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Strictly proportional random voting mechanisms without dummy voters.

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I. Introduction

Voting as a concept is one of the most common ways to make collective decisions. Especially in the modern societies of the western world, which often build their identity on recognizing democracy as a very fundamental value, voting is perceived as the most natural method of resolving conflicts or finding solutions for problems that concern a group of people. If one considers indirect democracy, where some representatives receive the possibility to vote for some proposal, according to their result in an election, seats are usually distributed among the parties roughly in proportion to their relative performance. Similarly, the number of votes in the annual general meeting is most often proportional to the number of company's shares one possesses. Also in context of intra- or international associations, the number of votes is typically divided between the parties according to their population, area or some other relevant factor(s). In all these examples, there is some objective measure which is assumed to be fair, relevant or efficient. Then the votes are distributed proportionally, according to that objective measure. This practice appears to be based on the idea that voting power is reflected in the relative distribution of seats. However, this intuitive concept does not hold, especially if the number of parties is small.

Let's consider a simple example of 3 parties, having 45, 35 and 20 votes, respectively. The minimum quota needed to make a decision (which is assumed to have a positive value) is 51 (normal majority). In such a context, if one assumes that all parties have no special preferences concerning the coalition they form, it turns out that any two parties are equally likely to form a coalition. Each party is equally "needed" to form a coalition, despite the differences in the number of votes. Accordingly, their powers are exactly the same and equal to 1/3. This example shows the difference in terms of power may not exist that in spite of huge differences in the weights (number of votes) of the parties. One could also present examples indicating the opposite: very little difference in weights might translate to a great difference with respect to power.

The observation stated above proves that the common practice of distributing the number of votes in proportion to the objective measure may lead to significant distortions between the distribution of power and the objective measure itself. These distortions may have great economic consequences. For instance, shareholders may strategically choose to buy a certain number of shares rather than 50% + 1, because it already provides one with ample power and is not as costly. Countries or regions may refrain from joining certain organizations if they believe their voting power is significantly lower than the objective measure according to which the votes are distributed. In case of situations similar to the latter example, distortions in voting power may lead to loss in welfare compared to scenarios where power could be split in a way that is considered right, fair and satisfactory to all parties. The only problem is to find a mechanism which would reduce or – preferably – eliminate the distortions in voting power.

This work is aimed to overview the existing concepts of such mechanisms and present some new ideas. The crucial question related to distortions in voting power is the definition of power itself. In the literature there are at least several measures of power (called usually "power indices") which can be adapted to voting. Each bases on a slightly different concept of power, but they are highly correlated and the output is a single number between 0 and 1, which is intuitively interpretable. This work's aim is not to compare or evaluate those concepts. On the contrary, four most common power indices: Banzhaf index, Shapley-Shubik index, Holler-Packel index (known also as Public Good Index) and Packel-Deegan index are considered as equally relevant alternatives. This means, an overview of mechanisms allowing for non-distortion voting procedure is presented for all four indices. The overview includes the concepts of Berg and Holler (1986), whose solution consists in randomizing the minimum quota. This concept applies only to the Shapley-Shubik index and the absolute Banzhaf measure. Next, the alternative solution by Di Giannatale and Pasarelli (2011) is presented. This concept involves randomizing player's rights to vote rather than the quota and is explained for the Shapley-Shubik Index, but in fact it holds trivially also for any other power index. Both these solutions allow for dummy parties with positive probability, which appears to be an unattractive feature both in political context and overall if one considers risk-averse agents. Therefore, also own concept which does without use of dummy parties is introduced. It is based on randomized semi-dictator games and is valid for all the power indices. As the assumptions regarding the objective measure distribution need not always be satisfied, an additional sketch of a numerical solution is presented.

The work is organized as follows: chapter 2 defines the notion of power and presents the most popular measures of power applicable in the voting context. Chapter 3 provides a brief explanation why randomizing is necessary to construct a non-distortion mechanism of voting, and presents previous concepts of such mechanisms. Chapter 4 presents the concept of own solution which does not involve any dummy parties (as long as the objective measure distribution fulfills certain requirements) and an algorithm of a numerical solution to the mechanism creation problem. Chapter 5 concludes.

II. Power in voting and its measures.

As mentioned earlier, the most crucial requirement to analyze distortion-free voting mechanisms is to define power and its measures. Four most popular power measures: Banzhaf index, Shapley-Shubik index, Holler-Packel index (known also as Public Good Index) and Packel-Deegan index will be presented in this chapter. Clearly, each measure is based on a different set of axioms and therefore interprets power in a different way. The most general definition of power[1], used e.g. by Daubechies (2002), appears to be: *A player's power is defined as that player's ability to influence decisions.* In order to analyze power in terms of voting, some kind of voting framework is needed. The most natural one for this purpose is the weighted voting framework. It consists in providing each player (voter) with a specified number of votes, called his or her weight. In contrast to normal parliamentary procedure, which assumes that each member's vote carries equal weight, it is possible that voters have different weights. In addition, a quota is specified, and if the sum of the voting weights of the voters supporting some decision is greater or equal to the quota, the decision is made. The notation:

$$[q : w_1, w_2, . . . w_m]$$

indicates that there are n voters, with voting weights $w_1, w_2, . . . w_m$ and the quota is equal to q.

A game presented above can also be framed in form of:

$$\{W_1\}, \{W_2\}, \ldots \{W_k\}$$

where W_i is a winning coalition, i.e a set of players S such that $\sum_{j\in S} w_j \geq q$. If the decision is made, i.e a winning coalition is formed, its members enjoy that and assign positive value to such event:

$$v(S) = \begin{cases} 1 \; if \; S \in W \\ 0 \; if \; S \; \in W' \end{cases}$$

[1] see Bell et al. (1969) for a broader discussion about the concept of power

where W' is complement of set W. Note that in this work q is assumed to be strictly larger than $\sum_i^n w_i$, because otherwise multiple decisions could be made by multiple winning coalitions.

Let's consider an example of a voting [5: 4, 2, 1], which is equivalent to {A, B, C}, {A,B}, {A,C}. The latter notation shows clearly that player A is a member of 3 winning coalitions, whereas players B and C are members of 2 winning coalitions (and are crucial members of only one). Without any further information, one can create an ordinal power ranking: player A is expected to have a higher probability of being a member of a wining coalition than players B and C.

However, in order to obtain a cardinal measure of power, one needs to have additional information (or assumptions) concerning the decision-making situation. Depending on the latter, one may choose measures of voting power such as indices of: Banzhaf, Shapley-Shubik, Holler-Packel or Deegan-Packel. The Banzhaf index can be derived under the assumption that the players are individuals, the Shapley-Shubik index can be derived assuming players are symmetric in the sense that they behave as clones. The Holler-Packel index is based on the assumption that political outcomes are public goods which are nonrival to members of the winning coalition. Finally, the Deegan-Packel index is very similar to the Holler-Packel measure, but it assumes that consumption of the political outcomes is rival to the members of the winning coalition.

