DISSERTATION

Titel der Dissertation

Necessary Density Conditions for Frames on Homogeneous Groups

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Contents

Introduction

Frames are overcomplete systems of vectors in a Hilbert space that give rise to redundant series expansions for every Hilbert space element. Intuitively, frames should therefore be 'denser' than bases. On the other hand, Riesz sequences, which are bases only on the subspace spanned by their elements, should be 'less dense' than bases.

This suggestion seems quite natural, yet a rigorous mathematical formulation for abstract frames without specific structure has been challenging. The most general approach to quantify the overcompleteness of frames is due to Balan, Casazza, Heil and Landau [5]. They studied abstract frames whose index set can be mapped into a discrete abelian group and defined a notion of density for such sets. For frames and Riesz sequences satisfying some weak form of localization they showed the existence of a so-called 'critical density' or 'Nyquist density', that is, a threshold that yields a lower bound for the density of frames and simultaneously an upper bound for the density of Riesz sequences.

Prior to [5] there has been a long history of density theory, however, limited to special classes of frames. In fact, density considerations can be traced back to the sampling theory of bandlimited functions by Beurling and Landau [35], because sampling sets for functions in a reproducing kernel Hilbert space correspond to frames of the reproducing kernels. As a convenient notion of density for a discrete subset X of $\mathbb{R}$ Beurling suggested the asymptotic number of elements of X in an interval normalized by the length of the interval [7].

Beurling's definition, extended to $\mathbb{R}^n$, turned out to be an appropriate tool to derive necessary density conditions for various classes of frames indexed by discrete subsets of $\mathbb{R}^n$ [32], [40], [42].

An important example in this context is the density theorem for irregular Gabor frames due to Ramanathan and Steger [40]. It states that for a frame of the form $\mathcal{G} = \{e^{2\pi i \xi \cdot t} g(t - x)\}_{(x,\xi) \in \Lambda}$ the Beurling density of the index set $\Lambda \subseteq \mathbb{R}^{2n}$ has to be greater than or equal to one.

Their proof contained some fundamental new ideas, the Homogeneous Approximation Property and a Comparison Theorem, that laid out the path for proving general density theorems. These methods have been successfully varied and applied by many authors [12], [25], [32]. In particular, the aforementioned density for abstract frames by Balan, Casazza, Heil and Landau [5] was greatly inspired by these principles.

While these tools are perfectly suited for frames with 'commutative index sets', their limitations are visible when it comes to the density theory of wavelet frames.

Wavelet frames are indexed by discrete subsets of the affine group. Although some ingredients like the Homogeneous Approximation Property are understood for wavelet frames [**31**], natural notions of density adapted to the geometry of the affine group fail to produce Nyquist density criteria for wavelet frames [**30**], [**44**]. However, there are great differences in the structure and geometry of commutative groups and the affine group. It is not only the non-commutativity that enters the scene. The affine group is neither nilpotent nor unimodular nor does it have polynomial growth. These structural differences exhibit many possible reasons why a genuine density theory for wavelet frames might fail.

So there is quite a gap in the existing literature. What about a density theory for frames indexed by subsets of nilpotent groups, of unimodular groups or groups of polynomial growth? On what groups can a meaningful notion of density be defined such that Nyquist type density properties persist?

This thesis is a first step towards an answer of these questions.

We identify the class of homogeneous groups as particularly suitable for carrying out a density theory beyond commutativity. Homogeneous groups are nilpotent Lie groups endowed with a family of dilations. On the one hand, they are, in a sense, the slightest non-commutative generalization of $\mathbb{R}^n$ and many classical techniques for analysis on Euclidean spaces can still be applied. On the other hand, the representation theory of nilpotent Lie groups provides a rich source of examples for frames that are naturally indexed by discrete subsets of homogeneous groups.

The main objective of this thesis is to develop a (Nyquist type) density theory for frames indexed by discrete subsets of homogeneous groups.

We define a density on homogeneous groups in analogy to the Beurling density on $\mathbb{R}^n$, however, adapted to the geometry of homogeneous groups. Instead of counting the elements of a set X in intervals or cubes we count in balls with respect to a left-invariant metric that interacts with the dilations in a simple fashion. We show that the resulting density is independent of the particular choice of the metric and, more generally, that it can be computed by replacing the balls by group translates and dilates of relatively compact sets with non-empty interior and boundary of measure zero. This result is non-trivial even on $\mathbb{R}^n$ where it is due to Landau [**35**]. A subtle example reveals that the density really depends on the group structure.

Once we have settled on a definition of density, we carry out a density theory in the spirit of Ramanathan and Steger [**40**].

In analogy to [**5**], we proof a Comparison Theorem for abstract frames that are indexed by discrete subsets of homogeneous groups and satisfy some weak form of localization, namely a Homogeneous Approximation Property with respect to some reference system.

In a further step we investigate two important classes of examples that are outside the scope of the theory of Balan, Casazza, Heil and Landau [**5**], but can be tackled with our approach. These are, on the one hand, frames of reproducing kernels in connection with the sampling problem in shift-invariant spaces on homogeneous

groups, and on the other hand, so-called coherent frames, that is, frames in the orbit of projective square-integrable group representations.

Shift-invariant spaces are function spaces of the form

$$V^2(\Gamma, \varphi) = \{f = \sum_{\gamma \in \Gamma} c_\gamma L_\gamma \varphi : c = (c_\gamma)_{\gamma \in \Gamma} \in \ell^2(\Gamma)\},$$

where $\varphi \in L^2(G)$ is some suitable generator function and Γ is a lattice in the homogeneous group G. We assume that the left translates of the generator with respect to Γ form a Riesz basis for $V^2(\Gamma, \varphi)$ and show that every frame of reproducing kernels satisfies a Homogeneous Approximation Property with respect to this basis of translates. Then the above mentioned Comparison Theorem provides necessary density conditions for sampling sets, because sampling sets correspond to frames of reproducing kernels and the density of a lattice can be computed.

Coherent frames are frames of the form

$$\{\pi(\chi)g : \chi \in X\},$$

where π is a square-integrable projective representation of G on a separable Hilbert space $\mathcal{H}$, X is some discrete subset of G and $g \in \mathcal{H}$ is some suitable atom. We show that coherent frames possess an intrinsic Homogeneous Approximation Property, which was already observed in [28] for unitary group representations and special atoms. It follows that coherent frames automatically obey a Homogeneous Approximation Property with respect to every reference system with the same structure. To derive concrete necessary density conditions one therefore has to construct a specific (orthonormal) basis in the orbit of the corresponding representation, compute its density and apply the Comparison Theorem. We carry out these steps for some concrete projective representations of low-dimensional homogeneous groups and indicate how a general form of the density threshold could look like.

This thesis is organized as follows. Chapter 1 is an exposition of all necessary prerequisites for the analysis on homogeneous groups such as basic definitions and elementary properties of homogeneous groups, homogeneous norms, Haar measure and function spaces on homogeneous groups and certain discrete subsets. The reader is expected to be familiar with the basic concepts of Lie theory including the concordance between Lie groups and Lie algebras via the exponential map (confer, e.g., [46]).
In Chapter 2 the above mentioned analogue of Beurling's density for discrete subsets of homogeneous groups is introduced and justified.
In Chapter 3 we recall the notion of frames, Riesz sequences and related concepts and proof a general theorem for the comparison of the densities of abstract frames and Riesz sequences that are indexed by discrete subsets of homogeneous groups and satisfy a Homogeneous Approximation Property.
Finally, the last two chapters are devoted to the study of the above mentioned

example classes, the frames of reproducing kernels in connection with the sampling problem in shift-invariant spaces on homogeneous groups (Chapter 4) and frames in the orbit of projective square-integrable group representations (Chapter 5). In either case, we first collect definitions and basic properties, then establish a Homogeneous Approximation Property and subsequently employ the abstract Comparison Theorem from Chapter 3 to obtain necessary density thresholds.

Acknowledgements

CHAPTER 1

Homogeneous Groups

1.1. Homogeneous Groups

Review of Nilpotent Lie Groups. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$. Let G be the corresponding connected and simply connected Lie group and let $\exp : \mathfrak{g} \to G$ denote the associated exponential map.

For $X, Y \in \mathfrak{g}$ sufficiently close to $0 \in \mathfrak{g}$ set

$$(1.1) \qquad\qquad X * Y := \exp^{-1}(\exp X \exp Y).$$

The Campbell-Baker-Hausdorff formula states that $X * Y$ is given by an infinite linear combination of X and Y and their iterated commutators (for the precise formula see, e.g., [**13**], p.11). The first few low order terms are

$$(1.2) \qquad X * Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \dots,$$

the dots indicate expressions involving commutators of order four and more.

For (a class of) Lie algebras where iterated Lie brackets of higher order eventually vanish the Campbell-Baker-Hausdorff series reduces to a polynomial map and thereby reveals the global structure of the corresponding connected and simply connected Lie group.

Definition 1.1 (Nilpotent Lie algebra)**.** Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$. The *descending central series* of $\mathfrak{g}$ is defined inductively by

$$\mathfrak{g}_{(1)} = \mathfrak{g}, \quad \mathfrak{g}_{(j+1)} = [\mathfrak{g}, \mathfrak{g}_{(j)}].$$

We say that $\mathfrak{g}$ is *nilpotent* if there is an integer m such that $\mathfrak{g}_{(m+1)} = \{0\}$.
More precisely, if $\mathfrak{g}_{(m+1)} = \{0\}$ and $\mathfrak{g}_{(m)} \neq \{0\}$ we say that $\mathfrak{g}$ is *nilpotent of step m*.

Definition 1.2 (Nilpotent Lie group)**.** A *nilpotent Lie group* is a Lie group G whose Lie algebra is nilpotent.

We always assume that G is connected and simply connected.

If G is a (connected, simply connected) nilpotent Lie group with Lie algebra $\mathfrak{g}$, then the exponential map $\exp : \mathfrak{g} \to G$ is a global diffeomorphism and the Campbell-Baker-Hausdorff formula (1.2) holds for all $X, Y \in \mathfrak{g}$ (for details and proofs see, e.g., [**13**], p.13). The Campbell-Baker-Hausdorff series terminates after finitely

many terms and defines a binary operation $* : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that is polynomial in the coordinates. Furthermore, this map is actually a group law that endows $\mathfrak{g}$ with a Lie group structure $(\mathfrak{g}, *)$ whose associated Lie algebra is $\mathfrak{g}$ and such that $\exp_{(\mathfrak{g},*)} = \mathrm{Id}_{\mathfrak{g}}$ (see, e.g., [**8**], p.130, or [**34**], p.445).

By the definition in formula (1.1), it further follows that $\exp : (\mathfrak{g}, *) \to (G, \cdot)$ is a Lie group isomorphism.

Summarizing we get the following statement about the structure of nilpotent Lie groups.

Theorem 1.3. *Let G be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Let $(\mathfrak{g}, *)$ denote the Lie group with the underlying manifold $\mathfrak{g}$ and with the multiplication given by the Campbell-Baker-Hausdorff product. Then $\exp : (\mathfrak{g}, *) \to (G, \cdot)$ is a Lie group isomorphism.*

Many authors use this fact to identify G with $\mathfrak{g}$ via the exponential map.

Another important application of Theorem 1.3 is the possibility to transfer coordinates from $\mathfrak{g}$ to G and thereby identify G with $\mathbb{R}^n$.

Definition 1.4 (Exponential coordinates)**.** Fix an ordered basis $\{X_1, \dots, X_n\}$ for $\mathfrak{g}$ and identify the vector $(x_1, \dots, x_n)$ in $\mathbb{R}^n$ with the element $x = \exp(x_1 X_1 + \cdots + x_n X_n)$ in G.

We say that G is equipped with *exponential coordinates* or *canonical coordinates of the first kind*.

In this parametrization of G the multiplication is just given by the polynomials arising from the Campbell-Baker-Hausdorff formula.

Example 1.5 (Heisenberg group)**.** The smallest non-commutative nilpotent Lie algebra is the *Heisenberg algebra* $\mathfrak{h} = \mathbb{R}X_1 + \mathbb{R}X_2 + \mathbb{R}X_3$ with Lie brackets defined by

(1.3) $$[X_3, X_2] = X_1, [X_3, X_1] = [X_2, X_1] = 0.$$

If $X = x_1 X_1 + x_2 X_2 + x_3 X_3$ and $Y = y_1 X_1 + y_2 X_2 + y_3 X_3$, then

$$[X, Y] = \Big[\sum_{i=1}^{3} x_i X_i, \sum_{j=1}^{3} y_j X_j \Big] = \sum_{i,j=1}^{3} x_i y_j [X_i, X_j] = (x_3 y_2 - x_2 y_3) X_1.$$

Thus

$$
\begin{aligned}
X * Y &= X + Y + \frac{1}{2}[X, Y] \\
&= \big(x_1 + y_1 + \frac{1}{2}(x_3 y_2 - x_2 y_3)\big) X_1 + (x_2 + y_2) X_2 + (x_3 + y_3) X_3.
\end{aligned}
$$

The corresponding connected and simply connected Lie group $\mathbb{H}$ is called the *Heisenberg group*. In its realization in exponential coordinates $\mathbb{H}$ can be regarded

as $\mathbb{R}^3$ with product

$$(1.4) \qquad (x_1, x_2, x_3)(y_1, y_2, y_3) = \left(x_1 + y_1 + \frac{1}{2}(x_3 y_2 - x_2 y_3), x_2 + y_2, x_3 + y_3\right).$$

Another widely used realization of a connected and simply connected nilpotent Lie group G is via so-called *Malcev coordinates*.

Definition 1.6 (Malcev coordinates). An ordered basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$ is called a *(strong) Malcev basis*, if for each k, $1 \leq k \leq n$, the linear span

$$\mathfrak{g}_k := \operatorname{span}\{X_1, \ldots, X_k\}$$

is an ideal in $\mathfrak{g}$. Fix a Malcev basis and identify the vector $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ with the element

$$x = \exp(x_1 X_1) \ldots \exp(x_n X_n) = \exp(x_1 X_1 * \cdots * x_n X_n)$$

in G. These coordinates for G are called *Malcev coordinates*.

For illustration we calculate the realization of the Heisenberg group from Example 1.5 in Malcev coordinates.

Example 1.7 (Heisenberg group continued). By the definition of the Lie brackets for the Heisenberg algebra $\mathfrak{h}$ in (1.3), it follows that $\{X_1, X_2, X_3\}$ is a Malcev basis. Let $(x_1, x_2, x_3) \in \mathbb{R}^3$ correspond to $x = \exp(x_1 X_1) \exp(x_2 X_2) \exp(x_3 X_3)$, and $(y_1, y_2, y_3) \in \mathbb{R}^3$ correspond to $y = \exp(y_1 X_1) \exp(y_2 X_2) \exp(y_3 X_3)$.

To obtain the multiplication rule for $\mathbb{H}$ in Malcev coordinates we compute

$$
\begin{aligned}
xy &= \exp(x_1 X_1) \exp(x_2 X_2) \exp(x_3 X_3) \exp(y_1 X_1) \exp(y_2 X_2) \exp(y_3 X_3) \\
&= \exp(x_1 X_1 * x_2 X_2 * x_3 X_3 * y_1 X_1 * y_2 X_2 * y_3 X_3).
\end{aligned}
$$

By repeatedly employing the Campbell-Baker-Hausdorff formula, we calculate that

$$
\begin{aligned}
x_1 X_1 * x_2 X_2 * x_3 X_3 * y_1 X_1 * y_2 X_2 * y_3 X_3 &= (x_1 + y_1) X_1 * x_2 X_2 * x_3 X_3 * y_2 X_2 * y_3 X_3 \\
&= (x_1 + y_1) X_1 * x_2 X_2 * (x_3 X_3 * y_2 X_2 * (-x_3 X_3)) * (x_3 + y_3) X_3 \\
&= (x_1 + y_1) X_1 * x_2 X_2 * (y_2 X_2 + x_3 y_2 X_1)) * (x_3 + y_3) X_3 \\
&= (x_1 + y_1 + x_3 y_2) X_1 * (x_2 + y_2) X_2 * (x_3 + y_3) X_3.
\end{aligned}
$$

Therefore the multiplication for $\mathbb{H}$ in Malcev coordinates is given by

$$(1.5) \qquad (x_1, x_2, x_3)(y_1, y_2, y_3) = \left(x_1 + y_1 + x_3 y_2, x_2 + y_2, x_3 + y_3\right).$$

Homogeneous Groups. We consider homogeneous groups as defined in the book of Folland and Stein [**21**], which also serves as our main reference for this section.

Definition 1.8 (Homogeneous Lie algebra)**.** A family of *dilations* on a Lie algebra $\mathfrak{g}$ is a family $\{\delta_r\}_{r>0}$ of algebra automorphisms of $\mathfrak{g}$ of the form $\delta_r = e^{A\log r}$, where A is a diagonalizable operator on $\mathfrak{g}$ with positive eigenvalues.
A Lie algebra endowed with a family of dilations $\{\delta_r\}_{r>0}$ is called a *homogeneous Lie algebra*.

In particular, $\delta_{rs} = \delta_r\delta_s$ for all $r,s > 0$.

Without loss of generality we assume that the smallest eigenvalue of A is greater than or equal to one (otherwise replace δ_r by $\delta_{r^\alpha} = e^{\alpha A\log r}$ with $\alpha > 0$ suitably chosen).

Lemma 1.9. *[**21**] Every homogeneous Lie algebra is nilpotent.*

PROOF. Let $\mathfrak{g}$ be a homogeneous Lie algebra with a fixed family of dilations $\delta_r = e^{A\log r}$. Denote by $\sigma(A)$ the set of eigenvalues of A and set $W_a = \ker(A - a\,\mathrm{Id})$ for $a \in \mathbb{R}$. Then

$$\mathfrak{g} = \bigoplus_{a\in\sigma(A)} W_a$$

and $\delta_r|_{W_a} = r^a\,\mathrm{Id}$ for all $a \in \mathbb{R}$. If $X \in W_a$ and $Y \in W_{a'}$, then

$$\delta_r[X,Y] = [\delta_r X, \delta_r Y] = [r^a X, r^{a'} Y] = r^{a+a'}[X,Y],$$

because δ_r is an algebra automorphism. Thus

$$(1.6) \qquad\qquad\qquad [W_a, W_{a'}] \subseteq W_{a+a'}.$$

Since $a \geq 1$ for all $a \in \sigma(A)$, it follows from inclusion (1.6) that the j-th element of the descending central series satisfies

$$\mathfrak{g}_{(j)} \subseteq \bigoplus_{a\geq j} W_a,$$

so $\mathfrak{g}_{(j)} = \{0\}$ for j sufficiently large. Therefore $\mathfrak{g}$ is nilpotent.
□

From now on we fix a family of dilations $\delta_r = e^{A\log r}$ and denote by $a_1,\ldots,a_n$ the eigenvalues of A, listed in decreasing order and each eigenvalue ocurring as many times as its multiplicity, that is,

$$a_1 \geq a_2 \geq \cdots \geq a_n \geq 1.$$

Further fix an ordered basis $\{X_1,\ldots,X_n\}$ for $\mathfrak{g}$ consisting of corresponding eigenvectors, i.e.,

$$(1.7) \qquad\qquad\qquad\qquad AX_i = a_i X_i$$

for $i = 1, \ldots, n$.

Claim: *The basis $\{X_1, \ldots, X_n\}$ constructed in (1.7) is a Malcev basis for $\mathfrak{g}$.*

We need to show that for each k, $1 \leq k \leq n$, the linear span $\mathfrak{g}_k := \mathrm{span}\{X_1, \ldots, X_k\}$ is an ideal in $\mathfrak{g}$. For that let $Y = \sum_{i=1}^{k} y_i X_i \in \mathfrak{g}_k$ and let X_{i_0} be an arbitrary basis element. Then

$$[X_{i_0}, Y] = \sum_{i=1}^{k} y_i [X_{i_0}, X_i].$$

In view of inclusion (1.6), each Lie bracket $[X_{i_0}, X_i]$ is either zero or an eigenvector of A to an eigenvalue a_j that is strictly greater than a_k. But the eigenspace of every eigenvalue a_j with $a_j > a_k$ is contained in $\mathfrak{g}_k$ by construction. Thus $[X_{i_0}, Y] \in \mathfrak{g}_k$ as a linear combination of elements in $\mathfrak{g}_k$. Now for arbitrary $X \in \mathfrak{g}$ write $X = \sum_{i=1}^{n} x_i X_i$, then also

$$[X, Y] = \sum_{i=1}^{n} x_i [X, Y] \in \mathfrak{g}_k.$$

Interesting classes of homogeneous Lie algebras are the graded and stratified Lie algebras.

Definition 1.10. A *graded Lie algebra* is a Lie algebra $\mathfrak{g}$ that has a direct sum decomposition

$$(1.8) \qquad \mathfrak{g} = \bigoplus_{i=1}^{k} W_i$$

such that $[W_i, W_j] \subseteq W_{i+j}$ if $i + j \leq k$ and $[W_i, W_j] = \{0\}$ if $i + j > k$. A decompositon of that form is called a *gradation* of $\mathfrak{g}$.

A Lie algebra $\mathfrak{g}$ is called a *stratified Lie algebra* if it is graded and W_1 generates $\mathfrak{g}$ as an algebra. In this case a decomposition

$$\mathfrak{g} = \bigoplus_{i=1}^{k} W_i$$

with the property $[W_1, W_i] = W_{i+1}$ is called a *stratification* of $\mathfrak{g}$.

Every graded Lie algebra $\mathfrak{g}$ with decomposition $\mathfrak{g} = \bigoplus_{i=1}^{k} W_i$ possesses a natural family of dilations given by

$$(1.9) \qquad \delta_r \left(\sum_{i=1}^{k} w_i \right) = \sum_{i=1}^{k} r^i w_i, \quad w_i \in W_i.$$

Definition 1.11 (Homogeneous group)**.** A *homogeneous group* is a connected and simply connected Lie group G whose Lie algebra is endowed with a family of dilations $\{\delta_r\}_{r>0} = \{e^{A \log r}\}_{r>0}$. The number $D := \mathrm{trace}(A) = \sum_{i=1}^{n} a_i$ is called the *homogeneous dimension* of G.

By Lemma 1.9, every homogeneous group is nilpotent and therefore isomorphic to the Lie group $(\mathfrak{g}, *)$, where

$$X * Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \dots.$$

Since the dilations δ_r are algebra automorphisms on $\mathfrak{g}$, it follows that

$$(1.10) \qquad \delta_r(X * Y) = \delta_r X + \delta_r Y + \frac{1}{2}[\delta_r X, \delta_r Y] + \dots = \delta_r X * \delta_r Y.$$

In other words, the dilations δ_r are group automorphisms on the Lie group $(\mathfrak{g}, *)$.

Definition 1.12 (Group dilations)**.** Let G be a homogeneous group whose Lie algebra is endowed with a family of dilations $\{\delta_r\}_{r>0}$. The maps $\delta_r^G : G \to G$, defined by

$$\delta_r^G := \exp \circ \, \delta_r \circ \exp^{-1},$$

are called *dilations* of the group G.

As a composition of three isomorphisms, the group dilations δ_r^G are group automorphisms on G, that is,

$$(1.11) \qquad\qquad\qquad \delta_r^G(xy) = \delta_r^G(x)\delta_r^G(y)$$

for all $x, y \in G$.

If G is identified with $\mathbb{R}^n$ via exponential coordinates or Malcev coordinates with respect to a basis of eigenvectors of δ_r as in (1.7), then δ_r^G takes the explicit form

$$(1.12) \qquad\qquad \delta_r^G : \mathbb{R}^n \to \mathbb{R}^n, \ \delta_r^G(x_1, \dots, x_n) = (r^{a_1} x_1, \dots, r^{a_n} x_n).$$

Indeed, in the case of Malcev coordinates observe that

$$
\begin{aligned}
\delta_r^G(\exp(x_1 X_1) \dots \exp(x_n X_n)) &= \delta_r^G(\exp(x_1 X_1)) \dots \delta_r^G(\exp(x_n X_n)) \\
&= \exp(\delta_r(x_1 X_1)) \dots \exp(\delta_r(x_n X_n)) \\
&= \exp(r^{a_1} x_1 X_1) \dots \exp(r^{a_n} x_n X_n).
\end{aligned}
$$

Therefore the action of δ_r^G in Malcev coordinates is

$$\delta_r^G(x_1, \dots, x_n) = (r^{a_1} x_1, \dots, r^{a_n} x_n).$$

For the case of exponential coordinates an analogous argument applies.

Unless there is some risk of confusion, the group dilations δ_r^G will henceforth simply be denoted by δ_r.

We close this section with some examples.

Example 1.13. $\mathbb{R}^n$ with addition and the usual scalar multiplication is a homogeneous group.

Example 1.14 (Heisenberg group)**.** Recall from Example 1.5 the *Heisenberg algebra* $\mathfrak{h} = \mathbb{R}X_1 + \mathbb{R}X_2 + \mathbb{R}X_3$ with non-zero Lie bracket

$$[X_3, X_2] = X_1.$$

Set $W_1 = \mathrm{span}\{X_2, X_3\}$, $W_2 = \mathrm{span}\{X_1\}$, then $\mathfrak{h} = W_1 \oplus W_2$ is a stratification of $\mathfrak{h}$ and the natural dilations as defined in (1.9) are given by

$$(1.13) \qquad \delta_r(x_1 X_1 + x_2 X_2 + x_3 X_3) = r^2 x_1 X_1 + r(x_2 X_2 + x_3 X_3).$$

On the Heisenberg group $\mathbb{H}$ in its realization in exponential or Malcev coordinates with respect to the basis $\{X_1, X_2, X_3\}$ the corresponding dilations are

$$(1.14) \qquad \delta_r(x_1, x_2, x_3) = (r^2 x_1, r x_2, r x_3).$$

In this case the homogeneous dimension of $\mathbb{H}$ is $D = 4$.

Example 1.15. Consider the four-dimensional Lie algebra $\mathfrak{g}_4 = \mathbb{R}X_1 + \cdots + \mathbb{R}X_4$ with non-vanishing Lie brackets

$$[X_4, X_3] = X_2, [X_4, X_2] = X_1.$$

Set $W_1 = \mathrm{span}\{X_3, X_4\}$, $W_2 = \mathrm{span}\{X_2\}$ and $W_3 = \mathrm{span}\{X_1\}$, then

$$\mathfrak{g}_4 = W_1 \oplus W_2 \oplus W_3$$

is a stratification of $\mathfrak{g}_4$ and the natural dilations as defined in (1.9) are given by

$$\delta_r(x_1 X_1 + x_2 X_2 + x_3 X_3 + x_4 X_4) = r^3 x_1 X_1 + r^2 x_2 X_2 + r(x_3 X_3 + x_4 X_4).$$

If the corresponding connected and simply connected Lie group G_4 is identified with $\mathbb{R}^4$ via Malcev coordinates with respect to the basis $\{X_1, X_2, X_3, X_4\}$, then the multiplication law becomes

$$(x_1, x_2, x_3, x_4)(y_1, y_2, y_3, y_4) = (x_1 + y_1 + x_4 y_2 + \tfrac{1}{2}x_4^2 y_3, x_2 + y_2 + x_4 y_3, x_3 + y_3, x_4 + y_4)$$

and the natural dilations on G_4 are given by

$$\delta_r(x_1, x_2, x_3, x_4) = (r^3 x_1, r^2 x_2, r x_3, r x_4).$$

In this case the homogeneous dimension of G_4 is $D = 7$.

1.2. Homogeneous Norms and Leftinvariant Metrics

Throughout this section let G be a homogeneous group.

A *homogeneous norm* on G is a continuous function $|\,| : G \to [0, \infty)$ such that

(i) $|x| = 0$ if and only if $x = e$;

(ii) $|x^{-1}| = |x|$ for all $x \in G$;

(iii) $|\delta_r x| = r|x|$ for all $x \in G$ and $r > 0$.

Homogeneous norms always exist.

First consider $G = (\mathfrak{g}, *)$. Fix a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$ consisting of eigenvectors of δ_r, that is, $\delta_r X_i = r^{a_i} X_i$ for $i = 1, \ldots, n$. For $X = \sum_{i=1}^n x_i X_i$, set

$$|X| := \max\{|x_i|^{\frac{1}{a_i}} : 1 \leq i \leq n\}.$$

Then $|\,|$ defines a homogeneous norm on $(\mathfrak{g}, *)$, because

$$|\delta_r X| = \max\{|r^{a_i} x_i|^{\frac{1}{a_i}} : 1 \leq i \leq n\} = r \max\{|x_i|^{\frac{1}{a_i}} : 1 \leq i \leq n\} = r|X|.$$

For general G set $|x|_G := |\exp^{-1} x|$.

Any two homogeneous norms $|\,|$ and $|\,|'$ on G are equivalent in the sense that there exist constants $A, B > 0$ such that

$$A|x| \leq |x|' \leq B|x|$$

for all $x \in G$ (see, e.g., [**24**], p.3).

Furthermore, every homogeneous norm $|\,|$ on G is quasi-subadditive, that is,

(1.15) $$|xy| \leq C(|x| + |y|)$$

for some constant $C > 0$ and all $x, y \in G$ (see, e.g., [**21**], p.9).

However, on every homogeneous group there also exists a subadditive homogeneous norm, that is, a homogeneous norm which satisfies inequality (1.15) with $C = 1$ (see [**29**] for an explicit construction).

Thus we may henceforth assume that G is equipped with a fixed subadditive homogeneous norm $|\,|$.

Next we consider the left-invariant metric d induced by the homogeneous norm $|\,|$ on G, that is,

(1.16) $$d : G \times G \to \mathbb{R}^+, d(x, y) := |x^{-1} y|.$$

Indeed, if $|\,|$ is a subadditive homogeneous norm on G, then d satisfies the usual metric properties

(i) $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$ for all $x, y \in G$,

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in G$,

as well as left-invariance,

(iv) $d(ax, ay) = d(x, y)$ for all $a, x, y \in G$,

and homogeneity,

(v) $d(\delta_r x, \delta_r y) = r d(x, y)$ for all $x \in G$ and $r > 0$.

With respect to the metric d define the balls with radius $N > 0$ and center $x \in G$ as usual by

$$B_N(x) := \{y \in G : d(x, y) < N\}.$$

Observe that $B_N(x)$ is the left translate by x of $B_N(e)$,

(1.17)
$$B_N(x) = x B_N(e),$$

because, by the left-invariance of d,

$$x B_N(e) = \{xz : d(e, z) < N\} = \{xz : d(x, xz) < N\} = \{y : d(x, y) < N\} = B_N(x).$$

Similarly, it follows from the homogeneity of d that $B_N(e)$ is the image of $B_1(e)$ under δ_N,

(1.18)
$$B_N(e) = \delta_N(B_1(e)).$$

Finally let us remark that the balls constructed in this way are relatively compact, that means, for all $x \in G$ and $N > 0$, the closure $\overline{B_N(x)}$ is compact (see [21], p.9).

1.3. Haar Measure and Lebesgue Spaces

Haar measure on Homogeneous Groups. On every locally compact group G (in particular, on every Lie group) there exists a non-zero Radon measure λ that is left invariant, that is, it satisfies

$$\lambda(xE) = \lambda(E)$$

for every measurable set $E \subseteq G$ and every $x \in G$. Equivalently,

$$\int_G f(y^{-1}x) d\lambda(x) = \int_G f(x) d\lambda(x)$$

for every integrable function f on G and every $y \in G$. This measure λ is uniquely determined up to positive multiples and is called a *left Haar measure* for G. Further, every locally compact group possesses also a *right Haar measure*, that is, a non-zero Radon measure ν that satisfies

$$\nu(Ex) = \nu(E)$$

for every measurable set $E \subseteq G$ and every $x \in G$. In general, left and right Haar measures do not coincide. If a group G admits a non-zero Radon measure λ that is both left and right invariant, then G is called *unimodular* and λ is called a *bi-invariant Haar measure* on G.

Nilpotent Lie groups are unimodular and a bi-invariant Haar measure is given by the image measure of the Lebesgue measure under the exponential map [21].

