|_{\mathbb{C}^{k-1}\times\{0\}} = f_{k-1}$. Suppose further that for every $M, R > 0$ there is a $C > 0$ so that $\|f_k\|_{\Delta_{M,R}(k)} \leq C$ for all $k \in \mathbb{N}$, where $\Delta_{M,R}(k) := \{(z_1,\ldots,z_k) \in \mathbb{C}^k \;\big|\; |z_j| \leq \frac{M}{R^j} \;\forall 1 \leq j \leq k\}$. Then there is a unique holomorphic function $F\colon \mathcal{O}_1 \to \mathbb{C}$ so that that $F|_{\mathbb{C}^k} = f_k$, where we identify $\mathbb{C}^k$ with a subspace of $\mathcal{O}_1$ via $\iota_k\colon \mathbb{C}^k \to \mathcal{O}_1$, $\iota(z_1,\ldots,z_k) := z_1 + z_2 x^1 \cdots + z_k x^{k-1}$.*

Proof: Let $f_k = \sum_{\alpha \in \mathbb{N}^k} c_\alpha^k x^\alpha$. The fact that $f_k|_{\mathbb{C}^{k-1}\times\{0\}} = f_{k-1}$ yields $c_{(\alpha,0)}^{k+1} = c_\alpha^k$ for $\alpha \in \mathbb{N}^k$ and inductively we get

$$c_{\alpha,0,\ldots,0}^{k+n} = c_\alpha^k \text{ for } \alpha \in \mathbb{N}^k,\; n \in \mathbb{N}. \tag{4.5}$$

A monomial $\gamma \in \mathcal{M}_1$ with $\mathrm{supp}(\gamma) \subset \{0,\ldots,n\}$ can be identified with the vector $(\gamma(0),\ldots,\gamma(n))$. We set $c_\gamma = c_{\gamma(0),\ldots,\gamma(n)}^n$, which is well defined (which follows from the above equation) and $F = \sum_{\gamma \in \mathcal{M}_1} c_\gamma x^\gamma$. Let $k \in \mathbb{N}$ be fixed, let $(z_1, \ldots, z_k) \in \mathbb{C}^k$

$$(F \circ \iota_k)((z_1, \ldots, z_k)) = \sum_{\gamma \in \mathcal{M}_1} c_\gamma \left(z_1 + \ldots z_k x^{k-1}\right)^\gamma =$$

$$= \sum_{\operatorname{supp}(\gamma) \subset \{1, \ldots, n\}} c_\gamma \left(z_1 + \ldots z_k x^{k-1}\right)^\gamma = \sum_{\alpha \in \mathbb{N}^k} c^k_\alpha z^\alpha = f_k(z_1, \ldots, z_k)$$

Let $S, M > 0$. By assumption, there is a $C > 0$ independent of k so that $\|f_k\|_{\Delta_{M,S(k)}} \leq C$. For $\gamma \in \mathcal{M}_1$ with $\operatorname{supp}(\gamma) \subset \{0, \ldots, n\}$ we have

$$|c_\gamma| = |c^{n+1}_{\gamma(0), \ldots, \gamma(n)}| \leq C \frac{1}{\left(\frac{M}{R^0}\right)^{\gamma(0)} \cdots \left(\frac{M}{R^n}\right)^{\gamma(n)}} = C \frac{R^{\operatorname{wt}(\gamma)}}{M^{\operatorname{sh}(\gamma)}}$$

This shows that $F(a)$ converges for every $a \in \mathcal{O}^p_d$ and that F is holomorphic as the series $\sum_{\gamma \in \mathcal{M}_1} c_\gamma x^\gamma$ converges uniformly on the compact subset of $\mathcal{O}_1$. □

4.4 The space $\mathcal{H}(\mathcal{O}^p_d, \mathcal{O}^q_d)$

In the study of holomorphic functions between locally convex spaces, two other canonical topologies beside the compact open topology on $\mathcal{H}(E, F)$ appear - the Nachbin topology τ_ω and the topology τ_δ. However, we focus on the setting where both the definition and the image space are rings of convergent power series, in which the situation becomes far simpler than in the general theory and the three mentioned topologies coincide. We will state the relevant theorems as formulated by Dineen with respect to the topologies $\tau_{\text{co}}, \tau_\delta, \tau_\omega$ without actually defining these topologies (as they coincide with τ_{co} in our setting) and refer the interested reader to [Din99] for further information. In the literature usually only the space $\mathcal{H}(E)$ of scalar-valued functions holomorphic functions is studied. If E is a (DFM)-space and if F is complete, then we can identify $(\mathcal{H}(E, F), \tau_{\text{co}})$ with $\mathcal{L}_B\left(\mathcal{H}(E)'_b, F\right)$, which enables us to use these results to study also the vector-valued case.

Lemma 4.4.1. *Let U be an open subset of $\mathcal{O}^p_d$. The compact-open topology τ_{co} on $\mathcal{H}(U, \mathcal{O}^p_d)$ is generated by the system of semi-norms $\{\|\ \|_{K,\phi} \mid K \underset{\text{co}}{\subseteq} U,\ \phi \in \mathcal{H}\left(\mathbb{C}^d\right)^q\}$, where $\|F\|_{K,\phi} := \sup_{a \in K} p^\infty_\phi(F(a))$.*

Proof: See Theorem 1.2.7 and Lemma 4.1.7. □

Definition 4.4.2. For an open subset U of a locally convex space E let

$$G(U) := \{\phi \in \mathcal{H}(U)^* |\ \phi \text{ is } \tau_{\text{co}}\text{-continuous on the locally bounded subsets of } \mathcal{H}(U)\}.$$

We endow $G(U)$ with the topology of uniform convergence on locally bounded subsets of $\mathcal{H}(U)$, with which it becomes a complete locally convex space.

Lemma 4.4.3. *If U is an open subset of a (DFM)-space E, then $G(U) = (\mathcal{H}(U), \tau_{\text{co}})'_b = \left((\mathcal{H}(U), \tau_{\text{co}})', \tau_{\text{co}}\right)$ and $G(U)$ is again a (DFM)-space.*

Proof: If E is a (DFM)-space then $(\mathcal{H}(U), \tau_{\text{co}})$ is a Fréchet-Montel space (Theorem 3.2.8). In particular $(\mathcal{H}(U), \tau_{\text{co}})$ is a (k)-space and hence a function $f\colon (\mathcal{H}(U), \tau_{\text{co}}) \to \mathbb{C}$ is continuous if its restrictions to compact subsets are continuous. The locally bounded subsets of $(\mathcal{H}(U), \tau_{\text{co}})$ coincide with the τ_{co}-bounded ones and these coincide with the relatively compact subsets (Theorem 3.2.7), which implies that $G(U) = (\mathcal{H}(U), \tau_{\text{co}})'_b = \left((\mathcal{H}(U), \tau_{\text{co}})', \tau_{\text{co}}\right)$. □

Theorem 4.4.4 ([Din99, p.184])**.** *Let U be an open subset of a locally convex space E and let F be a complete locally convex space. For each $f \in \mathcal{H}(U, F)$ there exists a unique $\mathrm{J_F}(f) \in \mathcal{L}(G(U), F)$ so that $f = \mathrm{J_F}(f) \circ \delta_U$, where $\delta_U\colon U \to G(U)$ denotes the mapping $x \mapsto \epsilon_x$ and ϵ_x is the evaluation $f \mapsto f(x)$. The mapping $f \mapsto \mathrm{J_F}(f)$ establishes a linear topological isomorphism between the spaces $(\mathcal{H}(U, F), \tau_\delta)$ and $(\mathcal{L}(G(U), F), \tau_\omega)$.*

Proposition 4.4.5. *Let U be an open subset of a (DFM)-space E and let F be an arbitrary locally convex space. On $\mathcal{H}(E, F)$, the canonical topologies τ_{co}, τ_δ and τ_ω coincide.*

Proof: For any pair of spaces, $\tau_{co} \preceq \tau_\omega \preceq \tau_\delta$ (see [Din99, p.170]). If E is a (DFM)-space and F a normed space, then all three topologies coincide on $\mathcal{H}(U, F)$ by a result of Dineen [Din99, p.172, ex.3.20(b)]. For any locally convex space F, we have by definition $(\mathcal{H}(U, F), \tau_\delta) = \varprojlim_{\alpha \in \mathrm{cs}(F)} (\mathcal{H}(U, F_\alpha), \tau_\delta)$, where F_α is the quotient space $(F, \alpha)/\ker(\alpha)$ (see [Din99, p.11]). Let $\pi_\alpha\colon F \to F_\alpha$ be the canonical quotient map and let $\hat{\pi}_\alpha\colon (\mathcal{H}(U, F), \tau_{\text{co}}) \to (\mathcal{H}(U, F_\alpha), \tau_{\text{co}})$, $f \mapsto \pi_\alpha \circ f$. It is easy to see that $\hat{\pi}_\alpha$ is continuous (for any $\alpha \in \mathrm{cs}(F)$) and since $(\mathcal{H}(U, F_\alpha), \tau_{co}) = (\mathcal{H}(U, F_\alpha), \tau_\delta)$ it follows that $\mathrm{id}\colon (\mathcal{H}(U, F_\alpha), \tau_{co}) \to (\mathcal{H}(U, F_\alpha), \tau_\delta)$ is continuous. The projective description $(\mathcal{H}(U, F), \tau_\delta) = \varprojlim_{\alpha \in \mathrm{cs}(F)} (\mathcal{H}(U, F_\alpha), \tau_\delta)$ yields then that $\mathrm{id}\colon (\mathcal{H}(U, F), \tau_{co}) \to (\mathcal{H}(U, F), \tau_\delta)$ is continuous, which means that $\tau_\delta \preceq \tau_{\text{co}}$ and we obtain that $\tau_{\text{co}} = \tau_\delta$. □

Summarizing the above results in our setting we obtain:

Theorem 4.4.6. *If E is a (DFM)-space and F a complete locally convex space, then*

$$(\mathcal{H}(U, F), \tau_{\text{co}}) \cong \mathcal{L}_B\left(\mathcal{H}(U)'_b, F\right)$$

If we additionally assume that F is reflexive, then

$$(\mathcal{H}(U, F), \tau_{\text{co}}) \cong \mathcal{L}_B(F'_b, (\mathcal{H}(U), \tau_{\text{co}}))$$

Proof: It remains to show that under the additional assumption that F is reflexive the second linear topological isomorphy holds, i.e. we need to show that $\mathcal{L}_B\left(\mathcal{H}(U)'_b, F\right) \cong \mathcal{L}_B(F'_b, \mathcal{H}(U))$. Suppose that E and F are both reflexive. By [FW68, p.84] the barrelledness of the target space implies that dual operators

$$\Phi_1\colon \mathcal{L}_B(E, F) \to \mathcal{L}_B(F'_b, E'_b), \ \Phi_1(T) = T'$$
$$\Phi_2\colon \mathcal{L}_B(F'_b, E'_b) \to \mathcal{L}_B\left((E'_b)'_b, (F'_b)'_b\right), \ \Phi_2(T) = T'$$

are continuous. If H is a LCVS, then let $\iota_H\colon H \to (H_b')_b'$ be the mapping $x \mapsto \epsilon_x$, where ϵ_x denotes the evaluation $x \mapsto f(x)$. E and F are reflexive which means that ι_E and ι_F are isomorphisms. Let Ψ be the isomorphism $\mathcal{L}_B\left((E_b')_b', (F_b')_b'\right) \to \mathcal{L}_B(E,F)$, $\Psi : S \mapsto \iota_F^{-1} \circ S \circ \iota_E$. Now it is easy to see that $(\Psi \circ \Phi_2) = \Phi_1^{-1}$, which show that

$$\mathcal{L}_B\left(E,F\right) \cong \mathcal{L}_B\left(F_b', E_b'\right)$$

Recall that $\mathcal{H}\left(U\right)$ is a Montel space and hence reflexive, which yields that

$$\mathcal{L}_B\left(\mathcal{H}\left(U\right)_b', F\right) \cong \mathcal{L}_B\left((F')_b, (\mathcal{H}\left(U\right), \tau_{\text{co}})\right). \qquad \square$$

Theorem 4.4.7. *Let E, F be locally convex spaces.*
(1) *If both E and F are reflexive, then $\mathcal{L}_B\left(E,F\right) \cong \mathcal{L}_B\left(F_b', E_b'\right)$.*
(2) [Sch71] *If E is bornological and F complete, then $\mathcal{L}_B\left(E,F\right)$ is complete.*
(3) *$\mathcal{L}_B\left(E,F\right)$ is a closed subspace of $\left(\mathscr{C}\left(E,F\right), \tau_{\text{co}}\right)$.*
(4) *If E is a Montel space and if F is a semi-Montel space, then $\mathcal{L}_B\left(E,F\right)$ is a semi-Montel space. Furthermore, the pointwise bounded subsets are exactly the relatively compact subsets of $\mathcal{L}_B\left(E,F\right)$.*
(5) [Sch71] *If E is semi-reflexive, and if E_b' and F are nuclear spaces, then $\mathcal{L}_B\left(E,F\right)$ is nuclear.*

Proof: For (2) and (6) we refer to [Sch71, p.117, Ex 8], [Sch71, p.173], respectively; and for (1) see the proof of the theorem above.
(3) If a net $(f_i)_{i\in I}$ of linear functions converges pointwise to a function f, then f is again linear.
(4) Let $\mathcal{B}$ be a pointwise bounded subset of $\mathcal{L}_B\left(E,F\right)$. By the theorem of Banach ([FW68, p.51]), $\mathcal{B}$ is equicontinuous. Its pointwise closure $\overline{\mathcal{B}}^{\tau_p}$ (taken in F^E) is again equicontinuous and τ_{co} reduces to τ_p on equicontinuous families ([Wil04, p.286]). Hence $(\overline{\mathcal{B}}^{\tau_p}, \tau_p) = (\overline{\mathcal{B}}^{\tau_p}, \tau_{\text{co}}) = (\overline{\mathcal{B}}^{\tau_{\text{co}}}, \tau_{\text{co}})$. As the pointwise limit of linear functions is again linear we have that $\overline{\mathcal{B}}^{\tau_p}$ is contained in $\mathcal{L}\left(E,F\right)$. Since F is semi-Montel, $\overline{\mathcal{B}}^{\tau_p}(x)$ is compact for every $x \in E$, which means (by Tychonov's theorem) that $(\overline{\mathcal{B}}^{\tau_p}, \tau_p)$ is compact. By assumption E is a Montel space, which yields $\mathcal{L}_B\left(E,F\right) = \left(\mathcal{L}\left(E,F\right), \tau_{\text{co}}\right)$ and as $(\overline{\mathcal{B}}^{\tau_p}, \tau_p) = (\overline{\mathcal{B}}^{\tau_{co}}, \tau_{\text{co}})$ we conclude that $\mathcal{B}$ is relatively compact in $\mathcal{L}_B\left(E,F\right)$. Note that a family $\mathcal{F}$ is bounded in $\mathcal{L}_B\left(E,F\right)$ iff it is uniformly bounded on the bounded sets of E, and is hence pointwise bounded - which shows that $\mathcal{L}_B\left(E,F\right)$ is a semi-Montel space.

$\square$

Corollary 4.4.8. *If E is a (DFM)-space and F a complete semi-Montel space, then $\left(\mathcal{H}\left(U,F\right), \tau_{\text{co}}\right)$ is a semi-Montel space.*

Proof: If E is a (DFM)-space, then $\mathcal{H}(U)$ and hence $\mathcal{H}(U)_b'$ is a Montel-space. Using Theorem 4.4.6 and Theorem 4.4.7(4) then gives the desired result. $\square$

So for a huge class of spaces questions about topological properties of $\mathcal{H}(E,F)$ can be reduced to questions about properties of $\mathcal{L}_B(V,W)$, where V,W are Fréchet spaces. In light of Theorem 4.4.7 one might expect that a lot of other linear-topological properties of Fréchet spaces such as barrelledness may be transferred to $\mathcal{L}_B(V, W)$. However, the issue is far more subtle than it appears at first sight, and additional assumptions on the spaces V, W are needed to establish positive results.

In our setting we have to deal with the space $\mathcal{L}_B\left(\mathcal{H}(\mathbb{C}^d), \mathcal{H}(\mathcal{O}_d)\right)$. The following (linear-topological invariant) properties of Fréchet spaces were introduced and studied by Vogt ([HH95]), and appear (among others) for example in the study of power series spaces [MV92].

Definition 4.4.9 ([HH95])**.** Let E be a Fréchet space with a fundamental system of semi-norms $\{\| \ \|_k\}_{k\in\mathbb{N}}$ and let $B_k := \{\|x\|_k \leq 1\}$. On E_b' we introduce the dual semi-norm $\|f\|_k^* := \sup\{|f(x)| \mid x \in B_k\}$.

Definition 4.4.10. If E is a LCVS, then $\mathcal{H}_\beta(E)$ denotes the vector space of holomorphic functions on E which are bounded on bounded sets equipped with the topology of uniform convergence on bounded sets. If E is a semi-Montel space, then $\mathcal{H}_\beta(E) \cong (\mathcal{H}(E), \tau_{\text{co}})$. We say that E has property (Ω) if

$$\forall p\ \exists q\ \forall k\ \exists d, C > 0 : \ \| \ \|_q^{*1+d} \leq C \| \ \|_k^* \| \ \|_p^{*d}$$

E is said to have property (DN) if

$$\exists p\ \forall q\ \exists k, C > 0 : \| \ \|_q^2 \leq C \| \ \|_k \| \ \|_p$$

We note that property (Ω) is valid for all power series spaces, while property (DN) holds only for power series spaces of infinite type [MV92].

Theorem 4.4.11 ([Vog84, p.369; Proposition 4.5])**.** *Let E and F be nuclear Fréchet spaces. If E has property (DN) and if F has property (Ω), then $\mathcal{L}_B(E, F)$ is bornological.*

Theorem 4.4.12 ([HH95, p.2])**.** *Let E be a Fréchet space. Then*
(1) *$\mathcal{H}_\beta(E_b')$ has property (DN) if E has property (DN).*
(2) *$\mathcal{H}_\beta(E_b')$ has property (Ω) if E_b' has an absolute basis.*

Proposition 4.4.13. *$(\mathcal{H}(\mathcal{O}_d^p, \mathcal{O}_d^q), \tau_{\text{co}})$ is a complete nuclear ultrabornological space.*

Proof: By Theorem 4.4.6 we have:

$$\mathcal{H}(\mathcal{O}_d^p, \mathcal{O}_d^q) \cong \mathcal{L}_B((\mathcal{O}_d^q)_b', \mathcal{H}(\mathcal{O}_d^p)) \cong \mathcal{L}_B\left(\mathcal{H}(\mathbb{C}^d)^q, \mathcal{H}(\mathcal{O}_d^p)\right) \cong \prod_{k=1}^{p} \mathcal{L}_B\left(\mathcal{H}(\mathbb{C}^d), \mathcal{H}(\mathcal{O}_d^p)\right)$$

Theorem 4.4.12 yields that $\mathcal{H}\left(\mathbb{C}^d\right)$ satisfies property (DN) as $\mathbb{C}^d$ satisfies (DN) as it is a normed space. We have seen in Lemma 4.3.6 that $\mathcal{O}_d^p$ has an absolute basis and so we can apply the above result to conclude that $\mathcal{H}(\mathcal{O}_d^p)$ has property (Ω). Theorem 4.4.11 then implies that $\mathcal{H}(\mathcal{O}_d^p, \mathcal{O}_d^q)$ is bornological. By Theorem 4.4.7 we have that $\mathcal{H}(\mathcal{O}_d^p, \mathcal{O}_d^q)$ is complete and thus ultrabornological, as every Mackey-complete bornological locally convex space is ultrabornological [Gac04, p.105]. Theorem 4.4.7 (5) yields that $\mathcal{H}(\mathcal{O}_d^p, \mathcal{O}_d^q)$ is nuclear. $\square$

Having showed that $\mathcal{L}_B\left(\mathcal{H}(\mathbb{C}^d),\mathcal{H}(\mathcal{O}_d)\right)$ is a Montel-space (complete barrelled nuclear space are Montel spaces [FW68, p.155]) and in the light of Webb's theorem that a Montel (DF)-space is sequential, it makes sense to investigate the question whether $\mathcal{L}_B\left(\mathcal{H}(\mathbb{C}^d),\mathcal{H}(\mathcal{O}_d)\right)$ is a (DF)-space. However, it turns out the answer to this question is negative, which follows from a result by Bierstedt and Bonet and the fact that a Fréchet space is a (DF)-space iff it is normable ([Jar81, p.259]).