In order to present the four power indices formally and in a little more detail, one needs to specify some more general notation. n is the total number of players. N is the number of all winning coalitions in a given game. Player i is called a critical member of S (alternatively: has a swing in S) if he or she turns a winning coalition: ∈W to a losing coalition: S\{i}∈ W'. Coalitions where i has a swing are called critical coalitions with respect to i. C_i indicates the set of all critical coalitions with respect to player i. Player i is called pivotal if he or she turns a losing coalition: S\{i}∈ W' which consists of the first s-1 players into a winning coalition: S∈W, such that player i is the s^{th} member of the coalition. The difference between a critical member of S and a pivotal member of S is that in the latter case the sequence of players in the coalition matters. Minimal Winning Coalition (MWC) is a coalition in which all members are critical. MC is the set of all MWC in the game. Decisiveness of player i, c_i, is defined as the number of MWC whose player i is a member. η_i is the number of swings of player i in the game, $\eta_i = |C_i|$. If $\eta_i = 0$, player i is called a dummy, i.e. player i can never turn a losing coalition into a winning coalition and vice versa. It is the case iff player i is not a member of any MWC. Player i is called a dictator if {i} is the only MWC in the game. A new notion, introduced in this work will be used for the specification of a distortion-free voting mechanism without dummy players: player i is called semi-dictator if $\forall MWC$ {i}∈ MWC.

The Banzhaf (1964) index, originally introduced by Penrose (1946) and rediscovered also by Rae (1969) and Coleman (1971), became part of the mainstream literature in 1970's[2]. It examines all winning coalitions, regardless of the order in which they may be formed. Power is derived from having a swing in a winning coalition. An absolute Banzhaf measure in game g is given by:

$$B_i(g) = \frac{\eta_i(g)}{2^{n-1}}$$

Banzhaf index is obtained by normalizing $B_i(g)$, so that $\sum_{i=1}^n B_i(g) = 1$:

$$B'_i(g) = \frac{\eta_i(g)}{N} \qquad (1)$$

[2] see Felsenthal and Machover (1998) for a history of this measure of voting power.

Banzhaf index is based on a following property of voters: each voter's probability of voting "yes" for a given decision is drawn independently from the uniform distribution on [0,1]. In other words, there is no correlation between the voters. Such a situation is analyzed as one of the possible voters' attitude towards alternative decisions. Straffin (1977) calls it independence assumption. The author shows that such an assumption is equivalent to saying that all voters vote in favor of any decision with probability ½. Accordingly, the Banzhaf index is an a priori measure of power – it evaluates the distribution of voting power behind a Rawlsian "veil of ignorance".

An alternative voters' attitude towards alternative decisions is a base of the Shapley-Shubik index. Each voter's probability of voting "yes" for a given decision is drawn from the uniform distribution on [0,1], but not independently. In other words, all voters have the same probability of voting for a given proposal, although it is different for different proposals. It can be interpreted as voters were judging decisions by some common standard, and the probability of voting yes is proposal's acceptability level according to that standard. Such an assumption is called homogeneity assumption by Straffin (1977)[3].

The Shapley-Shubik index (SS) is based on the concept of the pivotal voter. One needs to consider all coalitions S such that player i is a member of the coalition and then check if player i is pivotal in that coalition. The exact formula is as follows:

$$SS(i) = \sum_{\substack{i \in S \\ S \in W \\ S\setminus\{i\} \in W'}} \frac{(|S|-1)!(n-|S|)!}{n} \tag{2}$$

The implicit assumption of SS is that every sequence of players in a coalition is equally likely. The problem it brings is that it gives different weights to coalitions with different number of members.

The Public Good Index, called also Holler-Packel index, uses some of the logic of the Banzhaf index. It examines all the MWC and the power is derived from the fact of having a swing in those coalitions. The underlying assumption is that all MWCs are equally likely to form. The index is given by:

$$PGI(i) = \frac{c_i}{MC} \tag{3}$$

PGI explicitly treats accepting the proposal as a public good, which is non-rival in consumption and freeriding is not possible. Accordingly, if the public good is provided, each member of the winning coalition receives full value of the coalition. Power is considered to be derived only from decisive sets (MWCs). As Holler and Napel (2004) put it: All other coalitions [except for MWCs] are either losing or contain at least one member which does not contribute to winning. If coalitions of the second type are formed, then it is by luck, similarity of preferences, tradition, etc. – but not because of power[4].

The Deegan-Packel index (DP) is based on the same assumptions as PGI. The only difference is that consumption of the public good is rival to the members of the winning coalition. In other words, sharing the victory reduces utility derived from that victory, i.e. forming a MWC with one other player is better than with several players. This can be interpreted in the SS manner: each MWC is equally likely, but only the pivotal member obtains the value of the coalition. All sequences of any MWC's formation are obviously equally likely. The formula for the Deegan-Packel index is given by:

[3] see Straffin (1978) for considerations concerning different distributions of voters' probability of voting "yes".
[4] Holler and Napel (2004), page 8 (page 100 in the journal)

$$DP(i) = \sum_{S \in c_i} \frac{1}{MC * |S|} \qquad (4)$$

Now that all the most influential power indices have been introduced, note that the aim of this work is not to make any evaluations or comparisons between them in order to judge which one is the most relevant or appropriate. Therefore it is abstracted from the postulates of a proper *a priori* measure of power: iso-invariance, ignoring dummies, vanishing for dummies, normalization and local monotonicity[5]. Especially the importance of local monotonicity, which is not satisfied by PGI and Deegan-Packel index[6], and controversies related to this issue are not discussed[7]. The question of superiority of any of the power indices is not of any interest to this work. Accordingly, in the next chapters all the presented measures will be treated neutrally, as equally relevant alternatives.

III. Non-distortion voting mechanisms.

Distortion in this work is defined as the difference between the player's desirable level of power according to some objective measure and the real power of that player (regardless which power measure is chosen) in the voting game. For example, one may treat a country's population as an objective measure according to which voting power should be distributed. If one considers e.g. 3 countries with populations: x_1, x_2, x_3 and assumes that desirable power distribution is proportional to the populations, countries' relative power should be equal to: x_1/X[8], x_2/X and x_3/X, respectively. In order to obtain such a distribution of powers, it is natural to set weights of players proportionally to their objective measures. However, if one considers a game [q: x_1, x_2, x_3], it is impossible to obtain the desired power distribution, regardless of what level of q is chosen and regardless of which measure of power is applied. Even manipulating weights cannot solve this problem. The reason is that, especially for a low n, the values that power indices may take are strongly discrete. If one considers a three-player voting game, one may only obtain one of the following distributions, depending on which power measure one uses:

- Bz: [1, 0, 0], [3/5, 1/5, 1/5], [1/2, 1/2, 0] and [1/3, 1/3, 1/3][9],
- SS: [1, 0, 0], [4/6, 1/6, 1/6], [1/2, 1/2, 0] and [1/3, 1/3, 1/3],
- PGI or DP: [1, 0, 0], [1/2, 1/4, 1/4], [1/2, 1/2, 0] and [1/3, 1/3, 1/3].

Obviously, when n increases, the number of values a power measure can take also increases. Lindner and Machover (2004) provide sufficient conditions for the statement of Penrose: the ratio between the powers of any 2 players, measured by Bz, converges to the ratio between their voting weights if the total number of voters goes to infinity. In other words, one obtains the so-called strict

[5] see Steffen (2002), Allingham (1975), Felsenthal and Machover (1998), Freixas and Gambarelli (1997), Laruelle (1999), or Straffin (1983) for further considerations and comparisons of various power measures with respect to these criteria.

[6] consider e.g. a game [51;35,20,15,15,15], where PGI distribution is {16/60,8/60,12/60,12/60,12/60} and DP distribution is {18/60,9/60,11/60,11/60,11/60}. Local monotonicity is violated as players 3,4 and 5, who have lower weight than player 2, have – according to these indices – more power than player 2.

[7] see Holler and Napel (2004), Brams and Fishburn (1995) or Steffen (2002) for further considerations and references.

