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MASTERARBEIT

Titel der Masterarbeit

"The splitting number and some of its neighbors "

Verfasserin

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Abstract

Diese Masterarbeit ist eine Zusammenstellung von Konsitenzresultaten im Bezug auf die Begrenzungs- und die Aufspaltungszahl. Im ersten Teil der Arbeit, ist ihre Unabhängigkeit bewiesen:

- Bezüglich der Aussage $Con(\mathfrak{s} = \aleph_1 < \mathfrak{b} = \kappa)$, presentiert diese Arbeit drei verschiedene Argumente. Alle verwenden das Hechler Modell ([3], [2] und [7]).
- Das "proper forcing", dessen abzählbar unterstützte Iteration, das Resultat $Con(\mathfrak{b} = \aleph_1 < \mathfrak{s} = \aleph_2)$ ergibt, wurde von Shelah in [19] entdeckt. Auf der zweiten Koordinate der Mathias Bedingungen, hat Shelah eine zusätzliche kombinatorische Struktur eingefügt, in Form eines logarithmischen Maßes, dessen Eigenschaften, die beinahe ω^ω-Begrenzbarkeit und die Existenz einer unaufgespalteten reellen Zahl gewährleisten.

Der zweite Teil der Arbeit bezieht sich auf eine Methode, die beliebige Werte der Invarianten ergibt, und zwar die Konstruktion von Matrix Iterationen. Diese Iterationen bildet man mit Hilfe der Shelah-Technik ([5]) für Erweiterung eines gegebenes Ultrafilters, unter Beibehaltung der vollständigen Einbettbarkeit und einer zusätzlichen kombinatorischen Eigenschaft:

(1) $Con(\mathfrak{b} = \aleph_1 < \mathfrak{s} = \mathfrak{c} = \kappa)$ erhält man, wenn man mit einer endlich unterstützten Cohen Iteration anfängt und sicherstellt, dass die hinzugefügte reelle Zahl unbegrenzt erhalten wird. Längere Iterationen ergeben immernoch $\mathfrak{b} = \aleph_1$.

(2) $Con(\mathfrak{b} = \mathfrak{a} = \kappa < \mathfrak{s} = \mathfrak{c} = \lambda)$ erhält man, wenn man mit einer endlich unterstützten Iteration des forcings, das maximale fast-disjunkte Familien einfügt, anfängt. Die kombinatorische Eigenschaft, die erhalten werden soll, bezieht sich auf fast-Disjunktheit.

(3) $Con(\mathfrak{b} = \kappa < \mathfrak{s} = \mathfrak{a} = \mathfrak{c} = \lambda)$ ist nur unter der Annahme, dass ein messbare Kardinalzahl existiert bekannt. Dann kann man die "ultrapower" einer partiellen Ordnung bilden. Diese Konstruktion spielt eine wichtige Rolle, weil sie die maximalen fast-disjunkte Familien zerstört, und deswegen $\mathfrak{a}$ steigt. Die Iteration unterscheidet sich von (1) dahingehend, dass diese Konstruktion in jedem dritten Schritt gemacht wird.

Abstract

This thesis is a survey on consistency results involving the bounding number and the splitting number. In the first part of the thesis, their independence is showed:

- For $Con(\mathfrak{s} = \aleph_1 < \mathfrak{b} = \kappa)$, this thesis gives three different arguments that appear in the literature (in [3], [2] and [7]), all involving the Hechler model.
- The proper forcing notion whose countable support iteration gives $Con(\mathfrak{b} = \aleph_1 < \mathfrak{s} = \aleph_2)$ was developed by Shelah in [19]. On the pure part of the Mathias conditions, Shelah added an additional combinatorial structure, in form of a logarithmic measure, whose properties ensure, that the almost ω^ω-bounding property is satisfied, and that an unsplit real is added.

The second part of the thesis is concentrated on the method used towards obtaining arbitrary spread between these cardinal invariants, namely, the construction of matrix iterations. These iterations are constructed using Shelah's technique ([5]) of extending a given ultrafilter, while preserving the complete embeddability and some combinatorial property:

(1) $Con(\mathfrak{b} = \aleph_1 < \mathfrak{s} = \mathfrak{c} = \kappa)$ is obtained by starting with a finite support iteration (of length $\aleph_1$) of Cohen forcing and then ensuring that the added unbounded real remains unbounded. Longer iterations still keep $\mathfrak{b} = \aleph_1$. For sidestepping this problem, the almost disjointness number is involved ([10]), since $\mathfrak{b} \leq \mathfrak{a}$ is provable in ZFC.

(2) $Con(\mathfrak{b} = \mathfrak{a} = \kappa < \mathfrak{s} = \mathfrak{c} = \lambda)$ is obtained by starting with a finite support iteration of a forcing adding m.a.d. families, and preserving a combinatorial property related to the almost disjointness.

(3) $Con(\mathfrak{b} = \kappa < \mathfrak{s} = \mathfrak{a} = \mathfrak{c} = \lambda)$ is only known above a measurable cardinal, where the ultrapower of a partial order exists. Raising the poset to the ultrapower plays an important role, since big m.a.d. families are destroyed, and therefore, $\mathfrak{a}$ is increased. One proceeds as in (1), with the only difference, that every third step, the partial order is raised to its ultrapower.

Contents

Chapter 1

Introduction

Cardinal invariants of the continuum are cardinal numbers describing combinatorial properties of the real line, in various models. The first theorem about cardinal invariants is the theorem of Cantor, which says that the continuum is strictly larger than the cardinality of a countable infinite set, i.e. $2^{\aleph_0} > \aleph_0$. This theorem has various applications, especially in real analysis, where often interesting properties of countable sets, which cannot be extended to sets of cardinality $\mathfrak{c}$, are studied. A **cardinal invariant**, also called **cardinal characteristic**, is the minimal cardinal number for which such a property of countable sets, becomes false. Clearly, these invariants will take values between $\aleph_1$ and $\mathfrak{c}$, so they are uninteresting under the Continuum Hypothesis. Actually, the interesting part is the study of relationships between these cardinal numbers.

The rest of this chapter will be a short exposition of historical notes, generalities and basic facts on forcing, and some easy ZFC provable inequalities between the cardinal invariants $\mathfrak{b}$, its dual $\mathfrak{d}$, $\mathfrak{s}$ and $\mathfrak{a}$.

The splitting number $\mathfrak{s}$ is the minimal cardinality of a splitting family, the bounding number $\mathfrak{b}$ is the minimal cardinality of an unbounded family, $\mathfrak{d}$ is the minimal cardinality of a dominating family and $\mathfrak{a}$ is the minimal cardinality of an infinite maximal almost disjoint family.

The second chapter will contain three different arguments for $Con(\mathfrak{s} < \mathfrak{b})$, all three involving the Dominating forcing (also called Hechler forcing).

After some generalities on proper forcing and an introduction to Mathias forcing, the third chapter of the thesis presents the first creature forcing that appeared in the literature. S.Shelah obtained this proper forcing notion by adding an additional combinatorial structure on the pure part of the Mathias conditions, in form of a logarithmic measure. Using the properties of this logarithmic measure, it is shown that this forcing still adds an unsplit real (as Mathias forcing does), but moreover, its countable support iteration keeps $\mathfrak{b}$ small (which is not true in the Mathias model). With this forcing construction, the consistency of $\mathfrak{b} = \aleph_1 < \mathfrak{s} = \aleph_2$ is settled, and hence, the independence of the invariants $\mathfrak{b}$ and $\mathfrak{s}$.

Until the end of the thesis, the goal of obtaining arbitrary large values for these invariants will be set.

The fourth chapter is concerned with the absoluteness of maximal antichains, especially in case of non-definable forcing notions. The central result of this section is due to Blass and Shelah, and gives the construction of an ultrafilter for Mathias Prikry forcing, extending a given ultrafilter and preserving a given unbounded real unbounded. This construction is the basis for the matrix iteration presented in the fifth chapter, witnessing $Con(\mathfrak{b} = \aleph_1 < \mathfrak{s} = \kappa)$. However, a higher iteration would not witness

$Con(\aleph_1 < \mathfrak{b} < \mathfrak{s} = \kappa)$.

To get $\mathfrak{b}$ larger than $\aleph_1$, the cardinal invariant $\mathfrak{a}$ plays an important role. This is because of the ZFC-provable inequality $\mathfrak{b} \leq \mathfrak{a}$ (presented in section 1.3 of this thesis). The model obtained in the sixth chapter (using a matrix iteration) witnesses $\kappa = \mathfrak{a} \leq \mathfrak{b}$ and $\mathfrak{s} = \lambda$. Therefore, we are to be able to conclude $Con(\mathfrak{b} = \mathfrak{a} = \kappa < \mathfrak{s} = \lambda)$. The construction is similar to the one presented in Chapter 4, the arguments generalize nicely, but instead of an unbounded real, some other combinatorial property has to be preserved.

For the consistency of $\mathfrak{b} = \kappa < \mathfrak{s} = \mathfrak{a} = \lambda$, the construction of the matrix iteration is done above a measurable cardinal μ, where the ultrapower of a partial order can be constructed. Taking the ultrapower destroys maximal almost disjoint families of cardinality $\geq \kappa$ (this suffices since $\kappa = \mathfrak{b} \leq \mathfrak{a}$), thus, the final model will witness $\mu < \mathfrak{b} = \kappa < \mathfrak{a} = \mathfrak{s} = \lambda$.

There are still open questions concerning this matter, for example, if the last consistency result can be obtained without the measurable cardinal assumption, or if strict inequality can be obtained between the three invariants (of course, regarding the ZFC-provable inequality $\mathfrak{b} \leq \mathfrak{a}$). Thus, there is space for future research.

1.1 Historical notes and the method of forcing

The Continuum Hypothesis (CH) is a hypothesis presented by the German mathematician Georg Cantor in 1878, saying that every set of reals is either countable or has the cardinality $\mathfrak{c}$ of the real numbers. It is easy to see that $\mathfrak{c} = 2^{\aleph_0}$, namely:

- $\mathfrak{c} \geq 2^{\aleph_0}$, since the Cantor set is a set of reals of cardinality $2^{\aleph_0}$.
- $\mathfrak{c} \leq 2^{\aleph_0}$, since the set of rational numbers $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$ (every real r is the supremum of all smaller rational numbers, thus $\mathfrak{c} \leq |\mathcal{P}(\mathbb{Q})|$).

Proving or refuting CH is the first of Hilbert's 23 problems presented in 1900. Cantor had the idea to study perfect sets, hoping that this would be sufficient to determine the validity of CH. He managed to show that every perfect set has the same cardinality as the reals, but later, Bernstein constructed an uncountable set, with the property that neither it nor its complement contains a perfect set. Thus, studying the perfect sets was clearly not enough for deciding CH.

Many other mathematicians studied various combinatorial structures on the reals, giving rise to cardinal invariants. For example, in 1899, René Baire studied the meager sets and showed that countably many meager sets cannot cover the real line. In 1909, Felix Hausdorff studied maximal linearly ordered subsets of $({}^{\mathbb{N}}\mathbb{R}, <^*)$, where $<^*$ is the eventual domination ordering (see definition 1.1 in this thesis), showed that these have the cardinality of the continuum and established his main result, namely, the existence of (ω_1, ω_1)-gaps. These famous results gave rise to the invariants $cov(\mathcal{M})$ and $\mathfrak{b}$. The splitting number $\mathfrak{s}$ appeared as an algebraic characterization of sequential compactness (*i.e.* 2^λ is sequentially compact iff $\lambda < \mathfrak{s}$). Recall that a topological space is sequentially compact if every infinite sequence has a convergent subsequence. For metric spaces compactness and sequential compactness are equivalent.

In 1963, Paul Cohen introduced the method of **forcing** and used it to show that AC is independent of ZF and that CH is independent of ZFC. Solovay then extended his method to a general flexible technique for obtaining models of large finite fragments of ZFC which satisfy some additional axioms. The main idea is to start with M, a countable transitive model of ZFC called ground model, and to adjoin a new set G, a generic set, such that the obtained model $M[G]$, called generic extension of M, is still a countable transitive model of ZFC and moreover, some additional statements are satisfied. Precisely which statements are satisfied is sensitive to the properties of the forcing notion.

In the ground model M, **forcing notions** are partially ordered sets $(\mathbb{P}, \leq)$ (*i.e.* $\leq$ is a reflexive, transitive relation on $\mathbb{P}$). Without loss of generality, one only considers posets with a largest element $\mathbf{1}$. The elements of the forcing notion are called **conditions**. If $p \leq q$ we say that p is **stronger** than q. Intuitively, p contains at least as much information as q.

For any two conditions p and q, if there is a common extension r, then they are compatible, otherwise incompatible.

An **antichain** is $A \subset \mathbb{P}$ with the property that any two conditions in A are incompatible.

A **dense set** is a set $D \subset \mathbb{P}$, such that every condition in $\mathbb{P}$ has an extension in D and a **predense set** is a set $D \subset \mathbb{P}$, such that any condition in $\mathbb{P}$ is compatible with a condition in D.

A set D is **predense below** p if any condition in $\mathbb{P}$ extending p is compatible with a condition in D.

A **filter** is $F \subset \mathbb{P}$, nonempty, closed under weaker conditions and with the property that any two elements of F have a common extension in F.

A **$\mathbb{P}$-generic filter over** M is a filter G which intersects all dense sets in M (or equivalently, every maximal antichain or every predense set in M).

The **generic extension** will be the minimal extension of M to a countable transitive model containing the same ordinals and a chosen $\mathbb{P}$-generic filter G. Whenever the partial order $\mathbb{P}$ is non-atomic (*i.e.* any condition has incompatible extensions), $G \notin M$, so the extension is a proper extension.

The extension, usually denoted by $M[G]$, contains all the evaluations with respect to G of $\mathbb{P}$-names in M.

$\mathbb{P}$-names are defined inductively as $\sigma := \{< \tau, p >: \tau$ is a $\mathbb{P}$-name and $p \in \mathbb{P}\}$.

The **evaluation** of the name σ with respect to G is defined recursively as $\sigma_G := \{\tau_G : \exists p \in G$ such that$< \tau, p > \in \sigma\}$.

Hence, $M[G] = \{\sigma_G : \sigma$ is a $\mathbb{P}$-name in $M\}$.

Intuitively, these **$\mathbb{P}$-names** describe how the objects in $M[G]$ are constructed from G by applying processes definable in M. M sees the construction, but knowing the exact object requires knowledge about the generic G and, as stated before, in all interesting cases $G \notin M$.

To argue that $M[G]$ extends M, there are some special names, independent of the generic filter, called **standard names**:

$$\text{for all } x \in M,\ \check{x} = \{<\check{y}, \mathbf{1}>: y \in x\}.$$

The generic filter G is in $M[G]$, since $\Gamma = \{<\check{p}, p>: p \in \mathbb{P}\}$ is a $\mathbb{P}$-name for G.

The **forcing relation** $\Vdash$ is a relation between elements of the partial order and statements in the forcing language. $p \Vdash \psi$ means that for all filters G which are $\mathbb{P}$-generic over M, if $p \in G$ then ψ is true in $M[G]$.

The two most important facts about forcing are that the forcing relation $\Vdash$ is definable in M and that all statements true in the generic extension are forced by some condition in the generic filter. The later is known as the forcing theorem. One can see this theorem as an analog to Gödel's completeness theorem: an equivalence between semantics and syntax.

In [16], one can find the following important properties of forcing:

- $p \Vdash \varphi$ and $q \leq p$ then $q \Vdash \varphi$
- no condition forces both φ and $\neg\varphi$
- if φ is a sentence, then for every p there is a $q \leq p$ deciding φ($i.e$ $q \Vdash \varphi$ or $q \Vdash \neg\varphi$)

  (this is the same as saying that $\{p : p \Vdash \varphi \text{ or } p \Vdash \neg\varphi\}$ is dense).
- $p \Vdash \neg\varphi$ iff for no $q \leq p$: $q \Vdash \varphi$
- $p \Vdash \varphi \wedge \psi$ iff $p \Vdash \varphi$ and $p \Vdash \psi$
- $p \Vdash \forall x \varphi$ iff $p \Vdash \varphi(\dot{a})$ for every $\mathbb{P}$-name $\dot{a}$ over M
- $p \Vdash \varphi \vee \psi$ iff $\forall q \leq p \exists r \leq q (r \Vdash \varphi \text{ or } r \Vdash \psi)$

- $p \Vdash \exists x \varphi$ iff $\forall q \leq p \exists r \leq q \exists \dot{a} \in M^{\mathbb{P}}(r \Vdash \varphi(\dot{a}))$
- If $p \Vdash \exists x \varphi$ then for some $\mathbb{P}$-name $\dot{a}$, $p \Vdash \varphi(\dot{a})$

1.2 Preserving cardinals

A basic fact in set theory is that being a cardinal is not an absolute property. When doing forcing one needs to ensure that cardinals are preserved to be able to get the desired results.

Forcing notions satisfying the **countable chain condition**, shortly written c.c.c. (*i.e.* every antichain is countable) preserve all cofinalities, and thus, all cardinals. Moreover, the c.c.c. property is preserved under finite support iterations (f.s.i).

Examples of c.c.c. forcing notions are the countable forcing notions and the σ-centered forcing notions. It is well known that Cohen forcing is countable and in fact, any countable forcing notion is forcing equivalent with Cohen forcing (See section 1.4).

Recall that $\mathbb{P}$ and $\mathbb{Q}$ are **forcing equivalent** if they give rise to the same generic extensions, that is, for every $\mathbb{P}$ generic filter G, there is a $\mathbb{Q}$-generic filter H such that $V[G] = V[H]$, and vice versa.

$\mathbb{P}$ is called **centered** if any finitely many elements of $\mathbb{P}$ have a common extension, and it is **σ-centered** if it can be written as a countable union of centered posets (*i.e.* $\mathbb{P} = \bigcup_{n \in \omega} \mathbb{P}_n$ where $\mathbb{P}_n$ are centered).

A wider class of forcing notions preserving cardinals are the proper forcing notions. A forcing notion is **proper** if for all uncountable cardinals λ, $\forall S \in [\lambda]^\omega$, stationary set in V, S remains stationary in the generic extension of V. The properness is preserved under countable support iterations (c.s.i.) of length $\leq \omega_2$, but a c.s.i of length $> \omega_2$ of non-atomic proper forcing notions collapses $\mathfrak{c}$. As an example of a well known open problem which seems to require $\mathfrak{c} \geq \aleph_3$ is the consistency of $\mathfrak{s}$ being a singular cardinal.

If, for some regular cardinal θ, the forcing notion satisfies the $\theta-$c.c. (*i.e.* all antichains have cardinality $< \theta$), then cofinalities and cardinals $\geq \theta$ are preserved. But this property does not say anything about cardinals and cofinalities $< \theta$.

If the forcing notion is **θ-closed** , then cofinalities and cardinals $\leq \theta$ are also preserved. We say $\mathbb{P}$ is **θ-closed** if for any decreasing sequence of conditions $\langle p_\xi : \xi < \gamma \rangle$ for some $\gamma < \theta$ there exists a condition $p \in \mathbb{P}$, such that $p \leq p_\xi \ \forall \xi < \gamma$.

1.3 Basic definitions and ZFC-provable inequalities

As mentioned in the introduction, this paper gives the arguments for the independence of the bounding and the splitting number, but also involves the almost disjointness number for the purpose of obtaining arbitrary spread. Since they are the main object of study, the definitions of $\mathfrak{b}$, $\mathfrak{s}$ and $\mathfrak{a}$ are introduced at this point, along with some simple ZFC-provable relations between these characteristics.

Recall that ω^ω is the set of all functions from natural numbers to natural numbers, $[\omega]^\omega$ is the set of all infinite subsets of ω and $[\omega]^{<\omega}$ is the set of all finite subsets of ω.

Definition 1.1 .

For $f, g \in \omega^\omega$, f **is eventually dominated by** g, written as $f \leq^* g$ iff

$$\exists k \forall n \geq k : g(n) \leq f(n).$$

$F \subseteq \omega^\omega$ is a **dominating family** if for every $f \in \omega^\omega$ there is $g \in F$ with $f \leq^* g$.

$B \subseteq \omega^\omega$ is an **unbounded family** if no $f \in \omega^\omega$ dominates all functions in B.

The **bounding number** $\mathfrak{b}$ is the minimal cardinality of an unbounded family.

The **dominating number** $\mathfrak{d}$ is the minimal cardinality of a dominating family.

Definition 1.2 .

*For $a, b \in [\omega]^\omega$, let $a \subseteq^* b$ (a is **almost included** in b) iff $a \setminus b$ is finite.*

*A **splitting family** is a family $S \subseteq [\omega]^\omega$ with the property that $\forall a \in [\omega]^\omega \exists b \in S$, such that both $a \cap b$ and $a \setminus b$ are infinite.*

*The **splitting number** $\mathfrak{s}$ is the minimal cardinality of a splitting family.*

*Dually the **reaping number** $\mathfrak{r}$ is the minimal cardinality of an unsplittable family(i.e. a family with the property that no single infinite subset of ω splits all members of the family).*

Definition 1.3 .

Two sets $a, b \in [\omega]^\omega$*, are* ***almost disjoint*** *if* $|a \cap b| < \aleph_0$*.*

A family $\mathcal{A} \subseteq [\omega]^\omega$ *is an* ***almost disjoint family*** *if any two distinct members of the family are almost disjoint.*

$\mathcal{A}$ *is a* ***m.a.d.*** *family (maximal almost disjoint family) if it is maximal with respect to the above property, or equivalently, if for any* $C \in [\omega]^\omega$*, there is some* $A \in \mathcal{A}$ *such that* $|C \cap A| = \omega$*.*

The ***almost disjointness number*** $\mathfrak{b}$ *is the minimal cardinality of an infinite m.a.d. family.*

For the rest of this section, we concentrate on ZFC-provable inequalities between the cardinal invariants $\mathfrak{b}, \mathfrak{d}, \mathfrak{a}$ and $\mathfrak{s}$. The article [4] is a very good survey on ZFC-provable inequalities between cardinal invariants.

- $\aleph_1 \leq \mathfrak{b}$

  Proof:

  One has to show that, given countably many functions in $\{g_n \in \omega^\omega : n \in \omega\}$, there is always an $f \in \omega^\omega$ dominating all of them.
  Define $f(n) := max_{i \leq n} g_i(n)$.

  ■

- $\mathfrak{b} \leq cf(\mathfrak{d})$

  Proof:

  Given D a dominating family of size $\mathfrak{d}$, it can be decomposed as

  $$D = \bigcup \{D_\xi : \xi < cf(\mathfrak{d})\}, \text{ where } \forall \xi : |D_\xi| < \mathfrak{d}.$$

  Since $\mathfrak{d}$ is the minimal cardinality of a dominating family, $\forall \xi : D_\xi$ is not dominating, *i.e* $\forall \xi \, \exists f_\xi$ not dominated by any $g \in D_\xi$. If $\{f_\xi : \xi < cf(\mathfrak{d})\}$

is bounded, say by f, then $\nexists g \in D : f \leq^* g$, contradicting the fact that D is dominating. Thus, $\{f_\xi : \xi < cf(\mathfrak{d})\}$ has to be unbounded.

■

- $\mathfrak{b}$ is regular.

  Proof:

  One has to argue that $cf(\mathfrak{b}) = \mathfrak{b}$.

  Let B be an unbounded family of size $\mathfrak{b}$ and assume towards a contradiction, $\mathfrak{b}$ is singular. By the above characterization of singular cardinals, $B = \bigcup_{\xi < cf(\mathfrak{b})} B_\xi$, where $\forall \xi < cf(\mathfrak{b}) : |B_\xi| < \mathfrak{b}$. Thus, each B_ξ is bounded, say by f_ξ.

  If $\{f_\xi : \xi < cf(\mathfrak{b})\}$ would be bounded by g, then the same function g would also bound B, which is a contradiction. Thus, $\mathfrak{b}$ is regular.

  ■

  Notation: $\omega^{\uparrow\omega}$ denotes the set of increasing functions in ω^ω

- $\mathfrak{s} \leq \mathfrak{d}$ and dually $\mathfrak{b} \leq \mathfrak{r}$

  Proof:

  The statement follows from the existence of functions $\Psi : \omega^{\uparrow\omega} \to [\omega]^\omega$ and $\Phi : [\omega]^\omega \to \omega^{\uparrow\omega}$, such that $\Phi(A) \leq^* f \Rightarrow \Psi(f)$ splits A.

  For $A \in [\omega]^\omega$ and $n \in \omega$, let

  $$\Phi(A)(n) := min(A \setminus n) + 1$$

  For $f \in \omega^{\uparrow\omega}$, define

  - $f^0(0) = 0$,
  - $f^{k+1}(0) = f(f^k(0))$.

Let $\Psi(f) := \bigcup_k [f^{2k}(0), f^{2k+1}(0)]$, the union of the even numbered intervals.

The functions Φ and Ψ are even continuous.

Assuming $\Phi(A) \leq^* f$ one has to show $\Psi(f)$ splits A.

By definition of $\leq^*$, it follows that for almost all n, $\Phi(A)(n) \leq f(n)$, and thus, taking $n := f^k(0)$, one gets $\Phi(A)(f^k(0)) \leq f(f^k(0)) = f^{k+1}(0)$. Hence, $\Psi(f)$ splits A, since A has nonempty intersection to all intervals $[f^k(0), f^{k+1}(0)]$.

■

- $\mathfrak{b} \leq \mathfrak{a}$.

  Proof:

  One has to show that, given any m.a.d. family of size $\mathfrak{a}$, there exists an unbounded family of the same size.

  Let $\mathcal{A} \subseteq [\omega]^\omega$ be a m.a.d. family, $|\mathcal{A}| = \mathfrak{a}$.

  Select $\{C_n : n \in \omega\}$ to contain any countably many members of $\mathcal{A}$ and denote by $\mathcal{A}'$ the rest of $\mathcal{A}$.

  Without loss of generality (by only making finite changes) one can assume:

  - $\forall n, m \in \omega$, $n \neq m \to C_n \cap C_m = \emptyset$ and
  - $\bigcup_{n\in\omega} C_n = \omega$.

  Hence, w.l.o.g, the $C_n's$ form a partition of ω.

  Organize the sets C_n on $\omega \times \omega$ (using a suitable bijection between ω and $\omega \times \omega$) such that each C_n is the column $\{n\} \times \omega$.

Every $A \in \mathcal{A}'$ is almost disjoint from all C_n's, thus only finitely many of its elements appear on each column. Thus, it makes sense to define $f_A \in \omega^\omega$ to be the function whose graph is the upper boundary of A.

Assume towards a contradiction, there is a function $g \in \omega^\omega$, eventually dominating all the f_A's. Then its graph is almost disjoint from all $A \in \mathcal{A}'$ and all $C_n : n \in \omega$. But this is impossible, since it would contradict the maximality of the a.d. family $\mathcal{A}$.

So, the f_A's constitute an unbounded family of size $\mathfrak{a}$.

■

1.4 An easy example: Cohen Forcing

Let M be a countable transitive model of ZFC. For $I, J \in M$ let $Fn(I, J)$ be the set of all finite partial functions from I to J, ordered by reverse inclusion and let G be a $Fn(I, J)$-generic filter. For I an arbitrary, infinite set and J containing more than one element, $f_G = \bigcup G$ is a total surjective function $f_G : I \to J$ and for J countable $Fn(I, J)$ is c.c.c.

Since $G \notin M$, $f_G \notin M$, otherwise the set $E = \{p : p \nsubseteq f_G\}$ would also be in M, which is impossible because E would be a dense set in M, disjoint from G. However, $f_G \in M[G]$ by absoluteness of the union operation.

$Fn(\omega, \omega_1)$ adds a surjective function $f_G : \omega \to \omega_1$ and thus, in the extension ω_1 becomes countable. This is an example of a forcing notion collapsing a cardinal.

Cohen forcing is $\mathbb{C}_I := Fn(I, 2)$. In particular, for $I = \omega$, $\mathbb{C} = Fn(\omega, 2)$

adds a new subset of ω called a **Cohen real**. Cohen forcing is clearly countable.

Recall that, for $\mathbb{P}$ and $\mathbb{Q}$ partial orders, $i : \mathbb{P} \to \mathbb{Q}$ is a **dense embedding** if

- $\forall p, p' \in \mathbb{P} : p \leq_{\mathbb{P}} p' \to i(p) \leq_{\mathbb{Q}} i(p')$,
- $\forall p, p' \in \mathbb{P} : p \perp_{\mathbb{P}} p' \to i(p) \perp_{\mathbb{Q}} i(p')$ and
- $i''\mathbb{P}$ is a dense subset of $\mathbb{Q}$.

If a dense embedding between two partial orders $\mathbb{P}$ and $\mathbb{Q}$ exists, then they are forcing equivalent. The existence of a dense embedding is sufficient, but not necessary.

Example : There is no dense embedding between $Fn(\omega, \omega)$ and $2^{<\omega}$ (there is not even an embedding satisfying the first two conditions).

Proof:

Assume towards a contradiction, there is $i : Fn(\omega, \omega) \to 2^{<\omega}$ dense. In $Fn(\omega, \omega)$, let s be the condition sending 0 to 0 and 4 to 0, and t the condition sending 2 to 0 and 3 to 0. They are compatible.

Assume their images are also compatible, which in case of 2^{ω} is the same as comparable.

Assume $i(s) \leq i(t)$ (the other case follows the same argument). But there exists $s' \leq s$, $s' \perp t$ (for example the condition sending 0 to 0, 2 to 1 and 4 to 0). Since $s' \leq s$, it follows $i(s') \leq i(t)$, but $s' \perp t$ implies that the images are incomparable, a contradiction.

■

Even if there is no dense embedding between them, $Fn(\omega, \omega)$ and $2^{<\omega}$ are equivalent, by the following fact:

Fact Any countable, non-atomic partial order $\mathbb{P}$ is forcing equivalent with $\mathbb{C}$.

Proof:

To conclude this fact, it suffices to show the existence of a dense embedding from $\{p \in Fn(\omega, \omega) : dom(p) \in \omega\}$ onto $\mathbb{P}$.

In case $\mathbb{P} = Fn(\omega, \omega)$, the inclusion is a dense embedding.

In the general case, one first looks at the conditions with domain 1 and maps them onto a countable (maximal) antichain in $\mathbb{P}$. Then, one maps the conditions with domain 2 to extensions of the conditions of the previous antichain (possible since $\mathbb{P}$ is non-atomic), and so on.

The result is a dense embedding. ■

Thinking of $\mathbb{C}$ as forcing with $\omega^{<\omega}$, one can prove the following fact:

Fact: The Cohen real c is **unbounded** over the ground model reals (*i.e.* $\forall f \in \omega^\omega \cap V : c \nleq^* f$).

Proof:

For $f \in \omega^\omega \cap V$ and $n \in \omega$, the sets

$$D_{f,n} = \{t \in \omega^{<\omega} : \exists m \geq n \ t(m) > f(m)\} \text{ are dense.}$$

Given any condition $s \in \omega^{<\omega}$, one can find an extension in $D_{f,n}$ in the following way:

Choose m larger than $max(|s|, n)$ and $t \in \omega^{<\omega}$ an extension of s of length $m + 1$ with $t(m) = f(m) + 1$.

■

Thinking of $\mathbb{C}$ as forcing with $2^{<\omega}$, one can prove the following fact:

Fact: The Cohen generic real **splits** all the ground model infinite subsets of ω.

Proof:

Let $A \in [\omega]^\omega \cap V$ be arbitrary, $n \in \omega$. The sets

$E_{A,n} = \{t \in 2^{<\omega} : \exists m_0, m_1 \in A \setminus n : t(m_0) = 0 \text{ and } t(m_1) = 1\}$ are dense in V.

Thus, by genericity, the Cohen real splits A.

■

Definition 1.4 *An iteration* $\langle \mathbb{P}_\beta : \beta < \alpha \rangle$ *is* **nontrivial** if

$$\forall \beta < \alpha \Vdash_{\mathbb{P}_\beta} \mathbb{Q}_\beta \text{ has a pair of incompatible conditions.}$$

Fact: Cohen reals are **always added** in nontrivial finite support iterations at limit stages of countable cofinality.

Proof:

We show that $\mathbb{P}_\omega$ adds a Cohen generic subset of ω.

By the fact that the iteration is nontrivial, there are $\dot{p}_n$ and $\dot{q}_n$, such that they are forced in $\mathbb{P}_n$ to be incompatible conditions in $\mathbb{Q}_n$.

In $\mathbb{P}_\omega$ there is a condition p'_n with support $n + 1$, such that $p'_n(n) = p_n$.

If G is $\mathbb{P}_\omega$ generic, then $\{n \in \omega : p'_n \in G\}$ is Cohen generic.

■

Other facts: Cohen forcing does not add dominating reals, since unbounded families remain unbounded in the extension, and does not add unsplit reals.