Lemma 1.16. *Let G be a (connected and simply connected) nilpotent Lie group with Lie algebra $\mathfrak{g}$ and exponential map $\exp : \mathfrak{g} \to G$. If μ denotes the Lebesgue measure on $\mathfrak{g}$, then $\lambda := \mu \circ \exp^{-1}$ is a bi-invariant Haar measure on G.*

A function f on G is then integrable with respect ot λ if and only if $f \circ \exp$ is integrable with respect to the Lebesgue measure and the integral is given by

$$(1.19) \qquad \int_G f(x)d\lambda(x) = \int_{\mathfrak{g}} f(\exp(X))d\mu(X).$$

For a proof of Lemma 1.16 one considers exponential coordinates with respect to a Malcev basis for $\mathfrak{g}$. By the Campbell-Baker-Hausdorff formula, the differentials of the left and right translations in these coordinates are upper triangular matrices with ones on the diagonal, their determinants therefore identically one. For details see [**13**], p. 19.

In the following we assume that G is a homogeneous group. We review the behaviour of the Haar measure with respect to dilations. Recall that the dilations on G are defined as $\delta_r^G := \exp \circ \, \delta_r \circ \exp^{-1}$. By equation (1.19) and the change-of-variables formula one observes that

$$
(1.20) \qquad \begin{aligned}
\int_G f(\delta_r^G x)d\lambda(x) &= \int_G f(\exp(\delta_r(\exp^{-1} x)))d\lambda(x) \\
&= \int_{\mathfrak{g}} f(\exp(\delta_r X))d\mu(X) \\
&= \int_{\mathfrak{g}} r^{-D} f(\exp X)d\mu(X) \\
&= r^{-D} \int_G f(x)d\lambda(x)
\end{aligned}
$$

where D denotes the homogeneous dimension of G.
In particular, it follows that

$$(1.21) \qquad \lambda(\delta_r^G E) = r^D \lambda(E)$$

for every measurable set $E \subseteq G$.

We will henceforth simply write dx for the Haar measure on G and dX for the Lebesgue measure on $\mathfrak{g}$.

Remark 1.17. If G is identified with $\mathbb{R}^n$ via exponential or Malcev coordinates, then the Haar measure becomes the usual Lebesgue measure on $\mathbb{R}^n$.
For the realization of G in exponential coordinates this is just the statement of Lemma 1.16. But Malcev coordinates are related to exponential coordinates by a polynomial isomorphism with polynomial inverse whose Jacobian determinant is identically one (see [**13**], p. 18). Thus the claim follows.

Lebesgue Spaces. For $1 \leq p < \infty$ we denote by $L^p(G)$ the Lebesgue space with respect to the Haar measure on G, that is, the space of all measurable functions $f : G \to \mathbb{C}$ for which the norm

$$\|f\|_{L^p(G)} = \left(\int_G |f(x)|^p dx \right)^{\frac{1}{p}}$$

is finite. The space $L^\infty(G)$ consists of all measurable functions $f : G \to \mathbb{C}$ for which

$$\|f\|_{L^\infty(G)} = \operatorname{ess\,sup}_{x \in G} |f(x)| < \infty,$$

where the essential supremum is taken with respect to the Haar measure on G. As usual we identify functions in $L^p(G)$ that differ only on a set of measure zero.

Elementary Operations for Functions on G. For a function f on G and $x \in G$ the *left translation* is defined by

$$L_x f(y) = f(x^{-1}y),$$

and similarly the *right translation* by

$$R_x f(y) = f(yx).$$

Furthermore, the *involution* of a function f on G is given by

$$f^*(x) = \overline{f(x^{-1})},$$

and the *convolution* of two functions f and g on G by

(1.22) $$f * g(x) = \int_G f(y)g(y^{-1}x)dy$$

whenever the integral in (1.22) is defined.

For a function f on G and $r > 0$ the *dilation* is defined by

$$D_r f(y) = r^{\frac{D}{2}} f(\delta_r y),$$

where D denotes the homogeneous dimension of G. The normalization $r^{\frac{D}{2}}$ is chosen so that D_r becomes a unitary operator on $L^2(G)$. Indeed, by the properties of the Haar measure with respect to dilations (equation (1.20)), we get

$$\begin{aligned}
\|D_r f\|_{L^2(G)}^2 &= \int_G |D_r f(x)|^2 dx \\
&= \int_G r^D |f(\delta_r x)|^2 dx \\
&= \int_G |f(x)|^2 dx = \|f\|_{L^2(G)}^2.
\end{aligned}$$

For the later use we note how dilation interacts with involution and convolution. Namely,

$$(1.23) \qquad (D_r f)^*(x) \quad = \quad \overline{D_r f(x^{-1})} = r^{\frac{D}{2}} \overline{f(\delta_r(x^{-1}))}$$
$$= \quad r^{\frac{D}{2}} \overline{f((\delta_r x)^{-1})} = r^{\frac{D}{2}} f^*(\delta_r x) = D_r(f^*)(x)$$

and

$$(1.24) \qquad (D_r f * D_r g)(x) \quad = \quad \int_G D_r f(y) D_r g(y^{-1} x) dy$$
$$= \quad \int_G r^D f(\delta_r y) g(\delta_r(y^{-1} x)) dy$$
$$= \quad \int_G r^D f(\delta_r y) g((\delta_r y)^{-1} \delta_r x) dy$$
$$= \quad \int_G f(y) g(y^{-1} \delta_r x) dy = (f * g)(\delta_r x),$$

whenever f and g are such that the convolution (1.22) is defined.

1.4. Discrete Subsets of Homogeneous Groups

Definition 1.18. A subgroup Γ of G is called a *(uniform) lattice* if Γ is discrete and if the quotient $\Gamma \backslash G$ is compact. A set of representatives mod Γ is called a *fundamental domain* of Γ.

If U is a fundamental domain of a lattice Γ in G, then

$$(1.25) \qquad\qquad G = \Gamma U = \bigcup_{\gamma \in \Gamma} \gamma U$$

with $\gamma U \cap \gamma' U = \emptyset$ for $\gamma \neq \gamma'$. Since $\Gamma \backslash G$ is compact, the fundamental domain U can be chosen to be relatively compact.

Not every nilpotent Lie group admits a lattice. In fact, for nilpotent Lie groups the existence of a lattice depends on the structure constants of the associated Lie algebra being rational (confer, e.g., [**13**], p.200).

If we relax the conditions on a lattice and dispense with the group structure, we are led to the notion of a quasi-lattice.

Definition 1.19. A discrete subset $\Gamma \subseteq G$ is called a *quasi-lattice* if there exists a relatively compact Borel set U such that $G = \bigcup_{\gamma \in \Gamma} \gamma U$ and $\gamma U \cap \gamma' U = \emptyset$ for $\gamma \neq \gamma'$. Such a set U is called a *complement of* Γ.
For simplicity we assume that U contains the identity.

Quasi-lattices always exist in homogeneous groups. This is true more general in every connected simply connected nilpotent Lie group (cf. [**23**]). For the reader's convenience we recall the proof and especially emphasize that we can choose a connected complement with non-empty interior and boundary of measure zero.

First we recall a well-known factorization of G into lower-dimensional closed subgroups.

Lemma 1.20. *Let G be a connected simply connected nilpotent Lie group.*

  (a) *There exist a closed normal subgroup N of codimension one and a closed subgroup H of dimension one such that $G = NH$ and $N \cap H = \{e\}$.*
  (b) *If $N \times H$ is equipped with the product topology, then the map*

  $$\alpha : N \times H \to G, \ \alpha(n, h) = nh$$

  is a homeomorphism.
  (c) *The Haar measures λ_G, λ_N and λ_H of G, N and H can be normalized such that*

  $$\int_G f(x)d\lambda_G(x) = \int_N \int_H f(nh)d\lambda_H(h)d\lambda_N(n)$$

  for every $f \in L^1(G)$.

PROOF. (a) Fix a Malcev basis $\{X_1, \dots, X_n\}$ of the Lie algebra $\mathfrak{g}$ and set $\mathfrak{n} = \mathrm{span}\{X_1, \dots, X_{n-1}\}$ and $\mathfrak{h} = \mathbb{R}X_n$. Then $N := \exp(\mathfrak{n})$ and $H := \exp(\mathfrak{h})$ are closed connected simply connected subgroups of G, $N \cap H = \{e\}$, N is normal and $G = NH$ (confer, e.g., [**13**], p. 16, Proposition 1.2.7).

(b) By (a), every element $g \in G$ can be uniquely written in the form $g = nh$ with $n \in N$ and $h \in H$. Thus the map $\alpha : N \times H \to G$, $\alpha(n, h) = nh$ is a bijection and easily seen to be continuous. That α is even a homeomorphism follows by application of the Open Mapping Theorem for locally compact groups (see, e.g., [**43**], p. 60, 61).

(c) For $f \in L^1(G)$ and $x' = n'h' \in G$ we calculate

$$\int_N \int_H f(nhn'h')d\lambda_H(h)d\lambda_N(n) = \int_N \int_H f(nhn'h^{-1}hh')d\lambda_H(h)d\lambda_N(n)$$

$$= \int_H \int_N f(n(hn'h^{-1})hh')d\lambda_N(n)d\lambda_H(h).$$

Since N is normal and the Haar measures λ_N on N and λ_H on H are right-invariant, we get

$$\int_H \int_N f(n(hn'h^{-1})hh')d\lambda_N(n)d\lambda_H(h) = \int_H \int_N f(nhh')d\lambda_N(n)d\lambda_H(h)$$

$$= \int_H \int_N f(nh)d\lambda_N(n)d\lambda_H(h)$$

$$= \int_N \int_H f(nh)d\lambda_H(h)d\lambda_N(n).$$

Thus the integral

$$\int_N \int_H f(nh)d\lambda_H(h)d\lambda_N(n)$$

is invariant under right translation, so the assertion (c) follows from the uniqueness of the Haar measure.

$\square$

Proposition 1.21. *Let G be a connected simply connected nilpotent Lie group. Then there exists a quasi-lattice Γ in G and a connected complement U of Γ with non-empty interior and boundary of measure zero.*

PROOF. The proof is by induction on $n = \dim G$.
For the one-dimensional case take the isomorphic image of the lattice $\mathbb{Z} \subseteq \mathbb{R}$ with fundamental domain $[0, 1)$.

For the induction step we consider the factorization $G = NH$ as constructed in Lemma 1.20. Note that N and H are connected simply connected nilpotent Lie groups of dimension $n - 1$ and one respectively. By induction hypothesis, there exists a quasi-lattice Γ_0 in N and a connected complement U_0 with non-empty interior and boundary of measure zero in N, and a quasi-lattice Γ_1 in H and a connected complement U_1 with non-empty interior and boundary of measure zero in H. Define

$$\Gamma := \Gamma_1 \Gamma_0 = \{\gamma_1 \gamma_0 : \gamma_1 \in \Gamma_1, \gamma_0 \in \Gamma_0\},$$
$$U := U_0 U_1 = \{u_0 u_1 : u_0 \in U_0, u_1 \in U_1\}.$$

We claim that Γ is a quasi-lattice in G with complement U.
First observe that

$$\Gamma U = \Gamma_1 \Gamma_0 U_0 U_1 = \Gamma_1 N U_1 = N \Gamma_1 U_1 = NH = G,$$

because N is normal and Γ_0, Γ_1 are quasi-lattices for N and H.

To show that this covering $G = \Gamma U = \bigcup_{\gamma \in \Gamma} \gamma U$ is disjoint, we suppose that $\gamma U \cap \gamma' U \neq \emptyset$, i.e., that there exist $u, u' \in U$ such that $\gamma u = \gamma' u'$. By the definition of Γ and U, this means that

$$\gamma u = \gamma_1 \gamma_0 u_0 u_1 = \gamma_1' \gamma_0' u_0' u_1' = \gamma' u'$$

for some $\gamma_1, \gamma_1' \in \Gamma_1, \gamma_0, \gamma_0' \in \Gamma_0, u_1, u_1' \in U_1, u_0, u_0' \in U_0$.

Since N is normal and since $\Gamma_0 U_0$ and $\Gamma_1 U_1$ are coverings of N and H, we get

$$\gamma u = \gamma_1 \gamma_0 u_0 u_1 = (\gamma_1 \gamma_0 u_0 \gamma_1^{-1})(\gamma_1 u_1) = nh,$$

$$\gamma' u' = \gamma_1' \gamma_0' u_0' u_1' = (\gamma_1' \gamma_0' u_0' (\gamma_1')^{-1})(\gamma_1' u_1') = n'h'.$$

By the uniqueness of the factorization $G = NH$, it follows that $n = n'$ and $h = h'$, that is, $\gamma_1 \gamma_0 u_0 \gamma_1^{-1} = \gamma_1' \gamma_0' u_0' (\gamma_1')^{-1}$ and $\gamma_1 u_1 = \gamma_1' u_1'$. Since Γ_1 is a quasi-lattice in H, we conclude that $\gamma_1 = \gamma_1'$, and consequently $\gamma_0 u_0 = \gamma_0' u_0'$. Since Γ_0 is a quasi-lattice in N, we further obtain that $\gamma_0 = \gamma_0'$. Therefore $\gamma = \gamma'$. Hence we have $G = \bigcup_{\gamma \in \Gamma} \gamma U$ and $\gamma U \cap \gamma' U = \emptyset$ for $\gamma \neq \gamma'$.

To show that U has non-empty interior we note that, by induction hypothesis, there exists a non-empty subset $B_0 \subseteq U_0$ that is open in N and a non-empty subset $B_1 \subseteq U_1$ that is open in H. By Lemma 1.20 (b), it follows that $B_0 B_1$ is open in G with $B_0 B_1 \subseteq U_0 U_1$, so $U = U_0 U_1$ has non-empty interior.

Concerning the measure of the boundary of $U = U_0 U_1$ we note that, by Lemma 1.20 (b),

(1.26) $$\partial_G(U_0 U_1) \subseteq (\partial_N U_0)\overline{U_1} \cup \overline{U_0}(\partial_H U_1).$$

By induction hypothesis and Lemma 1.20 (c), it follows that

$$\lambda_G(\partial_G U) \leq \lambda_G((\partial_N U_0)\overline{U_1}) + \lambda_G(\overline{U_0}(\partial_H U_1))$$
$$= \lambda_N(\partial_N U_0)\lambda_H(\overline{U_1}) + \lambda_N(\overline{U_0})\lambda_H(\partial_H U_1) = 0.$$

Finally we remark that Γ is countable as the product of two countable sets, and U is connected and relatively compact as the product of two connected relatively compact sets. This completes the proof. $\square$

In the following we can therefore always assume that we deal with a connected complement U with non-empty interior and boundary of measure zero.

Translation and dilation of a quasi-lattice again give a quasi-lattice.

Lemma 1.22. *Let Γ be a quasi-lattice in G with complement U, let $g \in G$ and $r > 0$.*
(i) The set $g\Gamma$ is a quasi-lattice in G with complement U.
(ii) The set $\delta_r \Gamma$ is a quasi-lattice in G with complement $\delta_r U$.

Proof. (i) For arbitrary $x \in G$ write $x = gy$, where $y = g^{-1}x$. Since Γ is a quasi-lattice, there exist $\gamma \in \Gamma$ and $u \in U$ such that $y = \gamma u$, hence $x = g\gamma u$. If now $x \in g\gamma U \cap g\gamma' U$, then $y = g^{-1}x \in \gamma U \cap \gamma' U$. Thus $\gamma = \gamma'$, because Γ is a quasi-lattice.

(ii) Since δ_r is an automorphism on G, we have

$$G = \delta_r G = \delta_r \Big(\bigcup_{\gamma \in \Gamma} \gamma U \Big) = \bigcup_{\gamma \in \Gamma} (\delta_r \gamma) \delta_r U.$$

Suppose now that $(\delta_r \gamma \cdot \delta_r U) \cap (\delta_r \gamma' \cdot \delta_r U) \neq \emptyset$, or equivalently, since δ_r is a homomorphism, $\delta_r(\gamma U) \cap \delta_r(\gamma' U) \neq \emptyset$. Since δ_r is also bijective, it follows that $\gamma U \cap \gamma' U \neq \emptyset$ and hence $\gamma = \gamma'$, because Γ is a quasi-lattice.

$\square$

In the following we denote the cardinality of a subset $X \subseteq G$ by $|X|$.

Definition 1.23. A subset $X \subseteq G$ is called *relatively separated* if

$$(1.27) \qquad \max_{g \in G} |X \cap gU| < \infty$$

for some relatively compact subset U of G with non-empty interior.

In other words, a subset X is relatively separated if the number of elements of X that lie in any left translate of U is uniformly bounded.

For later use we state some well-known equivalent conditions.

Lemma 1.24. *For a subset $X \subseteq G$ the following statements are equivalent.*

(i) *X is relatively separated.*
(ii) *For every relatively compact subset V of G with non-empty interior*
$$\max_{g \in G} |X \cap gV| < \infty.$$
(iii) *For every relatively compact subset V of G with non-empty interior the sum $\sum_{\chi \in X} \mathbb{1}_{\chi V}$ is uniformly bounded on G, that is,*
$$\sup_{g \in G} \sum_{\chi \in X} \mathbb{1}_{\chi V}(g) < \infty.$$

Proof. $(i) \Rightarrow (ii)$: Let V be an arbitrary relatively compact subset of G with non-empty interior and let U be as in equation (1.27). Then $\bigcup_{g \in \overline{V}} gU^o$ is an open covering of $\overline{V}$. Since $\overline{V}$ is compact, there exists a finite subcover $\bigcup_{i=1}^n g_i U^o$ of $\overline{V}$. It follows that

$$
\begin{aligned}
\max_{g \in G} |X \cap gV| \ &\leq\ \max_{g \in G} |X \cap g\overline{V}| \leq \max_{g \in G} \Big| X \cap g \bigcup_{i=1}^n g_i U \Big| \\
&=\ \max_{g \in G} \Big| X \cap \bigcup_{i=1}^n g g_i U \Big| \leq \max_{g \in G} \sum_{i=1}^n |X \cap g g_i U| \\
&\leq\ \sum_{i=1}^n \max_{g \in G} |X \cap g g_i U| < \infty.
\end{aligned}
$$

$(ii) \Rightarrow (iii)$: Let V be an arbitrary relatively compact subset of G with non-empty interior. Then the set V^{-1} is also relatively compact with non-empty interior and

thus
$$\max_{g \in G} |X \cap gV^{-1}| < \infty.$$

Now

$$
\begin{aligned}
\sup_{g \in G} \sum_{\chi \in X} \mathbb{1}_{\chi V}(g) \ &= \ \max_{g \in G} |\{\chi \in X : g \in \chi V\}| \\
&= \ \max_{g \in G} |\{\chi \in X : \chi \in gV^{-1}\}| \\
(1.28) \qquad\qquad &= \ \max_{g \in G} |X \cap gV^{-1}| < \infty.
\end{aligned}
$$

$(iii) \Rightarrow (i)$: This implication also follows from equation (1.28).

$\square$

To verify that a subset of G is relatively separated it suffices to consider the left translates by elements of a lattice and count the elements therein.

Lemma 1.25. *Let Γ be a lattice in G with fundamental domain U. A subset $X \subseteq G$ is relatively separated if and only if*

$$(1.29) \qquad\qquad \max_{\gamma \in \Gamma} |X \cap \gamma V| < \infty$$

for some relatively compact subset V of G that contains the fundamental domain U.

PROOF. Let V be a relatively compact subset of G that contains the fundamental domain U and satisfies

$$\max_{\gamma \in \Gamma} |X \cap \gamma V| =: C < \infty.$$

Let K be an arbitrary relatively compact subset of G. Then the set UK is also relatively compact and hence bounded, that is,

$$(1.30) \qquad\qquad UK \subseteq B_N(e)$$

for some $N > 0$. Let $N_U > 0$ be such that $U \subseteq B_{N_U}(e)$ and let $R > N + N_U$. A translate γU intersects $B_N(e)$ only if $\gamma \in B_R(e)$. Indeed, if

$$x \in B_N(e) \cap \gamma U \subseteq B_N(e) \cap B_{N_U}(\gamma),$$

then

$$d(e, \gamma) \leq d(e, x) + d(x, \gamma) < N + N_U = R.$$

Therefore,

$$UK \subseteq B_N(e) \subseteq \bigcup_{\gamma \in \Gamma \cap B_R(e)} \gamma U.$$

Now let $g \in G$ be arbitrary. Since Γ is a lattice, we may write $g = \nu u \in \nu U$ for some unique $\nu \in \Gamma$. Then

$$(1.31) \qquad gK \subseteq \nu UK \subseteq \bigcup_{\gamma \in \Gamma \cap B_R(e)} \nu \gamma U \subseteq \bigcup_{\gamma \in \Gamma \cap B_R(e)} \nu \gamma V.$$

Let $n := |\Gamma \cap B_R(e)| < \infty$ denote the number of lattice points in $B_R(e)$. It follows that

$$
\begin{aligned}
|X \cap gK| &\leq \left| X \cap \bigcup_{\gamma \in \Gamma \cap B_R(e)} \nu\gamma V \right| \\
&\leq \sum_{\gamma \in \Gamma \cap B_R(e)} |X \cap \nu\gamma V| \\
&\leq n \max_{\gamma \in \Gamma} |X \cap \gamma V| = nC < \infty.
\end{aligned}
$$

Since $g \in G$ was arbitrary, we conclude that

$$
\max_{g \in G} |X \cap gK| \leq nC < \infty.
$$

Thus X is relatively separated.

$\square$

1.5. Wiener Amalgam Spaces

Throughout this section let G be a homogeneous group.

Definition 1.26. Let V be a relatively compact subset of G with non-empty interior and let $1 \leq p, q \leq \infty$. The *Wiener Amalgam Space* $W(L^p, L^q)$ consists of all functions $f : G \to \mathbb{C}$ for which the associated control function

$$
x \mapsto \|f \cdot L_x \mathbb{1}_V\|_{L^p(G)} = \|f \cdot \mathbb{1}_{xV}\|_{L^p(G)}
$$

belongs to $L^q(G)$. For $1 \leq q < \infty$ a norm on $W(L^p, L^q)$ is given by

$$
\|f\|_{W(L^p,L^q)} := \left(\int_G \|f \cdot \mathbb{1}_{xV}\|_p^q dx \right)^{\frac{1}{q}} = \left(\int_G \left(\int_{xV} |f(y)|^p dy \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}},
$$

for $q = \infty$ by

$$
\|f\|_{W(L^p,L^q)} := \sup_{x \in G} \|f \cdot \mathbb{1}_{xV}\|_p.
$$

The definition of the Wiener Amalgam Spaces $W(L^p, L^q)$ allows some flexibility.

Lemma 1.27.

(a) $W(L^p, L^q)$ *does not depend on the particular choice of V, i.e., different relatively compact subsets of G with non-empty interior define the same space and equivalent norms.*

(b) *If Γ is a quasi-lattice in G with complement U, then also*

$$
\|f\|_{W(L^p,\ell^q)} := \left(\sum_{\gamma \in \Gamma} \|f \cdot \mathbb{1}_{\gamma U}\|_p^q \right)^{\frac{1}{q}}
$$

defines an equivalent norm on $W(L^p, L^q)$.

For a proof see [**18**] or [**33**].

In this text we mainly deal with the space $W(C, L^q)$, the subspace of $W(L^\infty, L^q)$ consisting of continuous functions. In this case it is costumary to denote the control function by

$$f^{\#}(x) := \sup_{y \in xV} |f(y)|$$

and $f^{\#}$ is called the *(left) local maximum function* of f. The norm on $W(C, L^q)$ is then given by

$$\|f\|_{W(C,L^q)} = \|f^{\#}\|_{L^q(G)}.$$

If the defining set is chosen to be the complement U of a quasi-lattice Γ, then the equivalent discrete norm from Lemma 1.27 is computed as

$$\|f\|_{W(C,\ell^q)} = \|f^{\#}|_\Gamma\|_{\ell^q(\Gamma)} = \left(\sum_{\gamma \in \Gamma} \sup_{x \in \gamma U} |f(x)|^q \right)^{\frac{1}{q}}.$$

By abuse of notation, $f^{\#}|_\Gamma$ will be simply denoted by $f^{\#}$ and also called (left) local maximum function of f.

Wiener Amalgam Spaces are invariant under left translation and dilation.

Lemma 1.28.
  (a) *If $f \in W(C, L^q)$, then $L_z f \in W(C, L^q)$ for every $z \in G$.*
  (b) *If $f \in W(C, L^q)$, then $D_r f \in W(C, L^q)$ for every $r > 0$.*

PROOF. (a) For every $x \in G$,

$$
\begin{aligned}
(L_z f)^{\#}(x) &= \sup_{y \in xV} |L_z f(y)| = \sup_{y \in xV} |f(z^{-1}y)| \\
&= \sup_{y \in z^{-1}xV} |f(y)| = f^{\#}(z^{-1}x) = L_z(f^{\#})(x).
\end{aligned}
$$

Therefore

$$\|(L_z f)^{\#}\|_{L^q(G)} = \|L_z(f^{\#})\|_{L^q(G)} = \|f^{\#}\|_{L^q(G)} < \infty.$$

(b) Recall that the definition of $W(C, L^q)$ does not depend on the particular choice of the defining set (Lemma 1.27). We temporarily indicate the defining set in the local maximum function by a subscript, e. g.,

$$f_V^{\#}(x) = \sup_{y \in xV} |f(y)|.$$

For every $x \in G$,

$$
\begin{aligned}
(D_r f)_V^{\#}(x) &= \sup_{y \in xV} |D_r f(y)| = \sup_{y \in xV} |r^{\frac{D}{2}} f(\delta_r y)| \\
&= r^{\frac{D}{2}} \sup_{z \in \delta_r x \cdot \delta_r V} |f(z)| = r^{\frac{D}{2}} f_{\delta_r V}^{\#}(\delta_r x).
\end{aligned}
$$

By equation (1.20) it follows that

$$
\begin{aligned}
\|(D_r f)_V^{\#}\|_{L^q(G)}^q &= \int_G |(D_r f)_V^{\#}(x)|^q dx \\
&= r^{\frac{qD}{2}} \int_G |f_{\delta_r V}^{\#}(\delta_r x)|^q dx \\
&= r^{\frac{(q-2)D}{2}} \int_G |f_{\delta_r V}^{\#}(x)|^q dx \\
&= r^{\frac{(q-2)D}{2}} \|f_{\delta_r V}^{\#}\|_{L^q(G)}^q < \infty.
\end{aligned}
$$

□

Next we collect some inclusion relations that will be useful in the sequel [**17**], [**18**].

Lemma 1.29.
(a) $C_c(G) \subseteq W(L^\infty, L^1)$,
(b) $W(L^p, L^p) = L^p(G)$,
(c) If $p_1 \geq p_2, q_1 \leq q_2$, then $W(L^{p_1}, L^{q_1}) \subseteq W(L^{p_2}, L^{q_2})$ and

$$\|f\|_{W(L^{p_2}, L^{q_2})} \leq C\|f\|_{W(L^{p_1}, L^{q_1})}.$$

(d) In particular, $W(C, L^p) \subseteq L^p(G)$ and

$$\|f\|_{L^p(G)} \leq C\|f\|_{W(C, L^p)}.$$

We will also need the following convolution relation for Amalgam Spaces on unimodular groups [**17**].

Proposition 1.30. *Let G be a unimodular locally compact group and let $p_1, p_2, q_1, q_2 \in [1, \infty]$ be such that $\frac{1}{p} := \frac{1}{p_1} + \frac{1}{p_2} - 1 \geq 0$ and $\frac{1}{q} := \frac{1}{q_1} + \frac{1}{q_2} - 1 \geq 0$. Then*

$$W(L^{p_1}, L^{q_1}) * W(L^{p_2}, L^{q_2}) \subseteq W(L^p, L^q).$$

Corollary 1.31. *For every $q \geq 1$ we have*

$$W(L^\infty, L^1) * L^q(G) \subseteq W(L^\infty, L^q).$$

PROOF. By Lemma 1.29, $L^q(G) = W(L^q, L^q) \subseteq W(L^1, L^q)$. Thus Proposition 1.30 implies

$$W(L^\infty, L^1) * L^q(G) \subseteq W(L^\infty, L^1) * W(L^1, L^q) \subseteq W(L^\infty, L^q).$$

□

For later application we recall the following useful estimate [**28**].

Lemma 1.32. *Let X be a relatively separated subset of G. Let V be a symmetric relatively compact subset of G used to define the local maximum function. Set $C := \sup_{x \in G} \sum_{\chi \in X} \mathbb{1}_{\chi V}(x)$ and let $N_V > 0$ be such that $V \subseteq B_{N_V}(e)$. Then, for $N > N_V$, we have*

$$\sum_{\chi \in X \backslash B_N(e)} |f(\chi)|^2 \leq \frac{C}{\lambda(V)} \int_{G \backslash B_{N-N_V}(e)} f^{\#}(x)^2 dx.$$

PROOF. From $V = V^{-1}$ it follows that $\chi \in xV$ whenever $x \in \chi V$. Thus

$$|f(\chi)| \leq f^{\#}(x) \quad \forall x \in \chi V$$

and consequently

$$|f(\chi)|^2 \leq \frac{1}{\lambda(V)} \int_{\chi V} f^{\#}(x)^2 dx.$$

Summing over $\chi \in X \backslash B_N(e)$, we obtain

$$
\begin{aligned}
\sum_{\chi \in X \backslash B_N(e)} |f(\chi)|^2 &\leq \frac{1}{\lambda(V)} \sum_{\chi \in X \backslash B_N(e)} \int_{\chi V} f^{\#}(x)^2 dx \\
&= \frac{1}{\lambda(V)} \sum_{\chi \in X \backslash B_N(e)} \int_G \mathbb{1}_{\chi V}(x) f^{\#}(x)^2 dx \\
&= \frac{1}{\lambda(V)} \int_G \sum_{\chi \in X \backslash B_N(e)} \mathbb{1}_{\chi V}(x) f^{\#}(x)^2 dx \\
&\leq \frac{C}{\lambda(V)} \int_G \mathbb{1}_{G \backslash B_{N-N_V}(e)}(x) f^{\#}(x)^2 dx \\
&= \frac{C}{\lambda(V)} \int_{G \backslash B_{N-N_V}(e)} f^{\#}(x)^2 dx,
\end{aligned}
$$

because, for all $x \in G$,

$$\sum_{\chi \in X \backslash B_N(e)} \mathbb{1}_{\chi V}(x) \leq \sum_{\chi \in X \backslash B_N(e)} \mathbb{1}_{B_{N_V}(\chi)}(x) \leq C \, \mathbb{1}_{G \backslash B_{N-N_V}(e)}(x).$$

$\square$

CHAPTER 2

Density on Homogeneous Groups

2.1. Definition of Density

Let G be a homogeneous group with homogeneous dimension D and Haar measure λ. Fix a subadditive homogeneous norm $|\ |$ on G, denote the associated left-invariant metric as defined in (1.16) by d and the corresponding balls by $B_N(g)$.

The existence of a left-invariant homogeneous metric allows us to define an analogue of the Beurling density [7], [35], however, adapted to the geometry of the homogeneous group G.

Definition 2.1. The *upper density* of a subset $X \subseteq G$ is defined by

$$D^+(X) := \limsup_{N \to \infty} \max_{g \in G} \frac{|X \cap B_N(g)|}{\lambda(B_N(e))},$$

and its *lower density* by

$$D^-(X) := \liminf_{N \to \infty} \min_{g \in G} \frac{|X \cap B_N(g)|}{\lambda(B_N(e))}.$$

A subset X of G is said to have *uniform density* $D(X)$ if

$$D^+(X) = D^-(X) =: D(X).$$

Our goal in this section is to show that the definition of the upper and lower density does not depend on the particular choice of the homogeneous norm.

