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„Constructions of Quantum Fields with Anyonic Statistics“

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Abstract

From the principles of algebraic quantum field theory it follows that in low dimensions particles are not necessarily bosons or fermions, but their statistics can in general be governed by the braid group. Such particles are called anyons and their possible statistics is intimately related to their localization properties and their covariance with respect to rotations. This work is concerned with the explicit construction of quantum fields with anyonic statistics which are localized in various different regions on two- and three-dimensional Minkowski space, and we will analyze the connection between localization, statistics and spin. The reason why this is considerably more difficult than for bosons or fermions is the no-go theorem regarding free cone-localized anyons in $d = 2 + 1$. This problem is approached in this work from different directions leaving out some of the underlying assumptions one makes in the abstract algebraic quantum field theory. Despite a similar no-go theorem for free local anyons, it is in two dimensions possible to construct compactly localized quantum field nets with anyonic commutation relations for every mass $m \geq 0$ and every statistics parameter by using the theory of loop groups and implementable Bogoliubov transformations. This does not work in higher dimensions so in $d = 2 + 1$ we will first construct polarization free generators, which are only wedge-local, using a recent work about multiplicative deformations of free quantum fields on the Fock space. By generalizing this procedure to the charged case it is possible to extend the set of admissible deformations and end up with fields satisfying anyonic commutation relations, which are covariant w.r.t a Poincaré group representation with arbitrary real-valued spin. Another approach, which further demonstrates the connection between localization, statistics and spin of quantum field nets, is to focus first only on the rotational degrees of freedom and construct field operators on the circle. Using again implementable multiplication operators one can obtain a field net localized on intervals on the universal covering space of the circle. These field operators then have anyonic commutation relations depending on the winding number of the localization region and real-valued spin. By taking the tensor product with a local covariant quantum field theory on $\mathbb{R}^{2+1}$ it is possible to obtain an (interacting) anyon-like field net in three dimensions which is localized in paths of cones and covariant under translations and rotations.

Zusammenfassung

Aus den Prinzipien der algebraischen Quantenfeldtheorie folgt, dass Teilchen in niedrigen Dimensionen nicht notwendigerweise Bosonen oder Fermionen sein müssen, sondern im Allgemeinen auch Zopfgruppen-Statistik aufweisen können. Solche Teilchen, die sogenannten Anyonen, weisen einen engen Zusammenhang zwischen ihrer Statistik, ihrer Lokalisierung und der Kovarianz bezüglich Rotationen auf. Diese Arbeit beschäftigt sich mit der expliziten Konstruktion von Quantenfeldern mit anyonischer Statistik, die in unterschiedlichsten Gebieten auf dem zwei- und dreidimensionalen Minkowski-Raum lokalisiert sind. Insbesondere wird in diesem Rahmen auch die Beziehung zwischen Lokalisierung, Statistik und Spin der Quantenfelder analysiert. Der Grund für die Schwierigkeit solche Felder zu konstruieren ist das No-Go Theorem bezüglich freier Kegel-lokalisierter Anyonen in $d = 2 + 1$. Diese Problematik wird in der vorliegenden Arbeit von verschiedenen Seiten angegangen indem diverse Annahmen, die der abstrakten algebraischen Formulierung zu Grunde liegen, abgeschwächt werden. Trotz eines ähnlichen No-Go Theorems für freie lokale Anyonen ist es in zwei Dimensionen möglich, für jede Masse $m \geq 0$ und jeden Statistikparameter kompakt lokalisierte Quantenfelder mit anyonischen Vertauschungsrelationen zu definieren. Diese Konstruktion verwendet die Theorie der Loop Groups und der implementierbaren Bogoliubov Transformationen und ist in höheren Dimensionen im Allgemeinen nicht möglich. Daher werden in $d = 2 + 1$ unter Verwendung einer aktuellen Arbeit über multiplikative Deformationen von freien Quantenfeldern auf dem Fockraum zuerst polarisationsfreie Generatoren konstruiert, die lediglich in Keilen lokalisierbar sind. Durch eine Verallgemeinerung dieser Methode auf geladene Felder ist es möglich, die Menge der zulässigen Deformationen zu erweitern und Feldoperatoren zu erhalten, die anyonische Vertauschungsrelationen erfüllen und kovariant unter einer Darstellung der Poincarégruppe mit beliebigem reellwertigen Spin sind. Ein weiterer Zugang, der unter Anderem ebenfalls die Verbindung zwischen Lokalisierung, Statistik und Spin veranschaulicht, besteht darin zuerst nur die Rotations-Freiheitsgrade zu betrachten und Feldoperatoren auf dem Kreis zu konstruieren. Durch das erneute Verwenden von implementierbaren Multiplikationsoperatoren ist es möglich ein Feldnetz zu erhalten, das in Intervallen auf dem universellen Überlagerungsraum des Kreises lokalisiert ist. Die so konstruierten Feldoperatoren erfüllen anyonische Vertauschungsrelationen, die von der Windungszahl des Lokalisierungsintervalls abhängen, und können einen reellwertigen Spin aufweisen. Durch Bilden des Tensorprodukts mit einer lokalen kovarianten Quantenfeldtheorie auf $\mathbb{R}^{2+1}$ ist es dann möglich ein (wechselwirkendes) anyonisches Feldnetz in drei Dimensionen zu definieren, das in "Pfaden" von Kegeln lokalisiert und kovariant bezüglich Translationen und Rotationen ist.

Contents

I. Introduction and Motivation

In quantum mechanics particles of the same kind are usually assumed to be indistinguishable from which it can then be deduced that the wave functions of n particles need to have the same expectation values under relabelling of the individual particles. In three spatial dimensions this leads to symmetric (bosons) or anti-symmetric (fermions) wave functions corresponding to the two one-dimensional representations of the permutation group. This is the concept of statistics in the context of quantum mechanics.

Leinaas and Myrheim were the first the realize that these arguments have to be modified in two dimensions and that there is the theoretical possibility of particles that are neither bosons nor fermions [61]. The emergence of this new kind of particles can be understood by the more complicated topological structure of the configuration space of of n indistinguishable particles in two space dimensions. This is basically due to the fact that exchanging two particles in a clockwise or counter clockwise direction corresponds to topologically inequivalent operations (see also [71]). This should not be confused with the concept of *para-statistics* corresponding to higher dimensional representations of the permutation group (see e.g. the review [38] and references therein).

The statistics of n particles in low dimensions is then governed by the *braid group* B_n of n strands which is a non-abelian group with an infinite number of elements (in contrast to the $n!$ elements of the permutation group S_n). It is defined by its generating elements $\sigma_1, ..., \sigma_{n-1}$ which satisfy the relations

$$\begin{aligned} \sigma_k\, \sigma_{k+1}\, \sigma_k &= \sigma_{k+1}\, \sigma_k\, \sigma_{k+1} && \text{for } k = 1, ..., n-2 \\ \sigma_k\, \sigma_l &= \sigma_l\, \sigma_k && \text{if } |k-l| \geq 2, \end{aligned} \tag{I.1}$$

whereas the additional condition $\sigma_k^2 = 1$ would lead to the permutation group. An element of B_n can be pictured as a braiding of n strands as illustrated in figure I.1 for B_2.

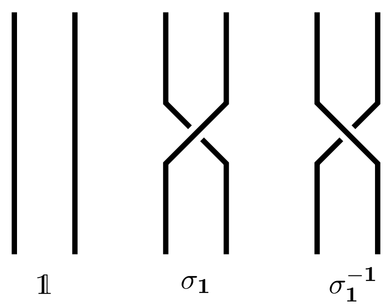


Figure I.1.: *Generators of* B_2

The braid group with an indefinite number of strands will be denoted by B_∞. These properties make the braid group much more complicated than the permutation group and a complete classification of its irreducible representations is still an open problem. In many cases it is, however, sufficient to consider one-dimensional representations which are characterized by a phase factor. Particles for which the statistics can be described by such one-dimensional representations have been called *"anyons"* by Wilczek [93], whereas in case the representation is non-abelian and higher dimensional the term *"plektons"* has been introduced in [30, 31]. In this work, however, we will only deal with abelian anyons.

Quantum mechanical models of anyons were analyzed first by Wilczek [93] and subsequently this generalization of the Bose/Fermi-alternative aroused great interest of physicists. One of the reasons for this was the supposed relevance of anyons for many phenomena in two-dimensional condensed matter physics, which are still only partly understood. In particular, there have been several suggestions to explain the fractional quantum Hall effect by models containing anyons (see e.g. [3, 45, 53]) and they were also supposed to play a role in high temperature superconductivity [53, 94]. More recently an application of (quasi) particles with braid group statistics in topological quantum computation has been suggested, using the fact that anyonic topological excitations are more stable with respect to small perturbations than normal excitations [47, 48].

In the 1980s Fredenhagen [27] and Fröhlich and Marchetti [35] considered the possibility of braid group statistics also in the context of algebraic quantum field theory (AQFT) in low dimensions. It turns out though that in 2+1 dimensions charged fields with generalized statistics cannot be localized in bounded regions but their localization regions need to extend to infinity in some direction. This is connected to the fact that the manifold of spacelike directions of three dimensional Minkowski space is not simply connected. The simplest spacetime regions of this kind are spacelike cones (which will be defined more precisely below) and it has been proved by Buchholz and Fredenhagen [9] that charged states in quantum field theory with a mass gap can always be localized in such cones. In this algebraic framework it was then also possible to prove several model independent results like the PCT theorem [70], the Bisognano-Wichmann theorem [70] and especially the spin-statistics theorem [69]. This theorem states that particles with anyonic statistics also carry a spin that can take any real value in contrast to bosons or fermions where the spin is always integer or half-integer (hence the name "any"ons because they can have "any" spin). Furthermore the concept of modular localization could be generalized to anyons and plektons [68] (and therefore to representations of the Lorentz group $\mathcal{L}_+^\uparrow$ with arbitrary spin) using again the non-trivial topology of the set of spacelike directions in 2+1 dimensions.

Despite all of these general results there are no well-defined explicit models of relativistic anyons in three dimensional Minkowski space which are physically and mathematically satisfying, especially from the viewpoint of algebraic quantum field theory. A well defined model complying with the basic axioms of relativistic quantum field theory would therefore be highly desirable, also because a non-relativistic quantum mechanical model should be obtainable as the limit of an underlying relativistic theory. In the case of bosonic or fermionic statistics the starting point for further non-trivial models is often a "free" (Wightman) field, creating one-particle states from the vacuum, but unfortunately this is impossible in the anyon case because of certain no-go theorems by Bros and Mund (see section II.2.3). This makes the rigorous construction of relativistic quantum fields with anyonic statistics particularly difficult. In this thesis we shall describe and clarify some old and new approaches in this direction.

The thesis is structured as follows. After recalling some basic principles of algebraic quantum field theory in chapter II we will first give a precise definition of what we mean by a relativistic covariant net of quantum fields with anyonic statistics. We will then see how the dimension of the underlying space and the requirement of anyonic commutation relations restrict the possible localization regions and the spin of these fields. At the end the no-go theorems by Bros and Mund will be recapitulated and extended to local quantum fields in 1+1 dimensions.

In chapter III the recently developed theory of wedge-local deformations of quantum fields [57] will be generalized to charged fields on two and three dimensional Minkowski space. It is then possible to arbitrarily change the commutation relations of the wedge-local generators and to make them covariant under a representation for general real-valued spin in 2+1 dimensions. For this to be possible one needs to generalize the localization regions from wedges to so-called "paths of wedges", depending additionally on some kind of winding number. Furthermore the class of admissible deformation functions could be enlarged for charged fields and the scattering states then also depend on the relative winding number of the localization regions.

After the construction of wedge-local one-particle generators we will show in chapter IV how local anyonic field nets can be constructed in 1+1 dimensions using the theory of implementable Bogoliubov transformations. While in the massive case so-called disorder operators on the Fock space can be used to change the commutation relations of the free field net, one has to go to inequivalent representations for $m = 0$ and leave the Fock space. It is then possible to explicitly construct a local covariant net of field algebras for every mass $m \geq 0$ and every statistics parameter $\lambda \in \mathbb{R}$. In accordance with the no-go theorem in two dimensions the resulting field operators create vectors with arbitrary high particle number from the vacuum.

Unfortunately the same procedure does not work in higher dimensions so to further demonstrate the connection between localization, statistics and spin we will use the method of implementable multiplication operators on the circle S_1 to construct in chapter V a local anyonic field net on the covering space $\widetilde{S_1}$ of the circle with arbitrary spin $s \in \mathbb{R}$. By taking tensor products of this net with a local theory on three dimensional Minkowski space one can then obtain a cone-local theory on $\mathbb{R}^{2+1}$ which still satisfies anyonic commutation relations and is covariant with respect to a representation of the translations and rotations.

The publication [74] in "Letters in Mathematical Physics" is based on chapter III of this thesis and a paper based on chapter V has also been submitted to the same journal (see also [75]).

To summarize, the main results of this thesis are the following.

- The generalization of the theory of multiplicative deformations to charged fields in 1+1 and 2+1 dimensions, leading to the possibility of wedge-local polarization free generators with anyonic commutation relations.
- The explicit and non-perturbative construction of local and covariant anyonic field operators in $d = 1 + 1$ for every mass $m \geq 0$ and every statistics parameter $\lambda \in \mathbb{R}$.
- The construction of a local quantum field net on the covering space of the circle with an arbitrary real-valued spin and commutation relations depending on the relative winding number. By taking tensor products this also leads to a cone-localized $\widetilde{E(2)}$-covariant net on $\mathbb{R}^{2+1}$ with anyonic commutation relations.

II. General Properties of Relativistic Anyons

Before describing the different constructions of quantum field nets with anyonic features we will analyze in this chapter some general properties that quantum field theories for anyons have to satisfy. We will focus especially on the possible localization regions that allow for non-standard commutation relations and how this affects the covariance with respect to rotations. For the "smallest" possible localization regions there then follow certain no-go theorems preventing the anyonic field operators from creating one-particle vectors from the vacuum. For the sake of completeness we will first give a very brief overview of the basic principles of algebraic quantum field theory needed to fully understand the concept of statistics in relativistic local quantum physics.

II.1. Basic Principles of Algebraic Quantum Field Theory

The algebraic formulation of QFT [43, 44] with its very clearly stated assumptions has turned out quite successful in analyzing the general structure of relativistic local quantum physics. There one focuses on the algebra of observables of a theory and in particular on its local structure. More precisely the starting point is a net $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ of von Neumann algebras, index by bounded spacetime regions $\mathcal{O}$. The algebras $\mathcal{A}(\mathcal{O})$ are usually considered to be in the (defining) vacuum representation π_0, i.e. they are algebras of bounded operators acting on a Hilbert space $\mathcal{H}_0$ where there exists a unique Poincaré invariant vacuum vector Ω. This net then has to satisfy the following physically motivated assumptions.

- *Isotony:* $\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$
- *Locality:* $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)'$ if $\mathcal{O}_1 \subset \mathcal{O}_2'$
- *Poincaré Covariance:* There is a unitary representation U_0 of the Poincaré group $\mathcal{P}_+^\uparrow$ on $\mathcal{H}_0$ such that for $g \in \mathcal{P}_+^\uparrow$

$$U_0(g)\mathcal{A}(\mathcal{O})U_0(g)^* = \mathcal{A}(g \cdot \mathcal{O}),$$

  where $g{\cdot}\mathcal{O}$ denotes the Poincaré transformed region $\mathcal{O}$. In addition the representation has to satisfy the *spectrum condition*, i.e. the joint spectrum of the generators of translations is contained in the closure of the forward light cone.

Here $\mathcal{O}'$ denotes the causal complement of a region $\mathcal{O}$ and $\mathcal{A}'$ is the commutant of $\mathcal{A}$. Furthermore the vacuum representation should be *cyclic*, i.e. $\bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O})\Omega$ is dense in $\mathcal{H}_0$, and *irreducible* meaning that $(\bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O}))' = \mathbb{C} \cdot \mathbb{1}$. Algebras for unbounded regions G can then simply be defined as the von Neumann algebra generated by all local algebras $\mathcal{A}(\mathcal{O})$ with $\mathcal{O} \subset G$.

Apart from the vacuum representation one also wants to describe charged particles in the theory, i.e. different superselection sectors. In the context of AQFT they arise as inequivalent irreducible representations of the observable algebra, where two representations π, π' on $\mathcal{H}_\pi, \mathcal{H}_{\pi'}$ are called equivalent if there exists a unitary $V : \mathcal{H}_{\pi'} \to \mathcal{H}_\pi$ such that $\pi = \operatorname{Ad} V \circ \pi'$. A superselection sector is then given by an equivalence class of irreducible representations. To select only the physically relevant representations one needs additional criteria. Usually one requires that there is a unitary representation U_π of the Poincaré group on $\mathcal{H}_\pi$ such that $\operatorname{Ad} U_\pi \circ \pi = \pi \circ \operatorname{Ad} U_0$ which also satisfies the spectrum condition (this is the selection criterion of Borchers). Additionally every representation considered should be *localizable* in some non-trivial[1] (convex) region C on Minkowski space w.r.t to the vacuum representation π_0, i.e.

$$\pi|_{\mathcal{A}(\mathcal{O})} \simeq \pi_0|_{\mathcal{A}(\mathcal{O})}, \quad \forall \mathcal{O} \subset C'. \tag{II.1}$$

Originally in the DHR criterion [24, 25] one required the representations to be localizable in double cones $\mathcal{O}$[2] but this was later generalized to cone-like regions in the massive case by Buchholz and Fredenhagen [9] and recently also in the massless case by Buchholz and Roberts [13].

To implement the concept of charge composition into this formalism one can now use the fact that under some technical assumptions (Haag duality) every localizable representation π is equivalent to a representation on $\mathcal{H}_0$ of the form

$$\pi \simeq \pi_0 \circ \rho,$$

where ρ is an *endomorphism* of the observable algebra. Since we took π_0 to be the defining representation of the observable algebra one could just omit the symbol π_0 and equivalently talk about endomorphisms instead of representations. Because of (II.1) this endomorphism is also localized in the region C, i.e. $\rho|_{\mathcal{A}(\mathcal{O})} = \mathbb{1}$ for all $\mathcal{O} \subset C'$. This now allows for a composition of charged sectors because the composition $\rho_1\rho_2 := \rho_1 \circ \rho_2$ of localized endomorphisms is again a localized endomorphism. Two of these endomorphisms are called equivalent if there exists a unitary *intertwiner* V such that $\rho_2 = V\rho_1 V^*$ and these equivalence classes of localizable endomorphisms now define our superselection sectors. These intertwiners now allow us to define the *statistics operator* of a sector ρ localized in a

[1] Non-trivial here means that C should not be empty or the whole Minkowski space.

[2] A double cone $\mathcal{O}$ in n dimensions can be defined as the non-empty intersection of a forward and a backward light cone.

region C. Consider for that purpose two “spectator morphisms” ρ_1, ρ_2 which are localized in spacelike separated regions C_1 and C_2 such that ρ, ρ_1 and ρ_2 are all equivalent. Then ρ_1 and ρ_2 commute and there exist intertwiners V_i between ρ and ρ_i (so-called “charge transporters”) which allow us to define the statistics operator ε_ρ according to

$$\varepsilon_\rho := \rho(V_1)^* V_2^* V_1 \rho(V_2).$$

This is an intertwiner of ρ^2 with itself and it turns out to depend only on the sector (i.e. the equivalence class of) ρ and possibly on the relative localization of C_1 w.r.t. C_2. One can now show that ε_ρ satisfies

$$\varepsilon_\rho\, \rho(\varepsilon_\rho)\, \varepsilon_\rho = \rho(\varepsilon_\rho)\, \varepsilon_\rho\, \rho(\varepsilon_\rho), \quad \text{and} \quad \rho^2(\varepsilon_\rho)\, \varepsilon_\rho = \varepsilon_\rho\, \rho^2(\varepsilon_\rho),$$

which leads to the fact that $\sigma_k \mapsto \rho^{k-1}(\varepsilon_\rho)$ yields a representation of the braid group B_∞. In four and higher dimensions it also follows that for localization in double cones or spacelike cones the statistics operator additionally satisfies $\varepsilon_\rho^2 = \mathbb{1}$ which leads to a representation of the permutation group and thus to Bose or Fermi statistics. In lower dimensions – depending on the form of the localization regions – this does not necessarily hold which is the reason for the occurrence of braid statistics (more detailed arguments for anyonic statistics in low dimensions will be given in the next section).

By construction all admissible representations $\pi = \pi_0 \circ \rho$ of the observable algebra now act on the same Hilbert space $\mathcal{H}_0$. So a vector $\Psi \in \mathcal{H}_0$ determines a state on the algebra $A \mapsto \langle \Psi, \pi_0\rho(A)\Psi \rangle$ only if we also specify the representation ρ in which the observables act. One is therefore led to consider pairs $(\rho, \Psi) \in \mathcal{H}_\rho := \{\rho\} \times \mathcal{H}_0$ as elements of the total Hilbert space $\mathcal{H} := \bigcup_\rho \mathcal{H}_\rho$. We can then consider the *field bundle* consisting of “generalized fields” $F(\rho, A)$, where A is an element of the observable algebra, which act on the Hilbert space according to

$$F(\rho_2, A) \cdot (\rho_1, \Psi) := (\rho_1\rho_2, \pi_0 \circ \rho_1(A)\Psi).$$

The commutation relations of such fields are then governed by the statistics operator so in the case of braid group statistics these charge carrying fields will also have plektonic commutation relations.

In the case where ρ is actually an *automorphism*, i.e. ρ^{-1} exists, one speaks of an Abelian sector which is the case for anyons in contrast to the more general case of plektons. The statistics operator is then a multiple of the identity $\varepsilon_\rho = \omega_\rho \cdot \mathbb{1}$ where $\omega_\rho \in \mathbb{C}$ is the *statistics phase*. In the simplest case one can consider a charge structure with charges $q \in \mathbb{Z}$ so after picking a reference morphism ρ one can reach all relevant sectors by applying ρ^q. The Hilbert space is then $\mathcal{H} = \bigoplus_{q\in\mathbb{Z}} \mathcal{H}_q$ with charged vectors $(q, \Psi) \in \mathcal{H}_q$. We then have a field algebra consisting of elements $F(c, A)$ which shift the charge of a vector by c and act on the Hilbert space according to $F(c, A)(q, \Psi) = (q + c, \rho^q(A)\Psi)$. This can be seen as the action of the observable A in the background charge q and addition of the charge c.

The aim of this work is now to directly construct such anyonic fields as well-defined operators on a Hilbert space without considering first an observable algebra and its possible representations (apart from the massless case in chapter IV). In the next section we will therefore first specify in more detail what we mean by an anyonic field algebra and from this general definition there already follow some restrictions on the possible localization regions and the representation of the rotations.

II.2. Quantum Fields for Anyons

Although the algebraic formulation of quantum field theory is conceptually very clear and rigorous and makes it possible to prove far-reaching theorems in vast generality, the stock of models for concrete realizations of the general principles is rather limited. More explicit examples of QFTs often focus on the direct construction of charge carrying fields or their expectation values and later identify those elements which are in the observable algebra.

In 2+1 dimensions most of these approaches to obtain models with anyon-like features start with a gauge theory action containing in addition a Higgs-like self-coupling $\lambda\phi^4$ with a scalar field ϕ and a Chern-Simons term $\mathcal{L}_{CS} \sim A \wedge dA$ with a gauge field A. This then leads to vortex-like solutions which can carry fractional charge and magnetic flux under the right conditions (see e.g. [4, 16, 35, 37, 60, 64, 87]). But there are also approaches where electrically and magnetically charged string-like fields are constructed without using a Chern-Simons term in the action [62]. These models have so far not been defined with full mathematical rigor in the sense that they are only defined perturbatively or one needs a lattice approximation to make them mathematically well-defined (see e.g. [33, 35] and references therein). Attempts to construct anyons directly on a lattice within the framework of quantum spin systems also exist [47, 48] and naturally have less problems being mathematically rigorous. The more explicit constructions on three dimensional Minkowski space, however, often lead to formal commutation relations which roughly look like

$$\phi(x_1)\phi(x_2) \sim e^{i\pi\lambda\,\mathrm{sign}[\arg(x_1-x_2)]}\,\phi(x_2)\phi(x_1),$$

where $\arg(x)$ denotes the angle of the vector $x \in \mathbb{R}^2$. After smearing the point-fields with test functions such commutation relations would lead to localization of the operators in smeared out "double strings", extending to infinity in two opposite directions, due to the two jumps the sign-function of an angle-variable has. Starting from representations of the Poincaré group and using infinite component fields a similar construction as for the usual free fields with half-integer spin is also possible for fields with arbitrary spin in 2+1 dimensions [39, 46].

In 1+1 dimensions massive quantum fields with anyon-like commutation relations are often constructed using so-called disorder operators (see e.g. [72, 73] for the general definition of a disorder operator) which basically act as a constant gauge transformation "on the right" and as another gauge transformation "on the left". Constructions using similar ideas can be found e.g. in [1,32,34,65,73,81,82,84]. The massless case is more complicated and one needs a more extensive analysis of the current algebra [12,19] on $\mathbb{R}$ or on the circle (see chapter IV for a more detailed treatment of the two dimensional case).

II.2.1. Anyonic Field Nets

In the following chapters we want to focus directly on the explicit and non-perturbative construction of anyon field algebras, instead of taking the detour over an observable algebra and its admissible representations. We want every step in the construction to be mathematically well-defined without referring to some perturbation theory or lattice approximations and the focus will be on clear-cut localization and covariance properties of the fields. For simplicity we therefor only consider abelian statistics with one $U(1)$-charge $q \in \mathbb{Z}$. This is an attempt to contribute to building a "bridge" between the abstract approach of AQFT having very transparent principles (where often no concrete examples are known) and more explicit constructions of quantum fields which are sometimes mathematically and conceptually less rigorous. In the following we will give the definition of a *field net for anyons* on two or three dimensional Minkowski space used in this work.

The net $B \mapsto \mathcal{F}(B)$ will be indexed by certain regions[3] $B \subset \mathbb{R}^{d+1}$ which we choose to be convex and causally complete[4] to exclude pathological cases where the localization regions are e.g. not simply connected. The algebras $\mathcal{F}(B)$ then have to satisfy the following minimal conditions.

- *Isotony:* For two regions B_1, B_2 satisfying $B_1 \subset B_2$ also the respective algebras satisfy $\mathcal{F}(B_1) \subset \mathcal{F}(B_2)$.

- The algebras $\mathcal{F}(B)$ are $\star$-algebras of operators acting on a Hilbert space $\mathcal{H}$ which is the direct sum of Hilbert spaces $\mathcal{H}_q$ for fixed charge q, i.e.

$$\mathcal{H} = \bigoplus_{q\in\mathbb{Z}} \mathcal{H}_q.$$

  Every $\mathcal{F}(B)$ contains elements F^c, which we simply call *fields*, carrying charge $c \in \mathbb{Z}$ meaning that they change the charge of a Hilbert space vector by c, i.e. $F^c\mathcal{H}_q \subset \mathcal{H}_{q+c}$.

- *Covariance:* On this Hilbert space there is a representation U of the universal covering group $\tilde{\mathcal{P}}_+^\uparrow$ of the Poincaré group $\mathcal{P}_+^\uparrow$ under which the net of algebras is covariant,

[3] We will see in the course of this section that this definition has to be slightly generalized in order to allow for cone-localized anyons in $d = 2 + 1$.

[4] An open subset B of Minkowski space is called *"causally complete"* if it satisfies $B'' = B$ where B' denotes the causal complement of the region B.

i.e. for $a \in \mathbb{R}^{d+1}$ and $\tilde{\Lambda} \in \tilde{\mathcal{L}}_+^\uparrow$ there holds

$$U(a,\tilde{\Lambda})\mathcal{F}(B)U(a,\tilde{\Lambda})^* = \mathcal{F}(\tilde{\Lambda}B + a),$$

where the action of $\tilde{\Lambda}$ on a region B will be specified in the concrete examples below. Furthermore this representation satisfies the *spectrum condition* meaning that the generators of translations have joint spectrum in the (closure of the) forward light cone $\overline{V^+}$.

- *Locality and Statistics:* For two spacelike separated regions B_1, B_2 all fields $F_1^{c_1} \in \mathcal{F}(B_1)$ and $F_2^{c_2} \in \mathcal{F}(B_2)$ satisfy commutation relations of the form

$$F_1^{c_1} F_2^{c_2} = e^{2\pi i \lambda\, c_1 c_2\, \mathcal{N}(B_1,B_2)}\, F_2^{c_2} F_1^{c_1}, \tag{II.2}$$

where the number $\lambda \in \mathbb{R}$ determines the statistics of the net and the integer $\mathcal{N}(B_1, B_2) \in \mathbb{Z}$ only depends on the localization regions but not the particular algebra elements $F_1^{c_1}, F_2^{c_2}$.

From isotony of the net it then immediately follows that $\mathcal{N}(B,B_1) = \mathcal{N}(B,B_2)$ if either $B_1 \subset B_2$ or $B_2 \subset B_1$. If the number $\mathcal{N}(B_1,B_2)$ is constant for all spacelike separated localization regions B_1, B_2 it obviously also follows that $e^{2\pi i\lambda\mathcal{N}} = \pm 1$ which means that anyonic statistics is only possible if the commutation relations of fields depend on their relative localization properties. An important observation is now that the requirement of isotony already highly restricts the possible dependence of the number $\mathcal{N}$ on the localization regions. To realize this we first define the notion of a *path of regions* $\tilde{B}$ by a finite sequence $(B_1, B_2, ..., B_N)$ where for every $i = 1, ..., N-1$ there either holds $B_i \subset B_{i+1}$ or $B_{i+1} \subset B_i$ [5].

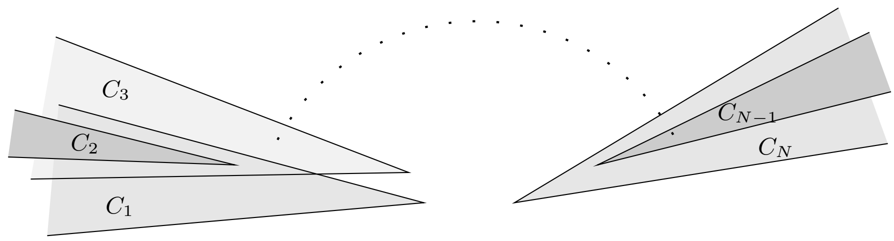


Figure II.1.: *Path of cones* $\tilde{C} = (C_1, C_2, ..., C_{N-1}, C_N)$.

Now consider three localization regions B, B_1 and B_2 such that B_1 and B_2 are spacelike separated from B. Assume that there is a path $\tilde{B}_{12}$ starting at B_1 with endpoint at B_2 such

[5]The definition of a path used here is not really useful for wedge regions because two wedges which are rotated w.r.t. each other can not satisfy $W_1 \subset W_2$ or $W_2 \subset W_1$. We will therefore later also introduce a slightly different concept of paths for regions extending to infinity in some directions. For spacelike cones these concepts will be equivalent.

that all elements of $\tilde{B}_{12}$ are still spacelike separated from B. It then follows immediately from the isotony of the net that the number $\mathcal{N}(B, B_i)$ is constant along the path so we get in particular the following proposition.

Proposition II.1. *The number $\mathcal{N}$ in the commutation relations of fields satisfies*

$$\mathcal{N}(B, B_x) = \mathcal{N}(B, B_y)$$

for all $B_x, B_y \subset B'$ for which there is a path connecting B_x with B_y staying entirely in the spacelike complement of B.

Remark: Apart from these localization dependent commutation relations it is always possible to change the *relative* commutation relations between two kinds of independent charged fields. Consider e.g. two scalar fields Φ_1, Φ_2 acting on the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ which carry charges Q_1 and Q_2 respectively, i.e. Φ_i raises the charge of a vector on $\mathcal{H}_i$ by one. One can then simply redefine one of the fields (or both of them) according to

$$\hat{\Phi}_1 := e^{2\pi i \lambda Q_2} \Phi_1, \quad \lambda \in \mathbb{R},$$

which leaves the commutation relations between fields of the same kind unchanged but results in an additional phase factor in the relative commutation relations

$$\hat{\Phi}_1 \Phi_2 = e^{2\pi i \lambda} \Phi_2 \hat{\Phi}_1.$$

Because of the possibility of this trivial redefinition we will *not* call such fields anyon fields since in this sense the commutation relations between independent charged fields are arbitrary.

From proposition II.1 and the definition of a field net there now follow some restrictions on possible localization regions and on the representation of the Poincaré group if we want to have fields with anyonic commutation relations.

II.2.2. Localization and Covariance

We now first want to analyze the possible localization properties of anyons in $d = 1+1$ and $d = 2+1$ dimensions. On two dimensional Minkowski space we see that even for compact localization, i.e. the localization regions are double cones $\mathcal{O}$, the causal complement of $\mathcal{O}$ decomposes into two disjoint regions which we will call the "left" and the "right" spacelike complement $\mathcal{O}'_L, \mathcal{O}'_R$. The commutation relations between two spacelike separated fields can therefore depend on which of the two fields is localized to the left (or right) of the other. This is a Poincaré invariant concept in 1+1 dimensions whereas in more dimensions one can always perform a rotation around π interchanging the notions of left and right.

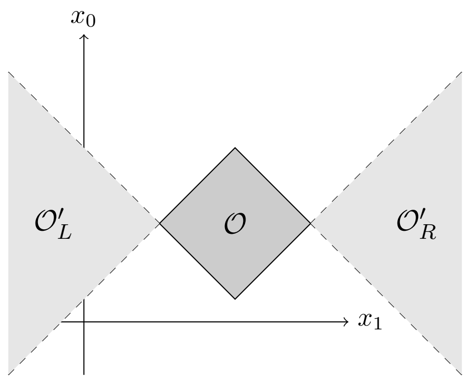


Figure II.2.: *Causal complement of a double cone $\mathcal{O}$.*

So for $d = 1 + 1$ there is the possibility for this kind of left-right statistics where the parameter $\mathcal{N}$ can take two values

$$\mathcal{N}(\mathcal{O}_1, \mathcal{O}_2) = sgn(\mathcal{O}_1, \mathcal{O}_2) := \begin{cases} +1 & \text{if } \mathcal{O}_1 \subset (\mathcal{O}'_2)_L \\ -1 & \text{if } \mathcal{O}_1 \subset (\mathcal{O}'_2)_R \end{cases},$$

depending on the relative localization of $\mathcal{O}_1$ w.r.t. $\mathcal{O}_2$.