Chapter 2

Arguments for $Con(\mathfrak{s} = \omega_1 < \mathfrak{b} = \lambda)$

As suggested in the title, the main goal of this chapter is to construct a model where the bounding number is λ, for some regular, uncountable cardinal λ, while the splitting number remains $\aleph_1$. One can increase $\mathfrak{b}$ by repeatedly adjoining dominating functions over a model of GCH via finite support iteration of a c.c.c. forcing notion. The first forcing notion that comes to mind for this purpose is **Hechler forcing**, for obvious reasons, also called **Dominating forcing**. One still has to argue why the resulting Hechler model has $\mathfrak{s} = \omega_1$.

This consistency result was actually first mentioned by B.Balcar, J.Pelant and P.Simon in their 1980 paper "The space of ultrafilters on $\mathbb{N}$ covered by nowhere dense sets". They not only use a different argument than the ones that will be presented in this thesis, but also they use a different forcing notion, namely the random forcing. Their argument goes backwards, starting with a model of $MA + \neg CH$, such that all invariants are large, and then $\aleph_1$

random reals are added. Since ω^ω of the extension is dominated by ω^ω of the ground model(*i.e.* the random forcing is ω^ω-bounding), $\mathfrak{b}$ remains large. On the other hand, $\mathfrak{s}$ will be $\aleph_1$, since the random reals form a set of positive outer measure, and thus, a splitting family.

2.1 Suslin forcing

In 1988, Shelah and Judah gave a general argument for the fact that $\mathfrak{s} = \omega_1$ in the Hechler model. Their result states that the ground model infinite subsets of ω always form a splitting family in the extension by a finite support iteration of a Suslin c.c.c. forcing notion (see [2]). This result not only says that the Hechler model has $\mathfrak{s} = \omega_1$, it says the same for Cohen, Random and Amoeba models as well. Since we need $\mathfrak{b}$ to be large, we add dominating reals, hence, the Hechler forcing catches our attention. This particular forcing even has a stronger property, called "strong preservation of splitting", which will also be presented in this thesis, in section 2.2.1.

DEFINITION 2.1 *A partial order $\mathbb{P}$ is called* ***Suslin c.c.c.*** *if it is c.c.c. and*

- $\mathbb{P} \subseteq \omega^\omega$
- $\leq \subseteq \omega^\omega \times \omega^\omega$
- $\perp_{\mathbb{P}} \subseteq \omega^\omega \times \omega^\omega$

are analytic ($\mathbf{\Sigma}^1_1$) sets.

Note that if $\mathbb{P}$ is Borel, then incompatibility is actually Borel. It is automatically also $\mathbf{\Pi}^1_1$, since the order is $\mathbf{\Sigma}^1_1$ and elements are incompatible if there is no common extension. The definition also requires it to be $\mathbf{\Sigma}^1_1$, and thus, $\perp$ is actually $\mathbf{\Delta}^1_1$, which is the same as Borel.

Examples of Suslin c.c.c. forcing notions are Cohen forcing $\mathbb{C}$, Hechler forcing $\mathbb{D}$, Amoeba forcing $\mathbb{A}$ and Random forcing $\mathbb{B}$. Recall that Amoeba forcing consists of open subsets of 2^ω of measure $< \frac{1}{2}$ ordered by reverse inclusion and Random forcing consists of Borel subsets of 2^ω (or $[0,1]$) of positive measure, ordered by inclusion.

Note that, by $\mathbf{\Sigma}^1_1$ **absoluteness**, the statements "$p \in \mathbb{P}$", " $q \leq_{\mathbb{P}} p$" and "$q \perp_{\mathbb{P}} p$" are absolute. Suslin c.c.c. forcing notions have even more nice properties, for example, being a maximal antichain is an absolute property, and therefore, genericity is downwards absolute.

Remark: $\mathbf{\Sigma}^1_n$-classes are upwards absolute (i.e. when $\mathbf{\Pi}^1_{n-1}$ is absolute in both directions then $\mathbf{\Sigma}^1_n$ is absolute) and $\mathbf{\Pi}$-classes are downwards absolute(in the analogous sense). Also note that $\mathbf{\Sigma}^1_1$-absoluteness implies $\mathbf{\Pi}^1_1$-absoluteness.

Lemma 2.2 *(See [6]) Let $M \subseteq N \models ZFC$ and $\mathbb{P} \in M$ a Suslin c.c.c. forcing notion. Then the property of being a maximal antichain can be written as $\varphi \wedge \psi$ where φ is a $\mathbf{\Sigma}^1_1$ formula and ψ is a $\mathbf{\Pi}^1_1$ formula. Therefore, this property is* ***absolute*** *between M and N.*

Proof:

Since $\mathbb{P}$ is c.c.c., antichains are countable and coded by reals.

Thus, let $A := \{x_n : n \in \omega\} \subseteq \mathbb{P}$.

A is a maximal antichain iff

- $\forall m \neq n : x_n \perp x_m$ ($\perp$ is $\mathbf{\Sigma}^1_1$ by definition of Suslin forcing) and

- $\forall y$ either $y \notin \mathbb{P}$ or $\exists n$, such that y and x_n are compatible $\Leftrightarrow \neg\exists y : y \in \mathbb{P} \wedge \forall n (y \perp x_n) (\mathbf{\Pi}^1_1)$

Note that for $\mathbb{P}$ Borel, being a maximal antichain is actually $\mathbf{\Pi}^1_1$, thus, the property of being a maximal antichains itself becomes $\mathbf{\Pi}^1_1$.

(Recall that the analytic sets correspond to statements with an existential quantifier over ω^ω, however there may be arbitrary quantification over ω).

■

Corollary 2.3 (Downwards absoluteness of genericity) *Let $M \subseteq N \models ZFC$ and $\mathbb{P} \in M$ a Suslin c.c.c. forcing notion.*

If G is $\mathbb{P}^N$-generic over N then $G \cap M$ is $\mathbb{P}^M$-generic over M.

Proof:

Let G be a generic filter over N.

Given A a maximal antichain in M, by the previous theorem, it remains maximal in N, so it has nonempty intersection with $G (\in N)$. Therefore, $G \cap M (= G \cap \mathbb{P}^M)$ is generic over M.

■

The following result states the connection between Suslin c.c.c. forcing notions and the cardinal invariant $\mathfrak{s}$, namely, that in every extension by a finite support iteration of Suslin c.c.c. forcing notion, the splitting number will remain small.

Theorem 2.4 (Judah, Shelah, 1988) *Let λ be regular, uncountable and $\mathbb{P}_\lambda$ be a finite support iteration of Suslin ccc forcing notions. Then the ground model infinite subsets of ω form a splitting family in the generic extension $\mathbb{V}^{\mathbb{P}_\lambda}$.*

In [2], one can find the following theorem:

Theorem 2.5 (Judah, Shelah, 1988) *Let $\mathbb{P}$ be a Suslin c.c.c. forcing notion. Then the ground model infinite subsets of ω form a splitting family in the generic extension $\mathbb{V}^{\mathbb{P}}$.*

Remark 2.6 This theorem does not imply the more general result stated in the preceding paragraph, since **f.s.i of Suslin c.c.c. forcing notions is obviously not Suslin c.c.c.**: one of the conditions in the definition of Suslin forcing notions was that the forcing can be coded by reals, thus, it must have size less than the continuum $\mathfrak{c}$ of the ground model. Clearly a finite support iteration of length λ, where λ is regular, uncountable, does not have size $\leq \mathfrak{c}$. But Theorem 2.4 holds, since the finite support iteration of Suslin c.c.c. forcing notions is in some sense "almost Suslin", meaning that all the properties needed in the proof of the Theorem 2.5 (for one Suslin c.c.c. forcing notion) also hold for the finite support iteration of Suslin c.c.c. forcing notions, and thus, the proof generalizes trivially.

The following lemmas can be found in [2] and conclude the proof of Theorem 2.5:

Lemma 2.7 *Given $\mathbb{P}$ a c.c.c. forcing notion and $\{x_\alpha : \alpha < \omega_1\}$ a family of ω_1 almost disjoint subsets of ω, let $\dot{x}$ be a $\mathbb{P}$-name for a subset of ω, such that $\Vdash_{\mathbb{P}} \exists\alpha < \omega_1 \ \dot{x} \subseteq x_\alpha$.*

Then there exists an $\alpha < \omega_1$, such that $\Vdash_{\mathbb{P}} |\dot{x} \cap x_\alpha| < \aleph_0$.

Proof: For each α choose, if possible, p_α, such that $p_\alpha \Vdash |\dot{x} \cap x_\alpha| = \aleph_0$.

Since $\{x_\alpha : \alpha < \omega_1\}$ is a family of almost disjoint sets, the p_α's are incompatible:

Assume towards a contradiction two of them were compatible, say p_α and p_β, for $\alpha \neq \beta$. We know $p_\alpha \Vdash |\dot{x} \cap x_\alpha| = \aleph_0$ and $p_\beta \Vdash |\dot{x} \cap x_\beta| = \aleph_0$.

Since $\Vdash_{\mathbb{P}} \exists \gamma < \omega_1 : \dot{x} \subseteq x_\gamma$, we can extend the conditions to decide γ.

But x_α and x_γ are almost disjoint, thus, $\alpha = \gamma$. The almost disjointness of x_β and x_γ gives $\beta = \gamma$ as well, thus, $\alpha = \beta$ is the only case allowing p_α and p_β to be compatible.

Thus, $\{p_\alpha : \alpha < \omega_1\}$ is an antichain.

Since $\mathbb{P}$ is c.c.c., the antichain is countable. Therefore, there must be an index $\alpha < \omega_1$, such that $\dot{x} \cap x_\alpha$ is finite.

■

The following lemma, characterizing meager sets, can be found in [2]:

Lemma 2.8 *(Characterization of meager sets)*

Whenever $F \subseteq 2^\omega$ is meager, there exists $x_F \in 2^\omega$ and $f_F \in \omega^\omega$, such that

$$F \subseteq \{x \in 2^\omega : \forall^\infty n : x \restriction [f_F(n), f_F(n+1)] \neq x_F \restriction [f_F(n), f_F(n+1)]\}$$

For the proof, see 2.2.4 in [2]

Lemma 2.9 *Given a countable model M, there is a family $\{c_\alpha : \alpha < \omega_1\} \subseteq [\omega]^\omega$, such that the c_α's are ω_1 many almost disjoint Cohen reals over M.*

Claim: Since M is countable, the set of Cohen reals over M is comeager.

Proof of Claim

Look at the Cohen algebra $\mathbb{C} = Borel/\mathcal{M}$, where $\mathcal{M}$ is the σ-ideal of meager. Let Co(M) be the set of Cohen reals over M. Then Co(M)= $2^\omega \setminus \bigcup\{X \in \mathcal{M}$ coded in $M\}$. Since $\mathcal{M}$ is a σ ideal, the countable union of Borel meager sets is also meager, thus, Co(M) is comeager. ■

Proof of Lemma

By the above claim, the set of Cohen reals over M is comeager.

Using the characterization of meager sets stated above and considering intervals, it follows that there are x_M and $\{I_s : s \in 2^{<\omega}\}$ pairwise disjoint, such that x is Cohen generic over M if x and x_M are equal on infinitely many intervals.

For $s \in 2^\omega$ define $c_s \in 2^\omega$, by:

$$c_s(i) = \begin{cases} x_M(i) & \text{if } \exists n \text{, such that } i \in I_{s\restriction n} \\ 0 & \text{otherwise} \end{cases}$$

Clearly these are all Cohen reals and almost disjoint (since $\{n : c_s(n) = c_{s'}(n) = 1\}$ is finite $\forall s \neq s'$).

■

Proof of Theorem 2.5

Recall what we have **to show:**

For every $\mathbb{P}$-name $\dot{x}$, there is $y \in V \cap [\omega]^\omega$, such that $\Vdash$ neither $\dot{x} \subseteq^* y$ nor $\dot{x} \subseteq^* \omega \setminus y$.

Assume towards a contradiction, there is a a $\mathbb{P}$-name $\dot{x}$ for an infinite subset of ω in the extension, such that none of the ground model reals splits $\dot{x}$. Thus $\forall y \in [\omega]^\omega \cap V$, $\Vdash \dot{x} \cap y$ or $\dot{x} \cap \omega \setminus y$ is finite.

Taking M a **countable** elementary substructure of $H(\kappa)$ containing P and $\dot{x}$ (here κ is large enough to have the desired properties), by the previous lemma, we find a family $\{c_\alpha : \alpha \in \omega_1\}$ of ω_1 almost disjoint Cohen reals over M .

Lemma 2.7 gives us the existence of c, a Cohen real over M such that, $\Vdash |\dot{x} \cap c| < \aleph_0$.

Let M_1 be the extension of M by this Cohen real c. **Cohen Extension**

Let G be a $\mathbb{P}$-generic filter over V.

Since $\mathbb{P}$ is **Suslin ccc**, we also know $G \cap M_1$ is $\mathbb{P}$-generic over M_1, since genericity is absolute for every c.t.m containing $\mathbb{P}$, so also for M_1.

Therefore, the evaluations of $\dot{x}$ are the same with respect to $G, G \cap M$ and $G \cap M_1$.

Let M_2 be $M_1[G \cap M_1]$. **Extension by the Suslin forcing**

Then $M_2 \models \dot{x}_{G \cap M_1} \subseteq^* c$ and therefore, $M_1 \models$ "$\Vdash_{\mathbb{P}} \dot{x} \subseteq^* c$".

Think of Cohen forcing $\mathbb{C}$ as forcing with $2^{<\omega}$. Thus, there exists $s \subseteq c$, such that

$$M \models \text{"} s \Vdash_{\mathbb{C}} \text{"} \Vdash_{\mathbb{P}} \ \dot{x} \subseteq^* \dot{c} \text{" "}$$

where $\dot{c}$ is a canonical name for a Cohen real.

Choose c' Cohen over M, such that $s \subseteq c'$ and $\omega \setminus (c \cup c')$ finite and extend M by c'.

Then $M[c'] \models$ "$\Vdash_{\mathbb{P}} \dot{x} \subseteq^* \dot{c'}$ ".

Using again the absoluteness of genericity for the Suslin forcing $\mathbb{P}$, $G \cap M[c']$ is $\mathbb{P}$ generic over $M[c']$ and $\dot{x}_G \cap M[c'] = \dot{x}_G$.

Thus, extending $M[c']$ by $G \cap M[c']$ we get a model for " $\dot{x}_G \subseteq^* c'$ ".

The above arguments say that in $V[G]$, $\dot{x}_G \subseteq^* c$, $\dot{x}_G \subseteq^* c'$ and $\omega \setminus (c \cup c')$ is finite.

But this means that $\dot{x}$ cannot be infinite, contradicting our assumption.

■

The proof generalizes to finite support iterations of Suslin c.c.c. forcing notions, since the only property needed is the absoluteness of genericity. Although the finite support iteration of Suslin c.c.c. forcing notions is not Suslin c.c.c., this property is preserved under finite support iterations, making the generalization trivial.

For the particular case when $\mathbb{P}_\lambda = \mathbb{D}_\lambda$, even a stronger property, called strong preservation of splitting, will be satisfied. Since for the scope of this thesis, other Suslin c.c.c. forcing notions are not relevant, we will also treat the stronger preservation result.

2.2 Hechler forcing

This section will contain the definition of the Hechler forcing and its basic properties. The Hechler model is the model obtained by iterating this forcing notion with finite support over a model of GCH.

DEFINITION 2.10 *The* ***Hechler forcing*** *is*

$$\mathbb{D} := \{(s, f) : s \in \omega^{<\omega}; f \in {}^{\omega}\omega;\ s \subseteq f\},$$

ordered by:

$(s, f) \leq (t, g)$ *iff*

- $t \subseteq s$,
- f *dominates* g *everywhere and*
- $\forall i$ *with* $|t| \leq i < |s| : g(i) \leq s(i)$

Facts:

- $\mathbb{D}$ is σ-centered, thus, has the countable chain condition (c.c.c.).

  Proof:

  The sets $D_s = \{(s, f) : f \in \omega^\omega, s \subseteq f\}$ are centered, since conditions with the same stem are compatible. It is easy to see that $\mathbb{D} = \cup_{n\in\omega} D_s$, thus $\mathbb{D}$ is σ-centered.

  ■

- Forcing with $\mathbb{D}$ adds a new function in ${}^\omega\omega$ which eventually dominates all the ground model functions (a dominating real).

  Proof:

  If G is a $\mathbb{D}$-generic filter, then $d = \cup\{s : (s, f) \in G \text{ for some } f\}$ is a dominating real, since

  $$D_f = \{(s, g) : \exists n \leq |s|, \text{ such that } f(m) < g(m) \forall m \geq n\} \text{ is dense.}$$

  ■

- Hechler forcing also adds a Cohen real.

  Proof:

  If d is the Hechler real, then c defined by $c(n) := d(n) \bmod 2$ is Cohen generic.

  ■

- Hechler forcing is Suslin c.c.c.

  Proof:

  To see that $\mathbb{D}$ is Suslin, identify $(s, f) \to (|s|, f)$ (since $\omega \times \omega^\omega \cong \omega^\omega$).

  The order is a closed relation.

Incompatibility is a union of clopen relations(s and t are incomparable or one extends the other and $t(n) < f(n)$ for some n).

Recall:

The basic clopen sets are $N_s = \{x \in \omega^\omega : s \subseteq x\}$.
Incomparability of s and t translates into disjointness of N_s and N_t, while $s \subseteq t$ is equivalent to $N_s \supseteq N_t$.
Also, recall that when we have an universal quantifier over the natural numbers followed by a clopen relation, the result is only closed.

■

The Hechler model is the model obtained by a finite support iteration of length λ of Hechler forcing over a model of GCH, for some regular, uncountable cardinal λ. The main theorem of the previous section says that extensions via f.s.i. of Suslin forcing, thus, also the Hechler model, have $\mathfrak{s} = \aleph_1$. Since Hechler forcing adjoins a dominating real, the Hechler model will have $\mathfrak{b}$ large, thus, $Con(\mathfrak{s} = \aleph_1 < \mathfrak{b} = \mathfrak{c} = \lambda)$ is obtained.

2.2.1 Strong preservation of splitting

As mentioned before, in the particular case of Hechler forcing, even a stronger property than "the ground model reals form a splitting family" is preserved during iterations. This result is known as "strong preservation of splitting" and can be found in [7].

When doing forcing, one does not only have to adjoin a real with certain properties, but also to ensure that undesired properties are not satisfied by the added reals. The latter is known as a **preservation theorem** and usually require two separate proofs, namely:

- the proof for the single-step forcing, sensitive to the forcing notion,
- the proof for the limit step, a general argument, showing that if all $\mathbb{P}_\alpha$, $\alpha < \delta$, have a certain property, then so does $\mathbb{P}_\delta$. This does not depend at all on the forcing notion we are iterating.

A different representation of the Hechler forcing consists of pairs (s, φ), such that $s \in \omega^{<\omega}$ and $\varphi : \omega^{<\omega} \to \omega$ with the same order. It is not known if the two forcing notions are forcing equivalent, but the finite support iterations have the same combinatorial properties.

This representation is more convenient to work with for the proof of following lemma:

Lemma 2.11 (Main Lemma) *(see [7]) Assume $\dot{A}$ is a $\mathbb{D}$-name for an infinite subset of ω.*

There are countably many ground model subsets of ω, say A_i , $i \in \omega$, such that, whenever $B \in [\omega]^\omega$ splits all the A_i's , then $\Vdash_{\mathbb{D}} B$ splits $\dot{A}$.

Proof:

This lemma is proved using a rank argument.

Such rank arguments were introduced by Baumgartner and Dordal and are common for establishing combinatorial properties of forcing notions adding dominating reals (for example, Hechler and Laver forcing). The original argument can be found at a later stage in this chapter, in the section treating the preservation of eventually narrow sequences in extensions by a single or a f.s.i. of Hechler forcing.

For $s \in \omega^{<\omega}$ and $n \in \omega$, say that s **favors** k to be the n-th element of $\dot{A}$ if there is no condition with first coordinate s, which forces that "k is not the n-th element of $\dot{A}$".

Define $rank_n(s)$ by recursion on the ordinals, as follows:

- $rank_n(s) = 0$ if for some k, s favors k to be n-th element of $\dot{A}$;
- for $\alpha > 0$: $rank_n(s) = \alpha$ if
  - there is no $\beta < \alpha$, such that $rank_n(s) = \beta$ and
  - there are infinitely many l, such that $rank_n(s^\frown l) < \alpha$.

Claim: $rank_n(s)$ is defined for all $s \in \omega^{<\omega}$ and $n \in \omega$, and thus, in particular, $rank_n(s) < \omega_1$.

Proof of the claim :

Assume $rank_n(s)$ is undefined for some s and some n.

Notice that for any $s \in \omega^{<\omega}$, if $rank_n(s)$ is undefined, then $rank_n(s^\frown l)$ is undefined for almost all l.

This allows us to recursively construct a function $\varphi : \omega^{<\omega} \to \omega$, such that whenever $s \subseteq t$ and $t(i) \geq \varphi(t \restriction i)$ for all $i \in |t| \setminus |s|$, then $rank_n(t)$ is undefined.

Consider the condition (s, φ).

Find $(t, \psi) \leq (s, \varphi)$ and k, such that (t, ψ) forces that k is the n-th element of $\dot{A}$.

Then clearly $rank_n(t) = 0$.

However, by the preceding paragraph, $rank_n(t)$ is undefined, giving the contradiction.

■

If s is such that $rank_n(s) = 0$ for infinitely many n, one can find $k_n \geq n$, such that s favors that k_n is the n-th element of $\dot{A}$.

Let $A_s := \{k_n : k_n \text{ is favored by } s \text{ to be the } n\text{-th element of } \dot{A}\}$.

If s and n are such that $rank_n(s) = 1$, there are infinitely many l, such that $rank_n(s^\frown l) = 0$. Then, as before, for each such l we may find k_l, such that $s^\frown l$ favors that k_l is the n-th element of $\dot{A}$.

Define $A_{s,n} := \{k_l : k_l \text{ is favored by } s^\frown l \text{ to be the } n\text{-th element of } \dot{A}\}$.

It is easy to see that for each k, the set $\{l : k_l = k\}$ must be finite, since otherwise, k would witnesses $rank_n(s) = 0$.

In particular, the collection $A_{s,n}$ of such k_l must be infinite.

Claim: If B splits all A_s and all $A_{s,n}$, then B is forced to split $\dot{A}$.

Let (s, φ) be condition and let $m \in \omega$.

We need to find an extension $(t, \psi) \leq (s, \varphi)$ and $m_0, m_1 \geq m$, such that $m_0 \in B$, $m_1 \notin B$, and (t, ψ) forces both m_0 and m_1 to belong to $\dot{A}$.

Since the construction of m_0 and m_1 is analogous, it suffices to produce one of them.

- First assume there are infinitely many n such that $rank_n(s) = 0$.

Since $B \cap A_s$ is infinite, we find $m_0 \geq m$ in this intersection.

By definition of A_s, there is some n, such that s favors that $k_n = m_0$ is the n-th element of $\dot{A}$, and thus, there is $(t, \psi) \leq (s, \varphi)$, such that $(t, \psi) \Vdash_{\mathbb{D}}$

$m_0 \in \dot{A}$.

- Next assume $rank_n(s) > 0$ for all but finitely many n.

Choose $n \geq m$ such that $rank_n(s) > 0$.

Claim: One can extend s to t such that $t(i) \geq \psi(t \upharpoonright i)$ for all $i \in |t| \setminus |s|$ and $rank_n(t) = 1$.

One proves, by induction on $rank_n(s)$, that this can be done:

- If $rank_n(s) = 1$, put $t = s$.
- If $rank_n(s) > 1$, then we can find l , by the definition of $rank_n$, such that $l \geq \varphi(s)$ and $1 \leq rank_n(s^\frown l) < rank_n(s)$.

  By the induction hypothesis, one finds $t \leq s^\frown l$ as required:

  Since $B \cap A_{t,n}$ is infinite, we find $l \geq \varphi(t)$ and $k = m_0 \geq m$, such that $m_0 \in B \cap A_{t,n}$. Thus, $t^\frown l$ favors that m_0 is the n-th element of $\dot{A}$.

  Hence, we can find a condition $(u, \psi) \leq (s, \varphi)$, such that $t^\frown l \subseteq u$ and $(u, \psi) \Vdash_{\mathbb{D}} m_0 \in \dot{A}$.

■

As stated before, the preservation at the limit step does not depend on the particular forcing notion. It is a general result, simply saying that, if a property holds for every $\alpha < \delta$, then it also holds for δ.

Denote the following property by $(*_\alpha)$:

$(*_\alpha)$: Whenever $\dot{A}$ is a $\mathbb{P}_\alpha$-name for an infinite subset of ω, there are countably A_i , $i \in \omega$, such that whenever $B \in [\omega]^\omega$ splits all the A_i's , then $\Vdash_\alpha B$ splits $\dot{A}$.

Lemma 2.12 (Preservation at limit step) *(see [7])*

If δ is a limit ordinal and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta \rangle$ is a finite support iteration of c.c.c. forcing notions, such that for all $\alpha < \delta$, the property $(*_\alpha)$ holds, then $(*_\delta)$ also holds.

Proof:

Since during finite support iterations of c.c.c. forcing notions, no reals are added at limit stages of uncountable cofinality, we may assume w.l.o.g that $cf(\delta) = \omega$.

Since the argument below generalizes to any cardinal of countable cofinality, we may assume, for simplicity, that $\delta = \omega$.

Let $\dot{A}$ be a $\mathbb{P}_\omega$-name for an infinite subset of ω and fix $n \in \omega$.

In $V^{\mathbb{P}_n}$, there is a decreasing sequence of conditions $p_k = p_{n,k}$ in the quotient forcing $\mathbb{P}_\omega \setminus \mathbb{P}_n$, such that p_k decides the k-th element of $\dot{A}$.

Say $p_k \Vdash_{[n,k)}$ "l_k is the k-th elem of $\dot{A}$".

Denote the set of all values by A_n, hence, $A_n = \{l_k : k \in \omega\}$.

Since $A_n \in V^{\mathbb{P}_n}$, in the ground model V, there is a $\mathbb{P}$- name for it, say $\dot{A}_n$.

We also know, by hypothesis, that for each $n \in \omega$, the property $(*_n)$ holds. Thus, there are countably many $A_{n,i} : i \in \omega$, such that whenever B splits all $A_{n,i}$, $i \in \omega$, then $\Vdash_n$ "B splits $\dot{A}_n$".

After unfixing n, we get there are countably many $\{A_{n,i} : i \in \omega, n \in \omega\}$

Claim: Whenever B splits all $A_{n,i}$, $i, n \in \omega$, then $\Vdash_n$ "B splits $\dot{A}$"

Proof of Claim:

Let $p \in \mathbb{P}_\omega$ and $m \in \omega$.

We need to find $q \leq p$ and $m_0, m_1 \geq m$, such that $m_0 \in B, m_1 \notin B$, and

$q \Vdash_\omega m_0, m_1 \in \dot{A}$.

As for the successor step, it suffices to find m_0.

Fix n, such that $p \in \mathbb{P}_n$ and work in $V^{\mathbb{P}_n}$.

We know that B splits A_n, thus, there is $m_0 \geq m$, such that $m_0 \in B \cap A_n$.

There is k such that $m_0 = l_k$ and $p_k \Vdash_{[n,\omega)} m_0 \in \dot{A}$.

In V , we have $\mathbb{P}_n$ -names $\dot{m}_0$ for m_0 and $\dot{p}_k$ for p_k.

By strengthening $p \in \mathbb{P}_n$ if necessary, we may assume p decides $\dot{m}_0$ to be m_0 and $\dot{p}_k$ to be p_k , a partial function with domain $[n, \omega)$, so that $q = p^\frown p_k$ is a condition. Then $q \Vdash_\omega m_0 \in \dot{A}$.

■

■

2.2.2 Preservation of eventually splitting sequences

Definition 2.13 *A **descending** $\subseteq^*$**-sequence** is a sequence $\langle a_\xi : \xi < \lambda \rangle$, such that if $\xi < \eta$ then $a_\eta \subseteq^* a_\xi$ but not $a_\xi \subseteq^* a_\eta$.*

*A sequence $\langle a_\xi : \xi < \lambda \rangle$ is an **eventually narrow sequence** if $\forall a \in [\omega]^\omega \exists \xi < \lambda$, such that $\forall \eta > \xi : a \setminus a_\eta$ is infinite.*

*A sequence $\langle a_\xi : \xi < \lambda \rangle$ is an **eventually splitting sequence** if $\forall a \in [\omega]^\omega \exists \xi < \lambda \forall \eta > \xi : a \cap a_\eta$ and $a \setminus a_\eta$ are both infinite.*

It is very easy to observe that $\langle a_\xi : \xi < \lambda \rangle$ is eventually splitting if and only if the sequence $\langle b_\xi : \xi < \lambda \rangle$ is eventually narrow, where $b_{2\xi} := a_\xi$ and $b_{2\xi+1} := \omega \setminus \xi$.

In their 1985 paper "Adjoining Dominating Functions" (see [3]), J.E.Baumgartner and P.Dordal presented an argument for $Con(\mathfrak{s} = \aleph_1 < \mathfrak{b} = \lambda)$. They argued that the Hechler forcing preserves eventually narrow sequences during finite support iterations of length λ (for λ regular, uncountable), therefore, they also preserve an eventually splitting sequence of length ω_1. This sequence contains the Cohen reals added by the Hechler forcing. It is well known and was proved in the previous chapter, that the Cohen real splits the ground model subsets of ω. Hence, $V^{\mathbb{D}_{\omega_1}}$ contains an eventually splitting sequence of length ω_1. The main difficulty is to prove the preservation of eventually narrow sequences during the iterations of Hechler forcing. Knowing this, the result follows trivially, since any eventually splitting sequence is eventually narrow.

J.E.Baumgartner and P.Dordal defined the derivatives of a dense open sets $D \subseteq \mathbb{D}$. Their argument can also be seen as a rank function on IS as follows:

For $s \in IS$, define $r_D(s)$ the rank of s as:

- $r_D(s) = 0$ if there is f, such that $(s, f) \in D$
- else $r_D(s) = min\{\alpha : \exists m \exists \{s_k : k \in \omega\} \in IS \cap \omega^m$, such that $r_D(s_k) < \alpha, s \subseteq s_k$ and $s_k(|s|) > k\}$

Note that the rank is defined for every $s \in IS$. The proof is analog to the rank argument presented in the previous section, for the strong preservation of splitting.

Theorem 2.14 *(see [3]) Any eventually narrow sequence remains eventually narrow in $V^{\mathbb{D}_\lambda}$.*

Proof: We distinguish two steps:

The successor step:

Assume $\langle a_\xi : \xi < \lambda \rangle$ is eventually narrow in V but is not preserved to be eventually narrow in $V^{\mathbb{D}}$. Thus, there is a condition $(s, f) \in \mathbb{D}$ and a name $\dot{a}$ for an infinite subset of ω, such that:

$$(s, f) \Vdash \forall \xi < \lambda \; \exists \eta > \xi : \dot{a} \subseteq^* a_\eta.$$

Let N be a countable elementary submodel of $H(\kappa)$ containing $\mathbb{D}$, f and $\dot{a}$, for some κ large enough such that $\mathbb{D} \in H(\kappa)$. Since N is countable and the sequence $\langle a_\xi : \xi < \lambda \rangle$ is eventually narrow, there is a witness $\xi < \lambda$, such that whenever $c \in N \cap [\omega]^\omega$, $c \setminus a_\eta$ is infinite.

Because in the extension, the sequence is not eventually narrow, there has to be some $n_0 \in \omega$ such that, extending (s, f) if necessary, the following holds:

$$(s, f) \Vdash \forall j \geq n_0, \text{ if } j \in \dot{a} \text{ then } j \in a_\eta.$$

Take $\dot{h}$ to be a canonical name, $\Vdash \dot{h}$ enumerates $\dot{a}$ in increasing order. Clearly, since it is definable from $\dot{a}$, $\dot{h} \in N$ and $\Vdash h(i) \geq i$ for all i.

For each $t \in IS$ such that $(t, f) \leq (s, f)$ and each $i \geq n_0$, let:

$$Z_t(i) := \{j : \forall g \in \omega^\omega \exists (t', g') \leq (t, g) \text{ such that } (t', g') \Vdash \dot{h}(i) = j\}$$

Claim: By an induction on the rank of the dense open set $D = \{p \in \mathbb{D} : \exists j$ such that $p \Vdash \dot{h}(i) = j\}$ one can prove that $Z_t(i) \neq \emptyset \; \forall i \geq n_0$ and thus, in particular, $Z_s(i)$ nonempty $\forall i \geq n_0$.

Proof of Claim

Fix $i \geq n_0$ and let $D = \{p \in P : \exists j \text{ such that } p \Vdash \dot{h}(i) = j\}$.

