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Complex-Valued Analytic Torsion on Compact Bordisms

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Contents

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Abstract

A compact Riemannian bordism is a compact manifold M of dimension m, with Riemannian metric g, whose boundary ∂M is the disjoint union of two closed submanifolds $\partial_+ M$ and $\partial_- M$, with absolute boundary conditions on $\partial_+ M$ and relative boundary conditions on $\partial_- M$. This thesis is concerned with the complex-valued analytic torsion on compact Riemannian bordisms.

Consider E, a flat complex vector bundle over M, with a Hermitian metric h and denote by $\Omega(M;E)$, the space of E-valued smooth differential forms on M. The Ray–Singer metric $\tau^{\mathsf{RS}}_{E,g,h}$, defined with the use of self-adjoint Laplacians $\Delta_{E,g,h}$, acting on smooth forms satisfying the boundary conditions above, is a Hermitian metric on the determinant line $\det(H(M,\partial_- M;E))$ of the cohomology groups $H(M,\partial_- M;E)$.

Assume E is endowed with a fiber-wise nondegenerate complex symmetric bilinear form b. We denote by $\beta_{E,g,b}$ the nondegenerate bilinear form on $\Omega(M;E)$ determined by g and b. The complex-valued analytic torsion $\tau^{\mathsf{RS}}_{E,g,b}$ considered as a nondegenerate bilinear form on the determinant line was first studied by Burghelea and Haller on closed manifolds in analogy with the Ray–Singer metric. In order to define $\tau^{\mathsf{RS}}_{E,g,b}$ one uses spectral theory of not necessarily self-adjoint Laplacians $\Delta_{E,g,b}$. In few words, one starts by regarding $\Omega_\Delta(M;E)(0)$ the generalized zero-eigenspace of $\Delta_{E,g,b}$, a finite dimensional cochain complex containing smooth forms only, which computes $H(M,\partial_- M;E)$. Then, one defines a nondegenerate bilinear form $\tau_{E,g,b}(0)$ on $\det(H(M,\partial_- M;E))$, by considering the restriction of $\beta_{E,g,b}$ as a nondegenerate bilinear form to $\Omega_\Delta(M;E)(0)$. Thus, $\tau^{\mathsf{RS}}_{E,g,b}$ is defined as the product of $\tau_{E,g,b}(0)$ with the non-zero complex number obtained as ζ-regularized determinant of $\Delta_{E,g,b}$.

The variation of the torsion with respect to smooth changes of the Riemannian metric and the bilinear form is encoded in the anomaly formulas. In order to obtain these formulas, we use the coefficient of the constant term in the heat trace asymptotic expansion for small time, associated to $\Delta_{E,g,b}$. Our method uses the anomaly formulas for the Ray–Singer metric obtained by Brüning and Ma.

CoEuler structures, the dual notion to Euler Structures of Turaev, were used by Burghelea and Haller to discuss the anomaly formulas for the torsion on closed manifolds. We extend the notion of coEuler structures to the situation of compact Riemannian bordisms. The space of coEuler structures is an affine space modeled by the cohomology group $H^{m-1}(M,\partial M;\mathbb{C})$.

Introduction

We denote by $(M, \partial_+M, \partial_-M)$ a compact Riemannian bordism. That is, M is a compact Riemannian manifold of dimension m, with Riemannian metric g, whose boundary ∂M is the disjoint union of two closed submanifolds ∂_+M and ∂_-M. For E a flat complex vector bundle over M, we study generalized Laplacians acting on E-valued smooth differential forms on M satisfying *absolute boundary conditions* on ∂_+M and *relative boundary conditions* on ∂_-M.

In this thesis, we study the *complex-valued* Ray–Singer torsion, or complex-valued analytic torsion, on $(M, \partial_+M, \partial_-M)$. The complex-valued Ray–Singer torsion was introduced and studied on closed manifolds by Burghelea and Haller in analogy to the Ray–Singer metric in [**BH07**], [**BH08**] and [**BH10**]. Our main result, Theorem 5.2.1, provides a variational formula, or anomaly formula, for the logarithmic derivative of the complex-valued Ray–Singer torsion on $(M, \partial_+M, \partial_-M)$ and its proof is based on the work by Brüning and Ma in [**BM06**] for the Ray–Singer metric on manifolds with boundary. As an intermediate step, we obtain anomaly formulas for the Ray–Singer metric on $(M, \partial_+M, \partial_-M)$, which coincide with the corresponding formulas obtained by Brüning and Ma in their recent paper [**BM11**], by different methods.

The Ray–Singer torsion was defined and studied by Ray and Singer in [**RS71**], [**RS73a**] and [**RS73b**], as a ζ-regularized product of all non-zero eigenvalues of a certain self-adjoint Laplacian. Ray and Singer first studied their torsion for unitary flat vector bundles on closed manifolds, by investigating the problem of describing the Franz–Reidemeister torsion (see [**Re**], [**Tu02**] and [**Ni03**]) in analytic terms. In particular, Ray and Singer proved that their torsion does not depend on the Riemannian metric. Later on, in [**BZ92**], Bismut and Zhang studied the analytic torsion for non necessarily unitary flat vector bundles over closed manifolds and they considered it as a Hermitian metric on certain determinant lines.

Let us first give some ingredients to recall the Ray–Singer metric, as it is done in [**BZ92**] on closed manifolds, and also in [**BM06**] and [**BM11**] on manifolds with boundary. One starts by considering *Hermitian Laplacians* $\Delta_{E,g,h}$ on $\Omega(M; E)$, the space of E-valued smooth differential forms on a manifold M, constructed by using a flat connection ∇^E, a Hermitian form h on E, and the Riemannian metric g on M. By imposing absolute (resp. relative) boundary conditions on ∂_+M (resp. ∂_-M) one specifies an elliptic boundary value problem. Boundary ellipticity (with respect to a cone), see for instance [**Gi84**] and [**Gi04**], permits one to extend $\Delta_{E,g,h}$ as a self-adjoint, densely defined and closed unbounded operator in the L^2-norm, see [**Mü78**]. One has a de-Rham–Hodge Theorem for self-adjoint Laplacians on manifolds with boundary: the kernel of $\Delta_{E,g,h}$ is of finite dimension and isomorphic to $H(M, \partial_- M; E)$, the cohomology of M relative to $\partial_- M$ (with local coefficients in E), see [**Mü78**], [**Lü93**], [**BM06**] and [**BM11**]. By means of this isomorphism, a Hermitian metric $\tau_{E,g,h}(0)$ on the determinant line $\det H(M, \partial_- M; E)$ of the cohomology $H(M, \partial_- M; E)$, is obtained. The Ray–Singer metric, denoted by $\tau^{\mathsf{RS}}_{E,g,h}$, is the Hermitian metric on $\det H(M, \partial_- M; E)$ given by

$$\tau^{\mathsf{RS}}_{E,g,h} := \tau_{E,g,h}(0) \cdot \prod_p \left(\det{}'\left(\Delta_{E,g,h,p}\right)\right)^{(-1)^p p},$$

where $\det{}'\left(\Delta_{E,g,h}\right)$ is the ζ-regularized product of all non-zero eigenvalues of the Laplacian, see for instance see [**Se69b**]. The (real-valued) product above computes the absolute value of the Reidemeister torsion, see [**BZ92**]. Moreover, in [**BZ92**] Bismut and Zhang proved that the Ray–Singer metric is a Riemannian invariant in odd dimensions and they computed corresponding anomaly formulas. The Ray–Singer metric on manifolds with boundary has been studied by several authors, see for instance [**RS71**], [**Mü78**], [**Mü93**], [**Lü93**], [**DF00**], [**BM06**], [**BM11**] and references therein. In particular, we are interested in the work of Brüning and Ma in [**BM06**], where they computed the variation of the analytic Ray–Singer torsion, with respect to smooth variations on the underlying Riemannian and Hermitian structures.

Assume now E admits a fiberwise nondegenerate symmetric bilinear form b. The complex-valued Ray–Singer torsion is defined as a bilinear form on $\det H(M, \partial_- M; E)$ and obtained in a very similar way as the Ray–Singer metric. Indeed, in this situation, a generalized Laplacian $\Delta_{E,g,b}$ on $\Omega(M; E)$ is considered, which we call *bilinear Laplacians*. We study these operators under absolute and relative boundary conditions as well as their spectral properties, ellipticity and regularity statements. The operator $\Delta_{E,g,b}$ extends to a, not necessarily self-adjoint, closed unbounded operator in the L^2-norm, it has compact resolvent and discrete spectrum, all its eigenvalues are of finite multiplicity, its (generalized) eigenspaces contain smooth differential forms only. Also, the bilinear form $\beta_{g,b}$ on $\Omega(M; E)$ induced by g and b is nondegenerate and restricts to each of eigenspaces as a nondegenerate bilinear form. In this context, we obtain in Proposition 3.3.11, a Hodge decomposition result for the bilinear Laplacian and in Proposition 3.3.12, we see that generalized 0-eigenspace of $\Delta_{E,g,b}$ is a sub-cochain complex in $\Omega(M; E)$ that computes (without necessarily being isomorphic to) relative cohomology $H(M, \partial_- M; E)$. We follow the approach in [**BH07**], to obtain a nondegenerate bilinear form $\tau_{E,g,b}(0)$ on the determinant line $\det(H(M, \partial_- M; E))$ by looking at the restriction of $\beta_{g,b}$ to the generalized 0-eigenspace of $\Delta_{E,g,b}$ as a nondegenerate bilinear form. The (inverse square of) the complex-valued Ray–Singer torsion for manifolds with boundary is defined by

$$\tau^{\mathsf{RS}}_{E,g,b} := \tau_{E,g,b}(0) \cdot \prod_p \left(\det{}'\left(\Delta_{E,g,b,p}\right)\right)^{(-1)^p p},$$

where the product above is now a non zero complex number with $\det'(\Delta_{E,g,b,p})$ being the ζ-regularized product of all non-zero eigenvalues of $\Delta_{E,g,b,p}$.

Let us also mention certain related work on the complex-valued Ray–Singer torsion. On closed manifolds, see Proposition 10.5 in [**SZ08**], the complex-valued Ray–Singer torsion was compared, up to a phase, with the Ray–Singer metric by Su and Zahng, by conveniently relating the underlying bilinear and Hermitian structures. On closed manifolds of odd dimension, by using the odd signature operator, Braverman and Kappeler defined in [**BK07a**] and [**BK07b**] the *refined analytic torsion*, as a refinement of the Ray-Singer torsion, and they proved that it computes, up to a phase, the Turaev torsion (see [**FT00**] and [**Tu90**]). The refined analytic torsion was also compared with the complex-valued Ray–Singer torsion by Braverman and Kappeler, see Theorem 1.4 in [**BK07c**]. The Ray–Singer analytic torsion has also been studied on the twisted (by an odd degree closed differential form) de-Rham complex by Mathai and Wu in [**MW11**] and Huang extended the refined analytic torsion on the twisted the de-Rham complex, see [**Hu10**]. In [**Su10**], the complex-valued Ray–Singer torsion on a twisted de-Rham complex was defined by Su and also compared with the refined analytic torsion constructed by Huang. In [**Ve09**], Vertman gave a (slightly) different refinement for the analytic torsion as the one in [**BK07a**] and [**BK07b**], in order to study it on manifolds with boundary. Then in [**Su09**], by using techniques from [**SZ08**], [**Ve09**] and [**Mü78**], Su generalized the complex-valued analytic Ray–Singer torsion to the situation in which $\partial_+M \neq \emptyset$ (or $\partial_-M \neq \emptyset$) and he compared it with the refined analytic torsion on manifolds with boundary defined in [**Ve09**].

The variation of the complex-valued Ray–Singer torsion on closed manifolds, with respect to smooth changes of g and b has been computed in [**BH07**]. Burghelea and Haller used their anomaly formulas to obtain a geometric invariant, by introducing appropriate correction terms to the torsion, see Theorem 4.2 in [**BH07**]. Moreover, Burghelea and Haller conjectured that this generalized complex-valued analytic torsion computes the complex-valued Reidemeister torsion including its phase. Theorem 5.10 in [**BH07**] gives a proof of this conjecture in some non-trivial cases by using analytic continuation from known results in [**Ch77**],[**Ch79**], [**Mü78**] and [**BZ92**]. Later on, this conjecture was proved in full generality by Su and Zhang in [**SZ08**].

We are interested in the variation of the complex-valued Ray–Singer torsion on compact bordisms. In odd dimensions and $\partial_+M \neq \emptyset$ (or $\partial_-M \neq \emptyset$), the complex-valued analytic Ray–Singer torsion, does depend *neither* on smooth variations of the Riemannian metric *nor* on smooth variations of the bilinear form, as long as these are compactly supported in the interior of M, see [**Su09**].

Our anomaly formulas are presented in Theorem 5.2.1 and they can also be found in the preprint [**Ma12**]. On the one hand, our formulas generalize the ones obtained by Burghelea and Haller in the closed situation, see [**BH07**]. On the other hand, they generalize those obtained by Su in odd dimensions, see [**Su09**]: they no longer require

g and b to be constant in a neighborhood of the boundary and both kind of boundary conditions are considered on complementary parts of the boundary respectively.

Structure of the thesis. The necessary background, notation, conventions and specific results needed along the thesis are presented in Chapter 1 and Chapter 2.

In Chapter 1, we recall elementary concepts serving as a background for the whole thesis. Since most of these notions are quite general, we do not include their proofs as these can be found elsewhere in the given literature. In Section 1.1, we give basic algebraic notions; in Section 1.2, we recall elements of Riemannian geometry; in Section 1.3, we give well-known facts on operator theory, in particular on unbounded operators and in section 1.4, some notions of analysis on manifolds is presented. The reader not feeling familiar with these subjects might find here some guide to further lecture. Otherwise, the reader is invited to skip this chapter and start with Chapter 2, only.

In Chapter 2, we discuss elliptic boundary value problems. Although these concepts account for much more general kind of boundary value problems, we restrict the presentation to boundary ellipticity (with respect to a cone) for operators of Laplace type under local boundary conditions. We indicate the results we need, whose proofs can be either found in the given literature or they are shortly presented for the sake of completeness. In Section 2.1, generalities on Laplace type operators and boundary operators are recalled. In Section 2.2 the notion of Shapiro–Lopatijnsky condition and of that of ellipticity with respect to a cone are given. From sections 2.3.1 to 2.3.3, we present results on the existence of elliptic estimates, notions of L^2-realizations for an operator of Laplace type (under elliptic boundary conditions) as well as results on the existence for their resolvent.

In Chapter 3, we study spectral properties for the bilinear Laplacian under absolute and relative boundary conditions on a compact Riemannian bordism, by using the material presented in Chapter 2. We start this chapter with a motivation: in Section 3.1, we give well-known Hodge–de-Rham decomposition results for the Hermitian Laplacian. In Section 3.2, we start by defining the generalized operator $\Delta_{E,g,b}$. In Section 3.2.1 we specify our boundary value problem under absolute and relative boundary conditions. In Section 3.2.2, we point out the role of the Hodge $\star$-operator by making Poincaré-Lefschetz duality between absolute and relative boundary conditions explicit. In Section 3.2.3, we give an explicit description of the boundary operators imposing absolute and relative boundary conditions in terms of local computable tensorial objects. In Section 3.3, we use the results from Chapter 2 to derive a Hodge–De-Rham decomposition result for the bilinear Laplacian on compact bordisms, see Proposition 3.3.11 and Proposition 3.3.12.

In Chapter 4, we are interested in the coefficient of the constant term in the heat trace asymptotic expansion:

$$\mathrm{Tr}_{\mathsf{L}^2}(\Psi \exp(t\Delta_{\mathcal{B}})) \sim \sum_{n=0}^{\infty} a_n(\Psi, \Delta_{\mathcal{B}}) t^{(n-m)/2},$$

when $t \to 0$, associated to $\Delta_{\mathcal{B}}$ a Laplace type operator, such as the bilinear or Hermitian Laplacian (under absolute and relative boundary conditions) and certain auxiliary bundle endomorphism Ψ. In Section 4.1.1, we present generalities about these coefficients. In particular, we give fundamental importance to the fact that these coefficients are computed in terms of locally computable endomorphism invariants expressible as polynomial functions in the jets of the symbols of the operators under consideration. In Section 4.1.2, we recall how Weyl's First Theorem of invariant theory, is used in the current setting, to express the coefficients in the asymptotic expansion, as universal polynomial in terms of locally computable geometric invariants. This is Proposition 4.1.5 and it is entirely based on the work by Gilkey in [**Gi84**], [**Gi04**] and references therein. We use Proposition 4.1.5 to prove Theorem 4.4.3, leading to Theorem 5.2.1 later on. Alternatively, the use of invariant theory (i.e., Proposition 4.1.5) in Lemma 4.4.1, can be avoided see Remark 4.4.2. In Section 4.2, we compute the coefficients of the constant terms for heat trace asymptotics for the Hermitian boundary value problem. We use the results of Brüning and Ma in [**BM06**] in the case $\partial_- M = \emptyset$, Poincaré duality and Proposition 4.1.5, to obtain the desired formulas when $\partial_+ M \neq \emptyset$ and $\partial_- M \neq \emptyset$, see Theorem 4.2.7. The formulas in Theorem 4.2.7 were also obtained with different methods by Bruning and Ma in their recent work on the gluing formulas for the Ray–Singer metric [**BM11**]. In Section 4.3, we have Proposition 4.3.3 giving the first key step towards to the computation for the corresponding coefficients in the asymptotic expansion for $\Delta_{E,g,b}$. In few words, Proposition 4.3.3 tells us how, for each point $x \in M$, we are able to construct a complex one-parameter family of bilinear boundary value problems and a real one-paramenter family of Hermitian boundary value problems, which agree in some small neighborhood of x for conveniently well-chosen values of the parameters. Section 4.4 presents Lemma 4.4.1, the second key step towards the computation for the coefficient of the constant term in the asymptotic expansion for $\Delta_{E,g,b}$, which exhibits the holomophic dependance of these coefficients on a complex paramenter. Then, in Theorem 4.4.3, we use these two key steps to compute the desired heat trace asymptotic coefficients.

For the reader's convenience, we sketch the main idea in the proof of Theorem 4.4.3. The heat trace asymptotic coefficients are obtained by integrating traces of endomorphism valued invariants over M. These invariants are in turn polynomials in tensorial objects computable using the local geometry of M only. So, *locally around each point of M, the coefficients of the constant term of the heat trace asymptotic expansion for the bilinear Laplacian are obtained by using the corresponding ones for the Hermitian Laplacian and an argument of analytic continuation.* The main point in this argument is given as follows. We prove that for each point in M, there exist an open neigbourhood U, a symmetric bilinear form $\widetilde{b}$ and a *flat* complex fiberwise defined *anti-linear* involution ν on $E|_U$, with the following feature: for *certain well-chosen* values $z \in \mathbb{C}$, with $|z|$ small enough, the one-parameter family of nondegenerate symmetric bilinear forms $b_z := b + z\widetilde{b}$

can be considered, by means of ν, as a real one-parameter family of Hermitian forms on $E|_U$. Thus, the known results from the Hermitian situation can be used. Finally, since the obtained formulas depend holomorphically on z, they also hold for all $z \in \mathbb{C}$, with $|z|$ small enough; in particular, for $z = 0$.

In Chapter 5, we define the complex-valued analytic torsion on a compact bordism, based on the results from Chapter 3 and we compute the corresponding anomaly with the results from Chapter 4. In Section 5.1.1, we recall some basic setting on finite dimensional graded complexes and their determinant lines; we explain how a given nondegenerate bilinear form on the complex determines a corresponding one on its determinant line. In Section 5.1.2 and Section 5.1.3, we use the results from Chapter 3 to obtain ζ-regularized determinants for the bilinear Laplacian. In Section 5.1.4 we define the complex-valued Ray–Singer torsion on a compact bordism. In Section 5.2, Theorem 5.2.1 is proved by using the approach given already [**BH07**], in the case of a closed manifold. That is, the computation of the logarithmic derivative of the complex-valued Ray–Singer torsion is translated into the computation of the coefficient of the constant term in the heat kernel asymptotic expansion corresponding to the bilinear Laplacian, from Chapter 4.

In Chapter 6, we define coEuler structures on $(M, \partial_+ M, \partial_- M)$, generalizing the work by Burghelea and Haller in [**BH06a**], [**BH06b**] and [**BH07**] on closed manifolds as dual to Euler structures introduced by Turaev in [**Tu90**], see also [**Tu02**]. In order to define the set of coEuler structures on a compact bordism, we need certain characteristic forms on the boundary. These differential forms are constructed from those defined in [**BM06**], which first appear in the anomaly formulas for the Ray–Singer metric and that we then used in Chapter 5 and Chapter 4 to write the corresponding formulas for the complex-valued Ray–Singer torsion. From Section 6.1.1 to Section 6.1.8, we recall in some detail how these characteristic forms were defined in [**BM06**] and then we adapt them to our situation. The necessary modification of these characteristic forms comes down to considering the inward (resp. outward) point normal geodesic unit vector fields on $\partial_+ M$ (resp. $\partial_- M$). Then, from Section 6.2.3 to Section 6.2.1, we use these characteristic forms to define coEuler Structures. We first consider the case $\chi(M, \partial_- M) = 0$ to define coEuler structures without base point. The space of CoEuler structures on $(M, \partial_+ M, \partial_- M)$ is seen as an affine space over the relative cohomology group $H^{m-1}(M, \partial M; \mathbb{C})$. Then, in Section 6.2.4, we study the case $\chi(M, \partial_- M) \neq 0$, to define coEuler structures with a base point. Finally, in Section 6.3, as on closed manifolds, we use coEuler structures to add correction terms to the complex-valued Ray–Singer torsion; these additional terms cancel out the variation of the complex-valued Ray–Singer torsion given in Theorem 5.2.1 so that, as on closed manifolds, we obtain a generalized version Ray–Singer torsion which depends on the flat connection, the homotopy class of the bilinear form and the coEuler structure only.

Next problems. A natural next step continuing the work in this thesis is to investigate the relation between the complex-valued Ray–Singer torsion and the combinatorial torsion, or Reidemeister torsion, in order to derive a Cheeger–Müller (Bismut–Zhang) type result (see [**BZ92**], [**Ch77**], [**Ch79**] and [**Mü78**]) for the complex-valued Ray–Singer torsion on a compact bordism. On closed manifolds, Burghelea and Haller formulated this problem in terms of Conjecture 5.1 in [**BH07**], which first was proved in the same paper for special non-trivial cases by using an argument of analytic continuation and then, in [**BH10**], for closed manifolds in odd dimensions, by extending the Witten–Helffer–Sjöstrand theory for the bilinear Laplacian and using the methods in [**BFK96**]. Later, Su and Zhang provided a proof of that conjecture in full generality, by adapting the methods from [**BZ92**] to the bilinear Laplacian. On manifolds with boundary, with the assumption that the Hermitian metric is flat and the Riemannian metric has product structure near the boundary, this comparison problem has been studied for the Ray–Singer metric, see for instance [**LR91**], [**Lü93**], [**Vi95**] and [**Has98**], where also gluing formulas were obtained, see Theorem 5.9 in [**Lü93**] and [**Vi95**]. More recently, Brüning and Ma obtained in [**BM11**], a Cheeger–Müller Theorem, see Theorem 0.1 in [**BM11**], for the the Ray–Singer metric on manifolds with boundary as well as gluing formulas, see Theorem 0.3 and Theorem 0.4 in [**BM11**]; these results were obtained without any assumption on the behavior of metric or Hermitian structure near the boundary, by applying the results from [**BZ94**] and [**BM11**]. A first attempt is to obtain analog formulas to those in (and in the generality of) Theorem 0.1 in [**BM11**] for the bilinear situation. Once these formulas are established, we would lead to conclude (as in Remark 5.3 in [**BH07**] for closed manifolds), that the (generalized) complex-valued analytic torsion is independent of the bilinear form, i.e., it depends on the flat connection ∇^E and the coEuler structure, only.

CHAPTER 1

Background

For the reader's convenience, this chapter contains the background needed for this thesis. The notions below are well-known and they can be found in several (under)-graduate textbooks. We provide the corresponding references at the begining of each section. These notions are recalled in the sake of completeness, as hints for further reading. If desired, this chapter can be completely skipped and the reader may start reading this thesis at Chapter 2.

In Section 1.1, we start with basic definitions, such as Hermitian and bilinear forms on finite dimensional vector spaces, complex conjugate and complexification of a vector space and continue with some elements of supergeometry such as superalgebras, supercommutators and supertraces. Section 1.1 ends with the statement of the first Theorem of Weyl's invariant theory as it is needed in Chapter 4. In Section 1.2, we provide basics from Riemannian geometry. We recall the notion of associated bundles, of a frame bundles, of structure group reduction and of orientation bundles. We deal with vector-valued differential forms, connections on vector bundles and their curvatures and de-Rham differential. Furthermore, we recall the Levi–Cività connection and Riemannian curvature, connection forms, curvature forms, the Christoffel symbols, the second fundamental form, collared neighborhoods, geodesic and normalized coordinate systems on manifolds with boundary, the volume form, the Hodge $\star$-operator, Stokes' Theorem, differential operators, their principal symbol and the notion of ellipticity. As a very useful result we have Lemma 1.2.1 which states how locally, in geodesic coordinates, higher order derivatives of the Riemannian metric can be described in terms of the curvature tensor and the second fundamental form. In Section 1.3, we include basic notions from operator theory dealing with bounded operators on Hilbert spaces such as the very important class of compact operators and trace-class operators. For unbounded operators on Hilbert spaces, we mention the notion of extension, commutativity, closedness and operator with compact resolvent. In this section, Theorem 1.3.1 provides a decomposition result for closed unbounded operators, used later in Chapter 2. Finally, in Section 1.4, some notions from analysis on manifolds are given, such ash Sobolev spaces (on manifolds with boundary) and generalized sections (or distributions), kernels, smoothing operators and the Schwartz kernel Theorem.

1.1. Algebraic background

The notions in this section can be found, for instance, in [**Br88**], [**Hal74**] and [**La02**].

1.1.1. Bilinear forms. Consider V a finite dimensional complex vector space with a complex symmetric bilinear form $b : V \times V \to \mathbb{C}$ and denote by $V' := \mathrm{Hom}(V, \mathbb{C})$ its algebraic dual vector space with induced dual bilinear form b'. A bilinear form b on V is **nondegenerate** if and only if the complex linear homomorphism $\hat{b} : V \to V'$, defined by $\hat{b}(v) := b(v, \cdot) : V \to \mathbb{C}$, is an isomorphism; for simplicity, we still denote $\hat{b}$ by b. Remark that b is nondegenerate if and only if b' is nondegenerate. The bilinear form b is **degenerate** if and only if there exists a non trivial vector $v_0 \in V$ with $b(v_0, v) = 0$ for all $v \in V$. For $u, v \in V$, we write that $u \perp_b v$, if they are b-**orthogonal** (or simply orthogonal), i.e., $b(u, v) = 0$. For each non-empty set $S \subset V$, we denote by $S^{\perp_b}$ the b-orthogonal subspace to S in V, of all $v \in V$ with $v \perp_b s$ for all $s \in S$.

1.1.2. Hermitian forms. By a **sesquilinear form** on a complex vector space V, we mean a map $h : V \times V \to \mathbb{C}$ being complex linear on the first argument and complex anti-linear on the second one. A **Hermitian form** on V is a sesquilinear form h, which satisfies $h(v, w) = \overline{h(w, v)}$. An **inner product** on V is a Hermitian form h which is **positive definite**: $h(v, v) \geqslant 0$ for all $v \in V$ and $h(v, v) = 0$ if and only if $v = 0$. If h is an inner product on V, then V is naturally endowed with the metric associated to h, which is called the **Hermitian metric** on V (associated h).

1.1.3. Complex conjugate vector space. Let V be a complex vector space and with $V_{\mathbb{R}}$ its underlying real vector space . The **complex conjugate** of V, denoted by $\overline{V}$, is the complex vector space having the same underlying real vector space as that of V, that is $\overline{V}_{\mathbb{R}} := V_{\mathbb{R}}$, but whose complex structure is obtained by *complex conjugating* the one in V. More precisely, the complex multiplication $\cdot$ in $\overline{V}$ is defined by $\mathbf{i} \cdot v := \bar{\mathbf{i}} v = -\mathbf{i} v$, for all $v \in V$. Equivalently, this can be described by means of a complex anti-linear involution $\tau : V \to \overline{V}$ with $\tau(v) = v$ and $\tau(\mathbf{i}v) = -\mathbf{i}\tau(v)$; the spaces $\overline{V}$ and V are isomorphic as real vector spaces, but as complex vector spaces, their complex structures are *intertwined* by τ. Every complex linear map $f : V \to V$ can be considered as a complex linear $f : \overline{V} \to \overline{V}$ as well. But, if $g : V \to V$ is a complex anti-linear map, then, by using τ, g is in one-to-one correspondence with the complex linear map $\overline{g} := \tau \circ g : V \to \overline{V}$. In particular, every sesquilinear form h on V can be considered as a complex bilinear map $h : V \times \overline{V} \to \mathbb{C}$.

1.1.4. Complexification. Let V be a complex or real finite dimensional vector space and $V_{\mathbb{R}}$ its underlying real vector space. The **complexification** of V is the complex vector space obtained by the tensor product $V^{\mathbb{C}} := {V_{\mathbb{R}}}^{\mathbb{C}} := V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. The complex multiplication is given by $\alpha(v \otimes \beta) := v \otimes (\alpha\beta)$ for all $v \in V$ and $\alpha, \beta \in \mathbb{C}$. Equivalently, the complexification $V^{\mathbb{C}}$ of V can be identified with space $V_{\mathbb{R}} \oplus V_{\mathbb{R}}$, seen as a complex vector space, whose complex scalar multiplication is defined by

$$(a + b\mathbf{i})(v, w) := (av - bw, bv + aw) \quad \text{for } a, b \in \mathbb{R};$$

in particular, the multiplication by $\mathbf{i}$ is given by

$$\mathbf{i}(v,w) = (-w, v).$$

In this picture, elements such as $v + \mathbf{i}w \in V^{\mathbb{C}}$ are seen as couples $(v,w) \in V_{\mathbb{R}} \oplus V_{\mathbb{R}}$ and the assignment $v \mapsto (v,0)$, provides the so-called $\mathbb{R}$-linear **standard embedding** of $V_{\mathbb{R}}$ into $V^{\mathbb{C}}$.

1.1.5. Superalgebras, $\mathbb{Z}_2$-graded tensor product. Let $\mathcal{A}$ be a unital algebra over $\mathbb{C}$. We say that $\mathcal{A}$ is a **superalgebra** if its underlying vector space is a $\mathbb{Z}_2$-graded vector space. That is,

$$\mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A}^1$$

where the product " $\cdot$ " respects the grading: $\mathcal{A}^i \cdot \mathcal{A}^j \subset \mathcal{A}^{(i+j)\mathsf{mod}2}$. If $\mathcal{A}$ and $\mathcal{B}$ are two unital superalgebras, besides the standard tensor product $\mathcal{A} \otimes \mathcal{B}$ with product $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$, we have the notion of **$\mathbb{Z}_2$-graded tensor product** denoted by $\mathcal{A} \widehat{\otimes} \mathcal{B}$ and defined as the superalgebra whose underlying vector space is also $\mathcal{A} \otimes \mathcal{B}$, but with a **$\mathbb{Z}_2$-graded product** given by

$$(a_1 \widehat{\otimes} b_1) \cdot (a_2 \widehat{\otimes} b_2) = (-1)^{\mathsf{deg}(b_1)\mathsf{deg}(a_2)} a_1 a_2 \otimes b_1 b_2.$$

The $\mathbb{Z}_2$-grading of $\mathcal{A} \widehat{\otimes} \mathcal{B}$ is given by

$$(\mathcal{A} \widehat{\otimes} \mathcal{B})^0 := (\mathcal{A}^0 \widehat{\otimes} \mathcal{B}^0) \oplus (\mathcal{A}^1 \widehat{\otimes} \mathcal{B}^1)$$

and

$$(\mathcal{A} \widehat{\otimes} \mathcal{B})^1 := (\mathcal{A}^0 \widehat{\otimes} \mathcal{B}^1) \oplus (\mathcal{A}^1 \widehat{\otimes} \mathcal{B}^0).$$

1.1.6. Supercommutators, supertraces. For $\mathcal{A}$ a superalgebra, the bilinear map

$$[\cdot,\cdot] : \mathcal{A} \times \mathcal{A} \to \mathcal{A}, \quad [a,b] := a \cdot b - (-1)^{\mathsf{deg}(a)\mathsf{deg}(b)} ba$$

satisfying

$$[a,b] + (-1)^{\mathsf{deg}(a)\mathsf{deg}(b)}[b,a] = 0 \quad \text{ and } \quad [a,[b,c]] = [[a,b],c] + (-1)^{\mathsf{deg}(a)\mathsf{deg}(b)}[b,[a,c]]$$

defines a **supercommutator** on $\mathcal{A}$. The couple $(\mathcal{A}, [\cdot,\cdot])$ is called a **Lie superalgebra**. A Lie superalgebra $\mathcal{A}$ is **supercommutative** if $[\cdot,\cdot] = 0$. A **supertrace** on a Lie superalgebra $(\mathcal{A}, [\cdot,\cdot])$ is a linear form $\mathrm{Tr}_{\mathbf{s}} : \mathcal{A} \to \mathbb{C}$, satisfying

$$\mathrm{Tr}_{\mathbf{s}}([a,b]) = 0.$$

1.1.7. Weyl's invariant theory. We recall Weyl's first Theorem of invariants. We adopt the approach and notation from Section 1.7 in [**Gi04**] and Section 2.5 in [**Gi84**], where much more details can be found (see also [**Pr07**] and [**FH91**]). Let V be a real vector space of dimension n, with a positive definite inner product g and denote by $\mathbb{GL}(V)$ the space of invertible linear maps from V into itself. Let $O(V) \subset \mathbb{GL}(V)$ consisting of all invertible linear maps $Q : V \to V$, which leave the inner product invariant, ie.,$g(Qv, Qw) = g(v,w)$, for all $v, w \in V$. For $V^{\times k}$ the k-fold cartesian vector space product of V with the natural action of $O(V)$, a **polynomial map** $f : V^{\times k} \to \mathbb{R}$

is a real valued function on V^k, such that $f(v_1,\dots,v_k)$ is polynomial in the variables $\{v_i|1\leqslant i\leqslant k\}$. In particular, a k-multilinear map is a polynomial map. A polynomial map f is said to be **invariant under the action** of $O(V)$, or **orthogonal invariant**, if

$$f(Qv_1,\cdots,Qv_k)=f(v_1,\cdots,v_k),\qquad \text{for all } (v_1,\cdots,v_k)\in V^k$$

and $Q\in O(V)$. The set of all such polynomial invariants is a real commutative unital algebra and it is denoted by $\mathcal{A}_k(V)$. Among such orthogonal invariants, we have the (symmetric) functions $g_{ij}:=g(v_i,v_j)$ for $1\leqslant i,j\leqslant k$. Furthermore, every orthogonal invariant $f\in\mathcal{A}_k(V)$ is expressible in terms of the invariants g_{ij} for $0\leqslant i,j\leqslant k$. More precisely, we have the following, see [**We46**].

Theorem 1.1.1. *(**Weyl's first theorem of invariants**) Let V be a real vector space of dimension n, with a positive definite inner product g, and $\mathcal{A}_k(V)$ be the algebra of polynomial invariants $f:V^{\times k}\to\mathbb{R}$ as above. For $\widetilde{\mathcal{A}}_k(V):=\mathbb{R}[g_{ij}]$ the free polynomial algebra generated by the the $\frac{1}{2}(k(k+1))$ formal symmetric variables $\{g_{ij}=g_{ji}\}_{1\leqslant i,j\leqslant k}$, consider the evaluation map $\mathsf{ev}(g_{ij})(v_1,\cdots,v_k):=g(v_i,v_j)$. Then, ev induces a natural surjective algebra homomorphism $\mathsf{ev}:\widetilde{\mathcal{A}}_k(V)\to\mathcal{A}_k(V)$.*

The relations among the generators of the algebra $\mathcal{A}_k(V)$ above are described by the **Second Fundamental Theorem of Weyl**. Just in words, these relations determine the kernel of the map ev in Theorem 1.1.1, as an ideal in $\mathbb{R}[g_{ij}]$ generated by certain *determinant functions*. For instance, the space $\mathcal{I}_{k,V}\subset\mathcal{A}_k(V)$ of all *multilinear maps* $f:V^{\times k}\to\mathbb{R}$ can be completely described by using Theorem 1.1.1, as $\mathcal{I}_{k,V}=\mathsf{Span}_{\sigma\in\Sigma_k}(p_{k,\sigma})$ where Σ_k is the group of permutations of the set of indices $\sigma:\{1,\dots,k\}\to\{1,\dots,k\}$ and each $p_{k,\sigma}(v_1,\dots,v_k):=g_{\sigma(1)\sigma(2)}\cdots g_{\sigma(k-1)\sigma(k)}$ is a multilinear orthogonal invariant map for each $\sigma\in\Sigma_k$. Any invariant multilinear map is obtained by contraction of indices in pairs, see Theorem 1.7.3 in [**Gi04**]. We omit the details describing these determinant functions in general, but we refer the reader to section 1.7 in [**Gi04**], section 5, chapter 11 in [**Pr07**] and the original work of Weyl [**We46**].

1.2. Riemannian geometry

In this thesis, M denotes a **compact Riemannian manifold**, by which is meant a compact smooth manifold with Riemannian metric g and smooth boundary ∂M. A **closed manifold** is to be understood as a compact manifold without boundary. We do not assume M to be orientable. The boundary ∂M, seen as a closed Riemannian submanifold of M, is endowed with the Riemannian metric g^∂ induced by that on M. For the material in this section, we refer the reader to [**AMR02**], [**BGV92**], [**Jo02**], [**Gi04**], [**Mo01**] and [**Ni07**].

1.2.1. Associated bundles. Here, we assume familiarity with the basic notions of (smooth) fiber, vector and principal bundles, otherwise we refer the reader for instance to [**BGV92**], [**Jo02**] and [**Mo01**]. Let P be a principal bundle over M with structure (Lie) group G. We simply refer to it as a principal G-bundle. Let E be a smooth manifold endowed with $\rho : G \to \mathsf{Diff}(E)$, a left action of G, where $\mathsf{Diff}(E)$ is the group of diffeomorphic transformations of E into itself. Then, the **associated bundle** $P \times_G E$ given by

$$P \times_G E := P \times E / \{(p \cdot g, f) \sim (p, \rho(g) f) \quad \text{for} \quad p \in P, g \in G \text{ and } f \in E\}$$

is a fiber bundle over M with fibre E.

1.2.2. Frame bundles of vector bundles. Let P be a principal G-bundle. If E is a vector space together with a linear representation of G, then $P \times_G E$ defines a vector bundle over M. In general, every complex (resp. real), vector bundle of rank k over M is obtained as an associated bundle for a certain principal bundle over M with structure group $\mathsf{GL}(k, \mathbb{C})$ (resp. $\mathsf{GL}(k, \mathbb{R})$); this principal bundle is called the **frame bundle**. More precisely, for $\pi : E \to M$ a complex vector bundle over M of rank k, its **frame bundle** is the principal bundle $\mathsf{p} : \mathsf{GL}(E) \to M$, whose fibre is given by $\mathsf{p}^{-1}(x) := \mathsf{GL}(\mathbb{C}^k; \pi^{-1}(x))$ with structure group $\mathsf{GL}(k, \mathbb{C})$ specifying the action

$$(p \cdot A)(v) := p(A \cdot v), \qquad \text{for } A \in \mathsf{GL}(k, \mathbb{C}),\ p : \mathbb{C}^k \to \pi^{-1}(x) \text{ and } v \in \mathbb{C}^k$$

and E being naturally isomorphic to $\mathsf{GL}(E) \times_{\mathsf{GL}(k;\mathbb{C})} \mathbb{C}^k$ as vector bundles over M; see for instance Proposition 1.4 in [**BGV92**]. As an important example, let us denote by

$$\mathsf{GL}(M) := \mathsf{GL}(TM)$$

the frame bundle over M corresponding to the tangent bundle $TM \to M$, with structure group $\mathsf{GL}(m, \mathbb{R})$.

1.2.3. Structure group reduction. For $P \to M$ a principal G-bundle and H a subgroup of G, P is said to be **induced from a principal H-bundle**, if there exists a principal H-bundle Q such that $Q \times_H G \cong P$ as principal bundles over M. For $H \subset G$ a subgroup of G and $P \to M$ a principal G-bundle over M with fiber F, if the associated principal G-bundle is induced from a principal H-bundle, one says that the **structure group** of the bundle can be **reduced** to H,

Let $\mathsf{GL}(m, \mathbb{R})$ be the group of real-valued invertible matrices of dimension m. Consider P any principal $\mathsf{GL}(m, \mathbb{R})$-bundle. For $\mathsf{O}(m; \mathbb{R})$ the orthogonal group of dimension m, denote by $\mathsf{Q}(m, \mathbb{R}) := \mathsf{GL}(m, \mathbb{R}) / \mathsf{O}(m, \mathbb{R})$ the quotient group. Since $\mathsf{Q}(m, \mathbb{R})$ is contractible, the bundle $P \times_{\mathsf{GL}(m,\mathbb{R})} \mathsf{Q}(m, \mathbb{R}) \to M$ admits a section, or equivalently, the principal $\mathsf{GL}(m, \mathbb{R})$-bundle P is induced from an $\mathsf{O}(m; \mathbb{R})$-bundle; that is, it admits a structure reduction to $\mathsf{O}(m; \mathbb{R})$ and hence any real vector bundle admits an Euclidiean metric. Analogously, every principal $\mathsf{GL}(m; \mathbb{C})$-bundle admits a structure reduction to $\mathsf{U}(m; \mathbb{C})$ and hence every complex vector bundle admits a Hermitian metric.

1.2.4. Orientation bundle. If E is a real vector bundle over M, then E is **orientable** if and only if the structure group of its frame bundle can be reduced from $\mathsf{O}(m,\mathbb{R})$ to $\mathsf{SO}(m,\mathbb{R})$. Consider $\mathsf{GL}(M)$ the frame bundle corresponding to the tangent bundle of M. Then, the **orientation bundle** Θ_M over a compact manifold M is the line bundle defined by $\Theta_M := \mathsf{GL}(M) \times_{\mathsf{O}(m;\mathbb{R})} \mathbb{Z}_2$, i.e., the fibered product of $\mathsf{GL}(M)$ as a principal $\mathsf{O}(m;\mathbb{R})$-bundle and (with fiber) the quotient group $\mathsf{O}(m;\mathbb{R})/\mathsf{SO}(m,\mathbb{R}) \cong \mathbb{Z}_2$.

1.2.5. Vector-valued differential forms. For TM the tangent bundle of M, we denote by

$$\mathfrak{X}(M) := \Gamma(M, TM)$$

the space of smooth vector fields on M, $T^*M \to M$ the cotangent bundle of M, $\Lambda(T^*M) \to M$ the exterior bundle of M and by

$$\Omega(M) := \Gamma(M, \Lambda(T^*M))$$

the space of smooth differential forms on M. These bundles are endowed with an inner product, induced by the Riemannian metric g of M, and simply denoted by $\langle\cdot,\cdot\rangle_g$ indistinctly when no confusion is expected. In a similar manner for $\pi : E \to M$ a complex vector bundles of rank k over M, with space of smooth sections $\Gamma(M;E)$, $\Lambda T^*M \otimes E$ is the tensor product vector bundle of ΛT^*M and E over M and

$$\Omega(M;E) := \Gamma(M; \Lambda T^*M \otimes E)$$

its space of smooth sections or E-valued smooth forms. By choosing a Hermitian structure h on E and using the Riemannian metric g on M, $\Omega(M;E)$ can be endowed with an inner product $\langle\cdot,\cdot\rangle_{g,h}$. The space $\Omega(M;E)$ is isomorphic to $\Omega(M) \otimes_{C^\infty(M)} \Gamma(M;E)$ so that it can be considered as $C^\infty(M)$-module or as a $\Omega(M)$-module. For a complex vector bundle E, we denote by $E' := \mathrm{End}(E;\mathbb{C})$ its dual vector bundle and by $\mathrm{End}(E) \cong E' \otimes E$ its bundle of endomorphisms. On $\mathrm{End}(E)$, the composition of endomorphisms is used to wedge $\mathrm{End}(E)$-valued forms on M and $\Omega(M;\mathrm{End}(E))$ can be regarded as a (graded) algebra. Then, $\mathrm{End}(E)$-valued forms can be considered as acting on E-valued forms so that $\Omega(M;E)$ can be considered as (graded) module over $\Omega(M;\mathrm{End}(E))$ as well.

1.2.6. Pull-back vector bundles. For N a closed submanifold of M we consider the bundles TN, T^*N and $\Lambda(T^*N)$. Moreover, given a bundle $\pi : E \to M$, we use the canonical embedding $i_N : N \to M$, to pull E back over N: the **pull-back** along i_N is

$$E|_N := i_N^* E := \{(e,n) \in E \times N | i_N(n) = \pi(e)\},$$

where $\pi|_N := i_N^*\pi : i_N^*E \to N$ is seen as the restriction bundle of π to N. In particular, for $\Lambda(T^*N) \otimes E|_N \to N$, the space of smooth $E|_N$-valued differentiable forms on N is denoted by

$$\Omega(N; E|_N) := \Gamma(N; \Lambda(T^*N) \otimes E|_N).$$

1.2.7. Linear connections. Let E be a complex vector bundle over M of rank k. A (complex) linear connection on E is a $\mathbb{C}$-bilinear map

$$\nabla^E : \mathfrak{X}(M) \times \Gamma(M;E) \to \Gamma(M;E), \quad (X,s) \mapsto \nabla^E_X(s),$$

which is also $C^\infty(M)$-linear with respect to $\mathfrak{X}(M)$ and satisfies the Leibniz rule on $\Gamma(M;E)$. The **covariant derivative along** a vector field $X \in \mathfrak{X}(M)$ is denoted by ∇^E_X. The connection ∇^E induces a **dual connection** on E' given by

$$\nabla^{E'}_X(t')(s) = X(t'(s)) - t'(\nabla^E_X(s)),$$

for all $s \in \Gamma(M;E)$, $t' \in \Gamma(E')$, and $X \in \mathfrak{X}(M)$ and also a connection on $\mathrm{End}(E)$ given by

$$\nabla^{\mathrm{End}(E)}_X(T)(s) = \nabla^E_X(T(s)) - T(\nabla^E_X(s)),$$

for all $X \in \mathfrak{X}(M)$, $T \in \Gamma(M; \mathrm{End}(E))$ and $s \in \Gamma(M;E)$.

1.2.8. Curvature of a connection. The curvature $\mathsf{R}^E \in \Omega^2(M; \mathrm{End}(E))$ corresponding to the connection ∇^E is defined by

$$\mathsf{R}^E(X,Y) := \nabla^E_X \nabla^E_Y - \nabla^E_Y \nabla^E_X - \nabla^E_{[X,Y]},$$

where $[X,Y]$ is the Lie bracket of vector fields X and Y in $\mathfrak{X}(M)$. If R^E is identically zero, then ∇^E is called a **flat connection**, and E a **flat vector bundle** . If E is a flat vector bundle, then there exists a locally constant trivializing atlas, i.e., a vector bundle atlas $\{U_j, \phi_j\}$, whose transition functions are locally constant functions.

1.2.9. The De Rham differential. A connection ∇^E on a complex vector bundle E over M determines $d_E : \Gamma(M;E) \to \Omega^1(M;E)$ a graded derivation on $\Omega(M;E)$, obtained by defining $d_E\,(s)(X) := \nabla^E_X(s)$, for $X \in \mathfrak{X}$ and $s \in \Gamma(M;E)$, and uniquely extending it to $\Omega(M;E)$ by requiring the (graded) Leibniz rule

$$d_E\,(\alpha \wedge v) = d\alpha \wedge v + (-1)^{\mathsf{deg}(\alpha)} \alpha \wedge d_E\,(v)$$

to hold for $\alpha \in \Omega(M)$ and $v \in \Omega(M;E)$. Moreover, if ∇^E is flat, then d_E defines a differential on $\Omega(M;E)$, also called the **de Rham differential** on E-valued smooth forms.

1.2.10. Connection 1-form. Each connection ∇^E can be uniquely locally described over an open neighborhood U of M in terms of $X \in \mathfrak{X}(M)$ and $s \in \Gamma(M;E)$, by a straightforward use of the Leibniz rule. Let $\nabla^E_X(s)|_U$ be the value of $\nabla^E_X(s)$ on U, which depends on X and s over U only. Then, if X and s are defined on U only, it makes completely sense to talk about $\nabla^E_X(s)|_U$ as a section of the bundle $E|_U$, with the connection obtained by restricting ∇^E to U. For $\{s_1, \ldots, s_k\}$ a frame of $E|_U$, we can write $\nabla^E_X s_j = \sum_{i=1}^k \omega^i_j(X) s_i$, for any $X \in \mathfrak{X}(U)$, where $\omega^i_j \in \Omega(U)$ are differential forms over U. By using the Leibniz identity only, the formulas above completely determine the value of the connection on arbitrary sections of E over U. Then, the collection of all ω^i_j can be seen as a matrix $\omega^E := (\omega^i_j)_{i,j} \in \mathbb{M}(k; \Omega^1(U))$, called the **connection 1-form**

$\{s_1, \ldots, s_k\}$ of $E|_U$. Now, for U' another open set of M, $\{s'_1, \ldots, s'_k\}$ a frame of E over U', consider the corresponding connection 1-form $(\omega^E)'$. If $h := (h^i_j)$ denotes the matrix of coordinate change between these frames, i.e. $s'_i = \sum_{j=1}^m h^i_j s_j$ over $U \cap U'$, then a direct computation involving the use of the Leibniz rule and basic algebraic manipulations leads to the transformation relation $(\omega^E)' = (dg)h^{-1} + h\omega^E h^{-1}$.

1.2.11. Curvature 2-form. For U an open set of M, consider the bundle $E|_U$ with $\{s_1, \ldots, s_k\}$ a frame of $E|_U$ and the corresponding induced connection. In a similar way, the curvature R^E can be described locally over U in terms of the chosen frame by

$$\mathsf{R}^E(X,Y)s_j = \sum_{j=1}^{k} \kappa^i_j(X,Y)s_i,$$

where $\kappa^i_j \in \Omega^2(U)$, for any $X, Y \in \mathfrak{X}(U)$. Then, the matrix $\kappa^E := (\kappa^i_j)_{i,j} \in \mathbb{M}(k; \Omega^2(U))$ is the **curvature 2-form of R^E with respect to the frame $\{s_1, \ldots, s_k\}$ of E over U**. For ω^E the connection 1-form over U, we have the relation

$$d\omega^E = -\omega^E \wedge \omega^E + \kappa^E.$$

Again consider for U' another open set of M, $\{s'_1, \ldots, s'_k\}$ a frame of E over U' and $(\kappa^E)'$ the corresponding curvature 2-form. If $h := (h^i_j)$ denotes the matrix of coordinate change between these frames, i.e. $s'_i = \sum_{j=1}^m h^i_j s_j$ over $U \cap U'$, then $(\kappa^E)' = h\kappa^E h^{-1}$ over $U \cap U'$. These relations translate explicitly the fact that the curvature is globally defined as an $\mathrm{End}(E)$-valued smooth 2-form over M.

1.2.12. Levi–Cività connection and Riemann curvature. For a Riemannian manifold M, there exists a unique a torsion free connection ∇^{TM} on TM that is compatible with the Riemannian metric. By being by a torsion free connection, we mean $[X,Y] = \nabla^{TM}_X Y - \nabla^{TM}_Y X$ for all $X, Y \in \mathfrak{X}(M)$, and by being compatible with the Riemannian metric, we mean $X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ for all $X, Y, Z \in \mathfrak{X}(M)$. This is the Levi–Cività connection and entirely determined by the so-called Kozul formula:

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]).$$

The Levi–Cività connection naturally induces connections on T^*M and on higher order tensor bundles of mixed type, which be denoted by indistinctly by ∇ (or ∇^g), when no confusion appears. Similarly, on the boundary, the Levi–Cività connection corresponding to the metric g^∂, is denoted by ∇^∂. The **Riemann curvature tensor**, denoted by R, is the curvature associated to ∇. The Riemann curvature tensor is the 4-tensor:

$$\mathsf{R}(X,Y,Z,W) = g\left(\left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\right) Z, W\right)$$

satisfying, together with the relation

$$\mathsf{R}(X,Y,Z,W) = \mathsf{R}(Z,W,X,Y) = -\mathsf{R}(Y,X,Z,W),$$

the **Bianchi identity**:

$$\mathsf{R}(X,Y,Z,W)+\mathsf{R}(Y,Z,X,W)+\mathsf{R}(Z,X,Y,W)=0.$$

Higher order covariant derivatives of R with respect to the the Levi–Cività connection, are denoted by $\nabla^k\mathsf{R}$.

1.2.13. Christoffel symbols and derivatives of the metric. Let us denote by

$$x=(x^1,\cdots,x^m)$$

a local coordinate chart of M. We denote by $\partial_j:=\partial_{x_j}$ and $dx^j:=g(\partial_j,\cdot)$ for $j\in\{1,\cdots,m\}$, the corresponding local coordinate frames for TM and T^*M respectively. The components of the metric with respect to the given local frame on TM are denoted by $g_{ij}=g(\partial_i,\partial_j)$, while g^{ij} indicate the components of the inverse matrix. In these coordinates, the Levi-Cività connection on TM (resp. T^*M) reads as

$$\nabla_{\partial_j}\partial_i=\Gamma^k_{ji}\partial_k \quad (\text{resp. } \nabla_{\partial_j}dx^i={\Gamma_j}^i{}_k d_k),$$

where

$$\Gamma_{jik}:=g(\nabla_{\partial_j}\partial_i,\partial_k) \quad \text{and} \quad \Gamma^k_{ji}:=g^{kl}\Gamma_{jil}$$

are the **Christoffel symbols** and satisfy the relations

$$\Gamma^k_{ji}=g^{kl}\Gamma_{jil} \quad \text{and} \quad {\Gamma_j}^i{}_k=-{\Gamma_{jk}}^i.$$

Similarly, the components of the curvature tensor R relative to the local coordinate frames ∂_i are given by

$$\mathsf{R}_{ijkl}:=g\left(\left(\nabla_i\nabla_j-\nabla_j\nabla_i\right)\partial_k,\partial_l\right).$$

The Christoffel symbols are expressible in terms of **first order derivatives** (or 1-jets) of the coefficients of the Riemannian metric:

$$\Gamma_{jik}=\frac{1}{2}(\partial_j g_{ik}+\partial_i g_{jk}-\partial_k g_{ji})$$

and subsequently the curvature form for the Levi-Cività connection can be expressed in terms of **second order derivatives** (2-jets) of the coefficients of the Riemannian metric

$$\mathsf{R}_{ijkl}=g_{\alpha l}\left(\partial_i{\Gamma_{jk}}^\alpha-\partial_j{\Gamma_{ik}}^\alpha+{\Gamma_{i\beta}}^\alpha{\Gamma_{jk}}^\beta-{\Gamma_{j\beta}}^\alpha{\Gamma_{ik}}^\beta\right).$$

1.2.14. Normal bundle. For M a manifold with boundary ∂M and $i:\partial M\hookrightarrow M$ the canonical embedding, the **normal bundle** of ∂M in M is the vector bundle over ∂M of rank 1, defined as the quotient

$$N(\partial M):=i^*TM/T\partial M$$

where $i^*TM=TM|_{\partial M}$ is the restriction of TM to ∂M. Remark that this definition does not require a Riemannian metric on M. However, if (M,g) is a Riemannian manifold, then the metric can be used to regard $N(\partial M)$ as a subbundle of $TM|_{\partial M}$, by identifying

$N(\partial M)$ with $T(\partial M)^{\perp_g}$, the orthogonal complement of $T\partial M$ in $TM|_{\partial M}$ with respect to g. In this situation, we have have a **splitting**

$$TM|_{\partial M} \cong T(\partial M) \oplus T(\partial M)^{\perp_g}$$

with

$$T(\partial M)^{\perp_g} \cong N(\partial M) \cong \partial M \times \mathbb{R};$$

where $\pi^{\partial} : TM|_{\partial M} \to T(\partial M)$ and $\pi^{\perp_g} : TM|_{\partial M} \to T(\partial M)^{\perp_g}$ are the corresponding projections.

1.2.15. Second Fundamtental form. For $X, Y \in \Gamma(\partial M, T(\partial M))$, denote by $\tilde{X}, \tilde{Y}$ be arbitrary extensions to a neighborhood of ∂M in M. If ∇ is the Levi–Cività connection on TM, then with respect to the splitting explained in Section 1.2.14, $\nabla_{\tilde{X}}\tilde{Y}$ can be written as $\nabla_{\tilde{X}}\tilde{Y} = \pi^{\partial}(\nabla_{\tilde{X}}\tilde{Y}) + \pi^{\perp_g}(\nabla_{\tilde{X}}\tilde{Y})$. Thus, the **second fundamental form** is the bundle map

$$\mathsf{L} : T(\partial M) \times T(\partial M) \to T(\partial M)^{\perp_g}, \quad \mathsf{L}(X, Y) := \pi^{\perp_g}(\nabla_{\tilde{X}}\tilde{Y}) := g(\nabla_{\tilde{X}}\tilde{Y}, \varsigma_{\mathsf{in}})\Big|_{\partial M} \cdot \varsigma_{\mathsf{in}}$$

where ς_{in} is the **inwards pointing geodesic unit normal** vector field to the boundary[1]. In fact, this definition does not depend on the extensions of X and Y, and that L is $C^{\infty}(M)$-bilinear and symmetric, in other words, L is symmetric $(0,2)$-tensor. The vanishing of the second fundamental form is translated into ∂M being **totally geodesic** in M, i.e., if the geodesics of $(\partial M, g^{\partial})$ are geodesics in (M, g) under the canonical embedding i. In particular, if L vanishes, then the metric g is product-like near the boundary.

1.2.16. Geodesic coordinates. Consider a compact Riemannian manifold (M, g). For each $x_0 \in M$, the **exponential map** at x_0 is the map $\exp_{x_0} : T_{x_0}M \to M$ defined by $\exp_{x_0}(X) := \gamma_X(1)$, where the curve $\gamma_X : [0, \infty) \to M$ indicates the geodesic starting at x_0, with constant velocity $\dot{\gamma_X}(0) = X$. The integral curve of X starting at x_0 is $\gamma_X(t) = \exp_{x_0}(tX)$. If $\mathbb{B}(0_{x_0}, \epsilon) \subset T_{x_0}M$ indicates the open ball in $T_{x_0}M$ centered at 0_{x_0} of radius $\epsilon > 0$, then there is an $\epsilon > 0$, for which $\exp_{x_0} : \mathbb{B}(0_{x_0}, \epsilon) \to M$, is a diffeomorphism on its image and we set $U := \exp_{x_0}(\mathbb{B}(0_{x_0}, \epsilon))$. With respect to the Riemannian metric, we fix $\mathbf{e} = (e_1, \ldots, e_n)$ an orthonormal basis of $T_{x_0}M$ and its associated coordinate chart $(x^1, \cdots, x^m)$. The local coordinate chart $(U, x^1, \cdots, x^m)$ obtained in this way is called a a **geodesic coordinate chart**. With respect to the local frame $\mathbf{e}$, each $X \in T_{x_0}M$ is written as $X = x^1 e_1 + \cdots + x^n e_n$ and therefore the geodesic curve at x_0 with velocity X, is $\gamma_X(t) := (tx^1, \cdots, tx^m)$. In these coordinates, $x_0 \in M$ is represented by $(0, \ldots, 0)$, $g_{ij}(x_0) = \delta_{ij}$, $\Gamma^k_{ij}(x_0) = 0$ and $\partial_k g_{ij}(x_0) = 0$. Moreover in these coordinates, higher order derivatives of the Riemannian metric can be expressed in terms of higher order derivatives of the curvature at $x_0 \in M$ (c.f. Lemma 1.2.1 in Chapter 4).

[1] Recall the Gauss formula: $\nabla_{\tilde{X}}\tilde{Y}\Big|_{\partial M} = \nabla^{\partial}_X Y + \mathsf{L}(X, Y)$ where ∇^{∂} is the Levi–Civita connection on the boundary.

1.2.17. Collared neighborhood. Let M be a compact manifold with Riemannian metric g and ς_{in} the inwards pointing geodesic unit normal vector field to ∂M. For each fixed point $y_0 \in \partial M$ consider the geodesic $\gamma_{y_0}(t)$ starting at y_0 with velocity

$$\frac{d}{dt}\gamma_{y_0}(0) = \varsigma_{\mathsf{in}}.$$

There is $\epsilon > 0$ such that for each y_0, the map $\gamma_{y_0}(t) \in M$ exists for each $t \in [0, \epsilon)$ and $U \supset \partial M$ an open neighborhood of ∂M in M, over which the map

$$\partial M \times [0, \epsilon) \to U, \quad (y_0, t) \mapsto \gamma_{y_0}(t)$$

is a diffeomorphism. The neighborhood U is called a **collared neighborhood** of ∂M in M.

1.2.18. Normalized coordinate system of the boundary. Given a local coordinate system $(y^1, \cdots, y^{m-1})$ for ∂M, the collared neighborhood induces a local coordinate system $x = (y^1, \cdots, y^{m-1}, x^m)$ near the boundary, called **normalized coordinate system**, where x^m measures the geodesic distance to the boundary. In these coordinates, the curves $x^m \mapsto (y, x^m)$ are unit speed geodesics orthonormal to the boundary. The associated coordinates frames for the tangent and cotangent bundles of the boundary are denoted by $\partial_\alpha := \partial_{y^\alpha}$ and $dy^\alpha = g(\partial_\alpha, \cdot)$ respectively; here the **greek** indices $\alpha, \beta, \ldots \in \{1, \cdots, m-1\}$, whereas **roman** indices $i, j \ldots \in \{1, \ldots, m\}$. Near the boundary, we denote by $\mathbf{e} = \{e_1, \cdots, e_m\}$ an arbitrary orthonormal frame for TM with $e_m := \varsigma_{\mathsf{in}}$ and as usual we use the metric to fix the corresponding local orthonormal coframe $\{e^1, \cdots, e^m\}$ on T^*M. At the boundary, higher order derivatives for the components of the Riemannian metric have a particular simplified form in these coordinates, see Lemma 1.2.1.

1.2.19. Local expression for the second fundamental form. Let us choose a local normalized coordinate chart around the boundary. The tensor field components of the second fundamental form L, relative to this chart are

$$\mathsf{L}_{\alpha\beta} := g\left(\nabla_{\partial_\alpha}\partial_\beta, e_m\right) = \Gamma_{\alpha\beta m},$$

where $\alpha, \beta \in \{1, \cdots, m-1\}$ and $e_m := \varsigma_{\mathsf{in}}$ is the inwards pointing geodesic unit normal vector field to ∂M. Since the curves $t \mapsto (y_0, t)$ are unit geodesics perpendicular to ∂M, we have

$$\nabla_{\partial_m}\partial_m = 0, \quad g_{mm}(y, 0) = 1 \quad \text{and} \quad g_{\alpha m}(y, 0) = 0.$$

In particular, the first derivative of the components of the metric along the normal direction, on a tubular neighborhood is

$$\partial_m g_{mm} = 0.$$

By using the formulas from 1.2.12 on the collared neighborhood, a straightforward computations leads to

$$\mathsf{L}_{\alpha\beta} = -\frac{1}{2}\partial_m g_{\alpha\beta}.$$

1.2.20. Jets of the Riemannian metric in geodesic coordinates. Higher order derivatives (or jets) of the Riemannian metric can be expressed in terms of geometric objects such as the curvature tensor and the second fundamental form. The following was originally proved by Atiyah-Bott and Patodi in [**ABP75**], see also Lemma 1.1.1 and Theorem 1.1.3 in [**Gi04**] and Lemma 1.11.4 in [**Gi01**].