[8] $X = x_1 + x_2 + x_3$

[9] assuming that the players are sorted from the most powerful to the least powerful.

proportionality between power and voting weights only if the number of players is infinite. For finite n, however, the distribution of power and the distribution of weights differ from each other[10].

Because of discretion of power indices' values one cannot obtain any value without taking a convex combination of the available values. Doing so one can obtain any value, as taking in particular [1, 0, …, 0] and [0,0, …, 1] guarantees obtaining any value between 0 and 1 for player 1 and player n. The natural way of receiving convex combinations of discrete distributions is randomizing between them. Shapley (1962) proved that strict proportionality for SS can be obtained if one sets weights equal to desired power measures and uses a random quota, which is uniformly distributed over the interval (0,q). Dubey and Shapley (1979) showed the same result for the Banzhaf index.

The most thorough papers dealing with mechanisms which yield strict proportionality are Holler (1985) and Berg and Holler (1986). The authors apply the randomized quota principle to discrete probability distributions and small n. They give a simple example of a game [d, 1, 2, 3, 4], where d is randomly drawn from a uniform distribution over {6, 7, 8, 9, 10}. The mechanism is presented in Table 1 which shows absolute values of the Banzhaf measure for each player and each d:

Table 1.

	Voting weights				Total power
	1	2	3	4	
d=	B_1	B_2	B_3	B_4	
6	1	3	3	5	12
7	1	1	3	5	10
8	1	1	3	3	8
9	0	2	2	2	6
10	1	1	1	1	4
∑	4	8	12	16	

One sees immediately that indeed the expected value of the Banzhaf measure for each player is strictly proportional to his or her weight. The authors show, moreover, that one need not randomize over all the values of d. In the given example, it is enough to randomize between d=6 and d=8 with equal probabilities in order to obtain exactly the same result, i.e. strict proportionality. As far as the given example is concerned, however, there is a slight issue, not discussed in the paper, which makes the solution somewhat imperfect. In particular, by using the absolute, rather than normalized, Banzhaf measure, one imposes a strong assumption that higher quotas result in less power to be split. This is captured by the last column of Table 1 (which is not contained in the original paper). For instance, by taking the simple average (as probability of drawing any quota is identical for all quotas) of absolute Banzhaf measures, one treats player one's power for d=6 as equivalent to the one for d=10. However, if one recalls that regardless of the quota the value of the game is always 1, it is reasonable to say that the surplus is constant across quotas and – therefore – the amount of power

[10] for further considerations see literature regarding *fair allocation of votes*, e.g. Laruelle and Widgre´n, (1998), Sutter (2000), Laruelle (2001) or Leech (2003)

should also be considered to be constant. That leads one to the normalized Banzhaf index. Its values for the given example are shown in Table 2.

Table 2.

	Voting weights			
	1	2	3	4
d=	B'_1	B'_2	B'_3	B'_4
6	1/12	3/12	3/12	5/12
7	1/10	1/10	3/10	5/10
8	1/8	1/8	3/8	3/8
9	0	2/6	2/6	2/6
10	1/4	1/4	1/4	1/4
Average	67/600	127/600	181/600	3/8

When one considers normalized Banzhaf indices – assuming the total amount of power to be split is constant regardless of the quota, weights or the number of players – it is clear that the solution by Dubey and Shapley (1979) does not work anymore. The average (expected) power of the players is not equal (although close to) their weights.

This problem does not exist when one considers the Shapley-Shubik index, because its construction guarantees that the total power is always the same across the quotas (but varies across the number of players). The same example, analyzed from the perspective of the Shapley-Shubik index, is presented in Table 3.

Table 3.

	Voting weights			
	1	2	3	4
d=	SS_1	SS_2	SS_3	SS_4
6	1	3	3	5
7	1	1	3	7
8	1	1	5	5
9	0	4	4	4
10	3	3	3	3
$\sum$	6	12	18	24

In this case there is no doubt about the correctness of the solution. The expected value of the Shapley-Shubik index for each player is strictly proportional to his or her weight. Here, however, it is not so straightforward to simplify the randomizing procedure. If one considers d to be distributed over only {6, 7, 8, 9}, it is impossible to obtain an appropriate probability vector (where no coordinate is negative) over d which yields strict proportional power. Also no solution exists for binary d. However, if one considers d distributed over {7, 8, 9, 10}, a probability vector (0.30, 0.15, 0.30, 0.25) makes the expected power strictly proportional to the players' weights.

As far as the remaining power indices are concerned, randomizing the quota does not yield strict proportionality between voters' weights and power measures. Therefore, the presented solution works only for the Shapley-Shubik index and the absolute Banzhaf measure (as shown earlier, it is not valid for the normalized index), but not for PGI and DP.

Another method of obtaining strict proportionality is randomizing players' voting rights. This mechanism, presented explicitly in Berg and Holler (1986) was omitted by Di Giannatale and Pasarelli (2011) and presented as their own idea. Surprisingly, the authors refer to the original paper and state that

> Instead of randomizing voting rights Berg and Holler (1986) propose to randomize the qualified majority threshold as a means to avoid discrepancy between the seat distribution in a committee and an exogenous distribution of voting power.[11]

In fact, Berg and Holler (1986) also show the method presented by Di Giannatale and Pasarelli (2011). Regardless of the authorship, the idea is simple. First one needs to specify the objective measure. Di Giannatale and Pasarelli (2011) analyze the voting mechanism from the perspective of an underlying economic game and state that the objective measure is the power distribution of the economic game. Obviously, this is just an example of an objective measure and one may as well abstract from any specific measures and simply analyze the strict proportionality per se. Nevertheless, once the objective measure is specified, one needs to sort the desirable distribution of powers in descending order:

$\alpha_1 \geq \alpha_2 \geq \dots \alpha_n$

where α_i the power measure of player i. The solution itself consists in creating n voting games, such that in game 1 only player 1 votes, in game 2 player 1 and player 2 vote, … and in game n all n players vote. Both Berg and Holler (1986) and Di Giannatale and Pasarelli (2011) specify that all those voting games are unanimity games:

Table 4.

voting rights	Unanimity games				
players	u_1	u_2	…	u_{n-1}	u_n
p_1	*	*	…	*	*
p_2		*	…	*	*
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
p_{n-1}			…	*	*
p_n			…		*

Accordingly, player 1 (the player with the most power) will vote with probability 1, no matter which of the games $u_1,\dots,u_n$ will be played. Player 2 has a chance to vote in all the games except for u_1. By iteration, player n votes only if game u_n is played. For such a framework, the only missing element is

[11] Di Giannatale and Pasarelli (2011), page 6.

the probability vector over the games. The vector yielding strict proportionality can be specified recursively by:

$$\mathrm{prob}(u_i) = \begin{cases} \mathrm{i} * (\,\alpha_i - \alpha_{i+1}) & \text{for i} < \mathrm{n} \\ \mathrm{i} * \alpha_i & \text{for i} = \mathrm{n} \end{cases} \qquad (5)$$

Theorem 1. Let P' be a vector of desired power measures P, specified by (1), (2), (3) or (4). If probabilities $prob(u_i)$ *of unanimity games* $u_1, \ldots, u_n$ *are specified by equation (5), the expected power measure of such a game will be equal to the desired power measure for all players:* $P'_i - P_i = 0$ *for every i.*

The proof, shown in Di Giannatale and Pasarelli (2011), is presented in the appendix.