Theorem 4.4.14 ([BB88, Proposition 4])**.** *Let λ_1 be a Koethe-sequence space of order one. If E is a locally complete LCVS, then $\mathcal{L}_B(\lambda_1, E)$ is a (DF)-space if and only if E is a (DF)-space.*

Recall that a map $F\colon \mathcal{O}_d^p \to \mathcal{O}_d$ is holomorphic iff it is bornological and if all its coefficient functions $F_\alpha\colon \mathcal{O}_d^p \to \mathbb{C}$ are holomorphical (see Theorem 4.3.2). The coefficient functions F_α are holomorphic iff they can be written as convergent power series $\sum_{\gamma\in\mathcal{M}_d^p} c_{\alpha,\gamma}x^\gamma$, and the bornologicity of F translates into certain growth conditions on the coefficients $c_{\alpha,\gamma}$ as presented below. We will use this description to show that $\mathcal{H}\left(\mathcal{O}_d^p,\mathcal{O}_d\right)$ is the projective limit of a sequence of (LB)-spaces, which shows for example that it is a webbed space. In chapter 5 we will make use of these results to show that the inductive description of certain subclasses of holomorphic functions carries the compact-open topology, so that we can transfer results obtained for $\left(\mathcal{H}\left(\mathcal{O}_d^p,\mathcal{O}_d\right),\tau_{\mathrm{co}}\right)$ on these subspaces. From now on - if not stated otherwise - we will consider $\mathcal{H}\left(\mathcal{O}_d^p,\mathcal{O}_d\right)$ always to be equipped with τ_{co}.

Definition 4.4.15. Let $d,p\in\mathbb{N}$ let fixed. A *formal power series* (of type (d,p)) is an expression of the form

$$\left(\sum_{\gamma\in\mathcal{M}_d^p} c_{\alpha,\gamma}x^\gamma\right)_{\alpha\in\mathbb{N}^d}$$

- i.e. a sequence of monomial series which are indexed by $\mathcal{M}_d^p$. By $\mathcal{P}_{\mathbb{N}^d,p}$ we are going to denote the set (of type (d,p)-) formal power series, which becomes a vector space under the usual operations of coefficient-wise addition and scalar multiplication. For $\gamma\in\mathcal{M}_d^p,\ \alpha\in\mathbb{N}^d$, we define

$$x_\alpha^\gamma\colon \mathcal{P}_d^p\to\mathcal{P}_d,\ a\mapsto a^\gamma\cdot x^\alpha$$

So we can write a formal power series $\left(\sum_{\gamma\in\mathcal{M}_d^p} c_{\alpha,\gamma}x^\gamma\right)_{\alpha\in\mathbb{N}^d}$ as

$$\sum_{\alpha\in\mathbb{N}^d,\gamma\in\mathcal{M}_d^p} c_{\alpha,\gamma}x_\alpha^\gamma$$

For $S\in\left(\mathbb{R}_+^d\right)^p$, $M\in\mathbb{R}_+^p$, $T\in\left(\mathbb{R}_+^d\right)^q$, and $F=\sum_{\alpha\in\mathbb{N}^d,\gamma\in\mathcal{M}_d^p} c_{\alpha,\gamma}x_\alpha^\gamma$, we set

$$\|F\|_{S,M,T} := \sup_{\alpha\in\mathbb{N}^d} T^\alpha \sup_{\gamma\in\mathcal{M}_d^p} |c_{\alpha,\gamma}|\,\frac{M^{\mathrm{sh}(\gamma)}}{S^{\mathrm{wt_v}(\gamma)}}$$

$$E_{S,M,T} := \left\{F\in\mathcal{P}_{\mathbb{N}^d,p}\ :\ \|F\|_{S,M,T}<\infty\right\}$$

$$B_{S,M,T} := \left\{ F \in \mathcal{P}_{\mathbb{N}^d,p} \;:\; \|F\|_{S,M,T} \leq 1 \right\}$$

Using the standard argument for ℓ^∞-type sequence spaces one sees that the spaces $E_{S,M,T}$ (which we consider from now on equipped with $\| \, \|_{S,M,T}$) are Banach-Spaces (see for example [MV92, p.326]. We note that

$$E_{S_1,M,T} \hookrightarrow E_{S_2,M,T} \text{ for } S_1 \leq S_2$$
$$E_{S,M_1,T} \hookrightarrow E_{S,M_2,T} \text{ for } M_2 \leq M_1$$
$$E_{S,M,T_1} \hookrightarrow E_{S,M,T_2} \text{ for } T_2 \leq T_1$$

Let $E_{S,M} := \varinjlim_T E_{S,M,T}$. By Theorem 4.3.16, Theorem 4.3.2 and Proposition 4.3.15 we have that

$$\mathcal{H}\left(\mathcal{O}_d^p, \mathcal{O}_d\right) = \bigcap_{S \in (\mathbb{R}_+^d)^p, M \in (\mathbb{R}_+)^p} E_{S,M}$$

We are going to show that in fact we even have the isomorphism

$$\left(\mathcal{H}\left(\mathcal{O}_d^p, \mathcal{O}_d\right), \tau_{\mathrm{co}}\right) \cong \varprojlim E_{S,M}$$

Lemma 4.4.16. *Let $S \in \left(\mathbb{R}_+^d\right)^p$, $M \in \mathbb{R}_+^p$. The net $(E_{S,M,T})_{T \in \mathbb{R}_+^d}$ is sequentially retractive.*

Proof: By B_T we are going to denote the closed unit ball in $E_{S,M,T}$. We are going to show that $(E_{S,M,T})_{T \in \mathbb{R}_+^d}$ satisfies property (M).
For $T_1 > T_2 > T_3 > 0$, we have $E_{S,M,T_1} \hookrightarrow E_{S,M,T_2} \hookrightarrow E_{S,M,T_3}$. Let $(F_k)_{k \in \mathbb{N}}$ be a sequence in B_{T_1}, $F_k = \left(\sum_{\gamma \in \mathcal{M}_d^p} c_{\alpha,\gamma}^k x^\gamma\right)_{\alpha \in \mathbb{N}^d}$, which converges with respect to $\| \, \|_{S,M,T_3}$ to $F = \left(\sum_{\gamma \in \mathcal{M}_d^p} c_{\alpha,\gamma} x^\gamma\right)_{\alpha \in \mathbb{N}^d} \in B_{T_1}$. Set $G_k = F - F_k \in 2 \cdot B_{T_1}$ and $g_{\alpha,\gamma}^k = c_{\alpha,\gamma}^k - c_{\alpha,\gamma}$. We claim that $G_k \to 0$ with respect to $\| \, \|_{S,M,T_2}$. Let $\epsilon > 0$ and choose α_0 large enough, so that $\left(\frac{T_2}{T_1}\right)^\alpha < \frac{\epsilon}{2}$ holds for $\alpha \geq \alpha_0$. Let $k_0 \in \mathbb{N}$ so that $\|G_k\|_{S,M,T_3} < \epsilon \left(\frac{T_3}{T_2}\right)^{\alpha_0}$ for $k \geq k_0$. Then $\|G_k\|_{S,M,T_2} \leq \epsilon$ for $k > k_0$:

$$\begin{aligned}
\sup_{\alpha > \alpha_0} T_2^\alpha \sup_{\gamma \in \mathcal{M}_d^p} \left|g_{\alpha,\gamma}^k\right| \frac{M^{\mathrm{sh}(\gamma)}}{S^{\mathrm{wt_v}(\gamma)}} &\leq \left(\frac{T_2}{T_1}\right)^{\alpha_0} \sup_{\alpha > \alpha_0} T_1^\alpha \sup_{\gamma \in \mathcal{M}_d^p} \left|g_{\alpha,\gamma}^k\right| \frac{M^{\mathrm{sh}(\gamma)}}{S^{\mathrm{wt_v}(\gamma)}} \leq \\
&\leq \frac{\epsilon}{2} \|G_k\|_{S,M,T_1} \leq \epsilon \\
\sup_{\alpha \leq \alpha_0} T_2^\alpha \sup_{\gamma \in \mathcal{M}_d^p} \left|g_{\alpha,\gamma}^k\right| \frac{M^{\mathrm{sh}(\gamma)}}{S^{\mathrm{wt_v}(\gamma)}} &\leq \left(\frac{T_2}{T_3}\right)^{\alpha_0} \sup_{\alpha \leq \alpha_0} T_3^\alpha \sup_{\gamma \in \mathcal{M}_d^p} \left|g_{\alpha,\gamma}^k\right| \frac{M^{\mathrm{sh}(\gamma)}}{S^{\mathrm{wt_v}(\gamma)}} \leq \\
&\leq \left(\frac{T_2}{T_3}\right)^{\alpha_0} \|G_k\|_{S,M,T_3} \leq \epsilon
\end{aligned}$$

Hence the topologies induced on B_{T_1} by E_{S,M,T_2}, E_{S,M,T_3}, respectively, coincide. Thus $(E_{S,M,T})_{T \in \mathbb{R}_+^d}$ satisfies property (M) and is sequentially retractive. □

Definition 4.4.17. Let $\mathfrak{X} = \left((E_n)_{n\in\mathbb{N}}, (\pi_n)_{n\in\mathbb{N}}\right)$ be a projective spectrum of LCVS, where $\pi_n \colon E_{n+1} \to E_n$. Set

$$\mathfrak{B}(\mathfrak{X}) := \left\{ (x_n)_{n\in\mathbb{N}} \in \prod_n E_n \;:\; \exists\, (y_n)_{n\in\mathbb{N}} \in \prod E_n \text{ with } x_n = y_n - \pi_n(y_{n+1}) \text{ for all } n \right\}$$
$$\operatorname{Proj}^1\mathfrak{X} := \prod_n E_n / \mathfrak{B}(\mathfrak{X})$$

Our main interest in Proj^1, which is called the *derived projective functor*, is as a useful tool for showing that the projective limit of a sequence of (separated) (LB)-spaces is ultrabornological.

Theorem 4.4.18 (Vogt [Wen03, Theorem 3.3.4]). *Let $\mathfrak{X} = (X_n)_{n\in\mathbb{N}}$ be a projective sequence of separated (LB)-spaces with $\operatorname{Proj}^1 = 0$. Then $\varprojlim X_n$ is ultrabornological.*

Theorem 4.4.19 (Retakh-Palamodov). *[[Wen03, p.27, Theorem 3.2.9]] For a projective sequence $\mathfrak{X} = (X_n)_{n\in\mathbb{N}}$ consisting of separated (LB)-spaces, the following are equivalent:*

(1) $\operatorname{Proj}^1 X = 0$.
(2) *There is a sequence of Banach discs $B_n \subset X_n$ such that*
   (a) $B_{n+1} \subset B_n$
   (b) $\forall N \in \mathbb{N}\ \exists M > N$ *such that* $X_M \subset \varprojlim X_n + B_N$

Theorem 4.4.20.

$$\operatorname{Proj}^1 (E_{S,M})_{S,M} = 0.$$

Proof: Recall that $E_{S_1,M_1} \subset E_{S_2,M_2}$ if $S_1 \le S_2,\ M_2 \le M_1$ and note that for any sequence $(S_n, M_n)_{n\in\mathbb{N}}$ with $\lim S_n = 0$ and $\lim M_n = \infty$ the projective sequence (E_{S_n,M_n}) is equivalent to the projective net $(E_{S,M})_{S,M}$ (which we order by inclusion). Without specifying any such sequence we just show that the net of unit balls $(B_{S,M,T})_{S,M}$ (for a fixed $T \in \mathbb{R}^d_+$) satisfies the conditions of Theorem 4.4.19. Let $S_1 < S_2$ and $M_2 < M_1$. Then it is easy to see that $B_{S_1,M_1,T} \subset B_{S_2,M_2,T}$, so the net $(B_{S,M,T})_{S,M}$ satisfies (2)(a) and it remains to show that it satisfies condition (2)(b) of Theorem 4.4.19. We are going to show that $E_{S_1,M_1} \subset \mathcal{H}(\mathcal{O}^p_d, \mathcal{O}_d) + B_{S_2,M_2,T}$ by showing that for $F \in E_{S_1,M_1}$ we can split off a suitable generalized textile map $\tilde{F}$ (which is an entire function; see Theorem 5.3.8) so that $F - \tilde{F} \in B_{S_2,M_2,T}$. Let $\|F\|_{S_1,M_1,R} \le K < \infty$. Note that

$$\|F\|_{S_2,M_2,T} = \sup_{\alpha\in\mathbb{N}^d} T^\alpha \sup_{\gamma\in\mathcal{M}^p_d} |c_{\alpha,\gamma}| \frac{M_2^{\mathrm{sh}(\gamma)}}{S_2^{\mathrm{wt_v}(\gamma)}} \le \sup_{\alpha\in\mathbb{N}^d} \left(\frac{T}{R}\right)^\alpha \sup_{\gamma\in\mathcal{M}^p_d} K \left(\frac{M_2}{M_1}\right)^{\mathrm{sh}(\gamma)} \left(\frac{S_1}{S_2}\right)^{\mathrm{wt_v}(\gamma)}$$

Set $Q = \frac{S_1}{S_2} \in (\mathbb{R}^d_+)^p$, $Q = ((Q_{1,j})_{1\le j\le d}, \dots, (Q_{p,j})_{1\le j\le d})$, let

$$U = (U_1, \dots, U_d) \in \mathbb{R}^d_+,\ U_k = \max_{i=1,\dots,p} Q_{i,k}$$

and choose $N \in \mathbb{N}$ so that $U^N < \frac{R}{T}$. Further, choose $\nu \in \mathbb{N}^d$ so that $U^\nu < \frac{1}{K}$. Then for $\gamma = (\gamma_1, \ldots, \gamma_p) \in \mathcal{M}_d^p$ with $\mathrm{wt}(\gamma) > N \cdot \alpha + \nu$ we obtain

$$\left(\frac{S_1}{S_2}\right)^{\mathrm{wt_v}(\gamma)} = Q_1^{\mathrm{wt}(\gamma_1)} \cdots Q_p^{\mathrm{wt}(\gamma_p)} \leq \tag{4.6}$$

$$\leq U^{\mathrm{wt}(\gamma_1)} \cdots U^{\mathrm{wt}(\gamma_p)} = U^{\mathrm{wt}(\gamma)} < U^{N\cdot\alpha+\nu} < \left(\frac{R}{T}\right)^{\alpha} \frac{1}{K} \tag{4.7}$$

Moreover, choose $N_0 \in \mathbb{N},\ \nu_0 \in \mathbb{N}^p$ so that $\left(\frac{M_1}{M_2}\right)^{(N_0,\ldots,N_0)} < \left(\frac{R}{T}\right)^{(1,\ldots,1)}$ and $\left(\frac{M_1}{M_2}\right)^{\nu_0} < \frac{1}{K}$. Set $A = (a_{i,j})_{i,j} \in M_{p\times d}(\mathbb{N}),\ a_{i,j} = N$ for all i, j. For $\gamma \in \mathcal{M}_d^p$ with $\gamma(0) > A \cdot \alpha + \nu_0$ we obtain

$$\left(\frac{M_2}{M_1}\right)^{\mathrm{sh}(\gamma)} \leq \left(\frac{M_2}{M_1}\right)^{\gamma(0)} < \left(\frac{R}{T}\right)^{\alpha} \frac{1}{K} \tag{4.8}$$

Set $h(\alpha) = N \cdot \alpha + \nu$ and $g(\alpha) = A \cdot \alpha + \nu_0$, $\tilde{F} = (\tilde{c}_{\alpha,\gamma})_{\alpha\in\mathbb{N}^d,\gamma\in\mathcal{M}_d^p}$, where

$$\tilde{c}_{\alpha,\gamma} = \begin{cases} c_{\alpha,\gamma} & \text{if } \mathrm{wt}(\gamma) \leq h(\alpha) \text{ and } \gamma(0) \leq g(\alpha) \\ 0 & \text{else} \end{cases}$$

and $G = F - \tilde{F}$. Then $\tilde{F}$ is a generalized textile map with growth vector (h, g) and since the coefficients satisfy the necessary Cauchy-type estimates $\tilde{F}\colon \mathcal{O}_d^p \to \mathcal{O}_d$ is a holomorphic (entire) function (see Theorem 5.3.8). It remains to show that $G \in B_{S_2,M_2,T}$:

$$\begin{aligned}\|G\|_{S_2,M_2,T} &= \sup_{\alpha\in\mathbb{N}^d} T^{\alpha} \sup_{\substack{\gamma\in\mathcal{M}_d^p \\ \gamma(0)>A\cdot\alpha+\nu_0 \text{ or } \mathrm{wt}(\gamma)>N\cdot\alpha+\nu}} |c_{\alpha,\gamma}| \frac{M_2^{\mathrm{sh}(\gamma)}}{S_2^{\mathrm{wt_v}(\gamma)}} \leq \\ &\leq \sup_{\alpha\in\mathbb{N}^d} \sup_{\substack{\gamma\in\mathcal{M}_d^p \\ \gamma(0)>A\cdot\alpha+\nu_0 \text{ or } \mathrm{wt}(\gamma)>N\cdot\alpha+\nu}} \left(\frac{T}{R}\right)^{\alpha} K \underbrace{\left(\frac{M_2}{M_1}\right)^{\mathrm{sh}(\gamma)}}_{(I)} \underbrace{\left(\frac{S_1}{S_2}\right)^{\mathrm{wt_v}(\gamma)}}_{(II)}\end{aligned}$$

Because of Equation 4.6 and Equation 4.8 we see that either $(I) < \left(\frac{R}{T}\right)^{\alpha} \frac{1}{K}$ or $(II) < \left(\frac{R}{T}\right)^{\alpha} \frac{1}{K}$, which shows that $\|G\|_{S_2,M_2,T} \leq 1$.
Hence $F = \tilde{F} + G \in \mathcal{H}\left(\mathcal{O}_d^p, \mathcal{O}_d\right) + B_{S_2,M_2,T}$, which completes the proof. □

Proposition 4.4.21. $\mathcal{H}\left(\mathcal{O}_d^p, \mathcal{O}_d^q\right) = \varprojlim_{S,M} \varinjlim_R E_{S,M,R}$

Proof: We are going to denote the projective topology of $\varprojlim_{S,M} \varinjlim_R E_{S,M,R}$ with τ_{proj}. For showing that $\mathrm{id}\colon \varprojlim_{S,M} \varinjlim_R E_{S,M,R} \to \mathcal{H}\left(\mathcal{O}_d^p, \mathcal{O}_d\right)$ is continuous it is enough to show that every τ_{proj}-null-sequence is a τ_{co}-null sequence, since $\varprojlim_{S,M} \varinjlim_R E_{S,M,R}$ is bornological by the preceding theorem. Let $F_n \xrightarrow{\tau_{\mathrm{proj}}} 0$, where $F_n = \sum_{\alpha,\gamma} F_{\alpha,\gamma}^n x_\alpha^\gamma$. Then $\forall S, M > 0\ \exists T > 0\colon\ \|F_n\|_{S,M,T} \to 0$ by Lemma 4.4.16. We need to show that $\|F_n - F\|_{H_{R,K,\phi}} \to 0$ for all $R \in (\mathbb{R}_+^d)^p, K \in \mathbb{R}_+^p$ and $\phi \in \mathcal{H}\left(\mathbb{C}^d\right)^q$ (see Lemma 4.4.1). For given R, K choose $0 < S < R$ and $R < M$ and $T \in \mathbb{R}_+^d$ so that $\|F_n\|_{S,M,T} < \infty$.