Lemma 1.2.1. *Let M be a compact Riemannian manifold and $x_0 \in \mathsf{Int}(M)$ a point in its interior. Let $(x^1, \cdots, x^m)$ be a geodesic coordinate system centered at $x_0 \in M$. Then*

$$\begin{aligned} g_{\mu\nu}(x_0) &= \delta_{\mu\nu}, \\ \partial_\sigma g_{\mu\nu}(x_0) &= 0 \\ \partial_\nu\partial_\epsilon g_{\mu\sigma}(x_0) &= \tfrac{1}{3}\left(\mathsf{R}_{\mu\nu\sigma\epsilon} - \mathsf{R}_{\mu\epsilon\nu\sigma}\right) \\ \partial_\nu\partial_\epsilon\partial_\alpha g_{\mu\sigma}(x_0) &= \tfrac{1}{3}\left(\partial_\alpha\mathsf{R}_{\mu\nu\sigma\epsilon} - \partial_\alpha\mathsf{R}_{\mu\epsilon\nu\sigma}\right), \end{aligned}$$

More generally, $(\partial_{\alpha_1}\cdots\partial_{\alpha_l} g_{\mu\nu})(x_0)$, arbitrarily higher order derivatives of the metric g at x_0, can be expressed as polynomials in the variables

$$\{\mathsf{R}, \nabla\mathsf{R}, \nabla^2\mathsf{R}\cdots\},$$

i.e., higher order covariant derivatives of the curvature at x_0. In order to account for the boundary, let $y_0 \in \partial M$ and $(y^1, \cdots, y^{m-1})$ be a geodesic coordinate system at y_0 so that (y, x^m) is a local coordinate chart of M, where x^m is the geodesic distance to the boundary. Then

$$\begin{aligned} g_{\mu m}(y_0) &= 0, \\ g_{mm}(y_0) &= 1, \\ g_{\mu\nu}(y_0) &= \delta_{\mu\nu}, \\ \partial_\sigma g_{\mu\nu}(y_0) &= 0, \\ \partial_m g_{\mu\nu}(y_0) &= -2\mathsf{L}_{\mu\nu} \end{aligned}$$

and more generally, $(\partial_{\alpha_1}\cdots\partial_{\alpha_l} g_{\mu\nu})(y_0)$, arbitrarily higher order derivatives of the metric of g at y_0, can be written as polynomials in the variables

$$\{\mathsf{R}, \nabla\mathsf{R}, \nabla^2\mathsf{R}\cdots, \mathsf{L}, \nabla^\partial\mathsf{L}, \nabla^{\partial\,2}\,\mathsf{L}\cdots\},$$

i.e., higher order covariant derivatives of the curvature and the second fundamental form at y_0.

1.2.21. Bundles of densities and volume forms. Let M be a compact manifold of dimension m and $s \in \mathbb{R}$. By using the frame bundle $\mathsf{GL}(M)$ corresponding to the tangent bundle of M, see Section 1.2.2, one can construct a vector bundle over M associated to each linear representation of $\mathsf{GL}(m, \mathbb{R})$. Among these associated bundles, we have the bundle of densities. The bundle $|\Lambda_M|^s \to M$ of **s-densities** over M is defined as the associated bundle to the frame bundle $\mathsf{GL}(M)$ with respect to the one dimensional representation $A \mapsto |\det(A)|^{-s}$ of $\mathsf{GL}(m, \mathbb{R})$. Sections of $|\Lambda_M|^s$ can be seen pointwise as functions

$$\alpha : \Lambda^m T_x M \backslash \{0\} \to \mathbb{R} \quad \text{with} \quad \alpha(\lambda X) = |\lambda|^s \alpha(X)$$

for each $\lambda \neq 0$, $X \in T_x M$ and $x \in M$. The bundle of s-densities is a trivializable bundle, but there is no a canonical trivialization.

We denote by $|\Lambda_M| := |\Lambda_M|^1 \to M$, the bundle of **1-densities**, whose sections are used for instegrating sections on non orientable manifolds: In order to do that, one uses the 1-density $\mathsf{vol}_g(M)$, given locally in terms of a local chart $(U, x_1, \dots, x_m)$ by

$$\mathsf{vol}_g(M)(\partial_{x_1}, \dots, \partial_{x_m}) = 1.$$

There is a unique real linear form $\int_M : \Gamma(M, |\Lambda_M|) \to \mathbb{R}$, called the **integral** over M, which is invariant under diffeomorphisms so that in local coordinates, exactly coincides with the Lebesgue integral: for $f := f \cdot \mathsf{vol}_g(M)(x)$ a smooth section of $|\Lambda_M|$ with compact support contained in U_α, we have

$$\int_M f := \int_{\overline{U}_\alpha} f(x) \cdot \mathsf{vol}_g(M) = \int_{\mathbb{R}^m} f_\alpha(x) dx^1 \cdots dx^m,$$

see for instance Proposition 1.23 in [**BGV92**].

Another example of associated bundle to the frame bundle $\mathsf{GL}(M)$ is the **bundle of volume forms** $\Lambda^m T^*M$, defined as the bundle associated to the one dimensional representation $A \mapsto \det(A)^{-1}$ of $\mathsf{GL}(m, \mathbb{R})$. There is a canonical line bundle isomorphism $\Phi : |\Lambda_M| \to \Lambda^m T^*M \otimes \Theta_M$, defined in such a way that for each $\rho \in |\Lambda_M|$, the map $\Phi(\rho) : TM \times \cdots \times TM \to \Theta_M$, is the skew symmetric m-linear map pointwise given by

$$\Phi(\rho)(e_1, \dots, e_m)_x = (x, \theta(e_1, \dots, e_m)_x \rho(e_1, \dots, e_m)_x),$$

where $\theta(e_1, \dots, e_m)_x$ is the orientation of $T_x M$ determined by the ordered set of linearly independent vectors $e_1, \dots, e_m$ at $T_x M$. We also denote by Φ the corresponding map of smooth sections. The bundle $\Lambda^m T^*M$ is trivializable if and only if M is orientable. If M is oriented, then $|\Lambda_M|$ is canonically isomorphic to $\Lambda^m T^*M$. The space of smooth sections of the bundle $\Lambda^m T^*M \otimes \Theta_M \to M$ is denoted by $\Omega(M; \Theta_M)$. These are also referred as forms on M **twisted** by Θ_M. There is a unique (twisted) De-Rham differential: $d_{\Theta_M} : \Omega^k(M, \Theta_M) \to \Omega^{k+1}(M, \Theta_M)$ such that for $v \in \Omega^k(M, \Theta_M)$ a form on M with values in Θ_M and $v = \alpha \otimes \sigma$, where σ is locally constant on a neighborhood U, then $d_{\Theta_M} v := d\alpha \otimes \sigma$ on U, where d is the De-Rham differential on $\Omega(M)$; for simplicity, we still write d for d_{Θ_M}, whenever no confusion appears.

In order to integrate (twisted) m-forms over M, we use the canonical isomorphism of line bundles isomorphism Φ above In view of Φ, the integral of $v \in \Omega^m(M; \Theta_M)$ is understood as the integral of the 1-density $\Phi^{-1}(v)$ over M. This construction permits us to identify the 1-density $\mathsf{vol}_g(M)$ with the (twisted) top-form $\Phi^{-1}(\mathsf{vol}_g(M)) \in \Omega^m(M; \Theta_M)$, called the **volume form** of (M, g). In the sequel we assume this identification to be made when it comes to integrate m-forms over M. If $\{X_1, \dots, X_m\} \subset T_x M$ is a basis of

T_xM and $\{e_1, \dots, e_m\}$ an orthonormal basis of T_xM, with the same orientation as the one specified by $\{X_1, \dots, X_m\}$ at x, then the value of $\mathsf{vol}_g(M)$ at x is

$$\mathsf{vol}_g(M)_x(X_1, \dots, X_m) := \sqrt{\det(g_x(X_i, X_j))}e^1 \wedge \cdots \wedge e^m.$$

1.2.22. Hodge $*$-operator. For a compact Riemannian manifold M with metric g, and $0 \leqslant k \leqslant m$, consider the bundles $\Lambda^k T^*M \to M$ endowed with the corresponding induced metrics $\langle \cdot, \cdot \rangle_g$. There exists a unique isometric isomorphism of vector bundles $\star_{k,m-k} : \Lambda^k T^*M \to \Lambda^{m-k} T^*M \otimes \Theta_M$, defined by $\alpha \wedge \star\alpha' = \langle \alpha, \alpha' \rangle_g \mathsf{vol}_g(M)$, where $\alpha, \alpha' \in \Lambda^k T^*M$. This operator is called the Hodge $\star$-operator

1.2.23. Stokes Theorem. Let $\Theta_{\partial M}$ be the orientation bundle of ∂M and consider $i : \partial M \hookrightarrow M$ the canonical embedding. In this thesis, the bundle $i^*\Theta_M$ is identified with the bundle $\Theta_{\partial M} \to \partial M$ by using the following convention: Near the boundary consider $-X$ any **outwards** pointing normal vector field to ∂M. Then, a section $\sigma \in \Omega^{m-1}(\partial M; \Theta_{\partial M})$ is identified with the section $-\alpha \wedge \sigma \in \Omega^m(M, \Theta_M)$, where $\alpha \in \Omega^1(M, \Theta_M)$ is a 1-form satisfying $\alpha(X) = 1$, near the boundary. In this way, we say that the (twisted) form σ on ∂M is **induced** by $-\alpha \wedge \sigma$ and we identify $\Theta_M|_{\partial M}$ with $\Theta_{\partial M}$. The Stokes' Theorem for non (necessarily orientable) compact manifolds states that $\int_M d_{\Theta_M} v = \int_{\partial M} i^* v$, for each $v \in \Omega(M; \Theta_M)$. In particular, if M is orientable and $-\alpha \wedge \sigma$ defines an orientation on M, then this convention is in accord with the **induced orientation** on ∂M, that is, the one specified by σ as a $(m-1)$-form on the boundary.

1.2.24. Differential operators. For F, G complex vector bundles over a compact Riemannian manifold M and $\otimes_{k,\mathsf{sym}} T^*M$ the k-fold symmetric tensor product bundle of T^*M, we consider $\mathrm{Hom}(\otimes_{k,\mathsf{sym}} T^*M \otimes F; G) \to M$, the **coefficient bundle** over M, with space of smooth sections $\Gamma(M; \mathrm{Hom}(\otimes_{k,\mathsf{sym}} T^*M \otimes F; G))$. With a connection $\nabla^F : \Gamma(M; F) \to \Gamma(M; T^*M \otimes F)$ on F and the Levi–Cività connection on T^*M seen as derivations, consider corresponding the induced connection on $\otimes_k T^*M \otimes F$,

$$\nabla^{\otimes k} := \nabla^{\otimes_k T^*M \otimes F} : \Gamma(M; \otimes_k T^*M \otimes F) \to \Gamma(M; \otimes_{k+1} T^*M \otimes F)$$

and denote by

$$\nabla^{F,g^k} : \Gamma(M; F) \to \Gamma(M; \otimes_k T^*M \otimes F),$$

the composition $\nabla^{\otimes_{k-1} T^*M \otimes F} \circ \cdots \circ \nabla^{T^*M \otimes F} \circ \nabla^F$ where $\nabla^{F,g^0} = \mathsf{Id}_F$. A **linear differential operator** $\mathsf{D} : \Gamma(M; F) \to \Gamma(M; G)$ of order $d \geqslant 0$ from F to G is a linear operator that can be written as

$$\mathsf{D} = \sum_{k=0}^{d} a_k \circ \nabla^{F,g^k}, \quad \text{where} \quad a_k \in \Gamma(M; \mathrm{Hom}(\otimes_{k,\mathsf{sym}} T^*M \otimes F, G)).$$

This definition is independent of ∇^F and the Riemannian metric g on M.

1.2.25. Ellipticity. For F, G complex vector bundles over M, consider a differential operator $\mathsf{D} : \Gamma(M; F) \to \Gamma(M; G)$ of order $d \geqslant 0$ from F to G. The **principal symbol** of D is the bundle map $\sigma_L(\mathsf{D}) : T^*M \to \mathrm{Hom}(F, G)$, invariantly defined by

$$\sigma_L(\mathsf{D})(\xi) := \mathbf{i}^d \cdot a_d(\xi \otimes \cdots \otimes \xi),$$

for $\xi \in T^*M$. The **symbolic spectrum** of D is the set

$$\mathsf{Spec}_\mathsf{L}(\mathsf{D}) := \{\lambda \in \mathbb{C} | \quad \exists \xi \in T^*M \backslash \{0\} \text{ s.t. } \det(\sigma_L(\mathsf{D})(\xi) - \lambda) = 0\}.$$

A differential operator $\mathsf{D} : \Gamma(M; F) \to \Gamma(M; G)$ is **elliptic** if $0 \notin \mathsf{Spec}_\mathsf{L}(\mathsf{D})$ or equivalently if $\sigma_L(\mathsf{D})(\xi) \in \mathrm{Hom}(F, G)$ is an isomorphism for all $\xi \in T^*M \backslash \{0\}$.

1.3. Operator theory

The material in this section can be found in [**Ka95**] and [**RS78**].

1.3.1. Bounded operators. Let $\mathcal{H}$ be a (complex separable) Hilbert space with Hermitian inner product $\langle \cdot, \cdot \rangle_\mathcal{H}$ and corresponding norm $\| \cdot \|_\mathcal{H}$. A subset $\mathcal{E} \subset \mathcal{H}$ is a **linear subspace** in $\mathcal{H}$, if for every $u, v \in \mathcal{E}$ and $\alpha \in \mathbb{C}$, then $u + \alpha v \in \mathcal{E}$. For a linear subspace $\mathcal{E}$, we denote by $\overline{\mathcal{E}} := \overline{\mathcal{E}}^{\| \cdot \|_\mathcal{H}}$, its closure in $\mathcal{H}$. A linear subspace $\mathcal{E}$ is **complete** if and only if $\mathcal{E} = \overline{\mathcal{E}}$, that is, $\mathcal{E}$ is a Hilbert subspace in $\mathcal{H}$. For $\mathcal{H}_1$ and $\mathcal{H}_2$ two Hilbert spaces, a linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ is **bounded** if there exists a constant $M < \infty$ such that $\|Au\|_{\mathcal{H}_2} \leqslant M \|u\|_{\mathcal{H}_1}$ and the **operator norm** of A, is defined by

$$\|A\|_{\mathcal{H}_1, \mathcal{H}_2} := \sup_{u \in \mathcal{H}_1; u \neq 0} \frac{\|Au\|_{\mathcal{H}_2}}{\|u\|_{\mathcal{H}_1}}.$$

The set of bounded linear operators is denoted by $\mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$, but if $\mathcal{H} = \mathcal{H}_1 = \mathcal{H}_2$, then this is denoted by $\mathbb{B}(\mathcal{H}) := \mathbb{B}(\mathcal{H}, \mathcal{H})$ and called the set of **(linear) bounded operators** on $\mathcal{H}$. Let $\mathsf{Im}(A)$ be the image of A and $\ker(A)$ its kernel. An operator $A \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$ is **invertible**, if $\ker(A) = \{0\}$ and $\mathsf{Im}(A) = \mathcal{H}_2$. If $A \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$ is invertible, by the inverse mapping Theorem, there exists a unique linear operator $A^{-1} \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_1)$ such that $A^{-1}Av = v$ for all $v \in \mathcal{H}_1$ and $AA^{-1}u = u$ for all $u \in \mathcal{H}_2$ and one says that A is an **isomorphism** (of Hilbert spaces). For each $A \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$, there exists a unique $A^* \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_1)$ satisfying $\langle Av, u \rangle_{\mathcal{H}_2} = \langle v, A^*u \rangle_{\mathcal{H}_1}$ for each $v \in \mathcal{H}_1$ and $u \in \mathcal{H}_2$ and called the **adjoint** to A. The space $\mathbb{B}(\mathcal{H})$ with operator norm $\| \cdot \|$ is an involutive Banach algebra over $\mathbb{C}$, the multiplication is the composition of operators, the involution is given by the adjoint operation and the relation $\|A^*A\| = \|A\|^2$ is satisfied, i.e., $\mathbb{B}(\mathcal{H})$ is a C^*-algebra.

1.3.2. Projections and decomposition of Hilbert spaces. Let A be a bounded operator on a Hilbert space $\mathcal{H}$. A linear subspace $\mathcal{E}$ is said to be **invariant** under A, if $A\mathcal{E} \subset \mathcal{E}$. An operator $P \in \mathbb{B}(\mathcal{H})$ satisfying $P^2 = P$ and $P^* = P$ is called an (orthogonal) **projection**. If P is a projection, then $I - P$ is a projection as well and there is a Hilbert space decomposition $\mathcal{H} = \mathcal{H}_P \oplus \mathcal{H}_{I-P}$ where $\mathcal{H}_P := \mathsf{Im}(P)$ and $\mathcal{H}_{I-P} := \mathsf{Im}(I - P)$ are Hilbert subspaces in $\mathcal{H}$ and invariant under P and $I - P$ respectively. Conversely if $\mathcal{H}$

decomposes as the direct sum of Hilbert subspaces $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, then there exists an orthogonal projection P with $\mathcal{H}_1 = \mathsf{Im}(P)$ and $\mathcal{H}_2 = \mathsf{Im}(I - P)$ respectively.

1.3.3. Unbounded operators. An unbounded linear operator from a Hilbert space $\mathcal{H}_1$ to a Hilbert space $\mathcal{H}_2$ is a couple $(T, \mathcal{D}(T))$, where $\mathcal{D}(T) \subset \mathcal{H}_1$ is a linear subspace of $\mathcal{H}_1$, called the **domain of definition** of T, and $T : \mathcal{D}(T) \to \mathcal{H}_2$ is a linear map satisfying

$$T(\alpha u + v) = \alpha T u + T v, \quad \text{for all} \quad u, v \in \mathcal{D}(T) \quad \text{and} \quad \alpha \in \mathbb{C}$$

.

Let $(T, \mathcal{D}(T))$ and $(S, \mathcal{D}(S))$ be two unbounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. The **addition** of T and S is the unbounded operator $(T + S, \mathcal{D}(T + S))$ where

$$\mathcal{D}(T + S) := \mathcal{D}(T) \cap \mathcal{D}(S)$$

and

$$(T + S)u := Tu + Su \quad \text{for all} \quad u \in \mathcal{D}(T + S).$$

The **composition** of two unbounded operator is the unbounded operator $(TS, \mathcal{D}(TS))$, where

$$\mathcal{D}(TS) := \{u \in \mathcal{D}(S) | Su \in \mathcal{D}(T)\}$$

and

$$(TS)u := T(Su), \quad \text{for all} \quad u \in \mathcal{D}(TS).$$

In general, the linear subspaces $\mathcal{D}(T + S)$ and $\mathcal{D}(TS)$ are not necessarily closed and they might consist of 0 only. If $A \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$, then $\mathcal{D}(T + A) = \mathcal{D}(T)$ and $\mathcal{D}(AT) = \mathcal{D}(T)$. The **commutator** $([T, S], \mathcal{D}([T, S]))$ of two unbounded operators $(T, \mathcal{D}(T))$ and $(S, \mathcal{D}(S))$ on $\mathcal{H}$ is defined by

$$[T, S]u := TSu - STu, \quad \text{for all} \quad u \in \mathcal{D}([T, S]),$$

where

$$\mathcal{D}([T, S]) := \{u \in \mathcal{D}(T) \cap \mathcal{D}(S) | Su \in \mathcal{D}(T) \text{ and } Tu \in \mathcal{D}(S)\}.$$

The operators $(T, \mathcal{D}(T))$ and $(S, \mathcal{D}(S))$ **commute** if $TSu = STu$ for all $u \in \mathcal{D}([T, S])$.

1.3.4. Extensions of unbounded operators. One says that two unbounded operators $(T, \mathcal{D}(T))$ and $(S, \mathcal{D}(S))$ from $\mathcal{H}_1$ to $\mathcal{H}_2$, define the same operator if $\mathcal{D}(T) = \mathcal{D}(S)$ and $Tu = Su$, for each $u \in \mathcal{D}(T)$. If $\mathcal{D}(T) \subset \mathcal{D}(S)$ and $Tu = Su$ for all $u \in \mathcal{D}(T)$, then $(S, \mathcal{D}(S))$ is an **extension** of $(T, \mathcal{D}(T))$ or, equivalently, $(T, \mathcal{D}(T))$ is a **restriction** of $(S, \mathcal{D}(S))$, in which case one writes $(T, \mathcal{D}(T)) \subset (S, \mathcal{D}(S))$. An unbounded operator $T : \mathcal{D}(T) \to \mathcal{H}_2$ is bounded on its domain $\mathcal{D}(T) \subset \mathcal{H}_1$, if there exists a constant $M < \infty$ such that $\|Tu\|_{\mathcal{H}_2} \leqslant M\|u\|_{\mathcal{H}_1}$ for all $u \in \mathcal{D}(T)$. If T is bounded on its domain $\mathcal{D}(T)$, then its operator norm is computed as the infimum of such M's on $\mathcal{D}(T)$. If T is an unbounded operator but bounded on its domain $\mathcal{D}(T)$, then T admits a **natural extension** $(\overline{T}, \mathcal{D}(\overline{T}))$, where $\mathcal{D}(\overline{T}) := \overline{\mathcal{D}(T)}$ is the closure of $\mathcal{D}(T)$ in $\mathcal{H}$ and the operator $\overline{T}$ is defined as follows: If $u \in \mathcal{D}(T)$ then $\overline{T}u := Tu$. If $u \in \overline{\mathcal{D}(T)} \backslash \mathcal{D}(T)$, then, one chooses a sequence $\{u_n\} \subset \mathcal{D}(T)$ converging to u in the $\|\cdot\|_{\mathcal{H}_1}$-norm so that, the sequence Tu_n converges $v \in \mathcal{H}_2$ in the $\|\cdot\|_{\mathcal{H}_2}$-norm, as T is bounded on $\mathcal{D}(T)$, and then one sets $\overline{T}u := v$. This definition does not depend on the choice of the sequence involved, as long as T is bounded on $\mathcal{D}(T)$. In addition, if $\mathcal{D}(T)$ is dense in $\mathcal{H}_1$, then $\overline{T} \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$.

1.3.5. Closed operators. An unbounded operator $(T, \mathcal{D}(T))$ from $\mathcal{H}_1$ to $\mathcal{H}_2$, is c**losed** if every sequence $\{u_n\}$ in $\mathcal{D}(T)$ that converges to u in $\mathcal{H}_1$ and that $\{Tu_n\}$ converges to v in $\mathcal{H}_2$, then $u \in \mathcal{D}(T)$ and $Tu = v$. An unbounded operator T bounded on its domain $\mathcal{D}(T)$, is closed if and only if $\mathcal{D}(T)$ is closed; in particular, every bounded operator is closed. If $(T, \mathcal{D}(T))$ is closed and $(S, \mathcal{D}(S))$ bounded on its domain with $\mathcal{D}(S) \supset \mathcal{D}(T)$, then $(T + S, \mathcal{D}(T + S))$ is closed.

Consider $\mathsf{Im}(T) := \{u \in \mathcal{H}_2 |$ there is $v \in \mathcal{D}(T)$ with $u = Tv\}$ the image of T and its kernel $\ker(T) := \{v \in \mathcal{D}(T) | Tv = 0\}$. An unbounded operator $(T, \mathcal{D}(T))$ from $\mathcal{H}_1$ to $\mathcal{H}_2$ is invertible on its image if $\ker(T) = \{0\}$ and there exists a unique linear operator $(T^{-1}, \mathcal{D}(T^{-1}))$ from $\mathcal{H}_2$ to $\mathcal{H}_1$, with $\mathcal{D}(T^{-1}) = \mathsf{Im}(T)$, $\mathsf{Im}(T^{-1}) = \mathcal{D}(T)$ such that their composition satisfy $T^{-1}Tv = v$ for all $v \in \mathcal{D}(T)$ and $TT^{-1}u = u$ for all $u \in \mathcal{D}(T^{-1})$. If $(T, \mathcal{D}(T))$ is invertible and closed, then $(T^{-1}, \mathcal{D}(T^{-1}))$ is closed. If $(T, \mathcal{D}(T))$ is an unbounded operator from $\mathcal{H}_2$ to $\mathcal{H}_3$ and $(S, \mathcal{D}(S))$, an unbounded operator from $\mathcal{H}_1$ to $\mathcal{H}_2$, are closed operators, and $T^{-1} \in \mathbb{B}(\mathcal{H}_3, \mathcal{H}_2)$, then $(TS, \mathcal{D}(TS))$, unbounded from $\mathcal{H}_1$ to $\mathcal{H}_3$, is also closed.

1.3.6. Graph norm. Let $(T, \mathcal{D}(T))$ be an unbounded operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. Consider the product Hilbert space $\mathcal{H}_1 \times \mathcal{H}_2$ endowed with the norm

$$\|(u, v)\|_{\mathcal{H}_1 \times \mathcal{H}_2} := (\|u\|^2_{\mathcal{H}_1} + \|v\|^2_{\mathcal{H}_2})^{1/2}.$$

The **graph** of $(T, \mathcal{D}(T))$ is the linear subspace

$$G(T) := \{(u, Tu) | u \in \mathcal{D}(T)\} \subset \mathcal{H}_1 \times \mathcal{H}_2.$$

The operator $(T, \mathcal{D}(T))$ is closed if and only if its graph $G(T)$ is a closed linear subspace in $\mathcal{H}_1 \times \mathcal{H}_2$ with respect to the norm $\|(\cdot, \cdot)\|_{\mathcal{H}_1 \times \mathcal{H}_2}$. If $(T, \mathcal{D}(T))$ is closed, then $G(T)$ is a Hilbert space, with inner product

$$\langle u, v \rangle_T := \langle u, v \rangle_{\mathcal{H}_1} + \langle Tu, Tv \rangle_{\mathcal{H}_2}$$

for $u, v \in \mathcal{D}(T)$. The associated norm to $\langle u, v \rangle_T$ is exactly the norm $\|(\cdot, \cdot)\|_{\mathcal{H}_1 \times \mathcal{H}_2}$ in $\mathcal{H}_1 \times \mathcal{H}_2$ restricted to the graph of T. This is the **graph norm** associated to T:

$$\|u\|_T = (\|u\|^2_{\mathcal{H}_1} + \|Tu\|^2_{\mathcal{H}_2})^{1/2}$$

for $u \in \mathcal{D}(T)$.

1.3.7. Closeable operators. An unbounded operator $(T, \mathcal{D}(T))$ from $\mathcal{H}_1$ to $\mathcal{H}_2$ is **closeable** , if it admits a closed extension $(S, \mathcal{D}(S))$; that is, if for every sequence $\{u_n\}$ in $\mathcal{D}(T)$ converging to 0 in the $\|\cdot\|_{\mathcal{H}_1}$-norm such that $\{Tu_n\}$ converges in the $\|\cdot\|_{\mathcal{H}_2}$-norm, then we necessarily have $\lim_{n\to 0} \|Tu_n\|_{\mathcal{H}_2} = 0$. T is closeable if and only if the closure of $\overline{G(T)}$ with respect to the $\|\cdot\|_T$-norm is itself a graph. If $(T, \mathcal{D}(T))$ is closeable, then there exists a unique operator $(\overline{T}, \mathcal{D}(\overline{T}))$ whose graph $G(\overline{T})$ is exactly the closure of $\overline{G(T)}$ with respect to $\|\cdot\|_T$. The operator $(\overline{T}, \mathcal{D}(\overline{T}))$, called **its closure (extension)**, is exactly the smallest closed extension of $(T, \mathcal{D}(T))$ and $u \in \mathcal{D}(\overline{T})$ if and only if $(u, \overline{T}u) \in \overline{G(T)}^{\|\cdot\|_T}$. In other words, $u \in \mathcal{D}(\overline{T})$ if and only if there exists a Cauchy sequence $u_n \in \mathcal{D}(T)$ converging to u such that (u_n, Tu_n) is a convergent sequence in $\mathcal{H}_1 \times \mathcal{H}_2$ in the graph norm. In this situation, one sets $\overline{T}u = \lim_{n\to\infty} Tu_n$. This notion generalizes that of closure for an operator T bounded on its domain.

1.3.8. The adjoint. Consider un unbounded operator $(T, \mathcal{D}(T))$ from $\mathcal{H}_1$ to $\mathcal{H}_2$ with $\mathcal{D}(T)$ dense in $\mathcal{H}_1$. The **adjoint** of T is the unbounded operator $(T^*, \mathcal{D}(T^*))$ from $\mathcal{H}_2$ to $\mathcal{H}_1$, where $\mathcal{D}(T^*)$ consists of all $u \in \mathcal{H}_2$ for which there exists $u^* \in \mathcal{H}_1$ such that $\langle u^*, v\rangle_{\mathcal{H}_1} = \langle u, Tv\rangle_{\mathcal{H}_2}$ for all $v \in \mathcal{D}(T)$ and T is defined by $T^*u := u^*$ if $u \in \mathcal{D}(T^*)$. The condition of $\mathcal{D}(T)$ being dense in $\mathcal{H}_1$ is necessary for T^* to be well defined. The linear subspace $\mathcal{D}(T^*)$ could a priori be trivial, but if T is closeable, then $\mathcal{D}(T^*)$ is dense in $\mathcal{H}_2$. If $(T, \mathcal{D}(T))$ is an unbounded operator on $\mathcal{H}$, with domain $\mathcal{D}(T)$ dense in $\mathcal{H}$, then $(T^*, \mathcal{D}(T^*)$ is a closed operator with $\ker(T^*) = \mathsf{Im}(T)^\perp$.

1.3.9. The resolvent and the spectrum. Let $(T, \mathcal{D}(T))$ be a closed unbounded operator on $\mathcal{H}$. The **resolvent set** of T is the set $\rho(T)$ consisting of all complex numbers $z \in \mathbb{C}$ such that $(T - z)$ is invertible with bounded inverse. In other words, z is in the resolvent set of T if and only if $\ker(T - z) = \{0\}$, $\mathsf{Im}(T - z) = \mathcal{H}_2$ and $(T - z)^{-1}$ is bounded. Remark that the set $\rho(T)$ is an open set in $\mathbb{C}$. For $z \in \rho(T)$, the bounded operator $\mathsf{R}_T(z) := (T - z)^{-1}$ is called the resolvent of T at z and it provides a bijection between $\mathcal{H}$ and $\mathcal{D}(T)$. The map $z \mapsto \mathsf{R}_T(z)$ is called the **resolvent** of T. The resolvent of T is holomorphic: for each $z \in \rho(T)$, the function $z \mapsto \mathsf{R}_T(z)$ admits a Taylor expansion in the operator norm. If $z_1, z_2 \in \rho(T)$ the operators $(T - z_1)^{-1}$ and $(T - z_2)^{-1}$ commute.

The **spectrum** of T is the closed subset $\mathsf{spec}(T) := \mathbb{C}\backslash\rho(T)$ in $\mathbb{C}$. Note that $z \in \mathsf{spec}(T)$ if and only if the operator $(T - z)$ is not injective or it is not surjective or it does not admit a bounded inverse. A complex number λ is an **eigenvalue** of $(T, \mathcal{D}(T))$ if $\exists u \in \mathcal{D}(T)$, $u \neq 0$, with $Tu = \lambda u$. All eigenvalues of T are contained in $\mathsf{spec}(T)$. If $\lambda \in \mathbb{C}$ is an eigenvalue of $(T, \mathcal{D}(T))$, then $\mathcal{H}_T(\lambda) := \{u \in \mathcal{D}(T) | Tu = \lambda u\}$, is the **eigen-space** of T corresponding to λ.

1.3.10. Compact operators. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two infinite dimensional Hilbert spaces. An operator $A \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$ is called **compact** if for any bounded sequence $\{v_n\}$ in $\mathcal{H}_1$ the sequence $\{Av_n\}$ contains a subsequence which converges in $\mathcal{H}_2$. The space $\mathbb{K}(\mathcal{H}_1, \mathcal{H}_2)$ of all compact operators from $\mathcal{H}_1$ to $\mathcal{H}_2$, is a closed (two sided) ideal in

$\mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$. The spectrum of a compact operator $K \in \mathbb{K}(\mathcal{H}_1, \mathcal{H}_2)$ is either a finite subset in $\mathbb{C}$ containing 0, or a countably infinite set in $\mathbb{C}$ with 0 as its only accumulation point. Each $\lambda \in \mathsf{spec}(K)$, with $\lambda \neq 0$, is an eigenvalue of K of finite multiplicity.

1.3.11. Fredholm operators. Let $A \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$ be a bounded linear operator. A is **Fredholm** if its kernel $\mathsf{Ker}(A)$ and its cokernel $\mathcal{H}_2/\mathsf{Im}(A)$ are of finite dimension. In particular $\mathsf{Im}(A)$ is closed in $\mathcal{H}_2$. A bounded operator A is Fredholm if and only there is $S \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_1)$ such that $SA - \mathsf{Id}_{\mathcal{H}_1} \in \mathbb{K}(\mathcal{H}_1)$ and $AS - \mathsf{Id}_{\mathcal{H}_2} \in \mathbb{K}(\mathcal{H}_2)$.

1.3.12. Trace class operators. An important class of compact operator is the so called trace class operators. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, $T \in \mathbb{K}(\mathcal{H}_1, \mathcal{H}_2)$ a compact operator and T^* its adjoint. Then, $\mathsf{Spec}(T^*T)$, the spectrum of $T^*T \in \mathbb{K}(H_1)$, consists of real nonnegative eigenvalues only. One says that the operator $T \in \mathbb{K}(\mathcal{H}_1, \mathcal{H}_2)$ is of **trace class** if the formal series $\sum_{\mu \in \mathsf{Spec}(T^*T)} \mu^{1/2}$ converges. If $T \in \mathbb{K}(\mathcal{H}_1, \mathcal{H}_2)$ is of trace class and $B \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_3)$ is a bounded operator, then the compact operator $BT \in \mathbb{K}(\mathcal{H}_1, \mathcal{H}_3)$ is also of trace class. Analogously, if $B \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $T \in \mathbb{K}(\mathcal{H}_2, \mathcal{H}_3)$ is of trace class, then $TB \in \mathbb{K}(\mathcal{H}_1, \mathcal{H}_3)$ is of trace class as well.

1.3.13. Unbounded operators with compact resolvent. A more general class of unbounded operators which possess similar properties as those in the bounded case are the **operators with compact resolvent** . If $(T, \mathcal{D}(T))$ is a closed operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ with compact resolvent, that is if $\mathsf{R}_T(z) \in \mathbb{K}(\mathcal{H}_1, \mathcal{H}_2)$, for some $z \in \mathbb{C}$, then $\mathsf{spec}(T)$ consists of isolated eigenvalues with finite multiplicity only and $\mathsf{R}_T(z)$ is compact for all $z \in \rho(T)$.

1.3.14. Commutativity and decomposition. Let $(T, \mathcal{D}(T))$ be an unbounded operator acting on $\mathcal{H}$ and $A \in \mathbb{B}(\mathcal{H})$ a bounded operator. We say that that T commutes with the bounded operator A, if for each $u \in \mathcal{D}(T)$ we have $Au \in \mathcal{D}(T)$ and $TAu = ATu$. Consider a decomposition of Hilbert spaces $\mathcal{H} = \mathcal{H}_P \oplus \mathcal{H}_{I-P}$ with $P \in \mathbb{B}(\mathcal{H}, \mathcal{H}_1)$ the bounded orthogonal projection on $\mathcal{H}_P$. The operator $(T, \mathcal{D}(T))$ is said to be **decomposable** according to the decomposition of $\mathcal{H}$ above, if T commutes with P, or in other words, if $P\mathcal{D}(T) \subset \mathcal{D}(T)$ and if T leaves invariant $\mathcal{H}_P$ and $\mathcal{H}_{I-P}$ in the sense that $T(\mathcal{D}(T) \cap \mathcal{H}_P) \subset \mathcal{H}_P$ and $T(\mathcal{D}(T) \cap \mathcal{H}_{I-P}) \subset \mathcal{H}_{I-P}$. If $(T, \mathcal{D}(T))$ is decomposable as above, then the restriction of T to $\mathcal{H}_P$, is denoted by $(T_{\mathcal{H}_P}, \mathcal{D}(T_{\mathcal{H}_P}))$, where $\mathcal{D}(T_{\mathcal{H}_P}) := \mathcal{D}(T) \cap \mathcal{H}_P$ and $T_{\mathcal{H}_P}u := Tu$, for all $u \in \mathcal{D}(T_{\mathcal{H}_P})$. The restriction of T to $\mathcal{H}_{I-P}$ is defined in the same way. If T is closed, then $T_{\mathcal{H}_P}$ and $T_{\mathcal{H}_{I-P}}$ are closed. The following result is Theorem 6.17 in [**Ka95**].

Theorem 1.3.1. *Let $(T, \mathcal{D}(T))$ be a closed unbounded operator acting on a Hilbert space $\mathcal{H}$. Suppose that $\mathsf{spec}(T)$ splits into two disjoint parts $\mathsf{spec}(T)_1$ and $\mathsf{spec}(T)_2$, such that $\mathsf{spec}(T)_1$ is a bounded subset in $\mathbb{C}$ that can be enclosed in the interior of a simple closed curve Γ and $\mathsf{spec}(T)_2$ in its exterior. Then, $\mathcal{H}$ decomposes as a direct sum of Hilbert spaces $\mathcal{H}_1 \oplus \mathcal{H}_2$ in such a way that $\mathsf{spec}(T)_1 = \mathsf{spec}(T_{\mathcal{H}_1})$ and $\mathsf{spec}(T)_2 = \mathsf{spec}(T_{\mathcal{H}_2})$,* *where* $(T_{\mathcal{H}_1}, \mathcal{D}(T_{\mathcal{H}_1}))$ *and* $(T_{\mathcal{H}_2}, \mathcal{D}(T_{\mathcal{H}_2}))$ *are the restrictions of* T *to the spaces* $\mathcal{H}_1$ *and* $\mathcal{H}_2$*, respectively. Moreover* $T_{\mathcal{H}_1} \in \mathbb{B}(\mathcal{H}_1)$ *and the (Riesz) projection corresponding to this subspace is given by* $P_{\mathcal{H}_1} : \mathcal{H} \to \mathcal{H}_1$ *the bounded operator given by*

$$P_{\mathcal{H}_1} := -\frac{1}{2\pi\mathbf{i}} \int_\Gamma (T-z)^{-1} dz : \mathcal{H} \to \mathcal{H}_1.$$

1.4. Analysis on manifolds

For the material in this section, we refer the reader to [**BGV92**], [**BW93**], [**Hö83**], [**Agr97**], [**FJ98**], [**Gi04**], [**Hö83**] and [**Gru96**].

1.4.1. Sobolev spaces. We first recall Sobolev spaces on $\mathbb{R}^n$. For $s \geqslant 0$, the **Sobolev space** $\mathsf{H}_s(\mathbb{R}^n)$ of order s of square integrable functions on $\mathbb{R}^n$ is defined as the Hilbert space

$$\mathsf{H}_s(\mathbb{R}^n) := \left\{ f \,\middle|\, f \in \mathsf{L}^2(\mathbb{R}^n) \text{ s.t. } \xi \mapsto (1+|\xi|^2)^{s/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx \in \mathsf{L}^2(\mathbb{R}^n) \right\},$$

with Sobolev s-norm

$$\|f\|_s := \frac{1}{(2\pi)^{n/2}} \left\| \xi \mapsto (1+|\xi|^2)^{s/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx \right\|_{\mathsf{L}^2(\mathbb{R}^n)}.$$

Then, every compact Riemannian manifold M of dimension m, with Riemannian metric g and closed boundary ∂M, can be embedded in an m-dimensional closed Riemannian manifold $\widetilde{M}$ with Riemannian metric $\widetilde{g}$). Let $\pi : F \to M$ be a complex vector bundle over M of rank k and $F|_{\partial M}$ the corresponding restriction bundle to the boundary. Let h be a fiberwise positive definite Hermitian metric on F. There exists a complex vector bundle $\widetilde{F}$ over $\widetilde{M}$ with a fiberwise positive definite Hermitian metric $\widetilde{h}$, such that the sub-bundle of $\widetilde{F}|_M$ coincides with F as bundles over M, and the Hermitian metric h on F coincides with the restriction of $\widetilde{h}$ to $\widetilde{F}|_M$. We denote by $\mathsf{L}^2(\widetilde{M}; \widetilde{F})$ the **space of square integrable sections** obtained by completing $\Gamma(\widetilde{M}; \widetilde{F})$ with respect to the L^2-norm associated to the inner product $\ll \cdot, \cdot \gg_{\widetilde{g}, \widetilde{h}}$ on $\Gamma(\widetilde{M}; \widetilde{F})$ induced by $\widetilde{g}$ and $\widetilde{h}$. Recall that the L^2-closure is independent on the underlying Riemannian and Hermitian choices.

In analogy with an s-norm on $\mathbb{R}^n$, there is the notion of an s-norm on $\Gamma(\widetilde{M}; \widetilde{F})$. Let $(\widetilde{U}_i, \widetilde{\phi}_i, \psi_i)$ be a vector bundle trivializating atlas, that is, the data $\{(\widetilde{U}_i, \widetilde{\phi}_i) | \widetilde{\phi}_i : \widetilde{\pi}^{-1}(\widetilde{U}_i) \to \widetilde{U}_i \times \mathbb{C}^k\}$ is a vector bundle trivialization of $\widetilde{\pi} : \widetilde{F} \to \widetilde{M}$ and $\widetilde{\psi}_i : \widetilde{U}_i \to \widetilde{\psi}_i(\widetilde{U}_i) \subset \mathbb{R}^m$ an associated coordinate chart. Let $\{\widetilde{\rho}_i\}$ be a subordinate partition of unity. On each $\widetilde{U}_i$, consider the function $\widetilde{\mathsf{pr}}_2 \circ \widetilde{\phi}_i \circ (\widetilde{\rho}_i \widetilde{u}) \circ \widetilde{\psi}_i^{-1} : \widetilde{\psi}_i(\widetilde{U}_i) \to \mathbb{C}^k$ where $\widetilde{\mathsf{pr}}_2$ is the projection in the second factor of $\widetilde{\phi}_i(u)$. The Sobolev s-norm of a section $\widetilde{u} \in \Gamma(\widetilde{M}; \widetilde{F})$ is defined by

$$\|\widetilde{u}\|_s^2 := \sum_i \|\widetilde{\mathsf{pr}}_2 \circ \widetilde{\phi}_i \circ (\widetilde{\rho}_i \widetilde{u}) \circ \widetilde{\psi}_i^{-1}\|_s^2,$$

where the s-norm on the right above is computed as for Sobolev spaces on $\mathbb{R}^n$, by using the L^2-norm induced by $\widetilde{g}$ and $\widetilde{h}$.

The Sobolev spaces $\mathsf{H}_s(\widetilde{M};\widetilde{F})$ of order $s \geqslant 0$ are obtained as completions of $\Gamma(\widetilde{M};\widetilde{F})$, with respect to the Sobolev s-norms $\|\cdot\|_s$. In words, these spaces consist of all $\widetilde{F}$-valued L^2-differential forms over $\widetilde{M}$, which in local coordinates correspond to $\widetilde{F}$-valued H_s-differential forms. Although the definition above is given in terms of vector bundle trivializations atlases and subordinate partition of unity, the topologies generated by each of these norms are equivalent. Finally, we point out here that $H_0(\widetilde{M};\widetilde{F}) \cong \mathsf{L}^2(\widetilde{M};\widetilde{F})$, $\Gamma(\widetilde{M};\widetilde{F}) = \cap_{s\in\mathbb{N}}\mathsf{H}_s(\widetilde{M};\widetilde{F})$ and $\Gamma(\widetilde{M};\widetilde{F}) \subset \mathsf{H}_s(\widetilde{M};\widetilde{F}) \subset \mathsf{L}^2(\widetilde{M};\widetilde{F})$ for $s \geqslant 0$.

Now we look at the boundary. Since ∂M is a closed Riemannian manifold, we use this construction to define the spaces $\mathsf{H}_s(\partial M; F|_{\partial M})$. In order to define the spaces $\mathsf{H}_s(M;F)$ we use the map $r_M : \mathsf{H}_s(\widetilde{M};\widetilde{F}) \to \mathsf{H}_s(M;F)$, taking sections $u \in \mathsf{H}_s(\widetilde{M},\widetilde{F})$ to $r_M u := u|_M$, their restrictions to M. Then we set $\mathsf{H}_s(M;F) := r_M \mathsf{H}_s\left(\widetilde{M};\widetilde{F}\right)$. In this situation, for $s \geqslant 0$, the relations $\Gamma(M;F) \subset \mathsf{H}_s(M;F) \subset \mathsf{L}^2(M;F)$ and $\Gamma(M;F) = \cap_{s\in\mathbb{N}}\mathsf{H}_s(M;F)$ hold as well.

1.4.2. Distributions. For F a complex vector bundle over M, consider F' its dual bundle and denote by $F'_M := F' \otimes |\Lambda_M|$ where The space of distributions $\Gamma^{-\infty}(M;F)$, also called generalized sections of the vector bundle F, is defined as the topological dual of the space of smooth sections $\Gamma(M;F'_M)$, endowed with the strong topology, i.e. uniform convergence of sections and their derivatives.

There is a canonical embedding $\Gamma(M,F) \hookrightarrow \Gamma^{-\infty}(M,F)$, identifying each $v \in \Gamma(M;F)$ with the functional $\rho_v :=< \cdot, v >$, where

$$< \cdot,\cdot >: \Gamma(M;F) \times \Gamma(M;F) \to \Gamma(M;|\Lambda_M|),$$

induced point-wise by the natural pairing ${F'_M}_y \otimes F_y \to \mathbb{C}$ for each $y \in M$, is given by

$$\int_M \rho(y)(w(y))\mathsf{vol}_g(M)(y),$$

for all $\rho \in \Gamma(M;F'_M)$ and $w \in \Gamma(M;F)$.

1.4.3. Smoothing operators. For E and F two complex vector bundles over M, a bounded linear operator $P : \Gamma(M;E) \to \Gamma^{-\infty}(M;F)$ is called a **generalized operator**. Among generalized operators, we have smoothing operators, which in certain sense allow us to *turn* generalized sections into smooth sections. More precisely, a generalized operator P as above is **smoothing** if it takes values in $\Gamma(M;F)$ and if it extends as a bounded linear map $P : \Gamma^{-\infty}(M;E) \to \Gamma(M;F)$. These operators can be described in terms of their kernels, for which we first recall the following notion.

For $i = 1,2$ consider the projection $\mathsf{pr}_i : M \times M \to M$ of $M \times M$ into its i-factor and define the bundle

$$F \boxtimes E'_M := \mathsf{pr}_1^* F \otimes \mathsf{pr}_2^* E'_M$$

as a vector bundle over $M \times M$, called the **big endomorphism bundle** over $M \times M$. The fiber of $F \boxtimes E'_M$ at $(x_1, x_2) \in M \times M$ is $F_{x_1} \otimes E'_{M\,x_2} \otimes |\Lambda_M|_{x_2}$.

1.4.4. Schwartz kernel. There is a one-to-one correspondence between the space of generalized sections of $F \boxtimes E'_M$ and the space of generalized operators from $\Gamma(M; E)$ to $\Gamma^{-\infty}(M; F)$:

$$\{K \in \Gamma^{-\infty}(M \times M; F \boxtimes E'_M)\} \longleftrightarrow \{P : \Gamma(M; E) \to \Gamma^{-\infty}(M; F)\},$$

Indeed, sections of $\Gamma(M \times M; (F \boxtimes E'_M)'_{M\times M})$ can be regarded as section of the bundle $\Gamma(M \times M; F'_M \boxtimes E)$ and reciprocally: There is a canonical identification

$$\left(F \boxtimes E'_M\right)'_{M\times M} \cong F'_M \boxtimes E$$

obtained by identifying $|\Lambda_{M\times M}|$, the density bundle of $M \times M$, with $|\Lambda_M| \otimes |\Lambda_M|$ and using that $|\Lambda_M|' \otimes |\Lambda_M| \cong \mathrm{End}(|\Lambda_M|)$ is canonically isomorphic with the trivial line bundle.

Therefore, for each generalized section $K \in \Gamma^{-\infty}(M \times M; F \boxtimes E'_M)$, i.e. a continuous map $K : \Gamma(M \times M; F'_M \boxtimes E) \to \mathbb{C}$, one defines the generalized operator

$$P_K : \Gamma(M; E) \to \Gamma^{-\infty}(M; F) \quad \text{by} \quad P_K(\phi)(\psi) := K(\psi \otimes \phi),$$

for $\phi \in \Gamma(M, E)$, $\psi \in \Gamma(M, F'_M)$ and $\psi \otimes \phi \in \Gamma(M \times M; F'_M \boxtimes E)$. In this manner, the distribution K, associated to the generalized operator P_K, is called the **kernel** of P_K. This correspondence gives a characterization for smoothing operators, expressed in the following result known as the Schwartz kernel Theorem, a proof which can be found for instance at page 70 in [**FJ98**].

Theorem 1.4.1. *A generalized operator P_K arising from a* ***smooth kernel****, i.e., K in $\Gamma(M \times M; F \boxtimes E'_M)$, is exactly a smoothing operator, i.e., $P_K : \Gamma^{-\infty}(M; E) \to \Gamma(M; F)$ is a bounded linear operator; in other words, the assignment*

$$\begin{array}{rcl} \Gamma(M \times M; F \boxtimes E'_M) & \to & \{\Gamma^{-\infty}(M; E) \to \Gamma(M; F)\} \\ K & \mapsto & P_K : \phi \mapsto P_K\phi \\ & & \qquad (P_K\phi)(x) := \int_{M \in y} (K(x, y)\phi(y)) \mathsf{vol}_g(M)(y) \end{array}$$

is an isomorphism.

CHAPTER 2

Elliptic boundary value problems

This chapter contains the background for Chapter 3 and Chapter 4. In Section 2.1, we discuss generalities of boundary value problems consisting of Laplace type operators, under local boundary conditions. In Section 2.2, the notions of Lopatijnski–Shapiro condition and boundary ellipticity (with respect to a cone), used to characterize boundary ellipticity for boundary value problems, are recalled. In general, boundary ellipticity is needed to obtain existence results for the solutions of a boundary value problem, and it guarantees the existence of elliptic estimates, see Section 2.3.1. In turn, elliptic estimates are used to study regularity for the solutions of the boundary value problem. In Section 2.3.2, elliptic estimates in order to study closed extensions of the corresponding generalized Laplacian, as unbounded operator in certain Sobolev spaces, see Proposition 2.3.3. These extensions are called Sobolev realizations for the boundary value problem, among which L^2-realizations are the object of our attention. We are interested in studying the spectral properties of L^2-realizations for certain boundary value problems In Section 2.3.3, based on known results for the resolvent of these operators, see Proposition 2.3.5, we obtain a characterization of the spectrum of these operators, see Proposition 2.3.7.

The material presented in this chapter can be found [**Agr97**], [**Agm65**], [**Se67**], [**BW93**], [**Gi84**], [**Gi04**], [**Gre71**], [**Gru96**], [**Hö83**], [**Ki01**] and [**Sh01**].

2.1. Operators of Laplace type and boundary operators

Let F, G be complex two complex vector bundles over a compact manifold M. Consider

$$\mathsf{D} : \Gamma(M;F)|_{\mathcal{B}} \longrightarrow \Gamma(M;G)$$

a differential operator acting on the space $\Gamma(M;F)|_{\mathcal{B}}$ of smooth sections of F satisfying appropriate boundary conditions. The data $(\mathsf{D}, \Gamma(M;F)|_{\mathcal{B}})$ will be referred as a **boundary value problem**. For given $u \in \Gamma(M;F)$ one would like to know whether there is a solution v in certain space of solutions, satisfying the specified boundary conditions and $\mathsf{D}v(x) = u(x)$ for $x \in M$. Subsequently, this leads to ask whether D, regarded as unbounded operator with domain of definition $\mathcal{D}(D)$, extends as a Fredholm operator to certain conveniently well-chosen Sobolev spaces. To deal with this problem, it is not enough to ask for D to be elliptic in the interior of M. In addition, one needs a local condition on the behavior of the solutions along the normal direction, in a tubular neighborhood, of ∂M in M. This condition is given by the Lopatijnsky–Shapiro condition, in Section 2.2. In this thesis, we are interested in elliptic boundary value problems consisting of Laplace type operators, under conveniently imposed absolute and relative boundary conditions on different parts of the boundary. Before getting there, we recall the reader known notions on generalized Laplacians and local boundary conditions.

Definition 2.1.1. *Let M be a compact manifold and F be a complex vector bundles over M. A differential operator $\mathsf{D} : \Gamma(M;F) \to \Gamma(M;F)$ of order $d = 2$ is of **Laplace type** (or **generalized Laplacian**) if $\sigma_L(\mathsf{D})(\xi) = \|\xi\|^2 \mathsf{id}_F$, for every $\xi \in T^*M$, where id_F is the identity in F.*

From its definition a Laplace type operator is elliptic. The following Lemma recalls that an operator of Laplace type acting on smooth sections of a complex vector bundle is entirely characterized by a linear connection and an endomorphism on the bundle.

Lemma 2.1.2. *Let M be a compact Riemannian manifold and F a complex vector bundle over M. Let $\mathsf{D} : \Gamma(M;F) \to \Gamma(M;F)$ be an operator of Laplace type. Then, there exists a unique connection ∇^{D} on F and a unique endomorphism E^{D} on F so that*

$$\mathsf{D} = \mathsf{D}(\nabla^{\mathsf{D}}, \mathsf{E}^{\mathsf{D}}) = -(\mathrm{Tr}_g(\nabla^{\mathsf{D},g} \circ \nabla^{\mathsf{D}}) + \mathsf{E}^{\mathsf{D}}), \tag{2.1}$$

where $\nabla^{\mathsf{D},g}$ is the connection induced by ∇^{D} on F and the Levi–Cività connection on ∇ on $T^*M \otimes F$ so that

$$\nabla^{\mathsf{D},g} \circ \nabla^{\mathsf{D}} : \quad F \xrightarrow{\nabla^{\mathsf{D}}} T^*M \otimes F \xrightarrow{\nabla^{\mathsf{D},g}} T^*M \otimes T^*M \otimes F$$

and the map Tr_g indicates the contraction of an element in $\Gamma(M;T^*M \otimes T^*M \otimes F)$ with the metric $g \in \Gamma(M;TM \otimes TM)$. If the local expression of D (with respect to local coordinate and trivializing bundle charts in M) is $\mathsf{D} = -(g^{\mu\nu}\mathsf{id}_F \partial_\mu \partial_\nu + a^\mu \partial_\mu + b)$, where $a^\mu, b \in \Gamma(M;\mathrm{End}(F))$, then the formulas for the 1-form connection ω^F associated to ∇^{D} and E are given by

$$\begin{array}{rcl} \omega^{\mathsf{D}}_\nu & = & \frac{1}{2}(g_{\nu\mu}a^\mu + g^{\alpha\beta}\Gamma_{\alpha\beta\nu} Id) \\ \mathsf{E}^{\mathsf{D}} & = & b - g^{\nu\mu}(\partial_\nu + \omega^{\mathsf{D}}_\nu \omega^{\mathsf{D}}_\mu - \omega^{\mathsf{D}}_\alpha \Gamma^\alpha_{\nu\mu}), \end{array} \tag{2.2}$$

where $\Gamma_{\alpha\mu\nu}$ are the Christoffel symbols.

Proof. See for instance Section 1.2.2 of [**Gi04**]. □

2.1.1. Generalities on boundary operators. We endow the bundle F with a connection ∇^F and denote by $\nabla^F_{\varsigma_{\mathsf{in}}}$ the covariant derivative along the inward unit geodesic normal vector field ς_{in}. Let $V := V_0 \oplus V_1 \to \partial M$ be the graded complex vector bundle over ∂M, with $V_i := F|_{\partial M}$. Sections of $V_i \to \partial M$ will be thought as arising from the i-th normal covariant derivative of a section of the bundle F, for $i \in \{0,1\}$. As an additional graded complex vector bundle over ∂M, we consider $W := W_0 \oplus W_1 \to \partial M$

with $\mathsf{rank}(W) = \mathsf{rank}(F)$. The **boundary operators** under consideration are given by

$$\mathcal{B} := B \circ \overline{\gamma} : \Gamma(M; F) \to \Gamma(\partial M; W) \tag{2.3}$$

where $\overline{\gamma}$ is the so-called **Cauchy data map** given by

$$\begin{array}{rcl} \overline{\gamma} : \Gamma(M; F) & \to & \Gamma(\partial M; V_0 \oplus V_1), \\ u & \mapsto & u|_{\partial M} \oplus (\nabla^F_{\varsigma_{\mathsf{in}}} u)|_{\partial M} \end{array} \tag{2.4}$$

and the operator B is a smooth (tangential) differential operator on the boundary. More precisely, in terms of the given grading B can be written as

$$B := \begin{pmatrix} B_{00} & 0 \\ B_{10} & B_{11} \end{pmatrix} : \Gamma(\partial M; V) \to \Gamma(\partial M; W), \tag{2.5}$$

where B_{ij} are differential operators such that

$$B_{ii} : \Gamma(\partial M; V_i) \to \Gamma(\partial M; W_i)$$

are differential operators of **0-th order** for $i \in \{0, 1\}$, and

$$\begin{array}{rcl} B_{10} : \Gamma(\partial M; V_0) & \to & \Gamma(\partial M; W_1) \\ v & \mapsto & b_{10} v + \Sigma_{i=1}^{m-1} b^i_{10} \nabla^F_{e_i} v \end{array}$$

is a differential operator of **first order** conveniently, where the coefficients b_{10}, b^i_{10}can be considered as 0-**order** differential operators on the boundary and $\nabla^F_{e_i}$ are covariant derivatives along tangential directions for $i = 1$ to $m - 1$. Then, the operator $\mathcal{B}$ in (2.3) can be more explicitly written as

$$\mathcal{B}v := \begin{pmatrix} \mathcal{B}^0 v \\ \mathcal{B}^1 v \end{pmatrix} := \begin{pmatrix} B_{00} v|_{\partial M} \\ b_{10} v|_{\partial M} + \sum_{a=1}^{m-1} b^a_{10} (\nabla^F_{e_a} v)|_{\partial M} + B_{11}((\nabla^F_{\varsigma_{\mathsf{in}}} v)|_{\partial M}). \end{pmatrix} \tag{2.6}$$

The **graded leading symbol** of the operator B in (2.3), is invariantly defined as the map $\sigma^*_L(B) : T^*(\partial M) \to \mathrm{Hom}(\Gamma(\partial M; V), \Gamma(\partial M; W))$, such that, if $B_{j,i}$ is the entry in the j-th row and i-th column of B, then

$$\sigma^*_L(B_{j,i})(\zeta) := \begin{cases} \sigma_L(B_{j,i})(\zeta) & \text{if} \quad \mathsf{order}(B_{ji}) = j - i \\ 0 & \text{if} \quad \mathsf{order}(B_{ji}) < j - i \end{cases}$$

for $\zeta \in T^*(\partial M)$, see [**Gi04**]. The graded principal symbol for a boundary operator B as in (2.5) is given by

$$\sigma^*_L(B)\zeta = \begin{pmatrix} B_{00} & 0 \\ \sigma_L(B_{10})\,\zeta & B_{11} \end{pmatrix} = \begin{pmatrix} B_{00} & 0 \\ \sqrt{-1} \sum_{a=1}^{m-1} b^a_{10} \zeta_a & B_{11} \end{pmatrix}. \tag{2.7}$$

2.2. Elliptic boundary value problems

Let $(\mathsf{D}, \Gamma(M; F)|_{\mathcal{B}})$ be a boundary value problem, where D is an operator of Laplace type acting on smooth sections of a complex vector bundle F and $\mathcal{B}$ a boundary operator as in Section 2.1.1. The aim of this section is to recall the notion of **boundary ellipticity with respect to a conical set** for $(\mathsf{D}, \Gamma(M; F)|_{\mathcal{B}})$, see Section 2.2.3.

2.2.1. Conical subsets. A subset $\mathcal{C}$ of $\mathbb{C}$ is a **conical subset** in $\mathbb{C}$, if $\lambda \in \mathcal{C}$, then $t\lambda \in \mathcal{C}$, for all $t \geqslant 0$. Some examples are **the rays of direction** θ

$$\mathcal{L}_\theta := \{re^{i\theta} \in \mathbb{C} \mid \ r \geqslant 0\}; \tag{2.8}$$

the **closed angles**, :

$$\mathcal{L}_{\theta,\epsilon} := \{re^{i\alpha} \in \mathbb{C} \mid \theta - \epsilon \leqslant \alpha \leqslant \theta + \epsilon \ \text{ and } r \geqslant 0\}, \text{for } \theta \in [0, 2\pi] \text{ and } \epsilon > 0, \tag{2.9}$$

where the amplitude ϵ is typically taken *small.* More generally **closed sectors** :

$$\mathcal{C}_\theta := \mathbb{C}\backslash\{\lambda \in \mathbb{C} \mid -\theta \leqslant \arg(\lambda) \leqslant \theta \ \text{ and } |\lambda| > 0\}, \text{for } \theta \in [0, \pi]; \tag{2.10}$$

In this thesis, we are interested in the cones

$$\mathcal{C}_\pi = \{0\} \quad \text{and} \quad \mathcal{C}_0 = \mathbb{C}\backslash(0, \infty). \tag{2.11}$$

2.2.2. Shapiro–Lopatijnski condition and boundary ellipticity. Let D be an operator of Laplace type acting on smooth sections of a vector bundle F. Let $\mathcal{C}$ denote a conical subset of $\mathbb{C}$. We expand D in a neighborhood of ∂M, as

$$\mathsf{D}(y, x^m) = \sum_{|(\beta,k)|\leqslant 2} p_{\beta,k}(y, x^m) \ {\nabla^F}^\beta \ {\nabla^F_{\varsigma_{\mathsf{in}}}}^k,$$

where $x := (y, x^m)$ with $y := (y^1, \dots, y^{m-1})$; for $\alpha = (\beta_1, \dots, \beta_{m-1}, k)$, a m-tuple of non negative integer numbers, we have written ${\nabla^F}^\beta := {\nabla^F_{e_1}}^{\beta_1} \cdots {\nabla^F_{e_{m-1}}}^{\beta_{m-1}}$. Take partial Fourier transform in the tangential variables only. In other words we replace the tangential derivatives ${\nabla^F_{e_i}}^\beta$ by $(\sqrt{-1})^{|\beta|}\zeta^\beta$, and suppress the lower order terms. Consider the differential operator $\sigma_L(\mathsf{D})_{(y,0)}(\zeta, i\partial_m)$ for each $(y, 0) \in \partial M$ and $0 \neq \zeta \in T^*(\partial M)$ fixed. We want to solve the following ordinary differential equation

$$\big(\sigma_L(\mathsf{D})_{(y,0)}(\zeta, i\partial_m) - \lambda\big) f(x^m) = 0, \tag{2.12}$$

such that the solutions f satisfy

$$\lim_{x^m\to\infty} |f(x^m)| = 0 \quad \text{for each} \quad (y, 0) \in \partial M, \quad 0 \neq \zeta \in T^*(\partial M) \quad \text{and} \quad \lambda \in \mathcal{C} \tag{2.13}$$

Definition 2.2.1. *Let $\mathcal{B} : \Gamma(M; F) \to \Gamma(W)$ be a boundary operator as in (2.6) and $\mathcal{C} \subset \mathbb{C}$ a conical set. The boundary value problem $(\mathsf{D}, \Gamma(M; E)_{\mathcal{B}})$ satisfies the **Shapiro–Lopatijnski condition** if for any non zero $(\zeta, \lambda) \in T^*(\partial M) \times \mathcal{C}$ and any $w \in W$, there exists a unique solution for the ODE in (2.12) such that (2.13) and the condition*

$$\sigma^*_L(B)(y, \zeta)\overline{\gamma} f = w, \tag{2.14}$$

are satisfied, for every $y \in \partial M$.