This method of obtaining strict proportionality is indeed very universal. Although both papers use this solution only for power measured by the Shapley-Shubik index, it actually works for all the power indices. The reason is that for any unanimity game u_i the power is split equally between players 1, 2, …, i. This property is captured by all the indices. Therefore, it is not important which power measure one considers to be the appropriate one, randomization of voting rights is compatible with it. Moreover, the form of the game u_i can also be manipulated a little without losing the desirable property of strict proportionality of the whole mechanism. In particular, as long as the quota is 100%, i.e. it is an unanimity game, the players' weights do not matter. Regardless of the choice of weights, the power is always equally split among the voters. However, if one specifies that weights of the voters are equal, there is no need to set the quota to 100% anymore. All the power indices yield identical power to players with equal weights, regardless of the quota. Accordingly, the method by Di Giannatale and Pasarelli (2011) would also work for normal majority or any other quota level, as long as the weights of the players are identical.

Both methods presented by Berg and Holler (1986), although mathematically sound and correct, do have some drawbacks with regard to applicability in the real world. As Holler and Napel (2004) put it:

> However, to randomize […] as an essential part of a decision rule invites a number of objections – in particular from those unlucky players who happen to have no say on a given issue. If the principle of randomized decision rules were accepted in general, it would even be possible to randomly choose a dictator, with probabilities corresponding to the desired a priori power allocation – arguably the most straightforward way of guaranteeing monotonicity and even strict proportionality. This illustrates the high price that monotonicity can have. It might be one reason why we often find that strict proportionality is not satisfied in reality and why power indices are needed to illustrate corresponding distortions.[12]

The strongest of the arguments against voting mechanisms involving randomizing appears to be the positive probability of some (or actually most) voters to be dummy (have no influence on the decision making at all). This issue calls for other solutions which guarantee strict proportionality and never makes any player a dummy. An example of such a solution is presented in chapter 4.

IV. New concepts of mechanism yielding strict proportionality and avoiding dummy players.

IV.1 Game space.

Before presenting the exact voting procedures yielding strict proportionality and avoiding dummy voters, it is necessary to show the space in which the solutions exist. Berg and Holler (1986) present

[12] Holler and Napel (2004), page 7 (page 99 in the journal)

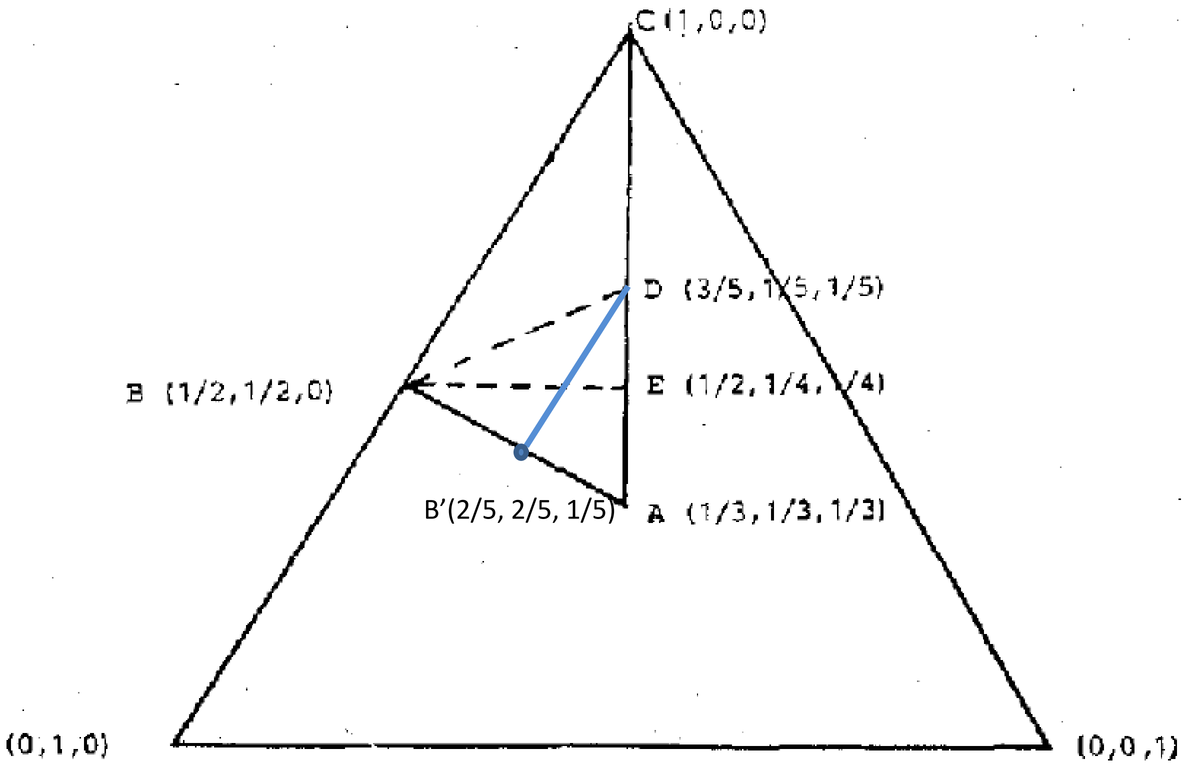
this space in a geometrical form. In order to analyze a 3-player voting game, the authors denote S as the 3-dimensional, ordered simplex:

$$S= \{x= (x_1, x_2, x_3): x_1 \geq x_2 \geq x_3 \geq 0, x_1+x_2+x_3=1\}$$

which corresponds to a triangle shown in Figure 1. Any voting game with weights w = (w_1, w_2, w_3) sorted in descending order and summing up to 1, can be represented by a point in S or by a point in the triangle ABC (the coordinates of points A, B, C and D are derived as all the possible distributions of power in a 3-player game, measured by the Banzhaf index).

$$S= \{x= (x_1, x_2, x_3): x_1 \geq x_2 \geq x_3 \geq 0, x_1+x_2+x_3=1\}$$

Figure 1.



source: Berg and Holler, page 7 (plus own work)

As long as the quota and the weights are fixed, it is only possible to obtain the power distribution corresponding to points A, B, C and D. If one allows for randomizing, one may obtain any point on the boundaries or inside the triangle ABC. This can be achieved by taking a linear combination of points A, B and C. In terms of the voting game that would mean e.g. that all three players vote (point A) with probability α, player 1 and 2 (point B) vote with probability β and player 1 only (point C) votes with probability 1-α-β. This solution in particular represents the mechanism by Di Giannatale and Pasarelli (2011). If the desired power distribution is within the triangle ABD, it is possible to achieve the desired point without having to resort to the dictator rule (not using C). It is sufficient to use the semi-dictator rule (point D), where player 1 is necessary, but not sufficient to create a winning coalition. But not using the dictator rule is somewhat insufficient for a solution to be socially acceptable and have a potential for political approval. The goal, as mentioned before, is never to involve a dummy player. Therefore, one would need to introduce point B' of coordinates: (2/5, 2/5, 1/5) and restrict the possible set of power distributions to the triangle AB'D. Then, if the desired power distribution is within the triangle AB'D, it is possible to achieve the desired point without ever making any of the players a dummy (not using B or C). This very concept – randomizing over players' weights leading at most to the semi-dictator rule – is the main point of this work.

IV.2 Assumptions.

In order to specify the solution, one first needs to precisely state the assumptions. As mentioned before, it is only possible to obtain a mechanism which yields strict proportionality and never involves dummy players if the desired power distribution of an ordered game lies within the triangle AB'D of Figure 1. Obviously, the coordinates of those points are only relevant if one considers a 3-player voting game. For the general case one needs that:

$$SDP \geq x_i \geq MP \qquad (6)$$

where SDP refers to the power of a semi-dictator and MP is the minimum power, i.e. the power of any non-semi-dictator player in the semi-dictator setup. These values differ across the power measures and depend on n.