Then D is open dense, so we may define the rank prove the lemma by an induction on the rank of the dense open set. Since the rank is defined for all $t \in IS$, this will suffice.

If $t \in D_0$, the lemma is trivial.

For limit ordinals a we are also done immediately, since $D_\alpha = \bigcup\{D_\beta : \beta < \alpha\}$.

The successor case is the only one requiring some work:

Suppose $t \in D_{\alpha+1}, (t, f) < (s, f)$, and $t \notin D_\alpha$.

Then there exists a countable sequence $\langle t_n : n \in \omega \rangle$ of elements of D_α, such that for some m we have:

- $|t_n| = m$ for all n and
- $t_n(|t|) \geq n$ for all n.

Since such a sequence must exist in $H(\kappa)$, without loss of generality $\langle t_n : n \in \omega \rangle \in N$.

Case 1: For some j, j belongs to infinitely many of the $Z_{t_n}(i)$.

Then clearly $j \in Z_t(i)$ and we are done.

Case 2. Otherwise, assume no such j exists. We claim that this case cannot occur.

Fix $j_n \in Z_{t_n}(i)$ minimal. Then by the assumption for this case $J = \{j_n : n \in \omega\}$ is infinite.

But also we may take $J \in N$, since J may be defined from $\langle t_n : n \in \omega \rangle$, and $\langle t_n : n \in \omega \rangle \in N$.

Now by the choice of a_ξ we know that $J \setminus a_\xi$, is infinite.

Choose n large enough so that $j_n \geq n_0$, $n \geq f(m-1)$, and $j_n \notin a_\xi$.

Then $(t_n, f) \leq (t, f) \leq (s, f)$, and since $j_n \in Z_{t_n}(i)$, there is some $(u, g) \leq (t_n, f)$ such that $(u, g) \Vdash \dot{h}(i) = j_n$.

But then, since $(s, f) \Vdash (\forall i \geq n_0)$, if $j \in \dot{a}$ holds, then $j \in a_\xi$ must hold as well, so we must have $j \in a_\xi$, a contradiction.

■

Choose $k_i \in Z_s(i)$, $\forall i \geq n_0$ (say the minimal one), and let $K := \{k_i : i \geq n_0\}$.

It is easy to see that K is infinite, since $k_i \geq i$ $\forall i \geq n_0$, and that $K \in N$, since it is defined form s. Thus, $K - a_\xi$ has to be infinite (recall that ξ was witnessing this for all infinite subsets of ω in N, so, in particular for K).

Looking at the definition of $Z_s(i)$, one sees

$$k_i \in K \setminus a_\xi \Rightarrow \exists g \in \omega^\omega, \exists s' \supseteq s : (s', g) \leq (s, f) \text{ and } (s', g) \Vdash \dot{h}(i)$$

This just says $(s', g) \Vdash k_i \in \dot{a} \setminus n_0$, and it is a contradiction, since it would have to also be in a_ξ.

Thus, eventually narrow sequences remain eventually narrow in the extension by Hechler forcing.

The limit step:

Let α be a limit ordinal. W.l.o.g. $cf(\alpha) = \omega$, otherwise no reals are introduced by $\mathbb{D}_\alpha$. Therefore, there is an increasing sequence $\langle \alpha_m : m \in \omega \rangle$, cofinal in α.

Assume again, there is a sequence which is eventually narrow in all previous models, but it is not preserved to be eventually narrow in $V^{\mathbb{D}_\alpha}$. Thus, there must be some condition $p \in \mathbb{D}$ and a name $\dot{a}$, such that $p \Vdash \dot{a}$ is infinite and $\forall \xi < \lambda \exists \eta \geq \xi : \dot{a} \subseteq^* a_\eta$.

Let G_α a $\mathbb{D}_\alpha$ generic filter over V and $p \in G_\alpha$. Then $G_\beta = G_\alpha \cap \mathbb{D}_\beta$ is $\mathbb{D}_\beta$ generic $\forall \beta < \alpha$.

For each $\xi < \lambda$, fix, if possible, $p_\xi \in G_\alpha$ and $n_\xi \in \omega$, such that

$$p_\xi \Vdash \forall i \geq n_\xi (i \in \dot{a} \Rightarrow i \in a_\xi).$$

It will be possible to define them for $\xi \in B$ for some B cofinal set in λ. Moreover, since $\mathbb{D}_\alpha$ is the direct limit of $\langle \mathbb{D}_\beta : \beta < \alpha \rangle$, we have $\forall \xi \in B : p_\xi \in G_{\alpha_m}$ for some m. Hence, there is $A \subseteq B$ also cofinal in λ and there are fixed m, n such that, $\forall \xi \in A : p_\xi \in G_{\alpha_m}$ and $n_\xi = n$ (*i.e.* A can be determined in $V[G_{\alpha_m}]$).

Let $b := \bigcap \{a_\xi \setminus n : \xi \in A\}$. Since $\Vdash \dot{a} \setminus n \subseteq b$, we can conclude that b must be infinite. But $b - a_\xi = \emptyset$, thus, finite for all $\xi \in A$, contradicting the fact that $\langle a_\xi : \xi < \lambda \rangle$ is eventually narrow in $V[G_{\alpha_m}]$.

■

Since eventually splitting sequences are eventually narrow, they are also preserved. The argument for the fact that $\mathfrak{s} = \omega_1$ in the Hechler model is now finished:

One uses the fact that the Hechler forcing also adds a Cohen real. Therefore, in $V^{\mathbb{D}_{\omega_1}}$, there is an eventually splitting sequence of length ω_1 containing these Cohen reals (recall that Cohen reals are splitting reals). The above result ensures that this sequence will remain eventually splitting in $V^{\mathbb{D}_\lambda}$ for λ regular, uncountable.

Chapter 3

Shelah's forcing for $Con(\mathfrak{b} = \omega_1 < \mathfrak{s} = \omega_2)$

As stated in the introduction, this thesis gives the arguments for the independence of $\mathfrak{b}$ and $\mathfrak{s}$. The previous chapter presented three ways to obtain $Con(\mathfrak{s} < \mathfrak{b})$. The other direction of the inequality is the difficult one. In 1985, Shelah modified the Mathias forcing argument by adding an additional combinatorial structure on the pure part of the conditions, and obtained a proper forcing notion, whose countable support iteration of length ω_2 gives $Con(\mathfrak{b} = \omega_1 < \mathfrak{s} = \omega_2)$ and $Con(\mathfrak{b} = \omega_1 < \mathfrak{a} = \omega_2)$. This is the first appearance of a creature forcing in the literature.

There are limitations to Shelah's forcing, namely, that being proper, it can't be used to make the continuum arbitrarily large, only $\mathfrak{c} = \aleph_2$ can be achieved.

To increase the splitting number $\mathfrak{s}$ one has to cofinally often add reals not split by the ground model reals. The first forcing notion that comes to mind for this purpose is Mathias forcing. It is proper, so the usual way to iterate it is with countable support. The problem is not only that, being proper, $\mathfrak{c} \geq \aleph_3$ cannot be achieved, but this forcing notion also adds a dominating real, and thus, a countable support iteration will also make $\mathfrak{b} = \mathfrak{c} = \omega_2$. Thus, Mathias model will not witness the desired consistency result. Since Shelah modified the Mathias forcing to get a more complicated forcing notion, whose iteration keeps $\mathfrak{b}$ small, being familiar with Mathias forcing helps a lot in following the arguments. Therefore, the first sections of this chapter will deal with generalities on proper forcing and the Mathias forcing.

3.1 Properness

The notion of properness was introduced and developed by S.Shelah, as a common property of forcing notions that can be iterated with countable support and do not collapse ω_1. This new concept was needed, since there are limitations of finite support iterations of c.c.c. forcing notions, namely:

- there may be no c.c.c. forcing notion with the desired combinatorial properties
- it may be required that no Cohen reals are added, and we know that Cohen reals are always added at limit stages of countable cofinality in finite support iterations of c.c.c. forcing notions, no matter what the particular forcing is.

Moreover, if one iterates with finite support a forcing notion that is not c.c.c., $\aleph_1$ is collapsed, hence, a new iteration method was implicitly needed.

The countable support iteration first appeared in Jensen's consistency proof of CH and in Laver's paper on the Borel Conjecture. As the name suggests, the idea of such an iteration is that the support is countable. This is equivalent to postulating that direct limits are taken at limit stages of uncountable cofinality, and inverse limits at the ones of countable cofinality (recall that in f.s.i., direct limits were taken at all limit stages).

There are many equivalent definitions of the notion of properness (see [3]), this thesis will only give two of them.

DEFINITION 3.1 *Let κ be a regular, uncountable cardinal.*

A set $C \subset \kappa$ is called ***closed unbounded (club)*** *if C is unbounded and contains all its limit points less than κ (i.e. $sup(C \cap \alpha) \in C$ for all limit ordinals $\alpha < \kappa$).*

A set $S \subset \kappa$ is ***stationary*** *if $S \cap C \neq \emptyset$ for all $C \subset \kappa$ club.*

A set $S \subset \kappa$ is ***closed*** *if any decreasing sequence of length less than κ of elements of C has a limit in C.*

The "club" property is an absolute property, however, being stationary is not. This leads to the following definition of properness:

DEFINITION 3.2 *A forcing notion $\mathbb{P}$ is said to be* ***proper*** *if $\forall\kappa$ uncountable cardinal and $\forall S \subseteq [\kappa]^{\leq\aleph_0}$ stationary, S remains stationary in the extension.*

The above definition is nice and short, but not very useful in applications. Therefore, an equivalent, model-theoretical definition will be introduced, a definition which proved itself to be, by far, the most useful one.

Recall that $H(\chi) = \{x : |trcl(x)| < \chi\}$, the collection of sets, hereditarily of size $< \chi$. It is a basic fact that for regular χ, $H(\chi) \models ZFC^-(ZFC$ without the power set axiom).

Let χ be a "large enough" regular cardinal (such that $H(\chi)$ encapsulates the relevant statements for the forcing construction) and N a countable elementary submodel of $H(\chi)$ containing the forcing notion $\mathbb{P}$.

Definition 3.3 *A condition $q \in \mathbb{P}$ is called $(N, \mathbb{P})$-**generic** if for all dense D, $D \subseteq \mathbb{P} \cap N$, $D \cap N$ is predense below q (i.e. any condition stronger than q is compatible with some condition in $D \cap N$).*

*A forcing notion $\mathbb{P}$ is called **proper** if for all large enough regular cardinals χ, for all countable models N, $N \prec H(\chi)$ containing $\mathbb{P}$ and for all conditions $p \in \mathbb{P} \cap N$, there is an extension q of p, which is $(N, \mathbb{P})$-generic.*

Note that an equivalent definition of $(N, \mathbb{P})$-generic conditions would be obtained if "dense" would be replaced by "maximal antichain‘", "predense" or "dense open" and that extensions of $(N, \mathbb{P})$-generic conditions are $(N, \mathbb{P})$-generic as well.

Fact: All c.c.c. forcing notions are proper.

Proof:

Claim: If $\mathbb{P}$ satisfies c.c.c., $\mathbf{1}_{\mathbb{P}}$ is $(N, \mathbb{P})$-generic and thus, by the above remark, all conditions are $(N, \mathbb{P})$-generic.

Take A a m.a.c, $A \in N$.

Since N is countable, A has to be the range of a function in N with domain ω, thus, $A \subseteq N$.

(as $f \in N$ and $\forall n \in \omega \cap N$, also $f(n) \in N$ for all n, so $A = ran(f) \subseteq N$)

Since A is also a m.a.c., $\Vdash_{\mathbb{P}} A \cap N \cap G \neq \emptyset$. So, $\mathbf{1}_{\mathbb{P}}$ is $(N, \mathbb{P})$-generic

■

Fact: The ω-closed forcing notions are also proper.

Proof:

Having $p \in \mathbb{P} \cap N$, enumerate the dense subsets of $\mathbb{P}$ that are in N by $\langle D_i : i \in \omega \rangle$.

Recursively construct p_i such that:

- $p_0 := p$,
- Let $p_{i+1} \leq p_i$, such that $p_{i+1} \in D_i \cap N$.

Since $\mathbb{P}$ is countably closed, the sequence $\langle p_i : i \in \omega \rangle$ has a lower bound q.

This q is $(N, \mathbb{P})$-generic since $q \Vdash \forall i : p_i \in \dot{G}$.

Thus, $\mathbb{P}$ is proper.

■

A particular case($A = \omega_1 \cap V$) of the following lemma gives the fact that ω_1 is always preserved when forcing with a proper forcing notion.

Lemma 3.4 *Every countable set A of ordinals in the extension is covered by a countable set of ordinals in the ground model. (i.e. For all $A \in V[G]$ countable, there is $B \in V$, also countable, covering A.)*

Proof:

Let $\dot{A}$ be a name for a countable set of ordinals A in the extension.

Thus, there must be a countable sequence of names $\langle \dot{\alpha}_n : n \in \omega \rangle$ for elements in $\dot{A}$.

For all n, take A_n to be a m.a.c. deciding $\dot{\alpha}_n$.

We will show that for any condition, there is an extension, forcing the existence of a countable set B covering A (which is the same as proving the density of the set of conditions forcing this statement).

Given $p \in \mathbb{P}$ and $N \prec H(\chi)$ countable, containing $\mathbb{P}, p$ and $\langle A_n : n \in \omega \rangle$, let $q \leq p$ be $(N, \mathbb{P})$-generic.

Define $B := \bigcup_{n \in \omega} \{\beta : \exists r \in A_n \cap N \text{ with } r \Vdash \alpha_n = \beta\}$.

B is countable since A_n is countable and the genericity of q implies that $A \subseteq B$.

■

Properness is preserved under countable support iterations, and thus, ω_1 is not collapsed. Moreover, if the iteration has length $\leq \omega_2$ and all iterands have size $\leq \aleph_1$, then all cardinals are preserved, since the limit satisfies the $\aleph_2$-cc.

However, the iterations of length $> \omega_2$ are known to collapse the continuum, therefore, using proper forcing, models with $\mathfrak{c} \geq \aleph_3$ cannot be obtained. This is a big impediment, since there are consistency results that require larger continuum (for example a model where $\mathfrak{s}$ is singular).

Another property preserved by countable support iterations of proper forcing notions is **the ω^ω-bounding property**, *i.e.* the property that no unbounded reals are added, every real in the extension is dominated by one of the ground model reals.

Since the previous lemma says that proper forcing notions preserve ω_1, the most obvious example of a forcing notion which is **not proper** is the one collapsing ω_1, namely $(Fn(\omega, \omega_1), \supseteq)$.

James Baumgartner introduced the property called Axiom A, an extension of c.c.c. and countably closed, but not covering all proper forcing notions. It is a very useful property, since many well known proper forcing notions satisfy it.

Definition 3.5 *A forcing notion* $(\mathbb{P}, <)$ *satisfies* ***Axiom A*** *if there is a collection of partial orderings* $\{\leq_n\}_n$ *of* $\mathbb{P}$ *such that*

i) $p \leq_0 q \Rightarrow p \leq q$ *and* $\forall n$: $p \leq_{n+1} q \Rightarrow p \leq_n q$

ii) for every fusion sequence (i.e. $\{p_n\}_n$ with $p_{i+1} \leq_i p_i$) there exists q, the fusion of the sequence(*i.e.* $q \leq_n p_n \ \forall n$)

iii) for every $D \subseteq \mathbb{P}$ dense, $\forall p \in \mathbb{P}, \forall n \in \omega$ there is $p' \leq_n p$ and $D_0 \subseteq D$ countable, with the property that D_0 is predense below p'.

To see that all c.c.c. forcing notions satisfy this property, put $p \leq_n q$ iff $p = q$ for all $n > 0$. In case of countably closed forcing notions, it suffices to take the same order $\leq_n = \leq$ for all $n \in \omega$. Thus, this is an alternative way of proving that c.c.c. and countably closed forcing notions are proper.

Some other important facts are that no new reals are added during countable support iterations at limit stages of uncountable cofinality and that in countable support iterations of length ω_2 of proper forcing notions, any set of reals of cardinality ω_1 is added at some initial stage of the iteration.

Lemma 3.6 *If the length of a countable support iteration has uncountable cofinality, every real in the final model already appears in some intermediate extension.*

Proof:

Let δ be a limit ordinal with $cf(\delta) > \omega$, $\dot{f}$ a $\mathbb{P}_\delta$-name for a real and $p \in \mathbb{P}_\delta$ an arbitrary condition.

Let $N \prec H(\chi)$ for large enough χ, containing $p, \dot{f}$.

Since $\mathbb{P}_\delta$ is proper, there is an extension q of p which is $(N, \mathbb{P})$-generic.

Define $\alpha := sup(\delta \cap N)$.

Claim: $q \Vdash \dot{f} \in V^{\mathbb{P}_\alpha}$

For all n let A_n be the maximal antichain deciding $\dot{f}(n)$.

Then $A_n \in N$ and $q \Vdash G \cap A_n \cap N \neq \emptyset$. But conditions in $A_n \cap N$ have support included in α, therefore, knowing G_α is enough for deciding $\dot{f}(n)$.

■

3.2 Mathias forcing

Definition 3.7 *The **Mathias forcing** is*

$$\mathbb{M} = \{(s, A) : s \in [\omega]^{<\omega}, A \in [\omega]^\omega, max\ s < min\ A\}.$$

with the extension relation

$$(s_1, A_1) \leq (s_2, A_2) \ \text{if}\ s_2 \subseteq s_1, A_1 \subseteq A_2, s_1 \setminus s_2 \subseteq A_2.$$

The finite set s is called **stem** and the infinite set A is called **pure part** of the Mathias condition (s, A). An extension which does not change the stem is a **pure extension**.

Fact: (See [7]) $\mathbb{M}$ satisfies **Axiom A** , so it is **proper**.

Proof:

Identify infinite sets A with the increasing enumeration of their elements $\{a^i : i \in \omega\}$ and define a decreasing sequence of partial orders $\{\leq_n\}_n$ on $\mathbb{M}$ by:

- $\leq_0 = \leq$
- $(t, B) \leq_i (s, A)$ iff

– the conditions have the same stem ($t = s$) and

– the first i elements of A and B are the same.

It is clear that every fusion sequence contains conditions with the same stem, hence, let $\langle (s, A_i) \rangle_i$ be such that $(s, A_{i+1}) \leq_i (s, A_i)$.

Let $A := \bigcap_i A_i$.

For all i, the condition (s, A) satisfies $(s, A) \leq_i (s, A_i)$, thus, it is the fusion of the given sequence.

Proving iii) is a bit more complicated.

Let $N \prec H(\chi)$ be countable (for χ large enough) and $(s, A) \in \mathbb{M}$.

Let D_n be an enumeration of (some of) the dense open sets in N and denote by $B^i = \{b^j : j > i\}$(all elements of B with after the i-th element).

Claim: There is a pure extension $(s, B) \in \mathbb{M}$ such that $\forall i$, whenever $n < i$ and $t \subseteq \{b^j : j < i\}$, the following holds:

if $(s \cup t, B^i)$ has a pure extension in D_n, then itself is already in D_n.

Note: This is the idea behind the preprocessed conditions treated later, so it should be kept in mind.

The condition (s, B) extending the given condition (s, A) is obtained as the fusion of a sequence A_i, so as long as one defines the A_i's in a convenient way, (s, B) will have the desired properties.

Let $A_0 = A$.

The infinite set A_{i+1} will be constructed from A_i in the following way:

List all the pairs (n, t) with $n < i+1$ and $t \subseteq \{a_i^j : j < i+1\}$ (*i.e.* elements of t are between the first $i+1$ elements of A_i), say indexed by $k < l$.

$A_{i+1} := A_{i,l}$ where $A_{i,k}$'s are decreasing and defined recursively on $k < l$.

- $A_{i,0} := A_i$
- $A_{i,k+1} := \{a_i^j : j < i+1\} \cup C$ if there is $C \subseteq A_{i,k}$ such that $(s \cup t_k, C) \in D_n$,
  $A_{i,k+1} := A_{i,k}$ otherwise.

Hence, $(s, A_{i+1}) \leq_i (s, A_i)$ and for $n < i, t \subseteq \{a_i^j : j < i\},$ if $(s \cup t, A_i^i)$ has an extension in D_n then $(s \cup t, A_i^i) \in D_n$, so the fusion (s, B) will be as required.

Since the above argument says that there is always such an extension (s, B) obtained as the fusion of a fusion sequence, the set D_0 containing the conditions of the form $(s \cup t_i, A_i^i)$ will be countable and predense below (s, B).

If the enumeration contains all dense open sets then (s, B) is $(N, \mathbb{M})$-generic.

■

Fact: $\mathbb{M}$ adds a real **not split** by the ground model reals.

Thus, a countable support iteration over a model of CH produces an extension with $\mathfrak{s} = \mathfrak{c} = \aleph_2$

Proof:

Considering any $A \in [\omega]^\omega \cap V$ and $p = (s, B) \in \mathbb{M}$, either $B \cap A$ or $B \cap A^c$ is infinite, and thus,

$$D_A = \{(s, B) : B \subseteq A \text{ or } B \subseteq A^c\} \text{ is dense.}$$

To see this, let $p = (s, C)$ be any condition.

One has to find an extension $q \leq p : q \in D_A$.

By the above remark, either $C \cap A$ or $C \cap A^c$ has to be infinite. Let C' be the one that is infinite and take $q = (s, C')$.

If G is a $\mathbb{M}$ generic filter over M, let

$$U_G = \bigcup\{s : \exists B \text{ such that } (s, B) \in G\}$$

Since any two conditions in G are pairwise compatible, $U_G \subseteq^* B$ for all B which appears as a pure part in Mathias conditions.

■

Mathias forcing is known in the literature to add an ultrafilter. But this is the same thing with adding an unsplit real:

An infinite set $A \subseteq \omega$ makes an ultrafilter on the family $S \subseteq [\omega]^\omega$ if $\forall B \in S$: either $A \subseteq B$ or $A \subseteq B^c$. S is a splitting family if and only if no A makes an ultrafilter on S.

At the end of the previous section, an important fact about countable support iterations of length ω_2 of proper forcing notions was stated, namely that any set of reals of cardinality ω_1 is added at some initial stage during the iteration. This implies the fact that $\mathfrak{s}$ can't be $\aleph_1$ in the final model, since any splitting family of size $\aleph_1$ is destroyed at the next stage, when a set A making an ultrafilter is added.

Fact: $\mathbb{M}$ also adds a **dominating real** , *i.e.* a real that dominates all ground model reals. Therefore, a countable support iteration of length ω_2 will yield an extension with $\mathfrak{b} = \omega_2$.

Proof:

For every $A \in [\omega]^\omega$, denote by f_A the enumerating function of A, namely

$$f_A(j) = \text{ the } j\text{-th element of } A.$$

Claim: The enumerating function of UG, denoted f_G, dominates all the ground model reals.

To see this, let $f \in \omega^\omega$ be any ground model function. One has to show that the set

$$D_f = \{(s, A) : \forall l \in \omega \ A(l) > f(|s| + l)\} \text{ is dense.}$$

Let $p = (s, B) \in \mathbb{M}$ be any condition.

Recursively one can find an $A \subseteq B$ infinite, $(s, A) \in D_f$.

Assuming $(s, f) \in G \cap D_f$, it is clear that s is an initial segment of U_G and $U_G \setminus s \subseteq A$.

Thus, $\forall l \in \omega : \ f_G(|s| + l) \geq A(l)$ and $A(l) \geq f(|s| + l)$. Hence, $f \leq^* f_G$.

■

3.3 Shelah's proper forcing

Since the Mathias forcing not only adds an unsplit real, but also a dominating real, the Mathias model will not witness $Con(\mathfrak{b} < \mathfrak{s})$. Just by adding an additional combinatorial structure on the pure part of Mathias conditions, Shelah obtains a proper forcing notion that still adds a real not split by the ground model reals, but which is also almost ω^ω-bounding and hence, keeps $\mathfrak{b}$ small. The combinatorial structure added is given in the form of a logarithmic measure on $[\omega]^{<\omega}$. This forcing is the first creature forcing that appeared in the literature. Excellent expositions of the material presented in this section are [1] and [12].

Definition 3.8 *A forcing notion $\mathbb{P}$ is called* ***weakly*** ω^ω ***bounding*** *if the ground model reals remain an unbounded family in any extension by $\mathbb{P}$, i.e.*

$$\forall f \in \omega^\omega \cap V[G], \exists g \in \omega^\omega \cap V \text{ such that } \{n \in \omega : f(n) \leq g(n)\} \text{ is infinite.}$$

Example: Cohen forcing is weakly ω^ω bounding.

Proof:

To see this, recall that Cohen forcing is countable and thus, its conditions can be enumerated as $\{c_n : n \in \omega\}$. Given $\dot{f}$ a name for a function, one should find a function $g \in V$ such that $\{n \in \omega : \dot{f}(n) \leq g(n)\}$ is infinite.

Defining $g(i)$ such that some extension of c_i forces $\dot{f}(i) = g(i)$, one gets the desired result.

■

The problem with this property is that it is not preserved during iterations.

Note that the ω^ω-bounding property is preserved, since $\leq$ is transitive. In case of weakly ω^ω bounding, the infinite sets can even be disjoint.

A simple example where this property is not preserved, since a dominating real is added, is to first add $\aleph_1$ Cohen reals and then to do Hechler forcing with conditions in V. Cohen forcing is weakly bounding and, although Hechler forcing adds a dominating real, it is also weakly bounding since every name for a real is already in some initial model $V^{\mathbb{C}\restriction\alpha}$ for some $\alpha < \omega_1$.

Thus, a stronger notion of unboundedness is needed to ensure that a witness for $\mathfrak{b} = \omega_1$ is preserved during iterations.

Definition 3.9 *$\mathbb{P}$ is called* ***almost*** ω^ω ***bounding*** *if for every $\mathbb{P}$-name $\dot{f}$ for a function in ω^ω and every condition $p \in \mathbb{P}$ there is a ground model function $g \in \omega^\omega$, such that for every $A \in [\omega]^\omega$, there is $q_A \leq p$ with*

$$q_A \Vdash \exists^\infty k \in A(\dot{f}(k) \leq \check{g}(k)\ )$$

Note that, by interchanging $\exists g \in \omega^\omega \cap M$ and $\forall A \in [\omega]^\omega$, one gets the weaker notion of weakly ω^ω bounding.

Example: The Cohen forcing is almost ω^ω bounding.

Proof:

Since Cohen forcing is countable, one can enumerate its conditions as

$$\mathbb{C} = \{c_n : n \in \omega\}.$$

Fix $\dot{f}$ a $\mathbb{C}$-name for a function in ω^ω and c_n a condition in $\mathbb{C}$.

One needs to define a ground model function $g \in \omega^\omega$ satisfying:

$$\forall A \in [\omega]^\omega \exists c_m \leq c_n \text{ such that } c_m \Vdash \exists^\infty k \in A : (\dot{f}(k) \leq \check{g}(k)).$$

Define $g(n) := max_{m \leq n}(min\{k : \exists q \leq c_m(q \Vdash \dot{f}(n) = k)\}) + 1$.

To see that this function works, it is enough to consider $A = \omega$.

Assume towards a contradiction, $\exists c_m \in \mathbb{C}$ and $n \geq m$ such that

$$c_m \Vdash g(k) < \dot{f}(k) \text{ for all } k \geq n,$$

in particular, $c_m \Vdash g(n) < \dot{f}(n)$.

Then, by definition of $g(n)$, we know $g(n) \geq min\{k : \exists q \leq c_m(q \Vdash \dot{f}(n) = k)\} + 1$.

Let q be the extension of c_m forcing the minimal value k. Then $q \Vdash \dot{f}(n) = k < g(n)$.

But this is a contradiction, since q, as an extension of c_m, has to force $g(n) < \dot{f}(n)$.

Lemma 3.10 *If P is weakly bounding and Q is almost ω^ω bounding, then $P * Q$ is weakly bounding. Therefore, the countable support iteration of almost ω^ω bounding forcing notions is weakly bounding.*

Proof:

Let $\dot{f}$ be a $P * Q$-name and $(p, q) \in P * Q$ be a condition forcing that $\dot{f}$ is a function in ω^ω.

One needs to show that in any extension by a $P{*}Q$ generic filter containing (p, q), there is a $h \in \omega^\omega \cap V$ weakly bounding $\dot{f}$.

Take G to be a P-generic filter containing p.

In the extension by this filter, $\dot{f}$ is a Q-name for a real.

Since Q is almost ω^ω bounding, given any $q' \in Q, q' \leq q$, there is $g \in V[G]$ almost bounding $\dot{f}$, *i.e.* for every $A \in [\omega]^\omega$, there is $q_A \leq q'$ with $q_A \Vdash \exists^\infty k \in A(\dot{f}(k) \leq g(k))$.

The hypothesis also says that P is weakly ω^ω-bounding, thus, for this $g \in V[G]$ there is $h \in V$ such that $B = \{n \in \omega : g(n) \leq h(n)\}$ is infinite in $V[G]$.

If H is a Q-generic filter over $V[G]$ containing q', then there is an infinite $B_0 \subseteq B$ with $f(n) \leq g(n)\ \forall n \in B_0$. Thus, $\dot{f}$ is weakly bounded by $h \in V$.

■

Assuming CH, Shelah found a proper, almost ω^ω bounding forcing notion of size $\aleph_1$ such that in every generic extension there is an infinite subset of ω not split by the ground model reals.

Definition 3.11 *Let $s \in [\omega]^{<\omega}$ and $h : \mathcal{P}(s) \to \omega$.*

The function h is a ***finite logarithmic measure on*** *s if whenever $x \subseteq s$ with $x = x_0 \cup x_1$ then $h(x_0) \geq h(x) - 1$ or $h(x_1) \geq h(x) - 1$, unless $h(x) = 0$.*

The value $h(s)$ is called the ***level*** *of the logarithmic measure h.*

*A set $e \subseteq s$ is said to be h-**positive** if $h(e) > 0$.*

Definition 3.12 Shelah's proper forcing notion *:*

*Let **Q** be the set of all pairs (u, T) where $u \in [\omega]^{<\omega}$ and $T = \langle (s_i, h_i) : i \in \omega \rangle$ is a sequence of finite logarithmic measures such that:*

i) *max* u<*min* s_0

ii) *max* $s_i <$ *min* s_{i+1}

iii) the sequence of the levels $\langle h_i(s_i) : i \in \omega \rangle$ *is unbounded(or strictly increasing)*

The underlying subset of ω *is* $int(T) = \bigcup_{i \in \omega} s_i$.

Let $T_l = \langle t_i^l : i \in \omega \rangle$ *where* $t_i^l = (s_i^l, h_i^l)$ *for* $l = 1, 2$. *Then* (u_2, T_2) ***extends*** (u_1, T_1), *written* $(u_2, T_2) \leq (u_1, T_1)$ *if*

i) $u_2 \supseteq u_1, u_2 \setminus u_1 \subseteq int(T_1)$ *and* $int(T_2) \subseteq int(T_1)$

Note that this is just the extension relation as Mathias conditions for $(u, int(T))$.

ii) *There is a sequence* $\langle B_i : i \in \omega \rangle$, *where* $\forall i \in \omega : B_i \in [\omega]^{<\omega}$ *such that*

- *max* $u_2 <$ *min* s_j^1 *for* $j := minB_0$
- *max* $B_i <$ *min* B_{i+1}
- $s_i^2 \subseteq \bigcup \{s_j^1 : j \in B_i\}$

iii) for every $e \subseteq s_i^2$ *which is* h_i^2*-positive, there is* $j \in B_i$ *such that* $s_j^1 \cap e$ *is* h_j^1*-positive.*

A condition with $u = \emptyset$ *is called a **pure condition** and, as before, extensions that do not change the stem are called **pure extensions**.*

For proving that Shelah's forcing notion is proper and has the almost ω^ω-bounding property, the notion of **preprocessed conditions** is needed.

DEFINITION 3.13 *Let $D \subseteq Q$ be a dense open set, $k \in \omega$ and $p = (u, T) \in Q$.*

*The condition p is **preprocessed for D and k** if $\forall v \subseteq k : v \supseteq u$, if $(v, \langle t_j : j > k \rangle)$ has a pure extension is D then $(v, \langle t_j : j > k \rangle)$ is already in D.*

Lemma 3.14 *Any extension of a preprocessed condition is also preprocessed for the same D and k.*

Proof:

Let $D \subseteq Q$ be a dense open set, $k \in \omega$ and $p = (u, T) \in Q$ a condition preprocessed for D and k. Let $q = (w, R) \in Q$ such that $q \leq p$ arbitrary.

To show: q is also preprocessed for D and k.

Let $v \subseteq k$, end-extending w such that $(v, \langle r_j : j > k \rangle)$ has a pure extension in D.

We need to show $(v, \langle r_j : j > k \rangle) \in D$.