Definition 2.2.2. *Let $\mathcal{B} : \Gamma(M; F) \to \Gamma(W)$ be a boundary operator as in (2.6) and $\mathcal{C} \subset \mathbb{C}$ a conical set. $(\mathsf{D}, \Gamma(M; F)|_{\mathcal{B}})$ is **elliptic with respect to the cone** $\mathcal{C}$ whenever it satisfies the Shapiro–Lopatinjski condition from Definition 2.2.1 and the symbolic spectrum of D, see Section 1.2.25, satisfies $\mathsf{Spec_L}(\mathsf{D}) \subset \mathbb{C}\backslash\mathcal{C}$.*

Remark 2.2.3. In particular, if $(\mathsf{D}, \Gamma(M;F)|_{\mathcal{B}})$ is elliptic with respect to cone $\mathcal{C}_0$ from (2.11), then $(\mathsf{D}, \Gamma(M;F)|_{\mathcal{B}})$ is elliptic with respect to $\{0\}$.

2.2.3. Boundary ellipticity for operators of Laplace type. Since we are interested in boundary value problems specified by operators of Laplace type, a characterization of boundary ellipticity (with respect to a cone) for such boundary value problems is useful. This is the statement of the following Lemma.

Lemma 2.2.4. *Let* $\mathsf{D} : \Gamma(M;F) \to \Gamma(M;F)$ *be an operator of Laplace-type and* $\mathcal{B}$ *be a boundary operator as (2.6). The boundary problem* $(\mathsf{D}, \Gamma(M;F)|_{\mathcal{B}})$ *is elliptic with respect to the cone* $\mathcal{C}_0 = \mathbb{C}\backslash(0,\infty)$ *if and only if the operator* $\boldsymbol{b}(\zeta,\lambda) : \Gamma(\partial M; F|_{\partial M}) \to \Gamma(\partial M; W)$ *given by*

$$\boldsymbol{b}(\zeta,\lambda)(v) := \begin{pmatrix} B_{00}v \\ \sqrt{-1}\sigma_L(B_{10})\zeta v - B_{11}\sqrt{|\zeta|^2-\lambda}\,v \end{pmatrix} \tag{2.15}$$

is an isomorphism for every $(0,0) \neq (\zeta,\lambda) \in T^*(\partial M) \times \mathcal{C}_0$.

Proof. This is Lemma 1.4.8 in [**Gi04**] and its proof is a direct translation of what it means for a boundary value problem, specified by an operator of Laplace type, to be elliptic with respect to a cone. □

2.2.4. Example: Mixed boundary conditions. In order to illustrate the notions above, we describe a type of boundary operators specifying so-called mixed boundary conditions, which are used in fields of index theory, PDE's theory, operator theory and physics, further details and examples can be found in [**BG92**], Section 1.5.3 in [**Gi04**], and Sections 4.5 and 4.6 [**Ki01**].

Let F be a vector bundle with connection ∇^F over a compact manifold M with boundary ∂M and Riemannian metric g. Let $F|_{\partial M} := i^*F$ be the pullback bundle along the natural embedding $i : \partial M \hookrightarrow M$. Near the boundary, consider a *collared neighborhood* U of ∂M in M and ς_{in} the inwards pointing geodesic unit normal vector field to the boundary. One starts by constructing an *involution* χ on F over U: Let $\chi \in \mathrm{End}(\partial M; F|_{\partial M})$ be such that $\chi^2 = \mathsf{id}_{F|_{\partial M}}$ and use the normal geodesics to the boundary to extend χ to a bundle endomorphism of F over U, with the condition $\nabla^F_{\varsigma_{\mathsf{in}}}\chi = 0$, so that $\chi^2 = \mathsf{id}_F$ holds over U. Next, over the collar we look at the decomposition of F in terms of the eigenvalues of χ, i.e., $+1$ and -1 and denote by $F_{\pm1}$ the (complementary) subbundles of F corresponding to the ±1-eigenvalues of χ respectively, with the corresponding spectral *projections*:

$$\Pi_{\pm1} := \frac{1}{2}(\mathsf{id}_F \pm \chi) : F \to F_{\pm1}.$$

Since $\nabla^F_{\varsigma_{\mathsf{in}}}\chi = 0$, we have

$$\Pi_{\pm1}\nabla^F_{\varsigma_{\mathsf{in}}} = \nabla^F_{\varsigma_{\mathsf{in}}}\Pi_{\pm1}.$$

Then, fix a bundle endomorphism S_{+1} of $F_{+1}|_{\partial M}$, which is first extended to a bundle endomorphism S of $F|_{\partial M}$, by setting 0 on $F_{-1}|_{\partial M}$ and then this is parallel extended along the normal geodesics, i.e. $\nabla^F_{\varsigma_{\mathsf{in}}}\mathsf{S} = 0$, to F over the collar U, so that

$$\mathsf{S}\Pi_{\pm 1} = \Pi_{\pm 1}\mathsf{S},$$

that is, S respects the splitting of $F = F_{+1} \oplus F_{-1}$ over U.

Definition 2.2.5. *Consider the involution χ and endomorphism S together with the projections $\Pi_\pm$ associated to the eigenvalues of χ as explained above. A section $v \in \Gamma(M;F)$ satisfies **mixed boundary conditions** if*

$$\mathcal{B}_{\mathsf{M}} v := \left(\Pi_{-1} v, \Pi_{+1}(\nabla^F_{\varsigma_{\mathsf{in}}} - \mathsf{S})v\right)|_{\partial M}$$

vanishes.

Remark 2.2.6. The operators imposing mixed boundary conditions given in Definition 2.2.5 above, are of the form given in (2.6), since for $v \in \Gamma(M;F)$, over the tubular neighborhood U, we have $(\Pi_\pm v)|_{\partial M} = \Pi_\pm(v|_{\partial M})$ so that $(\Pi_{+1}(\nabla^F_{\varsigma_{\mathsf{in}}} - \mathsf{S})v)|_{\partial M} = \Pi_{+1}(\nabla^F_{\varsigma_{\mathsf{in}}} - \mathsf{S})v|_{\partial M})$.

Remark 2.2.7. Dirichlet boundary conditions are obtained when $\Pi_{+1} = 0$, Robin boundary conditions when $\Pi_{-1} = 0$ and Neumann boundary conditions correspond to the case $\Pi_{-1} = 0$ and $\mathsf{S} = 0$.

Remark 2.2.8. Let F be a complex vector bundle over a compact manifold M. Consider the conical set $\mathcal{C}_0 = \mathbb{C}\backslash(0,\infty)$. Let $(\mathsf{D}, \Gamma(F;E)_{\mathcal{B}_{\mathsf{M}}})$ be a boundary value problem specified by an operator of Laplace type D acting on smooth sections of F under mixed boundary conditions. Then $(\mathsf{D}, \Gamma(F;E)_{\mathcal{B}_{\mathsf{M}}})$ is elliptic with respect to the cone $\mathcal{C}_0$. See for instance, Lemma 1.5.3 in **[Gi04]** whose proof is a direct application of Lemma 2.2.4.

2.3. The resolvent and spectrum of an elliptic boundary value problem

Let $(\mathsf{D}, \Gamma(M;F)|_{\mathcal{B}})$ be a boundary value problem, where $\mathsf{D} : \Gamma(M;F) \to \Gamma(M;F)$ is an operator of Laplace type and

$$\mathcal{B} := (\mathcal{B}^0, \mathcal{B}^1) : \Gamma(M,F) \to \Gamma(\partial M; W_0) \oplus \Gamma(\partial M; W_1)$$

a boundary operator as in (2.6), where W_0 and W_1 are vector bundles over ∂M and the operators

$$\mathcal{B}^i : \Gamma(M,F) \to \Gamma(\partial M, W_i)$$

are of order $i \in \{0,1\}$. We are interested in the spectral theory of D, considered as an (unbounded) operator in $\mathsf{L}^2(M;F)$ with domain of definition specified by the imposed boundary conditions in certain Sobolev spaces.

2.3.1. Estimates for elliptic boundary value problems.

Lemma 2.3.1. *The operator* $\mathsf{D} : \Gamma(M;F) \to \Gamma(M;F)$ *extends as a linear bounded operator*

$$\mathsf{D}_s := \mathsf{D}_{s+2,s} : \mathsf{H}_{s+2}(M;F) \to \mathsf{H}_s(M;F), \tag{2.16}$$

for each $s \geqslant 0$. *Each boundary operator* $\mathcal{B}^i$ *extends as a linear bounded operator*

$$\mathcal{B}^i_s := \mathcal{B}^i_{s+2,s+3/2-i} : \mathsf{H}_{s+2}(M,F) \to \mathsf{H}_{s+\frac{3}{2}-i}(\partial M, W_i), \tag{2.17}$$

for $i \in \{0,1\}$. *In other words, the operator*

$$\begin{array}{rcl} \mathbf{A} : \Gamma(M;F) & \longrightarrow & \Gamma(M;F) \oplus \Gamma(\partial M;W_0) \oplus \Gamma(\partial M;W_1) \\ u & \mapsto & (\mathsf{D}u, \mathcal{B}^0 u, \mathcal{B}^1 u) \end{array} \tag{2.18}$$

extends as bounded operator as

$$\begin{array}{rcl} \mathbf{A}_s : \mathsf{H}_{s+2}(M;F) & \to & \mathsf{H}_s(M;F) \oplus \mathsf{H}_{s+3/2}(\partial M;W_0) \oplus \mathsf{H}_{s+1/2}(\partial M;W_1) \\ u & \mapsto & \left(\mathsf{D}_s\, u, \mathcal{B}^0_s u, \mathcal{B}^1_s u\right), \end{array} \tag{2.19}$$

Proof. See for instance 20.1 in [**Hö83**] and Chapter 1 in [**Agr97**]. □

Ellipticity with respect to a cone permits one to answer the question whether or not the operator $\mathbf{A}_s$ in (2.19) is Fredholm. This is the following Lemma.

Lemma 2.3.2. *Let* $\mathsf{D} : \Gamma(M;F) \to \Gamma(M;F)$ *be an operator of Laplace-type and* $\mathcal{B}$ *be a boundary operator as (2.6). Suppose that* $(\mathsf{D}, \Gamma(M;F)|_{\mathcal{B}})$ *is elliptic with respect to the cone* $\{0\}$. *Then, for* $s \geqslant 0$, *the operator* $\mathbf{A}_s$ *in (2.19) is Fredholm and there exists a constant* $C_s > 0$, *for which the a priori estimate*

$$\|u\|_{s+2} \leqslant C_s \left(\| \mathsf{D}_s\, u\|_s + \|\mathcal{B}^0_s u\|_{s+3/2} + \|\mathcal{B}^1_s u\|_{s+1/2} + \|u\|_{L^2}\right)$$

holds.

Proof. This follows from Theorem 6.3.1 in [**Agr97**] and Theorem 20.1.2 in [**Hö83**]. □

2.3.2. Regularity for an elliptic boundary value problem. Let $(\mathsf{D}, \Gamma(M;F)|_{\mathcal{B}})$ be a boundary value problem, where $\mathsf{D} : \Gamma(M;F) \to \Gamma(M;F)$ is an operator of Laplace type and $\mathcal{B} := (\mathcal{B}^0, \mathcal{B}^1) : \Gamma(M,F) \to \Gamma(\partial M;W_0) \oplus \Gamma(\partial M;W_1)$ a boundary operator as in (2.6). The H_s**-realization** of D with respect to the boundary conditions, specified by the boundary operator $\mathcal{B}$, corresponds to consider the operator D as unbounded operator with domain of definition obtained as a convenient Sobolev closure of $\Gamma(M;F)|_{\mathcal{B}}$. More precisely, from Lemma 2.3.1, the operator

$$\mathsf{D}_s : \mathsf{H}_{s+2}(M;F) \to \mathsf{H}_s(M;F)$$

is bounded for each $s \geqslant 0$. Now look at the same operator D_s, as unbounded operator from $\mathsf{H}_s(M;F)$ to $\mathsf{H}_s(M;F)$, with domain of definition given by $\|\cdot\|_{s+2}$-closure of $\Gamma(M;F)|_{\mathcal{B}} \subset \Gamma(M;F)$:

$$\begin{aligned}(\mathsf{D}_{\mathcal{B},s}, \mathcal{D}(\mathsf{D}_{\mathcal{B},s})): \quad \mathsf{H}_s(M;F) &\to \mathsf{H}_s(M;F) \\ \mathsf{D}_{\mathcal{B},s}u &:= \mathsf{D}_s u, \text{for } u \in \mathcal{D}(\mathsf{D}_{\mathcal{B},s}), \\ \mathcal{D}(\mathsf{D}_{\mathcal{B},s}) &:= \overline{\Gamma(M;F)|_{\mathcal{B}}}^{\|\cdot\|_{s+2}} \\ &= \{u \in \mathsf{H}_{s+2}(M;F) \mid \mathcal{B}\, u = 0\}.\end{aligned} \tag{2.20}$$

We are particularly interested in the L^2**-realization** of D:

$$\begin{aligned}(\mathsf{D}_{\mathcal{B}}, \mathcal{D}(\mathsf{D}_{\mathcal{B}})): \quad \mathsf{L}^2(M;F) &\to \mathsf{L}^2(M;F) \\ \mathsf{D}_{\mathcal{B}}u &:= \mathsf{D}u, \text{for } u \in \mathcal{D}(\mathsf{D}_{\mathcal{B}}), \\ \mathcal{D}(\mathsf{D}_{\mathcal{B}}) &:= \overline{\Gamma(M;F)|_{\mathcal{B}}}^{\|\cdot\|_{2}} \\ &= \{u \in \mathsf{H}_{2}(M;F) \mid \mathcal{B}\, u = 0\}.\end{aligned} \tag{2.21}$$

Later on we simply write $\mathsf{D}_{\mathcal{B}}$ for the L^2-realization of $(\mathsf{D}, \Gamma(M;F)|_{\mathcal{B}})$, whenever the domain of definition of $\mathsf{D}_{\mathcal{B}}$ is unambiguously understood.

Proposition 2.3.3. *Consider $(\mathsf{D}, \Gamma(M;F)|_{\mathcal{B}})$ a boundary value problem which is elliptic with respect to the cone $\{0\}$, see Section 2.2, together with its H_s-realization as in (2.20). Then, if $u \in \mathcal{D}(\mathsf{D}_{\mathcal{B},s})$ is such that $\mathsf{D}_{\mathcal{B},s}u \in \mathsf{H}_{s+2}(M;F)$, then, in fact, $u \in \mathcal{D}(\mathsf{D}_{\mathcal{B},s+2})$, for $s \geqslant 0$. In particular, if $\mathsf{D}_{\mathcal{B},s}u \in \Gamma(M;F)$, then $u \in \Gamma(M;F)|_{\mathcal{B}}$. Moroever, the operator*

$$\mathsf{D}_{\mathcal{B},s} : \mathcal{D}(\mathsf{D}_{\mathcal{B},s}) \subset \mathsf{H}_s(M;F) \to \mathsf{H}_s(M;F) \quad \textit{is closed for all} \quad s \geqslant 0.$$

Proof. If $u \in \mathcal{D}(\mathsf{D}_{\mathcal{B},s})$ is such that $\mathsf{D}_{\mathcal{B},s}u \in \mathsf{H}_{s+2}(M;F)$, then by the estimates from Lemma 2.3.2, we have $\|u\|_{s+4} \leqslant C_s(\|\mathsf{D}_{\mathcal{B},s}u\|_{s+2} + \|u\|_{L^2})$; that is, $u \in \mathsf{H}_{s+4}(M;F)$. But $u \in \mathcal{D}(\mathsf{D}_{\mathcal{B},s+2})$, since $\mathcal{B}$ is bounded on $\mathsf{H}_{s+2}(M;F)$ and $\mathcal{B}u = 0$, see (2.17). In addition, if $\mathsf{D}_{\mathcal{B},s}u \in \Gamma(M;F)|_{\mathcal{B}}$, then by induction and Sobolev embedding, we have $u \in \Gamma(M;F)|_{\mathcal{B}}$. We now show that $(\mathsf{D}_{\mathcal{B},s}, \mathcal{D}(\mathsf{D}_{\mathcal{B},s}))$ is closed on $\mathsf{H}_s(M;F)$. Let $\{u_n\}$ be a sequence in $\mathcal{D}(\mathsf{D}_{\mathcal{B},s})$ converging to u in $\|\cdot\|_s$-norm such that $\{\mathsf{D}_{\mathcal{B},s}u_n\}$ converges to v in the $\|\cdot\|_s$-norm as well. First remark that $u \in \mathcal{D}(\mathsf{D}_{\mathcal{B},s})$; indeed, the sequence $\{u_n\}$ is also Cauchy with respect to the norm $\|\cdot\|_{s+2}$ because of the estimates from Lemma 2.3.2. Therefore, since the operator $\mathsf{D}_s : \mathsf{H}_{s+2}(M;F) \to \mathsf{H}_s(M;F)$ is bounded, we can write

$$\mathsf{D}_{\mathcal{B},s}u = \mathsf{D}_s(\lim_{n\to\infty} u_n|_{\|\cdot\|_{s+2}}) = \lim_{n\to\infty}(\mathsf{D}_s u_n)|_{\|\cdot\|_s} = v,$$

where $\lim\limits_{n\to\infty}(\cdot)|_{\|\cdot\|_s}$ indicates the limit with respect to the H_s-norm. This finishes the proof. $\square$

Remark 2.3.4. The generalized Laplacian can be considered as an unbounded operator $(\mathsf{D}, \mathcal{D}(\mathsf{D})) : \mathsf{H}_s(M;F) \to \mathsf{H}_s(M;F)$ with $\mathcal{D}(\mathsf{D}) := \Gamma(M;F)|_{\mathcal{B}} \subset \mathsf{H}_s(M;F)$ as domain of definition. Because of Proposition 2.3.3, this operator is closeable in the $\|\cdot\|_s$-norm. Consider $(\overline{\mathsf{D}}^s, \mathcal{D}(\overline{\mathsf{D}}^s))$ the $\|\cdot\|_s$-closure extension of $(\mathsf{D}, \mathcal{D}(\mathsf{D}))$, with domain of definition

$\mathcal{D}(\overline{\mathsf{D}}^s)$. Recall that $\mathcal{D}(\overline{\mathsf{D}}^s)$ formally consists of all $w \in \mathsf{H}_s(M;F)$, for which there is a sequence $\{w_n\}$ in $\Gamma(M;F)|_{\mathcal{B}}$ converging to w in the H_s-norm such that $\mathsf{D}w_n$ converges to some $v \in \mathsf{H}_s(M;F)$ in the H_s-norm. The operators $(\overline{\mathsf{D}}^s, \mathcal{D}(\overline{\mathsf{D}}^s))$ and $(\mathsf{D}_{\mathcal{B},s}, \mathcal{D}(\mathsf{D}_{\mathcal{B},s}))$ in (2.20), coincide. This follows from the existence of the elliptic estimates in Lemma 2.3.2: for $w \in \Gamma(M;F)$ the graph norm

$$\|w\|_{\mathsf{D},s+2} := \|w\|_s + \|\mathsf{D}w\|_s,$$

is equivalent to the norm $\|\cdot\|_{s+2}$ defining the Sobolev space $\mathsf{H}_{s+2}(M;F)$ as completion of $\Gamma(M;F)$.

2.3.3. The resolvent and spectrum of an elliptic boundary value problem. Consider $(\mathsf{D}, \Gamma(M;F)|_{\mathcal{B}})$ the boundary value problem where $\mathsf{D} : \Gamma(M;F) \to \Gamma(M;F)$ is a generalized Laplacian and $\mathcal{B} := (\mathcal{B}^0, \mathcal{B}^1) : \Gamma(M,F) \to \Gamma(\partial M; W_0) \oplus \Gamma(\partial M; W_1)$ is the boundary operator in (2.6).

Proposition 2.3.5. *For $(\mathsf{D}, \Gamma(M;F)|_{\mathcal{B}})$ the boundary value problem as above consider $(\mathsf{D}_{\mathcal{B},s}, \mathcal{D}(\mathsf{D}_{\mathcal{B},s}))$ its H_s-realization. Suppose that $(\mathsf{D}, \Gamma(M;F)|_{\mathcal{B}})$ is elliptic with respect to the ray $\mathcal{L}_{\theta_0}$, see (2.8), for some $\theta_0 \in [0, 2\pi[$. Then, there exist numbers r_0, $\epsilon > 0$ such that the resolvent $\mathcal{R}_{\mathsf{D}_{\mathcal{B},s}}(\lambda)$ of $\mathsf{D}_{\mathcal{B},s}$ exists for each $\lambda \in W_{r_0,\epsilon}$, where*

$$W_{r_0,\epsilon} := \{\lambda \in \mathbb{C}|\, |\lambda| \geqslant r_0 \text{ and } |arg(\lambda) - \theta_0| \leqslant \epsilon\}.$$

Moreover, if we write $\langle\mu\rangle := (1 + |\mu|^2)^{1/2}$, where $\mu = |\lambda|^{1/2}$, then, for each $s \geqslant 0$, there exists $C_s > 0$ such that the following estimates hold

$$\langle\mu\rangle^{s+2} \|\mathcal{R}_{\mathsf{D}_{\mathcal{B},s}}(\lambda)u\|_{L^2} + \|\mathcal{R}_{\mathsf{D}_{\mathcal{B}}}(\lambda)u\|_{s+2} \leqslant C_s(\langle\mu\rangle^s \|u\|_{L^2} + \|u\|_s), \tag{2.22}$$

uniformly in $W_{r_0,\epsilon}$, whenever $u \in \mathcal{D}(\mathsf{D}_{\mathcal{B},s})$.

Proof. See Theorem 3.3.2 and Corollary 3.3.3 in **[Gru96]** (see also Remark 3.3.4 and the discussion in section 1.5 of **[Gru96]**). ☐

Corollary 2.3.6. *Consider $(\mathsf{D}, \Gamma(M;F)|_{\mathcal{B}})$ the boundary value problem with its L^2-realization $(\mathsf{D}_{\mathcal{B}}, \mathcal{D}(\mathsf{D}_{\mathcal{B}}))$. If $(\mathsf{D}, \Gamma(M;F)|_{\mathcal{B}})$ is elliptic with respect to $\mathcal{C}$, a closed conical subset of $\mathbb{C}$, then there are constants $C > 0$ and $R \geqslant 0$ such that for all $\lambda \in \mathcal{C}$ with $|\lambda| > R$, we have*

$$\|(\mathsf{D}_{\mathcal{B}} - \lambda)^{-1}\|_{\mathsf{L}^2} \leqslant C\frac{1}{|\lambda|}.$$

Proof. For $s = 0$ in (2.22):

$$\langle\mu\rangle^2 \|\mathcal{R}_{\mathsf{D}_{\mathcal{B}}}(\lambda)u\|_{L^2} \leqslant \langle\mu\rangle^2 \|\mathcal{R}_{\mathsf{D}_{\mathcal{B}}}(\lambda)u\|_{L^2} + \|\mathcal{R}_{\mathsf{D}_{\mathcal{B}}}(\lambda)u\|_2 \leqslant 2C_0\|u\|_{L^2}.$$

That is,

$$\|\mathcal{R}_{\mathsf{D}_{\mathcal{B}}}(\lambda)\|_{L^2} \leqslant C\frac{1}{\langle\mu\rangle^2} \leqslant C\frac{1}{|\lambda|}.$$

□

Proposition 2.3.7. *Suppose that* $(\mathsf{D}, \Gamma(M;F)|_{\mathcal{B}})$ *is elliptic with respect to the cone* $\mathcal{C}_0 := \mathbb{C}\backslash(0,\infty)$. *Then, the unbounded operator* $(\mathsf{D}_{\mathcal{B}}, \mathcal{D}(\mathsf{D}_{\mathcal{B}}))$*is densely defined in the space* $\mathsf{L}^2(M;F)$*, possesses a non-empty resolvent set, its resolvent is compact and the spectrum of* $\mathsf{D}_{\mathcal{B}}$ *is discret and is described as follows. For every* $\theta > 0$*, there exists* $R > 0$ *such that* $\mathbb{B}_R(0)$*, the closed ball in* $\mathbb{C}$ *centered at 0 with radius* R*, contains at most a finite subset of* $\mathsf{Spec}(\mathsf{D}_{\mathcal{B}})$ *and, more importantly, the remaining part of the spectrum is entirely contained in the sector*

$$\Lambda_{R,\theta} := \{\, z \in \mathbb{C} | -\theta < \arg(z). < \theta \text{ and } |z| \geqslant R\} \tag{2.23}$$

Furthermore, for every $\lambda \notin \Lambda_{R,\theta}$ *large enough, there exists* $C > 0$*, for which*

$$\|(\mathsf{D}_{\mathcal{B}} - \lambda)^{-1}\|_{\mathsf{L}^2} \leqslant C/|\lambda|.$$

Proof. The domain $\mathcal{D}(\mathsf{D}_{\mathcal{B}})$ is dense in $\mathsf{L}^2(M;F)$, because the space $\Omega_c(M;F)$, consisting of smooth forms with compact support in the interior of M, is dense in $\mathsf{L}^2(M;F)$ and that $\Omega_c(M;F) \subset \mathcal{D}(\mathsf{D}_{\mathcal{B}})$. Since $(\mathsf{D}, \Gamma(M;F)|_{\mathcal{B}})$ is elliptic with respect to the cone $\mathcal{C}_0$, from Proposition 2.3.5 we know that the resolvent of $\mathsf{D}_{\mathcal{B}}$ exists. The estimate for the norm of the resolvent is given in Corollary 2.3.6. We now show that the resolvent is compact in $\mathsf{L}^2(M;F)$. That is, if $\{v_i\}$ is a bounded sequence in $\mathsf{Im}(\mathsf{D}_{\mathcal{B}} - \lambda)$ in the L^2-norm, then we need show that the sequence $\{(\mathsf{D}_{\mathcal{B}} - \lambda)^{-1}v_i\}$ admits a sub-sequence, which converges in the L^2-norm. Firstly, the sequence $\{(\mathsf{D}_{\mathcal{B}} - \lambda)^{-1}v_i\}$ is bounded in the H_2-norm as well: since $(\mathsf{D}_{\mathcal{B}} - \lambda)^{-1}v_i \in \mathcal{D}(\mathsf{D}_{\mathcal{B}})$, by using the elliptic estimates from Lemma 2.3.2 (see also the proof of Proposition 2.3.3) and that $(\mathsf{D}_{\mathcal{B}} - \lambda)^{-1}$ is bounded in the L^2-norm, one obtains

$$\|(\mathsf{D}_{\mathcal{B}} - \lambda)^{-1}v_i\|_{H_2} \leqslant C_1(\|v_i\|_{L^2} + \|(\mathsf{D}_{\mathcal{B}} - \lambda)^{-1}v_i\|_{L^2}) \leqslant C\|v_i\|_{L^2};$$

but the last term on the right hand side is bounded by assumption. Secondly, since $\mathsf{H}_2(M;F)$ is compactly embedded in $\mathsf{L}^2(M;F)$, the sequence $\{(\mathsf{D}_{\mathcal{B}} - \lambda)^{-1}v_i\}$, must possess a sub-sequence, which converges in the L^2-norm and so the resolvent is compact. Compactness of the resolvent implies the discreteness of the spectrum, with only possible accumulation point at infinity, see Theorem 6.29, chapter III, section 6 of [**Ka95**]. Finally, the existence of the angle $\Lambda_{R,\theta}$ follows from Proposition 2.3.5. □

CHAPTER 3

Generalized Laplacians on compact bordisms

In this chapter we provide the necessary spectral theory for certain generalized Laplacian in order to define the complex-valued analytic torsion in Chapter 5. We start with the following definition, see [**BFK99**] and [**Mi62**].

Definition 3.0.8. *A* ***compact Riemannian bordism*** *of dimension m is to be understood as the triplet*

$$(M, \partial_+ M, \partial_- M),$$

where M is a compact connected smooth Riemannian manifold of dimension m, whose boundary ∂M is the disjoint union of two closed submanifolds $\partial_+ M$ and $\partial_- M$. We denote by

$$(M, \partial_+ M, \partial_- M)' := (M, \partial_- M, \partial_+ M)$$

the ***dual bordism*** *to $(M, \partial_+ M, \partial_- M)$.*

Given a compact Riemannian bordism $(M, \partial_+ M, \partial_- M)$, consider E a flat complex vector bundle over M with flat connection ∇^E. Assume E to be endowed with a fiberwise nondegenerate symmetric bilinear form b –this is the case if and only if the bundle is the complexification of a real vector bundle. On closed manifolds, see [**BH07**] and [**BH10**], Burghelea and Haller introduced and studied generalized Laplacians $\Delta_{E,g,b}$ constructed by using the bilinear form b, the Riemannian metric g and the flat connection ∇^E. The operator $\Delta_{E,g,b}$ is also referred as the **bilinear Laplacian.** In this chapter we study (the spectral theory of) $\Delta_{E,g,b}$ acting on smooth sections of a flat complex vector bundle over a compact manifold with boundary under absolute boundary conditions on $\partial_+ M$ and relative boundary conditions on $\partial_- M$.

In Section 3.1, as a motivating example, we recall the analog problem of a boundary value problem specified by a self-adjoint Laplacian under absolute and relative boundary conditions on a compact Riemannian bordism. In Section 3.2, we review in some detail the construction of the bilinear Laplacian. After proving that the bilinear Laplacian is elliptic, see Lemma 3.2.3, we derive certain Green's formulas, see Lemma 3.2.4; these formulas provide correction terms accounting for the contribution from the boundary. These boundary terms vanish, for instance, by imposing absolute and relative boundary conditions, see Definition 3.2.5. In Section 3.2.1, absolute and relative boundary conditions are specified by the vanishing of certain boundary operators, see (3.14) and (3.16). In Section 3.2.2, we indicate how the boundary valued problem consisting of the bilinear Laplacian on $(M, \partial_+ M, \partial_- M)$ under absolute (resp. relative) boundary conditions can be interpreted as the **dual boundary value problem** specified by dual bilinear Laplacian on the dual bordism $(M, \partial_- M, \partial_+ M)$ under relative (resp. absolute) boundary conditions, by means of *Poincaré duality.* In Section 3.2.3, we (locally) describe the operators imposing absolute and relative boundary conditions in terms of invariant objects. In Section 3.3, we study spectral properties for the bilinear Laplacian on compact bordisms, where we use the results from Chapter 2. Although the bilinear Laplacian is not necessarily self-adjoint, it still possesses spectral properties close to the Hermitian Laplacian: for instance, in Proposition 3.3.1 we see that the boundary value problem studied in this thesis is an elliptic boundary value problem. In Section 3.3.3, we consider the bilinear Laplacian as unbounded operator in the L^2-norm with domain of definition specified by the boundary conditions in certain Sobolev space. In Section 3.3.4, the spectrum of the bilinear Laplacian is precisely described as a countable set in $\mathbb{C}$ having a similar behavior as the one corresponding to the bilinear Laplacian acting on closed manifolds. In Section 3.3.5, we are concerned with generalized eigenspaces of the bilinear Laplacian, in particular Proposition 3.3.4 gives a characterization of each of these as a finite dimensional vector spaces containing smooth forms only. In Section 3.3.6, we study the decomposition of the space of smooth forms, with respect to each generalized eigenspace. In particular, in Lemma 3.3.6, we see that the restriction of the bilinear Laplacian to the space of smooth forms satisfying boundary conditions in the complement of each generalized eigenspace is invertible. Then, Section 3.3.7 starts with a Hodge decomposition result for the bilinear Laplacian on smooth forms, see Corollary 3.3.10 and Proposition 3.3.11. Finally, Proposition 3.3.11 is used to prove that the space of smooth forms being in the generalized 0-eigenspace of the bilinear Laplacian still computes relative cohomology: its inclusion into the space of forms satisfying relative boundary conditions on $\partial_- M$ induces an isomorphism in cohomology, see Proposition 3.3.12.

3.1. Motivation: the Hermitian Laplacian

The construction of the analytic torsion, as first introduced by Ray and Singer on [**RS71**], is based on spectral information of certain self-adjoint Laplacians. As a motivation, we recall some facts around these self-adjoint Laplacians. Keep in mind the notation and notions from Section 2.2.

Let $E \to M$ a complex vector bundle over a compact Riemannian manifold M, with flat connection ∇^E and endowed with a Hermitian form h . The Riemannian metric g on M and the Hermitian form h on E induce an inner product $\langle v, w \rangle_{h,g}$ on each fiber which in turn defines by integration an inner product $\ll \cdot, \cdot \gg_{g,h}$ on $\Omega(M; E)$, the space of E-valued smooth differential forms on M, by the formula

$$\ll v, w \gg_{g,h} := \int_M \langle v, w \rangle_{h,g} \mathsf{vol}_g M,$$

for each $v, w \in \Omega(M;E)$. The de-Rham differential $d_E : \Omega(M;E) \to \Omega(M;E)$, associated to the connection ∇^E, possesses a unique formal adjoint with respect to $\ll \cdot,\cdot \gg_{g,h}$, which we denote by $d^*_{E,g,h}$. This operator being a codifferential on $\Omega(M;E)$ permits one to consider the Hermitian Laplacian

$$\Delta_{E,g,h} := d_E d^*_{E,g,h} + d^*_{E,g,h} d_E : \Omega(M;E) \to \Omega(M;E). \tag{3.1}$$

Now, for a compact Riemannian bordism $(M, \partial_+ M, \partial_- M)$, we denote by $\Omega(M;E)|^h_{\mathcal{B}}$ the space of E-valued smooth differential forms satisfying absolute boundary conditions on $\partial_+ M$ and relative boundary conditions on $\partial_- M$. More precisely, if $\star_h$ is the Hodge star operator induced by g and h, we set

$$\Omega(M;E)|^h_{\mathcal{B}} := \left\{ w \in \Omega(M;E) \;\middle|\; \begin{array}{cc} i^*_+ \star_h w = 0, & i^*_- w = 0 \\ i^*_+ d^*_{\bar{E}'\otimes\Theta_M,g,h'} \star_h w = 0, & i^*_- d^*_{E,g,h} w = 0 \end{array} \right\}, \tag{3.2}$$

where $d^*_{\bar{E}'\otimes\Theta_M,g,h'}$ indicates the formal adjoint to $d_{\bar{E}'\otimes\Theta_M}$ with respect to the inner product $\ll \cdot,\cdot \gg_{g,h'}$ on $\Omega(M;\bar{E}'\otimes\Theta_M)$ with $\bar{E}'$ being the dual of the complex conjugate bundle of E endowed with the Hermitian form h' dual to h. In the sense of Definition 2.2.2 from Chapter 2, the boundary value problem consisting of the Hermitian Laplacian in (3.1) under absolute and relative boundary conditions specified in (3.2) and denoted by

$$(\Delta, \Omega_{\mathcal{B}})|^{E,g,h}_{(M,\partial_+ M,\partial_- M)}, \tag{3.3}$$

is an elliptic boundary value problem. Therefore the operator

$$\Delta_{E,g,h} : \Omega(M;E)|^h_{\mathcal{B}} \subset \mathsf{L}^2(M;E) \to \Omega(M;E) \subset \mathsf{L}^2(M;E)$$

extends in the L^2-norm to a self-adjoint operator, denoted by $\Delta_{\mathcal{B},h}$, with domain of definition

$$\mathcal{D}(\Delta_{\mathcal{B},h}) = \overline{\Omega(M;E)|^h_{\mathcal{B}}}^{\mathsf{H}_2},$$

that is, the H_2-Sobolev closure of $\Omega(M;E)|^h_{\mathcal{B}}$; for these facts see [**Lü93**], [**Mü78**], [**Gi84**] and [**Gi04**].

Moreover, there exist well-known Hodge-decomposition results that we recall in the following Theorem.

Theorem 3.1.1. *Consider the Hermitian boundary value problem in (3.3). Let*

$$\mathcal{H}^q_{\Delta_{\mathcal{B}}}(M;E) := \ker(\Delta_{\mathcal{B},h}) \cap \Omega^q(M;E)|^h_{\mathcal{B}}$$

be the space of q-Harmonic forms satisfying absolute boundary conditions on $\partial_+ M$ and relative boundary conditions on $\partial_- M$ and set

$$\Omega(M;E)|^h_{\mathcal{B}^0} := \left\{ w \in \Omega(M;E) \;\middle|\; i^*_+ \star_h w = 0, \quad i^*_- w = 0 \right\}.$$

Then, every $w \in \Omega^q(M;E)|^h_{\mathcal{B}^0}$ can be uniquely written as $w = h + v_1 + v_2$ where

$$h \in \mathcal{H}^q_{\Delta_{\mathcal{B}}}(M;E), \quad v_1 \in d_E\left(\Omega^{q-1}(M;E)|^h_{\mathcal{B}^0}\right) \quad \text{and} \quad v_2 \in d^*_{E,g,h}\left(\Omega^{q+1}(M;E)|^h_{\mathcal{B}^0}\right).$$

Moreover, for $H^q(M, \partial_- M; E)$ the q-cohomology group relative to $\partial_- M$, the isomorphism

$$\mathcal{H}^q_{\Delta_{\mathcal{B}}}(M; E) \cong H^q(M, \partial_- M; E) \tag{3.4}$$

holds.

Proof. This is Theorem 1.10 in [**Lü93**], see also page 239 in [**Mü78**]. □

Then, as in the situation of a manifold without boundary, see [**BZ92**], the isomorphism in (3.4) is the first step in defining the Ray–Singer torsion on manifolds with boundary, under absolute and relative boundary conditions, as a Hermitian metric on $\det(H(M, \partial_- M; E))$, the determinant line associated to $H(M, \partial_- M; E)$. The Ray–Singer metric has been studied, by means of the Hermitian Laplacian, by many authors, see for instance [**RS71**], [**Lü93**], [**Ch77**], [**Ch79**] [**Mü78**], [**DF00**]. Also, in [**BM06**] Brüning and Ma studied the case $\partial_- M = \emptyset$ and later on in [**BM11**] the general one $\partial_- M \neq \emptyset$.

3.2. Bilinear Laplacians and absolute/relative boundary conditions

We consider a flat complex vector bundle E over a compact manifold M with Riemannian metric g. We denote by ∇^E the flat connection on E. Assume E is endowed with a fiberwise nondegenerate symmetric bilinear form b. The dual bundle of E is denoted by E' and it is naturally endowed with the corresponding dual connection $\nabla^{E'}$ and dual bilinear form b'. In the situation of a closed manifold, see [**BH07**], generalized Laplacians were constructed by using the data ∇^E and g but replacing a Hermitian structure by the considered bilinear form b on E. We study this problem on compact bordisms. With the use of b and g, one obtains the complex-valued bilinear form on $\Omega(M; E)$ given by

$$\beta_{g,b}(v, w) := \int_M b_g(v, w)\mathsf{vol}_g(M), \tag{3.5}$$

where, for each $x \in M$, we have on the fiber E_x the nondegenerate symmetric bilinear form

$$b_{g,x}(\alpha \otimes s, \alpha \otimes s')_x := \langle \alpha_x, \alpha'_x \rangle_g(x) b(x)(s_x, s'_x), \tag{3.6}$$

where $\alpha \otimes s$ and $\alpha' \otimes s'$ are elementary sections of the bundle $\Lambda^*(T^*M) \otimes E$ and $\langle \cdot, \cdot \rangle_g$ indicates the fiberwise inner product on $\Lambda^* T^* M$ induced by the metric g. In formula (3.5) the volume density associated to the Riemannian metric, $\mathsf{vol}_g(M) \in \Omega^m(M; \Theta_M)$, is used to integrate (3.6) over M. In analogy with the Hermitian situation, we have the following.

Definition 3.2.1. *For $0 \leqslant k \leqslant m$, the usual Hodge $\star$-operator on smooth k-forms: $\star_{g,k}$: $\Lambda^k(T^*M) \to \Lambda^{m-k}(M; \Theta_M)$, together with $b : E \to E'$, the bundle isomorphism induced* *by the nondegenerate bilinear form on E, determine a $C^\infty(M)$-linear isomorphism*

$$\star_{b,k} := \star_{g,k} \otimes b : \Omega^k(M;E) \to \Omega^{m-k}(M;E' \otimes \Theta_M), \tag{3.7}$$

which is also referred as the Hodge $\star$-operator.

Lemma 3.2.2. *The bilinear form $\beta_{g,b}$ defined in (3.5) is a nondegenerate symmetric bilinear form on $\Omega(M;E)$ that can be written as*

$$\beta_{g,b}(v,w) = \int_M \mathrm{Tr}(v \wedge \star_b w), \tag{3.8}$$

where Θ_M is the orientation bundle of M and $\mathrm{Tr} : \Omega(M, E \otimes E' \otimes \Theta_M) \to \Omega(M;\Theta_M)$ is the trace map induced by the canonical pairing between E and E'. The bilinear form $\beta_{g,b}$ continuously extends to a nondegenerate symmetric bilinear form on $\mathsf{L}^2(M;E)$.

Proof. First, it is clear that $\beta_{g,b}$ in (3.5) is globally defined as a symmetric bilinear form on $\Omega(M;E)$. Moroever $\beta_{g,b}$ is nondegenerate, since b_g is fiberwise nondegenerate. Indeed, for each $x \in M$ and $\epsilon > 0$, choose $f \in C^\infty(M)$, with $f(x) \neq 0$ and $\mathsf{supp}(f)$ compactly contained in the interior of the closed ball $\mathbb{B}_\epsilon(x)$, such that $\int_M f \mathsf{vol}_g(M) = 1$; hence, for each $v, w \in \Omega(M;E)$, the following holds

$$\begin{aligned} |b_{g,x}(v_x, f(x)w_x) - \beta_{g,b}(v, fw)| &= \left|b_{g,x}(v_x, f(x)w_x) - \textstyle\int_M b_g(v, fw)\mathsf{vol}_g(M)\right| \\ &= \left|b_{g,x}(v_x, f(x)w_x) - \textstyle\int_{\mathbb{B}_\epsilon(x)} b_g(v, fw)\mathsf{vol}_g(M)\right| < \epsilon. \end{aligned} \tag{3.9}$$

Now suppose there is $v \in \Omega(M;E)$, with $v \neq 0$, such that $\beta_{g,b}(v,w) = 0$, for all $w \in \Omega(M;E)$. That is, there is $v \in \Omega(M;E)$ and $x_0 \in M$ with $v_{x_0} \neq 0$ such that, $|b_{g,x_0}(v_{x_0}, w_{x_0})| \leqslant \epsilon/|f(x_0)|$ for each $w \in \Omega(M;E)$ and $\epsilon > 0$. Since b_{g,x_0} is nondegenerate on the fiber $\Lambda T^*_{x_0}M \otimes E_{x_0}$, we obtain $v_{x_0} = 0$, a contradiction and hence $\beta_{g,b}$ is nondegenerate on $\Omega(M;E)$.

Next, we show formula (3.8). For $v \wedge \star_b w \in \Omega(M; E \otimes E' \otimes \Theta_M)$, by using that $\alpha \wedge \star_g \alpha' = \langle \alpha, \alpha' \rangle_g \mathsf{vol}_g(M)$, for $\alpha, \alpha' \in \Omega(M;\Theta_M)$, and (3.6), we immediately have $\mathrm{Tr}(v \wedge \star_b w) = b_g(v,w)\mathsf{vol}_g(M)$, and hence $\beta_{g,b}(v,w) = \int_M \mathrm{Tr}(v \wedge \star_b w)$. Finally, $\beta_{g,b}$ continuously extends to a nondegenerate symmetric bilinear form on the L^2-closure of $\Omega(M;E)$, since $|\beta_{g,b}(v,w)| \leqslant C\|v\|_{\mathsf{L}^2}\|w\|_{\mathsf{L}^2}$ for all $v, w \in \Omega(M;E)$ for $C > 0$. □

For $E \to M$ a complex vector bundle over a compact manifold M with Riemannian metric g, with flat connection ∇^E and a nondegenerate symmetric bilinear form b, consider $d_E : \Omega^*(M;E) \to \Omega^{*+1}(M;E)$ the de-Rham differential on $\Omega(M;E)$ induced by the flat connection ∇^E. Moreover, by looking at the dual bundle E' of E, endowed with the flat connection $\nabla^{E'}$, and the orientation bundle Θ_M of M, with its canonic flat connection, consider

$$d_{E'\otimes\Theta} : \Omega^*(M;E'\otimes\Theta) \to \Omega^{*+1}(M;E'\otimes\Theta)$$

the associated De-Rham differential on the graded complex $\Omega^*(M;E'\otimes\Theta)$.

Lemma (Definition) 3.2.3. *Let $d^{\sharp}_{E,g,b,q} : \Omega^q(M;E) \to \Omega^{q-1}(M;E)$ be the operator given by*

$$d^{\sharp}_{E,g,b,q} := (-1)^q \left(\star_{b,m-(q-1)}\right)^{-1} d_{E'\otimes\Theta_M,m-q}\star_{b,q}, \tag{3.10}$$

where $\left(\star_{b,m-(q-1)}\right)^{-1}$ is the inverse of the operator $\star_{b,m-(q-1)}$ from (3.7), is a codifferential on $\Omega(M;E)$. In addition, the operators $d^{\sharp}_{E,g,b}$ and d_E permits one to define the operator $\Delta_{E,g,b,q} : \Omega^q(M;E) \to \Omega^q(M;E)$ as

$$\Delta_{E,g,b,q} := d_{E,q-1}d^{\sharp}_{E,g,b,q} + d^{\sharp}_{E,g,b,q+1}d_{E,q} \tag{3.11}$$

$\Delta_{E,g,b}$ is of Laplace type. In particular, if $\mathsf{Spec}_{\mathsf{L}}(\Delta_{E,g,b})$ is the symbolic spectrum of $\Delta_{E,g,b}$, see section 1.2.25, then $\mathsf{Spec}_{\mathsf{L}}(\Delta_{E,g,b}) \subset \mathbb{R}_+$. This generalized Laplacian is called the ***bilinear Laplacian****.*

Proof. Since $\Delta_{E,g,b}$ is a differential operator of order 2, so it remains to compute its principal symbol. We use ${d_E}^2 = 0$ to write $\Delta_{E,g,b} = (d_E + d^{\sharp}_{E,g,b})^2$ so that the principal symbol $\sigma_L(\Delta_{E,g,b}) = \sigma_L(d_E + d^{\sharp}_{E,g,b}) \circ \sigma_L(d_E + d^{\sharp}_{E,g,b})$. We denote by ext_ξ, (resp. by int_ξ), the exterior (resp. the interior) product by $\xi \in T^*_xM$ in $\Lambda^*T^*_xM \otimes E_x$, for each $x \in M$. Then

$$\sigma_L(d_E + d^{\sharp}_{E,g,b})(x,\xi) = \mathbf{i}(\mathsf{ext}_\xi - \mathsf{int}_\xi) \quad \text{on} \quad \Lambda^*T^*_xM \otimes E_x.$$

Since $(\mathsf{ext}_\xi - \mathsf{int}_\xi)^2 = -\|\xi\|^2\mathsf{Id}_x$, we obtain

$$\sigma_L(\Delta_{E,g,b})(x,\xi) = \mathbf{i}^2(-\|\xi\|^2\mathsf{Id}_x) = \|\xi\|^2\mathsf{Id}_x \quad \text{for all} \quad x \in M.$$

That is, $\Delta_{E,g,b}$ is of Laplace type. □

Lemma 3.2.4. *(Green's Formula). Let $E \to M$ be a complex vector bundle over a compact manifold M with Riemannian metric g, endowed with a flat connection ∇^E and a nondegenerate symmetric bilinear form b. For $v, w \in \Omega(M;E)$, we have*

(1) The operator $d^{\sharp}_{E,g,b}$ can be considered as the formal transposed to d_E in $\Omega(M;E)$ with respect to the bilinear form $\beta_{g,b}$ from (3.5). More precisely,

$$\beta_{g,b}(d_Ev, w) - \beta_{g,b}(v, d^{\sharp}_{E,g,b}w) = \int_{\partial M} i^*(\mathrm{Tr}(v \wedge \star_b w)).$$

(2) The bilinear Laplacian from (3.11) can be considered as formal symmetric with respect to the bilinear form $\beta_{g,b}$ from (3.5). More precisely, the difference $\beta_{g,b}(\Delta_Ev, w) - \beta_{g,b}(v, \Delta_Ew)$ is computed by the formula

$$\begin{aligned}&\textstyle\int_{\partial M} i^*(\mathrm{Tr}(d^{\sharp}_{E,g,b}v \wedge \star_b w)) - \int_{\partial M} i^*(\mathrm{Tr}(w \wedge \star_b d_Ev))\\ &\textstyle- \int_{\partial M} i^*(\mathrm{Tr}(d^{\sharp}_{E,g,b}w \wedge \star_b v)) + \int_{\partial M} i^*(\mathrm{Tr}(v \wedge \star_b d_Ew)).\end{aligned}$$

Proof. Let $d_{\Theta_M} : \Omega^*(M;\Theta_M) \to \Omega^{*+1}(M;\Theta_M)$ be the (twisted) De-Rham exterior derivative on $\Omega(M;\Theta_M)$ induced by the flat connection on Θ_M. Remember that d_{Θ_M}

and the differential

$$d_{E\otimes E'\otimes\Theta_M} : \Omega^*(M; E\otimes E'\otimes\Theta_M) \to \Omega^{*+1}(M; E\otimes E'\otimes\Theta_M)$$

are compatible with trace map $\mathrm{Tr} : \Omega(M; E\otimes E'\otimes\Theta_M) \to \Omega(M;\Theta_M)$ in the sense that

$$\mathrm{Tr}\circ d_{E\otimes E'\otimes\Theta_M} = d_{\Theta_M}\circ \mathrm{Tr}, \tag{3.12}$$

where $d_{E\otimes E'\otimes\Theta_M}$ is the differential associated to the connections $\nabla^{E'\otimes\Theta_M}$ and ∇^E induced by ∇^E on the corresponding bundles. For $v\wedge\star_b w$, where v, w are forms in $\Omega(M;E)$ of degree $p-1$ and p respectively, we compute by using the Leibniz Rule,

$$d_{E\otimes E'\otimes\Theta_M}(v\wedge\star_b w) = d_E v\wedge\star_b w + (-1)^{p-1} v\wedge d_{E'\otimes\Theta_M}(\star_b w);$$

remark that the second term on the right hand side of the last expression can be written in terms of $d^\sharp_{E,g,b}$:

$$d_{E\otimes E'\otimes\Theta_M}(v\wedge\star_b w) = d_E v\wedge\star_b w - v\wedge\star_b d^\sharp_{E'\otimes\Theta_M,g,b'} w.$$

Therefore, by taking the trace of the expression above, applying (3.12) on the left hand side and integrating over M, leads to

$$\int_M d_{\Theta_M}(\mathrm{Tr}(v\wedge\star_b w)) = \int_M \mathrm{Tr}(d_E v\wedge\star_b w) - \int_M \mathrm{Tr}(v\wedge\star_b d^\sharp_{E'\otimes\Theta_M,g,b'} w).$$

We use Stokes' Theorem (with the standard sign convention) on the left hand side and Lemma 3.2.2 to write the terms on right as

$$\int_{\partial M} i^*(\mathrm{Tr}(v\wedge\star_b w)) = \beta_{g,b}(d_E v, w) - \beta_{g,b}(v, d^\sharp_{E,g,b} w).$$

The Formula in (2) follows from the one in (1) by using symmetry of $\beta_{g,b}$. □

3.2.1. Absolute/relative boundary conditions on bordisms. Consider a compact Riemannian bordism $(M, \partial_+M, \partial_-M)$ and denote by $i_\pm : \partial_\pm M \hookrightarrow M$ the canonical embeddinngs of $\partial_\pm M$ into M. As above we look at a flat complex vector bundle $E \to M$ with a flat connection ∇^E and a symmetric nondegenerate bilinear form b. Let $\star_b$ be the Hodge $\star$-operator from Definition 3.2.1 and $d^\sharp_{E,g,b}$ the codifferential from Definition 3.2.3. We are interested in spectral properties of the bilinear Laplacian $\Delta_{E,g,b}$. We first need elliptic boundary conditions.

Definition 3.2.5. *A smooth form $w \in \Omega(M;E)$ satisfies* ***absolute/relative****-boundary conditions on the bordism $(M, \partial_+M, \partial_-M)$ if w satisfies absolute boundary conditions on ∂_+M and relative boundary conditions on ∂_-M. More precisely, the space of forms satisfying absolute/relative boundary conditions is given by*

$$\Omega(M;E)|_{\mathcal{B}} := \left\{ w\in\Omega(M;E) \;\middle|\; \begin{array}{cc} i_+^*\star_b w = 0, & i_-^* w = 0 \\ i_+^* d^\sharp_{E'\otimes\Theta_M,g,b}\star_b w = 0, & i_-^* d^\sharp_{E,g,b} w = 0 \end{array} \right\}. \tag{3.13}$$

The space $\Omega(M;E)|_{\mathcal{B}}$ in Definition 3.2.5 is described in terms of the vanishing set of certain boundary operators, which we introduce as follows.

Definition 3.2.6. *Let $E_\pm := i_\pm^* E \to \partial_\pm M$ the corresponding pull-back bundles along each of the canonical embeddings $i_\pm$. For $1 \leqslant q \leqslant m$, we define*

$$\begin{array}{llll}\mathcal{B}_{E,g,b}: & \Omega^q(M;E) & \longrightarrow & \Omega^{q-1}(\partial_+M;E_+) \oplus \Omega^q(\partial_+M;E_+) \\ & & & \quad\oplus\ \Omega^q(\partial_-M;E_-) \oplus \Omega^{q-1}(\partial_-M;E_-) \\ & w & \mapsto & (\mathcal{B}_+w, \mathcal{B}_-w),\end{array} \tag{3.14}$$

where the operators

$$\begin{array}{llll}\mathcal{B}_-: & \Omega^q(M;E) & \longrightarrow & \Omega^q(\partial_-M;E_-) \oplus \Omega^{q-1}(\partial_-M;E_-) \\ & w & \mapsto & (\mathcal{B}^0_-w, \mathcal{B}^1_-w) \\ \mathcal{B}_+: & \Omega^q(M;E) & \longrightarrow & \Omega^{q-1}(\partial_+M;E_+) \oplus \Omega^q(\partial_+M;E_+) \\ & w & \mapsto & (\mathcal{B}^0_+w, \mathcal{B}^1_+w)\end{array} \tag{3.15}$$

are respectively defined in terms of

$$\begin{array}{lcl}\mathcal{B}^0_-w := i_-^*w & & \mathcal{B}^1_-w := i_-^*d^\sharp_{E,g,b}w, \\ & \text{and} & \\ \mathcal{B}^0_+w := \star_b^{\partial M\,-1}\left(i_+^* \star_b w\right) & & \mathcal{B}^1_+w := \star_b^{\partial M\,-1}\left(i_+^* d^\sharp_{E'\otimes\Theta_M,g,b'} \star_b w\right).\end{array} \tag{3.16}$$

Notation 3.2.7. For a subspace $\mathsf{X} \subseteq \Omega(M;E)$, denote by

$$\mathsf{X}|_{\mathfrak{B}} := \{w \in \mathsf{X} | \mathfrak{B}w = 0\}$$

the space of smooth forms in X which satisfy the boundary conditions specified by the vanishing of the operator $\mathfrak{B} \in \{\mathcal{B}^0_\pm, \mathcal{B}^1_\pm, \mathcal{B}_\pm, \mathcal{B}\}$. Set

$$\mathsf{X}|_{\mathcal{B}^0} := \mathsf{X}|_{\mathcal{B}^0_-} \cap \mathsf{X}|_{\mathcal{B}^0_+}.$$

Lemma 3.2.8. *Consider the spaces introduced in Notation 3.2.7. Then the following assertions hold*

(a) $\mathsf{X}|_{\mathcal{B}} = \mathsf{X}|_{\mathcal{B}^0} \cap \mathsf{X}|_{\mathcal{B}^1_-} \cap \mathsf{X}|_{\mathcal{B}^1_+}$ *and* $\mathsf{X}|_{\mathcal{B}} \subset \mathsf{X}|_{\mathcal{B}^0} \subset \mathsf{X}|_{\mathcal{B}^0_-}$,
(b) d_E *leaves invariant the space* $\Omega(M;E)|_{\mathcal{B}^0_-}$: $d_E(\Omega(M;E)|_{\mathcal{B}^0_-}) \subset \Omega(M;E)|_{\mathcal{B}^0_-}$,
(c) $d_E(\Omega(M;E)|_{\mathcal{B}}) \subset \Omega(M;E)|_{\mathcal{B}^0}$, *and* $d^\sharp_{E,g,b}(\Omega(M;E)|_{\mathcal{B}}) \subset \Omega(M;E)|_{\mathcal{B}^0}$,
(d) *If* $v \in \Omega(M;E)|_{\mathcal{B}^0_-}$ *and* $w \in \Omega(M;E)|_{\mathcal{B}}$ *then* $\beta_{g,b}(d_Ev, d^\sharp_{E,g,b}w) = 0$,
(e) *If* $v, w \in \Omega(M;E)|_{\mathcal{B}^0}$, *then* $\beta_{g,b}(d_Ev, w) = \beta_{g,b}(v, d^\sharp_{E,g,b}w)$,
(f) *If* $v, w \in \Omega(M;E)|_{\mathcal{B}}$, *then* $\beta_{g,b}(\Delta_{E,g,b}v, w) = \beta_{g,b}(v, \Delta_{E,g,b}w)$.

Proof. The assertions in (a) and (b) follow straightforward from the definition of these spaces and that i^* and d_E commute. The rest of the proof is also straightforward, since the boundary operators above have been defined in a way the integrants on the right of formulas in Lemma 3.2.4 vanish. We write this in detail.

(c) We show the first inclusion. Remark that for $u \in \Omega(M;E)|_{\mathcal{B}}$ on the one hand we have,

$$i_-^*(d_E u) = d_E i_-^*(u) = 0$$

and on the other

$$i_+^*(\star_b d_E u) = \pm i_+^*(\star_b d_E \star_b^{-1} \star_b u) = \pm i_+^*(d^\sharp_{E'\otimes\Theta_M,g,b'} \star_b u) = 0;$$

hence $d_E u \in \Omega(M;E)|_{\mathcal{B}^0}$ if $u \in \Omega(M;E)|_{\mathcal{B}}$. We show the second inclusion. For $u \in \Omega(M;E)|_{\mathcal{B}}$, we have $i_-^*(d^\sharp_{E,g,b}u) = 0$, but also

$$i_+^*(\star_b d^\sharp_{E,g,b}u) = \pm i_+^*(\star_b(\star_b^{-1} d_{E'\otimes\Theta_M} \star_b)u) = \pm i_+^*(d_{E'\otimes\Theta_M} \star_b u) = \pm d_{E'\otimes\Theta_M} i_+^*(\star_b u) = 0,$$

which exactly means that $d^\sharp_{E,g,b}u \in \Omega(M;E)|_{\mathcal{B}^0}$ if $u \in \Omega(M;E)|_{\mathcal{B}}$.

(d) By Lemma 3.2.4 and ${d^\sharp_{E,g,b}}^2 = 0$, we have

$$\beta_{g,b}(d_E v, d^\sharp_{E,g,b}w) = \int_{\partial M} i^*(v \wedge \star_b d^\sharp_{E,g,b}w), \quad \text{for each} \quad v, w \in \Omega(M;E).$$

Since ∂M is the disjoint union of $\partial_- M$ and $\partial_+ M$, the integral over the boundary splits as

$$\int_{\partial M} i^*(v\wedge\star_b d^\sharp_{E,g,b}w) = \int_{\partial_- M} i_-^*(v\wedge\star_b d^\sharp_{E,g,b}w) + \int_{\partial_+ M} i_+^*(v\wedge\star_b d^\sharp_{E,g,b}w).$$

But, the integral over $\partial_- M$ vanishes, since $v \in \Omega(M;E)|_{\mathcal{B}^0_-}$ and the integral over $\partial_+ M$ vanishes, since $w \in \Omega(M;E)|_{\mathcal{B}}$ implies $i_+^* d_{E'\otimes\Theta,g,b'} \star_b w = 0$ so that

$$i_+^* \star_b d^\sharp_{E,g,b} = 0.$$

(e) Again from Lemma 3.2.4, we have

$$\begin{array}{rcl} \beta_{g,b}(d_E v, w) - \beta_{g,b}(v, d^\sharp_{E,g,b}(w)) & = & \int_{\partial M} i^*(v \wedge \star_b w) \\ & = & \int_{\partial_- M} i_-^*(v \wedge \star_b w) + \int_{\partial_+ M} i_+^*(v \wedge \star_b w). \end{array}$$

for every $v, w \in \Omega(M;E)$. Now, if $v, w \in \Omega(M;E)|_{\mathcal{B}^0}$, then the integral over $\partial_- M$ vanishes, because $i_-^*(v) = 0$ and the integral over $\partial_+ M$ vanishes as well, since $i_+^* \star_b w = 0$.

(f) This follows from (b), (c) and symmetry of $\beta_{g,b}$. See also proof of Lemma 3.2.4.

□

Notation 3.2.9. Let $(M, \partial_+ M, \partial_- M)$ be a given compact bordism with Riemannian metric g. Let E be a flat complex vector bundle with flat connection ∇^E and a fiberwise defined nondegenerate bilinear form b. The boundary value problem $(\Delta_{E,g,b}, \Omega(M;E)|_{\mathcal{B}})$ specified by the bilinear Laplacian $\Delta_{E,g,b}$ acting on the space $\Omega(M;E)|_{\mathcal{B}}$ in (3.13), characterized by the vanishing of the operator $\mathcal{B}_{E,g,b}$ in (3.14), will be denoted by

$$[\Delta, \Omega_{\mathcal{B}}]^{E,g,b}_{(M,\partial_+ M,\partial_- M)}. \tag{3.17}$$

In the same way, for $\mathfrak{B} \in \{\mathcal{B}^0_\pm, \mathcal{B}^1_\pm, \mathcal{B}_\pm, \mathcal{B}\}$, we denote by

$$[\Omega_{\mathfrak{B}}]^{E,g,b}_{(M,\partial_+ M,\partial_- M)} \tag{3.18}$$

the space $\Omega(M;E)|_{\mathfrak{B}}$ defined in Notation 3.2.7 corresponding to the data E, g, b and $(M, \partial_+M, \partial_-M)$.

Remark 3.2.10. In particular, if $\partial_+M = \partial M$ and $\partial_-M = \emptyset$ (resp. $\partial_+M = \emptyset$ and $\partial_-M = \partial M$), then $[\Delta, \Omega_{\mathcal{B}}]^{E,g,b}_{(M,\partial M,\emptyset)}$, (resp. $[\Delta, \Omega_{\mathcal{B}}]^{E,g,b}_{(M,\emptyset,\partial M)}$) denotes the boundary value problem where *only* absolute (resp. relative) boundary conditions are imposed on the boundary ∂M.