Let us start with the analysis of the Banzhaf index. Let A be the semi-dictator and n=3. Then, all the winning coalitions are: {AB, AC, ABC}. Player A has a swing in all the winning coalitions (by definition), but other players only have a swing in a 2-player coalition of player i and player A. Accordingly, because there are (n-1) two-player coalitions, the total number of swings is equal to the total number of coalitions plus n-1. That yields following formulas:

$$\mathrm{MP(Banzhaf)} = \frac{1}{N + (n-1)}$$

$$\mathrm{SDP(Banzhaf)} = \frac{N}{N + (n-1)}$$

Recall that n is the total number of players and N is the total number of winning coalitions. In order to specify the latter, let us increase n from 3 to 4 and list all the winning coalitions:

Table 5.

Winning coalitions:	n=3	n=4
	A (not a winning coalition!)	AD
	AB	ABD
	AC	ACD
	ABC	ABCD
		AB
		AC
		ABC
Total number of winning coalitions	3	7

One can see that by increasing n from 3 to 4 creates 4 more winning coalitions (marked yellow) in addition to the 3 existing ones. They are created by adding the new player to the existing winning coalitions. Additionally, there is one winning coalition created by adding the new player to the semi-dictator. Accordingly, by adding one new player to the coalition, one obtains N(n+1)=2*N(n)+1. This recursive formula with N(1)=0 leads to the general specification of N in the semi-dictator setup, as a function of n:

$$N(n) = 2^{n-1} - 1$$

This yields the total number of swings of: $2^{n-1} + n - 2$[13]. Plugging it back into MP and SDP gives:

$$\mathrm{MP(Banzhaf)} = \frac{1}{2^{n-1}+n-2} \quad (1')$$

$$\mathrm{SDP(Banzhaf)} = \frac{2^{n-1}-1}{2^{n-1}+n-2} \quad (1'')$$

The same analysis for the Shapley-Shubik index is much easier. There, non-semi-dictator player i is pivotal only if the first player in the coalition is player A and the second player is player i. This can happen only once for any player i. In any other case, player A is pivotal. Therefore, MP and SDP are specified as follows:

$$\mathrm{MP(SS)} = \frac{1}{n!} \quad (2')$$

$$\mathrm{SDP(SS)} = \frac{n!-(n-1)}{n!} \quad (2'')$$

In case of the Holler-Packel index, the analysis is even simpler. As this index only takes MWCs into consideration, it is easy to see that in case of a semi-dictator setup, there are n-1 MWCs (formed by player A and each player). Player A is a member of all of them and each non-semi-dictator player is a member of just one of them. Accordingly:

$$\mathrm{MP(PGI)} = \frac{1}{2(\mathrm{n}-1)} \quad (3')$$

$$\mathrm{SDP(PGI)} = \frac{1}{2} \quad (3'')$$

The Deegan-Packel index differs from the Public Good Index only with respect to sharing the value of the game. Accordingly, the power of each player in each coalition must be divided by the number of that coalition's members. However, in case of the semi-dictator setup, the MWCs always consist of two players. Therefore, in each coalition player A takes half of the power and the other player takes the other half. After normalization the measures are exactly the same:

$$\mathrm{MP(DP)} = \frac{1}{2(\mathrm{n}-1)} \quad (4')$$

$$\mathrm{SDP(DP)} = \frac{1}{2} \quad (4'')$$

With the main assumption: SDP≥x_i≥MP and the values of SDP and MP specified for all the power indices, it is possible to proceed to the formulation of the solution.

IV.3 Solution specification.

[13] interestingly, when one sets n=n+1, this formulation becomes $2^n + n - 1$, which is the shortest length of bitstring containing all bitstrings of given length (see: sequence A052944 in: The On-Line Encyclopedia of Integer Sequences, Rainer Rosenthal, Apr 30 2003)

As mentioned earlier, the idea is to randomize between games where one of the players is a semi-dictator and the other players are non-dummies. Let us specify the games as follows (assume that the players are sorted in descending order with respect to the desirable power):

Table 6.

voting rights	Semi-dictator games				
players	g_1	g_2	…	g_{n-1}	g_n
p_1	SD		…		
p_2		SD	…		
…	⋮	⋮	⋱	⋮	⋮
p_{n-1}			…	SD	
p_n			…		SD

SD indicates that in a given game player i is the semi-dictator. The missing element of the solution is now only the probability vector p_g over g=(g_1, g_2, …, g_n), such that the mixture over g yields power of player i equal to the desired power of player i [P'_i] (according to the objective measure) for all players. Such probability vector would obviously have a different shape for each measure of power. If one considers measuring power by the Banzhaf index, the probability of drawing each g_i should be:

$$prob(g_i) = \frac{(2^{n-1}-1)P'_i - 1}{2^{n-1}-n-1} \qquad (7)$$

In case of taking the Shapley-Shubik index, the adequate probability should be equal:

$$prob(g_i) = \frac{P'_i * n! - 1}{n! - n} \qquad (8)$$

If one considers PGI or DP index to be relevant, the probability vector should be specified as follows:

$$prob(g_i) = \frac{P'_i * 2(n-1) - 1}{n-2} \qquad (9)$$

Proposition 1. Let P' be a vector of desired power measures P, specified by (1), (2), (3) or (4). If probabilities $prob(g_i)$ of semi-dictator games g_1,…,g_n are specified by equation (7), (8), (9) or (9) (respectively to the power measure), the expected power measure of such a game will be equal to the desired power measure for all players: $P'_i - P_i = 0$ for every i.

The proof for all power measures is presented in the appendix.

Example: How could the proposed mechanism implemented in practice? Let's consider an example where one wants to obtain the power distribution of [0.4, 0.2, 0.15, 0.1, 0.1, 0.05]. The adequate probability distribution over semi-dictator games should then be: [0.401961, 0.20028, 0.14986, 0.0994398, 0.0994398, 0.0490196] for the Shapley-Shubik index (derived from (7)) and [0.456, 0.208, 0.146, 0.084, 0.084, 0.022] for the Banzhaf index (derived from (6)). If one is willing to implement the mechanism in a parliament of k seats, it is convenient to think of the randomization between semi-dictator games g_1, g_2, …, g_6 as a randomization of some fraction of seats. If there are k seats in the parliament, the quota is equal to q (which is at least equal to (k+1)/2) and there are n parties, the implementation of the mechanism could consist in parties obtaining a fixed number of seats (let us denote it by a) and a possibility of getting additional seats (let us denote it by b) with a certain probability given by the probability vector specified by [0.401961, 0.20028, 0.14986, 0.0994398, *0.0994398, 0.0490196] or [0.456, 0.208, 0.146, 0.084, 0.084, 0.022]. The fixed number of seats (a) and the drawn number of seats (b) need to fulfill following requirements:*

- *(n-1)*a=q-1* *(all the non-semi-dictator players cannot form a winning coalition without the semi-dictator)*
- *a+b=q-1* *(the semi-dictator cannot form a winning coalition alone)*
- *n*a+b=k* *(the fixed number of seats of all parties plus the drawn seats sum up to the size of the parliament)*

Solving the above system of equations for a and b, one obtains:

$$a = \frac{k}{2(n-1)}$$

$$b = \frac{2(n-1)(q-1)-k}{2(n-1)}$$

which gives reasonable values (a ≥1, b ≥1) for $\frac{k}{2(q-2)} + 1 \leq n \leq \frac{k}{2} + 1$. *If one assumes k=100 and q=51, the fixed number of seats (a) equals 10 and the drawn number of seats equals 40. Accordingly, each party's situation is as follows:*

