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Abstract

A stationary reflection principle in $\mathcal{P}_{\omega_1}(\kappa)$ is a statement of the form "*for every stationary subset S of $\mathcal{P}_{\omega_1}(\kappa)$, there exists a set $X \subseteq \kappa$ of size ω_1 such that $\omega_1 \subseteq X$ and $S \cap \mathcal{P}_{\omega_1}(X)$ is stationary in $\mathcal{P}_{\omega_1}(X)$*". In short, we say that S reflects to X. The main difference between these principles is the cofinality of the reflecting set X. Without restriction, we call this statement the Weak Reflection Principle. Requiring $\mathrm{cof}(\mathrm{ot}(X)) = \omega_1$, we obtain a stronger principle, namely the Reflection Principle. Interesting results can also be derived when we replace stationary with semistationary, a notion introduced by Shelah.

The goal of this thesis is to present all known results concerning the consistency of these stationary reflection principles in $\mathcal{P}_{\omega_1}(\kappa)$, where κ is a regular cardinal. Our main focus is on $\mathcal{P}_{\omega_1}(\omega_2)$.

We start by proving facts about a partial version of reflection. This kind of stationary reflection deals with reflecting points of a stationary subset of $\mathcal{P}_{\omega_1}(\omega_2)$. This theory was developed by Sakai. He proved its consistency without the use of large cardinals. We will use the techniques developed by König-Larson-Yoshinobu to show the failure of the cofinality ω case of the Partial Stationary Reflection Principles for $\mathcal{P}_{\omega_1}(\omega_2)$ under $2^{\omega_1} = \omega_2$ and for $\mathcal{P}_{\omega_1}(\omega_n)$ under CH and $2^{\omega_{n-1}} = \omega_n$ for $n > 2$.

Then we continue with the consistency of the stationary reflection principles mentioned above. For those results, we need large cardinals. The Weak and the Strong Reflection Principle for ω_2 are equiconsistent with a weakly

compact cardinal. Similarly, one can derive consistency for all cardinals by Lévy collapsing a supercompact to ω_2. Moreover, we show that we do not have to be concerned with the restriction of the cofinality of the reflecting set X to ω, since this statement is inconsistent. We mention a result by Sakai, who proved that under the assumption of a strongly compact cardinal, the Semistationary Reflection Principle is consistent.

In the final chapter we present the known implications between the Weak Reflection Principle, the Strong Reflection Principle, and the Semistationary Reflection Principle.

Zusammenfassung

Ein Reflexionsprinzip für stationäre Mengen in $\mathcal{P}_{\omega_1}(\kappa)$ ist eine Aussage der Form "*für jede stationäre Teilmenge S von $\mathcal{P}_{\omega_1}(\kappa)$ existiert eine Menge $X \subseteq \kappa$ der Größe ω_1, sodass $\omega_1 \subseteq X$ und $S \cap \mathcal{P}_{\omega_1}(X)$ in $\mathcal{P}_{\omega_1}(X)$ stationär ist*". Kurz gesagt, S wird von X reflektiert. Der größte Unterschied zwischen diesen Prinzipien ist die Kofinalität der reflektierenden Menge X. Ohne Einschränkung wird diese Aussage Schwaches Reflexionsprinzip genannt. Setzt man $\mathrm{cof}(\mathrm{ot}(X)) = \omega_1$ voraus, erhält man ein stärkeres Prinzip, das Reflexionsprinzip. Man kann interessante Resultate erzielen, wenn man stationär durch semistationär ersetzt, ein Begriff, der von Shelah eingeführt wurde.

Ziel dieser Arbeit ist es, die bekannten Resultate über die Widerspruchsfreiheit der Reflexionsprinzipien stationärer Mengen in $\mathcal{P}_{\omega_1}(\kappa)$ zu präsentieren, wobei κ eine reguläre Kardinalzahl bezeichnet. Unser Hauptaugenmerk richtet sich auf $\mathcal{P}_{\omega_1}(\omega_2)$.

Wir beginnen damit, Ergebnisse über eine partielle Version der Reflexion zu präsentieren. Diese Art von Reflexion behandelt Reflexionspunkte stationärer Teilmengen einer stationären Menge in $\mathcal{P}_{\omega_1}(\omega_2)$. Diese Theorie wurde von Sakai entwickelt. Er zeigte die Widerspruchsfreiheit der Prinzipien ohne Verwendung großer Kardinalzahlen. Wir verwenden Methoden, die von König-Larson-Yoshinobu entwickelt wurden, um zu zeigen, dass wenn $2^{\omega_1} = \omega_2$, die Restriktion der Kofinalität zu ω in $\mathcal{P}_{\omega_1}(\omega_2)$ fehlschlägt. Wir zeigen dies auch allgemeiner für partielle Reflexion in $\mathcal{P}_{\omega_1}(\omega_n)$ unter der Annahme von CH und $2^{\omega_{n-1}} = \omega_n$ für $n > 2$.

Danach behandeln wir die Widerspruchsfreiheit der oben genannnten Reflexionsprinzipien für stationäre Mengen. Um diese Resultate zu erhalten, benötigen wir große Kardinalzahlen. Die Widerspruchsfreiheit des Schwachen und Starken Reflexionsprinzips für ω_2 ist äquivalent zu der Widerspruchsfreiheit einer weakly compact cardinal. Auf ähnliche Weise kann man die Widerspruchsfreiheit dieser Reflexionsprinzipien für alle Kardinalzahlen ableiten, wenn man eine supercompact cardinal zu ω_2 kollabiert. Weiters zeigen wir, dass die Einschränkung der Kofinalität der reflektierenden Menge X auf ω irrelevant ist, da diese Aussage nicht widerspruchsfrei ist. Wir erwähnen ein Resultat von Sakai, das aussagt, dass unter der Annahme einer strongly compact cardinal das Reflexionsprinzip für semistationäre Mengen widerspruchsfrei ist.

Im letzten Kapitel präsentieren wir die bekannten Implikationen zwischen dem Schwachen Reflexionsprinzip, dem Starken Reflexionsprinzip und dem Semistationären Reflexionsprinzip.

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Chapter 1

Introduction

This introduction is meant to give the basic definitions and tools necessary for this thesis. The reason for studying stationary reflection principles is that they have interesting combinatorial consequences. The Weak Reflection Principle for ω_2, for example, implies $2^\omega \leq \omega_2$, $\sim\square_{\omega_2}$, and that every stationary subset of ω_2 with cofinality ω reflects to an ordinal in ω_2 with cofinality ω_1. Foreman-Magidor-Shelah were the ones who introduced this principle in the context of Martin's Maximum which shares similar consequences. The notion of semistationarity was introduced by Shelah [15] in connection to semiproperness of posets.

1.1 Stationary and semistationary sets

Clubs and stationary sets in $\mathcal{P}_{\omega_1}(\kappa)$ are natural extensions of these notions in a cardinal κ. We let $\mathcal{P}_{\omega_1}(\kappa) = \{a \subseteq \kappa \mid |a| < \omega_1\}$ and for a regular cardinal κ, let $\mathrm{cof}(\kappa)$ denote the class of all ordinals with cofinality κ. Instead of giving those definitions for a cardinal κ, we give them for an arbitrary set X.

Definition 1.1.1. *Let X be an arbitrary set. A subset C of $\mathcal{P}_{\omega_1}(X)$ is* closed and unbounded *(or* club*) in $\mathcal{P}_{\omega_1}(X)$ if the following two properties hold.*

(1) $\forall x \in \mathcal{P}_{\omega_1}(X)\ \exists y \in C\ x \subseteq y$, *and*

(2) for every increasing sequence $\langle x_n : n < \omega \rangle$ *of elements of* C,

$$\bigcup \{x_n : n < \omega\} \in C.$$

A subset S *of* $\mathcal{P}_{\omega_1}(X)$ *is called* stationary *if* S *intersects every club in* $\mathcal{P}_{\omega_1}(X)$.
For a set $Y \subseteq X$*, we say that* S reflects to Y *if* $S \cap \mathcal{P}_{\omega_1}(Y)$ *is stationary in* $\mathcal{P}_{\omega_1}(Y)$.

If it is clear from the context, we will omit in which space a set is club. Sometimes we will use the equivalent definition of club and stationary in terms of functions.

Definition 1.1.2. *For a function* $F : \mathcal{P}_\omega(X) \to X$ *and* $x \subseteq X$*, we say that* x *is* closed under F *if for every* $y \in \mathcal{P}_{\omega_1}(x)$, $F(y) \subseteq x$.

Fact 1.1.3 ([9])**.** *If* $C \subseteq \mathcal{P}_\omega(X)$ *is club, then there exists a function* $F : \mathcal{P}_\omega(X) \to X$ *such that every set in* $\mathcal{P}_{\omega_1}(X)$ *which is closed under* F *is in* C*. Then* $S \subseteq \mathcal{P}_{\omega_1}(X)$ *is stationary if for any function* $F : \mathcal{P}_\omega(X) \to X$ *there is a set* $b \in S$ *which is closed under* F.

One of the most important lemmata in connection to stationary sets is Fodor's Lemma, which states that for certain functions there are stationary sets on which they are constant.

Definition 1.1.4. *A partial function* $F : \mathcal{P}_{\omega_1}(X) \to X$ *is* regressive *if for all* a *in the domain of* F, $F(a)$ *is a member of* a.

Lemma 1.1.5 (**Fodor** ([5], 8.7))**.** *If* $S \subseteq \mathcal{P}_{\omega_1}(X)$ *is a stationary set and* $F : S \to X$ *a total regressive function, then there is a stationary set* $T \subseteq S$ *and a set* x *in* X *such that* $F(a) = x$ *for all* a *in* T.

In order to show ω_1-distributivity of a forcing poset, we will make frequent use of the following generalized version of the Δ-System Lemma.

Theorem 1.1.6 (**Δ-System Lemma** ([5], 9.19))**.** *Assume* GCH *and* $\kappa^{<\kappa} = \kappa$. *Suppose X is a collection of sets with cardinality less than κ and $|X| = \kappa^+$. Then there exists a collection $Z \subseteq X$ of size κ^+ and a set A such that $W \cap Y = A$ for any two distinct elements W, Y of Z.*

Semistationarity was introduced by Shelah [15] in close relation to semiproperness of posets, which we will define in 1.3.11.

Definition 1.1.7. *For countable sets x and y, we call y an* ω_1-extension *of x if $x \subseteq y$ and $x \cap \omega_1 = y \cap \omega_1$. We write $x \sqsubseteq y$.*

Definition 1.1.8. *Let X be a set such that $\omega_1 \subseteq X$. A set $S \subseteq \mathcal{P}_{\omega_1}(X)$ is* semistationary *if the set of ω_1-extensions $\{y \in \mathcal{P}_{\omega_1}(X) \mid \exists x \in S \; x \sqsubseteq y\}$ of elements of S is stationary in $\mathcal{P}_{\omega_1}(X)$.*

Note that for a set X of size ω_1, the notion of stationary and semistationary are the same on $\mathcal{P}_{\omega_1}(X)$. It is clear that if S denotes a stationary subset of $\mathcal{P}_{\omega_1}(X)$, then $S^* = \{y \in \mathcal{P}_{\omega_1}(X) \mid \exists x \in S \; x \sqsubseteq y\} \supseteq S$ is also stationary on $\mathcal{P}_{\omega_1}(X)$. Therefore S is also semistationary. If S is a semistationary subset of $\mathcal{P}_{\omega_1}(X)$, then $S^* = \{y \in \mathcal{P}_{\omega_1}(X) \mid \exists x \in S \; x \sqsubseteq y\}$ is stationary on $\mathcal{P}_{\omega_1}(X)$ by definition. Let h be a function from $\mathcal{P}_{\omega}(X)$ to X defining a club. Then there exists a set $y \in S^*$ which is closed under h. By definition of S^*, there is a set $x \in S$ such that $x \sqsubseteq y$, i.e. $x \subseteq y$ and $x \cap \omega_1 = y \cap \omega_1$. Since $|X| = \omega_1$, there is a bijection between X and ω_1. But x and y coincide on ω_1 and therefore x is also closed under h. Hence, S is also stationary in $\mathcal{P}_{\omega_1}(X)$. As a consequence, we can say that S reflects to X, S is stationary in $\mathcal{P}_{\omega_1}(X)$ or S is semistationary in $\mathcal{P}_{\omega_1}(X)$.

The next fact can be found in [11]. We use it to prove Lemma 1.1.10, a property unique for semistationary sets.

Fact 1.1.9. *Let κ be a regular cardinal and X and X' be sets with $\kappa \subseteq X \subseteq X'$.*

(1) If $C \subseteq \mathcal{P}_\kappa(X)$ is club then the set $\{x' \in \mathcal{P}_\kappa(X') \mid x' \cap X \in C\}$ is club in $\mathcal{P}_\kappa(X')$. Thus, if $S' \subseteq \mathcal{P}_\kappa(X')$ is stationary then the set $\{x' \cap X \mid x' \in S'\}$ is stationary in $\mathcal{P}_\kappa(X)$.

(2) If $C' \subseteq \mathcal{P}_\kappa(X')$ is club then the set $\{x' \cap X \mid x' \in C'\}$ contains a club in $\mathcal{P}_\kappa(X)$. Thus, if $S \subseteq \mathcal{P}_\kappa(X)$ is stationary then the set $\{x' \in \mathcal{P}_\kappa(X') \mid x' \cap X \in S\}$ is stationary in $\mathcal{P}_\kappa(X')$.

Lemma 1.1.10. *Let X and X' be sets with $\omega_1 \subseteq X \subseteq X'$.*

(1) If $S' \subseteq \mathcal{P}_{\omega_1}(X')$ is semistationary then the set $\{x' \cap X \mid x' \in S'\}$ is semistationary in $\mathcal{P}_{\omega_1}(X)$.

(2) If $S \subseteq \mathcal{P}_{\omega_1}(X)$ is semistationary then S is semistationary in $\mathcal{P}_{\omega_1}(X')$.

Proof. (1) Since S' is semistationary in $\mathcal{P}_{\omega_1}(X')$, the set

$$T' = \{y \in \mathcal{P}_{\omega_1}(X') \mid \exists x \in S'\ x \sqsubseteq y\}$$

is stationary in $\mathcal{P}_{\omega_1}(X')$. By Fact 1.1.9 (1), the set $\{y \cap X \mid y \in T\}$ is stationary in $\mathcal{P}_{\omega_1}(X)$. Since

$$\{y \cap X \mid y \in T\} \subseteq \{y \in \mathcal{P}_{\omega_1}(X) \mid \exists x' \cap X\ (x' \in S'\ \wedge\ x' \cap X \sqsubseteq y)\},$$

the right side is also stationary in $\mathcal{P}_{\omega_1}(X)$. But this is the set of ω_1-extensions of elements of $\{x' \cap X \mid x' \in S'\}$. Therefore, $\{x' \cap X \mid x' \in S'\}$ is semistationary.

(2) Let $S \subseteq \mathcal{P}_{\omega_1}(X)$ be semistationary and

$$T = \{y \in \mathcal{P}_{\omega_1}(X) \mid \exists x \in S\ x \sqsubseteq y\}.$$

Note that T is a subset of

$$T' = \{y \in \mathcal{P}_{\omega_1}(X') \mid \exists x \in S \; x \sqsubseteq y\}.$$

Since T is stationary in $\mathcal{P}_{\omega_1}(X)$, so is T'. Since

$$T' = \{y \in \mathcal{P}_{\omega_1}(X') \mid y \cap X \in T\},$$

T' is stationary in $\mathcal{P}_{\omega_1}(X')$ by 1.1.9 (2). Hence S is semistationary in $\mathcal{P}_{\omega_1}(X')$. $\square$

1.2 The principles

In this section we give an overview of the reflection principles we work with. Two versions of each principle can be found throughout the literature. We state both and prove to some extent that they are equivalent. These definitions can be easily generalized by replacing ω_2 with a larger cardinal.

Definition 1.2.1. *For a stationary set $S^* \subseteq \mathcal{P}_{\omega_1}(\omega_2)$ and $k \in \{0, 1\}$, let the* Partial Stationary Reflection Principle for ω_2 *or* $\mathrm{SR_k}(S^*)$ *denote the following principle:*

$\mathrm{SR_k}(S^*) \equiv$ *For every stationary set $S \subseteq S^*$ there is a set $X \subseteq \omega_2$ of size ω_1 with $\omega_1 \subseteq X$ and $\mathrm{cof}(\mathrm{ot}(X)) = \omega_k$ such that S reflects to X.*

The Weak Reflection Principle for ω_2 *or* $\mathrm{WRP}(\omega_2)$ *is the statement:*

$\mathrm{WRP}(\omega_2) \equiv$ *For every stationary set $S \subseteq \mathcal{P}_{\omega_1}(\omega_2)$ there is a set $X \subseteq \omega_2$ of size ω_1 with $\omega_1 \subseteq X$ such that S reflects to X.*

The Reflection Principle for ω_2 *or $RP(\omega_2)$ is the statement:*

$\mathrm{RP}(\omega_2) \equiv$ *For every stationary set $S \subseteq \mathcal{P}_{\omega_1}(\omega_2)$ there is a set $X \subseteq \omega_2$ of size ω_1 with $\omega_1 \subseteq X$ and $\mathrm{cof}(\mathrm{ot}(X)) = \omega_1$ such that S reflects to X.*

The Semistationary Reflection Principle for ω_2 *or* $\mathrm{SSR}(\omega_2)$ *is the statement:*

$$\mathrm{SSR}(\omega_2) \equiv \textit{For every semistationary set } S \subseteq \mathcal{P}_{\omega_1}(\omega_2) \textit{ there is a set } X \subseteq \omega_2 \textit{ of size } \omega_1 \textit{ with } \omega_1 \subseteq X \textit{ such that } S \cap \mathcal{P}_{\omega_1}(X) \textit{ is semistationary in } \mathcal{P}_{\omega_1}(X).$$

In fact, we can require X to be an ordinal and the statement is unchanged. Since this seems to make things easier, we will use the following versions of the reflection principles.

Definition 1.2.2. *For a stationary set $S^* \subseteq \mathcal{P}_{\omega_1}(\omega_2)$ and $k \in \{0,1\}$, let* $\mathrm{SR_k}(S^*)$ *denote the following principle:*

$$\mathrm{SR_k}(S^*) \equiv \textit{For every stationary set } S \subseteq S^* \textit{ there is an uncountable ordinal } \alpha \in \omega_2 \textit{ with } \mathrm{cof}(\alpha) = \omega_k \textit{ such that } S \textit{ reflects to } \alpha.$$

The Weak Reflection Principle for ω_2 *or* $\mathrm{WRP}(\omega_2)$ *is the statement:*

$$\mathrm{WRP}(\omega_2) \equiv \textit{For every stationary set } S \subseteq \mathcal{P}_{\omega_1}(\omega_2) \textit{ there is an uncountable ordinal } \alpha \in \omega_2 \textit{ such that } S \textit{ reflects to } \alpha.$$

The Reflection Principle for ω_2 *or* $\mathrm{RP}(\omega_2)$ *is the statement:*

$$\mathrm{RP}(\omega_2) \equiv \textit{For every stationary set } S \subseteq \mathcal{P}_{\omega_1}(\omega_2) \textit{ there is an ordinal } \alpha \in \omega_2 \textit{ with cofinality } \omega_1 \textit{ such that } S \textit{ reflects to } \alpha.$$

The Semistationary Reflection Principle for ω_2 *or* $\mathrm{SSR}(\omega_2)$ *is the statement:*

$$\mathrm{SSR}(\omega_2) \equiv \textit{For every semistationary set } S \subseteq \mathcal{P}_{\omega_1}(\omega_2) \textit{ there is an uncountable ordinal } \alpha \in \omega_2 \textit{ such that } S \cap \mathcal{P}_{\omega_1}(\alpha) \textit{ is semistationary in } \mathcal{P}_{\omega_1}(\alpha).$$

It suffices to show that the set version implies the ordinal version. First we show this for the Weak Reflection Principle, then for the Semistationary Reflection Principle.

Take a surjection $\sigma_\alpha : \omega_1 \to \alpha$ for each $\alpha < \omega_2$ and let the function $f : \omega_2 \times \omega_1 \to \omega_2$ be defined by $f(\alpha, \xi) = \sigma_\alpha(\xi)$ for each $(\alpha, \xi) \in \omega_2 \times \omega_1$.

Consider an arbitrary stationary set $S \subseteq \mathcal{P}_{\omega_1}(\omega_2)$. Without loss of generality, we may assume that every element of S is closed under f. Let X be a witness for the set version of WRP(ω_2) for S, i.e., X is a subset of $\mathcal{P}_{\omega_1}(\omega_2)$ such that $|X| = \omega_1 \subseteq X$ and $S \cap \mathcal{P}_{\omega_1}(X)$ is stationary in $\mathcal{P}_{\omega_1}(X)$. Since X is closed under f for stationary many elements, X is closed under f. Therefore, $\alpha \subseteq X$ for all $\alpha \in X$. Hence, X is an uncountable ordinal in ω_2.
As above it suffices to show that the set version of SSR(ω_2) implies the ordinal version. Let $\bar{\sup}(x) = \sup\{\alpha + 1 \mid \alpha \in x\}$. Take an arbitrary semistationary set $S \subseteq \mathcal{P}_{\omega_1}(\omega_2)$. Let $X \subseteq \omega_2$ be a witness for the set version of SSR(ω_2) for S and $X' := \bar{\sup}(X)$. Then X' is an uncountable ordinal in ω_2. By Lemma 1.1.10 (2) we get that $S \cap \mathcal{P}_{\omega_1}(X')$ is semistationary in $\mathcal{P}_{\omega_1}(X')$. Therefore $S \cap \mathcal{P}_{\omega_1}(X')$ is semistationary. Hence X' witnesses the ordinal version of SSR(ω_2).

1.3 Forcing

In this part of the introduction we will give an overview of most of the definitions and facts about forcing which are needed in later chapters. More on forcing can be found in the books of Jech [5] and Kunen [10].
A *forcing poset* or *forcing notion* is a partial order $(\mathbb{P}, <)$, which is reflexive and transitive. As usual, V denotes the set theoretic universe. Furthermore, we say that a cardinal θ is *sufficiently large*, if $\theta > 2^{|\mathbb{P}|}$. A *model* M is an elementary submodel of the structure $\langle H(\theta), \in, <, ...\rangle$ where $H(\theta)$ denotes the collection of sets hereditarily of cardinality less than θ, $<$ is an unspecific well-order of $H(\theta)$, and $H(\theta)$ contains all relevant objects. In particular, the model M contains $(\mathbb{P}, <)$. From now on we will abuse notation and let $\mathbb{P}$ denote the forcing notion.

Definition 1.3.1. *We say that a forcing notion* $\mathbb{P}$ *satisfies the* κ-chain condition *or* κ-c.c. *if every antichain in* $\mathbb{P}$ *has size less than* κ*. The* ω_1*-c.c. is the* countable chain condition *or* c.c.c.

Fact 1.3.2 ([5], 15.13)**.** *Let κ be a regular cardinal. If $\mathbb{P}$ satisfies the κ-c.c., then κ remains a regular cardinal in any generic extension by $\mathbb{P}$.*

Therefore if $\mathbb{P}$ satisfies the κ-c.c., all regular cardinals $\geq \kappa$ are preserved.