A somewhat exceptional position in 1+1 dimensions is held by wedge regions because they decompose into two distinct classes. Due to the missing rotations in the Lorentz group the left wedges and right wedges are not connected by any (proper orthochronous) Poincaré transformation but can only be transformed into each other by a reflection. Only a left and a right wedge can be spacelike separated so they can obviously never be connected by a path in the set of wedges. In this sense there are "left-fields" and "right-fields" for wedge localized field algebras in 1+1 dimensions, allowing for anyon-like commutation relations similar to the case where we have two independent fields carrying different charges. We will see an example of this in section III.3.1 about deformations on two dimensional Minkowski space.

In three (and more) dimensions compact localization does not lead to the possibility of anyonic commutation relations because there the causal complement $\mathcal{O}'$ of a double cone is connected. The same holds if the localization regions are *spacelike cones* where a spacelike cone C in n dimensions can be defined according to

$$C = a + \bigcup_{\mu \geq 0} \mu \mathcal{O},$$

where $a \in \mathbb{R}^n$ is its apex and $\mathcal{O}$ is a double cone, which is spacelike separated from the origin. For two given spacelike separated cones C and C_1 one can then reach every other cone in the spacelike complement of C by a path which starts at C_1 and stays inside C'. This is true even in $d = 2 + 1$ dimensions and we will see that the fields therefore have to carry an additional information in order to allow for anyonic commutation relations.

Localization regions that would allow for localization dependent commutation relations for the kind of field nets we are considering here are “tubes” in $d = 2 + 1$ or “thickened branes” in more dimensions. These are localization regions that cut space into two halves in the sense that their causal complement decomposes into two disjoint regions which then leads to a similar situation as for double cones in $d = 1 + 1$. Such regions can be obtained

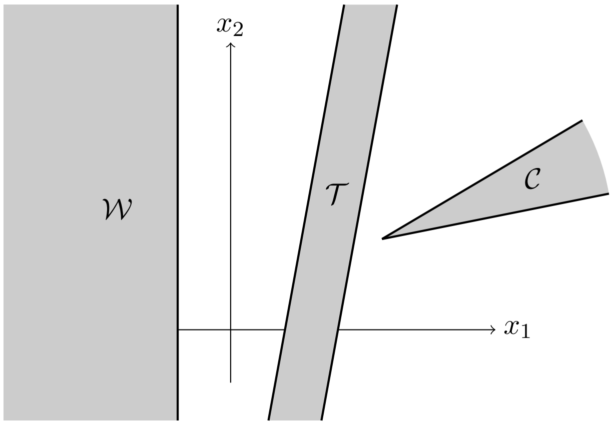


Figure II.3.: *Spatial projection of a wedge $\mathcal{W}$, a tube $\mathcal{T}$ and a spacelike cone $\mathcal{C}$ in $d = 2 + 1$*

e.g. by smearing the edge of a wedge to get an open set and then taking the causal completion and we will encounter an example of fields localized in tube-shaped regions in section III.3.3. However, such localization regions are still “too big” in the sense that if we have two spacelike separated regions, a small rotation (or boost in the wrong direction) of one of the tubes already destroys the spacelike separation. So the set of all possible causally separated regions for a given tube is small in this sense. Moreover, Buchholz and Fredenhagen [9] have shown that (under certain assumptions) charged particles can always be localized in spacelike cones so this should be the “worst” localization one has to consider.

As we have already argued before, localization in cones does not suffice for the possibility of anyonic commutation relations on three dimensional Minkowski space, so the field algebras will need to carry an additional information. One of the possibilities (which was originally considered in [9]) is to exclude a fixed reference direction e_0 from the set of possible localization regions. This then leads again to a situation where the causal complement of a spacelike cone C has two disconnected regions in the sense that they cannot be connected by a path of cones in $C' \cap e_0'$. The disadvantage of this approach is that at the end of every construction one has to make sure that all the relevant physical properties are independent of the choice of e_0.

A somewhat more elegant possibility is to consider the fields not to be simply localized in cones in $\mathbb{R}^{2+1}$ but to let the field net be indexed by *paths of cones* $\tilde{C}$ [29]. In order to give a precise definition of this localization concept we need to describe in more detail the notion of a path of localization regions. As already shown a path of cones from C_0 to C

can be defined as a sequence of cones starting at C_0 and ending at C where for neighboring elements one of them is always a subset of the other. Since we don't want our fields to depend on the explicit realization of such a path we introduce the notion of *equivalence classes* which are defined by the homotopy of paths. This means that two paths of cones are said to be equivalent if they have the same endpoints and result from each other by a finite number of operations of either omitting a region C_i or inserting a region C_i if the resulting sequence is still a path. From now on we will denote such an equivalence class, starting at a reference cone C_0 and ending at C, by the symbol $\tilde{C}$ (these concepts will be explained in a bit more detail in chapter III). To define a net using these paths we also need the notions of *inclusion* and *spacelike separation* which are defined in the following natural way. We write $\tilde{C}_1 \subset \tilde{C}_2$ if $C_1 \subset C_2$ and the paths are equivalent and we say that $\tilde{C}_1$ is spacelike separated from $\tilde{C}_2$ if the respective endpoints C_1 and C_2 are spacelike separated.

There is also an alternative definition of paths of cones which is more suitable for the generalization to wedges and to define the action of $\tilde{\mathcal{L}}_+^\uparrow$ on these paths. Consider the set of spacelike directions $H^d := \{e \in \mathbb{R}^d | e^2 = -1\}$, which is not simply connected in 2+1 dimensions. A generalized cone $\tilde{C}$ can then be defined as a pair

$$\tilde{C} = (C, \tilde{e}),$$

where $C \subset \mathbb{R}^{2+1}$ is an ordinary spacelike cone and $\tilde{e}$ is a homotopy class of paths $t \mapsto e(t)$ in H^3 starting at a reference direction e_0 and ending at a point e in C. We say that a direction $e \in H^3$ lies in a cone C if C contains e as an asymptotic direction, i.e. there exists a $\Delta \in \mathbb{R}$ such that $\delta e \in C, \forall \delta > \Delta$.

We can hence define a net of field algebras as indexed by such equivalence classes of paths, $\tilde{C} \mapsto \mathcal{F}(\tilde{C})$, where a localization region $\tilde{C}$ now carries the information of the winding number of the path in addition to the mere spacetime region C. Hence the fields can be interpreted as being localized in a *covering space* and for a given $\tilde{C}$ the set of all paths of cones which are spacelike to $\tilde{C}$ then decomposes again into disconnected components labelled by the homotopy classes of paths. The number $\mathcal{N}$ in the commutation relations between fields can now depend on the *relative winding number* of the respective localization regions and it is this phenomenon which underlies the possibility of braid group statistics in 2+1 dimensions [29]. The various notions introduced here will be described in more detail also in chapter III.

Up to now we only used the requirement of isotony for the field algebra and the rather general form of commutation relations for two fields and we didn't need the covariance with respect to some representation of the Poincaré group. We are going to specify these representations in the concrete examples below and in section III.3.2 we will see in particular how the universal covering of the Lorentz group acts on equivalence classes of paths of cones. The only thing that we want to dwell on a bit more in this chapter is the representation of a rotation around an angle of 2π and its connection to the commutation relations of fields. When working with the universal covering of the Lorentz group the 2π rotations, denoted simply by $U(2\pi)$ in the following, can be represented non-trivially. From the general assumptions of algebraic quantum field theory it follows, however, that on the one hand the observables should be invariant under 2π rotations and on the other hand they should be represented irreducibly on a given sector $\mathcal{H}_q$. Due to Schur's Lemma this already restricts the possible representers $U(2\pi)$ to a mere phase factor which means that for every sector with charge q we will have a real number $S_q \in \mathbb{R}$, the *spin of the sector* q, such that

$$U(2\pi) = \bigoplus_{q\in\mathbb{Z}} e^{2\pi i S_q}\, \mathbb{1}. \tag{II.3}$$

Further considerations in algebraic quantum field theory also show that the spins S_q additionally have to satisfy the condition $S_{-q} = S_q$, which simply means that particles have the same spin as their corresponding anti-particles [36, 69]. Moreover, the fact that the vacuum $\Omega \in \mathcal{H}_0$ should be invariant under 2π-rotation leads to $S_0 = 0$.

Now we have seen that in $d = 2 + 1$ dimensions the commutation relations of fields localized in spacelike separated paths of cones depend on the relative winding number $N(\tilde{C}_1, \tilde{C}_2)$ of the respective localization regions. Since such a winding number is defined by the homotopy classes of the paths involved, we see that if we rotate one of the localization regions around 2π the winding number changes by ± 1, depending on which of the cones was rotated. This allows us to proof the following lemma showing the connection between non-trivial behavior under 2π rotations and anyonic commutation relations.

Lemma II.1. *Assume we have a net of field algebras in $d = 2 + 1$, localized in paths of cones as defined in this section and the representation of rotations around 2π is given by* (II.3) *with S_q satisfying $S_{-q} = S_q$ and $S_0 = 0$. Then the S_q are quadratic in q, i.e.*

$$S_q = sq^2, \tag{II.4}$$

where the constant s is connected to the statistics parameter by $2s = \lambda$ and we will simply call it the spin of the theory.

Proof. Consider fields Φ_1, Φ_2 raising the charge by one which are localized in spacelike separated cones $\tilde{C}_1, \tilde{C}_2$. Rotating the first field around 2π leads to the commutation relation

$$\begin{aligned} U(2\pi)\Phi_1 U(2\pi)^*\ \Phi_2 &= e^{2\pi i\lambda\mathcal{N}(\tilde{\Lambda}_{2\pi}\tilde{C}_1,\tilde{C}_2)}\ \Phi_2\ U(2\pi)\Phi_1 U(2\pi)^* \\ &= e^{2\pi i\lambda} e^{2\pi i\lambda\mathcal{N}(\tilde{C}_1,\tilde{C}_2)}\Phi_2\ U(2\pi)\Phi_1 U(2\pi)^*, \end{aligned}$$

where $\tilde{\Lambda}_{2\pi}$ denotes a rotation around 2π within the universal cover of the Lorentz group. Using the shorthand notation $S_Q := \sum S_q P_q$, where P_q is the projector onto $\mathcal{H}_q$, and the fact that 2π-rotations are represented according to $U(2\pi) = e^{2\pi i S_Q}$ we can write

$$U(2\pi)\Phi_1 U(2\pi)^* = \Phi_1 e^{2\pi i(S_{Q+1} - S_Q)},$$

which leads to the commutation relation

$$\begin{aligned} U(2\pi)\Phi_1 U(2\pi)^* \; \Phi_2 &= \Phi_1 e^{2\pi i (S_{Q+1} - S_Q)} \; \Phi_2 \\ &= e^{2\pi i \lambda \mathcal{N}(\tilde{C}_1, \tilde{C}_2)} \Phi_2 \; \Phi_1 e^{2\pi i (S_{Q+1} - S_Q)} \; e^{2\pi i (S_{Q+2} - 2S_{Q+1} + S_Q)} \\ &= e^{2\pi i \lambda \mathcal{N}(\tilde{C}_1, \tilde{C}_2)} \Phi_2 \; U(2\pi)\Phi_1 U(2\pi)^* \; e^{2\pi i (S_{Q+2} - 2S_{Q+1} + S_Q)}. \end{aligned}$$

Comparing the two different expressions one concludes that S_q has to satisfy

$$S_{q+2} - 2S_{q+1} + S_q = \lambda, \quad \forall q \in \mathbb{Z}.$$

Solving this equation in combination with the conditions $S_0 = 0, S_{-q} = S_q$ then leads to the desired relation $S_q = \frac{\lambda}{2} q^2 =: sq^2$. □

So we conclude that, although it seems possible at first sight to chose arbitrary spins S_q for every charged sector $\mathcal{H}_q$, the dependence on the charge q is already determined by the form of the commutation relations (II.2), leading to this kind of quadratic spin addition rules (see also [28]). In addition one gets a kind of *spin-statistics relation* saying that the real parameter s in $U(2\pi) = e^{2\pi i s Q^2}$ is proportional to the statistics parameter λ in the exchange phase $e^{2\pi i \lambda \mathcal{N}}$ which determines the commutation relations.

In the coming chapters we will explicitly construct examples of field nets for anyons in two and three dimensions with various localization regions and arbitrary statistics parameter and spin. For fields with bosonic or fermionic commutation relations it is easy to write down (even compactly localized) examples in every dimension because of the existence of free Wightman fields for every half-integer spin. They are *polarization free generators* (PFGs), i.e. they create single particle states from the vacuum, and are therefore given by very simple expressions. For anyonic statistics this is not possible because of certain no-go theorems concerning free anyon fields.

II.2.3. No-Go Theorems

The first no-go theorem for "free" relativistic anyons was proved by Mund in [67]. It states that, under certain additional assumptions, there are no cone-localized polarization free generators with anyonic statistics in 2+1 dimensions, where an algebra element $\phi \in \mathcal{F}(\tilde{C})$ is called *polarization free* if it creates a one-particle state from the vacuum. We call a Hilbert space vector Ψ a *"one-particle state"* if its energy-momentum spectrum $\mathrm{spec}_p \Psi$ is a subset of the upper mass shell

$$H_m^+ := \{p \in \mathbb{R}^d | p^2 = m^2, p^0 \geq 0\}, \quad m \geq 0,$$

which is the lower boundary of the total energy-momentum spectrum *spec*(P) of the theory, apart from the origin $p = 0$ corresponding to the vacuum. The energy-momentum spectrum of a vector Ψ is defined as the set of points $p \in spec(P)$ such that for any neighborhood V of p, the spectral projector $E_V(P)$ does not map Ψ to zero. The additional assumptions needed are the *Reeh-Schlieder property* of the field net and a temperedness condition for one-particle generators. A field net is said to satisfy the Reeh-Schlieder property if the vacuum Ω is cyclic for every local algebra $\mathcal{F}(\tilde{C})$ (if C has a non-empty interior) and the temperedness condition states that for spacelike separated polarization free generators ϕ_1, ϕ_2 the vector $U(x)\phi_2\Omega$ is in the domain of ϕ_1 for every $x \in \mathbb{R}^3$ and the function

$$x \mapsto ||\phi_1 U(x)\phi_2\Omega||$$

is locally integrable and has a polynomial bound for large x. From these assumptions it then follows that the statistics of a field algebra containing such one-particle generators has to be bosonic or fermionic.

This no-go theorem has been strengthened considerably in the recent work of Bros and Mund [7] where they show that the two-particle scattering matrix for anyons always has to be non-trivial. More precisely let π be a massive single particle representation with braid group statistics, i.e. the sector π contains one-particle states, and $\overline{\pi}$ is the conjugate sector. Then for any open sets of momenta U_1, U_2, V_1, V_2 admitted by energy-momentum conservation, i.e. $(U_1 + U_2) \cap (V_1 + V_2) \neq \emptyset$, there is elastic two-particle scattering from $U_1 \times U_2$ into $V_1 \times V_2$ in the channel $\pi \times \overline{\pi} \to \pi \times \overline{\pi}$ [6].

From the work of Borchers et al. [6] it also follows that the existence of tempered wedge-localized polarization free generators already implies that there are indeed such subsets U_i, V_i for which there is no two-particle scattering. Together with the no-go theorem it therefore follows in particular that even the existence of *wedge-localized* tempered polarization free generators excludes anyonic statistics.

The ideas used in [67] to prove the non-existence of cone-localized PFGs for anyons in 2+1 dimensions can now also be used on two-dimensional Minkowski space to rule out the possibility of compactly localized anyon fields which create one-particle states from the vacuum. For this purpose consider a Wightman field ϕ_λ satisfying the commutation relations

$$\begin{aligned} \phi_\lambda(f)\phi_\lambda(g) &= e^{-2\pi i\lambda\epsilon(f,g)}\phi_\lambda(g)\phi_\lambda(f) \\ \phi_\lambda(f)\phi_\lambda^*(g) &= e^{2\pi i\lambda\epsilon(f,g)}\,\phi_\lambda^*(g)\phi_\lambda(f) \end{aligned} \tag{II.5}$$

if the test functions f and g have spacelike separated support where the factor $\epsilon(f, g) = \pm 1$ in the exponential measures if the test function g is supported in the right or the left spacelike complement of supp f. Similar to the situation in [67] we then get the following proposition.

[6] A more detailed treatment of single particle sectors with braid group statistics and the scattering theory for them can also be found in [7].

Proposition II.2. *Let ϕ_λ be a Wightman field satisfying the commutation relations* (II.5) *for some $\lambda \in \mathbb{R}$. If ϕ_λ is tempered, satisfies the Reeh-Schlieder property and generates one-particle vectors from the vacuum, then $\lambda \in \frac{\mathbb{N}}{2}$, i.e. ϕ_λ satisfies Bose- or Fermi statistics.*

Sketch of Proof. The impossibility to construct fields satisfying the above requirements follows from the same arguments as in the no-go theorem for string-localized free anyons in $d = 2+1$ in [67]. First one uses the fact that only a multiple of Ω is added to the commutation relations (II.5) if one translates the localization regions, such that they are not spacelike separated any more. Introducing the abbreviation $\omega := e^{2\pi i\lambda}$ and denoting a translated test function by $\alpha_x(f)$ for $x \in \mathbb{R}^2$ one can show that

$$\begin{aligned} &U(x)\phi_\lambda(f)U(-x)\,\phi_\lambda^*(g)\Omega - \omega^{\epsilon(f,g)}\,\phi_\lambda^*(g)\,U(x)\phi_\lambda(f)\Omega \\ &= \phi_\lambda(\alpha_x(f))\phi_\lambda^*(g)\Omega - \omega^{\epsilon(f,g)}\,\phi_\lambda^*(g)\phi_\lambda(\alpha_x(f))\Omega \\ &= C_{f,g}(x)\Omega. \end{aligned} \tag{II.6}$$

To prove this relation one uses the aforementioned temperedness condition to show that for two spacelike separated fields ϕ_1 and ϕ_2 the $\mathcal{H}$-valued function

$$(x,y) \mapsto U(x)\phi_1 U(-x)\,U(y)\phi_2 U(-y)\Omega$$

is a tempered distribution, whose Fourier transform has support contained in $(H_m^- \cup H_m^+) \times H_m^+$, where $H_m^\pm$ denotes the upper/lower mass shell. Defining the distributions F^+ and F^- by

$$\begin{gathered} U(x)\phi_1 U(-x)\,U(y)\phi_2 U(-y)\Omega =: F_{1,2}^+(x,y) + F_{1,2}^-(x,y) \\ \text{with } \ supp\widetilde{F^\pm} \subset H_m^\pm \times H_m^+, \end{gathered}$$

one gets that

$$\begin{aligned} F_{1,2}^-(x,y) &= \langle \Omega, U(x)\phi_1 U(-x) U(y)\phi_2 U(-y)\Omega\rangle \cdot \Omega \\ \mathrm{spec}_P\, F_{1,2}^+(x,y) &\subseteq H_m^+ + H_m^+, \end{aligned}$$

where $\mathrm{spec}_P\,\Psi$ denotes the spectral support of Ψ w.r.t. the energy-momentum operators. Using these spectral properties and the edge of the wedge theorem [90] then leads to $F_{1,2}^+(x,0) - \omega F_{2,1}^+(0,x) = 0$ which finally leads to equation (II.6). Moreover, the relation (II.6) extends from Ω to the dense subspace $\mathcal{F}(\mathrm{supp}\, f)'\Omega \cap \mathcal{F}(\mathrm{supp}\, g)'\Omega \cap \mathcal{F}(\mathrm{supp}\, f + x)'\Omega$, where $\mathcal{F}(\mathcal{O})$ denotes the algebra generated by the fields localized in $\mathcal{O}$. (For a more detailed and mathematically rigorous formulation of these arguments see [67].)
In the next step we choose two test functions f and g such that f is localized to the *left* of g. Then we know because of relation (II.6) that on a dense subspace we have

$$\Phi_\lambda(\alpha_x f)\Phi_\lambda(g)^* = \omega\ \Phi_\lambda(g)^*\Phi_\lambda(\alpha_x f) + C_{f,g}(x)$$

for all $x \in \mathbb{R}^2$. On the other hand the commutation relations (II.5) state that

$$\Phi_\lambda(\alpha_x f)\Phi_\lambda(g)^* = \omega^{-1}\Phi_\lambda(g)^*\Phi_\lambda(\alpha_x f),$$

holds for all x such that $\alpha_x f$ lies to the *right* of g. Combining these two equations we see that

$$\Phi_\lambda(g)^*\Phi_\lambda(\alpha_x f)(\omega^{-1} - \omega) = C_{f,g}(x)$$

for all x where $\alpha_x f$ is localized to the right of g. Now suppose that $\omega^2 \neq 1$ (i.e. $\omega^{-1} - \omega \neq 0$). Then $\Phi_\lambda(g)^*\Phi_\lambda(\alpha_x f)$ is proportional to the identity when acting on the vacuum vector and thus proportional to the identity on its whole domain of definition because the vacuum is a separating vector for the local algebras (which follows from the Reeh-Schlieder property). Therefore $\Phi_\lambda(g)^*\Phi_\lambda(\alpha_x f)$ is invariant under Poincaré transformations and together with locality this implies that the individual fields $\Phi_\lambda(\alpha_x f)$ and $\Phi_\lambda(g)^*$ are already proportional to the identity, which is inconsistent with anyonic commutation relations. This shows that $\omega^2 = 1$, which corresponds to Bose- or Fermi-statistics. $\square$

III. Wedge Local Fields from Deformations†

The proofs for all the no-go theorem stated above always rest upon the assumption that the fields are localizable either in compact regions in two dimensions or in spacelike cones in three dimensions, because only then it is possible to have three of them mutually spacelike separated. Therefore, in the next chapter we will try to construct field operators with anyonic commutation relations which have weaker localization properties, namely localization in wedges. For this purpose we will use the recent construction by Lechner [57] to obtain wedge-localized polarization-free generators on the Fock space as a deformation of a free Bose field. An operator is called "polarization-free" in this context if it creates only single-particle states from the vacuum, i.e. vectors for which the energy-momentum spectrum lies on the mass shell, which would not be possible for smaller localization regions.

III.1. Preliminaries: Wedge Local Deformation of Quantum Field Theories

Although quantum field theory is one of the most successful theories in explaining observations in high energy physics, there are serious difficulties in the mathematical construction of interacting theories. The rigorous non-perturbative construction of non-trivial models is still one of the major challenges in mathematical physics. One of the more recent approaches to gain further insight in this direction are *deformations of quantum field theories*. In general a deformation of a quantum field is defined as a continuous one-parameter family of new fields, where the original field is just a special case where the deformation parameter is zero. The difficult task is to find such deformations that preserve essential properties like the spectrum condition, Poincaré covariance and locality.

This relatively novel topic is of interest both from the point of view of mathematics as well as physics. It was originally motivated by quantum field theory on noncommutative (d-dimensional) Minkowski space [23, 91], which is defined by non-commuting position operators $x_0, ...x_{d-1}$ satisfying $[x_\mu, x_\nu] = iQ_{\mu\nu}$, where Q is an anti-symmetric $(d \times d)$-matrix. Grosse and Lechner [41, 42] were the first to consider free quantum fields Φ_Q on this space, which are defined by deforming the corresponding creation- and annihilation operators $a^*(p)$ and $a(p)$. The deformation manifests itself here in a modified algebraic

† Most of this chapter has been published in [74].

structure of the operators, which can be expressed by the commutation relations

$$a_Q^*(p)a_Q^*(q) = e^{-ipQq}\, a_Q^*(q)a_Q^*(p). \tag{III.1}$$

As expected in such a setting the deformed field then violates locality and covariance. However, if one considers a whole family of fields Φ_Q, with Q ranging over a whole Lorentz orbit of matrices, full Poincaré covariance and a weakened form of locality can be restored [41]. As it turns out the fields are localized in wedge-shaped regions, which are defined as Poincaré transforms of a standard right-wedge

$$W_R := \{(\boldsymbol{x} \in \mathbb{R}^d) : x_1 > |x_0|\}, \tag{III.2}$$

and appear in various contexts in quantum field theory, especially in the construction of integrable models [54, 56, 83, 85]. A crucial property of these wedge regions is that they are on the one hand large enough to make the explicit construction of observables localized in W possible, and on the other hand small enough such that the causal structure of Minkowski space can be completely described by such regions [92]. Furthermore the wedge-localization can be used to apply scattering theory to the deformed model and it has been shown that these simple Q-dependent fields lead to a non-trivial scattering matrix of the form $e^{ip_1Qp_2}$ and therefore describe an interacting quantum field theory [41].

Based on this idea Buchholz and Summers [15] generalized the concrete deformation of the free field to the so-called *warped convolutions* of localized operators using a unitary representation of the translation group. This deformation procedure does not rely on any connection to non-commutative space-time and can be applied to *any* quantum field theory in its vacuum representation and it can be shown that most of the results of Grosse and Lechner [41] still hold in this generalized setting. In fact one obtains a Poincaré covariant, wedge-localized quantum field theory with a non-trivial scattering matrix differing from the original one by certain momentum dependent factors. The deformed operators in this ansatz are defined by the convolution integral

$$A_Q := \int U(Qp)AU(Qp)^{-1}dE(p) = (2\pi)^{-d}\int dpdx\, e^{-ip\cdot x}\, U(Qp)A\, U(Qp)^{-1}U(x), \tag{III.3}$$

where A is an operator on a Hilbert space (from the original undeformed algebra), U a unitary representation of $\mathbb{R}^d$ with spectral measure dE and the anti-symmetric matrix Q denotes again the deformation parameter. The mathematically rigorous definition of this oscillatory integral is a highly non-trivial task [11] and a generalization has recently been analyzed by Lechner and Waldmann in [59]. In the course of this one can also study locally non-commutative spaces, where the non-commutativity of the position operators is restricted to small distances (of the order of the Planck length).

Furthermore it has been discovered in [42] that there is a connection between the deformation of Wightman quantum field theories and a modified ("twisted") tensor product on the underlying algebra of test functions. Namely it becomes apparent that the above deformation of the field operators $\Phi_Q(f)$ is equivalent to a deformed product on this Borchers-Uhlmann algebra $\underline{\mathscr{S}}$ containing f. Based on this conclusion and the fact that the deformation of the fields manifests itself in the S-matrix, more general deformations of the Borchers-Uhlmann algebra have been characterized by Lechner in [57]. They are specified by a linear homeomorphism ρ of $\underline{\mathscr{S}}$ with $\rho(1) = 1$, $\rho(f^*) = \rho(f)^*$ and some further conditions which ensure the consistency with wedge-locality. To simplify matters only multiplicative deformations on momentum space $\widetilde{\rho(f)}_n = \rho_n \cdot \tilde{f}_n$ have been considered and if one requires in addition the compatibility of ρ with a certain (vacuum) state ω on $\underline{\mathscr{S}}$ it is possible to go back to deformed quantum fields on a Hilbert space by using the GNS construction. A family of such fields is again covariant, localized in wedge regions and creates the same one-particle vectors from the vacuum.

A calculation of the scattering matrix also shows that there is still no particle production in this theory, but the two-particle scattering function S_2 turns out to be a more complicated function $S_2(p_1 Q p_2)$ in contrast to the simple phase factor $e^{ip_1 Q p_2}$ occurring for warped convolutions. In analogy to the formula for warped convolutions these deformations can also be described by an integral over the undeformed fields, where the representation $U(x)$ of the translations is getting replaced by a more general integral kernel $K(x)$, i.e.

$$\Phi_Q \sim \int dpdx\, e^{-ipx}\, U(x)\Phi U(-x) K(-Qp).$$

Deformations of this kind are particularly interesting with regard to integrable quantum field theories in 1+1 dimensions (see e.g. [54, 56, 83, 85]). Because, as it has also been shown in [57], it is possible to describe a large class of integrable models on 1+1 dimensional Minkowski space, namely those with a factorizing S-matrix satisfying $S(0) = 1$, by a deformation of a free hermitian scalar Bose field on the (symmetric) Fock space (by deforming free Fermi fields on an *anti-symmetric* Fock space, this procedure has recently been generalized in [2] to include also models with $S(0) = -1$). The deformed creation- and annihilation operators a_Q^* and a_Q then correspond precisely to the Zamolodchikov creation/annihilation operators which satisfy commutation relations of the form

$$a_Q^\dagger(\theta_1) a_{-Q}^\dagger(\theta_2) = S_2(\theta_1 - \theta_2)\, a_{-Q}^\dagger(\theta_2) a_Q^\dagger(\theta_1), \tag{III.4}$$

where θ_1, θ_2 are rapidity parameters (a more detailed definition can be found below) and S_2 is again the two-particle scattering function, used to define the deformation.

More precisely one considers multiplication operators

$$(T_R(Qp)\Psi)(p_1, ..., p_n) \sim \prod R(Qp \cdot p_i)\Psi(p_1, ..., p_n)$$

where Q is again an antisymmetric "deformation matrix" depending on a wedge W (see also [41]). These operators are then used to deform the creation and annihilation operators of the free field $\Phi(f) = a^*(f^+) + a(\bar{f}^+)$ according to

$$a_{R,Q}(p) := a(p)\, T_R(Qp)$$

etc. Here $f \in \mathscr{S}$ is a test function and f^+ denotes the restriction of its Fourier transform to the upper mass shell. One can then show that under certain conditions on the functions R the deformed fields $\Phi_{R,Q}(f) := a^*_{R,Q}(f^+) + a_{R,Q}(\bar{f}^+)$ are still Poincaré covariant Wightman fields which are no longer localizable in compact regions but they are localized in wedges. In two dimensions, however, it is possible to show that the algebras for double cones [1] still satisfy the Reeh-Schlieder property if the S-matrix of the model fulfills some kind of additional regularity condition [56].

In this work we want to generalize this procedure to a *charged* scalar field in low dimensions and we will see that in this case it is also possible to change the statistics of the fields in such a way that they satisfy anyonic commutation relations. Hence we obtain in this way anyonic fields which create single particle vectors from the vacuum and in this sense they are as close to free fields as one can get in the case of braided commutation relations. This is an interesting result because of the no-go theorem by Bros and Mund and the reason why we can circumvent it is that in our case the algebras for regions smaller than a wedge, e.g. spacelike cones, are "too small" in the sense that they don't generate a dense set in the Hilbert space when acting on the vacuum, i.e. they don't satisfy the Reeh-Schlieder property [2].

Before we give a detailed description of the deformation procedure, we will take a look at the converse problem, namely how warped convolutions affect field nets with braid group statistics.

III.2. Warped Convolution of Field Nets with Braid Group Statistics

As already mentioned it has been shown in [11, 15] how a local and covariant net

$$\mathcal{O} \to \mathcal{F}(\mathcal{O}) \;, \quad \mathcal{O} \text{ a double cone}$$

of *-algebras (a "quantum field theory" in the sense of AQFT) can be deformed using warped convolutions. This kind of deformations preserves certain properties of the theory

[1] In $d = 1 + 1$ double cones are obtained by intersection of two opposite translated wedges.

[2] In other words one could see the no-go theorem as a proof that in our case the algebras for regions smaller than a wedge can not satisfy Reeh-Schlieder.

like covariance, the Reeh-Schlieder property or the modular data, whereas locality of the net is changed to wedge-locality. We now want to apply this deformation scheme also to field nets with braid group statistics and see if the statistics (or the charge structure) of the fields is changed by it. This will include also theories with non-abelian anyons so-called "plektons" and for this we first need to recall some concepts and definitions from the theory of warped convolutions.

We assume the existence of a strongly continuous action U of the translation group $\mathbb{R}^d$ on some Hilbert space $\mathcal{H}$ with spectrum in the forward light cone V_+, acting on the elements of our algebra according to $\alpha_x(F) = U(x)FU(-x)$. Using the spectral projectors E of this representation,

$$U(x) = \int dE(p)e^{ixp},$$

and a skew-symmetric matrix Q, i.e. $pQq = -qQp, \forall p, q \in \mathbb{R}^d$, we can define the warped convolution of suitable elements of our algebra by

$$F_Q := \int dE(p)\, \alpha_{Qp}(F).$$

(For the exact definition of this integral and its properties see again [11].)

Now to proof wedge-locality of the resulting deformed net one uses the following important result. Let F, G be such that

$$[\alpha_{Qp}(F), \alpha_{-Qq}(G)] = 0$$

for all p, q in the spectrum of U. Then the deformed operators satisfy

$$[F_Q, G_{-Q}] = 0.$$

To see how this is related to wedge locality take elements $A \in \mathcal{F}(\mathcal{O}_1), B \in \mathcal{F}(\mathcal{O}_2)$ where $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated, which means that A and B commute. Since the family of wedges is causally separating for double cones, there exists in this case a wedge W such that $\mathcal{O}_1 \subset W$ and $\mathcal{O}_2 \subset W'$, where W' denotes the causal complement of W. Now for every W there is a skew-symmetric matrix Q, s.t.

$$QV_+ \subset W \ , \quad -QV_+ \subset W'.$$

(In fact under certain requirements on the skew-symmetric matrices Q there is a 1-1 correspondence between the Q's and wedges W, see [41].) In this case we also get

$$[\alpha_{Qp}(A), \alpha_{-Qq}(B)] = 0 \ , \quad \forall p, q \in V_+,$$

which then leads to

$$[A_Q, B_{-Q}] = 0$$

for this choice of A, B, Q. From these considerations one can then conclude that the deformed operators A_Q, B_{-Q} are localized in $\mathcal{O}_1 + W$ and $\mathcal{O}_2 + W'$ respectively and the deformed net is thus wedge-local (For a detailed treatment see again [11, 15]).

Now we want to generalize this result to the case where the fields carry arbitrary charges, are localized in infinitely extended spacelike cones C and their statistics is governed by the braid group. More precisely our algebra elements F are localized in (equivalence classes of) paths of spacelike cones $\tilde{C}$. Given two spacelike separated regions $\tilde{C}_1$ and $\tilde{C}_2$ one can define a winding number $N(\tilde{C}_1, \tilde{C}_2)$ of $\tilde{C}_2$ w.r.t. $\tilde{C}_1$ depending only on the equivalence classes of $\tilde{C}_1$ and $\tilde{C}_2$ (its exact definition will not be needed here and can be found in section III.3.2).