The extension relation in Q implies that v also end-extends u and that $\langle r_j : j > k \rangle$ extends $\langle t_j : j > k \rangle$. Therefore, any pure extension of $(v, \langle r_j : j > k \rangle)$ is also a pure extension of $(v, \langle t_j : j > k \rangle)$. Since $p = (u, T)$ is preprocessed for D and k, $(v, \langle t_j : j > k \rangle) \in D$.

D is open, *i.e* closed under extensions, therefore, $(v, \langle r_j : j > k \rangle) \in D$.

■

Lemma 3.15 *The forcing notion Q is Axiom A, thus, proper.*

Proof:

For *i)* one has to find a decreasing sequence $\{\leq_n\}_{n \in \omega}$ of suborders on Q . Define:

- $\leq_0 = \leq$
- $(u_2, T_2) \leq_{n+1} (u_1, T_1)$ if $u_1 = u_2, (u_2, T_2) \leq_n (u_1, T_1)$ and $\forall i \leq n : t_i^1 = t_i^2$
  (i.e. (u_2, T_2) is a pure extension of (u_1, T_1) and the first n measures and sets coincide).

It is clear that $\leq_n \subset \leq_m \ \forall m < n$.

For *ii)* one has to show that for every fusion sequence $\{p_n\}_{n \in \omega}$, the fusion of the sequence exists. By *i)*, the stem has to be the same for every condition in the fusion sequence. So, if $p_n = (u, T_n)$, then $p = (u, \langle t_i : i \in \omega \rangle)$ with $t_i = t_i^{i+1}$ is the fusion of the sequence (p takes the measure t_i that is common to all conditions that have indexes above i).

For *iii)* some results on preprocessed condition are needed.

Lemma 3.16 *Let $D \subseteq Q$ be a dense open set and $k \in \omega$. Any condition in Q has a $\leq_{k+1}$ extension that is preprocessed for D and k.*

Proof of lemma Let $p = (u, T) \in Q$, where $T = \langle t_i : i \in \omega \rangle$ be any condition and enumerate by $\{v_i : i < j\}$ all subsets of k end-extending u.

Define $\langle t_i^l : i > k \rangle$ inductively on $l \leq j$ in the following way:

- $t_i^0 := t_i$
  For $1 \leq l \leq j$:
- if $(v_l, \langle t_i^{l-1} : i > k \rangle)$ has a pure extension in D then $\langle t_i^l : i > k \rangle$ is the pure part of this extension
- otherwise $t_i^l := t_i^{l-1}$

Clearly $(u, \langle t_i^j : i \in \omega \rangle) \leq_{k+1} (u, T)$ where for $i < k : t_i^j := t_i$.

Claim: this condition is also preprocessed for D and k.

Let $v \subseteq k$, end-extending u. Suppose that $(v, \langle t_i^j : i > k \rangle)$ has a pure extension in D and show it is already in D.

Since the enumeration $\{v_i : i < j\}$ contains all subsets of k end-extending u, then v must be one of them. Say $v = v_l$ for some $l \leq j$. Thus, at stage l, the condition $(v_l, \langle t_i^{l-1} : i > k \rangle)$ had a pure extension in D and we defined this pure extension as $(v_l, \langle t_i^l : i > k \rangle) \in D$. Since D is open and $\langle t_i^j : i > k \rangle \leq \langle t_i^l : i > k \rangle$, it follows that the pure extension $(v = v_l, \langle t_i^j : i > k \rangle) \in D$.

Thus, $(u, \langle t_i^j : i \in \omega \rangle)$ is preprocessed for D and k. ■

Lemma 3.17 *Given $D \subseteq Q$ be a dense open set, any condition $p \in Q$ has an extension which is preprocessed for D and all $k \in \omega$.*

Proof of lemma

For every $k \in$, the previous lemma gives us the existence of a $\leq_{k+1}$ extension, preprocessed for D and k. Thus, we can find a fusion sequence $\{p_k\}_{k \in \omega}$($p_0 = p$), where each p_{k+1} is preprocessed for D and k.

Take q to be the fusion of the sequence $\{p_k\}_{k \in \omega}$. Then $\forall k \in \omega : q \leq_{k+1} p_{k+1}$, in particular, $q \leq p_{k+1}$, and since p_{k+1} is preprocessed for D and k, so is q. Hence, q is an extension of p, preprocessed for D and all $k \in \omega$

■

Back to Proof of *iii*):

Given $D \subseteq Q$ be a dense open set and $p \in Q$ arbitrary, by the above lemma, there is always a pure extension $q \leq p$ preprocessed for D and any $i \in \omega$.

Moreover, $\forall i \in \omega : q \leq_i p$.

Let $D_0 := \{p \in Q \text{ with } p = (v, \langle t_j : j > i \rangle) \text{ for } i \in \omega, v \subseteq i, v \text{ extends } u\}$.

Clearly D_0 is a countable subset of D.

It remains to show that D_0 is predense below q, *i.e* any extension of q is compatible with a condition in D_0.

Consider (v, R) any extension of q.

Since D is dense, there is $(v \cup w, R') \leq (v, R)$, $(v \cup w, R') \in D$. But for some $k \in \omega, k \supseteq w$: $(v \cup w, R') \leq (v \cup w, \langle t_j : j \geq k \rangle)$.

We know $(v \cup w, \langle t_j : j \geq k \rangle) \in D$ since q is preprocessed for D and k.

In particular, $(v \cup w, \langle t_j : j \geq k \rangle) \in D_0$ and compatible with (v, R). So, D_0 is indeed predense below q.

■

Lemma 3.18 *Q adds a real which is not split by the ground model reals.*

Proof:

Claim 1: Let $T = \langle (s_i, h_i) : i \in \omega \rangle$ be a pure condition and $A \in [\omega]^\omega$ any infinite set. Then one of the sequences $\langle h_i(s_i \cap A) \rangle_{i \in \omega}$ or $\langle h_i(s_i \cap A^c) \rangle_{i \in \omega}$ is unbounded.

Proof:

Recall that, if (x, h) is a logarithmic measure, $h(x) > 0$ and x is partitioned into $x_0 \cup x_1$, then, by definition, either $h(x_0) \geq h(x) - 1$ or $h(x_1) \geq h(x) - 1$.

Define

$$I_0 = \{i \in \omega : h_i(s_i \cap A) \geq h_i(s_i) - 1\} \text{ and}$$

$$I_1 = \{i \in \omega : h_i(s_i \cap A^c) \geq h_i(x_i) - 1\}.$$

By (iii) in the definition of the forcing notion Q, the sequence of levels is unbounded. Therefore, the Pigeonhole principle ensures that, at least one of the sets is infinite, and so, either $< h_i(x_i \cap A) >_{i\in\omega}$ or $< h_i(x_i \cap A^c) >_{i\in\omega}$ is unbounded.

■

Claim 2 : $D_A = \{(u, T) : int(T) \subseteq A \text{ or } int(T) \subseteq A^c\}$ is dense.

Proof:

To see this, fix A any infinite set. One has to show that, given T a pure condition, there exists $T' = \langle (h'_i, s'_i)_{i\in\omega} \rangle \leq T : T' \in D_A$.

By Claim 1, $\langle h_i(s_i \cap A)\rangle_{i\in\omega}$ or $\langle h_i(s_i \cap A^c)\rangle_{i\in\omega}$ is unbounded. Say $\langle h_i(s_i \cap A)\rangle_{i\in\omega}$ is unbounded (the proof is the same for the complement). So, for every $l \in \omega$, there is an $i \in \omega$ such that $h_i(s_i \cap A) \geq l$.

By induction, we define s'_j, h'_j and B_j in the following way:

In B_0, put an arbitrary i, $s'_0 = s_0 \cap A$ and $h'_0 = h_i$.

Assume we have already define s'_j, h'_j and B_j, and we want to define s'_{j+1}, h'_{j+1} and B_{j+1}. We assumed that $\langle h_i(s_i \cap A)\rangle_{i\in\omega}$ is unbounded so, for every $l = h'_j(s'_j)$ (which is equal to $h_i(s_i \cap A)$ for $i \in B_j$), there is an $k \in \omega$ such that $h_k(s_k \cap A) \geq l$. Then put k in B_{k+1}, take $s'_{j+1} = s_k \cap A$ and $h'_{j+1} = h_k$.

Clearly, the obtain condition is an extension of T with underlying set included in A, so the density of D_A is established.

■

Claim 3: The density of D_A implies that Q adds an unsplit real.

Proof:

First remark that, if (u, T) is a condition in Q, then $(u, int(T))$ is a Mathias condition.

For a Q-generic filter G, let $U_G := \bigcup\{u : (u, T) \in G \text{ for some } T\}$.

Since the elements of the generic filter are pairwise compatible, $int(U_G) \subseteq^* int(T)$ for every T that appears as a pure part of a condition in G. Hence, the generic real is an unsplit real. ■

■

As specified before, the reason why $\mathfrak{b}$ remains small in the extension of a model of CH obtained by iterating (with countable support, for ω_2 steps) the forcing notion Q, is that this forcing notion is **almost ω^ω-bounding** . For proving this, measures induced by positive sets play an important role.

DEFINITION 3.19 *Let $P \subseteq \mathcal{P}(s)$ be upwards closed (i.e. if $a \in P, a \subseteq b$ then $b \in P$).*

The logarithmic measure h induced by P is defined in the following way:

- *$h(e) \geq 0$ for every $e \in \mathcal{P}(s)$*
- *$h(e) > 0$ iff $e \in P$ (the elements of P are called* **positive sets**)
- For every $l \geq 1$,

$$h(e) \geq l+1 \text{ iff } |e| > 1 \text{ and whenever } e = e_0 \cup e_1, \text{ then } h(e_0) \geq l \text{ or } h(e_1) \geq l$$

Then $h(e) = l$ iff l is the maximal natural number such that $h(e) \geq l$.

An induced logarithmic measure is said to be **atomic** if there is a singleton $\{n\}$ with $h(\{n\}) > 0$.

From this point on, we will only consider non-atomic measures.

The following condition is sufficient for the measure induced by P to take arbitrary high values:

For every decomposition of ω into n pieces $\omega = \bigcup_{j<n} A_j$, there is a piece A_i with the property that $\mathcal{P}(A_i) \cap P \neq \emptyset$ (*i.e.* A_i contains a positive set).

The induced measure than takes arbitrary high values, since the above condition implies that for every $k \in \omega$ and for every decomposition $\omega = \bigcup_{j<n} A_j$, there is an $i < n$ and some $e \subseteq A_i$ such that $h(e) \geq k$. This implication can be prooved by induction. One assumes that the conclusion does not hold for the $(k+1)$-th step and uses König's Lemma to contradict the induction hypothesis.

For D a dense open set, $k \in \omega$ and $T = \langle t_l : l \in \omega \rangle$ a pure condition preprocessed for D and k, denote by $P_k(T, D)$ the family of all finite subsets $x \subseteq int(T)$ such that for some $l \in \omega$ the following hold:

- $x \cap$ int(t_l) is a positive set
- $\forall v \subseteq k, \exists w \subseteq x$ such that $(v \cup w, T) \in D$

The fact that the measure induced by $P_k(T, D)$ takes arbitrary high values is used for proving the following lemma, which can be found in [1] and [12], and is the main technical tool for proving that Q is almost ω^ω bounding.

Lemma 3.20 *Let $\dot{f}$ be a Q-name for a function in ω^ω and $p = (u, T)$ an arbitrary condition in Q. Then there is a pure extension $q = (u, R) \leq p$, with $R = \langle r_i = (x_i, g_i) : i \in \omega \rangle$ such that:*

$$\forall i \in \omega, \forall v \subseteq i \text{ end-extending } u \text{ and } \forall s \subseteq x_i \text{ which is } g_i\text{-positive, there is } w_v \subseteq s \text{ such that } (v \cup w_v, \langle r_j : j \geq i \rangle) \Vdash \dot{f}(i) = k \text{ for some } k \in \omega.$$

The proof can be found in [1].

Theorem 3.21 *(see [1]) The proper forcing notion Q is almost ω^ω-bounding.*

Proof:

Let $\dot{f}$ be an Q-name for a function in ω^ω. One has to show

$$\exists g \in V \cap \omega^\omega \text{ such that } \forall A \in [\omega]^\omega \exists q_A \in Q \text{ with } q_A \Vdash \exists^\infty k \in A(\dot{f}(k) \leq g(k)).$$

By the previous lemma, p has a pure extension $q = (u, R), R = \langle r_i = (x_i, g_i) : i \in \omega\rangle$ with $\forall i \in \omega, \forall v \subseteq i$ end-extending u and $\forall s \subseteq x_i$ which is g_i-positive, there is w_v s such that $(v \cup w_v, \langle r_j : j \geq i\rangle) \Vdash \dot{f}(i) = k$ for some $k \in \omega$.

For all $i \in \omega$ let $g(i) :=$ max $\{k : \exists v \subseteq i, \exists w \subseteq x_i : (v \cup w, \langle r_j : j \geq i+1\rangle) \Vdash \dot{f}(i) = k\}$.

Consider $A \in [\omega]^\omega$ arbitrary and let $q_A = (u, \langle r_j : j \in A\rangle)$.

Clearly q_A extends q, so it suffices to show that $q_A \Vdash$ for infinitely many $k \in A : \dot{f}(k) \leq g(k)$.

Let $n \in \omega$ and (v, R') an arbitrary extension of q_A.

Thus, by the extension relation is Q, there is $i \in A, i \geq n$ such that $v \subseteq i$ and $s = int(R') \cap x_i$ is g_i-positive.

Using the previous lemma for s, let $w \subseteq s$ be such that $(v \cup w, \langle r_j : j \geq i\rangle)$ decides a value for $\dot{f}(i)$.

Then $(v \cup w, R')$ extends (v, R') and $(v \cup w, \langle r_j : j \geq i\rangle)$, thus, it makes the same decision.

■

Chapter 4

Extending ultrafilters: towards matrix iterations

4.1 General result: preservation of embeddability

Embeddability issues are studied with the scope of comparing forcing extensions.

Fix M a countable transitive model of ZFC, $\mathbb{P}$ and $\mathbb{Q}$ two partial orders and an embedding $i : \mathbb{P} \to \mathbb{Q}$ in M. If i is a complete embedding and H is a $\mathbb{Q}$-generic filter, then $i^{-1}(H)$ is $\mathbb{P}$-generic over M and forcing with $\mathbb{Q}$ gives a bigger extension than forcing with $\mathbb{P}$($i.e.$ $M[i^{-1}(H)] \subset M[H]$).

Definition 4.1 *For $\mathbb{P}$ and $\mathbb{Q}$ partial orders, $i : \mathbb{P} \to \mathbb{Q}$ is a* ***complete embedding*** *if*

i) i preserves the order, i.e. $\forall p, p' \in \mathbb{P}(p' \leq p \to i(p') \leq i(p))$

ii) i preserves incompatibility, i.e. $\forall p, p' \in \mathbb{P}(p' \perp p \to i(p') \perp i(p))$

iii) for all $q \in \mathbb{Q}$ there is $p \in \mathbb{P}$ a ***reduction of*** *q* ***to*** *$\mathbb{P}$, i.e. such that $\forall p'(p' \leq p \to (i(p')||q)$.*

It is easy to see that any extension of a reduction is a reduction, so it is clearly not unique.

For $\mathbb{P}$ and $\mathbb{Q}$ partial orders, if such a complete embedding $i : \mathbb{P} \to \mathbb{Q}$ exists, one says $\mathbb{P}$ is **completely embedded** into $\mathbb{Q}$ and writes $\mathbb{P} < \circ\ \mathbb{Q}$. In most applications, the particular case $i = id$ will appear.

Note: $\mathbb{P} < \circ\ \mathbb{Q} \Leftrightarrow$ all maximal antichains of $\mathbb{P}$ are maximal antichains of $\mathbb{Q}$.

An often used and well known result concerning the **preservation of complete embeddability** is the following (see [6]) :

Lemma 4.2 *Let $\mathbb{P}, \mathbb{Q}$ be partial orders, $\mathbb{P} < \circ\ \mathbb{Q}$, $\dot{\mathbb{A}}$ a $\mathbb{P}$-name for a partial order, $\dot{\mathbb{B}}$ a $\mathbb{Q}$-name for a partial order such that $\Vdash_{\mathbb{Q}} \dot{\mathbb{A}} \subseteq \dot{\mathbb{B}}$ and all maximal antichains of $\dot{\mathbb{A}}$ in $V^{\mathbb{P}}$ are maximal antichains of $\dot{\mathbb{B}}$ in $V^{\mathbb{Q}}$.*

Then $\mathbb{P} * \dot{\mathbb{A}} < \circ\ \mathbb{Q} * \dot{\mathbb{B}}$.

An **application of this lemma** is the preservation of embeddability when forcing with a Suslin c.c.c. forcing notion. We know that maximal antichains remain maximal antichains, therefore, applying the previous lemma for the particular case that $\mathbb{A}$ and $\mathbb{B}$ are the same Suslin c.c.c. forcing notion, one gets the following result:

Corollary 4.3 *Let $\mathbb{P}, \mathbb{Q}$ be partial orders, $\mathbb{P} < \circ\ \mathbb{Q}$, $\dot{\mathbb{A}}$ a $\mathbb{P}$-name for a Suslin c.c.c. forcing coded in $V^{\mathbb{P}}$.*
Then $\mathbb{P} * (\dot{\mathbb{A}})^{V^{\mathbb{P}}} < \circ\ \mathbb{Q} * (\dot{\mathbb{A}})^{V^{\mathbb{Q}}}$

Proof: (of Lemma)

It suffices to show that all maximal antichains of $\mathbb{P} * \dot{\mathbb{A}}$ are maximal in $\mathbb{Q} * \dot{\mathbb{B}}$

Hence, let $\{(p_\alpha, \dot{a}_\alpha) : \alpha < \kappa\}$ be a maximal antichain of $\mathbb{P} * \dot{\mathbb{A}}$ and assume towards a contradiction, it is not maximal in $\mathbb{Q} * \dot{\mathbb{B}}$, $i.e$ there exists a condition $(q, \dot{b}) \in \mathbb{Q} * \dot{\mathbb{B}}$ which is incompatible with all elements of the given antichain.

We know $\mathbb{P} < \circ\, \mathbb{Q}$, so for every $\mathbb{Q}$-generic filter G there is a P-generic H such that $V[H] \subseteq V[G]$ (here $H \subseteq G \cap \mathbb{P}$, so $H \subseteq G$).

Let $\dot{\Omega}$ be a $\mathbb{P}$-name such that $\Vdash \dot{\Omega} = \{\alpha : p_\alpha \in \dot{H}\}$.

CLAIM: $\Vdash \{\dot{a}_\alpha : \alpha \in \dot{\Omega}\}$ *is a m.a.c of* $\dot{\mathbb{A}}$.

Proof of the claim

It is an **antichain:**

Assume towards a contradiction it is not an antichain. Thus, at least two of the conditions are compatible, say $\dot{a}_{\alpha_1} || \dot{a}_{\alpha_2}$.

$p_{\alpha_1} || p_{\alpha_2}$ since they have to be both in the filter H, therefore, it follows that $(p_{\alpha_1}, \dot{a}_{\alpha_1}) || (p_{\alpha_2}, \dot{a}_{\alpha_2})$ contradicting the fact that $(p_\alpha, \dot{a}_\alpha)$ formed an antichain.

So, $\Vdash \{\dot{a}_\alpha : \alpha \in \dot{\Omega}\}$ is an antichain of $\dot{\mathbb{A}}$.

Maximality :

Assume towards a contradiction $\Vdash \{\dot{a}_\alpha : \alpha \in \dot{\Omega}\}$ is not maximal in $\dot{\mathbb{A}}$. Then there is a condition $p \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{a}$ such that $p \Vdash \forall \alpha(\alpha \in \dot{\Omega} \rightarrow \dot{a} \perp \dot{a}_\alpha)$.

Since $(p, \dot{a}) \in \mathbb{P} * \dot{\mathbb{A}}$ and $\{(p_\alpha, \dot{a}_\alpha) : \alpha < \kappa\}$ is a maximal antichain of $\mathbb{P} * \dot{\mathbb{A}}$, there must be an $\alpha < \kappa$ such that $(p, \dot{a})$ is compatible with $(p_\alpha, \dot{a}_\alpha)$. Denote their common extension by $(p', \dot{a}')$.

Thus, $p' \Vdash (\dot{a}' \leq \dot{a}$ and $\dot{a}' \leq \dot{a}_\alpha)$.

Since $p' \leq p_\alpha$ it follows that $p' \Vdash \alpha \in \dot{\Omega}$

So we have $p' \Vdash$ "$\alpha \in \dot{\Omega}$ and $\dot{a}$ and $\dot{a}_\alpha$ are compatible", a contradiction with the choice of $\dot{a}$ (we had $p \Vdash \forall\alpha(\alpha \in \dot{\Omega} \rightarrow \dot{a} \perp \dot{a}_\alpha)$ and $p' \leq p$).

■

Let $b = \dot{b}[G]$, $a_\alpha = \dot{a}_\alpha[G] = \dot{a}_\alpha[H]$(for α such that $p_\alpha \in H$) and $\Omega = \dot{\Omega}[G] = \dot{\Omega}[H] = \{\alpha : p_\alpha \in H\}$.

The claim says that $\{a_\alpha : \alpha \in \Omega\}$ is a m.a.c. of $\mathbb{A}$, so by the hypothesis, it is also a m.a.c of $\mathbb{B}$(in $V[G]$), so there is $\alpha \in \Omega$ such that b is compatible with a_α.

We know that we can choose G in such a way that $q \in G$ (given any condition, there is always a generic filter containing this condition).

Recall that $(q, \dot{b})$ was the witness for the fact that $\{(p_\alpha, \dot{a}_\alpha)\}$ is not a m.a.c. of $\mathbb{Q} * \dot{\mathbb{A}}$.

Hence, there is a common extension of p_α and q in G, say q', which forces $\alpha \in \dot{\Omega}$ and $\dot{b}, \dot{a}_\alpha$ to be compatible ($p_\alpha \in G$ since $p_\alpha \in H$ and $H \subseteq G$ and any two elements of the generic set G are compatible).

Letting $\dot{b}'$ to be a $\mathbb{Q}$-name for the common extension of $\dot{b}$ and $\dot{a}_\alpha$, it follows that $(q', \dot{b}')$ is a common extension of $(q, \dot{b})$ and $(p_\alpha, \dot{a}_\alpha)$.

This is a contradiction since $(q, \dot{b})$ should be incompatible with all $(p_\alpha, \dot{a}_\alpha)$'s.

■

The following lemma just says that the complete embeddability is preserved in finite support iterations:

Lemma 4.4 *Assume* $\langle \mathbb{P}^0_\mu, \dot{\mathbb{Q}}^0_\mu : \mu < \xi \rangle$ *and* $\langle \mathbb{P}^1_\mu, \dot{\mathbb{Q}}^1_\mu : \mu < \xi \rangle$ *are finite support iterations with the property that* $\mathbb{P}^0_\mu < \circ\ \mathbb{P}^1_\mu$ *for all* $\mu < \xi$.

Then $\mathbb{P}^0_\xi < \circ\ \mathbb{P}^1_\xi$.

Proof:

Clearly $\mathbb{P}^0_\xi \subseteq \mathbb{P}^1_\xi$ and incompatibility is preserved.

Let $p \in \mathbb{P}^1_\xi$ be any condition. One has to find a reduction of p to $\mathbb{P}^0_\xi$.

Since $\mathbb{P}^1_\xi$ is the finite support iteration, there is an $\eta < \xi$ such that $p \in \mathbb{P}^1_\eta$.

The induction hypothesis $\mathbb{P}^0_\eta < \circ\ \mathbb{P}^1_\eta$ gives the existence of a reduction $q \in \mathbb{P}^0_\eta$ of p (as an element of $\mathbb{P}^1_\eta$) to $\mathbb{P}^0_\eta$.

Claim: q is a reduction of p (as an element of $\mathbb{P}^1_\xi$) to $\mathbb{P}^0_\xi$.

For this, one has to show that any extension of q is compatible with p.

Thus, let $r \leq q$ be a in $\mathbb{P}^0_\xi$ arbitrary.

One can write r as $r_0 \cup r_1$, where $r_0 \in \mathbb{P}^0_\eta$ and r_1 has support in $[\eta, \xi)$.

Since $q \in \mathbb{P}^0_\eta$, there must be the case that $r_0 \leq q$ and thus, since q is a reduction of p to $\mathbb{P}^0_\eta$, r_0 and p are compatible.

Denoting their common extension by r'_0, one gets that $r'_0 \cup r_1$ is a common extension of r and p.

Since r was arbitrary, it follows that q is a reduction of p to $\mathbb{P}^0_\xi$.

■

4.2 Mathias Prikry forcing $\mathbb{M}_{\mathcal{F}}$ and Laver Prikry forcing $\mathbb{L}_{\mathcal{F}}$

We now look at the non-definable c.c.c. context and we seek the necessary and sufficient conditions for maximal antichains to remain maximal for these forcing notions. In this study, we follow [6] and [20].

Simple non-definable c.c.c. forcing notions are the ones related to ultrafilters:

- $\mathbb{M}_{\mathcal{F}}$ (Mathias-Prikry forcing) and
- $\mathbb{L}_{\mathcal{F}}$ (Laver-Prikry forcing).

DEFINITION 4.5 ***Mathias Prikry forcing****:*

$\mathbb{M}_{\mathcal{F}} = \{(s, A)$ *such that* $s \in [\omega]^{<\omega}, A \in \mathcal{F}$, *max* $s <$ *min* $A\}$

Ordered by $(t, B) \leq (s, A)$ *if*

- $t \supseteq s$,
- $t \setminus s \subseteq A$ *and*
- $B \subseteq A$.

Some important **properties** of Mathias Prikry forcing are the following:

- $\mathbb{M}_{\mathcal{F}}$ is σ-centered, thus has the countable chain condition.
- $\mathbb{M}_{\mathcal{F}}$ adds a pseudo-intersection to $\mathcal{F}$, *i.e.* $m_{\mathcal{F}} \subseteq^* A \ \forall A \in \mathcal{F}$, namely

$$m_{\mathcal{F}} = \bigcup \{s : \text{for some } A, (s, A) \in M_{\mathcal{F}}\}.$$

So, Mathias-Prikry forcing $\mathbb{M}_{\mathcal{F}}$ adjoins, via finite approximations a new infinite subset of ω which is eventually contained in all members of the filter $\mathcal{F}$.

For carefully chosen filters, one can assure that the extension has certain properties, like for example the fact that no dominating real is added (see later), or the contrary, that dominating reals are added: if the ultrafilter $\mathcal{U}$ is either rapid, or not a P-point then $\mathbb{M}_{\mathcal{F}}$ adds a dominating real (this result is due to Canjar).

DEFINITION 4.6 ***Laver Prikry forcing***, *denoted* $\mathbb{L}_{\mathcal{F}}$ *consists of trees* $T \subseteq \omega^{<\omega}$ *such that*

- *there is a* ***stem***, *denoted* $stem(T)$ *(i.e. for every node* t*, either* $t \subseteq stem(T)$ *or* $stem(T) \subseteq t$*)*
- *for all nodes* $t \in T$ *with* $stem(T) \subseteq t$*:* $succ_T(s) \in \mathcal{F}$*.*

The order is inclusion.

Recall that $succ_T(t) = \{n : t^\frown n \in T\}$ is the set of successors of the node t.

Properties:

- $\mathbb{L}_{\mathcal{F}}$ is σ-centered, thus, has the countable chain condition.
- For G generic filter, $\mathbb{L}_{\mathcal{F}}$ adds a dominating real

$$l_{\mathcal{F}} = \bigcup\{stem(T) : T \in G\}$$

with the property that $ran(l_{\mathcal{F}}) \subseteq^* A \ \forall A \in \mathcal{F}$ (*i.e.* $ran(l_{\mathcal{F}})$ is a pseudo-intersection of $\mathcal{F}$).

Thus, both $\mathbb{M}_{\mathcal{F}}$ and $\mathbb{L}_{\mathcal{F}}$ canonically increase $\mathfrak{s}$.

Proposition 4.7 *If* $\mathcal{U}$ *is* ***Ramsey***, *then* $\mathbb{M}_{\mathcal{U}} \cong \mathbb{L}_{\mathcal{U}}$*.*

We are interested to solve the following problem:

Given $M \subseteq N$ two models of ZFC and filters $\mathcal{F} \in M$, $\mathcal{G} \in N$, $\mathcal{F} \subseteq \mathcal{G}$ we want to know when any maximal antichain $A \subseteq \mathbb{M}_{\mathcal{F}}$ (respectively $\mathbb{L}_{\mathcal{F}}$) in M is maximal antichain of $\mathbb{M}_{\mathcal{G}}$ (respectively $\mathbb{L}_{\mathcal{G}}$) in N.

We look at non-trivial cases, namely at filters $\mathcal{G}$ properly extending $\mathcal{F}$.

4.3 Absoluteness for Laver Prikry forcing

For Laver forcing, the condition $\mathcal{F}$ and $\mathcal{G}$ have to satisfy is a simple one, thus, the absoluteness of maximal antichains is much easier to obtain than for Mathias forcing.

Theorem 4.8 (see [6])

The following are equivalent:

1. *Every $\mathcal{F}$-positive set in M is $\mathcal{G}$-positive in N.*

2. *Every m.a.c. of $\mathbb{L}_{\mathcal{F}}$ in M is m.a.c of $\mathbb{L}_{\mathcal{G}}$ in N.*

We use the notation $\mathbb{L}_{\mathcal{F}} < \circ_M \mathbb{L}_{\mathcal{G}}$ to express the fact that every m.a.c. of $\mathbb{L}_{\mathcal{F}}$ in M is m.a.c of $\mathbb{L}_{\mathcal{G}}$ in N.

Note that this statement differs from complete embeddability since m.a.c. of $\mathbb{L}_{\mathcal{F}}$ are in M, not in N.

Recall some basic definitions concerning filters:

For a filter $\mathcal{F}$ denote by $\mathcal{F}^*$ the **dual ideal**, *i.e.* $\mathcal{F}^* = \{\omega \setminus X : X \in \mathcal{F}\}$.

A set X is **$\mathcal{F}$-positive** if it is $\mathcal{F}^*$-positive, *i.e.* X is not in the dual ideal.

In other words, $\mathcal{F}$-positive sets are not complements of any set in $\mathcal{F}$, they intersect all the sets in the filter $\mathcal{F}$.

Proof: **2.$\Rightarrow$ 1.**

Assume every m.a.c. of $\mathbb{L}_{\mathcal{F}}$ in M is m.a.c of $\mathbb{L}_{\mathcal{G}}$ in N, but there is X an $\mathcal{F}$-positive set in M that is not $\mathcal{G}$-positive in N.

Because X **is $\mathcal{F}$-positive**, X intersects all sets in $\mathcal{F}$, therefore, the set

$$D = \{T \in \mathbb{L}_{\mathcal{F}} : \mathit{stem}(T)(|\mathit{stem}(T)| - 1) \in X\} \text{ is } \textbf{dense} \text{ in } \mathbb{L}_{\mathcal{F}}.$$

Since X **is not $\mathcal{G}$-positive in** N, we have $\omega \setminus X \in \mathcal{G}$.

Therefore, the uniform tree with splitting in the complement $S := (\omega \setminus X)^{<\omega}$ is a Laver tree in $\mathbb{L}_{\mathcal{G}}$ and it is incompatible with all elements of D.

Thus, no maximal antichain $A \subseteq D$ of M remains a maximal antichain of $\mathbb{L}_{\mathcal{G}}$ in N ($S \perp T$, for all $T \in A$).

1.$\Rightarrow$ 2.

We know that every $\mathcal{F}$-positive set in M is $\mathcal{G}$-positive in N and need to show that every m.a.c. of $\mathbb{L}_{\mathcal{F}}$ in M is m.a.c of $\mathbb{L}_{\mathcal{G}}$ in N.

We show this using a **rank argument.**

The rank argument was introduced by J.E Baumgartner and P Dordal in their paper "Adjoining dominating functions" and was applied it to Hechler forcing in Chapter 2 of this thesis.

Let $A \in M$ be a maximal antichain in $\mathbb{L}_{\mathcal{F}}$.

By recursion on $\alpha < \omega_1$, define in M for $s \in \omega^{<\omega}$ when $\mathit{rank}(s) = \alpha$:

$\mathit{rank}(s) = 0$ if $\exists T \in A$ such that $\mathit{stem}(T) \subseteq s \in T$.

$rank(s) = \alpha$ if there is no $\beta < \alpha$ with $rank(s) = \beta$, and $\{n : rank(s^\frown n) < \alpha\}$ is $\mathcal{F}$-positive.

CLAIM: for every $s \in \omega^{<\omega}, rank(s)$ is defined (and thus, $< \omega_1$).

Proof of claim Assume towards a contradiction $rank(s)$ undefined for some s. Then the set $\{n : rank(s^\frown n)$ is undefined$\} \in \mathcal{F}$.