For that define

$$(2.1) \qquad\qquad D_B^+(X) := \limsup_{N \to \infty} \max_{g \in G} \frac{|X \cap g \cdot \delta_N B|}{\lambda(\delta_N B)},$$

$$(2.2) \qquad\qquad D_B^-(X) := \liminf_{N \to \infty} \min_{g \in G} \frac{|X \cap g \cdot \delta_N B|}{\lambda(\delta_N B)}.$$

If $B := B_1(e)$, then the quantities $D_B^+(X)$ and $D_B^-(X)$ are just the upper and lower densities from Definition 2.1 rewritten, because, for every $g \in G$ and $N \in \mathbb{N}$,

$$B_N(g) = g \cdot B_N(e) = g \cdot \delta_N(B_1(e))$$

by the left-invariance and homogeneity of the metric d.

Therefore the invariance of the upper and lower densities under a change of the homogeneous norm follows from the invariance of the quantities $D_B^+(X)$ and $D_B^-(X)$ under a change of the defining set B.

It turns out that $D_B^+(X)$ and $D_B^-(X)$ do not depend on the defining set B as long as B is chosen to be a relatively compact subset of G with non-empty interior and boundary of measure zero.

The corresponding statement for the Beurling densities in $\mathbb{R}^n$ is due to Landau [35]. He showed that the Beurling density computed by means of a compact set with boundary of measure zero is the same as computed by means of cubes.

We adapt Landau's ideas to our setting. As a counterpart of the unit cube in $\mathbb{R}^n$ we use a complement of some quasi-lattice in G.

In view of Proposition 1.21, a complement U of a quasi-lattice Γ in G is in the remainder of this section always assumed to be connected and to have non-empty interior and boundary of measure zero.

Proposition 2.2. *Let Γ be a quasi-lattice in G with complement U and let B be a relatively compact subset of G with non-empty interior and boundary of measure zero. Then, for every relatively separated subset X of G, we have*

$$D_U^-(X) = D_B^-(X) \quad and \quad D_U^+(X) = D_B^+(X).$$

For the proof of Proposition 2.2 we need two auxiliary lemmata.

Lemma 2.3. *Let Γ be a quasi-lattice in G with complement U and let B be a relatively compact subset of G with non-empty interior and boundary of measure zero. For every $\varepsilon > 0$ there exist an $r > 0$ and finite subsets $S \subseteq S' \subseteq G$ such that*
(i)

$$\bigcup_{\gamma \in S}(\gamma \cdot \delta_r U) \subseteq B \quad with \quad \lambda\Big(\bigcup_{\gamma \in S}(\gamma \cdot \delta_r U)\Big) > \lambda(B) - \varepsilon,$$

(ii)

$$B \subseteq \bigcup_{\gamma \in S'}(\gamma \cdot \delta_r U) \quad with \quad \lambda\Big(\bigcup_{\gamma \in S'}\gamma \cdot \delta_r U\Big) < \lambda(B) + \varepsilon,$$

and $(\gamma \cdot \delta_r U) \cap (\gamma' \cdot \delta_r U) = \emptyset$ for $\gamma \neq \gamma' \in S'$.

PROOF. Since the Haar measure λ is outer regular (see e.g. [15], [22]), we have

$$0 = \lambda(\partial B) = \inf\{\lambda(O) : O \supseteq \partial B \text{ open}\}.$$

Thus, given $\varepsilon > 0$, there exists some open set $O_\varepsilon \supseteq \partial B$ such that $\lambda(O_\varepsilon) < \varepsilon$.

Consider the distance function

$$x \mapsto \operatorname{dist}(x, O_\varepsilon^c) := \inf\{d(x,y) : y \in O_\varepsilon^c\},$$

which is a continuous function from G into $\mathbb{R}^+$. Since ∂B is compact, the minimum of $\operatorname{dist}(x, O_\varepsilon^c)$ on ∂B exists. Since $\partial B \cap O_\varepsilon^c = \emptyset$,

$$\eta := \frac{1}{2} \min\{\operatorname{dist}(x, O_\varepsilon^c) : x \in \partial B\} > 0.$$

For every $x \in \partial B$ we then have $B_\eta(x) \subseteq O_\varepsilon$. Thus also the "$\eta$-tube"
$\mathcal{E} := \bigcup_{x \in \partial B} B_\eta(x)$ is contained in O_ε and $\lambda(\mathcal{E}) \leq \lambda(O_\varepsilon) < \varepsilon$.
Now choose $r > 0$ such that $\delta_r U \subseteq B_{\frac{\eta}{2}}(e)$. By Lemma 1.22 (ii), the set $\Gamma' := \delta_r \Gamma$ is a quasi-lattice for G with complement $U' := \delta_r U$. Consider the following subsets of Γ':

$$S := \{\gamma \in \Gamma' : \gamma U' \subseteq B^o\},$$
$$S' := \{\gamma \in \Gamma' : \gamma U' \cap \overline{B} \neq \emptyset\},$$
$$S'' := \{\gamma \in \Gamma' : \gamma U' \cap \partial B \neq \emptyset\};$$

Clearly $S \subseteq S' \backslash S''$. Conversely, if $\gamma \in S' \backslash S''$, then $\gamma U'$ does not intersect the boundary of B and thus can be written as $\gamma U' = (\gamma U' \cap B^o) \cup (\gamma U' \cap \overline{B}^c)$. Since U' is connected, it follows that $\gamma U' = \gamma U' \cap B^o$, hence $\gamma \in S$. Therefore $S' \backslash S'' = S$, or equivalently, $S' \backslash S = S''$. Since Γ' is a quasi-lattice for G, we get

$$\bigcup_{\gamma \in S} \gamma U' \subseteq B \subseteq \bigcup_{\gamma \in S'} \gamma U',$$

and it remains to prove that

$$\lambda\Big(\bigcup_{\gamma \in S' \backslash S} \gamma U'\Big) = \lambda\Big(\bigcup_{\gamma \in S''} \gamma U'\Big) < \varepsilon.$$

For that purpose we need to show that, for every $\gamma \in S''$, we have

$$\gamma U' \subseteq \mathcal{E} = \bigcup_{x \in \partial B} B_\eta(x).$$

Let $\gamma \in S''$ be arbitrary and choose some $x \in \partial B$ such that $x \in \gamma U'$. From the definition of U' and the left-invariance of the metric, it follows that
$\gamma U' \subseteq \gamma B_{\frac{\eta}{2}}(e) = B_{\frac{\eta}{2}}(\gamma)$, thus $x \in B_{\frac{\eta}{2}}(\gamma)$.
Now let $y \in \gamma U' \subseteq B_{\frac{\eta}{2}}(\gamma)$ be arbitrary. By the triangle inequality, we get

$$d(x,y) \leq d(x,\gamma) + d(\gamma, y) < \frac{\eta}{2} + \frac{\eta}{2} = \eta,$$

thus $y \in B_\eta(x) \subseteq \mathcal{E}$. It follows that $\gamma U' \subseteq \mathcal{E}$. Because $\gamma \in S''$ was arbitrary, we conclude that

$$\bigcup_{\gamma \in S''} \gamma U' \subseteq \mathcal{E} \subseteq O_\varepsilon.$$

In particular,

$$\lambda\Big(\bigcup_{\gamma \in S''} \gamma U'\Big) \leq \lambda(\mathcal{E}) \leq \lambda(O_\varepsilon) < \varepsilon.$$

Finally we remark that the sets S, S', S'' are finite, because

$$\lambda(\bigcup_{\gamma \in S'} \gamma U') - \lambda(B) = \lambda((\bigcup_{\gamma \in S'} \gamma U')\backslash B) \le \lambda(\bigcup_{\gamma \in S''} \gamma U') < \varepsilon$$

and thus

$$\lambda(\bigcup_{\gamma \in S'} \gamma U') = |S'|\lambda(U') < \lambda(B) + \varepsilon < \infty$$

by the properties of the Haar measure.

$\square$

Following Landau [**35**] we may also approximate the complement U from inside and outside by finite unions of translates of a dilated relatively compact set with boundary of measure zero. In contrast to Lemma 2.3 the covering is no longer a disjoint union.

Lemma 2.4. *Let* Γ *be a quasi-lattice in* G *with complement* U *and let* B *be a relatively compact subset of* G *with non-empty interior and boundary of measure zero. For every* $\varepsilon > 0$ *there exist finite subsets* $S \subseteq S' \subseteq G$ *and* $r_\gamma > 0$ *for* $\gamma \in S'$ *such that*
(i)

$$\bigcup_{\gamma \in S}(\gamma \cdot \delta_{r_\gamma} B) \subseteq U \quad with \quad \lambda(\bigcup_{\gamma \in S} \gamma \cdot \delta_{r_\gamma} B) > \lambda(U) - \varepsilon,$$

and $(\gamma \cdot \delta_{r_\gamma} B) \cap (\gamma' \cdot \delta_{r_{\gamma'}} B) = \emptyset$ *for* $\gamma \neq \gamma' \in S$,
(ii)

$$U \subseteq \bigcup_{\gamma \in S'}(\gamma \cdot \delta_{r_\gamma} B) \quad with \quad \sum_{\gamma \in S'}\lambda(\gamma \cdot \delta_{r_\gamma} B) < \lambda(U) + \varepsilon.$$

PROOF. Without loss of generality we assume that $\lambda(U) = \lambda(B) = 1$ (otherwise replace U by $\delta_r U$ with $r = \lambda(U)^{-\frac{1}{D}}$ and B by $\delta_t B$ with $t = \lambda(B)^{-\frac{1}{D}}$).

(i) First we exhaust U with left translates of dilated B.

By induction we show that for every $n \in \mathbb{N}$ there exists a finite disjoint union E_n of left translates of dilated versions of B contained in U such that
$$(2.3) \qquad\qquad \lambda(U\backslash E_n) \le (1 - \rho^D)(1 - \alpha\rho^D)^n$$

for some $0 < \rho, \alpha < 1$ that are independent of n.

Since B is relatively compact, there exist $0 < \rho < 1$ and $x \in G$ such that
$$E_0 := x \cdot \delta_\rho B \subseteq U.$$

The measure of the remainder is
$$\lambda(U\backslash E_0) = \lambda(U) - \lambda(x \cdot \delta_\rho B) = 1 - \rho^D.$$

This settles the initial step $n = 0$.

Now by the induction hypothesis there exists a finite disjoint union E_{n-1} of left translates of dilated versions of B contained in U such that

$$(2.4) \qquad \lambda(U \backslash E_{n-1}) \leq (1 - \rho^D)(1 - \alpha\rho^D)^{n-1}$$

for some $\alpha < 1$ and ρ as above. The remainder $R_n := U \backslash E_{n-1}$ is a relatively compact subset of G whose boundary has measure zero. So by Lemma 2.3 (i) we may exhaust R_n from inside by a finite disjoint union of left translates of a sufficiently dilated U with measure arbitrarily close to $\lambda(R_n)$. Formally, there exist an $r_n > 0$ and a finite set $S_n \subseteq G$ such that

$$(2.5) \qquad \bigcup_{\gamma \in S_n} (\gamma \cdot \delta_{r_n} U) \subseteq R_n \quad \text{and} \quad \lambda\Big(\bigcup_{\gamma \in S_n} (\gamma \cdot \delta_{r_n} U)\Big) \geq \alpha\lambda(R_n).$$

By the left invariance and homogeneity of the Haar measure,

$$(2.6) \qquad \lambda\Big(\bigcup_{\gamma \in S_n} (\gamma \cdot \delta_{r_n} U)\Big) = \sum_{\gamma \in S_n} \lambda(\delta_{r_n} U) = |S_n| r_n^D \geq \alpha\lambda(R_n).$$

Since $x \cdot \delta_\rho B \subseteq U$,

$$(2.7) \qquad \bigcup_{\gamma \in S_n} ((\gamma\delta_{r_n} x) \cdot \delta_{r_n\rho} B) = \bigcup_{\gamma \in S_n} (\gamma \cdot \delta_{r_n}(x \cdot \delta_\rho B)) \subseteq \bigcup_{\gamma \in S_n} (\gamma \cdot \delta_{r_n} U) \subseteq R_n.$$

From inequality (2.6) it follows that

$$(2.8) \qquad \lambda\Big(\bigcup_{\gamma \in S_n} ((\gamma\delta_{r_n} x) \cdot \delta_{r_n\rho} B))\Big) = \sum_{\gamma \in S_n} \lambda(\delta_{r_n\rho} B) = |S_n| r_n^D \rho^D \geq \rho^D \alpha\lambda(R_n).$$

Now define

$$(2.9) \qquad E_n := E_{n-1} \cup \bigcup_{\gamma \in S_n} ((\gamma\delta_{r_n} x) \cdot \delta_{r_n\rho} B).$$

By estimate (2.8) and the induction hypothesis (2.4), it follows that

$$
\begin{aligned}
\lambda(U \backslash E_n) =& \lambda\Big(U \backslash \big(E_{n-1} \cup \bigcup_{\gamma \in S_n} ((\gamma\delta_{r_n} x) \cdot \delta_{r_n\rho} B))\big)\Big) \\
=& \lambda\Big((U \backslash E_{n-1}) \backslash \bigcup_{\gamma \in S_n} ((\gamma\delta_{r_n} x) \cdot \delta_{r_n\rho} B)\Big) \\
=& \lambda(R_n) - \lambda\Big(\bigcup_{\gamma \in S_n} ((\gamma\delta_{r_n} x) \cdot \delta_{r_n\rho} B)\Big) \\
\leq& \lambda(R_n) - \alpha\rho^D \lambda(R_n) \\
=& \lambda(R_n)(1 - \alpha\rho^D) \\
\leq& (1 - \rho^D)(1 - \alpha\rho^D)^{n-1}(1 - \alpha\rho^D) \\
=& (1 - \rho^D)(1 - \alpha\rho^D)^n.
\end{aligned}
$$

Given $\varepsilon > 0$ we can now choose n large enough such that $\lambda(U \backslash E_n) < \varepsilon$. Then E_n is a finite disjoint union of left translates of dilated versions of B with measure

$$\lambda(E_n) = \lambda(U \backslash (U \backslash E_n)) = 1 - \lambda(U \backslash E_n) > 1 - \varepsilon.$$

(ii) To cover U by left translates of dilated versions of B first choose $\sigma > 1$ and $y \in G$ such that $U \subseteq y \cdot \delta_\sigma B$. By the proof of (i), there exists a finite disjoint union E_k of left translates of dilated versions of B such that, for given $\varepsilon > 0$ and fixed $\beta > 1$,

$$\lambda(U \backslash E_k) < \frac{\varepsilon}{\sigma^D \beta - 1}.$$

Now use Lemma 2.3 (ii) to cover the remainder $R_{k+1} := U \backslash E_k$, which is a relatively compact subset of G whose boundary has measure zero, by a finite disjoint union of sufficiently dilated left translates of U, formally,

$$(2.10) \qquad R_{k+1} \subseteq \bigcup_{\gamma \in S'} (\gamma \cdot \delta_r U),$$

such that

$$(2.11) \qquad \lambda \Big(\bigcup_{\gamma \in S'} (\gamma \cdot \delta_r U) \Big) = |S'| r^D \leq \beta \lambda(R_{k+1}).$$

Since $U \subseteq y \cdot \delta_\sigma B$, we get

$$(2.12) \qquad R_{k+1} \subseteq \bigcup_{\gamma \in S'} (\gamma \cdot \delta_r U) \subseteq \bigcup_{\gamma \in S'} \big((\gamma \delta_r y) \cdot \delta_{r\sigma} B \big)$$

with

$$(2.13) \qquad \sum_{\gamma \in S'} \lambda \big((\gamma \delta_r y) \cdot \delta_{r\sigma} B \big) = \sigma^D |S'| r^D \leq \sigma^D \beta \lambda(R_{k+1}).$$

Altogether we get that

$$U = E_k \cup (U \backslash E_k) \subseteq E_k \cup \bigcup_{\gamma \in S'} \big((\gamma \delta_r y) \cdot \delta_{r\sigma} B \big)$$

and, by the choice of E_k,

$$\lambda(E_k) + \sum_{\gamma \in S'} \lambda \big((\gamma \delta_r y) \cdot \delta_{r\sigma} B \big) = \lambda(U \backslash R_{k+1}) + \sum_{\gamma \in S'} \lambda \big((\gamma \delta_r y) \cdot \delta_{r\sigma} B \big)$$
$$\leq 1 - \lambda(R_{k+1}) + \sigma^D \beta \lambda(R_{k+1})$$
$$= 1 + \lambda(R_{k+1})(\sigma^D \beta - 1) < 1 + \varepsilon.$$

Since E_k is a disjoint union of left translates of dilated versions of B, the statement of Lemma 2.4 (ii) follows.

$\square$

PROOF OF PROPOSITION 2.2. By Lemma 2.3, we may choose $r > 0$ small enough and a finite subset $S \subseteq G$ such that

$$\bigcup_{\gamma \in S}(\gamma \cdot \delta_r U) \subseteq B \text{ with } \lambda(\bigcup_{\gamma \in S} \gamma \cdot \delta_r U) = |S| r^D \lambda(U) > \lambda(B) - \varepsilon$$

and $(\gamma \cdot \delta_r U) \cap (\gamma' \cdot \delta_r U) = \emptyset$ for $\gamma \neq \gamma' \in S$.

For arbitrary $g \in G$ and $N > 0$ we estimate

$$\begin{aligned}
|X \cap g \cdot \delta_N B| &\geq |X \cap g \cdot \delta_N \big(\bigcup_{\gamma \in S}(\gamma \cdot \delta_r U)\big)| \\
&= |X \cap \bigcup_{\gamma \in S}(g \cdot \delta_N \gamma) \cdot \delta_{Nr} U| \\
&= \sum_{\gamma \in S} |X \cap (g \cdot \delta_N \gamma) \cdot \delta_{Nr} U|,
\end{aligned}$$

where the last equation follows because $\bigcup_{\gamma \in S}(g \cdot \delta_N \gamma) \cdot \delta_{Nr} U$ is a disjoint union as a translate and dilate of the disjoint union $\bigcup_{\gamma \in S}(\gamma \cdot \delta_r U)$.

Hence

$$\begin{aligned}
\min_{g \in G} |X \cap g \cdot \delta_N B| &\geq \min_{g \in G} \sum_{\gamma \in S} |X \cap (g \cdot \delta_N \gamma) \cdot \delta_{Nr} U| \\
&\geq \sum_{\gamma \in S} \min_{g \in G} |X \cap (g \cdot \delta_N \gamma) \cdot \delta_{Nr} U| \\
&= \sum_{\gamma \in S} \min_{g' \in G} |X \cap g' \cdot \delta_{Nr} U| \\
&= |S| \min_{g' \in G} |X \cap g' \cdot \delta_{Nr} U|
\end{aligned}$$

and, by the homogeneity properties of the Haar measure,

$$\begin{aligned}
\min_{g \in G} \frac{|X \cap g \cdot \delta_N B|}{\lambda(\delta_N B)} &\geq |S| \min_{g' \in G} \frac{|X \cap g' \cdot \delta_{Nr} U|}{\lambda(\delta_N B)} \\
&= |S| \min_{g' \in G} \frac{|X \cap g' \cdot \delta_{Nr} U|}{N^D \lambda(B)} \\
&= |S| \min_{g' \in G} \frac{|X \cap g' \cdot \delta_{Nr} U|}{N^D r^D \lambda(U)} \cdot \frac{r^D \lambda(U)}{\lambda(B)} \\
&= |S| \min_{g' \in G} \frac{|X \cap g' \cdot \delta_{Nr} U|}{\lambda(\delta_{Nr} U)} \cdot \frac{r^D \lambda(U)}{\lambda(B)}.
\end{aligned}$$

Finally we derive that

$$
\begin{aligned}
D_B^-(X) &= \liminf_{N\to\infty} \min_{g\in G} \frac{|X \cap g \cdot \delta_N B|}{\lambda(\delta_N B)} \\
&\geq \liminf_{N\to\infty} \min_{g'\in G} \frac{|X \cap g' \cdot \delta_{Nr} U|}{\lambda(\delta_{Nr} U))} \cdot \frac{|S| r^D \lambda(U)}{\lambda(B)} \\
&= D_U^-(X) \cdot \frac{|S| r^D \lambda(U)}{\lambda(B)} \\
&> D_U^-(X) \cdot \left(1 - \frac{\varepsilon}{\lambda(B)}\right)
\end{aligned}
$$

Since $\varepsilon > 0$ was arbitrary, it follows that $D_B^-(X) \geq D_U^-(X)$.

Using the approximation of U by dilated left translates of B obtained in Lemma 2.4 we can employ the same argument with the roles of U and B interchanged to get $D_U^-(X) \geq D_B^-(X)$. Therefore $D_U^-(X) = D_B^-(X)$.

The analogous argument, using the coverings of B and U obtained in Lemma 2.3 and Lemma 2.4 respectively, shows the equality $D_U^+(X) = D_B^+(X)$ for the upper density.

$\square$

Remark 2.5. For $\mathbb{R}^n$ with the usual addition and scalar multiplication, the densities in Definition 2.1 just recover the standard upper and lower Beurling densities, that is,

$$
D^+(X) = \limsup_{N\to\infty} \max_{y\in\mathbb{R}^n} \frac{|X \cap y + [0,N]^n|}{N^n},
$$

$$
D^-(X) = \liminf_{N\to\infty} \min_{y\in\mathbb{R}^n} \frac{|X \cap y + [0,N]^n|}{N^n}.
$$

2.2. 'Homogeneous' Density versus Beurling Density

By the results in Section 1.1, every homogeneous group G can be identified with $\mathbb{R}^n$ endowed with a group law that is polynomial in the coordinates and with a family of dilations.

As a set, every subset of G is therefore essentially a subset of $\mathbb{R}^n$. One may ask what happens if we just compute the usual Beurling density on $\mathbb{R}^n$. Does the density actually depend on the group structure that is imposed on $\mathbb{R}^n$?

In the present section we answer this question in the affirmative and show that differences already manifest on the simplest non-commutative homogeneous group, the Heisenberg group.

Recall from Example 1.7 and Example 1.14 that in Malcev coordinates the Heisenberg group $\mathbb{H}$ is $\mathbb{R}^3$ with the group law

$$(a, b, c) \cdot (u, v, w) = (a + u + cv, b + v, c + w)$$

and dilations

$$\delta_r(a, b, c) = (r^2 a, rb, rc).$$

In the following we present an example of a set $X \subseteq \mathbb{R}^3$ that, regarded as a subset of $\mathbb{H}$, has lower density different from the standard lower Beurling density on $\mathbb{R}^3$.

Define

$$X := \left(\mathbb{R}^3 \backslash \bigcup_{k=1}^{\infty} \left((0, 0, k^3) + [0, k)^3 \right) \right) \cap \mathbb{Z}^3.$$

Regarded as a subset of the vector space $\mathbb{R}^3$, the set X has lower Beurling density $D_{\mathbb{R}^3}^-(X) = 0$.

Indeed, by construction of X,

$$\min_{g \in \mathbb{R}^3} \frac{|X \cap g + [0, N]^3|}{N^3} = 0,$$

hence

$$D_{\mathbb{R}^3}^-(X) = \liminf_{N \to \infty} \min_{g \in \mathbb{R}^3} \frac{|X \cap g + [0, N]^3|}{N^3} = 0.$$

On the other hand, the lower density of $X \subseteq \mathbb{H}$ in the sense of Definition 2.1 satisfies $D_{\mathbb{H}}^-(X) = 1$.

In view of Proposition 2.2, we may choose $U = [0, 1)^3$ and compute the lower density of X as

$$D_{\mathbb{H}}^-(X) = \liminf_{N \to \infty} \min_{g \in \mathbb{H}} \frac{|X \cap g \cdot \delta_N U|}{\lambda(\delta_N U)}.$$

If $U = [0, 1)^3$, then $\delta_N U = [0, N^2) \times [0, N) \times [0, N)$ and

$$(2.14) \quad (a, b, c) \cdot \delta_N U$$
$$= \{(a + u + cv, b + v, c + w) : (u, v, w) \in [0, N^2) \times [0, N) \times [0, N)\}.$$

First count the integer triples $(l, m, n) \in \mathbb{Z}^3$ in $(a, b, c) \cdot \delta_N U$. By equation (2.14), there are N possible values for m and n and N^2 possible values for l such that $(l, m, n) \in (a, b, c) \cdot \delta_N U$. Thus, for every $g = (a, b, c) \in \mathbb{H}$,

$$(2.15) \qquad |\mathbb{Z}^3 \cap g \cdot \delta_N U| = N^4 = |\mathbb{Z}^3 \cap \delta_N U|.$$

Since $X \subseteq \mathbb{Z}^3$,

$$\min_{g \in \mathbb{H}} |X \cap g \cdot \delta_N U| = |\mathbb{Z}^3 \cap \delta_N U| - \max_{g \in \mathbb{H}} |(\mathbb{Z}^3 \backslash X) \cap g \cdot \delta_N U|.$$

For $N > 2$ we show the estimate

$$\max_{g \in \mathbb{H}} |(\mathbb{Z}^3 \backslash X) \cap g \cdot \delta_N U| = \max_{g \in \mathbb{H}} \left| \left(\mathbb{Z}^3 \cap \bigcup_{k=1}^{\infty} \left((0,0,k^3) + [0,k)^3 \right) \right) \cap g \cdot \delta_N U \right| \leq N^3.$$

We distinguish the following cases for $g = (a,b,c) \in \mathbb{H}$:

  (i)
$$(a,b,c) \cdot \delta_N U \cap \left((0,0,k^3) + [0,k)^3 \right) \neq \emptyset \text{ for some } k \geq N;$$

  (ii)
$$(a,b,c) \cdot \delta_N U \cap \left((0,0,k^3) + [0,k)^3 \right) \neq \emptyset \text{ for some } k < N;$$

  (iii)
$$(a,b,c) \cdot \delta_N U \cap \bigcup_{k=1}^{\infty} \left((0,0,k^3) + [0,k)^3 \right) = \emptyset.$$

In case (i), the translate $(a,b,c) \cdot \delta_N U$ intersects exactly one cube $(0,0,k^3) + [0,k)^3$, because $k \geq N$.

We need to count the integer triples $(l,m,n) \in \mathbb{Z}^3$ in

$$(a,b,c) \cdot \delta_N U \cap \left((0,0,k^3) + [0,k)^3 \right).$$

Since $N \leq k$, there are at most N possible values for m and n and N^2 possible values for l such that $(l,m,n) \in (a,b,c) \cdot \delta_N U \cap ((0,0,k^3) + [0,k)^3)$.

However, suppose that for some $(u,v,w) \in \delta_N U$,

$$(l,m,n) = (a,b,c) \cdot (u,v,w)$$
$$= (a+u+cv, b+v, c+w) \in [0,k) \times [0,k) \times [k^3, k^3+k) \cap \mathbb{Z}^3.$$

This assumption implies that

(2.16) $$c \geq k^3 - w > k^3 - N \geq k^3 - k.$$

But then, if $j \in \mathbb{Z} \backslash \{0\}$ and $u' \in [0, N^2)$, the element

$$(a,b,c) \cdot (u', v+j, w) = (a+u'+cv+cj, b+v+j, c+w)$$

cannot be in $[0,k) \times [0,k) \times [k^3, k^3+k)$, because

(2.17) $$|a+u'+cv| \leq |a+u+cv| + |u'-u| < k + N^2 \leq k + k^2$$

and therefore, by the inequalities (2.16) and (2.17),

$$|a + u' + cv + cj| \geq |c| - |a + u' + cv| \geq k^3 - k - k - k^2 > k.$$

Consequently there is at most one integer m such that a triple of the form $(l, m, n) \in \mathbb{Z}^3$ is in $(a, b, c) \cdot \delta_N U \cap [0, k) \times [0, k) \times [k^3, k^3 + k)$.

We conclude that

$$\left|(a, b, c) \cdot \delta_N U \cap \left((0, 0, k^3) + [0, k)^3\right) \cap \mathbb{Z}^3\right| = \left|(a, b, c) \cdot \delta_N U \cap (\mathbb{Z}^3 \backslash X)\right| \leq N^3.$$

In case (ii), the translate $(a, b, c) \cdot \delta_N U$ may intersect $\bigcup_{k=1}^{\infty} \left((0, 0, k^3) + [0, k)^3\right)$ in more than one cube with side length strictly less than N, say in the $m + 1$ cubes

$$(0, 0, k_0^3) + [0, k_0)^3, \ldots, (0, 0, (k_0 + m)^3) + [0, k_0 + m)^3,$$

where $k_0 + m < N$.

If $m = 0$, then

$$\left|(a, b, c) \cdot \delta_N U \cap (0, 0, k_0^3) + [0, k_0)^3\right| \leq k_0^3 \leq N^3.$$

If $m \geq 1$, then

$$\left|(a, b, c) \cdot \delta_N U \cap (\mathbb{Z}^3 \backslash X)\right| \leq \sum_{i=0}^{m} (k_0 + i)^3$$

$$= \sum_{i=0}^{m} \left(k_0^3 + 3ik_0^2 + 3i^2 k_0 + i^3\right)$$

$$= \sum_{i=0}^{m} k_0^3 + 3k_0^2 \sum_{i=0}^{m} i + 3k_0 \sum_{i=0}^{m} i^2 + \sum_{i=0}^{m} i^3.$$

Computing the sums we thus get

$$\left|(a, b, c) \cdot \delta_N U \cap (\mathbb{Z}^3 \backslash X)\right| \leq (m + 1)k_0^3$$

$$+ \frac{3}{2}m(m + 1)k_0^2$$

(2.18) $$+ \frac{1}{2}m(m + 1)(2m + 1)k_0$$

$$+ \frac{1}{4}m^2(m + 1)^2.$$

We want to estimate the expression (2.18) from above by N^3. Since $(a, b, c) \cdot \delta_N U$ intersects all the $m + 1$ cubes, it follows that

$$N > (k_0 + m)^3 - (k_0^3 + k_0) = 3mk_0^2 + (3m^2 - 1)k_0 + m^3,$$

hence

$$
\begin{aligned}
N^3 \;>\;& \left(3mk_0^2 + (3m^2 - 1)k_0 + m^3\right)^3 \\
=\;& 27m^3 k_0^6 \\
+\;& (81m^3 - 27m^2)k_0^5 \\
+\;& (118m^5 - 54m^3 + 9m)k_0^4 \\
+\;& (81m^6 - 27m^4 + 3m^3 - 1)k_0^3 \\
+\;& (36m^7 - 18m^5 + 3m^2)k_0^2 \\
+\;& (9m^8 - 3m^6)k_0 \\
+\;& m^9.
\end{aligned}
$$

(2.19)

The coefficient of each k_0^j, $j = 0, 1, \ldots, 6$, is in (2.18) smaller than in (2.19), so we also obtain in this case that

$$
\left|(a, b, c) \cdot \delta_N U \cap (\mathbb{Z}^3 \backslash X)\right| \leq N^3.
$$

In case (iii), the estimate $\left|(a, b, c) \cdot \delta_N U \cap (\mathbb{Z}^3 \backslash X)\right| \leq N^3$ is satisfied trivially.

Finally we conclude that

$$
\begin{aligned}
D_{\mathbb{H}}^-(X) \;&=\; \liminf_{N \to \infty} \min_{g \in \mathbb{H}} \frac{|X \cap g \cdot \delta_N U|}{\lambda(\delta_N U)} \\
&=\; \liminf_{N \to \infty} \left(\frac{|\mathbb{Z}^3 \cap \delta_N U|}{N^4} - \max_{g \in \mathbb{H}} \frac{|(\mathbb{Z}^3 \backslash X) \cap g \cdot \delta_N U|}{N^4} \right) \\
&=\; \liminf_{N \to \infty} \frac{N^4}{N^4} - \limsup_{N \to \infty} \max_{g \in \mathbb{H}} \frac{|(\mathbb{Z}^3 \backslash X) \cap g \cdot \delta_N U|}{N^4} \\
&\geq\; 1 - \lim_{N \to \infty} \frac{N^3}{N^4} = 1.
\end{aligned}
$$

2.3. Density of Quasi-lattices

In this section we compute the density of a quasi-lattice. For that we note some elementary inclusions.