For $a \in H_{R,K}$, we have

$$\begin{aligned}|(F(a))[\alpha]| = |\sum_{\gamma\in\mathcal{M}_d^p} F_{\alpha,\gamma}a^\gamma| &\leq \sum_{\gamma\in\mathcal{M}_d^p} \frac{\|F_n\|_{S,M,T}}{T^\alpha}\frac{S^{\mathrm{wt}(\gamma)}}{M^{\mathrm{sh}(\gamma)}}\frac{K^{\mathrm{sh}(\gamma)}}{R^{\mathrm{wt}(\gamma)}} \\ &\leq \frac{\|F_n\|_{S,M,T}}{T^\alpha}\sum_{\gamma\in\mathcal{M}_d^p}\left(\frac{K}{M}\right)^{\mathrm{sh}(\gamma)}\left(\frac{S}{R}\right)^{\mathrm{wt}(\gamma)} \leq \frac{\|F_n\|_{S,M,T}}{T^\alpha}\cdot C_1\end{aligned}$$

for some $C_1 \in \mathbb{R}_+$. For $\phi = \sum \phi_\alpha x^\alpha \in \mathcal{H}\left(\mathbb{C}^d\right)$ let $C_2 \in \mathbb{R}_+$ so that $|\frac{\phi_\alpha}{T^\alpha}| \leq C_2$ for all $\alpha \in \mathbb{N}^d$. Hence we obtain

$$\|F_n\|_{H_{R,K},\phi} = \sup_{a\in H_{R,K}}\sup_{\alpha\in\mathbb{N}^d} |\phi_\alpha \cdot (F(a))[\alpha]| \leq C_1\cdot C_2\,\|F_n\|_{S,M,T}$$

and thus $\|F_n\|_{H_{R,K},\phi} \to 0$. As $\phi, H_{R,K}$ were arbitrary, it follows that $F_n \xrightarrow{\tau_{\mathrm{co}}} 0$ and thus $\mathrm{id}\colon \varprojlim_{S,M}\varinjlim_R E_{S,M,R} \to \mathcal{H}\left(\mathcal{O}_d^p, \mathcal{O}_d^q\right)$ is continuous. The space $\varprojlim_{S,M}\varinjlim_R E_{S,M,R}$ is webbed and $\mathcal{H}\left(\mathcal{O}_d^p, \mathcal{O}_d\right)$ is ultrabornological, so the open mapping theorem can be applied to conclude that $\varprojlim_{S,M}\varinjlim_R E_{S,M,R} \cong \mathcal{H}\left(\mathcal{O}_d^p, \mathcal{O}_d\right)$. □

Corollary 4.4.22. *$\mathcal{H}\left(\mathcal{O}_d^p, \mathcal{O}_d^q\right)$ is a webbed space.*

In chapter 5 we will study certain classes of holomorphic functions with polynomial coefficients. The following theorem is tailored to this end and shows that ultrabornologicity is passed over to these classes. Note that in general ultrabornologicity is *not* passed over to closed subspaces - in fact there exist ultrabornological spaces with closed subspaces which are not even bornological ([Jar81, p.281]). However, ultrabornologicity is always passed over to quotients taken by closed subspaces ([Jar81, p.281]), which will be the key to prove the following theorem.

Theorem 4.4.23. *Let $p_{\alpha,\gamma}\colon \mathcal{H}\left(\mathcal{O}_d^p, \mathcal{O}_d\right) \to \mathbb{C}$, $\left(\sum_{\alpha\in\mathbb{N}^d,\gamma\in\mathcal{M}_d^p} c_{\alpha,\gamma}x_\alpha^\gamma\right) \mapsto c_{\alpha,\gamma}$, let $\mathcal{R} = \left\{p_{\alpha,\gamma} \mid \alpha \in \mathbb{N}^d,\ \gamma \in \mathcal{M}_d^p\right\}$ and let $\mathcal{Q}$ be an arbitrary subset of $\mathcal{R}$. Then:*

(1) *The projections $p_{\alpha,\gamma}$ are continuous.*
(2) *$H_\mathcal{Q} := \bigcap_{f\in\mathcal{Q}} \ker(f)$ is an ultrabornological closed subspace of $\mathcal{H}\left(\mathcal{O}_d^p, \mathcal{O}_d\right)$.*
(3) *$\mathcal{H}\left(\mathcal{O}_d^p, \mathcal{O}_d\right)$ is the topological direct sum $H_\mathcal{Q} \bigoplus H_{\mathcal{Q}^c}$.*

Proof: (1) Let $F_n = \left(\sum_{\gamma\in\mathcal{M}_d^p} c_{\alpha,\gamma}^n x^\gamma\right)_{\alpha\in\mathbb{N}^d}$ be a null sequence in $\mathcal{H}\left(\mathcal{O}_d^p, \mathcal{O}_d\right)$. For $S \in (\mathbb{R}_+^d)^p$, $M \in \mathbb{R}_+^p$, there exists a $T \in \mathbb{R}_+^d$ so that $\|F_n\|_{S,M,T} \to 0$, which implies that $0 = \lim_{n\to\infty} c_{\alpha,\gamma}^n$ by the definition of the norm. (2),(3) Clearly $H_\mathcal{Q}$ and $H_{\mathcal{Q}^c}$ are closed subspaces of $\mathcal{H}\left(\mathcal{O}_d^p, \mathcal{O}_d\right)$ and thus webbed. Since ultrabornologicity is passed over to quotients taken by closed subspaces we have that $\mathcal{H}\left(\mathcal{O}_d^p, \mathcal{O}_d\right)/H_{\mathcal{Q}^c}$ is ultrabornological. We show that the projection $p_\mathcal{Q}\colon E \to H_\mathcal{Q}$ is continuous by showing that its graph is closed, which yields that $H_\mathcal{Q} \bigoplus H_{\mathcal{Q}^c}$ is a topological direct sum. Let $(x_i, p_m(x_i)) \to (x, y)$. Note that

$$p_\mathcal{Q}\left(\sum_{\gamma\in\mathcal{M}_d^p,\alpha\in\mathbb{N}^d} c_{\alpha,\gamma}x_\alpha^\gamma\right) = \left(\sum_{p_{\alpha,\gamma}\in\mathcal{Q}^c} c_{\alpha,\gamma}x_\alpha^\gamma\right).$$

We have:

$$p_{\alpha,\gamma}(y) = \lim p_{\alpha,\gamma}\left(p_{\mathcal{Q}}(x_i)\right) = \begin{cases} 0 & \text{if } p_{\alpha,\gamma} \in \mathcal{Q} \\ \lim p_{\alpha,\gamma}(x_i) & \text{else} \end{cases}$$

$$p_{\alpha,\gamma}(y) = \begin{cases} 0 & \text{if } p_{\alpha,\gamma} \in \mathcal{Q} \\ p_{\alpha,\gamma}(x) & \text{else} \end{cases}$$

Hence $p_{\mathcal{Q}}(x) = y$ and thus $p_{\mathcal{Q}}$ is continuous, which implies (3). Consider the following commutative diagram to obtain (2), where $E := \mathcal{H}\left(\mathcal{O}_d^p, \mathcal{O}_d\right)$:

$$\begin{tikzcd} E \arrow[r, "p_{\mathcal{Q}}"] \arrow[d, "\pi"'] & H_{\mathcal{Q}} \\ E/H_{\mathcal{Q}^c} \arrow[ur, "\hat{p_{\mathcal{Q}}}"] & \end{tikzcd}$$

$\hat{p_{\mathcal{Q}}}$ is an isomorphism as $p_{\mathcal{Q}}$ is continuous, hence $H_{\mathcal{Q}}$ is ultrabornological. □

Theorem 4.4.24. *The set of monomials* $\left\{x_\alpha^\gamma \mid \alpha \in \mathbb{N}^d, \gamma \in \mathcal{M}_d^p\right\}$ *forms a basis for* $\mathcal{H}\left(\mathcal{O}_d^p, \mathcal{O}_d\right)$.

Proof: Let $F = \sum_{\alpha\in\mathbb{N}^d,\gamma\in\mathcal{M}_d^p} c_{\alpha,\gamma} x_\alpha^\gamma \in \mathcal{H}\left(\mathcal{O}_d^p, \mathcal{O}_d\right)$. We will use the projective description of $\mathcal{H}\left(\mathcal{O}_d^p, \mathcal{O}_d\right)$ as $\varprojlim E_{S,M}$ obtained in Proposition 4.4.21. So we have to show that the net of partial sums (the ordering on the finite sets being the inclusion)

$$\left(\sum_{(\alpha,\gamma)\in\mathcal{F}} c_{\alpha,\gamma} x_\alpha^\gamma\right)_{\mathcal{F} \underset{\text{finite}}{\subset} \mathbb{N}^d\times\mathcal{M}_d^p}$$

converges to F in all $E_{S,M}$. Let $S_1 \in (\mathbb{R}_+^d)^p$ and $M_1 \in \mathbb{R}_+^p$, and choose $S_2 \in (\mathbb{R}_+^d)^p$ and $M_2 \in \mathbb{R}_+^p$ with $S_2 < S_1$ and $M_2 > M_1$. There exists a $T_2 \in \mathbb{R}_+^d$ so that $C := \|F\|_{S_2,M_2,T_2} < \infty$. Let $T_1 \in \mathbb{R}_+^p$ with $T_1 < T_2$. We claim that the sum converges with respect to $\|\ \|_{S_1,M_1,T_1}$ to F. Let $\epsilon > 0$ and let $\alpha_0 \in \mathbb{N}^d$ so that the estimate $\left(\frac{T_1}{T_2}\right)^\alpha < \frac{\epsilon}{C}$ holds for all $\alpha \geq \alpha_0$. Recall that $\sum_{\gamma\in\mathcal{M}_d^p} \left(\frac{M_1}{M_2}\right)^{\mathrm{sh}(\gamma)} \left(\frac{S_2}{S_1}\right)^{\mathrm{wt_v}(\gamma)}$ is summable by Theorem 4.3.8. Hence there is a finite set $\mathcal{G} \subset \mathcal{M}_d^p$ so that

$$\sum_{\mathcal{M}_d^p\setminus\mathcal{G}} \left(\frac{M_1}{M_2}\right)^{\mathrm{sh}(\gamma)} \left(\frac{S_2}{S_1}\right)^{\mathrm{wt_v}(\gamma)} < \frac{\epsilon}{C}$$

Set $\mathcal{F} = \{\alpha \leq \alpha_0\} \times \mathcal{G}$. We claim that

$$\left\| F - \sum_{(\alpha,\gamma)\in\mathcal{F}} c_{\alpha,\gamma} x_\alpha^\gamma \right\|_{S_1,M_1,T_1} \leq \epsilon$$

Let $d_{\alpha,\gamma}$ denote the coefficients of $F - \sum_{(\alpha,\gamma)\in\mathcal{F}} c_{\alpha,\gamma} x_\alpha^\gamma$. We have:

$$\left\| F - \sum_{(\alpha,\gamma)\in\mathcal{F}} c_{\alpha,\gamma} x_\alpha^\gamma \right\|_{S_1,M_1,T_1} = \sup_{\alpha\in\mathbb{N}^d} T_1^\alpha \sup_{\gamma\in\mathcal{M}_d^p} |d_{\alpha,\gamma}| \frac{M_1^{\mathrm{sh}(\gamma)}}{S_1^{\mathrm{wt_v}(\gamma)}} =$$
$$= \sup_{\alpha\in\mathbb{N}^d} \underbrace{\left(\frac{T_1}{T_2}\right)^\alpha \sup_{\gamma\in\mathcal{M}_d^p} \underbrace{T_2^\alpha |d_{\alpha,\gamma}| \frac{M_2^{\mathrm{sh}(\gamma)}}{S_2^{\mathrm{wt_v}(\gamma)}}}_{E_2(\alpha,\gamma)} \underbrace{\left(\frac{M_1}{M_2}\right)^{\mathrm{sh}(\gamma)} \left(\frac{S_2}{S_1}\right)^{\mathrm{wt_v}(\gamma)}}_{E_3(\gamma)}}_{E_1(\alpha)}$$

Note that $E_2(\alpha,\gamma) \le C$ and that $E_3(\gamma) \le 1$ for all $\alpha \in \mathbb{N}^d, \gamma \in \mathcal{M}_d^p$. So if $\alpha \ge \alpha_0$ we have that $E_1(\alpha) \le \epsilon$ since we have that $\left(\frac{T_1}{T_2}\right)^\alpha < \frac{\epsilon}{C}$ in this case. If $\alpha < \alpha_0$ and if $\gamma \in \mathcal{F}$, then $d_{\alpha,\gamma} = 0$. If $\gamma \in \mathcal{M}_d^p \setminus \mathcal{F}$, we have that $E_3(\gamma) < \epsilon/C$ and hence that $E_2(\alpha,\gamma) \cdot E_3(\gamma) \le \epsilon$. This shows that $\left\| F - \sum_{(\alpha,\gamma)\in\mathcal{F}} c_{\alpha,\gamma} x_\alpha^\gamma \right\|_{S_1,M_1,T_1} \le \epsilon$ and clearly this estimate also holds for any finite subset $\widehat{\mathcal{F}}$ of $\mathbb{N}^d \times \mathcal{M}_d^p$ which contains $\mathcal{F}$. Thus $\left(\sum_{(\alpha,\gamma)\in\mathcal{F}} c_{\alpha,\gamma} x_\alpha^\gamma\right)_{\mathcal{F} \underset{\text{finite}}{\subset} \mathbb{N}^d\times\mathcal{M}_d^p}$ converges to F in E_{S_1,M_1,T_1} and hence in E_{S_1,M_1}. Since S_1 and M_1 were arbitrary, it follows that $\left(\sum_{(\alpha,\gamma)\in\mathcal{F}} c_{\alpha,\gamma} x_\alpha^\gamma\right)_{\mathcal{F} \underset{\text{finite}}{\subset} \mathbb{N}^d\times\mathcal{M}_d^p}$ converges to F in $\varprojlim_{S,M} E_{S,M} = \mathcal{H}\left(\mathcal{O}_d^p, \mathcal{O}_d\right)$.

□

Chapter 5
Textile maps

In this chapter we are going to study maps $F\colon \mathcal{O}_d^p \to \mathcal{O}_d^q$ whose coefficients have the same structure as those of substitution maps $\phi \mapsto H(x, \phi(x) - \phi(0))$ (called *tactile maps*), which we will call *textile maps*. It turns out that a textile maps is holomorphic if and only it preserves the boundedness of a polydisc $H_{S,M}$ – a criterion similar as for linear maps between normed spaces. This allows us to establish an inductive description of the space of holomorphic textile maps. The natural inductive topology turns it into a (DFS)-space, and the strong results of chapter 4 show that it coincides with the compact-open topology. Then we will turn our attention to certain generalizations of textile maps, for which we establish similar characterizations of holomorphy. In the final section we will investigate the Cauchy-Kovalevskaya-type differential equation $\delta_t u(x,t) = F(u(x,t))$, where the right side is a (holomorphic) textile map. We will show that this equation is always solvable in $\mathcal{P}_d^p$, but the solution might fail to be convergent. A positive result will be established for the subclass of tactilly bounded maps.

5.1 Special classes of Holomorphic functions

Definition 5.1.1. For $\gamma \in \mathcal{M}_d^p$ we set $\gamma! := \prod_{k=1}^p \prod_{\alpha\in\mathbb{N}^d} (\gamma_k(\alpha)!)$ and

$$\mu(\gamma) := \prod_{k=1}^{p} \frac{\mathrm{sh}(\gamma_k)!}{\prod\limits_{\alpha\in N^d} \gamma_k(\alpha)!} = \prod_{k=1}^{p} \mu(\gamma_k).$$

This is a generalization of the multinomial coefficient $\binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha_1!\cdots\alpha_d!}$ to our purpose. For $\gamma \in \mathcal{M}_d^p$ we set $\omega(\gamma) := (\mathrm{sh}(\gamma), \mathrm{wt}(\gamma)) \in \mathbb{N}^p \times \mathbb{N}^d$.

Lemma 5.1.2. *Let $k \in \mathbb{N}$, $a = \sum c_\alpha x^\alpha \in \mathcal{P}_d$ and $\delta \in \mathbb{N}^d$. Then*

$$(j_\delta(a))^k[\epsilon] = \sum_{\substack{\gamma\in\mathcal{M}_d \\ \omega(\gamma)=(k,\epsilon),\ \mathrm{supp}(\gamma)\le\delta}} \mu(\gamma) a^\gamma$$

and

$$(a)^k[\epsilon] = \sum_{\gamma\in\mathcal{M}_d,\ \omega(\gamma)=(k,\epsilon)} \mu(\gamma) a^\gamma$$

More generally, let $a = (a_1, ..., a_p) \in \mathcal{P}_d^p,\ \beta = (\beta_1, ..., \beta_p) \in \mathbb{N}^p, \epsilon \in \mathbb{N}^d$. *Then*

$$a^\beta[\epsilon] = \sum_{\substack{\gamma \in \mathcal{M}_d^p \\ \omega(\gamma)=(\beta,\epsilon)}} \mu(\gamma) a^\gamma$$

Proof: Let $(\alpha^j)_{1 \leq j \leq N}$ be an enumeration of the set $\{\alpha \leq \delta\} =: A$. By the multinomial theorem we get that

$$\left(\sum_{\alpha \leq \delta} c_\alpha x^\alpha\right)^k = \left(c_{\alpha^1} x^{\alpha^1} + ... + c_{\alpha^N} x^{\alpha^N}\right)^k = \sum_{\beta \in \mathbb{N}^N, |\beta|=k} \binom{k}{\beta} \prod_{i=1}^N c_{\alpha^i}^{\beta_i} x^{\alpha^i \cdot \beta_i} \tag{5.1}$$

Let $\pi\colon B := \{\beta \in \mathbb{N}^N :\ |\beta| = k\} \to \mathcal{M}_d$, where

$$(\pi(\beta))(\alpha) := \begin{cases} 0 & \text{if } \alpha \notin A \\ \beta_j & \text{if } \alpha = \alpha^j \in A \end{cases}$$

π is a bijection between B and $\{\gamma \in \mathcal{M}_d \mid\ \operatorname{sh}(\gamma) = k, \operatorname{supp}(\gamma) \leq \delta\} =: C$ and

$$\binom{k}{\beta} = \frac{k!}{\prod_{j=1}^N \beta_j!} = \frac{\operatorname{sh}(\pi(\beta))!}{\prod_{j=1}^N ((\pi(\beta))(\alpha^j))!} = \frac{\operatorname{sh}(\pi(\beta))!}{\prod_{\alpha \in \mathbb{N}^d} ((\pi(\beta))(\alpha))!} = \mu\left(\pi\left(\beta\right)\right).$$

So we can rewrite Equation 5.1 as

$$(j_\delta(a))^k = \sum_{\gamma \in C} \mu\left(\gamma\right) \prod_{\alpha \in \mathbb{N}^d} c_\alpha^{\gamma(\alpha)} x^{\alpha \cdot \gamma(\alpha)} = \sum_{\substack{\gamma \in \mathcal{M}_d \\ \operatorname{sh}(\gamma)=k, \operatorname{supp}(\gamma) \leq \delta}} \mu(\gamma) a^\gamma x^{\operatorname{wt}(\gamma)},$$

from which we see that

$$(j_\delta\left(a\right))^k\left[\epsilon\right] = \sum_{\substack{\gamma \in \widetilde{\mathcal{M}}_d,\ \operatorname{supp}(\gamma) \leq \delta \\ \omega(\gamma)=(k,\epsilon)}} a^\gamma \mu\left(\gamma\right)$$

and that

$$a^k[\epsilon] = (j_\epsilon\left(a\right))^k = \sum_{\substack{\gamma \in \widetilde{\mathcal{M}}_d \\ \omega(\gamma)=(k,\epsilon)}} a^\gamma \mu\left(\gamma\right).$$

Now let $a = (a_1, ..., a_p) \in \mathcal{P}_d^p,\ \beta = (\beta_1, ..., \beta_p) \in \mathbb{N}^p$. Then

$$a^\beta = \left(\sum_{\substack{\gamma_1 \in \mathcal{M}_d \\ \operatorname{sh}(\gamma_1)=\beta_1}} \mu(\gamma_1) a^{\gamma_1} x^{\operatorname{wt}(\gamma_1)}\right) \cdots \left(\sum_{\substack{\gamma_p \in \mathcal{M}_d \\ \operatorname{sh}(\gamma_p)=\beta_p}} \mu(\gamma_p) a^{\gamma_p} x^{\operatorname{wt}(\gamma_p)}\right) =$$

$$= \sum_{\gamma \in \mathcal{M}_d^p, \operatorname{sh}(\gamma)=\beta} \mu(\gamma_1) \cdots \mu(\gamma_p) a^{\gamma_1 + \cdots + \gamma_p} x^{\operatorname{wt}(\gamma_1) + \cdots + \operatorname{wt}(\gamma_p)} = \sum_{\substack{\gamma \in \mathcal{M}_d^p \\ \operatorname{sh}(\gamma)=\beta}} \mu(\gamma) a^\gamma x^{\operatorname{wt}(\gamma)}$$

So for $\epsilon \in \mathbb{N}^d$ we obtain

$$a^\beta[\epsilon] = \sum_{\substack{\gamma \in \mathcal{M}_d^p \\ \omega(\gamma)=(\beta,\epsilon)}} \mu(\gamma) a^\gamma \qquad \square$$

Definition 5.1.3. We are going to treat functions of the form $\phi \mapsto F(x, \phi - \phi(0))$ and such which show a similar analytical behavior. By definition these functions ignore the value $\phi(0)$, so it is convenient to set

$$\widetilde{\mathcal{M}}_d^p := \{f \in \mathcal{M}_d^p \mid f(0) = 0\}$$

in order to describe the coefficients of their power series expansion.

