universität wien

DIPLOMARBEIT

Titel der Diplomarbeit

Dynamics for games with continuous strategy sets

Verfasser

Mag. Johannes Mathä

angestrebter akademischer Grad

Magister der Naturwissenschaften (Mag.rer.nat)

Wien, im April 2013

Studienkennzahl lt. Studienblatt:	A 405
Studienrichtung lt. Studienblatt:	Mathematik
Betreuer:	Univ. Prof. Dr. Josef Hofbauer

Danksagungen

Ich möchte mich vor allem bei Prof. Dr. Josef Hofbauer für die hervorragende Betreuung und Unterstützung bedanken, sowohl für die fachliche Beratung als auch für die Geduld und Hilfestellung bei Problemen.
Natürlich möchte ich mich herzlich bei meinen Eltern bedanken, ohne die ich dieses Studium und vieles mehr in meinem Leben nicht geschafft hätte.
Zuletzt möchte ich mich bei meinen Geschwistern und guten Freunden bedanken, deren moralische Unterstützung oft mehr als notwendig war, vielen Dank! Im Speziellen danke ich meinem lieben Freund Alexander Haudum als Korrekturleser, thanks buddy!

Abstract

This thesis examines evolutionary game dynamics in the case of games with continuous strategy sets. The main goal of the study is the comparison of the *adaptive dynamics* and *best response dynamics* for games with such continuous strategy sets. In Chapter 1 the basic concepts of game theory are introduced and the theorem by Glicksberg, Fan, Debreu about the existence of Nash equilibria is proven. In Chapter 2 the stability conditions for adaptive dynamics systems and best response dynamics systems in the case of n-person games are discussed. In the case of $n = 2$ and strategy sets $S_i \subseteq \mathbb{R}$ the stability conditions for the two dynamics are equivalent whereas in the case of $n = 3$ this no longer is true. In addition the concept of *Cournot-tâtonnement* is introduced. In Chapter 3 the connection between stability of Nash equilibria of symmetric games and an ESS x being a CSS is examined. It is shown that in the one-dimensional case equivalence holds true in an important special case. This connection is also examined in the multidimensional case.

Deutsche Zusammenfassung

Diese Diplomarbeit untersucht evolutionäre Spieldynamiken für Spiele mit stetigen Strategieräumen. Im Zentrum der Arbeit steht der Vergleich zwischen der *adaptive dynamics* und der *best response dynamics* für Spiele mit solchen Strategieräumen. In Kapitel 1 werden grundlegende Konzepte der Spieltheorie sowie der Satz von Glicksberg, Fan und Debreu vorgestellt, der die Existenz von Nash-Gleichgewichten für Spiele mit solchen Strategieräumen zeigt. In Kapitel 2 werden die Stabilitätsbedingungen für Spiele mit adaptiver bzw. best-response-Dynamik gezeigt. Dabei zeigt sich, dass im Fall von zwei Spielern eine Äquivalenz zwischen den Stabilitätsbedingungen gilt, dies aber im Fall von drei Spielern jedoch nicht mehr gilt. In Kapitel 3 wird der Begriff von *continuously stable strategies* vorgestellt. Es wird gezeigt, dass im Fall von eindimensionalen Strategieräumen eine Äquivalenz von Stabilitätsbedingungen und CSS-Bedingungen in einem wichtigen Spezialfall gilt. Diese Resultate werden auch im mehrdimensionalen Fall untersucht.

Inhaltsverzeichnis

Introduction

The purpose of this thesis is to examine evolutionary game dynamics that are defined on continuous strategy sets. The origins of evolutionary game theory go back to famous biologists such as John Maynard Smith, George R. Price or William D. Hamilton. Rather than explaining and examining the behaviour of individuals choosing between certain choices a type of game gives them, the evolutionary game theory concentrates on whole populations of players using some strategy or type of behaviour. Successful strategies spread within the population. The payoffs for these player populations using some strategy depend on the actions of the coplayers, i.e. the frequencies of the strategies within the population. Since these frequencies change according to the payoffs, this yields a feedback loop. The dynamics of this feedback loop is the object of evolutionary game theory. (Hofbauer, Sigmund 2003).

The approach that is used here is a deterministic dynamical approach, rather than a stochastic one. When examining such dynamics, surveys and articles mostly start with discrete strategy sets. In this thesis we want to use a different ansatz. The strategy sets are compact and convex subsets of the $\mathbb{R}^n$, for the most part we will concentrate on the special case $n = 1$, i.e. intervals in $\mathbb{R}$.

This diploma thesis discusses two specific kinds of dynamics, namely the *adaptive dynamics* and the *best response dynamics*. The main goal is to discuss whether results about the stability of one of these dynamics imply results about stability of the other dynamics.

The first chapter briefly introduces the main concepts of game theory, such as the mathematical definition of the concept of a game, Nash equilibria and best replies. Finally a theorem about the existence of Nash equilibria for games with continuous strategy sets will be proven.

The center piece of this survey comes in Chapter 2, which first defines the two dynamics, i.e. adaptive dynamics and best response dynamics. The stability analysis of these two dynamics is being made in the cases of two and three players. It shall be examined if stability conditions for Nash equilibria in the adaptive dynamics case are equivalent to those in the best response dynamics case and whether these results are different in the cases of two or three players respectively. At the end of the chapter another approach to Nash equilibria and best response dynamics is presented, the so called *Cournot-tâtonnement*, a concept which is used in mathematical economics. Furthermore we are going to introduce the notion of *continuously stable strategies* (CSS) which was introduced by Eshel in 1983 (Eshel 1983) and continued to be an important concept for studying evolutionary game dynamics; it is a special class of *evolutionarily stable strategies* (ESS). In this thesis we discuss the connection between CSS and the stability of Nash equilibria for symmetric games. This connection is going to be examined in the case of one-dimensional strategy sets as well as in the multi-dimensional case. Here we concentrate on the CSS-notion, but many results for CSS are also equivalent to the concepts of *neighbourhood stability* and *risk-dominance* but are not going to be discussed here (these results are presented in (Cressman, Hofbauer 2003), (Cressman 2004) and (Cressman 2009)).

1 Basic properties

First we are going to introduce some basic properties for working with games on continuous strategy sets. We start with a basic definition of a strategic form game. Here we define this kind of games in a very general manner:

Definition 1
A *strategic form game* is a triplet

$$(I, (S_i)_{i\in I}, (u_i)_{i\in I})$$

such that

- I is a finite set of players, often $I = \{1, \dots, n\}$
- S_i is the strategy set for player i with strategies $x_i \in S_i$
- $u_i : S_1 \times S_2 \times \dots \times S_n \to \mathbb{R}$ is the payoff function of player i. We write $u_i(x_i, x_{-i}) = u_i(x_1, x_2 \dots, x_n)$ with $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$.

From now on we are going to assume that $(S_i)_{i\in I}$ are convex and compact subsets of $\mathbb{R}^{n_i}$.
Next we will introduce one of the fundamental concepts in game theory, the concept of *Nash equilibria*.

Definition 2
A *Nash equilibrium* of a strategic form game $(I, (S_i)_{i\in I}, (u_i)_{i\in I})$ is a strategy combination (or strategy profile) $\hat{x} \in S_1 \times S_2 \times \dots \times S_n$ such that for all $i \in I$:

$$u_i(\hat{x}_i, \hat{x}_{-i}) \geq u_i(x_i, \hat{x}_{-i}) \text{ for all } x_i \in S_i$$

We also want to introduce the concept of best responses.

Definition 3
Given a strategic form game $(I, (S_i)_{i\in I}, (u_i)_{i\in I})$. A strategy $x_i' \in S_i$ is called

a *best response* or *best reply* to $x_{-i} \in S_1 \times S_2 \times \dots S_{i-1} \times S_{i+1} \times \cdots \times S_n$ if

$$u_i(x_i', x_{-i}) \geq u_i(z_i, x_{-i})$$

for all $z_i \in S_i$. The set of all best replies is denoted $BR_i(x_{-i})$.

The next result that we want to obtain is a more general version of Nash's theorem about the existence of Nash equilibria. It proves the existence of Nash equilibria in very general strategic form games with continuous strategy sets. It was introduced by Glicksberg, Fan and Debreu in 1952. Before this theorem is shown, we present (without proof) a useful theorem which we are going to need for the existence theorem, the Kakutani fixed-point theorem

Theorem 1 (Kakutani)
Let A be a non-empty subset of a finite dimensional Euclidian space. Let $f : A \rightrightarrows A$ be a correspondence, with $x \in A \mapsto f(x) \subseteq A$ that fulfills the following conditions:

(i) A is a compact and convex set
(ii) $f(x)$ is non-empty for all $x \in A$
(iii) $f(x)$ is a convex-valued correspondence: for all $x \in A$, $f(x)$ is a convex set
(iv) $f(x)$ has a *closed graph*, which means that if $(x_n, y_n) \to (x, y)$ with $y_n \in f(x_n)$, then $y \in f(x)$

Then f has a fixed point, i.e. an $x \in A$ for which $x \in f(x)$.

Having this theorem we can prove the following result.[1]

Theorem 2 (Debreu, Glicksberg, Fan)
Given a strategic form game $(I, (S_i)_{i \in I}, (u_i)_{i \in I})$ satisfying for each $i \in I$ the following conditions:

[1]The proof follows (Fudenberg, Tirole 1991)

- S_i is compact and convex
- $u_i(s_i, s_{-i})$ is continuous
- $u_i(s_i, s_{-i})$ is concave in s_i

Then a Nash equilibrium exists.

Proof: We define the best response correspondence for player i:
$BR_i : S_{-i} \rightrightarrows S_i$,

$$BR_i(s_{-i}) = \{s_i' \in S_i | u_i(s_i', s_{-i}) \geq u_i(s_i', s_{-i}) \text{ for all } s_i \in S_i\}$$

The best response correspondence is defined as $BR(s) = (BR_i(s_i))_{i \in I}$ and $BR : S \rightrightarrows S$, with $S = S_1 \times S_2 \times \cdots \times S_n$
We will apply Kakutani's fixed-point theorem to the best response correspondence $BR : S \rightrightarrows S$. We will show that $BR(s)$ fulfills the conditions of Kakutani's theorem.

(i) Since $S = S_1 \times S_2 \times \cdots \times S_n$ is the finite product of convex, compact and non-empty sets S_i, the product S itself is convex, compact and non-empty

(ii) By definition

$$BR_i(s_i) = \arg\max_{s \in S_i} u_i(s, s_{-i})$$

By assumption S_i is non-empty and u_i is continuous. Therefore a maximum exists and it follows that $BR_i(s_i)$ is non-empty.