- *party A: 10 fixed seats and a 45.6% or 40.1961% (depending on whether one uses the Banzhaf or the Shapley-Shubik index) probability of getting 40 more seats and becoming the semi-dictator.*
- *party B: 10 fixed seats and a 20.8% or 20.028% (depending on whether one uses the Banzhaf or the Shapley-Shubik index) probability of getting 40 more seats and becoming the semi-dictator.*
- *party C: 10 fixed seats and a 14.6% or 14.986% (depending on whether one uses the Banzhaf or the Shapley-Shubik index) probability of getting 40 more seats and becoming the semi-dictator.*
- *party D: 10 fixed seats and a 8.4% or 9.94398% (depending on whether one uses the Banzhaf or the Shapley-Shubik index) probability of getting 40 more seats and becoming the semi-dictator.*
- *party E: 10 fixed seats and a 8.4% or 9.94398% (depending on whether one uses the Banzhaf or the Shapley-Shubik index) probability of getting 40 more seats and becoming the semi-dictator.*
- *party F: 10 fixed seats and a 2.2% or 4.90196% (depending on whether one uses the Banzhaf or the Shapley-Shubik index) probability of getting 40 more seats and becoming the semi-dictator.*

Note that in such a system each party has an expected power exactly equal to the desired power measure. Moreover no party is a dummy player regardless which party becomes the semi-dictator.

Of course the presented solution is limited by its assumption: SDP≥x_i≥MP, so it is impossible to use the solution in the example given above, if one considers PG or DP. When this assumption is not satisfied, it is difficult to adjust the solution with the aim of minimizing the number of the dummy players or the probability that there will be a dummy. If one considers an example of a desired power distribution [0.42, 0.30, 0.18, 0.10] (and considers the PGI as the relevant power measure), the last player clearly violates the assumption, as for n=4, MP(PGI)=1/6 and 0.10<1/6. It would be natural to consider an adjustment to the original solution where there is a possibility of player 4 to become a dummy with as low probability as possible. Then, player 4 would have to be the dummy with probability 40% and be a normal non-semi-dictator player with probability 60% (0.4*0+0.6*1/6=0.1). In such a situation, with probability 40%, players 1-3 would play a 3-player semi-dictator game, where the minimum power one can obtain is 0.25. Given player 4 is a dummy with probability of 40%, the minimum threshold of power for the other players changes. In particular, the lowest possible power for players 1-3 is now 40%*0.25+60%*1/6=0.2. Clearly, it now turns out that player 3 violates this modified threshold. Under such circumstances one needs to impose positive probability on player 3 becoming a dummy, but this would, in turn, increase the lower threshold for other players and so on.

Therefore, if one wants to have a system which minimizes the number of dummy players and the probability of drawing a game in which dummy players appear without making any assumptions about the desired power distribution, one needs to resort to numerical solutions. The exact code is not ready yet, because its programming is still in progress, but the sketch of an algorithm is presented below:

1) assign the objective measure of power $P'=[P'_1, P'_2, \ldots, P'_n]$, set the parliament size (k) and the quota (q)
2) find all feasible weight distributions such that $\sum_{i=1}^{n} w_{ji} = k$ as vectors $d_j=[w_{j1},w_{j2}, \ldots, w_{jn}]$
3) compute power indices of d_j $\forall j$, $P_j=[P_{j1}, P_{j2}, \ldots, P_{jn}]$
4) group the distributions d_j by their power indices P_j and assign indices l to each group
5) find all possible values of P_j and denote them by X_l
6) for each l, find all possible probability distributions such that prob(X)*X=P' and call them prob'(X)
7) find a set of probability distributions prob(X_l), such that prob (X)∈ prob'(X) so that the expected number of zeros in the matrix prob(X).*X is minimized. Call that probability distribution prob''(X).
8) once the optimal prob''(X) is found, for each l choose a set of d_j corresponding to X_l, such that the expected variance in w_{ji} across j is minimized and the variance in w_{ji} across i is minimized, with certain weights assigned to each minimization. That is equivalent to equalizing the number of fixed seats within each game and minimizing the number of randomly drawn seats. Denote the chosen set of d_j as d'_j.
9) set the number of fixed seats to $a_i = min_j(w_{ji})$, such that $w_{ji} \in d'_j$ and the total number of randomly drawn seats to $b = k - \sum_i a_i$.
10) set the possible drawing of seats to: $b_{ji} = [w_{j,i} - a_i]$ and apply the corresponding probability distribution prob(X_l).

V. Conclusions.

The idea of the voting mechanism which guarantees strict proportionality, presented in this work, does not require resorting to dummy voters. Resorting to dummy voters is often described as the most important argument against applying random voting mechanisms in real life. Risk aversion leads to firm disagreement to any system which makes it possible that one of the players has no real power, i.e. has no influence on decision making. Although such mechanisms, like the ones presented by Berg and Holler (1986), have an undeniable advantage of allowing the power distribution to replicate any distribution one considers to be objectively right, they are not widely applied in committees or parliaments. Such solutions are socially unacceptable, whereas fixed rules causing significant distortions in power distribution are commonly used. This indicates that distortions in power distribution are socially preferred to risk with regard to a posteriori power distribution. In this work the argument of Berg and Holler (1986), stating that resorting to dictator games is a "high price of monotonicity" due to objections of all the players without any say in a given case, is extended. Namely, it may be the expected number of dummy players which makes the distortion-free mechanisms so "costly". Therefore, probably using the voting system presented in this work, where no player can ever be a dummy, would be more socially acceptable than previous concepts. This is obviously a normative issue which can be settled by questionnaires or an analysis whether this distortion- and dummy-free mechanism will have applications in the future. Unfortunately, obtaining a solution without dummy players is costly in terms of assumptions regarding the desirable power distribution. The limitations, rather not likely to be binding in case of the Shapley-Shubik index, are probably extremely unlikely to be satisfied for PGI or the Deegan-Packel index, especially for small n. In case of unsatisfied assumptions one can only resort to numerical solutions.

It is clear that the propositions presented in this work have their standard drawbacks: as all random voting mechanisms they make it virtually impossible to form stable coalitions. Moreover, if one seriously considering applying such mechanisms in practice, one would have to introduce a perfect randomizing device. Credibility of such device would be absolutely crucial for the whole concept to make sense. In particular, the distinction between a priori and a posteriori power, which raises controversies in the literature, would become very relevant. For instance, if the probability distribution a priori was to be [0.5, 0.3, 0.2], it is not clear whether the randomizing device should provide such distribution of outcomes in expected terms, or it should guarantee such distribution a posteriori. In the first case, it would be very likely that after the whole period of time, when the random voting procedure was in place, the real distribution of frequency with which one became the semi-dictator was different than a priori probability distribution. That would mean that the actual distribution of power would – a posteriori – differ from the desirable power distribution. In case of guaranteeing a posteriori distribution of expected probabilities, there would be, in turn, a problem of fluctuations of a priori power. Considering the probability distribution mentioned above and assuming there are 10 decisions to be made (by voting), the actual outcomes of the randomizing device should be such that player 1 becomes the semi-dictator exactly 5 times, player 2 – 3 times and player 3 – 2 times. Then, suppose player 1 was chosen to be the dictator in the first 5 meetings. In such a case, his/her expected a priori power would be very low, whereas the other players' powers would be higher than desired. Such time dependency is problematic as it may lead to opportunistic proposal scheduling, etc.