3.2.2. Relative cohomology and Poincaré–Lefschetz duality. We freely use Notation 3.2.9. Given a bordism $(M, \partial_+M, \partial_-M)$ with Riemannian metric g, a flat connection ∇^E and a nondegenerate bilinear form on the bundle E, consider their dual versions: the bordism $(M, \partial_+M, \partial_-M)'$ with Riemannian metric g, the dual flat connection $\nabla^{E'}$ and the dual nondegenerate bilinear form b' on the bundle E'. We want to relate the boundary value problems corresponding to the corresponding data, by using the Hodge $\star$-operator $\star_b : \Omega^p(M;E) \to \Omega^{m-p}(M;E' \otimes \Theta_M)$. From the definition of the corresponding spaces of forms satisfying absolute/relative conditions in Definition (3.2.5), it is clear that a form $w \in \Omega^q(M;E)$ satisfies absolute/relative-boundary conditions on the bordism $(M, \partial_+M, \partial_-M)$ if and only if $\star_b w \in \Omega^{m-q}(M;E' \otimes \Theta_M)$ satisfies absolute/relative-boundary conditions on the dual bordism $(M, \partial_+M, \partial_-M)'$. Moreover, since

$$\star_b d^\sharp_{E,g,b} d_E = d_{E'\otimes\Theta_M} d^\sharp_{E'\otimes\Theta_M,g,b'} \star_b,$$

so that

$$\star_b \Delta_{E,g,b} = \Delta_{E'\otimes\Theta_M,g,b'} \star_b,$$

the operator $\star_b$ *intertwines* the boundary value problems

$$[\Delta, \Omega_{\mathcal{B}}]^{E,g,b}_{(M,\partial_+M,\partial_-M)} \quad \longleftrightarrow \quad [\Delta, \Omega_{\mathcal{B}}]^{E'\otimes\Theta_M,g,b'}_{(M,\partial_+M,\partial_-M)'}.$$

Moreover, notice that $[\Omega_{\mathcal{B}^0_-}]^{E,g,b}_{(M,\partial_+M,\partial_-M)}$, when considered as a subcochain complex in $(\Omega(M;E), d_E)$, computes the relative cohomology groups $H(M, \partial_-M; E)$. In the same way, $[\Omega_{\mathcal{B}^0_-}]^{E'\otimes\Theta_M,g,b'}_{(M,\partial_+M,\partial_-M)'}$ as a subcochain complex in $(\Omega(M;E'\otimes\Theta_M), d_{E'\otimes\Theta_M})$ computes the relative cohomology groups $H(M, \partial_+M; E' \otimes \Theta_M)$. Then, by computing the integral

$$\int_M \mathrm{Tr}(v \wedge \star_b w)$$

of representatives of relative cohomology classes $v \in \Omega^p(M;E)$ and $w \in \Omega^p(M;E'\otimes\Theta_M)$, the operator $\star_b$ induces a nondegenerate pairing in relative cohomology:

$$H^p(M, \partial_+M; E' \otimes \Theta_M) \times H^{m-p}(M, \partial_-M; E) \to \mathbb{C},$$

In other words, we have an isomorphism

$$H^p(M, \partial_+M; E' \otimes \Theta_M) \cong H^{m-p}(M, \partial_-M; E)' \tag{3.19}$$

also referred as the **Poincaré–Lefschetz** duality (for bordisms) in cohomology. Much more material on (co)bordism theory and Poincaré–Lefschetz duality can be found in chapter VII in [**Do95**] and chapter 5, section 2.8 in [**GMS98**], see also [**Mi62**], [**BT82**].

3.2.3. Invariant description of absolute/relative boundary conditions. We give a description of absolute/relative boundary conditions in terms of the local geometry around the boundary.

For simplicity we consider a manifold with boundary ∂M without distinguishing the roles of $\partial_\pm M$ unless it is explicitly needed. Let $E \to M$ be a complex vector bundle over M and ∇^E a connection on E and denote by $\nabla = \nabla^g$ the Levi–Cività connection on TM. In order to covariantly differentiate tensors of arbitrary type (i.e., elements in the mixed tensor algebra generated by TM, T^*M, E and E') we use the connection $\nabla^{E,g}$ obtained by extending the one on E by the Levi–Cività connection on TM. For short, we sometimes write $F := \Lambda T^*M \otimes E$. With ς_{in}, the inwards pointing geodesic unit normal vector field to the boundary, near the boundary we consider a *collared neighborhood* U of ∂M in M and *geodesic coordinates* $x = (x^1, \ldots, x^{m-1}, x^m)$. That is $\partial_m = \varsigma_{\mathsf{in}}$ and the coordinates $x^1, \ldots, x^{m-1}$ define a coordinate system at the boundary, so that $\{\partial_\alpha\}_{\alpha=1}^{m-1}$ is a coordinate frame of $T\partial M$. We designate by $\{dx^\alpha\}_{\alpha=1}^{m-1}$ the correspoding dual coordinate coframe of $T^*\partial M$ so that $\{dx^\alpha\}_{\alpha=1}^{m-1} \cup \{dx^m\}$ is coframe of T^*M. We use *Einstein convention on repeated indices* $i, j, \ldots \neq m$ and $\alpha, \beta, \ldots$ unless otherwise indicated.

Certain involutions and splittings. Over the collar, the metric can be written as

$$g(x) = g_{\alpha\beta}(x) dx^\alpha \otimes dx^\beta + dx^m \otimes dx^m. \tag{3.20}$$

For $\alpha \in T^*M$, we use the **left exterior** operator $\mathsf{ext}(\alpha) : \Lambda T^*M \to \Lambda T^{*+1}M$

$$\mathsf{ext}(\alpha)(\beta) := \alpha \wedge \beta \text{ for } \beta \in \Lambda T^*M,$$

and for $X \in TM$, $\mathsf{int}(X) : \Lambda T^*M \to \Lambda T^{*-1}M$ **left interior** operator defined by

$$\begin{aligned} \mathsf{int}(X)(f) &:= 0, \quad \text{for } f \in C^\infty(M), \\ \mathsf{int}(X)(\beta_1) &:= \beta_1(X), \quad \text{for } \beta_1 \in \Lambda^1 T^*M, \\ \mathsf{int}(X)(\beta_p \wedge \beta_q) &:= \mathsf{int}(X)(\beta_p) \wedge \beta_q + (-1)^p \beta_p \wedge \mathsf{int}(X)(\beta_q), \\ & \qquad \text{for } \beta_p \in \Lambda^p T^*M \text{ and } \beta_q \in \Lambda^q T^*M. \end{aligned}$$

In particular, with the short notation

$$\mathsf{ext}_j := \mathsf{ext}(dx^j) \text{ and } \mathsf{int}_j := \mathsf{int}(\partial_j),$$

we have

$$\mathsf{ext}_i \mathsf{int}_j + \mathsf{int}_j \mathsf{ext}_i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}. \tag{3.21}$$

Over the collar U, since ∇^g is compatible with Riemannian metric, we have

$$\nabla^g \mathsf{ext} = 0 \quad \text{and} \quad \nabla^g \mathsf{int} = 0,$$

so that, for each $i, j \in \{1, \ldots, m\}$, the relations

$$\begin{aligned} \nabla^g_{\partial_i} \mathsf{ext}_j &= \mathsf{ext}(\nabla^g_{\partial_i}(dx^j)) = \Gamma_{i\ k}^{\ j} \mathsf{ext}_k, \\ \nabla^g_{\partial_i} \mathsf{int}_j &= \mathsf{int}(\nabla^g_{\partial_i}(\partial_j)) = \Gamma_{ij}^{\ \ k} \mathsf{int}_k \end{aligned} \tag{3.22}$$

hold. In these coordinates, we have the decomposition

$$\Lambda(T^*M) \cong \Lambda(T^*\partial M) \oplus \Lambda(T^*\partial M)^\perp, \tag{3.23}$$

as bundles over U, where

$$\Lambda(T^*\partial M) \cong \mathsf{Span}_I(dx^I) \text{ and } \Lambda(T^*\partial M)^\perp \cong \mathsf{Span}_I(dx^m \wedge dx^I),$$

and $I = \{1 \leqslant \alpha_1 < \alpha_1 < \alpha_2 \cdots \alpha_p \leqslant m-1\}$ denotes a multi-index. Therefore, over the collar, the bundle ΛT^*M decomposes as

$$\begin{array}{rcl} (\Lambda T^*M)|_U & \cong & (\Lambda(T^*\partial M))\,|_U \oplus \left(\Lambda(T^*\partial M)^\perp\right)|_U \\ \alpha & \mapsto & (\alpha^{\mathsf{t}}, \alpha^{\mathsf{n}}), \end{array} \tag{3.24}$$

where, in terms of ext and int, using (3.21), one has

$$\begin{array}{rcl} \alpha^{\mathsf{t}} & := & \mathsf{int}_m \mathsf{ext}_m \alpha \\ \alpha^{\mathsf{n}} & := & \mathsf{ext}_m \mathsf{int}_m \alpha. \end{array} \tag{3.25}$$

That is, each form α can be locally written as

$$\alpha := (\mathsf{int}_m \mathsf{ext}_m + \mathsf{ext}_m \mathsf{int}_m)\alpha = \alpha^{\mathsf{t}} + dx^m \wedge \alpha^{\mathsf{n}}. \tag{3.26}$$

Next, according to the decomposition in (3.24) and (3.25), over U one defines the **involution**

$$\begin{array}{rrcl} \chi: & \Lambda T^*M & \to & \Lambda T^*M \\ & \alpha & \mapsto & (\mathsf{int}_m \mathsf{ext}_m - \mathsf{ext}_m \mathsf{int}_m)\alpha = \alpha^{\mathsf{t}} - dx^m \wedge \alpha^{\mathsf{n}} \end{array} \tag{3.27}$$

For $\nabla^g_{\partial_j}\chi$, the covariant derivatives of χ along ∂_j, a direct computation using (3.22), $\nabla^g_{\partial_m}(\partial_m) = 0$ and (3.20) (that is, $\Gamma_{m\ j}^{\ i} = 0$ for $i = m$ or $j = m$), leads to

$$\nabla^g_{\partial_m}\chi = 0 \quad \text{and} \quad \nabla^g_{\partial_\alpha}\chi = 2\mathsf{L}_{\alpha\beta}(\mathsf{ext}_\beta \mathsf{int}_m + \mathsf{ext}_m \mathsf{int}_\beta), \tag{3.28}$$

(see also Lemma 1.5.4 in [**Gi04**]). Remark that, the formulas (3.28) allows us to write any covariant derivatives of χ, with respect to $\nabla^g_{\partial_\alpha}$, along tangential direcctions, in terms of the second fundamental form L and the endomorphisms int and ext.

Next, we look at the eigen-values of χ, i.e. $+1$ and -1, in order to consider the corresponding decomposition of $\Lambda(T^*M)$ with spectral projections

$$\begin{aligned} \Pi_{\mathsf{t}}: \ \Gamma\left(M;\Lambda(T^*M)\right) &\to \Gamma(M;(\Lambda(T^*M))^{\mathsf{t}}) \\ \Pi_{\mathsf{t}} &:= \tfrac{1}{2}\left(1+\chi\right) = \mathsf{int}_m\mathsf{ext}_m \\ \Pi_{\mathsf{n}}: \ \Gamma\left(M;\Lambda(T^*M)\right) &\to \Gamma(M;(\Lambda(T^*M))^{\mathsf{n}}) \\ \Pi_{\mathsf{n}} &:= \tfrac{1}{2}\left(1-\chi\right) = \mathsf{ext}_m\mathsf{int}_m \end{aligned} \tag{3.29}$$

respectively. Remark that

$$\mathsf{int}_m = \mathsf{int}_m\Pi_{\mathsf{n}} = \Pi_{\mathsf{t}}\mathsf{int}_m \quad \text{and} \quad \mathsf{ext}_m\Pi_{\mathsf{t}} = \Pi_{\mathsf{n}}\mathsf{ext}_m. \tag{3.30}$$

Since over the collar $\nabla^g_m\chi = 0$ holds, we have

$$\nabla^g_{\partial_m}\Pi_{\mathsf{t}} = \Pi_{\mathsf{t}}\nabla^g_{\partial_m} \quad \text{and} \quad \nabla^g_{\partial_m}\Pi_{\mathsf{n}} = \Pi_{\mathsf{n}}\nabla^g_{\partial_m}. \tag{3.31}$$

Thus, by (3.28), tangentinal derivatives of Π_{n} and Π_{t} can be invariantly described in terms of the second fundamental form L, ext and int.

Invariant description for absolute/relative boundary conditions. For each $\alpha \in \Lambda T^*M$, we extend $\mathsf{ext}_\alpha : \Lambda T^*M \to \Lambda T^{*+1}M$ and $\mathsf{int}_\alpha : \Lambda T^*M \to \Lambda T^{*-1}M$ by the the identity on E so that we denote by

$$\mathsf{ext}_\alpha := \mathsf{ext}_\alpha \otimes \mathsf{Id}_E : \Lambda T^*M \otimes E \to \Lambda T^{*+1}M \otimes E$$

and

$$\mathsf{int}_\alpha := \mathsf{int}_\alpha \otimes \mathsf{Id}_E : \Lambda T^*M \otimes E \to \Lambda T^{*-1}M \otimes E.$$

In this situation, we have

$$\nabla^{E,g}\mathsf{ext} = 0 = \nabla^{E,g}\mathsf{int}. \tag{3.32}$$

Also, over the collar, the map in (3.27) induces an involution

$$\chi := \chi \otimes \mathsf{Id}_E : \Lambda T^*M \otimes E \to \Lambda T^*M \otimes E.$$

Then with the (3.28) we have

$$\nabla^{E,g}_{\partial_m}\chi = 0 \quad \text{and} \quad \nabla^{E,g}_{\partial_\alpha}\chi = 2\mathsf{L}_{\alpha\beta}(\mathsf{ext}_\beta\mathsf{int}_m + \mathsf{ext}_m\mathsf{int}_\beta) \otimes \mathsf{Id}_E, \tag{3.33}$$

so that $\chi^2 = \mathsf{Id}_{\Lambda T^*M\otimes E}$ locally around the boundary. Therefore χ allows to decomponse $\Lambda T^*M \otimes E$ in terms of the subbundles corresponding to the eigenvalues ± 1 of χ. We still denote by Π_{t} and Π_{t} the correspoding spectral projections. Remark that, over U, we obtain analog relations in analogy with (3.29) to (3.31).

The following Lemma gives an equivalent reformulation for the *absolute* boundary operator acting on E-valued differential forms in terms of the operators and Π_{n} and Π_{t}.

Lemma 3.2.11. *Consider the operators $\mathcal{B}_+$ and $\mathcal{B}_-$ from (3.15). Then,*

$$\begin{aligned} \mathcal{B}_+ v = 0 \quad &\Leftrightarrow \quad (\Pi_{\mathsf{n}} v, \Pi_{\mathsf{n}} d_E v)\,|_{\partial M} = 0, \\ \mathcal{B}_- v = 0 \quad &\Leftrightarrow \quad (\Pi_{\mathsf{t}} v, \Pi_{\mathsf{t}} d^\sharp_{E,g,b} v)|_{\partial M} = 0, \end{aligned} \tag{3.34}$$

respectively.

Proof. Let $i : \partial M \to M$ be the canonical embedding of ∂M into M. Remark that, for each E-valued smooth form v, we have $\mathsf{int}_m v = 0 \Leftrightarrow \Pi_{\mathsf{n}} v = 0$ and hence

$$i^* \mathsf{int}_m v = 0 \Leftrightarrow i^* \Pi_{\mathsf{n}} v = 0. \tag{3.35}$$

Let $\star^\partial_b$ denote the operator induced by $\star^\partial_g$ and b on the boundary. By using

$$\star^\partial_b \, i^* \mathsf{int}_m v = i^* \star_b v \tag{3.36}$$

in the formula (3.10) defining the operator $d^\sharp_{E\otimes\Theta_M,g,b}$, we can write the operator $\mathcal{B}_+$ defined by (3.15) and (3.16), as

$$\mathcal{B}_+ v = (i^* \mathsf{int}_m v, (-1)^{q+1} i^* \mathsf{int}_m (d_E v)) \quad \text{for all} \quad v \in \Omega^q(M; E). \tag{3.37}$$

Then, the statement for $\mathcal{B}_+$ follows from (3.37) and (3.35) above. The statement for $\mathcal{B}_-$ is clear by its definition, since for each E-valued smooth form v we have $i^* \Pi_{\mathsf{t}} v = i^* v$. $\square$

Proposition 3.2.12. *We have the following desrcription of absolute/relative boundary conditions in terms of (locally computable) tensorial objects.*

(a) Absolute boundary conditions, specified by the vanishing of $\mathcal{B}_+$ in (3.15), can be described in terms of the involution $\chi : \Lambda T^*M \otimes E \to \Lambda T^*M \otimes E$ and a (tangential) bundle endomorphism $\mathsf{S}_{\mathsf{abs}}$, locally computable on a (collared) neigbourhood of ∂M in terms of derivatives of the Riemannian metric and the second fundamental form and extended over the collar with the condition $\nabla_{\partial_m} \mathsf{S}_{\mathsf{abs}} = 0$. More precisely,

$$\mathcal{B}_+ u = 0 \iff \begin{array}{l} (\Pi_{\mathsf{n}} u)\,|_{\partial M} = 0 \quad \textit{and} \\ \left(\Pi_{\mathsf{t}} \left(\nabla^F_{\partial_m} + \mathsf{S}_{\mathsf{abs}}\right)(u)\right)\big|_{\partial M} = 0. \end{array} \tag{3.38}$$

(b) Relative boundary conditions, specified by the vanishing of $\mathcal{B}_-$ in (3.15), can be described in terms of the involution $\chi : \Lambda T^*M \otimes E \to \Lambda T^*M \otimes E$, the bundle endomorphism $b^{-1}\nabla^E_{\partial_m} b$, and a (tangential) bundle endomorphism $\mathsf{S}_{\mathsf{rel}}$, locally computable on a (collared) neigbourhood of ∂M in terms of derivatives of the Riemannian metric and the second fundamental form and extended over the collar with the condition $\nabla_{\partial_m} \mathsf{S}_{\mathsf{rel}} = 0$. More precisely,

$$\mathcal{B}_- u = 0 \iff \begin{array}{l} (\Pi_{\mathsf{t}} u)\,|_{\partial M} = 0 \quad \textit{and} \\ \left(\Pi_{\mathsf{n}} \left(\nabla^F_{\partial_m} + b^{-1}(\nabla^E_{\partial_m} b) + \mathsf{S}_{\mathsf{rel}}\right) u\right)\big|_{\partial M} = 0. \end{array} \tag{3.39}$$

Proof. (1) Consider the vanishing of $\mathcal{B}_+$ in the proof of Lemma 3.34 specifying absolute boundary conditions. The (vanishing of the) second component of $\mathcal{B}_+$ can be expressed, in geodesic coordinates over the collar U, by using

$$d_E = \sum_{i=1}^{m} dx^i \wedge \nabla^{E,g}_{\partial_i},$$

with the help of the Einstein convention on repeated indexes (with the exception of m), as

$$\begin{aligned} 0 &= i^*\mathsf{int}_m d_E u \\ &= i^*\left(\mathsf{int}_m\mathsf{ext}_m\left(\nabla^{E,g}_{\partial_m}u\right) + \mathsf{int}_m\mathsf{ext}_\alpha\left(\nabla^{E,g}_{\partial_\alpha}u\right)\right) \\ &= i^*\left(\Pi_\mathsf{t}\left(\nabla^{E,g}_{\partial_m}u\right) - \mathsf{ext}_\alpha\mathsf{int}_m\left(\nabla^{E,g}_{\partial_\alpha}u\right)\right) \\ &= i^*\left(\Pi_\mathsf{t}\left(\nabla^{E,g}_{\partial_m}u\right) - \mathsf{ext}_\alpha\mathsf{int}_m\Pi_\mathsf{n}\left(\nabla^{E,g}_{\partial_\alpha}u\right)\right) \\ &= i^*\left(\Pi_\mathsf{t}\left(\nabla^{E,g}_{\partial_m}u\right)\right) - i^*\left(\Pi_\mathsf{t}\mathsf{ext}_\alpha\mathsf{int}_m\left(\nabla^{E,g}_{\partial_\alpha}u\right)\right), \end{aligned} \tag{3.40}$$

whereas (the vanishing of) the first component in (3.37) is equivalent to

$$\begin{aligned} 0 &= i^*d_E(\mathsf{int}_m u) \\ &= i^*\left(\mathsf{ext}_m\left(\nabla^{E,g}_{\partial_m}(\mathsf{int}_m u)\right) + \mathsf{ext}_\alpha\left(\nabla^{E,g}_{\partial_\alpha}(\mathsf{int}_m u)\right)\right) \\ &= i^*\left(\mathsf{ext}_m\left(\nabla^{E,g}_{\partial_m}(\mathsf{int}_m u)\right) + \mathsf{ext}_\alpha\left(\left(\nabla^{E,g}_{\partial_\alpha}\mathsf{int}_m\right)u + \mathsf{int}_m(\nabla^{E,g}_{\partial_\alpha}u)\right)\right). \end{aligned} \tag{3.41}$$

and therefore, after projecting on the tangential part, we have

$$i^*\Pi_\mathsf{t}\left(\mathsf{ext}_\alpha\mathsf{int}_m(\nabla^{E,g}_{\partial_\alpha}u)\right) = -i^*\Pi_\mathsf{t}\left(\mathsf{ext}_\alpha\left(\left(\nabla^{E,g}_{\partial_\alpha}\mathsf{int}_m\right)u\right)\right) \tag{3.42}$$

That is, all terms containing $\nabla^{E,g}_{\partial_\alpha}$, i.e., derivatives of u along tangential directions, can be entirely expressed pointwise as a linear operator on u. By using the formulas (3.22) we can express the term on the right in the last line in 3.42 above in terms of the second fundamental form L, (c.f. Lemma 1.5.4 in [**Gi04**]):

$$\begin{aligned} -\Pi_\mathsf{t}\left(\mathsf{ext}_\alpha\left(\left(\nabla^{E,g}_{\partial_\alpha}\mathsf{int}_m\right)v\right)\right) &= -\Pi_\mathsf{t}\mathsf{ext}_\alpha\mathsf{int}_m(\nabla^{E,g}_\alpha v) \\ &= -\Pi_\mathsf{t}\mathsf{ext}_\alpha{\Gamma_{\alpha m}}^k\mathsf{int}_k(v) \\ &= -\Pi_\mathsf{t}\mathsf{ext}_\alpha{\Gamma_{\alpha m}}^\sigma\mathsf{int}_\sigma(v) \\ &= -\Pi_\mathsf{t}{\Gamma_{\alpha m}}^\sigma\mathsf{ext}_\alpha\mathsf{int}_\sigma\Pi_\mathsf{t}(v) \end{aligned}$$

and set

$$\mathsf{S}_\mathsf{abs} := -\Pi_\mathsf{t}\left({\Gamma_{\alpha m}}^\sigma\mathsf{ext}_\alpha\mathsf{int}_\sigma \otimes \mathsf{Id}_E\right)\Pi_\mathsf{t}. \tag{3.43}$$

Remark that, since ext_α commute with ext_m, with the relation in (3.32) (c.f. (3.22)), we obtain

$$\nabla^{E,g}_{\partial_m}\mathsf{S}_{abs} = 0, \quad \Pi_\mathsf{t}\mathsf{S}_\mathsf{abs} = \mathsf{S}_\mathsf{abs}\Pi_\mathsf{t} \quad \text{and} \quad \Pi_\mathsf{n}\mathsf{S}_\mathsf{abs} = \mathsf{S}_\mathsf{abs}\Pi_\mathsf{n} = 0. \tag{3.44}$$

Finally, we use (3.43) to write the expressions containing $\nabla^F_{\partial_\alpha}u$ on the right in the last line of (3.40) to get

$$\left(\Pi_\mathsf{t}\left(\nabla^F_{\partial_m} + \mathsf{S}\right)u\right)\big|_{\partial M} = 0.$$

That is exactly (3.38).

(2) Consider the dual bundle E', with the dual connection $\nabla^{E'}$ and dual bilinear form b'. Let $\nabla^{F'}$ be the connection on $F' := \Lambda(T^*M)\otimes E'\otimes\Theta_M$ induced by $\nabla^{E'}$ and the Levi-Cività connection on TM. We denote by $\mathcal{B}'_+$ the same operator from (3.15) and (3.16) imposing absolute boundary conditions but associated to the data F', $\nabla^{F'}$ and b'. From Section 3.2.2 recall that the Hodge $\star$-operator, intertwins absolute and relative boundary conditions. That is, the vanishing of the operator $\mathcal{B}_-$, on a smooth E-valued form u, is equivalent to the vanishing of $\mathcal{B}'_+$ on the smooth $E'\otimes\Theta_M$-valued form $\star_b u$. Thus, in this setting, with

$$\begin{aligned}
\Pi'_{\mathsf{t}} &:= \Pi_{\mathsf{t}}\otimes \mathsf{Id}_{E'\otimes\Theta_M},\\
\Pi'_{\mathsf{n}} &:= \Pi_{\mathsf{n}}\otimes \mathsf{Id}_{E'\otimes\Theta_M},\\
\chi' &:= \chi\otimes \mathsf{Id}_{E'\otimes\Theta_M},\\
\mathsf{S_{abs}}' &:= -\Pi'_{\mathsf{t}}\,(\Gamma_{\alpha m}{}^{\sigma}\mathsf{int}_\sigma\mathsf{ext}_\alpha\otimes \mathsf{Id}_{E'\otimes\Theta_M})\,\Pi'_{\mathsf{t}}
\end{aligned}$$

the computations performed above for (a) still hold; in particular, (3.38) reads as

$$\mathcal{B}'_+\star_b u=0 \iff \begin{array}{l}(\Pi'_{\mathsf{n}}\star_b u)\,|_{\partial M}=0 \quad\text{and}\\ \left(\Pi'_{\mathsf{t}}\left(\nabla^{F'}_{\partial_m}+\mathsf{S_{abs}}'\right)(\star_b u)\right)\Big|_{\partial M}=0.\end{array} \tag{3.45}$$

The spliting of $\Lambda(T^*M)$ in (3.23) is intertwined by the action of $\star_g$ and hence, over the collar, we have the bundle isomorphisms

$$\begin{aligned}
\star_b:&\quad \Lambda(T^*\partial M)\otimes E &\to&\quad \Lambda(T^*\partial M)^\perp\otimes E'\otimes\Theta_M\\
\star_b:&\quad \Lambda(T^*\partial M)^\perp\otimes E &\to&\quad \Lambda(T^*\partial M)\otimes E'\otimes\Theta_M
\end{aligned}$$

and

$$\begin{aligned}
\Pi_{\mathsf{n}} &= \star_b^{-1}\Pi'_{\mathsf{t}}\star_b\\
\Pi_{\mathsf{t}} &= \star_b^{-1}\Pi'_{\mathsf{n}}\star_b\,.
\end{aligned} \tag{3.46}$$

From (3.46), it follows

$$\Pi'_{\mathsf{n}}\star_b u=0 \Leftrightarrow \star_b^{-1}\Pi'_{\mathsf{n}}\star_b u=0\Leftrightarrow \Pi_{\mathsf{t}}u=0. \tag{3.47}$$

Now, by using (3.44) and (3.46) we obtain

$$\begin{aligned}
\Pi'_{\mathsf{t}}\left(\nabla^{F'}_{\partial_m}+\mathsf{S_{abs}}'\right)(\star_b u)=0 \quad&\Leftrightarrow\quad \star_b^{-1}\Pi'_{\mathsf{t}}\left(\nabla^{F'}_{\partial_m}+\mathsf{S_{abs}}'\right)(\star_b u)=0\\
&\Leftrightarrow\quad \Pi_{\mathsf{n}}\left(\star_b^{-1}\nabla^{F'}_{\partial_m}+\star_b^{-1}\mathsf{S_{abs}}'\right)(\star_b u)=0\\
&\Leftrightarrow\quad \Pi_{\mathsf{n}}\left(\nabla^{F}_{\partial_m}+\star_b^{-1}(\nabla^{F}_{\partial_m}\star_b)+\star^{-1}\mathsf{S_{abs}}'\star_b\right)u=0,\\
&\Leftrightarrow\quad \Pi_{\mathsf{n}}\left(\nabla^{F}_{\partial_m}+\star_b^{-1}(\nabla^{F}_{\partial_m}\star_b)+\mathsf{S_{rel}}\right)u=0,
\end{aligned} \tag{3.48}$$

where

$$\mathsf{S_{rel}} := -\Pi_{\mathsf{n}}\,(\Gamma_{\alpha m}{}^{\sigma}\mathsf{int}_\alpha\mathsf{ext}_\sigma\otimes\mathsf{Id}_E)\,\Pi_{\mathsf{n}} \tag{3.49}$$

encodes the tangential covariant derivatives in terms of the second fundamental form. Finally, by (3.47) and (3.48), we obtain (3.39).

□

Remark 3.2.13. Recall Definition 2.2.5 in Section 2.2.4. On the one hand, Proposition 3.2.12 tells us exactly that the operator $\mathcal{B}_+$ in (3.15) specifies mixed boundary conditions. On the other hand, assume $\nabla^E_{\partial_m} b = 0$; this assumption guarantees that $\mathcal{B}_-$ specifies mixed boundary conditions as well in the sense of Definition 2.2.5. In the general case, if we drop the condition on b, we cannot longer expect that $b^{-1}(\nabla^E_{\partial_m} b)$ is parallely transported along the normal geodesics over the collar. This can already been seen easily in the case $F = \Lambda T^*M$ with the $\nabla^{\phi,g}$, the Wittney connection, c.f. Lemma 1.5.5 in [**Gi04**]. However, this is not much of trouble for later considerations and we **do not** assume $\nabla^E_{\partial_m} b$ to vanish in general.

3.3. Hodge–De-Rham decomposition for the bilinear Laplacian

In general, the operator $\Delta_{E,g,b}$ considered as unbounded operator in $\mathsf{L}^2(M;E)$, with domain of definition $\Omega(M;E)|_{\mathcal{B}}$, is not self-adjoint. However, $\Delta_{E,g,b}$ being of Laplace type and this being an elliptic boundary value problem, its spectrum still possesses similar properties to that of a self-adjoint Laplacian. In fact, in this section we see that $\Delta_{\mathcal{B}}$ is densely defined in $\mathsf{L}^2(M;E)$, possesses a non-empty resolvent set, its resolvent is compact, its spectrum is discrete and therefore the generalized eigen-spaces of $\Delta_{\mathcal{B}}$ are of finite dimension. Elliptic estimates allow us to see that such spaces contain smooth forms only. Moreover, the restriction of $\Delta_{\mathcal{B}}$ to the space of smooth forms satisfying boundary conditions and orthogonal complement of each generalized eigen-space, is invertible. Then, we obtain a Hodge decomposition type result for the bilinear Laplacian action on smooth forms. In turn, that permits us to conclude that the (relative) cohomology of M can be computed by looking uniquely at the generalized 0-eigenspace of bilinear Laplacian. That is, the first step to define the complex-valued Ray–Singer torsion in Chapter 5.

3.3.1. Boundary ellipticity for the bilinear Laplacian. In the sense of Section 2.2, the boundary value problem $[\Delta, \Omega_{\mathcal{B}}]^{E,g,b}_{(M,\partial_+M,\partial_-M)}$ is an elliptic boundary value problem. More precisely, we have the following result.

Proposition 3.3.1. *The boundary value problem* $[\Delta, \Omega_{\mathcal{B}}]^{E,g,b}_{(M,\partial_+M,\partial_-M)}$ *is elliptic with respect to the cone* $\mathcal{C}_0 = \mathbb{C}\backslash(0,\infty)$.

Proof. As in Section 3.2.3, near the boundary consider ς_{in} the inwards pointing geodesic unit normal vector field to the boundary, together with a collared neighborhood U of ∂M in M and geodesic coordinates $x = (x^1, \ldots, x^{m-1}, x^m)$, such that, $\partial_m = \varsigma_{\mathsf{in}}$, the coordinates $x^1, \ldots, x^{m-1}$ define a coordinate system at the boundary, where $\{\partial_\alpha\}_{\alpha=1}^{m-1}$ is a coordinate frame of $T\partial M$. Let $\{dx^\alpha\}_{\alpha=1}^{m-1} \cup \{dx^m\}$ be the corresponding dual coordinate coframe. Let $\nabla^{E,g}$ be the connection in $\Lambda^*(T^*M) \otimes E$, induced by ∇^E and the Levi-Civita connection ∇. Remember that, on the collar U, every $\xi \in \Omega(M;E)$ decomposes as $\xi = \xi^{\mathsf{t}} + dx^m \wedge \xi^{\mathsf{n}}$, where ξ^{n} and ξ^{t} are tangential forms, see (3.23) and (3.24).

Since $\Delta_{E,g,b}$ is an operator of Laplace type, by Lemma 2.2.4, the operators $\Delta_{E,g,b}$ is elliptic with respect to the cone $\mathcal{C}_0$, if and only if the operator $\mathbf{b}(\zeta,\lambda)$, defined by formula (2.15) in terms of the graded leading symbol of $\mathcal{B}$, is invertible for each $\lambda\in\mathbb{C}$ with $0\neq(\zeta,\lambda)\in T^*M\times\mathcal{C}_0$. In order to prove that, remark that $\mathcal{B}$ is defined in terms of $\mathcal{B}_\pm$ locally respectively around $\partial_\pm M$, and denote by $\mathbf{b}_\pm$ the operators obtained by formula (2.15) corresponding to the boundary operators $\mathcal{B}_\pm$, respectively. Since $\partial_\pm M$ are mutally disjoint closed submanifolds, invertibility of $\mathbf{b}(\zeta,\lambda)$ is directly translated into invertibility of $\mathbf{b}_+(\zeta,\lambda)$ and $\mathbf{b}_-(\zeta,\lambda)$ on U, for each $\lambda\in\mathbb{C}$ with $0\neq(\zeta,\lambda)\in T^*M\times\mathcal{C}_0$.

So, let us start by describing the operator $\mathcal{B}_+$ on the collar U, by using formula (3.37), (c.f. proof of Lemma 3.2.11): for $u\in\Omega^q(M;E)$, we have locally over U

$$\mathcal{B}_+u=(i_+^*\mathsf{int}_m u,(-1)^{q+1}i_+^*\mathsf{int}_m(d_E u)). \tag{3.50}$$

Therefore, by using $d_E=\sum_{i=1}^m e^i\wedge\nabla^{E,g}_{\partial_i}$, the second component on the right hand side in the equality (3.50) can be developped as

$$i_+^*\mathsf{int}_m d_E u=i_+^*\left(\Pi_{\mathsf{t}}\left(\nabla^F_{\partial_m}u\right)-\mathsf{ext}_\alpha\mathsf{int}_m\left(\nabla^F_{\partial_\alpha}u\right)\right), \tag{3.51}$$

(c.f. (3.40)). In terms of (3.50) and (3.51), we can compute formula (2.15) so that the operator $\mathbf{b}_+(\zeta,\lambda)$ can be locally written as

$$\mathbf{b}_+(\zeta,\lambda)(u)=(\mathsf{int}_m u,\pm(-\sqrt{-1}\zeta\wedge\mathsf{int}_m u+\sqrt{|\zeta|^2-\lambda}\Pi_{\mathsf{t}}u)),$$

Now, it is clear that, on the collar, $\mathbf{b}_+(\zeta,\lambda)$ is an isomorphism, whenever $\lambda\in\mathcal{C}_0$. Indeed, since the respective bundles have the same rank, it is enough to see that $\mathbf{b}_+(\zeta,\lambda)$ is injective whenever $\lambda\in\mathcal{C}_0$. But, $\mathbf{b}_+(\zeta,\lambda)u=0$ implies $\sqrt{\|\zeta\|^2-\lambda}u=0$ and $\mathsf{int}_m u=0$. Since $(\zeta,\lambda)\neq(0,0)$ and $\lambda\in\mathcal{C}_0$, this means $u=0$.

Let us now describe the operator $\mathcal{B}_-$ on the collar U. In this case, we have,

$$\mathcal{B}_-u=(i_-^*\Pi_{\mathsf{t}}u,i_-^*\mathsf{int}_m(\Pi_{\mathsf{t}}d^\sharp_{E,g,b}u)). \tag{3.52}$$

The second component on the right hand side of (3.52) can be written, by using formulas (3.43), (3.49) and (3.48), as

$$i_-^*\mathsf{int}_m\Pi_{\mathsf{t}}d^\sharp_{E,g,b}u=i_-^*\Pi_{\mathsf{n}}\left(\nabla^{F'}_{\partial_m}-\star_b^{-1}\Pi_{\mathsf{t}}\mathsf{ext}_\alpha\left(\nabla^{F'}_{\partial_\alpha}\mathsf{int}_m\right)\star_b+\star_b^{-1}\left(\nabla^F_{\partial_m}\star_b\right)\right)u \tag{3.53}$$

Thus, the operator $\mathbf{b}_-(\zeta,\lambda)$ locally reads this time as

$$\mathbf{b}_-(\zeta,\lambda)(u)=(\Pi_{\mathsf{t}}u,(-\sqrt{-1}\zeta\wedge\mathsf{int}_m u+\sqrt{|\zeta|^2-\lambda}\Pi_{\mathsf{n}}u)).$$

Note here that the term $\star_b^{-1}\nabla^F_{\partial_m}\star_b$ appearing in (3.53) is of order zero and hence it does not contribute to the graded leading symbol of $\mathcal{B}_-$ needed in formula (2.15) to compute $\mathbf{b}_-(\zeta,\lambda)$. With the use of the same reasoning lines above the operator $\mathbf{b}_-(\zeta,\lambda)$ is an isomorphism, whenever $\lambda\in\mathcal{C}_0$. This completes the proof. $\square$

3.3.2. Elliptic estimates for the bilinear Laplacian. We freely use the notation and the results from Sections 2.1.1 and 2.3.1. By Proposition 3.3.1, for λ a fixed complex number, the boundary value problem $(\Delta-\lambda, \Omega(M;E)_{\mathcal{B}})$ is also elliptic with respect to the cone $\mathbb{C}\backslash(0,\infty)$ and therefore with respect to the cone $\{0\}$. Then, for each fixed $\lambda\in\mathbb{C}$, the operator

$$\begin{array}{rcl} \mathbf{A}_\lambda:\Omega^q(M;E) & \longrightarrow & \Omega^q(M;E)\oplus\Omega^q(\partial M;E|_{\partial M})\oplus\Omega^{q-1}(\partial M;E|_{\partial M}), \\ u & \mapsto & ((\Delta_{E,g,b}-\lambda)u, \mathcal{B}_{E,g,b}u) \end{array} \tag{3.54}$$

where the boundary operator

$$\mathcal{B}_{E,g,b}:=(\mathcal{B}^0_{E,g,b},\mathcal{B}^1_{E,g,b}) \tag{3.55}$$

is the same as the one in (3.14) with

$$\begin{array}{rcl} \mathcal{B}^0_{E,g,b}:\ \ \Omega^q(M;E) & \to & \Omega^q(\partial_-M;E)\oplus\Omega^{q-1}(\partial_+M;E) \\ u & \mapsto & (\mathcal{B}^0_+u,\mathcal{B}^0_-u) \end{array}$$

is of order 0, and

$$\begin{array}{rcl} \mathcal{B}^1_{E,g,b}:\ \ \Omega^q(M;E) & \to & \Omega^{q-1}(\partial_-M;E)\oplus\Omega^{q}(\partial_+M;E) \\ u & \mapsto & (\mathcal{B}^1_+u,\mathcal{B}^1_-u) \end{array}$$

is of order 1 (c.f. Sections 2.1.1 and 2.3.1). Then, by Lemma 2.3.1, for every $s\geqslant 0$, the operator $\mathbf{A}_\lambda$ in (3.54) admits an extension as bounded operator to each Sobolev space. Moreover, Lemma 2.3.2 tells us that that for each $s\geqslant 0$, the operator $\mathbf{A}_{s,\lambda}$ in (2.19) from Lemma 2.3.1 is Fredholm and there exists a constant $C_s>0$ for which the *a priori* estimate

$$\|u\|_{s+2}\leqslant C_s\left(\|(\Delta_{s+2,s}-\lambda)u\|_s+\|u\|_{L^2}\right) \tag{3.56}$$

holds on the corresponding space of forms satisfying boundary conditions.

3.3.3. L^2-realization for the bilinear Laplacian. We use the notation and results from Section 2.3.2. Consider the L^2-realization of this elliptic boundary value problem, see (2.21). By Proposition 2.3.3, the elliptic estimates for the bilinear Laplacian (3.56) implies that the unbounded operator

$$\Delta_{\mathcal{B}}:\mathcal{D}(\Delta_{\mathcal{B}})\subset\mathsf{L}^2(M;E)\to\mathsf{L}^2(M;E), \tag{3.57}$$

with domain of definition

$$\mathcal{D}(\Delta_{\mathcal{B}})\ \ :=\ \ \overline{\Omega(M;E)|_{\mathcal{B}}}^{\mathsf{H}_2(M;E)}, \tag{3.58}$$

is closed in the L^2-norm. The operator in (3.57) with domain of definition given by (3.58) coincides with the L^2-closure extension of

$$\Delta_{E,g,b}:\Omega(M;E)|_{\mathcal{B}}\subset\mathsf{L}^2(M;E)\to\Omega(M;E)\subset\mathsf{L}^2(M;E),$$

regarded as unbounded operator on $\mathsf{L}^2(M;E)$, see Remark 2.3.4.

3.3.4. The spectrum of the bilinear Laplacian. From Proposition 2.3.7 the operator $\Delta_{\mathcal{B}}$ given in (3.57) is densely defined in $\mathsf{L}^2(M;E)$, possesses a non-empty resolvent set, its resolvent is compact and its spectrum is discrete, which is described as follows. *For every $\theta > 0$, there exists $R > 0$ such that $\mathbb{B}_R(0)$ (the closed ball in $\mathbb{C}$ centered at 0 and radius R), contains at most a finite subset of $\mathsf{Spec}(\Delta_{\mathcal{B}})$ and the remaining part of the spectrum is entirely contained in the sector*

$$\Lambda_{R,\epsilon} := \{\, z \in \mathbb{C} | -\epsilon < \arg(z) < \epsilon \text{ and } |z| \geqslant R\}. \tag{3.59}$$

Furthermore, for every $\lambda \notin \Lambda_{R,\epsilon}$ large enough, there exists $C > 0$, for which

$$\|(\Delta_{\mathcal{B}} - \lambda)^{-1}\|_{\mathsf{L}^2} \leqslant C/|\lambda|.$$

3.3.5. Generalized eigenspaces and L^2-decomposition. In view of discreteness of the spectrum of $\Delta_{\mathcal{B}}$, for each $\lambda \in \mathsf{Spec}(\Delta_{\mathcal{B}})$, we choose $\gamma(\lambda)$ a closed counter clock wise oriented curve surrounding λ as the unique point in $\mathsf{Spec}(\Delta_{\mathcal{B}})$ and consider the **Riesz Projection** or **spectral projection** corresponding to λ:

$$\begin{array}{rrcl} \mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda): & \mathsf{L}^2(M;E) & \to & \mathcal{D}(\Delta_{\mathcal{B}}) \subset \mathsf{L}^2(M;E), \\ & w & \mapsto & -(2\pi i)^{-1}\int_{\gamma(\lambda)}(\Delta_{\mathcal{B}} - \mu)^{-1} w d\mu, \end{array} \tag{3.60}$$

where the integral above converges uniformly in the L^2-norm as the limit of Riemann sums, since the function $x \mapsto (\Delta_{\mathcal{B}} - x)^{-1}$ is analytic in a neighborhood of $\gamma(\lambda)$. Since the resolvent of $\Delta_{\mathcal{B}}$ is compact, the operator $\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)$ is compact, hence bounded, on $\mathsf{L}^2(M;E)$.

Notation 3.3.2. The image of $\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)$ in $\mathsf{L}^2(M;E)$ will be denoted by

$$\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda) := \mathsf{Im}\left(\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)\right) := \mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)(\mathsf{L}^2(M;E)) \tag{3.61}$$

Then, $(\mathsf{Id} - \mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)) : \mathsf{L}^2(M;E) \to \mathsf{L}^2(M;E)$ is the complementary spectral projection of $\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)$ in $\mathsf{L}^2(M;E)$ with image

$$\mathsf{Im}\left(\mathsf{Id} - \mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)\right) := (\mathsf{Id} - \mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda))(\mathsf{L}^2(M;E)) \tag{3.62}$$

and we set

$$\mathsf{Im}\left(\mathsf{Id} - \mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)\right)|_{\mathcal{B}} := \mathsf{Im}\left(\mathsf{Id} - \mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)\right) \cap \mathcal{D}(\Delta_{\mathcal{B}}), \tag{3.63}$$

where $\mathcal{D}(\Delta_{\mathcal{B}})$ is the domain of definition of $\Delta_{\mathcal{B}}$ in (3.58).

Lemma 3.3.3. *Consider the spaces introduced in Notation 3.3.2. The operators $\Delta_{\mathcal{B}}$ and $\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)$ commute: $\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)\Delta_{\mathcal{B}} \subset \Delta_{\mathcal{B}}\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)$; in other words, if $u \in \mathcal{D}(\Delta_{\mathcal{B}})$, then $\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)u \in \mathcal{D}(\Delta_{\mathcal{B}})$ and $\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)\Delta_{\mathcal{B}}u = \Delta_{\mathcal{B}}\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)u$. The space $\mathsf{L}^2(M;E)$ decomposes $\beta_{g,b}$-orthogonally*

$$\mathsf{L}^2(M;E) \cong \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda) \oplus \mathsf{Im}\left(\mathsf{Id} - \mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)\right), \tag{3.64}$$

such that

$$\begin{aligned} \mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)\mathcal{D}(\Delta_{\mathcal{B}}) &\subset \mathcal{D}(\Delta_{\mathcal{B}}), \\ \Delta_{\mathcal{B}}\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda) &\subset \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda), \\ \Delta_{\mathcal{B}}\left(\mathsf{Im}\,(\mathsf{Id}-\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda))|_{\mathcal{B}}\right) &\subset \mathsf{Im}\,(\mathsf{Id}-\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda))|_{\mathcal{B}}\,. \end{aligned} \tag{3.65}$$

The operator $\Delta_{\mathcal{B}}|_{\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)} : \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda) \to \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$, *that is, the restriction of* $\Delta_{\mathcal{B}}$ *to each* $\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$, *is bounded on* $\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$ *and*

$$\mathsf{Spec}(\Delta_{\mathcal{B}}|_{\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)}) = \{\lambda\}.$$

The operator

$$\Delta_{\mathcal{B}}|_{\mathsf{Im}\left(\mathsf{Id}-\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)\right)|_{\mathcal{B}}} : \mathsf{Im}\,(\mathsf{Id}-\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda))|_{\mathcal{B}} \to \mathsf{Im}\,(\mathsf{Id}-\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda))\,, \tag{3.66}$$

i.e., the restricition of $\Delta_{\mathcal{B}}$ *to* $\mathsf{Im}\,(\mathsf{Id}-\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda))|_{\mathcal{B}} \subset \mathsf{L}^2(M;E)$ *is an unbounded operator on* $\mathsf{L}^2(M;E)$. *The spectrum of* $\Delta_{\mathcal{B}}|_{\mathsf{Im}\left(\mathsf{Id}-\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)\right)|_{\mathcal{B}}}$ *is exactly* $\mathsf{Spec}\,(\Delta_{\mathcal{B}})\setminus\{\lambda\}$ *or in other words the operator in* $(\Delta_{\mathcal{B}}-\lambda)|_{\mathsf{Im}\left(\mathsf{Id}-\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)\right)|_{\mathcal{B}}}$ *is invertible.*

Proof. This is a direct application of Theorem 6.17, page 178 in [**Ka95**], presented as Theorem 1.3.1 in Section 1.3. The assertion for the $\beta_{g,b}$-orthogonality of such a decomposition follows from the Proposition 3.3.8 below, since $\Omega(M;E)$ is dense in $\mathsf{L}^2(M;E)$ and $\beta_{g,b}$ extends continuously to a nondegenerate symmetric bilinear form in the L^2-norm. □

Proposition 3.3.4. *Let* $\Delta_{\mathcal{B}}$ *be the* L^2*-realization of the bilinear Laplacian and* $\lambda \in \mathsf{Spec}\,(\Delta_{\mathcal{B}})$. *Then,* $\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda) \subset \Omega(M;E)|_{\mathcal{B}}$, *that is, it contains smooth differential only, which satisfy boundary conditions. The space* $\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$ *is of finite dimension and invariant under* d_E *and* $d^{\sharp}_{E,g,b}$. *Moreover, the operator* $\Delta_{\mathcal{B}}-\lambda$, *when restricted to* $\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$ *is nilpotent, i.e.,*

$$\exists N \in \mathbb{N} \textit{ s.t. } (\Delta_{E,g,b}-\lambda)^n\, w = 0, \forall n \geqslant N, \textit{ for each } w \in \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$$

In particular, for each $w \in \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$, *the form* $(\Delta_{E,g,b}-\lambda)^n\, w$ *satisfies boundary conditions for all* $n \geqslant 0$.

Proof. Since the resolvent of $\Delta_{\mathcal{B}}$ is compact and the operator $\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)$ is bounded on $\mathsf{L}^2(M;E)$, the space $\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$, i.e. the image of the spectral projection, is of finite dimension, see Theorem 6.29 in chapter III, Section 6 in [**Ka95**]. Now, from Lemma 3.3.3, the operator

$$\Delta_{\mathcal{B}}|_{\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)} : \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda) \to \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$$

is bounded, its spectrum contains λ only, $\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$ is of finite dimension and therefore $(\Delta_{\mathcal{B}}-\lambda)|_{\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)}$ is nilpotent. We now show the space $\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$ contains differential forms only. Indeed, we know that $\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda) \subset \mathcal{D}(\Delta_{\mathcal{B}}) = \overline{\Omega(M;E)_{\mathcal{B}}}^{\mathsf{H}_2} \subset \mathsf{H}_2(M;E)$ but also, by Lemma 3.3.3, that the operator $\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)$ commute with $\Delta_{\mathcal{B}}$ on $\mathcal{D}(\Delta_{\mathcal{B}})$ and that the space $\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$ is invariant under $\Delta_{\mathcal{B}}$.

Thus, if $w \in \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$, then $w \in \mathcal{D}(\Delta_{\mathcal{B}})$ and $\Delta_{\mathcal{B}} w \in \mathcal{D}(\Delta_{\mathcal{B}}) \subset \mathsf{H}_2(M;E)$ and therefore, by Proposition 2.3.3 where elliptic estimates have been used, we conclude $w \in \mathcal{D}(\Delta_{\mathcal{B},2}) = \overline{\Omega(M;E)_{\mathcal{B}}}^{\mathsf{H}_4} \subset \mathsf{H}_4(M;E)$; then by iterating this argument, we have

$$w \in \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda) \subset \mathcal{D}(\Delta_{\mathcal{B},s}) = \overline{\Omega(M;E)_{\mathcal{B}}}^{\mathsf{H}_{s+2}} \quad \text{for all } s \geqslant 0,$$

that is,

$$\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda) \subset \Omega(M;E)_{\mathcal{B}} \subset \Omega(M;E),$$

or in words, each generalized eigenspace contains smooth forms only. Now, if $w \in \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$, then we have $\mathsf{P}_{\mathcal{B}}(\lambda)(\Delta_{E,g,b} w) = \Delta_{E,g,b}(\mathsf{P}_{\mathcal{B}}(\lambda) w) = \Delta_{E,g,b} w$ so that $\Delta_{E,g,b} w \in \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$ and in particular $(\Delta_{E,g,b} - \lambda)^n w \in \Omega(M;E)|_{\mathcal{B}}$ for each $n \in \mathbb{Z}$. We now show that $\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$ is invariant under d_E and $d^{\sharp}_{E,g,b}$. Since the space $\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$ contains differential forms only, it suffices to show that $d_E w$ satisfies the boundary condition, whenever $w \in \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$. On $\partial_+ M$, the absolute part of the boundary, this immediately follows from $d_E^2 = 0$. Let us turn to $\partial_- M$, the relative part of the boundary. The Riesz projections are well defined as bounded operators and they commute with the Laplacian on its domain of definition, Lemma 3.3.3. That is, $\Delta_{E,g,b} w$ lies in $\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$ as well; in particular, it satisfies relative boundary conditions on $\partial_- M$, so that $i_-^*(\Delta_{E,g,b} w) = 0$. Together with $i_-^* d^{\sharp}_{E,g,b} w = 0$, this implies $i_-^* d^{\sharp}_{E,g,b} d_E w = 0$, hence $d_E w$ also satisfies relative boundary conditions. Finally, the corresponding statement for $d^{\sharp}_{E,g,b}$ follows by the duality between the absolute and relative boundary operators.

□

3.3.6. (Smooth) orthogonal complement for the generalized eigenspaces. Proposition 3.3.4 above justifies the choice of the symbol 'Ω' in the notation for the generalized eigenspaces in (3.61). The image of the projection $\mathsf{Id} - \mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)$ in $\mathsf{L}^2(M;E)$ does not only contain smooth forms. Here we are interested in smooth forms that are also in the image of $\mathsf{Id} - \mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)$.

Notation 3.3.5. We denote by

$$\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathbf{c}} := \Omega(M;E) \cap \mathsf{Im}\left(\mathsf{Id} - \mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)\right), \tag{3.67}$$

the space of smooth forms being in the complementary image of $\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)$. Moreover, with Notation 3.2.7, we set

$$\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathbf{c}}|_{\mathcal{B}} := \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathbf{c}} \cap \Omega(M;E)_{\mathcal{B}}, \tag{3.68}$$

to indicate the space of all smooth forms in $\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathbf{c}}$ satisfying boundary conditions.

Lemma 3.3.6. *Consider the space $\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathbf{c}}|_{\mathcal{B}}$ in (3.68). Then, the operator*

$$(\Delta_{\mathcal{B}} - \lambda)\,|_{\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathbf{c}}|_{\mathcal{B}}} : \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathbf{c}}|_{\mathcal{B}} \to \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathbf{c}}, \tag{3.69}$$

i.e., the restriction of $(\Delta_{\mathcal{B}} - \lambda)$ *to* $\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathsf{c}}|_{\mathcal{B}}$*, is invertible.*

Proof. We first show that the operator in (3.69) is injective. It is clear from their definitions, (3.68) and (3.63), that

$$\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathsf{c}}|_{\mathcal{B}} \subset \mathsf{Im}\,(\mathsf{Id} - \mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda))|_{\mathcal{B}}\,. \tag{3.70}$$

From Lemma 3.3.3, the operator

$$(\Delta_{\mathcal{B}} - \lambda)|_{\mathsf{Im}\left(\mathsf{Id}-\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)\right)\big|_{\mathcal{B}}} : \mathsf{Im}\,(\mathsf{Id} - \mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda))|_{\mathcal{B}} \to \mathsf{Im}\,(\mathsf{Id} - \mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)) \tag{3.71}$$

is invertible and hence the operator

$$(\Delta_{\mathcal{B}} - \lambda)\,|_{\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathsf{c}}|_{\mathcal{B}}} : \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathsf{c}}|_{\mathcal{B}} \to \mathsf{Im}\,(\mathsf{Id} - \mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda))$$

is injective. But since

$$(\Delta_{\mathcal{B}} - \lambda)\,(\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathsf{c}}|_{\mathcal{B}}) \subset \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathsf{c}} \subset \mathsf{Im}\,(\mathsf{Id} - \mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda))\,,$$

we conclude that the operator in (3.69) is injective. We now show surjectivity. Again, from Lemma 3.3.3, the operator in (3.71) is surjective. In particular, for each $w \in \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathsf{c}}$, there is $u \in \mathsf{Im}\,(\mathsf{Id} - \mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda))|_{\mathcal{B}}$. But from Proposition 2.3.3, proved by using elliptic estimates, it follows that $u \in \Omega(M;E)$, and hence $u \in \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathsf{c}}|_{\mathcal{B}}$, so that the operator in (3.69) is surjective. □

3.3.7. Hodge decomposition for the bilinear Laplacian on smooth forms. For $\lambda \in \mathsf{Spec}(\Delta_{\mathcal{B}})$ consider the corresponding spectral projection defined in (3.60) $\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda) : \mathsf{L}^2(M;E) \to \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$, with image $\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$, see (3.61), such that $\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)|_{\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)} = \mathsf{Id}$. From Proposition 3.3.4 we know that

$$\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda) \subset \Omega(M;E) \subset \mathsf{L}^2(M;E),$$

and hence

$$\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda) \subset \mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)\,(\Omega(M;E)) \subset \mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)\,\big(\mathsf{L}^2(M;E)\big) = \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda).$$

Therefore

$$\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda) = \mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)(\Omega(M;E)). \tag{3.72}$$

Lemma 3.3.7. *Let* $v, w \in \mathsf{L}^2(M;E)$*. Then* $\beta_{g,b}(\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)v, w) = \beta_{g,b}(v, \mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)w)$*.*

Proof. Since, the bilinear form $\beta_{g,b}$ continuously extends to a nondegenerate bilinear form on $\mathsf{L}^2(M;E)$ and $\Omega(M;E)$ is dense in this space, it is enough to prove the statement on smooth forms. Then for $v, w \in \Omega(M;E)$, we can write, see (3.60)

$$\beta_{g,b}(\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)v, w) \quad = \quad -(2\pi i)^{-1}\beta_{g,b}\left(\int_{\gamma(\lambda)}(\Delta_{\mathcal{B}} - \mu)^{-1} v d\mu, w\right).$$

Since the integral defining $\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)$ converges uniformly in the L^2-norm, we are allowed to take this integral out and write

$$\beta_{g,b}(\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)v,w) \quad = \quad -(2\pi i)^{-1}\int_{\gamma(\lambda)}\beta_{g,b}\left((\Delta_{\mathcal{B}}-\mu)^{-1}v,w\right)d\mu,$$

Since $\gamma(\lambda)\cap\mathsf{Spec}(\Delta_{\mathcal{B}})=\emptyset$, for each $\mu\in\gamma(\lambda)$, we have $(\Delta_{\mathcal{B}}-\mu)^{-1}w\in\mathcal{D}(\Delta_{\mathcal{B}})$, so that

$$\beta_{g,b}\left((\Delta_{\mathcal{B}}-\mu)^{-1}v,w\right)=\beta_{g,b}((\Delta_{\mathcal{B}}-\mu)^{-1}v,(\Delta_{\mathcal{B}}-\mu)(\Delta_{\mathcal{B}}-\mu)^{-1}w).$$

But from Lemma 3.3.6, both $(\Delta_{\mathcal{B}}-\mu)^{-1}v$ and $(\Delta_{\mathcal{B}}-\mu)^{-1}w$ belong in fact to the space $\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathbf{c}}|_{\mathcal{B}}$, so that we apply Lemma 3.2.8 to obtain

$$\begin{aligned}\beta_{g,b}\left((\Delta_{\mathcal{B}}-\mu)^{-1}v,w\right) &= \beta_{g,b}((\Delta_{\mathcal{B}}-\mu)^{-1}v,(\Delta_{E,g,b}-\mu)(\Delta_{\mathcal{B}}-\mu)^{-1}w)\\ &= \beta_{g,b}\left((\Delta_{E,g,b}-\mu)(\Delta_{\mathcal{B}}-\mu)^{-1}v,(\Delta_{\mathcal{B}}-\mu)^{-1}w\right)\\ &= \beta_{g,b}\left(v,(\Delta_{\mathcal{B}}-\mu)^{-1}w\right);\end{aligned}$$

that is

$$\beta_{g,b}(\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)v,w)=-(2\pi)^{-1}\int_{\gamma(\lambda)}\beta_{g,b}(v,(\Delta_{\mathcal{B}}-\mu)^{-1}w)d\mu$$

and hence

$$\beta_{g,b}(\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)v,w)=\beta_{g,b}(v,\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)w).$$

□

Proposition 3.3.8. *The space $\Omega(M;E)$ decomposes $\beta_{g,b}$-orthogonally as the direct sum:*

$$\Omega(M;E)\cong\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)\oplus\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathbf{c}}, \tag{3.73}$$

where $\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$ is given by (3.72) and $\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathbf{c}}$ by (3.67). In particular, if $\lambda,\mu\in\mathsf{Spec}(\Delta_{\mathcal{B}})$ with $\lambda\neq\mu$, then

$$\Omega_{\Delta_{\mathcal{B}}}(M;E)(\mu)\perp_{\beta}\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda).$$

In particular, $\beta_{g,b}$ restricts to each of these subspaces as nondegenerate symmetric bilinear form. Furthermore, with Notation 3.2.7, we have the $\beta_{g,b}$-orthogonal direct decomposition

$$\Omega(M;E)|_{\mathcal{B}^0_-}\cong\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)\oplus\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathbf{c}}|_{\mathcal{B}^0_-}, \tag{3.74}$$

which is invariant under the action of d_E.

Proof. Since $\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)\subset\Omega(M;E)\subset\mathsf{L}^2(\Omega(M,E))$, the decomposition in (3.73) follows from Lemma 3.3.3 and it only remains to show that

$$\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)\perp_{\beta}\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathbf{c}}.$$

So, let us take $v\in\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$ and $w\in\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathbf{c}}$, we have

$$\beta_{g,b}(v,w)=\beta_{g,b}(\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)v,w)=\beta_{g,b}(v,\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)w)=0,$$

where the second equality follows from Lemma 3.3.7 and the last one is true because w is in the image of the complementary projection of $\mathsf{P}_{\Delta_{\mathcal{B}}}(\lambda)$.

Since $\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda) \subset \Omega(M;E)|_{\mathcal{B}^0_-}$, the decomposition in (3.73) implies that $\beta_{g,b}$-orthogonal decomposition in (3.74) holds as well. We have already seen in Proposition 3.3.4 that $\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$ is invariant under d_E. But we have

$$d_E\left(\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathbf{c}}|_{\mathcal{B}^0_-}\right) \subset \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathbf{c}}|_{\mathcal{B}^0_-}$$

as well. Indeed, take

$$v \in \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathbf{c}}|_{\mathcal{B}^0_-} \quad \text{and} \quad w \in \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda).$$

Then, by using the Green's formulas from Lemma 3.2.4, we obtain

$$\beta_{g,b}(d_E v, w) = \beta_{g,b}(v, d^{\sharp}_{E,g,b} w) + \int_{\partial M} i^*(\mathrm{Tr}(v \wedge \star_b w)).$$

But $\beta_{g,b}(v, d^{\sharp}_{E,g,b} w) = 0$, since by Proposition 3.3.4 $d^{\sharp}_{E,g,b}$ leaves invariant $\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$, and

$$\int_{\partial M} i^*(\mathrm{Tr}(v \wedge \star_b w)) = \int_{\partial_+ M} i^*_+(\mathrm{Tr}(v \wedge \star_b w)) + \int_{\partial_- M} i^*_-(\mathrm{Tr}(v \wedge \star_b w)) = 0,$$

since $i^*_+ \star_b w = 0$ and $i^*_- v = 0$. Thus, $\beta_{g,b}(d_E v, w) = 0$ and therefore, by $\beta_{g,b}$-orthogonality of (3.73), it follows that

$$d_E v \in \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathbf{c}}.$$

Finally, since i^* commutes with d_E, we have $d_E v \in \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathbf{c}}|_{\mathcal{B}^0_-}$ as well. □

Corollary 3.3.9. *With Notation 3.2.7, consider the space $\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathbf{c}}|_{\mathcal{B}^0}$. Then, we have*

$$d_E(\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathbf{c}}|_{\mathcal{B}^0}) \perp_\beta \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$$
$$d^{\sharp}_{E,g,b}(\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathbf{c}}|_{\mathcal{B}^0}) \perp_\beta \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$$

Proof. If $u \in \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$ and $v \in \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)^{\mathbf{c}}|_{\mathcal{B}^0}$, then

$$\beta_{g,b}(u, d_E v) = \beta_{g,b}(d^{\sharp}_{E,g,b} u, v) = 0,$$

because of Lemma 3.2.8, invariance of $\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$ under $d^{\sharp}_{E,g,b}$ as stated in Proposition 3.3.4, and Proposition 3.3.8. The proof of the statement for $d^{\sharp}_{E,g,b}$ is analog. □

Corollary 3.3.10. *(Hodge decomposition) We have the $\beta_{g,b}$-orthogonal decomposition*

$$\Omega(M;E) \cong \Omega_{\Delta_{\mathcal{B}}}(M;E)(0) \oplus \Delta_{E,g,b}(\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathbf{c}}\,|_{\mathcal{B}}). \tag{3.75}$$

Proof. This follows from Proposition 3.3.8 and Lemma 3.3.6. □

Proposition 3.3.11. *Consider Notation 3.2.7. Then*

(i) The space $\Omega(M;E)$ decomposes $\beta_{g,b}$-orthogonally as

$$\begin{aligned}\Omega(M;E) \;&\cong\; \Omega_{\Delta_{\mathcal{B}}}(M;E)(0)\\ &\oplus\; d_E(d^{\sharp}_{E,g,b}(\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}})) \oplus d^{\sharp}_{E,g,b}(d_E(\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}}).\end{aligned}$$

(ii) The space $\Omega(M;E)|_{\mathcal{B}^0_-}$ decomposes $\beta_{g,b}$-orthogonally as

$$\begin{aligned}\Omega(M;E)|_{\mathcal{B}^0_-} &\cong \Omega_{\Delta_{\mathcal{B}}}(M;E)(0)\\ &\oplus d_E(\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}^0_-})\oplus d^{\sharp}_{E,g,b}(\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}}).\end{aligned}$$

(iii) The space $\Omega(M;E)|_{\mathcal{B}^0}$ decomposes $\beta_{g,b}$-orthogonally as

$$\begin{aligned}\Omega(M;E)|_{\mathcal{B}^0} &\cong \Omega_{\Delta_{\mathcal{B}}}(M;E)(0)\\ &\oplus d_E(\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}})\oplus d^{\sharp}_{E,g,b}(\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}}).\end{aligned}$$

Moreover, the restriction of $\beta_{g,b}$ to each of the spaces appearing above is nondegenerate.