(iii) Here we have to show that $BR(s)$ is a convex-valued correspondence. This follows from the fact that $u_i(s_i, s_{-i})$ is concave in s_i. Suppose not, then there exists some i and some $s_{-i} \in S_{-i}$ such that $BR_i(s_{-i}) \in \arg\max_{s \in S_i} u_i(s, s_{-i})$ is not convex.
This would mean that there exist $s_i', s_i'' \in S_i$ such that $s_i', s_i'' \in BR_i(s_{-i})$ and $\lambda s_i' + (1 - \lambda)s_i'' \notin BR_i(s_{-i})$ for some $\lambda \in (0, 1)$, this means in particular

$$\lambda u_i(s_i', s_{-i}) + (1 - \lambda)u_i(s_i'', s_{-i}) > u_i(\lambda s_i' + (1 - \lambda)s_i'', s_{-i})$$

which contradicts the concavity of u_i.

(iv) Here we have to show that $BR(s)$ has a closed graph; suppose it has not. Then there exists a sequence $(s_n, \hat{s}_n) \to (s, \hat{s})$ with $\hat{s}_n \in BR(s_n)$, but $\hat{s} \notin BR(s)$, i.e. there exists some i such that $\hat{s}_i \notin BR(s_{-i})$. This implies that there exists an $s'_i \in S_i$ and an $\epsilon > 0$ such that

$$u_i(s'_i, s_{-i}) > u_i(\hat{s}_i, s_{-i}) + 3\epsilon$$

By the continuity of u_i and the fact that $(s_{-i})_n \to s_{-i}$, we get for sufficiently large n

$$u_i(s'_i, (s_{-i})_n) \geq u_i(s'_i, s_{-i}) - \epsilon$$

Combining these two inequalities we get

$$u_i(s'_i, (s_{-i})_n) > u_i(\hat{s}_i, s_{-i}) + 2\epsilon \geq u_i((\hat{s}_i)_n, (s_{-i})_n) + \epsilon$$

where the second relation follows from the continuity of u_i. This is a contradiction to the assumption $(\hat{s}_i)_n \in BR_i((s_{-i})_n)$. $\square$

2 Dynamics for n-person Games

2.1 Definitions

In this chapter we are going to consider different dynamics for n-person games with continuous strategy spaces. These strategy sets are one-dimensional, i.e. intervals $S_i \subseteq \mathbb{R}$ for a player $i \in \{1, 2, \ldots, n\}$. The payoff functions u_i are therefore real valued functions defined on a subset $S_1 \times S_2 \times ... \times S_n$ of $\mathbb{R}^n$. We still have the assumption that the functions u_i are strictly concave in the i-th argument.

The first dynamics that we are going to discuss is the so called *best response dynamics*. Each player changes his strategy towards his best reply against the current profile of the other players. This leads to following definition:

Definition 1 The dynamics that is described by

$$\dot{x_i} = BR_i(x_{-i}) - x_i$$

where $i \in \{1, 2, ..., n\}$ and $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$

is called *best response dynamics*.
By abuse of notation we write BR_i for the best response function. Since the payoff functions u_i are strictly concave there is a unique best response to a strategy x_i.

Another approach is the *adaptive dynamics*. Each player changes his strategy continuously to improve his payoff, one can also call this a local best reply dynamics (Hofbauer, Sigmund 1990; Hofbauer, Sigmund 2003). This yields following definition:

Definition 2 The dynamics that is described by

$$\dot{x}_i = \frac{\partial u_i}{\partial x_i} \text{ where } i \in \{1, 2, ..., n\}$$

is called *adaptive dynamics*.

The rest points of the adaptive dynamics are Nash equilibria: let $\hat{x}$ be a rest point of the adaptive dynamics, i.e. $\partial_i u_i(\hat{x}_i, \hat{x}_{-i}) = 0$. Since u_i is strictly concave in the i-th argument it follows that $u_i(\hat{x}_i, \hat{x}_{-i}) > u_i(x_i, \hat{x}_{-i})$ for all $x_i \in S_i$, therefore $\hat{x}$ is an Nash equilibrium. If on the other hand $\hat{x}$ is an Nash equilibrium, i.e. $u_i(\hat{x}_i, \hat{x}_{-i}) > u_i(x_i, \hat{x}_{-i})$ it follows that $\partial_i u_i(\hat{x}_i, \hat{x}_{-i}) = 0$. Hence, every rest point of the adaptive dynamics is a Nash equilibrium and vice versa.

2.2 Stability analysis

2.2.1 Two player games

Since both dynamics are defined for nonlinear payoff functions, we have to linearize the dynamical systems around a Nash equilibrium in order to examine the stability of the systems. We will first look at the simple case of two players. The Jacobian matrix for the adaptive dynamics has the form

$$J_{AD}(x_1, x_2) = \begin{pmatrix} \partial_{11} u_1(x_1, x_2) & \partial_{12} u_1(x_1, x_2) \\ \partial_{12} u_2(x_1, x_2) & \partial_{11} u_2(x_1, x_2) \end{pmatrix}$$

if (x_1, x_2) is a Nash equilibrium. $\partial_{ij} u_k$ denote the second order partial derivatives with respect to i and j.

For the best response dynamics the case is a bit more complicated. The best response function $BR_i(x_{-i})$ is given only implicitly through the payoff functions u_i. The definition of the best response function yields

$$u_1(BR_1(x_2), x_2) \geq u_1(x_1, x_2)$$

for all $x_1 \in S_1$. It follows that $\partial_1(u_1(BR_1(x_2), x_2)) = 0$. We now consider a corollary of the implicit function theorem, which says that if for two functions $f(x), g(., x)$ the conditions $g(f(x), x) = 0$ and $\partial_1 g(f(x), x) \neq 0$ hold, the derivative of the implicit function $f(x)$ is given by $f'(x) = -\frac{\partial_2 g(f(x),x)}{\partial_1 g(f(x),x)}$. Using this corollary we get for the best response function.

$$\frac{d}{dx_2} BR_1(x_2) = -\frac{\partial_{12} u_1(x_1, x_2)}{\partial_{11} u_1(x_1, x_2)}$$

Now we can calculate the derivatives of $BR_i(x_{-i})$ and the Jacobian matrix for the best response dynamics has the form

$$J_{BR}(x_1, x_2) = \begin{pmatrix} -1 & -\frac{\partial_{12} u_1(x_1,x_2)}{\partial_{11} u_1(x_1,x_2)} \\ -\frac{\partial_{12} u_2(x_1,x_2)}{\partial_{22} u_2(x_1,x_2)} & -1 \end{pmatrix}$$

if (x_1, x_2) is a Nash equilibrium.
We assume here that u_i is not only strictly concave in x_i, but $\partial_{ii} u_i < 0$ which is a bit stronger.

In the general case of n players the Jacobians have the following form:

$$J_{AD}(x) = \begin{pmatrix} \partial_{11} u_1(x) & \partial_{12} u_1(x) & \dots & \partial_{1n} u_1(x) \\ \partial_{12} u_2(x) & \partial_{22} u_2(x) & \dots & \partial_{2n} u_2(x) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{1n} u_n(x) & \partial_{2n} u_n(x) & \dots & \partial_{nn} u_n(x) \end{pmatrix}$$

for a Nash equilibrium x. The Jacobian of the best response dynamics has the following form:

$$J_{BR}(x) = \begin{pmatrix} -1 & -\frac{\partial_{12} u_1(x)}{\partial_{11} u_1(x)} & \dots & -\frac{\partial_{1n} u_1(x)}{\partial_{11} u_1(x)} \\ -\frac{\partial_{12} u_2(x)}{\partial_{22} u_2(x)} & -1 & \dots & -\frac{\partial_{1n} u_2(x)}{\partial_{22} u_2(x)} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial_{1n} u_n(x)}{\partial_{nn} u_n(x)} & -\frac{\partial_{2n} u_n(x)}{\partial_{nn} u_n(x)} & \dots & -1 \end{pmatrix}$$

It is also possible to construct specific payoff functions out of a given matrix. For the adaptive dynamics one only has to demand that the diagonal elements are negative - which implies that the function is concave in the i-th argument; other than that one gets for a $n \times n$-matrix

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

the following payoff functions

$$u_1(x_1, x_2, ..., x_n) = \frac{a_{11}}{2} x_1^2 + a_{12} x_1 x_2 + a_{13} x_1 x_3 + ... + a_{1n} x_1 x_n$$

$$u_2(x_1, x_2, ..., x_n) = \frac{a_{22}}{2} x_2^2 + a_{21} x_1 x_2 + a_{23} x_2 x_3 + ... + a_{2n} x_2 x_n$$

$$\vdots$$

$$u_n(x_1, x_2, ..., x_n) = \frac{a_{nn}}{2} x_n^2 + a_{n1} x_1 x_n + a_{n2} x_2 x_n + ... + a_{(n-1),n} x_{n-1} x_n$$

with $a_{ij} \in \mathbb{R}$ and $a_{ii} < 0$.

For any given Jacobian matrix of the best response dynamics

$$\begin{pmatrix} -1 & b_{12} & \dots & b_{1n} \\ b_{21} & -1 & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & -1 \end{pmatrix}$$

one can also define the payoff functions:

$$v_1(x_1, x_2, ..., x_n) = -\frac{x_1^2}{2} + b_{12} x_1 x_2 + b_{13} x_1 x_3 + ... + b_{1n} x_1 x_n$$

$$v_2(x_1, x_2, ..., x_n) = -\frac{x_2^2}{2} + b_{21} x_1 x_2 + b_{23} x_2 x_3 + ... + b_{2n} x_2 x_n$$

$$\vdots$$

$$v_n(x_1, x_2, ..., x_n) = -\frac{x_n^2}{2} + b_{n1}x_1x_n + b_{n2}x_2x_n + ... + b_{(n-1),n}x_{n-1}x_n$$

with $b_{ij} \in \mathbb{R}$.

Taking another close look at the Jacobians of the two dynamics one can see that the matrices look very much alike: if one divides the row elements of J_{AD} by the respective diagonal element and then multiplies by (-1) one gets J_{BR}. This leads to the question whether these two dynamics may have similiar stability behaviour.

For stability analysis we start again with the simple case of two players and use the *Routh Hurwitz criterion*, which says that a 2×2 matrix A has eigenvalues with negative real part iff $\text{tr}(A) < 0$ and $\det(A) > 0$.