It can then be shown that generically the fields satisfy commutation relations of the following form (see e.g. [30, 31]), which is more general then in the abelian case we are considering in the rest of this work. For elements F, G, localized in spacelike separated regions $\tilde{C}_1, \tilde{C}_2$, there exist fields $F^j \in \mathcal{F}(\tilde{C}_1), G^i \in \mathcal{F}(\tilde{C}_2)$ and numbers $R_{ij}(F, G, n)$ depending only on the charge structure of the theory, the charges that F and G carry and on the winding number $n = N(\tilde{C}_1, \tilde{C}_2)$, such that

$$F\ G = \sum_{i,j} R_{ij}(F, G, n)\ G^i\ F^j.$$

The exact form of the exchange elements R can be calculated in a given theory but will not be of importance here.

What's more important is that also in the case of localization in two spacelike separated cones $\tilde{C}_1, \tilde{C}_2$ one can always find paths of wedges $\tilde{W}, \tilde{W}'$ such that

$$\tilde{C}_1 \subset \tilde{W}\ , \quad \tilde{C}_2 \subset \tilde{W}'\ , \quad \tilde{W}' = (W', \tilde{e}'),$$

for some path $\tilde{e}'$ ending in W' (since a wedge is just a special case of a spacelike cone with extremal opening angle the concept of a "path of wedges" is the same as for cones). Note that an inclusion of the form $\tilde{C} \subset \tilde{W}$ not only means that $C \subset W$ but also that the paths corresponding to $\tilde{C}$ and $\tilde{W}$ are equivalent. This shows that for spacelike separated $F \in \mathcal{F}(\tilde{C}_1)$ and $G \in \mathcal{F}(\tilde{C}_2)$ we can still find a matrix Q (corresponding to a wedge W with $C_1 \subset W$, $C_2 \subset W'$) such that

$$\alpha_{Qp}(F)\alpha_{-Qq}(G) = \sum_{i,j} R_{ij}(F, G, n)\, \alpha_{-Qq}(G^i)\alpha_{Qp}(F^j).$$

This is because the translation α, used in the definition of the deformation, does not affect the winding number n or the charges of F and G. Now in exactly the same way as in [11, 15] one can define deformed fields F_Q, G_Q and show that they satisfy

$$F_Q\, G_{-Q} = \sum_{i,j} R_{ij}(F, G, n)\, G^i_{-Q} F^j_Q.$$

We have thus seen that the deformation preserves the commutation relations of the fields also in the case of braid group statistics.

An important difference to the case of compact localization is that we cannot take any matrix Q to deform an operator $F \in \mathcal{F}(\tilde{C})$ but we have to make sure that the wedge corresponding to Q satisfies $C \subset W$. So the set of admissible wedges (or equivalently deformation matrices) which can be used to deform an algebra element is limited due to its extended cone-localization.

III.3. Construction of Wedge-Local Fields with Anyonic Commutation Relations

We are now going to show how the multiplicative deformations of [57] can be generalized to the charged case resulting in anyon-like commutation relations for the wedge-localized polarization free generators. The deformations we are going to construct are defined on the symmetric Fock space

$$\mathcal{H} = \mathcal{F}_s(\mathcal{H}_1) \tag{III.5}$$

over the doubled (i.e. "charged") one-particle space $\mathcal{H}_1 := \mathcal{H}_1^+ \oplus \mathcal{H}_1^-$. To allow for anyonic commutation relations we will only consider Minkowski space with dimensions $d = 1+1$ and $d = 1+2$. In these two cases we will use as one particle Hilbert space

$$\begin{aligned} \mathcal{H}_1^{\pm} &= L^2(\mathbb{R}^2, d\mu) \simeq L^2(\mathbb{R}, d\theta) \ , \qquad &&\text{for } d = 1+1, \\ \mathcal{H}_1^{\pm} &= L^2(\mathbb{R}^3, d\mu) \qquad\qquad\qquad\quad , &&\text{for } d = 2+1, \end{aligned} \tag{III.6}$$

where $d\mu(p) = \delta(p^2 - m^2)\theta(p_0)d^dp$ denotes the Lorentz invariant measure on the mass shell and we will only consider $m > 0$. In the two dimensional case we will work in the rapidity parametrization where the rapidity θ is connected to the momentum by

$$p(\theta) = m \begin{pmatrix} \cosh(\theta) \\ \sinh(\theta) \end{pmatrix}. \tag{III.7}$$

As is well-known in our case the Hilbert space $\mathcal{H}$ is isomorphic to the tensor product $\mathcal{F}_s(\mathcal{H}_1^+) \otimes \mathcal{F}_s(\mathcal{H}_1^-)$ and we will refer to the first tensor factor as the "*particle space*" and to the second as the "*anti-particle space*". Because of this tensor product structure we will consider vectors of the form $\Psi_n \otimes \Psi_m$ which we denote by Ψ_n^m for simplicity. On this doubled Fock space we now have a charge conjugation operator C, which simply exchanges the two factors, and a charge operator Q measuring the charge of a vector,

$$\begin{aligned} (C\Psi)_n^m(p_1, ..., p_{n+m}) &:= \Psi_m^n(p_{n+1}, ..., p_m, p_1, ..., p_n), \\ (Q\Psi)_n^m(p_1, ..., p_{n+m}) &:= (n-m)\Psi_n^m(p_1, ..., p_{n+m}). \end{aligned} \tag{III.8}$$

We also naturally have two sets of creation and annihilation operators, a, a^* and b, b^*, which are defined according to

$$(a(\varphi)\Psi)^m_n(p_1, ..., p_{n+m}) = \sqrt{n+1} \int d\mu(p)\, \overline{\varphi(p)}\, \Psi^m_{n+1}(p, p_1, ..., p_{n+m}),$$
$$(b(\varphi)\Psi)^m_n(p_1, ..., p_{n+m}) = \sqrt{m+1} \int d\mu(p)\, \overline{\varphi(p)}\, \Psi^{m+1}_n(..., p, p_{n+1}, ..., p_{n+m})$$

and $a^*(\varphi) := a(\varphi)^*$, $b^*(\varphi) := b(\varphi)^*$. Their distributional kernels satisfy the canonical commutation relations [3]

$$[a^\sharp(p), a^\sharp(p')] = 0 \;, \qquad [a^\sharp(p), b^\sharp(p')] = 0,$$
$$[a(p), a^*(p')] = \omega_p\, \delta(p - p') \;, \qquad [a(p), b^*(p')] = 0,$$

with $\omega_p = \sqrt{\mathbf{p}^2 + m^2}$. Also the corresponding charge conjugated relations hold, where a and b are interchanged. Using these creation and annihilation operators one can then define the free field and its charge conjugate,

$$\begin{aligned} \Phi(f) &:= a^*(f^+) + b(\bar{f}^+), \\ \Phi^*(f) &:= b^*(f^+) + a(\bar{f}^+) = \Phi(\bar{f})^* = C\Phi(f)C. \end{aligned} \tag{III.9}$$

Additionally we will have a unitary representation $U(a, \Lambda)$ of the Poincaré group, which will be defined later, together with an anti-unitary space-time reflection J acting in general according to

$$(J\Psi)^m_n(p_1, .., p_{n+m}) := e^{i\beta q}\, \overline{\Psi^m_n(-jp_1, ..., -jp_{n+m})}. \tag{III.10}$$

Here $q = n - m$ denotes the charge, j has been (arbitrarily) chosen to be the reflection at the x_2 axis, $j(x_0, x_1, x_2) = (-x_0, -x_1, x_2)$, and β is an additional parameter which will be specified below. In the rapidity parametrization in $d = 1 + 1$ this simplifies to

$$(J\Psi)^m_n(\theta_1, ..., \theta_{n+m}) = e^{i\beta q}\, \overline{\Psi^m_n(\theta_1, ..., \theta_{n+m})}.$$

III.3.1. Deformations in $d = 1 + 1$

Following [57] a deformation on Fock space is defined by considering multiplication operators $T_R(\theta)$ on $\mathcal{F}_s(\mathcal{H}^\pm_1)$ according to

$$(T_R(\theta)\Psi)_n(\theta_1, ..., \theta_n) := \prod_{j=1}^{n} R(\theta - \theta_j)\Psi_n(\theta_1, ..., \theta_n), \tag{III.11}$$

with a "*deformation function*" $\theta \mapsto R(\theta)$ which usually satisfies (among other properties concerning analyticity) $\overline{R(\theta)} = R(\theta)^{-1}$ and $R(-\theta) = \overline{R(\theta)}$. Now the condition $R(-\theta) = R(\theta)^{-1}$ would lead to $R(0) = \pm 1$ and as we will see later this would lead to

[3] Here and in the following $a^\sharp$ always stands for either a^* or a.

Bose- or Fermi commutation relations. We are therefore going to omit the second condition and consider deformation functions satisfying only $\overline{R(\theta)} = R(\theta)^{-1}$.

This can now be generalized to the charged situation at hand and to anyonic statistics as follows. On $\mathcal{F}_s(\mathcal{H}_1^+) \otimes \mathcal{F}_s(\mathcal{H}_1^-)$ we define

$$T_{R,r}(\theta) := e^{i\frac{\rho}{2}}(T_R(\theta) \otimes T_r(\theta)), \tag{III.12}$$

with functions R and r satisfying $R(-\theta) = e^{i\mu}\overline{R(\theta)}$ and $r(-\theta) = e^{i\nu}\overline{r(\theta)}$ and three yet undefined parameters $\mu, \nu, \rho \in \mathbb{R}$. Denoting the charge conjugation again by C and taking $\beta = 0$ in the definition of the space-time reflection J the operator $T_{R,r}$ transforms according to

$$CT_{R,r}(\theta)C = T_{r,R}(\theta), \qquad JT_{R,r}(\theta)J = T_{R,r}(\theta)^* = e^{-i\rho}T_{\bar{R},\bar{r}}(\theta).$$

This multiplication operator is now used to define deformed particle annihilation operators

$$a_{R,r}(\theta) := a(\theta)T_{R,r}(\theta) = e^{i\frac{\rho}{2}}(a_R(\theta) \otimes T_r(\theta)), \tag{III.13}$$

where $a_R(\theta)$ denotes the standard "Lechner deformed" operator on $\mathcal{F}_s(\mathcal{H}_1^+)$. The charge conjugated operator $b_{R,r}(\theta)$ then turns out to be

$$b_{R,r}(\theta) := Ca_{R,r}(\theta)C = b(\theta)T_{r,R}(\theta) = e^{i\frac{\rho}{2}}(T_r(\theta) \otimes b_R(\theta)). \tag{III.14}$$

Using the transformation properties of $T_{R,r}(\theta)$ the space-time reflected operators are $Ja_{R,r}(\theta)J = e^{-i\rho}a_{\bar{R},\bar{r}}(\theta)$ and the adjoint operators are of course defined as $a^*_{R,r}(\theta) := a_{R,r}(\theta)^*$.

To determine the locality properties of the deformed fields we will need the commutation relations between the various creation and annihilation operators and between them and the operators $T_{R,r}(\theta)$. A straightforward calculation yields

$$\begin{aligned} a(\theta)T_{R,r}(\theta') &= R(\theta'-\theta)T_{R,r}(\theta')a(\theta)\ , \quad & b(\theta)T_{R,r}(\theta') &= r(\theta'-\theta)T_{R,r}(\theta')b(\theta), \\ a^*(\theta)T_{R,r}(\theta') &= R(\theta'-\theta)^{-1}T_{R,r}(\theta')a^*(\theta)\ , \quad & b^*(\theta)T_{R,r}(\theta') &= r(\theta'-\theta)^{-1}T_{R,r}(\theta')b^*(\theta). \end{aligned}$$

This leads to the commutation relations

$$\begin{aligned} a^\sharp_{R,r}(\theta)a^\sharp_{\bar{R},\bar{r}}(\theta') &= e^{-i\mu}a^\sharp_{\bar{R},\bar{r}}(\theta')a^\sharp_{R,r}(\theta), \\ a^\sharp_{R,r}(\theta)b^\sharp_{\bar{R},\bar{r}}(\theta') &= e^{-i\nu}b^\sharp_{\bar{R},\bar{r}}(\theta')a^\sharp_{R,r}(\theta). \end{aligned} \tag{III.15}$$

The relations with a and b interchanged follow by charge conjugation and noting that the constants μ and ν are defined by the functions R and r and satisfy $\mu(\bar{R}) = -\mu(R)$. The commutation relations for mixed creation and annihilation operators turn out to be

$$\begin{aligned} a_{R,r}(\theta)a^*_{\bar{R},\bar{r}}(\theta') &= e^{i\mu}a^*_{\bar{R},\bar{r}}(\theta')a_{R,r}(\theta) + e^{i(\mu-\rho)}\delta(\theta-\theta')T_{R,r}(\theta)^2, \\ b_{R,r}(\theta)b^*_{\bar{R},\bar{r}}(\theta') &= e^{i\mu}b^*_{\bar{R},\bar{r}}(\theta')b_{R,r}(\theta) + e^{i(\mu-\rho)}\delta(\theta-\theta')T_{r,R}(\theta)^2. \end{aligned} \tag{III.16}$$

Note that the last term is different for the a's and b's if $R \neq r$! Using the trivial commutation relations between a and b^* we also get

$$a_{R,r}(\theta) b^*_{\bar{R},\bar{r}}(\theta') = e^{i\nu} b^*_{\bar{R},\bar{r}}(\theta') a_{R,r}(\theta), \tag{III.17}$$

and the charge conjugated relation with a and b interchanged.

Having computed all the necessary relations of the creation and annihilation operators we can now define the deformed fields. For a test function $f \in \mathscr{S}(\mathbb{R}^2)$ we define

$$\begin{aligned} \Phi_{R,r}(f) &:= a^*_{R,r}(f^+) + b_{R,r}(\bar{f}^+), \\ \Phi^*_{R,r}(f) &= b^*_{R,r}(f^+) + a_{R,r}(\bar{f}^+) = C\Phi_{R,r}(f)C, \end{aligned} \tag{III.18}$$

with $f^{\pm}(\theta) := \frac{1}{2\pi} \int d^2x f(\pm x) e^{ip(\theta)x}$. We also need the field for the reflected wedge

$$\widehat{\Phi}_{R,r}(f) := J\Phi_{R,r}(\alpha_j f)J = e^{i\rho} a^*_{\bar{R},\bar{r}}(f^+) + e^{-i\rho} b_{\bar{R},\bar{r}}(\bar{f}^+), \tag{III.19}$$

where the reflected test function $\alpha_j f$ is defined as $(\alpha_j f)(x) := \overline{f(-x)}$.
A straightforward computation, using the aforementioned relations for the deformed creation and annihilation operators, then shows that for the fields to satisfy simple commutation relations we need to set $\nu = -\mu$,[4] which then leads to

$$\Phi_{R,r}(f)\widehat{\Phi}_{R,r}(g) = e^{-i\mu}\,\widehat{\Phi}_{R,r}(g)\Phi_{R,r}(f). \tag{III.20}$$

Next we want to calculate the commutation relations between $\Phi_{R,r}(f)$ and $\widehat{\Phi}^*_{R,r}(g)$ which turn out to be

$$\begin{aligned} &\Phi_{R,r}(f)\widehat{\Phi}^*_{R,r}(g) - e^{i\mu}\,\widehat{\Phi}^*_{R,r}(g)\Phi_{R,r}(f) = \\ &e^{i\mu} \int d\theta g^+(\theta) f^-(\theta) T_{r,R}(\theta)^2 - e^{-2i\rho} \int d\theta f^+(\theta) g^-(\theta) T_{\bar{R},\bar{r}}(\theta)^2. \end{aligned} \tag{III.21}$$

For spacelike separated f and g with $supp(g)$ to the left of $supp(f)$ we want the right-hand side of equation (III.21) to vanish; thus we need to set $\rho = -\frac{\mu}{2}$ such that $e^{i\mu} = e^{-2i\rho}$. Then we can use the arguments in [54] to show that

$$(\Phi_{R,r}(f)\widehat{\Phi}^*_{R,r}(g) - e^{i\mu}\,\widehat{\Phi}^*_{R,r}(g)\Phi_{R,r}(f))\Psi^m_n \tag{III.22}$$

vanishes for all $\Psi^m_n \equiv \Psi_n \otimes \Psi_m$ in the domain of definition of Φ if the deformation functions R and r are analytic in the strip $S(0,\pi) := \{z \in \mathbb{C} : 0 < Im(z) < \pi\}$, bounded and continuous on its closure and satisfy the *"crossing relations"*

$$R(\theta + i\pi) = \overline{r(\theta)}\ , \qquad r(\theta + i\pi) = \overline{R(\theta)}. \tag{III.23}$$

[4] Note that if we would work with only one set of annihilation/creation operators on a single Fock space, the condition $e^{i\mu} = e^{-i\mu}$ would lead to $\mu = k\pi$ with $k \in 2\mathbb{Z} + 1$, i.e. to Bose- or Fermi statistics.

These conditions then allow us to shift the integration in (III.21) from θ to $\theta+i\pi$, because of the known analyticity properties of $f^{\pm}$ and $g^{\pm}$. For a more detailed argument see again [54] or the proof of Proposition III.1.

We have therefore shown that if there is a wedge W such that $supp(f) \subset W$ and $supp(g) \subset W'$ the fields satisfy

$$\Phi_{R,r}(f)\widehat{\Phi}^*_{R,r}(g) = e^{i\mu}\,\widehat{\Phi}^*_{R,r}(g)\Phi_{R,r}(f). \tag{III.24}$$

To summarize our construction let us compare it with the neutral case studied in [57] by writing down the input we need in both cases to define a deformation.

- In [57] a deformation was defined on the neutral bosonic Fock space by choosing a function $\mathcal{R} : S(0,\pi) \to \mathbb{C}$, which is analytic in $S(0,\pi)$, bounded on $\overline{S(0,\pi)}$ and satisfies

$$\mathcal{R}(-x) = \overline{\mathcal{R}(x)} = \mathcal{R}(x)^{-1}\ , \quad \forall x \in \mathbb{R} \tag{III.25}$$

  and the crossing relation

$$\mathcal{R}(i\pi - x) = \mathcal{R}(x)\ , \quad \forall x \in \mathbb{R}. \tag{III.26}$$

  The most general class of such functions has been calculated in [55] and it turns out that they are of the form

$$\mathcal{R}(\theta) = \pm e^{ia\sinh\theta}\prod_k \frac{\sinh\beta_k - \sinh\theta}{\sinh\beta_k + \sinh\theta} \tag{III.27}$$

  with some parameters $a, \{\beta_k\}$ satisfying certain additional conditions, where the β_k determine the zeroes of the function $\mathcal{R}$.

- In the charged case at hand we now have two functions R and r, analytic in $S(0,\pi)$ and bounded on its closure, which have to satisfy relations similar to (III.25) and (III.26), namely

$$\begin{aligned} e^{-i\mu}R(-x) = \overline{R(x)} = R(x)^{-1}\ , \quad \forall x \in \mathbb{R} \\ e^{i\mu}r(-x) = \overline{r(x)} = r(x)^{-1}\ , \quad \forall x \in \mathbb{R} \end{aligned} \tag{III.28}$$

  and the crossing relation in this case turns out to be

$$R(i\pi - x) = r(x)\ , \quad \forall x \in \mathbb{R}. \tag{III.29}$$

  Now of course one can always separate the phase factor $e^{\pm i\mu}$ from the deformation functions by defining $R =: e^{i\frac{\mu}{2}}R^+$ and $r =: e^{-i\frac{\mu}{2}}R^-$. The functions $R^{\pm}$ then satisfy the usual relations (III.25) without the phase factors present. But the aforementioned conditions can be simplified further by noting that because of the analyticity

of R^+ and R^- the crossing relation (III.29) can be used to *define* the function R^- in terms of R^+ by setting $R^-(\theta) := R^+(i\pi - \theta), \forall\theta \in S(0, \pi)$.
Therefore we are left with choosing a parameter $\mu \in \mathbb{R}$ and a single deformation function $R^+ : S(0, \pi) \to \mathbb{C}$, satisfying

$$\begin{aligned} R^+(-x) &= \overline{R^+(x)} = R^+(x)^{-1} \ , \quad \forall x \in \mathbb{R} \\ R^+(i\pi - x) &= \overline{R^+(i\pi + x)} = R^+(i\pi + x)^{-1} \ , \quad \forall x \in \mathbb{R} \end{aligned} \tag{III.30}$$

but *not* the crossing relation (III.26)! So there is no condition relating the values of R^+ on the upper boundary of $S(0, \pi)$ to those on the real boundary.

It is now an interesting question if the class of admissible deformation functions is actually larger in the charged case than in the neutral case and the answer to this question turns out to be yes. To show that there really are functions $F : \overline{S(0, \pi)} \to \mathbb{C}$ satisfying all of the above requirements for our R-functions but *not* the crossing symmetry $F(i\pi - x) = F(x), \forall x \in \mathbb{R}$, consider the family of functions

$$F_w(z) := i\frac{e^z\alpha - i\bar{\alpha}}{e^z\bar{\alpha} + i\alpha}, \quad \alpha = 1 - iw \ , \quad w \in \mathbb{R}, |w| < 1. \tag{III.31}$$

They are clearly analytic in $S(0, \pi)$ and a short calculation shows that for real x they also satisfy $F_w(-x) = \overline{F_w(x)} = F_w(x)^{-1}$ and $F_w(i\pi - x) = \overline{F_w(i\pi + x)} = F_w(i\pi + x)^{-1}$, but the condition $F_w(i\pi - x) = F_w(x)$ is not satisfied (for $w = 0$ at least $F_0(i\pi - x) = -F_0(x)$ still holds)!

So we could chose e.g. an arbitrary deformation function $\mathcal{R}$ from the neutral case and define

$$R^+(z) := F_w(z)\mathcal{R}(z) \ , \quad R^-(z) = F_w(i\pi - z)\mathcal{R}(z). \tag{III.32}$$

We believe that by taking products of functions F_w for different parameters w one would essentially obtain the most general class of deformation functions R^+ but a rigorous proof of this statement could not yet be formulated.

In the following we will see how these deformation functions for the charged fields are related to the two-particle scattering matrix of our model. Because we are now only interested in the momentum dependence of the S-matrix we set $\mu = 0$ which means that the deformed fields commute if the test functions have the right support properties.

Because of the simple structure of our deformed fields and because everything is on-shell in our setting the outgoing scattering states are of the form $a^*_{R,r}(f^+)a^*_{\bar{R},\bar{r}}(g^+)\Omega$ if the Fourier transforms of f and g have compact support and supp(g) is to the left of supp(f). (For the exact definition of the outgoing/incoming scattering states see section III.3.2) So because of $T_r(\theta)\Omega = \Omega$ the S-matrix for particle-particle (S_{pp}) and antiparticle-antiparticle (S_{aa}) scattering is formally just R^2. But in the charged case at hand we also have to consider states of the form $a^*_{R,r}(f^+)b^*_{\bar{R},\bar{r}}(g^+)\Omega$ which basically look like

$$(a^*_{R,r}(f^+)b^*_{\bar{R},\bar{r}}(g^+)\Omega)(\theta_1,\theta_2) \sim r(\theta_1-\theta_2)f^+(\theta_1)g^+(\theta_2) + r(\theta_2-\theta_1)f^+(\theta_2)g^+(\theta_1).$$

Therefore the S-matrix for particle-antiparticle scattering (S_{pa}) turns out to be r^2. We can summarize this by formally writing

$$\begin{aligned} S_{pp} \sim S_{aa} \sim R^2, \\ S_{pa} \sim S_{ap} \sim r^2. \end{aligned} \tag{III.33}$$

Recalling the relation $r(\theta) = \overline{R(\theta + i\pi)}$ we see that R^2 evaluated at the lower boundary of the strip $S(0,\pi)$ determines the scattering between the same kind of particles while R^2 at the upper boundary determines particle-antiparticle scattering.

In the next step we want to analyze the dependence on the additional parameter μ. Setting $\mu = 2\pi\lambda$, $\lambda \in \mathbb{R}$ the deformed fields satisfy anyonic commutation relations in the usual form, i.e.

$$\Phi\widehat{\Phi} = e^{-2i\pi\lambda}\widehat{\Phi}\Phi \ , \qquad \Phi\widehat{\Phi}^* = e^{2i\pi\lambda}\widehat{\Phi}^*\Phi. \tag{III.34}$$

In this case we can simplify the above deformation and rewrite it using the charge operator Q. For this purpose we take for simplicity a standard deformation function $\mathcal{R}$ satisfying $\mathcal{R}(-\theta) = \overline{\mathcal{R}(\theta)}$ and define

$$R(\theta) = e^{i\pi\lambda}\mathcal{R}(\theta) \qquad r(\theta) = e^{-i\pi\lambda}\mathcal{R}(\theta). \tag{III.35}$$

The corresponding multiplication operator $T_{R,r}(\theta) = e^{-i\pi\lambda/2}T_R(\theta) \otimes T_r(\theta)$ can then be written as

$$T_{R,r}(\theta) = (T_{\mathcal{R}}(\theta) \otimes T_{\mathcal{R}}(\theta))e^{i\pi\lambda(Q-1/2)}. \tag{III.36}$$

Note that by explicitly using the charge operator Q in the deformation we see that we are effectively using a different deformation function on every charge sector, i.e. $T_{R,r}$ depends on the charge of the vector we are applying it to (albeit in a rather trivial manner in the above example).

We can now again define the deformed fields according to

$$\begin{aligned} \Phi_{\mathcal{R},\lambda}(f) &:= a^*_{\mathcal{R},\lambda}(f^+) + b_{\mathcal{R},\lambda}(\bar{f}^+) = \Phi_{\mathcal{R}}(f)e^{-i\pi\lambda(Q+1/2)}, \\ \Phi^*_{\mathcal{R},\lambda}(f) &= b^*_{\mathcal{R},\lambda}(f^+) + a_{\mathcal{R},\lambda}(\bar{f}^+) = \Phi^*_{\mathcal{R}}(f)e^{i\pi\lambda(Q-1/2)}, \end{aligned} \tag{III.37}$$

and calculate the fields for the opposite wedge,

$$J\Phi_{\mathcal{R},\lambda}(\alpha_j f)J = \Phi_{\overline{\mathcal{R}},-\lambda}(f).$$

Using the commutation relations

$$Q\Phi_{\mathcal{R},\lambda}(f) = \Phi_{\mathcal{R},\lambda}(f)(Q+1) \ , \quad Q\Phi^*_{\mathcal{R},\lambda}(f) = \Phi^*_{\mathcal{R},\lambda}(f)(Q-1)$$

between the fields and the charge operator it is now easy to check that the fields satisfy

$$\begin{aligned}\Phi_{\mathcal{R},\lambda}(f)\Phi^*_{\overline{\mathcal{R}},-\lambda}(g) &= e^{2\pi i\lambda}\,\Phi^*_{\overline{\mathcal{R}},-\lambda}(g)\Phi_{\mathcal{R},\lambda}(f),\\ \Phi_{\mathcal{R},\lambda}(f)\Phi_{\overline{\mathcal{R}},-\lambda}(g) &= e^{-2\pi i\lambda}\,\Phi_{\overline{\mathcal{R}},-\lambda}(g)\Phi_{\mathcal{R},\lambda}(f),\end{aligned} \tag{III.38}$$

if f and g have again the right support properties.

Summing up we have seen that if we apply the deformations of [57] to a charged scalar field, we can change the deformations using the charge operator to obtain wedge-localized anyonic one-particle generators.

From the definition (III.37) we also see that the one-particle states the fields create are changed by a constant factor $e^{-i\pi\lambda/2}$ and also the S-matrix elements get multiplied with such exponential factors. Furthermore, the scattering states are no longer symmetric under permutations but inherit the braided symmetry from the fields which create them from the vacuum.

However, one could get the same result by choosing a deformation with $\mu = 0$ and instead take a representation J_λ of the reflections with $\beta = \pi\lambda q$, i.e.

$$J_\lambda := e^{i\pi\lambda Q^2} J = \bigoplus_{q=-\infty}^{\infty} e^{i\pi\lambda q^2} J P_q, \tag{III.39}$$

where J is just the representation used before, acting as complex conjugation, and P_q is the projection onto the charge q Hilbert space. Because the charge of a vector is invariant under Poincaré transformations this clearly still yields a representation of $\mathcal{P}_+$.

Now take any wedge-localized charged bosonic field ϕ and define the field for the opposite wedge as

$$\hat{\phi} := J_\lambda \phi J_\lambda. \tag{III.40}$$

A straightforward calculation then shows that these fields indeed satisfy commutation relations of the form (III.38). This construction is possible because we can choose a different representation of the Lorentz group for every charge q due to the charge structure of the Hilbert space. In this way it is possible to arbitrarily choose the commutation relations of the wedge-local fields, which is similar to the case described in the introduction with two different independent charged fields. Here we have "left" and "right" fields (localized in left- and right-wedges) which can only be transformed into each other by a reflection making them independent in this sense.

Hence we will now proceed to a more interesting construction, namely wedge-local fields with anyonic statistics in $d = 2 + 1$.

III.3.2. Deformations in $d = 2 + 1$

In the next step we will try to find a wedge-local deformation leading to braided commutation relations in $d = 2 + 1$. One of the reasons why this is considerably more complicated is the presence of the rotations (and boosts orthogonal to the edge of a wedge) in the Lorentz group in three dimensions and therefore there are not only left- and right wedges but a continuous family of possible directions of wedges. Moreover, it is known that for the fields to have definite commutation relations they have to carry further information in addition to the localization region. Therefore, we consider localization not only in wedges but in so-called *paths of wedges*, containing as an additional information a kind of winding number (for the definition of "paths of wedges" used in this chapter see e.g. also [68]). Just as in the case of spacelike cones such paths of wedges can be denoted by $\tilde{W} = (W, \tilde{e})$ and are defined by a wedge W and a homotopy class $\tilde{e}$ of paths in the manifold of space-like directions H^3, starting at a reference direction e_0 and ending at a point inside the wedge. To be concrete we chose e_0 as $e_0 = (0, 0, -1)$, which has the advantage that it is invariant under the reflection j.

As in the two-dimensional case we will work on the charged Fock-space, where we now have $\mathcal{H}_1^{\pm} = L^2(\mathbb{R}^3, d\mu)$ with the measure on the mass shell $d\mu(p) = d^3p\,\delta(p^2 - m^2)\theta(p_0)$ and for the sake of clarity we introduce the shorthand notation

$$\Psi_n^m(\underline{p}) := \Psi_n^m(p_1, ..., p_{n+m}), \quad \text{with } \underline{p} = (p_1, ..., p_{n+m}). \tag{III.41}$$

Representation of the Covering Group of $\mathcal{L}_+^{\uparrow}$

Because of the spin-statistics theorem in 2+1 dimensions [69] we want 2π-rotations to act non-trivially, i.e. $U(2\pi) \neq \pm 1$. We therefore consider representations of the universal covering of the Poincaré group $\tilde{\mathcal{P}}_+^{\uparrow}$, which is the semi-direct product of the translations with the universal covering of the Lorentz group $\tilde{\mathcal{L}}_+^{\uparrow}$. In three dimensions this group $\tilde{\mathcal{L}}_+^{\uparrow}$ can be identified with the set

$$\{(\gamma, \omega) | \gamma \in \mathbb{C}, |\gamma| < 1, \omega \in \mathbb{R}\}, \tag{III.42}$$

with corresponding group multiplication $(\gamma_1, \omega_1)(\gamma_2, \omega_2) = (\gamma_3, \omega_3)$, which is given by [5, p.594]

$$\begin{aligned} \gamma_3 &= (\gamma_2 + \gamma_1 e^{-i\omega_2})(1 + \gamma_1 \overline{\gamma_2} e^{-i\omega_2})^{-1}, \\ \omega_3 &= \omega_1 + \omega_2 - i \log\left[(1 + \gamma_1 \overline{\gamma_2} e^{-i\omega_2})(c.c.)^{-1}\right]. \end{aligned}$$

Identifying elements of this group with homotopy classes $\tilde{\Lambda}$ of paths $t \mapsto \Lambda(t) \in \mathcal{L}_+^{\uparrow}$ starting at the unit element and ending at $\Lambda \in \mathcal{L}_+^{\uparrow}$ we can define the action of $\tilde{\mathcal{L}}_+^{\uparrow}$ on paths of wedges $\tilde{W} = (W, \tilde{e})$ according to $\tilde{\Lambda} \cdot \tilde{W} = (\Lambda W, \tilde{\Lambda} \cdot \tilde{e})$, where $\tilde{\Lambda} \cdot \tilde{e}$ is the equivalence class of the path $t \mapsto \Lambda(t)e(t)$.

The translations act on the Fock-space in the usual way as $(U(a)\Psi)(p) = e^{iap}\Psi(p)$ and they obviously do not change winding numbers of localization regions. Therefore we will be interested in the representation of the covering of the Lorentz group most of the time. Because of the charge structure of our Fock-space, $\mathcal{H} = \oplus \mathcal{H}_q$, we consider a representation of $\tilde{\mathcal{P}}_+^\uparrow$ of the form $U = \oplus U_q$. The observables of a theory should commute with rotations around 2π and their restriction onto a subspace $\mathcal{H}_q$ with fixed charge should be irreducible. Therefore the 2π-rotations act as a multiple of the identity on vectors with fixed charge, i.e.

$$U_q(2\pi) = e^{2\pi i \mathcal{S}_q} \cdot \mathbb{1}, \tag{III.43}$$

where $\mathcal{S}_q$ is called the spin of the sector with charge q (determined only modulo 1). From the general theory of superselection sectors one knows that the spin is the same for a sector and its conjugate sector (see [36, 69]), therefore we must have $\mathcal{S}_q = \mathcal{S}_{-q}$. When restricted to the 1-(anti-) particle Hilbert space $\mathcal{H}_1^\pm$ we also want our representation of the Poincaré group to be one of the well-known irreducible unitary representations for a spin σ, defined according to,

$$U^{(\sigma)}(a, \tilde{\Lambda})\varphi)(p) := e^{ia\cdot p}\, e^{i\sigma\Omega(\tilde{\Lambda},p)}\varphi(\Lambda^{-1}p). \tag{III.44}$$

The factor $\Omega(\tilde{\Lambda}, p)$ is the Wigner-rotation, which can be expressed for $\tilde{\Lambda} = (\gamma, \omega)$ according to (see e.g. [68, Appendix B])

$$\Omega(\tilde{\Lambda}, p) = \omega - i \log\left[(1 - \gamma(p)\bar{\gamma}e^{-i\omega})(c.c)^{-1}\right] - i \log\left[\left(1 + \frac{\gamma - \gamma(p)e^{-i\omega}}{1 - \gamma(p)\bar{\gamma}e^{-i\omega}}\bar{\gamma}(\Lambda^{-1}p)\right).(c.c)^{-1}\right].$$

It satisfies the cocycle relation

$$\Omega(\tilde{\Lambda}\tilde{\Lambda}', p) = \Omega(\tilde{\Lambda}, p) + \Omega(\tilde{\Lambda}', \Lambda^{-1}p) \tag{III.45}$$

and for pure rotations $\tilde{r}$ it simplifies to $\Omega(\tilde{r}(\omega), p) = \omega$.