Theorem 5.1.4. *Let* $F(x, y) = \sum_{(\alpha,\beta)\in\mathbb{N}^d\times\mathbb{N}^p} c_{\alpha,\beta} x^\alpha y^\beta \in P_{d+p}$ *and* $\phi \in \mathcal{P}_d^p$*. For* $\epsilon \in \mathbb{N}^d$ *we set* $\widehat{F}_\epsilon(\phi) := F(x, \phi(x) - \phi(0))[\epsilon]$*. Then*

$$\widehat{F}_\epsilon(\phi) = \sum_{\gamma \in \widetilde{\mathcal{M}}_d^p, \mathrm{wt}(\gamma) \le \epsilon} F_{\epsilon,\gamma} \phi^\gamma \text{ where } F_{\epsilon,\gamma} := \mu(\gamma) c_{(\epsilon - \mathrm{wt}(\gamma), \mathrm{sh}(\gamma))}$$

If $|c_{\alpha,\beta}| \le \frac{M}{R_1^\alpha R_2^\beta}$ for $(R_1, R_2) = R \in \mathbb{R}_+^{d+p},\ M \in \mathbb{R}_+$, then

$$|F_{\epsilon,\gamma}| \le \frac{M \cdot \mu(\gamma)}{R_1^{\epsilon - \mathrm{wt}(\gamma)} R_2^{\mathrm{sh}(\epsilon)}} = \frac{M \cdot \mu(\gamma)}{R^{(\epsilon - \mathrm{wt}(\gamma), \mathrm{sh}(\gamma))}}$$

Proof:

$$\begin{aligned}
F_\epsilon(\phi) &= \sum_{\alpha \le \epsilon} c_{\alpha,\beta} (\phi^\beta)[\epsilon - \alpha] = \sum_{\alpha \le \epsilon} c_{\alpha,\beta} \sum_{\substack{\gamma \in \widetilde{\mathcal{M}}_d^p \\ \omega(\gamma)=(\beta,\epsilon-\alpha)}} \mu(\gamma)\phi^\gamma = \\
&= \sum_{\alpha \le \epsilon} \sum_{\beta \in \mathbb{N}^d} \sum_{\substack{\gamma \in \widetilde{\mathcal{M}}_d^p \\ \omega(\gamma)=(\beta,\epsilon-\alpha)}} \mu(\gamma) c_{(\epsilon - \mathrm{wt}(\gamma),\ \mathrm{sh}(\gamma))} \phi^\gamma = \\
&= \sum_{\alpha \le \epsilon} \sum_{\substack{\gamma \in \widetilde{\mathcal{M}}_d^p \\ \mathrm{wt}(\gamma)=\epsilon-\alpha}} \mu(\gamma) c_{(\epsilon - \mathrm{wt}(\gamma),\mathrm{sh}(\gamma))} \phi^\gamma = \\
&= \sum_{\substack{\gamma \in \widetilde{\mathcal{M}}_d^p \\ \mathrm{wt}(\gamma) \le \epsilon}} \phi^\gamma \underbrace{\mu(\gamma)\, c_{(\epsilon - \mathrm{wt}(\gamma),\ \mathrm{sh}(\gamma))}}_{=:F_{\epsilon,\gamma}} \square
\end{aligned}$$

Definition 5.1.5. A map $F\colon \mathcal{P}_d^p \to \mathcal{P}_d,\ F(\phi) = \sum_{\alpha\in\mathbb{N}^d} F_\alpha(\phi) x^\alpha$ is called *textile* if all coefficient functions F_α are polynomials on $\mathcal{P}_d^p$ of the form

$$F_\alpha(\phi) = \sum_{\substack{\gamma \in \widetilde{\mathcal{M}}_d^p \\ \mathrm{wt}(\gamma) \le \alpha}} F_{\alpha,\gamma} \phi^\gamma$$

Following the notation introduced in Definition 4.4.15 we can write a textile map F

in the form

$$\sum_{\mathrm{wt}(\gamma)\le\alpha} F_{\alpha,\gamma} x_\alpha^\gamma$$

The space of all textile maps $\mathcal{P}_d^p \to \mathcal{P}_d$ will be denoted by $\mathcal{T}^{d,p}$. Note that textile maps ignore the value of a power series at 0 (by definition), i.e. $F(\phi) = F(\phi - \phi(0))$ for any $F \in \mathcal{T}^{d,p},\ \phi \in \mathcal{P}_d^p$. Due to the condition $\mathrm{wt}(\gamma) \le \alpha$, we have that $F_\alpha(\phi) = F_\alpha(j_\alpha(\phi))$

An important example of textile maps are *tactile* maps: For $G \in \mathcal{P}_{d+p}$, the associated substitution map $\widehat{G}\colon \phi \mapsto G(x, \phi(x) - \phi(0))$ is called a *tactile* map. By Theorem 5.1.4 $\widehat{G}$ is a textile map. We denote the set of tactile maps $\mathcal{P}_d^p \to \mathcal{P}_d$ by $\mathcal{S}^{d,p}$. Tactile maps can be characterized by a certain interdependency between their coefficients (see Theorem 5.1.4).

Throughout this chapter, we consider $\mathcal{O}_d^p$ to be equipped with its (unique) (DFN)-topology (see Theorem 4.1.1). A map $F\colon \mathcal{O}_d^p \to \mathcal{O}_d$ is called textile (tactile) if it is the restriction of a textile (tactile) map $\widetilde{F}\colon \mathcal{P}_d^p \to \mathcal{P}_d$ with $\widetilde{F}(\mathcal{O}_d^p) \subset \mathcal{O}_d$. A textile map $F : \mathcal{O}_d^p \to \mathcal{O}_d$ is called *analytic* if it is continuous. We have already seen in Theorem 4.3.2 that a textile map $F\colon \mathcal{O}_d^p \to \mathcal{O}_d$ is continuous iff it is holomorphic, as its coefficient functions are polynomials and hence holomorphic function $\mathcal{O}_d^p \to \mathbb{C}$. In Theorem 5.1.10 we will give further characterizations of analyticity of textile maps. The set of analytic textile maps $\mathcal{O}_d^p \to \mathcal{O}_d$ will be denoted by $\mathcal{T}_A^{d,p}$ and the set of analytic tactile maps by $\mathcal{S}_A^{d,p}$. Lemma 5.1.7 shows that every tactile map which is induced from a convergent power series is analytic. The converse will be shown in Lemma 5.2.3. We say that a textile map $F \in \mathcal{T}^{d,p}$ is *tactilly bounded* if there exist $M \in \mathbb{R}_+, R \in \mathbb{R}_+^{d+p}$ so that the coefficients of F satisfy the estimate

$$|F_{\alpha,\gamma}| \le \frac{M \cdot \mu(\gamma)}{R^{(\alpha - \mathrm{wt}(\gamma), \mathrm{sh}(\gamma))}}.$$

This means that the coefficients of a tactilly bounded maps are bounded in the same way as those of an *analytic* tactile map (see Theorem 5.1.4). It turns out that this class is the right one if one seeks convergent solutions for the Cauchy-Kovalevskaya-type differential equation $\delta_t u(x,t) = F(x, u(x,t))$, if the defining function F is a textile map (see Theorem 5.4.2). We denote the family of tactilly bounded maps $\mathcal{P}_d^p \to \mathcal{P}_d$ by $\mathcal{T}_{\mathcal{B}}^{d,p}$.

Remark. The notions of textile and tactile maps were coined by H.Hauser, who proved rank theorems for tactile maps (see [HM94], [BH10]). Note that our definition of textile maps is more restrictive as we require that $F_{\alpha,\gamma} = 0$ if $\mathrm{wt}(\gamma) > \alpha$, while in [BH10] the authors study maps with arbitrary polynomial coefficient functions.

Definition 5.1.6. Let $d, p \in \mathbb{N}$ and $R, M \in \mathbb{R}_+$. Since geometric series are going to play a vital role, it is convenient to set

$$\mathfrak{g}_{d,R,M} = \sum_{\alpha\in\mathbb{N}^d} \frac{Mx^\alpha}{R^{|\alpha|}} \text{ and } \mathfrak{g}_{d,R,M,p} = (\mathfrak{g}_{d,R,M}, \ldots, \mathfrak{g}_{d,R,M}) \in \mathcal{O}_d^p.$$

We introduce the symmetrical polydiscs

$$H_{d,R,M} := \{ \sum_{\alpha\in\mathbb{N}^d} c_\alpha x^\alpha \ : \ |c_\alpha| R^{|\alpha|} \leq M\} \text{ and } H^p_{d,R,M} := (H_{d,R,M})^p \subset \mathcal{O}^p_d.$$

It is easy to see that the family of symmetrical polydiscs forms a fundamental system of bounded sets for $\mathcal{O}^p_d$.

Lemma 5.1.7. *Let $F = \sum_{(\alpha,\beta)\in\mathbb{N}^{d+p}} c_{\alpha,\beta} x^\alpha y^\beta \in \mathcal{P}_{d+p}$*

(1) *If $F \in \mathcal{O}_{d+p}$, then the associated tactile map $\widehat{F}\colon \mathcal{O}^p_d \to \mathcal{O}_d$ is analytic.*
(2) *Tactilly bounded maps are analytic.*
(3) *A textile map F is tactile iff $(\alpha_1 - \mathrm{wt}(\gamma_1), \mathrm{sh}(\gamma_1)) = (\alpha_2 - \mathrm{wt}(\alpha_2), \mathrm{sh}(\gamma_2))$ implies that $F_{\alpha_1,\gamma_1} = F_{\alpha_2,\gamma_2}$ (for all $\alpha_1, \alpha_2 \in \mathbb{N}^d$, $\gamma_1, \gamma_2 \in \widetilde{\mathcal{M}}^p_d$).*

Proof: (2) Let $F = \sum F_{\alpha,\gamma} x^\gamma_\alpha$ be a tactilly bounded map $\mathcal{P}^p_d \to \mathcal{P}_d$. By definition there exist $M \in \mathbb{R}_+, \mathbb{R} \in \mathbb{R}^{d+p}_+$ so that $|F_{\alpha,\gamma}| \leq \frac{M\mu(\gamma)}{R^{(\alpha-\mathrm{wt}(\gamma),\mathrm{sh}(\gamma))}}$. Let $G = \mathfrak{g}_{R,M} \in \mathcal{O}_{d+p}$. Let $K \in \mathbb{R}^p_+, S \in (\mathbb{R}^d_+)^p$ and $a \in H_{S,K}$. Then

$$|F_\alpha(a)| = |\sum_{\mathrm{wt}(\gamma)\leq\alpha} F_{\alpha,\gamma} a^\gamma| \leq \sum_{\mathrm{wt}(\gamma)\leq\alpha} |F_{\alpha,\gamma}|(|a|)^\gamma \leq$$
$$\overset{\text{Theorem 5.1.4}}{\leq} \widehat{G}_\alpha(|a|) \leq \widehat{G}_\alpha(\mathfrak{g}_{S,K}) \leq \frac{C}{T^\alpha}$$

for some $C \in \mathbb{R}_+, T \in \mathbb{R}^d$. Hence $F(H_{S,K}) \subset H_{T,C}$, which shows that $F\colon \mathcal{O}^p_d \to \mathcal{O}_d$ and that F is bornological and thus holomorphic (see Theorem 2.8.2). (1) By Lemma 5.1.7 tactile maps induced from convergent power series are tactilly bounded. (3) The necessity of the condition is the content of Theorem 5.1.4. Suppose that the above relation between the coefficients holds. Let $f\colon \mathrm{N}^\mathrm{d} \to \mathbb{N}$,

$$f(\alpha) := \begin{cases} 1 & \text{if } \alpha = e_1 \\ 0 & \text{else} \end{cases}$$

For $(\alpha,\beta) \in \mathbb{N}^d \times \mathbb{N}^p$, let $\nu(\beta) := (\beta_1 \cdot f, \dots, \beta_p \cdot f) \in M^p_d$ and $\epsilon(\alpha,\beta) := \alpha + \mathrm{wt}(\nu(\beta))$. Note that $(\epsilon(\alpha,\beta) - \mathrm{wt}(\nu(\beta)), \mathrm{sh}(\nu(\beta))) = (\alpha,\beta)$. Set $c_{\alpha,\beta} := F_{\epsilon(\alpha,\beta),\nu(\beta)}$ and let $G(x,y) := \sum c_{\alpha,\beta} x^\alpha y^\beta \in \mathcal{P}_{d+p}$. Then

$$\widehat{G}_{\alpha,\gamma} = c_{\alpha-\mathrm{wt}(\gamma),\mathrm{sh}(\gamma)} = F_{\alpha-\mathrm{wt}(\gamma)+\mathrm{wt}(\nu(\mathrm{sh}(\gamma))),\nu(\mathrm{sh}(\gamma))} = F_{\alpha,\gamma}.$$

This means that $F = \widehat{G}$, so F is tactile. □

Lemma 5.1.8 (Cauchy-Estimates for textile maps)**.** *Let $F\colon \mathcal{P}^p_d \to \mathcal{P}_d$ be a textile map, and suppose there exist $S \in (R^d_+)^p, K \in \mathbb{R}^p_+, R \in \mathbb{R}^d_+, M \in \mathbb{R}_+$ so that $F(H_{S,K}) \subset H_{R,M}$. Then we have the following estimates:*

(1) $|F_{\alpha,\gamma}| \leq \frac{M}{R^\alpha} \frac{S^{\mathrm{wt_v}(\gamma)}}{K^{\mathrm{sh}(\gamma)}}$
(2) *There exist $T, Q \in \mathbb{R}_+$ so that $|F_{\alpha,\gamma}| \leq \frac{1}{Q^{|\alpha|}} \frac{M}{T^{|\alpha|-|\mathrm{wt}(\gamma)|+|\mathrm{sh}(\gamma)|}}$*

Proof: As the coefficient functions of F are polynomials and hence holomorphic functions $\mathcal{O}^p_d \to \mathbb{C}$ we can apply Proposition 4.3.15 to conclude (1).

(2) Let $K = (K_i)_{i=1}^p$, $S = (S_{i,j})_{\substack{1\leq i\leq d\\ 1\leq j\leq p}}$. Set $K_0 = \min_{1\leq i\leq p} K_i$, $R_0 = \min_{1\leq i\leq d} R_i$, $S_0 = \max_{1\leq j\leq p}(\max_{1\leq i\leq d}(S_{i,j}))$. Choose $0 < T \leq \min\{K_0, R_0\}$ and set $Q := \frac{T}{S_0}$. Let $\alpha \in \mathbb{N}^d$ and $\gamma \in \widetilde{\mathcal{M}}_d^p$ with $\mathrm{wt}(\gamma) \leq \alpha$. Then

$$
\begin{aligned}
|F_{\alpha,\gamma}| &\leq \frac{M}{R^\alpha}\frac{S^{\mathrm{wt_v}(\gamma)}}{K^{\mathrm{sh}(\gamma)}} \leq \frac{M}{R_0^{|\alpha|}}\frac{S_0^{|\mathrm{wt}(\gamma)|}}{K_0^{|\mathrm{sh}(\gamma)|}} \leq \\
&\leq \frac{M}{T^{|\alpha|-|\mathrm{wt}(\gamma)|+|\mathrm{sh}(\gamma)|}} \underbrace{\left(\frac{S_0}{T}\right)^{|\mathrm{wt}(\gamma)|}}_{\leq\left(\frac{1}{Q}\right)^{|\alpha|}} \underbrace{\left(\frac{T}{K_0}\right)^{|\mathrm{sh}(\gamma)|}\left(\frac{T}{R_0}\right)^{|\alpha|}}_{\leq 1} \\
&\leq \frac{1}{Q^{|\alpha|}}\frac{M}{T^{|\alpha|-|\mathrm{wt}(\gamma)|+|\mathrm{sh}(\gamma)|}}
\end{aligned}
$$

Since $F_{\alpha,\gamma} = 0$ if $\mathrm{wt}(\gamma) \not\leq \alpha$, the estimate holds $\forall \gamma \in \widetilde{\mathcal{M}}_d^p$ $\square$

While it seems hard to give a direct growth estimate for the multinomial coefficient $\mu(\gamma)$, it is easy to estimate the growth rate by using the Cauchy estimates.