Considering this in regard to the adaptive dynamics, we can conclude that the system is stable iff

$$\text{tr}(J_{AD}) = \partial_{11}u_1(x_1, x_2) + \partial_{22}u_2(x_1, x_2) < 0$$

$$\det(J_{AD}) = (\partial_{11}u_1(x_1, x_2))(\partial_{22}u_2(x_1, x_2)) - (\partial_{12}u_1(x_1, x_2))(\partial_{12}u_2(x_1, x_2)) > 0$$

Due to the concavity of the payoff functions u_i the first condition is always fulfilled; the determinant however might not always be positive. But if we examine these two conditions for the best response dynamics, it is easy to see that if the adaptive dynamics system is stable, so is the best response dynamics system:

$$\text{tr}(J_{BR}) = -2$$

$$\det(J_{BR}) = 1 - \frac{(\partial_{12}u_1(x_1, x_2))(\partial_{12}u_2(x_1, x_2))}{(\partial_{11}u_1(x_1, x_2))(\partial_{22}u_2(x_1, x_2))}$$

The trace is obviously negative, but if we assume that the determinant is positive we get

$$1 - \frac{(\partial_{12}u_1(x_1, x_2))(\partial_{12}u_2(x_1, x_2))}{(\partial_{11}u_1(x_1, x_2))(\partial_{22}u_2(x_1, x_2))} > 0$$

Again due to the concavity of u_1 and u_2 the denominator $(\partial_{22}u_1(x_1, x_2))(\partial_{22}u_2(x_1, x_2))$ is positve. So by multiplying with $(\partial_{11}u_1(x_1, x_2))(\partial_{22}u_2(x_1, x_2))$ we get the following result:

$$(\partial_{11}u_1(x_1, x_2))(\partial_{22}u_2(x_1, x_2)) > (\partial_{12}u_1(x_1, x_2))(\partial_{12}u_2(x_1, x_2))$$

which is exactly the stability condition for the adaptive dynamics system with two players.

2.2.2 Games with three players

The case for three players is more complicated. The Jacobians then look like this:

$$J_{BR}(x) = \begin{pmatrix} -1 & -\frac{\partial_{12}u_1(x)}{\partial_{11}u_1(x)} & -\frac{\partial_{13}u_1(x)}{\partial_{11}u_1(x)} \\ -\frac{\partial_{12}u_2(x)}{\partial_{22}u_2(x)} & -1 & -\frac{\partial_{23}u_2(x)}{\partial_{22}u_2(x)} \\ -\frac{\partial_{13}u_3(x)}{\partial_{33}u_3(x)} & -\frac{\partial_{23}u_3(x)}{\partial_{33}u_3(x)} & -1 \end{pmatrix}$$

$$J_{AD}(x) = \begin{pmatrix} \partial_{11}u_1(x) & \partial_{12}u_1(x) & \partial_{13}u_1(x) \\ \partial_{12}u_2(x) & \partial_{11}u_2(x) & \partial_{23}u_2(x) \\ \partial_{13}u_3(x) & \partial_{23}u_3(x) & \partial_{33}u_3(x) \end{pmatrix}$$

In order to examine the stability behaviour of these two dynamics, we use the Routh Hurwitz criterion yet again. In the case of $n = 3$ however, the criteria get more complicated. A 3×3 matrix A has eigenvalues with negative real part (and therefore the dynamical system is stable) if and only if the following three inequalities are fulfilled:

$$\text{tr}(A) < 0$$
$$\det(A) < 0$$
$$\text{tr}(A)\text{M}(A) < \det(A)$$

where M(A) denotes the sum of 2×2 principal minors of the matrix A. For a matrix A

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

we define

$$\mathrm{M}(A) := ae - bd + ai - cg + ei - fh$$

Let's first discuss the traces:

$$\mathrm{tr}(J_{BR}) = -3$$

$$\mathrm{tr}(J_{AD}) = \partial_{11} u_1(x) + \partial_{22} u_2(x) + \partial_{33} u_3(x)$$

Like in the case of $n = 2$ we see that the trace of the best response Jacobian is obviously < 0 and the trace of the adaptive dynamics Jacobian is negative due to the concavity of u_i.

If one looks at the determinant condition we get a similar picture as in the two player case; the determinant terms however get more complex:

$$\det(J_{AD}) = (\partial_{11} u_1(x))(\partial_{22} u_2(x))(\partial_{33} u_3(x)) + (\partial_{13} u_1(x))(\partial_{12} u_2(x))(\partial_{23} u_3(x)) +$$

$$+ (\partial_{12} u_1(x))(\partial_{23} u_2(x))(\partial_{13} u_3(x)) - (\partial_{13} u_1(x))(\partial_{11} u_2(x))(\partial_{13} u_3(x)) -$$

$$- (\partial_{12} u_1(x))(\partial_{12} u_2(x))(\partial_{33} u_3(x)) - (\partial_{11} u_1(x))(\partial_{23} u_2(x))(\partial_{23} u_3(x))$$

Respectively for the best response dynamics:

$$\det(J_{BR}) = -1 - \frac{\partial_{13} u_1(x)}{\partial_{11} u_1(x)} \frac{\partial_{12} u_2(x)}{\partial_{22} u_2(x)} \frac{\partial_{23} u_3(x)}{\partial_{33} u_3(x)}$$

$$- \frac{\partial_{12} u_1(x)}{\partial_{11} u_1(x)} \frac{\partial_{23} u_2(x)}{\partial_{22} u_2(x)} \frac{\partial_{13} u_3(x)}{\partial_{33} u_3(x)} + \frac{\partial_{13} u_1(x)}{\partial_{11} u_1(x)} \frac{\partial_{13} u_3(x)}{\partial_{33} u_3(x)}$$

$$+ \frac{\partial_{12} u_1(x)}{\partial_{11} u_1(x)} \frac{\partial_{12} u_2(x)}{\partial_{22} u_2(x)} + \frac{\partial_{23} u_2(x)}{\partial_{22} u_2(x)} \frac{\partial_{23} u_3(x)}{\partial_{33} u_3(x)}$$

If we now assume that the system is stable for the adaptive dynamics once more, i.e. $\det(J_{AD}) < 0$ and we divide by $-(\partial_{11}u_1(x))(\partial_{22}u_2(x))(\partial_{33}u_3(x))$ we get the following term:

$$-1 - \frac{\partial_{13}u_1(x)}{\partial_{11}u_1(x)}\frac{\partial_{12}u_2(x)}{\partial_{22}u_2(x)}\frac{\partial_{23}u_3(x)}{\partial_{33}u_3(x)}$$

$$-\frac{\partial_{12}u_1(x)}{\partial_{11}u_1(x)}\frac{\partial_{23}u_2(x)}{\partial_{22}u_2(x)}\frac{\partial_{13}u_3(x)}{\partial_{33}u_3(x)} + \frac{\partial_{13}u_1(x)}{\partial_{11}u_1(x)}\frac{\partial_{13}u_3(x)}{\partial_{33}u_3(x)}$$

$$+\frac{\partial_{12}u_1(x)}{\partial_{11}u_1(x)}\frac{\partial_{12}u_2(x)}{\partial_{22}u_2(x)} + \frac{\partial_{23}u_2(x)}{\partial_{22}u_2(x)}\frac{\partial_{23}u_3(x)}{\partial_{33}u_3(x)} < 0$$

which is exactly the second stability condition for the best response dynamics. The direction of the inequality sign remains as it was before due to the concavity of u_i, which means that the second derivatives (and their product) are negative.

The by far most complex terms occur in the third condition. For the best response dynamic one gets the following condition:

$$-9 + 3\frac{\partial_{12}u_1(x)}{\partial_{11}u_1(x)}\frac{\partial_{12}u_2(x)}{\partial_{22}u_2(x)} + 3\frac{\partial_{23}u_2(x)}{\partial_{22}u_2(x)}\frac{\partial_{23}u_3(x)}{\partial_{33}u_3(x)}$$

$$+3\frac{\partial_{13}u_1(x)}{\partial_{11}u_1(x)}\frac{\partial_{13}u_3(x)}{\partial_{33}u_3(x)} < -1 - \frac{\partial_{13}u_1(x)}{\partial_{11}u_1(x)}\frac{\partial_{12}u_2(x)}{\partial_{22}u_2(x)}\frac{\partial_{23}u_3(x)}{\partial_{33}u_3(x)}$$

$$-\frac{\partial_{12}u_1(x)}{\partial_{11}u_1(x)}\frac{\partial_{23}u_2(x)}{\partial_{22}u_2(x)}\frac{\partial_{13}u_3(x)}{\partial_{33}u_3(x)} + \frac{\partial_{13}u_1(x)}{\partial_{11}u_1(x)}\frac{\partial_{13}u_3(x)}{\partial_{33}u_3(x)}$$

$$+\frac{\partial_{12}u_1(x)}{\partial_{11}u_1(x)}\frac{\partial_{12}u_2(x)}{\partial_{22}u_2(x)} + \frac{\partial_{23}u_2(x)}{\partial_{22}u_2(x)}\frac{\partial_{23}u_3(x)}{\partial_{33}u_3(x)}$$

Multiplying by $-(\partial_{11}u_1(x))(\partial_{22}u_2(x))(\partial_{33}u_3(x))$ and rearranging the terms delivers the following condition:

$$-8(\partial_{11}u_1(x))(\partial_{22}u_2(x))(\partial_{33}u_3(x)) + 4(\partial_{12}u_1(x))(\partial_{12}u_2(x))(\partial_{33}u_3(x)) +$$

$$+4(\partial_{23}u_2(x))(\partial_{23}u_3(x))(\partial_{11}u_1(x)) + 4(\partial_{13}u_1(x))(\partial_{13}u_3(x))(\partial_{22}u_2(x)) -$$

$$-(\partial_{13}u_1(x))(\partial_{12}u_2(x))(\partial_{23}u_3(x)) - (\partial_{12}u_1(x))(\partial_{23}u_2(x))(\partial_{13}u_3(x)) < 0$$

For the adaptive dynamics systems this condition gets even more complicated:

$$(\partial_{11}u_1(x) + \partial_{22}u_2(x) + \partial_{33}u_3(x))(\partial_{11}u_1(x)\partial_{22}u_2(x) + \partial_{11}u_1(x)\partial_{33}u_3(x))$$

$$\partial_{22}u_2(x)\partial_{33}u_3(x) - \partial_{12}u_1(x)\partial_{12}u_2(x) - \partial_{13}u_1(x)\partial_{13}u_3(x) - \partial_{23}u_2(x)\partial_{23}u_3(x)) <$$

$$< (\partial_{11}u_1(x))(\partial_{22}u_2(x))(\partial_{33}u_3(x)) + (\partial_{13}u_1(x))(\partial_{12}u_2(x))(\partial_{23}u_3(x)) +$$

$$+(\partial_{12}u_1(x))(\partial_{23}u_2(x))(\partial_{13}u_3(x)) - (\partial_{13}u_1(x))(\partial_{22}u_2(x))(\partial_{13}u_3(x)) -$$

$$-(\partial_{12}u_1(x))(\partial_{12}u_2(x))(\partial_{33}u_3(x)) - (\partial_{11}u_1(x))(\partial_{23}u_2(x))(\partial_{23}u_3(x))$$