Proof. We prove (i). From Corollary 3.3.10, every $u\in\Omega(M;E)$ can be written as

$$u=u_0+d_E(d^{\sharp}_{E,g,b}u)+d^{\sharp}_{E,g,b}(d_Eu)$$

with

$$u_0\in\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)\quad\text{and}\quad u\in\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}}.$$

That

$$d_E(d^{\sharp}_{E,g,b}(\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}}))\perp_{\beta_{g,b}} d^{\sharp}_{E,g,b}(d_E(\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}})),$$

follows from Lemma 3.2.8 and $d_E^2=0$. To see that (i) is a direct sum, we check that the intersection of the last two spaces on the right of (i) is trivial. So, take

$$u\in\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}},$$

and suppose there are

$$v,w\in\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}}\quad\text{with}\quad u=d_E(d^{\sharp}_{E,g,b}v)=d^{\sharp}_{E,g,b}(d_Ew).$$

Remark obviously that $\Delta_{E,g,b}u=0$ but also that $u\in\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)$, since

- (*) $i_-^*u=d_E(i_-^*d^{\sharp}_{E,g,b}v)=0$, as v satisfies boundary conditions,
- (*) $i_-^*d^{\sharp}_{E,g,b}u=i_-^*d^{\sharp}_{E,g,b}d^{\sharp}_{E,g,b}d_Ev=0$,
- (*) $i_+^*\star_b u=\pm d_E(i_+^*d^{\sharp}_{E'\otimes\Theta,g,b'}\star_b w)=0$, as w satisfies boundary conditions[1],
- (*) $i_+^*d^{\sharp}_{E'\otimes\Theta,g,b'}\star_b u=\pm i_+^*\star_b d_E(d_Ed^{\sharp}_{E,g,b}v)=0$.

Therefore, from Proposition 3.3.8, u must vanish, so that the sum in (i) is direct. This decomposition is clearly $\beta_{g,b}$-orthogonal.

Before heading to (ii) and (iii), we introduce some simplyfing notation.

- Consider the operator

  $$G_0:\Omega(M;E)\to\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}}$$

  defined by

  $$G_0(w):=\begin{cases}0 & \text{if } w\in\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)\\ \left(\Delta_{E,g,b}|_{\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}}\right)^{-1}w & \text{if } w\in\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\end{cases}.$$

[1]the signs $\pm$ in front depend on the degree of the forms, but are not relevant in our considerations

Notice that

$$\Delta_{E,g,b}G_0 = I - \mathsf{P}_{\Delta_{\mathcal{B}}}(0),$$

so that

$$w = \mathsf{P}_{\Delta_{\mathcal{B}}}(0)(u) + \Delta_{E,g,b}(G_0(u)) \quad \text{for every} \quad w \in \Omega(M;E)$$

and that (see Lemma 3.2.8 and the isomorphism in (3.3.6))

$$\begin{array}{rcccl} G_0 w & \in & \Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}}, & & \\ d^{\sharp}_{E,g,b}G_0 w & \in & \Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}^0} & \subset & \Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}^0_-}, \\ d_E G_0 w & \in & \Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}^0} & \subset & \Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}^0_-}. \end{array} \tag{3.76}$$

Now, we show that $\Omega(M;E)|_{\mathcal{B}^0_-}$ and $\Omega(M;E)|_{\mathcal{B}^0}$ decompose as stated. For (ii), we need check that

$$d_E G_0 w \in \Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}} \quad \text{if} \quad w \in \Omega(M;E)|_{\mathcal{B}^0_-}.$$

Indeed, if $w \in \Omega(M;E)|_{\mathcal{B}^0_-}$, then

(*) $i^*_-(d_E G_0 w) = 0$ and $i^*_+(\star_b(d_E G_0 w)) = 0$, since $d_E G_0 w \in \Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}^0}$.

(*) $i^*_+(d^{\sharp}_{E'\otimes\Theta_M,g,b'} \star_b (d_E G_0 w)) = 0$, because of the definition of $d^{\sharp}_{E,g,b}$ and that $d_E^2 = 0$.

(*)

$$\begin{array}{rcl} i^*_-(d^{\sharp}_{E,g,b} d_E G_0 w) & = & i^*_-((\Delta_{E,g,b} - d_E d^{\sharp}_{E,g,b})G_0 w) \\ & = & i^*_-((\mathsf{Id} - \mathsf{P}_{\Delta_{\mathcal{B}}}(0))w - d_E d^{\sharp}_{E,g,b}G_0 w) \\ & = & i^*_- w - i^*_-(\mathsf{P}_{\Delta_{\mathcal{B}}}(0)w) - i^*_-(d_E d^{\sharp}_{E,g,b}G_0 w) = 0 \end{array}$$

where we have used (3.76) and

(·) $i^*_- w = 0$ because $w \in \Omega(M;E)|_{\mathcal{B}^0_-}$.

(·) $i^*_-(\mathsf{P}_{\Delta_{\mathcal{B}}}(0)w) = 0$, because

$$\mathsf{P}_{\Delta_{\mathcal{B}}}(0)w \in \Omega_{\Delta_{\mathcal{B}}}(M;E)(0) \subset \Omega(M;E)|_{\mathcal{B}^0_-}.$$

(·) $i^*_-(d_E d^{\sharp}_{E,g,b}G_0 w) = 0$, as $G_0 w \in \Omega(M;E)|_{\mathcal{B}}$ and $i^*_-(d^{\sharp}_{E,g,b}G_0 w) = 0$.

The decomposition in (iii) is proved similarly: In view of (3.76), if $w \in \Omega(M;E)|_{\mathcal{B}^0}$, then

(*) $i^*_-(d^{\sharp}_{E,g,b}G_0 w) = 0$, and $i^*_+(\star_b d^{\sharp}_{E,g,b}G_0 w) = 0$, since

$$d^{\sharp}_{E,g,b}G_0 w \in \Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}^0}.$$

(*) clearly $i^*_-(d^{\sharp}_{E,g,b} d^{\sharp}_{E,g,b}G_0 w) = 0$.

(*) We have

$$\begin{aligned} i^*_+(d^{\sharp}_{E'\otimes\Theta_M,g,b'} \star_b d^{\sharp}_{E,g,b}G_0 w) &= \pm i^*_+(\star_b d_E d^{\sharp}_{E,g,b}G_0 w) \\ &= \pm i^*_+(\star_b(\Delta_{E,g,b} - d^{\sharp}_{E,g,b}d_E)G_0 w) \\ &= \pm i^*_+(\star_b(\mathsf{Id} - \mathsf{P}_{\Delta_{\mathcal{B}}}(0))w - \star_b d^{\sharp}_{E,g,b}d_E G_0 w) = 0 \end{aligned}$$

where the last equality follows from (3.76) and

(·) $i^*_+ \star_b w = 0$, because $w \in \Omega(M;E)|_{\mathcal{B}^0}$,

$(\cdot)$ $i_+^*(\star_b \mathsf{P}_{\Delta_{\mathcal{B}}}(0)w) = 0$, because

$$\mathsf{P}_{\Delta_{\mathcal{B}}}(0)w \in \Omega_{\Delta_{\mathcal{B}}}(M;E)(0) \subset \Omega(M;E)|_{\mathcal{B}^0},$$

and

$(\cdot)$ $i_+^*(d_{E'\otimes\Theta_M} \star_b d_E G_0 w) = d_{E'\otimes\Theta_M}\, i_+^*(\star_b d_E G_0 w) = 0$, as

$$d_E G_0 w \in \Omega(M;E)|_{\mathcal{B}^0}.$$

Since $\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}} \subset \Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}^0_-}$ so that

$$d_E(\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}}) \subset d_E(\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}^0_-}),$$

direct decomposition from (iii) follows from that of (ii), so it is enough to check directness of (ii). This is done in the following steps

(a) By Proposition 3.3.8, we have

$$d_E\left(\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}|_{\mathcal{B}^0_-}\right) \subset \Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}|_{\mathcal{B}^0_-},$$

thus

$$\Omega_{\Delta_{\mathcal{B}}}(M;E)(0) \cap d_E\left(\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}^0_-}\right) = \{0\}.$$

(b) From $\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}} \subset \Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}^0}$, Corollary 3.3.9 and Proposition 3.3.8 , it follows

$$\Omega_{\Delta_{\mathcal{B}}}(M;E)(0) \cap d^{\sharp}_{E,g,b}(\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}}) = \{0\}.$$

(c) We show

$$d_E\left(\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}^0_-}\right) \cap d^{\sharp}_{E,g,b}(\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}}) = \{0\}.$$

Suppose there is $0 \neq u \in \Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}$ such that

$$u = d_E v \quad \text{for} \quad v \in \Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}^0_-}$$

and

$$u = d^{\sharp}_{E,g,b} w \quad \text{for} \quad w \in d^{\sharp}_{E,g,b}(\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}}).$$

First, remark that

$$\Delta_{E,g,b} u = d_E d^{\sharp}_{E,g,b} d^{\sharp}_{E,g,b} d_E w + d^{\sharp}_{E,g,b} d_E d_E d^{\sharp}_{E,g,b} v = 0,$$

that is, $u \in \ker(\Delta_{E,g,b})$. But also, we have

$(*)$ $i_-^* u = i_-^* d_E v = 0$, since $v \in \Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}^0_-}$;

$(*)$ $i_-^* d^{\sharp}_{E,g,b} u = i_-^* d^{\sharp}_{E,g,b} d^{\sharp}_{E,g,b} w = 0$;

$(*)$ $i_+^* \star_b u = i_+^* \star_b d^{\sharp}_{E,g,b} w = \pm i_+^* \star_b \star_b^{-1} d_{E'\otimes\Theta,g,b'} \star_b w = \pm i_+^* d_{E'\otimes\Theta,g,b'} \star_b w = \pm d_E i_+^*(\star_b w) = 0$, since w satisfies boundary conditions;

$(*)$ $i_+^* d^{\sharp}_{E'\otimes\Theta,g,b'} \star_b u = \pm i_+^* \star_b d_E \star_b^{-1} \star_b u = \pm i_+^* \star_b d_E u = \pm i_+^* \star_b d_E d_E v = 0.$

These identities tell us that

$$u \in \Omega_{\Delta_{\mathcal{B}}}(M;E)(0)$$

as well, and therefore $u = 0$.

It remains to show the statements about the nondegeneracy of $\beta_{g,b}$. First, the same discussion following display (3.6) to prove that the defining formula of $\beta_{g,b}$ in (3.5) defines a nondegenerate bilinear form on $\Omega(M;E)$, holds to conclude that the $\beta_{g,b}$ is nondegenerate on $\Omega(M;E)|_{\mathcal{B}^0_-}$ and $\Omega(M;E)|_{\mathcal{B}^0}$. Next, from Lemma 3.2.8, the direct decompositions in (i), (ii) and (iii) are $\beta_{g,b}$-orthogonal. Thus, $\beta_{g,b}$ restricts to each space appearing on the right hand side of (i), (ii) and (iii) as a nondegenerate bilinear form as well. □

3.3.8. Hodge–De-Rham cohomology for bordisms. Recall Notation 3.2.7 and the results from Lemma 3.2.8. The cochain complex $(\Omega(M;E)|_{\mathcal{B}^0_-}, d_E)$ computes De-Rham cohomology of M relative to $\partial_- M$, with coefficients on E, see for instance **[BT82]**. Moreover, for $\lambda \in \mathsf{Spec}(\Delta_{\mathcal{B}})$, consider the inclusion of cochain complexes

$$\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda) \hookrightarrow \Omega(M;E)|_{\mathcal{B}^0_-}.$$

Now, remark that for $\lambda \neq 0$, the cohomology groups $H^*(\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)) = 0$. Indeed, on the one hand, from Proposition 3.3.4, we know that the spaces $\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$ are invariant under d_E and $d^\sharp_{E,g,b}$. Then, the operator $\Delta_{E,g,b}$ is cochain homotopic to 0. That is, the operator $\Delta_{E,g,b}$ induces 0 in cohomology. On the other hand, for $\lambda \neq 0$, the operator $\Delta_{E,g,b}$ is invertible on each sub-complex $\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)$ so that, it induces an isomorphism in cohomology. Thus, for $\lambda \neq 0$, we must have $H^*(\Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda)) = 0$. In other words, every generalized eigen space corresponding to a non-zero eigenvalue is *acyclic*. It remains to study the case $\lambda = 0$.

Proposition 3.3.12. *The inclusion $\Omega_{\Delta_{\mathcal{B}}}(M;E)(0) \hookrightarrow \Omega(M;E)|_{\mathcal{B}^0_-}$ induces an isomorphism in cohomology: $H^*(\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)) \cong H^*(M, \partial_- M, E)$.*

Proof. Since $\Omega_{\Delta_{\mathcal{B}}}(M;E)(0) \subset \Omega(M;E)|_{\mathcal{B}^0_-}$, the space $\Omega(M;E)|_{\mathcal{B}^0_-}$ admits a decomposition compatible with the decomposition of $\Omega(M;E)$ in Corollary 3.3.10. Thus, each $w' \in \Omega(M;E)|_{\mathcal{B}^0_-}$ can be uniquely written as $w' = w_0 + w$, where $w_0 \in \Omega_{\Delta_{\mathcal{B}}}(M;E)(0)$ and w belongs to the space

$$\Delta_{E,g,b}(\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}})\,|_{\mathcal{B}^0_-} := \Delta_{E,g,b}(\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}}) \cap \Omega(M;E)|_{\mathcal{B}^0_-}. \tag{3.77}$$

But the space in (3.77) is a cochain complex, since $\Delta_{E,g,b}(\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}})$ is contained in $\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}$ (see (3.69) in Lemma 3.3.6) and $\Omega(M;E)|_{\mathcal{B}^0_-}$ is invariant under the action of d_E (see Proposition 3.3.8). Thus, it is enough to show that every closed form w taken in the space (3.77) is exact. By Proposition 3.3.11.(ii), there are

$$w_1 \in \Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}^0_-} \quad \text{and} \quad w_2 \in \Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}}\,|_{\mathcal{B}}$$

such that

$$w = d_E w_1 + d^\sharp_{E,g,b} w_2.$$

First, we claim that $\beta_{g,b}(d^{\sharp}_{E,g,b}w_2, v_1) = 0$, for all $v_1 \in \Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}} \mid_{\mathcal{B}^0}$. Indeed, from Proposition 3.3.11.(iii), there exist

$$v_2, u_2 \in \Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}} \mid_{\mathcal{B}},$$

such that $v_1 = d_E v_2 + d^{\sharp}_{E,g,b}u_2$ and hence

$$\beta_{g,b}(d^{\sharp}_{E,g,b}w_2, d_E v_2 + d^{\sharp}_{E,g,b}u_2) = \beta_{g,b}(d^{\sharp}_{E,g,b}w_2, d_E v_2) + \beta_{g,b}(d^{\sharp}_{E,g,b}w_2, d^{\sharp}_{E,g,b}u_2) = 0. \tag{3.78}$$

Indeed, since u_2, v_2 and w_2 satisfy boundary conditions, we have that $d^{\sharp}_{E,g,b}w_2$, $d_E v_2$ and $d^{\sharp}_{E,g,b}u_2 \in \Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}} \mid_{\mathcal{B}^0}$, see Lemma 3.2.8.(c); hence by Lemma 3.2.8.(e) and by ${d^{\sharp}_{E,g,b}}^2 = 0$, we obtain that $\beta_{g,b}(d^{\sharp}_{E,g,b}w_2, d_E v_2) = 0$. But $\beta_{g,b}(d_E d^{\sharp}_{E,g,b}w_2, u_2)$, the second term on the right in (3.78), also vanishes, because w being close implies $d_E d^{\sharp}_{E,g,b}w_2 = 0$. Finally, since $d^{\sharp}_{E,g,b}w_2$ belongs to $\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\mathsf{c}} \mid_{\mathcal{B}^0}$ as well, and that $\beta_{g,b}$ restricted to this sub-space is also nondegenerate, see Proposition 3.3.11, from the claim above, we have $d^{\sharp}_{E,g,b}w_2 = 0$. That is, w is exact.

$\square$

CHAPTER 4

Heat trace asymptotics for generalized Laplacians

The aim of this chapter is to present the necessary material for the proof of the anomaly formulas for the complex-valued the Ray–Singer analytic torsion in Chapter 5. Our main result, Theorem 4.4.3. The reader might skip temporary this chapter, continue with Chapter 5 and then come back to this one when reading the proof of Theorem 5.2.1. However, the material and methods presented in the following sections being quite general, we decided to discuss them independently at this point. We sketch the structure of this chapter. In Section 4.1 we recall the definition of the heat operator associated to a boundary value problem and look at the heat trace asymptotic expansion associated to an operator of Laplace type under elliptic boundary conditions. The work by Gilkey in [**Gi84**] and [**Gi04**] is used to explain how first Weyl's of invariants can be applied to express the coefficients in the heat trace asymptotic expansion as universal polynomials locally computable in (higher order) covariant derivatives of tensorial objects, see Proposition 4.1.5. Then, we use the material presented in Section 4.1 to study the coefficients of the constant term in the heat trace asymptotic expansion associated to the bilinear Laplacian under absolute/relative boundary conditions (and certain bundle endomorphism). We compute these coefficients, by using the corresponding ones for a Hermitian Laplacian. In Section 4.2, Proposition 4.2.2 gives the infinitesimal version of the anomaly formulas obtained by Brüning and Ma in [**BM06**] for a Hermitian Laplacian with *absolute boundary conditions* only. In Proposition 4.2.5, we derive the corresponding formulas for the *dual problem*, i.e., the self-adjoint Laplacian with *relative boundary conditions* only. The proof of Proposition 4.2.5 is based on Lemma 4.2.3, which exhibits the relation between these dual boundary value problems. Theorem 4.2.7 provides the formulas for the coefficients of the constant term in the heat trace asymptotic expansion associated to a Hermitian Laplacian under absolute/relative boundary conditions. These formulas coincide with those obtained by Brüning and Ma in [**BM11**]. In Section 4.3, Proposition 4.3.3 gives the first key argument in the proof of Theorem 4.4.3. In few words, we prove that for each point in M, there exist an open neigbourhood U, a symmetric bilinear form $\widetilde{b}$ and a *flat* complex fiberwise defined *anti-linear* involution ν on $E|_U$, with the following feature: for *certain well-chosen* values $z \in \mathbb{C}$, with $|z|$ small enough, the one-parameter family of nondegenerate symmetric bilinear forms $b_z := b + z\widetilde{b}$ can be considered, by means of ν, as a real one-parameter family of Hermitian forms on $E|_U$. Thus, the known results from the Hermitian situation can be used. Theorem 4.4.3 relates the coefficients of the constant terms in the heat trace asymptotic expansions for bilinear boundary problems to the those corresponding to the Hermitian situation given in Theorem 4.2.7. In the proof of Theorem 4.4.3, we also use Lemma 4.4.1, which states in general that the coefficient in the heat trace asymptotic expansion depend holomorphically on a complex parameter z, as long as the bilinear metric does so. Finally Theorem 4.4.3 follows by a standard argument of analytic continuation, since the involved formulas depend holomorphically on the parameter z and therefore they must hold for all $z \in \mathbb{C}$ with $|z|$ small enough; in particular, for $z = 0$.

4.1. Heat trace asymptotics for generalized Laplacians

We recall the heat operator associated to a boundary value problem. We collect some facts about the coefficients in the heat trace asymptotic expansion associated to an operator of Laplace type with elliptic boundary conditions, see Proposition 4.1.2. These coefficients are computable by integrating endomorphisms-valued invariants locally computable as universal polynomials in higher order derivatives of the symbols of the operators under consideration. We are particularly interested in the coefficient of the constant term in the asymptotic expansion. In Section 4.1.2, we use the work by Gilkey based on [**Gi84**] and [**Gi04**], to explain how first Weyl's of invariants, see Theorem 1.1.1 in Section 1.1.7, is used in the current situation, to express the endomorphisms appearing in the asymptotic expansion as universal polynomials locally computable in (higher order) covariant derivatives of tensorial objects, see Proposition 4.1.5.

4.1.1. Heat trace asymptotics. Let F be a complex vector bundle over a compact manifold M and $(\mathsf{D}, \Gamma(M;F)|_{\mathcal{B}})$ a boundary value problem, where D is of Laplace type acting on smooth sections of F, together $\mathcal{B}$ a boundary operator imposing local boundary conditions such that $(\mathsf{D}, \Gamma(M;F)|_{\mathcal{B}})$ is elliptic with respect to the cone $\mathcal{C}_0 := \mathbb{C}\backslash(0,\infty)$. Consider $\mathsf{D}_{\mathcal{B}}$ the L^2-realization of D, with domain of definition $\mathcal{D}(\mathsf{D}_{\mathcal{B}}) \subset \mathsf{L}^2(M;F)$, as discused in Section 2.3.2.

Recall also the notions from Section 1.4. By Theorem 2.5.2 in [**Gre71**], (see also [**Se69b**], or more generally, for pseudo-differential boundary value problems, Chapter 4 in [**Gru96**]), that for each $u \in \Gamma(M;F)$ fixed, there exists a unique $u(t,x) \in \mathcal{C}^\infty(\mathbb{R}_+, \Gamma(M;F))$ providing a solution of **the heat equation** $(\partial_t + \mathsf{D})\, u(t,x) = 0$ satisfying the *boundary condition* $\mathcal{B}u(t,x) = 0$ for all $t > 0$ and the *initial condition* $u(0,\cdot) = u$ in the sense of distributions:

$$\lim_{t\to 0} \int_M \langle u(t,x), \rho(x)\rangle dx = \int_M \langle u(x), \rho(x)\rangle dx, \quad \text{for all} \quad \rho \in \Gamma(M;F'),$$

where $\langle\cdot,\cdot\rangle$ is induced by the natural pairing between F and its dual vector bundle F'.

Definition 4.1.1. *Let $(\mathsf{D}, \Gamma(M;F)|_{\mathcal{B}})$ be an elliptic boundary value problem, where D is of Laplace type acting on smooth sections of F, and $\mathcal{B}$ is a boundary operator imposing boundary conditions such that $(\mathsf{D}, \Gamma(M;F)|_{\mathcal{B}})$ is elliptic with respect to the* *cone* $\mathcal{C}_0 := \mathbb{C}\backslash(0,\infty)$. *For each* $t > 0$, *the operator*

$$\begin{array}{rrcl} e^{-t\mathsf{D}_{\mathcal{B}}}: & \Gamma(M;F) & \to & \Gamma^{-\infty}(M;F), \\ & u & \mapsto & u(t,\cdot) \end{array}$$

is called ***the heat operator*** *for the boundary value problem* $(\mathsf{D},\Gamma(M;F)|_{\mathcal{B}})$, *that associates to each smooth section of* F *a generalized section, or distribution, of* F. *The function* $u(t,x)$ *is called the* ***fundamental solution*** *associated to (the heat equation of) this boundary value problem.*

Proposition 4.1.2. *Let* $(\mathsf{D},\Gamma(M;F)|_{\mathcal{B}})$ *be a boundary value problem, where* D *is of Laplace type acting on smooth sections of a vector bundle* F *and* $\mathcal{B}$ *imposes boundary conditions so that* $(\mathsf{D},\Gamma(M;F)|_{\mathcal{B}})$ *is elliptic with respect to the cone* $\mathcal{C}_0 := \mathbb{C}\backslash(0,\infty)$. *Then following assertions hold.*

(1) *For* $t > 0$, *the heat operator extends to a bounded operator on* $\mathsf{L}^2(M;F)$ *and it is of trace class in the* L^2*-norm.*

(2) *The heat operator is a smoothing operator,* i.e., *the operator*

$$e^{-t\mathsf{D}_{\mathcal{B}}}: \Gamma^{-\infty}(M;F) \to \Gamma(M;F)$$

is linear and bounded, with smooth kernel $K_t(\mathsf{D},\mathcal{B}) \in \Gamma(M\times M; F\boxtimes F'_M)$ *such that for each* $u(t,\cdot) \in \Gamma^{-\infty}(M;F)$ *and* $K_t(\mathsf{D},\mathcal{B})(x,\cdot) \in \Gamma(M;F'_M)$ *we have*

$$e^{-t\mathsf{D}_{\mathcal{B}}}u(t,x) = \int_{M\ni y}[K_t(\mathsf{D},\mathcal{B})(x,y)](u(y))\mathsf{vol}_g(M)(y)$$

for each $x \in M$.

(3) *For each* $\psi \in \Gamma(M;\mathrm{End}(F))$, *the function*

$$\mathrm{Tr}_{\mathsf{L}^2}\left(\psi e^{-t\mathsf{D}_{\mathcal{B}}}\right) = \int_{M\ni y}\mathrm{Tr}_y\left[\psi(y)(K_t(\mathsf{D},\mathcal{B})(y,y))\right]\mathsf{vol}_g(M)(y)$$

admits a complete asymptotic expansion at $t\to 0$ *of the form*

$$\mathrm{Tr}_{\mathsf{L}^2}\left(\psi e^{-t\mathsf{D}_{\mathcal{B}}}\right) \sim \sum_{n=0}^{\infty} a_n(\psi,\mathsf{D},\mathcal{B})t^{(n-m)/2}, \tag{4.1}$$

where $a_n(\psi,\mathsf{D},\mathcal{B})$ *are the* ***heat trace asymptotic coefficients associated to*** ψ ***and*** $\mathsf{D}_{\mathcal{B}}$. *The asymptotic expansion in (4.1) is also referred as the* ***heat kernel asymptotic expansion associated to*** ψ ***and*** $\mathsf{D}_{\mathcal{B}}$.

(4) *There exist local endomorphism-valued invariants*

$$\mathfrak{e}_n(\mathsf{D}) \in \Gamma(M,\mathrm{End}(F)) \quad \textit{and} \quad \mathfrak{e}_{n,k}(\mathsf{D},\mathcal{B}) \in \Gamma(\partial M,\mathrm{End}(F|_{\partial M})) \tag{4.2}$$

so that

$$\begin{array}{rl} a_n(\psi,\mathsf{D},\mathcal{B}) = & \int_M \mathrm{Tr}\left(\psi\cdot\mathfrak{e}_n(\mathsf{D})\right)\mathsf{vol}_g(M) \\ & +\sum_{k=0}^{n-1}\int_{\partial M}\mathrm{Tr}\left({\nabla^F_{\varsigma_{\mathsf{in}}}}^k\psi\cdot\mathfrak{e}_{n,k}(\mathsf{D},\mathcal{B})\right)\mathsf{vol}^{\partial}_g(M), \end{array} \tag{4.3}$$

where ∇^F *is a fixed connection on* F.

(5) *The quantities $\mathfrak{e}_n(x,\mathsf{D})$ are locally computable as universal polynomials in finite order derivatives (jets) of the symbol of* D *with coefficients being smooth functions of the symbol of* D. *The $\mathfrak{e}_{n,k}(y,\mathsf{D},\mathcal{B})$ are locally computable as universal polynomials in finite order derivatives (jets) of the symbols of* D *and* $\mathcal{B}$ *with coefficients being smooth functions of the symbols of* D *and* $\mathcal{B}$.

(6) *The quantities $\mathfrak{e}_n(x,\mathsf{D})$ and $\mathfrak{e}_{n,k}(y,\mathsf{D},\mathcal{B})$ satisfy*

$$\begin{array}{rcl}\mathfrak{e}_n(x,c^2\mathsf{D}) & = & c^n\mathfrak{e}_n(x,\mathsf{D})\\ \mathfrak{e}_{n,k}(y,c^2\mathsf{D},\mathcal{B}) & = & c^{n-1-k}\mathfrak{e}_{n,k}(y,\mathsf{D},\mathcal{B}),\ \textit{for } k\in\{0,\ldots,n-1\},\end{array} \tag{4.4}$$

for each $c>0$.

Proof. The statements from (1) to (5) correspond exactly to Theorem 1.4.5 in [**Gi04**] (see also Theorem 1.3.5 and Lemma 1.3.6 in [**Gi04**]). For the original proofs, we refer the reader to Theorem 3 in [**Se69b**] and Lemma 1 and Lemma 2 in [**Se69a**] (for (5) see also displays (5) and (6) in [**Se69a**] concerning the invariants in the interior and displays (9) and (10) in [**Se69a**] concerning the invariants on the boundary). For the case of a closed manifold see Greiner [**Gre71**] and [**Se67**]. For (6), see Theorem 3.1.9 in [**Gi04**]. □

Notation 4.1.3. As in [**BGV92**], for m the dimension of M, we denote by

$$\operatorname*{LIM}_{t\to 0}\left(\operatorname{Tr}_{\mathsf{L}^2}\left(\psi e^{-t\mathsf{D}_{\mathcal{B}}}\right)\right):=a_m(\psi,\mathsf{D},\mathcal{B})$$

the **coefficient of the constant term** in the heat trace asymptotic expansion associated to ψ and $\mathsf{D}_{\mathcal{B}}$ in (4.1) above.

4.1.2. Invariant theory and the heat trace asymptotic expansion. Remember the material in Section 1.1.7, in particular, the notion of a polynomial function which is invariant under the action of the orthogonal group. We recall an example of a first application of Weyl's first theorem of invariants to characterize local invariants of a Riemannian manifold.

Example: Invariants of the Riemannian metric. By using Lemma 1.2.1, any local invariant of the Riemannian metric (obtained as polynomial in the higher order derivatives of the Riemannian metric with coefficients being smooth functions of the metric) can be expressed polynomially in terms of higher order covariant derivatives of the Riemann curvature tensor R and the second fundamental form L (with respect to ∇ and ∇^{∂} respectively). This, originally proved by Atiyah–Bott–Patodi in [**ABP75**] by using Weyl's first theorem of invariants (see Theorem 1.1.1), is presented in great detail in Section 1.7.2 in [**Gi04**], particularly see Lemmas 1.7.5 and 1.7.6 therein.

The invariant endomorphisms $\mathfrak{e}_n$ and $\mathfrak{e}_{n,k}$. Let $(\mathsf{D},\Gamma(M;F)|_{\mathcal{B}})$ be the boundary valued problem, where D is an operator of Laplace type (like the bilinear or Hermitian Laplacian) and $\mathcal{B}$ the operator imposing the boundary conditions in 3.2.6 (associated to the bilinear of Hermitian structures respectively). Let ∇^{D} be the connection on F and the bundle endomorphism $\mathsf{E}^{\mathsf{D}} \in \Gamma(M; \mathrm{End}(F))$ uniquely characterizing the operator of Laplace type D, see Lemma 2.1.2. We denote by R^{D} the curvature of ∇^{D}. Consider

$$\chi, \quad \mathsf{S}, \quad \text{and} \quad \mathsf{w}_{\varsigma_{\mathsf{in}}} := b^{-1}\nabla^{\mathsf{D}}_{\varsigma_{\mathsf{in}}} b \quad \in \Gamma(M, \mathrm{End}(F)) \tag{4.5}$$

the bundle endomorphisms characterizing the boundary operators $\mathcal{B}$ (imposing absolute/relative boundary conditions), see Proposition 3.2.12.

Consider the endomorphism invariants $\mathfrak{e}_n(\mathsf{D})$ and $\mathfrak{e}_{n,k}(\mathsf{D}, \mathcal{B})$ from (4.2) in Proposition 4.1.2. Then, as for the invariants of the Riemannian metric in the Example above, these local invariants can be expressed as universal polynomials in higher order covariant derivatives (or jets) of R, R^{D}, E^{D}, χ, S, $\mathsf{w}_{\varsigma_{\mathsf{in}}}$ and the second fundamental form L. This is achieved again by using (Weyl's first) Theorem 1.1.1. Let us somehow be more precise.

Notation 4.1.4. Let $\mathsf{e} := (e_1, \ldots, e_m)$ be an orthonormal frame of TM locally over some neighborhood of $x \in M$, such that, at the boundary, $e_m = \varsigma_{\mathsf{in}}$ is the inwards pointing unit normal vector field on ∂M; we use the indices $i, j, k \in \{1, \cdots, m\}$ to index this local frame. On the boundary, we consider the induced frame $(e_1 \cdots, e_{m-1})$ for $T(\partial M)$ over some neighborhood of $y \in \partial M$ and use the indices $a, b, c \in \{1, \cdots, m-1\}$. In this way, R_{ijkl} indicates the components of the curvature tensor R of the Levi Cività connection and $\mathsf{R}^{\mathsf{D}}_{ij}$ the components of the curvature R^{D} associated to the connection ∇^{D}, and $\mathsf{L}_{ab} := g(\nabla_{e_a} e_b, e_m)$ the components of the second fundamental form with respect to this frame. Furthermore, multiple covariant differentiation of tensors T (of general type), computed with respect to the connection ∇^{D} and the Levi-Cività connection ∇ on TM, is denoted by $\mathsf{T}_{;}$, that is, by using the symbol '**;**' as subscript. Analogously, the Levi-Cività connection ∇^{∂} on ∂M and the connection ∇^{D} permit to covariantly differentiate tensors defined on ∂M, along tangential directions, and in this case the notation '**:**' for multiple covariant tangential differentiation is chosen.

Proposition 4.1.5. *For an operator of Laplace type D, consider its characterizing connection ∇^{D} and bundle endomorphism E^{D} (see Lemma 2.1.2). For $\mathcal{B}$ the boundary operator from Definition 3.2.6 imposing absolute/relative boundary conditions, consider χ, S and $\mathsf{w}_{\varsigma_{\mathsf{in}}}$ the characterizing endomorphism bundles from (4.5). As in Notation 4.1.4, let $\mathsf{R}_{i_1i_2i_3i_4}$, $\mathsf{R}^{\mathsf{D}}_{i_1i_2}$ and $\mathsf{L}_{a_1a_2}$ be the components of the Riemann curvature tensor R, the curvature R^{D} and of the second fundamental form L respectively computed with respect to an specified orthonormal frame e in TM and the symbol '**;**' indicates covariant differentiation and '**:**' tangent covariant differentiation. Then, for the endomorphism invariants $\mathfrak{e}_n(\mathsf{D})$ and $\mathfrak{e}_{n,k}(\mathsf{D}, \mathcal{B})$ from (4.2), we have*

(1) The quantities $\mathfrak{e}_n(\mathsf{D})$ are locally computable as universal polynomials in the formal variables $\mathsf{R}_{i_1i_2i_3i_4;\ldots}$, $\mathsf{R}^{\mathsf{D}}_{i_1i_2;\ldots}$, $\mathsf{E}^{\mathsf{D}}{}_{;\ldots}$ and id.

(2) The quantities $\mathfrak{e}_{n,k}(\mathsf{D}, \mathcal{B})$ are locally computable as universal polynomials in the formal variables $\mathsf{R}_{i_1i_2i_3i_4;\ldots}$, $\mathsf{R}^{\mathsf{D}}_{i_1i_2;\ldots}$, $\mathsf{E}^{\mathsf{D}}{}_{;\ldots}$, $\mathsf{L}_{a_1a_2:\ldots}$, $\chi_{;\ldots}$, $\mathsf{S}_{;\ldots}$, $\mathsf{w}_{\varsigma_{\mathsf{in}};\ldots}$ and id.

Proof. We sketch the main ideas in the proof. For much more details, we refer the reader to the books of Gilkey. More precisely, see Lemma 3.1.10 and Lemma 3.1.11 from Section 3.1.8 in [**Gi04**], (see also Sections 1.7-1.8 and 2.2.4 in [**Gi04**] and Sections 1.7, 1.9 and 4.8 in [**Gi84**]).

Let us start with the invariants in the interior. Take into account Notation 4.1.4. For e a (local orthonormal) frame of TM, consider the set

$$\{\mathsf{R}_{i_1 i_2 i_3 i_4;\ldots}, \mathsf{R}^{\mathsf{D}}_{i_1 i_2;\ldots}, \mathsf{E}^{\mathsf{D}}{}_{;\ldots}\} \tag{4.6}$$

corresponding to the components of R, R^{D}, E^{D} and their multiple covariant derivatives, seen as of formal variables. Let $\mathfrak{e}(\mathsf{D}) = \mathfrak{e}(\mathsf{R}_{i_1 i_2 i_3 i_4;\ldots}, \mathsf{R}^{\mathsf{D}}_{i_1 i_2;\ldots}, \mathsf{E}^{\mathsf{D}}{}_{;\ldots})$ be a formal polynomial in the variables (4.6). These formal polynomials can be evaluated once a local orthonormal frame e is fixed. Then, $\mathfrak{e}(\mathsf{D})$ is said to be invariant if the value of $\mathfrak{e}(\mathsf{D})(\mathsf{e}) \in \mathrm{End}(F)$ is independent of the frame e and only depends on R, R^{D}, E^{D}. Let $\mathcal{E}_m(\mathsf{D})$ the set of all these invariant polynomials and for each positive integer n, define

$$\begin{aligned} \mathsf{weight}(\mathsf{R}_{i_1 i_2 i_3 i_4; i_5 \cdots i_n}) &:= 2+n, \\ \mathsf{weight}(\mathsf{R}^{\mathsf{D}}_{i_1 i_2; i_3 \cdots i_n}) &:= n, \\ \mathsf{weight}(\mathsf{E}^{\mathsf{D}}{}_{;i_1 \cdots i_n}) &:= 2+n. \end{aligned}$$

Let $\mathcal{E}_{m,n}(\mathsf{D}) \subset \mathcal{E}_m(\mathsf{D})$ be the space of all elements in $\mathcal{E}_m(\mathsf{D})$ which are invariant and homogeneous of *weight* n. Then, from (Weyl's) Theorem 1.1.1, any polynomial invariant endomorphism which is homogeneous of weight n belongs to $\mathcal{E}_{m,n}(\mathsf{D})$. It remains to explain why the quantities $\mathfrak{e}_n(\mathsf{D})$ appearing in (4.3) belong to $\mathcal{E}_{m,n}(\mathsf{D})$ as well. In order to see this, first remember that the symbol of D is a geometric invariant which does not depend on the choice of orthonormal frames nor on the connection. So, for each $x_0 \in M$, choose geodesic coordinates centered at x_0, in terms of which the jets of the metric at x_0 can be computed in terms of the variables $\mathsf{R}_{ijkl;\ldots}$, see Lemma 1.2.1. Next, construct a local frame of F around x_0 in the following way. Choose v_0 to be a frame at the fiber F_0 over x_0, and, with respect to ∇^{D}, parallely transport v_0 along all the geodesic rays leaving x_0; this guarantees that, locally around x_0, all covariant derivatives of ω^{D}, the connection 1-form associated to ∇^{D}, can be expressed in terms of R_{ijkl}, $\mathsf{R}^{\mathsf{D}}_{ij}$ and their multiple covariant derivatives as well. From Lemma 2.1.2, we know that the symbol of D can be described in terms of g_{ij}, ω^{D} and E^{D}. Thus, higher order derivatives of the symbol of D are all expressible, locally around x_0, in the variables in 4.6. Now, By (5) and (6) in Proposition 4.1.2. we know that $\mathfrak{e}_n(\mathsf{D})$ are endomorphism invariants locally computable as polynomials homogeneous of order n in the jets of the symbol of D. Henceforth each $\mathfrak{e}_n(\mathsf{D})$ is in turn expressible as a polynomial invariant homogenous of order n in the variables (4.6). Therefore by (Weyl's) Theorem 1.1.1, we have $\mathfrak{e}_n(\mathsf{D}) \in \mathcal{E}_{m,n}(\mathsf{D})$.

The treatment for the invariants $\mathfrak{e}_{n,k}(\mathsf{D}, \mathcal{B})$ on the boundary is similar. By Proposition 4.1.2 these invariants are local computable as universal polynomial in the jets of the symbols of D and $\mathcal{B}$. Hence, in addition to the formal variables considered for the invariants in the interior, in this case one also considers the formal variables coming from (higher order derivatives) of the second fundamental form L and the endomorphism χ, S and $\mathsf{w}_{\varsigma_{\mathrm{in}}}$ characterizing the absolute/relative boundary operator. Choose a geodesic coordinate system on ∂M, with respect to g^{∂} and extend it to a normalized coordinate system on a collared neighborhood. The jets of the metric are written in terms of higher order covariant derivatives of R and higher order tangential covariant derivatives of the second fundamental form, see Lemma 1.2.1 -remark also that higher order tangential derivatives of χ are related to the second fundamental form, see (3.28). On the boundary, since the inwards pointing geodesic unit normal vector field has a distinguish role, the structure group is reduced to $O(m-1,\mathbb{R})$. Let $\mathcal{E}^{\partial}_m(\mathsf{D},\mathcal{B})$ be the space of polynomial invariant functions in the formal variables

$$\{\mathsf{R}_{i_1i_2i_3i_4;\ldots},\mathsf{R}^{\mathsf{D}}_{i_1i_2;\ldots},\mathsf{E}^{\mathsf{D}}{}_{;\ldots},\mathsf{L}_{ab:\ldots},\chi_{;\ldots},\mathsf{S}_{;\ldots},\mathsf{w}_{\varsigma_{\mathrm{in}};\ldots}\}. \tag{4.7}$$

Then, by setting the degree of homogeneity (or weight) for the additional variables according with the relations for the polynomial invariants $\mathfrak{e}_{n,k}(\mathsf{D},\mathcal{B})$ in (4.4) and by (Weyl's) Theorem 1.1.1 of invariants, one has $\mathfrak{e}_{n,k}(\mathsf{D},\mathcal{B})\in\mathcal{E}^{\partial}_m(\mathsf{D},\mathcal{B})$. □

4.2. Heat trace asymptotics for the Hermitian Laplacian

We use the theory presented in Section 4.1 to study the coefficient $a_m(\psi,\Delta_{E,g,h},\mathcal{B})$ of the constant term in the heat trace asymptotic expansion in (4.3) associated to the Hermitian Laplacian $\Delta_{E,g,h}$ under absolute/relative boundary conditions, see Section 3.1, and certain well-chosen bundle endomorphism ψ. We first recall one of the main results by Brüning and Ma in [**BM06**], where the Hermitian Laplacian on a manifold with boundary under absolute boundary conditions was studied. Then, we use Poincaré duality to deduce the correspoding results for the Hermitian Laplacian on a manifold with boundary under relative boundary conditions.

Notation 4.2.1. **(Brüning–Ma)** In order to read the formulas appearing in Proposition 4.2.2, Proposition 4.2.5, Theorem 4.2.7 and Theorem 4.4.3, we need some characteristic forms appearing in the anomaly formulas from [**BM06**]. At this stage the specific knowledge of these characteristic forms is not needed, but in Chapter 6 we give a detailed construction of these forms. These forms are the *Euler form* $\mathbf{e}(M,g)\in\Omega^m(M;\Theta_M)$ associated to the metric g, and certain characteristic forms on the boundary such as $\mathbf{e_b}(\partial M,g), B(\partial M,g)\in\Omega^{m-1}(\partial M;\Theta_M)$ defined by the formulas (1.17), page 775 in [**BM06**], (see Definition 6.1.11 in Chapter 6). But also certain *secondary* forms or of *Chern–Simons type* $\widetilde{\mathbf{e}}(M,g_0,g_1)\in\Omega^{m-1}(M;\Theta_M)$ and $\widetilde{\mathbf{e}}_{\mathbf{b}}(\partial M,g_0,g_1)\in\Omega^{m-2}(\partial M;\Theta_M)$ corresponding to $\{g_s\}_s$ a smooth path of Riemannian metrics on M connecting the Riemannian metrics g_0 and g_1, defined in (1.45), page 780 in [**BM06**], (see (6.16) in Definition 6.1.12 in Chapter 6).

Proposition 4.2.2. ***(Brüning–Ma)*** *Keep in mind Notations 4.2.1 and 3.2.9. Let $(M,\partial M,\emptyset)$ be a compact Riemannian bordism. Consider $[\Delta,\Omega_{\mathcal{B}}]^{E,g,h}_{(M,\partial M,\emptyset)}$ the Hermitian* *boundary value problem and denote by* $\Delta_{\mathsf{abs},h}$ *its* L^2*-realization. Let us write* $\mathrm{Tr}_{\mathbf{s}}$ *to indicate the supertrace. For* $\phi \in \Gamma(M, \mathrm{End}(E))$ *we have*

$$(4.8) \qquad \mathop{\mathrm{LIM}}_{t\to 0}\left(\mathrm{Tr}_{\mathbf{s}}\left(\phi e^{-t\Delta_{\mathsf{abs},h}}\right)\right) = \int_M \mathrm{Tr}(\phi)\mathbf{e}(M,g) - (-1)^m \int_{\partial M} i^*\,\mathrm{Tr}(\phi)\mathbf{e_b}(\partial M, g).$$

Moreover, for $\xi \in \Gamma(M, \mathrm{End}(TM))$ *a symmetric endomorphism with respect to the metric* g*, we set*

$$(4.9) \qquad \Psi := \mathbf{D}^*\xi - \frac{1}{2}\,\mathrm{Tr}(\xi) \quad \in \Gamma(M, \mathrm{End}(\Lambda^* T^*M)),$$

where $\mathbf{D}^*\xi \in \Gamma(M, \mathrm{End}(\Lambda^*T^*M))$ *is obtained as the unique extension of* ξ *as a derivation on* $\Lambda^*(T^*M)$*. For* $\tau \in \mathbb{R}$ *taken small enough such that* $g + \tau g\xi$ *is a nondegenerate symmetric metric on* TM*, we have*

(4.10)

$$\begin{aligned}
\mathrm{LIM}_{t\to 0}\left(\mathrm{Tr}_{\mathbf{s}}\left(-\Psi e^{-t\Delta_{\mathsf{abs},h}}\right)\right) \;=\; & -2\int_M \tfrac{\partial}{\partial\tau}\big|_{\tau=0}\,\widetilde{\mathbf{e}}(M, g, g+\tau g\xi)\wedge\omega(\nabla^E, h) \\
& -2\int_{\partial M} \tfrac{\partial}{\partial\tau}\big|_{\tau=0}\,\widetilde{\mathbf{e}}_{\mathbf{b}}(\partial M, g, g+\tau g\xi)\wedge i^*\omega(\nabla^E, h) \\
& +\mathsf{rank}(E)\int_{\partial M} \tfrac{\partial}{\partial\tau}\big|_{\tau=0}\, B(\partial M, g+\tau g\xi),
\end{aligned}$$

where $\omega(\nabla^E, h) := -\frac{1}{2}\,\mathrm{Tr}(h^{-1}\nabla^E h)$ *is a real valued closed one form.*

Proof. We prove formula (4.8). First, each endomorphism $\phi \in \Gamma(M, \mathrm{End}(E))$ can be uniquely written as $\phi = \phi^{\mathsf{re}} + \mathbf{i}\phi^{\mathsf{im}}$ where $\phi^{\mathsf{re}}, \phi^{\mathsf{im}}$ are self-adjoint elements. Thus, it is enough to prove the formula (4.8) for ϕ self-adjoint. Now, suppose that $\phi_u := h_u^{-1}\frac{\partial h_u}{\partial u} \in \Gamma(M, \mathrm{End}(E))$, where h_u is a smooth one real parameter family of Hermitian forms on E with $h_0 = h$. Then, formula (4.8) exactly is the infinitesimal version of Brüning and Ma's formulas, see Theorem 4.6 in Section 4.3 and expression (5.72) in Section 5.5 in [**BM06**]. Next, suppose $\phi \in \Gamma(M, \mathrm{End}(E))$ to be an arbitrary self-adjoint element. Then, for u small enough, the family $h_u := h + uh\phi$ is a smooth family of Hermitian forms on E and $h_u^{-1}\frac{\partial h_u}{\partial u} = h_u^{-1}h\phi$ defines a smooth family of self-adjoint elements in $\Gamma(M, \mathrm{End}(E))$. Therefore, by using again Brüning and Ma's formulas for

$$h_0^{-1}\left(\frac{\partial h_u}{\partial u}\bigg|_{u=0}\right) = \phi,$$

the proof of (4.8) is complete. We now prove formula (4.10). Let g_u be a smooth family of Riemannian metrics on TM with $g_0 = g$ and denote by $\star_u$ the Hodge $\star$-operator corresponding to g_u. First, consider the case where $\xi_u := g_u^{-1}\frac{\partial g_u}{\partial u} \in \Gamma(M; \mathrm{End}(TM))$ and so, by formula (4.9), we obtain

$$\Psi_u := \mathbf{D}^*\left(g_u^{-1}\frac{\partial g_u}{\partial u}\right) - \frac{1}{2}\,\mathrm{Tr}\left(g_u^{-1}\frac{\partial g_u}{\partial u}\right) = -\star_u^{-1}\frac{\partial \star_u}{\partial u}$$

considered as a smooth family in $\Gamma(M, \mathrm{End}(\Lambda^*T^*M))$, for the last equality above see for instance Proposition 4.15 in [**BZ92**]. Then, formula (4.10) is the infinitesimal version of Brüning and Ma's results, see Theorem 4.6 in Section 4.3 and expressions

(5.74) and (5.75) in Section 5.5 in [**BM06**]. In the general case, take a symmetric $\xi \in \Gamma(M;\mathrm{End}(TM))$. Then, for u small enough the formula $g_u := g + ug\xi$ defines a smooth family of nondegenerate metrics on TM and hence $g_u^{-1}\frac{\partial g_u}{\partial u} = g_u^{-1}g\xi$ a smooth family of symmetric elements in $\Gamma(M,\mathrm{End}(TM))$. Hence we obtain a smooth family of symmetric endomorphisms $-\star_u^{-1}\frac{\partial \star_u}{\partial u}$ in $\Gamma(M,\mathrm{End}(\Lambda^*T^*M))$, for which we can use again Brüning and Ma's formulas. In particular, they must hold for $u = 0$ for which we have $g_0^{-1}\left(\left.\frac{\partial g_u}{\partial u}\right|_{u=0}\right) = \xi$, so that

$$\Psi_0 = \mathbf{D}^*(\xi) - \frac{1}{2}\mathrm{Tr}(\xi) = -\star^{-1}\left(\left.\frac{\partial \star_u}{\partial u}\right|_{u=0}\right).$$

That is, formula (4.10) holds. $\square$

The following uses Poincaré duality to relate boundary value problems under absolute and relative boundary conditions.

Lemma 4.2.3. *Recall Notation 3.2.9. Let $\bar{E}'$ be the dual of the conjugated complex vector bundle of E, endowed with the corresponding dual flat connection and dual Hermitian form. Consider the bordism $(M,\emptyset,\partial M)$ and its dual $(M,\emptyset,\partial M)' := (M,\partial M,\emptyset)$. We look at the Hermitian boundary value problem $[\Delta,\Omega_{\mathcal{B}}]^{E,g,h}_{(M,\emptyset,\partial M)}$ with L^2-realization denoted by $\Delta_{\mathsf{rel},h}$ and the its dual Hermitian boundary value problem $[\Delta,\Omega_{\mathcal{B}}]^{\bar{E}'\otimes\Theta_M,g,h'}_{(M,\emptyset,\partial M)'}$ with corresponding L^2-realization $\Delta'_{\mathsf{abs},h'}$. If ϕ, ξ and Ψ are as in Proposition 4.2.2, then*

$$\operatorname*{LIM}_{t\to 0}\left(\mathrm{Tr_s}\left(\phi e^{-t\Delta_{\mathsf{rel},h}}\right)\right) = (-1)^m \operatorname*{LIM}_{t\to 0}\left(\mathrm{Tr_s}\left(\phi^* e^{-t\Delta'_{\mathsf{abs},h'}}\right)\right), \tag{4.11}$$

where $\phi^* := h\phi h^{-1}$, and

$$\operatorname*{LIM}_{t\to 0}\mathrm{Tr_s}\left(\Psi e^{-t\Delta_{\mathsf{rel},h}}\right) = (-1)^{m+1}\operatorname*{LIM}_{t\to 0}\mathrm{Tr_s}\left(\Psi e^{-t\Delta'_{\mathsf{abs},h'}}\right). \tag{4.12}$$

Proof. We consider the complex vector bundle isomorphism between E and $\bar{E}'$ provided by the Hermitian metric on E (see page 286 in [**BT82**]), which we still denote by $h \in \Omega^0(M;\mathrm{End}(E,\bar{E}'))$. With respect to the induced connection on $\mathrm{End}(E,\bar{E}')$, consider $\nabla^E_X h \in \Omega^1(M;\mathrm{End}(E,\bar{E}'))$. By considering the Hermitian metric on E and the Riemannian metric on M, one obtains $\star_h := \star\otimes h : \Omega(M;E) \to \Omega(M;\bar{E}'\otimes\Theta_M)$ a *complex linear* isomorphism used to define

$$d^*_{E,g,h} := (-1)^q \star_h^{-1} d_{\bar{E}'\otimes\Theta_M}\star_h : \Omega^q(M;E) \to \Omega^{q-1}(M;E);$$

this is the formal adjoint to d_E with respect to the Hermitian product on $\Omega(M;E)$. Moreover, the formula $d_{\bar{E}'\otimes\Theta_M}d^*_{\bar{E}'\otimes\Theta_M,g,h'}\star_h = \star_h d^*_{E,g,h}\,d_E$ holds and therefore $\star_h\Delta_{E,g,h} = \Delta_{\bar{E}'\otimes\Theta_M,g,h'}\star_h$. As in Section 3.2.2 the operator $\star_h$ intertwines E-valued forms satisfying relative (resp. absolute) boundary conditions with $\bar{E}'$-valued forms satisfying absolute (resp. relative) boundary conditions. That is,

$$\Delta_{\mathsf{rel},h} = \star_h^{-1}\Delta'_{\mathsf{abs},h'}\star_h \tag{4.13}$$

and therefore $\phi\exp(-t\Delta_{\mathsf{rel},h}) = \star_h^{-1}\phi^*\exp(-t\Delta'_{\mathsf{abs},h'})\star_h$, where $\phi^* := h\phi h^{-1}$. Thus, since the supertrace vanishes on supercommutators of graded complex-linear operators and the degree of $\star_{h,q}$ is $m-q$, we obtain the formula

$$\operatorname{Tr}_{\mathbf{s}}(\phi e^{-t\Delta_{\mathsf{rel},h}}) = (-1)^m \operatorname{Tr}_{\mathbf{s}}(\phi^* e^{-t\Delta'_{\mathsf{abs},h'}})$$

and hence (4.11). We now turn to formula (4.12). First, remark that

$$\star_q\left(\mathbf{D}^*\xi - \tfrac{1}{2}\operatorname{Tr}(\xi)\right)\star_q^{-1} = -\mathbf{D}^*\xi + \tfrac{1}{2}\operatorname{Tr}(\xi). \tag{4.14}$$

We prove (4.14), by pointwise computing $\star_q\mathbf{D}^*\xi\star_q^{-1}$. Since ξ is a symmetric complex endomorphism of T_xM, we may choose an orthonormal frame $\{e_i\}_1^m$ such that $\xi e_i = \lambda_i e_i$. Then, for $\{e^{i_1}\wedge\cdots\wedge e^{i_q}\}_{1\leqslant i_1<\cdots<i_q\leqslant m}$ a positive definite oriented frame for $\Lambda^q T_x^*M$, the Hodge $\star$-operator is given by $\star_q\left(e^{i_1}\wedge\cdots\wedge e^{i_q}\right) = e^{j_1}\wedge\cdots\wedge e^{j_{m-q}}\in\Lambda^{m-q}T_x^*M$, where the ordered indices $(j_1,\dots,j_{m-q}) := (1,\dots,\widehat{i_1},\dots,\widehat{i_q},\dots,m)$ with $1\leqslant j_1<\ldots< j_{m-q}\leqslant m$, are obtained as the unique possible choice of ordered indices complementary to $\leqslant i_1<\cdots<i_q$. Therefore

$$\begin{aligned}
\star_q\mathbf{D}^*\xi\star_q^{-1}\left(e^{j_1}\wedge\cdots\wedge e^{j_{m-q}}\right) &= \star_q\mathbf{D}^*\xi\left(e^{i_1}\wedge\cdots\wedge e^{i_q}\right)\\
&= \star_q\textstyle\sum_{l=1}^q\left(e^{i_1}\wedge\cdots\wedge\xi(e^{i_l})\wedge\cdots\wedge e^{i_q}\right)\\
&= \star_q\textstyle\sum_{l=1}^q\lambda_{i_l}\left(e^{i_1}\wedge\cdots\wedge e^{i_l}\wedge\cdots\wedge e^{i_q}\right)\\
&= \textstyle\sum_{l=1}^q\lambda_{i_l}\left(e^{j_1}\wedge\cdots\wedge e^{j_{m-q}}\right)\\
&= \textstyle\sum_{l=1}^m\lambda_{i_l}\left(e^{j_1}\wedge\cdots\wedge e^{j_{m-q}}\right)-\sum_{l=1}^{m-q}\lambda_{j_l}\left(e^{j_1}\wedge\cdots\wedge e^{j_{m-q}}\right)\\
&= \textstyle\sum_{l=1}^m\lambda_{i_l}\left(e^{j_1}\wedge\cdots\wedge e^{j_{m-q}}\right)-\sum_{l=1}^{m-q}\left(e^{j_1}\wedge\cdots\wedge\lambda_{j_l}e^{j_l}\wedge\cdots\wedge e^{j_{m-q}}\right)\\
&= (\operatorname{Tr}\xi-\mathbf{D}^*\xi)\left(e^{j_1}\wedge\cdots\wedge e^{j_{m-q}}\right)
\end{aligned}$$

and we obtain (4.14), which in turn allows us to conclude

$$\begin{aligned}
\Psi(\star_q\otimes h)^{-1} &= \left(\left(\mathbf{D}^*\xi-\tfrac{1}{2}\operatorname{Tr}(\xi)\right)\otimes 1\right)(\star_q\otimes h)^{-1}\\
&= (\star_q\otimes h)^{-1}\left(\left(\star_q\left(\mathbf{D}^*\xi-\tfrac{1}{2}\operatorname{Tr}(\xi)\right)\star_q^{-1}\right)\otimes 1\right)\\
&= -(\star_q\otimes h)^{-1}\left(\left(\mathbf{D}^*\xi-\tfrac{1}{2}\operatorname{Tr}(\xi)\right)\otimes 1\right)\\
&= -(\star_q\otimes h)^{-1}\Psi.
\end{aligned}\tag{4.15}$$

Finally, we use (4.15) to pass to the complex conjugated; hence with (4.13) and duality between these boundary value problems we obtain

$$\Psi\exp\left(-t\Delta_{\mathsf{rel},h}\right) = \Psi\star_h^{-1}\exp\left(-t\Delta'_{\mathsf{abs},h'}\right)\star_h = -\star_h^{-1}\Psi\exp(-t\Delta'_{\mathsf{abs},h'})\star_h$$

thus, as for (4.11), we have

$$\operatorname{Tr}_{\mathbf{s}}(\Psi\exp(-t\Delta_{\mathsf{rel},h})) = -(-1)^m\operatorname{Tr}_{\mathbf{s}}(\Psi\exp(-t\Delta'_{\mathsf{abs},h'}))$$

□

Remark 4.2.4. The relations from Lemma 4.11 were also computed by Brüning and Ma in Theorem 3.4 in [**BM11**], by a different approach, in which they do *not* use the complex conjugate bundle $\overline{E}'$. Instead, they consider the Hodge $\star$-operator $\star_h$ to relate the Hermitian Laplacian acting under absolute boundary conditions on $\Omega(M;E)$ to the Hermitian Laplacian acting under relative boundary conditions on the space $\Omega(M;E\otimes\Theta_M)$. To do so, they use the Hermitian form h, to identify E with its dual. Then they split their proof into two cases according to the situation wether h is flat or not.