Theorem 5.1.9. *For $\alpha \in \mathbb{N}^d$ let $m_\alpha = \max\{\mu(\gamma) \mid \gamma \in \widetilde{\mathcal{M}}_d,\ \mathrm{wt}(\gamma) \leq \alpha\}$. The countable net $(m_\alpha)_{\alpha\in\mathbb{N}^d}$ grows geometrically or slower - i.e. there exist an $R \in \mathbb{R}_+^d$ so that $\sup_{\alpha\in\mathbb{N}^d} R^\alpha m_\alpha < \infty$.*

Proof: Consider the geometric series $G = \sum_{(\alpha,\beta)\in\mathbb{N}^d\times\mathbb{N}^d} x^\alpha y^\beta$ and let $g_{\alpha,\gamma}$ denote the coefficients of $\widehat{G}$. By Theorem 5.1.4 we know that $g_{\alpha,\gamma} = \mu(\gamma)$ if $\mathrm{wt}(\gamma) \leq \alpha$. Set $S = (1, \ldots, 1) \in \mathbb{R}_+^d$. We know that $\widehat{G}\colon \mathcal{O}_d \to \mathcal{O}_d$ is continuous, hence there exist $M \in \mathbb{R}_+$ and $R \in \mathbb{R}_+^d$ so that $\widehat{G}(H_{S,1}) \subset H_{R,M}$. The estimates from Lemma 5.1.8 then yield that $|g_{\alpha,\gamma}| \leq \frac{M}{R^\alpha}$, which shows that $m_\alpha R^\alpha \leq M$ for all $\alpha \in \mathbb{N}^d$. $\square$

Theorem 5.1.10. *For a textile map $F\colon \mathcal{P}_d^p \to \mathcal{P}_d$, the following are equivalent:*
(1) *$F\colon \mathcal{O}_d^p \to \mathcal{O}_d$ is holomorphic.*
(2) *$F\colon \mathcal{O}_d^p \to \mathcal{O}_d$ is continuous.*
(3) *$F\colon \mathcal{O}_d^p \to \mathcal{O}_d$ is bornological.*
(4) $\exists S \in (\mathbb{R}_+^d)^p,\ K \in \mathbb{R}_+^p : F(H_{S,K}) \subset H_{R,M}$
(5) $\exists S, K, R, M \in \mathbb{R}_+ :\ F(H_{d,S,K}^p) \subset H_{d,R,M}$
(6) $\exists S, K, R \in \mathbb{R}_+ :\ \sup_{\alpha\in\mathbb{N}^d} R^\alpha \sup_{\gamma\in\widetilde{\mathcal{M}}_d^p} |F_{\alpha,\gamma}| \frac{K^{\mathrm{sh}(\gamma)}}{S^{\mathrm{wt_v}(\gamma)}} < \infty$
(7) $\exists R, Q \in \mathbb{R}_+ :\ \forall \alpha \in \mathbb{N}^d\ \forall \gamma \in \widetilde{\mathcal{M}}_d^p\colon\ |F_{\alpha,\gamma}| \leq \frac{1}{Q^{|\alpha|}}\frac{M}{R^{|\alpha|-|\mathrm{wt}(\gamma)|+|\mathrm{sh}(\gamma)|}}$

Proof: The coefficient functions F_α are polynomials and thus holomorphic functions $\mathcal{O}_d^p \to \mathbb{C}$. Hence the equivalence (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) is just a special case of Theorem 4.3.2. (3) $\Rightarrow$ (4) is clear (4) $\Rightarrow$ (5) Let $K = (K_i)_{i=1}^p$, $S = (S_{i,j})_{\substack{1\leq i\leq d\\ 1\leq j\leq p}}$. Set $K_0 = \min_{1\leq i\leq p} K_i$, $S_0 = \max_{1\leq j\leq p}(\max_{1\leq i\leq d}(S_{i,j}))$, $R_0 = \min_{1\leq i\leq d} R_i$. Then $H_{d,S_0,K_0}^p \subset H_{S,K}$ and $H_{R,M} \subset H_{d,R_0,M}$, which yields the claim.
(5) $\Rightarrow$ (6) is Lemma 5.1.8 and (6) $\Rightarrow$ (7) follows out of the proof of Lemma 5.1.8.

(7) $\Rightarrow$ (3) Let $H(x,y) := \sum\limits_{(\alpha,\beta)\in\mathbb{N}^{d+p}} \frac{1}{R^{|(\alpha,\beta)|}} x^\alpha y^\beta$. Let $S, K > 0$ and let $a \in H^p_{d,S,K}$. Then

$$|F_\alpha(a)| \leq \sum_{\mathrm{wt}(\gamma)\leq\alpha} |F_{\alpha,\gamma}||a^\gamma| \leq \frac{1}{Q^{|\alpha|}} \sum_{\mathrm{wt}(\gamma)\leq\alpha} \frac{M}{R^{|\alpha|-|\mathrm{wt}(\gamma)|+|\mathrm{sh}(\gamma)|}} \mu(\gamma) \left(\mathfrak{g}_{d,S,K,p}\right)^\gamma =$$

$$\overset{\text{Theorem 5.1.4}}{=} \frac{M}{Q^{|\alpha|}} H(x, \mathfrak{g}_{d,S,K,p})[\alpha] \leq$$

$$\overset{\text{Lemma 5.1.7}}{\leq} \frac{M}{Q^{|\alpha|}} \frac{\tilde{K}}{T^{|\alpha|}} \text{ for some } \tilde{K}, t > 0.$$

Hence $F(H^p_{d,s,K}) \subset H_{d,T\cdot Q,\tilde{K}\cdot M}$, which means that F is bornological as the symmetrical polydiscs form a fundamental system of bounded sets for $\mathcal{O}^p_d$ $\square$

Remark. The above theorem shows that from a topological point of view textile maps behave similarly to linear maps between normed spaces: They are continuous iff they preserve the boundedness of a "ball" $H_{d,S,K}$.

Corollary 5.1.11. *A power series $F \in \mathcal{P}_n$ converges around 0 if and only if there exist $d + p = n$ so that $\widehat{F} \in \mathcal{S}^{d,p}_A$. In other words: F is holomorphic at 0 iff $\widehat{F}$ is holomorphic.*

Proof: ($\Leftarrow$) Let $F = \sum c_{\alpha,\beta} x^\alpha y^\beta \in \mathcal{P}_{d+p}$, and suppose that $\widehat{F} \in \mathcal{S}^{d,p}_A$. Then there exist $R, M, Q \in \mathbb{R}_+$ s.t. $|\widehat{F}_{\alpha,\gamma}| \leq \frac{1}{Q^{|\alpha|}} \frac{M}{R^{|\alpha|-|\mathrm{wt}(\gamma)|+|\mathrm{sh}(\gamma)|}}$. Let $f \colon \mathrm{N}^{\mathrm{d}} \to \mathbb{N}$,

$$f(\alpha) := \begin{cases} 1 & \text{if } \alpha = e_1 \\ 0 & \text{else} \end{cases}$$

For $(\alpha,\beta) \in \mathbb{N}^d \times \mathbb{N}^p$, let $\nu(\beta) := (\beta_1 \cdot f, \ldots, \beta_p \cdot f) \in M^p_d$ and

$$\epsilon(\alpha,\beta) := \alpha + \mathrm{wt}(\nu(\beta)) = \alpha + (|\beta|, 0, 0, \ldots, 0)$$

Hence

$$|c_{\alpha,\beta}| = |F_{\epsilon,\gamma}| \leq \frac{M}{Q^{|\epsilon|}} \frac{1}{R^{|\alpha|-|\mathrm{wt}(\gamma)|+|\mathrm{sh}(\gamma)|}} \leq \frac{M}{Q^{|\alpha|+|\beta|}} \frac{1}{R^{|\alpha|+|\beta|-|\beta|+|\beta|}} \leq \frac{M}{(Q\cdot R)^{|\alpha|+|\beta|}},$$

so $F \in \mathcal{O}^1_n$. ($\Rightarrow$) was shown in Lemma 5.1.7. $\square$

Corollary 5.1.12. *Let $F \in \mathcal{T}^{d,p}$. Suppose that F has non-negative coefficients and that F preserves the convergence of a geometric series, i.e. that there are $M \in \mathbb{R}_+, R \in (\mathbb{R}^d_+)^p$ so that $F(\mathfrak{g}_{R,M}) \in \mathcal{O}_d$. Then $F : \mathcal{O}^p_d \to \mathcal{O}_d$ and F is continuous.*

Proof: Let $a \in H_{R,M}$. Then $|F_\alpha(a)| \leq F_\alpha(|a|) \leq F_\alpha(\mathfrak{g}_{R,M}) \leq \frac{K}{S^\alpha}$ for some $S \in \mathbb{R}^d_+, K \in \mathbb{R}_+$. This yields that $F(H_{R,M}) \subset H_{S,K}$, which implies by Theorem 5.1.10 that F is holomorphic. $\square$

5.2 The topology of textile maps

Definition 5.2.1. Let d, p be fixed and let $F = \sum_{\mathrm{wt}(\gamma)\le\alpha} F_{\alpha,\gamma} x_\alpha^\gamma \in \mathcal{T} = \mathcal{T}_{d,p}$. For $R, Q, M \in \mathbb{R}_+$, we define:

$$\rho_{R,Q} : \mathcal{T} \to \mathbb{R}_+ \cup \{\infty\},\ \rho_{R,Q}(F) := \sup_{\alpha,\gamma} |F_{\alpha,\gamma}| R^{|\alpha|-|\mathrm{wt}(\gamma)|+|\mathrm{sh}(\gamma)|} Q^{|\alpha|}$$

$$\mathcal{T}_{R,Q} := \{F \in \mathcal{T} | \rho_{R,Q}(F) < \infty\},\ B_{R,Q,M} := \left\{F \in \mathcal{T} \,\middle|\, \rho_{R,Q}(F) \le M\right\}$$

Theorem 5.1.10 yields that

$$\mathcal{T}_A = \bigcup_{R,Q\in\mathbb{R}_+} \mathcal{T}_{R,Q}$$

We endow the spaces $\mathcal{T}_{R,Q}$ with the norm $\rho_{R,Q}$ with which they become Banach spaces and equip $\mathcal{T}_A$ with the inductive topology so that $\mathcal{T}_A = \varinjlim \mathcal{T}_{R,Q}$, where $(\mathcal{T}_{R,Q})_{R,Q\in\mathbb{R}_+}$ is ordered by inclusion.

Lemma 5.2.2. *Let* $0 < S < R$ *and* $0 < P < Q$.
(1) *The inclusion mapping* $\mathcal{T}_{R,Q} \to \mathcal{T}_{S,P}$ *is compact.*
(2) $\mathcal{T}_A$ *is a (DFS)-space.*
(3) $\mathcal{S}_A$ *is a closed subspace of* $\mathcal{T}$ *and hence a (DFS)-space.*

Proof: (1) Let $\left(F^k\right)_{k\in\mathbb{N}} \subset B_{R,Q},\ F^k = (\sum_{\mathrm{wt}(\gamma)\le\alpha} F^k_{\alpha,\gamma} x_\alpha^\gamma)$ be an arbitrary sequence. We need to show that $\left(F^k\right)_{k\in\mathbb{N}}$ possesses a subsequence which converges in $\mathcal{T}_{S,Q}$. As the coordinate sequences $(F^k_{\alpha,\gamma})_{k\in\mathbb{N}}$ are bounded we can extract for each coordinate a convergent subsequence. Using a diagonal argument we can extract a subsequence of $(F^k)_{k\in\mathbb{N}}$ which converges coordinate-wisely. So without loss of generality we may suppose that $(F^k)_{k\in\mathbb{N}}$ itself converges coordinate-wisely to a textile map $F = (\sum_{\mathrm{wt}(\gamma)\le\alpha} F_{\alpha,\gamma} x_\alpha^\gamma)$. The coordinate-wise convergence yields that $F \in B_{R,Q}$. Let $\epsilon > 0$ and choose $N_1 \in \mathbb{N}$ so that $\left(\frac{P}{Q}\right)^{N_1} \le \frac{\epsilon}{2}$. Let $N_2 > N_1$ so that

$$\forall k > N_2\ \forall |\alpha| \le N_1\ \forall\gamma : |F^k_{\alpha,\gamma} - F_{\alpha,\gamma}| \le \frac{\epsilon}{2} \frac{1}{R^{|\alpha|-|\mathrm{wt}(\gamma)|+|\mathrm{sh}(\gamma)|}} \frac{1}{Q^{|\alpha|}}$$

Then for $k > N_2$ we have $\rho_{S,P}(F^k - F) \le \epsilon$:

$$\rho_{S,P}(F^k - F) = \sup_{\alpha,\gamma} |F^k_{\alpha,\gamma} - F_{\alpha,\gamma}| S^{|\alpha|-|\mathrm{wt}(\gamma)|+|\mathrm{sh}(\gamma)|} P^{|\alpha|} =$$

$$= \sup_{\alpha,\gamma} \underbrace{|F^k_{\alpha,\gamma} - F_{\alpha,\gamma}| R^{|\alpha|-|\mathrm{wt}(\gamma)|+|\mathrm{sh}(\gamma)|} Q^{|\alpha|}}_{(I)} \underbrace{\left(\frac{S}{R}\right)^{|\alpha|-|\mathrm{wt}(\gamma)|+|\mathrm{sh}(\gamma)|}}_{(II)} \underbrace{\left(\frac{P}{Q}\right)^{|\alpha|}}_{(III)}$$

Note that $(I) \le \rho_{R,Q}(F^k - F) \le 2$, $(II) \le 1$ and $(III) \le 1$ for all $\alpha \in \mathbb{N}^d$ and $\gamma \in \widetilde{\mathcal{M}}^p_d$. If $|\alpha| \le N_1$, then $(I) \le \frac{\epsilon}{2}$, and if $|\alpha| \ge N_1$, then $(II) \le \frac{\epsilon}{2}$. This shows that $\rho_{S,P}(F^k - F) \le \epsilon$ for $k > N_2$, and hence F^k converges to F in $\mathcal{T}_{S,P}$. (2) is a simple consequence of (1), since the spectra $\left(\mathcal{T}_{1/n,1/n}\right)_{n\in\mathbb{N}}$ and $(\mathcal{T}_{R,Q})_{R,Q\in\mathbb{R}_+}$ are equivalent, yielding $\varinjlim_n \mathcal{T}_{1/n,1/n} = \mathcal{T}_A$

(3) Let $\left(F^k\right)_{k\in\mathbb{N}} \in \mathcal{S}_A$ converge to F in $\mathcal{T}_{A,}$. Let $(\alpha_1,\gamma_1),(\alpha_2,\gamma_2) \in \mathbb{N}^d \times \widetilde{\mathcal{M}}^p_d$ so that $(\alpha_1 - \mathrm{wt}(\gamma_1), \mathrm{sh}(\gamma_1)) = (\alpha_2 - \mathrm{wt}(\gamma_2), \mathrm{sh}(\gamma))$. Then $F_{\alpha_1,\gamma_1} = \lim_{k\to\infty} F^k_{\alpha_1,\gamma_1} = \lim_{k\to\infty} F^k_{\alpha_2,\gamma_2} = F_{\alpha_2,\gamma_2}$. By Lemma 5.1.7, this means that F is tactile, and since $F \in \mathcal{T}_A$ we conclude that $F \in \mathcal{S}_A$. □

Lemma 5.2.3. *$\mathcal{O}_{d+p} \cong \mathcal{S}^{d,p}_A$ as LCVS.*

Proof: The canonical map $\Phi\colon \mathcal{O}_n \to \mathcal{S}^{d,p}_A$ $F \mapsto \widehat{F}$ is a linear isomorphism by Theorem 5.1.11. Since the open mapping theorem holds for maps between (DFS)-spaces, it suffices to show that Φ is continuous, for which we use the sequential closed-graph theorem. Let $(F_n, \Phi(F_n)) \to (F, G)$, $F_n = \sum_{\alpha,\beta} c_{\alpha,\beta} x^\alpha y^\beta$, $\Phi(F_n) = \sum F^n_{\alpha,\gamma} x^\gamma_\alpha$. The projection $p_{\alpha,\beta}\colon F \mapsto c_{\alpha,\beta}$ and $p_{\alpha,\gamma}\colon \widehat{F} \to F_{\alpha,\gamma}$ are continuous on $\mathcal{O}_{d+p}$ and $\mathcal{S}^{d,p}_A$, respectively. This implies that the coefficients of G have to coincide with those of $\widehat{F}$, which yields $G = \Phi(F)$. □

Lemma 5.2.4.

(1) *The inductive topology of $\mathcal{T}^{d,p}_A$ coincides with the compact-open topology τ_{co}.*
(2) *$\mathcal{T}^{d,p}_A$ is a closed subspace of $\mathcal{H}(\mathcal{O}^p_d, \mathcal{O}_d)$.*
(3) *$\mathcal{T}^{d,p}_A$ is a (DFN)-space.*
(4) *The bounded sets of $\mathcal{T}^{d,p}_A$ are equicontinuous as a family of functions $\mathcal{O}^p_d \to \mathcal{O}_d$.*
(5) *The family $\{x^\gamma_\alpha \mid \alpha \in \mathbb{N}^d,\ \gamma \in \widetilde{\mathcal{M}}^p_d : \mathrm{wt}(\gamma) \le \alpha\}$ forms a basis for $\mathcal{T}^{d,p}_A$.*

Proof: As in Theorem 4.4.23 let $p_{\alpha,\gamma}$ denote the coordinate projections on $\mathcal{H}(\mathcal{O}^p_d, \mathcal{O}_d)$. Let $\mathcal{Q} = \{p_{\alpha,\gamma} \mid \mathrm{wt}(\gamma) > \alpha\}$. Then $\mathcal{T}^{d,p}_A = \bigcap_{p_{\alpha,\gamma}\in\mathcal{Q}} \ker p_{\alpha,\gamma}$ and thus $(\mathcal{T}^{d,p}_A, \tau_{\mathrm{co}})$ is an ultrabornological space by Theorem 4.4.23. Recall that Webb's form of the open mapping theorem (see for example [MV92, p.289]) states that if $T\colon E \to F$ is a continuous linear bijection, E a webbed space, F ultrabornological, then T is an isomorphism (i.e. T^{-1} is continuous). $\mathcal{T}^{d,p}_A$ is webbed and bornological, so for showing that $\mathrm{id}\colon \mathcal{T}^{d,p}_A \to (\mathcal{T}^{d,p}_A, \tau_{\mathrm{co}})$ is continuous it suffices to show that every set which is bounded in the inductive topology is also bounded in the compact-open topology. If $B \subset \mathcal{T}^{d,p}_A$ is bounded in the inductive topology, then there exist $R, Q > 0$ so that $B \subset B_{R,Q,M}$. The compact-open topology is generated by the family of seminorms $\left\{\|\ \|_{K,\phi} \;\middle|\; K \underset{co}{\subseteq} \mathcal{O}^p_d,\ \phi \in \mathcal{H}\left(\mathbb{C}^d\right)\right\}$, where $\|F\|_{K,\phi} := \sup_{a\in K}\ p^\infty_\phi(F(a))$. Let $\phi \in \mathcal{H}\left(\mathbb{C}^d\right)$, $K \underset{co}{\subseteq} \mathcal{O}^p_d$. Choose S, $\tilde{M}$ so that $K \subset H_{S,\tilde{M}}$. For $F \in B_{R,Q,M}$ and $a \in H_{S,\tilde{M}}$ we have that $|F_\alpha(a)| \le \frac{M}{Q^\alpha}\widehat{\mathfrak{g}}_{d+p,R,M}(K\mathfrak{g}_{S,\tilde{M}}) \le \frac{C}{T^\alpha}$ for some $C \in \mathbb{R}_+$, $T \in \mathbb{R}^d_+$. Therefore $p^\infty_\phi(F(a)) \le \sum |\phi_\alpha| \frac{C}{T^\alpha} \le \tilde{C}$, so $\|B_{R,Q,M}\|_{K,\phi} \le \tilde{C}$ and it follows that $B_{R,Q,M}$ is τ_{co}-bounded since K, ϕ were arbitrary. (2), (4) and (5) are a consequence of (1) and Theorem 4.4.23 and Theorem 4.4.24. (3) $\mathcal{H}(\mathcal{O}^p_d, \mathcal{O}_d)$ is a nuclear space by Proposition 4.4.13, and since nuclearity is passed over to subspaces (Theorem 2.6.13) we see that $\mathcal{T}^{d,p}_A$ is nuclear and hence a (DFN)-space by Theorem 2.7.8. □

5.2.1 Limit topologies on $\mathcal{S}_A$ and $\mathcal{T}_\mathcal{B}$

Definition 5.2.5. Let $d, p \in \mathbb{N}$ be fixed, $\mathcal{T}_\mathcal{B} = \mathcal{T}_\mathcal{B}^{d,p}$. We recall that a textile map $F = \sum F_{\alpha,\gamma} x_\alpha^\gamma$ is tactilly bounded iff there exists an $R \in \mathbb{R}_+$ so that $\sup_{\alpha,\gamma} |F_{\alpha,\gamma}| \frac{R^{|\alpha|-|\mathrm{wt}(\gamma)|+|\mathrm{sh}(\gamma)|}}{\mu(\gamma)} < \infty$. We define

$$\nu_{_R} \colon \mathcal{T} \to \mathbb{R}_+ \cup \{\infty\},\ \nu_{_R}(F) := \sup_{\alpha,\gamma} |F_{\alpha,\gamma}| \frac{R^{|\alpha|-|\mathrm{wt}(\gamma)|+|\mathrm{sh}(\gamma)|}}{\mu(\gamma)}$$
$$\mathcal{T}_{\mathcal{B}_R} := \{F \in \mathcal{T} \mid \nu_{_R}(F) < \infty\}$$

We note that $\mathcal{T}_\mathcal{B} = \bigcup_{R\in\mathbb{R}_+} \mathcal{T}_{\mathcal{B}_R}$. Let τ_{ind} be the inductive topology such that $(\mathcal{T}_\mathcal{B}, \tau_{\mathrm{ind}}) = \varinjlim\{(\mathcal{T}_{\mathcal{B}_R}, \nu_{_R}) : R \in \mathbb{R}_+\}$. Likewise we can introduce an inductive topology on $\mathcal{S}_A = \mathcal{S}_A^{d,p}$: We set

$$\mathcal{S}_{_R} := \{F \in \mathcal{S} \mid \nu_{_R}(F) < \infty\},$$

equip $\mathcal{S}_{_R}$ with $\nu_{_R}$ and denote by τ_{S} the inductive topology so that $(\mathcal{S}_A, \tau_{\mathrm{S}}) = \varinjlim\{(\mathcal{S}_{_R}, \nu_{_R}) : R \in \mathbb{R}_+\}$. We will see that while the inductive topology τ_{S} coincides with τ_{co} on $\mathcal{S}$, the limit topology τ_{ind} is strictly stronger than τ_{co} on $\mathcal{T}_\mathcal{B}$ and unlike $(\mathcal{S}_A, \tau_{\mathrm{S}})$ the larger space $(\mathcal{T}_\mathcal{B}, \tau_{\mathrm{ind}})$ fails to be a (DFS)-space.