If one multiplies the factors on the left side of the inequality (i.e. the trace and the sum of minors) one gets a sum with sixteen different terms. Some of them cancel out with the terms on the right side (the determinant). In the end one gets the following, rather complex inequality:

$$\partial_{11}u_1(x)[\partial_{11}u_1(x)\partial_{22}u_2(x) + \partial_{11}u_1(x)\partial_{33}u_3(x) + \partial_{13}u_1(x)\partial_{12}u_2(x) +$$

$$+\partial_{13}u_1(x)\partial_{13}u_3(x) + \partial_{23}u_2(x)\partial_{23}u_3(x)] + \partial_{22}u_2(x)[\partial_{11}u_1(x)\partial_{22}u_2(x) +$$

$$+\partial_{22}u_2(x)\partial_{33}u_3(x) + \partial_{13}u_1(x)\partial_{12}u_2(x) + \partial_{13}u_1(x)\partial_{13}u_3(x) + \partial_{23}u_2(x)\partial_{23}u_3(x)]$$

$$\partial_{33}u_3(x)[\partial_{11}u_1(x)\partial_{33}u_3(x) + \partial_{22}u_2(x)\partial_{33}u_3(x) + \partial_{13}u_1(x)\partial_{12}u_2(x) +$$

$$+\partial_{13}u_1(x)\partial_{13}u_3(x) + \partial_{23}u_2(x)\partial_{23}u_3(x)] + 2\partial_{11}u_1(x)\partial_{22}u_2(x)\partial_{33}u_3(x)$$

$$-\partial_{13}u_1(x)\partial_{12}u_2(x)\partial_{23}u_3(x) - \partial_{12}u_1(x))(\partial_{23}u_2(x))(\partial_{13}u_3(x) < 0$$

Rearranging the terms of the two inequalities does not lead to obvious equivalence, as it was the case with the trace and the determinant. This does not necessarily mean that in the case of stability of one system the other one might be unstable. But if this case should occur, what would that mean for the eigenvalues?

If we assume that the Jacobians are diagonalizable over $\mathbb{C}$, the 3×3 diagonal matrix looks like this

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

We now assume that this matrix has a negative trace and a negative determinant, but does not fulfill the third condition of the Routh Hurwitz criterion (we assume equality). The results concerning the eigenvalues are summarized in the following lemma.

Lemma 1 Given a diagonalized 3×3 matrix D of an arbitrary matrix A with eigenvalues λ_1, λ_2 and λ_3. The following two statements are equivalent:

(i) $\mathrm{tr}(D) < 0$, $\det(D) < 0$ and $\mathrm{tr}(D)\mathrm{M}(D) = \det(D)$, where M(D) is the sum of 2×2 principal minors defined as above
(ii) D has one real eigenvalue which is negative and two purely imaginary eigenvalues.

Proof:
(i) $\Rightarrow$ (ii):
If one considers a third order characteristic polynomial for a 3×3 matrix two cases can occur: the three zeros of the polynomial are all real or one zero is real and the other two are complex.

Case 1: $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$

In this case the following results hold

$$\mathrm{tr}(D) = \lambda_1 + \lambda_2 + \lambda_3 < 0$$

$$\det(D) = \lambda_1 \lambda_2 \lambda_3 < 0$$

$$\mathrm{tr}(D)\mathrm{M}(D) = (\lambda_1 + \lambda_2 + \lambda_3)(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) = \lambda_1\lambda_2\lambda_3 = \det(D)$$

If one multiplies the factors on the left side of the last equation and divide by $\lambda_1\lambda_2\lambda_3$ (assuming $\lambda_1\lambda_2\lambda_3 \neq 0$) one gets the following result

$$\frac{\lambda_1}{\lambda_2} + \frac{\lambda_1}{\lambda_3} + \frac{\lambda_2}{\lambda_1} + \frac{\lambda_2}{\lambda_3} + \frac{\lambda_3}{\lambda_1} + \frac{\lambda_3}{\lambda_2} + 2 = 0$$

Since the determinant is negative, only two cases can occur:

1. all the eigenvalues are negative
2. one eigenvalue is negative and the other two are positive

If we assume three negative eigenvalues, all the fractions were positive and the equation could therefore not hold. Hence two eigenvalues are positive and one is negative: wlog $\lambda_1, \lambda_2 > 0$ and $\lambda_3 < 0$. The negative trace yields

$$\lambda_3 < -\lambda_1 - \lambda_2$$

Inserting this into the equation we get

$$0 = \frac{\lambda_1}{\lambda_2} + \frac{\lambda_1}{\lambda_3} + \frac{\lambda_2}{\lambda_1} + \frac{\lambda_2}{\lambda_3} + \frac{\lambda_3}{\lambda_1} + \frac{\lambda_3}{\lambda_2} + 2 <$$

$$< \frac{\lambda_1}{\lambda_2} + \frac{\lambda_1}{\lambda_3} + \frac{\lambda_2}{\lambda_1} + \frac{\lambda_2}{\lambda_3} + \frac{-\lambda_1 - \lambda_2}{\lambda_1} + \frac{-\lambda_1 - \lambda_2}{\lambda_2} + 2 =$$

$$= \frac{\lambda_1 - \lambda_1 - \lambda_2}{\lambda_2} + \frac{\lambda_1}{\lambda_3} + \frac{\lambda_2 - \lambda_1 - \lambda_2}{\lambda_1} + \frac{\lambda_2}{\lambda_3} + 2 =$$

$$= \underbrace{\frac{\lambda_1}{\lambda_3}}_{<0} + \underbrace{\frac{\lambda_2}{\lambda_3}}_{<0} < 0$$

This leads to a contradiction, which means that three real eigenvalues cannot occur under these conditions.

Case 2: $\lambda_1, \lambda_2 \in \mathbb{C}\backslash\mathbb{R}$ and $\lambda_3 \in \mathbb{R}$

If this is the case, the two complex eigenvalues are complex conjugates. Then the eigenvalues take the following shape:

$$\lambda_1 = a + bi$$
$$\lambda_2 = a - bi$$
$$\lambda_3 = c$$

with $a, b, c \in \mathbb{R}$. Now using the Routh Hurwitz criterion yields:

$$\text{tr}(D) = 2a + c < 0$$

$$\det(D) = a^2c + b^2c < 0$$

$$\text{tr}(D)\text{M}(D) = (2a + c)(a^2 + b^2 + 2ac) = a^2c + b^2c = \det(D)$$

After rearranging the last equation we get:

$$2a^3 + a^2c + 2ab^2 + b^2c + 4a^2c + 2ac^2 = a^2c + b^2c$$

$$2a^3 + 2ab^2 + 4a^2c + 2ac^2 = 0$$

Here one must again distinguish between two possible cases:

$a \neq 0$: In this case we can divide by $2a$ and get

$$a^2 + b^2 + c^2 + 2ac = 0$$

This can be rewritten as

$$(a + c)^2 + b^2 = 0$$

which is a contradiction since $a, b \neq 0$.

$a = 0$: In this case we have

$$\operatorname{tr}(D) = c < 0$$

$$\det(D) = b^2 c < 0$$

$$\mathrm{M}(D) = b^2$$

This yields for the third condition of the Routh Hurwitz criterion

$$b^2 c = b^2 c$$

$$0 = 0$$

Hence if a 3×3 matrix fulfills the first two conditions but only holds for equality in the third condition, one eigenvalue is real (which is necessarily negative) and the other two are complex with vanishing real part.

(ii) $\Rightarrow$ (i):
We now assume that the three eigenvalues have the form $\lambda_1 = r$, $\lambda_2 = si$ and $\lambda_3 = -si$ with $r, s \in \mathbb{R}$ and $r < 0$. Then we have

$$\operatorname{tr}(D) = r < 0$$

$$\det(D) = rs^2 < 0$$

$$\operatorname{tr}(D)\mathrm{M}(D) = r(rsi - rsi + s^2) = rs^2 = \det(D)$$

And therefore equivalence holds. □

This is a typical case of a *Hopf bifurcation*. Such a bifurcation is characterized by the fact that by variation of a parameter a dynamical system loses its stability: a pair of complex conjugate eigenvalues (with negative real parts) of the linearization (i.e. the Jacobian matrix) around the fixed point of the dynamical system crosses the imaginary axis in the complex plane.

Hence at the bifurcation point the eigenvalues are purely imaginary.

So one could examine the stability behaviour of games where the eigenvalues of the Jacobians are such limited cases. This will be done in the next example.

Example 1 Given a game for three players with continuous strategy sets $S_1 = S_2 = S_3 = \mathbb{R}$. The payoff functions $u_i : \mathbb{R}^3 \to \mathbb{R}$ are defined by

$$\begin{aligned} u_1(x_1, x_2, x_3) &= -x_1^2 - 6x_1x_3 \\ u_2(x_1, x_2, x_3) &= -\frac{x_2^2}{2} + x_1x_2 + \frac{x_2x_3}{2} \\ u_3(x_1, x_2, x_3) &= -x_3^2 + 4x_2x_3 \end{aligned}$$

The Jacobian for the best response dynamics of this game has the following form

$$J_{BR} = \begin{pmatrix} -1 & 0 & -3 \\ 1 & -1 & \frac{1}{2} \\ 0 & 2 & -1 \end{pmatrix}$$

This matrix fulfills the first two conditions of the Routh Hurwitz criterion as

$$\mathrm{tr}(J_{BR}) = -3 < 0$$

$$\det(J_{BR}) = -6 < 0$$

But for the third condition only equality holds:

$$\mathrm{tr}(J_{BR})\mathrm{M}(J_{BR}) = (-3)2 = -6 = \det(J_{BR}) \qquad (*)$$

For the respective Jacobian matrix of the adaptive dynamics we get:

$$J_{AD} = \begin{pmatrix} -2 & 0 & -6 \\ 1 & -1 & \frac{1}{2} \\ 0 & 4 & -2 \end{pmatrix}$$

However, in this case the Routh Hurwitz criterion yields the following:

$$\text{tr}(J_{AD}) = -5 < 0$$

$$\det(J_{AD}) = -24 < 0$$

$$\text{tr}(J_{AD})\text{M}(J_{AD}) = (-5)6 = -30 < -24 = \det(J_{AD})$$

So we have found a three player game with concave payoff functions u_i which is stable for adaptive dynamics. But in the best response dynamics case one cannot decide whether the system is stable or not; however, small perturbations in the system destroy the equality in (*): suppose a small $\epsilon > 0$ and consider the following Jacobian of the best-response dynamics

$$J_{BR} = \begin{pmatrix} -1 & 0 & -3 \\ 1 & -1 & \frac{1}{2} + \epsilon \\ 0 & 2 & -1 \end{pmatrix}$$

we get the following results:

$$\text{tr}(J_{BR}) = -3 < 0$$

$$\det(J_{BR}) = -6 + 2\epsilon < 0$$

$$\text{tr}(J_{BR})\text{M}(J_{BR}) = (-3)(2 - 2\epsilon) = -6 + 6\epsilon > -6 + 2\epsilon = \det(J_{BR})$$

Hence, the system is unstable for the best response dynamics. In the adaptive dynamics case the system stays stable under small perturbations:

$$J_{AD} = \begin{pmatrix} -2 & 0 & -6 \\ 1 & -1 & \frac{1}{2} + \epsilon \\ 0 & 4 & -2 \end{pmatrix}$$

However, in this case the Routh Hurwitz criterion yields the following:

$$\text{tr}(J_{AD}) = -5 < 0$$

$$\det(J_{AD}) = -24 + 8\epsilon < 0$$

$$\mathrm{tr}(J_{AD})\mathrm{M}(J_{AD}) = (-5)(6-4\epsilon) = -30 + 24\epsilon < -24 + 8\epsilon = \det(J_{AD})$$

and therefore, for small $\epsilon > 0$ the system stays stable whereas in the best response dynamics case the system is unstable. Hence, equivalence like in the two player case cannot hold.