Proposition 4.2.5. *Recall Notation 3.2.9 and Notation 4.2.1. For the Riemannian bordism $(M,\emptyset,\partial M)$, consider the Hermitian boundary value problem $[\Delta,\Omega_{\mathcal{B}}]^{E,g,h}_{(M,\emptyset,\partial M)}$ with its L^2-realization denoted by $\Delta_{\mathsf{rel},h}$. If ϕ, ξ and Ψ are as in Proposition 4.2.2, then*

$$\mathrm{LIM}_{t\to 0}\left(\mathrm{Tr}_{\mathbf{s}}\left(\phi e^{-t\Delta_{\mathsf{rel},h}}\right)\right)=\int_M \mathrm{Tr}(\phi)\mathbf{e}(M,g)-\int_{\partial M} i^*\,\mathrm{Tr}(\phi)\mathbf{e}_{\mathbf{b}}(\partial M,g)$$

and

$$\begin{aligned}\mathrm{LIM}_{t\to 0}\left(\mathrm{Tr}_{\mathbf{s}}\left(-\Psi e^{-t\Delta_{\mathsf{rel},h}}\right)\right)&=-2\int_M \tfrac{\partial}{\partial\tau}\big|_{\tau=0}\widetilde{\mathbf{e}}(M,g,g+\tau g\xi)\wedge\omega(\nabla^E,h)\\&\quad+2(-1)^{m+1}\int_{\partial M}\tfrac{\partial}{\partial\tau}\big|_{\tau=0}\widetilde{\mathbf{e}}_{\mathbf{b}}(\partial M,g,g+\tau g\xi)\wedge i^*\omega(\nabla^E,h)\\&\quad+(-1)^{m+1}\mathsf{rank}(E)\int_{\partial M}\tfrac{\partial}{\partial\tau}\big|_{\tau=0}B(\partial M,g+\tau g\xi).\end{aligned}$$

Proof. Recall that $w\in\Omega^*(M;E)$ satisfies relative boundary conditions if and only if the smooth form $\star_h w\in\Omega^{m-*}(M;\bar{E}'\otimes\Theta_M)$ satisfies absolute boundary conditions on ∂M. Thus, the first formula follows from formula (4.11) in Lemma 4.2.3, and the results from Brüning and Ma for the Hermitian Laplacian stated in Proposition 4.2.2. The second formula follows from Lemma formula (4.12) in 4.2.3, Proposition 4.2.2 and $\omega(\nabla^E,h)=-\omega(\nabla^{E'},h')$, see for instance Section 2.4 in [**BH07**]. ☐

Lemma 4.2.6. *Recall Notation 3.2.9. For the bordism $(M,\partial M,\emptyset)$, consider the Hermitian boundary value problem $[\Delta,\Omega_{\mathcal{B}}]^{E,g,h}_{(M,\partial M,\emptyset)}$ with its L^2-realization $\Delta_{\mathsf{abs},h}$. For $(M,\emptyset,\partial M)$, consider the Hermitian boundary value problem $[\Delta,\Omega_{\mathcal{B}}]^{E,g,h}_{(M,\emptyset,\partial M)}$ together with its L^2-realization $\Delta_{\mathsf{rel},h}$. For $(M,\partial_+M,\partial_-M)$, consider $[\Delta,\Omega_{\mathcal{B}}]^{E,g,h}_{(M,\partial_+M,\partial_-M)}$ the Hermitian boundary value problem L^2-realization $\Delta_{\mathcal{B},h}$. If we chose the endomorphism $\psi_\pm\in\Gamma(M;\mathrm{End}(\Lambda^*(T^*M)\otimes E))$ in such a way that $\mathsf{supp}(\psi_\pm)\cap\partial_\mp M=\emptyset$, then*

$$\underset{t\to 0}{\mathrm{LIM}}\left(\mathrm{Tr}_{\mathbf{s}}\left(\psi_+e^{-t\Delta_{\mathcal{B},h}}\right)\right)=\underset{t\to 0}{\mathrm{LIM}}\left(\mathrm{Tr}_{\mathbf{s}}\left(\psi_+e^{-t\Delta_{\mathsf{abs},h}}\right)\right)$$

and

$$\underset{t\to 0}{\mathrm{LIM}}\left(\mathrm{Tr}_{\mathbf{s}}\left(\psi_-e^{-t\Delta_{\mathcal{B},h}}\right)\right)=\underset{t\to 0}{\mathrm{LIM}}\left(\mathrm{Tr}_{\mathbf{s}}\left(\psi_-e^{-t\Delta_{\mathsf{rel},h}}\right)\right).$$

Proof. This is a direct consequence of Lemma 4.1.5 and disjointness of ∂_+M and ∂_-M. ☐

Theorem 4.2.7. *Recall Notation 3.2.9 and Notation 4.2.1. For the compact Riemannian bordism $(M, \partial M, \emptyset)$, consider $[\Delta, \Omega_{\mathcal{B}}]^{E,g,h}_{(M,\partial_+ M,\partial_- M)}$, the Hermitian boundary value problem, with its corresponding L^2-realization $\Delta_{\mathcal{B},h}$. If ϕ, ξ and Ψ are as in Proposition 4.2.2, then*

$$\mathrm{LIM}_{t\to 0}\left(\mathrm{Tr}_{\mathbf{s}}\left(\phi e^{-t\Delta_{\mathcal{B},h}}\right)\right) = \int_M \mathrm{Tr}(\phi)\mathbf{e}(M,g) + (-1)^{m-1}\int_{\partial_+ M}\mathrm{Tr}(\phi) i_+^*\mathbf{e}_{\mathbf{b}}(\partial M, g) - \int_{\partial_- M}\mathrm{Tr}(\phi) i_-^*\mathbf{e}_{\mathbf{b}}(\partial M, g). \tag{4.16}$$

and

$$\begin{aligned}\mathrm{LIM}_{t\to 0}\left(\mathrm{Tr}_{\mathbf{s}}\left(-\Psi e^{-t\Delta_{\mathcal{B},h}}\right)\right) = &-2\int_M \tfrac{\partial}{\partial\tau}\big|_{\tau=0}\widetilde{\mathbf{e}}(M,g,g+\tau g\xi)\wedge\omega(\nabla^E,h)\\ &-2\int_{\partial_+ M}\tfrac{\partial}{\partial\tau}\big|_{\tau=0} i_+^*\widetilde{\mathbf{e}}_{\mathbf{b}}(\partial M,g,g+\tau g\xi)\wedge\omega(\nabla^E,h)\\ &+\mathsf{rank}(E)\int_{\partial_+ M}\tfrac{\partial}{\partial\tau}\big|_{\tau=0} i_+^* B(\partial M, g+\tau g\xi)\\ &-2(-1)^m\int_{\partial_- M}\tfrac{\partial}{\partial\tau}\big|_{\tau=0} i_-^*\widetilde{\mathbf{e}}_{\mathbf{b}}(\partial M,g,g+\tau g\xi)\wedge\omega(\nabla^E,h)\\ &+(-1)^{m+1}\mathsf{rank}(E)\int_{\partial_- M}\tfrac{\partial}{\partial\tau}\big|_{\tau=0} i_-^* B(\partial M, g+\tau g\xi).\end{aligned}$$

Proof. This follows from the result by Brüning and Ma in [**BM06**], stated in terms of Proposition 4.2.2 above, Proposition 4.2.5 and Lemma 4.2.6. More recently, Brüning and Ma gave also a proof of this statement, see Theorem 3.2 in [**BM11**], based on the methods developed in [**BM06**]. □

4.3. Involutions, bilinear and Hermitian forms

In Section 4.4, we compute the coefficients of the constant term in the heat trace asymptotic expansion associated to the bilinear boundary value problem, by using the corresponding results from Section 4.2 for the Hermitian boundary value problem. In order to do that, we first need relate both boundary value problems by using, in certain sense, a complex anti-linear involution on the bundle.

More precisely, we fix a Hermitian structure compatible with the bilinear as follows. Since E is endowed with a bilinear form b, there exists an anti-linear involution ν on E satisfying

$$\begin{aligned}&\overline{b(\nu e_1,\nu e_2)} = b(e_1,e_2) \quad \text{for all} \quad e_1,e_2\in E\\ \text{and}\quad &\\ &b(\nu e, e) > 0 \quad \text{for all} \quad e\in E \quad \text{with} \quad e\neq 0.\end{aligned} \tag{4.17}$$

Indeed, the fiberwise nondegenerate symmetric bilinear form b provides a reduction of the structure group of E to $O_k(\mathbb{C})$, where k is the rank of E; the natural inclusion $O_k(\mathbb{R}) \to O_k(\mathbb{C})$ is a homotopy equivalence and hence the structure group can further be reduced to $O_k(\mathbb{R})$; see for instance the proof of Theorem 5.10, in [**BH07**]. The existence of the complex anti-linear involution ν on E with the desires properties in (4.17) follows.

In this way, we obtain a fiberwise positive definite Hermitian form on E:

$$h(e_1, e_2) := b(e_1, \nu e_2), \tag{4.18}$$

which is compatible with the bilinear form b, by means of the complex-antilinear involution ν. Remark that, in general,

$$h^{-1}(\nabla^E\, h) = \nu^{-1}\left(b^{-1}(\nabla^E\, b)\right)\nu + \nu^{-1}(\nabla^E\, \nu)$$

since we do not require $\nabla^E \nu = 0$.

Thus, for a specified involution ν, we end up with a Hermitian form on $\Omega(M;E)$ that is compatible with $\beta_{g,b}$ in the sense

$$\ll v, w \gg_{g,h} = \beta_{g,b}(v, \nu w) \tag{4.19}$$

for $v, w \in \Omega(M;E)$. In [**SZ08**] and [**Su09**], given a bilinear form b, this involution has been exploited to study the bilinear Laplacian in terms of the Hermitian one associated to the compatible Hermitian form in (4.18), in both cases with and without boundary. However, our approach is a little different since we do not use a Hermitian form globally compatible with $\beta_{g,b}$ on $\Omega(M;E)$, but instead a local compatibility only, see section 4.4 below.

We now study the situation where ν is parallel with respect to ∇^E.

Lemma 4.3.1. *Let us consider $(M, \partial_+ M, \partial_- M)$ the compact Riemannian bordism together with the complex flat vector bundle E as above. Suppose E admits a nondegenerate symmetric bilinear form. Moreover, suppose there exists a complex anti-linear involution ν on E, satisfying the conditions in (4.17) and $\nabla^E \nu = 0$. Let h be the (positive definite) Hermitian form on E compatible with b defined by (4.18). Then, $\Delta_{E,g,b} = \Delta_{E,g,h}$ and $\mathcal{B}_{E,g,b} = \mathcal{B}_{E,g,h}$.*

Proof. Consider $\ll \cdot, \cdot \gg_{g,h}$ the Hermitian product on $\Omega(M;E)$ given by (4.19) and $d^*_{E,g,h}$, the formal adjoint to d_E with respect to this product, which in terms of the Hodge $\star$-operator can be written up to a sign as

$$d^*_{E,g,h} = \pm \star_h^{-1} d_E \star_h .$$

Remark that $\nabla^E \nu = 0$ implies that $d_E \nu = \nu d_E$ and hence, with $\star_h = \nu \circ \star_b$, we have

$$d^*_{E,g,h} = \pm \star_h^{-1} d_E \star_h = \pm \star_b^{-1} \nu^{-1} d_E \nu \star_b = \pm \star_b^{-1} d_E \star_b = d^\sharp_{E,g,b}, \tag{4.20}$$

and therefore the Hermitian and bilinear Laplacians coincide. We turn to the assertion for the corresponding boundary operators. On the one hand, the assertion is clear for $\mathcal{B}_{-E,g,b} = \mathcal{B}_{-E,g,h}$, because of (4.20) and (3.16). On the other hand, by using $\mathsf{int}_{\varsigma_{\mathsf{in}}}$, the (interior product) contraction along the vector field ς_{in}, for $v \in \Omega^p(M;E)$ the identity

$$\star_b^{\partial} i_+^* \mathsf{int}_{\varsigma_{\mathsf{in}}} v = i_+^* \star_b^M v$$

holds. Therefore the operator specifying absolute boundary can be written, independently of the Hermitian or bilinear forms, as

$$\mathcal{B}{+}^{p}_{E,g,b}v = (i_+^*\mathsf{int}_{\varsigma_{\mathsf{in}}}v, (-1)^{p+1}i_+^*\mathsf{int}_{\varsigma_{\mathsf{in}}}(d_E v)) = \mathcal{B}{+}^{p}_{E,g,h}v.$$

That finishes the proof. □

Lemma 4.3.2. *Let (M, g) be a compact Riemannian manifold and E a flat complex vector bundle over M. Assume E is endowed with a fiberwise nondegenerate symmetric bilinear form b. For each $x \in M$ there exist an open neighborhood U of x in M, a parallel anti-linear involution ν on $E|_U$ and a symmetric bilinear form $\widetilde{b}$ on E such that, for $z \in \mathbb{C}$, the family of fiberwise symmetric bilinear forms*

$$b_z := b + z\widetilde{b}, \tag{4.21}$$

has the following properties.

(i) *b_z is fiberwise nondegenerate for all $z \in \mathbb{C}$ with $|z| \leqslant \sqrt{2}$,*
(ii) *$\overline{b_{s-\mathbf{i}}(\nu e_1, \nu e_2)} = b_{s-\mathbf{i}}(e_1, e_2)$, for all $s \in \mathbb{R}$ and $e_i \in E|_U$,*
(iii) *$b_{s-\mathbf{i}}(e, \nu e) > 0$ for all $s \in \mathbb{R}$, $|s| \leqslant 1$ and $0 \neq e \in E|_U$.*

Proof. Since flat vector bundles are locally trivial, there exists a neighborhood V of x and a parallel complex anti-linear involution ν on $E|_V$. Moreover, since b is nondegenerate and ν a complex antilinear involution, we can assume without loss of generality that ν can be chosen to be compatible with b at the fiber E_x over x, such that

$$b_x(\nu e_1, \nu e_2) = \overline{b_x(e_1, e_2)} \quad \text{for all} \quad e_i \in E_x$$

and

$$b_x(\nu e, e) > 0 \quad \text{for all} \quad 0 \neq e \in E_x.$$

Consider

$$\begin{aligned} b^{\mathsf{Re}}(e_1, e_2) &:= \tfrac{1}{2}\left(b(e_1, e_2) + \overline{b(\nu e_1, \nu e_2)}\right), \\ b^{\mathsf{Im}}(e_1, e_2) &:= \tfrac{1}{2\mathbf{i}}\left(b(e_1, e_2) - \overline{b(\nu e_1, \nu e_2)}\right), \end{aligned}$$

as symmetric bilinear forms on $E|_V$. In particular, note that by construction

$$b|_V = b^{\mathsf{Re}} + \mathbf{i}b^{\mathsf{Im}} \quad \text{with} \quad b^{\mathsf{Im}}|_{E_x} = 0, \tag{4.22}$$

$$\overline{b^{\mathsf{Re}}(\nu e_1, \nu e_2)} = b^{\mathsf{Re}}(e_1, e_2) \quad \text{and} \quad \overline{b^{\mathsf{Im}}(\nu e_1, \nu e_2)} = b^{\mathsf{Im}}(e_1, e_2), \tag{4.23}$$

for all $e_i \in E|_V$. Now, choose an open neighborhood $U \subset V$ of x and a compactly supported smooth function $\lambda : V \to [0, 1]$ such that $\lambda|_U = 1$. Thus, by extending λ by zero to M, we set

$$\widetilde{b} := \lambda b^{\mathsf{Im}}, \tag{4.24}$$

as a globally defined symmetric bilinear form on E. Remark here that $\tilde{b}$ is not fiberwise nondegenerate on E. Using

$$b_{s-\mathbf{i}}|_U = \left(b + (s-\mathbf{i})\widetilde{b}\right)|_U = b|_U + (s-\mathbf{i})b^{\mathsf{Im}}|_U = b^{\mathsf{Re}}|_U + sb^{\mathsf{Im}}|_U$$

and (4.23) we immediately obtain (ii). In turn, (ii) implies

$$\overline{b_{s-\mathbf{i}}(\nu e, e)} = b_{s-\mathbf{i}}(\nu e, e)$$

and hence $b_{s-\mathbf{i}}(\nu e, e)$ is real for all $s \in \mathbb{R}$ and $e \in E|_U$. Finally, by the formula (4.21) defining b_z at x, we have $b^{\mathsf{Im}}|_x = 0$ and therefore

- $b_z|_x$ is nondegenerate,
- $b_{s-\mathbf{i}}|_x(\nu e, e) = b|_x(\nu e, e) > 0$ for all $0 \neq e \in E_x$,

from which (i) (resp. (iii)) follows by taking $|z| \leqslant \sqrt{2}$ (resp. $|s| \leqslant 1$) and then choosing the support of λ small enough around x. □

The following Proposition provides the key argument in the proof of Theorem 4.4.3.

Proposition 4.3.3. *Recall Notation 3.2.9. For the bordism $(M, \partial_+ M, \partial_- M)$ consider the boundary value problem $[\Delta, \Omega_{\mathcal{B}}]^{E,g,b}_{(M,\partial_+M,\partial_-M)}$ specified by the bilinear Laplacian under absolute/relative boundary conditions. Then, for each $x \in M$, there exist $\{b_z\}_{z\in\mathbb{C}}$ a family of fiberwise symmetric bilinear forms on E, and $\{h_s\}_{s\in\mathbb{R}}$ a family of fiberwise sesquilinear Hermitian forms on E such that*

(i) b_z is fiberwise nondegenerate for all $z \in \mathbb{C}$ such that $|z| \leqslant \sqrt{2}$.
(ii) h_s is fiberwise positive definite Hermitian form for all $s \in \mathbb{R}$ with $|s| \leqslant 1$.
(iii) For each $s \in \mathbb{R}$ with $|s| \leqslant 1$, consider $[\Delta, \Omega_{\mathcal{B}}]^{E,g,h_s}_{(M,\partial_+M,\partial_-M)}$ the corresponding Hermitian boundary value problem. Then, there exists a neighborhood U of x such that

$$\Delta_{E,g,b_{s-\mathbf{i}}}|_U = \Delta_{E,g,h_s}|_U \quad \text{and} \quad \mathcal{B}_{E,g,b_{s-\mathbf{i}}}|_U = \mathcal{B}_{E,g,h_s}|_U.$$

Proof. By Lemma 4.3.2.(i), for each $x \in M$, there exists a globally defined fiberwise symmetric bilinear form $\widetilde{b}$ on E such that the formula $b_z := b + z\widetilde{b}$ in (4.21) defines a family of fiberwise nondegenerate symmetric bilinear forms on E, satisfying the required property in (i). In addition, we know that for each $x \in M$, there exist an open neighborhood V of x and a parallel complex anti-linear involution ν on $E|_V$. By Lemma 4.3.2.(i)-(ii), we also know that we can find $U \subset V$ a small enough open neighborhood of x, such that $b_{s-\mathbf{i}}$ satisfies the conditions (i) and (ii) on $E|_U$, for $|s| \leqslant 1$. Hence, by using the formula in (4.18), we obtain a fiberwise positive definite Hermitian form compatible with $b_{s-\mathbf{i}}$ on $E|_U$ given by

$$h^U_s(e_1, e_2) := b_{s-\mathbf{i}}(\nu e_1, e_2).$$

Now we extend h^U_s to a (positive definite) Hermitian form on E as follows. We take h' any arbitrary Hermitian form on E and consider the finite open covering $\{U'_0, U'_1 \ldots, U'_N\}$

of M, with $U_0' := U$, together with a subordinate partition of unity $\{f_j\}_{U_j'}$. If $h_j' := h'|_{U_j}$, then

$$h_s := f_0 h_s^U + \sum_{j=1}^{N} f_j h_j'$$

globally defines a fiberwise positive definite Hermitian form on E, as the space of Hermitian forms on E is a convex space. This proves (ii). Then, (iii) follows from Lemma 4.3.1. $\square$

4.4. Heat trace asymptotic expansion for the bilinear Laplacian

In this section we finally are able to compute the coefficient of the constant term in the heat trace asymptotic expansion corresponding to the bilinear Laplacian under absolute/relative boundary conditions.

Lemma 4.4.1. *Let O be an open connected subset in $\mathbb{C}$ and $\{z \mapsto b_z\}_{z\in U}$ a holomorphic family of fiberwise nondegenerate symmetric bilinear forms on E. Recall Notation 3.2.9. For the bordism $(M, \partial_+M, \partial_-M)$ consider the family $\{[\Delta, \Omega_{\mathcal{B}}]^{E,g,b_z}_{(M,\partial_+M,\partial_-M)}\}_{z\in O}$, of boundary value problems for the corresponding bilinear Laplacians under absolute/relative boundary conditions, together with their L^2-realizations denoted by $\Delta_{\mathcal{B},b_z}$. For each $\psi \in \mathrm{End}(\Lambda T^*M \otimes E)$ and Notation 4.1.3, consider the coefficient of the constant term in the heat trace asymptotic expansion associated to ψ and $\Delta_{\mathcal{B}}$. Then, the map*

$$z \mapsto \operatorname*{LIM}_{t\to 0}\left(\mathrm{Tr}_{\mathbf{s}}\left(\psi e^{-t\Delta_{\mathcal{B},b_z}}\right)\right)$$

is holomorphic on O.

Proof. The value of $\mathrm{LIM}_{t\to 0}\left(\mathrm{Tr}_{\mathbf{s}}\left(\psi e^{-t\Delta_{\mathcal{B},b_z}}\right)\right)$ is computed by using the formula (4.3), which in turn requires the knowledge of the locally computable endomorphism invariants $\mathfrak{e}_m(\Delta_{E,g,b_z})$ and $\mathfrak{e}_{m,k}(\Delta_{E,g,b_z}, \mathcal{B}_{E,g,b_z})$. By compactness, we can assume without loss of generality that ψ is compactly supported in the interior of a sufficiently small open set U in M (or a collar neighborhood of ∂M in M). For each $z \in O$, we denote by ∇_z^{Δ} the connection on E and by E_z^{Δ} bundle endomorphism on E invariantly describing the Laplace type operator $\Delta_z := \Delta_{E,g,b_z}$, see Lemma 2.1.2; whereas S_z, χ_z and $\mathsf{w}_{\varsigma_{\mathsf{in}},z}$ indicate the bundle endomorphisms on $\Lambda T^*M \otimes E$ invariantly describing the absolute/relative boundary operators $\mathcal{B}_z := \mathcal{B}_{E,g,b_z}$ over a collar neighborhood near the boundary, see Section 2.2.4, Proposition 3.2.12 in Section 3.2.3 and (4.5). Moreover we denote by R the Riemann curvature tensor and by R_z^{Δ} the curvature of ∇_z^{Δ}. Recall Notation 4.1.4 in Section 4.1.2. By Proposition 4.1.5, $\mathfrak{e}_m(\Delta_z)$ are locally computable as universal polynomials in the variables $\mathsf{R}_{i_1i_2i_3i_4}$, $\mathsf{R}^{\Delta}_{z\,i_1i_2}$, E_z^{Δ} and finite number of their covariant derivatives, whereas the endomorphisms $\mathfrak{e}_{m,k}(\Delta_z, \mathcal{B}_z)$ are locally computable as universal polynomial in the variables $\mathsf{R}_{i_1i_2i_3i_4}$, L_{ab}, $\mathsf{R}^{\Delta}_{z\,i_1i_2}$, E_z^{Δ}, S_z, $\chi_z, \mathsf{w}_{\varsigma_{\mathsf{in}},z}$ and finite number of their covariant derivatives. Now remark that for $z \in O$, the function $z \mapsto b_z^{-1}$ is holomorphic, since the bilinear form b_z is nondegenerate and $z \mapsto b_z$ is holomorphic for $z \in O$. Then, by construction of the bilinear Laplacian and the boundary operators imposing absolute/relative boundary conditions, see Definition 3.2.3 and Definition 3.2.6, the assignments $z \mapsto \Delta_{E,g,b_z}$ and $z \mapsto \mathcal{B}_{E,g,b_z}$ are holomorphic in $z \in O$, since the mappings $z \mapsto \star \otimes b_z$, $z \mapsto \nabla^E b_z$ and $z \mapsto \star^{-1} \otimes b_z^{-1}$ are holomorphic. Therefore, the coefficients of the symbols of these operators are holomorphic functions of $z \in O$. In turn, the quantities E_z^{Δ}, S_z, χ_z, $\mathsf{w}_{\varsigma_{\mathrm{in}},z}$ and their covariant derivatives depend holomorphically on the coefficients of the symbols of Δ_{E,g,b_z} and $\mathcal{B}_{E,g,b_z}$ on O, see (2.2), (3.27), (3.28), (3.43) (3.46) and (3.49). Thus, the family $z \mapsto (\mathsf{E}_z^{\Delta}, \chi_z, \mathsf{S}_z, \mathsf{w}_{\varsigma_{\mathrm{in}},z})$ is holomorphic on O. This shows that the mappings $z \mapsto \mathrm{Tr}_{\mathbf{s}x}(\mathfrak{e}_m(\Psi, \Delta_z)_x)$ and $z \mapsto \mathrm{Tr}_{\mathbf{s}x}(\mathfrak{e}_{m,k}(\Psi, \Delta_z, \mathcal{B}_z)_x)$ are holomorphic on O for each $x \in U$. Finally, since the integral of a function depending holomorphically on a parameter z, also depends holomorphically on z, the function $z \mapsto \mathrm{LIM}_{t\to 0}\left(\mathrm{Tr}_{\mathbf{s}}\left(\psi e^{-t\Delta_{\mathcal{B},b_z}}\right)\right)$ depends holomorphically on $z \in O$; this is a consequence of Morera's Theorem, in the sense that uniform limits on compact sets of holomorphic functions are holomorphic, see for instance Chapter IV, Section 6 in [**Ga01**]. $\square$

Remark 4.4.2. The proof of Lemma 4.4.1 uses Proposition 4.1.5, in which the coefficients of the asymptotic expansion are locally computable as universal polynomials in tensorial variables as the curvature, second fundamental form, etc. Proposition 4.1.5 has been proved by using invariance theory as by Gilkey in [**Gi84**] and [**Gi04**]. However one could avoid the use of invariant theory completely by using immediately the results of from [**Se69a**], [**Se69b**] and [**Gre71**]. Indeed, the heat trace asymptotic coefficients can be computed inductively by using explicit formulas as a universal polynomial in terms of (finite number of the derivatives of) the coefficients of the symbol of Δ_{E,g,b_z}, whenever these are given in local coordinates around at $x \in M$, see Theorem 3 in [**Se69b**], formulas (3)-(6) and Lemma 1 in [**Se69a**], see also Section 2.6 in [**Gre71**]. In the same way, since $\mathfrak{e}_{m,k}(\Delta_{E,g,b_z}, \mathcal{B}_{E,g,b_z})$ are locally computable endomorphism invariants on the boundary, the value of $\mathrm{Tr}_{\mathbf{s}y}((\nabla_{\varsigma_{\mathrm{in}}}{}^k\psi)_y \cdot \mathfrak{e}_{m,k}(\Delta_{E,g,b_z}, \mathcal{B}_{E,g,b_z})_y)$ is expressible, by solving certain systems of ordinary differential equations inductively, as a universal polynomial in terms of (finite number of the derivatives of) the coefficients of the symbols of Δ_{E,g,b_z} and $\mathcal{B}_{E,g,b_z}$, whenever these are given in local coordinates around at $y \in \partial M$, see Theorem 3 in [**Se69b**], formulas (9)-(14) and Lemma 2 in [**Se69a**], see also Section 2.6 in [**Gre71**]. This is the way, we have proceed in [**Ma12**] in order to avoid the use of invariant theory, providing a direct proof of Lemma 4.4.1. We express our acknowledgments to the anonymous referees for having pointed this out. However, in this thesis we decided to keep the use of invariant theory *à la Gilkey* to give a better understanding of the structure of the coefficients in the heat trace asymptotic expansion in terms of the geometric invariants involved in our problem.

Theorem 4.4.3. *Recall Notations 3.2.9 and 4.2.1. For the bordism $(M,\partial_+M,\partial_-M)$, consider the boundary value problem $[\Delta,\Omega_{\mathcal{B}}]^{E,g,b}_{(M,\partial_+M,\partial_-M)}$ with its L^2-realization $\Delta_{\mathcal{B},b}$. If ϕ, ξ and Ψ are as in Proposition 4.2.2, then*

$$\begin{aligned}\mathrm{LIM}_{t\to 0}\left(\mathrm{Tr}_{\mathbf{s}}\left(\phi e^{-t\Delta_{\mathcal{B},b}}\right)\right) &= \int_M \mathrm{Tr}(\phi)\mathbf{e}(M,g)\\ &\quad+(-1)^{m-1}\int_{\partial_+M}\mathrm{Tr}(\phi)i_+^*\mathbf{e_b}(\partial M,g)-\int_{\partial_-M}\mathrm{Tr}(\phi)i_-^*\mathbf{e_b}(\partial M,g),\end{aligned}\tag{4.25}$$

and

$$\begin{aligned}\mathrm{LIM}_{t\to 0}\left(\mathrm{Tr}_{\mathbf{s}}\left(-\Psi e^{-t\Delta_{\mathcal{B},b}}\right)\right) &= -2\int_M \tfrac{\partial}{\partial\tau}\big|_{\tau=0}\widetilde{\mathbf{e}}(M,g,g+\tau g\xi)\wedge\omega(\nabla^E,b)\\ &\quad-2\int_{\partial_+M}\tfrac{\partial}{\partial\tau}\big|_{\tau=0}i_+^*\widetilde{\mathbf{e}}_{\mathbf{b}}(\partial M,g,g+\tau g\xi)\wedge\omega(\nabla^E,b)\\ &\quad+\mathsf{rank}(E)\int_{\partial_+M}\tfrac{\partial}{\partial\tau}\big|_{\tau=0}i_+^*B(\partial M,g+\tau g\xi)\\ &\quad-2(-1)^m\int_{\partial_-M}\tfrac{\partial}{\partial\tau}\big|_{\tau=0}i_-^*\widetilde{\mathbf{e}}_{\mathbf{b}}(\partial M,g,g+\tau g\xi)\wedge\omega(\nabla^E,b)\\ &\quad+(-1)^{m+1}\mathsf{rank}(E)\int_{\partial_-M}\tfrac{\partial}{\partial\tau}\big|_{\tau=0}i_-^*B(\partial M,g+\tau g\xi).\end{aligned}\tag{4.26}$$

Proof. By compactness of M, it suffices to show that each point $x\in M$ admits a neighborhood U so that the formulas above hold for all ϕ with $\mathsf{supp}(\phi)\subset U$ and ξ with $\mathsf{supp}(\xi)\subset U$. For each $x\in M$, choose

$$b_z=b+z\widetilde{b},\quad h_s\quad\text{and}\quad U$$

as in Proposition 4.3.3, with $\mathsf{supp}(\phi)\subset U$. By Proposition 4.3.3 (iii), we obtain

$$\underset{t\to 0}{\mathrm{LIM}}\,\mathrm{Tr}_{\mathbf{s}}\left(\phi e^{-t\Delta_{\mathcal{B},b_{s-\mathbf{i}}}}\right)=\underset{t\to 0}{\mathrm{LIM}}\,\mathrm{Tr}_{\mathbf{s}}\left(\phi e^{-t\Delta_{\mathcal{B},h_s}}\right),$$

for all $|s|\leqslant 1$, for these quantities depend on the geometry over U only. From Theorem 4.2.7, we have

$$\begin{aligned}\mathrm{LIM}_{t\to 0}\,\mathrm{Tr}_{\mathbf{s}}\left(\phi e^{-t\Delta_{\mathcal{B},b_{s-\mathbf{i}}}}\right) &= \int_M\mathrm{Tr}(\phi)\mathbf{e}(M,g)+(-1)^{m-1}\int_{\partial_+M}\mathrm{Tr}(\phi)i_+^*\mathbf{e_b}(\partial M,g)\\ &\quad-\int_{\partial_-M}\mathrm{Tr}(\phi)i_-^*\mathbf{e_b}(\partial M,g)\end{aligned}$$

for all $|s|\leqslant 1$. Now, since the function $z\mapsto \mathrm{LIM}_{t\to 0}\,\mathrm{Tr}_{\mathbf{s}}\left(\phi e^{-t\Delta_{\mathcal{B},b_z}}\right)$ depends holomorphically on z (see Lemma 4.4.1), that the right hand side of the equality above is constant in z, and that the domain of definition of z contains an accumulation point, these formulas are extended by analytically continuation to

$$\begin{aligned}\mathrm{LIM}_{t\to 0}\,\mathrm{Tr}_{\mathbf{s}}\left(\phi e^{-t\Delta_{\mathcal{B},b_z}}\right) &= \int_M\mathrm{Tr}(\phi)\mathbf{e}(M,g)+(-1)^{m-1}\int_{\partial_+M}\mathrm{Tr}(\phi)i_+^*\mathbf{e_b}(\partial M,g)\\ &\quad-\int_{\partial_-M}\mathrm{Tr}(\phi)i_-^*\mathbf{e_b}(\partial M,g),\end{aligned}$$

for all $|z| \leqslant \sqrt{2}$. After setting $z=0$ we obtain the desired identity in (4.25). Similarly, take ξ with $\mathsf{supp}(\xi) \subset U$, using Proposition 4.3.3 (iii), we obtain

$$\operatorname*{LIM}_{t\to 0} \mathrm{Tr}_{\mathbf{s}}\left(-\Psi e^{-t\Delta_{\mathcal{B},b_{s-\mathbf{i}}}}\right) = \operatorname*{LIM}_{t\to 0} \mathrm{Tr}_{\mathbf{s}}\left(-\Psi e^{-t\Delta_{\mathcal{B},h_s}}\right) \tag{4.27}$$

for all $|s| \leqslant 1$, for these quantities depend on the geometry over U only. Then, we apply Theorem 4.2.7 to the right hand side of the equality above and (4.27) is equivalent to

$$\begin{aligned} \mathrm{LIM}_{t\to 0} \mathrm{Tr}_{\mathbf{s}}\left(-\Psi e^{-t\Delta_{\mathcal{B},b_{s-\mathbf{i}}}}\right) &= -2\int_M \tfrac{\partial}{\partial\tau}\big|_{\tau=0} \widetilde{\mathbf{e}}(M,g,g+\tau g\xi)\wedge\omega(\nabla^E,b_{s-\mathbf{i}}) \\ &\quad -2\int_{\partial_+ M} \tfrac{\partial}{\partial\tau}\big|_{\tau=0} i_+^*\widetilde{\mathbf{e}}_{\mathbf{b}}(\partial M,g,g+\tau g\xi)\wedge\omega(\nabla^E,b_{s-\mathbf{i}}) \\ &\quad +\mathsf{rank}(E)\int_{\partial_+ M} \tfrac{\partial}{\partial\tau}\big|_{\tau=0} i_+^* B(\partial M,g+\tau g\xi) \\ &\quad -2(-1)^m\int_{\partial_- M} \tfrac{\partial}{\partial\tau}\big|_{\tau=0} i_-^*\widetilde{\mathbf{e}}_{\mathbf{b}}(\partial M,g,g+\tau g\xi)\wedge\omega(\nabla^E,b_{s-\mathbf{i}}) \\ &\quad +(-1)^{m+1}\mathsf{rank}(E)\int_{\partial_- M} \tfrac{\partial}{\partial\tau}\big|_{\tau=0} i_-^* B(\partial M,g+\tau g\xi), \end{aligned} \tag{4.28}$$

for all $|s| \leqslant 1$. Now, on the one hand the function $z \mapsto \mathrm{LIM}_{t\to 0} \mathrm{Tr}_{\mathbf{s}}\left(\phi e^{-t\Delta_{\mathcal{B},b_z}}\right)$ on the left of (4.28) depends holomorphically on z see Lemma 4.4.1. On the other hand the long expression on the right hand side of the equality above in (4.28) is also a holomorphic function in $z \in \mathbb{C}$ with $|z| \leqslant \sqrt{2}$, since it can be formally considered as the composition of constant functions (in z) and the function

$$z \mapsto \omega(\nabla^E, b_z) = -\frac{1}{2}\mathrm{Tr}(b_z^{-1}\nabla^E b_z),$$

which is holomorphic, since by Proposition 4.3.3 the bilinear form b_z in (4.21) is fiberwise nondegenerate for $|z| \leqslant \sqrt{2}$. Then the identity in (4.28) can be analytically extended to

$$\begin{aligned} \mathrm{LIM}_{t\to 0} \mathrm{Tr}_{\mathbf{s}}\left(-\Psi e^{-t\Delta_{\mathcal{B},b_{z-\mathbf{i}}}}\right) &= -2\int_M \tfrac{\partial}{\partial\tau}\big|_{\tau=0} \widetilde{\mathbf{e}}(M,g,g+\tau g\xi)\wedge\omega(\nabla^E,b_{z-\mathbf{i}}) \\ &\quad -2\int_{\partial_+ M} \tfrac{\partial}{\partial\tau}\big|_{\tau=0} i_+^*\widetilde{\mathbf{e}}_{\mathbf{b}}(\partial M,g,g+\tau g\xi)\wedge\omega(\nabla^E,b_{z-\mathbf{i}}) \\ &\quad +\mathsf{rank}(E)\int_{\partial_+ M} \tfrac{\partial}{\partial\tau}\big|_{\tau=0} i_+^* B(\partial M,g+\tau g\xi) \\ &\quad -2(-1)^m\int_{\partial_- M} \tfrac{\partial}{\partial\tau}\big|_{\tau=0} i_-^*\widetilde{\mathbf{e}}_{\mathbf{b}}(\partial M,g,g+\tau g\xi)\wedge\omega(\nabla^E,b_{z-\mathbf{i}}) \\ &\quad +(-1)^{m+1}\mathsf{rank}(E)\int_{\partial_- M} \tfrac{\partial}{\partial\tau}\big|_{\tau=0} i_-^* B(\partial M,g+\tau g\xi), \end{aligned} \tag{4.29}$$

for $z \in \mathbb{C}$ with $|z-\mathbf{i}| \leqslant \sqrt{2}$. Finally (4.26) follows from setting $z=i$ into (4.29) and then $b_0 = b$ follows from (4.21). □

CHAPTER 5

Complex-valued analytic torsions on compact bordisms

In this chapter we study the complex-valued Ray–Singer torsion on compact bordisms. We derive anomaly formulas expressing the variation of the torsion with respect to infinitesimal variation of the Riemannian metric and bilinear form.

This chapter is organized as follows. In Section 5.1, see Definition 5.1.1, we use the theory developped in Section 3.3 to define the complex-valued Ray–Singer torsion on a compact bordism, as a nondegenerate bilinear form on the determinant line $\det H(M, \partial_- M; E)$. In fact, this is based on the main result from Section 3.3.8, which allows us to compute the (relative) cohomology groups $H(M, \partial_- M; E)$ by looking at the generalized 0-eigen-space of $\Delta_{E,g,b}$ and subsequently permits us to obtain a nondegenerate bilinear form on $\det H(M, \partial_- M; E)$. In section 5.2, we obtain anomaly formulas for the complex-valued Ray–Singer torsion on a compact bordism, see Theorem 5.2.1. We prove this result with the approach already used in the case of a closed manifold in [**BH07**]. That is, the computation of the logarithmic derivative of the complex-valued Ray–Singer torsion is based on the knowledge of the coefficient of the constant term in the heat kernel asymptotic expansion corresponding to the bilinear Laplacian under absolute/relative boundary conditions. These coefficients were computed in Theorem 4.4.3 in Chapter 4.

5.1. Torsion on compact bordisms

In Section 5.1.1, the reader can find basic notions on finite dimensional graded complexes and their determinant lines. Some linear algebra to define the complex-valued analytic torsion is out-lined. We see how a given nondegenerate bilinear form on the graded complex determines a nondegenerate bilinear form on its determinant line. In Section 5.1.2 and Section 5.1.3, we use the results from Chapter 3 to obtain ζ-regularized determinants for the bilinear Laplacian. In Section 5.1.4, see Definition 5.1.4, we finally extend the definition of the complex-valued Ray–Singer torsion to the situation of a compact bordism.

5.1.1. Torsion on finite dimensional graded complexes. Let V be a finite dimensional complex vector space with $V' := \mathsf{Hom}(V; \mathbb{C})$ its dual. The determinant line of V is the top exterior product $\det V := \Lambda^{\mathsf{dim}(V)} V$. If $V^* = \oplus_{q=0} V^q$ is a finite dimensional graded complex vector space, then its graded determinant is defined by $\det V^* := \det V^{\mathsf{even}} \otimes \left(\det V^{\mathsf{odd}}\right)'$, where $V^{\mathsf{even}} := \oplus_{q=0} V^{2q}$ and $V^{\mathsf{odd}} := \oplus_{q=0} V^{2q+1}$ are ungraded vectors spaces. We collect certain well-known facts. Further details, as well as for determinant lines, can be found in [**BK07b**], [**BK07a**], [**BH10**] and [**KM76**].

Every short exact sequence $0 \to U^* \to V^* \to W^* \to 0$ of graded vector spaces, provides a canonic isomorphism of determinant lines

$$\det U^* \otimes \det W^* = \det V^*. \tag{5.1}$$

Moreover, there exists a canonic isomorphism

$$\det V^* \otimes \det V^{*+1} = \det V^* \otimes (\det V^*)' = \mathbb{C}. \tag{5.2}$$

For C^* a *finite dimensional* graded complex over $\mathbb{C}$ with differential $d : C^* \to C^{*+1}$, consider Z^* and B^* the sub-complexes of C^* consisting of cocycles in C^* and coboundaries in Z^* respectively. Let $H(C^*)$ be the associated cohomology groups. The complex C^* gives rise to the short exact sequences

$$0 \to B^* \to Z^* \to H(C^*) \to 0 \tag{5.3}$$

and

$$0 \to Z^* \to C^* \xrightarrow{d} B^{*+1} \to 0. \tag{5.4}$$

It follows from (5.3) and (5.1) that

$$\det B^* \otimes \det Z^* = \det H(C^*)$$

and from (5.4) that

$$\det Z^* \otimes \det C^* = \det B^{*+1}.$$

From (5.2), one gets a canonical identification

$$\det C^* = \det H(C^*). \tag{5.5}$$

In addition, consider (C^*, b) a complex C^* equipped with a a **graded nondegenerate symmetric bilinear form** b: *a nondegenerate symmetric bilinear form b, such that its restriction to each homogenous component C^q is a nondegenerate symmetric bilinear form and different homogenous components are b-orthogonal.* In turn, b induces a nondegenerate bilinear form on $\det C^*$ and by using the canonical isomorphism (5.5), one obtains a nondegenerate symetric bilinear form on $\det H(C^*)$ called the **torsion associated** to (C^*, b) and denoted by $\tau_{C^*,b}$.

5.1.2. Spectral cuts and Agmon's angle for the bilinear Laplacian. For $\theta \in [0, 2\pi]$, consider the complex ray $\mathsf{R}_\theta := \{\lambda \in \mathbb{C} | \mathsf{arg}(\lambda) = \theta\}$. From Lemma 3.2.3, we know for the symbolic spectrum $\mathsf{Spec}_\mathsf{L}(\Delta_{E,g,b}) \subset \mathbb{R}_+$, so that we can choose $\theta \in (0, 2\pi)$ with

$$\mathsf{R}_\theta \cap \mathsf{Spec}_\mathsf{L}(\Delta_{E,g,b}) = \emptyset; \tag{5.6}$$

in other words, for such θ, the operator $\sigma_L(\Delta_{E,g,b}) - \lambda$ is invertible for each $\lambda \in \mathsf{R}_\theta$. If θ is chosen so that (5.6) is satisfied, then R_θ is called **spectral cut** for $\sigma_L(\Delta_{E,g,b})$.

The operator $\Delta_{\mathcal{B}}$, the L^2-realization of $\Delta_{E,g,b}$, has compact resolvent and discrete spectrum consisting of eigen-values λ with finite algebraic multiplicity that accumulate at infinity only. But, from Proposition 2.3.7, we know more: *for each $\epsilon > 0$, there is a real number $R > 0$, large enough, with $\mathbb{B}_R(0)$, the closed ball centered in 0 of radius R, such that the set*

$$\mathsf{Spec}(\Delta_{\mathcal{B}}) \cap \mathbb{B}_R(0)$$

is finite and the rest of $\mathsf{Spec}(\Delta_{\mathcal{B}})$ *is entirely contained in the sector $\Lambda_{R,\epsilon}$, see (2.23).* Thus, there exist only finitely many points $\lambda \in \mathsf{Spec}(\Delta_{\mathcal{B}})$ with $\mathsf{Re}(\lambda) \leqslant 0$ and they all are contained in $\mathbb{B}_R(0)$.

Then we can choose an angle $\theta > 0$ and a conical neighborhood $\mathcal{R}_\theta$ of R_θ such that, for each complex ray $\mathsf{R}_{\theta'}$ contained in $\mathcal{R}_\theta$, we have

$$\mathsf{R}_{\theta'} \cap \mathsf{Spec}(\Delta_{\mathcal{B}}) \backslash \{0\} = \emptyset$$

and, for each large enough $S \geqslant R$, there exists a constant $C_{\mathsf{R}_{\theta'}} > 0$, for which

$$\|(\Delta_{\mathcal{B}} - \lambda)^{-1}\|_{\mathsf{L}^2} \leqslant C_{\mathsf{R}_{\theta'}} |\lambda|^{-1} \quad \text{for all } 0 \neq \lambda \in \mathsf{R}_{\theta'} \text{ with } |\lambda| \geqslant S,$$

see for instance Proposition 2.3.7. In the literature, see [**Agm65**], [**Se67**] and [**Se69b**], the ray R_θ is called of **minimimal growth** and θ an **Agmon's angle**.

5.1.3. Complex powers and ζ-regularized determinants. Further material related to the facts below can be found in [**Se69a**], [**Se69b**] (see also [**Gre71**] and [**Se67**], Section 8 in [**Agr97**] and more generally for Pseudo-differential operator, in Chapter 4, Section 4 in [**Gru96**]).

As in the preceding section, let $\theta \in (0, 2\pi)$ be an Agmon angle for the operator $\Delta_{\mathcal{B}}$. For $\lambda \in \mathbb{C}$, consider its complex powers with respect to the spectral cut R_θ; that is, the complex-valued function $\lambda \mapsto \lambda_\theta^{-z} := |\lambda|^{-z} e^{\mathbf{i}\cdot z \cdot \mathsf{arg}_\theta(\lambda)}$, where the argument $\mathsf{arg}_\theta(\lambda) \in (\theta - 2\pi, \theta)$ has been continuously determined on $\mathbb{C}\backslash\mathsf{R}_\theta$. In view of the discreteness of the spectrum, we can fix a number $R > 0$ small enough such that there is no non-zero eigenvalue of $\Delta_{\mathcal{B}}$ in $\mathbb{B}_R(0)$. Since we do not assume here injectivity of $\Delta_{\mathcal{B}}$, we need take a bit of caution and we proceed as in [**Se69b**].

For the operator $\Delta_{\mathcal{B}}$, consider its generalized 0-eigenspace $\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)$. We denote by $\Delta'_{\mathcal{B}}$ the **restriction** of $\Delta_{\mathcal{B}}$ to the space $\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\perp_\beta}\,|_{\mathcal{B}}$ of smooth differential forms which satisfy boundary conditions and are $\beta_{g,b}$-orthogonal to $\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)$, according to Notation 3.2.7..

Then, for $z \in \mathbb{C}$ with $\mathsf{Re}(z) > 0$, **complex powers of $\Delta'_{\mathcal{B}}$ with respect to the spectral cut** R_θ, can be defined by the formula

$$\Delta'^{\,-z}_{\mathcal{B},\theta} := \left\{ \begin{array}{ccc} \frac{\mathbf{i}}{2\pi} \int_{\Gamma_\theta} \lambda_\theta^{-z} (\Delta_{\mathcal{B}} - \lambda)^{-1} d\lambda, & \text{on} & \Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\perp_\beta} \\ 0 & \text{on} & \Omega_{\Delta_{\mathcal{B}}}(M;E)(0), \end{array} \right. \tag{5.7}$$

wtih the countour

$$\Gamma_\theta := \{\rho e^{i\theta} | \infty > \rho \geqslant R\} \cup \{R e^{it} | \theta \geqslant t \geqslant \theta - 2\pi\} \cup \{\rho e^{i(\theta - 2\pi)} | R \leqslant \rho < \infty\}.$$

The operator ${\Delta'_{\mathcal{B}\,\theta}}^{-1}$ can be seen as a *partial inverse* of $\Delta_{\mathcal{B}}$ on $\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\perp_\beta}$: it provides an inverse to $\Delta_{\mathcal{B}}$ on $\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)^{\perp_\beta}$ and it vanishes on $\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)$. The complex powers in (5.7) have been defined by using functional calculus. The Cauchy integral in (5.7) converges in the L^2-norm, because of the estimates for the resolvent in the conical set $\mathcal{R}_\theta$, see Proposition 2.3.7, Proposition 2.3.5 and Corollary 2.3.6. Moreover, as in the situation of a manifold without boundary, (see Proposition 10.1 in [**Sh01**]), these estimates guarentee the semigroup property for the complex powers

$$
{\Delta'_{\mathcal{B},\theta}}^{-z-z'} = {\Delta'_{\mathcal{B},\theta}}^{-z}\,{\Delta'_{\mathcal{B},\theta}}^{-z'} \tag{5.8}
$$

for $z, z' \in \mathbb{C}$ with $\mathsf{Re}(z) > 0$ and $\mathsf{Re}(z') > 0$ as well as

$$
{\Delta'_{\mathcal{B},\theta}}^{-k} = \left({\Delta'_{\mathcal{B},\theta}}^{-1}\right)^{k},
$$

for $k \in \mathbb{Z}$, see [**Se69a**] and [**Se69b**] and Section 4.4 in [**Gru96**]. Moreover, for $\theta \in (-2\pi, 0)$ an Agmon angle for $\Delta_{\mathcal{B}}$ and $z \in \mathbb{C}$ with $\mathsf{Re}(z) > \mathsf{dim}(M)/2$, the operator in (5.7) is of *Trace class* and the function

$$
z \mapsto \mathrm{Tr}\,{\Delta'_{\mathcal{B},\theta}}^{-z} \tag{5.9}
$$

extends to a *meromorphic* function on the complex plane which is *holomorphic* at $z = 0$, see [**Se69a**], [**Se69b**] and [**Agr97**] (see also Corollary 4.4.8 in [**Gru96**], [**Gre71**], [**Se67**] and [**Wo87**]).

Definition 5.1.1. *For $\Delta_{E,g,b,q}$, the bilinear Laplacian in degree q, its ζ-regularized determinant is defined by*

$$
\det{}'\left(\Delta_{E,g,b,q}\right) := \exp\left(-\left.\frac{\partial}{\partial z}\right|_{z=0} \mathrm{Tr}\left((\Delta'_{\mathcal{B},q})_\theta^{-z}\right)\right).
$$

Remark 5.1.2. From Proposition 2.3.7, the function $\det'$ in Definition 5.1.1 does not depend on the choice of the Agmon's angle, see for instance section 6.11 in [**BK07a**] (see also [**Se67**] and [**Sh01**]).

Lemma 5.1.3. *Consider $\Delta_{E,g,b,q}$ the bilinear Laplacian in degree q. Then, the formula*

$$
\Pi_q\left(\det{}'(\Delta_{E,g,b,q})\right)^{(-1)^q} = 1
$$

holds.

Proof. Fix θ an Agmon's angle and a corresponding spectral cut, but we drop θ in the notation. It is enough to prove that

$$
\sum_q (-1)^q \log(\det{}'(\Delta_{E,g,b,q})) = 0.
$$

The term on the left above can be written as

$$\begin{aligned} \textstyle\sum_q(-1)^q \log\left(\det{}'(\Delta_{E,g,b,q})\right) &= -\textstyle\sum_q(-1)^q \frac{\partial}{\partial z}\big|_{z=0} \operatorname{Tr}\left({\Delta'_{\mathcal{B},q}}^{-z}\right) \\ &= -\frac{\partial}{\partial z}\big|_{z=0} \operatorname{Tr}_{\mathbf{s}}\left({\Delta'_{\mathcal{B},q}}^{-z}\right) \end{aligned}$$

Consider $\Delta'_{\mathcal{B}}$ as unbounded operator on $\mathsf{L}^2(M;E)$, with domain of definition $\mathcal{D}(\Delta'_{\mathcal{B}})$. We look at the Dirac operator $\mathbb{D} := d_E + d^\sharp_{E,g,b}$ considered as a bounded operator from $\mathsf{H}_1(M;E)$ to $\mathsf{L}^2(M;E)$ and remark that

(i) The operator $\mathbb{D}^2$ coincides with $\Delta'_{\mathcal{B}}$ on $\mathcal{D}(\Delta'_{\mathcal{B}})$,
(ii) The operator $\Delta'_{\mathcal{B}}$ commutes with $\mathbb{D}$ on $\mathcal{D}(\Delta'_{\mathcal{B}})$.

Then, by using (5.8), (i) and (ii) above, for $z \in \mathbb{C}$ with $\mathsf{Re}(z) > 0$ large enough, we have

$$\begin{aligned} {\Delta'_{\mathcal{B}}}^{-z} &= {\Delta'_{\mathcal{B}}}^{-z-1}\Delta'_{\mathcal{B}} \\ &= {\Delta'_{\mathcal{B}}}^{-z-1}\mathbb{D}^2 \\ &= 1/2\left[{\Delta'_{\mathcal{B}}}^{-z-1}\mathbb{D},\mathbb{D}\right] \\ &= 1/2\left[{\Delta'_{\mathcal{B}}}^{-z/2-1}\mathbb{D},{\Delta'_{\mathcal{B}}}^{-z/2}\mathbb{D}\right]. \end{aligned}$$

For $\mathsf{Re}(z) > 0$ large enough, each of the powers of $\Delta'_{\mathcal{B}}$ in the supercommutators in the last line on the right above are Trace class operators in the L^2-norm. Therefore, since $\mathbb{D}$ is bounded, each of the terms in the supercommutators above, and hence the supercommutator itself, are of Trace class. Finally, since $\operatorname{Tr}_{\mathbf{s}}$ vanishes on supercommutators, $\operatorname{Tr}_{\mathbf{s}}({\Delta'_{\mathcal{B}}}^{-z}) = 0$.

□

5.1.4. Complex-valued Ray–Singer Torsion on bordisms. In [**BH07**], a generalization of the Ray–Singer metric by considering a fiberwise nondegenerate symmetric bilinear form on a flat complex vector bundle over a closed Riemannian manifold was given by Burghelea and Haller. Here, we study the corresponding problem on the bordism $(M, \partial_+M, \partial_-M)$. With the work in Chapter 3, we are able to give a definition for the complex-valued analytic Ray–Singer for bordisms. Remember that the inclusion $\Omega(M;E)|_{\mathcal{B}^0_-} \subset \Omega(M;E)$ computes relative cohomology $H(M,\partial_-M;E)$. By Proposition 3.3.4, the space $\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)$ is a finite dimension subcomplex in $\Omega(M;E)|_{\mathcal{B}^0_-}$. The restriction of $\beta_{g,b}$ to $\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)$ is a nondegenerate symmetric bilinear form in view of Proposition 3.3.8. The linear algebra from Section 5.1.1 now applies to obtain a nondegenerate bilinear form on $\det H^*\left(\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)\right)$. Then, by Proposition 3.3.12, $\det H^*\left(\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)\right) \cong \det H^*\left(M,\partial_-M;E\right)$, and hence a nondegenerate symmetric bilinear form

$$[\tau(0)]^{E,g,b}_{(M,\partial_+M,\partial_-M)} : \det H^*\left(M,\partial_-M;E\right) \times \det H^*\left(M,\partial_-M;E\right) \to \mathbb{C} \tag{5.10}$$

is obtained.

Definition 5.1.4. *Let $(M, \partial_+M, \partial_-M)$ be a compact Riemannian bordism and E be a complex flat vector bundle over M. Assume E is endowed with a fiberwise non-degenerate symmetric bilinear form b. Consider the bilinear Laplacian $\Delta_{E,g,b}$ acting on smooth forms satisfying absolute/relative boundary conditions. Then, the* ***complex-valued Ray–Singer torsion*** *is the bilinear form on* $\det H(M, \partial_-M; E)$ *defined by*

$$[\tau^{\mathsf{RS}}]^{E,g,b}_{(M,\partial_+M,\partial_-M)} := [\tau(0)]^{E,g,b}_{(M,\partial_+M,\partial_-M)} \cdot \prod_q \left(\det{}'\left(\Delta_{E,g,b,q}\right)\right)^{(-1)^q q},$$

where $\det{}'\left(\Delta_{E,g,b,q}\right)$ *given in Definition 5.1.1 and* $[\tau(0)]^{E,g,b}_{(M,\partial_+M,\partial_-M)}$ *is the bilinear form in (5.10).*

The complex-valued analytic torsion in Definition 5.1.4 is defined by

$$[\tau^{\mathsf{RS}}(\gamma)]^{E,g,b}_{(M,\partial_+M,\partial_-M)} := [\tau(\gamma)]^{E,g,b}_{(M,\partial_+M,\partial_-M)} \cdot \prod_q \left(\det{}^{\gamma}\left(\Delta_{E,g,b,q}\right)\right)^{(-1)^q q}, \tag{5.11}$$

where $[\tau(\gamma)]^{E,g,b}_{(M,\partial_+M,\partial_-M)}$ is the induced bilinear form on $\det H(M, \partial_-M; E)$ obtained this time by considering the restriction of $\beta_{g,b}$ to the finite dimension subcochain complex

$$\Omega_{\Delta_{\mathcal{B}}}(M;E)(\gamma) := \bigoplus_{\lambda \in \mathcal{F}^{\gamma}(\mathsf{Sp}(\Delta_{\mathcal{B}}))} \Omega_{\Delta_{\mathcal{B}}}(M;E)(\lambda), \tag{5.12}$$

where

$$\mathcal{F}^{\gamma}(\mathsf{Sp}(\Delta_{\mathcal{B}})) := \mathsf{Spec}(\Delta_{\mathcal{B}}) \cap \mathsf{Int}(\gamma)$$

is the finite set containing all the eigenvalues of $\Delta_{\mathcal{B}}$ lying in $\mathsf{Int}(\gamma)$, the interior of a simple closed curve γ, around 0 and with $\mathsf{Spec}(\Delta_{\mathcal{B}}) \cap \gamma = \emptyset$.

Let $\Delta^{\gamma}_{\mathcal{B}}$ be the restriction of $\Delta_{\mathcal{B}}$ to the space of smooth differential forms that are $\beta_{g,b}$-orthogonal to (5.12) and satisfy boundary conditions. Then the (ζ, γ)-regularized determinant of $\Delta_{E,g,b}$ is defined by

$$\det{}^{\gamma}(\Delta_{E,g,b}) := \exp\left(-\left.\frac{\partial}{\partial z}\right|_{z=0} \operatorname{Tr}\left((\Delta^{\gamma}_{\mathcal{B}})^{-z}_{\theta}\right)\right) \tag{5.13}$$

Lemma 5.1.5. *Let $[\tau^{\mathsf{RS}}]^{E,g,b}_{(M,\partial_+M,\partial_-M)}$ and $[\tau^{\mathsf{RS}}(\gamma)]^{E,g,b}_{(M,\partial_+M,\partial_-M)}$ be the complex-valued bilinear forms given by Definition 5.1.4 and (5.11) respectively. Then*

$$[\tau^{\mathsf{RS}}]^{E,g,b}_{(M,\partial_+M,\partial_-M)} = [\tau^{\mathsf{RS}}(\gamma)]^{E,g,b}_{(M,\partial_+M,\partial_-M)}. \tag{5.14}$$

Proof. The (L^2-realization for the) bilinear Laplacian on the compact Riemannian bordism $(M, \partial_+M, \partial_-M)$ under absolute/relative boundary conditions, see (3.59), possesses the same spectral properties as the bilinear Laplacian on a closed manifold, studied by Burghela and Haller. Therefore, the proof of Proposition 4.7 in **[BH07]** still holds in this situation. □

5.2. Anomaly formulas for the complex-valued Ray–Singer torsion

The following formulas generalize the ones obtained in [**BH07**] in the case without boundary and they are based on the corresponding ones for the Ray–Singer metric in [**BM06**]. They also coincide with the ones obtained by Su in odd dimensions, but they do not require that the smooth variations of g and b are compactly supported in the interior of M, see [**Su09**].

Theorem 5.2.1. *(**Anomaly formulas**) Recall Notation 4.2.1 Let $(M, \partial_+M, \partial_-M)$ be a Riemannian bordism and E be a complex flat vector bundle over M. Let g_u be a smooth one-parameter family of metrics on M and b_u a smooth one-parameter family of fiberwise nondegenerate symetric bilinear forms on E. We denote by $\dot{g}_u := \frac{\partial}{\partial u} g_u$ and $\dot{b}_u := \frac{\partial}{\partial u} b_u$ the corresponding infinitesimal variations. Let $[\tau^{\mathsf{RS}}]^{E,g_u,b_u}_{(M,\partial_+M,\partial_-M)}$ the associated family of complex-valued analytic torsions, see Definition 5.1.4. Then, we have the following logarithmic derivative*

$$\frac{\partial}{\partial w}\bigg|_u \left(\frac{[\tau^{\mathsf{RS}}]^{E,g_w,b_w}_{(M,\partial_+M,\partial_-M)}}{[\tau^{\mathsf{RS}}]^{E,g_u,b_u}_{(M,\partial_+M,\partial_-M)}} \right)^2 = \mathbf{E}(b_u, g_u) + \widetilde{\mathbf{E}}(b_u, g_u) + \mathbf{B}(g_u),$$

where

$$\begin{aligned}
\mathbf{E}(b_u, g_u) \quad &:= \quad \int_M \operatorname{Tr}(b_u^{-1}\dot{b}_u)\mathbf{e}(M, g) + (-1)^{m-1}\int_{\partial_+M} \operatorname{Tr}(b_u^{-1}\dot{b}_u)\mathbf{e_b}(\partial M, g_u) \\
&\qquad - \int_{\partial_-M} \operatorname{Tr}(b'^{-1}_u\dot{b}'_u))\mathbf{e_b}(\partial M, g_u), \\
\widetilde{\mathbf{E}}(b_u, g_u) \quad &:= \quad -2\int_M \tfrac{\partial}{\partial t}\big|_{t=0} \widetilde{\mathbf{e}}(M, g_u, g_u + t\dot{g}_u) \wedge \omega(\nabla^E, b_u) \\
&\qquad -2\int_{\partial_+M} \tfrac{\partial}{\partial t}\big|_{t=0} i_+^*\widetilde{\mathbf{e}}_\mathbf{b}(\partial M, g_u, g_u + t\dot{g}_u) \wedge \omega(\nabla^E, b_u) \\
&\qquad -2(-1)^m\int_{\partial_-M} \tfrac{\partial}{\partial t}\big|_{t=0} i_-^*\widetilde{\mathbf{e}}_\mathbf{b}(\partial M, g_u, g_u + t\dot{g}_u) \wedge \omega(\nabla^E, b_u), \\
\mathbf{B}(g_u) \quad &:= \quad \mathsf{rank}(E)\int_{\partial_+M} \tfrac{\partial}{\partial t}\big|_{t=0} i_+^* B(\partial M, g, g_u + t\dot{g}_u) \\
&\qquad +(-1)^{m+1}\mathsf{rank}(E)\int_{\partial_-M} \tfrac{\partial}{\partial t}\big|_{t=0} i_-^* B(\partial M, g, g_u + t\dot{g}_u),
\end{aligned}$$

and

$$\omega(\nabla^E, b) := -\frac{1}{2}\operatorname{Tr}(b^{-1}\nabla^E b)$$

is the complex-valued Kamber-Tondeur form, for which a detailed presentation is given in Section 2.4 in [**BH07**].

To prove the theorem above, the same procedure from (49) to (54) in [**BH07**] applies step-by-step to the bilinar Laplacian on manifolds with boundary. This uses Proposition 9.38 in [**BGV92**] giving the variation formula for the determinant of generalized

Laplacians, the telescopic sum cancellation by Ray and Singer in [**RS71**] and the same reasoning on finite dimensional complexes, see (53) and section 3 in [**BH07**]. This is justified since the bilinear Laplacian with absolute/relative boundary conditions possesses the same spectral properties as the one on a closed manifold, see (3.59) (see also Proposition 4.1.2). For the convenience of the reader we out-line the proof of Theorem 5.2.1.

Proof. We want to compute the variation of the torsion with respect to smooth variation of g and b. Let $U \subset \mathbb{R}$ be an open subset and $U \in u \mapsto (g_u, b_u)$ a smooth real one parameter families describing smooth variations of the Riemannian metric and bilinear form. For each $u \in U$, we denote by $\star_{b_u}$ the corresponding Hodge $\star$-operator associated to the Riemannian metric g and bilinear form b_u and by $\beta_u := \beta_{g_u,b_u}$ the associated non-degenerate symmetric bilinear form on $\Omega(M;E)$. Let $d^\sharp_{E,g_u,b_u}$ be the formal operator transposed to the differential d_E with respect to β_u and Δ_{E,g_u,b_u} be the symmetric bilinear Laplacian. We impose elliptic boundary conditions over the bordism $(M, \partial_+M, \partial_-M)$ in such a way that w satisfies absolute/relative boundary conditions if and only if $w \in \Omega_{\mathcal{B},u}(M;E) := \Omega_{\mathcal{B}_{E,g_u,b_u}}(M;E)$ for each $u \in U$. Let $\Delta_{\mathcal{B},u}$ be the associated L^2-realization of the elliptic boundary value problem $(\Delta_{E,g_u,b_u}, \mathcal{B}_{E,g_u,b_u})$ for each $u \in U$. Let γ be a simple closed curve around 0, such that the spectrum of $\Delta_{\mathcal{B},u}$ avoids γ for all $u \in U$. Finally, we denote by $[\tau^{\mathsf{RS}}]^{E,g_u,b_u}_{(M,\partial_+M,\partial_-M)}$ the complex-valued Ray–Singer torsion associated to $(M, \partial_+M, \partial_-M)$, E, g_u and b_u for each $u \in U$.