Motivated by the above considerations we define the full representation U on $\mathcal{H}$ according to

$$(U(a, \tilde{\Lambda})\Psi)_n^m(\underline{p}) := e^{ia\cdot\underline{p}}\, e^{is_q\Omega_n^m(\tilde{\Lambda},\underline{p})}\Psi_n^m(\Lambda^{-1}\underline{p}), \tag{III.46}$$

where we have introduced the notation

$$\Omega_n^m(\tilde{\Lambda}, \underline{p}) := \sum_{i=1}^{n} \Omega(\tilde{\Lambda}, p_i) - \sum_{j=n+1}^{n+m} \Omega(\tilde{\Lambda}, p_j). \tag{III.47}$$

From this we see that a 2π-rotation acts on vectors of charge q according to

$$U_q(2\pi) = e^{2\pi i q s_q} \cdot \mathbb{1} \stackrel{!}{=} e^{2\pi i \mathcal{S}_q} \cdot \mathbb{1},$$

leading to $\mathcal{S}_q = q s_q$ which implies that $s_{-q} = -s_q$. Following lemma II.1 our choice for s_q is $s_q = \lambda q$ with an unspecified parameter $\lambda \in \mathbb{R}$ and we will see that this choice allows our deformed fields to have anyonic commutation relations.

Restricting this representation to $\mathcal{H}_1$ then leads to

$$(U(a,\tilde{\Lambda})\Psi)^0_1(p) = e^{ia\cdot p}\, e^{is_1\Omega(\tilde{\Lambda},p)}\Psi^0_1(\Lambda^{-1}p) = e^{ia\cdot p}\, e^{i\lambda\Omega(\tilde{\Lambda},p)}\Psi^0_1(\Lambda^{-1}p),$$
$$(U(a,\tilde{\Lambda})\Psi)^1_0(p) = e^{ia\cdot p}\, e^{-is_{-1}\Omega(\tilde{\Lambda},p)}\Psi^1_0(\Lambda^{-1}p) = e^{ia\cdot p}\, e^{i\lambda\Omega(\tilde{\Lambda},p)}\Psi^1_0(\Lambda^{-1}p),$$

so we see that on $\mathcal{H}_1$ the representation really reduces to

$$U(a,\tilde{\Lambda}) \mid_{\mathcal{H}_1} = U^{(\lambda)}(a,\tilde{\Lambda}) \oplus U^{(\lambda)}(a,\tilde{\Lambda}).$$

Extension to $\tilde{\mathcal{P}}_+$:
Given the proper orthochronous Poincaré group $\mathcal{P}^{\uparrow}_+$ one can obtain the proper Poincaré group $\mathcal{P}_+$ by adjoining the reflection j at the x_2-axis, which satisfies the relations

$$j^2 = 1\ , \quad j\Lambda_1(t)j = \Lambda_1(t)\ , \quad jr(\omega)j = r(-\omega)\ , \quad j(a,1)j = (j\cdot a,1), \tag{III.48}$$

(which then imply $j\Lambda_2(t)j = \Lambda_2(-t)$), where Λ_1, Λ_2 are boosts in the direction of the x_1, x_2 axis respectively. This yields a disconnected group and its universal covering $\tilde{\mathcal{P}}_+$ can be defined by adjoining an element $\tilde{j}$ to $\tilde{\mathcal{P}}^{\uparrow}_+$ with the relations

$$\tilde{j}^2 = 1\ , \quad \tilde{j}(a,(\gamma,\omega))\tilde{j} = (j\cdot a,(\bar{\gamma},-\omega)). \tag{III.49}$$

Defining by $\tilde{j}\cdot\tilde{e}$ the equivalence class w.r.t. $j\cdot W$ of the path $t\mapsto j\cdot e(t)$ the element $\tilde{j}$ acts on $\tilde{W}$ according to

$$\tilde{j}\cdot\tilde{W} := (j\cdot W, \tilde{j}\cdot\tilde{e}), \tag{III.50}$$

where $\tilde{j}\cdot\tilde{e}$ is still a path starting at the reference direction e_0 because we have chosen e_0 invariant under j.

The Deformed Model

On the Hilbert space $\mathcal{H}$ we again define the free charged scalar field according to $\Phi(f) = a^*(f^+) + b(\bar{f}^+)$ with a and b defined as in the previous chapter and

$$f^{\pm}(p) = \frac{1}{(2\pi)^{3/2}}\int d^3x f(x) e^{\pm ix\cdot p}\Big|_{p\in H^+_m}.$$

This field is local and covariant with respect to the representation U only for $\lambda = 0$. Now we want to deform this field using multiplicative deformations such that the deformed field $\Phi_{\tilde{W}}(f)$ is covariant with respect to U for an *arbitrary* $\lambda\in\mathbb{R}$ and localized in $supp f + \tilde{W}$, where for simplicity we only consider wedges having the origin contained in their edge because of translation covariance. Taking a $\tilde{W}' = (W',\tilde{e}')$ with an arbitrary path $\tilde{e}'$ ending in W' and two test-functions f, g such that $supp(f) + W$ is causally separated from $supp(g) + W'$ we want the fields to satisfy

$$\Phi_{\tilde{W}}(f)\Phi^{\sharp}_{\tilde{W}'}(g) = e^{\mp 2\pi i\lambda k(\tilde{W},\tilde{W}')}\Phi^{\sharp}_{\tilde{W}'}(g)\Phi_{\tilde{W}}(f). \tag{III.51}$$

The statistics factor only depends on the two wedges $\tilde{W}, \tilde{W}'$ and we will see that it is related to the relative winding number $N(\tilde{W}, \tilde{W}')$ defined below according to $k(\tilde{W}, \tilde{W}') = -2N(\tilde{W}, \tilde{W}') - 1$.

A general multiplication operator on $\mathcal{H}$ is of the form

$$(T_{\tilde{W}}(p)\Psi)_n^m(\underline{p}) := A_{\tilde{W}}^{n,m}(p;\underline{p})\Psi_n^m(\underline{p}), \tag{III.52}$$

and such operators are now used to deform the creation and annihilation operators,

$$\begin{aligned} a_{\tilde{W}}(p) &:= T_{\tilde{W}}(p)a(p) \ , & a_{\tilde{W}}^*(p) &:= a_{\tilde{W}}(p)^* = a^*(p)T_{\tilde{W}}(p)^*, \\ b_{\tilde{W}}(p) &:= Ca_{\tilde{W}}(p)C \equiv T_{\tilde{W}}^c(p)b(p) \ , & b_{\tilde{W}}^*(p) &:= b_{\tilde{W}}(p)^*. \end{aligned} \tag{III.53}$$

The charge conjugated multiplication operator acts according to

$$(T_{\tilde{W}}^c(p)\Psi)_n^m(\underline{p}) = A_{\tilde{W}}^{m,n}(p;\underline{p}^c)\Psi_n^m(\underline{p}), \tag{III.54}$$

with $\underline{p}^c = (p_{n+1}, ..., p_{n+m}, p_1, ..., p_n)$. These definitions mean that for a one-particle vector φ the annihilation operator e.g. acts as

$$(a_{\tilde{W}}(\varphi)\Psi)_n^m(\underline{p}) = \sqrt{n+1}\int d\mu(p)\overline{\varphi(p)}A_{\tilde{W}}^{n,m}(p;\underline{p})\Psi_{n+1}^m(p,\underline{p}).$$

The deformed field is then defined as $\Phi_{\tilde{W}}(f) = a_{\tilde{W}}^*(f^+) + b_{\tilde{W}}(\bar{f}^+)$ and we want it to be covariant under $\tilde{\mathcal{P}}_+^\uparrow$, i.e.

$$\begin{gathered} U(a,\tilde{\Lambda})\Phi_{\tilde{W}}(f)U(a,\tilde{\Lambda})^{-1} = \Phi_{\tilde{\Lambda}\tilde{W}}(\alpha_{(a,\Lambda)}(f)), \\ \text{with } (\alpha_{(a,\Lambda)}(f))(x) = f(\Lambda^{-1}(x-a)). \end{gathered} \tag{III.55}$$

Therefore also the deformed annihilation operators have to satisfy $U(a,\tilde{\Lambda})a_{\tilde{W}}(\varphi)U(a,\tilde{\Lambda})^{-1} = a_{\tilde{\Lambda}\tilde{W}}(\alpha_{(a,\Lambda)}(\varphi))$ which leads to the following relation for the deformation functions,

$$e^{-i\lambda\Omega_n^m(\tilde{\Lambda},\underline{p})}e^{-i\lambda(q+1)\Omega(\tilde{\Lambda},p)}A_{\tilde{W}}^{n,m}(\Lambda^{-1}p;\Lambda^{-1}\underline{p}) = A_{\tilde{\Lambda}\tilde{W}}^{n,m}(p;\underline{p}). \tag{III.56}$$

The product structure of this covariance condition and the known deformation functions in [57] now motivate the following ansatz,

$$A_{\tilde{W}}^{n,m}(p;\underline{p}) := u_{\tilde{W}}^\lambda(p)^{q+1}\prod_{i=1}^{n}u_{\tilde{W}}^\lambda(p_i)R(Qp\cdot p_i)\prod_{j=n+1}^{n+m}\overline{u_{\tilde{W}}^\lambda(p_j)}R(Qp\cdot p_j), \tag{III.57}$$

where the functions u and R are defined in the following way.

- R is a "standard" deformation function in the sense of [57] as already described in the two-dimensional case, i.e. it satisfies $R(-a) = \overline{R(a)} = R(a)^{-1}$ and it has an analytic continuation into the upper half plane, continuous on its closure, to guarantee the right commutation relations of the deformed fields for space-like separation. The $Q \equiv Q(W)$ in the argument of R is a W dependent "deformation matrix", which is *anti-symmetric* w.r.t. the Lorentz inner product and its definition can be found e.g. in [41, 42]. Denoting by L_W a Lorentz transformation connecting W and W_0, i.e. $W = L_W W_0$ [5], the matrix $Q(W)$ is defined according to $Q(W) = L_W Q_0 L_W^{-1}$ where

$$Q_0 = \kappa \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \kappa > 0 \tag{III.58}$$

is the matrix belonging to the standard wedge W_0. This dependence on W of the matrix Q ensures the right covariance properties of the deformation functions and it follows in particular that $Q(W') = -Q(W)$.

- The functions $u^\lambda_{\tilde{W}}$ are intertwiners for the representation $U^{(\lambda)}$ of $\tilde{\mathcal{L}}^\uparrow_+$ which have to satisfy the relation

$$e^{-i\lambda\Omega(\tilde{\Lambda},p)} u^\lambda_{\tilde{W}}(\Lambda^{-1}p) = u^\lambda_{\Lambda\tilde{W}}(p). \tag{III.59}$$

To find such functions u satisfying this condition we first define the path for the standard wedge $\tilde{W}_0 = (W_0, \tilde{e}_0)$ with $W_0 = \{x \in \mathbb{R}^3 \mid x_1 > |x_0|\}$ and $\tilde{e}_0$ is a path starting at e_0 and staying inside W_0. We then consider equation (III.59) for $\tilde{W}_0$ and $\tilde{\Lambda} = \tilde{L}_{\tilde{W}}$, where $\tilde{L}_{\tilde{W}}$ is a Lorentz transformation connecting $\tilde{W}$ and $\tilde{W}_0$, i.e. $\tilde{L}_{\tilde{W}} \tilde{W}_0 = \tilde{W}$. We then get

$$u^\lambda_{\tilde{W}}(p) = e^{-i\lambda\Omega(\tilde{L}_{\tilde{W}},p)}\, u^\lambda_{\tilde{W}_0}\big(\tilde{L}^{-1}_{\tilde{W}} p\big), \tag{III.60}$$

which shows that the intertwiner for $\tilde{W}$ is determined by the intertwiner $u^\lambda_0 := u^\lambda_{\tilde{W}_0}$ for the standard wedge $\tilde{W}_0$. But this construction is not unique, because for every $\tilde{L}_{\tilde{W}}$ also $\tilde{L}_{\tilde{W}}\Lambda_1(t)$ is a $\tilde{\mathcal{L}}^\uparrow_+$ transformation mapping $\tilde{W}_0$ to $\tilde{W}$, because the boosts Λ_1 in 1-direction leave the standard wedge $\tilde{W}_0$ invariant. To restore uniqueness the functions u^λ_0 need to satisfy the consistency condition

$$u^\lambda_0(p) = e^{-i\lambda\Omega(\Lambda_1(t),p)} u^\lambda_0(\Lambda_1(-t)p). \tag{III.61}$$

To construct such functions we observe that, according to [68, Lemma C.1], the Wigner rotation factor $e^{-i\lambda\Omega(\Lambda_1(t),p)}$ can be written as

$$e^{-i\lambda\Omega(\Lambda_1(t),p)} = v(p)^\lambda v(\Lambda(-t)p)^{-\lambda} \ , \qquad v(p) := \frac{p_0 + m - p_1 + ip_2}{p_0 + m - p_1 - ip_2}. \tag{III.62}$$

This leads to the solution

$$u^\lambda_0(p) := f(p_2)^\lambda v(p)^\lambda \tag{III.63}$$

[5] Such a L_W always exists because the family of wedges has been defined as the orbit of W_0 under $\mathcal{L}^\uparrow_+$.

for u_0^λ where f is a yet undefined function of p_2 and thus invariant under x_1-boosts.[6] One can now easily check that our intertwiner function $u^\lambda_{\tilde{W}}(p) := e^{-i\lambda\Omega(\tilde{L}_{\tilde{W}},p)} u_0^\lambda(L^{-1}_{\tilde{W}}p)$ satisfies the relation (III.59).
Note that (apart from R which determines the momentum dependence of the S-matrix) the deformation function $A^{n,m}_{\tilde{W}}$ is fixed by covariance up to the function $f(p_2)$ which has to be chosen in such a way that the creation and annihilation operators satisfy the right commutation relations.

To calculate these relations we first need commutation relations between undeformed creation/annihilation operators and the deformation operators $T_{\tilde{W}}$. Acting on the charge q Hilbert space we get e.g.

$$\begin{aligned} T_{\tilde{W}}(p)a(p')\Big|_q &= \frac{u^\lambda_{\tilde{W}}(p)^{q-1}}{u^\lambda_{\tilde{W}}(p)^q}\Big(u^\lambda_{\tilde{W}}(p')R(Qp\cdot p')\Big)^{-1} a(p')T_{\tilde{W}}(p)\Big|_q \\ &= u^\lambda_{\tilde{W}}(p)^{-1}u^\lambda_{\tilde{W}}(p')^{-1}R(Qp\cdot p')^{-1}a(p')T_{\tilde{W}}(p) =: B_{\tilde{W}}(p,p')a(p')T_{\tilde{W}}(p), \\ T_{\tilde{W}}(p)a^*(p') &= B_{\tilde{W}}(p,p')^{-1}a^*(p')T_{\tilde{W}}(p). \end{aligned} \tag{III.64}$$

In the same way we can compute

$$T_{\tilde{W}}(p)b(p') = u^\lambda_{\tilde{W}}(p)\overline{u^\lambda_{\tilde{W}}(p')}^{-1}R(Qp\cdot p')^{-1}b(p')T_{\tilde{W}}(p) =: C_{\tilde{W}}(p,p')b(p')T_{\tilde{W}}(p). \tag{III.65}$$

All other relations now follow by charge conjugation and taking adjoints. Using these commutation relations we now obtain

$$\begin{aligned} a_{\tilde{W}}(p)a_{\tilde{W}'}(p') &= T_{\tilde{W}}(p)a(p)T_{\tilde{W}'}(p')a(p') \\ &= B_{\tilde{W}'}(p',p)^{-1}B_{\tilde{W}}(p,p')a_{\tilde{W}'}(p')a_{\tilde{W}}(p). \end{aligned} \tag{III.66}$$

Inserting the definition of B we get

$$B_{\tilde{W}'}(p',p)^{-1}B_{\tilde{W}}(p,p') = \frac{u^\lambda_{\tilde{W}'}(p')u^\lambda_{\tilde{W}'}(p)R(Qp'\cdot p)}{u^\lambda_{\tilde{W}}(p)u_{\tilde{W}}(p')R(Qp\cdot p')} = \frac{u^\lambda_{\tilde{W}'}(p)}{u^\lambda_{\tilde{W}}(p)}\frac{u^\lambda_{\tilde{W}'}(p')}{u^\lambda_{\tilde{W}}(p')}, \tag{III.67}$$

where the last equation follows by using the relation $R(-a) = R(a)^{-1}$. From the definition of the u's we see that

$$\frac{u^\lambda_{\tilde{W}'}(p)}{u^\lambda_{\tilde{W}}(p)} = e^{-i\lambda\left(\Omega(\tilde{L}_{\tilde{W}'},p)-\Omega(\tilde{L}_{\tilde{W}},p)\right)}u_0^\lambda\big(L^{-1}_{\tilde{W}'}p\big)u_0^\lambda\big(L^{-1}_{\tilde{W}}p\big)^{-1}, \tag{III.68}$$

and we are going to show that this term equals $e^{-i\pi\lambda k(\tilde{W},\tilde{W}')}$ with a $k\in\mathbb{Z}$, depending only on the wedges $\tilde{W},\tilde{W}'$ and not on the momenta p,p'.

[6] Note that every function on the mass shell which is invariant under boosts in the 1-direction is a function of p_2.

To calculate this expression we need to look more closely at the relation between $\tilde{L}_{\tilde{W}}$ and $\tilde{L}_{\tilde{W}'}$. First note that if $W = L_W W_0$ then $W' = L_W W_0' = L_W r(\pi) W_0$. Lifting this to paths of wedges we see that for every $\tilde{W}'$ there is an odd number $k \in 2\mathbb{Z}+1$ such that $\tilde{W}' = \tilde{L}_{\tilde{W}} \tilde{r}(k\pi) \tilde{W}_0$. This shows that every $\tilde{L}_{\tilde{W}'}$ is of the form $\tilde{L}_{\tilde{W}'} = \tilde{L}_{\tilde{W}} \tilde{r}(k\pi) \Lambda_1(t)$, where Λ_1 is a boost in x_1 direction. Using the fact that the intertwiners do not depend on the choice of this $\Lambda_1(t)$ we obtain

$$\frac{u^\lambda_{\tilde{W}'}(p)}{u^\lambda_{\tilde{W}}(p)} = e^{-i\lambda\left(\Omega(\tilde{L}_{\tilde{W}}\tilde{r}(k\pi),p) - \Omega(\tilde{L}_{\tilde{W}},p)\right)}\, u_0^\lambda\big(r(-k\pi)L^{-1}_{\tilde{W}}p\big)u_0^\lambda\big(L^{-1}_{\tilde{W}}p\big)^{-1}.$$

To calculate the exponential factor we need the cocycle relation (III.45) of the Wigner rotation, i.e.

$$\Omega(\tilde{\Lambda}\tilde{\Lambda}', p) = \Omega(\tilde{\Lambda}, p) + \Omega(\tilde{\Lambda}', \Lambda^{-1}p).$$

This leads to

$$\begin{aligned} \Omega\big(\tilde{L}_{\tilde{W}}\tilde{r}(k\pi), p\big) - \Omega(\tilde{L}_{\tilde{W}}, p) &= \Omega(\tilde{L}_{\tilde{W}}, p) + \Omega\big(\tilde{r}(k\pi), L^{-1}_{\tilde{W}}p\big) - \Omega(\tilde{L}_{\tilde{W}}, p) \\ &= \Omega\big(\tilde{r}(k\pi), L^{-1}_{\tilde{W}}p\big) = k\pi, \end{aligned} \tag{III.69}$$

where $k \equiv k(\tilde{W}, \tilde{W}')$ obviously only depends on the winding number of $\tilde{W}'$ w.r.t $\tilde{W}$.

Such a *winding number* $N(\tilde{C}_1, \tilde{C}_2)$ can be defined for general causally separated spacelike cones $\tilde{C}_1$ and $\tilde{C}_2$ (see e.g. [7]) in the following way[7]. Let $d\theta$ be the angle one-form in some fixed Lorentz frame, and for a path $\tilde{C} = (C, \tilde{e})$ let $\theta(\tilde{C})$ be the set of corresponding "accumulated angles", namely the interval

$$\theta(\tilde{C}) := \left\{ \int_e d\theta : e \in \tilde{e} \right\}. \tag{III.70}$$

Now given two paths $\tilde{C}_1, \tilde{C}_2$ with C_1 causally separated from C_2, one can define the relative winding number $N(\tilde{C}_1, \tilde{C}_2)$ of $\tilde{C}_2$ w.r.t. $\tilde{C}_1$ to be the unique integer n such that

$$\theta(\tilde{C}_2) + 2\pi n < \theta(\tilde{C}_1) < \theta(\tilde{C}_2) + 2\pi(n+1). \tag{III.71}$$

Considering two wedges $\tilde{W}$ and $\tilde{W}'$ one can now prove the following relation between $k(\tilde{W}, \tilde{W}')$ and $N(\tilde{W}, \tilde{W}')$.

Lemma III.1. *Let $k(\tilde{W}, \tilde{W}')$ and $N(\tilde{W}, \tilde{W}')$ be defined as before. Then the relation*

$$-k(\tilde{W}, \tilde{W}') = 2N(\tilde{W}, \tilde{W}') + 1 \tag{III.72}$$

holds.

[7] There are of course other equivalent definitions of a winding number for equivalence classes of paths, but we will not need them here.

Proof. As we have seen $\tilde{W}' = \tilde{L}_{\tilde{W}}\tilde{r}(k\pi)\tilde{W}_0$, with $\tilde{W} = \tilde{L}_{\tilde{W}}\tilde{W}_0$, which shows that $\theta(\tilde{W}') = \theta(\tilde{W}) + k\pi$. Using this in the definition of the winding number leads to

$$\theta(\tilde{W}') + 2\pi N < \theta(\tilde{W}) < \theta(\tilde{W}') + 2\pi(N+1)$$
$$\Rightarrow \theta(\tilde{W}) + k\pi + 2\pi N < \theta(\tilde{W}) < \theta(\tilde{W}) + k\pi + 2\pi(N+1).$$

From this it follows that

$$k + 2N < 0 < k + 2(N+1)$$

which immediately leads to $-k = 2N + 1$. □

Having established the relation between our $k(\tilde{W}, \tilde{W}')$ and the usual definition of the winding number $N(\tilde{W}, \tilde{W}')$ we return to the calculation of $\frac{u^\lambda_{\tilde{W}'}(p)}{u^\lambda_{\tilde{W}}(p)}$. To determine the second factor in (III.68) we need to compute the expression

$$\frac{u^\lambda_0\big(r(-k\pi)p\big)}{u^\lambda_0(p)} = \frac{u^\lambda_0\big(r(\pi)p\big)}{u^\lambda_0(p)},$$

where $r(\pi)$ acts on p simply as $r(\pi)(p_0, p_1, p_2) = (p_0, -p_1, -p_2)$. Inserting the definition of u^λ_0 and restricting the momentum p to the forward mass shell H^+_m leads via a straightforward calculation to

$$\frac{u^1_0(r(\pi)p)}{u^1_0(p)}\bigg|_{H^+_m} = \frac{f(-p_2)v(r(\pi)p)}{f(p_2)v(p)}\bigg|_{H^+_m} = \frac{f(-p_2)(m - ip_2)}{f(p_2)(m + ip_2)}.$$

In order to guarantee the right commutation relations between our annihilation operators we therefore have to choose a function $f : \mathbb{R} \to \mathbb{C}$ satisfying

$$\frac{f(-\zeta)(m - i\zeta)}{f(\zeta)(m + i\zeta)} = 1. \tag{III.73}$$

By studying the commutation relations between other creation and annihilation operators (e.g. between $a_{\tilde{W}}(p)$ and $b_{\tilde{W}'}(p')$) one realizes that f also has to satisfy $\bar{f} = f^{-1}$, i.e. $|f| = 1$. Using such an f in the definition of $u^\lambda_{\tilde{W}}$ one then finally arrives at the desired relation

$$\frac{u^\lambda_{\tilde{W}'}(p)}{u^\lambda_{\tilde{W}}(p)} = e^{-i\lambda\pi k(\tilde{W}, \tilde{W}')} \tag{III.74}$$

which leads to

$$a_{\tilde{W}}(p)a_{\tilde{W}'}(p') = e^{-2\pi i\lambda k(\tilde{W}, \tilde{W}')}a_{\tilde{W}'}(p')a_{\tilde{W}}(p). \tag{III.75}$$

In exactly the same way one can calculate

$$a_{\tilde{W}}(p)b_{\tilde{W}'}(p') = e^{2\pi i\lambda k(\tilde{W}, \tilde{W}')}b_{\tilde{W}'}(p')a_{\tilde{W}}(p) \tag{III.76}$$

and the mixed commutation relations

$$\begin{aligned} a_{\tilde{W}}(p)a^*_{\tilde{W}'}(p') &= e^{2\pi i\lambda k(\tilde{W}, \tilde{W}')}a^*_{\tilde{W}'}(p')a_{\tilde{W}}(p) + \omega_p\delta(p - p')T_{\tilde{W}}(p)T_{\tilde{W}'}(p)^*, \\ a_{\tilde{W}}(p)b^*_{\tilde{W}'}(p') &= e^{-2\pi i\lambda k(\tilde{W}, \tilde{W}')}b^*_{\tilde{W}'}(p')a_{\tilde{W}}(p). \end{aligned} \tag{III.77}$$

Again all the other commutation relations follow by charge conjugation and taking adjoints.

Up to now we have only taken into account covariance under the proper orthochronous group $\tilde{\mathcal{P}}_+^\uparrow$, but we also want our fields to have the correct transformation behavior under reflections at the edge of the wedge. Taking the standard wedge $\tilde{W}_0$ and the reflection $\tilde{j}$ at its edge we want the field to satisfy

$$J\Phi_{\tilde{W}_0}(f)J = \Phi_{\tilde{j}\tilde{W}_0}(\alpha_j(f)), \tag{III.78}$$

where again $\alpha_j(f)(x) = \overline{f(jx)}$ and J has been defined in (III.10). According to (III.50) the reflection $\tilde{j}$ acts on $\tilde{W}_0$ as

$$\tilde{j}\cdot\tilde{W}_0 = (-W_0, \tilde{j}\cdot\tilde{e}_0) = \tilde{r}(-\pi)\tilde{W}_0.$$

A straightforward calculation then shows that for equation (III.78) to hold we need to set $\beta = -2\pi\lambda$ in the definition (III.10) and the intertwiners u_0^λ have to satisfy $\overline{u_0^\lambda(-jp)} = u_0^\lambda(p)$. Because of $\tilde{j}\cdot\tilde{W}_0 = \tilde{r}(-\pi)\cdot\tilde{W}_0$ this leads to

$$\overline{u_0^\lambda(-jp)} = e^{-i\pi\lambda}u_{\tilde{j}\tilde{W}_0}(p).$$

This equation can be fulfilled if and only if the function f, used in the definition of u_0^λ, satisfies $f(-\zeta) = \overline{f(\zeta)}$ in addition to the previous relation (III.73). These two conditions now lead to the solution

$$f(\zeta) = \pm\frac{m - i\zeta}{\sqrt{m^2+\zeta^2}}, \tag{III.79}$$

which is unique up to a sign. With this choice the deformed field $\Phi_{\tilde{W}_0}$ then satisfies (III.78) with the space-time reflection J defined according to

$$(J\Psi)_n^m(\underline{p}) = e^{-2\pi i\lambda q}\,\overline{\Psi_n^m(-j\underline{p})}. \tag{III.80}$$

Summing up our construction we have seen that we can deform the CCR-algebra of "free" creation and annihilation operators in such a way that the deformed operators satisfy anyonic commutation relations and the resulting field is covariant under a spin λ representation of $\tilde{\mathcal{P}}_+$. This deformation was defined by simply using multiplication operators $(T_{\tilde{W}}(p)\Psi)_n^m(\underline{p}) = A_{\tilde{W}}^{n,m}(p;\underline{p})\Psi_n^m(\underline{p})$ which were chosen according to

$$A_{\tilde{W}}^{n,m}(p;\underline{p}) = u_{\tilde{W}}^\lambda(p)^{q+1}\prod_{i=1}^{n}u_{\tilde{W}}^\lambda(p_i)R(Qp\cdot p_i)\prod_{j=n+1}^{n+m}\overline{u_{\tilde{W}}^\lambda(p_j)}R(Qp\cdot p_j),$$

R ... arbitrary deformation function in the sense of [57],

$$u_{\tilde{W}}^\lambda(p) = e^{-i\lambda\Omega(\tilde{L}_{\tilde{W}},p)}u_0^\lambda(\tilde{L}_{\tilde{W}}^{-1}p),$$

$$u_0^\lambda(p) = \left(\frac{m-ip_2}{\sqrt{m^2+p_2^2}}\right)^\lambda\cdot v(p)^\lambda \quad , \qquad v(p) := \frac{p_0+m-p_1+ip_2}{p_0+m-p_1-ip_2}.$$

Having defined the deformation we can now state our main result, namely that the deformed field satisfies anyonic commutation relations.

Proposition III.1. *Consider paths of wedges $\tilde{W}$, $\tilde{W}'$ and test functions f, g such that*

$$\mathrm{supp}(f) + W \subset (\mathrm{supp}(g) + W')'. \tag{III.81}$$

Then the deformed fields $\Phi_{\tilde{W}}(f)$ and $\Phi_{\tilde{W}'}(g)$ satisfy the commutation relations

$$\begin{aligned} \Phi_{\tilde{W}}(f)\Phi_{\tilde{W}'}(g) &= e^{-2\pi i\lambda k(\tilde{W},\tilde{W}')}\Phi_{\tilde{W}'}(g)\Phi_{\tilde{W}}(f), \\ \Phi_{\tilde{W}}(f)\Phi^*_{\tilde{W}'}(g) &= e^{2\pi i\lambda k(\tilde{W},\tilde{W}')}\Phi^*_{\tilde{W}'}(g)\Phi_{\tilde{W}}(f), \end{aligned} \tag{III.82}$$

Proof. Using the above relations between the deformed creation and annihilation operators one immediately sees that the deformed field $\Phi_{\tilde{W}}(f) = a^*_{\tilde{W}}(f^+) + b_{\tilde{W}}(\bar{f}^+)$ with itself satisfies the commutation relation

$$\Phi_{\tilde{W}}(f)\Phi_{\tilde{W}'}(g) = e^{-2\pi i\lambda k(\tilde{W},\tilde{W}')}\Phi_{\tilde{W}'}(g)\Phi_{\tilde{W}}(f), \tag{III.83}$$

for arbitrary test functions f and g.

For the mixed commutation relations between $\Phi_{\tilde{W}}$ and $\Phi^*_{\tilde{W}'}$ we get

$$\begin{aligned} &\Phi_{\tilde{W}}(f)\Phi^*_{\tilde{W}'}(g) - e^{2\pi i\lambda k(\tilde{W},\tilde{W}')}\Phi^*_{\tilde{W}'}(g)\Phi_{\tilde{W}}(f) = \\ &\int d\mu(p)\left(f^-(p)g^+(p)\; T^c_{\tilde{W}}(p)T^c_{\tilde{W}'}(p)^* - f^+(p)g^-(p)\; e^{2\pi i\lambda k(\tilde{W},\tilde{W}')}T_{\tilde{W}'}(p)T_{\tilde{W}}(p)^*\right). \end{aligned} \tag{III.84}$$

To determine whether the right hand side vanishes for spacelike separated test functions we have to calculate how the operators $T^c_{\tilde{W}}(p)T^c_{\tilde{W}'}(p)^*$ and $T_{\tilde{W}'}(p)T_{\tilde{W}}(p)^*$ act on an arbitrary vector Ψ^m_n. This leads to

$$\begin{aligned} (T^c_{\tilde{W}}(p)T^c_{\tilde{W}'}(p)^*\Psi)^m_n(\underline{p}) &= u^\lambda_{\tilde{W}}(p)^{-q+1}\overline{u^\lambda_{\tilde{W}'}(p)^{-q+1}}\prod_{i=1}^{n}\overline{u^\lambda_{\tilde{W}}(p_i)}u^\lambda_{\tilde{W}'}(p_i)R(Qp\cdot p_i)^2 \\ &\quad\cdot\prod_{j=n+1}^{n+m}u^\lambda_{\tilde{W}}(p_j)\overline{u^\lambda_{\tilde{W}'}(p_j)}R(Qp\cdot p_j)^2, \\ (T_{\tilde{W}'}(p)T_{\tilde{W}}(p)^*\Psi)^m_n(\underline{p}) &= \overline{u^\lambda_{\tilde{W}}(p)^{q+1}}\; u^\lambda_{\tilde{W}'}(p)^{q+1}\prod_{i=1}^{n}\overline{u^\lambda_{\tilde{W}}(p_i)}u^\lambda_{\tilde{W}'}(p_i)\overline{R(Qp\cdot p_i)}^2 \\ &\quad\cdot\prod_{j=n+1}^{n+m}u^\lambda_{\tilde{W}}(p_j)\overline{u^\lambda_{\tilde{W}'}(p_j)}\,\overline{R(Qp\cdot p_j)}^2. \end{aligned}$$

One can see that the factors containing $u^\lambda(p_i)$ and $u^\lambda(p_j)$ are the same in both equations, so they are not causing any trouble. The deformation function R has been chosen in such a way that it has the right analytic properties, as in the two-dimensional case (cf. also [41]).