One can also construct examples where the opposite is true, meaning games with Nash equilibria that are stable for the best response dynamics but unstable for the adaptive dynamics:

Example 2 Given a game for three players with continuous strategy sets $S_1 = S_2 = S_3 = \mathbb{R}$. The payoff functions $u_i : \mathbb{R}^3 \to \mathbb{R}$ are defined by

$$\begin{aligned} u_1(x_1, x_2, x_3) &= -\frac{x_1^2}{2} - \frac{x_1 x_3}{2} \\ u_2(x_1, x_2, x_3) &= -\frac{x_2^2}{2} + 6x_1x_2 + 2x_2x_3 \\ u_3(x_1, x_2, x_3) &= -x_3^2 + 2x_2x_3 \end{aligned}$$

The Jacobian for the best response dynamics looks as follows

$$J_{BR} = \begin{pmatrix} -1 & 0 & -\frac{1}{2} \\ 6 & -1 & 2 \\ 0 & 1 & -1 \end{pmatrix}$$

which yields the following

$$\mathrm{tr}(J_{BR}) = -3 < 0$$

$$\det(J_{BR}) = -2 < 0$$

$$\mathrm{tr}(J_{BR})\mathrm{M}(J_{BR}) = (-3)1 < -2 = \det(J_{BR})$$

Therefore, the best response dynamics system is stable. However, for the adaptive dynamics system we get

$$J_{AD} = \begin{pmatrix} -1 & 0 & -\frac{1}{2} \\ 6 & -1 & 2 \\ 0 & 2 & -2 \end{pmatrix}$$

and the Routh-Hurwitz criteria yield:

$$\text{tr}(J_{AD}) = -4 < 0$$

$$\det(J_{AD}) = -4 < 0$$

$$\text{tr}(J_{AD})\text{M}(J_{AD}) = (-4)1 = -4 = \det(J_{AD}) \qquad (**)$$

which means that one cannot say whether the system is stable or not. But just like above one can show that small perturbations can destroy the stability in (**) and the system becomes unstable. Hence, equivalence cannot hold in the case of three players.

2.3 Stable equilibria and Cournot-tâtonnement

At last we look at another approach for stable equilibria and best response dynamics. Hervé Moulin (whose definitions and theorems are used in this chapter, see Moulin 1982) among others examined the Cournot tâtonnement process, which will be illustrated via an example later on. The Cournot tâtonnement process explores the dynamic consequence of the myopic, perfect competition-like behavioural assumption that each player maximizes his or her pay-offs by taking the strategies of the other players as fixed. Although it cannot be justified by rationality argument (since the myopic assumption that each player sticks to his or her strategy is constantly violated) it has a transparent descriptive power and allows us to distinguish stable resp. unstable Nash equilibria. (Moulin 1982).
We will start the discussion with an example:

Example 3: (Stability in a Cournot quantity setting duopoly)
The two players supply respectively the quantities x_1 and x_2 of the same commodity, for which the price is then settled as

$$p(\bar{x}) = 1 - \bar{x}$$
$$\text{where } \bar{x} = x_1 + x_2$$

We will consider two distinct assumptions on the cost function:

- *constant returns to scale* (CRS): the cost of producing y units of the commodity is $\frac{1}{2}y$ for both players.
- *increasing returns to scale* (IRS): the cost of producing y units is $\frac{1}{2}y - \frac{3}{4}y^2$ for both players.

Finally each player's maximal production capacity is $\frac{1}{2}$ (so that p and the costs are never negative).
The CRS-game is then:

$$S_1 = S_2 = [0, \tfrac{1}{2}]$$
$$u_i(x_1, x_2) = x_i(1 - \bar{x}) - \tfrac{1}{2}x_i \ , \ i = 1, 2$$

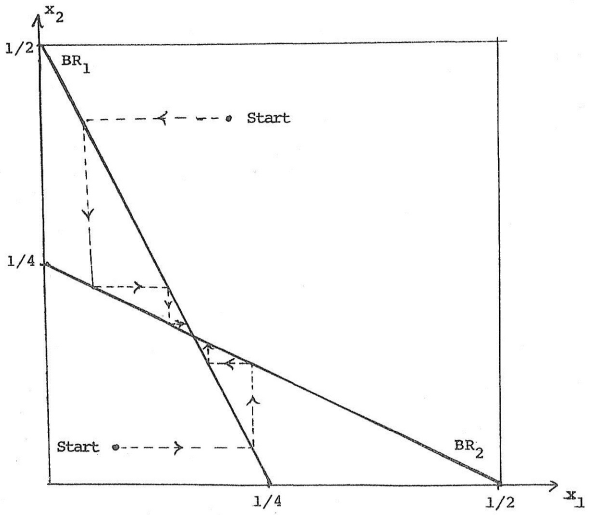

Figure 1: CRS-game

The best reply of player i to strategy x_j of player j is easily computed (as u_i is concave with respect to x_i).

$$BR_i = \{x_i = \alpha(x_j) | 0 \leq x_j \leq \tfrac{1}{2}\} \text{ with } \alpha(y) = \tfrac{1}{4} - \tfrac{1}{2}y$$

The unique Nash equilibrium is:

$$NE = BR_1 \cap BR_2 = \{(\frac{1}{6}, \frac{1}{6})\}$$

Starting from an (x_1^0, x_2^0) the Cournot tâtonnement goes by each player alternating picking a best reply strategy to the current strategy of the opponent:

$$(x_1^0, x_2^0) \to (x_1^1, x_2^0) = (\alpha(x_2^0), x_2^0) \in BR_1 \to (x_1^1, x_2^1) = (x_1^1, \alpha(x_1^1)) \in BR_2 \to ...$$

$$... \to (x_1^t, x_2^{t-1}) = (\alpha(x_2^{t-1}), x_2^{t-1}) \in BR_1 \to (x_1^t, x_2^t) = (x_1^t, \alpha(x_1^t)) \in BR_2 \to ...$$

So from any starting point (x_1^0, x_2^0) the sequences (x_1^t, x_2^t) as well as (x_1^t, x_2^{t-1}) converge to the Nash equilibrium $(\frac{1}{6}, \frac{1}{6})$. Hence $(\frac{1}{6}, \frac{1}{6})$ is a stable Nash equilibrium.

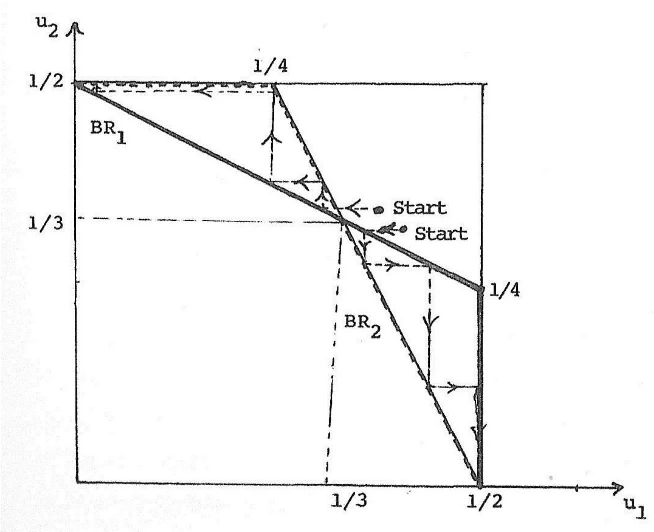

Figure 2: IRS-game

We will now consider the IRS-game, which has the following conditions. For the strategy sets $S_1 = S_2 = [0, 1]$ we have

$$u_i(x_1, x_2) = x_i(1 - \bar{x}) - (\tfrac{1}{2}x_i - \tfrac{3}{4}x_i^2), \;\; i = 1, 2$$

Again u_i is concave with respect to x_i and the best reply curves are:

$$BR_i = \{x_i = \beta(x_j) | 0 \leq x_j \leq \tfrac{1}{2}\} \;\; \text{where} \;\; \beta(y) = \begin{cases} \frac{1}{2} & \text{if } 0 \leq y \leq \frac{1}{4} \\ 1 - 2y & \text{if } \frac{1}{4} \leq y \leq \frac{1}{2} \end{cases}$$

Now we have three Nash equilibria:

$$NE = BR_1 \cap BR_2 = \{(\frac{1}{3}, \frac{1}{3}), (\frac{1}{2}, 0), (0, \frac{1}{2})\}$$

If one starts from any point $x^0 \neq (\frac{1}{3}, \frac{1}{3})$ the sequence always converges towards $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$. This holds even when x^0 is arbitrarily close to, but different from $(\frac{1}{3}, \frac{1}{3})$. In this case $(\frac{1}{3}, \frac{1}{3})$ is an unstable Nash equilibrium whereas $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ are both (locally) stable. (Moulin 1982)

After this motivating example we want to give a definition for Cournot-tâtonnement and stable equilibria. In n-person games the Cournot-tâtonnement can be given several definitions: the players can adjust their stratagies successively (in which their ordering does matter) or simultaneously. We will use the following definition:

Definition 3: Let S_i be endowed with some topology for all $i = 1, ..., n$. Consider a game with strategy sets S_i and payoff functions u_i. We assume that every player has a unique best reply strategy to any fixed strategies by the other players:

For all $i \in \{1, ..., n\}$ and all $x_j \in S_j$ there is a unique $r_i(x_{-i}) \in S_i$ such that $r_i(x_{-i}) \in BR_i(x_{-i})$. (1)

To any $x^0 \in S := \prod_{i=1}^{n} S_i$ we associate the (simultaneous) *Cournot-tâtonnement* starting at x^0, namely the following sequence $x^0, x^1, ..., x^t, ...$ of S:

$$x_i^t = r_i(x_{-i}^{t-1}),\ i = 1, ..., n,\ t = 1, 2, ... \tag{2}$$

We say that a Nash equilibrium x^* is (globally) stable for a game if for any position $x^0 \in S$ the Cournot-tâtonnement starting at x^0 converges to x^*.