Theorem 5.2.6. *Let $d, p \in \mathbb{N}$.*
(1) *$\mathcal{T}_\mathcal{B}$ is dense in $\mathcal{T}_A$.*
(2) *$(\mathcal{T}_\mathcal{B}, \tau_{\mathrm{ind}})$ is not a (DFS)-space.*
(3) *$(\mathcal{T}_\mathcal{B}, \tau_{\mathrm{ind}})$ is sequentially retractive.*
(4) *τ_{ind} is strictly finer then τ_{co} on $\mathcal{T}_\mathcal{B}$.*
(5) *$(\mathcal{S}_A, \tau_{\mathrm{S}}) \cong (\mathcal{S}_A, \tau_{\mathrm{co}})$.*

Proof: (1) Any textile map which has only finitely many coefficients $\neq 0$ is tactilly bounded, and hence for any $F \in \mathcal{T}_A$ the sequence of jets is contained in $\mathcal{T}_\mathcal{B}$. (2) For $\alpha \in \mathbb{N}^d$ let $\alpha^* \in \widetilde{\mathcal{M}}_d$ be the monomial with

$$\alpha^*(\beta) = \begin{cases} 1 & \text{if } \beta = \alpha \\ 0 & \text{else} \end{cases}$$

Let $e_n = (n, 0, \dots, 0) \in \mathbb{N}^d$, let $\gamma_n = ((e_n)^*, 0, \dots, 0) \in \widetilde{\mathcal{M}}_d^p$ and let F_n be the textile map with coefficients $(F^n_{\alpha,\gamma})$, where

$$F^n_{\alpha,\gamma} = \begin{cases} 1 & \text{if } \alpha = e_n \text{ and } \gamma = \gamma_n \\ 0 & \text{else} \end{cases}$$

For any $R \in \mathbb{R}_+$ we have $\nu_{_R}(F_n) = R$. Hence the sequence is contained in every unit (norm) ball $B_R := \{F \in \mathcal{T}_\mathcal{B} \mid \nu_{_R}(F) \leq 1\}$ if $R \leq 1$. However, $(F_n)_{n\in\mathbb{N}}$ cannot have a convergent subsequence in any $\mathcal{T}_{\mathcal{B}_R}$, as $\nu_{_R}(F_n - F_m) = R$ for $n \neq m$, for any $R \in \mathbb{R}_+$. We conclude that the inclusion $\iota \colon \mathcal{T}_{\mathcal{B}_R} \to \mathcal{T}_{\mathcal{B}_S}$ can never be compact for any $0 < S < R$, which shows that the inductive sequence $(\mathcal{T}_{\mathcal{B}_{1/n}})_{n\in\mathbb{N}}$ (which generates

$(\mathcal{T}_{\mathcal{B}}, \tau_{\text{ind}}))$ is not compact, and hence $\mathcal{T}_{\mathcal{B}}$ cannot be a (DFS)-space, since this would imply that any generating sequence of Banach spaces for $\mathcal{T}_{\mathcal{B}}$ is compact.
(3) We will show that $\left(\mathcal{T}_{\mathcal{B}_{1/n}}\right)_{n\in\mathbb{N}}$ has property (M), which is equivalent to $(\mathcal{T}_{\mathcal{B}}, \tau_{\text{ind}})$ being sequentially retractive (Theorem 2.2.11). For $0 < T < S < R$, the topologies induced by ν_S and ν_T coincide on B_R: By continuity of the embedding $\iota\colon \mathcal{T}_{\mathcal{B}_S} \to \mathcal{T}_{\mathcal{B}_T}$ it suffices to show that $id : (B_R, \nu_T) \to (B_R, \nu_S)$ is continuous. Let $(F^k)_{k\in\mathbb{N}}, F \in B_R$ so that $\nu_T\left(F^k - F\right) \to 0$. $G^k := F^k - F \in 2\cdot B_R$. We need to show that $\nu_S\left(G^k\right) \to 0$. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ s.t. $\left(\frac{S}{R}\right)^N < \epsilon/2$. Let $N_0 \in \mathbb{N} : \forall l > N_0 :\ \nu_T\left(G^l\right) < \epsilon/\left(\frac{S}{T}\right)^N$. Then for $l > N_0 : \nu_S\left(G^l\right) < \epsilon$. Let $\alpha \in \mathbb{N}^d, \gamma \in \widetilde{\mathcal{M}}_d^p$.
Case 1: $|\alpha| - |\text{wt}(\gamma)| + |\text{sh}(\gamma)| \le N$

$$|G^l_{\alpha,\gamma}| S^{|\alpha|-|\text{wt}(\gamma)|+|\text{sh}(\gamma)|} = \underbrace{|G^l_{\alpha,\gamma}| T^{|\alpha|-|\text{wt}(\gamma)|+|\text{sh}(\gamma)|}}_{\le \nu_T\left(G^k\right)} \underbrace{\left(\frac{S}{T}\right)^{|\alpha|-|\text{wt}(\gamma)|+|\text{sh}(\gamma)|}}_{\le\left(\frac{S}{T}\right)^N} \le \epsilon$$

Case 2: $|\alpha| - |\text{wt}(\gamma)| + |\text{sh}(\gamma)| > N$

$$|G^k_{\alpha,\gamma}| S^{|\alpha|-|\text{wt}(\gamma)|+|\text{sh}(\gamma)|} = \underbrace{\left|G^k_{\alpha,\gamma}\right| R^{|\alpha|-|\text{wt}(\gamma)|+|\text{sh}(\gamma)|}}_{\le 2} \underbrace{\left(\frac{S}{R}\right)^{|\alpha|-|\text{wt}(\gamma)|+|\text{sh}(\gamma)|}}_{<\epsilon/2} \le \epsilon$$

Taking the supremum yields $\nu_S\left(G^k\right) \le \epsilon$ for $k > N_0$. Hence $\nu_S\left(G^k\right) \to 0$.
(4) It is easy to see that $\text{id}\colon (\mathcal{T}_{\mathcal{B}}, \tau_{\text{ind}}) \to (\mathcal{T}_{\mathcal{B}}, \tau_{\text{co}})$ is bornological and thus continuous. By Theorem 2.2.12 we have that $(\mathcal{T}_{\mathcal{B}}, \tau_{\text{ind}})$ is a complete LCVS. But $(\mathcal{T}_{\mathcal{B}}, \tau_{\text{co}})$ cannot be complete, as it is a proper and dense subspace of $\mathcal{T}_A$.
(5) First we show that the spectrum $\{\mathcal{S}_R\ :\ R \in \mathbb{R}^d_+\}$ is compact, i.e. that $\iota\colon \mathcal{S}_R \to \mathcal{S}_T$ is compact for $0 < T < R$. Let $F_n = \sum c^n_{\alpha,\gamma} x^\gamma_\alpha \subset B_{\mathcal{S}_R} := \{F \in \mathcal{S} \mid \nu_R(F) \le 1\}$, which we may assume WLOG to converge coordinate-wisely to $F \in B_{\mathcal{S}_R}$.
Let $G_n = \sum c_{\alpha,\beta} x^\alpha y^\beta \in \mathcal{O}_{d+p}$ so that $\widehat{G_n} = F_n$. Let $\epsilon > 0$ and let $N_0 \in \mathbb{N}$ so that $\left(\frac{T}{R}\right)^{N_0} < \epsilon$. Let $A = \{(\delta_1, \delta_2) \in \mathbb{N}^{d+p}\ :\ |\delta_1| + |\delta_2| < N_0\}$. Recall that we have the relation $c^n_{\alpha-\text{wt}(\gamma),\text{sh}(\gamma)}\mu(\gamma) = F^n_{\alpha,\gamma}$. Let $K_0 \in \mathbb{N}$ so that

$$\left|c^k_{\delta_1,\delta_2} - c_{\delta_1,\delta_2}\right| < \epsilon$$

holds for all $k \ge K_0$, for all $(\delta_1, \delta_2) \in A$. For $n > K_0$ we have

$$\nu_T(F - F_n) = \sup \left(\frac{T}{R}\right)^{|\alpha|-|\text{wt}(\gamma)|+|\text{sh}(\gamma)|} R^{|\alpha|-|\text{wt}(\gamma)|+|\text{sh}(\gamma)|} \frac{|F^n_{\alpha,\gamma} - F_{\alpha,\gamma}|}{\mu(\gamma)} =$$

$$= \sup_{(\delta_1,\delta_2)\in\mathbb{N}^{d+p}} \left(\frac{T}{R}\right)^{|\delta_1|+|\delta_2|} R^{|\delta_1|+|\delta_2|} |c^n_{\delta_1,\delta_2} - c_{\delta_1,\delta_2}|$$

$$\sup_{(\delta_1,\delta 2)\notin A} \underbrace{\left(\frac{T}{R}\right)^{|\delta_1|+|\delta_2|}}_{\leq \left(\frac{T}{R}\right)^{N_0} \leq \epsilon} \underbrace{R^{|\delta_1|+|\delta_2|}|c^n_{\delta_1,\delta_2} - c_{\delta_1,\delta_2}|}_{\leq \nu_R(F^n - F) \leq 1} \leq \epsilon$$

$$\sup_{(\delta_1,\delta 2)\in A} \underbrace{\left(\frac{T}{R}\right)^{|\delta_1|+|\delta_2|} R^{|\delta_1|+|\delta_2|}|c^n_{\delta_1,\delta_2} - c_{\delta_1,\delta_2}|}_{\leq \epsilon} \leq \epsilon$$

Hence $\nu_T(F_n - F) < \epsilon$ for all $k \geq K_0$. We have shown that any sequence in $B_{\mathcal{S}_R}$ has a convergent subsequence which converges with respect to ν_T, which shows that $B_{\mathcal{S}_R}$ is compact in $\mathcal{S}_T$, hence ι is compact. Now we show that $\Psi\colon (\mathcal{S}_A, \tau_{\mathrm{s}}) \to \mathcal{O}_{d+p} \cong (\mathcal{S}_A, \tau_{\mathrm{co}})$, $\widehat{F} \mapsto F$ is continuous. The family of all $B_{\mathcal{S}_{R,M}} := \{F \in \mathcal{S} \mid \nu_R(F) \leq M\}$ is a fundamental system of bounded set of $(\mathcal{S}_A, \tau_{\mathrm{s}})$. If $\widehat{F} \in B_{\mathcal{S}_{R,M}}$, where $F = \sum c_{\alpha,\beta} x^\alpha y^\beta$, then $|c_{\delta_1,\delta_2}| = |F_{\alpha,\gamma}|(\mu(\gamma))^{-1} \leq M \frac{1}{R^{|\delta_1|+|\delta_2|}}$ for $(\delta_1,\delta_2) = (\alpha - \mathrm{wt}(\gamma), \mathrm{sh}(\gamma))$. This shows that Ψ is bornological and thus continuous, and as the open mapping theorem holds for pairs of (DFS)-spaces, Ψ is an isomorphism. □

5.3 Generalized textile maps

In this section we introduce a class of functions similar to textile maps, with a more relaxed growth condition concerning $\mathrm{wt}(\gamma)$.

Definition 5.3.1. For $N \in \mathbb{N}, \nu \in \mathbb{N}^d$ a function $F\colon \mathcal{P}_d^p \to \mathcal{P}_d$ is called a generalized textile maps if its coefficient functions F_α are of the form

$$F_\alpha(\phi) = \sum_{\substack{\gamma \in \widetilde{\mathcal{M}}_d^p \\ \mathrm{wt}(\gamma) \leq N\cdot\alpha+\nu}} c_{\alpha,\gamma}\phi^\gamma$$

The function $h\colon \mathbb{N}^d \to \mathbb{N}^d, h(\alpha) := N\cdot\alpha + \nu$ is the *growth-function* associated to F.

The key property of textile maps carries over to generalized textile maps: boundedness on a ball implies that the map preserves convergence and that it is analytic.

Proposition 5.3.2. *Let F be a generalized textile map with growth function $h(\alpha) = N\cdot\alpha + \nu$. TFAE:*

(1) *F is a holomorphic function $\mathcal{O}_d^p \to \mathcal{O}_d$.*
(2) $\exists S \in (\mathbb{R}_+^d)^p, K \in \mathbb{R}_+^p, R \in \mathbb{R}_+^d, M \in \mathbb{R}_+ :\ F(H_{S,K}) \subset H_{R,M}$.
(3) $\exists S \in (\mathbb{R}_+^d)^p, K \in \mathbb{R}_+^p, R \in \mathbb{R}_+^d :\ \sup_{\alpha\in\mathbb{N}^d} R^\alpha \sup_{\gamma\in\widetilde{\mathcal{M}}_d^p} |F_{\alpha,\gamma}| \frac{K^{\mathrm{sh}(\gamma)}}{S^{\mathrm{wt_v}(\gamma)}} < \infty$

Proof: (1) ⇒ (2) ⇒ (3) see Proposition 4.3.15.
(3) ⇒ (1) Let $\sup_{\alpha\in\mathbb{N}^d} R^\alpha \sup_{\gamma\in\widetilde{\mathcal{M}}_d^p} |F_{\alpha,\gamma}| \frac{K^{\mathrm{sh}(\gamma)}}{S^{\mathrm{wt_v}(\gamma)}} < M < \infty$. Set $U = R^{1/N}$ and let G be the textile map with coefficients

$$G_{\alpha,\gamma} := \begin{cases} \frac{M}{U^\alpha}\frac{S^{\mathrm{wt_v}(\gamma)}}{K^{\mathrm{sh}(\gamma)}} & \text{if } \mathrm{wt}(\gamma) \leq \alpha \\ 0 & \text{else} \end{cases}$$

The construction yields

$$|F_{\alpha,\gamma}| \leq \frac{1}{U^{\nu}} G_{N\cdot\alpha+\nu,\gamma}.$$

Let $C_1 \in (\mathbb{R}^d_+)^p, T_1 \in \mathbb{R}_+$. Then (by Theorem 5.1.10) there exist C_2, T_2 so that $G(H_{C_1,T_1}) \subset H_{C_2,T_2}$. So for $\alpha \in \mathbb{N}^d, \phi \in H_{C_1,T_1}$, we obtain $|F_\alpha(\phi)| \leq \frac{1}{U^\nu} G_{\alpha\cdot N+\nu}(|\phi|) \leq \tilde{C}\frac{1}{(T_2^N)^\alpha}$, where $\tilde{C} = \frac{1}{U^\nu T_2^\nu}$, and hence
$F(H_{C_1,T_1}) \subset H_{\tilde{C},T_2^N}$. We conclude that F is bornological and hence holomorphical. $\square$

Definition 5.3.3. A power series $F = \sum_{\alpha\in\mathbb{N}^d,\gamma\in\mathcal{M}^p_d} F_{\alpha,\gamma} x^\gamma_\alpha$ is called *locally textile* with growth function $h(\alpha) = N\cdot\alpha+\nu$ if $c_{\alpha,\gamma} = 0$ if $\mathrm{wt}(\gamma) \not\leq h(\alpha)$. We identify $\mathcal{M}^p_d$ with $\mathbb{N}^p \times \widetilde{\mathcal{M}}^p_d$, i.e. we identify $\gamma \in \mathcal{M}^p_d$ with $(\gamma(0), \widehat{\gamma})$, where $\gamma(0) = (\gamma_1(0), \ldots, \gamma_p(0))$ and $\hat{\gamma}(\alpha) = \gamma(\alpha)$ if $\alpha \neq 0$, $\hat{\gamma}(0) = 0$. By definition, $\mathrm{wt}(\gamma) = \mathrm{wt}(\hat{\gamma})$, which means that the weight condition $\mathrm{wt}(\gamma) \leq h(\alpha)$ does not impose any restriction on $\gamma(0)$. Note that unlike textile functions we allow locally textile functions to depend on the constant coefficient. Explicitly, let $\phi \in \mathcal{O}^p_d$, $\phi(0) =: \phi_0$. Then

$$F_\alpha(\phi) = \sum_{\substack{\gamma\in\mathcal{M}^p_d\\ \mathrm{wt}(\gamma)\leq h(\alpha)}} c_{\alpha,\gamma}\phi^\gamma = \sum_{\substack{\gamma\in\mathcal{M}^p_d\\ \mathrm{wt}(\gamma)\leq h(\alpha)}} c_{\alpha,\gamma}\phi_0^{\gamma(0)}\phi^{\hat{\gamma}} \leq \sum_{\substack{\delta\in\mathbb{N}^p,\hat{\gamma}\in\widetilde{\mathcal{M}}^p_d\\ \mathrm{wt}(\hat{\gamma})\leq h(\alpha)}} c_{\alpha,(\delta,\hat{\gamma})}\phi_0^\delta\phi^{\hat{\gamma}}$$

Clearly, the coefficients of a locally textile power series will fail to converge in general, unless we presuppose that $|\phi_0|$ is small enough. For $C \in \mathbb{R}^p_+$, we set
$\mathcal{P}^p_d(C) = \{\phi = (\phi_1, \ldots, \phi_p) \in \mathcal{P}^p_d |\ |\phi_k(0)| < C_k\}$ and $\mathcal{O}^p_d(C) = \mathcal{O}^p_d \cap \mathcal{P}^p_d(C)$.

Theorem 5.3.4. *Let F be a locally textile power series with growth function h and suppose that there exist $C \in \mathbb{R}^p_+, M \in \mathbb{R}_+, S \in (\mathbb{R}^d_+)^p, K \in \mathbb{R}^p_+$ such that*

$$|F_{\alpha,\gamma}| \leq \frac{1}{C^{\gamma(0)}}\frac{M}{R^\alpha}\frac{S^{\mathrm{wt_v}(\gamma)}}{K^{\mathrm{sh}(\gamma)}}.$$

Then F is a holomorphic function $\mathcal{O}^p_d(C) \to \mathcal{O}_d$.