Obviously, also the choice of the relevant power measure is non-trivial, but this work abstracts from that problem providing strictly proportional mechanisms without dummy players for all popular power indices. An interesting problem for further investigation is the (philosophical) question if, for certain application, it is desirable to use distortion-free voting mechanisms. If one considers democracy and parliaments, it is traditionally accepted that a party with more than 50% of the votes effectively becomes a dictator. If one applied the axiom that the a party's number of votes in the elections should be directly translated to its power in the parliament, a party with more than 50% of the votes would – depending on the power measure – become a dictator only occasionally, if at all. Therefore, one should always keep in mind the trade-off between distortions with regard to power and non-stability of coalitions or political non-continuity.

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Appendix.

Proof of theorem 1.[14]

By (1), (2), (3) or (4), depending on which power measure one considers, no expected distortion occurs (P_i=P'_i for all i) if prob(u_i) solve the following system of linear equations (m = 1, …, n):

$$\begin{cases} 1 * prob(u_1) + \frac{1}{2} * prob(u_2) + \cdots + \frac{1}{n} * prob(u_n) = P'_1 \\ \qquad\qquad \frac{1}{2} * prob(u_2) + \cdots + \frac{1}{n} * prob(u_n) = P'_2 \\ \qquad\qquad\qquad \ddots \qquad \cdots \qquad \vdots \quad \vdots \\ \qquad\qquad\qquad\qquad \frac{1}{n} * prob(u_n) = P'_n \end{cases}$$

where LHS of equation m specifies the expected power measure of player m and RHS of equation m specifies the desired power measure of player m. This system of linear equations has a unique solution regardless of P':

$$\begin{bmatrix} prob(u_1) \\ prob(u_2) \\ \vdots \\ prob(u_{n-1}) \\ prob(u_n) \end{bmatrix} = \begin{bmatrix} 1 * (P'_1 - P'_2) \\ 2 * (P'_2 - P'_3) \\ \vdots \\ (n-1) * (P'_{n-1} - P'_n) \\ n * P'_n \end{bmatrix}$$

For generic $\mathrm{prob}(u_i)$ this solution is equivalent to:

$$\mathrm{prob}(u_i) = \begin{cases} \mathrm{i} * (\,\alpha_i - \alpha_{i+1}) & \text{for i} < \text{n} \\ \mathrm{i} * \alpha_i & \text{for i} = \text{n} \end{cases}$$

Proof of proposition 1.

For the Banzhaf index:

As only semi-dictator games are considered, by (1), (1') and (1''), no expected distortion occurs (P_i=P'_i for all i) if prob(g_i) solve the following system of linear equations (m = 1, …, n):

$$\begin{cases} \frac{2^{n-1}-1}{2^{n-1}+n-2} * prob(g_1) + \frac{1}{2^{n-1}+n-2} * prob(g_2) + \cdots + \frac{1}{2^{n-1}+n-2} * prob(g_n) = P'_1 \\ \frac{1}{2^{n-1}+n-2} * prob(g_1) + \frac{2^{n-1}-1}{2^{n-1}+n-2} * prob(g_2) + \cdots + \frac{1}{2^{n-1}+n-2} * prob(g_n) = P'_2 \\ \qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad \ddots \qquad \vdots \qquad \vdots \quad \vdots \\ \frac{1}{2^{n-1}+n-2} * prob(g_1) + \frac{1}{2^{n-1}+n-2} * prob(g_2) + \cdots + \frac{2^{n-1}-1}{2^{n-1}+n-2} * prob(g_n) = P'_n \end{cases}$$

where LHS of equation m specifies the expected value of the Banzhaf index of player m and RHS of equation m specifies the desired power measure of player m. This system of linear equations has a unique solution, provided that $\frac{2^{n-1}-1}{2^{n-1}+n-2} \geq P'_i \geq \frac{1}{2^{n-1}+n-2}$ for all i:

[14] Di Giannatale and Pasarelli (2011), page 31.

$$\begin{bmatrix} prob(g_1) \\ prob(g_2) \\ \vdots \\ prob(g_{n-1}) \\ prob(g_n) \end{bmatrix} = \begin{bmatrix} \frac{(2^{n-1}-1)P'_1 - 1}{2^{n-1}-n-1} \\ \frac{(2^{n-1}-1)P'_2 - 1}{2^{n-1}-n-1} \\ \vdots \\ \frac{(2^{n-1}-1)P'_{n-1} - 1}{2^{n-1}-n-1} \\ \frac{(2^{n-1}-1)P'_n - 1}{2^{n-1}-n-1} \end{bmatrix}$$

For generic $\mathrm{prob}(g_i)$ this solution is equivalent to:

$$\mathrm{prob}(g_i) = \frac{(2^{n-1}-1)P'_i - 1}{2^{n-1}-n-1}$$

<u>For the Shapley-Shubik index:</u>

As only semi-dictator games are considered, by (2), (2') and (2''), no expected distortion occurs (P_i=P'_i for all i) if prob(g_i) solve the following system of linear equations (m = 1, …, n):

$$\begin{cases} \frac{n!-(n-1)}{n!} * prob(g_1) + \frac{1}{n!} * prob(g_2) + \cdots + \frac{1}{n!} * prob(g_n) = P'_1 \\ \frac{1}{n!} * prob(g_1) + \frac{n!-(n-1)}{n!} * prob(g_2) + \cdots + \frac{1}{n!} * prob(g_n) = P'_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad \ddots \qquad\qquad \vdots \qquad \vdots \;\; \vdots \\ \frac{1}{n!} * prob(g_1) + \frac{1}{n!} * prob(g_2) + \cdots + \frac{n!-(n-1)}{n!} * prob(g_n) = P'_n \end{cases}$$

where LHS of equation m specifies the expected value of the Shapley-Shubik index of player m and RHS of equation m specifies the desired power measure of player m. This system of linear equations has a unique solution, provided that $\frac{n!-(n-1)}{n!} \geq P'_i \geq \frac{1}{n!}$ for all i:

$$\begin{bmatrix} prob(g_1) \\ prob(g_2) \\ \vdots \\ prob(g_{n-1}) \\ prob(g_n) \end{bmatrix} = \begin{bmatrix} \frac{P'_1 * n! - 1}{n! - n} \\ \frac{P'_2 * n! - 1}{n! - n} \\ \vdots \\ \frac{P'_{n-1} * n! - 1}{n! - n} \\ \frac{P'_n * n! - 1}{n! - n} \end{bmatrix}$$