The only nontrivial factors left which could still cause problems are $u^\lambda_{\tilde{W}}(p)^{-q+1}\overline{u^\lambda_{\tilde{W}'}(p)^{-q+1}}$ and $\overline{u^\lambda_{\tilde{W}}(p)^{q+1}}\, u^\lambda_{\tilde{W}'}(p)^{q+1}$. But a straightforward calculation using the definition of $u^\lambda_{\tilde{W}}$ leads to

$$u^\lambda_{\tilde{W}}(p)^{-q+1}\overline{u^\lambda_{\tilde{W}'}(p)^{-q+1}} = e^{-i\pi\lambda k(\tilde{W},\tilde{W}')(-q+1)} = e^{2\pi i\lambda k(\tilde{W},\tilde{W}')}\overline{u^\lambda_{\tilde{W}}(p)^{q+1}}\, u^\lambda_{\tilde{W}'}(p)^{q+1}.$$

Inserting this into the commutation relation (III.84) one arrives at

$$\begin{aligned}
&\Big(\Phi_{\tilde{W}}(f)\Phi^*_{\tilde{W}'}(g)\Psi\Big)^m_n(\underline{p}) - \Big(e^{2\pi i\lambda k(\tilde{W},\tilde{W}')}\Phi^*_{\tilde{W}'}(g)\Phi_{\tilde{W}}(f)\Psi\Big)^m_n(\underline{p}) = \\
&\quad\int d\mu(p)\Big[f^-(p)g^+(p)\prod_{i=1}^{n+m} R(Qp\cdot p_i)^2 - f^+(p)g^-(p)\prod_{i=1}^{n+m}\overline{R(Qp\cdot p_i)}^2\Big] \\
&\quad\cdot e^{-i\pi\lambda k(\tilde{W},\tilde{W}')}\prod_{i=1}^{n}\overline{u^\lambda_{\tilde{W}}(p_i)}u^\lambda_{\tilde{W}'}(p_i)\prod_{j=n+1}^{n+m} u^\lambda_{\tilde{W}}(p_i)\overline{u^\lambda_{\tilde{W}'}(p_i)}\;\;\Psi^m_n(\underline{p}).
\end{aligned} \tag{III.85}$$

We now just have to show that the expression in the second line vanishes for all p_i if the test functions f and g have the right support properties (III.81). Due to the covariance of the field operators we have to consider this expression only for the standard wedge W_0, i.e. $Q = Q_0$ and f, g localized in W_0, W_0' respectively. Using ideas from the proof of Proposition 3.4 in [41] one introduces new coordinates on the mass shell such that

$$p = p(\theta, p_2) = \begin{pmatrix} m_\perp\cosh\theta \\ m_\perp\sinh\theta \\ p_2\end{pmatrix}, \qquad \int d\mu(p)\varphi(p) = \int \frac{1}{2}d\theta dp_2\;\varphi(p(\theta,p_2)), \tag{III.86}$$

where $m_\perp = \sqrt{m^2+p_2^2}$. Because f and g are assumed to have compact support their Fourier transforms $\tilde{f}$ and $\tilde{g}$ are entire analytic functions. Moreover $f^-(\theta+i\lambda, p_2)$ and $g^+(\theta+i\lambda,p_2)$ are bounded on the strip $0\le\lambda\le\pi$ (see again [41]) and the boundary values are related by

$$\begin{aligned} f^-(\theta+i\pi,p_2) &= f^+(\theta,-p_2),\\ g^+(\theta+i\pi,p_2) &= g^-(\theta,-p_2),\end{aligned}$$

where we used the obvious notation $f^\pm(\theta,p_2) = f^\pm(p(\theta,p_2))$. Furthermore, because κ in the definition of Q (III.58) satisfies $\kappa\ge 0$ and all the momenta are on the mass shell, it follows that

$$\mathrm{Im}(Qp(\theta+i\lambda)\cdot p_k) = \kappa m_\perp \sin\lambda\begin{pmatrix}\cosh\theta\\ \sinh\theta\end{pmatrix}\cdot\begin{pmatrix}p_k^0\\ p_k^1\end{pmatrix}\ge 0\ , \quad \text{for } 0\le\lambda\le\pi.$$

Therefore the functions $z\mapsto R(Qp(z)\cdot p_k))$ are analytic on the strip $S(0,\pi)$ and bounded on its closure. This allows us to shift the θ integration in the second line of (III.85) from $\mathbb{R}$ to $\mathbb{R}+i\pi$, which shows that the whole expression vanishes (See also [2] for a slightly more detailed treatment of these concepts).

We have thus shown that the deformed field satisfies the anyonic commutation relations

$$\Phi_{\tilde{W}}(f)\Phi_{\tilde{W}'}(g) = e^{-2\pi i\lambda k(\tilde{W},\tilde{W}')}\Phi_{\tilde{W}'}(g)\Phi_{\tilde{W}}(f),$$
$$\Phi_{\tilde{W}}(f)\Phi^*_{\tilde{W}'}(g) = e^{2\pi i\lambda k(\tilde{W},\tilde{W}')}\Phi^*_{\tilde{W}'}(g)\Phi_{\tilde{W}}(f),$$

if the localization regions $(\mathrm{supp}(f)+W)''$ and $(\mathrm{supp}(g)+W')''$ are spacelike separated. □

Summing up the results we have constructed field operators $\Phi_{\tilde{W}}(f)$ on the Hilbert space $\mathcal{F}_s(L^2(\mathbb{R}^3, d\mu)) \otimes \mathcal{F}_s(L^2(\mathbb{R}^3, d\mu))$ for every path of wedge $\tilde{W}$ and test function $f \in \mathscr{S}(\mathbb{R}^3)$. This family satisfies the following properties:

i) The fields are *polarization-free generators*, i.e.

$$\Phi_{\tilde{W}}(f)\Omega \in \mathcal{H}_1.$$

ii) *Covariance* under the representation (III.46) of $\tilde{\mathcal{P}}_+^\uparrow$ holds, i.e.

$$U(a,\tilde{\Lambda})\Phi_{\tilde{W}}(f)U(a,\tilde{\Lambda})^{-1} = \Phi_{\tilde{\Lambda}\tilde{W}}(\alpha_{(a,\Lambda)}(f)).$$

iii) Under the representation J of the reflection at the x_2-axis the field transforms according to

$$J\Phi_{\tilde{W}_0}(f)J = \Phi_{\tilde{j}\tilde{W}_0}(\alpha_j(f)).$$

iv) The fields are *localized in wedge regions* and satisfy *anyonic commutation relations*, depending on the relative winding number of $\tilde{W}$ and $\tilde{W}'$, i.e.

$$\Phi_{\tilde{W}}(f)\Phi^\sharp_{\tilde{W}'}(g) = e^{\mp 2\pi i\lambda k(\tilde{W},\tilde{W}')}\Phi^\sharp_{\tilde{W}'}(g)\Phi_{\tilde{W}}(f),$$

if $\mathrm{supp}(f) + W \subset (\mathrm{supp}(g) + W')'$.

Furthermore the *Reeh-Schlieder* property holds for wedges (because it holds for double cones in the free undeformed case), but what's more important is that it does *not* hold for regions smaller than a wedge, e.g. for spacelike cones. This follows from the recent work [7] by Bros and Mund, where they show (using results from [6]) that there can be no polarization-free generators for anyons if the Reeh-Schlieder property holds for spacelike cones.

Scattering States

We can now also define two-particle scattering states and their S-matrix by following the approach for wedge-localized operators in [6]. For this purpose we choose f and g in such a way that $\tilde{f}, \tilde{g}$ have non-overlapping compact support and $\mathrm{supp}\,\tilde{f}, \mathrm{supp}\,\tilde{g}$ intersect the upper but not the lower mass shell. Furthermore, we use the notation $p = (p^0, \mathbf{p})$, $\omega_p = \sqrt{\mathbf{p}^2 + m^2}$ and define the *velocity support* $\Gamma(f) \subset \mathbb{R}^3$ of a test function and its *time evolution* f_t according to

$$\begin{aligned} \Gamma(f) &:= \{(1, \mathbf{p}/\omega_p) : p \in \operatorname{supp} \tilde{f}\}, \\ f_t(x) &= \int dp \tilde{f}(p) e^{i(p^0-\omega_p)t} e^{-ip\cdot x}. \end{aligned} \tag{III.87}$$

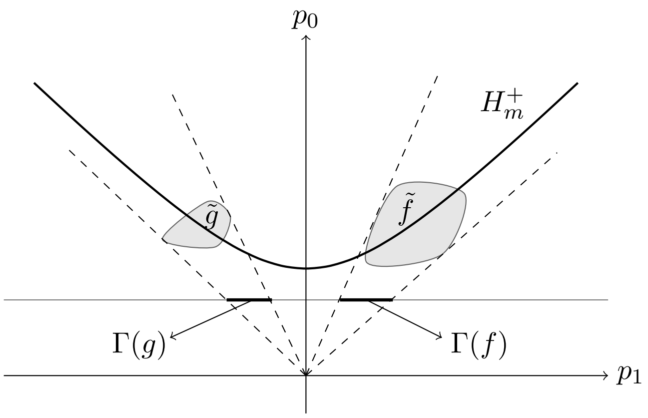


Figure III.1.: *Velocity supports of the test functions f and g*

It is well known in scattering theory that for large times t the fields $\Phi_{\tilde{W}}(f_t)$ and $\Phi_{\tilde{W}'}(g_t)$ are essentially localized in $W + t\Gamma(f)$ and $W' + t\Gamma(g)$ respectively, because asymptotically the support of f_t is contained in $t\Gamma(f)$. Now if the velocity supports of f, g are such that $\Gamma(f) - \Gamma(g) \subset W$ these localization regions are spacelike separated for positive times t. Therefore we can define outgoing two-particle scattering states as the limit

$$\lim_{t\to\infty} \Phi_{\tilde{W}}(f_t)\Phi_{\tilde{W}'}(g_t) =: (f^+, \tilde{W})_+ \times^{out} (g^+, \tilde{W}')_+ \tag{III.88}$$

and by using also $\Phi^*_{\tilde{W}}(f)$ we could similarly construct scattering states containing anti-particles.

For the incoming scattering states we have to exchange $\tilde{W}$ and $\tilde{W}'$ because for $t < 0$ the localization regions $W + t\Gamma(g)$ and $W' + t\Gamma(f)$ are spacelike separated. This leads to the definition

$$\lim_{t\to-\infty} \Phi_{\tilde{W}'}(f_t)\Phi_{\tilde{W}}(g_t) =: (f^+, \tilde{W}')_+ \times^{in} (g^+, \tilde{W})_+. \tag{III.89}$$

To compute these limits we use that the supports of f, g do not intersect the lower mass shell and that the time dependence of f_t is trivial on the upper mass shell. Thus the scattering states simplify to

$$\begin{aligned} (f^+, \tilde{W})_+ \times^{out} (g^+, \tilde{W}')_+ &= a^*_{\tilde{W}}(f^+) a^*_{\tilde{W}'}(g^+)\Omega, \\ (f^+, \tilde{W}')_+ \times^{in} (g^+, \tilde{W})_+ &= a^*_{\tilde{W}'}(f^+) a^*_{\tilde{W}}(g^+)\Omega. \end{aligned} \tag{III.90}$$

Of course these vectors inherit the nontrivial commutation relations from the fields which create them, leading e.g. to

$$(f^+, \tilde{W})_+ \times^{out} (g^+, \tilde{W}')_+ = e^{-2\pi i \lambda k(\tilde{W}, \tilde{W}')} \, (g^+, \tilde{W}')_+ \times^{out} (f^+, \tilde{W})_+. \tag{III.91}$$

From (III.90) the explicit form of the scattering states can be computed, namely

$$\begin{aligned} \Big((f^+, \tilde{W})_+ \times^{out} (g^+, \tilde{W}')_+\Big)(p_1, p_2) &= \frac{1}{\sqrt{2}}\Big(\mathfrak{R}(p_1, p_2) f^+(p_1) g^+(p_2) \; + \; (p_1 \leftrightarrow p_2)\Big), \\ \Big((f^+, \tilde{W}')_+ \times^{in} (g^+, \tilde{W})_+\Big)(p_1, p_2) &= \frac{1}{\sqrt{2}}\Big(\mathfrak{R}'(p_1, p_2) f^+(p_1) g^+(p_2) \; + \; (p_1 \leftrightarrow p_2)\Big), \end{aligned} \tag{III.92}$$

where $\mathfrak{R}$ and $\mathfrak{R}'$ are defined according to

$$\begin{aligned} \mathfrak{R}(p_1, p_2) &:= \Big(\overline{u^\lambda_{\tilde{W}}(p_1)}\Big)^2 \, \overline{u^\lambda_{\tilde{W}}(p_2)} \, \overline{u^\lambda_{\tilde{W}'}(p_2)} \, R(Qp_1 \cdot p_2), \\ \mathfrak{R}'(p_1, p_2) &:= \Big(\overline{u^\lambda_{\tilde{W}'}(p_1)}\Big)^2 \, \overline{u^\lambda_{\tilde{W}'}(p_2)} \, \overline{u^\lambda_{\tilde{W}}(p_2)} \, \overline{R(Qp_1 \cdot p_2)}. \end{aligned}$$

Now by taking test functions $f, g, h, k \in \mathscr{S}(\mathbb{R}^3)$ such that $\Gamma(f) - \Gamma(g) \subset W$ and $\Gamma(h) - \Gamma(k) \subset W$ we can calculate the two-particle S-matrix leading to

$$\begin{aligned} &\langle (f^+, \tilde{W})_+ \times^{out} (g^+, \tilde{W}')_+, (h^+, \tilde{W}')_+ \times^{in} (k^+, \tilde{W})_+ \rangle \\ &= \int d\mu(p_1) d\mu(p_2) \left(e^{2\pi i \lambda k(\tilde{W}, \tilde{W}')} R(p_1 Q p_2)^2\right) \overline{f^+(p_1) g^+(p_2)} h^+(p_1) k^+(p_2). \end{aligned} \tag{III.93}$$

Consequently we can see that the momentum dependence of the S-matrix is again determined by the square of the deformation function R^2 which in addition gets multiplied by a phase factor depending only on the relative winding number of the localization regions. Note that we would get the same result for anti-particle scattering because we used the same deformation function on the particle- and the anti-particle space. Of course we could also have written R^+ and R^- as in the two-dimensional case, but it is presently unclear to what extend these two functions could differ in $d > 2$.

III.3.3. $R = 1$: Localization in "Double Strings"

Instead of trying to generalize the construction and the respective deformation functions further, we can consider in dimension $d = 2 + 1$ the special case of a trivial deformation function $R = 1$ and thus only focus on the anyon-like properties of the resulting fields. In this case the multiplication operator used in the deformation takes the simple form

$$(T_{\tilde{W}}(p)\Psi)_n^m(\underline{p}) = u_{\tilde{W}}^\lambda(p)^{q+1} \prod_{i=1}^{n} u_{\tilde{W}}^\lambda(p_i) \prod_{j=n+1}^{n+m} \overline{u_{\tilde{W}}^\lambda(p_j)}\ \Psi_n^m(\underline{p}). \qquad \text{(III.94)}$$

Because the intertwiner $u_{\tilde{W}}^\lambda$ still appears in the definition of the field, it is still covariant under the unitary representation U of the cover of the Lorentz group $\mathcal{L}_+^\uparrow$ for spin $\lambda \in \mathbb{R}$. What has changed are the localization properties of the field. To see this remember that the wedge-locality of the deformed fields was essentially based on two ingredients: The analyticity properties of the deformation function R (which plays no role here because we consider $R = 1$) and the fact that the intertwiners were chosen in such a way that they satisfy

$$\frac{u_{\tilde{W}'}^\lambda(p)}{u_{\tilde{W}}^\lambda(p)} = e^{i\lambda\pi k(\tilde{W},\tilde{W}')},$$

where $N(\tilde{W},\tilde{W}') = -\frac{1}{2}(k(\tilde{W},\tilde{W}')+1)$ is the relative winding number of $\tilde{W}$ with respect to $\tilde{W}'$. Now such a relation also holds if we take, instead of $\tilde{W}'$, a wedge which differs from $\tilde{W}$ by an *even* number of 2π rotations. This is obvious because of the relation

$$u_{\tilde{r}(2\pi n)\tilde{W}}^\lambda(p) = e^{2\pi i\lambda n}\, u_{\tilde{W}}^\lambda(p).$$

We therefore see that the intertwiner actually satisfies

$$\frac{u_{\tilde{W}_2}^\lambda(p)}{u_{\tilde{W}_1}^\lambda(p)} = e^{i\lambda\pi k}\ , \quad \text{iff } \tilde{W}_2 = \tilde{r}(k\pi)\tilde{W}_1\ ,\ \forall\, k \in \mathbb{Z}. \qquad \text{(III.95)}$$

But this means that also the deformed fields satisfy

$$\Phi_{\tilde{W}_1}(f)\Phi_{\tilde{W}_2}^\sharp(g) = e^{-2\pi i\lambda k(\tilde{W}_1,\tilde{W}_2)}\ \Phi_{\tilde{W}_2}^\sharp(g)\Phi_{\tilde{W}_1}(f) \qquad \text{(III.96)}$$

not only if $\tilde{W}_2$ is spacelike separated from $\tilde{W}_1$ but also if the two wedges are "parallel", as long as the test functions f and g still have spacelike separated support.

These considerations show that the minimal localization region one can assign to the operator $\Phi_{\tilde{W}}(f)$ is a tube-like region which results from smearing the edge ∂W of W with the function f (this can also be seen as a "smeared double string", extending to infinity in two opposite directions). More precisely, because a path of wedges $\tilde{W} = (W, \tilde{e})$ also contains the information of a winding number, the fields are localized in "*generalized tubes*" (or "paths of tubes") $\widetilde{\mathcal{T}}(f,\tilde{W})$ which are specified by a test function f and a generalized wedge $\tilde{W}$ (or equivalently its edge). The following figure shows a sketch of the tube region resulting from smearing the edge of $\tilde{W}$ with the test function f.

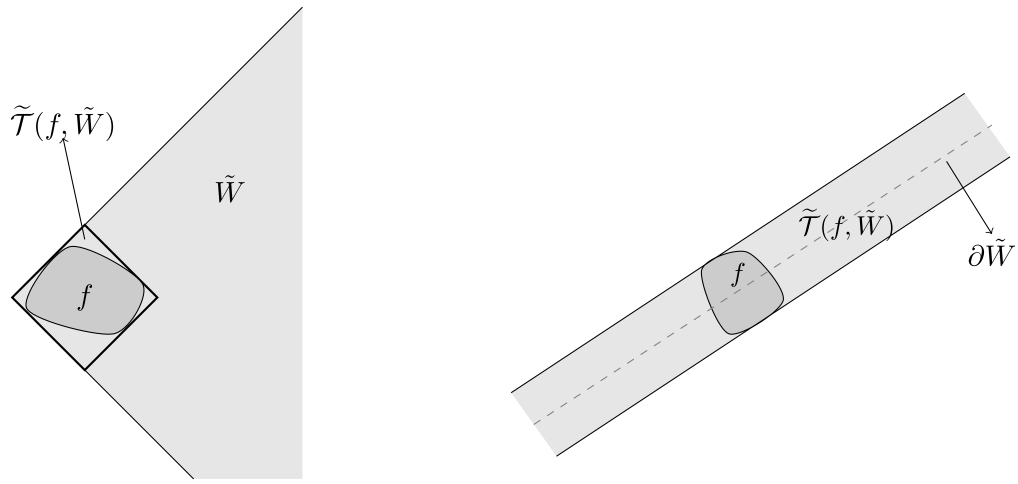


Figure III.2.: *Projections onto the x_0-x_1 and x_1-x_2 plane of the tube $\widetilde{\mathcal{T}}(f, \tilde{W})$ around the edge $\partial\tilde{W}$ of the wedge $\tilde{W}$*

So for the special case, where the deformation consists in only using the intertwiners $u^{\lambda}_{\tilde{W}}$ for the definition of the multiplication operator $T_{\tilde{W}}$, we obtain a field with the following properties.

i) *Covariance* under the unitary representation U of $\tilde{\mathcal{P}}^{\uparrow}_{+}$ for arbitrary real spin $\lambda \in \mathbb{R}$,

$$U(a, \tilde{\Lambda})\Phi_{\tilde{W}}(f)U(a, \tilde{\Lambda})^{-1} = \Phi_{\tilde{\Lambda}\tilde{W}}(\alpha_{(a,\Lambda)}(f))$$

ii) The fields are localized in *tube-shaped regions* and for spacelike separated tubes $\widetilde{\mathcal{T}}(f_1, \tilde{W}_1)$ and $\widetilde{\mathcal{T}}(f_2, \tilde{W}_2)$ the fields satisfy *anyonic commutation relations*

$$\Phi_{\tilde{W}_1}(f_1)\Phi^{\sharp}_{\tilde{W}_2}(f_2) = e^{\mp 2\pi i \lambda k(\tilde{W}_1, \tilde{W}_2)}\Phi^{\sharp}_{\tilde{W}_2}(f_2)\Phi_{\tilde{W}_1}(f_1).$$

iii) The fields are “free” in the sense that they create one-particle vectors from the vacuum. Moreover they lead to a *momentum independent* scattering matrix, which depends only on the relative winding number of localization regions.

Furthermore they of course still satisfy the Reeh-Schlieder property, now even for tube regions.

Remarks: i) Although this “double-string”-localization is an improvement in the size of localization regions, the set of possible tubes which are causally separated to a given tube is still very much restricted because only tubes that are parallel can be spacelike separated.

ii) Because the number $k(\tilde{W}_1, \tilde{W}_2) \in \mathbb{Z}$ appearing in the commutation relations can now be any integer, there are also fields which commute, namely those with vanishing relative winding number. Only after rotating one of the fields around a multiple of π they show their anyonic behavior.

III.4. Conclusion

The method of deforming a free hermitian scalar field to obtain new wedge-localized models with non-trivial S-matrix, described in detail in [57], has been generalized to a *charged* field on two- and three-dimensional Minkowski space. We have seen that working on a charged Hilbert space enables us to change the statistics of the deformed fields and (at least in two-dimensions) one can use a larger class of deformation functions in the definition of the deformation. It is believed that they exhaust the class of possible deformation functions in $d = 1 + 1$ but a proof could not yet be given. In $d = 2 + 1$ the fields are localized in so-called paths of wedges, have non-trivial commutation relations depending on the winding number of their localization regions and they are covariant with respect to a representation of $\tilde{\mathcal{L}}_+^\uparrow$ with spin $\lambda \in \mathbb{R}$.

The weakened localization in wedges instead of double cones or spacelike cones still allows to define two-particle scattering states and the wedge-algebras generate dense sets in $\mathcal{H}$ from the vacuum. In the case where the deformation function is just a constant it is possible to interpret the fields as being localized in generalized tubes, extending to infinity in two opposite directions. However, nothing is known about algebras for smaller spacetime regions like spacelike cones, except that they cannot satisfy the Reeh-Schlieder property due to the recent work by Bros and Mund [7].

Another open question is if the deformation functions defined in (III.32) are actually the most general which are possible in this case. We currently also do not know if these additional admissible functions can be generalized to the higher dimensional case.

In [57] the deformation was originally defined on the test function algebra (the so-called Borchers-Uhlmann algebra) and a representation on Fock space was then introduced via the GNS-construction. In the present work we defined the deformation on Fock space from the outset and it would be interesting to know if one can understand it as the GNS representation of a deformation of the underlying Borchers-Uhlmann algebra.

In addition one would like to generalize the deformation of a charged scalar field to a situation with a multi-component field where different particle species and charges are present, similar to the recent construction by Lechner and Schuetzenhofer [58]. In this case the commutation relations might be governed not only by phase factors, but by more general matrices yielding a non-abelian representation of the braid group.

IV. Compact Localization in 1+1 Dimensions

In the previous chapter we have shown that it is possible to use the approach of multiplicative deformations of free fields to construct wedge-local polarization free generators which have anyon-like features. But quantum fields describing anyons in the sense of AQFT should have better localization properties, namely they should be localizable in compact regions in $d = 1 + 1$ and spacelike cones in $d = 2 + 1$. However, the no-go theorems mentioned in the introduction tell us that such fields cannot be polarization free generators but rather have to be fully interacting.

A relatively simple method to construct localized operators on a Fock space, which generate vectors with arbitrary high particle number from the vacuum, is the approach of *implementable Bogoliubov transformations.* This allows us to first construct certain multiplication operators on a suitable one-particle position space and then implement them on the physical Fock space. Finding such one-particle operators leading to anyon-like implementers on the Fock space will be a significant simplification over the direct construction of the right operators on the full physical Hilbert space. Since this procedure is basically restricted to one spatial dimension we will focus on the 1+1 dimensional case in this chapter. A treatment of fermion current algebras and Schwinger terms in 3+1 dimensions can be found e.g. in [50]. First we will briefly outline the theory of quasifree second quantization [63] and implementable Bogoliubov transformations (using mainly the notation from [49]) and then apply it to massive and massless theories to construct nets of operator algebras showing anyonic features. It has originally been developed (and is most of the time used) for fermions, i.e. on the anti-symmetric Fock space, but it can also be done on an equal footing for bosons on the symmetric Fock space[1]. We will describe both formulations here and use them in the massive case to construct operators on the fermionic and bosonic Fock spaces. However, we will restrict ourselves to the fermionic case for $m = 0$ and in later constructions regarding anyons on the circle and winding numbers.

Explicit expressions and various proofs concerning Bogoliubov transformations in general can be found e.g. in [18, 40, 49, 78, 80]. Multiplication operators on the position space in the fermionic case are treated in more detail in [17, 20–22] and the bosonic case is also mentioned in [79, 81]. Extensive treatments of loop groups are found in [76, 86]. Imple-

[1] A supersymmetric formulation of the concepts is also possible [40].

mentable one-particle operators on the massive fermionic Fock space have also been used in [1, 82] to construct DHR endomorphisms with anyonic statistics. We will extend this to the explicit construction of field algebras for anyons and we will furthermore also show how the massless case can be treated.

IV.1. Mathematical Preliminaries: Implementable Bogoliubov Transformations

In the following constructions we will always consider an (auxiliary) one-particle Hilbert space which is a direct sum of two (particle and anti-particle) Hilbert spaces

$$\mathcal{H}_1 = \mathcal{H}_1^+ \oplus \mathcal{H}_1^-, \tag{IV.1}$$

where the subspaces $\mathcal{H}_1^+ \simeq \mathcal{H}_1^-$ will be some L^2-space, specified below in more detail. The corresponding projections onto $\mathcal{H}_1^+$ and $\mathcal{H}_1^-$ are denoted by P_+ and P_-. In the concrete cases below they will be the projections onto the positive and negative part of the spectrum of a Dirac operator (or Klein-Gordon operator in the boson case). The full physical Hilbert space will then be the fermionic (anti-symmetric) or bosonic (symmetric) Fock space over this one-particle Hilbert space,

$$\mathcal{H} = \mathcal{F}_{F/B}(\mathcal{H}_1) \simeq \mathcal{F}_{F/B}(\mathcal{H}_1^+) \otimes \mathcal{F}_{F/B}(\mathcal{H}_1^-), \tag{IV.2}$$

where the subscripts F and B denote the fermionic or bosonic case respectively. As is well known this Fock space is unitarily equivalent to a (unsymmetrized) tensor product of the particle- and anti-particle Hilbert spaces. Due to this tensor product structure we can naturally define the charge operator Q as

$$Q = N - M, \tag{IV.3}$$

where N is the particle number operator on the first tensor factor and M on the second one. The projections onto vectors with (anti-)particle number less or equal to n will be denoted by P_n. For every $\varphi = (\varphi^+, \varphi^-) \in \mathcal{H}_1$ we have creation and annihilation operators for particles $a^*(\varphi^+), a(\varphi^+)$ and anti-particles $b^*(\varphi^-), b(\varphi^-)$, satisfying canonical (anti-)commutation relations,

$$\begin{aligned} [a(\varphi_1^+), a^*(\varphi_2^+)]_{F/B} &= \langle \varphi_1^+, \varphi_2^+ \rangle \ , \qquad [a^\sharp(\varphi_1^+), a^\sharp(\varphi_2^+)]_{F/B} = 0, \\ [b(\varphi_1^-), b^*(\varphi_2^-)]_{F/B} &= \langle \varphi_1^-, \varphi_2^- \rangle \ , \qquad [b^\sharp(\varphi_1^-), b^\sharp(\varphi_2^-)]_{F/B} = 0, \end{aligned} \tag{IV.4}$$

where $[\cdot,\cdot]_F$ denotes the anti-commutator and $[\cdot,\cdot]_B$ the commutator. In addition they satisfy

$$a(\varphi^+)\Omega = b(\varphi^-)\Omega = 0 \ , \quad a(\varphi^+)^* = a^*(\varphi^+) \ , \quad b(\varphi^-)^* = b^*(\varphi^-), \tag{IV.5}$$

where Ω again denotes the unique vacuum vector which will be invariant under Poincaré transformations. Due to the estimates (which hold trivially for fermions)

$$||a^\sharp(\varphi^+)P_n|| \leq (n+1)||\varphi^+|| \ , \qquad ||b^\sharp(\varphi^-)P_m|| \leq (m+1)||\varphi^-||, \tag{IV.6}$$

the dense subspace of vectors with *finite particle number*

$$D_f := \{\psi \in \mathcal{F}_{F/B}(\mathcal{H}_1) \mid \ \exists \mathcal{N} < \infty : (N+M)\psi < \mathcal{N}\psi\}. \tag{IV.7}$$

provides a common invariant domain for all creation and annihilation operators. On this subspace the free field can then be defined for every $\varphi \in \mathcal{H}_1$ according to

$$\Phi(\varphi) := a^*(\varphi^+) + b(\overline{\varphi^-}) \ , \qquad \Phi^*(\varphi) = b^*(\overline{\varphi^-}) + a(\varphi^+) = \Phi(\varphi)^*. \tag{IV.8}$$

With this definition the field Φ is *linear* in the one-particle vector $\varphi \in \mathcal{H}_1$ and it also satisfies (anti-)commutation relations of the form

$$\begin{aligned} [\Phi^*(\varphi_1), \Phi(\varphi_2)]_{F/B} &= \langle \varphi_1, K_{F/B}\, \varphi_2 \rangle \\ [\Phi^\sharp(\varphi_1), \Phi^\sharp(\varphi_2)]_{F/B} &= 0, \end{aligned} \tag{IV.9}$$

where we introduced the matrix

$$K_{F/B} := P_+ \pm P_- \tag{IV.10}$$

which is just the identity in the fermionic case and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in the bosonic case.

Next consider a unitary operator U on the one-particle Hilbert space, which is *diagonal* w.r.t the decomposition $\mathcal{H}_1^+ \oplus \mathcal{H}_1^-$, i.e. it commutes with $P_\pm$. Such unitaries satisfying $[U, P_\pm] = 0$ will be given e.g. by the representers of Poincaré group elements. We define their second quantization by

$$\hat{\Gamma}(U) := \Gamma_+(U_{++}) \otimes \Gamma_-(\overline{U_{--}}), \tag{IV.11}$$

where $\Gamma_\pm$ is the usual second-quantization functor on the respective subspace, i.e.

$$\Gamma_\pm(U_{\pm\pm}) = \bigoplus_{n=0}^{\infty} \bigotimes\nolimits_{a/s}^{n} U_{\pm\pm} \, , \tag{IV.12}$$

and $\bigotimes_{a/s}^n$ denotes the n-fold anti-symmetric or symmetric tensor product. In particular it follows that the second quantization of a constant phase factor $e^{i\gamma}$ is $\hat{\Gamma}(e^{i\gamma}) = e^{i\gamma Q}$, where Q is again the charge operator. With respect to the free field this operator then satisfies

$$\hat{\Gamma}(U)\Phi(\varphi)\hat{\Gamma}(U)^* = \Phi(U\varphi). \tag{IV.13}$$

We therefore call the unitary operator $\hat{\Gamma}(U)$ the *implementer* of U on the Fock space.

For a general bounded operator V on $\mathcal{H}_1$ a transformation of the form

$$\Phi(\varphi) \mapsto \Phi(V\varphi) \tag{IV.14}$$

is called a *Bogoliubov transformation* if it leaves the commutation relations (IV.9) invariant. This is the case if and only if the operator V satisfies

$$VKV^* = V^*KV = K\ , \qquad \text{for } K \equiv K_{F/B}. \tag{IV.15}$$

This means that in the fermionic case it simply has to be unitary and in the bosonic case we call such an operator *pseudo-unitary.* For a diagonal operator U as before unitarity and pseudo-unitarity are the same and so the interesting cases arise if we consider (pseudo-) unitary operators which are *not* diagonal w.r.t $P_\pm$. For them the following well-known theorem holds, which dates back to the works of Shale and Stinespring [88,89]. The proofs for the various theorems needed in the following and a more extensive treatment can be found e.g. in [18, 78, 80].

Proposition IV.1. *For a (pseudo-)unitary operator V on the one-particle Hilbert space $\mathcal{H}_1$ a unitary implementer $\hat{\Gamma}(V)$ on the Fock space $\mathcal{F}_{a/s}(\mathcal{H}_1)$, satisfying*

$$\hat{\Gamma}(V)\Phi(\varphi)\hat{\Gamma}(V)^* = \Phi(V\varphi),$$

exists if and only if

$$V_{\pm\mp} \in \mathcal{B}_2, \tag{IV.16}$$

i.e. the off-diagonal components of V are Hilbert-Schmidt operators.

These implementable Bogoliubov transformations form a group which we will denote by $\mathcal{G}_{F/B}(\mathcal{H}_1)$. One immediately notices that, because of the definition of being an implementer, they can only be defined up to an arbitrary constant phase factor. So in explicit constructions one always has to specify a choice of phase. Another important fact (proved in [18]) is that in the fermionic case the group $\mathcal{G}_F(\mathcal{H}_1)$ decomposes into disconnected components, labelled by an integer $q(V)$ which turns out to be the *Fredholm index* of V_{--}, i.e.

$$q(V) := \dim\ker(V_{--}) - \dim\ker(V_{--}^*). \tag{IV.17}$$

More importantly it can be shown that the implementer $\hat{\Gamma}(V)$ shifts the charge of a vector exactly by the number $q(V)$ (see also [22]), so by considering unitaries V whose diagonal components have non-vanishing kernels we can construct charge-shifting operators on the Fock space.