Definition 1.3.3. *A forcing notion $\mathbb{P}$ is* κ-Knaster *for a regular uncountable cardinal κ, if for any sequence $\langle p_i : i < \kappa \rangle$ of conditions of $\mathbb{P}$ there exists a set $Z \subseteq \kappa$ of size κ such that for all $i < j$ in Z, the conditions p_i and p_j are compatible.*

It is easy to see that if a forcing notion is κ-Knaster, then it has the κ-c.c.

Definition 1.3.4. *A forcing notion $\mathbb{P}$ is* κ-distributive *if the intersection of less than κ-many open dense sets is dense.*

Note that κ-distributivity implies that for a generic filter G on $\mathbb{P}$ over V, if $x \subseteq V$ in $V[G]$ with $V[G] \Vdash |x| < \kappa$, then $x \in V$. In particular, no new bounded subsets of κ are added.

Definition 1.3.5. *For two forcing notions $\mathbb{P}$ and $\mathbb{Q}$ we define the* product *$\mathbb{P} \times \mathbb{Q}$ as the set of all pairs (p, q) such that $p \in \mathbb{P}$ and $q \in \mathbb{Q}$ with the following partial order:*

$$(p, q) \leq (p', q') \text{ if and only if } p \leq p' \text{ and } q \leq q'.$$

If G is a generic filter on $\mathbb{P} \times \mathbb{Q}$ let

$$G_1 = \{p \in \mathbb{P} \mid \exists q \ (p, q) \in G\}, \quad G_2 = \{q \in \mathbb{Q} \mid \exists p \ (p, q) \in G\}.$$

Then G_1 and G_2 are generic on $\mathbb{P}$ and $\mathbb{Q}$ respectively and $G = G_1 \times G_2$.

Fact 1.3.6 (**Product Lemma** ([5], 15.9))**.** *Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are two forcing notions in V. Then the following are equivalent:*

(1) $G \subseteq \mathbb{P} \times \mathbb{Q}$ is generic over V.

(2) $G = G_1 \times G_2$, $G_1 \subseteq \mathbb{P}$ *is generic over* V *and* $G_2 \subseteq \mathbb{Q}$ *is generic over* $V[G_1]$. *Furthermore,* $V[G] = V[G_1][G_2]$.

Consequently, if G_1 is generic over V and G_2 is generic over $V[G_1]$, then G_1 is generic over $V[G_2]$. Also $V[G_1][G_2] = V[G_2][G_1]$.
Now we describe a generalization of product forcing, namely iterated forcing.

Definition 1.3.7 (**Iteration of length** α)**.** *Let* $\alpha \geq 1$. *A forcing notion* $\mathbb{P}_\alpha$ *is an* iteration of length α *if it is a set of* α*-sequences which satisfies the following properties:*

(1) If $\alpha = 0$, *then there exists a forcing notion* $\mathbb{Q}_0$ *such that*

  (a) $\mathbb{P}_1$ *is the set of all 1-sequences* $\langle p(0)\rangle$ *with* $p(0) \in \mathbb{Q}_0$,

  (b) $\langle p(0)\rangle \leq_1 \langle q(0)\rangle$ *if and only if* $p(0) \leq q(0)$.

(2) If $\alpha = \beta + 1$, *then* $\mathbb{P}_\beta = \mathbb{P}_\alpha \restriction \beta = \{p \restriction \beta \mid p \in \mathbb{P}_\alpha\}$ *is an iteration of length* β *and there exists a forcing notion* $\dot{\mathbb{Q}}_\beta \in V^{\mathbb{P}}$ *such that*

  (a) $p \in \mathbb{P}_\alpha$ *if and only if* $p \restriction \beta \in \mathbb{P}_\beta$ *and* $\Vdash_\beta p(\beta) \in \dot{\mathbb{Q}}_\beta$,

  (b) $p \leq_\alpha q$ *if and only if* $p \restriction \beta \leq_\beta q \restriction \beta$ *and* $\Vdash_\beta p(\beta) \leq q(\beta)$.

(3) If α *is a limit ordinal, then for each* $\beta < \alpha$, $\mathbb{P}_\beta = \mathbb{P}_\alpha \restriction \beta$ *is an iteration of length* β *and*

  (a) the α*-sequence* $\langle 1, 1, \ldots, 1, \ldots\rangle$ *is in* $\mathbb{P}_\alpha$,

  (b) for $p \in \mathbb{P}_\alpha$ *and* $\beta < \alpha$, *if* $q \in \mathbb{P}_\beta$ *satisfies* $q \leq_{\mathbb{P}_\beta} p \restriction \beta$, *then there exists* $r \in \mathbb{P}_\alpha$ *such than for each* $\xi < \alpha$, $r(\xi) = q(\xi)$ *if* $\xi < \beta$ *and* $r(\xi) = p(\xi)$ *if* $\beta \leq \xi < \alpha$,

  (c) $p \leq_\alpha q$ *if and only if* $\forall \beta < \alpha\ p \restriction \beta \leq_\beta q \restriction \beta$.

A forcing iteration depends not only on the forcing notion $\dot{\mathbb{Q}}_\beta$ but also on the limit stages of the iteration. We distinguish between two sorts of limits.

Definition 1.3.8. *Suppose* $\mathbb{P}_\alpha$ *is an iteration of length* α *where* α *is a limit ordinal. Then* $\mathbb{P}_\alpha$ *is a* direct limit *if for every* α*-sequence* p*,*

$$p \in \mathbb{P}_\alpha \quad \textit{if and only if} \quad \exists \beta < \alpha \ p \upharpoonright \beta \in \mathbb{P}_\beta \textit{ and } \forall \xi \geq \beta \ p(\xi) = 1.$$

We call $\mathbb{P}_\alpha$ *an* inverse limit *if for every* α*-sequence* p*,*

$$p \in \mathbb{P}_\alpha \quad \textit{if and only if} \quad \forall \beta < \alpha \ p \upharpoonright \beta \in \mathbb{P}_\beta.$$

Countable support iterations will be an important tool in this thesis.

Definition 1.3.9. *The set* $\mathrm{supp}(p) = \{\beta < \alpha \mid \ \not\Vdash_\beta p(\beta) = 1\}$ *is the* support *of* $p \in \mathbb{P}_\alpha$*. Let* α *be an ordinal and* I *be the ideal on* α *consisting of all at most countable sets. A forcing iteration* $\mathbb{P}_\alpha$ *has* countable support *if for every limit ordinal* $\gamma \leq \alpha$*,*

$$p \in \mathbb{P}_\gamma \quad \textit{if and only if} \quad \forall \beta < \gamma \ p \upharpoonright \beta \in \mathbb{P}_\beta \textit{ and } \mathrm{supp}(p) \in I.$$

Theorem 1.3.10 ([5], 16.30)**.** *Suppose* κ *is a regular cardinal and* α *is a limit ordinal. Further, let* $\mathbb{P}_\alpha$ *be a forcing iteration such that for every* $\beta < \alpha$*,* $\mathbb{P}_\beta = \mathbb{P}_\alpha \upharpoonright \beta$ *satisfies the* κ*-chain condition. If* $\mathbb{P}_\alpha$ *is a direct limit and either* $\mathrm{cof}(\alpha) \neq \kappa$ *or* $\mathbb{P}_\beta$ *is a direct limit for a stationary set of ordinals* $\beta < \alpha$*, then* $\mathbb{P}_\alpha$ *satisfies the* κ*-chain condition.*

The idea of semiproperness and therefore semistationary sets was introduced by Shelah in [15]. We give the basic definitions.

Definition 1.3.11. *Suppose* $\mathbb{P}$ *is a forcing notion and* θ *a sufficiently large cardinal. For a countable elementary submodel* $N \prec \langle H(\theta), \in \rangle$*, a condition* q *is* $(N, \mathbb{P})$-semigeneric*, if for every name* $\dot\alpha \in N$ *with* $\Vdash$ *"*$\dot\alpha$ *is a countable ordinal",*

$$q \Vdash \exists \beta \in N \ \dot\alpha = \beta.$$

We call a forcing notion $\mathbb{P}$ semiproper*, if for every regular* $\theta > 2^{|\mathbb{P}|}$*, any* *countable elementary submodel $N \prec \langle H(\theta), \in \rangle$ containing $\mathbb{P}$, the following holds.*

$$\forall\ p \in \mathbb{P} \cap N\ \exists\ q \leq p\ q \textit{ is } (N, \mathbb{P}) - \textit{semigeneric}$$

Shelah showed that a forcing notion $\mathbb{P}$ is semiproper if and only if $\mathbb{P}$ preserves semistationarity. Interesting consequences of this statement can be found in [15] and [3].
Even though we will not use semigeneric conditions, we make frequent use of generic conditions and generic sequences.

Definition 1.3.12. *Suppose $\mathbb{P}$ is a forcing notion and M a countable elementary submodel of $\langle H(\theta), \in, <, ... \rangle$ for a sufficiently large θ. A condition p is $(M, \mathbb{P})$-*generic *(or M-*generic*), if for every antichain A in M, the set $A \cap M$ is predense below p.*

Definition 1.3.13. *Let $\mathbb{P}$ be a forcing notion and M a countable set. A descending sequence $\langle p_n : n < \omega \rangle$ of conditions in $\mathbb{P}$ is called an $(M, \mathbb{P})$-*generic sequence*, if $p_n \in M$ for every $n < \omega$ and for every dense open subset $D \subseteq M$ of $\mathbb{P}$ there exists $n < \omega$ with $p_n \in D$.*

Since we are often concerned with lower bounds of $(M, \mathbb{P})$-generic sequences, note that such lower bounds are $(M, \mathbb{P})$-generic conditions.

1.4 Large cardinals

We will now introduce those large cardinals, whose existence is needed for our consistency results. We start by defining the property all large cardinals satisfy, namely weak inaccessibility. The existence of weakly inaccessible cardinals is not provable in ZFC. For more information about large cardinals the reader is referred to [5] or [4].

Definition 1.4.1. *We call a cardinal $\kappa > \omega$* weakly inaccessible *if it is regular and limit.*

We call a cardinal $\kappa > \omega$ strongly inaccessible *if it is regular and for all* $\mu < \kappa$, $2^\mu < \kappa$.

We derive the consistency of our reflection principles from weakly, strongly, and supercompact cardinals. We give the definitions in terms of elementary embeddings. If j is an elementary embedding and κ is the critical point, then j is the identity map on all ordinals $\gamma < \kappa$ and $j(\kappa) > \kappa$.

Definition 1.4.2. *Assume that* $\kappa > \omega$. *Then*

(1) κ *is called* weakly compact, *if* κ *is inaccessible and for every transitive model* M *of* ZF *without the powerset axiom of size* κ *satisfying* $\kappa \in M$ *and* M *is closed under sequences of length* $< \kappa$, *there exists an elementary embedding* $j : M \to N$, *where* N *is transitive and* κ *is the critical point of* j.

(2) κ *is called* μ-strongly compact, *if there exists an elementary embedding* $j : V \to M$ *with critical point* κ, $j(\kappa) > \mu$, *and for any* $X \subseteq M$ *of size* $\leq \mu$, *there exists a* $Y \in M$ *such that* $Y \supseteq X$ *and* $(|Y| < j(\kappa))^M$.
We say that κ *is* strongly compact, *if it is* μ*-strongly compact for every ordinal* μ.

(3) κ *is called* μ-supercompact, *if there is an elementary embedding* $j : V \to M$ *with critical point* κ, $j(\kappa) > \mu$ *and* ${}^\mu M \subseteq M$, *i.e., every sequence of length* μ *of elements of* M *is in* M.
We say that κ *is* supercompact, *if it is* μ*-supercompact for every ordinal* μ.

We will make use of these elementary embeddings in the following way: First, we use the Lévy Collapse to collapse the large cardinal to ω_2. If $\mathbb{P} = \mathrm{Coll}(\omega_1, < \kappa)$, we can factorize $j(\mathbb{P}) = \mathrm{Coll}(\omega_1, < \kappa) * \mathrm{Coll}(\omega_1, [\kappa, j(\kappa))$ using standard arguments. In the extension, κ will no longer be a cardinal, but an uncountable ordinal in $j(\kappa) = \omega_2$. Therefore, if a set S is stationary in

$\mathcal{P}_{\omega_1}(\kappa)$ as it will be by assumption, then κ witnesses that there is an uncountable ordinal α in $j(\kappa) = \omega_2$ such that $j(S)$ reflects to α. By elementarity of j, there is a witness for the reflection principle in the original model.

Chapter 2

Partial Stationary Reflection

Instead of looking at stationary subsets of the space $\mathcal{P}_{\omega_1}(\omega_2)$, we are interested in the reflection points of a stationary set $S^* \subseteq \mathcal{P}_{\omega_1}(\omega_2)$. We denote this principle with $\mathrm{SR_k}(S^*)$, where $k \in \{0,1\}$ asserts that the cofinality of the reflecting set is ω_k. The first question is if the principles $\mathrm{SR_k}(S^*)$ are consistent from ZFC for $k \in \{0,1\}$. We give a positive answer which is due to Sakai [12]. Furthermore we want to know if a set S^* exists such that $\mathrm{SR_k}(S^*)$ holds. We will show that if $2^{\omega_1} = \omega_2$, then $\mathrm{SR_0}(S^*)$ does not hold for any stationary subset $S^* \subseteq \mathcal{P}_{\omega_1}(\omega_2)$ and under CH and $2^{\omega_{n-1}} = \omega_n$, $\mathrm{SR_0}(S^*)$ does not hold for $S^* \subseteq \mathcal{P}_{\omega_1}(\omega_n)$ for $n > 2$. This was proven in König-Larsen-Yoshinobu [7]. Recall that for a stationary subset $S^* \subseteq \mathcal{P}_{\omega_1}(\omega_2)$ and $k \in \{0,1\}$,we let $\mathrm{SR_k}(S^*)$ denote the following principle:
For every stationary subset $S \subseteq S^*$ there exists an uncountable ordinal α in ω_2 with cofinality ω_k such that S reflects to α.

2.1 $\mathrm{SR_k}(S^*)$ is consistent with ZFC

Sakai proved in [12] that for the consistency of partial stationary reflection no large cardinal is needed. The goal of this section is to prove the following theorem.

Theorem 2.1.1. *If* ZFC *is consistent, then so is* ZFC *with the existence of a stationary set* $S^* \subseteq \mathcal{P}_{\omega_1}(\omega_2)$ *such that* $\mathrm{SR_k}(S^*)$ *holds for* $k \in \{0, 1\}$.

The idea of proving Theorem 2.1.1 is to construct stationary sets $S_k^{\vec{C}}$ for $k \in \{0, 1\}$ from a $\square_{\omega_1}$-sequence which are maximal with respect to reflection. We define a countable support iteration of club shootings which destroys the stationarity of all non-reflecting subsets of $S_k^{\vec{C}}$. In order to preserve ω_1-distributivity and ω_2-c.c. at each stage of our iteration, we use T-complete and better forcing notions. This gives us absoluteness of stationarity in any generic extension.

Definition 2.1.2. *Let* $\mathbb{P}$ *be a forcing notion,* α *an ordinal* $\geq \omega_1$*, and* T *a subset of* $\mathcal{P}_{\omega_1}(\alpha)$*. We call* $\mathbb{P}$ *a* T-complete *forcing notion if it satisfies the following property:*
For a sufficiently large cardinal θ *and a countable elementary submodel* M *of* $\langle H(\theta), \in, \mathbb{P}, T\rangle$ *with* $M \cap \alpha \in T$*, every* $(M, \mathbb{P})$*-generic sequence has a lower bound in* $\mathbb{P}$*.*

We give one of the equivalent definitions of T-completeness via the following lemma.

Lemma 2.1.3. *Let* $\mathbb{P}$ *be a forcing notion,* α *an ordinal* $\geq \omega_1$*, and* T *be a subset of* $\mathcal{P}_{\omega_1}(\alpha)$*. Then* $\mathbb{P}$ *is* T*-complete if and only if the following holds:*
There exists a regular cardinal θ *with* $\mathbb{P},\ T \in H(\theta)$ *and an expansion* $\mathcal{M}$ *of the structure* $\langle H(\theta), \in\rangle$ *such that for every countable elementary submodel* M *of* $\mathcal{M}$ *with* $M \cap \alpha \in T$*, every* $(M, \mathbb{P})$*-generic sequence has a lower bound in* $\mathbb{P}$*.*

For the next result, we use a definition of stationary which can be found in [5]. A set $S \subseteq \mathcal{P}_{\omega_1}(H(\theta))$ is stationary, if for every model $\langle H(\theta), \in, ...\rangle$ there exists an M in S such that $M \prec \langle H(\theta), \in, ...\rangle$.

Lemma 2.1.4. *Let* α *be an ordinal* $\geq \omega_1$ *and* T *a stationary subset of* $\mathcal{P}_{\omega_1}(\alpha)$*. Then every* T*-complete forcing notion is* ω_1*-distributive.*

Proof. Let $\mathbb{P}$ be a T-complete forcing notion and $\{D_n \mid n < \omega\}$ a family of dense open subsets of $\mathbb{P}$. For an arbitrary $p \in \mathbb{P}$, we must find a condition $p' \leq p$ which is in $\bigcap_{n\in\omega} D_n$.
Let θ be a sufficiently large regular cardinal. By the stationarity of T, there exists a countable elementary submodel M of $\langle H(\theta), \in, \mathbb{P}, T\rangle$ such that $\{p\} \cup \{D_n \mid n < \omega\} \subseteq M$ and $M \cap \alpha \in T$. Take an $(M, \mathbb{P})$-generic sequence $\langle p_n : n < \omega\rangle$ with $p_0 = p$. Since $\mathbb{P}$ is T-complete, there exists a lower bound p' of $\langle p_n : n < \omega\rangle$. Clearly, $p' \leq p$ and $p' \in \bigcap_{n\in\omega} D_n$. □

T-completeness is preserved by countable support iterations.

Lemma 2.1.5. *Let α be an ordinal and T be a subset of $\mathcal{P}_{\omega_1}(\alpha)$. If $I = \langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi, \eta \leq \zeta\rangle$ is a countable support iteration of T-complete forcing notions for some ordinal ζ, then $\mathbb{P}_\zeta$ is T-complete.*

Proof. Let θ be a sufficiently large regular cardinal, M be a countable elementary submodel of $\langle H(\theta), \in, I, T\rangle$, and $\langle p_n : n < \omega\rangle$ an $(M, \mathbb{P}_\zeta)$-generic sequence. By Lemma 2.1.3 it suffices to show that $\langle p_n : n < \omega\rangle$ has a lower bound in $\mathbb{P}$.
Claim. For $\eta \in \zeta \cap M$ the sequence $\langle p_n \restriction \eta : n < \omega\rangle$ is $(M, \mathbb{P}_\eta)$-generic. Furthermore, if p' is a lower bound for $\langle p_n \restriction \eta : n < \omega\rangle$, then p' forces that $\langle p_n(\eta) : n < \omega\rangle$ is an $(M[\dot{G}_\eta], \dot{\mathbb{Q}}_\eta)$-generic sequence, where $\dot{G}_\eta$ is the canonical name for a $\mathbb{P}_\eta$-generic filter.
Proof of Claim. It is easy to see that $\langle p_n \restriction \eta : n < \omega\rangle$ is a descending sequence in $\mathbb{P}_\eta \cap M$. Take an arbitrary dense open set $D \subseteq M$ such that $D \in \mathbb{P}_\eta$. We have to show that there is an $n < \omega$ with $p_n \restriction \eta \in D$. Let $D^* := \{p \in \mathbb{P}_\zeta \mid p \restriction \eta \in D\}$. Note that D^* is a dense open set in $\mathbb{P}_\eta$ which belongs to M. By the $(M, \mathbb{P}_\zeta)$-genericity of the sequence $\langle p_n : n < \omega\rangle$, there is an $n < \omega$ with $p_n \in D^*$. Then $p_n \restriction \eta \in D$ for this n.
For the second part of the claim, it suffices to show the genericity of the sequence $\langle p_n(\eta) : n < \omega\rangle$. Let $\dot{D} \in M$ be a $\mathbb{P}_\eta$-name of a dense open subset of $\dot{Q}_\eta$. We must find $n < \omega$ with $p' \Vdash_\eta$ "$p_n(\eta) \in \dot{D}$".

Let

$$D^{**} := \{p \in \mathbb{P}_\zeta \mid p \restriction \eta \Vdash_\eta \text{"} p(\eta) \in \dot{D}\text{"}\}.$$

Clearly D^{**} is a dense open set in $\mathbb{P}_\eta$ and belongs to M. Therefore, there exists $n < \omega$ with $p_n \in D^{**}$. Then $p' \Vdash_\eta$ "$p_n(\eta) \in \dot{D}$" and $p' \leq p_n \restriction \eta$. This proves the claim.

Now we can construct a lower bound p' of the sequence $\langle p_n : n < \omega\rangle$. We define p' such that it is a function whose domain is ζ and $p'(\eta)$ is a $\mathbb{P}_\eta$-name for a condition of $\dot{\mathbb{Q}}_\eta$ for each $\eta < \zeta$. We choose $p'(\eta)$ by induction on $\eta < \zeta$:

(1) $p' \restriction \eta \Vdash_\eta$ "$p'(\eta)$ is a lower bound of $\langle p_n(\eta) : n < \omega\rangle$".

(2) $p'(\eta) = \dot{1}_\eta$ for every $\eta < \zeta \setminus M$.

We make sure that $\operatorname{supp}(p')$ is countable via (2), since M is countable. First note that if $\eta \leq \zeta$ and $p'(\eta)$ satisfies the induction hypotheses for each $\eta' < \eta$, then $p' \restriction \eta = \langle p'(\eta') : \eta' < \eta\rangle$ is a lower bound of $\langle p_n \restriction \eta : n < \omega\rangle$. Secondly, $p' \restriction \eta$ is an $(M, \mathbb{P}_\eta)$-generic condition by the claim.

Now we construct $p'(\eta)$. Suppose that for $\eta < \zeta$, $p' \restriction \eta$ has already been constructed. If $\eta \notin M$, let $p'(\eta) = \dot{1}_\eta$. Since $\operatorname{supp}(p_n)$ is a countable set belonging to M for each $n < \omega$ and $M \prec \langle H(\theta), \in\rangle$, we have that $\operatorname{supp}(p_n) \subseteq M$. Therefore $p_n(\eta) = \dot{1}_\eta$ for each $\eta < \omega$ and hence $p'(\eta)$ satisfies the first induction hypothesis.

For $\eta \in M$, let $\dot{G}_\eta$ be the canonical name for a $\mathbb{P}_\eta$- generic filter. By the claim, we have

$$p' \restriction \eta \Vdash_\eta \text{"}\langle p_n(\eta) : n < \omega\rangle \text{ is an } (M[\dot{G}_\eta], \dot{\mathbb{Q}}_\eta)\text{-generic sequence"}.$$

Furthermore,

$$p' \restriction \eta \Vdash_\eta \text{"} M[\dot{G}_\eta] \prec \langle H(\theta)^{V[\dot{G}_\eta]}, \in, \dot{\mathbb{Q}}_\eta, T\rangle \text{ and } M[\dot{G}] \cap \lambda = M \cap \lambda \in T\text{"}$$

by the $(M, \mathbb{P}_\eta)$-genericity of $p' \restriction \eta$. It is easy to see that the induction hypotheses are satisfied. Since we can now construct a lower bound p' of the sequence $\langle p_n : n < \omega \rangle$, the proof is complete. □

Definition 2.1.6. *Let $\mathbb{P}$ be a forcing notion. We say that $\mathbb{P}$ is* good, *if:*

(1) A condition $p \in \mathbb{P}$ is a function such that $|p| = \omega$ and $\operatorname{ran}(p) \subseteq \omega_1$.