Lemma 2.6. *Let Γ be a quasi-lattice in G with complement U. There exists some $N_U > 0$ such that, for $N > N_U$,*

(2.20) $$(\Gamma \cap B_{N-N_U}(e))U \subseteq B_N(e) \subseteq (\Gamma \cap B_{N+N_U}(e))U,$$

and consequently,

(2.21) $$|\Gamma \cap B_{N-N_U}(e)| \, \lambda(U) \leq \lambda(B_N(e)) \leq |\Gamma \cap B_{N+N_U}(e)| \lambda(U).$$

PROOF. The complement U is relatively compact, hence bounded, so we can choose some $N_U > 0$ such that $U \subseteq B_{N_U}(e)$.
If $x \in (\Gamma \cap B_{N-N_U}(e))U$, then $x = \gamma u$ for some $u \in U$ and $\gamma \in \Gamma$ with $|\gamma| \leq N - N_U$. Therefore

$$|x| = |\gamma u| \leq |\gamma| + |u| \leq (N - N_U) + N_U = N,$$

so $x \in B_N(e)$.

Now let $x \in B_N(e)$. Since Γ is a quasi-lattice in G, there exists a $\gamma \in \Gamma$ such that $x = \gamma u \in \gamma U$ for some $u \in U$. Thus $\gamma = xu^{-1}$ and

$$|\gamma| = |xu^{-1}| \leq |x| + |u^{-1}| = |x| + |u| \leq N + N_U.$$

We conclude that $x \in (\Gamma \cap B_{N+N_U}(e))U$. Therefore the inclusion (2.20) is established.

By the left-invariance of the Haar measure, it follows from the inclusion (2.20) that

$$\lambda(B_N(e)) \geq \lambda\big((\Gamma \cap B_{N-N_U}(e))U\big) = \lambda\Big(\bigcup_{\gamma \in \Gamma \cap B_{N-N_U}(e)} \gamma U \Big)$$

$$= \sum_{\gamma \in \Gamma \cap B_{N-N_U}(e)} \lambda(\gamma U) = |\Gamma \cap B_{N-N_U}(e)| \lambda(U),$$

because the translates γU of the complement U are disjoint.
The second inequality follows analogously.

$\square$

Proposition 2.7. *Let Γ be a quasi-lattice in G with complement U, then Γ has uniform density*

$$D^+(\Gamma) = D^-(\Gamma) = \frac{1}{\lambda(U)}.$$

PROOF. Let $g \in G$ be arbitrary. Then $\gamma \in B_N(g) = gB_N(e)$ if and only if $g^{-1}\gamma \in B_N(e)$. Therefore

$$D^+(\Gamma) = \limsup_{N\to\infty} \max_{g\in G} \frac{|g^{-1}\Gamma \cap B_N(e)|}{\lambda(B_N(e))},$$

$$D^-(\Gamma) = \liminf_{N\to\infty} \min_{g\in G} \frac{|g^{-1}\Gamma \cap B_N(e)|}{\lambda(B_N(e))}.$$

Recall that for every $g \in G$ the set $g^{-1}\Gamma$ is a quasi-lattice with complement U (Lemma 1.22). By inequality (2.21) in Lemma 2.6 we obtain that

$$\min_{g\in G} |g^{-1}\Gamma \cap B_{N-N_U}(e)|\lambda(U) \leq \lambda(B_N(e)) \leq \min_{g\in G} |g^{-1}\Gamma \cap B_{N+N_U}(e)|\lambda(U).$$

Using this inequality and the homogeneity properties of the Haar measure we estimate

$$
\begin{aligned}
D^-(\Gamma) &= \liminf_{N\to\infty} \min_{g\in G} \frac{|g^{-1}\Gamma \cap B_{N+N_U}(e)|}{\lambda(B_{N+N_U}(e))} \\
&= \liminf_{N\to\infty} \min_{g\in G} \frac{|g^{-1}\Gamma \cap B_{N+N_U}(e)|}{(N+N_U)^D \lambda(B_1(e))} \frac{\lambda(U)}{\lambda(U)} \\
&\geq \liminf_{N\to\infty} \frac{\lambda(B_N(e))}{(N+N_U)^D \lambda(B_1(e))} \frac{1}{\lambda(U)} \\
&= \liminf_{N\to\infty} \frac{N^D}{(N+N_U)^D} \frac{1}{\lambda(U)} = \frac{1}{\lambda(U)}
\end{aligned}
$$

and similarly

$$
\begin{aligned}
D^-(\Gamma) &= \liminf_{N\to\infty} \min_{g\in G} \frac{|g^{-1}\Gamma \cap B_{N-N_U}(e)|}{\lambda(B_{N-N_U}(e))} \frac{\lambda(U)}{\lambda(U)} \\
&\leq \liminf_{N\to\infty} \frac{N^D}{(N-N_U)^D} \frac{1}{\lambda(U)} = \frac{1}{\lambda(U)}.
\end{aligned}
$$

The claim concerning the upper density follows analogously.

$\square$

CHAPTER 3

Density and Frames

3.1. Frames and Riesz Sequences

In this section we collect the main properties of frames, Riesz sequences and related notions. Frames were introduced by Duffin and Schaeffer [16] and nowadays frame theory is an active area of research [9]. We follow the exposition in the books [11], [27] and [48].

Definition 3.1. A family $\mathcal{F} = \{f_i\}_{i \in I}$ in a separable Hilbert space $\mathcal{H}$ is called a *Bessel sequence* if there exists a constant $B > 0$ such that

$$(3.1) \qquad \sum_{i \in I} |\langle f, f_i \rangle|^2 \le B\|f\|^2$$

for all $f \in \mathcal{H}$. The constant B is called the *Bessel bound* for $\mathcal{F}$.

By definition, $\mathcal{F}$ is a Bessel sequence if and only if the *coefficient operator* C, defined by

$$Cf = \{\langle f, f_i \rangle\}_{i \in I},$$

is a bounded operator from $\mathcal{H}$ into $\ell^2(I)$ with operator norm $\|C\|_{op} \le B^{\frac{1}{2}}$.

For every finite sequence $c = (c_i)_{i \in I}$,

$$\langle C^*c, f \rangle = \langle c, Cf \rangle = \sum_{i \in I} c_i \overline{\langle f, f_i \rangle} = \sum_{i \in I} c_i \langle f_i, f \rangle = \Big\langle \sum_{i \in I} c_i f_i, f \Big\rangle,$$

so the adjoint operator $C^* : \ell^2(I) \to \mathcal{H}$ is given by

$$C^*c = \sum_{i \in I} c_i f_i$$

and C^* is bounded with the same operator norm.
Summarizing, we have the following characterization of Bessel sequences.

Lemma 3.2. *A family $\mathcal{F} = \{f_i\}_{i \in I}$ in $\mathcal{H}$ is a Bessel sequence with Bessel bound B if and only if the inequality*

$$(3.2) \qquad \Big\| \sum_{i \in I} c_i f_i \Big\|^2 \le B\|c\|^2$$

holds for every sequence $c = (c_i)_{i \in I} \in \ell^2(I)$.

The operator C^* is often called the *reconstruction operator*.

Definition 3.3. A family $\mathcal{F} = \{f_i\}_{i \in I}$ in a separable Hilbert space $\mathcal{H}$ is called a *Riesz-Fischer sequence* if for every sequence $c = (c_i)_{i \in I} \in \ell^2(I)$ there exists at least one function $f \in \mathcal{H}$ such that

$$\langle f, f_i \rangle = c_i, i \in I. \tag{3.3}$$

To put it differently, a family $\mathcal{F}$ is a Riesz-Fischer sequence if and only if the associated coefficient operator $C : \mathcal{H} \to \ell^2(I)$ is surjective.

We recall another characterization of Riesz-Fischer sequences proved in [**48**].

Lemma 3.4. *A family $\mathcal{F} = \{f_i\}_{i \in I}$ in $\mathcal{H}$ is a Riesz-Fischer sequence if and only if there exists a constant $A > 0$ such that the inequality*

$$A\|c\|^2 \leq \left\| \sum_{i \in I} c_i f_i \right\|^2 \tag{3.4}$$

holds for every finite sequence $c = (c_i)_{i \in I}$.

Definition 3.5. A family $\mathcal{F} = \{f_i\}_{i \in I}$ in a separable Hilbert space $\mathcal{H}$ is called a *Riesz sequence* if there exist constants $A, B > 0$ such that the inequalities

$$A\|c\|^2 \leq \left\| \sum_{i \in I} c_i f_i \right\|^2 \leq B\|c\|^2 \tag{3.5}$$

hold for every finite sequence $c = (c_i)_{i \in I}$.

A Riesz sequence $\mathcal{F} = \{f_i\}_{i \in I}$ is called a *Riesz basis* for $\mathcal{H}$, if $\overline{\text{span}}\{f_i\}_{i \in I} = \mathcal{H}$.

If $\mathcal{F} = \{f_i\}_{i \in I}$ is a Riesz sequence, then inequality (3.5) implies in particular that

$$A^{\frac{1}{2}} \leq \|f_i\| \leq B^{\frac{1}{2}}$$

for all $i \in I$, in other words, every Riesz sequence is uniformly bounded below and above in norm.

A convenient characterization for a system $\mathcal{F} = \{f_i\}_{i \in I}$ to be a Riesz sequence is given in terms of the associated Gram matrix.

Lemma 3.6. *A family $\mathcal{F} = \{f_i\}_{i \in I}$ in a separable Hilbert space $\mathcal{H}$ is a Riesz sequence if and only if the associated Gram matrix G, defined by $G_{ij} = \langle f_j, f_i \rangle$, $i, j \in I$, is a bounded invertible operator on $\ell^2(I)$.*

PROOF. For every finite sequence $c = (c_i)_{i \in I}$,

$$\langle Gc, c \rangle \;=\; \sum_{i,j \in I} \langle f_j, f_i \rangle c_j \overline{c_i} = \left\| \sum_{i \in I} c_i f_i \right\|^2.$$

Thus inequality (3.5) is equivalent to the boundedness and invertibility of G.

$\square$

To check boundedness properties of infinite matrices Schur's test is a helpful criterion. We recall the statement and direct the interested reader to [**11**] or [**27**] for a proof.

Lemma 3.7 (Schur's test)**.** *Let* $A = (a_{ij})_{i,j \in I}$ *be an infinite matrix such that* $a_{ij} = \overline{a_{ji}}$ *and*

$$(3.6) \qquad \sum_{i \in I} |a_{ij}| \leq K \quad \forall j \in I.$$

Then the operator A *defined by the matrix-vector multiplication* $(Ac)_i = \sum_{j \in I} a_{ij} c_j$ *is bounded on* $\ell^2(I)$ *with operator norm at most* K.

Definition 3.8. A family $\mathcal{F} = \{f_i\}_{i \in I}$ in a separable Hilbert space $\mathcal{H}$ is called a *frame* if there exist constants $A, B > 0$ such that

$$(3.7) \qquad A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2$$

for all $f \in \mathcal{H}$. The numbers A and B are called *frame bounds* for $\mathcal{F}$.

Note that a frame with frame bounds A, B is in particular a Bessel sequence with Bessel bound B.

Lemma 3.9. *Every Riesz basis is a frame.*

For a proof see, e.g., [**11**]. The converse is not true. A frame that is not a Riesz basis is said to be *overcomplete*.

The *frame operator* $S : \mathcal{H} \to \mathcal{H}$ associated to a frame $\mathcal{F} = \{f_i\}_{i \in I}$ is defined as

$$Sf := C^*Cf = \sum_{i \in I} \langle f, f_i \rangle f_i.$$

We collect some important properties of frames [**11**], [**27**].

Proposition 3.10. *Let* $\mathcal{F} = \{f_i\}_{i \in I}$ *be a frame for* $\mathcal{H}$ *with frame bounds* A, B.
(a) The frame operator S *is a positive invertible operator satisfying* $AI \leq S \leq BI$.
(b) The family $\tilde{\mathcal{F}} = \{S^{-1}f_i\}_{i \in I}$ *is a frame for* $\mathcal{H}$ *with frame bounds* B^{-1}, A^{-1}.
(c) Every $f \in \mathcal{H}$ *has* frame expansions *of the form*

$$(3.8) \qquad f = \sum_{i \in I} \langle f, S^{-1}f_i \rangle f_i$$

and

$$(3.9) \qquad f = \sum_{i \in I} \langle f, f_i \rangle S^{-1}f_i,$$

where both sums converge unconditionally in $\mathcal{H}$.

The frame $\tilde{\mathcal{F}} = \{S^{-1}f_i\}_{i \in I}$ is called the *canonical dual frame* of $\mathcal{F}$.

More general, a frame $\tilde{\mathcal{F}} = \{\tilde{f}_i\}_{i \in I}$ is called a *dual frame* of $\mathcal{F} = \{f_i\}_{i \in I}$ if

$$(3.10) \qquad f = \sum_{i \in I} \langle f, \tilde{f}_i \rangle f_i = \sum_{i \in I} \langle f, f_i \rangle \tilde{f}_i$$

for all $f \in \mathcal{H}$.

For an arbitrary frame there may be many dual frames. In fact, if a frame is overcomplete, then there always exist dual frames other than the canonical dual frame (see, e.g., [**11**]). For Riesz bases, however, the dual frame is unique and has some additional properties [**11**].

Lemma 3.11. *Let $\mathcal{F} = \{f_i\}_{i \in I}$ be a Riesz basis for $\mathcal{H}$. Then the canonical dual frame $\tilde{\mathcal{F}} = \{S^{-1}f_i\}_{i \in I} = \{\tilde{f}_i\}_{i \in I}$ is the unique sequence in $\mathcal{H}$ satisfying*

$$(3.11) \qquad f = \sum_{i \in I} \langle f, \tilde{f}_i \rangle f_i = \sum_{i \in I} \langle f, f_i \rangle \tilde{f}_i$$

for all $f \in \mathcal{H}$. Moreover, $\tilde{\mathcal{F}}$ is also a Riesz basis for $\mathcal{H}$, and $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are biorthogonal, that is, $\langle f_i, \tilde{f}_j \rangle = \delta_{ij}$.

In this case $\tilde{\mathcal{F}}$ is called the *dual Riesz basis* of $\mathcal{F}$.

Since $\mathcal{F} = \{f_i\}_{i \in I}$ is complete, that is, $\overline{\mathrm{span}}\{f_i\}_{i \in I} = \mathcal{H}$, the dual Riesz basis $\tilde{\mathcal{F}}$ is also the unique sequence in $\mathcal{H}$ that is biorthogonal to $\mathcal{F}$.

3.2. Homogeneous Approximation Property and Density

In this section we consider frames $\{f_\chi\}_{\chi \in X}$ in a separable Hilbert space $\mathcal{H}$ that are indexed by a discrete subset X of a homogeneous group G. In this abstract setting the index χ of a frame element f_χ has no a priori connotation. However, if we impose some form of 'localization', achieved by the so-called Homogeneous Approximation Property, we may think of the vector f_χ living near χ in G.

The Homogeneous Approximation Property was first observed as a property inherent to Gabor frames by Ramanathan and Steger [**40**] and subsequently established also for frames of windowed exponentials [**25**] and wavelet frames [**31**]. An abstraction for general frames without special structure was provided in the fundamental paper of Balan, Casazza, Heil and Landau [**5**].

Recall that in a separable Hilbert space $\mathcal{H}$ the distance of an element $f \in \mathcal{H}$ to a closed linear subspace $V \subseteq \mathcal{H}$ is given by

$$\mathrm{dist}(f, V) := \inf_{v \in V} \|f - v\| = \|f - P_V f\|,$$

where P_V denotes the orthogonal projection onto V.

Definition 3.12. Let X and Y be relatively separated subsets of G and let $\mathcal{H}$ be a separable Hilbert space. Let $\mathcal{F} = \{f_\chi\}_{\chi \in X}$ be a frame for $\mathcal{H}$ with dual frame $\tilde{\mathcal{F}} = \{\tilde{f}_\chi\}_{\chi \in X}$ and let $\mathcal{E} = \{e_v\}_{v \in Y}$ be a set in $\mathcal{H}$.
The frame $\mathcal{F}$ has the *Homogeneous Approximation Property* with respect to the reference set $\mathcal{E}$, if for every $\varepsilon > 0$ there exists some $N_\varepsilon > 0$ such that, for every $v \in Y$,

$$\mathrm{dist}(e_v, \mathrm{span}\{\tilde{f}_\chi : \chi \in X \cap B_{N_\varepsilon}(v)\}) < \varepsilon.$$

The following theorem for the comparison of densities is an important ingredient for deriving necessary density conditions for frames and Riesz sequences. The proof relies on the double-projection technique of Ramanathan and Steger [**40**] and the homogeneity properties of the Haar measure.

Theorem 3.13 (Comparison Theorem)**.** *Let X and Y be relatively separated subsets of G. Assume that $\mathcal{F} = \{f_\chi\}_{\chi \in X}$ is a frame for $\mathcal{H}$ and that $\mathcal{E} = \{e_v\}_{v \in Y}$ is a Riesz sequence in $\mathcal{H}$.*
If $\mathcal{F}$ has the Homogeneous Approximation Property with respect to $\mathcal{E}$, then

$$D^-(Y) \leq D^-(X) \quad and \quad D^+(Y) \leq D^+(X).$$

PROOF. Let $\tilde{\mathcal{F}} = \{\tilde{f}_\chi\}_{\chi \in X}$ be a dual frame for $\mathcal{F}$, and let $\tilde{\mathcal{E}} = \{\tilde{e}_v\}_{v \in Y}$ be the Riesz basis in $\overline{\mathrm{span}}(\mathcal{E})$ that is biorthogonal to $\mathcal{E}$. The elements of a Riesz sequence are uniformly bounded in norm, so

$$C := \sup_{v \in Y} \|\tilde{e}_v\| < \infty.$$

Given $\varepsilon > 0$, choose $N_\varepsilon > 0$ such that the Homogeneous Approximation Property is satisfied with $\frac{\varepsilon}{C}$, that is,

$$\mathrm{dist}(e_v, \mathrm{span}\{\tilde{f}_\chi : \chi \in X \cap B_{N_\varepsilon}(v)\}) < \frac{\varepsilon}{C}$$

for all $v \in Y$. Fix an arbitrary element $g \in G$ and a radius $N > 0$, and define the subspaces

$$V := \mathrm{span}\{e_v : v \in Y \cap B_N(g)\}$$

and

$$W := \mathrm{span}\{\tilde{f}_\chi : \chi \in X \cap B_{N+N_\varepsilon}(g)\}$$

of $\mathcal{H}$. Since X and Y are relatively separated, the subspaces V and W are finite-dimensional with $\dim V = |Y \cap B_N(g)|$ and $\dim W \leq |X \cap B_{N+N_\varepsilon}(g)|$. Let P_V and P_W denote the orthogonal projections of $\mathcal{H}$ onto V and W, and define a map

$$T : V \to V, \ T := P_V P_W.$$

Since the domain of T is V and P_V and P_W are projections, we may write the operator T as

$$T = P_V P_W P_W P_V = (P_W P_V)^*(P_W P_V),$$

so T is a positive operator on V. We estimate the trace of T from above and below. Every eigenvalue λ of T satisfies $\lambda \leq \|T\| \leq \|P_V\|\|P_W\| \leq 1$. Hence we get the upper estimate

$$\text{(3.12)} \qquad \text{trace}(T) \leq \text{rank}(T) \leq \dim W \leq |X \cap B_{N+N_\varepsilon}(g)|.$$

To obtain a lower bound note that $\{e_\upsilon : \upsilon \in Y \cap B_N(g)\}$ is a Riesz basis for V and its dual Riesz basis in V is given by $\{P_V \tilde{e}_\upsilon : \upsilon \in Y \cap B_N(g)\}$. From the biorthogonality we get

$$
\begin{aligned}
\text{trace}(T) \quad &= \sum_{\upsilon \in Y \cap B_N(g)} \langle Te_\upsilon, P_V \tilde{e}_\upsilon \rangle = \sum_{\upsilon \in Y \cap B_N(g)} \langle P_V Te_\upsilon, \tilde{e}_\upsilon \rangle \\
&= \sum_{\upsilon \in Y \cap B_N(g)} \langle e_\upsilon, \tilde{e}_\upsilon \rangle + \sum_{\upsilon \in Y \cap B_N(g)} \langle (P_V P_W - I)e_\upsilon, \tilde{e}_\upsilon \rangle \\
\text{(3.13)} \qquad &\geq |Y \cap B_N(g)| - \sum_{\upsilon \in Y \cap B_N(g)} |\langle (P_V P_W - I)e_\upsilon, \tilde{e}_\upsilon \rangle|.
\end{aligned}
$$

We further estimate each term as

$$
\begin{aligned}
|\langle (P_V P_W - I)e_\upsilon, \tilde{e}_\upsilon \rangle| \quad &\leq \quad \|(P_V P_W - I)e_\upsilon\|\|\tilde{e}_\upsilon\| \\
&\leq \quad C \cdot \|P_V P_W e_\upsilon - P_V e_\upsilon\| \\
&\leq \quad C \cdot \|P_W e_\upsilon - e_\upsilon\| \\
&= \quad C \cdot \text{dist}(e_\upsilon, \text{span}\{\tilde{f}_\chi : \chi \in X \cap B_{N+N_\varepsilon}(g)\}).
\end{aligned}
$$

Since $B_{N_\varepsilon}(\upsilon) \subseteq B_{N+N_\varepsilon}(g)$ for $\upsilon \in Y \cap B_N(g)$ and thus

$$\text{span}\{\tilde{f}_\chi : \chi \in X \cap B_{N_\varepsilon}(\upsilon)\} \subseteq \text{span}\{\tilde{f}_\chi : \chi \in X \cap B_{N+N_\varepsilon}(g)\},$$

the Homogeneous Approximation Property implies that

$$
\begin{aligned}
|\langle (P_V P_W - I)e_\upsilon, \tilde{e}_\upsilon \rangle| \quad &\leq \quad C \cdot \text{dist}(e_\upsilon, \text{span}\{\tilde{f}_\chi : \chi \in X \cap B_{N_\varepsilon}(\upsilon)\}) \\
\text{(3.14)} \qquad &< \quad C\frac{\varepsilon}{C} = \varepsilon.
\end{aligned}
$$

Combining the estimates (3.13) and (3.14) we obtain the inequality

$$\text{(3.15)} \qquad \text{trace}(T) \geq |Y \cap B_N(g)| - \sum_{\upsilon \in Y \cap B_N(g)} \varepsilon = (1-\varepsilon)|Y \cap B_N(g)|.$$

Putting things together we find that

$$(1-\varepsilon)|Y \cap B_N(g)| \leq \text{trace}(T) \leq |X \cap B_{N+N_\varepsilon}(g)|$$

and, as a consequence,

$$\text{(3.16)} \qquad (1-\varepsilon)\frac{|Y \cap B_N(g)|}{N^D \lambda(B_1(e))} \leq \frac{|(X \cap B_{N+N_\varepsilon}(g))|}{(N+N_\varepsilon)^D \lambda(B_1(e))} \frac{(N+N_\varepsilon)^D}{N^D}.$$

Since $g \in G$ was arbitrary, we get

$$(1-\varepsilon)\min_{g \in G}\frac{|Y \cap B_N(g)|}{N^D \lambda(B_1(e))} \leq \min_{g \in G}\frac{|(X \cap B_{N+N_\varepsilon}(g))|}{(N+N_\varepsilon)^D \lambda(B_1(e))} \frac{(N+N_\varepsilon)^D}{N^D}.$$

Taking the lower limit now yields

$$(1 - \varepsilon)D^-(Y) \leq D^-(X).$$

Since $\varepsilon > 0$ was arbitrary, we obtain

$$D^-(Y) \leq D^-(X).$$

Taking the maximum over all $g \in G$ and the upper limit, we similary get

$$D^+(Y) \leq D^+(X).$$

$\square$

To derive efficient necessary density bounds one usually compares with the density of a Riesz basis, which is both a Riesz sequence and a frame. If the Riesz basis is additionally indexed by a (quasi-) lattice, then its density can be computed according to Proposition 2.7 and thereby yields a concrete threshold.

Corollary 3.14. *Let X be a relatively separated subset of G. Let Γ be a quasi-lattice in G with complement U and assume that $\mathcal{E} = \{e_\gamma\}_{\gamma \in \Gamma}$ is a Riesz basis in $\mathcal{H}$.*

(a) If $\mathcal{F} = \{f_\chi\}_{\chi \in X}$ is a frame for $\mathcal{H}$ that satisfies the Homogeneous Approximation Property with respect to $\mathcal{E}$, then

$$D^-(X) \geq \frac{1}{\lambda(U)}.$$

(b) If $\mathcal{F} = \{f_\chi\}_{\chi \in X}$ is a Riesz sequence in $\mathcal{H}$ such that $\mathcal{E}$ satisfies the Homogeneous Approximation Property with respect to $\mathcal{F}$, then

$$D^+(X) \leq \frac{1}{\lambda(U)}.$$

CHAPTER 4

Sampling and Interpolation in Shift-Invariant Spaces

The sampling and interpolation problem for bandlimited functions on $\mathbb{R}^n$ is at the origin of Beurling's definition of density [7], [35]. Beurling's density has also been used to derive necessary density conditions in so-called shift-invariant spaces, which can be seen as a generalization of the space of bandlimited functions [1]. Bandlimited functions have been studied in the more general context of stratified Lie groups [23], [39], and only recently also generalizations of shift-invariant spaces to certain nilpotent Lie groups were proposed [6], [14].

In this chapter we study shift-invariant spaces on homogeneous groups. As a first application of the abstract Comparison Theorem in the previous chapter, we derive necessary density conditions for sampling and interpolation in terms of the density defined in Chapter 2. This is done via a translation of the sampling and interpolation problem into a question of frames and Riesz sequences of reproducing kernels, as in the theory on $\mathbb{R}^n$ [1], [2].

Throughout this chapter we assume that G is a homogeneous group that admits a lattice Γ in G.

4.1. Definitions and Prerequisites

Given a lattice $\Gamma \subseteq G$ and a so-called generator $\varphi \in L^2(G)$ we consider the shift-invariant space of the form

$$V^2(\Gamma, \varphi) = \{f = \sum_{\gamma \in \Gamma} c_\gamma L_\gamma \varphi : c = (c_\gamma)_{\gamma \in \Gamma} \in \ell^2(\Gamma)\},$$

where $L_\gamma f(x) = f(\gamma^{-1} x)$ denotes the left translation operator.

Standard assumptions on the generator function are membership in the Wiener Amalgam Space $W(C, L^1)$ and the so-called Riesz basis property.

We say that $\varphi \in L^2(G)$ has the *Riesz basis property with respect to* Γ if the set of left translations $\{L_\gamma \varphi\}_{\gamma \in \Gamma}$ forms a Riesz sequence in $L^2(G)$, i.e., if there exist constants $A, B > 0$ such that, for all $c \in \ell^2(\Gamma)$, we have

$$A\|c\|_{\ell^2(\Gamma)} \leq \big\| \sum_{\gamma \in \Gamma} c_\gamma L_\gamma \varphi \big\|_{L^2(G)} \leq B\|c\|_{\ell^2(\Gamma)}.$$

In the following we fix a lattice Γ in G with fundamental domain U and abbreviate $V^2(\Gamma, \varphi)$ by $V^2(\varphi)$.

On $\mathbb{R}^n$, a well-known criterion for the Riesz basis property of a function $\varphi \in L^2(\mathbb{R}^n)$ is stated in the Fourier domain [2]. A set of translates $\{\varphi(\cdot - k)\}_{k \in \mathbb{Z}^n}$ is a Riesz sequence in $L^2(\mathbb{R}^n)$ if and only if there exist constants $A, B > 0$ such that

$$A \le \sum_{k \in \mathbb{Z}^n} |\hat{\varphi}(\xi + k)|^2 \le B \quad \text{a.e. } \xi.$$

For functions on nilpotent Lie groups the transition to the Fourier domain is more complicated and involves the group Fourier transform and the Plancherel theory for nilpotent Lie groups. In this direction, characterizations of the Riesz basis property for functions on certain nilpotent Lie groups have been announced only recently in [6], [14].

In the setting of homogeneous groups we can construct a suitable class of generators $\varphi \in W(C, L^1)$ that have the Riesz basis property in a more elementary way.

By Lemma 3.6, the system $\{L_\gamma \varphi\}_{\gamma \in \Gamma}$ is a Riesz sequence if and only if the associated Gram matrix G, given by $G_{\nu\gamma} := \langle L_\gamma \varphi, L_\nu \varphi \rangle$, $\gamma, \nu \in \Gamma$, is a bounded invertible operator on $\ell^2(\Gamma)$.

The action of G on a sequence $b \in \ell^2(\Gamma)$ is

$$
\begin{aligned}
Gb(\nu) &= \sum_{\gamma \in \Gamma} \langle L_\gamma \varphi, L_\nu \varphi \rangle b(\gamma) \\
&= \sum_{\gamma \in \Gamma} \langle \varphi, L_{\gamma^{-1}\nu} \varphi \rangle b(\gamma) \\
&= \|\varphi\|_{L^2(G)}^2 b(\nu) + Tb(\nu),
\end{aligned}
$$

(4.1)

where

$$Tb(\nu) := \sum_{\nu \ne \gamma \in \Gamma} \langle \varphi, L_{\gamma^{-1}\nu} \varphi \rangle b(\gamma).$$

We want to use Schur's test (Lemma 3.7) to estimate the operator norm of T. The matrix entries of T are $T_{\nu\gamma} = \langle \varphi, L_{\gamma^{-1}\nu} \varphi \rangle$ for $\nu \ne \gamma$ and $T_{\nu\nu} = 0$. Thus we have to estimate, for every $\nu \in \Gamma$,

(4.2)

$$
\begin{aligned}
\sum_{\nu \ne \gamma \in \Gamma} |\langle \varphi, L_{\gamma^{-1}\nu} \varphi \rangle| &= \sum_{e \ne \gamma \in \Gamma} |\langle \varphi, L_\gamma \varphi \rangle| \\
&= \sum_{e \ne \gamma \in \Gamma} \left| \int_G \varphi(x) \overline{\varphi(\gamma^{-1}x)} dx \right| \\
&= \sum_{e \ne \gamma \in \Gamma} \left| \int_G \varphi(x) \varphi^*(x^{-1}\gamma) dx \right| \\
&= \sum_{e \ne \gamma \in \Gamma} |\varphi * \varphi^*(\gamma)|.
\end{aligned}
$$

If we take $\varphi \in W(C, L^1)$ such that

$$\text{(i)} \quad \|\varphi\|_{L^2(G)} = 1$$

and

$$\text{(ii)} \quad \sum_{e \neq \gamma \in \Gamma} |\varphi * \varphi^*(\gamma)| < 1,$$

then Schur's test (Lemma 3.7) implies that $\|T\| < 1$ and it follows from equation (4.1) for the action of the Gram matrix G that

$$G = I + T, \quad \|T\| < 1,$$

or equivalently,

$$\|G - I\| < 1.$$

This gives the invertibility of the Gram matrix G and thereby the Riesz basis property for φ.

To find a generator $\varphi \in W(C, L^1)$ that has the Riesz basis property we therefore need to construct a function $\varphi \in W(C, L^1)$ that satisfies the properties (i) and (ii). The idea is to take a sufficiently dilated version of some arbitrary normalized function in $W(C, L^1)$.