In the boson case it already follows from pseudo-unitarity and the Hilbert-Schmidt condition that the diagonal parts $V_{\pm\pm}$ have bounded inverses (as operators on $\mathcal{H}_1^\pm$), so $\ker V_{--} = \ker V_{--}^* = \emptyset$ and therefore $q(V) = 0$. This is the crucial difference between the theory of bosonic- and fermionic Bogoliubov transformations. so in this sense there are more Bogoliubov transformations for fermions than for bosons (which is one of the reasons why we will only consider the fermionic case later). Moreover, denoting by $\mathcal{G}_F^0(\mathcal{H}_1)$ the set of all unitaries V on $\mathcal{H}_1$ such that V_{--} has a bounded inverse, there is even a one-to-one correspondence between $\mathcal{G}_F^0(\mathcal{H}_1)$ and $\mathcal{G}_B(\mathcal{H}_1)$ [49].

In [78] Ruijsenaars explicitly constructed the implementers $\hat{\Gamma}(V)$ for $V \in \mathcal{G}_B$, $V \in \mathcal{G}_F^0$ and $V \in \mathcal{G}_F$. Although these explicit expressions are not needed most of the time we will state some of them here for the sake of completeness.

Consider first the case where $\ker V_{--} = \ker V_{++} = \emptyset$. For such a V we define its conjugate Z according to [49, 81]

$$\begin{aligned} Z_{++} &:= V_{++} - V_{+-}V_{--}^{-1}V_{-+}, \qquad & Z_{+-} &:= V_{+-}V_{--}^{-1}, \\ Z_{-+} &:= -V_{--}^{-1}\, V_{-+}, & Z_{--} &:= V_{--}^{-1}. \end{aligned} \tag{IV.18}$$

This is a bounded operator with Hilbert-Schmidt off-diagonal elements. Now a unitary operator on Fock space implementing the transformation $V \in \mathcal{G}_{F/B}^0(\mathcal{H}_1)$ can then be defined by [81]

$$\begin{aligned} \hat{\Gamma}(V) &:= N_{F/B}(V)\ E_c(Z) \\ E_c(Z) &:= :\exp(Z_{+-}a^*b^* + (Z_{++} - P_+)a^*a \mp (Z_{--} - P_-)bb^* \mp Z_{-+}ba): \end{aligned} \tag{IV.19}$$

where the upper sign again refers to the fermionic and the lower sign to the bosonic case. Here : (...) : denotes normal ordering and $N_{F/B}(V)$ is a normalization constant

$$N_{F/B}(V) := \det\left(1 \pm (Z_{+-})^* Z_{+-}\right)^{\mp 1/2}, \tag{IV.20}$$

where det(...) denotes the Fredholm determinant which is well defined here because $Z_{\pm\mp}$ are Hilbert-Schmidt operators. Expressions of the form Aa^*b are always short for

$$Aa^*b := \int d\theta d\theta' A(\theta, \theta') a^*(\theta) b(\theta'), \tag{IV.21}$$

where $A(\theta, \theta')$ is the integral kernel of the operator A (which will always exist in the cases we are going to consider). Acting on the vacuum vector Ω the implementer simplifies to

$$\hat{\Gamma}(V)\Omega = N_{F/B}(V) \exp(Z_{+-}a^*b^*)\Omega \tag{IV.22}$$

which shows that $\hat{\Gamma}(V)$ creates vectors with arbitrary high particle number from the vacuum meaning in particular that $\hat{\Gamma}(V)$ does not leave the domain D_f invariant. Fortunately in [78] Ruijsenaars showed that there actually is a dense domain $\tilde{D} \in \mathcal{F}_{F/B}(\mathcal{H}_1)$ (namely the maximal domain on which all creation and annihilation operators can be defined) such that $\hat{\Gamma}(V)\tilde{D} = \tilde{D}, \forall V \in \mathcal{G}_{F/B}^0$, so in the following we will consider all the expressions regarding implementers as being defined on this domain.

Another interesting case (apart from $V \in \mathcal{G}^0(\mathcal{H}_1)$) that we will encounter are operators which raise or lower the charge by one. So consider $V \in \mathcal{G}_F(\mathcal{H}_1)$ such that $q(V) = 1$, i.e.

$$\ker(V_{--}) = \{\lambda e_- \mid \lambda \in \mathbb{C}\}, \qquad \ker(V_{--}^*) = \emptyset, \tag{IV.23}$$

where e_- is a unit vector in $\mathcal{H}_1^-$. Setting $e_0 := Ve_- \in \mathcal{H}_1^+$ the explicit form of the implementer of V turns out to be [22]

$$\hat{\Gamma}(V) = N_F(V)\left[a^*(e_0)E_c(V) + E_c(Z)b(\overline{e_-})\right], \tag{IV.24}$$

which obviously raises the charge of a vector by one. Note that the definition of Z contains the operator V_{--}^{-1} so we have to restrict it to $\ker(V_{--})^\perp$ and set it equal to zero on $\ker V_{--}$ to make the definition meaningful. For a more extensive treatment of this case see again [78].

Next consider a bounded operator A on $\mathcal{H}_1$, satisfying

$$KA^*K = A, \tag{IV.25}$$

i.e. it is self-adjoint in the fermion case and we call it pseudo-self-adjoint in the boson case. Then the following proposition holds [22, 40].

Proposition IV.2. *For a (pseudo-)self-adjoint operator A the unitary operator e^{itA} is in $\mathcal{G}_{F/B}(\mathcal{H}_1)$ for all $t \in \mathbb{R}$ if and only if*

$$A_{\pm\mp} \in \mathcal{B}_2, \tag{IV.26}$$

i.e. its off-diagonal elements are Hilbert-Schmidt operators.

Such (pseudo-)self-adjoint operators form the Lie algebra $\mathfrak{g}_{F/B}(\mathcal{H}_1)$ of the group $\mathcal{G}_{F/B}(\mathcal{H}_1)$ and due to continuity in the parameter t the operators e^{itA} have vanishing Fredholm index $q(e^{itA}) = 0$ and thus their implementers leave the charged sectors invariant. Moreover there exists an essentially self-adjoint operator $d\hat{\Gamma}(A)$ on the Fock space such that the implementer can be written as

$$\hat{\Gamma}(e^{itA}) = e^{itd\hat{\Gamma}(A)}. \tag{IV.27}$$

The phase choice for $\hat{\Gamma}(e^{itA})$ then fixes the arbitrary constant in the definition of $d\hat{\Gamma}(A)$ and the explicit form of $d\hat{\Gamma}(A)$ is [22, 40]

$$d\hat{\Gamma}(A) = d\Gamma_+(A_{++}) - d\Gamma_-(A_{--}^T) \pm A_{+-}a^*b^* + A_{-+}ba. \tag{IV.28}$$

With respect to the free field this operator then satisfies

$$[d\hat{\Gamma}(A), \Phi(\varphi)] = \Phi(A\varphi)\ , \qquad \forall\varphi \in \mathcal{H}_1. \tag{IV.29}$$

We can see now e.g. that the charge operator Q is in this sense the "second quantization" of the identity operator, namely

$$d\hat{\Gamma}(\mathbb{1}) = d\Gamma_+(P_+) - d\Gamma_-(P_-) = N - M = Q, \tag{IV.30}$$

which is in accordance with the fact that $\hat{\Gamma}(e^{i\gamma}) = e^{i\gamma Q}$ for $\gamma \in \mathbb{R}$.

In explicit constructions it is important to be able to compute commutation relations between different implementers. This can be done by using the following lemma, for which a proof can be found e.g. in [22, 40, 49].

Lemma IV.1. *Consider $A, B \in \mathfrak{g}_{F/B}(\mathcal{H}_1)$. Then their implementers satisfy commutation relations of the form*

$$[d\hat{\Gamma}(A), d\hat{\Gamma}(B)] = d\hat{\Gamma}([A, B]) - iS(A, B) \cdot \mathbb{1}, \tag{IV.31}$$

where $S(A, B)$ is the so-called Schwinger-term*,*

$$S(A, B) := \pm i Tr(A_{-+}B_{+-} - B_{-+}A_{+-}). \tag{IV.32}$$

Remark: In our convention $S(A, B)$ is a real number, but often one calls $C(A, B) = -iS(A, B)$ the Schwinger-term, which is purely imaginary.

So we see that even if A and B commute their implementers do not necessarily commute on the Fock space. In particular we get for $[A, B] = 0$,

$$\hat{\Gamma}(e^{iA})\hat{\Gamma}(e^{iB}) = e^{iS(A,B)}\hat{\Gamma}(e^{iB})\hat{\Gamma}(e^{iA}). \tag{IV.33}$$

The above relations provide the abstract current algebras for fermions and bosons. The Schwinger-term obeys 2-cocycle relations making it a 2-cocycle of the Lie algebra $\mathfrak{g}_{F/B}(\mathcal{H}_1)$ and the implementers $d\hat{\Gamma}(\cdot)$ provide a representation of a central extension of $\mathfrak{g}_{F/B}(\mathcal{H}_1)$. Note that we have by construction $\langle \Omega, d\hat{\Gamma}(A)\Omega\rangle = 0$ which is equivalent to a phase choice such that $\langle \Omega, \hat{\Gamma}(e^{iA})\Omega\rangle = 1$.

Apart from the current algebra relations there is also an explicit formula for commutation relations when one of the unitaries involved is a charge shift. The following lemma has e.g. been proved in [22].

Lemma IV.2. *Let $V \in \mathcal{G}_{F/B}(\mathcal{H}_1)$ be a (pseudo-)unitary charge shift and $A \in \mathfrak{g}_{F/B}(\mathcal{H}_1)$ a (pseudo-)self-adjoint operator. If A and V commute, then their implementers satisfy*

$$\hat{\Gamma}(e^{iA})\hat{\Gamma}(V) = e^{i\langle \hat{\Gamma}(V)\Omega, d\hat{\Gamma}(A)\hat{\Gamma}(V)\Omega\rangle}\, \hat{\Gamma}(V)\hat{\Gamma}(e^{iA}). \tag{IV.34}$$

In the concrete examples below we will need to be able to explicitly calculate such commutation relations between implementers. In this context the following lemma is very useful, which deals with operators creating one-particle vectors from the vacuum on Fock space.

Lemma IV.3. *Unitaries $V \in \mathcal{G}_F(\mathcal{H}_1)$ with $q(V) = 1$, i.e. V is of the form* (IV.23)*, which in addition satisfy $V_{-+} = 0$ create one-particle vectors from the vacuum, more precisely*

$$\hat{\Gamma}(V)\Omega = e_0, \qquad \text{with } e_0 = Ve_-. \tag{IV.35}$$

Proof. On the vacuum $\hat{\Gamma}(V)$ acts as

$$\hat{\Gamma}(V)\Omega = N_F(V)a^*(e_0)e^{Z_{+-}a^*b^*}\Omega.$$

From unitarity of V and the assumption that $V_{-+} = 0$ it follows that $V_{+-}V_{--}^* = 0$. Inserting $\mathbb{1} = V_{--}^{-1}V_{--}$ into this equation and using that (also due to unitarity of V) there holds $V_{--}V_{--}^* = \mathbb{1}$ we see that also $V_{-+}V_{--}^{-1} = 0$, but this expression just equals $Z_{+-} = 0$. Together with $Z_{-+} = -V_{--}^{-1}V_{-+} = 0$ this shows that the off-diagonal elements of the conjugate Z are zero, which also leads to $N_F(V) = 1$. We therefore get

$$\hat{\Gamma}(V)\Omega = a^*(e_0)\Omega = e_0,$$

completing the proof. □

One last result that will be needed in the constructions concerns the covariance of the unitary implementers w.r.t. Poincaré transformations. As already mentioned they will be represented by unitary operators which are diagonal w.r.t $P_\pm$ and their implementers leave the vacuum invariant, i.e. $\hat{\Gamma}(U)\Omega = \Omega$. In this case the following lemma holds.

Lemma IV.4. *For $V \in \mathcal{G}_{F/B}(\mathcal{H}_1)$ and a unitary $U \in \mathcal{G}^0_{F/B}(\mathcal{H}_1)$ such that $\hat{\Gamma}(U)\Omega = \Omega$ there holds*

$$\hat{\Gamma}(U)\hat{\Gamma}(V)\hat{\Gamma}(U)^* = \hat{\Gamma}(UVU^*). \tag{IV.36}$$

Proof. Because $\hat{\Gamma}(U)\hat{\Gamma}(V)\hat{\Gamma}(U)^*$ and $\hat{\Gamma}(UVU^*)$ implement the same Bogoliubov transformation they have to be equal up to a phase, i.e. $\hat{\Gamma}(U)\hat{\Gamma}(V)\hat{\Gamma}(U)^* = e^{i\gamma}\hat{\Gamma}(UVU^*)$. But by taking vacuum expectation values and using the assumption that $\hat{\Gamma}(U)\Omega = \Omega$ one sees that

$$\langle \Omega, \hat{\Gamma}(U)\hat{\Gamma}(V)\hat{\Gamma}(U)^*\Omega\rangle = \langle \Omega, \hat{\Gamma}(V)\Omega\rangle = e^{i\gamma}\langle \Omega, \hat{\Gamma}(UVU^*)\Omega\rangle.$$

But by definition $\hat{\Gamma}$ has been chosen such that $\langle \Omega, \hat{\Gamma}(\cdot)\Omega\rangle = 1$ which shows that also $e^{i\gamma} = 1$. □

From the uniqueness (up to a phase) of the implementers it follows that relations of the form

$$\begin{aligned} \hat{\Gamma}(U)\hat{\Gamma}(V) &= e^{i\sigma(U,V)}\,\hat{\Gamma}(UV) \\ \hat{\Gamma}(U)\hat{\Gamma}(V) &= e^{i[\sigma(U,V)-\sigma(V,U)]}\,\hat{\Gamma}(V)\hat{\Gamma}(U)\ , \quad \text{for } [U,V] = 0, \end{aligned} \tag{IV.37}$$

with a 2-cocycle $\sigma(\cdot,\cdot)$ have to hold for arbitrary unitaries $U, V \in \mathcal{G}_{F/B}(\mathcal{H}_1)$. But expressions for this cocycle can in general be complicated and will be calculated in the concrete examples below if needed. Explicit expressions for various commutation relations and cocycles can be found e.g. in [49].

IV.1.1. Multiplication operators

In the theory of a charged Dirac particle of mass $m \geq 0$ in $d = 1+1$ dimensions one usually starts with a position space $\check{\mathcal{H}}_1$ on which a Dirac Hamiltonian $\check{H}$ acts. The operators $P_\pm$ are then the projections onto the subspaces on which the spectrum of $\check{H}$ is purely positive or negative respectively. After Fourier transformation

$$\mathcal{F} : \check{\mathcal{H}}_1 := L^2(\mathbb{R}, dx) \oplus L^2(\mathbb{R}, dx) \longrightarrow \hat{\mathcal{H}}_1 := L^2(\mathbb{R}, dp) \oplus L^2(\mathbb{R}, dp)$$

the Hamiltonian acts as multiplication with the hermitian matrix

$$\hat{H}(p) = \begin{pmatrix} p & m \\ m & -p \end{pmatrix} \tag{IV.38}$$

and the momentum operator $\hat{P}$ simply multiplies with $p \cdot \mathbb{1}$. This Hamiltonian satisfies $\hat{H}^2 = \omega_p^2 \cdot \mathbb{1}$ where the energy is $\omega_p := \sqrt{p^2 + m^2}$ which allows us to write the projections $P_\pm$ as

$$P_\pm = \frac{1}{2}\left(\mathbb{1} \pm \frac{\hat{H}(p)}{\omega_p}\right), \tag{IV.39}$$

satisfying indeed $P_\pm P_\pm = P_\pm$ and $P_\pm P_\mp = 0$. One can then diagonalize this Hamiltonian on a suitable one-particle space $\mathcal{H}_1 = \mathcal{H}_1^+ \oplus \mathcal{H}_1^-$ with a unitary map $\mathcal{W} : \hat{\mathcal{H}}_1 \to \mathcal{H}_1$ such that

$$H := \mathcal{W}\hat{H}\mathcal{W}^{-1} = \begin{pmatrix} \omega_p & 0 \\ 0 & -\omega_p \end{pmatrix} \quad \text{and } P = \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix}. \tag{IV.40}$$

After diagonalization the Hamiltonian is positive on $\mathcal{H}_1^+$ and *negative* on $\mathcal{H}_1^-$. The second quantization map $d\hat{\Gamma}(H) = d\Gamma_+(H_{++}) - d\Gamma_-(H_{--})$ then corrects this leading to a positive definite energy operator on the Fock space.

The same procedure can be done in the bosonic case for the Klein-Gordon Hamiltonian

$$\hat{H}_B = \begin{pmatrix} 0 & 1 \\ p^2 + m^2 & 0 \end{pmatrix},$$

satisfying also $\hat{H}_B^2 = \omega_p^2 \cdot \mathbb{1}$ and acting on the space $L^2(\mathbb{R}, \omega_p dp) \otimes L^2(\mathbb{R}, \frac{dp}{\omega_p})$. Since this does not provide a deeper insight into the topic and we won't need the theory for bosons in most of the work we will from now on restrict to the fermionic case. Explicit expressions for the diagonalization map $\mathcal{W}$ (in the Fermi- and the Bose case) can be found e.g. in [22, 79, 81].

Remark: For $m = 0$ the projectors in the fermionic case simplify to

$$P_\pm = \begin{pmatrix} \Theta(\pm p) & 0 \\ 0 & \Theta(\mp p) \end{pmatrix}, \tag{IV.41}$$

where Θ denotes the Heaviside function. They then commute with the Hamiltonian and the two chiral components completely decouple. One can therefore restrict the attention to one of the subspaces $L^2(\mathbb{R})$ in many respects. On position space the components of $P_\pm$ then equal the projections onto the Hardy spaces of functions holomorphic in the upper or lower half-plane.

In the following we will consider operators which act as multiplication with a function on position space, i.e.

$$(\check{A}\check{\varphi})(x) := \alpha(x)\check{\varphi}(x), \qquad \check{\varphi} \in L^2(\mathbb{R})^2, \tag{IV.42}$$

which then leads to a convolution operator $\hat{A}\hat{\varphi} = \tilde{a} \star \hat{\varphi}$ on momentum space. This has first of all the advantage that all of these operators commute, which will then allow us to calculate commutation relations of the respective implementers. Secondly, although such multiplication operators are simple to handle on the one-particle level, they can still lead to highly non-trivial operators on the Fock space creating vectors with arbitrary high particle number from the vacuum.

Using again the unitary operator $\mathcal{W}$ we can then define the operator A on the proper one-particle space $\mathcal{H}_1$ as

$$A := \mathcal{W}\hat{A}\mathcal{W}^* = \begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix}.$$

As we have seen in the previous section this leads to an implementable Bogoliubov transformation if and only if A_{+-} and A_{-+} are Hilbert-Schmidt operators. To check this we have to calculate

$$||A_{+-}||_2^2 = Tr\left(A_{+-}^* A_{-+}\right) = Tr\left(A^* P_+ A P_-\right), \tag{IV.43}$$

which leads to

$$\begin{aligned} Tr(A^*P_+AP_-) &\propto \int dpdq\, |\tilde{\alpha}(p-q)|^2 \; tr\left[\left(\mathbb{1} + \frac{H(q)}{\omega_q}\right)\left(\mathbb{1} - \frac{H(p)}{\omega_p}\right)\right] \\ &\propto \int dpdq\, |\tilde{\alpha}(p-q)|^2 \; \left(1 - \frac{pq+m^2}{\omega_p\,\omega_q}\right) \\ &= \int dp\, |\tilde{\alpha}(p)|^2 \; I_m(p), \end{aligned} \tag{IV.44}$$

where tr denotes the matrix-trace and we have introduced

$$I_m(p) := \int dq \left(1 - \frac{(p+q)q + m^2}{\omega_{p+q}\,\omega_q}\right). \tag{IV.45}$$

Now it can be shown (see [22] for a similar calculation) that there is a constant C such that the function $I_m(p)$ satisfies

$$0 \le I_m(p) \le C\,|p|\,, \quad \forall m \ge 0. \tag{IV.46}$$

These considerations now allow us to identify a large class of multiplication operators leading to implementable Bogoliubov transformations.

Lemma IV.5. *If the function α corresponding to the multiplication operator A according to* (IV.42) *is an element of the Sobolev space $H_1(\mathbb{R})$ then the off-diagonal elements of A are Hilbert-Schmidt operators.*

Proof. The space $H_1(\mathbb{R})$ consists of continuous L^2-functions such that their derivative is also L^2. In Fourier space this condition means that $\int dp\,(p^2+1)\,|\tilde{\alpha}(p)|^2 < \infty$ which immediately implies that $\int dp\,|p|\,|\tilde{\alpha}(p)|^2 < \infty$. □

For real valued functions α the corresponding operator A is of course self-adjoint and for such operators we can also explicitly calculate the Schwinger term $S(A,B)$.

Lemma IV.6. *For operators $\check{A} = \alpha \cdot \mathbb{1}, \check{B} = \beta \cdot \mathbb{1}$, multiplying with real-valued functions α, β, the Schwinger term $S(A,B)$ vanishes.*

Proof. According to lemma IV.1 the Schwinger term has the form

$$S(A,B) \propto Tr[A_{-+}B_{+-} - B_{-+}A_{+-}] \propto \operatorname{Im} Tr[A_{-+}B_{+-}] = \operatorname{Im} Tr[AP_+BP_-]$$

Inserting the definition of the projectors $P_\pm$ we get

$$\begin{aligned} Tr[AP_+BP_-] &\propto \int dpdq\,\tilde{\alpha}(p-q)\tilde{\beta}(q-p)\,tr\left[\left(\mathbb{1} + \frac{H(q)}{\omega_q}\right)\left(\mathbb{1} - \frac{H(p)}{\omega_p}\right)\right] \\ &= \int dpdq\,\tilde{\alpha}(p-q)\tilde{\beta}(q-p)\,\left[2 - \frac{tr\,(H(p)H(q))}{\omega_p\,\omega_q}\right] \end{aligned}$$

Now we see that the second factor containing the trace is real-valued and symmetric under the exchange of p and q. Furthermore, because α and β are real, their Fourier transforms satisfy $\tilde{\alpha}(q-p) = \overline{\tilde{\alpha}(p-q)}$. This shows that the whole expression is a real number and hence $\operatorname{Im} Tr(A_{-+}B_{+-}) = 0$. □

Apart from the operators being diagonal on the position space we only needed that the Hamiltonian satisfies $tr(H) = 0$ so this result obviously also holds in the bosonic case.

The above lemmas hold for arbitrary mass $m \geq 0$ but there is still a significant difference between the massive and the massless case if we look at the behavior for small p. Namely for $m > 0$ the function $I_m(p)$ satisfies

$$I_m(p) \sim c\,p^2, \quad \text{for } p^2 \ll m^2, \tag{IV.47}$$

whereas for $m = 0$ the function I_0 simplifies to

$$I_0(p) \propto |p| \tag{IV.48}$$

for *all* $p \in \mathbb{R}$.

Now consider a real-valued function α which looks like a smeared sign function, i.e.

$$\alpha(x) = \int dy \; \mathrm{sgn}(x-y)\chi(y),$$

where χ is a real-valued smooth function with compact support such that $\int dx\chi(x) = 1$. This of course defines a self-adjoint multiplication operator on $L^2(\mathbb{R})^2$ which, however, is *not* an element of the Sobolev space $H_1(\mathbb{R})$. The Fourier transform then looks like $\tilde{\alpha}(p) \sim \frac{1}{p}\tilde{\chi}(p)$ and the integral in the Hilbert-Schmidt condition turns out to be

$$\int dp \, |\tilde{\alpha}(p)|^2 I_m(p) = \int dp \, \frac{I_m(p)}{p^2} |\tilde{\chi}(p)|^2.$$

For $m = 0$ this clearly diverges because $I_0(p) \sim |p|$ and we get an infrared singularity at $p = 0$. But for $m > 0$ we know that $I_m(p)$ goes to zero *quadratically* thus cancelling the divergence resulting in a finite integral. The multiplication operator corresponding to this function therefore leads to an implementable Bogoliubov transformation $e^{i\pi\lambda\alpha}$ only in the massive case. This is the main reason for the difference we will encounter in the construction of anyonic quantum fields in the massive and the massless case. Because for $m > 0$ we will see that we can use these sign-functions to define "disorder operators" which can then be used to construct field nets on the Fock space with arbitrary commutation relations.

Apart from diagonal self-adjoint operators we also want to have operators leading to charge shifts on the Fock space. For this purpose consider a non-diagonal multiplication operator of the form

$$V = \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}, \tag{IV.49}$$

which is clearly unitary if $v = e^{if}$ with a real-valued function f. Then the following lemma holds, which can also be found in [22].

Lemma IV.7. *If the function v satisfies*

$$v - 1 \in H_1(\mathbb{R}) \tag{IV.50}$$

then the corresponding operator V is in $\mathcal{G}_F(\mathcal{H}_1)$.

Proof. For the sake of simplicity (and because we will only need such operators in the massless case) we only give the proof for $m = 0$ here. As we have seen in this case the projectors simplify significantly and the integral kernel of the off-diagonal term turns out to be

$$V_{+-}(p,q) = (P_+ V P_-)(p,q) = \begin{pmatrix} \tilde{v}(p-q)\Theta(p)\Theta(-q) & 0 \\ 0 & 0 \end{pmatrix}$$

because $\delta(p-q)\Theta(p)\Theta(-q) = 0$. The Hilbert-Schmidt condition then takes the form

$$Tr(V_{+-}^* V_{+-}) \propto \int dpdq\, |\tilde{v}(p-q)|^2 \Theta(p)\Theta(-q).$$

Now define a function u according to $u := v - 1$ such that $\tilde{v} = \tilde{u} + \sqrt{2\pi}\,\delta$. This immediately leads to

$$\begin{aligned} \int dpdq\, |\tilde{v}(p-q)|^2 \Theta(p)\Theta(-q) &= \int dpdq\, |\tilde{u}(p-q)|^2 \Theta(p)\Theta(-q) \\ &= \int_0^\infty dp\, p\, |\tilde{u}(p)|^2 < \infty \end{aligned}$$

because $u = v - 1 \in H_1(\mathbb{R})$. □

From the condition that $v - 1 = e^{if} - 1 \in H_1(\mathbb{R})$ it follows in particular that

$$e^{if(x)} \to 1, \text{ for } x \to \pm\infty,$$

which means that the function f has to satisfy

$$f(\infty) - f(-\infty) = 2\pi\, n(f) \tag{IV.51}$$

where $n(f) \in \mathbb{Z}$ is usually called the *"winding number"* of the function f. So we see that an operator of the form (IV.49) is only implementable if the function f has integer winding number. The significance of this number now lies in the following lemma for which a proof and a more detailed description can also be found in [22].

Lemma IV.8. *Consider a unitary operator of the form*

$$V = \begin{pmatrix} e^{if} & 0 \\ 0 & 1 \end{pmatrix}$$

where f is a smooth real-valued function with integer winding number. Then the implementer $\hat{\Gamma}(V)$ shifts the charge of a vector on Fock space exactly by the number

$$q(V) = n(f).$$

This holds for any mass $m \geq 0$ but since we will need such charge shifts only in the massless case we will investigate them further in the chapter about $m = 0$.

IV.2. The Massive Case: Order- and Disorder Operators

In this section we are going to show how "kink functions" of the form $\alpha \sim \int dy\, \mathrm{sgn}(x-y)\chi(y)$ can be used to construct disorder operators on the bosonic and fermionic Fock space if the mass is positive. Multiplying them with free fields then leads to a field algebra which is still localized in compact regions on 1+1 dimensional Minkowski space and covariant under the same representation of the Poincaré group as the free field. The only thing that gets changed by this procedure is the commutation relation of space-like separated fields which is governed by a phase factor $e^{\pm 2\pi i\lambda}$ depending only on which of the two localization regions is to the left/right of the other. The constant $\lambda \in \mathbb{R}$ which we will introduce can be an arbitrary real number so we can continuously interpolate between bosonic and fermionic commutation relations.

The unsmeared operators that arise from pure sign functions have already been used by Ruijsenaars in [81] in the context of integrable models, especially the fermionic and bosonic Federbush model and the massless Thirring model. Because of the discontinuity of the sign function the resulting operators are not implementable but in [81] it is shown how they can still be considered as quadratic forms on the Fock space. Similar operators have been used in [1] and [82] for the construction of DHR endomorphisms which generate the charged sectors of the free Dirac field and exhibit braid group statistics. In this sense the statistics of massive particles in $d = 1+1$ is arbitrary which is not the case for $m = 0$ because the respective operators are not implementable there.

Here we will pursue a similar idea and explicitly construct localized and covariant field algebras which have anyonic commutation relations. The theory of implementable Bogoliubov transformations allows us to make sure that every step is mathematically well defined.

The theory of a massive charged boson or fermion on 1+1 dimensional Minkowski space is most conveniently formulated on the one-particle Hilbert space

$$\mathcal{H}_1 := L^2(\mathbb{R}, d\theta) \oplus L^2(\mathbb{R}, d\theta) = \mathcal{H}_1^+ \oplus \mathcal{H}_1^- \tag{IV.52}$$

where θ is the rapidity parameter, connected to the momentum on the mass shell through[2]

$$\begin{pmatrix} \omega_p(\theta) \\ p(\theta) \end{pmatrix} := \begin{pmatrix} m\cosh\theta \\ m\sinh\theta \end{pmatrix}.$$

One can now find a unitary map $\mathcal{W} : L^2(\mathbb{R}, dp)^2 \to \mathcal{H}_1$ which diagonalizes the Dirac Hamiltonian (IV.38) such that

$$\begin{aligned}(H\varphi)^\pm(\theta) = (\mathcal{W}\hat{H}\mathcal{W}^{-1}\varphi)^\pm(\theta) = \pm m\cosh(\theta)\, \varphi^\pm(\theta),\\ (P\varphi)^\pm(\theta) = \pm m\sinh(\theta)\, \varphi^\pm(\theta),\end{aligned}$$

[2] Note that we use a slightly different notation here than in chapter III where p denoted the energy-momentum on $\mathbb{R}^{1+1}$ or $\mathbb{R}^{2+1}$, whereas in this chapter $p \in \mathbb{R}$ is just the spatial momentum.

where P is again the momentum operator and we will denote vectors in $L^2(\mathbb{R}, dp)^2$ by $\hat{\varphi}$. Motivated by [81] we can define this map according to

$$\begin{aligned}(\mathcal{W}\hat{\varphi})^\alpha(\theta) &= \sum_{\beta=\pm} W(\alpha\theta)_{\alpha\beta} \cdot \hat{\varphi}^\beta(\alpha p(\theta)),\\ (\mathcal{W}^{-1}\varphi)^\alpha(p) &= \sum_{\beta=\pm} W^\dagger(p)_{\alpha\beta} \cdot \varphi^\beta(\beta\theta(p))\end{aligned} \tag{IV.53}$$

with $\theta(p) := \operatorname{arsinh}(\frac{p}{m})$ and the matrices

$$\begin{aligned}W(\theta) &= \sqrt{\frac{m}{2}} \begin{pmatrix} e^{\theta/2} & e^{-\theta/2} \\ -e^{-\theta/2} & e^{\theta/2} \end{pmatrix},\\ W^\dagger(p) &= \frac{1}{\sqrt{2}\,\omega_p} \begin{pmatrix} \sqrt{\omega_p + p} & -\sqrt{\omega_p - p} \\ \sqrt{\omega_p - p} & \sqrt{\omega_p + p} \end{pmatrix}.\end{aligned} \tag{IV.54}$$

The same can be done for the Klein-Gordon Hamiltonian (IV.1.1) with a map $\mathcal{W} : L^2(\mathbb{R}, \omega_p dp) \oplus L^2(\mathbb{R}, \frac{dp}{\omega_p}) \to \mathcal{H}_1$ where the matrices $W(\theta)$ and $W^\dagger(p)$ have to be chosen according to

$$\begin{aligned}W(\theta) &= \frac{1}{\sqrt{2}} \begin{pmatrix} m\cosh\theta & 1 \\ -m\cosh\theta & 1 \end{pmatrix},\\ W^\dagger(p) &= \frac{1}{\sqrt{2}} \begin{pmatrix} \omega_p^{-1} & -\omega_p^{-1} \\ 1 & 1 \end{pmatrix}.\end{aligned} \tag{IV.55}$$

The representation U_1 of the Poincaré group $\mathcal{P}_+^\uparrow$ on the one-particle Hilbert space $\mathcal{H}_1$ is then given by

$$\begin{aligned}(U_1(a,\Lambda)\varphi)^\pm(\theta) &:= e^{\pm i(a^0\omega_p(\theta) - a^1 p(\theta))}\varphi^\pm(\theta - \mu)\ ,\\ \text{where } \mathcal{L}_+^\uparrow \ni \Lambda \equiv \Lambda(\mu) &= \begin{pmatrix} \cosh\mu & \sinh\mu \\ \sinh\mu & \cosh\mu \end{pmatrix}, \quad a \in \mathbb{R}^2.\end{aligned} \tag{IV.56}$$

Consider now as in the previous section a bounded self-adjoint multiplication operator on position space of the form

$$(\check{A}\check{\varphi})(x) = \alpha(x)\check{\varphi}(x), \qquad \text{with } \alpha(x) = \operatorname{sgn}(x).$$

After Fourier transformation we get a convolution operator $\hat{A}\hat{\varphi} = \frac{1}{\sqrt{2\pi}}\tilde{\alpha} \star \hat{\varphi}$ with

$$\tilde{\alpha}(p) = -i\sqrt{\frac{2}{\pi}}\,\mathcal{P}\left(\frac{1}{p}\right),$$

where $\mathcal{P}(\cdot)$ denotes the Cauchy principal value, i.e. the operator $\mathcal{P}\left(\frac{1}{p}\right)$ acts on a vector $\hat{\varphi}$ as

$$⨍ dq \frac{1}{q}\hat{\varphi}(p-q) = \lim_{\varepsilon \to 0} \left(\int_{-\infty}^{-\varepsilon} dq \frac{1}{q}\hat{\varphi}(p-q) + \int_{\varepsilon}^{\infty} dq \frac{1}{q}\hat{\varphi}(p-q) \right).$$

Mapping this operator to our Hilbert space $\mathcal{H}_1$ using $\mathcal{W}$ we still get a convolution operator which in the fermionic case has the integral kernel

$$A_F(\theta) = (\mathcal{W}\hat{A}\mathcal{W}^{-1})(\theta) = \frac{1}{2\pi i}\begin{pmatrix} \mathcal{P}\sinh(\frac{\theta}{2})^{-1} & -\cosh(\frac{\theta}{2})^{-1} \\ \cosh(\frac{\theta}{2})^{-1} & -\mathcal{P}\sinh(\frac{\theta}{2})^{-1} \end{pmatrix} \tag{IV.57}$$

and in the boson case this yields

$$A_B(\theta) = \frac{1}{2\pi i}\begin{pmatrix} \mathcal{P}\coth(\frac{\theta}{2}) & -\tanh(\frac{\theta}{2}) \\ \tanh(\frac{\theta}{2}) & -\mathcal{P}\coth(\frac{\theta}{2}) \end{pmatrix}. \tag{IV.58}$$

In the following, if a statement does not depend on whether we are in the Bose- or Fermi case, we will omit the indices F/B in the respective expressions. The fact that in both cases the operator A is still a convolution operator on $\mathcal{H}_1$ is due to the special form of the function α and it also immediately shows that A is *not* implementable because the off-diagonal elements cannot be Hilbert-Schmidt. Furthermore, by definition, the operator A acts as $\pm\mathbb{1}$ on vectors of the form $\varphi_{r/l} = \mathcal{W}\hat{\varphi}_{r/l}$, if $\operatorname{supp}\check{\varphi}_r$ is in $\mathbb{R}_+$ and $\operatorname{supp}\check{\varphi}_l$ in $\mathbb{R}_-$ and it satisfies the following properties.