(2) For $p, q \in \mathbb{P}$, $p \leq q$ if and only if $p \supseteq q$

(3) For all $p, q \in \mathbb{P}$ if $p \restriction (\operatorname{dom}(p) \cap \operatorname{dom}(q)) = q \restriction (\operatorname{dom}(p) \cap \operatorname{dom}(q))$ then p and q are compatible.

We say that $\mathbb{P}$ is better *if $\mathbb{P}$ also satisfies the following property:*

(4) If $\langle p_n : n < \omega \rangle$ is a descending sequence in $\mathbb{P}$ with a lower bound, then $\bigcup_{n \in \omega} p_n \in \mathbb{P}$.

The following lemma can be shown by using the Δ-System Lemma. We skip its proof.

Lemma 2.1.7. *Every good forcing notion has the $(2^\omega)^+$-c.c.*

If we assume CH, then for a stationary set T, countable support iterations of T-complete better forcing notions satisfy the ω_2-c.c.

Lemma 2.1.8. *Let α be an ordinal and T be a subset of $\mathcal{P}_{\omega_1}(\alpha)$. If for some ordinal ζ, $I = \langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi, \eta \leq \zeta \rangle$ is a countable support iteration of T-complete better forcing notions, then $\mathbb{P}_\zeta$ has the $(2^\omega)^+$-c.c.*

Proof. We show that the forcing notion $\mathbb{P}_\zeta$ restricted to a certain dense set D is good. Then $\mathbb{P}_\zeta$ has the $(2^\omega)^+$-c.c. by Lemma 2.1.7. Let

$$D := \{p \in \mathbb{P}_\zeta \mid \forall \eta < \zeta \ \exists q \in V \ p(\eta) = \check{q}\}.$$

We show that D is dense in $\mathbb{P}_\zeta$, i.e., for an arbitrary $p \in \mathbb{P}_\zeta$ we find a $p' \in D$ which is below p. Let θ be a sufficiently large regular cardinal and M a countable elementary submodel of $\langle H(\theta), \in, I, T\rangle$ with $p \in M$. Such an M exists because T is stationary. Take an $(M, \mathbb{P}_\zeta)$-generic sequence $\langle p_n : n < \omega\rangle$ such that $p_0 \leq p$. The required p' will be a lower bound of $\langle p_n : n < \omega\rangle$ constructed the same way as in the proof of Lemma 2.1.5.

By induction on $\eta < \zeta$, we pick a $\mathbb{P}_\eta$-name $p'(\eta)$ for a condition in $\dot{\mathbb{Q}}_\eta$. The induction hypotheses remain the same as in the proof of 2.1.5. So assume that $\eta < \zeta$ and that $\mathbb{P}' \restriction \eta$ has already been constructed. The definition of $p'(\eta)$ splits into two cases. The first case is that $\eta \notin M$. Then let $\mathbb{P}'(\eta) = \dot{1}_\eta$. Since we can assume that $\dot{1}_\eta = \check{\emptyset}$ for all $\eta < \zeta$, $p'(\eta) = \check{\emptyset}$. For the second case $\eta \in M$ we claim the following.

Claim. For all $n < \omega$ there exists $q_n \in V$ with $p' \restriction \eta \Vdash_\eta$ "$p_n(\eta) = \check{q}_n$".

Proof of Claim. Fix $n < \omega$. Now look at the set

$$A := \{p \in \mathbb{P}_\eta \mid \exists q \in V\ p \Vdash_\eta \text{"}p_n(\eta) = \check{q}\text{"}\}.$$

Since $\mathbb{P}_\eta$ is ω_1-distributive by Lemma 2.1.4 and 2.1.5, the set A is a dense open subset of $\mathbb{P}_\eta$ and $A \in M$. Using the claim in the proof of Lemma 2.1.5, the sequence $\langle p_m \restriction \eta : m < \omega\rangle$ is $(M, \mathbb{P}_\eta)$-generic. Therefore there exists $m < \omega$ with $p_m \in A$. But then $p' \restriction \eta \in A$ because p' is a lower bound of $\langle p_m : m < \omega\rangle$. Hence there exists $q_n \in V$ such that $p' \restriction \eta \Vdash_\eta$ "$p_n(\eta) = \check{q}_n$", which proves the claim.

Now we can construct $p'(\eta)$. For each $n < \omega$ let q_n be as in the claim above and $q' := \bigcup_{n<\omega} q_n$. Repeating the argument from the proof of Lemma 2.1.5, one can show that $p' \restriction \eta$ forces that the sequence $\langle p_n(\eta) : n < \omega\rangle$ has a lower bound in $\dot{\mathbb{Q}}_\eta$. Since $\mathbb{Q}_\eta$ is better, $p' \restriction \eta$ forces that q' is a lower bound of $\langle p_n(\eta) : n < \omega\rangle$. Now let $p'(\eta) = \check{q}'$. By construction, $p' \leq p$ and $p \in D$. Hence D is dense in $\mathbb{P}_\eta$.

We define the forcing notion $\mathbb{P}^D_\zeta$ which is $\mathbb{P}_\zeta$ restricted to D (or at least isomorphic to this forcing notion). First, we observe that by ω_1-distributivity of $\mathbb{P}_\eta$, $p(\eta)$ is a countable function from the ordinals to ω_1 for each $p \in D$. A condition $p^* \in \mathbb{P}^D_\zeta$ for $p \in D$ is a partial function from $\zeta \times \mathrm{ON}$ to ω_1 whose

domain is equal to $\{(\eta,\alpha) \mid \alpha \in \mathrm{dom}(p(\eta))\}$. Define $p^*(\eta,\alpha) = p(\eta)(\alpha)$ for all $(\eta,\alpha) \in \mathrm{dom}(p^*)$.
Then $\mathbb{P}^D_\zeta = \{p^* \mid p \in D\}$ and for $p^*, q^* \in \mathbb{P}^D_\zeta$, we let $p^* \leq q^*$, if p^* extends q^* as a function. By construction, $\mathbb{P}^D_\zeta$ is good. Then $\mathbb{P}^D_\zeta$ has the $(2^\omega)^+$-c.c. by Lemma 2.1.7. Since D is dense in $\mathbb{P}_\zeta$, the forcing notion $\mathbb{P}_\zeta$ also has the $(2^\omega)^+$-c.c. and the proof of this lemma is complete. □

For the proof of Theorem 2.1.1 we need certain subsets $S^{\vec{C}}_0$ and $S^{\vec{C}}_1$ of $\mathcal{P}_{\omega_1}(\omega_2)$, where $\vec{C} = \langle C_\alpha : \alpha \in \mathrm{Lim}(\omega_2)\rangle$ denotes a $\square_{\omega_1}$-sequence. We show that those sets are maximal with respect to reflection. As usual, when x is a set, then $\mathrm{Lim}(x)$ denotes the set of limit points of x. We will prove that every subset of $\mathcal{P}_{\omega_1}(\omega_2) \setminus S^{\vec{C}}_0$ does not reflect to any uncountable ordinal in $E^{\omega_2}_\omega$ and every subset of $\mathcal{P}_{\omega_1}(\omega_2) \setminus S^{\vec{C}}_1$ does not reflect to any ordinal in $E^{\omega_2}_{\omega_1}$.

Definition 2.1.9. *For an uncountable cardinal κ and a set $E \subseteq \mathrm{Lim}(\kappa^+)$, let $\square^E_\kappa$ denote the following principle:*
$\square^E_\kappa \equiv$ *There exists a sequence $\langle C_\alpha : \alpha \in E\rangle$ such that for every $\alpha, \alpha' \in E$*

(1) C_α is a club subset of α,

(2) if $\mathrm{cof}(\alpha) < \kappa$ then $\mathrm{ot}(C_\alpha) < \kappa$, and

(3) if $\alpha' \in \mathrm{Lim}(C_\alpha)$ then $C_{\alpha'} = C_\alpha \cap \alpha'$.

We call a sequence $\langle C_\alpha : \alpha \in E\rangle$ satisfying the properties (1)-(3) a $\square^E_\kappa$-sequence. In the case where $E = \mathrm{Lim}(\kappa^+)$ we omit the superscript and write $\square_\kappa$.

Definition 2.1.10. *For a $\square_{\omega_1}$-sequence $\vec{C} = \langle C_\alpha : \alpha \in \mathrm{Lim}(\omega_2)\rangle$ let*
$S^{\vec{C}}_0 :=$ *the set of all $x \in \mathcal{P}_{\omega_1}(\omega_2)$ such that*

(1) $x \cap \omega_1 \in \omega_1$ and $\sup(x) \notin x$,

(2) $\mathrm{ot}(C_{\sup(x)}) < x \cap \omega_1$,

(3) $C_{\sup(x)} \subseteq x$.

$S_1^{\vec{C}}$:= *the set of all* $x \in \mathcal{P}_{\omega_1}(\omega_2)$ *such that*

(1) $x \cap \omega_1 \in \omega_1$ *and* $\sup(x) \notin x$,

(2) $\mathrm{ot}(C_{\sup(x)}) = x \cap \omega_1$,

(3) $C_{\sup(x)} \subseteq x$.

Lemma 2.1.11. *Let* $\vec{C} = \langle C_\alpha : \alpha \in \mathrm{Lim}(\omega_2)\rangle$ *be a* $\square_{\omega_1}$*-sequence. Then the following holds:*

(1) $S_0^{\vec{C}} \cap \mathcal{P}_{\omega_1}(\alpha)$ *contains a club in* $\mathcal{P}_{\omega_1}(\alpha)$ *for every* $\alpha \in E^{\omega_2}_{\omega} \setminus \omega_1$.

(2) $S_1^{\vec{C}} \cap \mathcal{P}_{\omega_1}(\alpha)$ *contains a club in* $\mathcal{P}_{\omega_1}(\alpha)$ *for every* $\alpha \in E^{\omega_2}_{\omega_1}$.

In particular, both $S_0^{\vec{C}}$ *and* $S_1^{\vec{C}}$ *are stationary in* $\mathcal{P}_{\omega_1}(\omega_2)$.

Proof. (1) Fix $\alpha \in E^{\omega_2}_{\omega} \setminus \omega_1$. Since $\mathrm{cof}(\alpha) = \omega$, the order type of C_α is countable by property (2) of the definition of a $\square_{\omega_1}$-sequence $\vec{C}$. Let C be the set of all $x \in \mathcal{P}_{\omega_1}(\alpha)$ such that $C_\alpha \subseteq x$ and $\mathrm{ot}(C_\alpha) < x \cap \omega_1 \in \omega_1$. Then C is club in $\mathcal{P}_{\omega_1}(\alpha)$. Since $\sup(x) \leq \alpha$, we have $C \subseteq S_0^{\vec{C}}$.
(2) Fix $\alpha \in E^{\omega_2}_{\omega_1}$ and an enumeration $\langle d_i : i < \omega_1\rangle$ of the sets in C_α. We define C as the set of all $x \in \mathcal{P}_{\omega_1}(\alpha)$ such that $x \cap \omega_1$ is a countable limit ordinal, $\sup(x) = d_{x\cap\omega_1} \notin x$ and $\{d_i \mid i \in x \cap \omega_1\} \subseteq x$. Then C is club in $\mathcal{P}_{\omega_1}(\alpha)$. Now we need to show that $C \subseteq S_1^{\vec{C}}$. For every $x \in C$ property (1) of the definition of $S_1^{\vec{C}}$ is satisfied by definition of C. For (2) and (3) note that $C_{\sup(x)} = C_{d_{x\cap\omega_1}} = \{d_i \mid i \in x \cap \omega_1\}$ by the coherency of $\vec{C}$. Then $C_{\sup(x)} \subseteq x$ by definition of C and $\mathrm{ot}(C_{\sup(x)}) = x \cap \omega_1$. Hence $C \subseteq S_1^{\vec{C}}$. □

Next we define the club shooting $\mathbb{Q}(S)$, which is a forcing notion designed to destroy the stationarity of a given set S by adding a generic function under which S is not closed. In our iteration we use $\mathbb{Q}(S)$ for each non-reflecting subset S of S^*. By the previous section we need to show that $\mathbb{Q}(S)$ is T-complete and better.

Definition 2.1.12. *Let S be a subset of $\mathcal{P}_{\omega_1}(\omega_2)$. Define $\mathbb{Q}(S)$ as the forcing poset consisting of conditions p satisfying:*

(1) p is a function of the form $p : a^p \times a^p \to \omega_1$, where a^p is a countable subset of ω_2,

(2) for every x in S, if $x \subseteq a^p$, then x is not closed under p.

Let $q \leq p$, if q extends p as a function, that is, if $a^p \subseteq a^q$ and $q \restriction (a^p \times a^p) = p$. For a countable set $a \subseteq \omega_2$, we write a^2 for $a \times a$.

We summarize the properties of $\mathbb{Q}(S)$.

Lemma 2.1.13. *Let S be a subset of $\mathcal{P}_{\omega_1}(\omega_2)$.*

(1) For every $x \in \mathcal{P}_{\omega_1}(\omega_2)$, the set $\{p \in \mathbb{Q}(S) \mid x \subseteq a^p\}$ is dense in $\mathbb{Q}(S)$.

(2) Let G be a generic filter on $\mathbb{Q}(S)$ over V. Then $\bigcup G$ is a total function, $\bigcup G : \omega_2^V \times \omega_2^V \to \omega_2^V$. Furthermore, there are no sets $y \in S$ which are closed under $\bigcup G$.

(3) $\mathbb{Q}(S)$ is better.

Proof. (1) For a set $y \in \mathcal{P}_{\omega_1}(\omega_2)$ and a condition $p \in \mathbb{Q}(S)$ we must find $p' \leq p$ with $y \subseteq a^{p'}$. We define p' as a function from $a^{p'} \times a^{p'}$ to ω_1, where $a^{p'} = a^p \cup y$. Take $\xi \in \omega_1 \setminus a^{p'}$ and let

$$p'(b) = \begin{cases} p(b) & \text{if } b \in a^p \times a^p \\ \xi & \text{otherwise} \end{cases}$$

Now we have to show that if $x \in S$ and $x \subseteq a^{p'}$, then x is not closed under p'. Let x be such a set. In the first case $x \subseteq a^p$. Since $p \in \mathbb{Q}(S)$, the set x is not closed under p by the definition of $\mathbb{Q}(S)$. Thus x is not closed under p' which extends p. In the second case $x \not\subseteq a^p$. Then there exists a $b \in (x \times x) \setminus (a^p \times a^p)$ such that $p'(b) = \xi \notin a^{p'}$. The fact that $x \subseteq a^{p'}$ implies

$p'(b) \notin x$. Hence x is not closed under p'.
(2) Clear from (1).
(3) It is easy to see that $\mathbb{Q}(S)$ satisfies conditions (1) and (2) from the definition of better. It remains to check (3) and (4). To prove property (3), take conditions $p, q \in \mathbb{Q}(S)$ and assume that

$$p \restriction (\mathrm{dom}(p) \cap \mathrm{dom}(q)) = q \restriction (\mathrm{dom}(p) \cap \mathrm{dom}(q)).$$

In order to show compatibility we must find a common extension r of p and q. Let $a^r = (a^p \cup a^q)$ and r be the function from $a^r \times a^r$ to ω_1 defined as follows: Fix $\xi \in \omega_1 \setminus (a^r)$ and let

$$r(b) = \begin{cases} p(b) & \text{if } b \in a^p \times a^p \\ q(b) & \text{if } b \in a^q \times a^q \\ \xi & \text{otherwise.} \end{cases}$$

Then r is well-defined because p and q coincide on $\mathrm{dom}(p) \cap \mathrm{dom}(q)$. Finally, we must show that if $x \in S$ and $x \subseteq a^r$, then x is not closed under r. Assume that $x \in S$ and $x \subseteq a^r$. If $x \subseteq a^p$ or $x \subseteq a^q$, then we can repeat the argument from the proof of (1).
For the final case assume $x \nsubseteq a^p$ and $x \nsubseteq a^q$. Take $\alpha \in x \setminus a^p$ and $\beta \in x \setminus a^q$ and let $b := (\alpha, \beta)$. Then $b \in x \times x$ but $b \notin a^p \times a^p$ and $b \notin a^q \times a^q$. Therefore, $r(b) = \xi \notin x$ and x is not closed under r.
To prove property (4) of the definition of better, we take a descending sequence $\langle p_n : n < \omega \rangle$ in $\mathbb{Q}(S)$ with a lower bound p'. Then $\bigcup\limits_{n<\omega} p_n$ is a restriction of p' to $(\bigcup\limits_{n<\omega} a^{p_n}) \times (\bigcup\limits_{n<\omega} a^{p_n})$. Hence $\bigcup\limits_{n<\omega} p_n$ is clearly in $\mathbb{Q}(S)$. $\square$

Iterated club shootings are T-complete for certain stationary subsets T of $\mathcal{P}_{\omega_1}(\omega_2)$. The following statement will be a sufficient condition for $\mathbb{Q}(S)$ to be T-complete.

Definition 2.1.14. *For sets $S, T \subseteq \mathcal{P}_{\omega_1}(\omega_2)$, let $\Phi(S, T)$ denote the following*

principle:
There exists a regular cardinal $\theta > 2^{\omega_2}$ and an expansion $\mathcal{M}$ of the structure $\langle H(\theta), \in \rangle$ such that for every countable elementary substructure M of $\mathcal{M}$ with $M \cap \omega_2 \in T$, we have $S \cap \mathcal{P}(M) \subseteq M$.

Lemma 2.1.15. *Let $S, T \subseteq \mathcal{P}_{\omega_1}(\omega_2)$ and assume that $\Phi(S, T)$ holds. Then $\mathbb{Q}(S)$ is T-complete.*

Proof. Let θ and $\mathcal{M}$ be witnesses for $\Phi(S, T)$, M a countable elementary submodel of $\mathcal{M}$ with $M \cap \omega_2 \in T$, and $\langle p_n : n < \omega \rangle$ an $(M, \mathbb{Q}(S))$-generic sequence. By Lemma 2.1.3 and the definition of being better, it suffices to show that $p' = \bigcup_{n<\omega} p_n$ is a condition in $\mathbb{Q}(S)$. Let $a^{p'} = \bigcup_{n<\omega} a^{p_n}$. Then $a^{p'} \in \mathcal{P}_{\omega_1}(\omega_2)$ and p' is a function from $a^{p'} \times a^{p'}$ to ω_1. We need to show that if $x \in S$ and $x \subseteq a^{p'}$, then x is not closed under p'. Assume that $x \in S$ and $x \subseteq a^{p'}$. Since each a^{p_n} is a countable set belonging to $M \prec \langle H(\theta), \in \rangle$, we have $a^{p_n} \subseteq M$. Therefore, $a^{p'} \subseteq M$ and hence $x \subseteq M$. Then $\Phi(S, T)$ implies that $x \in M$ and the set $D := \{p \in \mathbb{Q}(S) \mid x \subseteq a^p\}$ belongs to M. Also D is dense open in $\mathbb{Q}(S)$ by Lemma 2.1.13 (1). By $(M, \mathbb{Q}(S))$-genericity there exists $n < \omega$ with $p_n \in D$. Since $x \subseteq a^{p_n}$ by the definition of D and $p_n \in \mathbb{Q}(S)$, x is not closed under p_n. Hence x is not closed under p' which extends p_n. □

The next question is for which stationary set $T \subseteq \mathcal{P}_{\omega_1}(\omega_2)$ the iteration of the club shootings will be T-complete. We will present such a set and prove its stationarity by using the following lemma due to Shelah. As usual, let $E^{\omega_2}_{\omega} = \{\alpha \in \omega_2 \mid \mathrm{cof}(\alpha) = \omega\}$.

Lemma 2.1.16. *If $\langle S_i : i < \omega_1 \rangle$ is a sequence of stationary subsets of $E^{\omega_2}_{\omega}$, then the set*

$$T = \{x \in \mathcal{P}_{\omega_1}(\omega_2) \mid x \cap \omega_1 \in \omega_1 \ \wedge \ \sup(x) \notin x \ \wedge \ \sup(x) \in S_{x \cap \omega_1}\}$$

is stationary in $\mathcal{P}_{\omega_1}(\omega_2)$.

Definition 2.1.17. *Let $\vec{C} = \langle C_\alpha : \alpha \in \mathrm{Lim}(\omega_2)\rangle$ be a $\square_{\omega_1}$-sequence. We define*
$T^{\vec{C}}$ to be the set of all $x \in \mathcal{P}_{\omega_1}(\omega_2)$ such that

(1) $x \cap \omega_1 \in \omega_1$ and $\sup(x) \notin x$,

(2) $\mathrm{ot}(C_{\sup(x)}) > x \cap \omega_1$.

Lemma 2.1.18. *Let $\vec{C}$ be a $\square_{\omega_1}$-sequence. Then $T^{\vec{C}}$ is stationary.*

Proof. Let $\vec{C} = \langle C_\alpha : \alpha \in \mathrm{Lim}(\omega_2)\rangle$ be a $\square_{\omega_1}$-sequence. For every $i \in \omega_1$ let $S_i := \{\alpha \in E^{\omega_2}_\omega \mid \mathrm{ot}(C_\alpha) > i\}$. Since $S_i \cap \beta$ contains a club subset in β for every $\beta \in E^{\omega_2}_{\omega_1}$, the set S_i is a stationary subset of $E^{\omega_2}_\omega$.
Note that the set $T^{\vec{C}} := \{x \in \mathcal{P}_{\omega_1}(\omega_2) \mid x \cap \omega_1 \wedge \sup(x) \notin x \wedge \sup(x) \in S_{x\cap\omega_1}\}$ is also stationary by Lemma 2.1.16. □

Next we need to refine the sets $S^{\vec{C}}_k$ and $T^{\vec{C}}$ in the following way.