For that let $\psi \in W(C, L^1)$ with $\|\psi\|_{L^2(G)} = 1$. Denote by $\psi^{\#}(\gamma) := \sup_{x \in \gamma U} |\psi(x)|$ the local maximum function and let $N_U > 0$ be such that $U \subseteq B_{N_U}(e)$. Since $\psi \in W(C, L^1) \subseteq L^1(G)$ and G is unimodular, the function ψ^* is also in $L^1(G)$. Corollary 1.31 implies that $\psi * \psi^* \in W(C, L^1) * L^1(G) \subseteq W(C, L^1)$, so we can find some $N > 2N_U > 0$ such that

$$(4.3) \qquad \sum_{\gamma \in \Gamma \setminus B_{N-N_U}(e)} (\psi * \psi^*)^{\#}(\gamma) < 1.$$

Now let $r > 0$ such that

$$(4.4) \qquad \delta_r \gamma \notin B_N(e) \quad \forall \gamma \in \Gamma, \gamma \neq e.$$

Set $\varphi := D_r \psi$ and $\Gamma' := \delta_r \Gamma$. The equations (1.23) and (1.24) imply that

$$
\begin{aligned}
\sum_{e \neq \gamma \in \Gamma} |\varphi * \varphi^*(\gamma)| \ &= \ \sum_{e \neq \gamma \in \Gamma} |D_r \psi * (D_r \psi)^*(\gamma)| \\
&= \ \sum_{e \neq \gamma \in \Gamma} |D_r \psi * D_r(\psi^*)(\gamma)| \\
&= \ \sum_{e \neq \gamma \in \Gamma} |\psi * \psi^*(\delta_r \gamma)| \\
(4.5) \qquad &= \ \sum_{\gamma' \in \Gamma' \setminus B_N(e)} |\psi * \psi^*(\gamma')| \\
&\leq \ \sum_{\nu \in \Gamma \setminus B_{N-N_U}(e)} \sum_{\gamma' \in \Gamma' \cap \nu U} |\psi * \psi^*(\gamma')|.
\end{aligned}
$$

If $\gamma', \eta' \in \Gamma' \cap \nu U$ for some $\nu \in \Gamma$, then $\gamma', \eta' \in \nu U \subseteq \nu B_{N_U}(e) = B_{N_U}(\nu)$ and thus

$$d(e, (\gamma')^{-1}\eta') = d(\gamma', \eta') \leq d(\gamma', \nu) + d(\nu, \eta') < 2N_U < N.$$

Condition (4.4) now implies that $(\gamma')^{-1}\eta' = e$ and therefore $\gamma' = \eta'$. In other words, for every $\nu \in \Gamma$ the translate νU contains at most one element γ' of Γ'. We continue equation (4.5) and conclude that

$$
\begin{aligned}
\sum_{e \neq \gamma \in \Gamma} |\varphi * \varphi^*(\gamma)| &\leq \sum_{\nu \in \Gamma \setminus B_{N-N_U}(e)} \sum_{\gamma' \in \Gamma' \cap \nu U} |\psi * \psi^*(\gamma')| \\
&\leq \sum_{\nu \in \Gamma \setminus B_{N-N_U}(e)} \sup_{x \in \nu U} |\psi * \psi^*(x)| \\
&= \sum_{\nu \in \Gamma \setminus B_{N-N_U}(e)} (\psi * \psi^*)^{\#}(\nu) < 1.
\end{aligned}
$$

Finally we remark that $\varphi = D_r \psi \in W(C, L^1)$ by Lemma 1.28 (b).

Next we collect some elementary properties of shift-invariant spaces on G. These results follow essentially like the corresponding statements for shift-invariant spaces on $\mathbb{R}^n$ (compare, e.g., [**2**]).

Lemma 4.1. *If $\varphi \in W(C, L^1)$ and $c \in \ell^2(\Gamma)$, then the function $f = \sum_{\gamma \in \Gamma} c_\gamma L_\gamma \varphi$ belongs to $W(C, L^2)$ and*

(4.6) $$\|f\|_{W(C, \ell^2)} \leq \|c\|_{\ell^2(\Gamma)} \|\varphi\|_{W(C, \ell^1)}.$$

PROOF. Let $f^{\#}(\gamma) := \sup_{x \in U} |f(\gamma x)|$ and $\varphi^{\#}(\gamma) := \sup_{x \in U} |\varphi(\gamma x)|$ be the (left) local maximum functions of f and φ. We have

$$
\begin{aligned}
f^{\#}(\nu) &= \sup_{x \in U} |f(\nu x)| \\
&= \sup_{x \in U} \left| \sum_{\gamma \in \Gamma} c_\gamma L_\gamma \varphi(\nu x) \right| \\
&\leq \sum_{\gamma \in \Gamma} |c_\gamma| \sup_{x \in U} |\varphi(\gamma^{-1}\nu x)| \\
&= \sum_{\gamma \in \Gamma} |c_\gamma| \varphi^{\#}(\gamma^{-1}\nu) = (|c| * \varphi^{\#})(\nu).
\end{aligned}
$$

Now Young's inequality implies that

$$\||c| * \varphi^{\#}\|_{\ell^2(\Gamma)} \leq \|c\|_{\ell^2(\Gamma)} \|\varphi^{\#}\|_{\ell^1(\Gamma)}.$$

Hence

$$
\begin{aligned}
\|f\|_{W(L^\infty, \ell^2)} &= \|f^{\#}\|_{\ell^2(\Gamma)} \leq \||c| * \varphi^{\#}\|_{\ell^2(\Gamma)} \\
&\leq \|c\|_{\ell^2(\Gamma)} \|\varphi^{\#}\|_{\ell^1(\Gamma)} = \|c\|_{\ell^2(\Gamma)} \|\varphi\|_{W(C, \ell^1)}.
\end{aligned}
$$

To see that $f = \sum_{\gamma \in \Gamma} c_\gamma L_\gamma$ is continuous let $\sigma : \mathbb{N} \to \Gamma$ be an arbitrary enumeration of Γ. Set

$$f_N := \sum_{n=1}^{N} c_{\sigma(n)} L_{\sigma(n)} \varphi = \sum_{\gamma \in \sigma(\{1,..,N\})} c_\gamma L_\gamma \varphi$$

and denote by

$$c_N := c \cdot \mathbb{1}_{\sigma(\{1,..,N\})}$$

the sequence with terms $c_{N,\gamma} = c_\gamma$ if $\gamma \in \sigma(\{1,..,N\})$ and $c_{N,\gamma} = 0$ if $\gamma \notin \sigma(\{1,..,N\})$. By Lemma 1.29 and equation (4.6), it follows that

$$\begin{aligned}
\|f - f_N\|_{L^\infty(G)} &\leq C \|f - f_N\|_{W(C,\ell^\infty)} \\
&\leq C' \|f - f_N\|_{W(C,\ell^2)} \\
&= C' \Big\| \sum_{\gamma \in \Gamma} (c_\gamma - c_{N,\gamma}) L_\gamma \varphi \Big\|_{W(C,\ell^2)} \\
&\leq C'' \|c - c_N\|_{\ell^2(\Gamma)} \to 0
\end{aligned}$$

as $N \to \infty$. Thus f is continuous as the uniform limit of the continuous functions f_N.

$\square$

Corollary 4.2. *Let $\varphi \in W(C, L^1)$, then*

$$V^2(\varphi) \subseteq W(C, L^2) \subseteq L^2(G).$$

If φ additionally has the Riesz basis property, then, for all $f \in V^2(\varphi)$, we have the norm equivalence

$$\|f\|_{L^2(G)} \asymp \|c\|_{\ell^2(\Gamma)} \asymp \|f\|_{W(C,\ell^2)}.$$

PROOF. Recall from Lemma 1.29 that $W(C, L^2) \subseteq L^2(G)$ with $\|f\|_{L^2(G)} \leq C \|f\|_{W(C,\ell^2)}$. Lemma 4.1 now implies the inclusion $V^2(\varphi) \subseteq W(C, L^2) \subseteq L^2(G)$. If $f = \sum_{\gamma \in \Gamma} c_\gamma L_\gamma \varphi \in V^2(\varphi)$, then Lemma 4.1 and the Riesz basis property of φ imply that

$$\|f\|_{L^2(G)} \leq C \|f\|_{W(C,\ell^2)} \leq C \|c\|_{\ell^2(\Gamma)} \|\varphi\|_{W(C,\ell^1)} \leq C' \|f\|_{L^2(G)} \|\varphi\|_{W(C,\ell^1)}.$$

$\square$

Lemma 4.3. *Suppose that $\varphi \in V^2(\varphi)$ has the Riesz basis property. Then there exists a unique $\tilde{\varphi} \in V^2(\varphi)$, the so-called* dual generator, *such that $\langle L_\nu \tilde{\varphi}, L_\gamma \varphi \rangle = \delta_{\nu,\gamma}$ for all $\gamma, \nu \in \Gamma$. Consequently,*

$$f = \sum_{\gamma \in \Gamma} \langle f, L_\gamma \tilde{\varphi} \rangle L_\gamma \varphi = \sum_{\gamma \in \Gamma} \langle f, L_\gamma \varphi \rangle L_\gamma \tilde{\varphi}$$

for all $f \in V^2(\varphi)$ ($\{L_\gamma \tilde{\varphi}\}_{\gamma \in \Gamma}$ is the dual Riesz basis *for $\{L_\gamma \varphi\}_{\gamma \in \Gamma}$ in $V^2(\varphi)$).*

PROOF. Since $\{L_\gamma\varphi\}_{\gamma\in\Gamma}$ is a Riesz basis for $V^2(\varphi)$, there exists a unique sequence $\{g_\nu\}_{\nu\in\Gamma}$ in $V^2(\varphi)$ such that $\langle g_\nu, L_\gamma\varphi\rangle = \delta_{\nu,\gamma} = \delta_\nu(\gamma)$ for all $\gamma, \nu \in \Gamma$ (Lemma 3.11).

Set $\tilde\varphi := g_e \in V^2(\varphi)$, then, for all $\gamma, \nu \in \Gamma$,

$$\langle L_\nu\tilde\varphi, L_\gamma\varphi\rangle = \langle\tilde\varphi, L_{\nu^{-1}\gamma}\varphi\rangle = \delta_e(\nu^{-1}\gamma) = L_\nu\delta_e(\gamma) = \delta_\nu(\gamma).$$

By uniqueness of the dual Riesz basis, $\{g_\nu\}_{\nu\in\Gamma} = \{L_\gamma\tilde\varphi\}_{\gamma\in\Gamma}$.

$\square$

In the following we additionally assume that the generator φ satisfies

$$(4.7) \qquad \inf_{x\in G} \sum_{\gamma\in\Gamma} |L_\gamma\varphi(x)|^2 \geq \alpha > 0.$$

Condition (4.7) prevents pathological examples of spaces $V^2(\varphi)$ where all functions in $V^2(\varphi)$ vanish simultaneously on some subset of G.

Lemma 4.4. *If $\varphi \in W(C, L^1)$ satisfies condition (4.7), then there exists some $N > 0$ such that, for every $\nu \in \Gamma$,*

$$(4.8) \qquad \inf_{x\in\nu U} \sum_{\gamma\in\Gamma\cap B_N(\nu)} |L_\gamma\varphi(x)|^2 \geq \frac{\alpha}{2} =: \alpha' > 0.$$

PROOF. Since $\varphi \in W(C, L^1)$, the local maximum function $\varphi^\#$ defined by $\varphi^\#(\gamma) = \sup_{x\in U} |\varphi(\gamma x)|$ is in $\ell^1(\Gamma) \subseteq \ell^2(\Gamma)$. So we can choose $N > 0$ such that

$$(4.9) \qquad \sum_{\gamma\in\Gamma\setminus B_N(e)} \varphi^\#(\gamma)^2 < \frac{\alpha}{2}.$$

Let $\nu \in \Gamma$ be arbitrary and let $x = \nu u \in \nu U$. Then

$$(4.10) \qquad \begin{aligned} \sum_{\gamma\in\Gamma\setminus B_N(\nu)} |L_\gamma\varphi(x)|^2 &= \sum_{\gamma\in\Gamma\setminus B_N(\nu)} |\varphi(\gamma^{-1}\nu u)|^2 \\ &= \sum_{\gamma\in\Gamma\setminus B_N(\nu)} |\varphi((\nu^{-1}\gamma)^{-1}u)|^2 \\ &= \sum_{\eta\in\Gamma\setminus B_N(e)} |\varphi(\eta^{-1}u)|^2 \\ &= \sum_{\eta\in\Gamma\setminus B_N(e)} |\varphi(\eta u)|^2 \\ &\leq \sum_{\eta\in\Gamma\setminus B_N(e)} \sup_{y\in U}|\varphi(\eta y)|^2 \\ &= \sum_{\eta\in\Gamma\setminus B_N(e)} \varphi^\#(\gamma)^2 < \frac{\alpha}{2}. \end{aligned}$$

By the assumption (4.7) and inequality (4.10) we therefore obtain that

$$\sum_{\gamma \in \Gamma \cap B_N(\nu)} |L_\gamma \varphi(x)|^2 \;=\; \sum_{\gamma \in \Gamma} |L_\gamma \varphi(x)|^2 - \sum_{\gamma \in \Gamma \setminus B_N(\nu)} |L_\gamma \varphi(x)|^2$$

$$\geq\; \alpha - \frac{\alpha}{2} = \frac{\alpha}{2}.$$

Since $x \in \nu U$ was arbitrary, it follows that

$$\inf_{x \in \nu U} \sum_{\gamma \in \Gamma \cap B_N(\nu)} |L_\gamma \varphi(x)|^2 \geq \frac{\alpha}{2}.$$

$\square$

4.2. Sampling and Interpolation

Definition 4.5. (i) A set $X \subseteq G$ is called a *set of sampling* for $V^2(\varphi)$, if there exist constants $0 < A \leq B < \infty$ such that, for all $f \in V^2(\varphi)$,

$$A\|f\|^2 \leq \sum_{\chi \in X} |f(\chi)|^2 \leq B\|f\|^2.$$

(ii) A set $X \subseteq G$ is called a *set of interpolation* for $V^2(\varphi)$, if for every sequence $a = (a_\chi)_{\chi \in X}$ in $\ell^2(X)$ there exists a function $f \in V^2(\varphi)$ such that $f(\chi) = a_\chi$ for all $\chi \in X$.

As in the theory of shift-invariant spaces on $\mathbb{R}^n$, the space $V^2(\varphi)$ turns out to be a reproducing kernel Hilbert space, which allows a transition from sets of sampling and interpolation to frames and Riesz sequences of reproducing kernels. Again large parts of the theory can be conducted in analogy to the theory on $\mathbb{R}^n$ [1], [2].

Proposition 4.6. *Let $\varphi \in W(C, L^1)$ and assume that φ has the Riesz basis property and satisfies condition (4.7).*
A subset $X \subseteq G$ is relatively separated if and only if there exists a constant $B > 0$ such that

(4.11)
$$\sum_{\chi \in X} |f(\chi)|^2 \leq B\|f\|_{L^2(G)}^2$$

for all $f \in V^2(\varphi)$.

PROOF. ($\Rightarrow$) Since X is relatively separated, there are at most K sampling points χ in each translate γU of U, so

$$\sum_{\chi \in X} |f(\chi)|^2 \;=\; \sum_{\gamma \in \Gamma} \sum_{\chi \in X \cap \gamma U} |f(\chi)|^2$$

$$\leq\; \sum_{\gamma \in \Gamma} K \sup_{x \in \gamma U} |f(x)|^2$$

$$=\; K \sum_{\gamma \in \Gamma} \sup_{x \in U} |f(\gamma x)|^2 = K\|f\|_{W(C,\ell^2)}^2.$$

Now Corollary 4.2 implies that

$$\sum_{\chi \in X} |f(\chi)|^2 \leq K\|f\|^2_{W(C,\ell^2)} \leq \tilde{K}\|f\|^2_{L^2(G)}.$$

($\Longleftarrow$) By Lemma 4.4, there exists some $N > 0$ such that, for every $\nu \in \Gamma$,

$$(4.12) \qquad \inf_{x \in \nu U} \sum_{\gamma \in \Gamma \cap B_N(\nu)} |L_\gamma \varphi(x)|^2 \geq \alpha' > 0.$$

Let $n := |\Gamma \cap B_N(e)|$ denote the number of lattice points in $B_N(e)$. For every $\nu \in \Gamma$,

$$|\Gamma \cap B_N(\nu)| = |\nu\Gamma \cap \nu B_N(e)| = |\nu(\Gamma \cap B_N(e))| = |\Gamma \cap B_N(e)| = n.$$

Let B denote the constant in inequality (4.11). For arbitrary $\nu \in \Gamma$ we show that

$$(4.13) \qquad |X \cap \nu U| \leq \frac{Bn}{\alpha'}\|\varphi\|^2_{L^2(G)}.$$

First we use inequality (4.11) for $f = L_\gamma \varphi \in V^2(\varphi)$ to obtain that

$$\sum_{\chi \in X \cap \nu U} |\varphi(\gamma^{-1}\chi)|^2 \leq \sum_{\chi \in X} |\varphi(\gamma^{-1}\chi)|^2 \leq B\|L_\gamma \varphi\|^2_{L^2(G)} = B\|\varphi\|^2_{L^2(G)}$$

for every $\gamma \in \Gamma$. It follows that

$$(4.14) \qquad \sum_{\gamma \in \Gamma \cap B_N(\nu)} \sum_{\chi \in X \cap \nu U} |\varphi(\gamma^{-1}\chi)|^2 \leq \sum_{\gamma \in \Gamma \cap B_N(\nu)} B\|\varphi\|^2_{L^2(G)}$$

$$= nB\|\varphi\|^2_{L^2(G)}.$$

On the other hand, it follows from property (4.12) that

$$(4.15) \qquad \sum_{\gamma \in \Gamma \cap B_N(\nu)} \sum_{\chi \in X \cap \nu U} |\varphi(\gamma^{-1}\chi)|^2 = \sum_{\chi \in X \cap \nu U} \sum_{\gamma \in \Gamma \cap B_N(\nu)} |\varphi(\gamma^{-1}\chi)|^2$$

$$\geq \sum_{\chi \in X \cap \nu U} \alpha' = |X \cap \nu U|\alpha'.$$

Now the inequalities (4.14) and (4.15) combined yield the desired upper bound (4.13). Since $\nu \in \Gamma$ was arbitrary,

$$\max_{\nu \in \Gamma} |X \cap \nu U| \leq \frac{Bn}{\alpha'}\|\varphi\|^2_{L^2(G)}.$$

Therefore X is relatively separated by Lemma 1.25.

$\square$

A Hilbert space $\mathcal{H}$ of continuous functions on G is called a *reproducing kernel Hilbert space*, if for every $x \in G$ the point evaluation $f \mapsto f(x)$ is a bounded linear functional on $\mathcal{H}$.

The Riesz Representation Theorem then implies that for every $x \in G$ there exists a unique function $K_x \in \mathcal{H}$ satisfying $f(x) = \langle f, K_x \rangle$. The functions $\{K_x\} \subseteq \mathcal{H}$ are called the *reproducing kernels*.

Proposition 4.7. *Let $\varphi \in W(C, L^1)$. Assume that φ has the Riesz basis property and denote the dual generator by $\tilde{\varphi}$.*

(a) *The space $V^2(\varphi)$ is a reproducing kernel Hilbert space.*
(b) *The kernel functions K_x, $x \in G$, are explicitly given by*

$$(4.16) \qquad K_x = \sum_{\gamma \in \Gamma} \overline{\varphi(\gamma^{-1}x)} L_\gamma \tilde{\varphi}.$$

(c) *The map $x \mapsto K_x$, $G \to V^2(\varphi)$ is continuous.*

PROOF. (a) It follows from Proposition 4.6 applied to the set $X = \{x\}$ that

$$|f(x)| \leq C\|f\|_{L^2(G)}$$

for all $f \in V^2(\varphi)$.

(b) Applying Proposition 4.6 to the set $X = \{\gamma^{-1}x : \gamma \in \Gamma\}$ we obtain that the sequence $(\varphi(\gamma^{-1}x))_{\gamma \in \Gamma}$ belongs to $\ell^2(\Gamma)$. It follows that the function $K_x := \sum_{\gamma \in \Gamma} \overline{\varphi(\gamma^{-1}x)} L_\gamma \tilde{\varphi}$ belongs to $V^2(\varphi)$ and that, for $f = \sum_{\nu \in \Gamma} c_\nu L_\nu \varphi \in V^2(\varphi)$,

$$\begin{aligned}
\langle f, K_x \rangle &= \left\langle \sum_{\nu \in \Gamma} c_\nu L_\nu \varphi, \sum_{\gamma \in \Gamma} \overline{\varphi(\gamma^{-1}x)} L_\gamma \tilde{\varphi} \right\rangle \\
&= \sum_{\nu \in \Gamma} \sum_{\gamma \in \Gamma} c_\nu \varphi(\gamma^{-1}x) \langle L_\nu \varphi, L_\gamma \tilde{\varphi} \rangle \\
&= \sum_{\nu \in \Gamma} c_\nu L_\nu \varphi(x) = f(x)
\end{aligned}$$

by the biorthogonality of $\{L_\nu \varphi\}_{\nu \in \Gamma}$ and $\{L_\gamma \tilde{\varphi}\}_{\gamma \in \Gamma}$.

(c) Let $x_0 \in G$ and $\varepsilon > 0$ be arbitrary. We need to find some $\delta > 0$ such that

$$\|K_x - K_{x_0}\|_{L^2(G)} < \varepsilon$$

for all $x \in B_\delta(x_0)$. Using the formula (4.16) for the kernel functions K_x we note that

$$(4.17) \qquad \begin{aligned}
\|K_x - K_{x_0}\|_{L^2(G)} &= \left\| \sum_{\gamma \in \Gamma} \overline{(\varphi(\gamma^{-1}x) - \varphi(\gamma^{-1}x_0))} L_\gamma \tilde{\varphi} \right\|_{L^2(G)} \\
&\leq \|\tilde{\varphi}\|_{L^2(G)} \sum_{\gamma \in \Gamma} |\varphi(\gamma^{-1}x) - \varphi(\gamma^{-1}x_0)|.
\end{aligned}$$

Since $\varphi \in W(C, L^1)$, there exists some $N > 0$ such that

$$(4.18) \qquad \sum_{\gamma \in \Gamma \setminus B_N(e)} \sup_{u \in U} |\varphi(\gamma u)| < \frac{\varepsilon}{3\|\tilde{\varphi}\|_2}.$$

Consider the ball $B_1(x_0)$ of radius one around x_0. Since $B_1(x_0)$ is relatively compact, there exists some $N_0 > 0$ such that

$$(4.19) \qquad B_1(x_0) \subseteq \bigcup_{\nu \in \Gamma \cap B_{N_0}(e)} \nu U.$$

Let $R > N_0 + N$. For all $\nu \in B_{N_0}(e)$ it follows that

$$B_N(e) \subseteq B_R(\nu^{-1}).$$

Indeed, if $x \in B_1(x_0)$ and $\nu \in B_{N_0}(e)$, then also $\nu^{-1} \in B_{N_0}(e)$ and thus

$$d(\nu^{-1}, x) \le d(\nu^{-1}, e) + d(e, x) \le N_0 + N < R.$$

By inclusion (4.19), each $x \in B_1(x_0)$ can be written as $x = \nu u$ for some $\nu \in B_{N_0}(e)$, $u \in U$. Thus

$$
(4.20) \qquad
\begin{aligned}
\sum_{\gamma \in \Gamma \backslash B_R(e)} |\varphi(\gamma^{-1} x)| &\le \sum_{\gamma \in \Gamma \backslash B_R(e)} \sup_{u \in U} |\varphi(\gamma^{-1} \nu u)| \\
&= \sum_{\gamma \in \Gamma \backslash B_R(e)} \sup_{u \in U} |\varphi((\nu^{-1}\gamma)^{-1} u)| \\
&= \sum_{\eta \in \Gamma \backslash B_R(\nu^{-1})} \sup_{u \in U} |\varphi(\eta^{-1} u)| \\
&\le \sum_{\eta \in \Gamma \backslash B_N(e)} \sup_{u \in U} |\varphi(\eta^{-1} u)| < \frac{\varepsilon}{3\|\tilde\varphi\|_2}.
\end{aligned}
$$

We now continue our initial estimate by splitting the infinite sum in inequality (4.17) into a finite part and a small remainder, that is,

$$\|K_x - K_{x_0}\|_2 \le \|\tilde\varphi\|_2 \sum_{\gamma \in \Gamma} |\varphi(\gamma^{-1} x) - \varphi(\gamma^{-1} x_0)|$$

$$= \|\tilde\varphi\|_2 \Big(\sum_{\gamma \in \Gamma \cap B_R(e)} |\varphi(\gamma^{-1} x) - \varphi(\gamma^{-1} x_0)| + \sum_{\gamma \in \Gamma \backslash B_R(e)} |\varphi(\gamma^{-1} x) - \varphi(\gamma^{-1} x_0)| \Big)$$

$$\le \|\tilde\varphi\|_2 \Big(\sum_{\gamma \in \Gamma \cap B_R(e)} |\varphi(\gamma^{-1} x) - \varphi(\gamma^{-1} x_0)| + \sum_{\gamma \in \Gamma \backslash B_R(e)} |\varphi(\gamma^{-1} x)| + \sum_{\gamma \in \Gamma \backslash B_R(e)} |\varphi(\gamma^{-1} x_0)| \Big)$$

$$< \frac{2\varepsilon}{3} + \|\tilde\varphi\|_2 \sum_{\gamma \in \Gamma \cap B_R(e)} |\varphi(\gamma^{-1} x) - \varphi(\gamma^{-1} x_0)|,$$

where the last inequality follows from the above estimate (4.20). Since the generator φ is continuous, we can choose $0 < \delta < 1$ such that

$$\sum_{\gamma \in \Gamma \cap B_R(e)} |\varphi(\gamma^{-1} x) - \varphi(\gamma^{-1} x_0)| < \frac{\varepsilon}{3\|\tilde\varphi\|_2}$$

for all $x \in B_\delta(x_0)$. Putting things together we finally conclude that

$$\|K_x - K_{x_0}\|_2 < \varepsilon$$

for all $x \in B_\delta(x_0)$.

$\square$

With the notion of reproducing kernels we can restate Proposition 4.6 as follows.

Corollary 4.8. *Let $\varphi \in W(C, L^1)$ and assume that φ has the Riesz basis property and satisfies condition (4.7). A subset $X \subseteq G$ is relatively separated if and only if the associated family $\mathcal{K} = \{K_\chi\}_{\chi \in X}$ of reproducing kernels is a Bessel sequence in $V^2(\varphi)$.*

PROOF. Since $f(\chi) = \langle f, K_\chi \rangle$ for all $\chi \in X$, the inequality (4.11) in Proposition 4.6 is precisely the Bessel condition for $\mathcal{K} = \{K_\chi\}_{\chi \in X}$.

$\square$

Proposition 4.9. *Let $\varphi \in W(C, L^1)$ and assume that φ has the Riesz basis property. If a subset $X \subseteq G$ is a set of interpolation for $V^2(\varphi)$, then X is relatively separated.*

PROOF. Since $f(\chi) = \langle f, K_\chi \rangle$ for all $\chi \in X$, a subset $X \subseteq G$ is a set of interpolation for $V^2(\varphi)$ if and only if the family $\mathcal{K} = \{K_\chi\}_{\chi \in X}$ is a Riesz-Fischer sequence in $V^2(\varphi)$ (compare Definition 4.5 (ii) and Definition 3.3). By Lemma 3.4, the property of $\mathcal{K} = \{K_\chi\}_{\chi \in X}$ forming a Riesz-Fischer sequence in $V^2(\varphi)$ is equivalent to the existence of a constant $A > 0$ such that the lower Riesz inequality

$$(4.21) \qquad A\|c\|^2 \leq \Big\| \sum_{\chi \in X} c_\chi K_\chi \Big\|_{L^2(G)}^2$$

holds for every finite sequence $c = (c_\chi)_{\chi \in X}$. In particular, if X is a set of interpolation for $V^2(\varphi)$, then

$$(4.22) \qquad \|K_\chi - K_{\chi'}\|_{L^2(G)} \geq A^{\frac{1}{2}}\sqrt{2}$$

for all $\chi \neq \chi' \in X$. By the continuity of the map $x \mapsto K_x$, $G \to V^2(\varphi)$ (Proposition 4.7 (c)), it follows that $d(\chi, \chi') \geq \delta$ for some $\delta > 0$. Thus X is relatively separated.

$\square$

Furthermore, we can also translate the properties of X being a set of sampling or interpolation into statements about the associated family $\mathcal{K} = \{K_\chi\}_{\chi \in X}$ of reproducing kernels.

Proposition 4.10. *Let $\varphi \in W(C, L^1)$ and assume that φ has the Riesz basis property. Let $\mathcal{K} = \{K_\chi\}_{\chi \in X}$ be the family of reproducing kernels associated to a subset X of G.*

(a) *The set X is a set of sampling for $V^2(\varphi)$ if and only if the family $\mathcal{K} = \{K_\chi\}_{\chi \in X}$ is a frame for $V^2(\varphi)$.*

(b) *The set X is a set of interpolation for $V^2(\varphi)$ if and only if the family $\mathcal{K} = \{K_\chi\}_{\chi \in X}$ is a Riesz sequence in $V^2(\varphi)$.*

PROOF. (a) follows directly from the definitions, because $f(\chi) = \langle f, K_\chi \rangle$ for $\chi \in X$.

(b) As already noted in the previous proof, a subset $X \subseteq G$ is a set of interpolation for $V^2(\varphi)$ if and only if the family $\mathcal{K} = \{K_\chi\}_{\chi \in X}$ obeys the lower Riesz inequality. Proposition 4.9 in this case implies that the set X is relatively separated, a property which further implies the Bessel condition for the associated family $\mathcal{K} = \{K_\chi\}_{\chi \in X}$ of reproducing kernels (Corollary 4.8). But the Bessel condition for $\mathcal{K} = \{K_\chi\}_{\chi \in X}$ is equivalent to the upper Riesz inequality by Lemma 3.2. Thus the assertion follows.

$\square$

4.3. Homogeneous Approximation Property and Density

Now that we have converted the sampling and interpolation problem into a question about frames and Riesz sequences indexed by discrete subsets of G, we want to employ the abstract Comparison Theorem from Chapter 3 (Theorem 3.13) to derive necessary density conditions for sampling and interpolation. To meet the assumptions made in Theorem 3.13 we need to establish the Homogeneous Approximation Property.

Proposition 4.11 (Homogeneous Approximation Property of $\mathcal{E}$). *Let $\varphi \in W(C, L^1)$ and suppose that the set of left translates $\mathcal{E} = \{L_\gamma \varphi\}_{\gamma \in \Gamma}$ forms a Riesz basis for $V^2(\varphi)$ with dual Riesz basis $\tilde{\mathcal{E}} = \{L_\gamma \tilde{\varphi}\}_{\gamma \in \Gamma}$. Let $\mathcal{K} = \{K_\chi\}_{\chi \in X}$ be a set of reproducing kernels in $V^2(\varphi)$. Then $\mathcal{E}$ has the Homogeneous Approximation Property with respect to $\mathcal{K}$, that is, for every $\varepsilon > 0$ there is an $N_\varepsilon > 0$ such that for each $\chi \in X$ we have*

$$\mathrm{dist}(K_\chi, \mathrm{span}\{L_\gamma \tilde{\varphi} : \gamma \in \Gamma \cap B_{N_\varepsilon}(\chi)\}) < \varepsilon.$$

PROOF. Since $\varphi \in W(C, L^1)$, the (left) local maximum function $\varphi^\#$ defined by $\varphi^\#(\gamma) = \sup_{x \in U} |\varphi(\gamma x)|$ is in $\ell^1(\Gamma) \subseteq \ell^2(\Gamma)$.
Let $N_U > 0$ be such that $U \subseteq B_{N_U}(e)$. Given $\varepsilon > 0$, choose $N_\varepsilon > N_U > 0$ such that

$$\sum_{\gamma \in \Gamma \backslash B_{N_\varepsilon - N_U}(e)} \varphi^\#(\gamma)^2 < A\varepsilon^2,$$

where A is the lower frame bound for $\mathcal{E} = \{L_\gamma \varphi\}_{\gamma \in \Gamma}$.