For generic $\mathrm{prob}(g_i)$ this solution is equivalent to:

$$\mathrm{prob}(g_i) = \frac{P'_i * n! - 1}{n! - n}$$

<u>For the Public Good Index and the Deegan-Packel index :</u>

As only semi-dictator games are considered, by (3), (3') and (3'') or (4), (4') and (4''), no expected distortion occurs (P_i=P'_i for all i) if prob(g_i) solve the following system of linear equations (m = 1, …, n):

$$\begin{cases} \frac{1}{2} * prob(g_1) + \frac{1}{2(\mathrm{n}-1)} * prob(g_2) + \cdots + \frac{1}{2(\mathrm{n}-1)} * prob(g_n) = P'_1 \\ \frac{1}{2(\mathrm{n}-1)} * prob(g_1) + \frac{1}{2} * prob(g_2) + \cdots + \frac{1}{2(\mathrm{n}-1)} * prob(g_n) = P'_2 \\ \vdots \qquad \vdots \qquad \ddots \qquad \vdots \qquad \vdots \quad \vdots \\ \frac{1}{2(\mathrm{n}-1)} * prob(g_1) + \frac{1}{2(\mathrm{n}-1)} * prob(g_2) + \cdots + \frac{1}{2} * prob(g_n) = P'_n \end{cases}$$

where LHS of equation m specifies the expected value of the Shapley-Shubik index of player m and RHS of equation m specifies the desired power measure of player m. This system of linear equations has a unique solution, provided that $\frac{1}{2} \geq P'_i \geq \frac{1}{2(n-1)}$ for all i:

$$\begin{bmatrix} prob(g_1) \\ prob(g_2) \\ \vdots \\ prob(g_{n-1}) \\ prob(g_n) \end{bmatrix} = \begin{bmatrix} \frac{P'_1 * 2(n-1) - 1}{n-2} \\ \frac{P'_2 * 2(n-1) - 1}{n-2} \\ \vdots \\ \frac{P'_{n-1} * 2(n-1) - 1}{n-2} \\ \frac{P'_n * 2(n-1) - 1}{n-2} \end{bmatrix}$$

For generic $\mathrm{prob}(g_i)$ this solution is equivalent to:

$$\mathrm{prob}(g_i) = \frac{P'_i * 2(n-1) - 1}{n-2}$$

Abstract

Majority voting is considered to be the right and just way to make collective decisions and is therefore a common method of resolving conflicts or finding solutions for problems that concern a group of people. If one considers e.g. indirect democracy, it turns out that making the distribution of votes in the parliament proportional to the outcome of the elections yields distortions with regard to power. Some parties get more relative power than the relative number of votes they received, whereas some parties get less power. These distortions are a negative phenomenon which makes the distribution of power biased with respect to the will of the nation, causes possibilities for strategic behaviors, etc. However, it is impossible to avoid such distortions unless one uses a randomizing device for determining voting weights or voting quota. On the other hand, when such randomization is implemented, parties may, with positive probability, become dummies. That means they may have no voting power at all. Such a situation is inefficient, as long as risk-averse agents are concerned. It is also considered socially and politically unacceptable. This work presents an overview of the existing mechanisms which yield distortion-free transformation of votes into voting power and also develops a mechanism which does not resort to dummy players. The notion of a semi-dictator setup is introduced: player X is called the semi-dictator if player X is a member of all feasible winning coalitions, but cannot form a winning coalition alone. The new distortion-free mechanism consists in randomizing between semi-dictator setups. This guarantees that no agent may ever be a dummy player. It also imposes certain restrictions on the desired power distribution. The solution and its assumptions regarding the desired power distribution are developed for the major power indices: Banzhaf, Shapley-Shubik, Holler-Packel and Deegan-Packel. For cases where the desired power distribution does not satisfy the assumptions, a sketch of a numerical solution is presented.

Zusammenfassung

Das Mehrheitswahlsystem wird als eine richtige und gerechte Methode betrachtet, mithilfe deren gemeinsame Entscheidungen getroffen werden können. Deswegen wird es gewöhnlich verwendet, um Konflikte bzw. Probleme, die eine Gruppe von Menschen betreffen, zu lösen. Wenn man allerdings Stimmen in einem Parlament proportional zu einem Wahlergebnis verteilt, stellt sich beispielsweise im Falle von unmittelbarer Demokratie heraus, dass dies zu Verzerrungen in Bezug auf die Macht führt. Folglich bekommen einige Parteien mehr jeweilige Macht als sie laut der jeweiligen Anzahl der bekommenen Stimmen erhalten sollen. Mittlerweile bekommen andere Parteien weniger Macht. Solche Verzerrungen sind eine negative Erscheinung , die dazu führt, dass die Macht tendenziös verteilt ist, was gegen den Willen der Nation ist und den Spielraum für strategisches Verhalten lässt usw. Es ist jedoch möglich, solche Verzerrungen zu vermeiden, sofern man keine Geräte anwendet, die die Stimmenwichtigkeit und Stimmenanzahl zufällig einschätzen. Andererseits, es ist hochwahrscheinlich, dass die Parteien unecht werden, wenn man mit den Stimmen zufällig umgeht. Das bedeutet, dass sie kein Stimmrecht haben. Solche Situation ist ineffizient, solange bis die risikomeidenden Faktoren in Betracht gezogen werden. Das ist auch sozial und politisch unakzeptabel. Die vorliegende Arbeit ist ein Überblick über die Mechanismen, die darauf abgezielt sind, eine Transformation der nicht verzerrten Stimmen in das Stimmrecht zu liefern. Die Arbeit entwickelt einen Mechanismus, der auf die unechten Spieler nicht zurückgreift. Es wurde auch der Begriff „Semi-Dictator- Setup“ eingeführt. Ein Spieler X wird als Semi-Dictator genannt, wenn er ein Mitglied aller ausführbaren Siegkoalitionen ist, aber selbst kann keine Siegkoalition bilden. Der neue Mechanismus besteht darin, dass man nach dem Zufallsprinzip unter den Semi-Dictator Setups handelt. Das garantiert, dass keiner Spieler unecht wird. Die gewünschte Machtverteilung wird auch zu einem gewissen Maße eingeschränkt. Die Lösung und ihre Annahme im Hinblick auf die gewünschte Machtverteilung wurde für die meisten Machtindizien entwickelt: Banzhaf, Shapley-Shubik, Holler-Packel und Deegan-Packel. Im Falle, wenn die gewünschte Machtverteilung nicht zufriedenstellend ist, wird eine Skizze der numerischen Lösungen dargestellt.

Key words

voting, power index, distortion-free transformation, semi-dictator

Curriculum Vitae

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Education

Warsaw School of Economics, Poland
Bachelor in Finance and Accounting, Specialty: Investment Banking, July 2010
Average of the grades: 4,7/5,5, final grade: 5/5,5

Universidad de Navarra - Pamplona, Spain (Erasmus programme)
Economics, bilingual, January-May 2010

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University of Warsaw, Poland

Master program in international economics, 2010-2012

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University of Vienna, Austria

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Professional experience

- Summer 2009, 2010: Fund raiser for an NGO “Bund Naturschutz”, Germany
- September 2008 – may 2011: English and maths teacher for a 15-year-old secondary school student
- June – August 2011: Supporting staff at “Belvoir Terrace” in Lenox, MA, USA

Publications

“Between unemployment and inflation, keynes and smith in the face of the crisis in the USA” in: *pociag do pracy I inne studia przypadku do nauczania ekonomii,* pod red. b. czarnego, warszawa 2010, ISBN 978-83-61713-76-0 pp. (in polish)

Other activities and achievements

June 2011: Being granted the CEEPUS scholarship, which enabled me to make a double-degree program and study at the University of Vienna in the academic year 2011-2012

June 2011: Finishing the 1st year of education at University of Warsaw as the best student in my year

2010 – 2011: A member of students' scientific association of economics at Warsaw School of Economics

September 2009: 2nd place in a competition to create a case study supporting classes of economics, organized by National School of Public Administration and National Bank of Poland; the work was later published (see: publications)

2007 – today: being granted a scholarship for good academic performance every semester, first at Warsaw School of Economics, then at the University of Warsaw and University of Vienna

January 2008 – January 2009: Member of the board of directors responsible for PR in Soli Deo (Catholic student organization)

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Languages

Polish: Mother Tongue

English: C1 according to CEFR; * Certificate in Advanced English (Cambridge), 2005; my whole master program is in English

German: B2 According to CEFR; * Test onDaf – B2, 2009

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Traveling, motor sports, Bicycle tourism, photography