Definition 2.1.19. *Fix a surjection $\sigma_\alpha : \omega_1 \to \alpha$ for each $\alpha \in \omega_2$. We call a sequence $\vec{\sigma} = \langle \sigma_\alpha : \alpha \in \omega_2\rangle$ a* surjection system. *For such a surjection system $\vec{\sigma}$, a $\square_{\omega_1}$-sequence $\vec{C}$, and $k \in \{0, 1\}$ we define*

$$S^{\vec{C},\vec{\sigma}}_k := \{x \in S^{\vec{C}}_k \mid \forall \alpha \in x\ x \cap \alpha = \sigma_\alpha``(x \cap \omega_1)\}$$

Lemma 2.1.20. *Let $\vec{C} = \langle C_\alpha : \alpha \in \mathrm{Lim}(\omega_2)\rangle$ be a $\square_{\omega_1}$-sequence, θ a sufficiently large regular cardinal, and M a countable elementary submodel of $\langle H(\theta), \in, \vec{C}\rangle$. Furthermore let α' be an ordinal in $E^{\omega_2}_\omega$ with $\alpha' < \sup(M \cap \omega_2)$, $\alpha' \notin M$, and $\sup(M \cap \alpha') = \alpha'$. Then $\mathrm{ot}(C_{\alpha'}) = M \cap \omega_1$.*

Proof. Define $\beta' = \min(M \setminus \alpha)$. Then $\beta' \in M \cap \omega_2$ and $\sup(M \cap \beta') = \alpha' < \beta'$. Furthermore, $\beta' \in E^{\omega_2}_{\omega_1}$ by elementarity of M. We enumerate $C_{\beta'}$ increasingly. Let $\langle \beta_i : i < \omega_1\rangle$ denote this enumeration. Now we prove $\sup(M \cap \beta') = \beta_{M\cap\omega_1}$. Since $C_{\beta'}$ is in M by elementarity of M, the set $\{\beta_i \mid i \in M \cap \omega_1\}$ is a subset of M. Hence,

$$\sup(M \cap \beta') \geq \sup\{\beta_i \mid i \in M \cap \omega_1\} = \beta_{M\cap\omega_1}.$$

So suppose $\sup(M \cap \beta') > \beta_{M\cap\omega_1}$. Then there exists $\beta \in M \cap \beta'$ such that $\beta \geq \beta_{M\cap\omega_1}$. Let γ denote the least ordinal less than ω_1 with $\beta_\gamma \geq \beta$. Thus $\gamma \geq M \cap \omega_1$ since $\beta \geq \beta_{M\cap\omega_1}$. However, $\gamma \in M \cap \omega_1$ by the elementarity of M, which is a contradiction. Hence, $\sup(M \cap \beta') \leq \beta_{M\cap\omega_1}$.
Therefore $\sup(M \cap \beta') = \beta_{M\cap\omega_1}$. By definition of α, we obtain $\alpha' = \beta_{M\cap\omega_1}$. Then the coherency of $\vec{C}$ implies $C_{\alpha'} = \{\beta_i \mid i \in < \cap\omega_1\}$. Thus

$$\mathrm{ot}(C_{\alpha'}) = M \cap \omega_1.$$

□

Lemma 2.1.21. *Let* $\vec{C} = \langle C_\alpha : \alpha \in \mathrm{Lim}(\omega_2)\rangle$ *be a* $\square_{\omega_1}$*-sequence and* $\vec{\sigma} = \langle \sigma_\alpha : \alpha \in \omega_2 \setminus \omega_1\rangle$ *a surjection system.*

(1) If S *is a subset of* $S_0^{\vec{C},\vec{\sigma}}$ *which does not reflect to any ordinal in* $E^{\omega_2}_{\omega} \setminus \omega_1$*, then* $\mathbb{Q}(S)$ *is* $T^{\vec{C}}$*-complete.*

(2) If S *is a subset of* $S_1^{\vec{C},\vec{\sigma}}$ *which does not reflect to any ordinal in* $E^{\omega_2}_{\omega_1}$*, then* $\mathbb{Q}(S)$ *is* $T^{\vec{C}}$*-complete.*

Proof. (1) It suffices to show that $\Phi(S,T)$ holds by Lemma 2.1.15. So let θ be a sufficiently large regular cardinal and M a countable elementary submodel of $\langle H(\theta), \in, S, \vec{C}, \vec{\sigma}\rangle$ with $M \cap \omega_2 \in T$. We show that $S \cap \mathcal{P}(M) \subseteq M$, i.e., if we take $x \in S$ and $x \subseteq M$ we show that $x \in M$. First note that $x \cap \omega_1 \leq M \cap \omega_1 \in \omega_1$.
Claim 1. $\sup(x) \in M$.
Proof of Claim 1. Assume that $x \notin M$. We show that $M \cap \omega_1 \leq \mathrm{ot}(C_{\sup(x)})$. In the first possible case $\sup(x) = \sup(M \cap \omega_1)$. Then $M \cap \omega_1 < \mathrm{ot}(C_{\sup(x)})$ by the second part of the definition of $T^{\vec{C}}$ since $M \cap \omega_2 \in T^{\vec{C}}$. In the second case $\sup(x) < \sup(M \cap \omega_2)$. Then $M \cap \omega_2 = \mathrm{ot}(C_{\sup(x)})$ by the previous lemma. Also $x \cap \omega_1 > \mathrm{ot}(C_{\sup(x)})$ since $x \in S_0^{\vec{C},\vec{\sigma}}$. Therefore $M \cap \omega_1 \leq \mathrm{ot}(C_{\sup(x)}) < x \cap \omega_1$ which is a contradiction to $x \subseteq M$. This proves Claim 1.
Claim 2. $x \cap \omega_1 < M \cap \omega_1$.

Proof of Claim 2: Assume that $x \cap \omega_1 = M \cap \omega_1$. At this point we make use of the surjection system $\vec{\sigma}$. For each $\alpha \in M \cap \omega_2$, $M \cap \alpha = \sigma_\alpha"(M \cap \omega_1)$ by elementarity of M. Thus

$$M \cap \sup(x) = \bigcup_{\alpha \in x} \sigma_\alpha"(M \cap \omega_1) = \bigcup_{\alpha \in x} \sigma_\alpha"(x \cap \omega_1) = x.$$

The last equality follows from $x \in S_0^{\vec{C},\vec{\sigma}}$. Since $S \cap \mathcal{P}_{\omega_1}(\sup(x))$ is non-stationary by assumption and $\sup(x) \in M$ by Claim 1, there exists a function $f : \mathcal{P}_\omega(\sup(x)) \to \sup(x)$ in M such that every set in $S \cap \mathcal{P}_{\omega_1}(\sup(x))$ is not closed under f. But $x = M \cap \sup(x)$ and therefore x is closed under f by elementarity of M. Since $x \in S \cap \mathcal{P}_{\omega_1}(\sup(x))$ this is a contradiction and proves Claim 2.

Now $x = \bigcup\{\sigma_\alpha"(x \cap \omega_1) \mid \alpha \in C_{\sup(x)}\}$ because $x \in S_0^{\vec{C},\vec{\sigma}}$. Therefore x is definable in $\langle H(\theta), \in, \vec{C}, \vec{\sigma}\rangle$ with parameters $x \cap \omega_1$ and $\sup(x)$. But both of these parameters belong to M by Claim 1 and 2 and $M \prec \langle H(\theta), \in, \vec{C}, \vec{\sigma}\rangle$. Hence $x \in M$.

(2) Again we show that $\Phi(S, T)$ holds. So let θ, M and x be as in the proof of (1). We show that $x \in S$.

Claim 1'. $\sup(x) \in M$.

Proof of Claim 1'. Assume that $\sup(x) \notin M$. We show that $\sup(x) < \sup(M \cap \omega_2)$. If not, $\sup(x) = \sup(M \cap \omega_2)$ and $M \cap \omega_1 < \text{ot}(C_{\sup(x)})$ by the second part of the definition of $T^{\vec{C}}$ because $M \cap \omega_2 \in T$ and $x \in S_1^{\vec{C},\vec{\sigma}}$. This contradicts $x \subseteq M$.

Then $M \cap \omega_1 = \text{ot}(C_{\sup(x)})$ by Lemma 2.1.20. Then $M \cap \omega_1 = x \cap \omega_1$ because $x \in S_1^{\vec{C},\vec{\sigma}}$. Just like in the proof of Claim 2 one can show that $M \cap \sup(x) = x$. Let $\alpha = min(M \setminus \sup(x))$. Then $\alpha \in E^{\omega_2}_{\omega_1}$ and $S \cap \mathcal{P}_{\omega_1}(\alpha)$ is non-stationary by assumption. Since $\alpha \in M \prec \langle H(\theta), \in, \vec{C}, \vec{\sigma}\rangle$ there exists a function $f : \mathcal{P}_\omega(\alpha) \to \alpha$ in M such that every set in $S \cap \mathcal{P}_{\omega_1}(\alpha)$ is not closed under f. But $x = M \cap \sup(x) = M \cap \alpha$ and therefore x is closed under f by elementarity of M. This is a contradiction to $x \in S$ and we proved Claim 1'.

Observe that $x \cap \omega_1 = \mathrm{ot}(C_{\sup(x)}) \in M \cap \omega_1$ by Claim 1' and elementarity of M. The rest of the proof is similar to the proof of (1). $\square$

Now we are able to prove Theorem 2.1.1 by combining all of the lemmata above and prove the following theorem.

Theorem 2.1.22. *Suppose that* GCH *and* $\square_{\omega_1}$ *hold and let* $\vec{C}$ *be a* $\square_{\omega_1}$*-sequence. Then there exists an* ω_1*-distributive and* ω_2*-c.c. forcing extension in which* $\mathrm{SR_k}(S_k^{\vec{C}})$ *holds for* $k \in \{0, 1\}$*.*

Proof. Take a surjection system $\vec{\sigma}$ in V. By iterating the club shootings from Definition 2.1.12, we obtain a countable support iteration destroying the stationarity of all non-reflecting stationary subsets of $S_0^{\vec{C},\vec{\sigma}}$ and $S_1^{\vec{C},\vec{\sigma}}$. Note that $S_k^{\vec{C},\vec{\sigma}}$ and $T^{\vec{C}}$ are absolute between all ω_1-distributive and ω_2-c.c. forcing extensions of V.

By combining all the lemmata above, we can construct a countable support iteration $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi, \eta < \omega_3 \rangle$ which satisfies the following properties:

(1) $\mathbb{P}_\xi$ is ω_1-distributive and has the ω_2-c.c. for each $\xi < \omega_3$.

(2) If $\eta < \omega_3$, $\Vdash_{\mathbb{Q}_\eta}$ "$\dot{\mathbb{Q}}_\eta = \mathbb{Q}(\dot{S})$" for some $\mathbb{P}_\eta$-name $\dot{S}$ such that either
$\Vdash_{\mathbb{Q}_\eta}$ "$\dot{S} \subseteq S_0^{\vec{C}}$ and $\dot{S}$ does not reflect to any uncountable ordinal in $E_\omega^{\omega_2}$", or
$\Vdash_{\mathbb{Q}_\eta}$ "$\dot{S} \subseteq S_1^{\vec{C}}$ and $\dot{S}$ does not reflect to any ordinal in $E_{\omega_1}^{\omega_2}$".
Therefore $\Vdash_{\mathbb{Q}_\eta}$ "$\dot{\mathbb{Q}}_\eta$ is T-complete, better and $|\dot{\mathbb{Q}}_\eta| \leq \omega_2$".

(3) If $\xi < \omega_3$ and $\dot{S}$ is a $\mathbb{P}_\xi$-name such that either
$\Vdash_{\mathbb{P}_\xi}$ "$\dot{S} \subseteq S_0^{\vec{C}}$ and $\dot{S}$ does not reflect to any uncountable ordinal in $E_\omega^{\omega_2}$", or
$\Vdash_{\mathbb{P}_\xi}$ "$\dot{S} \subseteq S_1^{\vec{C}}$ and $\dot{S}$ does not reflect to any ordinal in $E_{\omega_1}^{\omega_2}$"
then there exists an ordinal $\eta \in \omega_3 \setminus \xi$ such that $\Vdash_{\mathbb{Q}_\eta}$ "$\dot{\mathbb{Q}}_\eta = \mathbb{Q}(\dot{S})$".

Then the limit of this iteration $\mathbb{P}_{\omega_3}$ is ω_1-distributive and ω_2-c.c. Now take a generic filter G on $\mathbb{P}_{\omega_3}$ over V. Then the following holds in $V[G]$:

(1) If $S \subseteq S_0^{\vec{C},\vec{\sigma}}$ and S does not reflect to any ordinal in $E^{\omega_2}_{\omega} \setminus \omega_1$ then S is non-stationary.

(2) If $S \subseteq S_1^{\vec{C},\vec{\sigma}}$ and S does not reflect to any ordinal in $E^{\omega_2}_{\omega_1}$ then S is non-stationary.

Therefore, $SR_k(S_k^{\vec{C},\vec{\sigma}})$ holds for $k \in \{0,1\}$ in $V[G]$. Since $S_k^{\vec{C}} \setminus S_k^{\vec{C},\vec{\sigma}}$ is non-stationary, $SR_k(S_k^{\vec{C}})$ holds for $k \in \{0,1\}$ in $V[G]$. □

2.2 $\mathrm{SR}_0(S^*)$ fails under $2^{\omega_1} = \omega_2$

When trying to shed some new light on the question whether the Weak Reflection Principle for ω_n implied the Reflection Principle for ω_n, König-Larson-Yoshinobu were able to do so under certain assumptions. For $n > 2$, they obtained the result by assuming CH and $2^{\omega_{n-1}} = \omega_n$. In the case of $n = 2$, CH could be dropped and only $2^{\omega_1} = \omega_2$ was needed. In the course of those proofs, König-Larson-Yoshinobu showed in [7] that under the above mentioned assumptions $\mathrm{SR}_0(S^*)$ failed. In the last section we constructed a model such that $\mathrm{SR}_0(S^*)$ holds for a subset S^* of $\mathcal{P}_{\omega_1}(\omega_2)$. For this result we assumed GCH and $\square_{\omega_1}$. In the generic extension however, 2^{ω_1} became ω_3. Therefore, the assumption $2^{\omega_1} = \omega_2$ does not contradict the following proofs. We start with the more general setting of $\mathcal{P}_{\omega_1}(\omega_n)$.

Lemma 2.2.1. *Assume* CH *and* $1 \leq n < \omega$. *Then for every* $S^* \subseteq \mathcal{P}_{\omega_1}(\omega_n)$, *there is a set* $S \subseteq S^*$ *which is cofinal in* S^* *and does not reflect to any set of size* ω_1.

Definition 2.2.2. *Let* λ *be a regular cardinal. For a set* $S^* \subseteq \mathcal{P}_{\omega_1}(\lambda)$ *and a set* $x \in \mathcal{P}_{\omega_1}(\lambda)$ *we denote* $S^*(x)$ *as the set of all supersets of* x *in* S^*. *We call the union of all supersets of* x *in* S^*, *i.e.,*

$$\bigcup S^*(x) = \bigcup \{y \in S^* \mid x \subseteq y\}$$

the S^*-coverage *of* x.

For the proof of Lemma 2.2.1 we want to partition S^* in the following way.

Lemma 2.2.3. *Every set* $S^* \subseteq \mathcal{P}_{\omega_1}(\lambda)$ *can be partitioned into two sets* S_0^* *and* S_1^* *such that*

(1) S_0^* *has no* $\subsetneq$*-increasing chains of length* ω_1*, and*

(2) each $x \in S_1^*$ *has uncountable* S_1^**-coverage.*

Proof. For a set $S^* \subseteq \mathcal{P}_{\omega_1}(\lambda)$ we iteratively remove all sets with countable coverage. So let

$$S_\alpha^* = \begin{cases} S^* & \text{if } \alpha = 0 \\ \{x \in S_\beta^* \mid x \text{ has uncountable } S_\beta^*\text{-coverage}\} & \text{if } \alpha = \beta + 1 \\ \bigcap\limits_{\beta<\alpha} S_\beta^* & \text{if } \alpha \text{ is limit} \end{cases}$$

Since at some stage there are no more sets with countable coverage left, there must be an ordinal ∞ with $S_\infty^* = S_{\infty+1}^*$. Therefore let $S_1^* = S_\infty^*$ and $S_0^* = S^* \setminus S_1^*$. Then every set in S_1^* has uncountable S_1^*-coverage. To see (1), take a $\subsetneq$-increasing sequence $\vec{s}$ of length ω_1. Every member of $\vec{s}$ has uncountable S^*-coverage. Hence this sequence will not be removed by the iteration above. Therefore $\vec{s}$ is in S_1^* and S_0^* does not contain any such sequences of length ω_1. $\square$

The next two lemmata show how to thin out the set S^* after partitioning. We make use of the fact that a set containing no increasing sequences of length $\omega + 1$ cannot be stationary.

Lemma 2.2.4. *Let* λ *be a regular cardinal and* $S^* \subseteq \mathcal{P}_{\omega_1}(\lambda)$*. Suppose* $\langle x_\alpha : \alpha < \xi \rangle$ *is a possibly incomplete list of members of* S^**. For every* $\alpha < \xi$ *assume that* T_α *is a* $\subseteq$*-cofinal subset of* $S^*(x_\alpha)$ *which does not contain any* *continuous, $\subsetneq$-increasing chains of length $\omega+1$. Define a sequence $\langle T'_\alpha : \alpha < \xi\rangle$ inductively as follows. Let*

$$T'_\alpha = \{y \in T_\alpha \mid \forall\beta < \alpha\ \forall x \in T'_\beta\ (y \not\subseteq x\ \wedge\ x \not\subseteq y)\}.$$

Let $T' = \bigcup_{\alpha<\xi} T'_\alpha$. Then T' a cofinal subset of $\bigcup_{\alpha<\xi} S^*(x_\alpha)$ which contains no continuous, $\subsetneq$-increasing sequence of length $\omega+1$.

Proof. We prove this lemma by induction on ξ. First we check that T' is cofinal in $\bigcup_{\alpha<\xi} S^*(x_\alpha)$. Take $x \in S^*(x_\alpha)$. We must find a $y \in T'$ with $x \subseteq y$. Since T_α is cofinal in $S^*(x_\alpha)$, we may assume that $x \in T_\alpha$. If x is also in T'_α, we are done. So suppose that $x \notin T'_\alpha$ and x is not in any member of $\bigcup_{\beta<\alpha} T'_\beta$ (or else we would have already found a $y \in T'$ with $x \subseteq y$). By the definition of T'_α there must be a $z \in T_\beta$ for some $\beta < \alpha$ with $z \subseteq x$ (again, if $x \subseteq z$ we would be done). Since $\bigcup_{\gamma<\alpha} T_\gamma \subseteq \bigcup_{\alpha<\xi} S^*(x_\alpha)$ and $\bigcup_{\gamma<\alpha} T'_\gamma$ is cofinal in $\bigcup_{\alpha<\xi} S^*(x_\alpha)$ by the induction hypothesis, $\bigcup_{\gamma<\alpha} T'_\gamma$ is also cofinal in $\bigcup_{\gamma<\alpha} T_\gamma$. At this point note that $T_\beta \cap S^*(z)$ is cofinal in $S^*(z)$. Therefore there must be a $y \in \bigcup_{\gamma<\alpha} T'_\gamma$ with $y \subseteq x$.

Now we need to show that T' does not contain any continuous, $\subsetneq$-increasing sequences of length $\omega+1$. This follows from the fact that no T_α contains such a sequence and therefore neither does T'_α by construction. □

Lemma 2.2.5. *Assume* CH *and $1 \leq n < \omega$. Let $S^* \subseteq \mathcal{P}_{\omega_1}(\omega_n)$ be such that every member in S^* has uncountable S^*-coverage. Then there is a cofinal set $S \subseteq S^*$ which contains no continuous, $\subsetneq$-increasing sequence of length $\omega+1$.*

Proof. The proof is by induction on n.

Claim. If every member of $S^*(x)$ has S^*-coverage of cardinality ω_2, then there exists a cofinal subset $S(x)$ of S^* which does not contain any continuous, $\subsetneq$-increasing sequence of length $\omega+1$.

Proof of Claim. By using CH we can enumerate $S^*(x) = \{x_\alpha \mid \alpha < \omega_n\}$.

For every $\alpha < \omega_n$ choose $y_\alpha \in S^*(x_\alpha)$ with $y_\alpha \nsubseteq \bigcup_{\beta<\alpha} y_\beta$. So each y_α contains a new ordinal. Then the set $S(x) = \{y_\alpha \mid \alpha < \omega_n\}$ does not contain any continuous, $\subsetneq$-increasing sequences of length $\omega + 1$. This proves the claim. Now we can thin out $S^*(x)$ as desired. Either every superset of x in S^* has S^*-coverage of cardinality ω_n then this works by the claim, or x has S^*-coverage of cardinality less than ω_n, then we use the induction hypothesis. In either case the set $x \in S^*$ is cofinal in S^*. Therefore we can apply Lemma 2.2.4, which finishes the proof. $\square$

Proof of Lemma 2.2.1: The goal is to find a subset of S^* which does not reflect to any set of size ω_1. First partition S^* into two sets as in Lemma 2.2.3. Then we can apply Lemma 2.2.5 to S_1^* to find a cofinal set $S \subseteq S_1^*$ which does not contain any continuous, $\subsetneq$-increasing sequence of length $\omega+1$. Thus $S_0^* \cup S$ is as desired. $\square$

Definition 2.2.6. *Suppose λ is a cardinal and E is a stationary subset of λ. By $\curlywedge^*(\kappa, E)$ we denote the assertion that there exists a sequence $\langle C_\alpha : \alpha \in E\rangle$ satisfying*

(1) C_α is a club subset of $\mathcal{P}_\kappa(\alpha)$ for all α in E, and

(2) for all clubs $D \subseteq \mathcal{P}_\kappa(\lambda)$ there exists a club $C \subseteq \lambda$ such that for all $\alpha \in C \cap E$, $C_\alpha \leq^* D$.

In (2), $A \leq^* B$ if and only if there is a set x of size less than κ such that if $x \subseteq y \in A$ then $y \in B$ for all y. The sequence $\langle C_\alpha : \alpha \in E\rangle$ is called a tail club guessing sequence*.*

The principle $\curlywedge^*(\kappa, E)$ is a weaker form of $\diamondsuit^*(E)$, since it implies $\curlywedge^*(\kappa, E)$ for all κ, but the converse it not true. Furthermore, $\curlywedge^*(\kappa, E)$ is preserved by κ-c.c. forcing notions. Note that the logical strength depends on the set E and increases with the size of E. When $E = E^{\lambda^+}_{<\kappa} = \{\alpha \in [\lambda, \lambda^+) \mid \omega \leq \mathrm{cof}(\alpha) < \kappa\}$, the following theorem gives an equivalence to $\curlywedge^*(\kappa, E^{\lambda^+}_{<\kappa})$, which shows that its logical strength is pretty weak. The proof of the next theorem can be found in [7].

Theorem 2.2.7. *Suppose $\kappa \leq \lambda$. Then the following are equivalent:*

(1) $\curlywedge^*(\kappa, E^{\lambda^+}_{<\kappa})$.

(2) There exists a club $F \subseteq \mathcal{P}_\kappa(\lambda^+)$ such that for each club $D \subseteq \mathcal{P}_\kappa(\lambda^+)$ there exists a club $C \subseteq \lambda^+$ such that for all $\alpha \in C \cap E^{\lambda^+}_{<\kappa}$, $F \cap \mathcal{P}_\kappa(\alpha) \leq^* D$.

(3) The number of clubs in $\mathcal{P}_\kappa(\lambda)$ is λ^+. Furthermore the collection of these clubs is cofinal with respect to $\leq^*$.

Property (3) is easily obtained by cardinal arithmetic, i.e., if $2^\lambda = \lambda^+$, then there are λ^+-many club subsets of $\mathcal{P}_\kappa(\lambda)$. The collection of all those clubs is cofinal with respect to $\leq^*$, since it generates the club filter. For the failure of $\curlywedge^*(\kappa, E^{\lambda^+}_{<\kappa})$ take a model in which λ^{++}-many Cohen-subsets of λ are added. This is possible if λ is regular.

A similar characterization as in Theorem 2.2.7 cannot be given when $\curlywedge^*$ is defined on ordinals with higher cofinality. The tail guessing principle is much stronger when $E = E^{\lambda^+}_\lambda$ and cannot be derived from GCH.