For each $\chi \in X$ we have the expansion

$$K_\chi = \sum_{\gamma \in \Gamma} \langle K_\chi, L_\gamma \varphi \rangle L_\gamma \tilde{\varphi}.$$

Therefore

$$
\begin{aligned}
\operatorname{dist}(K_\chi, \operatorname{span}\{L_\gamma \tilde\varphi : \gamma \in \Gamma \cap B_{N_\varepsilon}(\chi)\})^2 &\leq \left\| K_\chi - \sum_{\gamma \in \Gamma \cap B_{N_\varepsilon}(\chi)} \langle K_\chi, L_\gamma\varphi \rangle L_\gamma\tilde\varphi \right\|^2_{L^2(G)} \\
&= \left\| \sum_{\gamma \in \Gamma \backslash B_{N_\varepsilon}(\chi)} \langle K_\chi, L_\gamma\varphi \rangle L_\gamma\tilde\varphi \right\|^2_{L^2(G)} \\
&\leq \frac{1}{A} \sum_{\gamma \in \Gamma \backslash B_{N_\varepsilon}(\chi)} |\langle K_\chi, L_\gamma\varphi \rangle|^2 \\
&= \frac{1}{A} \sum_{\gamma \in \Gamma \backslash B_{N_\varepsilon}(\chi)} |\varphi(\gamma^{-1}\chi)|^2.
\end{aligned}
$$

(4.23)

Since Γ is a lattice with fundamental domain U, we can assign to each $\chi \in X$ a unique element $\nu_\chi \in \Gamma$ such that $\chi \in \nu_\chi U$.
Recall that $\nu_\chi U \subseteq \nu_\chi B_{N_U}(e) = B_{N_U}(\nu_\chi)$, so $\chi \in B_{N_U}(\nu_\chi)$. Thus, for every $N > N_U$,

$$ B_{N-N_U}(\nu_\chi) \subseteq B_N(\chi) $$

and consequently

$$ \Gamma \backslash B_{N_\varepsilon}(\chi) \subseteq \Gamma \backslash B_{N_\varepsilon - N_U}(\nu_\chi). $$

Now continue the above estimate (4.23) as follows,

$$
\begin{aligned}
\operatorname{dist}(K_\chi, \operatorname{span}\{L_\gamma \tilde\varphi : \gamma \in \Gamma \cap B_{N_\varepsilon}(\chi)\})^2 &\leq \frac{1}{A} \sum_{\gamma \in \Gamma \backslash B_{N_\varepsilon}(\chi)} |\varphi(\gamma^{-1}\chi)|^2 \\
&\leq \frac{1}{A} \sum_{\gamma \in \Gamma \backslash B_{N_\varepsilon - N_U}(\nu_\chi)} |\varphi(\gamma^{-1}\chi)|^2 \\
&\leq \frac{1}{A} \sum_{\gamma \in \Gamma \backslash B_{N_\varepsilon - N_U}(\nu_\chi)} \varphi^\#(\gamma^{-1}\nu_\chi)^2 \\
&= \frac{1}{A} \sum_{\gamma \in \Gamma \backslash (\nu_\chi \cdot B_{N_\varepsilon - N_U}(e))} \varphi^\#((\nu_\chi^{-1}\gamma)^{-1})^2 \\
&= \frac{1}{A} \sum_{\eta \in \Gamma \backslash B_{N_\varepsilon - N_U}(e)} \varphi^\#(\eta^{-1})^2 \\
&= \frac{1}{A} \sum_{\eta \in \Gamma \backslash B_{N_\varepsilon - N_U}(e)} \varphi^\#(\eta)^2 < \varepsilon.
\end{aligned}
$$

$\square$

If the set $\mathcal{K} = \{K_\chi\}_{\chi \in X}$ of reproducing kernels forms a frame for $V^2(\varphi)$, then we may similarly derive the Homogeneous Approximation Property of $\mathcal{K}$ with respect to $\mathcal{E}$.

Proposition 4.12 (Homogeneous Approximation Property of $\mathcal{K}$). *Let $\varphi \in W(C, L^1)$ and suppose that φ satisfies condition (4.7). Further suppose that the set of left translates $\mathcal{E} = \{L_\gamma \varphi\}_{\gamma \in \Gamma}$ forms a Riesz basis for $V^2(\varphi)$.*
If the set of reproducing kernels $\mathcal{K} = \{K_\chi\}_{\chi \in X}$ is a frame for $V^2(\varphi)$ with dual frame $\{\tilde{K}_\chi\}_{\chi \in X}$, then $\mathcal{K}$ has the Homogeneous Approximation Property with respect to $\mathcal{E}$, that is, for every $\varepsilon > 0$ there is an $N_\varepsilon > 0$ such that for each $\gamma \in \Gamma$ we have

$$\mathrm{dist}(L_\gamma \varphi, \mathrm{span}\{\tilde{K}_\chi : \chi \in X \cap B_{N_\varepsilon}(\gamma)\}) < \varepsilon.$$

PROOF. By Corollary 4.8, the assumption that the family $\mathcal{K} = \{K_\chi\}_{\chi \in X}$ is a frame for $V^2(\varphi)$ implies that the index set $X \subseteq G$ is relatively separated. Thus

$$\max_{\gamma \in \Gamma} |X \cap \gamma U| =: K < \infty.$$

Since $\varphi \in W(C, L^1)$, the (left) local maximum function $\varphi^\#$ defined by $\varphi^\#(\gamma) = \sup_{x \in U} |\varphi(\gamma x)|$ is in $\ell^1(\Gamma) \subseteq \ell^2(\Gamma)$. Let $N_U > 0$ be such that $U \subseteq B_{N_U}(e)$. Given $\varepsilon > 0$, choose $N_\varepsilon > N_U > 0$ such that

$$\sum_{\gamma \in \Gamma \setminus B_{N_\varepsilon - N_U}(e)} \varphi^\#(\gamma)^2 < \frac{A\varepsilon^2}{K},$$

where A is the lower frame bound for the frame of reproducing kernels $\{K_\chi\}_{\chi \in X}$.

For each $\gamma \in \Gamma$ we have the frame expansion

$$L_\gamma \varphi = \sum_{\chi \in X} \langle L_\gamma \varphi, K_\chi \rangle \tilde{K}_\chi = \sum_{\chi \in X} \varphi(\gamma^{-1}\chi) \tilde{K}_\chi.$$

Therefore

$$\text{dist}(L_\gamma\varphi, \text{span}\{\tilde{K}_\chi : \chi \in X \cap B_{N_\varepsilon}(\gamma)\})^2 \leq \left\| L_\gamma\varphi - \sum_{\chi\in X\cap B_{N_\varepsilon}(\gamma)} \langle L_\gamma\varphi, K_\chi\rangle \tilde{K}_\chi \right\|^2_{L^2(G)}$$

$$= \left\| \sum_{\chi\in X\setminus B_{N_\varepsilon}(\gamma)} \langle L_\gamma\varphi, K_\chi\rangle \tilde{K}_\chi \right\|^2_{L^2(G)}$$

$$\leq \frac{1}{A} \sum_{\chi\in X\setminus B_{N_\varepsilon}(\gamma)} |\langle L_\gamma\varphi, K_\chi\rangle|^2$$

$$\leq \frac{1}{A} \sum_{\nu\in\Gamma\setminus B_{N_\varepsilon-N_U}(\gamma)} \sum_{\chi\in X\cap\nu U} |\varphi(\gamma^{-1}\chi)|^2$$

$$\leq \frac{K}{A} \sum_{\nu\in\Gamma\setminus B_{N_\varepsilon-N_U}(\gamma)} \varphi^\#(\gamma^{-1}\nu)^2$$

$$= \frac{K}{A} \sum_{\eta\in\Gamma\setminus B_{N_\varepsilon-N_U}(e)} \varphi^\#(\eta)^2 < \varepsilon^2.$$

$\square$

Corollary 4.13. *Let $\varphi \in W(C, L^1)$ and assume that φ has the Riesz basis property and satisfies condition (4.7).*

(a) *If $X \subseteq G$ is a set of interpolation for $V^2(\varphi)$, then*

$$D^+(X) \leq \frac{1}{\lambda(U)}.$$

(b) *If $X \subseteq G$ is a set of sampling for $V^2(\varphi)$, then*

$$D^-(X) \geq \frac{1}{\lambda(U)}.$$

PROOF. Let $\mathcal{K} = \{K_\chi\}_{\chi\in X}$ be the family of reproducing kernels corresponding to X and let $\mathcal{E} = \{L_\gamma\varphi\}_{\gamma\in\Gamma}$ be the Riesz basis of left translates spanning $V^2(\varphi)$.

(a) By Proposition 4.11, $\mathcal{E}$ has the Homogeneous Approximation Property with respect to $\mathcal{K}$. Since $X \subseteq G$ is a set of interpolation for $V^2(\varphi)$, the family $\mathcal{K} = \{K_\chi\}_{\chi\in X}$ is a Riesz sequence in $V^2(\varphi)$ (Proposition 4.10). Therefore the Comparison Theorem (Theorem 3.13) applied to the frame $\mathcal{E}$ and the Riesz sequence $\mathcal{K}$ implies that

$$D^+(X) \leq D^+(\Gamma) = \frac{1}{\lambda(U)}.$$

(b) If $X \subseteq G$ is a set of sampling for $V^2(\varphi)$, then the family $\mathcal{K}$ of reproducing kernels is a frame for $V^2(\varphi)$ (Proposition 4.10). In this case $\mathcal{K}$ has the Homogeneous Approximation Property with respect to the Riesz basis $\mathcal{E}$ (Proposition 4.12). Again the Comparison Theorem (Theorem 3.13), now applied to compare the density of the frame $\mathcal{K}$ to that of the Riesz basis $\mathcal{E}$, gives the desired result

$$D^-(X) \geq D^-(\Gamma) = \frac{1}{\lambda(U)}.$$

$\square$

CHAPTER 5

Coherent Frames

The most prominent examples of frames in applications, the Gabor frames and wavelet frames, are generated from a single vector under the action of a square-integrable (projective) group representation of a locally compact group.

Generally, frames that arise as subsets of the orbit of a (projective) square-integrable group representation are called *coherent frames* [**19**].

The existence of coherent frames for (square-) integrable group representations of locally compact groups was settled in the context of general coorbit theory [**26**] (see also [**10**] for a version with projective representations).
A Homogeneous Approximation Property for frames in the orbit of square-integrable unitary representations was established in [**28**]. However, on general locally compact groups a suitable notion of density to derive a theorem for the comparison of the densities of frames and Riesz sequences is missing.
In the realm of homogeneous groups we can work with the density defined in Chapter 2 and employ the abstract Comparison Theorem derived in Chapter 3.

5.1. Definitions and Prerequisites

First we collect some necessary definitions and prerequisites about projective representations [**36**], [**47**].
Let G be a homogeneous group. Let $\mathcal{H}$ be a separable Hilbert space and denote by $\mathcal{U}(\mathcal{H})$ the group of unitary operators on $\mathcal{H}$.

Definition 5.1. A *(continuous) projective representation* of G on $\mathcal{H}$ is a strongly continuous mapping $\pi : G \to \mathcal{U}(\mathcal{H})$ such that

    (i) $\pi(e) = I$;

    (ii) There exists a continuous function $\mu : G \times G \to \mathbb{T}$ such that
$$\pi(xy) = \mu(x, y)\pi(x)\pi(y)$$
      for all $x, y \in G$.

The mapping π is called a *unitary representation* of G on $\mathcal{H}$ if
$$\mu(x, y) = 1$$

for all $x, y \in G$.

We will always deal with continuous projective representations and simply call them projective representations.

The function μ, which is often called the *multiplier of* π, must satisfy
  (a) $\mu(x, e) = \mu(e, x) = 1$ for all $x \in G$,

  (b) $\mu(x, yz)\mu(y, z) = \mu(xy, z)\mu(x, y)$ for all $x, y, z \in G$.

It follows from (a) and (b), by taking $y = x^{-1}, z = x$, that
  (c) $\mu(x, x^{-1}) = \mu(x^{-1}, x)$ for all $x \in G$.

Further note that
$$(5.1) \qquad \pi(x)^* = \pi(x)^{-1} = \mu(x, x^{-1})\pi(x^{-1})$$
for every $x \in G$, because $\pi(x)$ is unitary and, by (c),
$$I = \pi(e) = \pi(xx^{-1}) = \mu(x, x^{-1})\pi(x)\pi(x^{-1}),$$
$$I = \pi(e) = \pi(x^{-1}x) = \mu(x, x^{-1})\pi(x^{-1})\pi(x).$$

Definition 5.2. A linear subspace W of $\mathcal{H}$ is said to be π-*invariant* if $\pi(x)W \subseteq W$ for all $x \in G$. A projective representation π of G on $\mathcal{H}$ is called *irreducible* if the only closed π-invariant subspaces are $\{0\}$ and $\mathcal{H}$.

A projective representation π of G on $\mathcal{H}$ is *square integrable*, if for every $f, g \in \mathcal{H}$ the representation coefficient $V_g f : G \to \mathbb{C}$, defined by
$$V_g f(x) := \langle f, \pi(x)g \rangle,$$
is in $L^2(G)$.

Note that for every $f, g \in \mathcal{H}$ the representation coefficient $V_g f$ is a continuous function on G, because the map $\pi : G \to \mathcal{U}(\mathcal{H})$ is strongly continuous.

Definition 5.3. Two projective representations π_1, π_2 of G on Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ respectively are called *projectively equivalent* if there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ and a function $\nu : G \to \mathbb{T}$ such that, for all $x \in G$,
$$(5.2) \qquad \pi_2(x)U = \nu(x)U\pi_1(x).$$
Two unitary representations π_1, π_2 of G on Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ are called *unitarily equivalent* if equation (5.2) holds with $\nu(x) = 1$ for all $x \in G$.

The representation coefficient obeys the following covariance property.

Lemma 5.4. *Let π be a projective representation of G with multiplier μ on a Hilbert space $\mathcal{H}$. Then*

(5.3)
$$V_g(\pi(y)f)(x) = \overline{\mu(y, y^{-1}x)}L_y(V_g f)(x)$$

for all $x, y \in G$ and $f, g \in \mathcal{H}$.

PROOF. Let $x, y \in G$ and $f, g \in \mathcal{H}$. Since $\pi(y)^* = \pi(y)^{-1} = \mu(y, y^{-1})\pi(y^{-1})$, we calculate

$$
\begin{aligned}
V_g(\pi(y)f)(x) &= \langle \pi(y)f, \pi(x)g \rangle \\
&= \langle f, \pi(y)^{-1}\pi(x)g \rangle \\
&= \langle f, \mu(y, y^{-1})\pi(y^{-1})\pi(x)g \rangle \\
&= \langle f, \mu(y, y^{-1})\overline{\mu(y^{-1}, x)}\pi(y^{-1}x)g \rangle \\
&= \overline{\mu(y, y^{-1})}\mu(y^{-1}, x)\langle f, \pi(y^{-1}x)g \rangle \\
&= \overline{\mu(y, y^{-1})}\mu(y^{-1}, x)V_g f(y^{-1}x) \\
&= \overline{\mu(y, y^{-1})}\mu(y^{-1}, x)L_y(V_g f)(x).
\end{aligned}
$$

(5.4)

Using the properties (a) and (b) of the multiplier μ we get

$$\mu(y, y^{-1}) = \mu(e, x)\mu(y, y^{-1}) = \mu(yy^{-1}, x)\mu(y, y^{-1}) = \mu(y, y^{-1}x)\mu(y^{-1}, x).$$

Therefore we can simplify equation (5.4) to

$$V_g(\pi(y)f)(x) = \overline{\mu(y, y^{-1}x)}L_y(V_g f)(x).$$

$\square$

We single out the subspace of $\mathcal{H}$ that is mapped into the space $W(C, L^2)$ under V_g.

Definition 5.5. Let π be a square-integrable projective representation of G on a Hilbert space $\mathcal{H}$. Define

$$\mathcal{N} := \{f \in \mathcal{H} : V_g f \in W(C, L^2) \text{ for all } g \in \mathcal{H}\}.$$

Lemma 5.6. *Let π be a square-integrable irreducible projective representation of G on a Hilbert space $\mathcal{H}$. The set $\mathcal{N}$ forms a dense subspace of $\mathcal{H}$.*

PROOF. We show that $\mathcal{N}$ is a non-trivial π-invariant subspace of $\mathcal{H}$. The irreducibility of π then implies that $\mathcal{N}$ is a dense subspace of $\mathcal{H}$.

Let $0 \neq \varphi \in C_c(G)$ and $0 \neq f \in \mathcal{H}$ be arbitrary. Similar to a construction in [20] we set

$$f' := \pi(\varphi)f = \int_G \varphi(y)\pi(y)f \, dy.$$

We claim that $V_g f'$ belongs to $W(C, L^2)$ for all $g \in \mathcal{H}$, that is, $f' \in \mathcal{N}$, and so $\mathcal{N} \neq \{0\}$.

From the definitions and the covariance property of V_g (Lemma 5.4) it follows that, for arbitrary $g \in \mathcal{H}$,

$$
\begin{aligned}
(5.5) \qquad V_g f'(x) &= \langle \pi(\varphi)f, \pi(x)g \rangle \\
&= \int_G \varphi(y)\langle \pi(y)f, \pi(x)g \rangle dy \\
&= \int_G \varphi(y) V_g(\pi(y)f)(x) dy \\
&= \int_G \varphi(y) V_g f(y^{-1}x)\overline{\mu(y, y^{-1}x)} dy.
\end{aligned}
$$

Then

$$
\begin{aligned}
(5.6) \qquad |V_g f'(x)| &= \left| \int_G \varphi(y) V_g f(y^{-1}x)\overline{\mu(y, y^{-1}x)} dy \right| \\
&\leq \int_G |\varphi(y)| \, |V_g f(y^{-1}x)| \, dy \\
&= \left(|\varphi| * |V_g f| \right)(x).
\end{aligned}
$$

Since π is square integrable, i.e., $V_g f \in L^2(G)$, and $\varphi \in C_c(G) \subseteq W(L^\infty, L^1)$, it follows from Corollary 1.31 that

$$
|\varphi| * |V_g f| \in W(L^\infty, L^1) * L^2(G) \subseteq W(L^\infty, L^2).
$$

By equation (5.6), $|V_g f'| \leq |\varphi| * |V_g f| \in W(L^\infty, L^2)$. Since $V_g f'$ is continuous, $V_g f' \in W(C, L^2)$. Since $g \in \mathcal{H}$ was arbitrary, $f' \in \mathcal{N}$. Thus $\mathcal{N} \neq \{0\}$.

To see that $\mathcal{N}$ is π-invariant, let $f \in \mathcal{N}$ be arbitrary. We need to show that $\pi(y)f \in \mathcal{N}$ for all $y \in G$. So for arbitrary $g \in \mathcal{H}$ and $y \in G$ we need to show that $V_g(\pi(y)f) \in W(C, L^2)$. By Lemma 5.4 we have, for $x \in G$,

$$
V_g(\pi(y)f)(x) = \overline{\mu(y, y^{-1}x)} L_y(V_g f)(x)
$$

and therefore

$$
|V_g(\pi(y)f)| = |L_y(V_g f)|.
$$

Since $V_g f \in W(C, L^2)$ and $W(C, L^2)$ is invariant under left translation (Lemma 1.28 (a)), it follows that also $V_g(\pi(y)f) \in W(C, L^2)$, that is, $\pi(y)f \in \mathcal{N}$. $\qquad \square$

5.2. Homogeneous Approximation Property and Density

Coherent Bessel sequences are indexed by relatively separated sets.

Lemma 5.7. *Let π be a square-integrable irreducible projective representation of G on a separable Hilbert space $\mathcal{H}$. If $0 \neq g \in \mathcal{H}$ and $X \subseteq G$ are such that $\mathcal{G} = \{\pi(\chi)g : \chi \in X\}$ is a Bessel sequence in $\mathcal{H}$, then the set X is relatively separated.*

PROOF. The proof is similar to the second part of the proof of Proposition 4.6. Choose some $f \in \mathcal{H}$ with $\|f\|_{\mathcal{H}} = 1$. Note that $\|\pi(y)f\|_{\mathcal{H}} = 1$ for all $y \in G$ and recall from Lemma 5.4 that

$$|\langle \pi(y)f, \pi(\chi)g\rangle| = |\langle f, \pi(y^{-1}\chi)g\rangle| = |V_g f(y^{-1}\chi)|.$$

The representation coefficient $V_g f$ is not identically zero and continuous on G, so it must be bounded away from zero on a ball $B_R(a)$ for some $R > 0$ and $a \in G$, that is,

$$(5.7) \qquad \varepsilon := \inf_{z \in B_R(a)} |V_g f(z)| > 0.$$

We argue by contradiction and assume that X is not relatively separated. Then for arbitrary $n \in \mathbb{N}$ there exists some $g \in G$ such that

$$|X \cap B_R(g)| \geq n.$$

If $\chi \in B_R(g) = gB_R(e)$, then $g^{-1}\chi \in B_R(e)$ and hence $ag^{-1}\chi \in B_R(a)$. Therefore,

$$(5.8) \qquad n\varepsilon^2 \leq \sum_{\chi \in X \cap B_R(g)} |V_g f(ag^{-1}\chi)|^2 \leq \sum_{\chi \in X} |V_g f(ag^{-1}\chi)|^2.$$

On the other hand, if B denotes the Bessel bound for the system $\mathcal{G} = \{\pi(\chi)g : \chi \in X\}$, then

$$(5.9) \qquad \sum_{\chi \in X} |V_g f(ag^{-1}\chi)|^2 = \sum_{\chi \in X} |\langle \pi(ga^{-1})f, \pi(\chi)g\rangle|^2$$
$$\leq B\|\pi(ga^{-1})f\|_{\mathcal{H}}^2 = B.$$

Since $n \in \mathbb{N}$ was arbitrary, the inequalities (5.8) and (5.9) combined yield a contradiction.

$\square$

Coherent frames possess some intrinsic Homogeneous Approximation Property as was already noted by Gröchenig for unitary group representations and a special class of atoms [28]. We provide a modification of his proof to our setting of projective representations of homogeneous groups and remove the assumption that was imposed on the atom in [28].

Proposition 5.8 (Homogeneous Approximation Property)**.** *Let π be a square-integrable irreducible projective representation of G on a separable Hilbert space $\mathcal{H}$. Let $g \in \mathcal{H}$ and $X \subseteq G$ be such that $\mathcal{G} = \{\pi(\chi)g : \chi \in X\}$ is a frame for $\mathcal{H}$ and denote its dual frame by $\tilde{\mathcal{G}} = \{g_\chi : \chi \in X\}$.*
Then $\mathcal{G}$ possesses the Homogeneous Approximation Property, that is, for every $f \in \mathcal{H}$ and $\varepsilon > 0$ there exists a constant $N = N(f, \varepsilon)$ such that, for every $x \in G$,

$$\mathrm{dist}\big(\pi(x)f, \mathrm{span}\{g_\chi : \chi \in X \cap B_N(x)\}\big) < \varepsilon.$$

PROOF. We first prove the Homogeneous Approximation Property for the dense subspace $\mathcal{N}$ (Definition 5.5) and then extend to all of $\mathcal{H}$ by continuity.

Let $f \in \mathcal{N}$ be arbitrary. Let V be the symmetric relatively compact subset of G used to define the local maximum function, that is,

$$f^{\#}(x) = \sup_{y \in xV} |f(y)|,$$

and let $N_V > 0$ be such that $V \subseteq B_{N_V}(e)$.

Since the index set X of the frame $\mathcal{G} = \{\pi(\chi)g : \chi \in X\}$ is automatically relatively separated by Lemma 5.7, it follows from Lemma 1.24 that

$$C := \sup_{x \in G} \sum_{\chi \in X} \mathbb{1}_{\chi V}(x) < \infty.$$

By the definition of the subspace $\mathcal{N}$, the representation coefficient $V_g f$ belongs to $W(C, L^2)$, put differently, $(V_g f)^{\#} \in L^2(G)$. So given $\varepsilon > 0$, we can choose $N = N(f, \varepsilon)$ such that

(5.10) $$\int_{G \backslash B_{N-N_V}(e)} (V_g f)^{\#}(x)^2 dx < \frac{A\lambda(V)\varepsilon^2}{C},$$

where A is the lower frame bound of the frame $\mathcal{G}$.

For arbitrary $y \in G$ consider the frame expansion of $\pi(y)f$ with respect to the frame $\mathcal{G}$, that is,

$$\pi(y)f = \sum_{\chi \in X} \langle \pi(y)f, \pi(\chi)g \rangle g_\chi.$$

Then

$$\text{dist}\big(\pi(y)f, \text{span}\{g_\chi : \chi \in X \cap B_N(y)\}\big)^2$$

$$\leq \left\| \pi(y)f - \sum_{\chi \in X \cap B_N(y)} \langle \pi(y)f, \pi(\chi)g \rangle g_\chi \right\|^2$$

$$= \left\| \sum_{\chi \in X \setminus B_N(y)} \langle \pi(y)f, \pi(\chi)g \rangle g_\chi \right\|^2$$

$$\leq \frac{1}{A} \sum_{\chi \in X \setminus B_N(y)} |\langle \pi(y)f, \pi(\chi)g \rangle|^2$$

$$= \frac{1}{A} \sum_{\chi \in X \setminus B_N(y)} |\langle f, \pi(y^{-1}\chi)g \rangle|^2$$

$$= \frac{1}{A} \sum_{\chi \in X \setminus B_N(y)} |V_g f(y^{-1}\chi)|^2$$

$$= \frac{1}{A} \sum_{v \in (y^{-1}X) \setminus B_N(e)} |V_g f(v)|^2.$$

Note that the set $y^{-1}X = \{y^{-1}\chi : \chi \in X\}$ is also relatively separated and

$$\sup_{x \in G} \sum_{v \in y^{-1}X} \mathbb{1}_{vV}(x) = \sup_{x \in G} \sum_{\chi \in X} \mathbb{1}_{y^{-1}\chi V}(x) = \sup_{x \in G} \sum_{\chi \in X} \mathbb{1}_{\chi V}(x) = C.$$

Thus we may apply Lemma 1.32 and obtain, by the choice of N, that

$$\text{dist}\big(\pi(y)f, \text{span}\{g_\chi : \chi \in X \cap B_N(y)\}\big)^2$$

$$\leq \frac{1}{A} \sum_{v \in y^{-1}X \setminus B_N(e)} |V_g f(v)|^2$$

$$\leq \frac{C}{A\lambda(V)} \int_{G \setminus B_{N-N_V}(e)} (V_g f)^\#(x)^2 dx < \varepsilon^2.$$

So the Homogeneous Approximation Property is established for $f \in \mathcal{N}$.

Now let $f \in \mathcal{H}$ be arbitrary. Since $\mathcal{N}$ is dense in $\mathcal{H}$ (Lemma 5.6), we can choose $\tilde{f}$ in $\mathcal{N}$ such that $\|f - \tilde{f}\|_{\mathcal{H}} < \frac{\varepsilon}{2}$. Take $N := N(f, \varepsilon)$ to be the natural number $N(\tilde{f}, \frac{\varepsilon}{2})$ that satisfies the Homogeneous Approximation Property for $\tilde{f}$ and $\frac{\varepsilon}{2}$, that is,

$$N := N(f, \varepsilon) := N(\tilde{f}, \tfrac{\varepsilon}{2}).$$

With the temporary notation $W := \mathrm{span}\{g_\chi : \chi \in X \cap B_N(y)\}$ we get

$$\mathrm{dist}\big(\pi(y)f, W\big) = \Big|\mathrm{dist}\big(\pi(y)f, W\big) - \mathrm{dist}\big(\pi(y)\tilde{f}, W\big) + \mathrm{dist}\big(\pi(y)\tilde{f}, W\big)\Big|$$

$$\leq \Big|\mathrm{dist}\big(\pi(y)f, W\big) - \mathrm{dist}\big(\pi(y)\tilde{f}, W\big)\Big| + \mathrm{dist}\big(\pi(y)\tilde{f}, W\big)$$

$$\leq \|\pi(y)f - \pi(y)\tilde{f}\|_{\mathcal{H}} + \mathrm{dist}\big(\pi(y)\tilde{f}, W\big)$$

$$= \|f - \tilde{f}\|_{\mathcal{H}} + \mathrm{dist}\big(\pi(y)\tilde{f}, W\big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

because dist is continuous and $\pi(y)$ is unitary.

$\square$

Corollary 5.9. *Let π be a square-integrable irreducible projective representation of G on a separable Hilbert space $\mathcal{H}$.*
Let $g \in \mathcal{H}$ and $X \subseteq G$ be such that $\mathcal{G} = \{\pi(\chi)g : \chi \in X\}$ is a frame for $\mathcal{H}$ and let $\phi \in \mathcal{H}$ and $Y \subseteq G$ be such that $\mathcal{E} = \{\pi(\upsilon)\phi : \upsilon \in Y\}$ is a Riesz sequence for $\mathcal{H}$.
Then $\mathcal{G}$ has the Homogeneous Approximation Property with respect to $\mathcal{E}$ and

$$D^-(Y) \leq D^-(X) \quad and \quad D^+(Y) \leq D^+(X).$$

PROOF. Given $\varepsilon > 0$, set $N_\varepsilon := N(\phi, \varepsilon)$. Then, for all $\upsilon \in Y$,

$$\mathrm{dist}\big(\pi(\upsilon)\phi, \mathrm{span}\{g_\chi : \chi \in X \cap B_{N_\varepsilon}(\upsilon)\}\big) < \varepsilon.$$

Thus $\mathcal{G}$ has the Homogeneous Approximation Property with respect to $\mathcal{E}$ in the sense of Definition 3.12. By the Comparison Theorem 3.13 it now follows that

$$D^-(Y) \leq D^-(X) \quad and \quad D^+(Y) \leq D^+(X).$$

$\square$

5.3. Examples

Some words about the assumptions on the representations are in order. For a (projective) representation π of a unimodular group G on a Hilbert space $\mathcal{H}$ the existence of a 'well-spread' subset $X \subseteq G$ and an atom $g \in \mathcal{H}$ such that the system $\{\pi(\chi)g : \chi \in X\}$ is a frame for $\mathcal{H}$ implies that the representation π has to be square-integrable (see, e.g., [**4**]).

However, for a connected and simply connected nilpotent Lie group N no irreducible unitary representation π can be square-integrable. If Z is the center of N, then it follows from Schur's lemma (see, e.g., [**15**], p.130) that

$$\pi|_Z = \chi \mathrm{I}$$

for some one-dimensional representation $\chi : Z \to \mathbb{T}$, called the central character. For $0 \neq f, g \in \mathcal{H}$ and $z \in Z, x \in N$ we then have

(5.11) $\qquad |\langle f, \pi(zx)g\rangle| = |\chi(z)||\langle f, \pi(x)g\rangle| = |\langle f, \pi(x)g\rangle|,$

so the representation coefficient $V_g f$ is constant on Z-cosets. Since $Z \cong \mathbb{R}^k$ for some $k \in \mathbb{N}$, the representation coefficient $V_g f$ is not in $L^2(N)$.

Nevertheless one can still hope to find projective square-integrable representations of connected and simply connected nilpotent Lie groups.

In the classical representation theory of nilpotent Lie groups it is customary to extend the notion of square-integrability in the following way [**37**].

Definition 5.10. An irreducible unitary representation π of N on $\mathcal{H}$ is said to be *square integrable modulo the center Z*, if for every $f, g \in \mathcal{H}$ the function $|V_g f|$, which is constant on Z-cosets by equation (5.11), is in $L^2(N/Z)$.

Furthermore, every unitary representation π of N gives rise to a projective representation of N/Z in a natural way.

Let $q : N \to N/Z$ denote the quotient map, and for $x \in N$ let $\dot{x} = q(x) = xZ$ denote the left coset. Choose a (continuous) section s for q, that is, a map $s : N/Z \to N$ such that $q \circ s = \mathrm{id}_{N/Z}$.

Lemma 5.11. *If π is a unitary representation of N on a Hilbert space $\mathcal{H}$, then the map*

$$(5.12) \qquad \tilde{\pi} : N/Z \to \mathcal{U}(\mathcal{H}), \ \tilde{\pi}(\dot{x}) := \pi(s(\dot{x}))$$

is a projective representation of N/Z on $\mathcal{H}$.