Notice that a stable Nash equilibrium necessarily is the unique Nash equilibrium of a game; for if the initial position is a Nash equilibrium the Cournot-tâtonnement is a constant sequence.
Sufficient conditions for a Nash equilibrium to be globally stable are hard to obtain and can be quite restrictive. However if one weakens the condition by requiring that the Cournot-tâtonnement only starts near x^* we are able to characterize almost completely the (locally) stable Nash equilibrium. (Moulin 1982)

Definition 4: Consider a game with strategy sets S_i, payoff functions u_i and unique best reply strategies to any fixed strategies by the other players. We say that a Nash equilibrium x^* is *locally stable* for this game if there existes for all $i = 1, ..., n$ a neighbourhood V_i of x_i such that assumption (1) holds on V_i (with $i = 1, \ldots n$) and x^* is stable for a restricted game with strategy sets V_i.

To derive a computational characterization of local stability, one assumes that for all $i \in I := \{1, 2, \ldots, n\}$, S_i is a subset of an euclidian space E_i and we fix a Nash equilibrium x^* such that x_i^* is an interior point of S_i, all $i \in I$. We assume moreover that the payoff functions u_i are twice continuously differentiable in a neighbourhood of x_i and that the second derivative $\frac{\partial^2 u_i}{\partial x_i^2}$ is a negative definite operator at x^* (hence (1) holds in a suitable neighbourhood of x^*).

We define a linear operator T from $E_I = \prod\limits_{i \in I} E_i$ into itself:
for all $e \in E_I$

$$T_i(e) = \sum_{j \in I \setminus \{i\}} (\frac{\partial^2 u_i}{\partial x_i^2})^{-1} (\frac{\partial^2 u_i}{\partial x_i \partial x_j})(e_j) \tag{3}$$

where all the above derivatives are taken at x^*. (Moulin 1982)
If one considers $T_i(e)$ as a matrix it has a strong similarity to the Jacobian J_{BR} of the best response dynamics. The only difference are the entries in the main diagonal: for the Cournot-tâtonnement the entries are zeros, whereas for the best response dynamics all the entries are (-1). One can show that from stability for Cournot-tâtonnement it follows that the best response dynamics is stable.
With this repertoire we can formulate the following theorem:

Theorem 1 (Moulin)
Suppose that the modulus of all eigenvalues of T is strictly less than 1. Then x^* is a locally stable Nash equilibrium.
Suppose that x^* is a locally stable Nash equilibrium. Then the modulus of all eigenvalues of T is less than or equal to 1.

Proof: A proof for this theorem can be found in (Moulin 1982).

More useful, however, is the following corollary of this theorem, which gives a good characterization for local stability in the case $n = 2$.

Corollary 1:
Suppose $n = 2$ and S_1, S_2 are one dimensional. Let x^* be a Nash equilibrium of a game with strategy sets S_1, S_2 and payoff functions u_1, u_2 such that:

- x_i^* is an interior point of S_i
- u_i is C^2-differentiable in a neighbourhood of x^*
- $\frac{\partial^2 u_i}{\partial x_i^2}(x^*) < 0$

Then we have:

$$\left|\frac{\partial^2 u_1}{\partial x_1 \partial x_2} \frac{\partial^2 u_2}{\partial x_1 \partial x_2}\right| < \left|\frac{\partial^2 u_1}{\partial x_1^2} \frac{\partial^2 u_2}{\partial x_2^2}\right| \Rightarrow x^* \text{ is locally stable.} \tag{4}$$

$$\left|\frac{\partial^2 u_1}{\partial x_1 \partial x_2} \frac{\partial^2 u_2}{\partial x_1 \partial x_2}\right| > \left|\frac{\partial^2 u_1}{\partial x_1^2} \frac{\partial^2 u_2}{\partial x_2^2}\right| \Rightarrow x^* \text{ is not locally stable.} \tag{5}$$

where all these derivatives are taken at x^*.

This stability condition is similar to the condition as in the continuous case for adaptive dynamics (which can be rewritten as the stability condition for the best response dynamics). The difference lies in the absolute value: whereas the stability conditions for best response dynamics and adaptive dynamics do not require the absolute values, this is the case for the corollary.
Under the assumption of the corollary, the best reply sets BR_i are two C^1 curves that intersect at x^*. The inequalities in (4) and (5) simply compare the modulus of the slopes $s_i = \frac{\partial_{1i} u_i}{\partial_{i2} u_i}$ to BR_i, $i = 1, 2$.

$$|s_1| > |s_2| \Rightarrow x^* \text{ is locally stable.} \tag{6}$$

$$|s_1| < |s_2| \Rightarrow x^* \text{ is not locally stable.} \tag{7}$$

Local stability for Cournot-tâtonnement is also related to another concept. One says that a game is locally dominance-solvable at x^* if there exists a rectangular neighbourhood V_N of x^* such that in the restriction of the game to V_N, the successive elimination of dominated strategies shrinks V_N to x^* in the limit. Then under the assumption of Theorem 1, one can prove that:

- If the modulus of all eigenvalues of T is strictly less than 1, then the game is locally dominance-solvable at x^*.
- If the game is locally dominance-solvable at x^* then the modulus of all eigenvalues of T is less than or equal to 1. (Moulin 1982)

Moulin also showed connections between dominance solvabilty and nice behaviour of best response dynamics.

3 Symmetric Games and Continuously Stable Strategies

3.1 Continuously Stable Strategies

The concept of *continuously stable strategies* (CSS) was introduced by Eshel in the early 1980s and developed as a means to predict the long-run behaviour of individuals in a single-species system without an explicit description of how this system evolves when individual fitness is modelled by payoffs in a symmetric game with a continuous set S of pure strategies. (Cressman 2009) The basis of the term is the concept of *evolutionarily stable strategies* (ESS) which were introduced for finite games by Maynard Smith and Price (1973). The obvious extension to games with continuous strategy sets is not good enough. For some but not necessarily all ESSs of the model, if a large enough majority of the population chooses a strategy close enough to the ESS, then only those mutant strategies which are even closer to the ESS will be selectively advantageous. Those kind of ESSs are called CSS. (Eshel 1983) This chapter follows the definition and examples from Eshel.

We now consider a large population playing different strategies. If some small part of the population doesn't play a Nash equilibrium strategy and plays a dissident strategy instead, they are going to be penalized. The further question is whether these dissident strategies might still invade the majority strategy, because one cannot assume that every nonstrict Nash equilibrium is proof against invasion. This yields the following definition:

Definition 1 Given a game with payoff function $u : S^2 \to \mathbb{R}$ with $S \subseteq \mathbb{R}^n$. A strategy x is called an *evolutionarily stable strategy* (ESS) if for any other strategy $y \neq x$ either one of the following two conditions is fulfilled:

$$u(x, x) > u(y, x)$$

or

$$u(x,x) = u(y,x) \text{ and } u(x,y) > u(y,y)$$

The first condition is called a *strict Nash equilibrium.*

One might ask another question. If a large enough majority of the population prefers a strategy y which is sufficiently close to x (for example $|x-y| < \delta$), will it be advantageous for each individual in this population to choose a strategy closer to x, rather than further apart from x? This might not automatically be the case, so if an ESS fulfills this condition we have a more specific class of ESS. We therefore define

Definition 2: An ESS x is said to be a *continuously stable strategy* if there is a value $\varepsilon > 0$ such that for any strategy y in an ε vicinity of x there is a positive value $\delta > 0$ such that for any strategy z at a δ vicinity of y, $u(z,y) > u(y,y)$ if and only if $|z-x| < |y-x|$.

This condition - stronger than ESS - guarantees a positive answer to the question above. However, it is not easy to work with it. In the one-dimensional case (i.e. intervals $S \subseteq \mathbb{R}$) there are equivalent conditions for CSS which are easier to handle. We assume that the payoff function u is continuous and has all second order derivatives. An immediate necessary condition for a strategy $\hat{x}$ being an ESS is

$$\frac{\partial}{\partial x} u(x,\hat{x})\Big|_{x=\hat{x}} = 0 \tag{1}$$

and

$$\frac{\partial^2}{\partial x^2} u(x,\hat{x})\Big|_{x=\hat{x}} \leq 0 \tag{2}$$

Eshel (Eshel 1983) proved the following theorem which gives equivalent conditions for an ESS being a CSS:

Theorem 1 (Eshel)
(i) A necessary condition for an ESS $\hat{x}$ being a CSS is that at the point $x = y = \hat{x}$

$$\frac{\partial^2}{\partial x \partial y} u + \frac{\partial^2}{\partial x^2} u \leq 0 \tag{3}$$

(ii) A sufficient condition for an ESS $\hat{x}$ being a CSS is that both equations (2) and (3) hold as strict inequalities.

Proof: Let $\hat{x}$ be any ESS then we know equation (1) holds. Thus from the continuity of the second order derivative it follows that a positive value $\varepsilon > 0$ exists so that for all θ with $|\theta| < \varepsilon$

$$\frac{\partial u(x, \hat{x} + \theta)}{\partial x}\Big|_{x=\hat{x}+\theta} = (\theta\frac{\partial}{\partial x} + \theta\frac{\partial}{\partial y})(\frac{\partial u(x,y)}{\partial x})\Big|_{x=y=\hat{x}} + \mathrm{o}(\theta) \tag{4}$$

$$= \theta(\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial x \partial y})\Big|_{x=y=\hat{x}} + \mathrm{o}(\theta)$$

(i) Suppose (3) does not hold, namely

$$\frac{\partial^2}{\partial x \partial y}u + \frac{\partial^2}{\partial x^2}u > 0$$

and consider a strategy y with $\hat{x} - \varepsilon < y < \hat{x}$. From equation (4) one then knows that $\frac{\partial u}{\partial x}\Big|_{x=y} < 0$. This means a positive value $\delta = \delta(y) > 0$ exists so that for $y - \delta < z < y$ the inequality $u(z, y) > u(y, y)$ must hold, while for $y < z < y + \delta$ the inequality $u(z, y) < u(y, y)$ holds. Therefore, if the entire population chooses a strategy y, slightly smaller than $\hat{x}$ ($\hat{x}-\varepsilon < y < \hat{x}$) then a slight modification from the population consensus y becomes adavantageous if and only if $z < y$, i.e. further apart from the ESS $\hat{x}$. Hence $\hat{x}$ is not a CSS. (Analogous for $\hat{x} < y < \hat{x} + \varepsilon$.)