Theorem 2.2.8. *Suppose* CH *and $\curlywedge^*(\omega_1, E^{\omega_n}_\omega)$ hold for some $2 \leq n < \omega$. If $B \subseteq \mathcal{P}_{\omega_1}(\omega_n)$ is stationary, then there is a stationary set $A \subseteq B$ such that $\{x \in A \mid \sup(x) = \alpha\}$ does not reflect to any set of size ω_1 for all $\alpha \in E^{\omega_n}_\omega$.*

Proof. Take the club $F \subseteq \mathcal{P}_{\omega_1}(\omega_n)$ from Theorem 2.2.7 (2). Without loss of generality, we may assume that each member of F has limit order type with supremum in $E^{\omega_n}_\omega$. For every $\xi \in E^{\omega_n}_\omega$, let $F^\xi = \{x \in F \mid \sup(x) = \alpha\}$. Now take some stationary $B \subseteq \mathcal{P}_{\omega_1}(\omega_n)$. We can assume that $B \subseteq F$. Next we want to apply Lemma 2.2.1 for every $\xi \in E^{\omega_n}_\omega$ to obtain a $\subseteq$-cofinal set $A^\xi \subseteq B \cap F^\xi$ which does not reflect to any set of size ω_1. The union of all these A^ξ's will be as desired. So let

$$A = \bigcup_{\xi \in E^{\omega_n}_\omega} A^\xi.$$

We need to show that A is stationary. Let $D \subseteq \mathcal{P}_{\omega_1}(\omega_n)$ be a club. By definition of F, we can assume that $D \subseteq F$. Next we define a club D' as the set of all $x \in \mathcal{P}_{\omega_1}(\omega_n)$ containing increasing ordinals γ_i for $i < \omega$, which satisfy the following properties:

(1) $\sup_{i<\omega}(\gamma_i) = \sup(x)$,

(2) for every $i < \omega$, $D \cap \mathcal{P}_{\omega_1}(\gamma_i)$ is club in $\mathcal{P}_{\omega_1}(\gamma_i)$, and

(3) each element of $F \cap \mathcal{P}_{\omega_1}(\gamma_i)$ containing $x \cap \gamma$ is in D.

If we choose $x \in B \cap D'$ with $\sup(x) = \xi$, then $x \in B \cap F^\xi$ Therefore there exists a $y \in A^\xi$ which contains x. Since $x \in D'$, we have $y \cap \gamma_i \in D$ for every $i < \omega$. Hence $y \in D \cap A$. □

Theorem 2.2.9. *Assume* CH *and* $2^{\omega_{n-1}} = \omega_n$. *Then* $\mathrm{SR}_0(S^*)$ *fails for* $S^* \subseteq \mathcal{P}_{\omega_1}(\omega_n)$.

Proof. Follows from Theorem 2.2.7 and 2.2.8. □

Theorem 2.2.10. *Assume* $2^{\omega_1} = \omega_2$. *Then* $\mathrm{SR}_0(S^*)$ *fails for* $S^* \subseteq \mathcal{P}_{\omega_1}(\omega_2)$.

Proof. We will show that for every stationary subset S of $\mathcal{P}_{\omega_1}(\omega_2)$, there exists a stationary subset T of S which does not reflect to any uncountable ordinal in ω_2 with cofinality ω. Let

$$A = \{\alpha \in \omega_2 \cap \mathrm{cof}(\omega) \mid S \cap \mathcal{P}_{\omega_1}(\alpha) \text{ is stationary in } \mathcal{P}_{\omega_1}(\alpha)\}.$$

If A is non-stationary, let C denote a club in ω_2 with $A \cap C = \emptyset$. Then $T = \{x \in S \mid \sup(x) \in C\}$.

So suppose A is stationary. If α is in A, it is straightforward to find an unbounded subset of $S \cap \mathcal{P}_{\omega_1}(\alpha)$ which does not contain any continuous, increasing sequence of length $\omega + 1$. Therefore, the above lemmata are not needed for this proof and we can drop CH in the assumptions. The result can be obtained by repeating the proof of Theorem 2.2.8. $\square$

Chapter 3

Stationary Reflection Principles

3.1 Consistency of WRP and RP

The consistency strength of the Weak Reflection Principle for ω_2 and the Reflection Principle for ω_2 is a weakly compact cardinal. We obtain this result by collapsing a weakly compact cardinal to ω_2. Following Veličković [16], we show that those reflection principles are in fact equiconsistent with a weakly compact cardinal. This makes crucial use of Jensen's work [6].
Using the same arguments, collapsing a supercompact cardinal to ω_2, one can obtain the (Weak) Reflection Principle for all cardinals.

Theorem 3.1.1. *The consistency of* ZFC + *"there exists a weakly compact cardinal" implies the consistency of*

(1) ZFC + *"the Reflection Principle for* ω_2*", and therefore*

(2) ZFC + *"the Weak Reflection Principle for* ω_2*".*

Proof. Suppose κ is a weakly compact cardinal. Let $\mathbb{P} = \mathrm{Coll}(\omega_1, <\kappa)$ denote the Lévy Collapse and G a generic filter on $\mathbb{P}$ over V. In $V[G]$, take an arbitrary stationary set $S \subseteq \mathcal{P}_{\omega_1}(\kappa)$. We need to show that there exists an ordinal $\alpha < \kappa$ with cofinality ω_1 such that $S \cap \mathcal{P}_{\omega_1}(\alpha)$ is stationary.

Let $\langle M, \in \rangle$ be a transitive elementary submodel of $\langle V, \in \rangle$ of size κ containing all relevant objects. By weak compactness of κ, there exists an elementary embedding $j : M \to N$ with critical point κ, (i.e., $j(\kappa) > \kappa$) for some transitive model $\langle N, \in \rangle$. By standard arguments, we can decompose $j(\mathbb{P})$ to $\mathbb{P} \times \mathbb{Q}$, where $\mathbb{Q} = \mathrm{Coll}(\omega_1, [\kappa, j(\kappa)))$. Note that $\mathbb{Q}$ is σ-closed in V and M. By absoluteness, $\mathbb{Q}$ remains σ-closed in $M[G]$, where G denotes a generic filter on $\mathbb{P}$ over M. Next let H be a generic filter on $\mathbb{Q}$ over $M[G]$. In $M[G*H]$, we can extend $j : M \to N$ to $j' : M[G] \to N[G*H]$. Clearly, $S = j'(S) \cap \mathcal{P}_{\omega_1}(\kappa)$. Since $\mathbb{Q}$ is σ-closed, S remains stationary in $M[G*H]$. By the same argument, S is stationary in $N[G*H]$. Then κ witnesses that in $N[G*H]$ there exists an ordinal $\alpha < j(\kappa)$ of cofinality ω_1 such that $j'(S) \cap \mathcal{P}_{\omega_1}(\alpha)$ is stationary. By elementarity of j', there is $\alpha < \kappa$ with cofinality ω_1 in $M[G]$ such that $S \cap \mathcal{P}_{\omega_1}(\alpha)$ is stationary. By elementarity of M, the same statement holds in $V[G]$. □

Theorem 3.1.2. *The following are equiconsistent:*

(1) ZFC + *"there exists a weakly compact cardinal",*

(2) ZFC + $\mathrm{WRP}(\omega_2)$*.*

By proving Theorem 3.1.1, we have taken care of one direction of this theorem. For the converse, we define a weaker form of a $\square_\kappa$-sequence in order to use Jensen's work [6]. He proved that if κ is a regular cardinal which is not weakly compact, then there exists such a weaker $\square_\kappa$-sequence.

Definition 3.1.3. *Let κ be a regular cardinal $\geq \omega_2$. We call a sequence $\langle C_\alpha : \alpha \in \mathrm{Lim}(\kappa) \rangle$ a $\square^*_\kappa$-sequence if it satisfies the following properties:*

(1) C_α is club in α for every $\alpha \in \mathrm{Lim}(\kappa)$,

(2) $C_\beta = C_\alpha \cap \beta$ for every $\beta \in \mathrm{Lim}(C_\alpha)$,

(3) $\neg \exists C \subseteq \kappa$ (C is club $\wedge$ $\forall \alpha \in \mathrm{Lim}(C)$ $C_\alpha = C \cap \alpha$).

The main tool for the construction in our proof is the following two-player game, which we will revisit in the proof of Theorem 3.3.1.

Definition 3.1.4. *We define the two-player game $\mathcal{G}_\alpha$ as follows. Let $\kappa > \omega_1$ be a regular cardinal, F a function $F : \mathcal{P}_\omega(\kappa) \to \kappa$, and α an ordinal $< \omega_1$.*

$$\begin{array}{lccccc} I: I_0, \zeta_0 & I_1, \zeta_1 & \dots & I_n, \zeta_n & \dots \\ II: \quad \eta_0 & \eta_1 & \dots & \eta_n & \dots \end{array}$$

At stage n Player I picks an interval I_n in κ and an ordinal $\zeta_n \in I_n$. Player II responds with an ordinal $\eta_n < \kappa$. Then Player I has to choose I_{n+1} such that $\inf(I_{n+1}) > \eta_n$. Player I wins, if by letting $y = \mathrm{cl}_F(\{\zeta_n \mid n < \omega\} \cup \alpha)$ we have $y \subseteq \bigcup\{I_n \mid n < \omega\}$ and $y \cap \omega_1 = \alpha$.
Since $\mathcal{G}_\alpha$ is an open game for Player II, the Gale-Stewart Theorem implies that one of the players has a winning strategy.

The proof of the following theorem can be found in [16].

Lemma 3.1.5. *$A_F = \{\alpha < \omega_1 \mid II$ has a winning strategy in $\mathcal{G}_\alpha\}$ is non-stationary.*

The following theorem is crucial to proving Theorem 3.1.2.

Theorem 3.1.6. *Let κ be a regular cardinal and assume that $\square^*_\kappa$ holds. Then there exists a stationary set $S \subseteq \mathcal{P}_{\omega_1}(\kappa)$ such that S does not reflect to any $\alpha \in \kappa$.*

Proof. Let $\vec{C} = \langle C_\alpha : \alpha \in \mathrm{Lim}(\kappa)\rangle$ be a $\square^*_\kappa$-sequence and

$$S = \{x \in \mathcal{P}_{\omega_1}(\kappa) \mid \alpha = \bigcup x \notin x \ \wedge \ \sup(C_\alpha \cap x) < \alpha\}.$$

Since $\vec{C}$ is a $\square^*_\kappa$-sequence, C_α is a club in α for every limit $\alpha \in \kappa$. Therefore $S \cap \mathcal{P}_{\omega_1}(\alpha)$ is non-stationary. Thus we need to prove the following claim to finish the proof.

Claim. S is stationary.

Proof of Claim. For a function $F : \mathcal{P}_\omega(\kappa) \to \kappa$ we have to find a set $x \in S$ which is closed under F. We use a winning strategy for player I in the game $\mathcal{G}_0$ to recursively construct a sequence $\langle I_g : g \in \kappa^{<\omega}\rangle$ of intervals in κ and a sequence $\langle \delta_g : g \in \kappa^{<\omega}\rangle$ satisfying:

(1) For every $d \in \kappa^{<\omega}$ $\delta_d \in I_d$,

(2) For every $d \in \kappa^{<\omega}$ and $\alpha, \beta \in \kappa$ $[\alpha < \beta \to \sup(I_d \cap \alpha) < \inf(I_d \cap \beta)]$,

(3) For every $f \in \kappa^\omega$ $\mathrm{cl}_F\{\delta_{f\restriction n} \mid n < \omega\} \subseteq \bigcup\{I_{f\restriction n} \mid n < \omega\}$.

Now fix a sufficiently large regular cardinal θ and an elementary submodel $M \prec H(\theta)$ containing every relevant object. Furthermore let $M \cap \kappa = \alpha \in \kappa$ and $\mathrm{cof}(\alpha) = \omega$. Fix a sequence $\langle \alpha_n : n < \omega\rangle$ converging to α. We build a sequence of functions $f \in \alpha^\omega$ inductively such that the following holds:

(1) $\forall n < \omega$ $\delta_{f\restriction n} \geq \alpha_n$,

(2) $\forall n > 0$ $I_{f\restriction n} \cap C_\alpha = \emptyset$.

For the inductive step we use the following claim.

Subclaim. Let $\langle I_\xi : \xi < \kappa\rangle$ be a sequence in M where I_ξ are intervals in κ such that if $\zeta < \xi$ then $\sup(I_\zeta) < \inf(I_\xi)$. Then there exists $\xi < \alpha$ with $I_\xi \cap C_\alpha = \emptyset$.

Proof of Subclaim. If not, then $\forall \xi < \alpha$ $J_\xi \cap C_\alpha \neq C_\alpha$. Hence

$$M \vdash \forall \xi < \kappa\ \exists \gamma < \kappa\ \forall \zeta < \xi\ J_\xi \cap C_\gamma \neq \emptyset.$$

Then $H(\theta)$ satisfies the same statement by elementarity of M. Let D be the set of limit points of I_γ. We now prove that $\forall \beta, \gamma \in D$ if $\beta < \gamma$ then $C_\beta = C_\gamma \cap \beta$:

For such β and γ, fix $\zeta < \kappa$ with $\beta, \gamma < \inf(I_\zeta)$ and $\delta < \kappa$ with $\forall \xi < \zeta$ $I_\xi \cap C_\gamma \neq \emptyset$. Thus β and γ are limit points of C_δ. Since C_δ belongs to a

$\square^*_\kappa$-sequence, $C_\beta = C_\delta \cap \beta$ and $C_\gamma = C_\delta \cap \gamma$ by property (2) of $\square^*_\kappa$. Hence $C_\beta = C_\gamma \cap \beta$. But then the set D is a contradiction to property (3) of the definition of $\square^*_\kappa$. This proves the subclaim.
To finish the proof of the claim let $x = \mathrm{cl}_F\{\delta_{f\restriction n} \mid n < \omega\}$. Note that $\sup(C_\alpha \cap x) < \alpha = \sup(x)$. Therefore $x \in S$ and x is closed under F. Hence S is stationary and the proof of the claim and therefore the proof of the theorem is complete. $\square$

Proof of Theorem 3.1.2: $\neg$(1) implies $\neg$(2): By Jensen's result, there exists a $\square^*_{\omega_2}$-sequence. Using Theorem 3.1.6, it follows that there exists a set $S \subseteq \mathcal{P}_{\omega_1}(\omega_2)$ which does not reflect. Hence, WRP(ω_2) fails. $\square$

3.2 Consistency of SSR

Recall that for a cardinal $\lambda \geq \omega_2$, the *Semistationary Reflection Principle for* λ denotes the following principle:
For every semistationary $S \subseteq \mathcal{P}_{\omega_1}(\lambda)$, *there exists* $X \subseteq \lambda$ *such that* $|X| = \omega_1 \subseteq X$ *and* $S \cap \mathcal{P}_{\omega_1}(X)$ *is semistationary in* $\mathcal{P}_{\omega_1}(X)$.
The consistency of the Semistationary Reflection Principles can be taken care of in two steps. The first case is SSR(ω_2). We will show in the next chapter that for ω_2 the Semistationary Reflection Principle is equivalent to the Weak Reflection Principle. Therefore, SSR(ω_2) is also equiconsistent with a weakly compact cardinal.
For a larger cardinal λ, Shelah has obtained a model in which SSR(λ) holds by using a λ-strongly compact cardinal. We refer the reader to [15]. However, Sakai proved in [13] that collapsing a λ-strongly compact cardinal does not suffice to show WRP(λ). The key to this result is the following theorem.

Theorem 3.2.1. *Suppose that* λ *is a supercompact cardinal. Then there exists a generic extension in which* λ *is a strongly compact cardinal and* WRP(λ^+) *fails.*

3.3 $\mathrm{SR}_0(\mathcal{P}_{\omega_1}(\omega_2))$ is inconsistent

The next result shows that it suffices to consider the Weak Reflection Principle for ω_2 and the Reflection Principle for ω_2, since the following reflection principle is inconsistent.
For every stationary set $S \subseteq \mathcal{P}_{\omega_1}(\omega_2)$, there exists an uncountable ordinal α in ω_2 with cofinality ω such that $S \cap \mathcal{P}_{\omega_1}(\alpha)$ is stationary in $\mathcal{P}_{\omega_1}(\alpha)$.
As usual $E^{\omega_2}_{\omega_1} = \{\alpha \in \omega_2 \mid \mathrm{cof}(\alpha) = \omega_1\}$.

Theorem 3.3.1. *The statement $\mathrm{SR}_0(\mathcal{P}_{\omega_1}(\omega_2))$ is inconsistent.*

Proof. First, let $\langle C_\alpha : \alpha \in E^{\omega_2}_{\omega_1}\rangle$ be a sequence of the following sets:

(1) C_α is an unbounded set of α, and

(2) $\mathrm{ot}(C_\alpha) = \omega$.

We call such a sequence a ladder system. Next we define a set witnessing the failure of $\mathrm{SR}_0(\mathcal{P}_{\omega_1}(\omega_2))$. Let $X = \{x \in \mathcal{P}_{\omega_1}(\omega_2) \mid C_{\sup(x)} \not\subseteq x\}$.
Claim. The set X is stationary in $\mathcal{P}_{\omega_1}(\omega_2)$.
Proof of Claim. Here we can repeat the proof of Theorem 3.1.6. The only difference is the proof of the subclaim. For that we can get a contradiction as follows. Recall that we assumed $\forall \xi < \alpha \; J_\xi \cap C_\alpha \neq C_\alpha$. Hence

$$M \vdash \forall \xi < \kappa \; \exists \gamma < \kappa \; \forall \zeta < \xi \; J_\xi \cap C_\gamma \neq \emptyset.$$

But then the order type of C_γ would be at least ξ, because the intervals J_ξ are pairwise disjoint. Therefore $\mathrm{ot}(C_\gamma)$ cannot be equal to ω for every $\xi \in (\omega, \kappa)$.
For the failure of $\mathrm{SR}_0(\mathcal{P}_{\omega_1}(\omega_2))$ look at $X \cap \mathcal{P}_{\omega_1}(\alpha)$ for any $\alpha \in \omega_2$ with cofinality ω. For any such α, there are club many y in $\mathcal{P}_{\omega_1}(\alpha)$ such that $C_\alpha \subseteq y$. Therefore $X \cap \mathcal{P}_{\omega_1}(\alpha)$ is non-stationary for any α in $E^{\omega_2}_{\omega_1}$. $\square$

3.4 A remark on bigger spaces

We have now considered the consistency of many stationary reflection principles in $\mathcal{P}_{\omega_1}(\lambda)$. One might ask about those principles for the uncountable space $\mathcal{P}_{\omega_2}(\lambda)$. Feng and Magidor considered this question in [2]. They started with the following principle, which is consistent.

Assume that λ is a cardinal and $\lambda \geq \omega_2$. For every stationary set $S \subseteq \mathcal{P}_{\omega_1}(H(\lambda))$, there exists an $X \subseteq H(\lambda)$ of size ω_1 such that $\omega_1 \subseteq X$ and S reflects to X.

So we ask if one could extend this principle to the following.

Assume that λ is a cardinal and $\lambda \geq \omega_3$. For every stationary set $S \subseteq \mathcal{P}_{\omega_2}(H(\lambda))$, there exists an $X \subseteq H(\lambda)$ of size ω_2 such that $\omega_2 \subseteq X$ and S reflects to X.

In this context, S reflects to X if $S \cap \mathcal{P}_{\omega_2}(X)$ is stationary in $\mathcal{P}_{\omega_2}(X)$. Feng and Magidor proved that this extension is false for sufficiently large cardinals λ.

Theorem 3.4.1 ([2], 2.1)**.** *Suppose λ is a regular cardinal and $\lambda \geq (2^{\omega_2})^{++}$. Then there exists a stationary set $S \subseteq \mathcal{P}_{\omega_2}(H(\lambda))$ such that for any set $X \subseteq H(\lambda)$ of size ω_2, $\omega_2 \subseteq X$ implies that $S \cap \mathcal{P}_{\omega_2}(X)$ is non-stationary in $\mathcal{P}_{\omega_2}(X)$.*

3.5 A result on cardinal arithmetic

In the beginning of the introduction we gave a result on cardinal arithmetic as motivation for considering reflection principles. As an example, we will prove that the Weak Reflection Principle for ω_2 implies $2^\omega \leq \omega_2$. This is due to Todorčević, but we follow Shelah [14].

Theorem 3.5.1. WRP(ω_2) *implies* $2^\omega \leq \omega_2$.

Proof. Suppose that $2^\omega > \omega_2$. We will find a stationary set $S \subseteq \mathcal{P}_{\omega_1}(\omega_2)$ which does not reflect, i.e., $S \cap \mathcal{P}_{\omega_1}(\alpha)$ is not stationary in $\mathcal{P}_{\omega_1}(\alpha)$ for each uncountable α in ω_2. We define h_α for every ordinal α to be a one to one function from $|\alpha|$ to α. Let $V' = L[\langle h_\alpha : \alpha < \omega_2 \rangle]$. Note that in this new model V' both ω_1 and ω_2 remain the same. Then in V' there are at most ω_2^V countable subsets of ω_2^V. By a fact from Baumgartner-Taylor [1], every club C in $\mathcal{P}_{\omega_1}(\omega_2)$ has size equal to 2^ω which is greater than ω_2 by assumption. Therefore, the set $S = \{x \in \mathcal{P}_{\omega_1}(\omega_2) \mid x \notin V'\}$ is a stationary subset of $\mathcal{P}_{\omega_1}(\omega_2)$. But for each $\alpha < \omega_2$, we can define a club $C_\alpha \subseteq V'$ using h_α. Since S and C_α are disjoint for each $\alpha < \omega_2$, we have found a non-reflecting set as desired. $\square$

Chapter 4

Implications between Reflection Principles

In this chapter we compare the Weak Reflection Principle to the Reflection Principle and the Semistationary Reflection Principle. Clearly, the Reflection Principle implies the Weak Reflection Principle for any cardinal. As far as the converse is concerned, two things have been proven. Krueger showed in [8] that under the assumption of a supercompact cardinal, we can separate the two principles. We give an outline of this proof. Furthermore, König-Larsen-Yoshinobu proved in [7] that if $2^{\omega_1} = \omega_2$, the Weak Reflection Principle implies the Reflection Principle for ω_2. We can also derive this result for ω_n for arbitrary n under the assumption that CH holds and $2^{\omega_{n-1}} = \omega_n$. In comparison to the Semistationary Reflection Principle, the Weak Reflection Principle turns out to be stronger for any cardinal greater than ω_2. Since we are mainly interested in ω_2, we give a proof of their equivalence and refer the reader to Sakai's paper [13] for more information.

4.1 $\mathrm{WRP}(\omega_2)$ and $\neg\,\mathrm{RP}(\omega_2)$

In 2011, Krueger asserted that under the assumption of a supercompact cardinal, the Weak Reflection Principle does not imply the Reflection Principle for ω_2. It is an open question, if a supercompact is really necessary to obtain this result. Krueger proved the following theorem by adapting Sakai's club shooting to negate the Reflection Principle. However, this would not work to maintain the Weak Reflection Principle, since Sakai's proof depends on the $\square_{\omega_1}$-sequence and the Weak Reflection Principle for ω_2 implies $\sim\square_{\omega_1}$. Therefore, Krueger combined Sakai's work with classical methods for constructing $\mathrm{WRP}(\omega_2)$, i.e., using an elementary embedding to get a generic extension in which $\mathrm{WRP}(\omega_2)$ holds.

Theorem 4.1.1. *Let κ be a κ^+-supercompact cardinal and assume that $2^\kappa = \kappa^+$. Then there exists a forcing poset $\mathbb{P}$ which collapses κ to become ω_2 and*

$$\Vdash_{\mathbb{P}} \mathrm{WRP}(\omega_2) \wedge \neg\,\mathrm{RP}(\omega_2)\,.$$

The idea of the proof is to define the forcing notion $\mathbb{P}$ in a way that whenever K is a generic filter over V, then in $V[K]$ there is a set S satisfying:

(1) S is a stationary subset of $\mathcal{P}_{\omega_1}(\omega_2)$,

(2) S does not reflect to any ordinal in $\omega_2 \cap \mathrm{cof}(\omega_1)$,

(3) every stationary subset of S reflects to an uncountable ordinal in $\omega_2 \cap \mathrm{cof}(\omega)$,

(4) every stationary subset of $\mathcal{P}_{\omega_1}(\omega_2)\setminus S$ reflects to an uncountable ordinal in $\omega_2 \cap \mathrm{cof}(\omega_1)$.