Recursively build tree $S \in \mathbb{L}_\mathcal{F}$ such that $stem(S) = s$ and for all $t \supset s \in S$, $rank(t)$ is undefined.

Let $T \in A$ be compatible with S with common extension U.

Then $stem(T) \subseteq stem(U) \in U \subseteq T$ so that $rank(stem(U)) = 0$ and also $stem(S) \subseteq stem(U) \in U \subseteq S$ so that $rank(stem(U))$ undefined.

■

Recall that, by hypothesis, we know that every $\mathcal{F}$-positive set in M is $\mathcal{G}$-positive in N and we need to show that every m.a.c. of $\mathbb{L}_\mathcal{F}$ in M is m.a.c of $\mathbb{L}_\mathcal{G}$ in N.

Let $S \in N$ be a condition in $\mathbb{L}_\mathcal{G}$(arbitrary).

Put $s = stem(S)$.

By **induction on** $rank(s)$, show **there is** $T \in A$ **compatible with** S**.** This implies that A (which was an arbitrary m.a.c. of $\mathbb{L}_\mathcal{F} \in M$) remains m.a.c. of $\mathbb{L}_\mathcal{G}$ in N.

if $rank(s) = 0$:

there is $T \in A$ such that $stem(T) \subseteq s \in T$. Compatibility is straightforward.

if $rank(s) > 0$:

Consider $\{n : rank(s^\frown n) < rank(s)\}$.

This set is $\mathcal{F}$-positive and, by assumption, still $\mathcal{G}$-positive.

Hence, there is $n \in succ_S(s)$ with $rank(s^\frown n) < rank(s)$ and we know that the induction hypothesis is true for smaller ranks. Thus, considering $S^{[s^\frown n]} = \{t \in S : t \subseteq s \text{ or } s^\frown n \subseteq t\}$. (or both $s^\frown n$), we get a sub-tree of S with stem of smaller rank, namely $s^\frown n$.

By induction hypothesis, there is $T \in A$ compatible with $S_{s^\frown n}$. But then T is also compatible with S.

■

Corollary 4.9 (Shelah) *If $\mathcal{U}$ is an ultrafilter in M and $\mathcal{V}$ is an ultrafilter in N extending $\mathcal{U}$ then every m.a.c. of $\mathbb{L}_{\mathcal{U}}$ in M is still m.a.c of $\mathbb{L}_{\mathcal{V}}$ in N.*

Proof:

$\mathcal{U}$ is an ultrafilter, so for every set X, either X or its complement $\omega \setminus X$ is in $\mathcal{U}$. In this case $\mathcal{U}$-positive means $\in \mathcal{U}$. ■

4.4 Absoluteness for Mathias Prikry forcing

The same theorem as for Laver forcing is not true, even the special case in the corollary fails:

Remark 4.10 *Given $M \subseteq N$ be models of ZFC, if $\mathcal{U}$ is not Ramsey and $\exists c$ Cohen in N over M, then there are $\mathcal{V} \supseteq \mathcal{U}$ and a m.a.c. of $\mathbb{M}_{\mathcal{U}}$ that is not maximal of $\mathbb{M}_{\mathcal{V}}$.*

What we do know is the following:

Lemma 4.11 (Blass, Shelah) *(See [20]) Given $M \subseteq N$ be models of ZFC, an ultrafilter $\mathcal{U}$ and $c \in N \cap \omega^\omega$ unbounded over M, there exists an ultrafilter $\mathcal{V}$ extending $\mathcal{U}$ such that any maximal antichain $A \subseteq \mathbb{M}_\mathcal{U}$ in M is maximal antichain of $\mathbb{M}_\mathcal{V}$ in N and for all $\dot{f}$ $\mathbb{M}_\mathcal{U}$-names for a function in ω^ω, $\Vdash_{\mathbb{M}_\mathcal{V}}$ $c \not\leq^* \dot{f}$*

Remark 4.12 :

- "for all $\dot{f}$ $\mathbb{M}_\mathcal{U}$-names for a function in ω^ω, $\Vdash_{\mathbb{M}_\mathcal{U}}$ $c \not\leq^* \dot{f}$ means that c is still unbounded in N over $M[m_\mathcal{U}]$, where $m_\mathcal{U}$ is the $\mathbb{M}_\mathcal{U}$-generic real constructed from the $\mathbb{M}_\mathcal{V}$-generic real $m_\mathcal{V}$.

- $\mathcal{U} \subseteq \mathcal{V}$ implies $\mathbb{M}_\mathcal{U} \subseteq \mathbb{M}_\mathcal{V}$.

- antichains remain antichains since $\leq$ and $\perp$ agree for $\mathbb{M}_\mathcal{U}$ and $\mathbb{M}_\mathcal{V}$, so the only point is the maximality.

- However, the fact that maximal antichains are preserved (denoted $\mathbb{M}_\mathcal{U} <\circ_M \mathbb{M}_\mathcal{V}$) does not imply complete embeddability $\mathbb{M}_\mathcal{U} <\circ \mathbb{M}_\mathcal{V}$, only implies that $\mathbb{M}_\mathcal{U}$-names in M are still $\mathbb{M}_\mathcal{V}$-names in N (both in N would be $<\circ$).

- the proof of this lemma gives the construction of an ultrafilter $\mathcal{V}$ such that $\Vdash_{\mathbb{M}_\mathcal{V}}$ "$\mathbb{M}_\mathcal{U}$ does not add a dominating real".

- because no dominating real is added it is clear that $\mathcal{V}$ is not rapid and a "p-point" relative to the corresponding models (namely for any countable $\mathcal{A} \subseteq \mathcal{U}$ in M, there is a pseudo-intersection $B \in \mathcal{V}$, $B \in N$).

Proof:

Work in N:

For $A \subseteq \mathbb{M}_{\mathcal{U}}$ maximal antichain $A \in M$ and s finite subset of ω one says:

$C \subseteq \omega$ is **forbidden by** A, s if (s, C) is incompatible with all conditions in A.

These sets will assure that maximal antichains remain maximal antichains, in the sense that whenever we have a counterexample to the maximality of an antichain in M, it will be such a forbidden set in N.

Having $\dot{f}$ an $\mathbb{M}_{\mathcal{U}}$-name for a function in M, there are maximal antichains $B_n^{\dot{f}}$ and functions $g_n^{\dot{f}} : B_n^{\dot{f}} \to \omega$ such that for each $p \in B_n^{\dot{f}} : p \Vdash_{\mathbb{M}_{\mathcal{U}}} \dot{f}(n) = g_n^{\dot{f}}(p)$. The function $g_n^{\dot{f}}$ gives a partition of the antichain.

For $t \in [\omega]^{<\omega}$, we say $D \subseteq \omega$ is **forbidden by** $\dot{f}, t$ if $\forall n$ (t, D) is incompatible with all conditions $p \in B_n^{\dot{f}}$ satisfying $g_n^{\dot{f}}(p) < c(n)$.

Intuitively: $(t, D) \Vdash_{\mathbb{M}_{\mathcal{V}}}$ for all n $\dot{f}(n) \geq c(n)$, but this definition is not possible at this stage since the ultrafilter $\mathcal{V}$ was not defined yet. This type of forbidden sets will assure that c is still unbounded in the corresponding extensions, in the sense discussed in the remark.

Take $\mathcal{I}$ the **ideal generated by the forbidden sets** (in N).

CLAIM $N \models \mathcal{U} \cap \mathcal{I} = \emptyset$

Claim $\Rightarrow$ Lemma.

Using **Zorn's Lemma** one can construct $\mathcal{V}$ an ultrafilter extending $\mathcal{U}$ and still having empty intersection with $\mathcal{I}$.

Recall Zorn's Lemma: Suppose a partially ordered set P has the property that every totally ordered subset has an upper bound in P. Then the set P

contains at least one maximal element.

Clearly $\mathcal{V}$ is a filter.

Claim: $\mathcal{V}$ is an **ultrafilter**

Proof:

Assume towards a contradiction $\mathcal{V}$ is not an ultrafilter, *i.e.* $\exists X \subseteq \omega$ such that $X \notin \mathcal{V}$ and $\omega \setminus X \notin \mathcal{V}$.

We also have the ideal $\mathcal{I}$, so either $X \in \mathcal{I}$ or $X \notin \mathcal{I}$.

If $X \notin \mathcal{V}$ and $X \notin \mathcal{I}$ then $X \cap U = \emptyset$ for some $U \in \mathcal{V}$ (this has to be the reason why is not in $\mathcal{V}$).

But then $U \subseteq \omega \setminus X$, thus, $\omega \setminus X \in \mathcal{V}$, a contradiction.

If $X \in \mathcal{I}$ then $\omega \setminus X \notin \mathcal{I}$, so we can do the above argument with $\omega \setminus X$ instead of X and we get $X \in \mathcal{V}$, again a contradiction.

■

Note that, to conclude $\omega \setminus X \notin \mathcal{I}$ given that $X \in \mathcal{I}$, we used the fact that $\mathcal{I}$ is an ideal, only taking the forbidden sets would have not been enough. Also note that the above is just the proof that maximal filters are the same as ultrafilters, with the additional case distinction for the ideal.

Proof of Claim:

Assume towards a contradiction that $\mathcal{U} \cap \mathcal{I} \neq \emptyset$.

This implies that there are forbidden sets $C_i : i < k$ and $D_j : j < k$ such that if $E := (\bigcup_{i<k} C_i) \cup (\bigcup_{i<k} D_i)$, we must have that E is in $\mathcal{U} \cap M$. Without

loss of generality, we take the same number of sets k. We only need the fact that these are finitely many to be able to apply the compactness argument.

The fact that the sets C_i's and D_i's are forbidden is witnessed, say by A_i and s_i resp. $\dot{f}_i$ and t_i.

Without loss of generality:

- C_i's and D_i's are pairwise disjoint (this fact is needed for building the partition).
- $min(E) > max(s_i)$ and $min(E) > max(t_i)$ for all i.

Given $t \in [\omega]^{<\omega}$, (s, C) a condition in $\mathbb{M}_{\mathcal{U}}$ and $A \subseteq \mathbb{M}_{\mathcal{U}}$ we say that

- t is **permitted** by (s, C) if $s \subseteq t \subseteq s \cup C$.
- t is permitted by A if there is a condition in A that permits t.

The following subclaim only mentions objects in M and is absolute, therefore, we argue in M.

Subclaim: There is a function $h \in \omega^\omega$ in M, with $h(n) > n$ such that whenever $E \cap [n, h(n))$ is partitioned into $2k$ many pieces, at least one piece (name it r) has the following properties:

- $\forall i < k$ there is $t \subseteq r$ such that $s_i \cup t$ is permitted by A_i.
- $\forall i < k$ there is $t \subseteq r$ such that $t_i \cup t$ is permitted by some p in the antichain $B_n^{\dot{f}_i}$ with $g_n^{\dot{f}_i}(p) < h(n)$.

Proof: of subclaim:

Assume towards a contradiction there is a counterexample to the subclaim, say n. This means there is no **no h(n)** satisfying the conditions above.

We will choose for this arbitrary n, $h(n)$ in such a way that the conditions in the subclaim are satisfied, therefore, we get a contradiction.

Consider $E \setminus n$.

By a compactness argument or equivalently König's Lemma, we could partition $E \setminus n$ into $2k$ pieces, none satisfying the conclusions of the subclaim, since:

Knowing that for all possible values for $h(n)$ there is one partition of $E \cap [n, h(n))$ into $2k$ many pieces such that no piece of the partition satisfies the conditions in the subclaim, applying the compactness argument, we get the existence of a partition of $E \setminus n$ into $2k$ many pieces, none satisfy the conditions in the subclaim (a coherent partition).

Whenever we have a partition of $E \setminus n$, one of the pieces must be in the ultrafilter $\mathcal{U}$, since we know $E \in \mathcal{U}$. The fact that $E \in \mathcal{U}$ implies $E \setminus n \in \mathcal{U}$ since $E \cap n$ is not in $\mathcal{U}$, and we know that, whenever we decompose a set in the ultrafilter as the disjoint union of two other sets, one of them must also be in the ultrafilter. Denote the piece which is in $\mathcal{U}$ by X.

A_i is m.a.c. in M, therefore, (s_i, X) is compatible with one of the conditions, say p in A_i, *i.e.* there is a common extension. But this just means there is $t \subseteq X$ such that $s_i \cup t$ is permitted by p.

Also, $B_n^{\dot{f}_i}$ is m.a.c. in M, therefore, (t_i, X) is compatible with one of its elements, say q. Thus, there is $t \subseteq X$ such that $t_i \cup t$ is permitted by q.

Note that we need $h(n)$ large enough such that the finite piece r as in the subclaim is in $E \cap [n, h(n))$ and $\dot{f}(n) < h(n)$. X could be infinite, but we can choose r a finite subset of X satisfying the same requirements since we only have to cover $2k$ many sets and to dominate k many values $\dot{f}(n)$.

Therefore, choosing $h(n)$ large enough, $E \cap [n, h(n))$ has the desired properties.

■

To conclude the **Claim** :

Fix n.

Consider the partition given by $\{C_i \cap [n, h(n)), D_i \cap [n.h(n)) : i < k\}$ (this is where the pairwise disjointness assumption is needed).

Consider a piece of the form $C_i \cap [n, h(n))$.

There is no $t \subseteq C_i \cap [n, h(n))$ with $s_i \cup t$ permitted by A_i, since (s_i, C_i) is incompatible with all members of A_i (recall that C_i's were forbidden by A_i, s_i).

So, such a piece does not have the properties of r in the subclaim.

Therefore, there must be a piece of the form $D_i \cap [n, h(n))$ as in the subclaim.

Subclaim implies there is $t \subseteq D_i \cap [n, h(n))$ such that $t_i \cup t$ is permitted by some p in the antichain $B_n^{\dot{f}_i}$ such that $g_n^{\dot{f}_i}(p) < h(n)$.

On the other hand, (t_i, D_i) is incompatible with all conditions q in the antichain $B_n^{\dot{f}_i}$ such that $g_n^{\dot{f}_i}(q) < c(n)$. Since p is not such a q, we have $g_n^{\dot{f}_i}(p) \geq c(n)$.

This two last results imply $c(n) < h(n)$, so after unfixing n the unboundedness of c over M is contradicted.

■

Chapter 5

The matrix iteration for $Con(\mathfrak{b} = \aleph_1 < \mathfrak{s} = \kappa)$

The goal of this chapter is to prove the following theorem:

Theorem 5.1 The Blass-Shelah model, (See [20])

Let $\kappa = \kappa^\omega$ be a regular, uncountable cardinal. It is consistent that $\mathfrak{s} = \mathfrak{c} = \kappa$ and $\mathfrak{b} = \aleph_1$.

The idea of the proof is to start with a finite support iteration on length $\mu = \aleph_1$ of Cohen forcing, and then, in the obtained model, to iterate Mathias Prikry forcing with finite support, for some appropriate ultrafilters. Actually, at each level $\gamma < \omega_1$ there will be a finite support iteration of Mathias Prikry forcing. The ultrafilters $\dot{\mathcal{U}}_\alpha^\gamma$ in this iteration will be the ultrafilters given by Lemma 4.11 in previous chapter. Such non-linear iterations are called matrix iterations.

We will start by explaining what happens in the successor step in the matrix iteration described above.

5.1 Zoom into the iteration - the successor step

The method of matrix iterations appeared in the literature in the 1989 paper "Ultrafilters with small generating sets", written by S.Shelah and A.Blass. In this paper, they established $Con(\mathfrak{u} = \kappa < \mathfrak{d} = \lambda)$ for arbitrary regular, uncountable cardinals κ and λ, where $\mathfrak{u}$ is the invariant called "the ultrafilter number". A.Blass specifies in the paper that the presented technique was actually developed by Shelah in 1984 and that his contribution was to fill in the details and write the paper.

Recall the general result about preservation of complete embeddability (Lemma 4.2), presented in the previous chapter:

Given $\mathbb{P}, \mathbb{Q}$ be partial orders, $\mathbb{P} < \circ\ \mathbb{Q}$, $\dot{\mathbb{A}}$ a $\mathbb{P}$-name for a partial order, $\dot{\mathbb{B}}$ a $\mathbb{Q}$-name for a partial order such that $\Vdash_{\mathbb{Q}} \dot{\mathbb{A}} \subseteq \dot{\mathbb{B}}$ and all maximal antichains of $\dot{\mathbb{A}}$ in $V^{\mathbb{P}}$ are maximal antichains of $\dot{\mathbb{B}}$ in $V^{\mathbb{Q}}$ we can conclude $\mathbb{P}*\dot{\mathbb{A}} < \circ\ \mathbb{Q}*\dot{\mathbb{B}}$.

The main lemma of the previous section gave the construction of an ultrafilter $\mathcal{V}$ extending $\mathcal{U}$ (which implies $\mathbb{M}_{\mathcal{U}} \subseteq \mathbb{M}_{\mathcal{V}}$), such that maximal antichains of $\mathbb{M}_{\mathcal{U}}$ in M are preserved to be maximal antichains of $\mathbb{M}_{\mathcal{V}}$ in N (denoted by $\mathbb{M}_{\mathcal{U}} < \circ_M \mathbb{M}_{\mathcal{V}}$).

So we can plug $\dot{\mathbb{M}}_{\mathcal{U}}$ and $\dot{\mathbb{M}}_{\mathcal{V}}$ as $\dot{\mathbb{A}}$ and $\dot{\mathbb{B}}$ in the above lemma to get the following:

$$
\begin{array}{ccccccc}
 & & & \mathbb{M}_{\mathcal{V}} & & & \\
 & & N = V^{\mathbb{Q}} & \longrightarrow & N[m_{\mathcal{V}}] & & \\
 & \mathbb{Q} & & & & \mathbb{Q} * \mathbb{M}_{\mathcal{V}} & \\
c \in N & \circ & & & & \circ & c \text{ remains unbounded} \\
\text{unbounded over } M & \vee & & & & \vee & \text{over } M[m_{\mathcal{U}}] \\
 & \mathbb{P} & & & & \mathbb{P} * \mathbb{M}_{\mathcal{U}} & \\
 & & & \mathbb{M}_{\mathcal{U}} & & & \\
 & & M = V^{\mathbb{P}} & \longrightarrow & M[m_{\mathcal{U}}] & &
\end{array}
$$

The main lemma also assures that the given unbounded real c remains unbounded over the extension by the $\mathbb{M}_{\mathcal{U}}$-generic real $m_{\mathcal{U}}$ constructed from the $\mathbb{M}_{\mathcal{V}}$-generic real $m_{\mathcal{V}}$. Thus, starting with $\mathbb{P} <\circ\ \mathbb{Q}$, we can add a Cohen real over $V^{\mathbb{P}}$ just by choosing $\mathbb{Q}$ to be the Cohen forcing.

Considering $M = V^{\mathbb{P}}$ and $N = V^{\mathbb{Q}}$, the crucial lemma in tells us that, given any ultrafilter $\mathcal{U} \in M = V^{\mathbb{P}}$, we can construct an ultrafilter $\mathcal{V} \in N = V^{\mathbb{Q}}$ such that, when we do the two step iteration with the corresponding Mathias Prikry forcing, we preserve the complete embeddability ($\mathbb{P} * \mathbb{M}_{\mathcal{U}} <\circ\ \mathbb{Q} * \mathbb{M}_{\mathcal{V}}$) and the unbounded real c.

For the first step we take $\mathbb{P}$ to be the trivial forcing $\{\mathbf{1}\}$, $\mathbb{Q}$ to be the Cohen forcing $\mathbb{C}$, and $\mathcal{U}$ an arbitrary ultrafilter in $V^{\mathbb{P}}$. We will construct an ultrafilter $\mathcal{V}$.

For the second step, $\mathbb{P}$ will be the old $\mathbb{Q}$ and the new $\mathbb{Q}$ will again be Cohen forcing $\mathbb{C}$ and the starting ultrafilter will be the $\mathcal{V}$ we just constructed. Thus, we vertically do a finite support iteration of Cohen forcing and we build an ascending sequence of ultrafilters with the property that the two step iteration with Mathias Prikry with the corresponding ultrafilters preserve the complete embeddability and the Cohen reals unbounded.

The fact that we still have complete embeddability an unbounded reals allows us to repeat the process. Thus, we can choose in $M[m_{\mathcal{U}}]$ another ultrafilter containing $m_{\mathcal{U}}$ and apply the above again. Thus, horizontally we will do a finite support iteration, every time forcing with $\mathbb{M}_{\mathcal{U}}$ for the corresponding ultrafilter.

5.2 The iteration for $Con(\mathfrak{b} = \aleph_1 < \mathfrak{s} = \mathfrak{c} = \kappa)$

$$
\begin{array}{ccccccccccc}
 & *\mathbb{M}_{\mathcal{U}_0} & & *\,\mathbb{M}_{\mathcal{U}_1} & & *\,\mathbb{M}_{\mathcal{U}_2} & & *\,\mathbb{M}_{\mathcal{U}_\beta} & & *\,\mathbb{M}_{\mathcal{U}_\alpha} & *\,\mathbb{M}_{\mathcal{U}_\kappa} \\
\mathbb{P}_0 & - & \mathbb{P}_1 & - & \mathbb{P}_2 & - - - & \mathbb{P}_\beta & - & \mathbb{P}_\alpha & - - - & \mathbb{P}_\kappa \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & *\mathbb{M}_{\mathcal{U}_0^\gamma} & & *\,\mathbb{M}_{\mathcal{U}_1^\gamma} & & *\,\mathbb{M}_{\mathcal{U}_2^\gamma} & & *\,\mathbb{M}_{\mathcal{U}_\beta^\gamma} & & *\,\mathbb{M}_{\mathcal{U}_\alpha^\gamma} & *\,\mathbb{M}_{\mathcal{U}_\kappa^\gamma} \\
\mathbb{P}_0^\gamma & - & \mathbb{P}_1^\gamma & - & \mathbb{P}_2^\gamma & - - - & \mathbb{P}_\beta^\gamma & - & \mathbb{P}_\alpha^\gamma & - - - & \mathbb{P}_\kappa^\gamma \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & *\mathbb{M}_{\mathcal{U}_0^\delta} & & *\,\mathbb{M}_{\mathcal{U}_1^\delta} & & *\,\mathbb{M}_{\mathcal{U}_2^\delta} & & *\,\mathbb{M}_{\mathcal{U}_\beta^\delta} & & *\,\mathbb{M}_{\mathcal{U}_\alpha^\delta} & *\,\mathbb{M}_{\mathcal{U}_\kappa^\delta} \\
\mathbb{P}_0^\delta & - & \mathbb{P}_1^\delta & - & \mathbb{P}_2^\delta & - - - & \mathbb{P}_\beta^\delta & - & \mathbb{P}_\alpha^\delta & - - - & \mathbb{P}_\kappa^\delta \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & *\mathbb{M}_{\mathcal{U}_0^1} & & *\,\mathbb{M}_{\mathcal{U}_1^1} & & *\,\mathbb{M}_{\mathcal{U}_2^1} & & *\,\mathbb{M}_{\mathcal{U}_\beta^1} & & *\,\mathbb{M}_{\mathcal{U}_\alpha^1} & *\,\mathbb{M}_{\mathcal{U}_\kappa^1} \\
\mathbb{P}_0^1 & - & \mathbb{P}_1^1 & - & \mathbb{P}_2^1 & - - - & \mathbb{P}_\beta^1 & - & \mathbb{P}_\alpha^1 & - - - & \mathbb{P}_\kappa^1 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & *\mathbb{M}_{\mathcal{U}_0^0} & & *\,\mathbb{M}_{\mathcal{U}_1^0} & & *\,\mathbb{M}_{\mathcal{U}_2^0} & & *\,\mathbb{M}_{\mathcal{U}_\beta^0} & & *\,\mathbb{M}_{\mathcal{U}_\alpha^0} & *\,\mathbb{M}_{\mathcal{U}_\kappa^0} \\
\mathbb{P}_0^0 & - & \mathbb{P}_1^0 & - & \mathbb{P}_2^0 & - - - & \mathbb{P}_\beta^0 & - & \mathbb{P}_\alpha^0 & - - - & \mathbb{P}_\kappa^0
\end{array}
$$

Recall the **theorem** we want to prove:

Let $\kappa = \kappa^\omega$ be regular, uncountable. It is consistent that $\mathfrak{s} = \mathfrak{c} = \kappa$ and $\mathfrak{b} = \aleph_1$.

Proof:

Recall the Idea : matrix iteration with $\aleph_1 \times \kappa$ and ultrafilters given by the crucial lemma.

Let $\kappa = \kappa^\omega$ be regular, uncountable.

Add $\aleph_1$ many Cohen reals $c_\gamma : \gamma < \omega_1$ over V(via $\mathbb{C}_{\aleph_1}$, the finite support iteration of Cohen forcing of length $\aleph_1$), since we need to have the unbounded reals that are going to be preserved unbounded when applying lemma.

For $\gamma < \omega_1$, denote by V_γ the model resulting after adding the first γ many of the Cohen reals (*i.e.* $V_\gamma = V[\ \{c_\delta : \delta < \gamma\}\ ]$ and $V_0 = V$).

In the model V_{ω_1}, perform a finite support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \kappa\rangle$ such that

$$\text{for all } \alpha, \Vdash_\alpha \text{"}\dot{\mathbb{Q}}_\alpha \cong \dot{\mathbb{M}}_{\mathcal{U}_\alpha} \text{ for some appropriate } \dot{\mathcal{U}}_\alpha\text{"}.$$

Actually, for each $\gamma < \omega_1$ we also have, in V_γ, a finite support iteration $\langle \mathbb{P}^\gamma_\alpha, \dot{\mathbb{Q}}^\gamma_\alpha : \alpha < \kappa\rangle$ such that

$$\text{for all } \alpha, \Vdash^\gamma_\alpha \text{"}\dot{\mathbb{Q}}^\gamma_\alpha \cong \dot{\mathbb{M}}_{\mathcal{U}^\gamma_\alpha} \text{ for some appropriate } \dot{\mathcal{U}}^\gamma_\alpha\text{"},$$

in such a way that:

(1) $\mathbb{P}_\alpha \Vdash$ "$\dot{\mathcal{U}}_\alpha$ is the increasing union of $\dot{\mathcal{U}}^\gamma_\alpha$" ,

(2) for $\alpha \leq \kappa, \gamma < \omega_1$, maximal antichains of $\mathbb{P}^\gamma_\alpha$ in V_γ are maximal of $\mathbb{P}_{\omega_1}$ in V_{ω_1} and

(3) for $\alpha \leq \kappa, \gamma < \omega_1$, in V_{ω_1},whenever $\dot{f} \in V_\gamma$ is a $\mathbb{P}^\gamma_\alpha$-name for a function in ω^ω, we have $\Vdash_\alpha c_\gamma \not\leq^* \dot{f}$.

Recursion on α:

a) $\alpha = 0$. There is nothing to show.

b) $\alpha = \beta + 1$:

Assume we have $\mathbb{P}_\beta$ and $\mathbb{P}^\gamma_\beta$ satisfying (1.) – (3.).

(2) for β implies the downwards absoluteness of genericity:

if G_β is $\mathbb{P}_\beta$-generic over V_{ω_1}, it is also $\mathbb{P}^\gamma_\beta$-generic over V_γ for all γ.

Work in $V_{\omega_1}[G_\beta]$.

Recursively build ultrafilters $\mathcal{U}^\gamma_\beta$ in $V_{\omega_1}[G_\beta]$.

- $\gamma = 0$:

  Let $\mathcal{U}^0_\beta$ be an arbitrary ultrafilter of $V_0[G_\beta]$.

- $\gamma = \delta + 1$

  Apply the lemma giving the construction of an ultrafilter with

  $$M = V_\delta[G_\beta] \text{ and } N = V_{\delta+1}[G_\beta].$$

  The assumption (3) for β states that $c_\delta \in N$ is still unbounded over M.

  Hence, the lemma gives the existence of an ultrafilter $\mathcal{U}^\gamma_\beta (= \mathcal{U}^{\delta+1}_\beta)$ extending $\mathcal{U}^\delta_\beta$ such that maximal antichains of $\mathbb{M}_{\mathcal{U}^\delta_\beta}$ in M remain maximal antichains of $\mathbb{M}_{\mathcal{U}^{\delta+1}_\beta}$ in N and for all $\dot{f}$ $\mathbb{M}_{\mathcal{U}^\delta_\beta}$-name for a real in M, $\Vdash_{\mathbb{M}_{\mathcal{U}^\gamma_\beta}} c_\delta \not\leq^* \dot{f}$.

– For γ limit:

If $cf(\gamma) \geq \omega_1$, then $\mathcal{U}^\gamma_\beta = \bigcup_{\delta<\gamma} \mathcal{U}^\delta_\beta$.

Claim: This is again an ultrafilter.

Proof of claim:

Assume we have X such that neither X nor its complement $\omega \setminus X$ is in $\dot{\mathcal{U}}_\alpha$.

Note that we can go down to $\mathcal{U}^\gamma_\alpha$, since we know that every real in the final model can be found in an intermediate model, since no new reals are added at limit stages of uncountable cofinality($V^\mu_\alpha \cap \omega^\omega = \bigcup_{\gamma<\mu} V^\gamma_\alpha \cap \omega^\omega$).

But surely either X or its complement $\omega \setminus X$ is in $\mathcal{U}^\gamma_\alpha$, since $\mathcal{U}^\gamma_\alpha$ is an ultrafilter.

Therefore, it is in $\mathcal{U}_\alpha$.

■

If $cf(\gamma) = \omega$, then extend $\bigcup_{\delta<\gamma} \mathcal{U}^\delta_\beta$ to an ultrafilter $\mathcal{U}^\gamma_\beta$ in $V_\gamma[G_\beta]$ such that:

$\forall \delta < \gamma$ all m.a.c. $A \subseteq \mathbb{M}_{\mathcal{U}^\delta_\beta}$, $A \in V_\delta[G_\beta]$ are maximal in $\mathbb{M}_{\mathcal{U}^\gamma_\beta}$.

This time it is not an ultrafilter anymore since new reals may be added, so we can not go down to some $\mathcal{U}^\gamma_\alpha$ as before. The argument for the fact that we can always extend it to an ultrafilter is similar to crucial lemma, but simpler.

This completes the construction of $\mathcal{U}^\gamma_\beta$, thus, of $\mathbb{Q}_\alpha$ for successor α.

It remains to argue that (2) and (3) hold.

For (2): If $G_\beta * H$ is $\mathbb{P}_\beta * \dot{\mathbb{Q}}_\alpha$-generic over V_{ω_1}(that is $\mathbb{P}_\alpha$-generic over V_{ω_1}) then, by construction, it is also $\mathbb{P}^\gamma_\alpha$-generic over V_γ for all $\gamma < \omega_1$.

Similar argument for (3).

c) α limit, then:

if $cf(\alpha) \geq \omega_1$, then there is nothing to show:

Given a m.a.c $A \subseteq \mathbb{P}^\gamma_\alpha$ in V_γ, there is a $\beta < \alpha$ such that $A \subseteq \mathbb{P}^\gamma_\beta$ (we know the forcing is c.c.c., thus, A is countable) and A is a maximal antichain in $\mathbb{P}_\beta$ (by induction hyp. (2)), therefore, A is maximal in $\mathbb{P}_\alpha$ too.

Similar argument for (3).

Case $cf(\alpha) = \omega$.

Let $A \subseteq \mathbb{P}^\gamma_\alpha$ be a m.a.c. in V_γ. We need to show that A is still a m.a.c. of $\mathbb{P}_\alpha$.

Consider any condition $p \in \mathbb{P}_\alpha \cap V_{\omega_1}$.

We know there is a $\beta < \alpha$ such that $p \in \mathbb{P}_\beta$ (just by def. of f.s.i: the support is finite, bounded by α, therefore, $\exists \beta < \alpha$ with $p \in \mathbb{P}_\beta$).

We can look at the iteration using the quotient forcing and see $\mathbb{P}^\gamma_\alpha$ as $\mathbb{P}^\gamma_\beta * \dot{\mathbb{R}}^\gamma_{[\beta,\alpha)}$.

Thus, conditions in $\mathbb{P}^\gamma_\alpha$ can thus, be seen as pairs $(q, \dot{r})$ in $\mathbb{P}^\gamma_\beta * \dot{\mathbb{R}}^\gamma_{[\beta,\alpha)}$.

Let $A_\beta := \{q : \exists\ \dot{r}$ such that $(q, \dot{r}) \in A\}$ (the projection of A to $\mathbb{P}^\gamma_\beta$).

Then $A_\beta \subseteq \mathbb{P}^\gamma_\beta \cap V_\gamma$ is predense (it is not necessary an antichain, but maximal).

The induction hypothesis for (2) says that m.a.c. are preserved, therefore, in particular, predense sets are preserved predense. Thus, just by definition of predense, there is $q \in A_\beta$ with $q||p$.

Choose $p' \in \mathbb{P}_\beta$ to be their common extension.