Proof. Define

$$\zeta := \zeta_s : N \to Z, \ \zeta(x) := s(\dot{x})^{-1} x.$$

Then every $x \in N$ can be written as $x = s(\dot{x})\zeta(x) \in s(N/Z) \cdot Z$.
It follows that

$$(5.13) \qquad s((\dot{xy})) = xy\zeta(xy)^{-1} = s(\dot{x})s(\dot{y})\zeta(x)\zeta(y)\zeta(xy)^{-1}.$$

Therefore

$$\begin{aligned}
\tilde{\pi}(\dot{x}\dot{y}) = \tilde{\pi}((\dot{xy})) &= \pi(s((\dot{xy}))) \\
&= \pi(s(\dot{x})s(\dot{y})\zeta(x)\zeta(y)\zeta(xy)^{-1}) \\
&= \chi(\zeta(x)\zeta(y)\zeta(xy)^{-1})\pi(s(\dot{x}))\pi(s(\dot{y})) \\
&= \chi(\zeta(x)\zeta(y)\zeta(xy)^{-1})\tilde{\pi}(\dot{x})\tilde{\pi}(\dot{y}).
\end{aligned}$$

So $\tilde{\pi}$ is a (continuous) projective representation of N/Z with multiplier

$$\mu(\dot{x}, \dot{y}) = \chi(\zeta(x)\zeta(y)\zeta(xy)^{-1}).$$

$\square$

Furthermore, one can show that different choices of the section yield projectively equivalent projective representations (see, e.g., [**3**]).

Note that unitary representations of a connected and simply connected nilpotent Lie group N that are square integrable modulo the center Z yield square-integrable projective representations of the group N/Z, which is again a connected and simply connected nilpotent Lie group.

Thus we can resort to the well-understood representation theory of nilpotent Lie groups to find suitable examples for our analysis. The irreducible unitary representations of nilpotent Lie groups are classified by Kirillov theory [**13**], those which are square-integrable modulo the center are characterized by the results of Moore and Wolf [**37**].

For nilpotent Lie groups of dimension up to 6 all the irreducible unitary representations were explicitly calculated by Nielsen and listed in [**38**] together with other relevant data used for their construction.

In the following we select some suitable low-dimensional examples of nilpotent Lie groups from [**38**] and derive necessary density conditions for the coherent frames and Riesz sequences in the orbit of the associated square-integrable projective representations.

Not surprisingly, we want to make use of the Homogeneous Approximation Property for coherent frames (Proposition 5.8) and employ the resulting Comparison Theorem for coherent systems (Corollary 5.9). To obtain good density thresholds we need to compare with the density of a Riesz basis. So the main task is to find a Riesz basis in the orbit of the given square-integrable projective representation and calculate its density.

First we review the well-known example of (irregular) Gabor frames. These are frames in the orbit of the projective representations of $\mathbb{R}^2$ that are deduced from the Schrödinger representations of the Heisenberg group.

Example 1. Recall the Heisenberg algebra $\mathfrak{h} = \mathbb{R}X_1 + \mathbb{R}X_2 + \mathbb{R}X_3$ with non-vanishing Lie bracket
$$[X_3, X_2] = X_1.$$

The corresponding connected and simply connected Lie group $\mathbb{H}$ in Malcev coordinates is $\mathbb{R}^3$ with the multiplication law

$$(x_1, x_2, x_3)(y_1, y_2, y_3) = (x_1 + y_1 + x_3 y_2, x_2 + y_2, x_3 + y_3).$$

The irreducible unitary representations of $\mathbb{H}$ on $L^2(\mathbb{R})$ that are square-integrable modulo the center $Z = \mathbb{R} \times \{0\} \times \{0\}$ are parametrized by $\xi \in \mathbb{R}, \xi \neq 0$, and given by

$$\pi_\xi(x_1, x_2, x_3)\phi(t) = e^{2\pi i \xi(x_1 - x_2 t)}\phi(t - x_3).$$

The quotient group $\mathbb{H}/Z$ is isomorphic to $\mathbb{R}^2$ with the usual addition.
The representations π_ξ give rise to irreducible square-integrable projective representations of $\mathbb{R}^2$ on $L^2(\mathbb{R})$, which we also call π_ξ, via

$$\pi_\xi(x_1, x_2)\phi(t) = e^{-2\pi i \xi x_1 t}\phi(t - x_2).$$

Lemma 5.12. *Let $\phi = \mathbb{1}_Q$, where $Q = [0, 1]$, and $X_\xi = \frac{1}{\xi}\mathbb{Z} \times \mathbb{Z}$.
The system $\{\pi_\xi(\chi)\phi : \chi \in X_\xi\}$ forms an orthonormal basis for $L^2(\mathbb{R})$.*

PROOF. [**11**], p. 71, Example 3.5.3

□

Lemma 5.13. *The set X_ξ is a lattice in $\mathbb{R}^2$ with density $D(X_\xi) = |\xi|$.*

PROOF. This is well-known (and also follows from Proposition 2.7).

□

Since an orthonormal basis is both a frame and a Riesz sequence, Corollary 5.9 together with Lemma 5.12 and Lemma 5.13 implies the following density thresholds.

Corollary 5.14. *Let $g \in L^2(\mathbb{R})$ and $X \subseteq \mathbb{R}^2$.*

(a) *If the set $\{\pi_\xi(x)g : x \in X\}$ is a frame for $L^2(\mathbb{R})$, then*
$$D^-(X) \geq |\xi|.$$
(b) *If the set $\{\pi_\xi(x)g : x \in X\}$ is a Riesz sequence in $L^2(\mathbb{R})$, then*
$$D^+(X) \leq |\xi|.$$
(c) *If the set $\{\pi_\xi(x)g : x \in X\}$ is a Riesz basis for $L^2(\mathbb{R})$, then*
$$D^-(X) = D^+(X) = |\xi|.$$

The statement of Corollary 5.14 somehow differs from the density theory of Balan, Casazza, Heil and Landau [**5**], where the critical density that separates frames from Riesz sequences is always equal to one, regardless of the specific structure of the investigated frames and Riesz sequences. This is due to the fact that in [**5**] a normalized version of the Beurling density is used.
As an illustration of how the dependence of the critical density on the parameter of the representation already implicitly occurs in the existing literature, we review the connection of Gabor frames with Gaussian window to sampling and interpolation in the Bargmann-Fock Spaces (confer, e.g., [**27**], p. 53).

Sampling and Interpolation in the Bargmann-Fock Spaces.

Definition 5.15. For $\xi > 0$ the *Bargmann-Fock space* $\mathcal{F}_\xi$ is defined to be the Hilbert space

$$\mathcal{F}_\xi = \{F \text{ entire on } \mathbb{C} : \xi \int_{\mathbb{C}} |F(z)|^2 e^{-\pi\xi|z|^2} dz < \infty\}$$

with inner product

$$\langle F, G \rangle_{\mathcal{F}_\xi} = \xi \int_{\mathbb{C}} F(z)\overline{G(z)}e^{-\pi\xi|z|^2} dz.$$

Definition 5.16. The *Bargmann transform* of a function f on $\mathbb{R}$ is the function $B_\xi f$ on $\mathbb{C}$ defined by

$$B_\xi f(z) = (2\xi)^{\frac{1}{4}} \int_{\mathbb{R}} f(t)e^{2\pi\xi tz - \pi\xi t^2 - \frac{\pi\xi}{2}z^2} dt.$$

The Bargmann transform B_ξ is a unitary operator from $L^2(\mathbb{R})$ onto $\mathcal{F}_\xi$ (see, e.g., [**49**], p.222).

Let $\varphi_\xi(t) = (2\xi)^{\frac{1}{4}} e^{-\pi\xi t^2}$. If we write $z = x_2 + ix_1$, then

(5.14) $\langle f, \pi_\xi(x_1, x_2)\varphi_\xi \rangle = e^{\pi i\xi x_1 x_2} B_\xi f(z)e^{-\frac{\pi\xi}{2}|z|^2}.$

Indeed,

$$e^{\pi i\xi x_1 x_2} B_\xi f(z)e^{-\frac{\pi\xi}{2}|z|^2}$$

$$= e^{\pi i\xi x_1 x_2} e^{-\frac{\pi\xi}{2}(x_1^2 + x_2^2)}(2\xi)^{\frac{1}{4}} \int_{\mathbb{R}} f(t)e^{2\pi\xi t(x_2 + ix_1) - \pi\xi t^2 - \frac{\pi\xi}{2}(x_2 + ix_1)^2} dt$$

$$= (2\xi)^{\frac{1}{4}} \int_{\mathbb{R}} f(t)e^{2\pi i\xi x_1 t}e^{2\pi\xi x_2 t}e^{-\pi\xi t^2}e^{-\pi\xi x_2^2} dt$$

$$= (2\xi)^{\frac{1}{4}} \int_{\mathbb{R}} f(t)e^{2\pi i\xi x_1 t}e^{-\pi\xi(t - x_2)^2} dt$$

$$= \int_{\mathbb{R}} f(t)\overline{e^{-2\pi i\xi x_1 t}\varphi_\xi(t - x_2)} dt$$

$$= \langle f, \pi_\xi(x_1, x_2)\varphi_\xi \rangle.$$

Definition 5.17. (i) A set Λ of complex numbers is a *set of sampling* for $\mathcal{F}_\xi$ if there exist positive numbers A and B such that, for all $F \in \mathcal{F}_\xi$,

$$(5.15) \qquad A\|F\|^2_{\mathcal{F}_\xi} \leq \sum_{z \in \Lambda} |F(z)|^2 e^{-\pi\xi|z|^2} \leq B\|F\|^2_{\mathcal{F}_\xi}.$$

(ii) A set Λ of complex numbers is a *set of interpolation* for $\mathcal{F}_\xi$ if for every sequence $a = (a_z)_{z \in \Lambda} \in \ell^2(\Lambda)$ there exists a function $F \in \mathcal{F}_\xi$ such that $e^{-\frac{\pi\xi}{2}|z|^2} F(z) = a_z$ for all $z \in \Lambda$.

As a consequence of Corollary 5.14 and equation (5.14) we recover Seip's necessary density conditions for sampling and interpolation in the Bargmann-Fock spaces [**41**], [**42**].

Corollary 5.18. (a) *If a subset $\Lambda \subseteq \mathbb{C}$ is a set of sampling for $\mathcal{F}_\xi$, then*

$$D^-(\Lambda) \geq \xi.$$

(b) *If a subset $\Lambda \subseteq \mathbb{C}$ is a set of interpolation for $\mathcal{F}_\xi$, then*

$$D^+(\Lambda) \leq \xi.$$

PROOF. (a) Since the Bargmann transform $B_\xi : L^2(\mathbb{R}) \to \mathcal{F}_\xi$ is unitary, we can rewrite the sampling inequality (5.15) as

$$A\|f\|^2_{L^2(\mathbb{R})} = A\|B_\xi f\|^2_{\mathcal{F}_\xi} \leq \sum_{z \in \Lambda} |B_\xi f(z)|^2 e^{-\pi\xi|z|^2} \leq B\|B_\xi f\|^2_{\mathcal{F}_\xi} = B\|f\|^2_{L^2(\mathbb{R})}$$

for all $f \in L^2(\mathbb{R})$. By equation (5.14), it follows that

$$A\|f\|^2_{L^2(\mathbb{R})} \leq \sum_{x_2+ix_1 \in \Lambda} |\langle f, \pi_\xi(x_1, x_2)\varphi_\xi\rangle|^2 \leq B\|f\|^2_{L^2(\mathbb{R})}$$

for all $f \in L^2(\mathbb{R})$. This means that the system $\{\pi_\xi(x_1, x_2)\varphi_\xi : x_2 + ix_1 \in \Lambda\}$ is a frame for $L^2(\mathbb{R})$ and thus $D^-(\Lambda) \geq \xi$ by Corollary 5.14.

(b) If Λ is a set of interpolation for $\mathcal{F}_\xi$, then we can similarly to (a) use the unitary operator B_ξ and equation (5.14) to obtain that for every sequence $a = (a_z)_{z \in \Lambda} \in \ell^2(\Lambda)$ there exists a function $f \in L^2(\mathbb{R})$ such that

$$\langle f, \pi_\xi(x_1, x_2)\varphi_\xi\rangle = a_z$$

for all $z = x_2 + ix_1 \in \Lambda$. In other words, the system $\{\pi_\xi(x_1, x_2)\varphi_\xi : x_2 + ix_1 \in \Lambda\}$ is a Riesz-Fischer sequence in $L^2(\mathbb{R})$. By Lemma 3.4, there exists a constant $A > 0$ such that

$$\|\pi_\xi(x_1, x_2)\varphi_\xi - \pi_\xi(x_1', x_2')\varphi_\xi\| \geq A\sqrt{2}$$

for all $z = x_2 + ix_1 \neq z' = x_2' + ix_1' \in \Lambda$. Since π_ξ is strongly continuous, also

$$|z - z'| \geq \delta > 0$$

for all $z \neq z' \in \Lambda$ and some $\delta > 0$. Thus Λ is (relatively) separated. It follows that the system $\{\pi_\xi(x_1, x_2)\varphi_\xi : x_2 + ix_1 \in \Lambda\}$ is a Bessel sequence in $L^2(\mathbb{R})$, because

$$
\begin{aligned}
\sum_{x_2+ix_1\in\Lambda} |\langle f, \pi_\xi(x_1,x_2)\varphi_\xi\rangle|^2 &= \sum_{x_2+ix_1\in\Lambda} |V_{\varphi_\xi}f(x_1,x_2)|^2 \\
&= \sum_{\gamma\in\mathbb{Z}^2} \sum_{x_2+ix_1\in\gamma+[0,1]^2} |V_{\varphi_\xi}f(x_1,x_2)|^2 \\
&\leq \sum_{\gamma\in\mathbb{Z}^2} C \max_{x_2+ix_1\in\gamma+[0,1]^2} |V_{\varphi_\xi}f(x_1,x_2)|^2 \\
&= C\|V_{\varphi_\xi}f\|^2_{W(C,L^2)} \\
&\leq C'\|V_{\varphi_\xi}\varphi_\xi\|^2_{W(C,L^1)}\|f\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

where the last inequality follows from [27], Theorem 12.2.1. We conclude that the system $\{\pi_\xi(x_1, x_2)\varphi_\xi : x_2 + ix_1 \in \Lambda\}$ is a Riesz sequence in $L^2(\mathbb{R})$ (Lemma 3.2) and thus $D^+(\Lambda) \leq \xi$ by Corollary 5.14.

$\square$

Whereas in the case of Gabor frames one can compare with the standard orthonormal basis

$$\{e^{2\pi ikt}\mathbb{1}_{[0,1]}(t-l) : k, l \in \mathbb{Z}\}$$

of $L^2(\mathbb{R})$, for other representations similar reference bases need to be constructed first.

In the following examples we construct an explicit orthonormal basis in the orbit of a given square-integrable projective representation. For that we use two elementary facts for a system of orthogonal functions $\{f_\chi\}_{\chi\in X}$ in $L^2(\Omega)$, where Ω is some measure space:

(i) If $\{e_\upsilon\}_{\upsilon\in Y}$ is an orthonormal basis in $L^2(\Omega)$ and $e_\upsilon \in \overline{\mathrm{span}}\{f_\chi\}_{\chi\in X}$ for all $\upsilon \in Y$, then also $\{f_\chi\}_{\chi\in X}$ is an orthonormal basis in $L^2(\Omega)$.

(ii) If $\{f_\chi\}_{\chi\in X}$ is an orthonormal basis in $L^2(\Omega)$ and $m \in L^\infty(\Omega)$, $|m| = 1$, then the system $\{mf_\chi\}_{\chi\in X}$ is also an orthonormal basis in $L^2(\Omega)$.

Example 2. Consider the Lie algebra $\mathfrak{g}_{5,3} = \mathbb{R}X_1 + \cdots + \mathbb{R}X_5$ from [38] with non-vanishing Lie brackets

$$[X_5, X_4] = X_3, [X_5, X_3] = X_1, [X_4, X_2] = X_1.$$

The corresponding connected and simply connected Lie group $G_{5,3}$ in Malcev coordinates is $\mathbb{R}^5$ with the multiplication law

$$(x_1, x_2, x_3, x_4, x_5)(y_1, y_2, y_3, y_4, y_5)$$
$$= (x_1 + y_1 + x_4 y_2 + x_5 y_3 + \tfrac{1}{2} x_5^2 y_4, x_2 + y_2, x_3 + y_3 + x_5 y_4, x_4 + y_4, x_5 + y_5).$$

The irreducible unitary representations of $G_{5,3}$ on $L^2(\mathbb{R}^2)$ that are square-integrable modulo the center $Z = \mathbb{R} \times \{0\} \times \{0\} \times \{0\} \times \{0\}$ are parametrized by $\xi \in \mathbb{R}, \xi \neq 0$, and given by

$$\pi_\xi(x_1, \ldots, x_5)\phi(s, t) = e^{2\pi i \xi(x_1 - x_2 x_4 + x_4 s - x_3 t + \frac{1}{2} x_4 t^2)} \phi(s - x_2, t - x_5).$$

We want to study the projective representations of the quotient group $G_{5,3}/Z$ derived from the representations π_ξ as in Lemma 5.11.
By deleting the coordinates of the center in the above multiplication law for $G_{5,3}$ and relabeling the remaining coordinates x_j by x_{j-1}, we are led to the multiplication law

$$(x_1, x_2, x_3, x_4)(y_1, y_2, y_3, y_4) = (x_1 + y_1, x_2 + y_2 + x_4 y_3, x_3 + y_3, x_4 + y_4)$$

of the group $\mathbb{R} \times \mathbb{H}$, which is isomorphic to the quotient group $G_{5,3}/Z$.

The unitary representations π_ξ thus give rise to square-integrable projective representations of $\mathbb{R} \times \mathbb{H}$ on $L^2(\mathbb{R}^2)$, which we also call π_ξ, via

$$\pi_\xi(x_1, x_2, x_3, x_4)\phi(s, t) = e^{2\pi i \xi(-x_1 x_3 + x_3 s - x_2 t + \frac{1}{2} x_3 t^2)} \phi(s - x_1, t - x_4).$$

Because of projective equivalence, we may for brevity omit the occurring phase factor and work in the following with the projective representations

$$\pi_\xi(x_1, x_2, x_3, x_4)\phi(s, t) = e^{2\pi i \xi(x_3 s - x_2 t + \frac{1}{2} x_3 t^2)} \phi(s - x_1, t - x_4).$$

Lemma 5.19. *Let $\phi = \mathbb{1}_Q$, where $Q = [0,1]^2$, and $X_\xi = \mathbb{Z} \times \frac{1}{\xi}\mathbb{Z} \times \frac{1}{\xi}\mathbb{Z} \times \mathbb{Z}$. The system $\{\pi_\xi(\chi)\phi : \chi \in X_\xi\}$ forms an orthonormal basis for $L^2(\mathbb{R}^2)$.*

PROOF. First we show the orthogonality.
Let $\chi = (k, \frac{1}{\xi}l, \frac{1}{\xi}m, n) \in X_\xi, \chi' = (k', \frac{1}{\xi}l', \frac{1}{\xi}m', n') \in X_\xi$. Then

$$\langle \pi_\xi(\chi)\phi, \pi_\xi(\chi')\phi \rangle = \int_\mathbb{R} \int_\mathbb{R} \pi_\xi(\chi)\phi(s,t)\overline{\pi_\xi(\chi')\phi(s,t)}dsdt$$

$$= \int_\mathbb{R} \int_\mathbb{R} e^{2\pi i \xi(\frac{1}{\xi}m - \frac{1}{\xi}m')s} e^{2\pi i \xi(\frac{1}{\xi}l' - \frac{1}{\xi}l)t} e^{\pi i \xi(\frac{1}{\xi}m - \frac{1}{\xi}m')t^2}$$

$$\cdot \mathbb{1}_Q(s-k, t-n)\mathbb{1}_Q(s-k', t-n')dsdt$$

$$= \delta_{k,k'}\delta_{n,n'} \int_n^{n+1} \int_k^{k+1} e^{2\pi i(m-m')s} e^{2\pi i(l'-l)t} e^{\pi i(m-m')t^2} dsdt$$

$$= \delta_{k,k'}\delta_{n,n'} \int_n^{n+1} \left(\int_k^{k+1} e^{2\pi i(m-m')s} ds \right) e^{2\pi i(l'-l)t} e^{\pi i(m-m')t^2} dt$$

$$= \delta_{k,k'}\delta_{n,n'}\delta_{m,m'} \int_n^{n+1} e^{2\pi i(l'-l)t} dt$$

$$= \delta_{k,k'}\delta_{n,n'}\delta_{m,m'}\delta_{l,l'}$$

$$= \delta_{\chi,\chi'},$$

because the integer translates of the characteristic function $\mathbb{1}_Q$ are disjoint for distinct pairs of integers and the functions $\{f(\cdot) = e^{2\pi i l \cdot} : l \in \mathbb{Z}\}$ are orthogonal on $L^2([k, k+1])$ and $L^2([n, n+1])$ respectively.

Next we prove the completeness of the system $\{\pi_\xi(\chi)\phi : \chi \in X_\xi\}$ in $L^2(\mathbb{R}^2)$. For this purpose we show that every element of the standard orthonormal basis

$$\mathcal{E} = \{e^{2\pi i l t} e^{2\pi i m s}\mathbb{1}_Q(s-k, t-n) : k, l, m, n \in \mathbb{Z}\}$$

has an expansion with respect to the given orthogonal system

$$\{\pi_\xi(\chi)\mathbb{1}_Q(s,t) : \chi \in X_\xi\} = \{e^{2\pi i m s} e^{2\pi i l t} e^{\pi i m t^2}\mathbb{1}_Q(s-k, t-n) : k, l, m, n \in \mathbb{Z}\}$$

in $L^2(\mathbb{R}^2)$.

So for some fixed basis element $e^{2\pi i l_0 t} e^{2\pi i m_0 s}\mathbb{1}_Q(s-k_0, t-n_0)$ in $\mathcal{E}$ we are looking for an expansion of the form

$$e^{2\pi i l_0 t} e^{2\pi i m_0 s}\mathbb{1}_Q(s-k_0, t-n_0) = \sum_{k,l,m,n \in \mathbb{Z}} a_{klmn} e^{2\pi i m s} e^{2\pi i l t} e^{\pi i m t^2}\mathbb{1}_Q(s-k, t-n).$$

If $k \neq k_0$ or $n \neq n_0$ or $m \neq m_0$, set $a_{klmn} = 0$. To find the remaining coefficients $a_{k_0 l m_0 n_0}$ consider the series expansion of the function $e^{2\pi i l_0 t} e^{-\pi i m_0 t^2}$ in terms of the basis $\{e^{2\pi i l t} : l \in \mathbb{Z}\}$ on $L^2([n_0, n_0 + 1])$, that is,

$$e^{2\pi i l_0 t} e^{-\pi i m_0 t^2} = \sum_{l \in \mathbb{Z}} c_l e^{2\pi i l t}.$$

Then

$$e^{2\pi i l_0 t} e^{2\pi i m_0 s} = \sum_{l \in \mathbb{Z}} c_l e^{2\pi i l t} e^{\pi i m_0 t^2} e^{2\pi i m_0 s}$$

on $L^2([k_0, k_0 + 1] \times [n_0, n_0 + 1])$. Therefore the choice $a_{k_0 l m_0 n_0} = c_l$ provides the desired result.

Since every basis element in $\mathcal{E}$ belongs to $\overline{\text{span}}\{\pi_\xi(\chi)\phi : \chi \in X_\xi\}$, the completeness of the system $\{\pi_\xi(\chi)\phi : \chi \in X_\xi\}$ in $L^2(\mathbb{R}^2)$ now follows from Property (i).

$\square$

Lemma 5.20. *The set X_ξ is a lattice in $\mathbb{R} \times \mathbb{H}$ with density $D(X_\xi) = \xi^2$.*

PROOF. We claim that X_ξ is a lattice in $\mathbb{R} \times \mathbb{H}$ with fundamental domain $U_\xi := A_\xi([0, 1)^4)$, where A_ξ is defined by

$$A_\xi := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\xi} & 0 & 0 \\ 0 & 0 & \frac{1}{\xi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Clearly X_ξ is a subgroup in $\mathbb{R} \times \mathbb{H}$, so it remains to show that every element $x \in \mathbb{R} \times \mathbb{H}$ can be uniquely written as $x = \chi v$ with $\chi \in X_\xi$ and $v \in U_\xi$.
Let $x = (x_1, x_2, x_3, x_4) \in \mathbb{R} \times \mathbb{H}$ be arbitrary. Set

$$
\begin{aligned}
n &= \lfloor x_4 \rfloor \in \mathbb{Z}, & z &= x_4 - \lfloor x_4 \rfloor \in [0, 1), \\
m &= \lfloor \xi x_3 \rfloor \in \mathbb{Z}, & w &= \xi x_3 - \lfloor \xi x_3 \rfloor \in [0, 1), \\
l &= \lfloor \xi x_2 - nw \rfloor \in \mathbb{Z}, & v &= \xi x_2 - nw - \lfloor \xi x_2 - nw \rfloor \in [0, 1), \\
k &= \lfloor x_1 \rfloor \in \mathbb{Z}, & u &= x_1 - \lfloor x_1 \rfloor \in [0, 1),
\end{aligned}
$$

and $\chi = (k, \frac{1}{\xi}l, \frac{1}{\xi}m, n) \in X_\xi$, $v = (u, \frac{1}{\xi}v, \frac{1}{\xi}w, z) \in U_\xi$. Then

$$\chi v = (k + u, \tfrac{1}{\xi}(l + v + nw), \tfrac{1}{\xi}(m + w), n + z) = x.$$

Since this choice is unique, X_ξ is a lattice with fundamental domain U_ξ.

By Lemma 2.7, the density of a lattice is computed as the reciprocal of the volume of its fundamental domain. Therefore,

$$D(X_\xi) = \frac{1}{\lambda(U_\xi)} = \frac{1}{\lambda(A_\xi([0, 1)^4))} = \frac{1}{|\det A_\xi|} = \frac{1}{\frac{1}{\xi^2}} = \xi^2.$$

$\square$

Since an orthonormal basis is both a frame and a Riesz sequence, Corollary 5.9 together with Lemma 5.19 and Lemma 5.20 implies the following density thresholds.

Corollary 5.21. *Let $g \in L^2(\mathbb{R}^2)$ and let $X \subseteq \mathbb{R} \times \mathbb{H}$.*

(a) *If the set $\{\pi_\xi(x)g : x \in X\}$ is a frame for $L^2(\mathbb{R}^2)$, then*
$$D^-(X) \geq \xi^2.$$

(b) *If the set $\{\pi_\xi(x)g : x \in X\}$ is a Riesz sequence in $L^2(\mathbb{R}^2)$, then*
$$D^+(X) \leq \xi^2.$$

(c) *If the set $\{\pi_\xi(x)g : x \in X\}$ is a Riesz basis for $L^2(\mathbb{R}^2)$, then*
$$D^-(X) = D^+(X) = \xi^2.$$

Example 3. Consider the Lie algebra $\mathfrak{g}_{6,23} = \mathbb{R}X_1 + \cdots + \mathbb{R}X_6$ from [**38**] with non-vanishing Lie brackets

$$[X_6, X_5] = X_4, [X_6, X_4] = X_2, [X_6, X_3] = -X_1, [X_5, X_4] = X_1, [X_5, X_3] = X_2.$$

The corresponding connected and simply connected Lie group $G_{6,23}$ in Malcev coordinates is $\mathbb{R}^6$ with the multiplication law

$$(x_1, x_2, x_3, x_4, x_5, x_6)(y_1, y_2, y_3, y_4, y_5, y_6)$$
$$= (x_1 + y_1 + x_5 y_4 - x_6 y_3 + x_5 x_6 y_5 + \tfrac{1}{2} x_6 y_5^2, x_2 + y_2 + x_5 y_3 + x_6 y_4 + \tfrac{1}{2} x_6^2 y_5,$$
$$x_3 + y_3, x_4 + y_4 + x_6 y_5, x_5 + y_5, x_6 + y_6).$$

The irreducible unitary representations of $G_{6,23}$ on $L^2(\mathbb{R}^2)$ that are square-integrable modulo the center $Z = \mathbb{R} \times \mathbb{R} \times \{0\} \times \{0\} \times \{0\} \times \{0\}$ are parametrized by $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, $\xi_1^2 + \xi_2^2 \neq 0$, and given by

$$\pi_\xi(x_1, \ldots, x_6)\phi(s, t)$$
$$= e^{2\pi i((x_1 - \frac{1}{2}x_5^2 x_6 - x_4 s + x_5 x_6 s - \frac{1}{2}x_6 s^2 + x_3 t)\xi_1 + (x_2 - \frac{1}{2}x_5 x_6^2 - x_3 s + \frac{1}{2}x_6^2 s - x_4 t + x_5 x_6 t - x_6 st)\xi_2)} \phi(s - x_5, t - x_6).$$

We want to study the projective representations of the quotient group $G_{6,23}/Z$ derived from the representations π_ξ as in Lemma 5.11.

By deleting the coordinates of the center in the above multiplication law for $G_{6,23}$ and relabeling the remaining coordinates x_j by x_{j-2}, we are led to the multiplication law

$$(x_1, x_2, x_3, x_4)(y_1, y_2, y_3, y_4) = (x_1 + y_1, x_2 + y_2 + x_4 y_3, x_3 + y_3, x_4 + y_4)$$

of the group $\mathbb{R} \times \mathbb{H}$, which is isomorphic to the quotient group $G_{6,23}/Z$.

The representations π_ξ give rise to irreducible square-integrable projective repre-sentation of $\mathbb{R} \times \mathbb{H}$ on $L^2(\mathbb{R}^2)$, which we also call π_ξ, via

$$\pi_\xi(x)\phi(s,t) = e^{2\pi i((-x_2 s + x_3 x_4 s - \frac{1}{2}x_4 s^2 + x_1 t)\xi_1 + (-x_1 s + \frac{1}{2}x_4^2 s - x_2 t + x_3 x_4 t - x_4 s t)\xi_2)}\phi(s - x_3, t - x_4).$$

Note that as in the previous example we have omitted occurring phase factors.

Set
$$M_\xi := \begin{pmatrix} \xi_2 & \xi_1 \\ -\xi_1 & \xi_2 \end{pmatrix}, \quad \text{then } \pi_\xi \text{ can be rewritten as}$$

$$\pi_\xi(x)\phi(s,t)$$
$$= e^{-2\pi i\left((x_1 \xi_2 + x_2 \xi_1 - x_3 x_4 \xi_1 - \frac{1}{2}x_4^2 \xi_2)s + (-x_1 \xi_1 + x_2 \xi_2 - x_3 x_4 \xi_2)t\right)} e^{-\pi i\left(x_4 \xi_1 s^2 + 2 x_4 \xi_2 s t\right)}\phi(s - x_3, t - x_4)$$
$$= e^{-2\pi i\left(\langle M_\xi(x_1,x_2)^T, (s,t)^T\rangle - (x_3 x_4 \xi_1 + \frac{1}{2}x_4^2 \xi_2)s - x_3 x_4 \xi_2 t\right)} e^{-\pi i\left(x_4 \xi_1 s^2 + 2 x_4 \xi_2 s t\right)}\phi(s - x_3, t - x_4).$$

For $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, $\xi_1^2 + \xi_2^2 \neq 0$ we therefore define

$$(5.16) \qquad A_\xi := \begin{pmatrix} \frac{\xi_2}{\xi_1^2 + \xi_2^2} & -\frac{\xi_1}{\xi_1^2 + \xi_2^2} & 0 & 0 \\ \frac{\xi_1}{\xi_1^2 + \xi_2^2} & \frac{\xi_2}{\xi_1^2 + \xi_2^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} M_\xi^{-1} & 0 \\ 0 & I \end{pmatrix}.$$

Note that $\det A_\xi = \frac{1}{\xi_1^2 + \xi_2^2} \neq 0$, so A_ξ is invertible.

Lemma 5.22. *Let $\phi = \mathbb{1}_Q$, where $Q = [0,1]^2$, and set $X_\xi := A_\xi(\mathbb{Z}^4)$, where A_ξ is the matrix defined in (5.16). The system $\{\pi_\xi(\chi)\phi : \chi \in X_\xi\}$ forms an orthonormal basis for $L^2(\mathbb{R}^2)$.*

PROOF. First we show orthogonality.