Clearly, conditions (1) and (2) imply $\neg\,\mathrm{RP}(\omega_2)$. We show that (3) and (4) suffice to ensure $\mathrm{WRP}(\omega_2)$. Take an arbitrary stationary set $T \subseteq \mathcal{P}_{\omega_1}(\omega_2)$. Since the union of two non-stationary sets is non-stationary, either $T \cap S$ or

$T \setminus S$ must be stationary. In the first case, $T \cap S$ reflects to an uncountable ordinal α in ω_2 with cofinality ω by (3). In the second case, $T \setminus S$ reflects to an ordinal α in ω_2 with cofinality ω_1 by (4). Hence in either case, there exists an uncountable ordinal α in ω_2 such that T reflects to α.
We assume that V is a model in which there exists a κ^+-supercompact cardinal κ. In V, we define a forcing notion $\mathbb{P}$ which will be of the form

$$\mathbb{P} \equiv \mathrm{Coll}(\omega_1, < \kappa) * \dot{\mathbb{P}}_{\omega_2} * \dot{\mathbb{Q}}.$$

First, we force with the Lévy Collapse to make κ equal to ω_2. In this generic extension, we define the forcing poset $\mathbb{P}_{\omega_2}$ which adds a generic stationary set $S \subseteq \mathcal{P}_{\omega_1}(\omega_2)$ which does not reflect to any ordinal in $\omega_2 \cap \mathrm{cof}(\omega_1)$. In the third step of our iteration, we define a forcing notion $\mathbb{Q}$ which destroys the stationarity of any subset of S which does not reflect to any uncountable ordinal in $\omega_2 \cap \mathrm{cof}(\omega)$. In the final extension we will have a set S as desired in (1), (2), (3) and (4).

4.1.1 Adding a generic stationary set

The forcing notion $\mathbb{P}_{\omega_2}$ is a special case of the forcing poset we introduce next. Since after forcing with $\mathbb{P}_{\omega_2}$ we want to extend the elementary embedding we obtain from the supercompactness of κ, the next definition gives a generalized version of this forcing notion.

Definition 4.1.2. *Let $\alpha < \beta \leq \omega_2$ be ordinals such that β has uncountable cofinality. Let A be a subset of $\mathcal{P}_{\omega_2}(\alpha)$. Furthermore assume that $\mathcal{P}_{\omega_2}(\alpha) \setminus A$ is stationary, if $\mathrm{cof}(\alpha) = \omega_1$. We define the forcing poset $\mathbb{P}(\alpha, A, \beta)$ as the set of pairs (X, F) satisfying:*

(1) X is a countable subset of $\mathcal{P}_{\omega_1}(\beta)$,

(2) for every b in X, b is not a subset of α,

(3) F *denotes a function whose domain is a countable subset of* $[\alpha, \beta) \cap \mathrm{cof}(\omega_1)$,

(4) *for every* ξ *in* $\mathrm{dom}(F)$, $F(\xi)$ *is a* $\subseteq$*-increasing and continuous sequence* $\langle a_i^\xi : i \leq \gamma_\xi \rangle$ *of countable subsets of* ξ, *for some* $\gamma_\xi < \omega_1$,

(5) *for every* ξ *in* $\mathrm{dom}(F)$ *and* $i \leq \gamma_\xi$, a_i^ξ *is not in* $X \cup A$,

(6) *for every* ξ *in* $\mathrm{dom}(F)$, $a_{\gamma_\xi}^\xi$ *is a subset of* $\bigcup X$.

We define an order on $\mathbb{P}(\alpha, A, \beta)$ *by letting* $(X', F') \leq (X, F)$ *if:*

(1) $X \subseteq X'$

(2) *for every* b *in* $X' \setminus X$, b *is not a subset of* $\bigcup X$,

(3) $\mathrm{dom}(F) \subseteq \mathrm{dom}(F')$,

(4) *for every* ξ *in* $\mathrm{dom}(F)$, $F(\xi)$ *is an initial segment of* $F'(\xi)$.

For an ordinal $\beta \leq \omega_1$ *with uncountable cofinality, let* $\mathbb{P}_\beta$ *denote* $\mathbb{P}(0, \emptyset, \beta)$.

For a condition (X, F), the set X is an approximation of the generic stationary set $S \subseteq \mathcal{P}_{\omega_1}(\beta)$, which is added by the forcing. The function F approximates an array of clubs which witnesses that S satisfies (2) of the requirements above, i.e., it does not reflect to any ordinal in $[\alpha, \beta) \cap \mathrm{cof}(\omega_1)$. Therefore, $\mathbb{P}_{\omega_2}$ adds a generic stationary set S which satisfies (1) and (2).
The motivation for defining $\mathbb{P}_{\omega_2}$ in a more generalized way is the following. For an ordinal $\eta < \omega_2$ with cofinality ω_1, let H denote a generic filter on $\mathbb{P}_\eta$ over V. Then we can define the set $S = \bigcup\{X \mid \exists F \ (X, F) \in H\}$ in $V[H]$, which is a subset of $\mathcal{P}_{\omega_1}(\eta)$. We will prove that $\mathcal{P}_{\omega_1}(\eta) \setminus S$ is stationary in $\mathcal{P}_{\omega_1}(\eta)$ and that $\mathbb{P}_{\omega_2}$ is forcing equivalent to the two-step iteration $\mathbb{P}_\eta * \mathbb{P}(\eta, S, \omega_2)$.

Since we often consider generic sequences of conditions, we define a lower bound on the sequence of functions of these conditions.

Definition 4.1.3. *Let $\langle (X_n, F_n) : n < \omega \rangle$ be a descending sequence of conditions in $\mathbb{P}(\alpha, A, \beta)$. Then the* infinum *of the sequence of functions $\langle F_n : n < \omega \rangle$ is a function K whose domain is equal to $\bigcup\{\mathrm{dom}(F_n) : n < \omega\}$, which is a countable subset of $[\alpha, \beta) \cap \mathrm{cof}(\omega_1)$. For ξ in $\mathrm{dom}(K)$, $\bigcup\{F_n(\xi) : n < \omega, \xi \in \mathrm{dom}(F_n)\}$ is an increasing, continuous sequence of the form $\langle x_i^\xi : i < \gamma_\xi \rangle$ where $\gamma_\xi < \omega_1$. If γ_ξ is a successor ordinal, let $K(\xi)$ be $\langle x_i^\xi : i \leq \gamma_\xi - 1 \rangle$. If γ_ξ is a limit ordinal, define $a_{\gamma_\xi} := \bigcup\{a_i^\xi : i < \gamma_\xi\}$ and let $K(\xi)$ be the sequence $\langle a_i^\xi : i \leq \gamma_\xi \rangle$.*

The next proposition shows how to construct a lower bound for a sequence of conditions. The proof can be found in [8].

Proposition 4.1.4. *Let θ be a sufficiently large cardinal and N an elementary substructure of $H(\theta)$ containing $\mathbb{P}(\alpha, A, \beta)$. If $\mathrm{cof}(\alpha) = \omega_1$, assume that $N \cap \alpha$ is not in A. Let $\langle (X, F) : n < \omega \rangle$, X' and F' be as above. Suppose Y is a countable subset of $\mathcal{P}_{\omega_1}(\beta)$ such that $X' \subseteq Y$, and for every y in $Y \setminus X'$, either $y = N \cap \beta$, or there is an uncountable ordinal η in $[\alpha, \beta) \cap \mathrm{cof}(\omega)$ such that $y = N \cap \eta$. Then (Y, F') is a condition in $\mathbb{P}(\alpha, A, \beta)$ and $(Y, F') \leq (X_n, F_n)$ for every $n < \omega$.*

Theorem 4.1.5. *The forcing poset $\mathbb{P}(\alpha, A, \beta)$ is ω_1-distributive.*

Proof. For a collection of dense sets $\{D_n \mid n < \omega\}$ of $\mathbb{P}(\alpha, A, \beta)$ and a condition (X, F), we have to find a condition $(Y, K) \leq (X, F)$, which belongs to $\bigcap\{D_n \mid n < \omega\}$. Let θ be a sufficiently large regular cardinal and N a countable elementary submodel of $H(\theta)$, which contains all relevant objects. Furthermore let $N \cap \alpha \notin A$ if $\mathrm{cof}(\alpha) = \omega_1$, which is possible since in that case, $\mathbb{P}(\alpha, A, \beta) \setminus A$ is stationary in $\mathcal{P}_{\omega_1}(\alpha)$. We construct (Y, K) as follows. Pick an N-generic sequence $\langle (X_n, F_n) : n < \omega \rangle$ below (X, F). Let $Y = \bigcap\{X_n \mid n < \omega\}$ and K be the infinum of the sequence $\langle F_n : n < \omega \rangle$. Then (Y, K) is a condition in $\mathbb{P}(\alpha, A, \beta)$ with $(Y, K) \leq (X_n, F_n)$ for all $n < \omega$ by Proposition 4.1.4. In particular, $(Y, K) \leq (X, F)$. By assumption, D_n is in N for each $n < \omega$. So fix an $n < \omega$. Then (X_m, F_m) is in D_n for some $m < \omega$, since (Y, K) is in D_n. Hence (Y, K) is in $\bigcap\{D_n \mid n < \omega\}$. □

The proof of the next theorem uses the Δ-System Lemma repeatedly. We use the same arguments as in the proof which states that the forcing poset $\mathbb{Q}$ is ω_1-distributive, which we will give in detail.

Theorem 4.1.6. *If $2^\omega = \omega_1$, then the forcing poset $\mathbb{P}(\alpha, A, \beta)$ is ω_2-c.c..*

As hinted, the next definition gives us the required set, which is added by the forcing notion $\mathbb{P}(\alpha, A, \beta)$. We prove that it is a stationary subset of $\mathcal{P}_{\omega_1}(\beta)$, which does not reflect to any ordinal in $[\alpha, \beta)$ with cofinality ω_1.

Definition 4.1.7. *Let $\dot{S}(\alpha, A, \beta)$ be a $\mathbb{P}(\alpha, A, \beta)$-name such that*

$$\Vdash \dot{S}(\alpha, A, \beta) = \bigcup\{X \mid \exists F\ (X, F) \in \dot{H}\},$$

where $\dot{H}$ denotes the canonical name for the generic filter.
We write $\dot{S}_\beta$ for $\dot{S}(0, \emptyset, \beta)$.

Lemma 4.1.8. *In $V[H]$, for every ξ in $[\alpha, \beta) \cap \mathrm{cof}(\omega_1)$, the set $\{a_i^\xi \mid i < \omega_1\}$ is a club subset of $\mathcal{P}_{\omega_1}(\beta)$ which is disjoint from $S(\alpha, A, \beta)$. Hence $S(\alpha, A, \beta)$ does not reflect to any ordinal in $[\alpha, \beta) \cap \mathrm{cof}(\omega_1)$.*

Proof. Clearly, $\{a_i^\xi \mid i < \omega_1\}$ is club in $\mathcal{P}_{\omega_1}(\xi)$ for every ξ in $[\alpha, \beta) \cap \mathrm{cof}(\omega_1)$. Assume towards a contradiction that $S \cap \{a_i^\xi \mid i < \omega_1\}$ is non-empty. Then there is a b in S such that $b = a_i^\xi$ for some $i < \omega_1$. Now fix (X, F) in H such that $b \in X$. Then there is a condition (Y, K) below (X, F) such that b is in the sequence $F(\xi)$. Hence $b \in Y$, which contradicts property (5) of the definition of $\mathbb{P}(\alpha, A, \beta)$. □

Proposition 4.1.9. *The forcing poset $\mathbb{P}(\alpha, A, \beta)$ forces that $\dot{S}(\alpha, A, \beta)$ is a stationary subset of $\mathcal{P}_{\omega_1}(\beta)$.*

Proof. Let (X, F) be a condition in $\mathbb{P}(\alpha, A, \beta)$. Assume (X, F) forces that $\dot{C}$ is a club in $\mathcal{P}_{\omega_1}(\beta)$. We must find a condition $(Z, K) \leq (X, F)$ and a set b with $(Z, K) \Vdash b \in \dot{S} \cap \dot{C}$. Let θ, N and (Y, K) be as in the proof of Theorem 4.1.5. We define $Z = Y \cup \{N \cap \beta\}$. Then (Z, K) is a condition in $\mathbb{P}(\alpha, A, \beta)$ by Proposition 4.1.4. Recall that $(Z, K) \leq (X, F)$ and that (Z, K) is an N-generic condition. Since $N \cap \beta$ is in Y, $(Y, K) \Vdash N \cap S$. Furthermore, $\dot{C}$ is in N by assumption on N. Therefore, $\mathbb{P}(\alpha, A, \beta)$ forces that for a canonical $\mathbb{P}(\alpha, A, \beta)$-name $\dot{H}$ of the generic filter, $N[\dot{H}] \cap \beta \in \dot{C}$. Since (Z, K) is N-generic, $(Z, K) \Vdash N[\dot{H}] \cap \beta = N \cap \beta$. Hence, $(Z, K) \Vdash N \cap \beta \in \dot{S} \cap \dot{C}$. □

The next lemma is used to show that $\mathbb{P}(\alpha, A, \beta)$ preserves stationary sets disjoint from $S(\alpha, A, \beta)$. We state it without proof, which can be found in [8].

Lemma 4.1.10. *Let $\theta \geq \omega_1$ be a regular cardinal.*

(1) $\mathbb{P}(\alpha, A, \beta)$ forces that there are stationary many N in $\mathcal{P}_{\omega_1}(H(\theta))$ such that $\dot{S}(\alpha, A, \beta) \cap \mathcal{P}(N \cap \beta) \subseteq N$.

(2) Suppose $\mathrm{cof}(\alpha) = \omega_1$, and let T be a stationary subset of $\mathcal{P}_{\omega_1}(\alpha)$ which is disjoint from S. Then $\mathbb{P}(\alpha, A, \beta)$ forces that there are stationary many N in $\mathcal{P}_{\omega_1}(H(\theta))$ such that $\dot{S}(\alpha, A, \beta) \cap \mathcal{P}(N \cap \beta) \subseteq N$ and $N \cap \alpha \in T$.

Corollary 4.1.11. *Suppose that $\mathrm{cof}(\alpha) = \omega_1$. Let T be a stationary subset of $\mathcal{P}_{\omega_1}(\alpha)$ which is disjoint from A. Then $\mathbb{P}(\alpha, A, \beta)$ forces that T is stationary in $\mathcal{P}_{\omega_1}(\alpha)$.*

Proof. For a generic filter H on $\mathbb{P}(\alpha, A, \beta)$ over V, define a club subset C of $\mathcal{P}_{\omega_1}(\alpha)$ in $V[H]$. Let θ be a sufficiently large cardinal. Note that in V, θ is a regular cardinal $\geq \omega_2$. Using Lemma 4.1.10, we can find a countable elementary substructure N of $H(\theta)$ such that $C \in N$ and $N \cap \alpha \in T$. Since $C \in N$, $N \cap \alpha$ is in C, which completes the proof. □

Corollary 4.1.12. *The forcing poset* $\mathbb{P}(\alpha, A, \beta)$ *forces that* $\mathcal{P}_{\omega_1}(\beta) \setminus \dot{S}(\alpha, A, \beta)$ *is stationary in* $\mathcal{P}_{\omega_1}(\beta)$.

Proof. For a generic filter H on $\mathbb{P}(\alpha, A, \beta)$ over V, let $F : \mathcal{P}_\omega(\beta) \to \beta$ be a function defined in $V[H]$. Fix a regular cardinal θ much larger than β. Just like in the proof of the previous lemma, we can use Lemma 4.1.10 to get a countable elementary substructure N of $H(\theta)$ such that $F \in N$ and $S \cap \mathcal{P}(N \cap \beta) \subseteq N$. Then $N \cap \beta$ is closed under F by elementarity of N. Since $N \cap \beta$ is in $\mathcal{P}(N \cap \beta)$ but not in N, we have that $N \cap \beta$ is not in S. Hence $\mathcal{P}_{\omega_1}(\beta) \setminus S$ is stationary in $\mathcal{P}_{\omega_1}(\beta)$. $\square$

Now we have successfully added a stationary subset of $\mathcal{P}_{\omega_1}(\beta)$ which does not reflect to any ordinal in $[\alpha, \beta)$ with cofinality ω_1. By using the forcing poset $\mathbb{P}_{\omega_2}$, we have obtained a set satisfying (1) and (2) of the requirements for the proof of Theorem 4.1.1. We end this section by giving two lemmata for technical purposes.

Lemma 4.1.13. *For each set* b *in* $\mathcal{P}_{\omega_1}(\beta)$, *there are densely many conditions* (Y, K) *in* $\mathbb{P}(\alpha, A, \beta)$ *with* $b \subseteq \bigcup Y$.

Proof. Suppose (X, F) is an arbitrary condition. Consider a countable set $d \subseteq \beta$ with $b \cup (\bigcup X) \cup \{\alpha\} \subsetneq d$. It is easy to see that $(X \cup \{d\}, F)$ is a condition in $\mathbb{P}(\alpha, A, \beta)$ which is below (X, F). Then $b \subseteq \bigcup(X \cup \{d\})$. $\square$

Lemma 4.1.14. *Suppose* (X, F) *is a condition in* $\mathbb{P}(\alpha, A, \beta)$. *Then*

$$(X, F) \Vdash \dot{S}(\alpha, A, \beta) \cap \mathcal{P}(\bigcup X) = X.$$

Proof. It is clear that $(X, F) \Vdash X \subseteq \dot{S}(\alpha, A, \beta) \cap \mathcal{P}(\bigcup X)$. Let H be a $\mathbb{P}(\alpha, A, \beta)$-generic filter containing (X, F). Assume towards a contradiction that there is a member x of $\dot{S}(\alpha, A, \beta) \cap \mathcal{P}(\bigcup X)$, which is not in X. Then we can take a condition (Y, K) in H such that x is in Y. Now consider a condition (Z, L) in H which is below (X, F) and (Y, K). But then x is in $Z \setminus X$ and $x \subseteq \bigcup X$. This contradicts $(Z, L) \leq (X, F)$. $\square$

4.1.2 Extending an elementary embedding I

Next we give an idea how to factorize the forcing notion $\mathbb{P}_\beta$ as the two-step iteration $\mathbb{P}_\eta * \mathbb{P}(\eta, \dot{S}_\eta, \beta)$.

Definition 4.1.15. *Suppose (X, F) is a condition in $\mathbb{P}_\beta$ and for every $\xi \in \mathrm{dom}(F)$ we have $F(\xi) = \langle a_i^\xi : i < \gamma_\xi \rangle$. Then define*

(1) $X_\eta = \{b \in X \mid b \subseteq \eta\}$,

(2) $F_\eta = F \restriction \eta$,

(3) $X^\eta = \{b \in X \mid b \nsubseteq \eta\}$,

(4) $F^\eta = F \restriction [\eta, \beta)$.

Furthermore let $\mathbb{P}_{\eta,\beta}$ be the suborder of $\mathbb{P}_\beta$ consisting of conditions (X, F) with $\bigcup X_\eta = (\bigcup X^\eta) \cap \eta$.

Theorem 4.1.16. *If $i : \mathbb{P}_{\eta,\beta} \to \mathbb{P}_\eta * \mathbb{P}(\eta, \dot{S}_\eta, \beta)$ is defined by $i(X, F) = (X_\eta, F_\eta) * (\check{X}^\eta, \check{F}^\eta)$, then i is an isomorphism of $\mathbb{P}_{\eta,\beta}$ onto a dense subset of the two-step iteration $\mathbb{P}_\eta * \mathbb{P}(\eta, \dot{S}_\eta, \beta)$.*

Since one can show that $\mathbb{P}_{\eta,\beta}$ is dense, Theorem 4.1.16 implies that $\mathbb{P}_\beta$ is forcing equivalent to $\mathbb{P}_\eta * \mathbb{P}(\eta, \dot{S}_\eta, \beta)$. Recall that the reason for this factorization was to extend the elementary embedding j we obtain from supercompactness of the cardinal κ. The next theorem summarizes all properties of the extension of j, which will also be denoted by j for simplicity. For the proof we refer the reader to [8].

Theorem 4.1.17. *Suppose that $\bar{G} * \bar{H}$ is a $j(\mathrm{Coll}(\omega_1, < \kappa) * \dot{\mathbb{P}}_\kappa)$-generic filter over V. By letting $G = \bar{G} \cap \mathrm{Coll}(\omega_1, < \kappa)$ and $H = \bar{H} \cap \mathbb{P}_\kappa$, we have that G is a $\mathrm{Coll}(\omega_1, < \kappa)$-generic filter over V and H is a $\mathbb{P}_\kappa$-generic filter over $V[G]$. Moreover, the elementary embedding $j : V \to M$ can be extended in $V[G * H]$ to $j : V[G * H] \to M[\bar{G} * \bar{H}]$ with $j(G * H) = \bar{G} * \bar{H}$. Furthermore,* *(1)* $M[G * H]^{\kappa^+} \cap V[G * H] \subseteq M[G * H]$,

(2) if H' *is a* $\mathbb{P}(\kappa, \dot{S}_\kappa, j(\kappa))$*-generic filter over* $M[\bar{G}][H]$*, then* $M[\bar{G} * \bar{H}] = M[\bar{G}][H][H']$,

(3) $j(S_\kappa) = S_{j(\kappa)}$, $S_{j(\kappa)} = S_\kappa \cup S(\kappa, S_\kappa, j(\kappa))$, $S_\kappa = \{b \in S_{j(\kappa)} \mid b \subseteq \kappa\}$*, and* $S(\kappa, S_\kappa, j(\kappa)) = \{b \in S_{j(\kappa)} \mid b \not\subseteq \kappa\}$.

4.1.3 Adding a club

The next goal is to find a forcing poset designed to destroy the stationarity of all non-reflecting subsets of S. In Chapter 2 we have reviewed a forcing notion $\mathbb{Q}(T)$ due to Sakai which adds a club disjoint from T. Recall the following definition.

Definition 4.1.18. *Let* T *be a subset of* $\mathcal{P}_{\omega_1}(\omega_2)$*. Define* $\mathbb{Q}(T)$ *as the forcing poset consisting of conditions* p *satisfying:*

(1) p *is a function of the form* $p : a^p \times a^p \to \omega_1$*, where* a^p *is a countable subset of* ω_2*,*

(2) for every x *in* T*, if* $x \subseteq a^p$*, then* x *is not closed under* p*.*

Let $q \leq p$*, if* q *extends* p *as a function, that is, if* $a^p \subseteq a^q$ *and* $q \restriction (a^p \times a^p) = p$*. For a countable set* $a \subseteq \omega_2$*, we write* a^2 *for* $a \times a$*.*

Now we want to know how to iterate this club shooting. We work in the extension of V by a generic filter on $\mathbb{P}_{\omega_2}$ and define a forcing notion $\mathbb{Q}$, which forces (3) of the requirements for the proof of Theorem 4.1.1, i.e., every stationary subset of S_{ω_2} reflects to an uncountable ordinal in ω_2 with cofinality ω. Similar to Chapter 2, the forcing notion $\mathbb{Q}$ is an iteration of club shootings $\mathbb{Q}(T)$ for each subset T of S_{ω_2}, which does not reflect to any uncountable ordinal in ω_2 with cofinality ω.
The following definitions are for technical purposes to define $\mathbb{Q}$.