Let $\dot{r}$ be such that $(q, \dot{r}) \in A$. We see $(p', \dot{r})$ is a common extension of $(q, \dot{r})$ and $p = (p, 1)$.

Thus, A still m.a.c. in $\mathbb{P}_\alpha$.

We still need to show that **c remains unbounded.**

Let $\dot{f}$ be a $\mathbb{P}^\gamma_\alpha$ name for a real in V_γ.

For $\beta < \alpha$ we have a $\mathbb{P}^\gamma_\beta$-name $\dot{f}_\beta$ in V_γ, such that

$$\Vdash^\gamma_\beta \ \exists \ \langle \dot{r}^n_\beta : n \in \omega \rangle \text{ decreasing, } \dot{r}^n_\beta \in \mathbb{R}^\gamma_{[\beta,\alpha)} \text{ such that } \dot{r}^n_\beta \ \Vdash^\gamma_{\beta,\alpha} \dot{f}_\beta \restriction n = \dot{f} \restriction n.$$

By induction hypothesis for (3): $\Vdash_\beta c_\gamma \not\leq^* \dot{f}_\beta$.

Assume towards contradiction, there are $p \in P_\alpha$ and $n_0 \in \omega$ such that $p \Vdash_\alpha c_\gamma(n) \leq \dot{f}(n)$ for all $n \geq n_0$.

Choose $\beta < \alpha$ such that $p \in \mathbb{P}_\beta$.

By the above we find $n \geq n_0$ and $q \in \mathbb{P}_\beta$ such that $q \leq p$ and $q \Vdash_\beta c_\gamma(n) > \dot{f}_\beta(n)$.

Then, $\quad q \Vdash_\beta \quad "\dot{r}^{n+1}_\beta \ \ \Vdash_{\beta,\alpha} \ \ \dot{f}(n) = \dot{f}_\beta(n) < c_\gamma(n)"$.

Note that $(q, \dot{r}^{n+1}_\beta) \leq p$, but they force contradictory statements.

This shows (3) for α.

Let $\mathcal{U} = \bigcup_{\alpha<\kappa} \mathcal{U}_\alpha$ in $V_{\omega_1}[G_\kappa]$.

$\mathfrak{b} = \aleph_1$ is witnessed by the $\aleph_1$-many Cohen reals and $\mathfrak{s} = \kappa$ by construction. The proofs are similar to the ones presented in Chapter 6, in Lemma 6.15.

■

One may ask why we could not get $Con(\mathfrak{b} = \lambda < \mathfrak{s} = \kappa)$ for $\lambda < \kappa$ arbitrary regular, uncountable cardinals using the above arguments. The problem is that Cohen reals are added in limit stages of countable cofinality during finite support iterations, so $\mathfrak{b} = \aleph_1$ is witnessed, even if the iteration is longer.

Chapter 6

The matrix iteration for $Con(\mathfrak{b} = \mathfrak{a} = \kappa < \mathfrak{s} = \lambda)$

This consistency result first appeared in 2011, in the paper "Mad families, splitting families, and large continuum", written by V. Fischer and J. Brendle ([10]). The result holds in a model, constructed using a matrix iteration. The idea of the construction is a generalization of the extension of the ultrafilters for Mathias forcing, presented in the previous chapter. There, it was argued that an iteration longer than $\aleph_1$ can't be used to get $Con(\mathfrak{b} = \kappa < \mathfrak{s} = \lambda)$, for κ, λ regular, uncountable cardinals. To be able to get this result, the cardinal invariant $\mathfrak{a}$, called the almost disjointness number, must be treated as well. $\mathfrak{b} \leq \mathfrak{a}$ is a ZFC result, while $\mathfrak{a} \leq \mathfrak{b}$ will hold in the constructed model.

6.1 A forcing notion for adding a mad family

The forcing notion presented in this section was developed by S. Hechler in 1971 (see [14]) . He used it to show the existence of models with arbitrary large maximal almost disjoint families.

Definition 6.1 *Let* $P_\gamma := \{p : F_p \times n_p \to 2$, *where* $F_p \in [\gamma]^{<\omega}$ *and* $n_p \in \omega\}$, *ordered by:* $p \leq q$ *iff* $q \subseteq p$ *and* $\forall i \in n_q \setminus n_p : |q^{-1}(1) \cap F_p \times \{i\}| \leq 1$.

Lemma 6.2 P_γ *is ccc.*

Proof:

Assume towards a contradiction, an uncountable antichain exists, *i.e.* uncountably many incompatible conditions $\{p_i : i < \omega_1\}$, therefore, one has uncountably many domains $F_p \times n_p$.

W.l.o.g we can refine to the same n, since there are only countably many $n \in \omega$, so the incompatibility must be related to the F_p's.

Therefore, we have F_{p_i}'s form an uncountable family of finite subsets of ω.

By the Δ-system Lemma this family of finite sets has an uncountable subfamily such that the sets in the subfamily intersect to a finite F.

For the conditions to be incompatible, they have to have different values on F, which is impossible, since there aren't uncountably many possible values. So an uncountable antichain cannot exist.

■

Let G be a $\mathbb{P}_\gamma$-generic filter. Then, for all $\alpha < \gamma$, define

$$A_\alpha := \{i : \exists p \in G : p(\alpha, i) = 1\}.$$

Definition 6.3 The Quotient forcing:

Let $\gamma < \delta$ *be an ordinal and let* G *be a* P_γ *generic filter over the ground model* V.

In $V[G]$, *define* $P_{[\gamma,\delta)}$ *to be the poset of all pairs* (p, H), *where*

- H *is a finite subset of* γ

- $p : F_p \times n_p \to 2$, *where* $F_p \in [\delta \setminus \gamma]^{<\omega}$ *and* $n_p \in \omega$.

The order: $(q, K) \leq (p, H)$ *if*

- $q \leq_{P_\delta} p$
- $H \subseteq K$
- $\forall \alpha \in F_p$, $\beta \in H$ *and* $i \in n_q \setminus n_p$:, *if* $i \in A_\beta$, *then* $q(\alpha, i) = 0$
  (i.e. for all $\alpha \in F_p$, $\beta \in H$, *then* $p \Vdash \dot{A}_\alpha \cap \dot{A}_\beta \subseteq n_p$*).*

REMARK 6.4 $P_\delta = P_\gamma * \dot{P}_{[\gamma,\delta)}$, *i.e.* P_δ is forcing equivalent with the two step iteration of P_γ and $\dot{P}_{[\gamma,\delta)}$.

Fact: 1.) if $p \in P_\gamma$, then for all $\alpha \in F_p$ $(p \Vdash \dot{A}_\alpha \restriction n_p = p \restriction \{\alpha\} \times n_p)$
Fact: 2.) for all $\alpha, \beta \in F_p (p \Vdash \dot{A}_\alpha \cap \dot{A}_\beta \subseteq n_p)$.

Lemma 6.5 *Let* G *a* P_γ*-generic filter and for all* $\alpha < \gamma$, *let* A_α *be as before. Then:*

1) P_γ *adds an a.d. family, namely,* $\mathcal{A}_\gamma := \{A_\alpha : \alpha < \gamma\}$.

2) for $\gamma \geq \omega_1$, *regular,* $\mathcal{A}_\gamma$ *is a m.a.d. family.*

Proof:

1). The almost disjointness:

Assume towards a contradiction that the $\mathcal{A}_\gamma$ family is not almost disjoint. Hence, $\exists p \in G$ a condition and $\alpha, \beta \in F_p$ such that $p \Vdash A_\alpha \cap A_\beta$ is infinite. Thus, for all n, in particular, for n_p, we know $p \Vdash \exists i > n_p : i \in A_\alpha \cap A_\beta$.

By a basic property of forcing, there must be an extension $q \leq p$ and $i_0 > n_p$ such that

$$q \Vdash i_0 \in A_\alpha \cap A_\beta.$$

Choose $q \in G$ such that $q(\alpha, i_0) = q(\beta, i_0) = 1$.

Since p and q are both in the generic filter G, they have to be compatible, therefore, a common extension, say $r \in G$, has to exist.

Since $r \leq q$ one gets $r(\alpha, i_0) = r(\beta, i_0) = 1$, but $\alpha, \beta \in F_p$ and $i_0 > n_p$, thus, $|r^{-1}(1) \cap F_p \times \{i_0\}| \geq 2$, contradicting $r \leq p$.

2).The maximality for $\gamma \geq \omega_1$:

Let G be a P_γ-generic filter over V.

One has to show that for all $\mathbb{P}\gamma$-names $\dot{C}$ for infinite subsets of ω: $|\dot{C} \cap \dot{A}_\alpha| = \omega$ for all $\alpha < \gamma$.

Since $\gamma \geq \omega_1$ is regular (by assumption), $\dot{C}$ is actually a P_δ-name, for some $\delta < \gamma$.

Since $P_\gamma = P_\delta * \dot{P}_{[\delta,\gamma)}$, we can consider V^{P_γ} as the ground model and only look at the quotient, thus G may be assumed to be a $P_{[\delta,\gamma)}$-generic. Thus, we have to show:

$$\forall C \in V : V[G] \models |C \cap \dot{A}_\alpha| = \aleph_0 \text{ for some } \alpha < \gamma$$
$$\forall C \in V : \exists p \in G : p \Vdash |C \cap \dot{A}_\alpha| = \aleph_0 \text{ for some } \alpha < \gamma.$$

Since the set

$$D_{C,\alpha,k} := \{q \in \mathbb{P}_\gamma : \exists i \in n_q, i \geq k \text{ such that } C \restriction [i, n_q] = q \restriction [i, n_q] \times \{\alpha\}\}$$

is dense, any C infinite subset of ω, will meet all A_α's infinitely often and thus, in particular, the almost disjoint family is maximal.

■

6.2 Another crucial lemma

The aim of this section is to define an analogous of the crucial lemma related to the extension of ultrafilters for Mathias forcing, presented in the previous chapter. Recall that this lemma gave a construction of an ultrafilter $\mathcal{V}$ extending any given ultrafilter $\mathcal{U}$, in such a way that any maximal antichain $A \subseteq \mathbb{M}_{\mathcal{U}}$ in M is maximal antichain of $\mathbb{M}_{\mathcal{V}}$ in N and, moreover, a given unbounded real c ($c \in N$, unbounded over M) remains unbounded over $M[m_{\mathcal{U}}]$, where $m_{\mathcal{U}}$ is the $\mathbb{M}_{\mathcal{U}}$-generic real constructed from the $\mathbb{M}_{\mathcal{V}}$-generic real $m_{\mathcal{V}}$.

In this section, following [10], we will also be interested to preserve another combinatorial property, this time related to the almost disjointness number $\mathfrak{a}$, to ensure both $\mathfrak{b}$ and $\mathfrak{a}$ are small, while $\mathfrak{s}$ is increased.

Definition 6.6 *Let $M \subseteq N$ be models of ZFC, $\mathcal{B} = \{B_\alpha : \alpha < \gamma\} \subseteq M \cap [\omega]^\omega$, $A \in N \cap [\omega]^\omega$.*

Then $\begin{pmatrix} M, N \\ * \\ \mathcal{B}, A \end{pmatrix}$ holds if for every $h : \omega \times [\gamma]^{<\omega} \to \omega$, $h \in M$ and for every $m \in \omega$, there are $n \geq m$ and $F \in [\gamma]^{<\omega}$, such that $[n, h(n, F)) \setminus \bigcup_{\alpha \in F} B_\alpha \subseteq A$.

Lemma 6.7 *Let $M \subseteq N$ be models of ZFC, $\mathcal{B} = \{B_\alpha : \alpha < \gamma\} \subseteq M \cap [\omega]^\omega$ and $A \in N \cap [\omega]^\omega$.Let $\mathcal{I}(\mathcal{B})$ be the ideal generated by the finite sets and the elements of $\mathcal{B}$.*

If $\begin{pmatrix} M, N \\ * \\ \mathcal{B}, A \end{pmatrix}$ holds and $B \in M \cap [\omega]^\omega$, $B \notin \mathcal{I}(\mathcal{B})$, then $|A \cap B| = \aleph_0$.

Proof: Assume towards a contradiction, there is $B \notin \mathcal{I}(\mathcal{B})$ with $|A \cap B| < \aleph_0$. Hence, there must be an $n \in \omega$ such that $A \cap B \subseteq n$.

Let $m \geq n$ and F a finite subset of γ.

The union $\bigcup_{\alpha \in F} B_\alpha$ consists of finitely many sets in the ideal $\mathcal{I}(\mathcal{B})$, so it is itself in the ideal.

Since $\mathcal{I}(\mathcal{B})$ was defined as the ideal generated by sets in $\mathcal{B}$ and the finite sets, whenever X is a set in $\mathcal{I}(\mathcal{B})$, $Y \subseteq^* X$ implies that Y is also in $\mathcal{I}(\mathcal{B})$.

Thus, since our hypothesis says that $B \notin \mathcal{I}(\mathcal{B})$, it follows that B can't be almost included in a set in the ideal, in particular, $B \not\subseteq^* \bigcup_{\alpha \in F} B_\alpha$. But this just means $B \setminus \bigcup_{\alpha \in F} B_\alpha$ is infinite, hence, it has an element grater than the given m.

Denote this this element by $k_{m,F}$.

Define $h(m,F) := \begin{cases} k_{m,F}+1 & \text{for all } m \geq n \text{ and for all } F \in [\gamma]^{<\omega} \\ 0 & \text{if } m < n \end{cases}$

The function h is in M and clearly $[m, h(m,F)) \setminus \bigcup_{\alpha \in F} B_\alpha \nsubseteq A$ for all $m \geq n, F \in [\gamma]^{<\omega}$, contradicting $\begin{pmatrix} & M, N \\ * & \\ & \mathcal{B}, A \end{pmatrix}$. Thus, $A \cap B$ can't be finite.

■

The following lemma states, that the A_α added by P_γ satisfy the $*$-property, in the following sense:

Lemma 6.8 *Let $G_{\gamma+1}$ be $P_{\gamma+1}$-generic filter, $G_\gamma = G_{\gamma+1} \cap P_\gamma$ and $\mathcal{A}_\gamma = \{A_\alpha : \alpha < \gamma\}$, where for all $\alpha < \gamma$, $A_\alpha := \{i : \exists p \in G : p(\alpha, i) = 1\}$.*

Then $\begin{pmatrix} & V[G_\gamma], V[G_{\gamma+1}] \\ * & \\ & \mathcal{A}_\gamma \quad , \quad A_\gamma \end{pmatrix}$ holds.

Proof:

Let $h : \omega \times [\gamma]^{<\omega} \to \omega$ in $V[G_\gamma]$

Let (p, H) be a condition in $\mathbb{P}_{[\gamma,\gamma+1]}$ and let m be a natural number.

Then $\operatorname{dom}(p) = \gamma \times n_p$, where $n_p \in \omega$.

Define an extension $(q, K) \leq_{\mathbb{P}_{[\gamma,\gamma+1]}} (p, H)$:

Let n be a natural number, grater than both n_p and m.

The condition q will have $n_q := h(n, H)$ and, as finite subset, $\{\gamma\}$.

Thus, $\operatorname{dom}(q) = \{\gamma\} \times n_q$.

Let $K = H$.

$$\text{Define } q(\gamma, i) := \begin{cases} p(\gamma, i) & \text{if } i < n_p \\ 0 & \text{if } i \in [n_p, n) \\ 1 & \text{if } i \in [n, n_q) \text{ and } i \notin \bigcup_{\alpha \in H} A_\alpha \\ 0 & \text{if } i \in [n, n_q) \text{ and } i \in \bigcup_{\alpha \in H} A_\alpha. \end{cases}$$

The condition (q, K), as defined above, is an extension of (p, H) and moreover,

$$(q, K) \Vdash [n, h(n, H)) \setminus \bigcup_{\alpha \in H} A_\alpha \subseteq A_\gamma.$$

(Just by the definition of the quotient forcing).

■

Recall the crucial lemma for the extension of ultrafilters in Blass Shelah's model:

Lemma (Blass, Shelah)

Given $M \subseteq N$ be models of ZFC, an ultrafilter $\mathcal{U}$ and $c \in N \cap \omega^\omega$ unbounded over M, there exists an ultrafilter $\mathcal{V}$ extending $\mathcal{U}$ such that

1. *any maximal antichain $A \subseteq \mathbb{M}_\mathcal{U}$ in M is maximal antichain of $\mathbb{M}_\mathcal{V}$ in N*
2. *for all $\dot{f}$ $\mathbb{M}_\mathcal{U}$-names for a function in ω^ω, $\Vdash_{\mathbb{M}_\mathcal{V}} c \not\leq^* \dot{f}$.*

In analogy, J.Brendle and V.Fischer stated and proved the following lemma:

Lemma 6.9 (Brendle, Fischer, 2011) *Given $M \subseteq N$ be models of ZFC, an ultrafilter $\mathcal{U}$, a family $\mathcal{B} = \{B_\alpha : \alpha < \gamma\} \subseteq M \cap [\omega]^\omega$ and $A \in N \cap [\omega]^\omega$ such that $\begin{pmatrix} M, N \\ * \\ \mathcal{B}, A \end{pmatrix}$ holds, there exists an ultrafilter $\mathcal{V}$ extending $\mathcal{U}$ such that:*

1. *any maximal antichain* $A \subseteq \mathbb{M}_{\mathcal{U}}$ *in* M *is maximal antichain of* $\mathbb{M}_{\mathcal{V}}$ *in* N

2. $\begin{pmatrix} & M[G], N[G] \\ * & \\ & \mathcal{B}\ ,\ A \end{pmatrix}$ *holds, where* G *is* $\mathbb{M}_{\mathcal{V}}$*-generic over* N

Proof:

Work in N:

For $A \subseteq \mathbb{M}_{\mathcal{U}}$ maximal antichain $A \in M$ and s finite subset of ω one says:

$C \subseteq \omega$ is **forbidden by** A, s if (s, C) is incompatible with all conditions in A.

These sets will assure that maximal antichains remain maximal antichains, in the sense that whenever we have a counterexample to the maximality of an antichain in M, it will be such a forbidden set in N.

Having $\dot{f}$ an $\mathbb{M}_{\mathcal{U}}$-name for a function in M, there are

- m.a.c. $B^{\dot{f}}_{n,F} \subseteq \mathbb{M}_{\mathcal{U}}$ and
- functions $g^{\dot{f}}_{n,F} : B^{\dot{f}}_{n,F} \to \omega$ (the function $g^{\dot{f}}_{n,F}$ is partitioning the antichain)

such that for each $p \in B^{\dot{f}}_{n,F} : p \Vdash_{\mathbb{M}_{\mathcal{U}}} \dot{f}(n, F) = g^{\dot{f}}_{n,F}(p)$.

For $t \in [\omega]^{<\omega}$, we say $D \subseteq \omega$ is **forbidden by** $\dot{f}, t$ if

for all n and F : (t, D) is incompatible with all conditions $p \in B^{\dot{f}}_{n,F}$ satisfying $[n, g^{\dot{f}}_{n,F}(p)) \setminus \bigcup_{\alpha \in F} B_\alpha \subseteq A$.

Intuitively: $(t, D) \Vdash_{\mathbb{M}_{\mathcal{V}}}$ for all n, F : $[n, \dot{f}(p)) \setminus \bigcup_{\alpha \in F} B_\alpha \nsubseteq A$.

This is just an intuition, since, at this stage, the above does not make sense: the ultrafilter $\mathcal{V}$ was not defined yet.

This type of forbidden sets will ensure the $*$-property in the corresponding extensions.

Take $\mathcal{I}$ the **ideal generated by the forbidden sets** (in N).

CLAIM $N \models \mathcal{U} \cap \mathcal{I} = \emptyset$.

Claim $\Rightarrow$ Lemma.

Using **Zorn's Lemma** one can construct $\mathcal{V}$ an ultrafilter extending $\mathcal{U}$ and still having empty intersection with $\mathcal{I}$.

Clearly $\mathcal{V}$ is a filter.

Claim: $\mathcal{V}$ is an **ultrafilter**.

Proof:

Assume towards a contradiction $\mathcal{V}$ is not an ultrafilter, *i.e.* $\exists X \subseteq \omega$ such that $X \notin \mathcal{V}$ and $\omega \setminus X \notin \mathcal{V}$.

We also have the ideal $\mathcal{I}$, so either $X \in \mathcal{I}$ or $X \notin \mathcal{I}$.

If $X \notin \mathcal{V}$ and $X \notin \mathcal{I}$, then $X \cap U = \emptyset$ for some $U \in \mathcal{V}$ (this has to be the reason why is not in $\mathcal{V}$).

But then $U \subseteq \omega \setminus X$, thus, $\omega \setminus X \in \mathcal{V}$, a contradiction.

If $X \in \mathcal{I}$, then $\omega \setminus X \notin \mathcal{I}$, so we can do the above argument with $\omega \setminus X$ instead of X and we get $X \in \mathcal{V}$, again a contradiction.

$\blacksquare$

Proof of Claim:

Assume towards a contradiction that $\mathcal{U} \cap \mathcal{I} \neq \emptyset$.

This implies that there are forbidden sets $C_i : i < k$ and $D_j : j < k$ such that if $E := (\bigcup_{i<k} C_i) \cup (\bigcup_{i<k} D_i)$, we must have that E is in $\mathcal{U} \cap M$. Without loss of generality, we take the same number of sets k. We only need the fact that these are finitely many to be able to apply the compactness argument.

Denote by A_i and s_i the witnesses, for the fact that the sets C_i and are forbidden, and by $\dot{f}_i$ and t_i the witnesses for D_i.

Without loss of generality:

- C_i's and D_i's are pairwise disjoint (we will build a partition) and
- $min(E) > max(s_i)$ and $min(E) > max(t_i)$ for all i.

Given $t \in [\omega]^{<\omega}$, (s, C) a condition in $\mathbb{M}_{\mathcal{U}}$ and $A \subseteq \mathbb{M}_{\mathcal{U}}$ we say that

- t is **permitted** by (s, C) if $s \subseteq t \subseteq s \cup C$.
- t is permitted by A if there is a condition in A that permits t.

The following subclaim only mentions objects in M and is absolute, therefore, we argue in M.

Subclaim:

There is a function $h : \omega \times [\gamma]^{<\omega} \to \omega$in M, with $h(n, F) > n$ such that whenever $E \cap [n, h(n, F))$ is partitioned into $2k$ many pieces, at least one piece (name it r) has the following properties:

- $\forall i < k$ there is $t \subseteq r$ such that $s_i \cup t$ is permitted by A_i.

- $\forall i < k$ there is $t \subseteq r$ such that $t_i \cup t$ is permitted by some p in the antichain $B^f_{n,F}$ with $g^f_{n,F}(p) < h(n, F)$.

Proof: of subclaim:

First note that p is compatible with (t, D) iff there is $u \subseteq D$ such that p permits $t \cup u$.

Assume towards a contradiction there is a counterexample to the subclaim, say (n, F). This means, there is no $h(n, F)$ satisfying the conditions above.

We will choose, for this arbitrary pair (n, F), the value $h(n, F)$ in such a way that the conditions in the subclaim are satisfied, therefore, we get a contradiction.

Consider $E \setminus n$.

By a compactness argument or equivalently König's Lemma, we could partition $E \setminus n$ into $2k$ pieces, none satisfying the conclusions of the subclaim, since:

Knowing that for all possible values for $h(n, F)$ there is a partition of $E \cap [n, h(n, F))$ into $2k$ many pieces such that no piece of the partition satisfies the conditions in the subclaim, applying the compactness argument, we get that there is a partition of $E \setminus n$ into $2k$ many pieces such that no piece of the partition satisfies the conditions in the subclaim.

Whenever we have a partition of $E \setminus n$, one of the pieces must be in the ultrafilter $\mathcal{U}$, since we know $E \in \mathcal{U}$. The fact that $E \in \mathcal{U}$ implies $E \setminus n \in \mathcal{U}$ since $E \cap n$ is not in $\mathcal{U}$, and we know that, whenever we decompose a set in the ultrafilter as the disjoint union of two other sets, one of them must also be in the ultrafilter. Denote the piece which is in $\mathcal{U}$ by X.

A_i is m.a.c. in M, therefore, (s_i, X) is compatible with one of the conditions, say p in A_i, *i.e.* there is a common extension. But this just means there is $t \subseteq X$ such that $s_i \cup t$ is permitted by p.

Also, $B^{\dot{f}}_{n,F}$ is m.a.c. in M, therefore, (t_i, X) is compatible with one of its elements, say q. Thus, there is $t \subseteq X$ such that $t_i \cup t$ is permitted by q.

Note that we need $h(n, F)$ large enough such that the finite piece r as in the subclaim is in $E \cap [n, h(n, F))$ and has the required property. X could be infinite, but we can choose r a finite subset of X satisfying the same requirements.

Therefore, choosing $h(n, F)$ large enough, $E \cap [n, h(n, F))$ has the desired properties.

■

To conclude the **Claim** :

Fix n and F.

Consider the partition given by $\{C_i \cap [n, h(n, F)), D_i \cap [n, h(n, F)) : i < k\}$ (this is where the pairwise disjointness assumption is needed).

Consider a piece of the form $C_i \cap [n, h(n, F))$.

There is no $t \subseteq C_i \cap [n, h(n, F))$ with $s_i \cup t$ permitted by A_i, since (s_i, C_i) is incompatible with all members of A_i (recall that C_i's were forbidden by

$A_i, s_i)$.

So, such a piece does not have the properties of r in the subclaim.

Therefore, there must be a piece of the form $D_i \cap [n, h(n,F))$ as in the subclaim.

Subclaim implies there is $t \subseteq D_i \cap [n, h(n,F))$ such that $t_i \cup t$ is permitted by some p in the antichain $B^{\dot{f}}_{n,F}$ such that $g^{\dot{f}}_{n,F}(p) < h(n,F)$.

On the other hand, (t_i, D_i) is incompatible with all conditions q in the antichain $B^{\dot{f}}_{n,F}$ such that $[n, g^{\dot{f}}_{n,F}(q)) \setminus \bigcup_{\alpha \in F} B_\alpha \subseteq A$. Since p is not such a q, we have $[n, g^{\dot{f}}_{n,F}(p)) \setminus \bigcup_{\alpha \in F} B_\alpha \nsubseteq A$.

Unfixing n and F, $\begin{pmatrix} & M, N \\ * & \\ & \mathcal{B}, A \end{pmatrix}$ is contradicted, so the assumption was false.
■

Lemma 6.10 *Let $M \subseteq N$ be models of ZFC, $\mathbb{P}$ a poset, $\mathbb{P} \subseteq M$ and G a $\mathbb{P}$-generic filter over M (thus, G is also $\mathbb{P}$-generic over N).*

If $c \in N \cap \omega^\omega$ unbounded over M, then c(which is also in $N[G]$) is unbounded over $M[G]$.

Proof:

Assume towards a contradiction c, as an element of $N[G]$ is not unbounded over $M[G]$. Thus, there is $d \in \omega^\omega \cap M[G]$ such that $\forall m \in \omega\ \exists n \geq m : N[G] \models c(n) < d(n)$.

Let $\dot{d}$ be a $\mathbb{P}$-name for d.

We want to construct a function f_0 in N, such that $\forall m \in \omega\ \exists n \geq m : c(n) < f_0(n)$.

For the $\mathbb{P}$-name $\dot{d}$, there are conditions p_n deciding the values $k_n \in \omega$ of $\dot{d}(n)$, *i.e.* $p_n \Vdash \dot{d}(n) = k_n$.

In M, define f_0 as follows:

$$f_0(n) := \begin{cases} 0, & \text{if } n < m, \\ k_n, & \text{if } n \geq m. \end{cases}$$

The function f_0 contradicts the fact that c, as an element of N was unbounded over M, thus, the assumption was false.

■

Lemma 6.11 *Let $M \subseteq N$ be models of ZFC, $\mathbb{P}$ a poset, $\mathbb{P} \subseteq M$ and G a $\mathbb{P}$-generic filter over M (thus, G is also $\mathbb{P}$-generic over N). For $\mathcal{B} = \{B_\alpha : \alpha < \gamma\} \subseteq M \cap [\omega]^\omega$ and $A \in N \cap [\omega]^\omega$:*

$$\text{if } \begin{pmatrix} M, N \\ * \\ \mathcal{B}, A \end{pmatrix} \text{ holds, then } \begin{pmatrix} M[G], N[G] \\ * \\ \mathcal{B} \;,\; A \end{pmatrix} \text{ holds as well.}$$

Proof:

Assume towards a contradiction, $\begin{pmatrix} M, N \\ * \\ \mathcal{B}, A \end{pmatrix}$ holds, but $\begin{pmatrix} M[G], N[G] \\ * \\ \mathcal{B} \;,\; A \end{pmatrix}$ does not.

If $\begin{pmatrix} M[G], N[G] \\ * \\ \mathcal{B} \;,\; A \end{pmatrix}$ fails, then there are $h \in M[G]$, $h : \omega \times [\gamma]^{<\omega} \to \omega$ and $m \in \omega$, such that $\forall n \geq m$, $\forall F \in [\gamma]^{<\omega}$: $N[G] \models [n, h(n, F)) \setminus \bigcup_{\alpha \in F} B_\alpha \nsubseteq A$.

Let $\dot{h}$ in M be the $\mathbb{P}$-name for the function h above.

Let p be a condition in G and $m \in \omega$ such that

$$p \Vdash_N \forall n \geq m, \forall F \in [\gamma]^{<\omega} : [n, \dot{h}(n, F)) \setminus \bigcup_{\alpha \in F} B_\alpha \nsubseteq A$$

However for all $n \geq m$, $F \in [\gamma]^{<\omega}$, there are $p_{n,F} \leq p$ (in M) and $k_{n,F} \in \omega$ such that $p_{n,F} \Vdash_M \dot{h}(n, F) = k_{n,F}$.

Then $p_{n,F} \Vdash_N ([n, k_{n,F}) \setminus \bigcup_{\alpha \in F} B_\alpha \nsubseteq A)$ and so, $N \models ([n, k_{n,F}) \setminus \bigcup_{\alpha \in F} B_\alpha \nsubseteq A)$.

In M define $h_0 : \omega \times [\gamma]^{<\omega} \to \omega$ as follows:

- $h_0 \upharpoonright m \times [\gamma]^{<\omega} = 0$ and
- for all $n \geq m, F \in [\gamma]^{<\omega}$, let $h_0(n, F) = k_{n,F}$.

Then h_0 gives a contradiction to $\begin{pmatrix} & M, N \\ * & \\ & \mathcal{B}, A \end{pmatrix}$.

■

Lemma 6.12 *Given $\langle \mathbb{P}^0_n, \dot{\mathbb{Q}}^0_n : n \in \omega \rangle$ and $\langle \mathbb{P}^1_n, \dot{\mathbb{Q}}^1_n : n \in \omega \rangle$ finite support iterations with the property that $P^0_n <\circ\ P^1_n$ for all $n \in \omega$*

For $c \in V^1_0 \cap \omega^\omega$, if for all $n \in \omega$, c, as an element of V^1_n, is unbounded over V^0_n, then c, as an element of V^1_ω, is unbounded over V^0_ω

The proof is similar to the one for the following lemma.

Lemma 6.13 *Given $\langle \mathbb{P}^0_n, \dot{\mathbb{Q}}^0_n : n \in \omega \rangle$ and $\langle \mathbb{P}^1_n, \dot{\mathbb{Q}}^1_n : n \in \omega \rangle$ finite support iterations with the property that $P^0_n <\circ\ P^1_n$ for all $n \in \omega$.*

For $\mathcal{A}_\gamma = \{A_\gamma : \gamma < \alpha\} \subseteq V^0_0 \cap [\omega]^\omega$ and $A \in V^1_0 \cap [\omega]^\omega$:

if $\begin{pmatrix} & V^0_n, V^1_n \\ * & \\ & \mathcal{A}_\gamma, A \end{pmatrix}$ holds for all $n \in \omega$, then $\begin{pmatrix} & V^0_\omega, V^1_\omega \\ * & \\ & \mathcal{A}_\gamma, A \end{pmatrix}$ holds as well.

Proof:

Assume towards a contradiction $\begin{pmatrix} & V^0_n, V^1_n \\ * & \\ & \mathcal{A}_\gamma, A \end{pmatrix}$ holds, but $\begin{pmatrix} & V^0_\omega, V^1_\omega \\ * & \\ & \mathcal{A}_\gamma, A \end{pmatrix}$ does not.

Let $h : \omega \times [\alpha]^{<\omega} \to \omega$ be a function in V^0_ω such that for some $m \in \omega$, for all $n \geq m$ and for all $F \in [\alpha]^{<\omega}$

$$V^1_\omega \models [n, h(n, F)) \setminus \bigcup_{\gamma \in F} A_\gamma \not\subseteq A.$$

Then there are a P^0_ω -name $\dot{h}$, $p \in P^1_\omega$ such that $p \Vdash [k, \dot{h}(k,F)) \setminus \bigcup_{\gamma \in F} A_\gamma \not\subseteq A$ for all $k \geq m$ and $F \in [\alpha]^{<\omega}$. Since p has finite support, there is $n \in \omega$ such that $p \in P^1_n$.