(ii) Assume that the equations (2) and (3) hold as strict inequalities. From equation (4) it follows that for $y < \hat{x}$, and x sufficiently small close to y

$$u(x, y) > u(y, y) \text{ if } y < x < \hat{x}$$

and

$$u(x, y) < u(y, y) \text{ if } x < y < \hat{x}$$

while for $y < \hat{x}$ we get

$$u(x, y) > u(y, y) \text{ if } \hat{x} < x < y$$
$$u(x, y) > u(y, y) \text{ if } \hat{x} < y < x$$

Hence $\hat{x}$ is a CSS. □

3.2 Symmetric Games for two players with one-dimensional strategy space

In this section we want to return to the dynamics described in the previous chapter. We now want to examine the case of symmetric games. In this first subsection we will assume once more games with continuous strategy sets $S \subseteq \mathbb{R}$ (i.e. S is an interval in $\mathbb{R}$). The payoff functions u_1 and u_2 are continuous payoff functions who are concave in the i-th argument. Furthermore, we are going to demand symmetric payoff functions, i.e. for the functions $u_1(x_1, x_2)$ and $u_2(x_1, x_2)$ we want that

$$u_1(x_1, x_2) = u_2(x_2, x_1)$$

Considering this condition it suffices to name the payoff functions $u_1(x_1, x_2) = u_2(x_2, x_1) = v(x_1, x_2)$.

We now want to look at the adaptive dynamics and best response dynamics in the symmetric case. We start with the adaptive dynamics:

$$\dot{x_1} = \frac{\partial u_1}{\partial x_1}(x_1, x_2) = \partial_1 u_1(x_1, x_2)$$

$$\dot{x_2} = \frac{\partial u_2}{\partial x_2}(x_1, x_2) = \partial_2 u_2(x_1, x_2)$$

As we have assumed symmetry we get

$$u_1(x_1, x_2) = u_2(x_2, x_1)$$

and therefore for the first derivative in the i-th argument respectively

$$\partial_1 u_1(x_1, x_2) = \partial_2 u_2(x_2, x_1) \qquad (*)$$

In the special case $x_1 = x_2 = x$ this equation yields

$$\dot{x} = \partial_1 u_1(x, x) = \partial_2 u_2(x, x)$$

So obviously the dynamics is invariant on the set $\{x_1 = x_2\}$ and we write from now on

$$\dot{x} = \partial_1 v(x, x)$$

which is the one-dimensional version of the adaptive dynamics for symmetric games. We now begin with the stability analysis in this case. Near a Nash equilibrium $(\hat{x}, \hat{x})$ the Jacobian Matrix for the adaptive dynamics has the following form:

$$J_{AD} = \begin{pmatrix} \partial_{11} v(\hat{x}, \hat{x}) & \partial_{12} v(\hat{x}, \hat{x}) \\ \partial_{12} v(\hat{x}, \hat{x}) & \partial_{11} v(\hat{x}, \hat{x}) \end{pmatrix}$$

The fact that $\partial_{11} v(\hat{x}, \hat{x}) = \partial_{22} v(\hat{x}, \hat{x})$ simply follows from (*). The characteristic equation for this Jacobian looks as follows:

$$\lambda^2 - 2\lambda \partial_{11} v(\hat{x}, \hat{x}) + (\partial_{11} v(\hat{x}, \hat{x}))^2 - (\partial_{12} v(\hat{x}, \hat{x}))^2 = 0$$

This yields the following eigenvalues

$$\lambda_1 = \partial_{11} v + \partial_{12} v \quad (5)$$

$$\lambda_2 = \partial_{11} v - \partial_{12} v \quad (6)$$

In the one-dimensional case on the set $\{(x_1, x_2) \in \mathbb{R}^2 | x_1 = x_2\}$ only the first eigenvalue λ_1 is of relevance, it derives from a straigh-forward linearization of $\dot{x} = \partial_1 v(x, x)$ (using the chain rule). Due to the concavity of v we know that $\partial_{11} v(\hat{x}, \hat{x}) < 0$. The dynamical system for the adaptive dynamics in the two-dimensional case is stable if and only if both eigenvalues have negative real part (since both eigenvalues are real in this case, stability holds if the eigenvalues are negative). This is the case if equations (5) and (6) < 0, i.e.

$$\partial_{11} v(\hat{x}, \hat{x}) + \partial_{12} v(\hat{x}, \hat{x}) < 0 \quad (7)$$

$$\partial_{11}v(\hat{x},\hat{x}) - \partial_{12}v(\hat{x},\hat{x}) < 0 \tag{8}$$

As $\partial_{11}v(\hat{x},\hat{x}) < 0$ this can only hold if

$$|\partial_{11}v| > |\partial_{12}v| \tag{9}$$

If we again consider the one-dimensional case, the stability condition looks very much alike. Then, stability holds if and only if

$$\partial_{11}v + \partial_{12}v < 0 \tag{10}$$

The symmetric version of the best response dynamics is defined in the following way. In the standard case we have

$$\dot{x_1} = BR_1(x_2) - x_1$$

$$\dot{x_2} = BR_2(x_1) - x_2$$

As we are now in the symmetric case the best response function $BR_i(x_{-i})$ is uniquely defined and therefore we get for the special case $x_1 = x_2 = x$

$$\dot{x} = BR(x) - x \tag{11}$$

With an analogous procedure as in the previous chapter we can write the Jacobian matrix for the best response dynamics in the following form:

$$J_{BR} = \begin{pmatrix} -1 & -\frac{\partial_{12}v(\hat{x},\hat{x})}{\partial_{11}v(\hat{x},\hat{x})} \\ -\frac{\partial_{12}v(\hat{x},\hat{x})}{\partial_{11}v(\hat{x},\hat{x})} & -1 \end{pmatrix}$$

for a Nash equilibrium $(\hat{x},\hat{x})$ and a continuously differentiable payoff function $v(x,y)$ that is concave in both arguments.

For stability analysis we will again use the *Routh Hurwitz criterion* in the case $n = 2$. The first condition $\mathrm{tr}(J_{BR}) < 0$ is always fulfilled as $\mathrm{tr}(J_{BR}) = -2$

for every game. The second condition, $\det(J_{BR}) > 0$, is fulfilled if

$$1 - (\frac{\partial_{12}v(\hat{x},\hat{x})}{\partial_{11}v(\hat{x},\hat{x})})^2 > 0$$

Which can be rewritten as

$$(\frac{\partial_{12}v(\hat{x},\hat{x})}{\partial_{11}v(\hat{x},\hat{x})})^2 < 1 \tag{12}$$

This inequality only holds if

$$|\partial_{11}v| > |\partial_{12}v|$$

which is exactly the stability condition (9) for the adaptive dynamics and interestingly for the Cournot-tâtonnement as well. Hence we have shown that the best response dynamics system is stable if and only if this is the case for the adaptive dynamics system. This is hardly surprising as we have shown that this equivalence holds for any games with $n = 2$, not just for symmetric games.

If we however assume the one-dimensional case (11) on the set $\{(x_1, x_2) \in \mathbb{R}^2 | x_1 = x_2\}$, the condition is slightly different. A linearization yields

$$-\frac{\partial_{12}v(\hat{x},\hat{x})}{\partial_{11}v(\hat{x},\hat{x})} - 1 \tag{13}$$

It follows that the one-dimensional system is stable if and only if (13) is < 0, i.e.

$$\partial_{11}v + \partial_{12}v < 0$$

which is exactly stability condition (10).

3.3 CSS for symmetric games with one-dimensional strategy space

The next question we want to examine is the following. If one of these dynamical system possesses an ESS, might this ESS be a CSS? In order to check whether this is the case we have to examine whether the conditions from Theorem 1 hold for the respective dynamics, i.e. whether equations (2) and (3) hold for strict inequalities:

$$\frac{\partial^2}{\partial x_1^2} v(\hat{x}, \hat{x}) < 0 \tag{14}$$

$$\frac{\partial^2}{\partial x_1 \partial x_2} v(\hat{x}, \hat{x}) + \frac{\partial^2}{\partial x_1^2} v(\hat{x}, \hat{x}) < 0 \tag{15}$$

for a strict Nash equilibrium $(\hat{x}, \hat{x})$ (meaning $\hat{x}$ is an ESS).

We will first look at the adaptive dynamics case. Equation (14) is always fulfilled since we have required that $v(x, y)$ is concave in both arguments, therefore the second derivative $\partial_{11} v(\hat{x}, \hat{x}) < 0$.
Let us now consider equation (15). This equation looks exactly like the stability condition (7) for adaptive dynamics. Hence if the adaptive dynamics system is stable, then the ESS $\hat{x}$ for a strict Nash equilibrium $(\hat{x}, \hat{x})$ is automatically a CSS.
Now one could ask whether this is true for the other direction, i.e. if $\hat{x}$ is a CSS, it follows that the adaptive dynamics system is stable.
While (15) guarantees that one eigenvalue (λ_1 in (5)) is lower than zero, this might not hold for the eigenvalue λ_2 in (6). If

$$\partial_{12} v(\hat{x}, \hat{x}) < \partial_{11} v(\hat{x}, \hat{x}) < 0$$

the stability condition (9) does not hold and therefore the dynamical system is not stable. So take for instance the following example:

Example 2 Consider a game with symmetric payoff functions $u_1(x_1, x_2)$, $u_2(x_1, x_2) : \mathbb{R}^2 \to \mathbb{R}$

$$u_1(x_1, x_2) = -x_1^2 - 6x_1x_2$$

$$u_2(x_1, x_2) = -x_2^2 - 6x_1x_2$$

The Jacobian matrix for the adaptive dynamics has the following form

$$J_{AD} = \begin{pmatrix} -2 & -6 \\ -6 & -2 \end{pmatrix}$$

We get the following eigenvalues: $\lambda_1 = -8$, $\lambda_2 = 4$. Hence the system is not stable, but $\frac{\partial^2}{\partial x_1 \partial x_2} v(\hat{x}, \hat{x}) + \frac{\partial^2}{\partial x_1^2} v(\hat{x}, \hat{x}) = -8 < 0$ and $\frac{\partial^2}{\partial x_1^2} v(\hat{x}, \hat{x}) = -2 < 0$, which means that $\hat{x}$ is a CSS.