Lemma IV.9. *Let U_1 be the representation* (IV.56)*, Λ a Lorentz boost and $a \in \mathbb{R}^2$. Then A satisfies*

i) $[A, U_1(\Lambda)] = 0$,

ii) $[U_1(a)A\,U_1(a)^*, A] = 0$, *if* $a \cdot a < 0$,

iii) $KA^*K = A$, *i.e. A_F is self-adjoint and A_B is pseudo-self-adjoint.*

Proof. Property *i)* holds because A is a convolution operator and a boost simply acts as a shift in the rapidity. *ii)* also follows immediately from the symmetry of *cosh* and the anti-symmetry of *sinh*, *tanh* and *coth*. To show *iii)* we note that for a spacelike $a \in \mathbb{R}^2$ there is a boost Λ_0 projecting it onto the time zero plane, i.e. $\Lambda_0 a = (0, a_0)$. From covariance w.r.t boosts *i)* it then follows that

$$[U_1(a)AU_1(-a), A] = U_1(\Lambda_0)\,[U_1(a_0)AU_1(-a_0), A]\,U_1(\Lambda_0)^*.$$

But for purely spatial a_0 the operator $U_1(a_0)AU_1(-a_0)$ is again a multiplication operator on position space, multiplying with the translated function $sgn(x - a_0)$ and therefore it commutes with the untranslated A. □

As we have seen in IV.1.1 we could now make the operator A implementable by simply considering a smeared function instead of the discontinuous sgn. The implementer of $e^{i\lambda A}$ would then lead to what we will call a disorder operator on Fock space which could be used together with the free field to define a local covariant field net with anyonic commutation relations. However, in the following we will formulate these operators and their properties in a slightly different way which is more in line with the formulation of chapter III and makes use of the concepts of the theory of *modular localization* [8,26,68].

To describe this in more detail consider first test functions from Schwartz space $\mathscr{S}(\mathbb{R}^2)$ and we will be interested in particular in test functions with compact support. As we have already seen in chapter III one can then map every $f \in \mathscr{S}(\mathbb{R}^2)$ to a vector in $\mathcal{H}_1$ through

$$f \longmapsto f_B := \begin{pmatrix} f^+ \\ f^- \end{pmatrix}, \qquad \text{where } f^\pm(\theta) := \frac{1}{2\pi} \int d^2a\, f(\pm a) e^{i(a^0\omega_p(\theta) - a^1 p(\theta))}. \tag{IV.59}$$

In the fermion case we will need an additional intertwiner in the definition of the map $f \mapsto f_F$ such that the corresponding free field

$$\Phi_F(f) := \Phi(f_F) = a^*(f_F^+) + b(\overline{f_F^-})$$

satisfies anti-commutation relations. Hence we define

$$f_F(\theta) := v(\theta) f_B(\theta) \ , \qquad \text{with } v(\theta) := e^{\theta/2} \tag{IV.60}$$

Note that this intertwiner obviously has an analytic continuation to the entire complex plane and it satisfies

$$\begin{aligned} v(\theta - \mu) &= e^{-\mu/2} v(\theta), \\ v(\theta \pm i\pi)^2 &= -v(\theta)^2, \end{aligned} \tag{IV.61}$$

which ensures the corresponding covariance and locality properties of the free Dirac field. Defining an action α of $\mathcal{P}_+^\uparrow$ on the test functions by

$$(\alpha_{(b,\Lambda)} f)(a) := f(\Lambda^{-1}(a - b)) \tag{IV.62}$$

one can easily compute that

$$\begin{aligned} \left(\alpha_{(a,\Lambda)} f\right)_B &= U_1(a, \Lambda) f_B, \\ \left(\alpha_{(a,\Lambda)} f\right)_F &= e^{\mu/2}\, U_1(a, \Lambda) f_F. \end{aligned}$$

Now if we take test functions which have compact support, their localization properties translate to the Hilbert space in the following way. For an $f \in \mathscr{S}(\mathbb{R}^2)$ with support in the standard right wedge $W_R = \{x \in \mathbb{R}^2 : x_1 > |x_0|\}$ the definition of $f^\pm$ leads to the fact that f^+/f^- has an *$\mathcal{H}_1$-valued analytic continuation* into the strip $S(-\pi, 0)/S(0, \pi)$ respectively, where we define

$$S(a, b) := \{z \in \mathbb{C} : a < \operatorname{Im} z < b\}.$$

This means e.g. that $f^-(\theta + i\mu)$ is an analytic function for $0 < \mu < \pi$ and

$$\theta \mapsto f^-(\theta + i\mu) \in \mathcal{H}_1, \quad \forall \mu \in (0, \pi).$$

In addition the *boundary values* in the Bose/Fermi case satisfy

$$\begin{aligned} f_B^{\pm}(\theta \mp i\pi) &= f_B^{\mp}(\theta), \\ f_F^{\pm}(\theta \mp i\pi) &= \mp i f_F^{\mp}(\theta). \end{aligned} \tag{IV.63}$$

Similar results hold for $\operatorname{supp} f \subset W_L = W_R' = -W_R$ with $S(-\pi, 0)$ and $S(0, \pi)$ interchanged. Now we can use these analyticity properties of localized test functions to define a (modular) localization structure on the charged one-particle Hilbert space $\mathcal{H}_1$. Namely we call a vector $\varphi \in \mathcal{H}_1$ localized in the right or left wedge W_R, W_L precisely if it satisfies the above properties and the subset of wedge-localized vectors will be denoted by $\mathcal{K}(W_R)$ or $\mathcal{K}(W_L)$ respectively. Now remember that the operator A came from multiplication with a sign-function on position space, i.e. it acts as $+\mathbb{1}$ "on the right" and as $-\mathbb{1}$ "on the left". This behavior translates to the Hilbert space $\mathcal{H}_1$ and its localization structure in the following way.

Lemma IV.10. *On vectors $\varphi_{R/L} \in \mathcal{H}_1$ localized in the right/left wedge in the sense described in the previous paragraph the operator A acts according to*

$$A\varphi_{R/L} = \pm \varphi_{R/L}.$$

Proof. We will give here a proof that does not rely on any position space arguments or the map $\mathcal{W}$ but only on the analyticity properties of the integral kernel $A(\theta - \theta')$ and the localization structure on $\mathcal{H}_1$. First note that the Cauchy principal value can be rewritten according to

$$\mathcal{P}\left(\frac{1}{p}\right) = \lim_{\varepsilon \to 0} \frac{1}{p \pm i\varepsilon} \pm i\pi\delta(p).$$

For the diagonal components of the integral kernel of A this implies

$$\mathcal{P}\sinh\left(\frac{\theta}{2}\right)^{-1} = \lim_{\epsilon \to 0} \sinh\left(\frac{\theta \pm i\epsilon}{2}\right)^{-1} \pm 2\pi i\, \delta(\theta),$$

and the same holds for $coth = \frac{cosh}{sinh}$. Furthermore the functions $cosh^{-1}$ and $tanh$ are analytic in the strip $S(-\frac{\pi}{2}, \frac{\pi}{2})$ and have the boundary values

$$\cosh^{-1}(\theta \pm \frac{i\pi}{2}) = \mp i \sinh(\theta)^{-1}, \qquad \tanh(\theta \pm i\frac{\pi}{2}) = \coth(\theta).$$

We will demonstrate the proof only for A_B and φ_R localized in W_R since the same reasoning applies to the other cases as well. Using the above-mentioned relations we can then calculate

$$\begin{aligned} (A_B\varphi_R)^+(\theta) &= \frac{1}{2\pi i}\left[⨍ d\theta' \coth\left(\frac{\theta - \theta'}{2}\right)\varphi_R^+(\theta') - \int d\theta' \tanh\left(\frac{\theta - \theta'}{2}\right)\varphi_R^-(\theta')\right] \\ &= \frac{1}{2\pi i}\left[\lim_{\varepsilon \to 0}\int d\theta' \left(\coth\left(\frac{\theta - \theta' + i\varepsilon}{2}\right) + 2\pi i\delta(\theta - \theta')\right)\varphi_R^+(\theta') \right. \\ &\qquad \left. - \int d\theta' \tanh\left(\frac{\theta - \theta'}{2}\right)\varphi_R^-(\theta')\right] \\ &= \varphi_R^+(\theta) + \frac{1}{2\pi i}\lim_{\varepsilon \to 0}\int d\theta' \left[\coth\left(\frac{\theta - \theta' + i\varepsilon}{2}\right)\varphi_R^+(\theta') - \tanh\left(\frac{\theta - \theta'}{2}\right)\varphi_R^-(\theta')\right] \end{aligned}$$

Now remember that for φ_R localized in the right wedge φ_R^- has an analytic continuation into the strip $S(0,\pi)$ and $\varphi_R^-(\theta + i\mu) \to 0$ for $\theta \to \pm\infty$. This means that we can shift the integration in the last term under the integral from θ' to $\theta' + i\pi - i\varepsilon$. This leads to

$$\begin{aligned}\int d\theta' \tanh\left(\frac{\theta-\theta'}{2}\right)\varphi_R^-(\theta') &= \lim_{\epsilon\to 0}\int d\theta' \tanh\left(\frac{\theta-\theta'-i\pi+i\epsilon}{2}\right)\varphi_R^-(\theta'+i\pi-i\epsilon)\\ &= \lim_{\epsilon\to 0}\int d\theta' \coth\left(\frac{\theta-\theta'+i\epsilon}{2}\right)\varphi_R^+(\theta'-i\epsilon)\end{aligned}$$

which cancels the second term in the above integral for $\epsilon \to 0$ and hence shows that indeed $(A_B\varphi_R)^+ = +\varphi_R^+$. The same argument can be used for the component φ_R^- because of the simple form of the operator A_B. □

We have seen that A acts as $\pm\mathbb{1}$ on vectors which are localized in the standard right- or left wedge in the sense explained above. By using the representation of the translations we can now define for any $a \in \mathbb{R}^2$ operators $A(a) := U_1(a)AU_1(-a)$ which act as $\pm\mathbb{1}$ with respect to any other translated wedge $W_a = W_R + a$. This family of operators is covariant and local in the sense of Lemma IV.9 and by taking exponentials $e^{i\pi\lambda A(a)}$ we would get what Ruijsenaars calls "classical field operators" in [81]. They could be made implementable by considering a smoothed out sign function in the first place but we will see that we can simply smear the operator A on the one-particle Hilbert space with a test function on $\mathbb{R}^2$ with compact support.

For this purpose consider a smooth real-valued function $\chi \in C_0^\infty(\mathbb{R}^2)$ with support in a double cone $\mathcal{O} \subset \mathbb{R}^2$ and satisfying $\int d^2a\,\chi(a) = 1$. Using this function we then define the smeared operator A_χ according to

$$A_\chi := \int d^2a\,\chi(a)\,U_1(a)AU_1(-a). \tag{IV.64}$$

A straightforward calculation using the definition of U_1 leads to the new integral kernel

$$(A_\chi)_{\alpha\beta}(\theta_1,\theta_2) = A_{\alpha\beta}(\theta_1-\theta_2)\,\tilde{\chi}\big(\alpha\omega(\theta_1)-\beta\omega(\theta_2), \alpha p(\theta_1)-\beta p(\theta_2)\big) \tag{IV.65}$$

of the smeared operator. In two dimensions the causal complement of a double cone $\mathcal{O}$ has two connected components, $\mathcal{O}_R'$ and $\mathcal{O}_L'$, which are wedge-shaped regions. So from the definition (IV.64) one sees that A_χ acts as $+\mathbb{1}$ on vectors localized in $\mathcal{O}_R'$ and as $-\mathbb{1}$ in $\mathcal{O}_L'$. We therefore call $\mathcal{O}$ the localization region of the operator A_χ (whereas the unsmeared A would be "pointlike localized") and by using smearing functions with different supports we obtain operators for different localization regions.

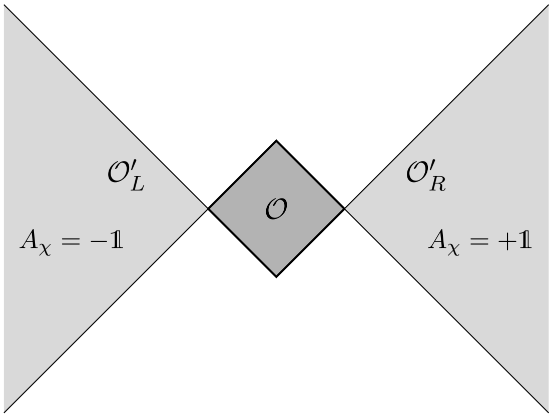


Figure IV.1.: *Action of A_χ in the causal complement of $\mathcal{O} \supset \operatorname{supp}\chi$*

This family now satisfies the following important properties.

Lemma IV.11. *For every $\chi \in C_0^\infty(\mathbb{R}^2)$ let A_χ be defined according to* (IV.64). *The family $\chi \mapsto A_\chi$ then satisfies*

i) Covariance*:*

$$U_1(a,\Lambda)A_\chi U_1(a,\Lambda)^{-1} = A_{\alpha_{(a,\Lambda)}\chi}$$

$A_{\alpha_{(a,\Lambda)}\chi}$ is localized in the Poincaré transformed double cone $(a,\Lambda)\cdot\mathcal{O}$ if supp$\chi \subset \mathcal{O}$.

ii) Locality*: $[A_{\chi_1}, A_{\chi_2}] = 0$ if suppχ_1 and suppχ_2 are spacelike separated*

iii) A_χ is (pseudo-)self-adjoint, i.e. $K(A_\chi)^*K = A_\chi$

iv) A_χ acts as $+\mathbb{1}/-\mathbb{1}$ on vectors localized in $\mathcal{O}'_R/\mathcal{O}'_L$, for supp$\chi \subset \mathcal{O}$.

Proof. i) and *iv)* follow immediately from the definition (IV.64) and by using Lemma IV.9 together with the condition $\int \chi = 1$. *iii)* The relation $K(A_\chi)^*K = A_\chi$ holds because K commutes with U_1 and the smearing function χ is real-valued. *ii)* Locality can also be proved by a straightforward calculation noting that

$$A_{\chi_1}A_{\chi_2} = \int d^2x d^2y\, \chi_1(x)\, \chi_2(y)\, U_1(x)AU_1(-x)U_1(y)AU_1(-y),$$

which equals $A_{\chi_2}A_{\chi_1}$ if and only if $(x-y)^2 < 0$ for all $x \in supp(\chi_1)$ and $y \in supp(\chi_2)$, because of Lemma IV.9. □

So we see that the smeared operators keep all the nice properties of the unsmeared ones and next we are going to show that A_χ defines in fact an implementable Bogoliubov transformation, because the smooth function χ makes sure that the off-diagonal elements are Hilbert-Schmidt operators.

Lemma IV.12. *For every $\chi \in C_0^\infty(\mathbb{R}^2)$ the smeared operators A_χ are in $\mathfrak{g}(\mathcal{H}_1)$, i.e. the off-diagonal elements are Hilbert-Schmidt operators, $(A_\chi)_{+-}, (A_\chi)_{-+} \in \mathcal{B}_2(\mathcal{H}_1)$.*

Proof. i) To check the Hilbert-Schmidt condition for $(A_\chi)_{+-}$ we have to show that

$$\begin{aligned}\|(A_\chi)_{+-}\|_2^2 &= \int d\theta_1 d\theta_2\, |(A_\chi)_{+-}(\theta_1,\theta_2)|^2 \\ &= \int d\theta_1 d\theta_2\, |\tilde{\chi}(\omega(\theta_1)+\omega(\theta_2), p(\theta_1)+p(\theta_2))|^2\, |A_{+-}(\theta_1-\theta_2)|^2 < \infty.\end{aligned}$$

In both the Fermi and the Bose case the function A_{+-} satisfies $|A_{+-}(\theta)|^2 \le 1$ so we can only focus on the $\tilde{\chi}$ part. Because $\chi \in C_0^\infty(\mathbb{R}^2)$ we know that $\tilde{\chi} \in \mathscr{S}(\mathbb{R}^2)$ and for Schwartz functions the bound

$$|\tilde{\chi}(\omega,p)|^2 \le \frac{c}{1+\omega^2+p^2}$$

holds for some $c \in \mathbb{R}$. Using this we get

$$\begin{aligned}\|(A_\chi)_{+-}\|^2 &\le c \int d\theta_1 d\theta_2\, \frac{1}{1+|\sinh\theta_1+\sinh\theta_2|^2+|\cosh\theta_1+\cosh\theta_2|^2} \\ &\le c \int d\theta_1 d\theta_2\, \frac{1}{\cosh\theta_1+\cosh\theta_2}.\end{aligned}$$

But the integral $\int dxdy \frac{1}{\cosh x+\cosh y}$ is just π^2 which makes the whole expression finite and completes the proof. □

Note that the above proof also works for Schwartz functions $\chi \in \mathscr{S}(\mathbb{R}^2)$ which do not have compact support. Therefore for every test function $\chi \in \mathscr{S}(\mathbb{R}^2)$ the operator A_χ is a bounded, (pseudo-)self-adjoint operator with Hilbert-Schmidt off-diagonal elements.
This allows us to define for every $\lambda \in \mathbb{R}$ the unitary operator

$$V_\chi^\lambda := e^{i\pi\lambda A_\chi}, \tag{IV.66}$$

which is then an implementable Bogoliubov transformation $V_\chi^\lambda \in \mathcal{G}(\mathcal{H}_1)$ and the properties of A_χ regarding covariance and locality carry over to V_χ^λ. In particular for $\chi \in C_0^\infty(\mathbb{R}^2)$ with support in $\mathcal{O}$, it acts as the constant gauge transformation

$$V_\chi^\lambda \varphi = \begin{cases} e^{+i\pi\lambda}\varphi, & \varphi \text{ localized in } \mathcal{O}'_R \\ e^{-i\pi\lambda}\varphi, & \varphi \text{ localized in } \mathcal{O}'_L \end{cases} \tag{IV.67}$$

on vectors localized in the causal complement of $\mathcal{O}$. We call such an operator a *"one-particle disorder operator"* and its implementer on Fock space can be used to change the commutation relations of the free Wightman field for bosons and fermions. For a test function $f \in \mathcal{S}(\mathbb{R}^2)$ these fields are defined according to

$$\Phi_{F/B}(f) = \Phi(f_{F/B}) = a^*(f^+_{F/B}) + b(\overline{f^-_{F/B}}) \tag{IV.68}$$

and they are covariant and local in the sense that

$$\begin{aligned} &U(a,\Lambda)\Phi_B(f)U(a,\Lambda)^{-1} = \Phi_B(\alpha_{(a,\Lambda)}f), \\ &U(a,\Lambda)\Phi_F(f)U(a,\Lambda)^{-1} = e^{-\mu/2}\Phi_F(\alpha_{(a,\Lambda)}f), \\ &[\Phi^*_{F/B}(f), \Phi_{F/B}(g)]_{F/B} = 0, \qquad \text{for } supp(f) \subset (supp(g))', \end{aligned} \tag{IV.69}$$

where U is just the second quantization of U_1, i.e. $U = \hat{\Gamma}(U_1)$. Because $V_\chi^\lambda \in \mathcal{G}_{F/B}(\mathcal{H}_1)$ there exists a unitary implementer $\hat{\Gamma}(V_\chi^\lambda)$ such that

$$\hat{\Gamma}(V_\chi^\lambda)\Phi(f_{F/B})\hat{\Gamma}(V_\chi^\lambda)^* = \Phi(V_\chi^\lambda f_{F/B}).$$

If we now take functions f and χ which are supported in spacelike separated regions, i.e. $suppf \subset \mathcal{O}' \subset (supp\chi)'$, the operator $\hat{\Gamma}(V_\chi^\lambda)$ satisfies

$$\hat{\Gamma}(V_\chi^\lambda)\Phi_{F/B}(f)\hat{\Gamma}(V_\chi^\lambda)^* = \begin{cases} e^{+i\pi\lambda}\,\Phi_{F/B}(f), & \text{if } suppf \subset \mathcal{O}'_R \\ e^{-i\pi\lambda}\,\Phi_{F/B}(f), & \text{if } suppf \subset \mathcal{O}'_L, \end{cases} \tag{IV.70}$$

which means that its adjoint action $Ad\hat{\Gamma}(V_\chi^\lambda)$ is just the constant gauge transformation $e^{\pm i\pi\lambda}$ when restricted to fields localized in the right/left spacelike complement of $\mathcal{O}$ respectively. The operator $\hat{\Gamma}(V_\chi^\lambda)$ is therefore a *"disorder operator"* for the free field algebras on the Hilbert space $\mathcal{H}$ and it is localized in the double cone $\mathcal{O}$ (for the general notion of disorder operators for nets of operator algebras see e.g. [72, 73]).

In the next step we are going to use these operators to define fields satisfying anyonic commutation relations, but we first need to make sure that the disorder operators actually commute for spacelike separated localization regions. For this purpose we only need to calculate the Schwinger term of the operators A_χ because we already know that the one-particle operators V_χ^λ commute.

Lemma IV.13. *For smearing functions χ_1, χ_2 which have spacelike separated supports the Schwinger term $S(A_{\chi_1}, A_{\chi_2})$ vanishes.*

Proof. Remembering that the Schwinger term for two operators $A, B \in \mathfrak{g}(\mathcal{H}_1)$ has been defined according to

$$S(A,B) \propto Tr[A_{-+}B_{+-} - B_{-+}A_{+-}] \propto \operatorname{Im} Tr[A_{-+}B_{+-}],$$

we can write

$$S(A_{\chi_1}, A_{\chi_2}) = \int d^2x d^2y\, \chi_1(x)\chi_2(y) S(A(x), A(y)).$$

This expression is zero if the term $S(A(x), A(y))$ vanishes for every $x, y \in \mathbb{R}^2$ such that $(x-y)^2 < 0$ because we assumed that χ_1 and χ_2 have spacelike separated support. Now by the same reasoning as in the proof of Lemma IV.9 there is a Poincaré transformation leaving the trace invariant such that $S(A(x), A(y)) = S(A(x_0), A)$, where x_0 is a purely spatial vector. But then $A(x_0)$ comes from an operator on position space multiplying with a translated sign function and by lemma IV.6 we already know that the Schwinger term vanishes for such operators. □

Fixing the parameter $\lambda \in \mathbb{R}$ we can now define for every pair of test functions f and χ with support in a common double cone $\mathcal{O}$ a new composite field according to

$$\Phi_{F/B}^\lambda(f,\chi) := \Phi_{F/B}(f)\,\hat{\Gamma}(V_\chi^\lambda). \tag{IV.71}$$

This definition is motivated by the following proposition regarding the statistics of spacelike separated fields.

Proposition IV.3. *Consider functions $f_1, \chi_1 \in \mathscr{S}(\mathbb{R}^2)$ and $f_2, \chi_2 \in \mathscr{S}(\mathbb{R}^2)$ with supports in $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively such that $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated. The fields $\Phi_1 := \Phi^\lambda(f_1, \chi_1)$ and $\Phi_2 := \Phi^\lambda(f_2, \chi_2)$ then satisfy*

$$\Phi_1 \, \Phi_2 = \pm e^{2\pi i \lambda \, \varepsilon(\mathcal{O}_1, \mathcal{O}_2)} \, \Phi_2 \, \Phi_1, \tag{IV.72}$$

where the sign ± 1 depends on whether we started with a bosonic or fermionic field and $\varepsilon(\mathcal{O}_1, \mathcal{O}_2)$ is defined for spacelike separated regions according to

$$\varepsilon(\mathcal{O}_1, \mathcal{O}_2) = \begin{cases} +1, & \text{if } \mathcal{O}_1 \subset (\mathcal{O}_2)'_L \\ -1, & \text{if } \mathcal{O}_1 \subset (\mathcal{O}_2)'_R \end{cases}. \tag{IV.73}$$

Proof. This follows immediately by commutativity of the operators $\hat{\Gamma}(V_\chi^\lambda)$, the disorder property (IV.70) and the (anti-)locality of the free Wightman field $\Phi(f)$. □

In addition the field Φ^λ is also covariant w.r.t. the representation $U = \hat{\Gamma}(U_1)$, i.e.

$$\begin{aligned} U(a,\Lambda)\Phi_B^\lambda(f,\chi)U(a,\Lambda)^{-1} &= \Phi_B^\lambda\left(\alpha_{(a,\Lambda)}f, \alpha_{(a,\Lambda)}\chi\right), \\ U(a,\Lambda)\Phi_F^\lambda(f,\chi)U(a,\Lambda)^{-1} &= e^{-\mu/2}\,\Phi_F^\lambda\left(\alpha_{(a,\Lambda)}f, \alpha_{(a,\Lambda)}\chi\right). \end{aligned} \tag{IV.74}$$

By construction the operators Φ^λ raise the charge of a Fock space vector by one and by considering the conjugate field

$$(\Phi^\lambda)^*(f,\chi) = \hat{\Gamma}(V_\chi^\lambda)^* \Phi(f)^*$$

we also get operators which can lower the charge. Moreover a straightforward calculation shows that the field and its conjugate satisfy commutation relations of the form

$$\Phi_1 \Phi_2^* = \pm e^{-2\pi i \lambda \varepsilon(\mathcal{O}_1, \mathcal{O}_2)} \Phi_2^* \Phi_1, \tag{IV.75}$$

if Φ_1 is localized in $\mathcal{O}_1$ and Φ_2^* in $\mathcal{O}_2$.

Summing up we have seen that it is possible to define for every double cone operators with anyonic statistics by "deforming" the massive free Klein-Gordon or Dirac field with unitary disorder operators. The new fields are still compactly localized and the statistics factor $e^{\pm 2\pi i \lambda}$ of two spacelike separated fields explicitly depends on which of the fields is localized to the left or right of the other. This is characteristic for anyonic commutation relations in two dimensions because only there the notion of a right or left spacelike complement is a meaningful Poincaré invariant concept.

IV.3. The Massless Case: Localized Morphisms

In the case $m = 0$ it is well known that the subspaces of positive and negative chirality (i.e. the left and right movers) decouple for the free theories in $d = 1 + 1$ we are considering here. In the Dirac case this is evident because there the Hamiltonian is simply a diagonal matrix. The one-particle Hilbert space then splits into two subspaces that are invariant under the Hamiltonian H and the projections $P_\pm$. Furthermore the parametrization via the rapidity takes the form

$$\begin{pmatrix} \omega_p(\theta) \\ p(\theta) \end{pmatrix} = \begin{pmatrix} e^\theta \\ e^\theta \end{pmatrix}, \tag{IV.76}$$

for chirality $+1$ (i.e. on the Hilbert space for right moving particles) and

$$\begin{pmatrix} \omega_p(\theta) \\ p(\theta) \end{pmatrix} = \begin{pmatrix} e^{-\theta} \\ -e^{-\theta} \end{pmatrix}, \tag{IV.77}$$

for chirality -1. As shown also in [81] mapping the multiplication operator $(\check{A}\check{\varphi})(x) = sgn(x)\check{\varphi}(x)$ to the one-particle Hilbert space for zero mass leads to the same operators $A_{F/B}$ we defined in the previous chapter for $m > 0$. Without doing the explicit calculation this is plausible because the integral kernels $A_{F/B}(\theta)$ depend only on the dimensionless parameter θ. However, as we have already seen in the introduction to this chapter, considering a smeared sign function, which would lead to a disorder operator on Fock space, does not result in an implementable Bogoliubov transformation if the mass is equal to zero. Moreover, smearing the operator A on the one-particle space $\mathcal{H}_1$ with a two-dimensional test function $\chi \in \mathcal{S}(\mathbb{R}^2)$ does also not help making the off-diagonal elements Hilbert-Schmidt operators. This can be illustrated by considering the integral

$$\int d\theta_1 d\theta_2 \, |\tilde{\chi}(\omega(\theta_1) + \omega(\theta_2), p(\theta_1) + p(\theta_2))|^2 \leq c \int d\theta_1 d\theta_2 \frac{1}{1 + 2(e^{s\theta_1} + e^{s\theta_2})^2}$$

which clearly diverges for $s = \pm 1$ in contrast to the massive case.

These considerations show that in the massless case we cannot change the statistics of the free field algebras acting on the respective Fock spaces by simply adjoining disorder operators to the free fields. So we will need a different strategy here and in the following we are going to focus on the fermionic case only, so the operators in consideration will be self-adjoint or unitary. Besides, we will first construct a chiral theory localized in intervals on $\mathbb{R}$ which is covariant under translations and dilations. After that we can "glue together" two such theories to obtain a massless net of field algebras localized in double cones on $\mathbb{R}^{1+1}$ and covariant under the full Poincaré group in $1 + 1$ dimensions. This can be done in the following way.

First we note that double cones in $\mathbb{R}^{1+1}$ are in one-to-one correspondence with pairs of intervals on $\mathbb{R}$. This relationship is demonstrated in the following picture where the double cone $\mathcal{O}_{1,2}$ is defined through the intervals I_1 and I_2 on the line.

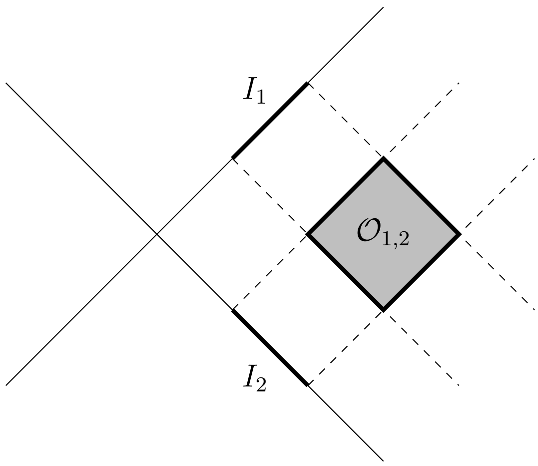


Figure IV.2.: *Correspondence between two intervals and a double cone*

Translated intervals $I_1 + x_+$, $I_2 + x_-$ then lead to a translated double cone

$$\mathcal{O} + \frac{1}{2}\begin{pmatrix} x_+ + x_- \\ x_+ - x_- \end{pmatrix} \subset \mathbb{R}^{1+1},$$

where $x_\pm := x_0 \pm x_1$ are *light-cone coordinates.* In these coordinates the Minkowski scalar product turns out to be

$$x_0 y_0 - x_1 y_1 = \frac{1}{2}(x_+ y_- + x_- y_+) \tag{IV.78}$$

and in particular $\omega_p^2 - p^2 = p_+ p_- = 0$ which means that on the light cone one of the coordinates p_+ or p_- vanishes.

Next we consider the matrix Λ_μ for a boost with boost-parameter $\mu \in \mathbb{R}$ which is given by

$$\Lambda_\mu = \begin{pmatrix} \cosh\mu & \sinh\mu \\ \sinh\mu & \cosh\mu \end{pmatrix}.$$

We can see that a boost acts on the light-cone coordinates according to

$$\Lambda_\mu \begin{pmatrix} p_+ + p_- \\ p_+ - p_- \end{pmatrix} = \begin{pmatrix} e^{\mu} p_+ + e^{-\mu} p_- \\ e^{\mu} p_+ - e^{-\mu} p_- \end{pmatrix},$$

i.e. on p_+ and p_- it just acts as a dilation multiplying with $e^{\pm\mu}$. This shows in particular that the dilated intervals $I_1^{\mu}, I_2^{-\mu}$, where $I^{\pm\mu} := e^{\pm\mu} I$, correspond to the boosted double cone $\Lambda_\mu \mathcal{O}_{1,2}$. So in order to recover a Poincaré covariant theory on $\mathbb{R}^2$ we need two theories on $\mathbb{R}$ which are covariant under translations and dilations. From the Poincaré group relation $\mathcal{U}(\Lambda)\mathcal{U}(a)\mathcal{U}(\Lambda)^* = \mathcal{U}(\Lambda a)$ for $a \in \mathbb{R}^2$ it then follows that these representations need to satisfy

$$U^d(\mu) U(w) U^d(\mu)^* = U\left(e^{\mu} w\right), \quad \mu, w \in \mathbb{R} \tag{IV.79}$$

where U^d denotes the representation of the dilations and the representation of the translations is simply denoted by U.

Now imagine one has such a local net of field algebras $I \mapsto \mathcal{F}(I)$ for anyons on the line acting on a Hilbert space $\mathcal{H}$, which is covariant with respect to a representation of the translations and dilations. We can then simply define the theory on $\mathbb{R}^{1+1}$ as the tensor product of two theories on $\mathbb{R}$ according to

$$\begin{aligned} \mathscr{H} &:= \mathcal{H} \otimes \mathcal{H}, \\ \mathcal{U}(a_0, a_1) &:= U(a_-/2) \otimes U(a_+/2), \\ \mathcal{U}(\Lambda_\mu) &:= U^d(\mu) \otimes U^d(-\mu), \\ \mathscr{F}(\mathcal{O}_{1,2}) &:= \mathcal{F}(I_1) \otimes \mathcal{F}(I_2), \end{aligned} \tag{IV.80}$$

where $\mathcal{O}_{1,2}$ is again the double cone corresponding to the intervals I_1 and I_2 (see also [77] for an analysis of the statistics of space-time fields as a product of light-cone fields). The unitaries $\mathcal{U}$ then yield a representation of $\mathcal{P}_+^\uparrow$ and to get a physically relevant theory we need to make sure that the generators of the translations on $\mathscr{H}$ have joint spectrum in the forward light cone. In addition, to call the resulting theory "massless" this spectrum should actually "fill up" the whole closure of the light cone $\overline{V^+}$.