Let G^1_n be a P^1_n -generic filter containing p and let $h = h/G^0_n$ be the quotient name, where $G^0_n = G^1_n \cap P^0_n$.

Let $R^{l,n}_\omega$ be the quotient poset P^l_n/G^l_n in $V^l_n = V[G^l_n]$.

Then $h' \in V^0_n$ and for all $k \geq m$, $F \in [\alpha]^{<\omega}$

$$V^1_n \models_{R^{1,n}_\omega} [k, h'(k,F)) \setminus \bigcup_{\gamma \in F} A_\gamma \not\subseteq A.$$

Then for all for all $k \geq m$, $F \in [\alpha]^{<\omega}$, find $p_{k,F}$ a condition in the quotient $R^{0,n}_\omega$ and $x_{k,F} \in \omega$, such that $p_{k,F} \Vdash h'(k,F) = x_{k,F}$.

Let $h_0 \restriction m \times [\alpha]^{<\omega} = 0$.

Then $h_0 \in V^0_n$ and

$$[k, h_0(k,F)) \setminus \bigcup_{\gamma \in F} A_\gamma \not\subseteq A \text{ for all } k \geq m,\ F \in [\alpha]^{<\omega} ,$$

contradicting $\begin{pmatrix} V^0_n, V^1_n \\ * \\ \mathcal{A}_\gamma, A \end{pmatrix}$.

■

6.3 The matrix for $\mathfrak{b} = \mathfrak{a} = \kappa < \mathfrak{s} = \lambda$

Take $f : \{\eta < \lambda : \eta = 1 \bmod 2\} \to \kappa$ **onto** with the property that $\forall \alpha < \kappa$: $f^{-1}(\alpha)$ is **cofinal in** λ.

Using recursion, one defines a matrix consisting of finite support iterations

$$\langle \langle \mathbb{P}^\alpha_\zeta : \alpha \leq \kappa, \zeta \leq \lambda \rangle \langle \dot{\mathbb{Q}}^\alpha_\zeta : \alpha \leq \kappa, \zeta < \lambda \rangle \rangle.$$

For all $\alpha \leq \kappa, \zeta \leq \lambda$, let V^α_ζ be the extension of the ground V by $\mathbb{P}^\alpha_\zeta$.

During the construction, we want the following properties to be satisfied:

1) for all $\zeta \leq \lambda$, for all $\beta < \alpha \leq \kappa$: $\mathbb{P}^{\beta}_{\zeta} <\circ \ \mathbb{P}^{\alpha}_{\zeta}$

2) for all $\zeta \leq \lambda$, for all $\alpha < \kappa$: $\begin{pmatrix} & V^{\alpha}_{\zeta}, V^{\alpha+1}_{\zeta} \\ * & \\ & \mathcal{A}_{\alpha}, A_{\alpha} \end{pmatrix}$.

- $\zeta = 0$, then

  for all $\alpha \leq \kappa$:

  let $\mathbb{P}^{\alpha}_{0}$ is the poset for adding an a.d. family $\mathcal{A}_{\alpha} = \{A_{\beta} : \beta < \alpha\}$(see Definition 6.1 and (1) in Lemma 6.5).

  Recall that for $\alpha \geq \omega_1$, the family $\mathcal{A}_{\alpha}$ is maximal in V^{α}_{0}, by (2) in Lemma 6.5.

  By Remark 6.4, for all $\beta < \alpha$, $\mathbb{P}^{\alpha}_{0}$ can be written as the two step iteration of $\mathbb{P}^{\beta}_{0}$ and the quotient forcing, thus, $\mathbb{P}^{\beta}_{0} <\circ \ \mathbb{P}^{\alpha}_{0}$.

  By Lemma 6.8, the sets $\{A_{\beta} : \beta < \alpha\}$ added by $\mathbb{P}^{\alpha}_{0}$ satisfy the $*$-property, hence, $\begin{pmatrix} & V^{\alpha}_{0}, V^{\alpha+1}_{0} \\ * & \\ & \mathcal{A}_{\alpha}, A_{\alpha} \end{pmatrix}$ holds.

- $\zeta = \eta + 1$

  Suppose that for all $\alpha \leq \kappa$, $\mathbb{P}^{\alpha}_{\eta}$ has been defined and satisfies the required properties (1) and (2) as above.

  - $\zeta = 1 \bmod 2$, then $\Vdash_{\mathbb{P}^{\alpha}_{\eta}} \dot{Q}^{\alpha}_{\eta} = \mathbb{M}_{\dot{\mathcal{U}}^{\alpha}_{\eta}}$, where $\dot{\mathcal{U}}^{\alpha}_{\eta}$ is a $\mathbb{P}^{\alpha}_{\eta}$-name for an ultrafilter, with the property that for all $\beta < \alpha$, $\dot{\mathcal{U}}^{\alpha}_{\eta}$ is forced in $\mathbb{P}^{\alpha}_{\eta}$ to extend $\dot{\mathcal{U}}^{\beta}_{\eta}$.

    More precisely:

    if $\alpha = 0$, then:

    Let $\dot{\mathcal{U}}^{0}_{\eta}$ be a $\mathbb{P}^{0}_{\eta}$-name for any ultrafilter, containing the reals added by the previous Mathias Prikry forcing notions (except for

$\eta = 0$, where any ultrafilter can be chosen) and let $\dot{\mathbb{Q}}^0_\eta$ be a $\mathbb{P}^0_\eta$-name for $\mathbb{M}_{\dot{\mathcal{U}}^0_\eta}$.

Then define $\mathbb{P}^0_\zeta$ to be $\mathbb{P}^0_\eta * \dot{\mathbb{Q}}^0_\eta$.

If $\alpha = \beta + 1$,

Let $\dot{\mathcal{U}}^\alpha_\eta$ be a $\mathbb{P}^\alpha_\eta$-name for an ultrafilter forced in $\mathbb{P}^\alpha_\eta$ to extend the already defined ultrafilter $\dot{\mathcal{U}}^\beta_\eta$.

This extension is constructed in the crucial lemma 6.9, to have the required properties, namely:

(i) for all $\beta < \alpha \leq \kappa : \Vdash_{\mathbb{P}^\alpha_\eta} \dot{\mathcal{U}}^\beta_\eta \subseteq \dot{\mathcal{U}}^\alpha_\eta$,

(ii) m.a.c of $\mathbb{M}_{\dot{\mathcal{U}}^\beta_\eta}$ in V^β_η remain m.a.c of $\mathbb{M}_{\dot{\mathcal{U}}^\alpha_\eta}$ in V^β_α and

(iii) $\begin{pmatrix} & V^\beta_\zeta, V^{\beta+1}_\zeta \\ * & \\ & \mathcal{A}_\beta, A_\beta \end{pmatrix}$ holds (recall that $V^{\beta+1}_\zeta = V^\alpha_\zeta$).

Then $\mathbb{P}^\alpha_\zeta = \mathbb{P}^\alpha_\eta * \dot{\mathbb{Q}}^\alpha_\eta$, where $\Vdash_{\mathbb{P}^\alpha_\eta} \dot{\mathbb{Q}}^\alpha_\eta = \mathbb{M}_{\dot{\mathcal{U}}^\alpha_\eta}$.

One also has $\mathbb{P}^\beta_\zeta <\circ\ \mathbb{P}^\alpha_\zeta$, where $\mathbb{P}^\beta_\zeta = \mathbb{P}^\beta_\eta * \dot{\mathbb{Q}}^\beta_\eta$, by Lemma 4.2.

If α is a limit ordinal:

Assume $\forall \beta < \alpha\ \dot{\mathcal{U}}^\beta_\eta$ has been defined and $\Vdash_{\mathbb{P}^\beta_\eta} \dot{\mathbb{Q}}^\beta_\eta = \mathbb{M}_{\dot{\mathcal{U}}^\beta_\eta}$.

$cf(\alpha) > \omega$, then take $\dot{\mathcal{U}}^\alpha_\eta$ to be a $\mathbb{P}^\alpha_\eta$-name for the union $\bigcup_{\beta<\alpha} \mathcal{U}^\beta_\eta$, which is again an ultrafilter (see previous chapter).

Let $\mathbb{P}^\alpha_\zeta := \mathbb{P}^\alpha_\eta * \dot{\mathbb{Q}}^\alpha_\eta$, where $\Vdash_{\mathbb{P}^\alpha_\eta} \dot{\mathbb{Q}}^\alpha_\eta = \mathbb{M}_{\dot{\mathcal{U}}^\alpha_\eta}$.

$cf(\alpha) = \omega$, then extend the union $\bigcup_{\beta<\alpha} \mathcal{U}^\beta_\eta$ to an ultrafilter, as in the matrix iteration for $Con(\mathfrak{b} = \omega_1 < \mathfrak{s} = \kappa)$ presented in the previous chapter, namely, such that:

(i) $\forall \beta < \alpha : \Vdash_{\mathbb{P}^\alpha_\eta} \dot{\mathcal{U}}^\beta_\eta \subseteq \dot{\mathcal{U}}^\alpha_\eta$

(ii) m.a.c. of $\mathbb{M}_{\dot{\mathcal{U}}^\beta_\eta}$ in V^β_η remain m.a.c. of $\mathbb{M}_{\dot{\mathcal{U}}^\alpha_\eta}$ in V^α_η.

Note that for all $\beta < \alpha$: $\mathbb{P}^\beta_\zeta <\circ\ \mathbb{P}^\alpha_\zeta$, by Lemma 4.2.

– $\zeta = 0 \bmod 2$:

if $\alpha \leq f(\eta)$ just take $\dot{\mathbb{Q}}^\alpha_\eta$ to be the **trivial forcing**.

if $\alpha > f(\eta)$ take $\dot{\mathbb{Q}}^\alpha_\eta$ to be a $\mathbb{P}^\alpha_\eta$-name for the dominating forcing $\mathbb{D}^{V^{f(\eta)}_\eta}$.

Take $\mathbb{P}^\alpha_\zeta := \mathbb{P}^\alpha_\eta * \dot{\mathbb{Q}}^\alpha_\eta$.

One still has to argue that for all $\beta < \alpha$: $\mathbb{P}^\beta_\zeta <\circ\ \mathbb{P}^\alpha_\zeta$.

Three cases are distinguished:

If $\alpha < \beta \leq f(\eta)$

then $\mathbb{P}^\alpha_\zeta = \mathbb{P}^\alpha_\eta$ and $\mathbb{P}^\beta_\zeta = \mathbb{P}^\alpha_\eta$.

Thus, by induction hypothesis, $\mathbb{P}^\alpha_\zeta <\circ\ \mathbb{P}^\beta_\zeta$.

If $\alpha \leq f(\eta) < \beta$,

then $\mathbb{P}^\alpha_\zeta = \mathbb{P}^\alpha_\eta$.

One has $\mathbb{P}^\alpha_\zeta <\circ\ \mathbb{P}^\beta_\zeta$, since $\mathbb{P}^\alpha_\zeta = \mathbb{P}^\alpha_\eta <\circ\ \mathbb{P}^\beta_\eta <\circ\ \mathbb{P}^\beta_\eta * \dot{\mathbb{Q}}^\beta_\eta = \mathbb{P}^\beta_\zeta$.

If $f(\eta) < \alpha < \beta$,

then $\mathbb{P}^\alpha_\zeta <\circ\ \mathbb{P}^\beta_\zeta$ holds by Lemma 4.2.

Lemma 6.11 gives that for all $\alpha \leq \kappa$ $\begin{pmatrix} V^\alpha_\zeta, V^{\alpha+1}_\zeta \\ * \\ \mathcal{A}_\alpha, A_\alpha \end{pmatrix}$ holds as well.

- If ζ is a limit ordinal and $\mathbb{P}^\alpha_\eta$ and $\dot{\mathbb{Q}}^\alpha_\eta$ were defined for all $\eta < \zeta$, then for all $\alpha \leq \kappa$, one takes $\mathbb{P}^\alpha_\zeta$ be the finite support iteration $\langle \mathbb{P}^\alpha_\eta, \dot{\mathbb{Q}}^\alpha_\eta :$ $\eta < \zeta \rangle$.

Lemma 4.4 gives $\mathbb{P}^\alpha_\zeta < \circ\ \mathbb{P}^\beta_\zeta$, and Lemma 6.13 gives $\begin{pmatrix} & V^\alpha_\zeta, V^{\alpha+1}_\zeta \\ * & \\ & \mathcal{A}_\alpha, A_\alpha \end{pmatrix}$ for all $\alpha \leq \kappa$.

Note that in the constructed matrix iteration $\mathbb{P}^\alpha_\zeta < \circ\ \mathbb{P}^\beta_\eta$ holds for all $\alpha < \beta \leq \kappa$ and all $\eta < \zeta \leq \lambda$.

Lemma 6.14 *Given $\zeta \leq \lambda$, one has that:*

(1) For all $p \in \mathbb{P}^\kappa_\zeta$ $\exists \alpha < \kappa$ such that $p \in \mathbb{P}^\alpha_\zeta$.

(2) For all $\mathbb{P}^\kappa_\zeta$-name for a real $\dot{f}$ $\exists \alpha < \kappa$ such that $\dot{f}$ is a $\mathbb{P}^\alpha_\zeta$-name.

Proof:

The proof is done by simultaneous induction on ζ.

First note that, since κ is a regular, uncountable cardinal and $\mathbb{P}^\kappa_\zeta$ is ccc, (2) follows from (1).

- Assume $\zeta = 0$. Then (1) follows from the fact that $\mathbb{P}^\kappa_0$ can be written as a two step iteration, since $\mathbb{P}^\kappa_0$ is the forcing for adding a mad family introduced in 6.1.

- Assume $\zeta = \eta + 1$, a successor ordinal and $p \in \mathbb{P}^\kappa_\zeta$. Then $p = (p_0, \dot{p}_1)$, for some $p_0 \in \mathbb{P}^\kappa_\eta$ and $\dot{p}_1$ such that $\Vdash_{\mathbb{P}^\kappa_\eta} \dot{p}_1 \in \dot{\mathbb{Q}}^\kappa_\eta$.

    - If $\zeta \equiv 1 \bmod 2$, then $\dot{\mathbb{Q}}^\kappa_\eta$ is a name for the Mathias forcing, therefore, $\dot{p}_1$ is a name for a Mathias condition $(s, \dot{A})$, where $s \in [\omega]^{<\omega}$ and $\Vdash_{\mathbb{P}^\kappa_\eta} \dot{A} \in \dot{\mathcal{U}}^\kappa_\eta$.

    - If $\zeta \equiv 0 \bmod 2$, then $\dot{\mathbb{Q}}^\kappa_\eta$ is either a name the trivial forcing, thus, $\dot{p}_1$ is trivial, or a name for the dominating forcing, and therefore,

$\dot{p}_1$ is a name for a Hechler condition $(s', \dot{f})$, where $s' \in \omega^{<\omega}$ and $\dot{f}$ a $\mathbb{P}^{\kappa}_{\eta}$-name for a function in ω^{ω}.

In either case, the induction hypothesis (2) implies the existence of some $\alpha_1 < \kappa$, such that $\dot{p}_1$ is a $\mathbb{P}^{\alpha_1}_{\eta}$-name, and the hypothesis (1) implies the existence of an $\alpha_0 < \kappa$, such that $p_0 \in \mathbb{P}^{\alpha_0}_{\eta}$.

Taking α :=max$\{\alpha_0, \alpha_1\}$, one can conclude $p = (p_0, \dot{p}_1) \in \mathbb{P}^{\alpha}_{\eta}$.

- Assume ζ is a limit ordinal and p a condition in $\mathbb{P}^{\kappa}_{\zeta}$.

  Since p has finite support, p must actually be a $\mathbb{P}^{\kappa}_{\eta}$-condition, for some $\eta < \zeta$. For this $\mathbb{P}^{\kappa}_{\eta}$-condition, one can now apply the induction hypothesis and conclude the existence of some $\alpha < \kappa$, such that $p \in \mathbb{P}^{\alpha}_{\eta}$. Therefore, in particular, $p \in \mathbb{P}^{\alpha}_{\zeta}$.

■

Lemma 6.15 *In* V^{κ}_{λ}, $\mathfrak{b} = \mathfrak{a} = \kappa$ *and* $\mathfrak{s} = \lambda$.

Proof:

- $\mathfrak{a} \leq k$

  Claim: The family $\mathcal{A}_{\kappa} = \{A_{\alpha} : \alpha < \kappa\} \in V^{\kappa}_0$ remains maximal in V^{κ}_{λ}.

  Proof:

  Assume towards a contradiction that the above family is not maximal in V^{κ}_{λ}. Hence, there is B an infinite subset of ω in V^{κ}_{λ}, almost disjoint from all the members of the family, *i.e.* $\forall \alpha < \kappa : |B \cap A_{\alpha}| < \omega$.

  Since $B \in [\omega]^{<\omega} \cap V^{\kappa}_{\lambda}$, one can find, by the previous lemma, an $\alpha < \kappa$, such that $B \in [\omega]^{<\omega} \cap V^{\alpha}_{\lambda}$.

  However, $B \notin \mathcal{I}(\mathcal{A}_{\alpha})$ and $\left(* \begin{matrix} V^{\alpha}_{\lambda}, V^{\alpha+1}_{\lambda} \\ \mathcal{A}_{\alpha}, A_{\alpha} \end{matrix} \right)$ hold, thus, $|B \cap A_{\alpha}| = \omega$, by Lemma 6.7, contradicting the assumption.

  ■

- $\mathfrak{b} \geq k$

  Let $\mathcal{B} \subseteq \omega^\omega \cap V_\lambda^\kappa$ be a family of reals, of cardinality $< \kappa$.

  Claim: $\mathcal{B}$ is not unbounded.

  Proof:

  Since $\mathcal{B} \subseteq \omega^\omega \cap V_\lambda^\kappa$, by the previous lemma, there is $\alpha < \kappa$ and $\zeta < \lambda$ such that $\mathcal{B} \subseteq \omega^\omega \cap V_\zeta^\alpha$.

  The function f was chosen in such a way, that $f^{-1}(\alpha)$ is cofinal in λ, thus, for ζ, there is $\zeta' > \zeta$, with $f(\zeta') = \alpha$. But then, $\mathbb{P}_{\zeta'+1}^{\alpha+1}$ adds a real dominating $\omega^\omega \cap V_{\zeta'}^\alpha$. Since $V_\zeta^\alpha \subseteq V_{\zeta'}^\alpha$, the same real dominates $\omega^\omega \cap V_\zeta^\alpha$ and, therefore, $\mathcal{B}$ can't be unbounded.

  ■

  By combining the two results above with the ZFC result $\mathfrak{b} \leq \mathfrak{a}$, we can conclude $\kappa \leq \mathfrak{b} \leq \mathfrak{a} \leq \kappa$, therefore, $\mathfrak{b} = \mathfrak{a} = \kappa$ holds in V_λ^κ.

- $\mathfrak{s} = \lambda$

  Assume $S \subseteq V_\lambda^\kappa \cap [\omega]^\omega$ is a family of cardinality $< \lambda$.

  Claim: S is not splitting.

  Proof:

  Since $S \subseteq V_\lambda^\kappa \cap [\omega]^\omega$, there is $\zeta < \lambda$ a successor ordinal, say $\zeta = \eta + 1$, $\zeta \equiv 1 \bmod 2$, such that $S \subseteq V_\eta^\kappa \cap [\omega]^\omega$. But then, at this stage of the iteration, a Mathias real not split by S is added, and therefore, S can't be splitting.

  ■

■

Chapter 7

Prequisites for $\mathrm{Con}(\mathfrak{b} = \kappa < \mathfrak{s} = \mathfrak{a} = \lambda)$ above a measurable cardinal

7.1 Measurable cardinals

A **principal filter** on a set S is a filter F with the property that there is $X_0 \subseteq S$, $X_0 \neq \emptyset$ such that $F = \{X \subseteq S : X_0 \subseteq X\}$.

A filter F is **κ-complete** if it is closed under intersections of less than κ many sets, i.e. whenever, for some $\gamma < \kappa$, $\{X_\alpha : \alpha > \gamma\}$ is a family of subsets of S, such that $X_\alpha \in F$ for all $\alpha < \kappa$, then the intersection of the family $\bigcap_{\alpha<\gamma} X_\alpha$ is also in F.

Definition 7.1 *An uncountable cardinal κ is called **measurable** if there exists a κ-complete nonprincipal ultrafilter on κ.*

Note that, if $\mathcal{U}$ is a κ-complete nonprincipal ultrafilter on κ, then every set in $\mathcal{U}$ has cardinality at least κ, since otherwise, it could be written as a union of $\leq \kappa$ many singletons.

Theorem 7.2 *Every measurable cardinal is inaccessible.*

Proof:

(i) Every measurable is **regular**.

Otherwise one contradicts the κ-completeness: a cardinal κ can always be written as a union of $cf(\kappa)$ many sets, each of cardinality $< \kappa$.

Recall that a cardinal κ is **singular** if it can be written as the union of $< \kappa$ many sets, each of cardinality $< \kappa$ (which implies that $cf(\kappa) < \kappa$).

(ii) Every measurable is a **strong limit**.

Assume towards a contradiction there is a measurable cardinal, which is not a strong limit, *i.e.* there must be an $\lambda < \kappa$ such that $2^\lambda \geq \kappa$. We want to get a contradiction.

Consider S a set of functions $f : \lambda \to 2$ of cardinality κ (such S exists since $2^\lambda \geq \kappa$).

For every $\alpha < \lambda$, since U is an ultrafilter, either $\{f \in S : f(\alpha) = 0\} \in U$ or $\{f \in S : f(\alpha) = 1\} \in U$.

Denote by X_α the one that is in U and by ε_α the value of $f(\alpha)$(either 0 or 1).

Let $X := \bigcap_{\alpha<\lambda} X_\alpha$.

The κ-completeness implies $X \in U$. But X has exactly one element, the function f with $f(\alpha) = \varepsilon_\alpha$.

This is a contradiction with the fact that U is nonprincipal.

■

Definition 7.3 *If j is an elementary embedding, the* ***critical point*** *of j is the least α such that $j(\alpha) > \alpha$, if it exists.*

Lemma 7.4 *If $\mathcal{U}$ is a κ-complete ultrafilter on a regular cardinal κ, then the critical point of the corresponding elementary embedding is κ.*

DEFINITION 7.5 *A filter $\mathcal{D}$ is a* ***normal filter*** *if any regressive function defined on a filter-set is constant on a filter-set (i.e $\forall A \in F$ and $\forall f : A \to \kappa$, $f(\alpha) < \alpha\ \forall \alpha \in A \setminus \{0\}$ there is $\alpha < \kappa$ with $f^{-1}[\alpha] \in F$).*

Lemma 7.6 *Any normal ultrafilter on κ is κ-complete.*

Thus, "κ is a measurable" is equivalent with "there exists an elementary embedding with critical point κ" and also equivalent with "there exists a normal ultrafilter on κ."

7.2 Ultrapowers of partial orders

In 1999, Shelah made the ingenious observation, that given κ a measurable cardinal, witnessed by the κ-complete ultrafilter $\mathcal{D}$, $\mu > \kappa$ a regular cardinal and $\mathbb{P}$ a forcing notion adding μ dominating reals by finite support(*e.g.* Hechler forcing), then forcing with the ultrapower $\mathbb{P}^\kappa/\mathcal{D}$ destroys any m.a.d family in the intermediate extension via $\mathbb{P}$. In the ccc case we have $\mathbb{P} <\circ \mathbb{P}^\kappa/\mathcal{D}$, so we can look at the ultrapower as first forcing with $\mathbb{P}$ and then with the quotient. But forcing with the ultrapower also preserves the witnesses for $\mathfrak{b} = \mathfrak{d} = \mu$ introduced by $\mathbb{P}$. Thus, Shelah obtained $Con(\mathfrak{d} < \mathfrak{a})$ by raising the posets to the ultrapower λ times, for $\lambda > \mu$ regular cardinal.

For the material in this chapter, we follow [8], [9], [14] and [10].

Assume $\mathbb{P}$ is a partial order and κ a measurable cardinal, witnessed by the κ-complete ultrafilter $\mathcal{D}$.

One can define an equivalence relation on the class of functions in $\mathbb{P}^\kappa$, by identifying functions that agree on an ultrafilter set (*i.e.* $f =_{\mathcal{D}} g$ iff $\{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in \mathcal{D}$).

Definition 7.7 *The* **ultrapower of the poset** $\mathbb{P}$ *is*

$$\mathbb{P}^\kappa/\mathcal{D} := \{[f] : f \in \mathbb{P}^\kappa\},$$

where $[f] = \{g \in \mathbb{P}^\kappa : \{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in \mathcal{D}\}$ *is the equivalence class of* f.

The **order on** $\mathbb{P}^\kappa/\mathcal{D}$ *is:* $[f] \leq [g]$ *iff* $\{\alpha < \kappa : f(\alpha) \leq g(\alpha)\} \in \mathcal{D}$ *and does not depend on the representatives chosen.*

7.3 $\mathbb{P}$ versus $\mathbb{P}^\kappa/\mathcal{D}$

In this section we want to study the relations between the forcing poset $\mathbb{P}$ and its ultrapower poset $\mathbb{P}^\kappa/\mathcal{D}$. We start with an easy remark, namely that raising a partial order to the ultrapower is only interesting, when $|\mathbb{P}| \geq \kappa$ (since otherwise $\mathbb{P} \cong \mathbb{P}^\kappa/\mathcal{D}$), and we continue by studying complete embeddability issues and the connection between the chain condition of $\mathbb{P}$ and the chain condition of its ultrapower $\mathbb{P}^\kappa/\mathcal{D}$.

Remark 7.8 *If* $|\mathbb{P}| < \kappa$, *then* $\mathbb{P} \cong \mathbb{P}^\kappa/\mathcal{D}$.

Proof:

To show $\mathbb{P} \cong \mathbb{P}^\kappa/\mathcal{D}$, one has to find a one-to-one function between $\mathbb{P}$ and its ultrapower $\mathbb{P}^\kappa/\mathcal{D}$, containing the equivalence classes of functions $f : \kappa \to \mathbb{P}$.

Fix f such a function.

For each $p \in \mathbb{P}$, we can define the set $D_p = \{\alpha < \kappa : f(\alpha) = p\}$.

Then $\kappa = \dot{\bigcup}_{p\in\mathbb{P}} D_p$ (κ is written as the disjoint union of $< \kappa$ many sets).

Thus, $\exists! p \in \mathbb{P} : D_p \in \mathcal{D}$. This p exists since $\kappa \in \mathcal{D}$, and whenever we write a set in the ultrafilter as a disjoint union of $< \kappa$ many sets, exactly one has to be in the ultrafilter. So, for the fixed f we found a unique p.

W.l.o.g. (only changing the values on a small set), the function can be taken to be the constant function $f(\alpha) = p \; \forall \alpha \in \kappa$. So, the one-to-one function is the function $p \mapsto [f]$.

■

Lemma 7.9 *For $\mathbb{P}$ a poset and $\mathbb{P}^\kappa/\mathcal{D}$ its ultrapower the following hold:*

i) $\mathbb{P} \subseteq \mathbb{P}^\kappa/\mathcal{D}$.

Proof:

Identify $p \in \mathbb{P}$ with the class $[f]$ of constant functions $f(\alpha) = p \; \forall \alpha < \kappa$. This function is an injective function from $\mathbb{P}$ into $\mathbb{P}^\kappa/\mathcal{D}$. ■

ii) $\mathbb{P} < \circ \; \mathbb{P}^\kappa/\mathcal{D}$ iff $\mathbb{P}$ is κ-cc.

Note: Having $\mathbb{P} < \circ \; \mathbb{P}^\kappa/\mathcal{D}$, we know any generic of $\mathbb{P}^\kappa/\mathcal{D}$ defines a generic for $\mathbb{P}$, and we can see $\mathbb{P}^\kappa/\mathcal{D}$ as a two step iteration $\mathbb{P} * \dot{\mathbb{Q}}$. Recall that $\mathbb{P} < \circ \; \mathbb{P}^\kappa/\mathcal{D}$ iff m.a.c. of $\mathbb{P}$ remain m.a.c. of $\mathbb{P}^\kappa/\mathcal{D}$.

Proof:

$\Rightarrow$

Assume $\mathbb{P} < \circ \; \mathbb{P}^\kappa/\mathcal{D}$, but $\mathbb{P}$ is not κ-cc. Thus, there is a maximal antichain of size $> \kappa$, say $\{p_\alpha : \alpha < \lambda\}$, for some $\lambda \geq \kappa$. Define the function $f : \kappa \to \mathbb{P}$ by $f(\alpha) = p_\alpha$.

Verify $[f] \perp p_\alpha$ for all $\alpha < \lambda$.

$\Leftarrow$

Assume $\mathbb{P}$ is κ-cc but $\mathbb{P} \not< \circ \; \mathbb{P}^\kappa/\mathcal{D}$.

First note that the antichains of $\mathbb{P}$ remain antichains in $\mathbb{P}^\kappa/\mathcal{D}$, since $\leq_{\mathbb{P}^\kappa/\mathcal{D}} \restriction \mathbb{P} = \leq_{\mathbb{P}}$ and $p \perp_{\mathbb{P}} q$ entails $p \perp_{\mathbb{P}^\kappa/\mathcal{D}} q$. Therefore, the only point is the maximality.

For $\lambda < \kappa$, let $A = \{p_\alpha : \alpha < \lambda\}$ be a maximal antichain in $\mathbb{P}$. Assume A is not maximal in $\mathbb{P}^\kappa/\mathcal{D}$. Then we could find $[f]$ such that $[f] \perp p_\alpha$ for all α.

Defining $A_\alpha := \{\beta : f(\beta) \perp p_\alpha\}$ for all α, by κ-completeness, $\bigcap_\alpha A_\alpha \in \mathcal{D}$, so, in particular, nonempty, contradicting the maximality of A in $\mathbb{P}$.

■

iii) *P is $\mu - cc$ for $\mu < \kappa \Rightarrow \mathbb{P}^\kappa/\mathcal{D}$ is also $\mu - cc$.*

In particular, if P is ccc $\Rightarrow \mathbb{P}^\kappa/\mathcal{D}$ is also ccc.

Proof:

Fix μ. Assume $\mathbb{P}^\kappa/\mathcal{D}$ is not $\mu - cc$, *i.e.* there are $[f_\gamma] : \gamma < \mu$, μ-many incompatible conditions in $\mathbb{P}^\kappa/\mathcal{D}$.

For $\gamma, \delta < \mu$, $\gamma \neq \delta$ define:

$$Y_{\gamma,\delta} := \{\alpha : f_\gamma(\alpha) \perp f_\delta(\alpha)\} \in \mathcal{D}.$$

By κ-completeness of the ultrafilter $\mathcal{D}$, the intersection $Y = \bigcap_{\gamma,\delta<\mu} Y_{\gamma,\delta} \in \mathcal{D}$.

If $\alpha \in Y$, then $\alpha \in \bigcap_{\gamma,\delta<\mu} Y_{\gamma,\delta}$, thus, $\{f_\gamma(\alpha) : \gamma < \mu\}$ is an antichain of length μ in $\mathbb{P}$, contradicting the fact that $\mathbb{P}$ has μ-cc.

■

Note that if $\mathbb{P}$ is κ-cc but not $\mu - cc$ for $\mu < \kappa \Rightarrow \mathbb{P}^\kappa/\mathcal{D}$ is not κ-cc.

7.4 The c.c.c. case

The case we are interested in, is when $\mathbb{P}$ is ccc, thus, $\mathbb{P}^\kappa/\mathcal{D}$ also ccc, and moreover, we know $\mathbb{P}^\kappa/\mathcal{D} = \mathbb{P} * \dot{\mathbb{Q}}$. We want to understand better what the quotient forcing $\dot{\mathbb{Q}}$ is, namely what kind of reals it adds and under which hypotheses.

The first question we ask is **how m.a.c. in $\mathbb{P}^\kappa/\mathcal{D}$ connect to m.a.c in $\mathbb{P}$.**

Take $\{[f_n] : n \in \omega\}$ a maximal antichain in $\mathbb{P}^\kappa/\mathcal{D}$.

Look at $\{\alpha : \{f_n(\alpha) : n \in \omega\}$ m.a.c. in $\mathbb{P}\}$.

The set $\{\alpha : \{f_n(\alpha) : n \in \omega\}$ m.a.c. in $\mathbb{P}\}$ is in $\mathcal{D}$, so changing the values of $f_n(\alpha)$ on only a small set, we can assume w.l.o.g. that $\forall \alpha : \{f_n(\alpha) : n \in \omega\}$ m.a.c. in $\mathbb{P}$.