Proof:

$$|F_\alpha(\phi)| \leq \sum_{\substack{\delta\in\mathbb{N}^p,\hat{\gamma}\in\widetilde{\mathcal{M}}^p_d\\ \mathrm{wt}(\hat{\gamma})\leq h(\alpha)}} |F_{\alpha,\gamma}||\phi_0^\delta||\phi^{\hat{\gamma}}| \leq \sum_{\substack{\delta\in\mathbb{N}^p,\hat{\gamma}\in\widetilde{\mathcal{M}}^p_d\\ \mathrm{wt}(\hat{\gamma})\leq h(\alpha)}} \left|\frac{\phi(0)^\delta}{C^\delta}\right| \frac{M}{R^\alpha}\frac{S^{\mathrm{wt_v}(\gamma)}}{K^{\mathrm{sh}(\gamma)}}|\phi|^{\hat{\gamma}} \leq$$

$$\leq \prod_{k=1}^p \frac{1}{1-\frac{\phi_k(0)}{C_k}} \sum_{\substack{\hat{\gamma}\in\widetilde{\mathcal{M}}^p_d\\ \mathrm{wt}(\hat{\gamma})\leq h(\alpha)}} \frac{M}{R^\alpha}\frac{S^{\mathrm{wt_v}(\gamma)}}{K^{\mathrm{sh}(\gamma)}}|\phi|^{\hat{\gamma}}$$

The estimate shows that for each α the power series $F_\alpha(\phi)$ converges uniformly on compact subsets of $\mathcal{O}^p_d(C)$, which means that the coefficients are holomorphic functions. The same estimate yields the bornologicity of F, and thus F is holomorphical. $\square$

Definition 5.3.5. Let $F \in \mathcal{O}_{d+p}$ and $\phi \in \mathcal{O}_d^p$. We define the formal composition $F(x, \phi(x))$ coefficient-wise: Let $\epsilon \in \mathbb{N}^d$. Then $F(x, \phi(x))[\epsilon] := \sum_{\alpha,\beta} c_{\alpha,\beta}(\phi(x))^\beta[\epsilon]$. Of course this definition makes only sense for certain power series, as specified in the next lemma. If F is holomorphic in a neighborhood of the compact polydisc Δ_{R_1,R_2} centered at zero, if $\phi = (\phi_1, \dots, \phi_p)$ is holomorphic at zero and if $|\phi_k(0)| < R_{2,k}$ for all $1 \le k \le p$, then the coefficients of the formal composition $F(x, \phi(x))$ coincide with the coefficients of the taylor series at 0 of the analytic function $F(x, \phi(x))$, which is a consequence of Faà di Bruno's formula.

Lemma 5.3.6. *Let $F(x, y) = \sum_{\alpha,\beta} c_{\alpha,\beta} x^\alpha y^\beta \in \mathcal{O}_{d+p}$, let $R_1 \in R_+^d$, $R_2 = (R_{2,1}, \dots, R_{2,p}) \in \mathbb{R}_+^p$ and $M \in \mathbb{R}_+$ so that*

$$|c_{\alpha,\beta}| \le \frac{M}{R_1^\alpha R_2^\beta}$$

Then the formal composition $\widetilde{F}\colon \phi \mapsto F(x, \phi(x))$ is a well-defined operator $\mathcal{P}_d^p(R_2) \to \mathcal{P}_d$ and the restriction $\widetilde{F}|\colon \mathcal{O}_d^p(R_2) \to \mathcal{O}_d$ is a holomorphic locally textile map. The coefficient functions are

$$\widetilde{F}_\epsilon(\phi) := \left(\widetilde{F}(\phi)\right)[\epsilon] := \sum_{\gamma \in \mathcal{M}_d^p,\ \mathrm{wt}(\gamma) \le \epsilon} F_{\epsilon,\gamma} a^\gamma, \ \text{where } F_{\epsilon,\gamma} = c_{\epsilon - \mathrm{wt}(\gamma), \mathrm{sh}(\gamma)} \cdot \mu(\gamma).$$

We obtain the estimates

$$|F_{\epsilon,\gamma}| \le \frac{M}{R_1^{\epsilon - \mathrm{wt}(\gamma)}} \frac{\mu(\gamma)}{R_2^{\mathrm{sh}(\gamma)}}$$

$$\text{and} \quad |\widetilde{F}_\epsilon(\phi)| \le \sum_{\gamma \in \widetilde{\mathcal{M}_d^p}, \mathrm{wt}(\gamma) \le \epsilon} \frac{1}{R_1^{\epsilon - \mathrm{wt}(\gamma)}} \frac{1}{R_2^{\mathrm{sh}(\gamma)}} \mu(\gamma) \phi^\gamma \prod_{k=1}^p \left(\frac{1}{1 - \frac{\phi_k(0)}{R_{2,k}}} \right)^{\mathrm{sh}(\gamma_k)+1}.$$

Proof:

$$F(x, \phi(x))[\epsilon] = \sum_{\alpha,\beta} c_{\alpha,\beta} (\phi(x))^\beta [\epsilon - \alpha] = \sum_{\alpha,\beta} c_{\alpha,\beta} \sum_{\substack{\gamma \in \mathcal{M}_d^p \\ \omega(\gamma) = (\beta, \epsilon - \alpha)}} \phi^\gamma \mu(\gamma) = \tag{5.2}$$

$$= \sum_{\gamma \in \mathcal{M}_d^p, \mathrm{wt}(\gamma) \le \epsilon} \underbrace{c_{\epsilon - \mathrm{wt}(\gamma), \mathrm{sh}(\gamma)} \mu(\gamma)}_{=: F_{\epsilon,\gamma}} \phi^\gamma \tag{5.3}$$

We readily receive the estimate

$$|F_{\epsilon,\gamma}| \le \frac{M \mu(\gamma)}{R_1^{\epsilon - \mathrm{wt}(\gamma)} \cdot R_2^{\mathrm{sh}(\gamma)}} \tag{5.4}$$

For $\delta \in \mathbb{N}^p$ let $\delta^\star = (\delta_1^\star, \dots, \delta_p^\star) \in \mathcal{M}_d^p$ be the monomial with

$$\delta_k^\star(\alpha) = \begin{cases} \delta_k & \text{if } \alpha = 0 \\ 0 & \text{else} \end{cases}$$

We set

$$\binom{\mathrm{sh}(\gamma)+\delta}{\delta} := \prod_{k=1}^{p} \binom{\mathrm{sh}(\gamma_k)+\delta_k}{\delta_k}$$

It is easy to see that for $\gamma = (\gamma_1, \ldots, \gamma_p) \in \widetilde{\mathcal{M}}_d^p$ and $\delta \in \mathbb{N}^p$ we have the identity $\mu(\gamma + \delta^\star) = \mu(\gamma) \cdot \binom{\mathrm{sh}(\gamma)+\delta}{\delta}$. Using Equation 5.3 and Equation 5.4 we obtain

$$\begin{aligned}
|F(x,\phi(x))[\epsilon]| &\leq \sum_{\gamma \in \widetilde{\mathcal{M}}_d^p, \mathrm{wt}(\gamma) \leq \epsilon} \frac{M}{R_1^{\epsilon - \mathrm{wt}(\gamma)}} \frac{1}{R_2^{\mathrm{sh}(\gamma)}} \sum_{\delta \in \mathbb{N}^p} |\phi^\gamma| \mu(\gamma + \delta^\star) \left(\frac{|\phi(0)|}{R_2} \right)^\delta = \\
&= \sum_{\gamma \in \widetilde{\mathcal{M}}_d^p, \mathrm{wt}(\gamma) \leq \epsilon} \frac{M}{R_1^{\epsilon - \mathrm{wt}(\gamma)}} \frac{\mu(\gamma)}{R_2^{\mathrm{sh}(\gamma)}} \sum_{\delta \in \mathbb{N}^p} |\phi^\gamma| \binom{\mathrm{sh}(\gamma)+\delta}{\delta} \left(\frac{|\phi(0)|}{R_2} \right)^\delta = \\
&= \sum_{\gamma \in \widetilde{\mathcal{M}}_d^p, \mathrm{wt}(\gamma) \leq \epsilon} \frac{M}{R_1^{\epsilon - \mathrm{wt}(\gamma)}} \frac{\mu(\gamma)}{R_2^{\mathrm{sh}(\gamma)}} \prod_{k=1}^{p} \sum_{\delta_k = 0}^{\infty} \binom{\mathrm{sh}(\gamma_k)+\delta_k}{\delta_k} \left(\frac{|\phi_k(0)|}{R_{2,k}} \right)^{\delta_k}
\end{aligned}$$

Now we rewrite the binomial series to obtain

$$|F(x,\phi(x))[\epsilon]| \leq \sum_{\gamma \in \widetilde{\mathcal{M}}_d^p, \mathrm{wt}(\gamma) \leq \epsilon} \frac{1}{R_1^{\epsilon - \mathrm{wt}(\gamma)}} \frac{1}{R_2^{\mathrm{sh}(\gamma)}} \mu(\gamma) \phi^\gamma \prod_{k=1}^{p} \left(\frac{1}{1 - \frac{\phi_k(0)}{R_{2,k}}} \right)^{\mathrm{sh}(\gamma_k)+1}$$

□

Definition 5.3.7. We say that a locally textile map $F = \sum_{\alpha,\gamma} F_{\alpha,\gamma} x_\alpha^\gamma$ is *tactilly bounded* if $F_{\alpha,\gamma} = 0$ if $\mathrm{wt}(\gamma) > \alpha$ and if there exist $R_1 \in \mathbb{R}_+^d$, $R_2 \in \mathbb{R}_+^p$ and $M \in \mathbb{R}_+$ so that

$$|F_{\alpha,\gamma}| \leq \frac{M}{R_1^{\alpha - \mathrm{wt}(\gamma)}} \frac{\mu(\gamma)}{R_2^{\mathrm{sh}(\gamma)}}$$

for all $\alpha \in \mathbb{N}^d, \gamma \in \mathcal{M}_d^p$.

Imposing a linear growth condition on γ_0 and thus restoring the triangle scheme for the coefficients $F_{\alpha,\gamma}$ leads to another class of holomorphic functions for which a characterization of holomorphy as in Theorem 5.1.10 is possible:

Theorem 5.3.8. *Let F be a locally convergent textile power series with growth function h and suppose that there is a linear function $g\colon \mathbb{N}^d \to \mathbb{N}^p$, $g(\alpha) = L\alpha + \nu_0$, $L \in M_{p \times d}(\mathbb{N}), \nu_0 \in \mathbb{N}^p$, so that*

$$F_\alpha(\phi) = \sum_{\substack{\delta \in \mathbb{N}^d, \hat{\gamma} \in \widetilde{\mathcal{M}}_d^p \\ \mathrm{wt}(\hat{\gamma}) \leq h(\alpha),\ \gamma(0) \leq g(\alpha)}} F_{\alpha,\gamma} \phi^\gamma$$

Then F is called a generalized textile map *with growth vector (h, g).*

The following are equivalent:
(1) F *is a holomorphic function* $\mathcal{O}_d^p \to \mathcal{O}_d$.
(2) $\exists S \in (\mathbb{R}_+^d)^p, K \in \mathbb{R}_+^p, R \in \mathbb{R}_+^d, M \in \mathbb{R}_+ :\ F(H_{S,K}) \subset H_{R,M}$.
(3) $\exists S \in (\mathbb{R}_+^d)^p, K \in \mathbb{R}_+^p, R \in \mathbb{R}_+^d :\ \sup_{\alpha\in\mathbb{N}^d} R^\alpha \sup_{\gamma\in\widetilde{\mathcal{M}}_d^p} |F_{\alpha,\gamma}| \frac{K^{\mathrm{sh}(\gamma)}}{S^{\mathrm{wt_v}}} < \infty$

Proof: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) See Proposition 4.3.15.
(3) $\Rightarrow$ (1) Let $\sup_{\alpha\in\mathbb{N}^d} R^\alpha \sup_{\gamma\in\mathcal{M}_d^p} |F_{\alpha,\gamma}| \frac{K^{\mathrm{sh}(\gamma)}}{S^{\mathrm{wt_v}(\gamma)}} < M < \infty$. Let $\tilde{F}$ be the generalized textile map whose coefficients are

$$\tilde{F}_\alpha = \sum_{\substack{\hat{\gamma}\in\widetilde{\mathcal{M}}_d^p \\ \mathrm{wt}(\hat{\gamma})\le h(\alpha)}} \frac{M}{R^\alpha} \frac{S^{\mathrm{wt_v}(\hat{\gamma})}}{K^{\mathrm{sh}(\hat{\gamma})}} x_\alpha^\gamma$$

Let $(w_1(\alpha),\ldots,w_p(\alpha))^{\mathrm{t}} = g(\alpha)$. We estimate F by $\tilde{F}$:

$$\begin{aligned} |F_\alpha(\phi)| &\le \sum_{\substack{\gamma_0\in\mathbb{N}^p,\hat{\gamma}\in\widetilde{\mathcal{M}}_d^p \\ \mathrm{wt}(\hat{\gamma})\le h(\alpha),\ \gamma_0\le g(\alpha)}} \frac{M}{r^\alpha} \frac{s^{\mathrm{wt_v}(\hat{\gamma})}}{K^{\mathrm{sh}(\hat{\gamma})}} \left| \frac{\phi_0^{\gamma_0}}{M^{\gamma_0}} \right| |\phi|^{\hat{\gamma}} \le \\ &\le \prod_{k=1}^p \frac{|\phi_k(0)|^{w_k(\alpha)+1}-1}{|\phi_k(0)|-1} \tilde{F}_\alpha(|\phi|) \end{aligned}$$

Let $T \in (\mathbb{R}_+^d)^p, C \in \mathbb{R}_+^p$, WLOG $C_k > 1$ for all $1 \le k \le p$. Choose $T_2, \tilde{C}$ so that $\tilde{F}(H_{T,C}) \subset H_{T_2,\tilde{C}}$. Set $E = (1,\ldots,1) \in \mathbb{N}^p$. Then, for $\phi \in H_{T,C}$, we obtain

$$|F_\alpha(\phi)| \le \prod_{k=1}^p \frac{|\phi_k(0)|^{w_k(\alpha)+1}-1}{|\phi_k(0)|-1} \tilde{F}_\alpha(|\phi|) \le C^{g(\alpha)} \tilde{F}_\alpha(|\phi|) \le C^{\nu_0} C^{L\alpha} \frac{\tilde{C}}{T_2^\alpha}$$

The estimate yields that $F\colon \mathcal{O}_d^p \to \mathcal{O}_d$ is bornological and thus holomorphical. □

5.4 A Cauchy-Kovalevskaya-type theorem

Definition 5.4.1. For $u(x,t) = (u_1,\ldots,u_p) \in \mathcal{P}_{d+1}^p = (\mathcal{P}_d(\mathcal{P}_1))^p$,
$u_k(x,t) = \sum_{(\alpha,j)\in\mathbb{N}^d\times\mathbb{N}} c_{\alpha,j}^k x^\alpha t^j = \sum_{\alpha\in\mathbb{N}^d} u_\alpha^k(t)\, x^\alpha$, $F = \sum_{\alpha,\gamma} F_{\alpha,\gamma} x_\alpha^\gamma$ and $\gamma \in \mathcal{M}_d^p$ we formally set

$$\begin{aligned} u(x,t)^\gamma &:= \prod_{1\le k\le p} \prod_{\alpha\in\mathbb{N}^d} \left(u_\alpha^k(t)\right)^{\gamma_k(\alpha)} \\ F_\alpha(u(x,t)) &:= \sum_{\gamma\in\mathcal{M}_d^p} F_{\alpha,\gamma}(u(x,t))^\gamma \\ F(u(x,t)) &:= \sum_{\alpha\in\mathbb{N}^d} F_\alpha(u(x,t))\, x^\alpha \end{aligned}$$

Note that $F_\alpha(u(x,t))$ may be ill-defined (divergent) for general F, u.

Theorem 5.4.2. *Let $(F_k)_{1\leq k\leq p}$ be locally textile maps with growth function $h(\alpha)=\alpha$, i.e.*

$$F_k(a)[\alpha] = \sum_{\substack{\gamma\in\mathcal{M}_d^p \\ \mathrm{wt}(\gamma)\leq\alpha}} F^k_{\alpha,\gamma} a^\gamma$$

Further let $w \in \mathcal{P}_d^p$ with $w(0)=0$ and consider the Cauchy-Problem

$$\delta_t u(x,t) = F(u(x,t)), \quad u(x,0) = w \tag{5.5}$$

If all F_k are locally textile maps which are tactilly bounded and if $w\in\mathcal{O}_d^p$ then there exists a unique solution $u(x,t)\in\mathcal{O}_{d+1}^p$.

Proof: Let $u(t) = (u_1(t),\dots,u_p(t))$, $u_k(t) = \sum_{\alpha\in\mathbb{N}^d} c^k_\alpha(t)x^\alpha$, where $c^k_\alpha(t) = \sum_{j\in\mathbb{N}} c^k_{\alpha,j}t^j$, and $w=(w_1,\dots,w_p)$, where $w_k = \sum_{\alpha\in\mathbb{N}^d} d^k_\alpha x^\alpha$. The condition $w(0)=0$ yields that $c^k_{0,0}=0$ for all $1\leq k\leq p$, which is critical for the fact that the other coefficients can be calculated polynomially. For γ in $\mathcal{M}_d^p$ let

$$\hat\gamma\in\widetilde{\mathcal{M}}_d^p,\ \hat\gamma(\alpha) = \begin{cases}\gamma(\alpha) & \text{if } \alpha\neq 0\\ 0 & \text{for } \alpha = 0\end{cases}$$

We observe that

$$\begin{aligned}
&\delta_t u(x,t) = F(u(x,t)) \Leftrightarrow \forall k\in\mathbb{N}\ \forall\alpha\in\mathbb{N}^d:\ \delta_t c^k_\alpha(t) = F^k_\alpha(u(x,t))\\
&\Leftrightarrow \forall k\in\mathbb{N}\ \forall\alpha\in\mathbb{N}^d\ \forall j\in\mathbb{N}: c^k_{\alpha,j+1}\cdot(j+1) = F^k_\alpha(u(x,t))[j]\\
&\Leftrightarrow \forall k\in\mathbb{N}\ \forall\alpha\in\mathbb{N}^d\ \forall j\in\mathbb{N}: c^k_{\alpha,j+1} = \frac{1}{1+j}\sum_{\substack{\gamma\in\mathcal{M}_d^p\\ \mathrm{wt}(\gamma)\leq\alpha}} F^k_{\alpha,\gamma}\left(u(x,t)^{\hat\gamma}c_0(t)^{\gamma(0)}\right)[j]
\end{aligned}$$

Hence, for every $\alpha\in\mathbb{N}^d$

$$c^k_{\alpha,j+1} = P^k_{\alpha,j}\left((F^k_{\alpha,\gamma})_{\substack{\gamma\in\mathcal{M}_d^p\\ |\gamma(0)|\leq j}}, (c^m_{\beta,l})_{\substack{1\leq m\leq p, 0\leq l\leq j,\\ 0\leq\beta\leq\alpha}}\right) \tag{5.6}$$

where $P^k_{\alpha,j}$ is a polynomial with non-negative coefficients. Proceeding iteratively in this manner we obtain

$$c^k_{\alpha,j} = Q^k_{\alpha,j}\left(\left(F^n_{\epsilon,\gamma}\right)_{\substack{1\leq n\leq p,\ |\gamma(0)|\leq j\\ \epsilon\leq\alpha,\ \gamma\in\mathcal{M}_d^p}}, \left(d^k_\beta\right)_{\substack{1\leq k\leq p\\ 0\leq\beta\leq\alpha}}\right) \tag{5.7}$$

where $Q^k_{\alpha,j}$ is a polynomial with non-negative coefficients depending only on the initial data w and the textile operator F, which shows that Equation 5.5 has a solution in $\mathcal{P}_d^p$ for any $w\in\mathcal{P}_d^p$ with $w(0)=0$. Now suppose that the initial data w is a convergent power series and that F is tactilly bounded, i.e. that there are $s\in\mathbb{R}_+$ so that