Definition 4.1.19. *Let $f : \omega_3 \to \omega_3 \times \omega_3$ be a surjective function satisfying that whenever $f(\alpha) = (i, j)$, then $i \leq \alpha$. This f is used as a bookkeeping function.*

Definition 4.1.20. *Let $\mathbb{R}$ denote the set of all non-empty partial functions r of the form $r : (a^r)^2 \to \omega_1$, where a^r is a countable subset of ω_2. Let $s \leq_{\mathbb{R}} r$ if s extends r as a function, that is, $a^r \subseteq a^s$ and $s \restriction (a^r) = r$.*

Definition 4.1.21. *For every ordinal α in ω_2, fix a surjective function $\sigma_\alpha : \omega_1 \to \alpha$. Let E be the set of all b in $\mathcal{P}_{\omega_1}(\omega_2)$ such that $\mathrm{ot}(b)$ is a limit ordinal and b is closed under σ_α for all α in b.*
Note that E is a club subset of $\mathcal{P}_{\omega_1}(\omega_2)$.

The forcing poset $\mathbb{Q}$ will be defined satisfying the following recursion hypotheses. Note that we work in $V[H]$. In particular, we define the sequence of forcing posets $\langle \mathbb{Q}_\alpha : \alpha \leq \omega_3 \rangle$ such that each $\mathbb{Q}_\alpha$ satisfies the recursion hypotheses below. We will use the bookkeeping function to enumerate all subsets of S_{ω_2}, which do not reflect to an uncountable ordinal with cofinality ω. We define the sequence $\langle \dot{T}^i_j : i, j \leq \omega_3 \rangle$ of those subsets as follows. For every $\alpha < \omega_3$ and $f(\alpha) = (i, j)$, we let $T(\alpha) = \dot{T}^i_j$. Then $\mathbb{Q}$ is defined as the forcing poset $\mathbb{Q}_{\omega_3}$.

Recursion Hypotheses: For all $\alpha \leq \omega_3$:

(1) If p is in $\mathbb{Q}_\alpha$, then p is a partial function $p : \alpha \to \mathbb{R}$ whose domain is countable, and for all p and q in $\mathbb{Q}_\alpha$, $q \leq p$ if and only if $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and for all η in $\mathrm{dom}(p)$, let $q(\eta) \leq_{\mathbb{R}} p(\eta)$.

(2) Let $\beta < \alpha$. Then

  (a) for all q in $\mathbb{Q}_\alpha$, $q \restriction \beta$ is in $\mathbb{Q}_\beta$,

  (b) $\mathbb{Q}_\beta \subseteq \mathbb{Q}_\alpha$,

  (c) if q is in $\mathbb{Q}_\alpha$ and $s \leq q \restriction \beta$ is in $\mathbb{Q}_\beta$, then letting $t = s \cup (q \restriction [\beta, \alpha))$, t is in $\mathbb{Q}_\alpha$ and $t \leq s$, q in $\mathbb{Q}_\alpha$, and

(d) the inclusion map $\mathbb{Q}_\beta \to \mathbb{Q}_\alpha$ is a complete embedding.

(3) $\mathbb{Q}_\alpha$ is ω_1-distributive and ω_2-c.c.

(4) If $\alpha < \omega_3$, then the sequence $\langle \dot{T}_i^\alpha : i < \omega_3 \rangle$ is an enumeration of all nice $\mathbb{Q}_\alpha$-names $\dot{T}$ for a subset of $\mathcal{P}_{\omega_1}(\omega_2)$ such that $\mathbb{Q}_\alpha$ forces $\dot{T} \subseteq S_{\omega_2} \cap E$ and $\dot{T}$ does not reflect to any uncountable ordinal in $\omega_2 \cap \mathrm{cof}(\omega)$.

One can easily show that the recursion hypotheses (1), (2) and (4) are satisfied by the next definition. Since checking (3) is more work, we will tend to this separately.

Definition 4.1.22. *We define $\mathbb{Q}_\alpha$. The definition splits into three cases.*
- α *is equal to* 0
Let $\mathbb{Q}_0$ consist of the empty function.
- α *is a limit ordinal*
Let $\mathbb{Q}_\alpha$ be the set of all partial functions $p : \alpha \to \mathbb{R}$, whose domain is countable such that for all $\beta < \alpha$, $p \restriction \beta$ is in $\mathbb{Q}_\beta$. Let $q \leq p$ in $\mathbb{Q}_\alpha$ if $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and for all η in $\mathrm{dom}(p)$, $q(\eta) \leq_{\mathbb{R}} p(\eta)$.
- α *is a successor ordinal*
Let $\alpha = \beta + 1$ and f be the bookkeeping function from Definition 4.1.19, that is, $f(\beta) = (i, j)$. Then $i \leq \beta$ and $j < \omega_3$, so $\dot{T}_j$ is defined and is equal to $\dot{T}(\beta)$.
Let $\mathbb{Q}_\alpha$ consist of all partial functions $p : \alpha \to \mathbb{R}$, whose domain is countable such that $p \restriction \beta$ is in $\mathbb{Q}_\beta$ and if β is in $\mathrm{dom}(p)$, we have $p \restriction \beta \Vdash_{\mathbb{Q}_\beta} p(\beta) \in \mathbb{Q}(\dot{T}(\beta))$. Let $q \leq p$ if $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and for all η in $\mathrm{dom}(p)$, $q(\eta) \leq_{\mathbb{R}} p(\eta)$.

The next lemma completes the recursive definition of $\mathbb{Q}$.

Lemma 4.1.23. *The forcing poset $\mathbb{Q}_{\beta+1}$ is isomorphic to a dense subset of $\mathbb{Q}_\beta * \mathbb{Q}(\dot{T}(\beta))$. Therefore $\mathbb{Q}_{\beta+1}$ is forcing equivalent to $\mathbb{Q}_\beta * \mathbb{Q}(\dot{T}(\beta))$.*

Notation 4.1.24. *Let p be a partial function $p : \alpha \to \mathbb{R}$, whose domain is countable. For every ξ in $\mathrm{dom}(p)$, let a^p_ξ denote $a^{p(\xi)}$. Particularly, a^p_ξ is the non-empty countable set a with $\mathrm{dom}(p(\xi)) = a^2$.*

The following lemma follows directly from the definition of the $\mathbb{Q}_\alpha$'s.

Lemma 4.1.25. *Let p be in $\mathbb{Q}_\alpha$. Suppose β is in $\mathrm{dom}(p)$ and x is a countable subset of a^p_β which is closed under $p(\beta)$. Then $p \restriction \beta \Vdash x \notin \dot{T}(\beta)$.*

Lemma 4.1.26. *Let p be a partial function $p : \alpha \to \mathbb{R}$ with a countable domain. If p is not in $\mathbb{Q}_\alpha$, then there exist β, t, and x such that:*

(1) β is in $\mathrm{dom}(p)$,

(2) $p \restriction \beta$ is in $\mathbb{Q}_\beta$,

(3) $t \leq p \restriction \beta$ in $\mathbb{Q}_\beta$,

(4) x is a countable subset of a^p_β which is closed under $p(\beta)$,

(5) $t \Vdash_\beta x \in \dot{T}(\beta)$.

Proof. We prove this lemma by induction on α. Clearly, it holds when $\alpha = 0$. So suppose this statement is true for $\mathbb{Q}_\eta$ for all $\eta < \alpha$.
Case 1. $\alpha = \alpha_0 + 1$
Assume that p is a partial function of the form $p : \alpha \to \mathbb{R}$ whose domain is countable and $p \notin \mathbb{Q}_\alpha$. Then either $p \restriction \alpha_0 \notin \mathbb{Q}_\alpha$ or $p \restriction \alpha_0 \in \mathbb{Q}_\alpha$, $\alpha_0 \in \mathrm{dom}(p)$, and $p \restriction \alpha_0$ does not force that $p(\alpha_0) \in \mathbb{Q}(\dot{T}(\alpha_0))$. In the first case, we can apply the induction hypothesis to $p \restriction \alpha_0$ in $\mathbb{Q}_{\alpha_0}$ and we are done. In the second case, there exists $t_0 \leq p \restriction \alpha_0$, which forces that $p(\alpha_0) \notin \mathbb{Q}(\dot{T}(\alpha_0))$. The fact that $p(\alpha_0)$ is in $\mathbb{R}$ implies that t_0 forces that there is a countable set $x \subseteq a^{p(\alpha_0)}$, which is closed under $p(\alpha_0)$ and is a member of $\dot{T}(\alpha_0)$. We can apply Recursion Hypothesis (3) to the forcing poset $\mathbb{P}_{\alpha_0}$ and therefore it is ω_1-distributive. This asserts that t_0 forces that there exists such a set x in the ground model. So pick a condition $t \leq t_0$ and a set x such that t forces

x satisfies all properties above. Then x is a countable subset of $a^p_{\alpha_0}$ which is closed under $p(\alpha_0)$. Furthermore, t forces that x is in $\dot{T}(\alpha_0)$.

Case 2. α is limit

Suppose that $p \notin \mathbb{Q}_\alpha$. Then there exists $\gamma < \alpha$ with $p \restriction \gamma \notin \mathbb{Q}_\gamma$ by definition of $\mathbb{Q}_\alpha$. Again, we can apply the induction hypothesis to $p \restriction \gamma$ in $\mathbb{Q}_\gamma$ and the proof is complete. □

Lemma 4.1.27. *Let p and q be conditions in $\mathbb{Q}_\alpha$ such that for all β in $\mathrm{dom}(p) \cap \mathrm{dom}(q)$, $p(\beta) \restriction (a^p_\beta \cap a^q_\beta)^2 = q(\beta) \restriction (a^p_\beta \cap a^q_\beta)^2$. Then p and q are compatible. Furthermore, there is a condition $r \leq p, q$ such that*

(1) $\mathrm{dom}(r) = \mathrm{dom}(p) \cup \mathrm{dom}(q)$,

(2) for all β in $\mathrm{dom}(p) \setminus \mathrm{dom}(q)$, $r(\beta) = p(\beta)$, *and*

(3) for all β in $\mathrm{dom}(q) \setminus \mathrm{dom}(p)$, $r(\beta) = q(\beta)$.

In particular, if p and q are conditions in $\mathbb{Q}_\alpha$ such that for all β in $\mathrm{dom}(p) \cap \mathrm{dom}(q)$, *$a^p_\beta \cap a^q_\beta$ is empty, then p and q are compatible.*

Lemma 4.1.28. *Let z be a countable subset of α and let b be a countable subset of ω_2. Then there are densely many conditions t in $\mathbb{Q}_\alpha$ such that $z \subseteq \mathrm{dom}(t)$ and for all β in z, $b \subseteq a^t_\beta$.*

Definition 4.1.29. *We call a condition p in $\mathbb{Q}_\alpha$* square*, if there is a set a such that for all β in $\mathrm{dom}(p)$, we have $a^p_\beta = a$. Then a^p denotes this set.*

Proposition 4.1.30. *The forcing poset $\mathbb{Q}_\alpha$ is ω_1-distributive.*

Proof. Suppose $\{D_n \mid n < \omega\}$ is a family of dense open subsets of $\mathbb{Q}_\alpha$ and p is a condition in $\mathbb{Q}_\alpha$. We must find a condition $q \leq p$ such that $q \in \bigcap\{D_n \mid n < \omega\}$. Let θ be a sufficiently large cardinal. Now we can apply Lemma 4.1.10 to $\mathbb{P}_{\omega_2}$ to find stationary many N in $\mathcal{P}_{\omega_1}(H(\theta))$ which satisfy $S_{\omega_2} \cap \mathcal{P}(N \cap \omega_2) \subseteq N$. Note that this is in the model $V[H]$, where H denotes a generic filter on $\mathbb{P}_{\omega_2}$ over V. So we can pick an N in $\mathcal{P}_{\omega_1}(H(\theta))$ which is an elementary submodel of $H(\theta)$ containing all relevant objects as elements and $S_{\omega_2} \cap \mathcal{P}(N \cap \omega_2) \subseteq N$.

Next take an N-generic sequence $\langle p_n : n < \omega \rangle$ in $\mathbb{Q}_\alpha$ with $p_0 = p$. Then it is easy to see that $\bigcup\{\mathrm{dom}(p_n) \mid n < \omega\} = N \cap \alpha$. For every $\beta \in N \cap \alpha$, we have $\bigcup\{a_\beta^{p_n} \mid n < \omega\} = N \cap \omega_2$.

We construct q in the following way. Let $\mathrm{dom}(q) = \bigcup\{\mathrm{dom}(p_n) \mid n < \omega\}$ and for each $\beta \in \mathrm{dom}(q)$, let $q(\beta) = \bigcup\{p_n(\beta) \mid n < \omega\}$. Therefore q is a partial function $q : \alpha \to \mathbb{R}$ whose domain is countable. To finish this proof, we need to show that q is a condition in $\mathbb{Q}_\alpha$. Because then q is below p_n for all $n < \omega$ and hence is in $\bigcap\{D_n \mid n < \omega\}$.

So assume that q is not a condition in $\mathbb{Q}_\alpha$. By Lemma 4.1.26, we can fix β, t, and x such that $\beta \in \mathrm{dom}(q) = N \cap \alpha$, $q \restriction \beta \in \mathbb{Q}_\beta$, $t \leq q \restriction \beta$ in $\mathbb{Q}_\beta$, x is a countable subset of $a_\beta^q = N \cap \omega_2$ which is closed under $q(\beta)$, and $t \Vdash_\beta x \in \dot{T}(\beta)$. Observe that $q \restriction \beta \leq p_n \restriction \beta$ for all $n < \omega$. Since $t \Vdash_\beta x \in \dot{T}(\beta)$, the set x is in S_{ω_2}. Furthermore, $x \subseteq N \cap \omega_2$. By Lemma 4.1.10, x is in $S_{\omega_2} \cap \mathcal{P}(N \cap \omega_2)$. Hence x is in N by assumption on N.

To prove the contradiction, let D denote the set of conditions s in $\mathbb{Q}_\alpha$ such that β is in $\mathrm{dom}(s)$ and $x \subseteq a_\beta^s$. Then D is dense open by Lemma 4.1.28 and in N by elementarity of N. Fix $n < \omega$ with $p_n \in D$. Then x is closed under $p_n(\beta)$, since x is closed under $q(\beta)$ and $q(\beta) \restriction (a_\beta^{p_n})^2 = p_n(\beta)$. But then $p_n \restriction \beta \Vdash_\beta x \notin \dot{T}(\beta)$. Since $t \Vdash_\beta x \in \dot{T}(\beta)$, the conditions $p_n \restriction \beta$ and t must be incompatible. This is a contradiction, because $t \leq q \restriction \beta \leq p_n \restriction \beta$. □

Proposition 4.1.31. *The forcing notion $\mathbb{Q}_\alpha$ is ω_2-c.c.*

Proof. We show that the forcing notion $\mathbb{Q}_\alpha$ is ω_2-Knaster. Suppose $\langle p_i : i < \omega_2 \rangle$ is a sequence of conditions in $\mathbb{Q}_\alpha$. Without loss of generality, we may assume that p_i is square for each $i < \omega_2$. Note that CH holds in $V[H]$. Therefore, we can apply the Δ-System Lemma to the sequence $\langle \mathrm{dom}(p_i) : i < \omega_2 \rangle$, which consists of countable sets, to find an unbounded set $Z_0 \subseteq \omega_2$ and a countable set b such that for all $i < j$ in Z_0, we have $\mathrm{dom}(p_i) \cap \mathrm{dom}(p_j) = b$. We use the Δ-System Lemma again to the sequence $\langle a^{p_i} : i \in Z_0 \rangle$ to find an unbounded set $Z_1 \subseteq Z_0$ and a countable set c such that for all $i < j$ in Z_1, we have $a^{p_i} \cap a^{p_j} = c$.

Using CH, we can show that for each $i \in Z_1$, there are at most ω_1 many possibilities for a sequence $\langle p_i(\beta) \upharpoonright c^2 : \beta \in b \rangle$. Thus, fix an unbounded set $Z_2 \subseteq Z_1$ such that for all $i < j$ in Z_2 and all $\beta \in b$, we have $p_i(\beta) \upharpoonright c^2 = p_j(\beta) \upharpoonright c^2$. So let $i < j$ in Z_2. Thus if $\beta \in \mathrm{dom}(p_i) \cap \mathrm{dom}(p_j) = b$, then $p_i(\beta) \upharpoonright (a^{p_i} \cap a^{p_j})^2 = p_i(\beta) \upharpoonright c^2 = p_j(\beta) \upharpoonright (a^{p_i} \cap a^{p_j})^2$. Then p_i and p_j are compatible by Lemma 4.1.27. $\square$

4.1.4 Preserving the stationarity of S_{ω_2}

Proposition 4.1.32. *The forcing poset $\mathbb{Q}$ forces that S_{ω_2} is stationary in $\mathcal{P}_{\omega_1}(\omega_2)$.*

Proof. We will work in the model $V[H]$, where H is a generic filter over V. So it suffices to show in V that the forcing notion $\mathbb{P}_{\omega_2} * \dot{\mathbb{Q}}$ forces that $\dot{S}_{\omega_2}$ is stationary.

Now take a condition $(X, F) * \dot{p}$ in $\mathbb{P}_{\omega_2} * \dot{\mathbb{Q}}_\alpha$ and assume that $(X, F) * \dot{p}$ forces that $\dot{h} : \mathcal{P}_\omega(\omega_2) \to \omega_2$ is a function. To show that S_{ω_2} is stationary, we must find a condition $(Y, K) * \dot{q}$ below $(X, F) * \dot{p}$ and a set z such that $(Y, K) * \dot{q}$ forces that z is a member of $\dot{S}_{\omega_2}$ which is closed under $\dot{h}$. Since $(X, F) * \dot{p}$ forces that there are club many ordinals $\tau < \omega_2$ which are closed under $\dot{h}$, we can choose a condition $(X', F') * \dot{p}' \leq (X, F) * \dot{p}$ and an uncountable ordinal τ in ω_2 with cofinality ω such that $(X', F') * \dot{p}'$ forces τ is closed under $\dot{h}$.

For each $\alpha < \omega_3$, the iteration $\mathbb{P}_{\omega_2} * \dot{\mathbb{Q}}$ forces that $\dot{T}(\alpha)$ is a subset of $\mathcal{P}_{\omega_1}(\omega_2)$ which does not reflect to any uncountable ordinal in $\omega_2 \cap \mathrm{cof}(\omega)$. Then in particular, $\mathbb{P}_{\omega_2} * \dot{\mathbb{Q}}_\alpha$ forces that $\dot{T}(\alpha)$ does not reflect to τ. So pick a $\mathbb{P}_{\omega_2} * \dot{\mathbb{Q}}$-name $\dot{f}_\alpha$ for a function $\dot{f}_\alpha : \mathcal{P}_\omega(\tau) \to \tau$ such that $\mathbb{P}_{\omega_2} * \dot{\mathbb{Q}}_\alpha$ forces that no set in $\dot{T}(\alpha) \cap \mathcal{P}_{\omega_1}(\tau)$ is closed under $\dot{f}_\alpha$.

Let θ be a sufficiently large cardinal and N a countable elementary substructure of $H(\theta)$ containing all relevant objects, particularly the sequence of functions $\langle \dot{f}_\alpha : \alpha < \omega_3 \rangle$. The goal is to define a condition $(Y, K) * \dot{q} \leq (X', F') * \dot{p}'$

which is N-generic for $\mathbb{P}_{\omega_2} * \dot{\mathbb{Q}}$.

Therefore, pick an N-generic sequence $\langle (X_n, F_n) * \dot{p}_n : n < \omega \rangle$ with $(X_0, F_0) * \dot{p}_0 = (X', F') * \dot{p}'$. Define K as the infinum of the sequence $\langle F_n : n < \omega \rangle$ as in Definition 4.1.3. Further, let $Y = \bigcup\{X_n : n < \omega\} \cup \{N \cap \tau\}$. Then (Y, K) is an N-generic condition in $\mathbb{P}_{\omega_2}$ and $(Y, K) \leq (X_n, F_n)$ for all $n < \omega$ due to Proposition 4.1.4.

Since $N \cap \tau$ is in Y, $(Y, K) \Vdash N \cap \tau \in \dot{S}_{\omega_2}$. By Lemma 4.1.13, we obtain $\bigcup Y = N \cap \omega_2$. Then using Lemma 4.1.14, we have $(Y, K) \Vdash \dot{S}_{\omega_2} \cap \mathcal{P}(N \cap \omega_2) = Y$.

Let H be a $\mathbb{P}_{\omega_2}$-generic filter over V containing the condition (Y, K). Then $N[H] \cap V = N$ by N-genericity of (Y, K). If we let $p_n = \dot{p}_n^H$ for each $n < \omega$ then $\langle p_n < \omega \rangle$ is an $N[H]$-generic sequence for $\mathbb{Q}$. By Lemma 4.1.5 and $N[H]$-genericity, $\bigcup\{\text{dom}(p_n) \mid n < \omega\} = N[H] \cap \omega_3 = N \cap \omega_3$ and for every β in $\bigcup\{\text{dom}(p_n) \mid n < \omega\}$, we have $\bigcup\{a_\beta^{p_n} \mid n < \omega\} = N[H] \cap \omega_2 = N \cap \omega_2$.

Now we define a lower bound q of the sequence $\langle p_n < \omega \rangle$. Let the $\text{dom}(q) = \bigcup\{\text{dom}(p_n) \mid n < \omega\}$ and for each β in $\text{dom}(q)$, define $q(\beta)$ as the union of the set of functions $\{p_n(\beta) : n < \omega\}$. It is easy to see that q is a partial function $q : \omega_3 \to \mathbb{R}$ whose domain is countable. Moreover, $\text{dom}(q) = N \cap \omega_3$, and for each β in $\text{dom}(q)$, we have $a_\beta^p = N \cap \omega_2$.

Next we need to prove that q is a condition in $\mathbb{Q}$. Then q is clearly below p_n for all $n < \omega$. Assume towards a contradiction that q is not in $\mathbb{Q}$. Using Lemma 4.1.26, we can fix β, t and x such that β is in $\text{dom}(q)$, $q \restriction \beta$ is in $\mathbb{Q}_\beta$, $t \leq q \restriction \beta$ in $\mathbb{Q}_\beta$, x is a countable subset of a_β^q which is closed under $q(\beta)$, and $t \Vdash_\beta x \in \dot{T}(\beta)$. Note here that β is in $N \cap \omega_3$ and x is in $S_{\omega_2} \cap \mathcal{P}(N \cap \omega_2) = Y$. It is clear that $q \restriction \beta \leq p_n \restriction \beta$ for all $n < \omega$. In particular, $q \restriction \beta$ is $N[H]$-generic for $\mathbb{Q}_\beta$.