The second thing we look at is **how $\mathbb{P}$-names for reals in ω^ω relate to the $\mathbb{P}^\kappa/\mathcal{D}$-names for reals**.

Given $\dot{f}$ a $\mathbb{P}$-name for a real in ω^ω, by the ccc, there are $\{p_{n,i} : n, i \in \omega\}$ and $\{k_{n,i} : n \in \omega\}$, such that :

- $\{p_{n,i} : i \in \omega\}$ is a (countable) maximal antichain of $\mathbb{P}$, for all $n \in \omega$ and
- $p_{n,i} \Vdash_{\mathbb{P}} \dot{f}(n) = k_{n,i}$.

This completely describes $\dot{f}$.

Taking $\{p^\alpha_{n,i} : n, i \in \omega$ and $\alpha < \kappa\} \subseteq \mathbb{P}$ and $\{k^\alpha_{n,i} : n, i \in \omega$ and $\alpha < \kappa\} \subseteq \omega$ such that $\{p^\alpha_{n,i} : i \in \omega\}$ is a maximal antichain for all n and α, one gets $\mathbb{P}$-names $\dot{f}^\alpha$.

Form

$$[p_{n,i}] = \{p^\alpha_{n,i} : \alpha < \kappa\}/\mathcal{D} \in \mathbb{P}^\kappa/\mathcal{D} \text{ and}$$
$$k_{n,i} = [k_{n,i}] = \{k^\alpha_{n,i} : \alpha < \kappa\}/\mathcal{D} \in \omega^\kappa/\mathcal{D} = \omega.$$

Using the fact that $\mathcal{D}$ is κ-complete, in $\mathbb{P}^\kappa/\mathcal{D}$ $\{[p_{n,i}] : i \in \omega\}$ is a maximal antichain for all $n \in \omega$.

Therefore, one can construct a $\mathbb{P}^\kappa/\mathcal{D}$-name for a real $[\dot{f}]$, which is in some sense, the **mean** of the $\mathbb{P}$-names $\dot{f}^\alpha$, such that $[p_{n,i}] \Vdash_{\mathbb{P}^\kappa/\mathcal{D}} [\dot{f}](n) = k_{n,i}$. Sometimes, one writes $[\dot{f}] = \langle \dot{f}^\alpha : \alpha < \kappa \rangle/\mathcal{D}$.

Every $\mathbb{P}^\kappa/\mathcal{D}$-name is actually of the above form:

In the ultrapower, which is again ccc, let $[\dot{f}]$ be a $\mathbb{P}^\kappa/\mathcal{D}$-name. As above, one gets $\{[p_{n,i}] : i, n \in \omega\}$ and $\{k_{n,i} = [k_{n,i}] : i, n \in \omega\}$.

Defining $k^\alpha_{n,i} := k_{n,i}$, $p^\alpha_{n,i} = p_{n,i}(\alpha)$ and $A := \{\alpha : \{p^\alpha_{n,i} : i \in \omega\}$ is m.a.c. for all n$\}$, one gets $A \in \mathcal{D}$, therefore, w.l.o.g. $\forall n, \alpha : \{p_{n,i}(\alpha) : i \in \omega\}$ m.a.c. in $\mathbb{P}$.

The values forced for $[\dot{f}](i)$ may be assumed to be the same, since they are the same on an ultrafilter set.

Hence, we may assume $[p_{n,i}] = \{p^\alpha_{n,i} : \alpha < \kappa\}/\mathcal{D} \in \mathbb{P}^\kappa/\mathcal{D}$ and $k_{n,i} = [k_{n,i}] = \{k^\alpha_{n,i} : \alpha < \kappa\}/\mathcal{D} \in \omega^\kappa/\mathcal{D} = \omega$ have been constructed as above.

Given the $\mathbb{P}^\kappa/\mathcal{D}$-name $[\dot{f}]$, one actually has κ-many $\mathbb{P}$-names $\dot{f}^\alpha$, given by stipulating $p_{n,i}(\alpha) \Vdash_{\mathbb{P}} \dot{f}^\alpha(i) = k_{n,i}$.

So, $[\dot{f}] = \langle \dot{f}_\alpha : \alpha < \kappa \rangle/\mathcal{D}$, *i.e.* the names and the sequences are in direct correspondence.

Lemma 7.10 Preservation of unbounded reals under taking ultrapowers

Let $\mathbb{P} <\circ\ \mathbb{Q}$, $c \in V^{\mathbb{Q}}$ *such that* $\forall f \in V^{\mathbb{P}} \cap \omega^\omega$: $\Vdash_{\mathbb{P}} c \not\leq^* f$.

Then $\forall f \in V^{\mathbb{P}^\kappa/\mathcal{D}} \cap \omega^\omega$*:* $\Vdash_{\mathbb{Q}^\kappa/\mathcal{D}} c \not\leq^* f$*.*

Proof: Assume towards a contradiction, there is $\dot{f}$ a $\mathbb{P}^\kappa/\mathcal{D}$-name for a real and $[q]$ a condition in $\mathbb{Q}^\kappa/\mathcal{D}$, such that $\exists k \in \omega$ with the property that

$$\forall i \geq k\colon\ [q] \Vdash_{\mathbb{Q}^\kappa/\mathcal{D}} \dot{c}(i) \leq \dot{f}(i).$$

As argued before, there are $\{[p_{n,i}] : n, i \in \omega\}$ and $\{k_{n,i} : n, i \in \omega\} \subseteq \omega$, such that

- $\{[p_{n,i}] : i \in \omega\}$ is a m.a.c for all $n \in \omega$ and
- $\forall n, i \in \omega : [p_{n,i}] \Vdash_{\mathbb{P}^\kappa/\mathcal{D}} \dot{f}(i) = k_{n,i}$.

By the above arguments, in $\mathbb{P}$, for all $\alpha < \kappa$, there are m.a.c. $\{p^\alpha_{n,i} : i \in \omega\}$ such that $[p_{n,i}] = \langle\{p^\alpha_{n,i} : \alpha < \kappa\rangle/\mathcal{D}$, and a $\mathbb{P}$-name $\dot{f}^\alpha$ such that $p^\alpha_{n,i} \Vdash_{\mathbb{P}} \dot{f}^\alpha(i) = k_{n,i}$.

The set $A := \{\alpha : q(\alpha) \Vdash_{\mathbb{Q}} \dot{c}(i) \leq \dot{f}^\alpha(i)\}$ is in $\mathcal{D}$, thus, nonempty.

Let $\alpha \in A$. Then one gets a contradiction with the unboundedness of c over $V^{\mathbb{P}}$, since $\dot{f}^\alpha$ is a $\mathbb{P}$-name for a real with the property that $\forall i \geq k$: $q(\alpha) \Vdash_{\mathbb{Q}} \dot{c}(i) \leq \dot{f}^\alpha(i)$.

■

Lemma 7.11 Preservation of compete embeddability under taking ultrapowers

Given $\mathbb{P}, \mathbb{Q}$ *two ccc forcing notions,* $\mathbb{P} < \circ\ \mathbb{Q}$*. Then* $\mathbb{P}^\kappa/\mathcal{D} < \circ\ \mathbb{Q}^\kappa/\mathcal{D}$*.*

$$\begin{array}{ccc} \mathbb{Q} & -> & \mathbb{Q}^\kappa/\mathcal{D} \\ \circ & & \circ \\ \vee & & \vee \\ \mathbb{P} & -> & \mathbb{P}^\kappa/\mathcal{D} \end{array}$$

Proof:

Because $\mathbb{P}, \mathbb{Q}$ are ccc, so are $\mathbb{P}^\kappa/\mathcal{D}$ and $\mathbb{Q}^\kappa/\mathcal{D}$, so antichains are all countable.

We have to show that m.a.c. of $\mathbb{P}^\kappa/\mathcal{D}$ are m.a.c. of $\mathbb{Q}^\kappa/\mathcal{D}$.

We know $\mathbb{P} < \circ\ \mathbb{Q}$, so in each coordinate, the maximal antichain is preserved by this complete embedding(m.a.c of $\mathbb{P}$ are m.a.c. of $\mathbb{Q}$).

On an ultrafilter set of coordinates, it is maximal, thus, remains maximal. (Łoś)

■

7.5 Destroying big m.a.d. families

The following lemma says that no family which is almost disjoint in $V^{\mathbb{P}}$ and has cardinality bigger than κ will be m.a.d. in $V^{\mathbb{P}^\kappa/\mathcal{D}}$. In particular, even if the a.d. family is maximal in the extension by $\mathbb{P}$, surely it is not maximal in the extension by $\mathbb{P}^\kappa/\mathcal{D}$.

Thus, if $\mathbb{P} \Vdash$ "$\mathfrak{a} \geq \kappa$", then no m.a.d. family of $V^{\mathbb{P}}$ will be m.a.d. in $V^{\mathbb{P}^\kappa/\mathcal{D}}$:

$\mathfrak{a} \geq \kappa$ means that all m.a.d families have size at least κ, so applying the lemma, we know that all these m.a.d. families are destroyed.

Lemma 7.12 *If $\mathbb{P} \Vdash$ "$\dot{\mathcal{A}}$ is an a.d. family and $|\dot{\mathcal{A}}| \geq \kappa$", then $\mathbb{P}^\kappa/\mathcal{D} \Vdash \dot{\mathcal{A}}$ is not m.a.d".*

Proof:

Let $\mu \geq \kappa$ and $\dot{\mathcal{A}}$ a $\mathbb{P}$-name for an a.d. family $\dot{\mathcal{A}} = \{\dot{A}_\alpha : \alpha < \mu\}$.

The idea of the proof is to look at the first κ-many names $\langle \dot{A}_\alpha : \alpha < \kappa \rangle$ and to take the mean of these $\mathbb{P}$-names, namely to define the $\mathbb{P}^\kappa/\mathcal{D}$-name $[\dot{A}] := \langle \dot{A}_\alpha : \alpha < \kappa \rangle/\mathcal{D}$. Then, one can show that this $\mathbb{P}^\kappa/\mathcal{D}$-name $[\dot{A}]$ is almost disjoint from all $\dot{A}_\alpha : \alpha < \mu$.

For all $\alpha < \mu$, identify the sets $\dot{A}_\alpha$ with their characteristic function. Thus, as before, $\dot{A}_\alpha$ is decided by $\mathbb{P}$-conditions $p^\alpha_{n,i}$ and values $k^\alpha_{n,i} \in 2$, where $\{p^\alpha_{n,i} : i \in \omega\}$ is a maximal antichain, for all $n \in \omega, \alpha < \mu$.

Hence, the following hold:

$$p^\alpha_{n,i} \Vdash n \in \dot{A}^\alpha \text{ iff } k^\alpha_{n,i} = 1,$$
$$p^\alpha_{n,i} \Vdash n \notin \dot{A}^\alpha \text{ iff } k^\alpha_{n,i} = 0.$$

Define

- $[p_{n,i}] := \langle p^\alpha_{n,i} \rangle/\mathcal{D} \in \mathbb{P}^\kappa/\mathcal{D}$ and
- $[k_{n,i}] = \langle k^\alpha_{n,i} \rangle/\mathcal{D} \in 2^\kappa/\mathcal{D} = 2$.

We know $\{[p_{n,i}] : i \in \omega\}$ is a m.a.c. in $\mathbb{P}^\kappa/\mathcal{D}$.

Let $\dot{A} := \langle \dot{A}_\alpha : \alpha < \kappa \rangle/\mathcal{D}$ be the corresponding $\mathbb{P}^\kappa/\mathcal{D}$-name.

The following claim establishes that the family is not maximal in $\mathbb{P}^\kappa/\mathcal{D}$:

Claim $\Vdash_{\mathbb{P}^\kappa/\mathcal{D}} |[\dot{A}] \cap \dot{A}_\beta| < \omega$ for all $\beta < \mu$.

Proof of Claim

Fix $\beta < \mu$ and denote $\dot{A}_\beta$ by $\dot{B}$.

For all $\alpha < \kappa$, except for possibly one (β), there are

- m.a.c. $\{q_i^\alpha : i \in \omega\} \subseteq \mathbb{P}$ and
- $\{n_i^\alpha : i \in \omega\} \subseteq \omega$.

such that $q_i^\alpha \Vdash_{\mathbb{P}} \dot{B} \cap \dot{A}_\alpha \subseteq n_i^\alpha$.

Define

- $[q_i] := \langle q_i^\alpha \rangle / \mathcal{D} \in \mathbb{P}^\kappa/\mathcal{D}$ and
- $[n_i] := \langle n_i^\alpha \rangle / \mathcal{D} \in \omega^\kappa/\mathcal{D} = \omega$.

Since $\{[q_i] : i \in \omega\}$ is a m.a.c, it suffices to show $[q_i] \Vdash_{\mathbb{P}^\kappa/\mathcal{D}} [\dot{A}] \cap \dot{B} \subseteq n_i$.

Assume otherwise. Then one can find $i, l \geq n_i$ and $[r] \leq [q_i]$ such that

$$[r] \Vdash_{\mathbb{P}^\kappa/\mathcal{D}} l \in [\dot{A}] \cap \dot{B}.$$

W.l.o.g., there is j such that $[r] \leq p_{l,j}$ and $k_{l,j} = 1$.

Say $[r] = \langle r^\alpha \rangle / \mathcal{D}$.

Since $\{\alpha : r^\alpha \leq q_i^\alpha, r^\alpha \leq p_{l,j}^\alpha, k_{l,j}^\alpha = 1, n_i^\alpha = n_i \text{ and } r^\alpha \Vdash_{\mathbb{P}} l \in \dot{B}\} \in \mathcal{D}$, it is, in particular, nonempty. Choose α in this set. Since $r^\alpha \leq p_{l,j}^\alpha$ and $k_{l,j}^\alpha = 1$, we know $r^\alpha \Vdash_{\mathbb{P}} l \in \dot{A}_\alpha \cap \dot{B}$. But this is a contradiction, since $r^\alpha \leq q_i^\alpha$, $n_i^\alpha \leq l$ and $q_i^\alpha \Vdash_{\mathbb{P}} \dot{B} \cap \dot{A}_\alpha \subseteq n_i^\alpha$.

■ ■

Chapter 8

The matrix for $\mu < \mathfrak{b} = \kappa < \mathfrak{s} = \mathfrak{a} = \lambda$

Take $f : \{\eta < \lambda : \eta = 1 \bmod 3\} \to \kappa$ **onto** with the property that $\forall \alpha < \kappa$: $f^{-1}(\alpha)$ is **cofinal in** λ.

Following [10], as in Chapter 6, one defines a matrix consisting of finite support iterations

$$\langle \langle \mathbb{P}^{\alpha}_{\zeta} : \alpha \leq \kappa, \zeta \leq \lambda \rangle \langle \dot{\mathbb{Q}}^{\alpha}_{\zeta} : \alpha \leq \kappa, \zeta < \lambda \rangle \rangle.$$

For all $\alpha \leq \kappa, \zeta \leq \lambda$, let V^{α}_{ζ} be the extension of the ground V by $\mathbb{P}^{\alpha}_{\zeta}$.

During the construction, we want the following properties to be satisfied:

$$1) \text{ for all } \zeta \leq \lambda, \text{ for all } \beta < \alpha \leq \kappa : \mathbb{P}^{\beta}_{\zeta} < \circ \ \mathbb{P}^{\alpha}_{\zeta}$$

$$2) \text{ for all } \zeta \leq \lambda, \text{ for all } \alpha < \kappa : (*V^{\alpha}_{\zeta}, V^{\alpha+1}_{\zeta}, c_{\alpha+1})$$

- $\zeta = 0$, then

  for all $\alpha \leq \kappa$ let $\mathbb{P}^{\alpha}_{0}$ to be the c.c.c. poset for adding α Cohen reals $\{c_{\gamma}\}_{\gamma<\alpha}$, namely, a finite support iteration of Cohen forcing. Thus, the property (1) above holds.

Recall that the Cohen reals are unbounded, so the (2) holds as well.

- If $\zeta = \eta + 1$

Suppose that for all $\alpha \leq \kappa$, $\mathbb{P}^{\alpha}_{\eta}$ has been defined and satisfies the required properties.

  - If $\zeta = 1$ mod 3, then define $\dot{\mathbb{Q}}^{\alpha}_{\eta}$ as before, by induction on α:

    if $\alpha = 0$, then

    Let $\dot{\mathcal{U}}^{0}_{\eta}$ be a $\mathbb{P}^{0}_{\eta}$-name for any ultrafilter and $\dot{\mathbb{Q}}^{0}_{\eta}$ be a $\mathbb{P}^{0}_{\eta}$-name for $\mathbb{M}_{\dot{\mathcal{U}}^{0}_{\eta}}$.

    Then $\mathbb{P}^{0}_{\zeta} := \mathbb{P}^{0}_{\eta} * \dot{\mathbb{Q}}^{0}_{\eta}$.

    If $\alpha = \beta + 1$,

    Let $\dot{\mathcal{U}}^{\alpha}_{\eta}$ be a $\mathbb{P}^{\alpha}_{\eta}$-name for an ultrafilter forced in $\mathbb{P}^{\alpha}_{\eta}$ to extend the already defined ultrafilter $\dot{\mathcal{U}}^{\beta}_{\eta}$.

    This extension is constructed in the crucial lemma 4.11, to have the required properties, namely:

    (i) for all $\beta < \alpha \leq \kappa : \Vdash_{\mathbb{P}^{\alpha}_{\eta}} \dot{\mathcal{U}}^{\beta}_{\eta} \subseteq \dot{\mathcal{U}}^{\alpha}_{\eta}$

    (ii) m.a.c of $\mathbb{M}_{\dot{\mathcal{U}}^{\beta}_{\eta}}$ in V^{β}_{η} remain m.a.c of $\mathbb{M}_{\dot{\mathcal{U}}^{\alpha}_{\eta}}$ in V^{β}_{α} and

    (iii) $(*V^{\beta}_{\zeta}, V^{\beta+1}_{\zeta}, c_{\beta+1})$ holds (recall that $V^{\beta+1}_{\zeta} = V^{\alpha}_{\zeta}$) .

    Then $\Vdash_{\mathbb{P}^{\alpha}_{\eta}} \dot{\mathbb{Q}}^{\alpha}_{\eta} = \mathbb{M}_{\dot{\mathcal{U}}^{\alpha}_{\eta}}$ and $\mathbb{P}^{\alpha}_{\zeta} := \mathbb{P}^{\alpha}_{\eta} * \dot{\mathbb{Q}}^{\alpha}_{\eta}$.

    One also has $\mathbb{P}^{\beta}_{\zeta} <\circ\ \mathbb{P}^{\alpha}_{\zeta}$, by Lemma 4.2.

    If α is a limit ordinal:

    Assume $\forall \beta < \alpha\ \dot{\mathcal{U}}^{\beta}_{\eta}$ has been defined and $\Vdash_{\mathbb{P}^{\beta}_{\eta}} \dot{\mathbb{Q}}^{\beta}_{\eta} = \mathbb{M}_{\dot{\mathcal{U}}^{\beta}_{\eta}}$.

$cf(\alpha) > \omega$, then take $\dot{\mathcal{U}}^\alpha_\eta$ to be a $\mathbb{P}^\alpha_\eta$-name for the union $\bigcup_{\beta<\alpha} \mathcal{U}^\beta_\eta$, which is again an ultrafilter.

Let $\mathbb{P}^\alpha_\zeta := \mathbb{P}^\alpha_\eta * \dot{\mathbb{Q}}^\alpha_\eta$, where $\Vdash_{\mathbb{P}^\alpha_\eta} \dot{\mathbb{Q}}^\alpha_\eta = \mathbb{M}_{\dot{\mathcal{U}}^\alpha_\eta}$.

$cf(\alpha) = \omega$, then , as before, extend $\bigcup_{\beta<\alpha} \mathcal{U}^\beta_\eta$ to an ultrafilter, such that:

(i) $\forall \beta < \alpha : \Vdash_{\mathbb{P}^\alpha_\eta} \dot{\mathcal{U}}^\beta_\eta \subseteq \dot{\mathcal{U}}^\alpha_\eta$

(ii) m.a.c. of $\mathbb{M}_{\dot{\mathcal{U}}^\beta_\eta}$ in V^β_η remain m.a.c. of $\mathbb{M}_{\dot{\mathcal{U}}^\alpha_\eta}$ in V^α_η.

Note that for all $\beta < \alpha$: $\mathbb{P}^\beta_\zeta <\circ\ \mathbb{P}^\alpha_\zeta$, by Lemma 4.2.

– $\zeta = 2 \bmod 3$, then

if $\alpha \leq f(\eta)$ just take $\dot{\mathbb{Q}}^\alpha_\eta$ to be the **trivial forcing**.

if $\alpha > f(\eta)$ take $\dot{\mathbb{Q}}^\alpha_\eta$ to be a $\mathbb{P}^\alpha_\eta$-name for the dominating forcing $\mathbb{D}^{V^{f(\eta)}_\eta}$.

Take $\mathbb{P}^\alpha_\zeta := \mathbb{P}^\alpha_\eta * \dot{\mathbb{Q}}^\alpha_\eta$

One still has to argue that for all $\beta < \alpha$: $\mathbb{P}^\beta_\zeta <\circ\ \mathbb{P}^\alpha_\zeta$.

Three cases are distinguished:

If $\alpha < \beta \leq f(\eta)$

then $\mathbb{P}^\alpha_\zeta = \mathbb{P}^\alpha_\eta$ and $\mathbb{P}^\beta_\zeta = \mathbb{P}^\alpha_\eta$.

Thus, by induction hypothesis, $\mathbb{P}^\alpha_\zeta <\circ\ \mathbb{P}^\beta_\zeta$.

If $\alpha \leq f(\eta) < \beta$

then $\mathbb{P}^\alpha_\zeta = \mathbb{P}^\alpha_\eta$.

One has $\mathbb{P}^\alpha_\zeta <\circ\ \mathbb{P}^\beta_\zeta$, since $\mathbb{P}^\alpha_\zeta = \mathbb{P}^\alpha_\eta <\circ\ \mathbb{P}^\beta_\eta <\circ\ \mathbb{P}^\beta_\eta * \dot{\mathbb{Q}}^\beta_\eta = \mathbb{P}^\beta_\zeta$.

If $f(\eta) < \alpha < \beta$

then $\mathbb{P}^{\alpha}_{\zeta} <\circ\ \mathbb{P}^{\beta}_{\zeta}$ holds by Lemma 4.2

Lemma 6.10 gives that for all $\alpha \leq \kappa$, $(*V^{\alpha}_{\zeta}, V^{\alpha+1}_{\zeta}, c_{\alpha+1})$ holds as well.

- $\zeta = 0 \bmod 3$, then **take ultrapowers** .

  for all $\alpha \leq \kappa$, let $\dot{\mathbb{Q}}^\alpha_\eta$ be a $\mathbb{P}^\alpha_\eta$-name for the quotient poset of $((\mathbb{P}^\alpha_\eta)^\mu)/\mathcal{D}$ and $\mathbb{P}^\alpha_\eta$ and define $\mathbb{P}^\alpha_\zeta := \mathbb{P}^\alpha_\eta * \dot{\mathbb{Q}}^\alpha_\eta$.

  The property (1) holds by induction hypothesis and Lemma 7.11, which says that the complete embeddability is preserved under raking ultrapowers.

  For (2), one applies the inductive hypothesis and Lemma 7.10, which says that the unbounded reals are preserved under taking ultrapowers.

- If ζ is a limit ordinal and $\mathbb{P}^\alpha_\eta$ and $\dot{\mathbb{Q}}^\alpha_\eta$ were defined for all $\eta < \zeta$, then

  for all $\alpha \leq \kappa$, one takes $\mathbb{P}^\alpha_\zeta$ be the finite support iteration $\langle \mathbb{P}^\alpha_\eta, \dot{\mathbb{Q}}^\alpha_\eta : \eta < \zeta \rangle$.

  Lemma 4.4 gives $\mathbb{P}^\alpha_\zeta <\circ\ \mathbb{P}^\beta_\zeta$ and Lemma 6.12 gives $(*V^\alpha_\zeta, V^{\alpha+1}_\zeta, c_{\alpha+1})$ $\forall \alpha \leq \kappa$.

Lemma 8.1 *In the matrix iteration above, the following hold:*

(1) For all $p \in \mathbb{P}^\kappa_\zeta$ $\exists \alpha < \kappa$ such that $p \in \mathbb{P}^\alpha_\zeta$.

(2) For all $\mathbb{P}^\kappa_\zeta$-name for a real $\dot{f}$ $\exists \alpha < \kappa$ such that $\dot{f}$ is a $\mathbb{P}^\alpha_\zeta$-name.

Proof:

- Assume $\zeta = 0$. Then (1) follows from the fact that $\mathbb{P}^\kappa_0$ can be written as a two step iteration, since $\mathbb{P}^\kappa_0$ is the forcing for adding κ Cohen reals

- Assume ζ is a limit ordinal and p a condition in $\mathbb{P}^\kappa_\zeta$.

  Since p has finite support, p must actually be a $\mathbb{P}^\kappa_\eta$-condition, for some $\eta < \zeta$. For this $\mathbb{P}^\kappa_\eta$-condition, one can now apply the induction hypothesis and conclude the existence of some $\alpha < \kappa$, such that $p \in \mathbb{P}^\alpha_\eta$. Therefore, in particular, $p \in \mathbb{P}^\alpha_\zeta$.

- Assume $\zeta = \eta + 1$, a successor ordinal and $p \in \mathbb{P}^\kappa_\zeta$. Then $p = (p_0, \dot{p}_1)$, for some $p_0 \in \mathbb{P}^\kappa_\eta$ and $\dot{p}_1$ such that $\Vdash_{\mathbb{P}^\kappa_\eta} \dot{p}_1 \in \dot{\mathbb{Q}}^\kappa_\eta$.

  - If $\zeta \equiv 1 \bmod 3$, then $\dot{\mathbb{Q}}^\kappa_\eta$ is a name for the Mathias forcing, therefore, $\dot{p}_1$ is a name for a Mathias condition $(s, \dot{A})$, where $s \in [\omega]^{<\omega}$ and $\Vdash_{\mathbb{P}^\kappa_\eta} \dot{A} \in \dot{\mathcal{U}}^\kappa_\eta$.
  - If $\zeta \equiv 2 \bmod 3$, then $\dot{\mathbb{Q}}^\kappa_\eta$ is either a name the trivial forcing, thus, $\dot{p}_1$ is trivial, or a name for the dominating forcing, and therefore, $\dot{p}_1$ is a name for a Hechler condition $(s', \dot{f})$, where $s' \in \omega^{<\omega}$ and $\dot{f}$ a $\mathbb{P}^\kappa_\eta$-name for a function in ω^ω.

    In either case, the induction hypothesis (2) implies the existence of some $\alpha_1 < \kappa$, such that $\dot{p}_1$ is a $\mathbb{P}^{\alpha_1}_\eta$-name, and the hypothesis (1) implies the existence of an $\alpha_0 < \kappa$, such that $p_0 \in \mathbb{P}^{\alpha_0}_\eta$.

    Taking $\alpha :=\max\{\alpha_0, \alpha_1\}$, one can conclude $p = (p_0, \dot{p}_1) \in \mathbb{P}^\alpha_\eta$.

  - $\zeta = 0 \bmod 3$,

    This is the only case that is different from Lemma 6.14, corresponding to the step where the ultrapowers are taken.

    Assume $[f] \in (P^\kappa_\eta)^\mu/\mathcal{D}$. Then $[f] = \langle f(\gamma) : \gamma < \mu \rangle/\mathcal{D}$, where for all $\gamma < \mu : f(\gamma) \in \mathbb{P}^\kappa_\eta$.

By the induction hypothesis and the fact that κ is regular $\kappa > \mu$, one can conclude the existence of an $\alpha < \kappa$, with $f(\gamma) \in \mathbb{P}^{\alpha}_{\eta}$, for all $\gamma < \mu$. Thus, $[f]$ is in the ultrapower of $\mathbb{P}^{\alpha}_{\eta}$, but $(P^{\alpha}_{\eta})^{\mu}/\mathcal{D} = \mathbb{P}^{\alpha}_{\zeta}$, hence, $[f] \in \mathbb{P}^{\alpha}_{\zeta}$.

■

Lemma 8.2 *$\mathfrak{b} = \kappa$ and $\mathfrak{a} = \mathfrak{s} = \lambda$ hold in V^{κ}_{λ}.*

Proof:

- $\mathfrak{b} = \kappa$.

  Proof:

  Let $\dot{f} \in \omega^{\omega} \cap V^{\kappa}_{\lambda}$. Then $\exists \zeta < \lambda, \alpha < \kappa$ such that f is actually in $\omega^{\omega} \cap V^{\alpha}_{\zeta}$.

  Since $(*V^{\alpha}_{\zeta}, V^{\alpha+1}_{\zeta}, c_{\alpha+1})$ holds, in $V^{\alpha+1}_{\zeta}$, $c_{\alpha+1} \not\leq^{*} f$, therefore, the same holds in V^{κ}_{λ}. Hence, in V^{κ}_{λ}, $\{c_{\alpha+1} : \alpha < \kappa\}$ is unbounded, yielding $\mathfrak{b} \leq \kappa$

  If $\mathcal{B}$ is a family of cardinality $< \kappa$, $\mathcal{B} \subseteq V^{\kappa}_{\lambda} \cap \omega^{\omega}$, one can again find $\zeta < \lambda, \alpha < \kappa$ such that $\mathcal{B}$ is actually a subset of $\omega^{\omega} \cap V^{\alpha}_{\zeta}$.

  By the fact that $f^{-1}(\alpha)$ is cofinal in λ, $\exists \zeta' > \zeta$ with $f(\zeta') = \alpha$.

  But then, $\mathbb{P}^{\alpha+1}_{\zeta'+1}$ adds a real dominating $\omega^{\omega} \cap V^{\alpha}_{\zeta'}$. Since $V^{\alpha}_{\zeta} \subseteq V^{\alpha}_{\zeta'}$, the same real dominates $\omega^{\omega} \cap V^{\alpha}_{\zeta}$ and, therefore, $\mathcal{B}$ can't be unbounded, yielding $\mathfrak{b} \geq \kappa$.

  Therefore, in V^{κ}_{λ}, $\mathfrak{b} = \kappa$.

  ■

- $\mathfrak{a} = \lambda$.

  By the above and the ZFC result $\mathfrak{b} \leq \mathfrak{a}$, we can conclude $\mathfrak{a} \geq \kappa$ in V^{κ}_{λ}.

Claim: If for some γ, $\kappa \leq \gamma < \lambda$, $\mathcal{A} \subseteq V_\lambda^\kappa \cap [\omega]^\omega$ is an almost disjoint family of cardinality γ, then it can't be maximal.

Proof:

Assume $\mathcal{A} \subseteq V_\lambda^\kappa \cap [\omega]^\omega$ as above, maximal almost disjoint. We know there is $\zeta < \lambda$, $\zeta = \eta + 1$, $\zeta = 0$ mod 3 such that $\mathcal{A}$ is actually a subset of $V_\eta^\kappa \cap [\omega]^\omega$. At this stage, the ultrapower of the poset is taken.

Then, by Lemma 7.12, we know that such a family cannot be maximal in V_ζ^κ, since taking ultrapowers destroys big m.a.d. families.

Thus, there is a infinite subset of ω in V_ζ^κ having infinite intersection with all the members of the family $\mathcal{A}$, contradicting its maximality.

■

- $\mathfrak{s} = \lambda$.

  Assume $S \subseteq V_\lambda^\kappa \cap [\omega]^\omega$ is a family of cardinality $< \lambda$.

  Claim: S is not splitting.

  Proof:

  Since $S \subseteq V_\lambda^\kappa \cap [\omega]^\omega$, there is $\zeta < \lambda$ a successor ordinal, say $\zeta = \eta + 1$, $\zeta \equiv 1$ mod 3, such that $S \subseteq V_\eta^\kappa \cap [\omega]^\omega$. But then, at this stage of the iteration, a Mathias real not split by S is added, and therefore, S can't be splitting.

  ■

■

The matrix iteration techniques have various other applications, they do not only apply to the cardinal invariants presented in this thesis. As an example, one can consider Diego Mejia's paper " Matrix iterations and Cichon's diagram ", where the author constructs a non-linear iteration in the study of measure and category, with the scope of obtaining different values for the cardinals in Cichon's diagram. It might also be interesting to consider an analogue of the matrix iteration of a higher dimension.

Even taking only the cardinal invariants $\mathfrak{b}, \mathfrak{a}$ and $\mathfrak{s}$ into consideration, one can see that the subject is open to further research. There are still interesting open questions, like obtaining the consistency of the strict inequalities $\mathfrak{b} < \mathfrak{a} < \mathfrak{s}$ and $\mathfrak{b} < \mathfrak{s} < \mathfrak{a}$, or obtaining $Con(\mathfrak{b} < \mathfrak{s} = \mathfrak{a})$ without the measurable cardinal assumption.

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