By Definition the complex-valued Ray–Singer torsion 5.1.4, $[\tau^{\mathsf{RS}}]^{E,g_u,b_u}_{(M,\partial_+M,\partial_-M)}$ is constructed as the product of $[\tau(0)]^{E,g_u,b_u}_{(M,\partial_+M,\partial_-M)}$ (i.e., the restriction of the bilinear form β_u to the finite dimensional subspace $\Omega_{\Delta_{\mathcal{B}},u}(M;E)(0)$) and the regularized product of all non-zero eigenvalues of $\Delta_{\mathcal{B},u}$. Since the bilinear Laplacian is not necessarily self-adjoint, the dimension of $\Omega_{\Delta_{\mathcal{B}},u}(M;E)(0)$ is not locally constant under smooth variations $u \in U$. Thus, in view of Lemma 5.1.5, instead of taking the defining expression for the torsion in Definition 5.1.4, we consider it as given by (5.11), that is

$$[\tau^{\mathsf{RS}}]^{E,g_u,b_u}_{(M,\partial_+M,\partial_-M)} = [\tau(\gamma)]^{E,g_u,b_u}_{(M,\partial_+M,\partial_-M)} \prod_q \left(\det{}^\gamma\left(\Delta_{E,g,b,q,u}\right)\right)^{(-1)^q q},$$

i.e., being constructed as the restriction of the bilinear form β_u to the finite dimensional subcochain complex $\Omega_{\Delta_{\mathcal{B},u}}(\gamma) := \Omega_{\Delta_{\mathcal{B},u}}(M;E)(\gamma)$, see (5.12), obtained as the union of the generalized eigen-spaces corresponding to the eigen-values in the interior of γ. For a fixed $u \in U$, we compute the logarithmic derivative of the complex number

$$\frac{[\tau^{\mathsf{RS}}]^{E,g_w,b_w}_{(M,\partial_+M,\partial_-M)}}{[\tau^{\mathsf{RS}}]^{E,g_u,b_u}_{(M,\partial_+M,\partial_-M)}},$$

with respect to the parameter w at u. That is, one needs to compute the logarithmic derivatives of

$$\frac{[\tau(\gamma)]^{E,g_w,b_w}_{(M,\partial_+M,\partial_-M)}}{[\tau(\gamma)]^{E,g_u,b_u}_{(M,\partial_+M,\partial_-M)}} \quad \text{and} \quad \frac{\prod_q \left(\det{}^{\gamma}\left(\Delta_{E,g,b,q,w}\right)\right)^{(-1)^q q}}{\prod_q \left(\det{}^{\gamma}\left(\Delta_{E,g,b,q,u}\right)\right)^{(-1)^q q}}$$

with respect to w at u. To compute the logarithmic derivatives of the numbers above, we proceed as in the closed situation in [**BH07**]. We start by considering specific linear bundle endomorphisms on $\Lambda T^*M\otimes E$. Fix $u_0 \in U$, and for each $u \in U$ define a symmetric bundle endomorphism $G_u \in \Gamma(M, \mathsf{Aut}(TM))$ by the condition

$$g_u(X,Y) = g_{u_0}(G_uX, Y) = g_{u_0}(X, G_uY)$$

and denote by $\mathbf{D}^*G_u^{-1}$ its natural extension to $\Gamma(M, \mathsf{Aut}(\Lambda T^*M))$. In the same way, for each $u \in U$, define a symmetric bundle endomorphism $B_u \in \Gamma(M, \mathsf{Aut}(E))$ by the condition

$$b_u(e,f) = b_{u_0}(B_ue, f) = b_{u_0}(e, B_uf).$$

Then for $u \in U$ define $A_u \in \Gamma(M, \mathsf{Aut}(\Lambda T^*M \otimes E))$ by the formula

$$A_u := \det(G_u)^{1/2}\mathbf{D}^*G_u^{-1} \otimes B_u.$$

Remark that, by construction, we get

$$\beta_{g_u,b_u}(v,w) = \beta_{g_{u_0},b_{u_0}}(A_uv, w) = \beta_{g_{u_0},b_{u_0}}(v, A_uw),$$

for $v, w \in \Omega(M;E)$. We restrict now β_u to $\Omega_{\mathcal{B},u}(M;E)$ for $u \in U$. This guarantees, see Lemma 3.2.4, that[1]

$$d^{\sharp}_{E,g_u,b_u} = A_u^{-1} d^{\sharp}_{E,g_{u_0},b_{u_0}} A_u,$$

over $\Omega_{\mathcal{B},u}(M;E)$ for each $u \in U$. In this way, we are interested in the bundle endomorphism $A_u^{-1}\dot{A}_u \in \Gamma(M; \mathrm{End}\left(\Lambda^*T^*M \otimes E\right))$ encoding the infinitesimal variation of the metric and that of the bilinear form at u. More explicitly,

$$A_u^{-1}\dot{A}_u = -\left(\mathbf{D}^*\left(g_u^{-1}\dot{g}_u\right) - \frac{1}{2}\operatorname{Tr}\left(g_u^{-1}\dot{g}_u\right)\right) \otimes \mathsf{Id} + \mathsf{Id} \otimes \left(b_u^{-1}\dot{b}_u\right), \tag{5.15}$$

where $\mathbf{D}^*\left(g_u^{-1}\dot{g}_u\right)$ is the extension of $g_u^{-1}\dot{g}_u \in \Gamma(M; \mathrm{End}(TM))$ as a derivation on Λ^*T^*M (e.g Section 4.2). Let P_u denote the spectral projection on $\Omega_{\Delta_{\mathcal{B},u}}(M;E)(\gamma)$ and Q_u the spectral projection onto the generalized eigenspaces of $\Delta_{\mathcal{B},u}$ corresponding to the eigenvalues in the exterior of γ. Then, in terms of these operators and analogue to the situation for the torsion on a closed manifold, see [**BH07**], we have

$$\left.\frac{\partial}{\partial w}\right|_u \left(\frac{[\tau(\gamma)]^{E,g_w,b_w}_{(M,\partial_+M,\partial_-M)}}{[\tau(\gamma)]^{E,g_u,b_u}_{(M,\partial_+M,\partial_-M)}}\right) = \operatorname*{LIM}_{t\to 0} \mathrm{Tr}_{\mathbf{s}}\left(A_u^{-1}\dot{A}_u\mathsf{P}_u e^{t\Delta_{\mathcal{B},u}}\right), \tag{5.16}$$

[1] Remark here this identity still holds for w with $i_-^*w = 0$ but also for such forms with compact supported in the interior of M.

and

$$\frac{\partial}{\partial w}\bigg|_u \left(\prod_q \left(\det{}^\gamma\left(\Delta_{E,g,b,q,w}\right)\right)^{(-1)^q q}\right) = \operatorname*{LIM}_{t\to 0} \operatorname{Tr}_{\mathbf{s}}\left(A_u^{-1}\dot{A}_u \mathsf{Q}_u e^{t\Delta_{\mathcal{B},u}}\right) \tag{5.17}$$

We out-line the proofs of (5.17) and (5.16). We start with (5.17). For $|w-u|\in\mathbb{R}$ is small enough, the projection $\mathsf{P}_w|_{\Omega_u(\gamma)}:\Omega_{\Delta_{\mathcal{B}},u}(\gamma)\to\Omega_{\Delta_{\mathcal{B}},w}(\gamma)$ is actually an isomorphism of complexes. Then, the following diagram commutes

$$\begin{array}{ccccc}
\det\Omega_{\Delta_{\mathcal{B}},u}(\gamma) & \longrightarrow & \det H(\Omega_{\Delta_{\mathcal{B}},u}(\gamma)) & \xrightarrow{\text{H-DR}} & \det H(M,\partial_- M)\\
\Big\downarrow{\scriptstyle \det(\mathsf{P}_w|_{\Omega_u(\gamma)})} & & \Big\downarrow{\scriptstyle \det(H(\mathsf{P}_w|_{\Omega_u(\gamma)}))} & & \Big\downarrow{\scriptstyle \det(H(\mathsf{P}_w|_{\Omega_u(\gamma)}))}\\
\det\Omega_{\Delta_{\mathcal{B}},w}(\gamma) & \longrightarrow & \det H(\Omega_{\Delta_{\mathcal{B}},w}(\gamma)) & \xrightarrow{\text{H-DR}} & \det H(M,\partial_- M)
\end{array}$$

For $|w-u|$ is small enough, the nondegenerate bilinear forms $(\mathsf{P}_w|_{\Omega_{\Delta_{\mathcal{B}},u}(\gamma)})^*\beta_{E,g_w,b_w}$ and $\beta_{E,g_u,b_u}|_{\Omega_{\Delta_{\mathcal{B}},u}(\gamma)}$ are considered as isomorphisms from $\Omega_{\Delta_{\mathcal{B}},u}(\gamma)$ to its dual so that

$$(\beta_{E,g,b}|_{\Omega_{\Delta_{\mathcal{B}},u}(\gamma)})^{-1}(\mathsf{P}_w|_{\Omega_{\Delta_{\mathcal{B}},u}(\gamma)})^*\beta_{E,g_w,b_w}$$

is an automorphism of $\Omega_{\Delta_{\mathcal{B}},u}(\gamma)$. Thus, the change of the torsion is computed as the induced nondegenerate bilinear form on the determinant line corresponding to the change of the bilinear forms $(\mathsf{P}_w|_{\Omega_{\Delta_{\mathcal{B}},u}(\gamma)})^*\beta_{E,g_w,b_w}$ and $\beta_{E,g_u,b_u}|_{\Omega_{\Delta_{\mathcal{B}},u}(\gamma)}$ on $\Omega_{\Delta_{\mathcal{B}},u}(\gamma)$:

$$\frac{[\tau^{\mathrm{RS}}]^{E,g_w,b_w}}{[\tau^{\mathrm{RS}}]^{E,g_u,b_u}} = \det_{\mathbf{s}}\left((\beta_{E,g,b}|_{\Omega_{\Delta_{\mathcal{B}},u}(\gamma)})^{-1}(\mathsf{P}_w|_{\Omega_{\Delta_{\mathcal{B}},u}(\gamma)})^*\beta_{E,g_w,b_w}\right)$$

Since for $e,f\in\Omega_{\Delta_{\mathcal{B},u}}(M;E)(\gamma)$ we have $\beta_{g_w,b_w}(e,f)=\beta_{g_u,b_u}(A_u e,f)=\beta_{g_u,b_u}(e,A_u f)$, we obtain

$$(\beta_{E,g,b}|_{\Omega_{\Delta_{\mathcal{B}},u}(\gamma)})^{-1}(\mathsf{P}_w|_{\Omega_{\Delta_{\mathcal{B}},u}(\gamma)})^*\beta_{E,g_w,b_w} = \mathsf{P}_u A_u^{-1}A_w P_w|_{\Omega_{\Delta_{\mathcal{B}},u}(\gamma)}$$

Therefore,

$$\frac{\partial}{\partial w}|_u\left(\frac{[\tau^{\mathrm{RS}}]^{E,g_w,b_w}}{[\tau^{\mathrm{RS}}]^{E,g_u,b_u}}\right) = \operatorname{Tr}_{\mathbf{s}}\left(\mathsf{P}_u A_u^{-1}\dot{A}_u\mathsf{P}_u + \mathsf{P}_u A_u^{-1}A_u\dot{\mathsf{P}}_u\right) = \operatorname{Tr}_{\mathbf{s}}(\mathsf{P}_u A_u^{-1}\dot{A}_u\mathsf{P}_u)$$

for $\operatorname{Tr}_{\mathbf{s}}\mathsf{P}_u^2=\text{const}$ implies $\operatorname{Tr}_{\mathbf{s}}\mathsf{P}_u\dot{\mathsf{P}}_u=0$. That proves (5.17). We sketch the proof of (5.16). Consider the complementary orthogonal projection $\mathsf{Q}_u:=\mathsf{Id}-\mathsf{P}_u$. We use the variation formula for the determinant of generalized Laplacians, see Proposition 9.38 **[BGV92]** to compute the logarithmic derivative

$$\begin{aligned}
\tfrac{\partial}{\partial w}|_u\left(\textstyle\prod_q\left(\det{}^\gamma\left(\Delta_{E,g,b,q,u}\right)\right)^{(-1)^q q}\right) &= \textstyle\sum_q(-1)^q q\left(\tfrac{\partial}{\partial u}\det{}^\gamma(\Delta_{\mathcal{B},u,q})\right)\\
&= \textstyle\sum_q(-1)^q q\left(\mathrm{LIM}_{t\to 0}\operatorname{Tr}(\dot{\Delta}_{\mathcal{B},u,q}(\Delta_{\mathcal{B},u,q})^{-1}\mathsf{Q}_{u,q}\exp(-t\Delta_{\mathcal{B},u,q}))\right)\\
&= \mathrm{LIM}_{t\to 0}\left(\operatorname{Tr}_{\mathbf{s}}(\mathsf{N}\dot{\Delta}_{\mathcal{B},u,q}(\Delta_{\mathcal{B},u,q})^{-1}\mathsf{Q}_{u,q}\exp(-t\Delta_{\mathcal{B},u,q}))\right),
\end{aligned}$$

where $\mathsf{N}v=qv$ for each $v\in\Omega^q(M;E)$. By using $\dot{\Delta}_{\mathcal{B},u}=[d_E,\dot{d}^\sharp_{E,g_u,b_u}]$, $[\mathsf{N},d_E]=d_E$, $[d_E,\Delta_{\mathcal{B},u}]=0$, $[d_E,\mathsf{Q}_u]=0$, $\dot{d}^\sharp_{E,g_u,b_u}=[d^\sharp_{E,g_u,b_u},A_u^{-1}\dot{A}_u]$, and that $\operatorname{Tr}_{\mathbf{s}}$ vanishes on supercommutators, we get

$$
\begin{aligned}
\mathrm{Tr}_{\mathbf{s}}(\mathsf{N}\dot{\Delta}_{\mathcal{B},u,q}(\Delta_{\mathcal{B},u,q})^{-1}\mathsf{Q}_{u,q}\exp(-t\Delta_{\mathcal{B},u,q}))
&= \mathrm{Tr}_{\mathbf{s}}(\mathsf{N}d_E\dot{d}^{\sharp}_{E,g_u,b_u}(\Delta_{\mathcal{B},u,q})^{-1}\mathsf{Q}_{u,q}\exp(-t\Delta_{\mathcal{B},u,q}))\\
&\quad + \mathrm{Tr}_{\mathbf{s}}(\mathsf{N}\dot{d}^{\sharp}_{E,g_u,b_u}d_E(\Delta_{\mathcal{B},u,q})^{-1}\mathsf{Q}_{u,q}\exp(-t\Delta_{\mathcal{B},u,q}))\\
&= \mathrm{Tr}_{\mathbf{s}}(d_E\dot{d}^{\sharp}_{E,g_u,b_u}(\Delta_{\mathcal{B},u,q})^{-1}\mathsf{Q}_{u,q}\exp(-t\Delta_{\mathcal{B},u,q}))\\
&= \mathrm{Tr}_{\mathbf{s}}(d_E d^{\sharp}_{E,g_u,b_u}A_u^{-1}\dot{A}_u(\Delta_{\mathcal{B},u,q})^{-1}\mathsf{Q}_{u,q}\exp(-t\Delta_{\mathcal{B},u,q}))\\
&\quad - \mathrm{Tr}_{\mathbf{s}}(d_E A_u^{-1}\dot{A}_u d^{\sharp}_{E,g_u,b_u}(\Delta_{\mathcal{B},u,q})^{-1}\mathsf{Q}_{u,q}\exp(-t\Delta_{\mathcal{B},u,q}))\\
&= \mathrm{Tr}_{\mathbf{s}}(A_u^{-1}\dot{A}_u(d_E d^{\sharp}_{E,g_u,b_u} + d^{\sharp}_{E,g_u,b_u}d_E)(\Delta_{\mathcal{B},u,q})^{-1}\mathsf{Q}_{u,q}\exp(-t\Delta_{\mathcal{B},u,q}))\\
&= \mathrm{Tr}_{\mathbf{s}}(A_u^{-1}\dot{A}_u\mathsf{Q}_{u,q}\exp(-t\Delta_{\mathcal{B},u,q})),
\end{aligned}
$$

which proves (5.16). The contributions in (5.17) and (5.16) add up together to compute the total variation of the torsion $[\tau(\gamma)]^{E,g_u,b_u}_{(M,\partial_+M,\partial_-M)}$, with respect to infinitesimal changes in u. Then

$$
\frac{\partial}{\partial w}\bigg|_u \left(\frac{[\tau^{\mathsf{RS}}]^{E,g_w,b_w}_{(M,\partial_+M,\partial_-M)}}{[\tau^{\mathsf{RS}}]^{E,g_u,b_u}_{(M,\partial_+M,\partial_-M)}}\right) = \operatorname*{LIM}_{t\to 0}\mathrm{Tr}_{\mathbf{s}}\left(A_u^{-1}\dot{A}_u e^{-t\Delta_{\mathcal{B},u}}\right) \tag{5.18}
$$

Formula (5.18) generalizes formula (54) in **[BH07]** to manifolds with with boundary and it tells us that the variation of the torsion is obtained as in the closed case, by computing the term in the right hand side of the equality in (5.18). This term corresponds to the coefficient of the constant term in the heat trace asymptotic expansion associated to the boundary value problem $[\Delta,\Omega_{\mathcal{B}}]^{E,g,b}_{(M,\partial_+M,\partial_-M)}$ and the bundle endomorphism $A_u^{-1}\dot{A}_u$ given in (5.15). The right hand side of the equality in (5.18) is computed by using Theorem 4.4.3 where we set $\phi = b_u^{-1}\dot{b}_u$ and $\xi = g_u^{-1}\dot{g}_u$.

□

CHAPTER 6

CoEuler structures and the analytic torsion on bordisms

In this chapter we define coEuler structures on bordisms, generalizing in this way the corresponding notion on closed manifolds in [**BH06a**], [**BH06b**] and [**BH07**]. We use coEuler structures to encode the variation of the complex-valued analytic torsion on compact bordisms.

In Section 6.1 we give the background needed to define coEuler structures. In [**BM06**], Brüning and Ma studied certain characteristic forms on the boundary of a compact Riemannian manifold. These forms appear in the anomaly formulas for the Ray–Singer metric (see Theorem 0.1 in [**BM06**] and also Theorem 3.4 in [**BM11**]), and then in the anomaly formulas for the complex-valued Ray–Singer torsion obtained in Theorem 5.2.1 in Chapter 5. From Section 6.1.1 to Section 6.1.8 we recall in some detail how these characteristic forms are obtained and we slightly modify them to the situation of a compact Riemannian bordism $(M, \partial_+M, \partial_-M)$. The characteristic forms used to define coEuler structures, have been obtained by slightly modifying those in [**BM06**]: In [**BM06**], the vector field ς_{in}, the inwards pointing geodesic unit normal vector field (to each point at) the boundary, is used to construct the characteristic forms appearing in their anomaly formulas, but those given in Definition 6.1.11 are constructed instead with a geodesic unit normal vector field ς, which *distinguishes* the roles of ∂_-M and ∂_+M: it points *inwards* on ∂_+M and *outwards* on ∂_-M, see (6.10). We denote this vector field by ς_{in}. In Definition 6.1.12, we define the **relative Euler form** on the bordism $(M, \partial_+M, \partial_-M)$, as the couple

$$\underline{\mathbf{e}}(M, \partial_+M, \partial_-M, g) := (\mathbf{e}(M, g), \mathbf{e}_\partial(\partial_+M, \partial_-M, g))$$

where $\mathbf{e}(M, g)$ is the Euler form associated to the metric g and $\mathbf{e}_\partial(\partial_+M, \partial_-M, g)$ is a characteristic form on the boundary constructed by using ς. Also with the help of ς, in Definition 6.1.12, certain secondary (of Chern–Simons' type) relative forms on bordisms are defined. Lemma 6.1.13, essentially proved in [**BM06**], presents some properties that these forms satisfy. In Section 6.1.8, we recall the Gauss–Bonnet–Chern Theorem in terms of the relative Euler form and we explain how this is obtained from Theorem 4.2.7.

In Section 6.2, we define coEuler structures. We split the presentation into two parts. In Section 6.2.1, we start with coEuler structures without a base point. To do that, we assume $\chi(M, \partial_+M) = 0$ (or equivalently $\chi(M, \partial_-M) = 0$) so that the set of coEuler structures on a closed Riemannian manifold is an *affine space* over the cohomology group $H^{m-1}(M;\mathbb{C})$ and in that situation, CoEuler structures are represented by couples (α, g) consisting of a smooth differential form α and a Riemannian metric g with $d\alpha = \mathbf{e}(M,g)$, under an equivalence relation defined by using the secondary Chern–Simons' forms. In our case, the space of CoEuler structures on $(M, \partial_+M, \partial_-M)$ is an affine space over the relative cohomology group $H^{m-1}(M, \partial M;\mathbb{C})$, whose classes are represented by couples $(\underline{\alpha}, g)$, where $\underline{\alpha}$ is in this case, a relative form (see Definition 6.1.1), with $\mathbf{d}\underline{\alpha} = \underline{\mathbf{e}}(M, \partial_+M, \partial_-M, g)$, under an equivalence relation specified by using this time secondary relative Chern–Simons' forms. A coEuler structure on the bordism $(M, \partial_+M, \partial_-M)$ is in a one-to-one correspondence with a coEuler structure on its dual bordism $(M, \partial_-M, \partial_+M)$, by means of a so-called *flip* map, compatible with Poincaré duality and affine over the involution $(-1)^m$ in relative cohomology, see Section 6.2.2. In Section 6.2.3, we derive Proposition 6.2.5, which gives the infinitesimal variation of the integrals of the relative form $\underline{\alpha} \wedge \omega(\nabla^E, b)$, where $\omega(\nabla^E, b)$ is the closed one-form of Kamber–Tondeur, with respect to a smooth variations in the Riemannian metric and bilinear structures. The variational formula (6.24) from Proposition 6.2.5 is used in Section 6.3 to cancel out the variation of the complex-valued Ray–Singer torsion. In Section 6.2.4, we study the case $\chi(M, \partial_\pm M) \neq 0$, by incorporating a base point. In analogy with the situation on closed manifolds, we define base-pointed coEuler structures, generalizing to this setting the results from Section 6.2.1. In particular, Proposition 6.2.8 generalizes Proposition 6.2.5, by using a regularization procedure for the integral of $\underline{\alpha} \wedge \omega(\nabla^E, b)$, where $\underline{\alpha}$ is a relative form with a *singularity* in the interior; this regularization is explained in subsection 6.2.5.

In Section 6.3, we define a generalized version for the complex-valued analytic torsion on compact bordisms, by adding *correction* terms to the complex-valued Ray–Singer analytic torsion, see Definition 6.3.1. These correction terms, expressed in terms of coEuler structures, are incorporated to cancel out the variation of the complex-valued Ray–Singer torsion with respect to smooth variations of the Riemannian metric and bilinear structures, given in Theorem 5.2.1. In analogy with the situation on closed manifolds, the generalized complex-valued analytic torsion depends on the flat connection, the homotopy class of the bilinear form and on the coEuler structure only.

6.1. Background setting

Consider a compact connected Riemannian m-dimensional manifold M with boundary ∂M, denote by Θ_M the orientation bundle of M and by $\Theta_M^{\mathbb{C}}$ its complexification bundle.

6.1.1. Relative de-Rham cohomology: without base point.

Consider the $\mathbb{Z}$-graded differential cochain complex

$$\Omega(M, \partial M; \Theta_M^{\mathbb{C}}) := \oplus_{q=1}^{m} \Omega^q(M, \partial M; \Theta_M^{\mathbb{C}}), \tag{6.1}$$

where

$$\Omega^q(M, \partial M; \Theta_M^{\mathbb{C}}) := \Omega^q(M; \Theta_M^{\mathbb{C}}) \oplus \Omega^{q-1}(\partial M; \Theta_M^{\mathbb{C}})$$

with the differential map $\mathbf{d} : \Omega^q(M, \partial M; \Theta_M^{\mathbb{C}}) \to \Omega^{q+1}(M, \partial M; \Theta_M^{\mathbb{C}})$ given by

$$\mathbf{d}(\alpha, \alpha_\partial) := (d\alpha, i^* w - d^\partial \alpha_\partial), \tag{6.2}$$

see page 78 in [**BT82**].

Definition 6.1.1. *Elements* $\underline{\alpha} := (\alpha, \alpha_\partial) \in \Omega^q(M, \partial M; \Theta_M^{\mathbb{C}})$ *are called* ***relative forms****.*

Definition 6.1.1 is motivated by the fact that the complex in (6.1) computes the *relative cohomology* groups $H^q(M, \partial M; \Theta_M^{\mathbb{C}})$ in degree q.

Notation 6.1.2. For $\underline{\alpha} \in \Omega^q(M, \partial M, \Theta_M)$ and $w \in \Omega^{m-q}(M)$, we have the pairing

$$\int_{(M,\partial M)} \underline{\alpha} \wedge w := \int_M \alpha \wedge w - \int_{\partial M} \alpha_\partial \wedge i^* w,$$

which induces a nondegenerate pairing $\langle \cdot, \cdot \rangle$ in cohomology:

$$\begin{aligned} \langle \cdot, \cdot \rangle : \quad & H^*(M, \partial M, \Theta_M^{\mathbb{C}}) \times H^{m-*}(M; \mathbb{C}) \to \mathbb{C} \\ & \langle [(\alpha, \alpha_\partial)], [w] \rangle := \textstyle\int_{(M,\partial M)} (\alpha, \alpha_\partial) \wedge w. \end{aligned}$$

c.f. Section 3.2.2.

Lemma 6.1.3. *Suppose M is a compact connected manifold. Then,*

$$H^m\left(M, \partial M; \Theta_M^{\mathbb{C}}\right) \cong H^0(M; \mathbb{C}) \cong \mathbb{C}.$$

Proof. This follows from non-degeneracy from $\langle \cdot, \cdot \rangle$ (see Notation 6.1.2) and connectedness of M, □

6.1.2. Relative de-Rham cohomology: with base point. For $x_0 \in M \backslash \partial M$ a base point in the interior of M, consider

$$\dot{M} := M \backslash \{x_0\}, \tag{6.3}$$

together with the inclusions

$$\partial M \subset \dot{M} \subset M, \tag{6.4}$$

and the vector spaces

$$\Omega^q(\dot{M}, \partial M; \Theta_M^{\mathbb{C}}) := \Omega^q(\dot{M}; \Theta_M^{\mathbb{C}}) \oplus \Omega^{*-1}(\partial M; \Theta_M^{\mathbb{C}}). \tag{6.5}$$

The space $\Omega^q(\dot{M}, \partial M; \Theta_M^{\mathbb{C}})$ endowed with the same differential map $\mathbf{d}$ as in (6.2) is also a complex, whose cohomology groups are denoted by $H^q(\dot{M}, \partial M; \Theta_M^{\mathbb{C}})$.

Lemma 6.1.4. *Let M be a compact connected Riemannian manifold of dimension m with boundary ∂M. For $x_0 \in M\backslash\partial M$ a base point in the interior of M, consider the pointed space $\dot{M}$ in (6.3). Then*

$$\begin{array}{rcl} H^m(\dot{M}, \partial M; \Theta^{\mathbb{C}}_M) & \cong & 0 \\ H^{m-1}\left(M, \partial M; \Theta^{\mathbb{C}}_M\right) & \cong & H^{m-1}(\dot{M}, \partial M; \Theta^{\mathbb{C}}_M). \end{array}$$

Proof. We sketch the main ideas in the proof. First, if $H_c(\dot{M})$ indicates the cohomology of $\dot{M}$ with compact supports, then by Poincaré–Lefschetz duality, we have $H^m(\dot{M}, \partial M; \Theta^{\mathbb{C}}_M) \cong H^m_c(\dot{M}) = 0$. In order to show the second equality above in the statement, consider the long exact sequence in (top degree) cohomology, associated to the inclusion of spaces in (6.4), see[**BT82**]:

$$\cdots\to H^{m-1}(M,\dot{M})\to H^{m-1}(M,\partial M)\underbrace{\to}_{a} H^{m-1}(\dot{M},\partial M)\underbrace{\to}_{\partial_{m-1}} H^m(M,\dot{M})\underbrace{\to}_{b} H^m(M,\partial M)\to H^m(\dot{M},\partial M)\to 0,$$

where by simplicity, we have omitted writing the coefficient bundle $\Theta^{\mathbb{C}}_M$. By Poincaré–Lefschetz duality, we have $H^m(M, \partial M; \Theta^{\mathbb{C}}_M) \cong H^m_c(M) = \mathbb{C}$ and for the local cohomology groups, by excision, we have in general

$$H^k(M, \dot{M}; \Theta^{\mathbb{C}}_M) \cong H^k(\mathbb{C}^m, \mathbb{C}^m\backslash\{0\}) = \begin{cases} \mathbb{C} & k = m \\ 0 & \text{else} \end{cases}.$$

Thus, since by exactness b is surjective, b is also a bijective by dimensional reasons. Moreover, by exactness b being injective the map ∂_{m-1} is zero. Thus, since the local cohomology group $H^{m-1}(M, \dot{M}; \Theta^{\mathbb{C}}_M)$ vanishes, a is bijective by exactness. This proves the statement.

□

6.1.3. Berezin integral and Pfaffian. We adopt the notation from [**BM06**] and [**BZ92**], see also Section 1.1.5. For A and B two $\mathbb{Z}_2$ graded unital algebras, $A\widehat{\otimes}B$ denotes their $\mathbb{Z}_2$-graded tensor product and set $A := A\widehat{\otimes}I$, $\widehat{B} := I\widehat{\otimes}B$ and $\wedge := \widehat{\otimes}$, such that

$$A \wedge \widehat{B} = A\widehat{\otimes}B.$$

For W and V finite dimensional vector spaces of dimension n and l respectively where W is endowed with a Hermitian product $\langle\cdot,\cdot\rangle$ and V' the dual of V, the Berezin integral on elements of the $\mathbb{Z}_2$-graded tensor product $\Lambda V' \wedge \widehat{\Lambda(W')}$ is

$$\begin{array}{rcl} \int^B : \Lambda V' \wedge \widehat{\Lambda(W')} & \to & \Lambda V' \otimes \Theta_W \\ \alpha \wedge \widehat{\beta} & \mapsto & C_B \beta_{g,b}(w_1, \ldots, w_n)\alpha, \end{array}$$

where $\{w_i\}_{i=1}^n$ is an orthonormal basis of W, Θ_W is the orientation bundle of W and the constant $C_B := (-1)^{n(n+1)/2}\pi^{-n/2}$. Let $\{w^i\}_{i=1}^n$ be the corresponding dual basis in W'. If K is an antisymmetric endomorphism of W, then it is identified with a unique element $\mathbf{K} := \langle \cdot, K\cdot\rangle \in \Lambda(W')$ given by $\mathbf{K} := \frac{1}{2}\sum_{1\leqslant i,j\leqslant n}\langle w_i, Kw_j\rangle \widehat{w^i} \wedge \widehat{w^j}$. The **Pfaffian** of $\mathbf{K}/2\pi$ is then defined by

$$\mathbf{Pf}\,(\mathbf{K}/2\pi) := \int^B \exp(\mathbf{K}/2\pi).$$

Remark that $\mathbf{Pf}\,(\mathbf{K}/2\pi) = 0$, if n is odd. By standard fiberwise considerations the map $\mathbf{Pf}$ is extended for vector bundles over M. In particular, we look at

$$\int^{B_M} : \Gamma(M; \Lambda T^*M \wedge \widehat{\Lambda T^*M}) \to \Gamma(M; \Lambda T^*M \otimes \Theta_M)$$

and

$$\int^{B_{\partial M}} : \Gamma(\partial M; \Lambda T^*\partial M \wedge \Lambda(\widehat{T^*\partial M})) \to \Gamma(\partial M; \Lambda T^*\partial M \otimes \Theta_{\partial M}).$$

Thes Berezin integrals $\int^{B_M}$ and $\int^{B_{\partial M}}$ above can be compared by using the standard convention for the induced orientation bundle on the boundary discussed in Section 1.2, so that the relation $\int^{B_M} \gamma \wedge \widehat{\beta} \wedge \widehat{e^m} = \pi^{-1/2}\int^{B_{\partial M}} \gamma \wedge \widehat{\beta}$ holds, for $\gamma \in \Omega(M)$ and $\widehat{\beta} \in \Gamma(\partial M; \Lambda(\widehat{T^*(\partial M)}))$.

6.1.4. Deformation spaces on manifolds with boundary. Let $\{g_s := g_s^{TM}\}_{s\in\mathbb{R}}$ be smooth families of Riemannian metrics on TM and $\{g_s^\partial := g_s^{T\partial M}\}_{s\in\mathbb{R}}$ the induced family of metrics on $T\partial M$. Let $\nabla_s := \nabla^{TM}_{g_s}$ and $\mathsf{R}_s := \mathsf{R}^{TM}_{g_s}$ be the Levi-Cività connections and curvatures on TM associated to the metrics g_s, together with $\nabla^\partial_s := \nabla^{T\partial M}_{g^\partial_s}$ and $\mathsf{R}^\partial_s := \mathsf{R}^{T\partial M}_{g^\partial_s}$ the Levi-Cività connections and curvatures on $T\partial M$ associated to the metrics g^∂_s.

Consider the deformation space $\widetilde{M} := M \times \mathbb{R}$ with

$$\pi_{\widetilde{M}} : \widetilde{M} \to \mathbb{R} \text{ and } \mathbf{p}_M : \widetilde{M} \to M,$$

its canonical projections and the deformation space $\widetilde{\partial M} := \partial M \times \mathbb{R}$ with

$$\pi_{\widetilde{\partial M}} : \widetilde{\partial M} \to \mathbb{R} \text{ and } \mathbf{p}_{\partial M} : \widetilde{\partial M} \to \partial M$$

its canonical projections. If $\widetilde{i} := i \times \mathbf{id}_{\mathbb{R}} : \widetilde{\partial M} \to \widetilde{M}$ is the natural embedding induced by $i : \partial M \to M$, then $\pi_{\widetilde{\partial M}} = \pi_{\widetilde{M}} \circ \widetilde{i}$.

By construction, the fibers of $\pi_{\widetilde{M}} : \widetilde{M} \to \mathbb{R}$ are compact and diffeomorphic to M and those of $\pi_{\widetilde{\partial M}} : \widetilde{\partial M} \to \mathbb{R}$ are compact and diffeomorphic to ∂M.

Consider the pull-back of the tangent bundle $TM \to M$ along $\mathbf{p}_M : \widetilde{M} \to M$ as a subbundle of $T\widetilde{M}$ and denote it by

$$\mathcal{TM} := \mathbf{p}_M^* TM \to \widetilde{M}, \tag{6.6}$$

whereas its dual vector bundle is denoted by $\mathcal{T}^*\mathcal{M} \to \widetilde{M}$. Analogously, the pull-back of the tangent bundle $T\partial M \to \partial M$ along $\mathbf{p}_{\partial M} : \widetilde{\partial M} \to \partial M$, seen as subbundle of $T\widetilde{\partial M}$, is denoted by

$$\mathcal{T}\partial\mathcal{M} := \mathbf{p}_{\partial M}^* T\partial M \to \widetilde{\partial M} \tag{6.7}$$

and dual $\mathcal{T}^*\partial\mathcal{M} \to \widetilde{\partial M}$.

Let $T^H\widetilde{M} \to \widetilde{M}$ be a horizontal subbundle of $T\widetilde{M} \to \widetilde{M}$ such that

$$T^H\widetilde{M} \oplus \mathcal{T}\mathcal{M} \cong T\widetilde{M}, \quad \text{with} \quad T^H\widetilde{M} \cong T\mathbb{R}$$

as vector bundles over $\widetilde{M}$. The (orientable) normal bundle to ∂M in M, is identified with the orthogonal complement of $T\partial M$ in TM. This is illustrated as follows

$$\begin{tikzcd}
 & \mathcal{T}\mathcal{M} := \mathbf{p}_{\partial M}^* TM \arrow[d] & TM \arrow[d, "\mathbf{q}_{TM}"] \\
 & \widetilde{M} := M \times \mathbb{R} \arrow[r, "\mathbf{p}_M"] \arrow[dl, "\pi_{\widetilde{M}}"'] & M \\
\mathbb{R} & & \\
 & \widetilde{\partial M} := \partial M \times \mathbb{R} \arrow[r, "\mathbf{p}_{\partial M}"] \arrow[ul, "\pi_{\widetilde{\partial M}}"] \arrow[uu, "\widetilde{i}"] & \partial M \arrow[uu, "i"']
\end{tikzcd}$$

As explained in Section 6.1.3 above, we identify the smooth sections

$$\Gamma(\widetilde{M}; \Lambda(T^*\widetilde{M})) \ni w \quad \leftrightarrow \quad w \wedge 1 \in \Gamma(\widetilde{M}; \Lambda(T^*\widetilde{M}) \wedge \widehat{\Lambda(T^*\widetilde{M})}),$$

and we set

$$\widehat{w} := 1 \wedge \widehat{w} \in \Gamma(\widetilde{M}; \Lambda(T^*\widetilde{M}) \wedge \widehat{\Lambda(T^*\widetilde{M})}).$$

We endow the bundle $\mathcal{T}\mathcal{M}$ in (6.6) naturally with a Riemannian metric $g^{\mathcal{T}\mathcal{M}}$ such that

(a) For each $s \in \mathbb{R}$, $g^{\mathcal{T}\mathcal{M}}|_{M\times\{s\}} = g_s$, that is, at each fiber $M \times \{s\}$, the metrics $g^{\mathcal{T}\mathcal{M}}$ and g_s coincide.

(b) The metric $g^{\mathcal{T}\mathcal{M}}$ is compatible with the connection

$$\nabla^{\mathcal{T}\mathcal{M}} := \mathbf{p}_M^* \nabla_s + ds \wedge \left(\frac{\partial}{\partial s} + \frac{1}{2} g_s^{-1} \frac{\partial}{\partial s} g_s \right), \tag{6.8}$$

The curvature tensor associated to $\nabla^{\mathcal{T}\mathcal{M}}$ is denoted by

$$\mathsf{R}^{\mathcal{T}\mathcal{M}} := \mathbf{p}_M^* \mathsf{R}_s + ds \wedge \left(\frac{\partial}{\partial s} \nabla_s - \frac{1}{2} \left[\nabla_s, g_s^{-1} \frac{\partial}{\partial s} g_s \right] \right), \tag{6.9}$$

see section 1.5, (1.44) and Definition 1.1 in [**BM06**], (see also (4.50) and (4.50) in [**BZ92**]). In the same way $\mathcal{T}\partial\mathcal{M}$ is equipped with the metric $g^{\mathcal{T}\partial\mathcal{M}}$, compatible connection $\nabla^{\mathcal{T}\partial\mathcal{M}}$ and curvature $\mathsf{R}^{\mathcal{T}\partial\mathcal{M}}$.

6.1.5. Normalized vector fields and adapted frames on bordisms. Consider a compact Riemannian bordism $(M, \partial_+ M, \partial_- M)$. Let ς_{in} denote the *inwards* pointing unit normal geodesic vector field at the boundary and set

$$\varsigma_{\mathsf{out}} := -\varsigma_{\mathsf{in}} \quad \text{and} \quad \varsigma := \begin{cases} \varsigma_{\mathsf{in}} & \text{on } \in \partial_+ M \\ \varsigma_{\mathsf{out}} & \text{on } \in \partial_- M \end{cases}. \tag{6.10}$$

For $\widetilde{M} := M \times \mathbb{R}$, consider the vector bundle $\mathcal{TM} \to \widetilde{M}$ in (6.6) endowed with the Riemannian metric $g^{\mathcal{TM}}$. As in [**BM06**], we consider $\{e_i\}_1^m$ local orthonormal frames of $\mathcal{TM}$ and $\{e^i\}_{i=1}^m$ its dual frame on $\mathcal{T}^*\mathcal{M}$, with the property that *near the boundary* we have

$$e_m(y, s) := \varsigma_{\mathsf{in}} \quad \text{for each } y \in \partial M \text{ and } s \in \mathbb{R}.$$

so that $\{e_\alpha\}_{1\leqslant\alpha\leqslant m-1}$ is a local orthonormal frame for the vector bundle $\mathcal{T}\partial\mathcal{M} \to \widetilde{\partial M}$ in (6.7), where $\widetilde{\partial M} := \partial M \times \mathbb{R}$, and $\{e_\alpha\}_{1\leqslant\alpha\leqslant m-1} \cup \{e_m\}$ is a local orthonormal frame of $\mathcal{TM}|_{\widetilde{\partial M}} \to \widetilde{\partial M}$.

6.1.6. Certain characteristic forms on manifolds with boundary. Consider a compact Riemannian bordism $(M, \partial_+ M, \partial_- M)$. Consider normalized orthonogonal local frames $\{e^i\}_{i=1}^m$ as in Section 6.1.5. Consider the vector bundle $\mathcal{TM} \to \widetilde{M}$ endowed with the connection $\nabla^{\mathcal{TM}}$ in (6.8). Then, the corresponding curvature $\mathsf{R}^{\mathcal{TM}}$, see (6.9), considered as a smooth section of $\Lambda^2(T^*\widetilde{M}) \wedge \widehat{\Lambda^2(\mathcal{T}^*\widetilde{M})} \to \widetilde{M}$, can be expanded in terms of the frame above as

$$\mathbf{R}^{\mathcal{TM}} := \frac{1}{2} \sum_{1\leqslant k,l\leqslant m} g^{\mathcal{TM}}\left(e_k, \mathsf{R}^{\mathcal{TM}} e_l\right) \widehat{e^k} \wedge \widehat{e^l}.$$

The following definitions are inspired and strongly based on [**BM06**].

Definition 6.1.5. *Let ψ be a smooth unit normal vector field on the boundary. On the boundary, we set*

$$\begin{aligned}
i^*\mathbf{R}^{\mathcal{TM}} &:= \tfrac{1}{2} \sum_{1\leqslant k,l\leqslant m} g^{\mathcal{TM}}(e_k, i^*\mathsf{R}^{\mathcal{TM}} e_l)\widehat{e^k} \wedge \widehat{e^l} \\
&\in \Gamma(\widetilde{\partial M}; \Lambda^2(T^*\widetilde{\partial M}) \wedge \widehat{\Lambda^2(\mathcal{T}^*\mathcal{M})}),
\\[1em]
\mathbf{R}^{\mathcal{TM}}|_{\partial M} &:= \tfrac{1}{2} \sum_{1\leqslant\alpha,\beta\leqslant m-1} g^{\mathcal{TM}}(e_\alpha, i^*\mathsf{R}^{\mathcal{TM}} e_\beta)\widehat{e^\alpha} \wedge \widehat{e^\beta} \\
&\in \Gamma(\widetilde{\partial M}; \Lambda^2(T^*\widetilde{\partial M}) \wedge \Lambda^2(\widehat{\mathcal{T}^*(\partial\mathcal{M})}))),
\\[1em]
\mathbf{R}^{\mathcal{T}\partial\mathcal{M}} &:= \tfrac{1}{2} \sum_{1\leqslant\alpha,\beta\leqslant m-1} g^{\mathcal{T}\partial\mathcal{M}}(e_\alpha, i^*\mathsf{R}^{\mathcal{TM}} e_\beta)\widehat{e^\alpha} \wedge \widehat{e^\beta} \\
&\in \Gamma(\widetilde{\partial M}; \Lambda^2(T^*\widetilde{\partial M}) \wedge \Lambda^2(\widehat{\mathcal{T}^*(\partial\mathcal{M})}))).
\end{aligned}$$

(6.11)

$$\mathbf{S}_\psi := \tfrac{1}{2} \sum_{\beta=1}^{m-1} \left(\sum_{\alpha=1}^{m-1} g^{\mathcal{TM}}(\nabla_{e_\alpha}^{\mathcal{TM}} \psi, e_\beta) e^\alpha \right) \wedge \widehat{e^\beta} \quad \in \quad \Gamma(\widetilde{\partial M}; T^*\widetilde{\partial M} \wedge \Lambda^1(\widehat{\mathcal{T}^*(\partial\mathcal{M})})).$$

Remark 6.1.6. The form $\mathbf{S}_\psi$ in Definition 6.1.5 is slightly more general as the original one considered in [**BM06**]. For instance, think of ψ to be taken $\pm\varsigma$, or even ς_{in} or ς_{out} along the *whole* boundary ∂M. The definition of $\mathbf{S}_\psi$ is compatible with the corresponding one by Brüning and Ma in [**BM06**] and [**BM11**] in the following sense. If $\psi := \varsigma_{\mathsf{in}}$, then $\mathbf{S}_{\varsigma_{\mathsf{in}}}(\widetilde{\partial M})$ corresponds to $\dot{\mathbf{S}}$ in [**BM06**] and [**BM11**].

Definition 6.1.7. *Let ψ be a smooth unit normal vector field on the boundary. Consider the forms on the boundary from Definition 6.1.5 and the Berezin integrals $\int^{B_M}$ and $\int^{B_{\partial M}}$, from Section 6.1.3. We set*

$$\begin{array}{rcl}
\mathbf{e}(\widetilde{M},\nabla^{\mathcal{TM}}) &:=& \int^{B_M}\exp\left(-\frac{1}{2}\mathbf{R}^{\mathcal{TM}}\right),\\
\mathbf{e}(\widetilde{\partial M},\nabla^{\mathcal{T}\partial\mathcal{M}}) &:=& \int^{B_{\partial M}}\exp\left(-\frac{1}{2}\mathbf{R}^{\mathcal{T}\partial\mathcal{M}}\right),\\
\mathbf{e}_{\mathbf{b},\psi}(\widetilde{\partial M},\nabla^{\mathcal{TM}}) &:=& (-1)^{m-1}\int^{B_{\partial M}}\exp\left(-\frac{1}{2}(\mathbf{R}^{\mathcal{TM}}|_{\partial M})\right)\sum_{k=0}^{\infty}\frac{\mathbf{S}_\psi^k}{2\Gamma(\frac{k}{2}+1)},\\
B_\psi(\widetilde{\partial M},\nabla^{\mathcal{TM}}) &:=& -\int_0^1\frac{du}{u}\int^{B_{\partial M}}\exp\left(-\frac{1}{2}\mathbf{R}^{\mathcal{T}\partial\mathcal{M}}-u^2\mathbf{S}_\psi^2\right)\sum_{k=1}^{\infty}\frac{(u\mathbf{S}_\psi)^k}{2\Gamma(\frac{k}{2}+1)},
\end{array}$$

In particular, the forms from Brüning in Ma defined in [**BM06**] *are*

$$\begin{array}{rcl}
\mathbf{e}_{\mathbf{b}}(\widetilde{\partial M},\nabla^{\mathcal{TM}}) &:=& (-1)^{m-1}\int^{B_{\partial M}}\exp\left(-\frac{1}{2}(\mathbf{R}^{\mathcal{TM}}|_{\partial M})\right)\sum_{k=0}^{\infty}\frac{\mathbf{S}_{\mathsf{in}}^k}{2\Gamma(\frac{k}{2}+1)},\\
B(\widetilde{\partial M},\nabla^{\mathcal{TM}}) &:=& -\int_0^1\frac{du}{u}\int^{B_{\partial M}}\exp\left(-\frac{1}{2}\mathbf{R}^{\mathcal{T}\partial\mathcal{M}}-u^2\mathbf{S}_\psi^2\right)\sum_{k=1}^{\infty}\frac{(u\mathbf{S}_{\mathsf{in}})^k}{2\Gamma(\frac{k}{2}+1)},
\end{array}$$

Lemma 6.1.8. *Let ψ be a smooth unit normal vector field (without singularities) on the boundary. The forms from Definition 6.1.7 verify*

$$\begin{array}{rcl}
\mathbf{e}_{\mathbf{b},\psi}(\widetilde{\partial M},\nabla^{\mathcal{TM}}) &=& (-1)^{m-1}\mathbf{e}_{\mathbf{b},-\psi}(\widetilde{\partial M},\nabla^{\mathcal{TM}})\\
B_\psi(\widetilde{\partial M},\nabla^{\mathcal{TM}}) &=& (-1)^{m-1}B_{-\psi}(\widetilde{\partial M},\nabla^{\mathcal{TM}})
\end{array}$$

Proof. First, note that $\mathbf{S}_\psi = -\mathbf{S}_{-\psi}$. We compute $\mathbf{e}_{\mathbf{b},\psi}(\widetilde{\partial M},\nabla^{\mathcal{TM}})$ by recalling that Berezin integrals *see* top degrees terms only:

$$\begin{array}{rcl}
\mathbf{e}_{\mathbf{b},\psi}(\widetilde{\partial M},\nabla^{\mathcal{TM}}) &=& (-1)^{m-1}\int^{B_{\partial M}}\exp\left(-\frac{1}{2}(\mathbf{R}^{\mathcal{TM}}|_{\partial M})\right)\sum_{k=0}^{\infty}\frac{\mathbf{S}_\psi^k}{2\Gamma(\frac{k}{2}+1)},\\
&=& (-1)^{m-1}\int^{B_{\partial M}}\sum_{l=0}^{\infty}\frac{-\frac{1}{2}(\mathbf{R}^{\mathcal{TM}}|_{\partial M})^l}{l!}\sum_{k=0}^{\infty}\frac{(-1)^k\mathbf{S}_{-\psi}^k}{2\Gamma(\frac{k}{2}+1)},\\
&=& (-1)^{m-1}\int^{B_{\partial M}}\sum_{l,k=0}^{\infty}\frac{-\frac{1}{2}(\mathbf{R}^{\mathcal{TM}}|_{\partial M})^l}{l!}\frac{(-1)^k\mathbf{S}_{-\psi}^k}{2\Gamma(\frac{k}{2}+1)},\\
&=& (-1)^{m-1}\int^{B_{\partial M}}\sum_{l=0}^{\infty}\frac{-\frac{1}{2}(\mathbf{R}^{\mathcal{TM}}|_{\partial M})^l}{l!}\frac{(-1)^{m-(2l+1)}\mathbf{S}_{-\psi}^{m-(2l+1)}}{2\Gamma(\frac{m-(2l+1)}{2}+1)},\\
&=& (-1)^{m-1}\int^{B_{\partial M}}\sum_{l=0}^{\infty}\frac{-\frac{1}{2}(\mathbf{R}^{\mathcal{TM}}|_{\partial M})^l}{l!}\frac{(-1)^{m-1}\mathbf{S}_{-\psi}^{m-(2l+1)}}{2\Gamma(\frac{m-(2l+1)}{2}+1)},\\
&=& \int^{B_{\partial M}}\sum_{l=0}^{\infty}\frac{-\frac{1}{2}(\mathbf{R}^{\mathcal{TM}}|_{\partial M})^l}{l!}\frac{\mathbf{S}_{-\psi}^{m-(2l+1)}}{2\Gamma(\frac{m-(2l+1)}{2}+1)},\\
&=& \int^{B_{\partial M}}\sum_{l,k=0}^{\infty}\frac{-\frac{1}{2}(\mathbf{R}^{\mathcal{TM}}|_{\partial M})^l}{l!}\frac{\mathbf{S}_{-\psi}^k}{2\Gamma(\frac{k}{2}+1)},\\
&=& (-1)^{m-1}\mathbf{e}_{\mathbf{b},-\psi}(\widetilde{\partial M},\nabla^{\mathcal{TM}})
\end{array}$$

and analogously for the forms $B_{\mp\psi}(\widetilde{\partial M},\nabla^{\mathcal{TM}})$. $\square$

Definition 6.1.9. *Given a bordism $(M, \partial_+ M, \partial_- M)$, define the functions $\Pi_\pm : \partial M \to \mathbb{R}$ respectively by*

$$\Pi_+(y) := \begin{cases} 1 & \text{if } y \in \partial_+ M \\ 0 & \text{if } y \in \partial_- M \end{cases} \quad \text{and} \quad \Pi_-(y) := \begin{cases} 0 & \text{if } y \in \partial_+ M \\ 1 & \text{if } y \in \partial_- M. \end{cases} \tag{6.12}$$

Let $\psi := \varsigma$ specified by the vector field in (6.10). By using the forms in Definition 6.1.7, we set

$$\begin{aligned}
\mathbf{e}_\partial(\widetilde{\partial_+ M}, \widetilde{\partial_- M}, \nabla^{\mathcal{TM}}) &:= i_+^* \left(\mathbf{e}_{\mathbf{b},\varsigma}(\widetilde{\partial M}, \nabla^{\mathcal{TM}})\right) \Pi_{+1} - i_-^* \left(\mathbf{e}_{\mathbf{b},\varsigma}(\widetilde{\partial M}, \nabla^{\mathcal{TM}})\right) \Pi_{-1} \\
\mathbf{e}_\partial(\widetilde{\partial_- M}, \widetilde{\partial_+ M}, \nabla^{\mathcal{TM}}) &:= i_-^* \left(\mathbf{e}_{\mathbf{b},-\varsigma}(\widetilde{\partial M}, \nabla^{\mathcal{TM}})\right) \Pi_{-1} - i_+^* \left(\mathbf{e}_{\mathbf{b},-\varsigma}(\widetilde{\partial M}, \nabla^{\mathcal{TM}})\right) \Pi_{+1} \\
B(\widetilde{\partial_\pm M}, \widetilde{\partial_\mp M}, \nabla^{\mathcal{TM}}) &:= B_{\pm\varsigma}(\widetilde{\partial M}, \nabla^{\mathcal{TM}})
\end{aligned}$$

Lemma 6.1.10. *For the forms given in Definition 6.1.9, the relations*

$$\begin{aligned}
\mathbf{e}_\partial(\widetilde{\partial_+ M}, \widetilde{\partial_- M}, \nabla^{\mathcal{TM}}) &= (-1)^m \mathbf{e}_\partial(\widetilde{\partial_- M}, \widetilde{\partial_+ M}, \nabla^{\mathcal{TM}}) \\
B(\widetilde{\partial_+ M}, \widetilde{\partial_- M}, \nabla^{\mathcal{TM}}) &= (-1)^{m-1} B(\widetilde{\partial_- M}, \widetilde{\partial_+ M}, \nabla^{\mathcal{TM}})
\end{aligned}$$

hold.

Proof. This is clear from construction. □

Definition 6.1.11. *Let $(M, \partial_+ M, \partial_- M)$ be a Riemannian bordism. Consider the forms given in Definition 6.1.7 and Definition 6.1.9. For each $s \in \mathbb{R}$, we set*

$$\begin{aligned}
\mathbf{e}(M, g_s) &:= \mathbf{e}(\widetilde{M}, \nabla^{\mathcal{TM}})\Big|_{M \times \{s\}}, \\
\mathbf{e}(\partial M, g_s) &:= \mathbf{e}(\widetilde{\partial M}, \nabla^{\mathcal{T}\partial\mathcal{M}})\Big|_{\partial M \times \{s\}}, \\
\mathbf{e}_{\mathbf{b},\psi}(\partial M, g_s) &:= \mathbf{e}_{\mathbf{b},\psi}(\widetilde{\partial M}, \nabla^{\mathcal{T}\partial\mathcal{M}})\Big|_{\partial M \times \{s\}}, \\
\mathbf{e}_\partial(\partial_+ M, \partial_- M, g_s) &:= \mathbf{e}_\partial(\widetilde{\partial_+ M}, \widetilde{\partial_- M}, \nabla^{\mathcal{TM}})\Big|_{\partial M \times \{s\}}, \\
B(\partial_+ M, \partial_- M, g_s) &:= B(\widetilde{\partial_+ M}, \widetilde{\partial_- M}, \nabla^{\mathcal{TM}})\Big|_{M \times \{s\}}.
\end{aligned}$$

In particular, the forms from Brüning in Ma defined in [**BM06**] *and used in Chapter 4 and Chapter 5 to describe the anomaly formulas are*

$$\begin{aligned}
\mathbf{e}_{\mathbf{b}}(\partial M, g_s) &:= \mathbf{e}_{\mathbf{b}}(\widetilde{\partial M}, \nabla^{\mathcal{TM}})\Big|_{M \times \{s\}}, \\
B(\partial M, g_s) &:= B(\widetilde{\partial M}, \nabla^{\mathcal{TM}})\Big|_{M \times \{s\}},
\end{aligned}$$

see Definition 6.1.7

6.1.7. Relative Chern–Simons forms on bordisms.

Definition 6.1.12. *Let $(M, \partial_+ M, \partial_- M)$ be a Riemannian bordism. Consider the forms from Definition 6.1.11. We define the* ***relative Euler form***

$$\underline{\mathbf{e}}(M, \partial_+ M, \partial_- M, g) := (\mathbf{e}(M, g), \mathbf{e}_\partial(\partial_+ M, \partial_- M, g)) \in \Omega^m(M, \partial M; \Theta_M). \tag{6.13}$$

Moreover, with

$$\begin{array}{rcl} \widetilde{\mathbf{e}}\,(M, g_0, g_\tau) & := & \int_0^\tau \mathbf{incl}_s^* \left(\iota\left(\frac{\partial}{\partial s}\right) \mathbf{e}(\widetilde{M}, \nabla^{\mathcal{TM}})\right) ds \\ & \in & \Omega^{m-1}(M, \Theta_M), \\ \widetilde{\mathbf{e}}_\partial\,(\partial_+ M, \partial_- M, g_0, g_\tau) & := & \int_0^\tau \mathbf{incl}_s^* \left(\iota\left(\frac{\partial}{\partial s}\right) \mathbf{e}_\partial\,(\widetilde{\partial_+ M}, \widetilde{\partial_- M}, \nabla^{\mathcal{TM}})\right) ds \\ & \in & \Omega^{m-2}(\partial M, \Theta_M) \\ \widetilde{\mathbf{e}}_{\mathbf{b}}\,(\partial M, g_0, g_\tau) & := & \int_0^\tau \mathbf{incl}_s^* \left(\iota\left(\frac{\partial}{\partial s}\right) \mathbf{e}_{\mathbf{b}}\,(\widetilde{\partial M}, \nabla^{\mathcal{TM}})\right) ds \\ & \in & \Omega^{m-2}(\partial M, \Theta_M) \end{array} \tag{6.14}$$

where $\mathbf{incl}_s : M \to \widetilde{M}$ is the inclusion map given by $\mathbf{incl}_s(x) = (x, s)$ for $x_0 \in M$ and $s \in \mathbb{R}$, the relative form

$$\underline{\widetilde{\mathbf{e}}}(M, \partial_+ M, \partial_- M, g_0, g_\tau) \in \Omega^{m-1}(M, \partial M; \Theta_M)$$

defined by

$$\underline{\widetilde{\mathbf{e}}}(M, \partial_+ M, \partial_- M, g_0, g_\tau) := \left(\widetilde{\mathbf{e}}\,(M, g_0, g_\tau)\,, -\,\widetilde{\mathbf{e}}_\partial\,(\partial_+ M, \partial_- M, g_0, g_\tau)\right) \tag{6.15}$$

where $\widetilde{\mathbf{e}}\,(M, g_0, g_\tau)$ and $\widetilde{\mathbf{e}}_\partial\,(\partial_+ M, \partial_- M, g_0, g_\tau)$ are the forms of Chern–Simons type given in (6.14), will be called the ***secondary relative Euler form*** *associated to the bordism $(M, \partial_+ M, \partial_- M)$. In particular, the original secondary relative Euler forms from Brüning and Ma in* [**BM06**]*, is given by*

$$\underline{\widetilde{\mathbf{e}}}(M, \partial M, g_0, g_\tau) := \left(\widetilde{\mathbf{e}}\,(M, g_0, g_\tau)\,, -\,\widetilde{\mathbf{e}}_{\mathbf{b}}\,(\partial M, g_0, g_\tau)\right) \tag{6.16}$$

Lemma 6.1.13. ***(Brüning–Ma)*** *The relative Euler form $\underline{\mathbf{e}}(M, \partial_+ M, \partial_- M, g)$ in (6.13) associated to the metric g is closed in the relative cochain complex $\Omega(M, \partial M; \Theta_M)$ and modulo exact forms, does not depend on the choice of g. In other words, its class in cohomology*

$$[\underline{\mathbf{e}}(M, \partial_+ M, \partial_- M)] := [\underline{\mathbf{e}}(M, \partial_+ M, \partial_- M, g)] \tag{6.17}$$

is independent of g. The relative secondary Euler form $\underline{\widetilde{\mathbf{e}}}(M, \partial_+ M, \partial_- M, g_0, g_1)$ associated to a couple of Riemannian metrics g_0, g_1 in M, see (6.15), does not depend on the choice of the path of metrics, so that, it defines a secondary relative Euler class in the sense of Chern–Simons. If $\{g_s\}$ is a smooth path of Riemannian metrics connecting g_0 to g_1, then

$$\mathbf{d}\underline{\widetilde{\mathbf{e}}}(M, \partial_+ M, \partial_- M, g_0, g_1) = \underline{\mathbf{e}}(M, \partial_+ M, \partial_- M, g_1) - \underline{\mathbf{e}}(M, \partial_+ M, \partial_- M, g_1). \tag{6.18}$$

The secondary Euler form $\underline{\widetilde{\mathbf{e}}}(M, \partial_+M, \partial_-M, g_0, g_1)$, modulo exact forms, does not depend on the choice of the path of metrics, so that, it defines a secondary relative Euler class in the sense of Chern–Simons. Moreoever, up to exact forms in relative cohomology, the following relations hold

$$\begin{aligned} &\underline{\widetilde{\mathbf{e}}}(M, \partial_+M, \partial_-M, g_0, g_\tau) = -\underline{\widetilde{\mathbf{e}}}(M, \partial_+M, \partial_-M, g_\tau, g_0) \\ &\underline{\widetilde{\mathbf{e}}}(M, \partial_+M, \partial_-M, g_0, g_\tau) = \underline{\widetilde{\mathbf{e}}}(M, \partial_+M, \partial_-M, g_0, g_s) + \underline{\widetilde{\mathbf{e}}}(M, \partial_+M, \partial_-M, g_s, g_\tau) \end{aligned} \tag{6.19}$$

Proof. Since ∂_+M and ∂_+M are disjoint closed submanifolds, the statements above are exactly Theorem 1.9 in [**BM06**]. The identities in (6.19) follow straightforward from the definition of $\underline{\widetilde{\mathbf{e}}}(M, \partial_+M, \partial_-M, g_0, g_\tau)$ in (6.14). □

6.1.8. Gauss–Bonnet–Chern Theorem.

Theorem 6.1.14. *(**Brüning–Ma**) Let $(M; \partial_+M, \partial_-M)$ be a compact Riemannian bordism of dimension m and metric g. Let $\chi(M, \partial_-M)$ be the Euler characteristic relative to ∂_-M. If $\underline{\mathbf{e}}(M, \partial_+M, \partial_-M, g)$ is the relative Euler form given in (6.13), then*

$$\chi(M, \partial_-M) = \int_{(M,\partial M)} \underline{\mathbf{e}}(M, \partial_+M, \partial_-M, g)$$

Proof. Consider $E := M \times \mathbb{C}$, the trivial bundle over M, with a Hermitian metric h on $\mathbb{C}$. Let $\Delta_{E,g,h}$ the Hermitian Laplacian acting on $\Omega(M; E)|^h_{\mathcal{B}}$ the space of E-valued smooth forms satisfying absolute/boundary conditions on the bordism $(M, \partial_+M, \partial_-M)$, see Section 3.1. Let $\Delta_{\mathcal{B},h}$ be the corresponding L^2-realization for this boundary value problem. Since $\Delta_{\mathcal{B},h}$ is self-adjoint, from the McKean–Singer Theorem we know that the function $\mathrm{Tr}_{\mathbf{s}}(\exp(-t\Delta_{\mathcal{B},h}))$, i.e., the supertrace corresponding to the heat operator associated to $\Delta_{\mathcal{B},h}$ is independent of t. Thus, for t large, the heat operator $\exp(-t\Delta_{\mathcal{B},h,q})$ converges to the spectral projection onto the kernel of $\Delta_{\mathcal{B},h,q}$ in each degree q, so that

$$\lim_{t\to\infty} \mathrm{Tr}_{\mathbf{s}}(\exp(-t\Delta_{\mathcal{B},h})) = \chi(M, \partial_-M).$$

Therefore, for t small, we must have

$$\mathop{\mathrm{LIM}}_{t\to 0} (\mathrm{Tr}_{\mathbf{s}}(\exp(-t\Delta_{\mathcal{B},h}))) = \chi(M, \partial_-M).$$

By the local index Theorem, $\mathrm{LIM}_{t\to 0}(\mathrm{Tr}_{\mathbf{s}}(\exp(-t\Delta_{\mathcal{B},h})))$ can be analytically computed as the integral of certain characteristic classes. In our case this is directely obtained by setting $\phi = \mathsf{id}$ in the first formula of Theorem 4.2.7 (see also Theorem 3.4 in [**BM11**]).