If, however, one considers the one-dimensional case on $\{(x_1, x_2) \in \mathbb{R}^2 | x_1 = x_2\}$ we get the equivalence, since the stability condition (10) and the CSS-condition (15) are identical. Condition (14) again follows from the concavity of v.

Let us now return to the best response dynamics. We want to know if stability of the dynamical system also implies an ESS $\hat{x}$ being a CSS, as it was the case with the adaptive dynamics.
We again assume that $(\hat{x}, \hat{x})$ is a strict Nash equilibrium for an ESS $\hat{x}$. As in the adaptive dynamics case, condition (14) is always true due to the concavity of $v(x, y)$. The stability condition (12) (resp. the equivalent condition (9)) guarantees that $\frac{\partial^2}{\partial x_1 \partial x_2} v(\hat{x}, \hat{x}) + \frac{\partial^2}{\partial x_1^2} v(\hat{x}, \hat{x}) < 0$ because even in the case of $\frac{\partial^2}{\partial x_1 x_2} v(\hat{x}, \hat{x}) > 0$ the sum is still negative and condition (15) is fulfilled. Hence we again have that stability of the dynamical system implies the CSS condition.

The question whether equivalence holds has a similar answer as in the case of adaptive dynamics. Since both dynamics have the same stability condition

$$|\partial_{11} v| > |\partial_{12} v|$$

the same arguments as above yield the same result as before. In the case that

$$\partial_{12}v(\hat{x},\hat{x}) < \partial_{11}v(\hat{x},\hat{x}) < 0$$

the system is unstable but the CSS condition is fulfilled.

Example 3 Given the same game as in Example 1.

$$u_1(x_1,x_2) = -x_1^2 - 6x_1x_2$$

$$u_2(x_1,x_2) = -x_2^2 - 6x_1x_2$$

The Jacobian matrix for the best response dynamics has the following form:

$$J_{BR} = \begin{pmatrix} -1 & -3 \\ -3 & -1 \end{pmatrix}$$

We get the following eigenvalues: $\lambda_1 = -4$, $\lambda_2 = 2$. Hence the system is not stable, but $\frac{\partial^2}{\partial x_1 \partial x_2}v(\hat{x},\hat{x}) + \frac{\partial^2}{\partial x_1^2}v(\hat{x},\hat{x}) = -8 < 0$ and $\frac{\partial^2}{\partial x_1^2}v(\hat{x},\hat{x}) = -2 < 0$, which means that $\hat{x}$ is a CSS.

For the one-dimensional case on $\{(x_1,x_2) \in \mathbb{R}^2 | x_1 = x_2\}$ the case is simpler. The system is stable if and only if (11) is < 0 which is only a rewritten version of (9). Since the CSS-condition (15) and stability condition (9) are identical and (14) follows from the concavity of v, we get the equivalence: if an ESS $\hat{x}$ is a CSS, the one-dimensional best response dynamics system is stable and vice versa.

3.4 CSS for multi-dimensional symmetric games

The multi-dimensional case is slightly more complex than the one-dimensional. The strategy sets S_i are now convex and compact subsets of $\mathbb{R}^n$. Hence in the case of two players the payoff functions $u_i(x_1,x_2)$ are now functions with $x_1 \in S_1$ and $x_2 \in S_2$ with $S_1, S_2 \subseteq \mathbb{R}^n$. We look at symmetric games, i.e. $S_1 = S_2 = S$ and $u_1(x,y) = u_2(y,x) = v(x,y),\ x,y \in S \subseteq \mathbb{R}^n$.

First, we want to find a multi-dimensional interpretation of the CSS notion. Consider a convex and compact set $S \subseteq \mathbb{R}^n$ containing a neighbourhood of x^* (in particular, x^* is in the interior of S). Following Lessard (1990) and Cressman (2009), the Taylor expansion up to second order of $u(x', x)$ about x^* is now

$$u(x', x) = u(x^*, x^*) + \nabla_1 u(x^*, x^*) \cdot (x' - x^*) + \nabla_2 u(x^*, x^*) \cdot (x - x^*) +$$

$$+\frac{1}{2}[(x' - x^*) \cdot A(x' - x^*) + 2(x' - x^*) \cdot B(x - x^*) + (x - x^*) \cdot C(x - x^*)]$$

where A, B, C are appropriate $n \times n$ matrices of second order partial derivatives evaluated at x^*.
In particular

$$A = \nabla_{11} u$$
$$B = \nabla_{12} u$$

where $\nabla_{11}(\hat{x}, \hat{x})$ and $\nabla_{12}(\hat{x}, \hat{x})$ are $n \times n$-matrices and $\nabla_{11}(\hat{x}, \hat{x})$ is the Hessian matrix. The Hessian is symmetric whereas $\nabla_{12} v$ in general might not be symmetric. Lessard (1990) showed that x^* is a CSS if and only if the matrices A and $A+B$ are negative definite matrices, i.e. $x^t A x < 0$ and $x^t (A+B) x < 0$ respectively for all $x \neq 0$. This characterization is going to be used in this chapter.

We first look for the symmetric version of the adaptive dynamics. The (symmetric version of the) adaptive dynamics is given by:

$$\dot{x} = \nabla_1 v(x, x)$$

where v is the payoff function like in the case with one-dimensional strategy sets. In order to examine the stability of the system we linearize the symmetric version of the system around a Nash equilibrium $\hat{x}$ and get

$$\nabla_{11}v(\hat{x},\hat{x}) + \nabla_{12}v(\hat{x},\hat{x})$$

Let us now suppose that some x is a CSS, then $\nabla_{11}v(\hat{x},\hat{x}) + \nabla_{12}v(\hat{x},\hat{x})$ and $\nabla_{11}v(\hat{x},\hat{x})$ are negative definite. A matrix C is negative definite, if its quadratic form x^tCx is < 0 for all $x \neq 0$. If the quadratic form is negative then all the real parts of the eigenvalues of the Matrix C are negative. Since we know that $\nabla_{11}v(\hat{x},\hat{x}) + \nabla_{12}v(\hat{x},\hat{x})$ and $\nabla_{11}v(\hat{x},\hat{x})$ are negative definite it follows that the real parts of the eigenvalues are negative and therefore the adaptive dynamics is stable.

Now we take a look at the best response dynamics. Under the same assumptions as in the adaptive dynamics case, we get for two players the following system:

$$\dot{x} = BR(x) - x$$

For examining the stability condition, we linearize this equation around a Nash equilibrium $\hat{x}$ and get - analogous to the one-dimensional case - the following:

$$-(\nabla_{11}v(\hat{x}))^{-1} \cdot \nabla_{12}v(\hat{x}) - \mathbb{I}_n$$

where v denotes the payoff function and $\mathbb{I}_n$ the identity matrix.
With an analogous argument as in the adaptive dynamics case one can show that if an ESS x fulfills the CSS-condition then the best response dynamics is stable.

4 Conclusion

The purpose of this thesis was to examine the connection between the *adaptive dynamics* and *best response dynamics* in various circumstances. First we duscussed general n-person games on continuous subsets, in particular convex and compect subsets of $\mathbb{R}^d$, but in this case we concentrated on the special case of $d = 1$ where the continuous strategy sets are intervals in $\mathbb{R}$. It was shown that in the case of two players the stability conditions were equivalent, which means that if a Nash equilibrium is stable for the adaptive dynamics then it is also stable for the best response dynamics. In the case of three players this no longer is true; there exist games where this equivalence is invalid, i.e. games where Nash equilibria are stable for the adaptive dynamics and unstable for the best response dynamics. At the end of Chapter 2 the Cournot-tâtonnement was presented, another approach for stable equilibria and best response dynamics. It was shown that the stability conditions for Nash equilibria by using Cournot-tâtonnement are stronger than the stability conditions for the best response dynamics.
In Chapter 3 the concept of continuously stable strategies was introduced. It was discussed whether there was a connection between an ESS being a CSS and the stability of symmetric games with adaptive dynamics and best response dynamics respectively. In the case of one-dimensional strategy spaces this holds true if one considers the invariant set $\{(x_1, x_2) \in \mathbb{R}^2 | x_1 = x_2\}$. In this case the stability conditions of the adaptive dynamics (or the best response dynamics) and the CSS-conditions are equivalent. We also looked at the case of multi-dimensional strategy spaces; in this case only one direction can be shown: if a strategy x is a CSS then it follows that x is asymptotically stable under both the adaptive dynamics and the best response dynamics.

5 Bibliography

Cressman, Ross and Josef Hofbauer (2003) Measure dynamics on a one-dimensional continuous strategy space: Theoretical foundations for adaptive dynamics. Theoretical Population Biology 67, p. 47-59.

Cressman, Ross (2004) Coevolution, adaptive dynamics, and the replicator equation for a single species with a continuous strategy space. http://webctupdates.wlu.ca/documents/41242/coevolution-adaptive-dynamics-nov04.pdf. Access date April 8, 2013.

Cressman, Ross (2009) Continuously Stable Strategies, Neighborhood Superiority and Two-Player Games with Continuous Strategy Space. International Journal of Game Theory 38/2. p. 221-247

Eshel, Ilan and Uzi Motro (1981) Kin selection and Strong Evolutionary Stability of Mutual Help. Theoretical Population Biology 19/3 p. 420-433

Eshel, Ilan (1983) Evolutionary and Continuous Stability. Journal for Theoretical Biology 103, p. 99-111.

Fudenberg, Drew and Jean Tirole. (1991) Game Theory. MIT Press, Cambridge.

Hofbauer, Josef and Karl Sigmund (1990) Adaptive Dynamics and Evolutionary Stability. Applied Mathematics Letters 3/4. p. 75-79.

Hofbauer, Josef and Karl Sigmund (1998) Evolutionary games and population dynamics. Cambridge University Press, Cambridge.

Hofbauer, Josef and Karl Sigmund (2003) Evolutionary game dynamics. Bulletin of the American mathematical Society 40/4, p. 479-519.

Lessard, Sabin (1990) Evolutionary Stability: One Concept, Several Meanings. Theoretical Population Biology 37/1, p.157-190.

Moulin, Hervé (1982) Game theory for the social sciences. New York University Press, New York and London

6 List of Figures

Curriculum vitae

Personal Data

Name	Johannes Mathä
Date of Birth	January 5, 1987
Nationality	Austria

Education

4 years Volksschule Rohrbach
8 years Bundesrealgymnasium Rohrbach, Matura
2005-2011:
German Philology at University of Vienna
2006-2013:
Mathematics at University of Vienna

Selected Work

2008-2010:
Tutor for Linguistics and Language History
at Institute for German Philology at University of Vienna
since 2008:
Freelancer at Adalbert-Stifter-Institute in Linz