$w \in H^p_{d,s,M}$ and $C \in \mathbb{R}^p_+, R = (R_1, R_2) \in \mathbb{R}^{d+p}_+$ so that

$$|F^k_{\alpha,\gamma}| \leq \frac{C \cdot \mu(\gamma)}{R_1^{\alpha-\mathrm{wt}(\gamma)}} \frac{1}{R_2^{\mathrm{sh}(\gamma)}}$$

for all α, γ, k. Set $g(x) = M \cdot \sum_{\alpha \neq 0} \frac{x^\alpha}{s^{|\alpha|}}$, $H(x,y) = K \cdot \sum \left(\frac{1}{R}\right)^{(\alpha,\beta)} x^\alpha y^\beta \in \mathcal{O}_{d+p}$ and $G(x,y) = (H(x,y), \ldots, H(x,y)) \in \mathcal{O}^p_{d+p}$. Let ω be the solution to the analytic Cauchy-Kovalevskaya-Problem

$$\delta_t v(x,t) = G(x, v(x,t)),\ v(x,0) = (g, \ldots, g) \tag{5.8}$$

This is equivalent to ω solving

$$\delta_t v(x,t) = \widetilde{G}(v(x,t)),\ v(x,0) = (g, \ldots, g) \tag{5.9}$$

The Cauchy-Kovalesvkaya Theorem yields that ω is analytic, hence there exist $t, K > 0$ so that for all $(\alpha, j) \in \mathbb{N}^{d+1}, 1 \leq m \leq p:$

$$0 \leq \omega^m_{\alpha,j} \leq \frac{K}{t^{|\alpha|+j}}$$

Since $\widetilde{G}$ is a locally textile map, we can calculate the coefficients $\omega^k_{\alpha,j}$ also by evaluating the polynomials $Q^k_{\alpha,j}$ at the coefficients of $\widetilde{G}$ and g, which enables us to bound the modulus of the coefficients of u by the coefficients of ω:

$$\begin{aligned}
\left|c^k_{\alpha,j}\right| &= \left| Q^k_{\alpha,j}\left(\left(F^n_{\epsilon,\gamma}\right)_{\substack{1\leq n\leq p,\ |\gamma(0)|\leq j \\ \epsilon\leq\alpha,\ \gamma\in\mathcal{M}^p_d}}, \left(d^m_\beta\right)_{\substack{1\leq m\leq p \\ 0\leq\beta\leq\alpha}} \right) \right| \leq \\
&\leq Q^k_{\alpha,j}\left(\left(|F^n_{\epsilon,\gamma}|\right)_{\substack{1\leq n\leq p,\ |\gamma(0)|\leq j \\ \epsilon\leq\alpha,\ \gamma\in\mathcal{M}^p_d}}, \left(|d^m_\beta|\right)_{\substack{1\leq m\leq p \\ 0\leq\beta\leq\alpha}} \right) \leq \\
&\leq Q^k_{\alpha,j}\left(\left(\widetilde{G}^n_{\epsilon,\gamma}\right)_{\substack{1\leq n\leq p,\ |\gamma(0)|\leq j \\ \epsilon\leq\alpha,\ \gamma\in\mathcal{M}^p_d}}, \left(g^m_\beta\right)_{\substack{1\leq m\leq p \\ 0\leq\beta\leq\alpha}} \right) = \omega^k_{\alpha,j} \leq \frac{K}{t^{|\alpha|+j}}
\end{aligned}$$

Hence we obtain the estimate

$$\left|c^m_{\alpha,j}\right| \leq \frac{K}{t^{|\alpha|+j}},$$

which means that $u(x,t) \in \mathcal{O}^p_{d+1}$. □

Corollary 5.4.3. *Let $w \in \mathcal{O}_d^p$ with $w(0) = 0$.*
(1) *Let F be tactilly bounded textile map. Then the Cauchy-problem*

$$\delta_t u(x,t) = F(u(x,t)),\ u(x,0) = w$$

has a unique analytic solution.
(2) *Let $G \in \mathcal{O}_{d+p}^p$. Then the Cauchy-problem*

$$\delta_t u(x,t) = G(x, u(x,t) - u(0,t)),\ u(x,0) = w$$

has a unique analytic solution.

Proof: (1) follows directly from Theorem 5.4.2.
(2) We may rewrite the differential equation as

$$\delta_t u(x,t) = \widehat{G}(u(x,t)),\ u(x,0) = w.$$

$\widehat{G}$ is tactilly bounded, so (2) is a direct consequence of (1). □

Example 5.4.4. For analytic textile maps F the solution to Theorem 5.4.2 might fail to be convergent, even if the coefficient functions of F are linear and the equation is homogeneous. Let $q \in (0,1)$ and let F be the textile map $\mathcal{P}_1 \to \mathcal{P}_1$ with

$$F_n(\sum_{k\in\mathbb{N}} c_k x^k) = \frac{a_n}{q^n} + \frac{1}{q^n}$$

Applying Theorem 5.1.10 it is easy that $F\colon \mathcal{O}_1 \to \mathcal{O}_1$ is analytic.
Let $v(x,t) = \sum_{(n,k)\in\mathbb{N}^2} v_{n,k} x^n t^k$ be the solution in $\mathcal{P}_2$ to

$$\delta_t v(x,t) = F(v(x,t)),\ v(x,0) = 0$$

We see that

$$\begin{aligned} v_{n,0} &= 0 \\ v_{n,1} &= \frac{1}{q^n} \\ v_{n,j+1} &= \frac{1}{j+1}\frac{1}{q^n} v_{n,j} \end{aligned}$$

and hence

$$v_{n,k} = \frac{1}{k!}\left(\frac{1}{q^n}\right)^k \text{ if } k > 0,\ v_{n,0} = 0$$

But v does not converge in any neighborhood of zero. Let $R > 0$ and choose $k \in N$ s.t. $q^k < R$. Then

$$v_{n,k} R^{n+k} = \frac{R^k}{k!}\left(\frac{R}{q^k}\right)^n \xrightarrow{n\to\infty} \infty$$

Bibliography

[AK68] Ichiro Amemiya and Yukio Kōmura. Über nicht-vollständige Montelräume. *Math. Ann.*, 177:273–277, 1968.

[AS68] V. I. Averbuh and O. G. Smoljanov. Different definitions of derivative in linear topological spaces. *Uspehi Mat. Nauk*, 23(4 (142)):67–116, 1968.

[BB88] Klaus Dieter Bierstedt and Jośe Bonet. Dual density conditions in (DF)-spaces. I. *Result. Math.*, 14(3-4):242–274, 1988.

[BD78] Philip J. Boland and Séan Dineen. Holomorphic functions on fully nuclear spaces. *Bull. Soc. Math. France*, 106(3):311–336, 1978.

[BH10] Clemens Bruschek and Herwig Hauser. Arcs, cords, and felts—six instances of the linearization principle. *Amer. J. Math.*, 132(4):941–986, 2010.

[BM03] Alan F. Beardon and David Minda. On the pointwise limit of complex analytic functions. *Am. Math. Mon.*, 110(4):289–297, 2003.

[BZ79] Joseph Becker and William R. Zame. Applications of functional analysis to the solution of power series equations. *Math. Ann.*, 243:37–54, 1979.

[CO86] B. Cascales and J. Orihuela. Metrizability of precompact subsets in (LF)-spaces. *Proc. R. Soc. Edinb., Sect. A*, 103:293–299, 1986.

[Din75] Seán Dineen. Surjective limits of locally convex spaces and their application to infinite dimensional holomorphy. *Bull. Soc. Math. France*, 103(4):441–509, 1975.

[Din99] Seán Dineen. *Complex analysis on infinite dimensional spaces.* Springer Monographs in Mathematics. London: Springer. xv, 543 p., 1999.

[DL08] PawełDomański and Michael Langenbruch. Vector valued hyperfunctions and boundary values of vector valued harmonic and holomorphic functions. *Publ. Res. Inst. Math. Sci.*, 44(4):1097–1142, 2008.

[Dom12] Paweł Domański. Notes on real analytic functions and classical operators. Blasco, Óscar (ed.) et al., Topics in complex analysis and operator theory. Third winter school in complex analysis and operator theory, Valencia, Spain, February 2–5, 2010. Providence, RI: American Mathematical Society (AMS). 3-47 (2012)., 2012.

[Eng89] Ryszard Engelking. *General topology*, volume 6 of *Sigma Series in Pure Mathematics*. Heldermann Verlag, Berlin, second edition, 1989. Translated from the Polish by the author.

[Flo70] K. Floret. Bases in sequentially retractive limit-spaces. *Stud. Math.*, 38:221–226, 1970.

[Flo71] K. Floret. Lokalkonvexe Sequenzen mit kompakten Abbildungen. *J. Reine Angew. Math.*, 247:155–195, 1971.

[Flo73] Klaus Floret. Folgenretraktive Sequenzen lokalkonvexer Räume. *J. Reine Angew. Math.*, 259:65–85, 1973.

[Flo80] Klaus Floret. Some aspects of the theory of locally convex inductive limits. Functional analysis: surveys and recent results II, Proc. Conf., Paderborn 1979, North-Holland Math. Stud. 38, 205-237 (1980)., 1980.

[Fra67] S. P. Franklin. Spaces in which sequences suffice. II. *Fund. Math.*, 61:51–56, 1967.

[FW68] K. Floret and J. Wloka. *Einführung in die Theorie der lokalkonvexen Räume.* Berlin-Heidelberg-New York: Springer-Verlag. VII, 194 S. , 1968.

[Gac04] Florian Gach. Topological versus bornological concepts in infinite dimensions. Master's thesis, University of Vienna, 2004.

[Gor] Anthony Goreham. *Sequential convergence in topological spaces.* PhD thesis, http://at.yorku.ca/p/a/a/o/51.htm.

[HH95] Nguyen Minh Ha and Le Mau Hai. Linear topological invariants of spaces of holomorphic functions in infinite dimension. *Publ. Mat., Barc.*, 39(1):71–88, 1995.

[HM94] Herwig Hauser and Gerd Müller. A rank theorem for analytic maps between power series spaces. *Inst. Hautes Études Sci. Publ. Math.*, (80):95–115 (1995), 1994.

[Jar81] Hans Jarchow. *Locally convex spaces.* , 1981.

[KC73] R. J. Knowles and T. A. Cook. Incomplete reflexive spaces without Schauder bases. *Proc. Cambridge Philos. Soc.*, 74:83–86, 1973.

[Kel75] John L. Kelley. *General topology. 2nd ed.* Graduate Texts in Mathematics. 27. New York - Heidelberg - Berlin: Springer-Verlag. XIV, 298 p., 1975.

[KM97] Andreas Kriegl and Peter W. Michor. *The convenient setting of global analysis.* Providence, RI: American Mathematical Society, 1997.

[Kra] Steven Krantz. On limits of sequences of holomorphic functions. *arxiv.org.*

[KŚ92] Dorota Krassowska and Wiesław Śliwa. When $(E, \sigma(E, E'))$ is a DF-space? *Comment. Math. Univ. Carolin.*, 33(1):43–44, 1992.

[KS02] Jerzy Kąkol and Stephen A. Saxon. Montel (DF)-spaces, sequential (LM)-spaces and the strongest locally convex topology. *J. Lond. Math. Soc., II. Ser.*, 66(2):388–406, 2002.

[Kuc01] Jan Kucera. Sequential completeness of LF-spaces. *Czech. Math. J.*, 51(1):181–183, 2001.

[Mor93] Mitsuo Morimoto. *An introduction to Sato's hyperfunctions. Transl. from the Japanese by Mitsuo Morimoto.* Translations of Mathematical Monographs. 129. Providence, RI: American Mathematical Society (AMS). vii, 273 p., 1993.

[MV92] Reinhold Meise and Dietmar Vogt. *Introduction to functional analysis. (Einführung in die Funktionalanalysis.).* Vieweg Studium. 62, Aufbaukurs Mathematik. Braunschweig: Friedr. Vieweg & Sohn. ix, 416 S. , 1992.

[PCB87] Pedro Pérez Carreras and José Bonet. *Barrelled locally convex spaces.* North-Holland Mathematics Studies, 131. Notas de Matemática (113). Amsterdam etc.: North-Holland. XV, 512 p., 1987.

[Sch71] H.H. Schaefer. *Topological vector spaces.* Graduate Texts in Mathematics. 3. 3rd printing corrected. New York- Heidelberg-Berlin: Springer-Verlag. XI, 294 p. , 1971.

[staa] http://math.stackexchange.com/q/233522/25717.

[stab] math.stackexchange.com/questions/113240/.

[Val82] Manuel Valdivia. *Topics in locally convex spaces.* North-Holland Mathematics Studies, 67. Notas de Matematica, 85. Amsterdam - New York - Oxford: North-Holland Publishing Company. XIII, 510 p., 1982.

[Vog84] Dietmar Vogt. Some results on continuous linear maps between Fréchet spaces. Functional analysis: surveys and recent results III, Proc. Conf., Paderborn/Ger. 1983, North-Holland Math. Stud. 90, 349-381 (1984)., 1984.

[Web68] J.H. Webb. Sequential convergence in locally convex spaces. *Proc. Camb. Philos. Soc.*, 64:341–364, 1968.

[Wen96] Jochen Wengenroth. Acyclic inductive spectra of Fréchet spaces. *Stud. Math.*, 120(3):247–258, 1996.

[Wen03] Jochen Wengenroth. *Derived functors in functional analysis.* Lecture Notes in Mathematics. 1810. Berlin: Springer. viii, 134 p. , 2003.

[Wil04] Stephen Willard. *General topology. Reprint of the 1970 original.* Mineola, NY: Dover Publications, 2004.

Abstract

This thesis deals with holomorphic functions $\mathcal{O}_d^p \to \mathcal{O}_d$, where $\mathcal{O}_d$ denotes the ring of convergent power series in d variables. In the first two chapters the necessary concepts from functional analysis and topology are developed. The representation of $\mathcal{O}_d$ as union $\bigcup_{S\in\mathbb{R}_+^d} \ell^\infty(S)$ of weighted Banach spaces yields a natural inductive topology, where $\ell^\infty(S)$ is the Banach space of power series for which $\sup_{\alpha\in\mathbb{N}^d} |c_\alpha S^\alpha|$ is finite. It turns out that $\mathcal{O}_d$ is a (DFS)-space, which seems to be the best setting for the usage of concepts of infinite-dimensional calculus, as different approaches coincide and smooth functions are always continuous, which is in general false. In chapter three we give an overview of two concepts of holomorphicity. Chapter four then specifically deals with $\mathcal{O}_d$ and the holomorphic functions on it. We extend the result by Dineen and Boland, that holomorphic functions $\mathcal{O}_d \to \mathbb{C}$ can be expanded into monomial series, to the vector-valued case $\mathcal{O}_d \to \mathcal{O}_d$ and establish some results on the space $(\mathcal{H}(\mathcal{O}_d^p, \mathcal{O}_d), \tau_{\mathrm{co}})$. The last chapter treats a special class of holomorphic functions $\mathcal{O}_d^p \to \mathcal{O}_d$, whose Taylor coefficients have a similar structure as those of substitution maps
$\phi(x) \mapsto F(x, \phi(x))$. We start by studying such maps that ignore the constant term $\phi(0)$ – which we call textile maps – which behave similar to linear maps in normed spaces: they are continuous if and only if they preserve the boundedness of a "ball". The same condition also implies that maps of this class are entire functions. It is then shown that the space of these maps equipped with the compact-open topology is a (DFS)-space and the results established before are then generalized to broader classes. Finally we turn our attention to the differential equation $\delta_t u(x,t) = F(u(x,t))$, where the right side is a generalized textile map, and show that it is analytically solvable for analytical initial conditions. A consequence of this result is that $\delta_t u(x,t) = F(x, u(x,t))$ (where F is a convergent power series) remains analytically solvable if the coefficients of the right side (considered as a holomorphic function $\mathcal{O}_d^p \to \mathcal{O}_d^p$) are continuously perturbated.

Zusammenfassung

Diese Arbeit beschäftigt sich mit holomorphen Funktionen $\mathcal{O}_d^p \to \mathcal{O}_d$, wobei $\mathcal{O}_d$ für den Ring konvergenter Potenzreihen in d Variablen stehe. In den ersten beiden Kapiteln werden die hierfür notwendigen Konzepte aus Funktionalanalysis und Topologie erarbeitet. Die Darstellung $\mathcal{O}_d = \bigcup_{S\in\mathbb{R}_+^d} \ell^\infty(S)$ ergibt eine natürliche Topologisierung von $\mathcal{O}_d$ als induktiven Limes, wobei $\ell^\infty(S)$ der Banachraum aller Potenzreihen $\sum c_\alpha x^\alpha$ ist, für welche die gewichtete Supremumsnorm $\sup_{\alpha\in\mathbb{N}^d} |c_\alpha S^\alpha|$ endlich ist. Es zeigt sich, dass $\mathcal{O}_d$ mit dieser Topologie ein (DFS)-Raum wird. Räume dieser Klasse erscheinen als besonders geeigneter Rahmen für die Verwendung von Konzepten unendlichdimensionaler Analysis, da hier verschiedene Zugänge übereinstimmen und glatte Funktionen stetig sind, was im Allgemeinen falsch ist. In Kapitel drei wird ein kurzer Überblick über zwei Konzepte holomorpher Funktionen zwischen lokalkonvexen Räumen geschaffen. In Kapitel vier wird dann speziell auf $\mathcal{O}_d$ eingegangen, und ein Resultat von Boland und Dineen, wonach jede holomorphe Funktion $\mathcal{O}_d^p \to \mathbb{C}$ in eine Taylorreihe bestehend aus Monomen entwickelt werden kann, auf holomorphe Funktionen $\mathcal{O}_d^p \to \mathcal{O}_d$ verallgemeinert. Im letzten Kapitel wird dann eine Klasse holomorpher Funktionen, deren Koeffizienten eine ähnliche Struktur wie jene von Substitutionsabbildungen $\phi(x) \mapsto F(x, \phi(x))$ besitzen, betrachtet. Zunächst werden Abbildungen dieser Klasse, welche den konstanten Term $\phi(0)$ ignorieren – die wir als textile Abbildungen bezeichnen – untersucht. Diese zeigen ein ähnliches Verhalten wie lineare Abbildungen zwischen normierten Räumen: sie sind stetig genau dann wenn sie auf einer "Kugel" beschränkt sind und die selbe Bedingung ist hinreichend dafür, dass Abbildungen dieser Klasse ganze Funktionen sind. Ausgestattet mit der kompakt-offenen Topologie wird der Raum dieser Abbildungen zu einem (DFS)-Raum. Diese Resultate werden dann auf allgemeinere Klassen ausgeweitet. Im letzten Teil dieses Kapitels betrachten wir die Differentialgleichung $\delta_t u(x,t) = F(u(x,t))$, wobei F eine verallgemeinerte textile Abbildung ist. Es wird gezeigt, dass diese bei analytischer Anfangsbedingung analytisch lösbar ist. Dieses Resultat kann so interpretiert werden, dass die Differentialgleichung $\delta_t u(x,t) = F(x, u(x,t))$ analytisch lösbar bleibt, wenn die Koeffizienten der rechten Seite (aufgefasst als holomorphe Abbildung $\mathcal{O}_d^p \to \mathcal{O}_d^p$) stetig perturbiert werden.

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Curriculum Vitae

Personal data

Name:	Sebastian Woblistin
Date of birth:	20^{th} August 1985
Place of birth:	Vienna
E-mail:	sebastian@woblistin.net

Education

2012 - present:	Bachelor studies in Physics at the University of Vienna
Summer term 2010:	Erasmus Outgoing at the department of Mathematics in Coimbra, Portugal
2009-2010:	Bachelor program in Romance studies (Portuguese)
2006-present:	Diploma studies in Mathematics at the University of Vienna
2005-2009:	Diploma studies in Economics at the Vienna University of Economics and Business
1999-2004:	Vienna Business School Hamerlingplatz