Therefore, by Lemma 6.1.8 (see also Remark 6.1.6), we have

$$\begin{aligned}
\chi(M,\partial_-M) &= \int_M \mathbf{e}(M,g)+(-1)^{m-1}\int_{\partial_+M} i_+^*\mathbf{e}_{\mathbf{b},\varsigma_{\mathsf{in}}}(\partial M,g)-\int_{\partial_-M} i_-^*\mathbf{e}_{\mathbf{b},\varsigma_{\mathsf{in}}}(\partial M,g)\\
&= \int_M \mathbf{e}(M,g)-(-1)^{m}\int_{\partial_+M} i_+^*\mathbf{e}_{\mathbf{b},\varsigma_{\mathsf{in}}}(\partial M,g)-(-1)^{m-1}\int_{\partial_-M} i_-^*\mathbf{e}_{\mathbf{b},\varsigma_{\mathsf{out}}}(\partial M,g)\\
&= \int_M \mathbf{e}(M,g)-(-1)^{m}\int_{\partial_+M} i_+^*\mathbf{e}_{\mathbf{b},\varsigma_{\mathsf{in}}}(\partial M,g)-(-1)^{m}\int_{\partial_-M} i_-^*\big(-\mathbf{e}_{\mathbf{b},\varsigma_{\mathsf{out}}}(\partial M,g)\big)\\
&= (-1)^m\Big(\int_M \mathbf{e}(M,g)-\int_{\partial_+M} i_+^*\mathbf{e}_\partial(\partial_+M,\partial_-M,g)-\int_{\partial_-M} i_-^*\mathbf{e}_\partial(\partial_+M,\partial_-M,g)\Big)\\
&= (-1)^m\big(\int_M \mathbf{e}(M,g)-\int_{\partial M}\mathbf{e}_\partial(\partial_+M,\partial_-M,g)\big)\\
&= (-1)^m\int_{(M,\partial M)}\underline{\mathbf{e}}(M,\partial_+M,\partial_-M,g).
\end{aligned}$$

Remark also that if $\chi(M,\partial_+M)$ is the Euler characteristic relative to ∂_+M, then

$$\begin{aligned}
\chi(M,\partial_-M)&=(-1)^m\big(\int_M \mathbf{e}(M,g)-\int_{\partial M}\mathbf{e}_\partial(\partial_+M,\partial_-M,g)\big)\\
&=(-1)^m(-1)^m\big(\int_M \mathbf{e}(M,g)-\int_{\partial M}\mathbf{e}_\partial(\partial_-M,\partial_+M,g)\big)\\
&=(-1)^m\chi(M,\partial_+M).
\end{aligned}$$

□

6.2. CoEuler structures

6.2.1. CoEuler structures without base point. We extend the notion of coEuler structures in [**BH07**] to the case of compact Riemannian bordisms $(M,\partial_+M,\partial_-M)$.

Lemma 6.2.1. *Recall Definitions 6.1.12 and 6.1.11 together with the pairing $\langle\cdot,\cdot\rangle$ from Notation 6.1.2. Let $\underline{\mathbf{e}}(M,\partial_+M,\partial_-M,g)$ be the relative form given in (6.13). Suppose that $\chi(M,\partial_+M)=0$, i.e. the Euler Characteristic relative to ∂_+M vanishes. Then the set*

$$\mathbf{E}^*(M,\partial_+M,\partial_-M;\mathbb{C}) := \left\{(g,\underline{\alpha})\ \middle|\ \begin{array}{rcl}\underline{\alpha} & \in & \Omega^{m-1}(M,\partial M;\Theta_M^{\mathbb{C}})\\ \mathbf{d}\underline{\alpha} & = & \underline{\mathbf{e}}(M,\partial_+M,\partial_-M,g)\end{array}\right\} \tag{6.20}$$

is not empty, on which we define the following relation. We say that $(g,\underline{\alpha})\sim^{cs}(g',\underline{\alpha}')$ in $\mathbf{E}^*(M,\partial_+M,\partial_-M;\mathbb{C})$, if and only if

$$\underline{\alpha}'-\underline{\alpha}=\widetilde{\underline{\mathbf{e}}}(M,\partial_+M,\partial_-M,g,g')\in\Omega^{m-1}(M,\partial M;\Theta_M^{\mathbb{C}})/\mathbf{d}\Omega^{m-2}(M,\partial M;\Theta_M^{\mathbb{C}}),$$

where $\widetilde{\underline{\mathbf{e}}}(M,\partial_+M,\partial_-M,g,g')$ is the secondary form defined in (6.15). The relation $\sim^{cs}$ is an equivalence relation on $\mathbf{E}^*(M,\partial_+M,\partial_-M;\mathbb{C})$.

Proof. By Theorem 6.1.14, the relative Euler form $\underline{\mathbf{e}}(M,\partial_+M,\partial_-M,g)$ defined in (6.13) satisfies $\langle[\underline{\mathbf{e}}(M,\partial_+M,\partial_-M,g)],[1]\rangle=0$. Since $\langle\cdot,\cdot\rangle$ is nondegenerate, the relative form $\underline{\mathbf{e}}(M,\partial_+M,\partial_-M,g)$ is exact in relative cohomology. That is, there exists a relative form $\underline{\alpha}\in\Omega^{m-1}(M,\partial M;\Theta_M)$ such that $\mathbf{d}\underline{\alpha}=\underline{\mathbf{e}}(M,\partial_+M,\partial_-M,g)$. Hence the space $\mathbf{E}^*(M,\partial_+M,\partial_-M;\mathbb{C})$ is not empty. The relation $\sim^{cs}$ satisfies the reflexivity property, since $\widetilde{\underline{\mathbf{e}}}(M,\partial_+M,\partial_-M,g,g)=0$. Symmetry and transitivity of $\sim^{cs}$ are implied by Lemma 6.1.13. □

Definition 6.2.2. *Let $\mathbf{E}^*(M,\partial_+M,\partial_-M;\mathbb{C})$ be the space defined in (6.20). The set of* ***coEuler structures*** *on a compact Riemannian bordism $(M,\partial_+M,\partial_-M)$ is defined as the quotient*

$$\mathfrak{Eul}^*(M,\partial_+M,\partial_-M;\mathbb{C}) := \mathbf{E}^*(M,\partial_+M,\partial_-M;\mathbb{C})/\sim^{cs}; \tag{6.21}$$

the equivalence class of $(g,\underline{\alpha})$ will be denoted by $[g,\underline{\alpha}]$.

Lemma 6.2.3. *Let $H^{m-1}(M,\partial M;\Theta_M^{\mathbb{C}})$ the cohomology groups in degree $m-1$ relative to ∂M and coefficients in $\Theta_M^{\mathbb{C}}$. For a closed relative form $\underline{\beta}\in\Omega^{m-1}(M,\partial M;\Theta_M^{\mathbb{C}})$, denote by $[\underline{\beta}]$ its corresponding class in relative cohomology. Consider Υ, the action of $H^{m-1}(M,\partial M;\Theta_M^{\mathbb{C}})$ on the space of coEuler structures $\mathfrak{Eul}^*(M,\partial_+M,\partial_-M;\mathbb{C})$ from Definition 6.2.2, given by*

$$\begin{aligned}\Upsilon: H^{m-1}(M,\partial M;\Theta_M^{\mathbb{C}})\times\mathfrak{Eul}^*(M,\partial_+M,\partial_-M;\mathbb{C}) &\to \mathfrak{Eul}^*(M,\partial_+M,\partial_-M;\mathbb{C})\\ \big([\underline{\beta}],[g,\underline{\alpha}]\big)&\mapsto[g,\underline{\alpha}-\underline{\beta}].\end{aligned}\tag{6.22}$$

Then, Υ is well defined, independent of each choice of representatives, free and transitive on $\mathfrak{Eul}^*(M,\partial_+M,\partial_-M;\mathbb{C})$.

Proof. For $[\underline{\beta}]\in H^{m-1}(M,\partial M;\Theta_M^{\mathbb{C}})$, a class in relative cohomology represented by the closed relative form $\underline{\beta}\in\Omega^{m-1}(M,\partial M,\Theta_M^{\mathbb{C}})$, consider its action on the coEuler structure $[g,\underline{\alpha}]$, represented by the couple $(g,\underline{\alpha})$. Remark the couple $(g,\underline{\alpha}-\underline{\beta})$ is an element is $\mathbf{E}^*(M,\partial_+M,\partial_-M;\mathbb{C})$, because

$$\mathbf{d}(\underline{\alpha}-\underline{\beta})=\mathbf{d}\underline{\alpha}-\mathbf{d}\underline{\beta}=\mathbf{d}\underline{\alpha}=\underline{\mathbf{e}}(M,\partial_+M,\partial_-M,g)$$

Let us prove that Υ does not depend on the choice of representatives. The map Υ is independent of the choice of representative for the coEuler class. Indeed, let $(g',\underline{\alpha}')$ be representing the same class as $(g,\underline{\alpha})$ in $\mathfrak{Eul}^*(M,\partial_+M,\partial_-M;\mathbb{C})$ for which we have $\Upsilon_{\underline{\beta}}(g',\underline{\alpha}')=(g',(\underline{\alpha}'-\underline{\beta}))$. Since $(\underline{\alpha}'-\underline{\beta})-(\underline{\alpha}-\underline{\beta})=(\underline{\alpha}'-\underline{\alpha})=\widetilde{\underline{\mathbf{e}}}(M,\partial_+M,\partial_-M,g,g')$ modulo relative exact forms, we have

$$\Upsilon_{\underline{\beta}}([g,\underline{\alpha}])=\Upsilon_{\underline{\beta}}([g',\underline{\alpha}']).$$

The map Υ is also independent of the choice of the representative for the class in cohomology $[\underline{\beta}]$. Indeed, different choices for the cohomology class of $\underline{\beta}$ are obtained by adding coboundaries in $\Omega^{m-1}(M,\partial_+M;\Theta_M^{\mathbb{C}})$, that is $\underline{\beta}+\mathbf{d}\underline{\beta}'$. But for these forms we have

$$\Upsilon_{\underline{\beta}}([g,\underline{\alpha}])=\Upsilon_{\underline{\beta}+\mathbf{d}\underline{\beta}'}([g,\underline{\alpha}]),$$

since the equivalence relation $\sim^{cs}$ is given up to relative exact forms only, see Lemma 6.2.1. So, we have proved Υ is well defined and independent of every choice of representatives.

The same argument is used to see that the group $H^{m-1}(M,\partial_+M,\partial_-M;\Theta_M^{\mathbb{C}})$ acts freely on $\mathfrak{Eul}^*(M,\partial_+M,\partial_-M;\mathbb{C})$. Indeed, if $\underline{\beta}$ is such that $[g,\underline{\alpha}-\underline{\beta}]=[g,\underline{\alpha}]$, then

$$\underline{\beta}=\underline{\widetilde{\mathbf{e}}}(M,\partial_+M,\partial_-M,g,g)+\mathbf{d}\underline{\beta}',$$

but, since the first term on the right hand side in the equality above vanish, the relative form $\underline{\beta}$ is necessarily exact. We show this action is transitive on $\mathfrak{Eul}^*(M,\partial_+M,\partial_-M;\mathbb{C})$: for two classes $[g,\underline{\alpha}]$ and $[g',\underline{\alpha}']$, we can choose the relative form

$$\underline{\beta}:=(\underline{\alpha}-\underline{\alpha}')+\underline{\widetilde{\mathbf{e}}}(M,\partial_+M,\partial_-M,g,g')$$

By Lemma 6.1.13, the relative form $\underline{\beta}$ is closed:

$$\begin{aligned}\mathbf{d}\underline{\beta} &= \underline{\mathbf{e}}(M,\partial_+M,\partial_-M,g)-\underline{\mathbf{e}}(M,\partial_+M,\partial_-M,g')+\mathbf{d}\underline{\widetilde{\mathbf{e}}}(M,\partial_+M,\partial_-M,g,g')\\ &= 0\end{aligned}$$

Finally by construction, we have $\Upsilon_{[\underline{\beta}]}([g,\underline{\alpha}])=[g,\underline{\alpha}-\underline{\beta}]=[g',\underline{\alpha}']$. □

6.2.2. The flip map for coEuler Structures. Consider a compact Riemannian bordism $(M,\partial_+M,\partial_-M)$ together with its dual $(M,\partial_-M,\partial_+M)$ and the corresponding spaces of coEuler structures $\mathfrak{Eul}^*(M,\partial_\pm M,\partial_\mp M;\mathbb{C})$, see Definition 6.2.2. In view of Lemma 6.1.10, there is a natural involution

$$\begin{array}{rrcl}\nu: & \mathfrak{Eul}^*(M,\partial_+M,\partial_-M;\mathbb{C}) & \to & \mathfrak{Eul}^*(M,\partial_-M,\partial_+M;\mathbb{C})\\ & [g,\underline{\alpha}] & \mapsto & [g,(-1)^m\underline{\alpha})]\end{array}\tag{6.23}$$

which is affine over the involution in relative cohomology

$$(-1)^m\cdot\mathsf{id}:H^{m-1}(M,\partial M;\Theta_M^{\mathbb{C}})\to H^{m-1}(M,\partial M;\Theta_M^{\mathbb{C}}).$$

Remark 6.2.4. If M is a closed Riemannian manifold, i.e., both ∂_+M and ∂_-M are empty, then $\mathfrak{Eul}^*(M,\emptyset,\emptyset;\mathbb{C})$, affine over $H^{m-1}(M;\Theta_M^{\mathbb{C}})$, coincides with $\mathfrak{Eul}^*(M;\mathbb{C})$, the set of coEuler structures on a manifold without boundary (see [**BH07**] and [**BH06a**]). If M is closed and of odd dimension, then the involution ν, being affine over $-\mathsf{id}$, possesses a unique fixed point in $\mathfrak{Eul}^*(M;\mathbb{C})$, which corresponds to the canonic coEuler structure

$$\mathfrak{e}^*_{\mathsf{can}}:=[g,(\alpha_{\mathsf{can}}=0,\alpha_\partial=0)]$$

where $\alpha_{\mathsf{can}}=0$, because for odd dimensional closed manifolds $\mathbf{e}(M,g)=0$ and forms $\alpha_\partial=0$, see section 2.2 in [**BH07**].

6.2.3. Variation formulas for coEuler structures without base point. The following result generalizes the formula (56) in [**BH07**] and it used to encode the variation of the complex-valued analytic torsion on bordisms, see Section 6.3.

Proposition 6.2.5. *Let $(M,\partial_+M,\partial_-M)$ be a compact Riemannian bordism. Assume $\chi(M,\partial_+M)=0$. Consider $\{(g_u,\underline{\alpha}_u)\}_u$ a smooth real one-parameter family of Riemannian metrics g_u and relative forms $\underline{\alpha}_u$, representing $[g_u,\underline{\alpha}_u]\in\mathfrak{Eul}^*(M,\partial_+M,\partial_-M;\mathbb{C})$, i.e., the same coEuler structure. For g_u consider the forms $\underline{\mathbf{e}}(M,\partial_+M,\partial_-M,g_u)\in$*

*$\Omega^m(M,\partial M;\Theta_M)$ and $B(\partial_+ M,\partial_- M,g_u)\in\Omega^{m-1}(\partial M;\Theta_M)$ from Definition 6.1.11 as well as the relative Chern-Simon's form $\underline{\widetilde{\mathbf{e}}}(M,\partial_+ M,\partial_- M,g_u,g_w)\in\Omega^{m-1}(M,\partial M;\Theta_M)$ from Definition 6.1.12. Let E be a complex flat vector bundle over M with flat connection ∇^E, endowed with a smooth family of nondegenerate symmetric bilinears forms b_u. Let $\omega(\nabla^E,b_u):=-\frac{1}{2}\operatorname{Tr}(b_u^{-1}\nabla^E b_u)\in\Omega^1(M;\mathbb{C})$ be the **Kamber–Tondeur** form associated to b_u and ∇^E. Recall the integral $\int_{(M,\partial M)}$ from Notation 6.1.2. Then, the formulas*

(6.24)
$$\begin{aligned}&\tfrac{\partial}{\partial u}\int_{(M,\partial M)}2\underline{\alpha}_u\wedge\omega(\nabla^E,b_u)\\&\qquad=-(-1)^m\int_{(M,\partial M)}\underline{\mathbf{e}}(M,\partial_+ M,\partial_- M,g_u)\operatorname{Tr}\left(b_u^{-1}\dot b_u\right)\\&\qquad\quad+2\int_{(M,\partial M)}\tfrac{\partial}{\partial\tau}\Big|_{\tau=0}\underline{\widetilde{\mathbf{e}}}(M,\partial_+ M,\partial_- M,g_u,g_u+\tau\dot g_u)\wedge\omega(\nabla^E,b_u)\end{aligned}$$

and

$$\frac{\partial}{\partial u}\int_{\partial M}B(\partial_+ M,\partial_- M,g_u)=\int_{\partial M}\frac{\partial}{\partial\tau}\bigg|_{\tau=0}B(\partial_+ M,\partial_- M,g+\tau\dot g_u). \tag{6.25}$$

hold.

Proof. First, remark that

$$\begin{aligned}\tfrac{\partial}{\partial u}\underline{\alpha}_u&=\tfrac{\partial}{\partial w}\big|_u(\underline{\alpha}_w-\underline{\alpha}_u)\\&=\tfrac{\partial}{\partial w}\big|_u\underline{\widetilde{\mathbf{e}}}(M,\partial_+ M,\partial_- M,g_u,g_w)\\&=\tfrac{\partial}{\partial\tau}\big|_0\underline{\widetilde{\mathbf{e}}}(M,\partial_+ M,\partial_- M,g_u,g_u+\tau\dot g_u),\end{aligned}$$

and analogously

$$\tfrac{\partial}{\partial u}B(\partial_+ M,\partial_- M,g_u)=\tfrac{\partial}{\partial\tau}\big|_0B(\partial_+ M,\partial_- M,g+\tau\dot g_u).$$

Also, we have, see [**BH07**]:

$$\begin{aligned}\tfrac{\partial}{\partial u}\operatorname{Tr}(b_u^{-1}\nabla^E b_u)&=\operatorname{Tr}\big(-b_u^{-1}\dot b_u b_u^{-1}\nabla^E b_u\big)+\operatorname{Tr}\big(b_u^{-1}\nabla^E\dot b_u\big)\\&=\operatorname{Tr}\big(-b_u^{-1}(\nabla^E b_u)b_u^{-1}\dot b_u\big)+\operatorname{Tr}\big(b_u^{-1}\nabla^E\dot b_u\big)\\&=\operatorname{Tr}\big((\nabla^E b_u^{-1})\dot b_u\big)+\operatorname{Tr}\big(b_u^{-1}\nabla^E\dot b_u\big)\\&=\operatorname{Tr}\big(\nabla^E(b_u^{-1}\dot b_u)\big)\\&=d\operatorname{Tr}\big(b_u^{-1}\dot b_u\big).\end{aligned}$$

Therefore, since for each u, the couple $[g_u,\underline{\alpha}_u]$ represents the same coEuler structure, we obtain, modulo exact relative forms

$$\begin{aligned}&\tfrac{\partial}{\partial u}\int_{(M,\partial M)}2\underline{\alpha}_u\wedge\omega(\nabla^E,b_u)\\&=\int_{(M,\partial M)}\partial_w|_u(\underline{\alpha}_w)\wedge2\omega(\nabla^E,b_u)+\int_{(M,\partial M)}\underline{\alpha}_w\wedge\partial_w|_u(-\operatorname{Tr}(b_w^{-1}\nabla^E b_u))\\&=2\int_{(M,\partial M)}\tfrac{\partial}{\partial\tau}\big|_0\underline{\widetilde{\mathbf{e}}}(M,\partial_+ M,\partial_- M,g_u,g_u+\tau\dot g_u)\wedge\omega(\nabla^E,b_u)+\underbrace{\int_{(M,\partial M)}-\underline{\alpha}_u\wedge d\operatorname{Tr}\big(b_u^{-1}\dot b_u\big)}_{(*)};\end{aligned}$$

with $\underline{\alpha}_u=(\alpha_u,\sigma_u)$, $\mathbf{d}\underline{\alpha}_u=\underline{\mathbf{e}}(M,\partial_+M,\partial_-M,g_u)$ and Stokes' Theorem, the second term on the right above becomes

$$\begin{aligned}(*)&=-(-1)^m\Big(\textstyle\int_M d\alpha_u\operatorname{Tr}\big(b_u^{-1}\dot b_u\big)-\Big(\int_{\partial M}i^*\big(\alpha_u\operatorname{Tr}\big(b_u^{-1}\dot b_u\big)\big)-\int_{\partial M}\big(d^\partial\sigma_u i^*\operatorname{Tr}\big(b_u^{-1}\dot b_u\big)\big)\Big)\Big)\\&=-(-1)^m\Big(\textstyle\int_M d\alpha_u\operatorname{Tr}\big(b_u^{-1}\dot b_u\big)-\int_{\partial M}(i^*\alpha_u-d^\partial\sigma_u)i^*\operatorname{Tr}\big(b_u^{-1}\dot b_u\big)\Big)\\&=-(-1)^m\textstyle\int_{(M,\partial M)}\mathbf{d}\underline{\alpha}\operatorname{Tr}\big(b_u^{-1}\dot b_u\big)\\&=-(-1)^m\textstyle\int_{(M,\partial M)}\underline{\mathbf{e}}(M,\partial_+M,\partial_-M,g_u)\operatorname{Tr}\big(b_u^{-1}\dot b_u\big).\end{aligned}$$

The proof is complete. □

6.2.4. CoEuler structures with base point. We drop the conditions on the relative Euler characteristics $\chi(M,\partial_+M)$ to define coEuler structures. As in the case of a closed manifold, we do this by considering a base point and defining the set of coEuler structures based at a point.

Consider $(g,\underline{\alpha})$, where $\underline{\alpha}\in\Omega^{m-1}(\dot M,\partial M;\Theta_M^{\mathbb{C}})$, see (6.5), and define

$$\mathbf{E}^*_{x_0}(M,\partial_+M,\partial_-M;\mathbb{C}):=\left\{(g,\underline{\alpha})\;\middle|\;\begin{array}{rcl}\underline{\alpha}&\in&\Omega^{m-1}(\dot M,\partial M;\Theta_M^{\mathbb{C}})\\ \mathbf{d}\underline{\alpha}&=&\underline{\mathbf{e}}(M,\partial_+M,\partial_-M,g)\end{array}\right\}.\tag{6.26}$$

In view of $H^m(\dot M,\partial M;\Theta_M^{\mathbb{C}})\cong 0$, see Lemma 6.1.4, these sets are non-empty. Then, as for the case without base point, $(g,\underline{\alpha})\sim^{cs}(g',\underline{\alpha}')$ in $\mathbf{E}^*_{x_0}(M,\partial_+M,\partial_-M;\mathbb{C})$ if and only if

$$\underline{\alpha}'-\underline{\alpha}=\widetilde{\underline{\mathbf{e}}}(M,\partial_+M,\partial_-M,g,g')\quad\text{in}\quad\Omega^{m-1}(\dot M,\partial M;\Theta_M^{\mathbb{C}})/\mathbf{d}\Omega^{m-2}(\dot M,\partial M;\Theta_M^{\mathbb{C}}).\tag{6.27}$$

The relation in (6.27) is an equivalence relation for the same reasons as in the case without base point. The corresponding quotient space

$$\mathfrak{Eul}^*_{x_0}(M,\partial_+M,\partial_-M;\mathbb{C}):=\mathbf{E}^*_{x_0}(M,\partial_+M,\partial_-M;\mathbb{C})/\sim^{cs}\tag{6.28}$$

is called the space of **coEuler structures based at** x_0 **on** $(M,\partial_+M,\partial_-M)$ and an equivalence class is denoted by $[g,\underline{\alpha}]$. Furthermore, the action of $H^{m-1}(M,\partial M;\Theta_M^{\mathbb{C}})$ on $\mathfrak{Eul}^*_{x_0}(M,\partial_+M,\partial_-M;\mathbb{C})$ defined by

$$\begin{aligned}\Upsilon:H^{m-1}(M,\partial M;\Theta_M^{\mathbb{C}})\times\mathfrak{Eul}^*_{x_0}(M,\partial_+M,\partial_-M;\mathbb{C})&\to\mathfrak{Eul}^*_{x_0}(M,\partial_+M,\partial_-M;\mathbb{C})\\ \big([\underline{\beta}],[g,\underline{\alpha}]\big)&\mapsto[g,\underline{\alpha}-\underline{\beta}]\end{aligned}\tag{6.29}$$

is well defined and independent of each choice of representatives, see Lemma 6.2.3; this action also is free and transitive since $H^{m-1}(M,\partial M)\cong H^{m-1}(\dot M,\partial M)$, see Lemma 6.1.4. Finally, the *flip* map

$$\begin{array}{rrcl}\nu: & \mathfrak{Eul}^*_{x_0}(M,\partial_+M,\partial_-M;\mathbb{C}) & \to & \mathfrak{Eul}^*_{x_0}(M,\partial_-M,\partial_+M;\mathbb{C})\\ & [g,\underline{\alpha}] & \mapsto & [g,(-1)^m\underline{\alpha})]\end{array}\tag{6.30}$$

intertwines the spaces $\mathfrak{Eul}^*_{x_0}(M,\partial_\pm M,\partial_\mp M;\mathbb{C})$ and it is affine over the involution in relative cohomology

$$(-1)^m\mathsf{id}:H^{m-1}(\dot M,\partial M;\Theta_M^{\mathbb{C}})\to H^{m-1}(\dot M,\partial M;\Theta_M^{\mathbb{C}}).$$

6.2.5. Variational formula for coEuler structures with base point. The main of this section is to give an analog to Proposition 6.2.5 in the case of coEuler structures with base point. Let $\alpha \in \Omega^{m-1}(\dot{M};\Theta_M^{\mathbb{C}})$ be a smooth differential form on M, with possible singularity $x_0 \in \mathbf{int}(M)$, the interior of M and $\dot{M} := M\backslash\{x_0\}$. For ω a closed 1-form on M, we make sense of integrals of the type $\int_M \alpha\wedge\omega$, by means of a **regularization procedure** as described in the remaining of this section.

The **local degree** of α at the singularity x_0, see for instance section 11 in [**BT82**], is given by

$$\mathsf{deg}_{x_0}(\alpha) := \lim_{\delta\to 0}\int_{\partial(\mathbb{B}^m(\delta,x))} i^*\alpha, \tag{6.31}$$

where $\partial(\mathbb{B}^m(\delta,x))$ indicates the boundary of the m-dimensional closed ball $\mathbb{B}^m(\delta,x)$ centered at x_0 and radius $\delta>0$. With the standard sign convention involved in Stokes' Theorem, $\partial(\mathbb{B}^m(\delta,x))$ is oriented with respect to the unit outwards point vector field normal to $\mathbb{B}^m(\delta,x)$.

Lemma 6.2.6. *Let α be a smooth form in $\Omega^{m-1}(\dot{M};\Theta_M^{\mathbb{C}})$ such that $d\alpha$ and α_∂ are smooth and without singularities in M. For ω a smooth closed 1-form on M, choose a smooth function $f\in C^\infty(M)$ such that the 1-form*

$$\omega' := \omega - df$$

is smooth on M and vanishes on a small neighborhood of x_0. Then the complex-valued function

$$\mathcal{S}(\underline{\alpha},\omega,f) := \int_{(M,\partial M)} \underline{\alpha}\wedge\omega' + (-1)^m\int_{(M,\partial M)} \mathbf{d}\underline{\alpha}\wedge f - f(x_0)\mathsf{deg}_{x_0}(\alpha), \tag{6.32}$$

does not depend on the choice of f and satisfies the following assertions.

(1) If $\underline{\beta}\in\Omega^{m-1}(M,\partial M;\Theta_M^{\mathbb{C}})$, i.e., without singularities, then

$$\mathcal{S}(\underline{\beta},\omega) = \int_{(M,\partial M)} \underline{\beta}\wedge\omega.$$

In particular, $\mathcal{S}(\mathbf{d}\underline{\gamma},\omega)=0$ for all $\underline{\gamma}\in\Omega^{m-2}(M,\partial M;\Theta_M^{\mathbb{C}})$.

(2) $\mathcal{S}(\omega,\underline{\alpha})$ is linear in $\underline{\alpha}$ and in ω.

(3) $\mathcal{S}(\underline{\alpha},dh) = (-1)^m\int_{(M,\partial M)}\mathbf{d}\underline{\alpha}\wedge h - h(x)\mathsf{deg}_{x_0}(\alpha)$.

Proof. Without loss of generality assume $\mathcal{X}(\alpha)=\{x\}$. We want to know how the function $\int_{(M,\partial M)}\underline{\alpha}\wedge\omega'$ changes, with respect to f. Let us take $f_1, f_2\in C^\infty(M)$ two functions as above, such that the corresponding one forms ω_1',ω_2' vanish on a small open neighborhood of x_0, so that $d(f_2-f_1)=0$ locally around x_0; that means f_2-f_1 is constant[1] on a small neighborhood of x_0. Now, consider the variation

$$\begin{array}{rcl}\Delta & = & \int_{(M,\partial M)} \underline{\alpha}\wedge(w_2'-w_1')\\ & = & \int_{M\setminus\{x\}} \alpha\wedge(w_2'-w_1')-\int_{\partial M} \alpha_\partial\wedge i^*(w_2'-w_1')\\ & = & -\int_{M\setminus\{x\}} \alpha\wedge d(f_2-f_1)+\int_{\partial M} \alpha_\partial\wedge i^*(d(f_2-f_1)),\end{array}$$

We develop both terms on the right of the last equality above. The first one, the integral over M, can be re written as

$$-\int_{M\setminus\{x\}} \alpha\wedge d(f_2-f_1)=-(-1)^{m-1}\int_{M\setminus\{x\}} d(\alpha(f_2-f_1))+(-1)^{m-1}\int_{M\setminus\{x\}} d\alpha\wedge(f_2-f_1),$$

whereas the second one, the integral over the boundary becomes

$$\begin{array}{rcl}\int_{\partial M} \alpha_\partial\wedge d^\partial i^*(f_2-f_1) & = & (-1)^{m-2}\underbrace{\int_{\partial M} d^\partial(\alpha_\partial\wedge i^*(f_2-f_1))}_{=0}-(-1)^{m-2}\int_{\partial M} d^\partial\alpha_\partial\wedge i^*(f_2-f_1)\\ & = & (-1)^{m-1}\int_{\partial M} d^\partial\alpha_\partial\wedge i^*(f_2-f_1)\end{array}$$

and therefore

$$\begin{array}{rcl}\Delta & = & -(-1)^{m-1}\Big(\int_{M\setminus\{x\}} d(\alpha(f_2-f_1))-\int_{M\setminus\{x\}} d\alpha\wedge(f_2-f_1)-\int_{\partial M} d^\partial\alpha_\partial\wedge i^*(f_2-f_1)\Big)\\ & = & -(-1)^{m-1}\Big(\int_{M\setminus\{x\}} d(\alpha(f_2-f_1))-\int_{M} d\alpha\wedge(f_2-f_1)-\int_{\partial M} d^\partial\alpha_\partial\wedge i^*(f_2-f_1)\Big),\end{array}$$

where we have used

$$\int_{M\setminus\{x\}} d\alpha\wedge(f_2-f_1)=\int_{M} d\alpha\wedge(f_2-f_1),$$

since by assumption, the form $d\alpha$ does not have singularities on M. Hence, to make sense of Δ, we now make sense of the integral $\int_{M\setminus\{x\}} d(\alpha(f_2-f_1))$. This integral can be computed as the limit:

$$\int_{M\setminus\{x\}} d(\alpha(f_2-f_1)):=\lim_{\delta\to 0}\int_{M\setminus\mathbb{B}(\delta,x)} d(\alpha(f_2-f_1))$$

where $\mathbb{B}(\delta,x)$ is the closed ball centered at x_0 of radius $\delta>0$ and with boundary $\partial(\mathbb{B}(\delta,x))$ endowed with the orientation specified by the unit outwards pointing vector field normal to $\mathbb{B}(\delta,x)$. Then, by using Stokes' Theorem with the standard convention, the limit above can be computed as

$$\begin{array}{rcl}\int_{M\setminus\{x\}} d(\alpha(f_2-f_1)) & = & \lim_{\delta\to 0}\int_{M\setminus\mathbb{B}(\delta,x)} d(\alpha(f_2-f_1))\\ & = & \lim_{\delta\to 0}\int_{\partial(M\setminus\mathbb{B}(\delta,x))} i^*(\alpha(f_2-f_1))\\ & = & \lim_{\delta\to 0}\Big(\int_{\partial M} i^*(\alpha(f_2-f_1))+\int_{-\partial(\mathbb{B}^m(\delta,x))} i^*(\alpha(f_2-f_1))\Big)\\ & = & \int_{\partial M} i^*(\alpha(f_2-f_1))+\lim_{\delta\to 0}\int_{-\partial(\mathbb{B}^m(\delta,x))} i^*(\alpha(f_2-f_1)),\end{array}$$

where $-\partial(\mathbb{B}^m(\delta,x))$ indicates the sphere with opposite orientation as $\partial(\mathbb{B}(\delta,x))$.Now, we look at the second term on the right of the equality above. Since f_2-f_1 is constant on a small neighborhood of x_0, we have, for $\delta'>0$ small enough,

$$\begin{array}{rcl}\lim_{\delta\to 0}\int_{-\partial(\mathbb{B}^m(\delta,x))} i^*(\alpha(f_2-f_1)) & = & (f_2-f_1)(x')\lim_{\delta\to 0}\int_{-\partial(\mathbb{B}^m(\delta,x))} i^*\alpha \quad \text{for all } x'\in\mathbb{B}(\delta',x),\\ & = & (-1)^m(f_2-f_1)(x)\mathsf{deg}_{x_0}(\alpha),\end{array}$$

[1]If we choose $f_2(x)=f_1(x)=0$, then $f_2-f_1=0$ around x_0.

where the sign $(-1)^m$ above comes from the standard convention taken for the Stokes' Theorem. Hence

$$\int_{M\setminus\{x\}} d(\alpha(f_2-f_1))=\int_{\partial M} i^*(\alpha(f_2-f_1))+(-1)^m(f_2-f_1)(x)\mathsf{deg}_{x_0}(\alpha).$$

Therefore the variation Δ becomes

$$\begin{aligned}\Delta &= -(-1)^{m-1}\Big[\int_{\partial M} i^*(\alpha(f_2-f_1))+(-1)^m(f_2-f_1)(x)\mathsf{deg}_{x_0}(\alpha)-\int_M d\alpha\wedge(f_2-f_1)-\int_{\partial M} d^\partial\alpha_\partial\wedge i^*(f_2-f_1)\Big]\\
&= -(-1)^{m-1}\Big[\int_{\partial M}(i^*\alpha-d^\partial\alpha_\partial)\wedge i^*(f_2-f_1)-\int_M d\alpha\wedge(f_2-f_1)+(-1)^m(f_2-f_1)(x)\mathsf{deg}_{x_0}(\alpha)\Big]\\
&= -(-1)^{m-1}\Big[-\int_{(M,\partial M)}\mathbf{d}\underline{\alpha}\wedge(f_2-f_1)+(-1)^m(f_2-f_1)(x)\mathsf{deg}_{x_0}(\alpha)\Big]\\
&= -\Big((-1)^m\int_{(M,\partial M)}\mathbf{d}\underline{\alpha}\wedge(f_2-f_1)-(f_2-f_1)(x)\mathsf{deg}_{x_0}(\alpha)\Big),\end{aligned}$$

and so

$$\begin{aligned}\mathcal{S}_{f_2}(\underline{\alpha},\omega)-\mathcal{S}_{f_1}(\underline{\alpha},\omega) &=\Delta+\Big((-1)^m\int_{(M,\partial M)}\mathbf{d}\underline{\alpha}\wedge(f_2-f_1)-(f_2-f_1)(x)\mathsf{deg}_{x_0}(\alpha)\Big)\\ &=0,\end{aligned}$$

so $\mathcal{S}_f(\underline{\alpha},\omega)$ does not depend on the choice of f. Remark that linearity of $\mathcal{S}(\underline{\alpha},\omega)$ with respect to ω immediately follows also from its the independence of f. The remaining assertions in (1) and (2) follow from similar considerations as above, we omit the details. Let us turn to assertion (3). For a smooth function h, we compute

$$\begin{aligned}\mathcal{S}_f(\underline{\alpha},\omega+dh) &= \int_{(M,\partial M)}\underline{\alpha}\wedge(\omega+dh-df)+(-1)^m\int_{(M,\partial M)}\mathbf{d}\underline{\alpha}\wedge f-f(x_0)\mathsf{deg}_{x_0}(\alpha),\\
&= \int_{(M,\partial M)}\underline{\alpha}\wedge(\omega+d(h-f))+(-1)^m\int_{(M,\partial M)}(\mathbf{d}\underline{\alpha}\wedge(f-h)+\mathbf{d}\underline{\alpha}\wedge h)\\
&\qquad -(f(x_0)-h(x))\mathsf{deg}_{x_0}(\alpha)-h(x)\mathsf{deg}_{x_0}(\alpha),\\
&= \int_{(M,\partial M)}\underline{\alpha}\wedge(\omega-d(f-h))+(-1)^m\int_{(M,\partial M)}\mathbf{d}\underline{\alpha}\wedge(f-h)-(f-h)(x)\mathsf{deg}_{x_0}(\alpha)+\\
&\qquad (-1)^m\int_{(M,\partial M)}\mathbf{d}\underline{\alpha}\wedge h-h(x)\mathsf{deg}_{x_0}(\alpha),\\
&= \mathcal{S}_{f-h}(\underline{\alpha},\omega)+(-1)^m\int_{(M,\partial M)}\mathbf{d}\underline{\alpha}\wedge h-h(x)\mathsf{deg}_{x_0}(\alpha).\end{aligned}$$

that is,

$$\begin{aligned}(-1)^m\int_{(M,\partial M)}\mathbf{d}\underline{\alpha}\wedge h-h(x)\mathsf{deg}_{x_0}(\alpha) &= \mathcal{S}_f(\underline{\alpha},\omega+dh)-\mathcal{S}_{f-h}(\underline{\alpha},\omega)\\
&= \mathcal{S}_f(\underline{\alpha},\omega+dh)-\mathcal{S}_f(\underline{\alpha},\omega)\\
&= \mathcal{S}_f(\underline{\alpha},dh)\\
&= \mathcal{S}(\underline{\alpha},dh),\end{aligned}$$

where the second equality above holds, since $\mathcal{S}$ is independent of f and the third one because $\mathcal{S}$ is linear on ω. □

Corollary 6.2.7. *Let α be as in Lemma 6.2.6. Then, we have the formula*

$$\mathsf{deg}_{x_0}(\alpha)=(-1)^m\int_{(M,\partial M)}\mathbf{d}\underline{\alpha}.$$

Proof. Let ω, f, $\underline{\alpha}$ and $\mathcal{X}(\alpha)$ be as above and consider f_0 to be a constant function on M. Then we compute

$$\begin{aligned}\mathcal{S}_{f+f_0}(\underline{\alpha},\omega) &= \int_{(M,\partial M)} \underline{\alpha}\wedge\omega' + (-1)^m \int_{(M,\partial M)} \mathbf{d}\underline{\alpha}\wedge f + (-1)^m f_0 \int_{(M,\partial M)} \mathbf{d}\underline{\alpha} - \big(f(x_0)\mathsf{deg}_{x_0}(\alpha) + f_0 \mathsf{deg}_{x_0}(\alpha)\big),\\ &= \mathcal{S}_f(\underline{\alpha},\omega) + f_0\Big((-1)^m \int_{(M,\partial M)} \mathbf{d}\underline{\alpha} - \mathsf{deg}_{x_0}(\alpha)\Big).\end{aligned}$$

But, from Lemma 6.2.6 above, we know $\mathcal{S}_{f+f_0}(\underline{\alpha},\omega) = \mathcal{S}_f(\underline{\alpha},\omega)$, and hence the last term on the right above vanishe, so that the desired relation between the total degree of the form α and $\underline{\alpha}$ follows. □

The formula obtained Corollary 6.2.7 computes the total degree of α in terms of the relative form $\underline{\alpha}$. We use this formula to conclude the following, which generalizes formula (6.24) in Proposition 6.2.5.

Proposition 6.2.8. *Consider a compact Riemannian bordism $(M, \partial_+ M, \partial_- M)$, together with the relative Euler form $\underline{\mathbf{e}}(M, \partial_+ M, \partial_- M, g)$ as defined in (6.13). For $x_0 \in$ $\mathbf{int}(M)$, a base point in the interior of M, consider the space $\mathfrak{Eul}^*_{x_0}(M, \partial_+ M, \partial_- M; \mathbb{C})$ of coEuler structures based at x_0, see (6.28). Let $\mathfrak{e}^* \in \mathfrak{Eul}^*_{x_0}(M, \partial_+ M, \partial_- M; \mathbb{C})$ be represented by $(g, \underline{\alpha})$, where $\underline{\alpha} := (\alpha, \alpha_\partial)$ is a relative form with $\alpha \in \Omega^{m-1}(\dot{M}; \Theta^{\mathbb{C}}_M)$ having a unique singularity at x_0. Assume $d\alpha$ and α_∂ are smooth, i.e. without singularities in M. For $\omega \in \Omega^1(M)$, a smooth closed 1-form on M, choose a smooth function $f \in C^\infty(M)$ such that the 1-form $\omega' := \omega - df \in \Omega^1(M)$ and vanishes on a small neighborhood of x_0. Then*

$$\begin{aligned}\mathcal{S}_f(\underline{\alpha},\omega) = &\int_{(M,\partial M)} \underline{\alpha}\wedge\omega' + (-1)^m \int_{(M,\partial M)} \underline{\mathbf{e}}(M,\partial_+ M,\partial_- M,g)\wedge f\\ &- f(x_0)\chi(M,\partial_- M)\end{aligned} \tag{6.33}$$

In particular, if $\mathfrak{e}^*$ is represented by $(g, \underline{\alpha})$ and $(g', \underline{\alpha}')$, then

$$\mathcal{S}(\underline{\alpha}',\omega) - \mathcal{S}(\underline{\alpha},\omega) = \int_{(M,\partial M)} \underline{\widetilde{\mathbf{e}}}(M,\partial_+ M,\partial_- M,g,g')\wedge\omega'. \tag{6.34}$$

Proof. Under these assumption, from Corollary 6.2.7, we have

$$\mathsf{deg}_{x_0}(\alpha) = (-1)^m \int_{(M,\partial M)} \mathbf{d}\underline{\alpha} = (-1)^m \int_{(M,\partial M)} \underline{\mathbf{e}}(M,\partial_+ M,\partial_- M,g) = \chi(M,\partial_- M),$$

where the last equality follows from Gauss–Bonnet–Chern Theorem. Therefore, (6.33) follows from the definition of $\mathcal{S}$ in (6.32). Finally, formula (6.34) follows from (6.33) and the defining relation (6.27).

□

6.3. Generalized complex-valued analytic torsion

In this section, we extend Theorem 4.2, in **[BH07]** to the situation of a compact bordism. The *generalized* complex-valued Ray–Singer torsion on closed manifolds was constructed in Theorem 4.2, in **[BH07]**, by adding appropriate correction terms to the complex-valued torsion in order to cancel out the infinitesimal variation to the complex-valued analytic torsion. These correction terms were introduced using coEuler structures, once the anomaly formulas for the torsion was computed. The procedure in the situation on a compact bordism is carried out in a similar fashion. In fact, the required correction terms are constructed by using this time the notion of coEuler structures on compact bordisms, from Section 6.2, and the anomaly formulas obtained in Theorem 5.2.1.

Theorem (Definition) 6.3.1. *Let $(M, \partial_+ M, \partial_- M)$ be a compact Riemannian bordism with Riemannian metric g. Suppose $\chi(M, \partial_- M) = 0$ (or equivalently $\chi(M, \partial_- M) = 0$). Let $\mathfrak{e}^* \in \mathfrak{Eul}^*(M, \partial_+ M, \partial_- M; \mathbb{C})$ a the coEuler structure (without base point), see Section 6.2.1. Let E be a complex flat vector bundle over M, with flat connection ∇^E. Assume E is endowed with a complex nondegenerate symmetric bilinear form b. As a bilinear form on $\det(H(M, \partial_- M))$, we define*

(6.35)

$$[\tau]^{E,\mathfrak{e}^*,[b]}_{(M,\partial_+ M,\partial_- M)} \quad := \quad [\tau^{\mathsf{RS}}]^{E,g,b}_{(M,\partial_+ M,\partial_- M)} \cdot \exp\Big(2\int_{(M,\partial M)} \underline{\alpha} \wedge \omega(E,b) - \mathsf{rank}(E) \int_{\partial M} B(\partial_+ M, \partial_- M, g)\Big)$$

where

- *$[\tau^{\mathsf{RS}}]^{E,g,b}_{(M,\partial_+ M,\partial_- M)}$ is the complex-valued Ray–Singer torsion on $(M, \partial_+ M, \partial_- M)$, see Definition 5.1.4,*
- *the couple $(g, \underline{\alpha})$, for $\underline{\alpha} \in \Omega^{m-1}(M, \partial M; \Theta^{\mathbb{C}}_M)$, represents the coEuler structure $\mathfrak{e}^* \in \mathfrak{Eul}^*(M, \partial_+ M, \partial_- M; \mathbb{C})$,*
- *$B(\partial_+ M, \partial_- M, g)$ is the characteristic form given in Definition 6.1.11,*
- *$[b]$ indicates the homotopy class of b,*
- *$\omega(E, b) \in \Omega^1(M; \Theta_M)$ is the Kamber–Tondeur form associated to ∇^E and b.*

Proof. We have to prove that $[\tau]^{E,\mathfrak{e}^*,[b]}_{(M,\partial_+ M,\partial_- M)}$ in (6.35) is indeed well defined, i.e., it is independent of the choice of representatives for the coEuler structure and it depends on ∇^E and the homotopy class $[b]$ of b only.

Let $\{(g_w, \underline{\alpha}_w)\}$ be a real one-parameter smooth path (of Riemannian metrics g_w on M and of relative forms $\underline{\alpha}_w \in \Omega^{m-1}(M, \partial M; \Theta^{\mathbb{C}}_M))$ representing the *same* coEuler structure $\mathfrak{e}^* \in \mathfrak{Eul}^*(M, \partial_+ M, \partial_- M; \mathbb{C})$. Let $\{b_w\}$ be a one real parameter smooth path of nondegenerate symmetric bilinear forms on E. Consider the corresponding family $[\tau]^{E,(g,\underline{\alpha}_w),b_w}_{(M,\partial_+ M,\partial_- M)}$ of bilinear forms given by the formula in (6.35), for which its logarithmic derivative with respect to w vanishes. Indeed, this derivative consists of two contributions: the variation of the exponential depending on the coEuler structures, computed in Proposition 6.2.5 and the variation of the complex-valued analytic torsion, computed in Theorem 5.2.1; but these two contributions cancel each other out Remark here that the terms appearing in the anomaly formulas in Theorem 5.2.1 use the original characteristic forms in [**BM06**]. But the only difference between using the forms $B(\partial_+ M, \partial_- M, g)$, $\mathbf{e}_\partial(\partial_+ M, \partial_- M, g)$ and $\widetilde{\mathbf{e}}_\partial(\partial_+ M, \partial_- M, g)$ instead of $B(\partial M, g)$, $\mathbf{e}_\partial(\partial M, g)$ and $\widetilde{\mathbf{e}}_\partial(\partial M, g)$

respectively, is that by construction certain signs $(-1)^m$ are suppressed, as it can easily be checked. $\square$

6.3.1. Direct sums. Consider the bordism $(M, \partial_+ M, \partial_- M)$, two flat complex vector bundles E_1 and E_2 over M, endowed with the fiberwise nondegenerate symmetric bilinear forms b_1 and b_2 respectively and the $E_1 \oplus E_2$ the flat sum vector bundle endowed with connection $\nabla^{E_1 \oplus E_2}$ and nondegenerate complex symmetric bilinear form $b_1 \oplus b_2$. In this situation, look at the bilinear Laplacian $\Delta_{E_1 \oplus E_2, g, b_1 \oplus b_2}$ acting on $E_1 \oplus E_2$-valued smooth forms over M, under absolute/relative boundary conditions. Since

$$\Omega(M; E_1 \oplus E_2) = \Omega(M; E_1) \oplus \Omega(M; E_2)$$

and

$$\Omega(M; E_1 \oplus E_2)_{\mathcal{B}} = \Omega(M; E_1)_{\mathcal{B}} \oplus \Omega(M; E_2)_{\mathcal{B}}$$

we have

$$\Delta_{E_1 \oplus E_2, g, b_1 \oplus b_2} = \Delta_{E_1, g, b_1} \oplus \Delta_{E_2, g, b_2}$$

as well as for their L^2-realizations

$$[\Delta_{\mathcal{B}}]^{E_1 \oplus E_2, g, b_1 \oplus b_2}_{(M, \partial_+ M, \partial_- M)} = [\Delta_{\mathcal{B}}]^{E_1, g, b_1}_{(M, \partial_+ M, \partial_- M)} \oplus [\Delta_{\mathcal{B}}]^{E_2, g, b_2}_{(M, \partial_+ M, \partial_- M)}$$

and hence

$$\det{}' \left(\Delta_{E_1 \oplus E_2, g, b_1 \oplus b_2}\right) = \det{}' \left(\Delta_{E_1, g, b_1}\right) \cdot \det{}' \left(\Delta_{E_2, g, b_2}\right).$$

Thus, with the canonic isomorphism of complex lines

$$\det H^*(M, \partial_- M; E_1 \oplus E_2) \cong \det H^*(M, \partial_- M; E_1) \otimes \det H^*(M, \partial_- M; E_2),$$

and the identity

$$\omega_{E_1 \oplus E_2, g, b_1 \oplus b_2} = \omega_{E_1, g, b_1} + \omega_{E_2, g, b_2},$$

for the corresponding Kamber–Tondeur forms, see Section 2.4 in [**BH07**], we obtain

$$[\tau]^{E_1 \oplus E_2, \mathfrak{e}^*, [b_1 \oplus b_2]}_{(M, \partial_+ M, \partial_- M)} = [\tau]^{E_1, \mathfrak{e}^*, [b_1]}_{(M, \partial_+ M, \partial_- M)} \otimes [\tau]^{E_2, \mathfrak{e}^*, [b_2]}_{(M, \partial_+ M, \partial_- M)}. \tag{6.36}$$

6.3.2. Generalized complex-valued analytic torsion and Poincaré duality. Let $(M, \partial_+ M, \partial_- M)'$ be the dual bordism of $(M, \partial_+ M, \partial_- M)$ and E' the dual bundle of E endowed with the dual connection and b' the nondegenerate symmetric bilinear form dual to b on E. By Poincaré–Lefschetz duality, see (3.19), there is a canonic isomorphism of determinant line bundles

$$\det \left(H\left(M, \partial_+ M; E' \otimes \Theta_M\right)\right) \cong \left(\det \left(H(M, \partial_- M; E)\right)\right)^{(-1)^{m+1}}, \tag{6.37}$$

see for instance [**KM76**], [**Mi62**] and [**Mi66**]. The bilinear Laplacians $\Delta_{E,g,b,q}$ and $\Delta_{E' \otimes \Theta_M, g, b', m-q}$, as well as the corresponding boundary conditions are intertwined by the isomorphism $\star_g \otimes b : \Omega^q(M; E) \to \Omega^{m-q}(M; E' \otimes \Theta_M)$. This implies that their L^2-realizations $\Delta_{\mathcal{B}_q}$ and $\Delta_{\mathcal{B}'_{m-q}}$ are isospectral, and therefore

$$\det{}'(\Delta_{E,g,b,q}) = \det{}'(\Delta_{E' \otimes \Theta_M, g, b', m-q}). \tag{6.38}$$

By definition of the torsion in (6.35), the isomorphism in (6.38), the identity in (6.37), the formula $\Pi_q \left(\det'(\Delta_{E,g,b,q})\right)^{(-1)^q} = 1$ proved in Lemma 5.1.3, the relation between the forms $B(\partial_+ M, \partial_- M, g)$ and $B(\partial_- M, \partial_+ M, g)$ from Lemma 6.1.10, and

$$\omega(E' \otimes \Theta_M, b') = -\omega(E, b), \tag{6.39}$$

see Section 2.4 in **[BH07]**, we obtain

$$[\tau]^{E'\otimes\Theta_M,\nu(\mathfrak{e}^*),[b']}_{(M,\partial_+ M,\partial_- M)'} = \left([\tau]^{E',\mathfrak{e}^*,[b]}_{(M,\partial_+ M,\partial_- M)}\right)^{(-1)^{m+1}}, \tag{6.40}$$

where $\nu : \mathfrak{Eul}^*(M, \partial_+ M, \partial_- M; \mathbb{C}) \to \mathfrak{Eul}^*(M, \partial_- M, \partial_+ M; \mathbb{C})$ is the map in (6.23), intertwining the corresponding coEuler structures. The formula in (6.40) exhibits the behavior of generalized complex-valued torsion on the bordism $(M, \partial_+ M, \partial_- M)$ under Poincaré–Lefschetz duality, generalizing this situation in the case without boundary, see formula (31) in **[BH07]**.

6.3.3. Without conditions on $\chi(M, \partial_\pm M)$. Let $(M, \partial_+ M, \partial_- M)$ be a compact Riemannian bordism and E a complex flat vector bundle over M with flat connection ∇^E. We assume it is endowed with a complex nondegenerate symmetric bilinear form b and consider $\omega(E, b)$ the Kamber—Tondeur form. For $x_0 \in \mathbf{int}(M)$, let

$$\mathfrak{e}^*_{x_0} \in \mathfrak{Eul}^*_{x_0}(M, \partial_+ M, \partial_- M; \mathbb{C})$$

be a coEuler structures based at x_0, see (6.28), represented by $(g, \underline{\alpha})$, where $\underline{\alpha} := (\alpha, \alpha_\partial)$ is a relative form with

$$\alpha \in \Omega^{m-1}(\dot{M}; \Theta^{\mathbb{C}}_M) \quad \text{and} \quad \dot{M} := M \backslash \{x_0\}.$$

Let $b_{(\det E_{x_0})^{-\chi(M,\partial_- M)}}$ be the induced bilinear form on $(\det E_{x_0})^{-\chi(M,\partial_- M)}$. Consider $\tau^{\mathsf{RS}}_{E,g,b}$ the complex-valued Ray–Singer torsion on $(M, \partial_+ M, \partial_- M)$, see 5.2.1, and $\mathcal{S}$ the function regularizing $\int_{(M,\partial M)}$ studied in Proposition 6.2.8. We define

$$\tau^{\mathsf{an}}_{E,\mathfrak{e}^*_{x_0},[b]} := \tau^{\mathsf{RS}}_{E,g,b} \cdot \mathsf{e}^{2\mathcal{S}(\underline{\alpha},\omega(E,b)) - \mathsf{rank}(E)\int_{\partial M} B(\partial_+ M,\partial_- M,g)} \otimes b_{(\det E_{x_0})^{-\chi(M,\partial_- M)}}, \tag{6.41}$$

regarded as a bilinear form on $\det(H(M, \partial_- M)) \otimes (\det E_{x_0})^{-\chi(M,\partial_- M)}$,

Theorem 6.3.2. *The bilinear form in (6.41) is independent of the choice of representative for the coEuler structure and depends on the connection and the homotopy class $[b]$ of b only.*

Proof. On the one hand, if b is fixed and we only look at changes of the metric, then the variation of $\tau^{\mathsf{an}}_{E,(g,(\underline{\alpha},\theta)),b}$ with respect to the metric compensates the variation of the function $\mathcal{S}(\underline{\alpha}, \omega(E, b))$, which is explicitly given by formula (6.34) in Proposition 6.2.8. On the other hand, when g and $\mathfrak{e}^*_{x_0}$ are kept constant and we allow b to smoothly change from b_1 to b_2, then the variation of the Kamber–Tondeur form is given by

$$\omega(E,b_2) - \omega(E,b_1) = -\tfrac{1}{2}\det((b_1^{-1}b_2)^{-1})d\det\big(b_1^{-1}b_2\big) = -\tfrac{1}{2}d\log\det((b_1^{-1}b_2)^{-1}),$$

where the last equality holds, since b_2 and b_1 are homotopic and therefore the function

$$\det((b_1^{-1}b_2)^{-1}) : M \to \mathbb{C}\backslash\{0\},$$

is homotopic to the constant function 1, which in turn allows to find a function

$$\log\det((b_1^{-1}b_2)^{-1}) : M \to \mathbb{C},$$

with

$$d\log\det((b_1^{-1}b_2)^{-1}) = \det((b_1^{-1}b_2)^{-1}) d\det(b_1^{-1}b_2).$$

This, with $f = \operatorname{Tr}((b_1^{-1}b_2)^{-1})$ and Lemma 6.2.6, implies that

$$2\mathcal{S}_f(\underline{\alpha},\omega(E,b_2)) - 2\mathcal{S}_f(\underline{\alpha},\omega(E,b_1)) = 2\mathcal{S}_f\left(\underline{\alpha}, d\log\det\left(\left(b_1^{-1}b_2\right)^{-1}\right)\right)$$

$$= -(-1)^m \int_{(M,\partial M)} \underline{\mathbf{e}}(M,\partial_+M,\partial_-M,g)\log\det\left(\left(b_1^{-1}b_2\right)^{-1}\right) + \log\det\left(\left(b_1^{-1}b_2\right)^{-1}\right)(x_0)\chi(M,\partial_-M)$$

$$= -(-1)^m \int_{(M,\partial M)} \underline{\mathbf{e}}(M,\partial_+M,\partial_-M,g)\operatorname{Tr}\left(\left(b_1^{-1}b_2\right)^{-1}\right) + \operatorname{Tr}\left(\left(b_1^{-1}b_2\right)^{-1}\right)(x_0)\chi(M,\partial_-M),$$

where the additional term $\operatorname{Tr}((b_1^{-1}b_2)^{-1})(x_0)\chi(M,\partial_-M)$ cancels the variation of the induced bilinear form on $(\det E_{x_0})^{-\chi(M,\partial_-M)}$ given by

$$\left(b_{1_{(\det E_{x_0})^{-\chi(M,\partial_-M)}}}\right)^{-1} b_{2_{(\det E_{x_0})^{-\chi(M,\partial_-M)}}} = \det(b_1^{-1}b_2)^{-\chi(M,\partial_-M)}.$$

□

Zusammenfassung

Ein kompakter riemannscher Bordismus ist eine (differenzierbare) kompakte Mannigfaltigkeit M der Dimension m mit riemannscher Metrik g, deren Rand ∂M genau die disjunkte Vereinigung der zwei geschlossenen Mannigfaltigkeiten $\partial_+ M$ und $\partial_- M$ ist, mit absoluten (bzw. relativen) Randbedingungen auf $\partial_+ M$ (bzw. $\partial_- M$). Diese Dissertation befasst sich mit der komplexwertigen analytischen Torsion auf kompakten Bordismen.

Sei E ein flaches komplexes Vektorbündel über M und h eine hermitische Metrik auf E. Um die Ray–Singer Metrik $\tau^{\mathsf{RS}}_{E,g,h}$ als eine hermitische Metrik auf dem Determinantenbündel $\det(H(M, \partial_- M; E))$ der De-Rham Komologie $H(M, \partial_- M; E)$ zu definieren, studiert man selbstadjungierte Laplace-Operatoren $\Delta_{E,g,h}$ die auf E-wertigen glatten Differentialformen $\Omega(M; E)$, mit den obigen Randbedingungen, wirken. Nun nehmen wir an, dass E mit einer faserweisen nicht-ausgearteten komplexen symmetrischen Bilinearform b ausgestattet ist. Sei $\beta_{E,g,b}$ die von b und g induzierte Bilinearform auf $\Omega(M; E)$. Die komplexwertige Ray–Singer Torsion $\tau^{\mathsf{RS}}_{E,g,b}$ ist eine nicht-ausgeartete komplexe Bilinearform auf dem Determinanten-Linienbündel $\det(H(M, \partial_- M; E))$, die von Burghelea und Haller auf geschlossenen Mannigfaltigkeiten in Analogie zu der Ray–Singer Metrik eingeführt wurde. Um $\tau^{\mathsf{RS}}_{E,g,b}$ zu definieren, betrachten wir nicht-selbstadjungierte Laplace-Operatoren $\Delta_{E,g,b}$. Wir erhalten ein Hodge–De-Rham Zerlegungsresultat, das besagt, dass der verallgemeinerte Nulleigenraum des Laplace-Operators $\Delta_{E,g,b}$ endlich dimensional ist, nur glatte Formen enthält und uns erlaubt die Kohomologie Gruppen $H(M, \partial_- M; E)$ zu berechnen. Dann induziert die Einschränkung von $\beta_{g,b}$ auf dem verallgemeinerten Nulleigenraum des Laplace-Operators $\Delta_{E,g,b}$ eine nicht-ausgeartete komplexe symmetrische Bilinearform $\tau_{E,g,b}(0)$ auf $\det(H(M, \partial_- M; E))$. Dann wird $\tau^{\mathsf{RS}}_{E,g,b}$ als das Produkt von $\tau_{E,g,b}(0)$ mit der ζ-regularisierten Determinante von $\Delta_{E,g,b}$ definiert.

Die Variation der Torsion $\tau^{\mathsf{RS}}_{E,g,b}$, bezogen auf glatte Veränderungen der Metrik und der Bilinearform, ist als *Anomalienformel* bezeichnet. Für die Berechnung dieser Formel braucht man die Koeffizienten des konstanten Terms in der asymptotischen Expansion für die Wärmeleitung des Operators $\Delta_{E,g,b}$. Wir berechnen diese Koeffizienten, indem wir die von Brüning und Ma gefundenen Formeln für die Ray–Singer Metrik benutzen.

Schließlich definieren wir coEuler Strukturen auf einem kompakten riemannschen Bordismus. Im Rahmen einer geschlossenen Mannigfaltigkeit sind CoEuler Strukturen von Burghelea und Haller studiert worden. In unserem Fall wird der Raum von coEuler Strukturen als ein affiner Raum über die Gruppe $H^{m-1}(M, \partial M; \mathbb{C})$ definiert. Diese können als duale Objekte für die Euler-Strukturen von Turaev angesehen werden.

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2002-2003	**Erasmus exchange in Mathematics** Mathematics department, University of Pisa; Italy
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2004-2006 **Research assistant**
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Teaching Experience

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- *Introduction to Cryptography.* Winter, 04/05
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- *Fractal Geometry.* Winter, 05/06
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Theses

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- *Non-propagation dans des systèmes quantiques aux hamiltoniens asymptotiquement périodiques.*
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  Research Group in Differential Geometry.
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  International Winter School in Geometry and Physics.
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  Noncommutative Geometry Seminar.
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