For every
$$\chi = \left(\frac{1}{\xi_1^2 + \xi_2^2}(\xi_2 k - \xi_1 l), \frac{1}{\xi_1^2 + \xi_2^2}(\xi_1 k + \xi_2 l), m, n\right) \in X_\xi,$$

the representation π_ξ becomes

$$\pi_\xi(\chi)\phi(s,t) = e^{-2\pi i\left((k - mn\xi_1 - \frac{1}{2}n^2\xi_2)s + (l - mn\xi_2)t\right)} e^{-\pi i\left(n\xi_1 s^2 + 2n\xi_2 s t\right)}\phi(s - m, t - n).$$

For $\chi, \chi' \in X_\xi$ we thus get

$$
\begin{aligned}
\left\langle \pi_\xi(\chi)\phi, \pi_\xi(\chi')\phi \right\rangle &= \int_{\mathbb{R}} \int_{\mathbb{R}} \pi_\xi(\chi)\phi(s,t)\overline{\pi_\xi(\chi')\phi(s,t)}\,ds\,dt \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-2\pi i\left((k-k')-(mn-m'n')\xi_1 - \frac{1}{2}(n^2 - n'^2)\xi_2\right)s}\, e^{-2\pi i\left((l-l')-(mn-m'n')\xi_2\right)t} \\
&\qquad\qquad \cdot e^{-\pi i\left((n-n')\xi_1 s^2 + 2(n-n')\xi_2 st\right)}\mathbb{1}_Q(s-m,t-n)\mathbb{1}_Q(s-m',t-n')\,ds\,dt \\
&= \delta_{m,m'}\delta_{n,n'}\int_n^{n+1}\int_m^{m+1} e^{-2\pi i(k-k')s}e^{-2\pi i(l-l')t}\,ds\,dt \\
&= \delta_{(k,l,m,n),(k',l',m',n')} = \delta_{A_\xi^{-1}(\chi),A_\xi^{-1}(\chi')} = \delta_{\chi,\chi'},
\end{aligned}
$$

because the integer translates of the characteristic function $\mathbb{1}_Q$ are disjoint for distinct pairs of integers and the functions $\{f(\cdot) = e^{2\pi ik\cdot} : k \in \mathbb{Z}\}$ are orthogonal on $L^2([m, m+1])$ and $L^2([n, n+1])$ respectively.

Next we prove the completeness of the system $\{\pi_\xi(\chi)\mathbb{1}_Q(s,t) : \chi \in X_\xi\}$ in $L^2(\mathbb{R}^2)$. As in Example 2 we show that every element of the standard orthonormal basis

$$
\mathcal{E} = \{e^{2\pi iks}e^{2\pi ilt}\mathbb{1}_Q(s-m,t-n) : k,l,m,n \in \mathbb{Z}\}
$$

has an expansion with respect to the given orthonormal system

$$
\left\{ e^{-2\pi i\left((k-mn\xi_1 - \frac{1}{2}n^2\xi_2)s + (l-mn\xi_2)t\right)} e^{-\pi i\left(n\xi_1 s^2 + 2n\xi_2 st\right)}\mathbb{1}_Q(s-m,t-n) : k,l,m,n \in \mathbb{Z} \right\}
$$

in $L^2(\mathbb{R}^2)$. Fix $m_0, n_0 \in \mathbb{Z}$ and set

$$
m(s,t) := e^{-2\pi i\left((-m_0 n_0 \xi_1 - \frac{1}{2}n_0^2\xi_2)s - m_0 n_0 \xi_2 t\right)} e^{-\pi i\left(n_0\xi_1 s^2 + 2n_0\xi_2 st\right)}.
$$

By Property (ii), the system

$$
\left\{ e^{-2\pi iks}e^{-2\pi ilt}m(s,t) : k,l \in \mathbb{Z} \right\}
$$

is an orthonormal basis in $L^2([m_0, m_0+1] \times [n_0, n_0+1])$, because it is just the well-known orthonormal basis $\{e^{-2\pi iks}e^{-2\pi ilt} : k,l \in \mathbb{Z}\}$ multiplied by a function of modulus one.

For fixed $k_0, l_0 \in \mathbb{Z}$ we thus have an expansion of the element $e^{2\pi ik_0 s}e^{2\pi il_0 t}$ of the form

$$
e^{2\pi ik_0 s}e^{2\pi il_0 t} = \sum_{k,l\in\mathbb{Z}} b_{kl} e^{-2\pi iks}e^{-2\pi ilt}e^{-2\pi i\left((-m_0 n_0 \xi_1 - \frac{1}{2}n_0^2\xi_2)s - m_0 n_0 \xi_2 t\right)} e^{-\pi i\left(n_0\xi_1 s^2 + 2n_0\xi_2 st\right)}
$$

on $L^2([m_0, m_0+1] \times [n_0, n_0+1])$.

It follows that every basis element $e^{2\pi ik_0 s}e^{2\pi il_0 t}\mathbb{1}_Q(s-m_0,t-n_0)$ in $\mathcal{E}$ has an expansion with respect to the system $\{\pi_\xi(\chi)\mathbb{1}_Q(s,t) : \chi \in X_\xi\}$ in $L^2(\mathbb{R}^2)$, so $\{\pi_\xi(\chi)\mathbb{1}_Q(s,t) : \chi \in X_\xi\}$ is an orthonormal basis by Property (i).

$\square$

Lemma 5.23. *The set X_ξ is a quasi-lattice in $\mathbb{R} \times \mathbb{H}$ with density $D(X_\xi) = \xi_1^2 + \xi_2^2$.*

PROOF. We claim that X_ξ is a quasi-lattice in $\mathbb{R} \times \mathbb{H}$ with complement $U_\xi := A_\xi([0,1)^4)$. We need to show that every element $x \in \mathbb{R} \times \mathbb{H}$ can be uniquely written as $x = \chi v$ with $\chi \in X_\xi$ and $v \in U_\xi$. Let $x = (x_1, x_2, x_3, x_4) \in \mathbb{R} \times \mathbb{H}$ be arbitrary. Set

$$n = \lfloor x_4 \rfloor \in \mathbb{Z}, \qquad\qquad\qquad z = x_4 - \lfloor x_4 \rfloor \in [0,1),$$
$$m = \lfloor x_3 \rfloor \in \mathbb{Z}, \qquad\qquad\qquad w = x_3 - \lfloor x_3 \rfloor \in [0,1),$$
$$l = \lfloor -\xi_1 x_1 + \xi_2(x_2 - nw) \rfloor \in \mathbb{Z},$$
$$v = -\xi_1 x_1 + \xi_2(x_2 - nw) - \lfloor -\xi_1 x_1 + \xi_2(x_2 - nw) \rfloor \in [0,1),$$
$$k = \lfloor \xi_2 x_1 + \xi_1(x_2 - nw) \rfloor \in \mathbb{Z},$$
$$u = \xi_2 x_1 + \xi_1(x_2 - nw) - \lfloor \xi_2 x_1 + \xi_1(x_2 - nw) \rfloor) \in [0,1),$$

and let

$$\chi = \left(\tfrac{1}{\xi_1^2 + \xi_2^2}(\xi_2 k - \xi_1 l), \tfrac{1}{\xi_1^2 + \xi_2^2}(\xi_1 k + \xi_2 l), m, n \right) \in A_\xi(\mathbb{Z}^4) = X_\xi,$$

$$v = \left(\tfrac{1}{\xi_1^2 + \xi_2^2}(\xi_2 u - \xi_1 v), \tfrac{1}{\xi_1^2 + \xi_2^2}(\xi_1 u + \xi_2 v), w, z \right) \in A_\xi([0,1)^4) = U_\xi.$$

Then

$$\chi v = \left(\tfrac{1}{\xi_1^2 + \xi_2^2}(\xi_2(k+u) - \xi_1(l+v)), \tfrac{1}{\xi_1^2 + \xi_2^2}(\xi_1(k+u) + \xi_2(l+v)) + nw, m+w, n+z \right) = x.$$

Since this choice is unique, X_ξ is a quasi-lattice in $\mathbb{R} \times \mathbb{H}$ with complement U_ξ.

By Lemma 2.7, the density of a quasi-lattice is computed as the reciprocal of the volume of its complement. Therefore,

$$D(X_\xi) = \frac{1}{\lambda(U_\xi)} = \frac{1}{\lambda(A_\xi([0,1)^4))} = \frac{1}{|\det A_\xi|} = \xi_1^2 + \xi_2^2.$$

$\square$

Since an orthonormal basis is both a frame and a Riesz sequence, Corollary 5.9 together with Lemma 5.22 and Lemma 5.23 implies the following density thresholds.

Corollary 5.24. *Let $g \in L^2(\mathbb{R}^2)$ and $X \subseteq \mathbb{R} \times \mathbb{H}$.*

(a) *If the set $\{\pi_\xi(x)g : x \in X\}$ is a frame for $L^2(\mathbb{R}^2)$, then*
$$D^-(X) \geq \xi_1^2 + \xi_2^2.$$

(b) *If the set $\{\pi_\xi(x)g : x \in X\}$ is a Riesz sequence in $L^2(\mathbb{R}^2)$, then*
$$D^+(X) \leq \xi_1^2 + \xi_2^2.$$

(c) *If the set $\{\pi_\xi(x)g : x \in X\}$ is a Riesz basis for $L^2(\mathbb{R}^2)$, then*
$$D^-(X) = D^+(X) = \xi_1^2 + \xi_2^2.$$

Example 4. Consider the Lie algebra $\mathfrak{g}_{6,24} = \mathbb{R}X_1 + \cdots + \mathbb{R}X_6$ from **[38]** with non-vanishing Lie brackets

$$[X_6, X_5] = X_4, [X_6, X_4] = X_3, [X_6, X_3] = X_2, [X_5, X_4] = X_1.$$

The corresponding connected and simply connected Lie group $G_{6,24}$ in Malcev coordinates is $\mathbb{R}^6$ with the multiplication law

$$(x_1, x_2, x_3, x_4, x_5, x_6)(y_1, y_2, y_3, y_4, y_5, y_6)$$
$$= (x_1 + y_1 + x_5 y_4 + x_5 x_6 y_5 + \tfrac{1}{2} x_6 y_5^2, x_2 + y_2 + x_6 y_3 + \tfrac{1}{2} x_6^2 y_4 + \tfrac{1}{6} x_6^3 y_5,$$
$$x_3 + y_3 + x_6 y_4 + \tfrac{1}{2} x_6^2 y_5, x_4 + y_4 + x_6 y_5, x_5 + y_5, x_6 + y_6).$$

The irreducible unitary representations of $G_{6,24}$ on $L^2(\mathbb{R}^2)$ that are square-integrable modulo the center $Z = \mathbb{R} \times \mathbb{R} \times \{0\} \times \{0\} \times \{0\} \times \{0\}$ are parametrized by $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, $\xi_1 \xi_2 \neq 0$, and given by

$$\pi_\xi(x)\phi(s, t) = e^{2\pi i((x_1 - \frac{1}{2} x_5^2 x_6 - x_4 s + x_5 x_6 s - \frac{1}{2} x_6 s^2)\xi_1)}$$
$$\cdot e^{2\pi i((x_2 - \frac{1}{6} x_5 x_6^3 + \frac{1}{6} x_6^3 s - x_3 t + \frac{1}{2} x_5 x_6^2 t + \frac{1}{2} x_4 t^2 - \frac{1}{2} x_5 x_6 t^2 - \frac{1}{2} x_6^2 s t + \frac{1}{2} x_6 s t^2)\xi_2)} \phi(s - x_5, t - x_6).$$

We want to study the projective representations of the quotient group $G_{6,24}/Z$ derived from the representations π_ξ as in Lemma 5.11.

By deleting the coordinates of the center in the above multiplication law for $G_{6,24}$ and relabeling the remaining coordinates x_j by x_{j-2}, we are led to the multiplication law

$$(x_1, x_2, x_3, x_4)(y_1, y_2, y_3, y_4) = (x_1 + y_1 + x_4 y_2 + \tfrac{1}{2} x_4^2 y_3, x_2 + y_2 + x_4 y_3, x_3 + y_3, x_4 + y_4)$$

of the group G_4 (confer Example 1.15), which is isomorphic to the quotient group $G_{6,24}/Z$.

The representations π_ξ give rise to irreducible square-integrable projective representation of G_4 on $L^2(\mathbb{R}^2)$, which we also call π_ξ, via

$$\pi_\xi(x)\phi(s, t) = e^{-2\pi i\left((x_2 \xi_1 - x_3 x_4 \xi_1 - \frac{1}{6} x_4^3 \xi_2)s + (x_1 \xi_2 - \frac{1}{2} x_3 x_4^2 \xi_2)t\right)}$$
$$\cdot e^{-\pi i(x_4 \xi_1 s^2 - (x_2 - x_3 x_4)\xi_2 t^2 + x_4^2 \xi_2 s t - x_4 \xi_2 s t^2))} \phi(s - x_3, t - x_4).$$

Note that as in the previous examples we have omitted occurring phase factors.

Lemma 5.25. *Let $\phi = \mathbb{1}_Q$, where $Q = [0,1]^2$, and $X_\xi := \frac{1}{\xi_2}\mathbb{Z} \times \frac{1}{\xi_1}\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. The system $\{\pi_\xi(\chi)\phi : \chi \in X_\xi\}$ forms an orthonormal basis for $L^2(\mathbb{R}^2)$.*

PROOF. First we show orthogonality.
Let $\chi = (\frac{1}{\xi_2}k, \frac{1}{\xi_1}l, m, n) \in X_\xi$, $\chi' = (\frac{1}{\xi_2}k', \frac{1}{\xi_1}l', m', n') \in X_\xi$. Then

$$\langle \pi_\xi(\chi)\phi, \pi_\xi(\chi')\phi \rangle = \int_\mathbb{R} \int_\mathbb{R} \pi_\xi(\chi)\phi(s,t)\overline{\pi_\xi(\chi')\phi(s,t)}\,ds\,dt$$

$$= \int_\mathbb{R} \int_\mathbb{R} e^{-2\pi i \left((l-l')-(mn-m'n')\xi_1 - \frac{1}{6}(n^3-n'^3)\xi_2\right)s} e^{-2\pi i \left((k-k')-\frac{1}{2}(mn^2-m'n'^2)\xi_2\right)t}$$

$$\cdot\, e^{-\pi i \left((n-n')\xi_1 s^2 - (l-l')\frac{\xi_2}{\xi_1}t^2 + (mn-m'n')\xi_2 t^2 + (n^2-n'^2)\xi_2 st - (n-n')\xi_2 st^2\right)}$$

$$\cdot\, \mathbb{1}_Q(s-m, t-n)\mathbb{1}_Q(s-m', t-n')\,ds\,dt$$

$$= \delta_{m,m'}\delta_{n,n'} \int_n^{n+1} \int_m^{m+1} e^{-2\pi i(l-l')s} e^{-2\pi i(k-k')t} e^{\pi i(l-l')\frac{\xi_2}{\xi_1}t^2}\,ds\,dt$$

$$= \delta_{m,m'}\delta_{n,n'} \int_n^{n+1} \left(\int_m^{m+1} e^{-2\pi i(l-l')s}\,ds\right) e^{-2\pi i(k-k')t} e^{\pi i(l-l')\frac{\xi_2}{\xi_1}t^2}\,dt$$

$$= \delta_{m,m'}\delta_{n,n'}\delta_{l,l'} \int_n^{n+1} e^{-2\pi i(k-k')t}\,dt$$

$$= \delta_{m,m'}\delta_{n,n'}\delta_{l,l'}\delta_{k,k'}$$

$$= \delta_{\chi,\chi'},$$

because the integer translates of the characteristic function $\mathbb{1}_Q$ are disjoint for distinct pairs of integers and the functions $\{f(\cdot) = e^{2\pi ik\cdot} : k \in \mathbb{Z}\}$ are orthogonal on $L^2([m, m+1])$ and $L^2([n, n+1])$ respectively.

Next we prove the completeness of the system $\{\pi_\xi(\chi)\mathbb{1}_Q(s,t) : \chi \in X_\xi\}$ in $L^2(\mathbb{R}^2)$. As in the previous examples we show that every element of the standard orthonormal basis

$$\mathcal{E} = \{e^{-2\pi ikt}e^{-2\pi ils}\mathbb{1}_Q(s-m, t-n) : k, l, m, n \in \mathbb{Z}\}$$

has an expansion with respect to the given orthonormal system

$$\left\{e^{-2\pi i(l-mn\xi_1 - \frac{1}{6}n^3\xi_2)s} e^{-2\pi i(k-\frac{1}{2}mn^2\xi_2)t}\right.$$

$$e^{-\pi i(n\xi_1 s^2 - l\frac{\xi_2}{\xi_1}t^2 + mn\xi_2 t^2 + n^2\xi_2 st - n\xi_2 st^2)}$$

$$\left.\mathbb{1}_Q(s-m, t-n) : k, l, m, n \in \mathbb{Z}\right\}$$

in $L^2(\mathbb{R}^2)$. Fix a basis element $e^{-2\pi ik_0 t}e^{-2\pi il_0 s}\mathbb{1}_Q(s-m_0, t-n_0) \in \mathcal{E}$.

Consider the series expansion of the function $e^{-2\pi ik_0 t}e^{-\pi il_0 \frac{\xi_2}{\xi_1}t^2}$ in terms of the basis $\{e^{-2\pi ikt} : k \in \mathbb{Z}\}$ on $L^2([n_0, n_0+1])$, that is,

$$e^{-2\pi ik_0 t}e^{-\pi il_0 \frac{\xi_2}{\xi_1}t^2} = \sum_{k\in\mathbb{Z}} c_k e^{-2\pi ikt}.$$

It follows that

$$e^{-2\pi i k_0 t}e^{-2\pi i l_0 s} = \sum_{k\in\mathbb{Z}} c_k e^{-2\pi i k t}e^{\pi i l_0 \frac{\xi_2}{\xi_1}t^2}e^{-2\pi i l_0 s}$$

on $L^2([m_0, m_0+1]\times[n_0, n_0+1])$. To put it differently, every element of the standard orthonormal basis $\{e^{-2\pi i k t}e^{-2\pi i l s} : k, l \in \mathbb{Z}\}$ for $L^2([m_0, m_0+1]\times[n_0, n_0+1])$ has an expansion with respect to the orthogonal system $\{e^{-2\pi i k t}e^{\pi i l \frac{\xi_2}{\xi_1}t^2}e^{-2\pi i l s} : k, l \in \mathbb{Z}\}$ on $L^2([m_0, m_0+1]\times[n_0, n_0+1])$. By Property (i), we therefore conclude that the system $\{e^{-2\pi i k t}e^{\pi i l \frac{\xi_2}{\xi_1}t^2}e^{-2\pi i l s} : k, l \in \mathbb{Z}\}$ is an orthonormal basis in $L^2([m_0, m_0+1]\times[n_0, n_0+1])$. Now set

$$m(s, t) := e^{-2\pi i\left((-m_0 n_0\xi_1 - \frac{1}{6}n_0^3\xi_2)s - \frac{1}{2}m_0 n_0^2\xi_2 t\right)}e^{-\pi i\left(n_0\xi_1 s^2 + m_0 n_0\xi_2 t^2 + n_0^2\xi_2 st - n_0\xi_2 st^2\right)}.$$

By Property (ii), the system

$$\left\{e^{-2\pi i k t}e^{\pi i l \frac{\xi_2}{\xi_1}t^2}e^{-2\pi i l s}m(s, t) : k, l \in \mathbb{Z}\right\}$$

is also an orthonormal basis in $L^2([m_0, m_0+1]\times[n_0, n_0+1])$, because it is just the orthonormal basis $\{e^{-2\pi i k t}e^{\pi i l \frac{\xi_2}{\xi_1}t^2}e^{-2\pi i l s} : k, l \in \mathbb{Z}\}$ multiplied by a function of modulus one.

It follows that the basis element $e^{-2\pi i k_0 t}e^{-2\pi i l_0 s}\mathbb{1}_Q(s - m_0, t - n_0) \in \mathcal{E}$ has an expansion with respect to the system $\{\pi_\xi(\chi)\mathbb{1}_Q(s, t) : \chi \in X_\xi\}$ in $L^2(\mathbb{R}^2)$, so $\{\pi_\xi(\chi)\mathbb{1}_Q(s, t) : \chi \in X_\xi\}$ is an orthonormal basis by Property (i).

$$\square$$

Lemma 5.26. *The set X_ξ is a quasi-lattice in G_4 with density $D(X_\xi) = |\xi_1\xi_2|$.*

PROOF. We claim that X_ξ is a quasi-lattice in G_4 with complement $U_\xi := A_\xi([0, 1)^4)$, where A_ξ is defined by

$$A_\xi := \begin{pmatrix} \frac{1}{\xi_2} & 0 & 0 & 0 \\ 0 & \frac{1}{\xi_1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We need to show that every element $x \in G_4$ can be uniquely written as $x = \chi v$ with $\chi \in X_\xi$ and $v \in U_\xi$. Let $x = (x_1, x_2, x_3, x_4) \in G_4$ be arbitrary. Set

$n = \lfloor x_4\rfloor \in \mathbb{Z},$ $z = x_4 - \lfloor x_4\rfloor \in [0, 1),$

$m = \lfloor x_3\rfloor \in \mathbb{Z},$ $w = x_3 - \lfloor x_3\rfloor \in [0, 1),$

$l = \lfloor \xi_1(x_2 - nw)\rfloor \in \mathbb{Z},$ $v = \xi_1(x_2 - nw) - \lfloor \xi_1(x_2 - nw)\rfloor \in [0, 1),$

$k = \lfloor \xi_2(x_1 - \frac{1}{\xi_1}nv - \frac{1}{2}n^2 w)\rfloor \in \mathbb{Z},$

    $u = \xi_2(x_1 - \frac{1}{\xi_1}nv - \frac{1}{2}n^2 w) - \lfloor \xi_2(x_1 - \frac{1}{\xi_1}nv - \frac{1}{2}n^2 w)\rfloor \in [0, 1),$

and $\chi = (\frac{1}{\xi_2}k, \frac{1}{\xi_1}l, m, n) \in X_\xi$, $\upsilon = (\frac{1}{\xi_2}u, \frac{1}{\xi_1}v, w, z) \in U_\xi$. Then

$$\chi\upsilon = \left(\frac{1}{\xi_2}(k+u) + \frac{1}{\xi_1}nv + \frac{1}{2}n^2w, \frac{1}{\xi_1}(l+v) + nw, m+w, n+z \right) = x.$$

Since this choice is unique, X_ξ is a quasi-lattice in G_4 with complement U_ξ.

By Lemma 2.7, the density of a quasi-lattice is computed as the reciprocal of the volume of its complement. Therefore,

$$D(X_\xi) = \frac{1}{\lambda(U_\xi)} = \frac{1}{\lambda(A_\xi([0,1)^4))} = \frac{1}{|\det A_\xi|} = |\xi_1\xi_2|.$$

$\square$

Since an orthonormal basis is both a frame and a Riesz sequence, Corollary 5.9 together with Lemma 5.25 and Lemma 5.26 implies the following density thresholds.

Corollary 5.27. *Let $g \in L^2(\mathbb{R}^2)$ and $X \subseteq G_4$.*

(a) *If the set $\{\pi_\xi(x)g : x \in X\}$ is a frame for $L^2(\mathbb{R}^2)$, then*
$$D^-(X) \geq |\xi_1\xi_2|.$$

(b) *If the set $\{\pi_\xi(x)g : x \in X\}$ is a Riesz sequence in $L^2(\mathbb{R}^2)$, then*
$$D^+(X) \leq |\xi_1\xi_2|.$$

(c) *If the set $\{\pi_\xi(x)g : x \in X\}$ is a Riesz basis for $L^2(\mathbb{R}^2)$, then*
$$D^-(X) = D^+(X) = |\xi_1\xi_2|.$$

5.4. Outlook

The classical Kirillov theory provides a construction of all irreducible unitary representations of a nilpotent Lie group, namely as representations induced by certain characters of closed subgroups.

Let N be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$. Denote by $\mathfrak{n}^*$ the linear dual of $\mathfrak{n}$ and let $\xi \in \mathfrak{n}^*$. A subalgebra $\mathfrak{m}$ of $\mathfrak{n}$ is called *maximal subordinate* to ξ if $\mathfrak{m}$ is of maximal dimension such that $\xi([\mathfrak{m}, \mathfrak{m}]) = 0$. If $\mathfrak{m}$ is a maximal subordinate subalgebra to $\xi \in \mathfrak{n}^*$, then the map

$$(5.17) \qquad\qquad \chi_\xi(\exp(X)) = e^{2\pi i \xi(X)}, \ X \in \mathfrak{m}.$$

defines a one-dimensional representation, a so-called *character*, of the closed subgroup $M = \exp(\mathfrak{m})$ of N, because $\xi([\mathfrak{m}, \mathfrak{m}]) = 0$. The induced representation

$$(5.18) \qquad\qquad \pi_\xi := \mathrm{Ind}(\chi_\xi) := \mathrm{Ind}_M^N(\chi_\xi)$$

(see, e.g., [22], [45] for the inducing construction) is an irreducible unitary representation of N and (up to equivalence) independent of the particular choice of the maximal subordinate subalgebra $\mathfrak{m}$. Furthermore, every irreducible unitary

representation of a nilpotent Lie group N is (up to equivalence) obtained as a representation induced by a one-dimensional representation χ_ξ as in (5.17) and there is a bijection between the equivalence classes of irreducible unitary representations of N and the orbits in $\mathfrak{n}^*$ under the so-called coadjoint representation. For details and proofs we refer the interested reader to the standard reference for representation theory of nilpotent Lie groups [**13**].

We are particularly interested in the description of those irreducible unitary representations π_ξ that are square-integrable modulo the center.

Fix a Malcev basis $\{X_1, \ldots, X_n\}$ for $\mathfrak{n}$ and denote the center of $\mathfrak{n}$ by $\mathfrak{z}$. Let $k \in \mathbb{N}$ be such that $\mathfrak{z} = \mathrm{span}\{X_1, \ldots, X_k\}$ and let

$$\mathfrak{z}^* = \{\xi \in \mathfrak{n}^* : \xi(X_i) = 0, k < i \leq n\}$$

For $\xi \in \mathfrak{n}^*$ consider the matrix

(5.19)
$$B(\xi) = \Big(\xi([X_i, X_j])\Big)_{k+1 \leq i,j \leq n}$$

and define the Pfaffian $\mathrm{Pf}(\xi)$ by

$$\mathrm{Pf}(\xi)^2 = \det B(\xi).$$

By the results of Moore and Wolf [**37**], the induced representation π_ξ for $\xi \in \mathfrak{n}^*$ is square-integrable modulo the center if and only if $\mathrm{Pf}(\xi) \neq 0$. Furthermore, if $\tilde{\xi} \in \mathfrak{n}^*$ with $\mathrm{Pf}(\tilde{\xi}) \neq 0$, then all elements in the subspace $\tilde{\xi} + \mathfrak{z}^\perp = \tilde{\xi} + \{\xi \in \mathfrak{n}^* : \xi|_{\mathfrak{z}} \equiv 0\}$ of $\mathfrak{n}^*$ lead to equivalent induced representations. So the irreducible unitary representations of N that are square-integrable modulo the center are parametrized by the subset $\{\xi \in \mathfrak{z}^* : \mathrm{Pf}(\xi) \neq 0\}$ of $\mathfrak{n}^*$ (for details see [**13**] or [**37**]).

Let $\{X_1^*, \ldots, X_n^*\} \subseteq \mathfrak{n}^*$ denote the dual basis to $\{X_1, \ldots, X_n\}$. Revisiting the examples of the previous section in the light of the representation theoretic background one can check that the representations π_ξ studied there are just the representations induced by the elements $\xi = \xi_1 X_1^* + \cdots + \xi_k X_k^* \in \mathfrak{z}^*$ with $\mathrm{Pf}(\xi) \neq 0$ and that the value for the critical density equals $|\mathrm{Pf}(\xi)|$.

The same holds true for all the relevant examples in [**38**], so we are lead to the following statement.

Theorem 5.28. *Let G be a homogeneous group isomorphic to a quotient group N/Z for some connected and simply connected nilpotent Lie group N of dimension at most 6 and let $s : G \to N$ be a continuous section. Let $\pi_\xi := \mathrm{Ind}(\chi_\xi) \circ s$ for some $\xi \in \mathfrak{z}^*$ with $\mathrm{Pf}(\xi) \neq 0$. Then there exists an orthonormal basis for $\mathcal{H}$ of the form $\{\pi_\xi(\chi)\phi : \chi \in X_\xi\}$, where $\phi \in \mathcal{H}$ and X_ξ is a subset of G with uniform density*

$$D(X_\xi) = |\mathrm{Pf}(\xi)|.$$

Corollary 5.29. *Let G and π_ξ be as in Theorem 5.28. Let $g \in \mathcal{H}$ and $X \subseteq G$.*

(a) *If the set $\{\pi_\xi(x)g : x \in X\}$ is a frame for $\mathcal{H}$, then*
$$D^-(X) \geq |\mathrm{Pf}(\xi)|.$$

(b) *If the set $\{\pi_\xi(x)g : x \in X\}$ is a Riesz sequence in $\mathcal{H}$, then*
$$D^+(X) \leq |\mathrm{Pf}(\xi)|.$$

(c) *If the set $\{\pi_\xi(x)g : x \in X\}$ is a Riesz basis for $\mathcal{H}$, then*
$$D^-(X) = D^+(X) = |\mathrm{Pf}(\xi)|.$$

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Abstract

This thesis is concerned with necessary density conditions for frames and Riesz sequences indexed by discrete subsets of homogeneous groups. We define a density on homogeneous groups in analogy to the Beurling density on $\mathbb{R}^n$, however, adapted to the geometry of homogeneous groups. Employing this density, we present a theorem for the comparison of the densities of frames and Riesz sequences indexed by discrete subsets of homogeneous groups. It is a first non-commutative extension of previous results like the density theorem for irregular Gabor frames of Ramanathan and Steger and its generalization to abstract frames with 'commutative index sets' by Balan, Casazza, Heil und Landau.

The comparison theorem is used to derive necessary density conditions for sampling and interpolation in shift-invariant spaces on homogeneous groups. This is done via the correspondence of sampling sets and frames of reproducing kernels.

Further, necessary density conditions for frames and Riesz sequences in the orbit of projective square-integrable group representations are investigated with the help of the comparison theorem. For some concrete examples of projective representations of low-dimensional homogeneous groups we construct orthonormal bases in the orbit and thereby deduce explicit thresholds for the density of frames and Riesz sequences in the respective orbits.

Zusammenfassung

Diese Dissertation beschäftigt sich mit notwendigen Dichtebedingungen für Frames
und Rieszfolgen, die durch diskrete Teilmengen von homogenen Liegruppen in-
diziert sind.

In Analogie zur Beurlingdichte auf $\mathbb{R}^n$ wird ein wohldefinierter Dichtebegriff auf
homogenen Liegruppen eingeführt. Unter Verwendung dieser Dichte wird ein Satz
zum Vergleich der Dichten von Frames und Rieszfolgen, die durch diskrete Teil-
mengen von homogenen Liegruppen indiziert sind, präsentiert. Dabei handelt es
sich um eine Verallgemeinerung des Dichtesatzes für Gaborframes von Ramanathan
und Steger sowie dessen Weiterentwicklung für abstrakte Frames mit 'kommuta-
tiven Indexmengen' durch Balan, Casazza, Heil und Landau.

Dieser Vergleichssatz wird verwendet um notwendige Dichtebedingungen für Ab-
tasten und Interpolation in Shift-invarianten Räumen auf homogenen Gruppen
herzuleiten. Dies geschieht mittels eines Zusammenhangs zwischen Abtastmengen
und Frames bestehend aus den zugehörigen reproduzierenden Kernen.

Weiters werden mithilfe des Vergleichssatzes notwendige Dichtebedingungen für
Frames und Rieszfolgen im Orbit projektiver quadrat-integrierbarer Darstellun-
gen von homogenen Gruppen untersucht. Für einige konkrete Beispiele projek-
tiver quadrat-integrierbarer Darstellungen niedrigdimensionaler homogener Grup-
pen werden Orthonormalbasen im Orbit konstruiert und dadurch explizite Ab-
schätzungen für die Dichte von Frames und Rieszfolgen im Orbit der jeweiligen
Darstellung gewonnen.

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