Since x is in $Y = \bigcup\{X_n : n < \omega\} \cup \{N \cap \tau\}$ and $\bigcup\{X_n : n < \omega\} \subseteq N$, we have that x is either in N or $x = N \cap \tau$. First assume that x is in N. Let D be the dense subset of conditions s in $\mathbb{Q}$ with $\beta \in \text{dom}(s)$ and $x \subseteq a_\beta^s$. Then the elementarity of N implies that D is in N. Now fix $n < \omega$ such that p_n is in D. We obtain that $x \subseteq p_n$. Since x is closed under $q(\beta)$ and

$q(\beta) \restriction (a_\beta^{p_n})^2 = p_n(\beta)$, the set x is closed under $p_n(\beta)$. Lemma 4.1.25 implies that $p_n \restriction \beta \Vdash_\beta x \notin \dot{T}(\beta)$. Since t forces that $x \in \dot{T}(\beta)$, the conditions $p_n \restriction \beta$ and t must be incompatible. But $t \leq q \restriction \beta \leq p_n \restriction \beta$, which is a contradiction. The second case is $x = N \cap \tau$. Recall that $\mathbb{P}_{\omega_2} * \dot{\mathbb{Q}}$ forces that $\dot{f}_\beta : \mathcal{P}_\omega(\tau) \to \tau$ is a function such that there is no set in $\dot{T}(\beta) \cap \mathcal{P}_{\omega_1}(\tau)$ which is closed under $\dot{f}_\beta$. Since β is in N, the function $\dot{f}_\beta$ is also in N by elementarity of N. Suppose I_β is a $\mathbb{Q}_\beta$-generic filter over $V[H]$ which contains t. By choice of t, we obtain $t \leq q \restriction \beta$. Since $q \restriction \beta$ is $N[H]$-generic, $N[H][I_\beta] \cap V[H] = N[H]$. Furthermore, $N[H][I_\beta] \cap \tau = N[H] \cap \tau = N \cap \tau$. Define $f_\beta = \dot{f}_\beta^{H * I_\beta}$. Since the function $\dot{f}_\beta$ is in N, we have f_β is in $N[H][I_\beta]$. Then $N[H][I_\beta] \cap \tau = N \cap \tau$ is closed under f_β by elementarity of $N[H][I_\beta]$. But since t is in I_β, the set $x = N \cap \tau$ is in $T(\beta) = \dot{T}(\beta)^{I_\beta}$. So x is a set in $T(\beta) \cap \mathcal{P}_{\omega_1}(\tau)$ and x is closed under f_β, which is a contradiction by choice of f_β. Therefore, q is a condition in $\mathbb{Q}$.

Suppose $\dot{q}$ is a $\mathbb{P}_{\omega_2}$-name such that the condition (Y, K) forces that $\dot{q}$ is in $\dot{\mathbb{Q}}$ and $\dot{q} \leq \dot{p}_n$ for all $n < \omega$. Then the condition $(Y, K) * \dot{q}$ is N-generic for $\mathbb{P}_{\omega_2} * \dot{\mathbb{Q}}$ and below $(X, F) * \dot{p}$. Now we need to show that $(Y, K) * \dot{q}$ forces that $N \cap \tau$ is a member of $\dot{S}_{\omega_2}$ which is closed under the function $\dot{h}$. Since $\dot{h}$ belongs to N, the condition $(Y, K) * \dot{q}$ forces that $N[\dot{H} * \dot{I}] \cap \omega_2$ is closed under $\dot{h}$, where $\dot{H} * \dot{I}$ is the canonical $\mathbb{P}_{\omega_2} * \dot{\mathbb{Q}}$-name for the generic filter. Hence, $(Y, K) * \dot{q}$ forces that $N[\dot{H} * \dot{I}] \cap \tau$ is closed under $\dot{h}$. We obtain that $(Y, K) * \dot{q} \Vdash N[\dot{H} * \dot{I}] \cap \tau = N \cap \tau$ by N-genericity of $(Y, K) * \dot{q}$. But $N \cap \tau$ is a member of Y and therefore, $(Y, K) * \dot{q}$ forces that $N \cap \tau$ is in $\dot{S}_{\omega_2}$ and closed under $\dot{h}$. □

Theorem 4.1.33. *The forcing notion $\mathbb{Q}$ forces that S_{ω_2} is a stationary set in $\mathcal{P}_{\omega_1}(\omega_2)$, which does not reflect to any ordinal in $\omega_2 \cap \mathrm{cof}(\omega_1)$ and for every stationary subset T of S_{ω_2}, there exists an uncountable ordinal τ in $\omega_2 \cap \mathrm{cof}(\omega)$ such that T reflects to τ.*

Proof. Take a condition p in $\mathbb{Q}$ such that p forces that $\dot{U} \subseteq S_{\omega_2}$ such that $\dot{U}$ does not reflect to any uncountable ordinal in $\omega_2 \cap \mathrm{cof}(\omega)$. The goal is to prove that p forces that $\dot{U}$ is non-stationary. As the forcing poset $\mathbb{Q}$ is ω_1-distributive, E is still club in any generic extension by $\mathbb{Q}$. Therefore, it suffices to show that $\dot{U} \cap E$ is non-stationary.

Next fix a nice $\mathbb{Q}$-name $\dot{T}$ for a subset of $\mathcal{P}_{\omega_1}(\omega_2)$ such that $p \Vdash_{\mathbb{Q}} \dot{T} = \dot{U} \cap E$. Furthermore, let $\mathbb{Q}$ force that if p is not in the generic filter, then $\dot{T}$ is the empty set. Now for some sequence of antichains $\langle A_x : x \in \mathcal{P}_{\omega_1}(\omega_2)\rangle$ of $\mathbb{Q}$, we have that $\dot{T}$ is equal to $\bigcup\{A_x \times \{\check{x}\} \mid x \in \mathcal{P}_{\omega_1}(\omega_2)\}$. Next we do a simple counting argument. Since $\mathbb{Q}$ has the ω_2-c.c., the cardinality of each A_x is at most ω_1. As $\mathcal{P}_{\omega_1}(\omega_2)$ has cardinality ω_2 and $\mathbb{Q} = \bigcup\{\mathbb{Q}_\alpha \mid \alpha < \omega_3\}$, we can fix $\beta < \omega_3$ such that for all $x \in \mathcal{P}_{\omega_1}(\omega_2)$, the antichain A_x is a subset of $\mathbb{Q}_\beta$. Thus $\dot{T}$ is a nice $\mathbb{Q}_\beta$-name.

For every τ in $\omega_2 \cap \mathrm{cof}(\omega)$, we have that $\Vdash_{\mathbb{Q}}$ "$\dot{T}$ does not reflect to τ". Hence we can pick a nice $\mathbb{Q}$-name $\dot{C}_\tau$ for a club in $\mathcal{P}_{\omega_1}(\tau)$ which is disjoint from $\dot{T}$. Repeating the argument above, we can choose $\xi < \omega_3$ larger than β such that $\dot{C}_\tau$ is a $\mathbb{Q}_\xi$-name for all such τ. Clearly, $\mathbb{Q}_\xi$ forces that $\dot{C}_\tau \cap \dot{T}$ is empty for all uncountable τ in $\omega_2 \cap \mathrm{cof}(\omega)$.

We have shown that $\dot{T}$ is a nice $\mathbb{Q}_\xi$-name for a subset of $\mathcal{P}_{\omega_1}(\omega_2)$ such that $\mathbb{Q}_\xi$ forces that $\dot{T} \subseteq S_{\omega_2} \cap E$. Further $\dot{T}$ does not reflect to τ for all uncountable τ in $\omega_2 \cap \mathrm{cof}(\omega)$. By recursion hypothesis (4), there exists $i < \omega_3$ with $\dot{T} = \dot{T}^\xi_i$. Now fix an ordinal $\gamma < \omega_3$ such that $f(\gamma) = (\xi, i)$. Then $\dot{T}(\gamma) = \dot{T}^\xi_i = \dot{T}$. By Lemma 4.1.23, the forcing poset $\mathbb{Q}_{\gamma+1}$ adds a club in $\mathcal{P}_{\omega_1}(\omega_2)$ which is disjoint from $\dot{T}$. As the inclusion map $\mathbb{Q}_{\gamma+1} \to \mathbb{Q}$ is a complete embedding, $\mathbb{Q}$ adds a club which is disjoint from $\dot{T}$. Since $p \Vdash \dot{T} = \dot{U} \cap E$, the condition p forces that $U \cap E$ is non-stationary. □

4.1.5 Extending an elementary embedding II

Next we show how to define a projection mapping $\pi : j(\mathbb{Q}) \to \mathbb{Q}$ in $M[\bar{G} * \bar{H}]$. One actually has to prove the existence of such a j, but we leave this to the reader. Let $\bar{I}$ be a generic filter on $j(\mathbb{Q})$ over $V[\bar{G} * \bar{H}]$. The incentive is to extend j in $V[\bar{G} * \bar{H} * \bar{I}]$ such that $j(G * H * I) = \bar{G} * \bar{H} * \bar{I}$, where I is the filter on $\mathbb{Q}$ generated by $\pi[\bar{I}]$.

When extending the elementary embedding, we always have to keep track of the cardinal κ. In $M[G*H]$ and $V[G*H]$, κ is equal to ω_2 and its successor κ^+ is equal to ω_3 by assumption on κ. Nevertheless, in the model $M[\bar{G}*\bar{H}]$ we have that $j(\kappa)$ is equal to ω_2. Note that κ and κ^+ are both ordinals with cofinality ω_1 in (ω_1, ω_2). Here, κ^+ denotes the ordinal successor, since κ is no longer a cardinal in this extension.

One can easily deduce from Theorem 4.1.17 and the Recursion Hypotheses (1) and (4) that the sequences $\langle \mathbb{Q}_\alpha : \alpha \leq \kappa^+ \rangle$ and $\langle \dot{T}(\alpha) : \alpha < \kappa \rangle$ are in $M[G*H]$.

Definition 4.1.34. *For each $\alpha \leq \kappa^+$, we define a map π_α with domain $j(\mathbb{Q}_\alpha)$ in $M[\bar{G}*\bar{H}]$ as follows. For a condition q in $j(\mathbb{Q}_\alpha)$, let the domain of π_α be equal to $j^{-1}(\mathrm{dom}(q) \cap j[\alpha])$, and for every γ in this domain, let $\pi_\alpha(q)(\gamma) = q(j(\gamma)) \restriction \kappa^2$.*

Note that for a condition q in $j(\mathbb{Q}_\alpha)$, q is a partial function from $j(\alpha)$ to $j(\mathbb{R})$ whose domain is countable. Hence, $\mathrm{dom}(q) \cap j[\alpha]$ is a countable subset of $j(\alpha)$. Furthermore, $j^{-1}(\mathrm{dom}(q) \cap j[\alpha])$ is a countable subset of α. By the fact that $M[\bar{G}*\bar{H}]$ is an extension of M by an ω_1-distributive forcing notion, the set $j^{-1}(\mathrm{dom}(q) \cap j[\alpha])$ is in M. Thus, $j^{-1}(\mathrm{dom}(q) \cap j[\alpha])$ is in $V[G*H]$. The next lemma shows that this definition is well-defined. The following proposition and theorem summarize the properties of the mapping π and can be found in [8].

Lemma 4.1.35. *In $M[\bar{G}*\bar{H}]$, $\mathbb{R} = \{s \in j(\mathbb{R}) : a^s \subseteq \kappa\}$. Therefore if t is in $j(\mathbb{R})$, then $t \restriction \kappa^2$ is in $\mathbb{R}$.*

Proposition 4.1.36. *For all $\alpha \leq \kappa^+$, π_α is a projection mapping $\pi_\alpha : j(\mathbb{Q}_\alpha) \to \mathbb{Q}_\alpha$. Furthermore, if $t \leq \pi_\alpha(q)$ in $\mathbb{Q}_\alpha$, then $j(t)$ and q are compatible in $j(\mathbb{Q}_\alpha)$.*

Definition 4.1.37. *We define $\pi = \pi_{\kappa^+}$. Therefore, π is a map from $j(\mathbb{Q})$ to $\mathbb{Q}$.*

Theorem 4.1.38. *Suppose that $\bar{I}$ is a $j(\mathbb{Q})$-generic filter over $V[\bar{G} * \bar{H}]$. If $I = \pi[\bar{I}]$, then the elementary embedding j can be lifted to $j : V[G * H * I] \to M[\bar{G} * \bar{H} * \bar{I}]$ with $j(G * H * I) = \bar{G} * \bar{H} * \bar{I}$.*

4.1.6 Preserving stationary sets

This is the last step to acquire Krueger's result. It remains to show that the stationarity of sets disjoint from S is preserved by our forcing iteration.

Lemma 4.1.39. $M[\bar{G} * \bar{H} * \bar{I}] = M[G * H * I][G'][H'][\bar{I}]$.

Proof. By Theorem 4.1.17. $M[\bar{G} * \bar{H}] = M[\bar{G}][H][H']$. If we define $G' = G \cap \mathrm{Coll}(\omega_1, [\kappa, j(\kappa)))$, then $M[\bar{G}] = M[G][G']$. Hence

$$M[\bar{G} * \bar{H}] = M[G][G'][H][H'].$$

Therefore,

$$M[\bar{G} * \bar{H} * \bar{I}] = M[G][G'][H][H'][\bar{I}].$$

Since $\pi : j(\mathbb{Q}) \to \mathbb{Q}$ is a projection mapping and $\pi[\bar{I}] = I$, we obtain

$$M[G][G'][H][H'][\bar{I}] = M[G][G'][H][H'][I][\bar{I}].$$

Hence

$$M[\bar{G} * \bar{H} * \bar{I}] = M[G][G'][H][H'][I][\bar{I}].$$

By using the Product Lemma repeatedly, we can reverse the order of the extensions. Because $\mathbb{P}_\kappa$ is defined in $M[G]$,

$$M[G][G'][H] = M[G][H][G'].$$

Hence,

$$M[\bar{G} * \bar{H} * \bar{I}] = M[G][H][G'][H'][I][\bar{I}].$$

Since we defined $\mathbb{Q}$ in the model $M[G][H]$, we have

$$M[G][H][G'][H'][I] = M[G][H][I][G'][H'].$$

Thus,

$$M[\bar{G} * \bar{H} * \bar{I}] = M[G * H * I][G'][H'][\bar{I}].$$

□

The next theorem completes the properties we require from the set S. We refer the reader to [8] for the long proof.

Theorem 4.1.40. *In $V[G*H*I]$, suppose T is a stationary subset of $\mathcal{P}_{\omega_1}(\kappa)$ which is disjoint from S_κ. Then T remains a stationary subset of $\mathcal{P}_{\omega_1}(\kappa)$ in $M[\bar{G} * \bar{H} * \bar{I}]$.*

4.2 $\mathrm{WRP}(\omega_2)$ implies $\mathrm{RP}(\omega_2)$ if $2^{\omega_1} = \omega_2$

In this section we revisit Section 2.2. As a corollary, we obtain that the Weak Reflection Principle for ω_2 implies the Reflection Principle for ω_2 under the assumption that $2^{\omega_1} = \omega_2$. For ω_n, we use CH and $2^{\omega_{n-1}} = \omega_n$.

Corollary 4.2.1. *Suppose* CH *holds and $2^{\omega_{n-1}} = \omega_n$ for some $2 \leq n < \omega$. Then* $\mathrm{WRP}(\omega_n)$ *implies* $\mathrm{RP}(\omega_n)$.

Corollary 4.2.2. *Suppose $2^{\omega_1} = \omega_2$. Then* $\mathrm{WRP}(\omega_2)$ *implies* $\mathrm{RP}(\omega_2)$.

Proof of Corollaries 4.2.1 and 4.2.2. Let S be a stationary subset of $\mathcal{P}_{\omega_1}(\omega_n)$. By Theorems 2.2.9 and 2.2.10 respectively, there exists a stationary set $T \subseteq S$, which does not reflect to any uncountable ordinal in $\omega_n \cap \mathrm{cof}(\omega)$. By $\mathrm{WRP}(\omega_n)$, there is an uncountable ordinal α in ω_2 such that T reflects to α. By the choice of T the cofinality of α must be ω_1. Therefore, S also reflects to an uncountable ordinal in ω_n with cofinality ω_1. □

4.3 $\mathrm{WRP}(\omega_2)$ and $\mathrm{SSR}(\omega_2)$

In this section we compare the Semistationary Reflection Principle to the Weak Reflection Principle and prove that they are equivalent for ω_2. For bigger cardinals we can separate them using a supercompact.

Theorem 4.3.1. *For a cardinal $\lambda \geq \omega_2$, $\mathrm{WRP}(\lambda)$ implies $\mathrm{SSR}(\lambda)$.*

Proof. Assume that $\mathrm{WRP}(\lambda)$ holds. Take an arbitrary semistationary set $S \subseteq \mathcal{P}_{\omega_1}(\lambda)$. Let T be the set of ω_1-extensions of elements of S, i.e. $T = \{y \in \mathcal{P}_{\omega_1}(\lambda) \mid \exists x \in S\ x \sqsubseteq y\}$. Since S is semistationary, T is stationary. By $\mathrm{WRP}(\lambda)$ there exists $X \subseteq \lambda$ such that $|X| = \omega_1 \subseteq X$ and $T \cap \mathcal{P}_{\omega_1}(X)$ is stationary in $\mathcal{P}_{\omega_1}(X)$. But $T \cap \mathcal{P}_{\omega_1}(X) = \{y \in \mathcal{P}_{\omega_1}(X) \mid \exists x \in S\ x \sqsubseteq y\} \subseteq \{y \in \mathcal{P}_{\omega_1}(\lambda) \mid \exists x \in S \cap \mathcal{P}_{\omega_1}(X)\ x \sqsubseteq y\}$, which is the set of ω_1-extensions of elements of $S \cap \mathcal{P}_{\omega_1}(X)$. Hence $S \cap \mathcal{P}_{\omega_1}(X)$ is semistationary. □

Next we are going to show that at stage ω_2, those reflection principles are equivalent. This is actually false for all cardinals $\geq \omega_3$. We follow Sakai [13].

Theorem 4.3.2. $\mathrm{SSR}(\omega_2)$ *implies* $\mathrm{WRP}(\omega_2)$ *and therefore they are equivalent.*

For the proof of Theorem 4.3.2 we need the following lemma.

Lemma 4.3.3. *Let λ and κ be cardinals such that κ is regular and $\omega_2 \leq \kappa \leq \lambda$. Suppose for a set $S \subseteq \mathcal{P}_{\omega_1}(\lambda)$ there exists $X \in \mathcal{P}_\kappa(\lambda)$ such that $X \cap \kappa \in \kappa$ and $S \cap \mathcal{P}_{\omega_1}(X)$ is semistationary. Let $X^* \in \mathcal{P}_\kappa(\lambda)$ satisfy*

(1) $\omega_1 \subseteq X^* \cap \kappa \in \kappa$ and $S \cap \mathcal{P}_{\omega_1}(X^*)$ is semistationary,

(2) for every $X \in \mathcal{P}_\kappa(\lambda)$, if $\omega_1 \subseteq X \cap \kappa \in \kappa$ and $S \cap \mathcal{P}_{\omega_1}(X)$ is semistationary then $\bar{\sup}(X^*) \leq \bar{\sup}(X)$.

Then

$$S_0 := \{y \in \mathcal{P}_{\omega_1}(X^*) \mid \exists x \in S \cap \mathcal{P}_{\omega_1}(X^*)\ x \sqsubseteq y\ \wedge\ \bar{\sup}(x) = \bar{\sup}(y)\}$$

is stationary in $\mathcal{P}_{\omega_1}(X^*)$.

Proof. Assume that S_0 is non-stationary. Let $S_1 := \{y \in \mathcal{P}_{\omega_1}(X^*) \mid \exists x \in S \cap \mathcal{P}_{\omega_1}(X^*)\ x \sqsubseteq y \ \wedge\ \bar{\sup}(x) < \bar{\sup}(y)\}$. Since $S_0 \cup S_1$ is the set of all ω_1-extensions of elements of S, this union is stationary. By the assumption and the fact that the union of two non-stationary sets is non-stationary, S_1 must be stationary. For each $y \in S_1$, choose $x_y \in S$ with $x_y \sqsubseteq y$ and $\bar{\sup}(x) < \bar{\sup}(y)$. Let $\xi_y \in y$ with $\bar{\sup}(x_y) \leq \xi_y$. By Fodor's Lemma there exists $\xi^* \in X^*$ such that $S^* := \{y \in S_1 \mid \xi_y = \xi^*\}$ is stationary. Let $X' := X^* \cap \xi^*$. Since $X^* \in \mathcal{P}_\kappa(\lambda)$, we get that $X^* \in \mathcal{P}_\kappa(\lambda)$. Clearly, $\omega_1 \subseteq X' \cap \kappa \in \kappa$. Furthermore $\bar{\sup}(X') < \bar{\sup}(X^*)$.
Now we show that $S \cap \mathcal{P}_{\omega_1}(X')$ is semistationary, which is a contradiction to property (2) of X^*. Note that for $y \in S^*$, $x_y \in \mathcal{P}_{\omega_1}(X')$ since $x_y \in X^*$ and $\bar{\sup}(x) \leq \xi^*$. Therefore, $x_y \sqsubseteq y \cap X'$. Hence,

$$\{y \cap X' \mid y \in S^*\} \subseteq \{y \in \mathcal{P}_{\omega_1}(X') \mid \exists x \in S \cap \mathcal{P}_{\omega_1}(X')\ x \sqsubseteq y\}.$$

Since S^* is stationary in $\mathcal{P}_{\omega_1}(X')$ the left side is stationary in $\mathcal{P}_{\omega_1}(X^*)$. Thus the right side is stationary, which implies that $S \cap \mathcal{P}_{\omega_1}(X')$ is semistationary. □

Proof of 4.3.1. Suppose that SSR(ω_2) holds. Let S be a stationary subset of $\mathcal{P}_{\omega_1}(\omega_2)$. For each $\alpha \in [\omega_1, \omega_2)$ fix a bijection $\pi_\alpha : \omega_1 \to \alpha$. Without loss of generality we may assume for each $x \in S$ that $\omega_1 < \bar{\sup}(x)$ and x is closed under π_α and π_α^{-1} for each $\alpha \in x \setminus \omega_1$.
Let α' be the least ordinal in ω_2 to which S reflects and S_0 be the set of all $y \subseteq \mathcal{P}_{\omega_1}(\alpha')$ such that

(1) there exists $x \in S \cap \mathcal{P}_{\omega_1}(\alpha')$ with $x \sqsubseteq y$ and $\bar{\sup}(x) = \bar{\sup}(y)$, and

(2) y is closed under π_α and π_α^{-1} for every $\alpha \in y \setminus \omega_1$.

Then S_0 is stationary in $\mathcal{P}_{\omega_1}(\alpha')$ by Lemma 4.3.3. For every $y \in S_0$ choose a set $x_y \in S \cap \mathcal{P}_{\omega_1}(\alpha')$ witnessing (1). But if $y \in S_0$ then

$$y \cap \alpha = \pi_\alpha(y \cap \omega_1) = \pi_\alpha(x_y \cap \omega_1) = x_y \cap \alpha$$

for every $\alpha \in x_y \setminus \omega_1$. Then $\bar{\sup}(y) = \bar{\sup}(x_y)$ implies $y = x_y$ for each $y \in S_0$. Therefore $S_0 \subseteq S \cap \mathcal{P}_{\omega_1}(\alpha')$ and hence $S \cap \mathcal{P}_{\omega_1}(\alpha')$ is stationary.

□

We conclude this section with the theorem refuting the equivalence of $\mathrm{SSR}(\lambda)$ and $\mathrm{WRP}(\lambda)$ for $\lambda > \omega_2$. The proof can be found in [13].

Theorem 4.3.4 (Sakai)**.** *Let κ be a supercompact cardinal. Then there exists a generic extension in which* $\mathrm{SSR}(\lambda)$ *holds for all $\lambda \geq \omega_2$ but* $\mathrm{WRP}(\lambda)$ *fails for every $\lambda \geq \omega_3$.*

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