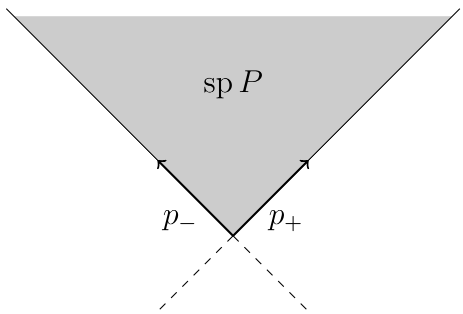


Figure IV.3.: *Spectrum of the generators of translations in a massless theory.*

Furthermore the fields in $\mathscr{F}$ inherit the statistics of the chiral algebras $\mathcal{F}$ in the following sense. On a Hilbert space $\mathcal{H} = \oplus \mathcal{H}_q$ we will construct field operators $F^c(I)$ which change the charge of a vector by the number $c \in \mathbb{Z}$ and are localized in the interval $I \subset \mathbb{R}$. For non-overlapping intervals I_1 and I_2 these operators will then satisfy commutation relations of the form

$$F^{c_1}(I_1) F^{c_2}(I_2) = e^{i\pi\nu c_1 c_2\, \varepsilon(I_1, I_2)} F^{c_2}(I_2) F^{c_1}(I_1),$$

where $\nu \in \mathbb{R}$ is a real parameter determining the statistics of the net and $\varepsilon = \pm 1$ is a sign depending on the relative localization of the intervals I_1, I_2. An important observation now is that for two double cones which are spacelike separated also the intervals corresponding to them have to be non-overlapping.

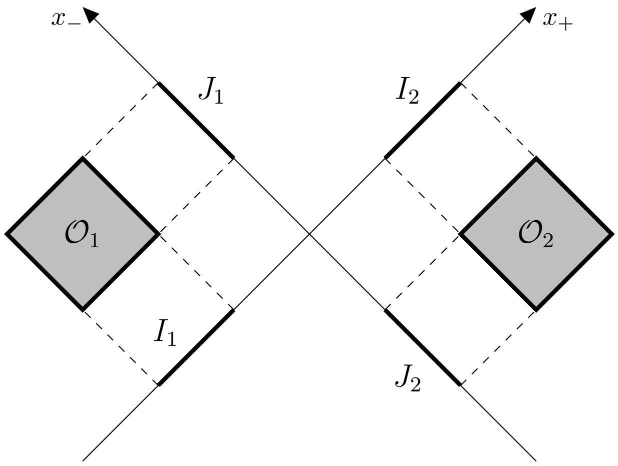


Figure IV.4.: $\mathcal{O}_2$ *in the right spacelike complement of* $\mathcal{O}_1$ *and corresponding intervals*

Consequently, for $I_1 \cap I_2 = J_1 \cap J_2 = \emptyset$, we can define two fields acting on $\mathscr{H}$ according to

$$\Phi_1 := F^{b_1}(I_1) \otimes F^{c_1}(J_1),$$
$$\Phi_2 := F^{b_2}(I_2) \otimes F^{c_2}(J_2),$$

localized in the spacelike separated double cones $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively. Their statistics is then defined by the commutation relations of the fields F in the sense that

$$\Phi_1 \Phi_2 = e^{i\pi\nu(b_1 b_2 - c_1 c_2)\varepsilon(\mathcal{O}_1, \mathcal{O}_2)} \Phi_2 \Phi_1,$$

where the number ε again measures if $\mathcal{O}_2$ lies in the right or left spacelike complement of $\mathcal{O}_1$.

Summing up we have seen that we can define a Poincaré covariant massless theory for anyons on 1+1 dimensional Minkowski space if we have given a translation and dilation covariant field net on the line with non-negative generator of the translations. In the following we are going to explain how such a net of operator algebras can be defined, using again the theory of loop groups [76,86] and implementable Bogoliubov transformations.

Our position space is $L^2(\mathbb{R}, dx)$ and the Dirac operator on it is simply the generator of translations $\check{H} = -i\frac{d}{dx}$. Fourier transformation[3] leads to a "diagonalization" of $\check{H}$ on the one-particle Hilbert space

$$\mathcal{H}_1 = L^2\left(\mathbb{R}, \frac{dp}{|p|}\right) \simeq \mathcal{H}_1^+ \oplus \mathcal{H}_1^-, \tag{IV.81}$$

where the projections $P_\pm$ onto $\mathcal{H}_1^\pm$ are given by

$$(P_\pm \tilde{\varphi})(p) = \Theta(\pm p)\tilde{\varphi}(p). \tag{IV.82}$$

[3] Remember that we use the convention $\tilde{\varphi}(p) = \frac{1}{\sqrt{2\pi}} \int dx \varphi(x) e^{-ixp}$ for the Fourier transformation.

On $\mathcal{H}_1$ we define a unitary representation of the translations U_1 and the dilations U_1^d according to

$$\begin{aligned}(U_1(w)\tilde{\varphi})(p) &:= e^{-iwp}\tilde{\varphi}(p), \quad w \in \mathbb{R},\\ (U_1^d(\mu)\tilde{\varphi})(p) &:= \tilde{\varphi}\left(e^{\mu}p\right), \quad \mu \in \mathbb{R}.\end{aligned} \tag{IV.83}$$

It can be seen immediately that this definition satisfies the requirement

$$U_1^d(\mu)U_1(w)U_1^d(\mu)^* = U_1\left(e^{\mu}w\right),$$

and that the representations U_1 and U_1^d commute with the projectors $P_\pm$. If we now consider a multiplication operator

$$(V\varphi)(x) := v(x)\varphi(x), \tag{IV.84}$$

the Hilbert-Schmidt condition for the off-diagonal elements reads

$$\begin{aligned}||V_{+-}||_2^2 &\propto \int dpdq\, \Theta(p)\Theta(-q)|\tilde{v}(p-q)|^2\\ &= \int_0^\infty p\,|\tilde{v}(p)|^2.\end{aligned} \tag{IV.85}$$

So by lemma IV.7 and the subsequent discussion we know that the operator V is an implementable unitary if v is of the form $v(x) = e^{if(x)}$ with a smooth real-valued function f satisfying

$$f(\infty) - f(-\infty) = 2\pi q(f),$$

where the "winding number" $q(f)$ has to be an integer. Besides, from lemma IV.8 we know that the implementer $\hat{\Gamma}(e^{if})$[4] changes the charge of a Fock space vector by the number $q(f)$ [17, 18, 22]. Smooth functions α satisfying $q(\alpha) = 0$, i.e. $\alpha(\infty) = \alpha(-\infty)$ are then elements of the Lie algebra $\mathfrak{g}_F(\mathcal{H}_1)$.

All these implementable multiplication operators obviously commute on the one-particle level and in the simple setting we are working here it is now possible to explicitly calculate the phase factors in the relations

$$\begin{aligned}\hat{\Gamma}(e^{if})\hat{\Gamma}(e^{ig}) &= e^{i\sigma(f,g)}\,\hat{\Gamma}(e^{i(f+g)}),\\ \Rightarrow \hat{\Gamma}(e^{if})\hat{\Gamma}(e^{ig}) &= e^{i\check{\sigma}(f,g)}\,\hat{\Gamma}(e^{ig})\hat{\Gamma}(e^{if}),\end{aligned} \tag{IV.86}$$

where the real number $\check{\sigma}(f,g) := \sigma(f,g) - \sigma(g,f)$ is independent of the phase choice for the implementers and determines their commutation relations. The cocycle $e^{i\sigma(f,g)}$, however, depends on this choice of phase factor and we won't need its explicit form here. The only relation which will be important in calculations is the 2-cocycle identity which amounts to

$$\sigma(f,g) + \sigma(f+g,h) = \sigma(f,g+h) + \sigma(g,h) \tag{IV.87}$$

[4] In the following we will always identify a multiplication operator with the function it multiplies with to simplify the notation.

and the fact that it is possible to choose the phases in such a way that $\sigma(f,f)=0$, which we will always assume in the following. To calculate an explicit expression for $\check{\sigma}(f,g)$ for arbitrary f and g we first examine the case $\alpha,\beta\in\mathfrak{g}(\mathcal{H}_1)$.

Lemma IV.14. *For $\alpha,\beta\in\mathfrak{g}(\mathcal{H}_1)$, i.e. $q(\alpha)=q(\beta)=0$, the number $\check{\sigma}(\alpha,\beta)$ is just the Schwinger term which takes the simple form*

$$\check{\sigma}(\alpha,\beta)=\frac{1}{2\pi}\int dx\ \alpha(x)\beta'(x)\ =\frac{1}{4\pi}\int dx\left(\alpha(x)\beta'(x)-\alpha'(x)\beta(x)\right). \tag{IV.88}$$

Proof. Using the definition $\check{\sigma}(\alpha,\beta)=iTr(\alpha_{-+}\beta_{+-}-\beta_{-+}\alpha_{+-})$ of the Schwinger term from lemma IV.1 and the projections (IV.82) we calculate

$$\begin{aligned}iTr(\alpha_{-+}\beta_{+-}-\beta_{-+}\alpha_{+-})&=\frac{i}{2\pi}\int_{-\infty}^{0}dp\int_0^\infty dq\left(\tilde{\alpha}(p-q)\tilde{\beta}(q-p)-\tilde{\beta}(p-q)\tilde{\alpha}(q-p)\right)\\&=\frac{i}{2\pi}\int_0^\infty dq\int_{-\infty}^{-q}dp\left(\tilde{\alpha}(p)\tilde{\beta}(-p)-\tilde{\beta}(p)\tilde{\alpha}(-p)\right)\\&=-\frac{i}{2\pi}\int_0^\infty dp\,p\left(\tilde{\alpha}(p)\tilde{\beta}(-p)-\tilde{\beta}(p)\tilde{\alpha}(-p)\right)\\&=-\frac{i}{2\pi}\int dp\,p\,\tilde{\alpha}(p)\tilde{\beta}(-p)=-\frac{1}{2\pi}\int dp\,\widetilde{(\alpha')}(p)\tilde{\beta}(-p),\end{aligned}$$

where we used $\int_0^\infty dq\int_q^\infty dpI(p)=\int_0^\infty dp\,pI(p)$ to get from the second line to the third. Inverse Fourier transformation then leads to the simple expression in position space

$$\begin{aligned}-\frac{1}{2\pi}\int dp\,\widetilde{(\alpha')}(p)\tilde{\beta}(-p)&=-\frac{1}{2\pi}\int dx\,\alpha'(x)\beta(x)\\&=\frac{1}{4\pi}\int dx\left(\alpha(x)\beta'(x)-\alpha'(x)\beta(x)\right),\end{aligned}$$

where potential boundary terms vanish because $q(\alpha)=q(\beta)=0$. □

In the next step we will take a closer look at charge shifting operators. Consider for this purpose the orthonormal basis $\{b_n|n\in\mathbb{Z}\}$ for $L^2(\mathbb{R},dx)$ with

$$b_n(x):=\frac{1}{\sqrt{\pi}}\frac{1}{x-i}e^{in\eta(x)},\qquad \eta(x):=\pi+2\arctan(x), \tag{IV.89}$$

where we call η the *"standard kink"* (see [22]). It can then be checked that the Fourier transform of this basis satisfies

$$\operatorname{supp}\tilde{b}_n\subset\begin{cases}\mathbb{R}_+, & \text{if } n\geq 0\\ \mathbb{R}_-, & \text{if } n<0\end{cases}, \tag{IV.90}$$

so we can define an orthonormal basis e_n for $\mathcal{H}_1$ according to $e_n(p):=\sqrt{|p|}\,\tilde{b}_n(p)$ such that $e_n\in\mathcal{H}_1^+$ for $n\geq 0$ and $e_n\in\mathcal{H}_1^-$ for $n<0$.

If we now consider the multiplication operator $V := e^{i\eta}$ it can easily be seen that this leads to an implementable Bogoliubov transformation because η is a smooth function with $q(\eta) = 1$ and furthermore V acts as a shift operator for the basis b_n in the sense that

$$Vb_n = b_{n+1}.$$

This shows in particular that the component V_{-+} vanishes and $e_{-1} \in \ker V_{--}$ so by lemma IV.3 we know that the implementer $\hat{\Gamma}(e^{i\eta})$ creates the one-particle vector e_0 from the vacuum. This now allows us to calculate the commutation relations between $\hat{\Gamma}(e^{i\eta})$ and an operator of the form $\hat{\Gamma}(e^{i\alpha})$ with $\alpha \in \mathfrak{g}(\mathcal{H}_1)$ because lemma IV.2 tells us that $\check{\sigma}(\alpha, \eta) = \langle \hat{\Gamma}(e^{i\eta})\Omega, d\hat{\Gamma}(\alpha)\hat{\Gamma}(e^{i\eta})\Omega \rangle$ which in this case simplifies to

$$\check{\sigma}(\alpha, \eta) = \langle e_0, \alpha\, e_0 \rangle = \frac{1}{\pi} \int dx\, \frac{\alpha(x)}{1+x^2} = \frac{1}{2\pi} \int dx\, \alpha(x)\eta'(x).$$

We are now ready to compute the exchange factor $\check{\sigma}(f, g)$ for arbitrary implementable functions f and g.

Lemma IV.15. *For smooth real-valued functions f, g with $q(f), q(g) \in \mathbb{Z}$ the phase factor $\check{\sigma}(f, g)$ takes the form*

$$\check{\sigma}(f, g) = \frac{1}{4\pi} \left[\int dx\, \big(f(x)g'(x) - f'(x)g(x)\big) + f(-\infty)g(\infty) - f(\infty)g(-\infty) \right]. \quad \text{(IV.91)}$$

Proof. First note that we can use the standard kink η to write the functions f, g as $f = q(f)\eta + \alpha$ and $g = q(g)\eta + \beta$ with some $\alpha, \beta \in \mathfrak{g}(\mathcal{H}_1)$. Next we can use the linearity of $\check{\sigma}$, which already follows from the 2-cocycle identity of σ, to write

$$\check{\sigma}(f, g) = q(g)\check{\sigma}(\alpha, \eta) - q(f)\check{\sigma}(\beta, \eta) + \check{\sigma}(\alpha, \beta).$$

Now we already calculated each of these individual terms which leads us to

$$\check{\sigma}(f, g) = \frac{1}{4\pi} \int dx\, \left[\alpha(x)\left(\beta'(x) + 2q(g)\eta'(x)\right) - \beta(x)\left(\alpha'(x) + 2q(f)\eta'(x)\right)\right]$$

and partial integration, $\int dx\, \alpha(x)\eta'(x) = 2\pi f(-\infty) - \int dx\, \alpha'(x)\eta(x)$, then results in

$$\begin{aligned}
\check{\sigma}(f, g) =& \frac{1}{4\pi} \int dx\, \left[\alpha(x)\left(\beta'(x) + q(g)\eta'(x)\right) - q(g)\alpha'(x)\eta(x)\right] + \frac{1}{2} f(-\infty)q(g) \\
&- \frac{1}{4\pi} \int dx\, \left[\beta(x)\left(\alpha'(x) + q(f)\eta'(x)\right) - q(f)\beta'(x)\eta(x)\right] - \frac{1}{2} g(-\infty)q(f) \\
=& \frac{1}{4\pi} \left[\int dx\, (\alpha(x) + q(f)\eta(x))\left(\beta'(x) + q(g)\eta'(x)\right) + f(-\infty)g(\infty) \right] \\
&- \Big(\text{term with } f \leftrightarrow g \text{ and } \alpha \leftrightarrow \beta\Big).
\end{aligned}$$

Reinserting the definition of α and β then leads to the desired relation. □

Remark: The various expressions e.g. for the Schwinger term or the Hilbert-Schmidt condition do not depend on the choice of measure ($\frac{dp}{|p|}$ in our case) because the splitting of $\mathfrak{h}_1$ does not depend on it.

For the sake of simplicity we will in the following mostly work with functions which satisfy $f(-\infty) = 0$, since a constant phase factor $e^{if(-\infty)} = e^{i\cdot const}$ is always implementable trivially as $\hat{\Gamma}(e^{i\cdot const}) = e^{i\cdot const\cdot Q}$ and can be factored out[5]. This means in particular that $2\pi q(f) = f(\infty)$ and Lie algebra elements $\alpha \in \mathfrak{g}(\mathcal{H}_1)$ then go to zero at infinity. Additionally the term $f(-\infty)g(\infty) - f(\infty)g(-\infty)$ in the expression for $\check{\sigma}(f,g)$ vanishes. Now consider two implementable functions f_1 and f_2 which vary only in finite regions,

$$\operatorname{supp} f_1' \subset I_1 \subset \mathbb{R}, \qquad \operatorname{supp} f_2' \subset I_2 \subset \mathbb{R},$$

where I_1 and I_2 are non-overlapping intervals in $\mathbb{R}$, i.e. $I_1 \cap I_2 = \emptyset$. In this case the exchange phase $\check{\sigma}(f_1, f_2)$ simplifies considerably, namely we get

$$\check{\sigma}(f_1, f_2) = \pi q_1 q_2\, sgn(f_1, f_2), \tag{IV.92}$$

where $sgn(f_1, f_2) = \pm 1$ is a sign depending on the relative location of the intervals I_1, I_2. This can be checked immediately by noting that

$$\int dx\, \big(f_1(x) f_2'(x) - f_1'(x) f_2(x)\big) = \begin{cases} +2\pi q_2 f_1(\infty), & \text{if } I_1 \text{ to the left of } I_2 \\ -2\pi q_1 f_2(\infty), & \text{if } I_1 \text{ to the right of } I_2 \end{cases}.$$

For the commutation relations of the implementers this then implies

$$\hat{\Gamma}(e^{if_1})\hat{\Gamma}(e^{if_2}) = (-1)^{q_1 q_2}\hat{\Gamma}(e^{if_2})\hat{\Gamma}(e^{if_1}), \tag{IV.93}$$

meaning that they either commute or anti-commute depending on the charges q_1, q_2 they carry. We can therefore identify $\operatorname{supp} f'$ as the *"localization region"* of an operator e^{if} or $\hat{\Gamma}(e^{if})$ respectively. So because only charge shifts with integer winding number are implementable on the Fock space we can only end up with fermionic (or bosonic) field nets, in contrast to the massive case where disorder operators could be defined for any parameter $\lambda \in \mathbb{R}$.

To obtain fields with anyonic commutation relations we now first want to "take a step back" and look at the above construction from a more abstract algebraic point of view. For functions α with $q(\alpha) = 0$, i.e. $\alpha \in \mathfrak{g}(\mathcal{H}_1)$, we know that their implementers $\hat{\Gamma}(e^{i\alpha})$ leave the *vacuum Hilbert space*

$$\mathcal{H}_0 = \mathcal{F}_a(\mathcal{H}_1)|_{Q=0} \tag{IV.94}$$

[5] Another motivation for this is that in the vacuum representation, which we will consider below, the factor $e^{i\cdot const\cdot Q}$ is the identity so adding a constant to a function f does not change the implementer in the vacuum sector.

invariant. With the cocyle $\sigma(\alpha,\beta)$ from before we then get a representation of the *Weyl algebra* on this vacuum Hilbert space. Using the fact that $\check{\sigma}(\alpha,\beta)=0$ if $\operatorname{supp}\alpha\cap\operatorname{supp}\beta=\emptyset$ we can again assign a localization region to every element of this algebra such that operators with support in non-intersecting intervals commute. Together with the second-quantized representation of the translations

$$U_0(w) := \hat{\Gamma}(U_1(w)) \;, \qquad \text{restricted to } \mathcal{H}_0$$

we have a local translation covariant net of algebras in the vacuum representation, which we call our "observable algebras". The algebra $\mathcal{A}(I)$ for a given interval $I\subset\mathbb{R}$ is then generated by the elements $\hat{\Gamma}(e^{i\alpha})$ for all $\alpha\in\mathfrak{g}(\mathcal{H}_1)$ such that $\operatorname{supp}(\alpha)\subset I$.

If we pretend for a moment that charge shifts $\hat{\Gamma}(e^{if})$ with non-integer winding number $q(f)$ were implementable, they would lead to relations of the form

$$\begin{aligned}\hat{\Gamma}(e^{if})^*\,\hat{\Gamma}(e^{i\alpha})\,\hat{\Gamma}(e^{if}) &= e^{i\check{\sigma}(\alpha,f)}\,\hat{\Gamma}(e^{i\alpha}),\\ \hat{\Gamma}(e^{if})^*\,U_0(w)\,\hat{\Gamma}(e^{if}) &= \hat{\Gamma}(e^{i(f_w-f)})\,U_0(w).\end{aligned} \tag{IV.95}$$

The important observation now is that although the implementers $\hat{\Gamma}(e^{if})$ do not exist as unitary operators on the Fock space, the map $\alpha\mapsto\check{\sigma}(\alpha,f)\in\mathbb{R}$ is still defined for every $\alpha\in\mathfrak{g}(\mathcal{H}_1)$ as long as f is a smooth bounded function with well-defined limits $f(\pm\infty)=\lim_{x\to\pm\infty}f(x)$. Motivated by this we can now define for every function f, which is admissible in this sense, an *automorphism* ρ_f of our observable algebra on the vacuum Hilbert space $\mathcal{H}_0$ by simply defining

$$\rho_f(\hat{\Gamma}(e^{i\alpha})) := e^{i\check{\sigma}(\alpha,f)}\hat{\Gamma}(e^{i\alpha}). \tag{IV.96}$$

This can also be seen as defining a new *representation* of the algebra (see section II.1) and from the definition of $\check{\sigma}$ and the considerations about "localization regions" of functions mentioned above it follows that

$$\rho_f(\hat{\Gamma}(e^{i\alpha})) = \hat{\Gamma}(e^{i\alpha}) \;, \quad \text{if } \operatorname{supp}\alpha\cap\operatorname{supp}f'=\emptyset,$$

which means that ρ_f simply acts as the identity on elements localized in the complement of $\operatorname{supp}f'$. We therefore say that the automorphism ρ_f is *localized in* $\operatorname{supp}f'$ and we only want to consider morphisms which are localized in a finite interval. This kind of localization concept for morphisms of nets of algebras (or representations equivalently) has been introduced in the DHR theory of localized charges [24, 25] and later extended in [9] to non-compact localization regions. However, we are not interested here in the whole algebraic structure and possible representations of the Weyl algebra but simply utilize the general theory of AQFT as a guide to explicitly construct local field algebras with anyonic statistics acting on a concrete Hilbert space.

In the next step we want to reduce the huge redundancy in this formalism by identifying those functions which lead to unitarily equivalent morphisms. For this purpose we introduce the concept of *intertwiners* and we call two localized automorphisms ρ_f and ρ_g unitarily equivalent if there exists an element $V_{f,g}$ of a local observable algebra such that

$$V_{f,g}\, \rho_f(\hat{\Gamma}(e^{i\alpha}))V_{f,g}^* = \rho_g(\hat{\Gamma}(e^{i\alpha})), \quad \forall\alpha \in \mathfrak{g}(\mathcal{H}_1). \tag{IV.97}$$

The following lemma then classifies the equivalence classes of all automorphisms of the form IV.96.

Lemma IV.16. *Two localized morphisms ρ_f and ρ_g are unitarily equivalent if and only if the winding numbers of f and g coincide. In this case an intertwiner between ρ_f and ρ_g can be defined (up to an arbitrary phase factor) according to*

$$V_{f,g} = \hat{\Gamma}(e^{i(f-g)}). \tag{IV.98}$$

Proof. If $q(f) = q(g)$ then $(f-g) \in \mathfrak{g}(\mathcal{H}_1)$ so the one-particle operator $e^{i(f-g)}$ is clearly implementable on the vacuum Hilbert space and it can easily be checked that $V_{f,g}$ is indeed a unitary intertwiner between ρ_f and ρ_g.

On the other hand, if $q(f) \neq q(g)$, assume that there exists a unitary operator V in some local algebra such that

$$V e^{i\breve{\sigma}(\alpha,f)}\hat{\Gamma}(e^{i\alpha})V^* = e^{i\breve{\sigma}(\alpha,g)}\hat{\Gamma}(e^{i\alpha}), \quad \forall\alpha \in \mathfrak{g}(\mathcal{H}_1) \text{ with compact support.}$$

Taking the vacuum expectation value of this expression and using that $\langle\Omega, \hat{\Gamma}(e^{i\alpha})\Omega\rangle = 1$ this leads to

$$\langle\Omega, V\,\hat{\Gamma}(e^{i\alpha})V^*\Omega\rangle = e^{i\breve{\sigma}(\alpha,g-f)}.$$

Now consider a smooth function α^ν with compact support such that $\alpha^\nu(x) = \nu \in \mathbb{R}$ for $|x| < 1$ and the sequence of functions $\alpha_n^\nu(x) := \alpha^\nu(x/n)$ for $n \in \mathbb{N}$. The algebra element $\hat{\Gamma}(e^{i\alpha_n^\nu})$ then commutes with every element localized in the interval $(-n, n)$ so in the limit $n \to \infty$ it commutes with *every* local algebra. This means in particular that

$$\langle\Omega, V\,\hat{\Gamma}(e^{i\alpha_n^\nu})V^*\Omega\rangle \underset{n\to\infty}{\longrightarrow} 1.$$

But on the other hand we get

$$\breve{\sigma}(\alpha_n^\nu, g-f) \underset{n\to\infty}{\longrightarrow} \frac{1}{4\pi}\int dx\, \alpha_n^\nu(x)\left[g'(x) - f'(x)\right] \underset{n\to\infty}{\longrightarrow} \frac{\nu}{2}[q(g) - q(f)].$$

Since this works for every $\nu \in \mathbb{R}$ a unitary intertwiner V between ρ_f and ρ_g can only exist if $q(f) = q(g)$. □

We call such an equivalence class a "sector" of our theory and the family of sectors we obtain in this way is thus labelled by the winding number $q \in \mathbb{R}$.

Inspired by the fictitious relations (IV.95) we can now in addition define for every ρ_f on $\mathcal{H}_0$ a unitary representation U_f of the translations according to

$$\begin{aligned} U_f(w) &:= Y_f(w)U_0(w) \ , \qquad \forall w \in \mathbb{R} \\ Y_f(w) &:= e^{-i\sigma(f,f_w)}\hat{\Gamma}(e^{i(f_w - f)}), \end{aligned} \tag{IV.99}$$

where Y_f evidently is an intertwiner between ρ_f and the translated morphism ρ_{f_w}. The additional phase factor $e^{-i\sigma(f,f_w)}$ makes sure that U_f really is a representation of $\mathbb{R}$ and it satisfies

$$U_f(w)\rho_f(\hat{\Gamma}(e^{i\alpha}))U_f(w)^* = \rho_f(\hat{\Gamma}(e^{i\alpha_w})).$$

Denoting by $\operatorname{Ad} U$ the adjoint action of a unitary U this can also be written as

$$\operatorname{Ad} U_f \circ \rho_f = \rho_f \circ \operatorname{Ad} U_0.$$

The object that is most interesting for our construction is the *statistics operator* ε for a sector ρ_f [24], which will later determine the commutation relations of charge carrying fields. For its definition we first choose two admissible "spectator functions" h_1 and h_2 with the same winding number as f (i.e. ρ_{h_1}, ρ_{h_2} and ρ_f are unitarily equivalent) and which have non-intersecting localization regions. With the respective intertwiners V_{f,h_1}, V_{f,h_2} one can then define the statistics operator $\varepsilon(\rho_{h_1}, \rho_{h_2})$ according to

$$\varepsilon(\rho_{h_1}, \rho_{h_2}) = \rho_f(V_{f,h_1}^{-1})V_{f,h_2}^{-1}V_{f,h_1}\rho_f(V_{f,h_2}). \tag{IV.100}$$

In the simple setting we are working here one can now easily calculate this operator and show that it is independent of the function f appearing in the above definition.

Lemma IV.17. *Let h_1 and h_2 be admissible spectator functions with the same winding number $q(h_1) = q(h_2) = q \in \mathbb{R}$, localized in non-intersecting intervals. The statistics operator defined in equation* (IV.100) *then takes the form*

$$\varepsilon(\rho_{h_1}, \rho_{h_2}) = e^{i\tilde{\sigma}(h_1,h_2)} \cdot \mathbb{1} = e^{i\pi q^2\, sgn(h_1,h_2)} \cdot \mathbb{1}.$$

Proof. Straightforward calculation using the definitions of ρ and V. □

This shows that the statistics parameter only depends on the winding number q of the sector in consideration and the relative localization of h_1 and h_2.

Up to now all the relevant representations are defined on the same Hilbert space $\mathcal{H}_0$ and are inequivalent for different winding numbers. To obtain a net of field algebras with fixed statistics defined on a separable Hilbert space we will from now on consider a fixed *reference morphism* ρ_h where the function h has winding number $\gamma \in \mathbb{R}$. Sectors with higher charge $q \in \mathbb{Z}$ can then be reached by applying this morphism q times, which we write as $\rho_h^q = \rho_h \underbrace{\circ \cdots \circ}_{q} \rho_h$. We then define a *"physical Hilbert space"* according to

$$\mathcal{H} := \bigoplus_{q \in \mathbb{Z}} \mathcal{H}_q, \qquad \mathcal{H}_q \simeq \mathcal{H}_0 \tag{IV.101}$$

and we denote vectors in the sector with charge q by (q, Ψ) where Ψ is an element of $\mathcal{H}_0$. On this Hilbert space we can now define a representation π of the observable algebra acting on $\mathcal{H}_q$ according to

$$\pi(\hat{\Gamma}(e^{i\alpha}))(q, \Psi) := (q, \rho_h^q(\hat{\Gamma}(e^{i\alpha}))\Psi), \tag{IV.102}$$

On the total Hilbert space this is simply a direct sum of the representations ρ_h^q for all $q \in \mathbb{Z}$. In the same fashion we define a representation U of the translations by

$$U(w)(q, \Psi) := (q, U_{qh}(w)\Psi), \tag{IV.103}$$

where U_{qh} is just the representation defined in (IV.99).

Due to the direct sum structure the Hilbert spaces for fixed charge are invariant under the representation π. Apart from the charge neutral observables we also want to have elements which shift the charge of a vector. We therefore define on $\mathcal{H}$ for every $c \in \mathbb{Z}$ and admissible function f with winding number γ a *field operator* $F^c(f)$ according to

$$\begin{aligned} F^c(f)(q, \Psi) &:= \left(q + c, \rho_h^q(V^*_{ch,cf})\Psi\right), \\ \text{where} \quad \rho_h^q(V^*_{ch,cf}) &= e^{iqc\check{\sigma}(f,h)}\, \hat{\Gamma}(e^{ic(f-h)}). \end{aligned} \tag{IV.104}$$

This definition is motivated by the reduced field bundle formalism which can be found for example in [31]. Such an operator evidently shifts the charge of a vector by the number $c \in \mathbb{Z}$ and a straightforward calculation shows that the commutation relations of these fields are governed by the statistics operator.

Lemma IV.18. *Field operators $F^{c_1}(f_1)$ and $F^{c_2}(f_2)$ defined on $\mathcal{H}$ as in* (IV.104) *satisfy commutation relations of the form*

$$F^{c_1}(f_1)F^{c_2}(f_2) = e^{ic_1c_2\check{\sigma}(f_1,f_2)}\, F^{c_2}(f_2)F^{c_1}(f_1),$$

which are independent of the reference morphism ρ_h.

Proof. Acting on a vector $(q, \Psi) \in \mathcal{H}_q$ we get

$$F^{c_1}(f_1)F^{c_1}(f_2)(q, \Psi) = \left(q + c_1 + c_2, \rho_h^{q+c_2}(V^*_{c_1h,c_1f_1})\rho_h^q(V^*_{c_2h,c_2f_2})\Psi\right)$$

and for the operator acting on Ψ in this expression the definitions of V and ρ lead to

$$\rho_h^{q+c_2}\left(V^*_{c_1h,c_1f_1}\right)\rho_h^q\left(V^*_{c_2h,c_2f_2}\right) = e^{i(q+c_2)c_1\check{\sigma}(f_1,h)}e^{iqc_2\check{\sigma}(f_2,h)}\,\hat{\Gamma}(e^{ic_1(f_1-h)})\hat{\Gamma}(e^{ic_2(f_2-h)}).$$

By commuting the operators $\hat{\Gamma}(e^{ic_1(f_1-h)})$ and $\hat{\Gamma}(e^{ic_2(f_2-h)})$ and rearranging the phase factors this equals

$$\begin{aligned}
&e^{i(qc_2\check{\sigma}(f_2,h)+c_1c_2\check{\sigma}(f_1,h)+qc_1\check{\sigma}(f_1,h))}e^{ic_1c_2\check{\sigma}(f_1-h,f_2-h)}\,\hat{\Gamma}(e^{ic_2(f_2-h)})\hat{\Gamma}(e^{ic_1(f_1-h)})\\
&= e^{ic_1c_2\check{\sigma}(f_1,f_2)}\, e^{i(q+c_1)c_2\check{\sigma}(f_2,h)}\hat{\Gamma}(e^{ic_2(f_2-h)})\, e^{iqc_1\check{\sigma}(f_1,h)}\hat{\Gamma}(e^{ic_1(f_1-h)})\\
&= e^{ic_1c_2\check{\sigma}(f_1,f_2)}\,\rho_h^{q+c_1}\left(V^*_{c_2h,c_2f_2}\right)\rho_h^{q}\left(V^*_{c_1h,c_1f_1}\right),
\end{aligned}$$

which completes the proof. □

This also shows that the localization region of a field operator is determined by the function used in its definition. Namely, if we consider functions f_1, f_2 with non-overlapping localization regions, i.e. $\operatorname{supp} f_1' \cap \operatorname{supp} f_2' = \emptyset$ we can again use that fact that in this situation

$$\check{\sigma}(f_1,f_2) = \pi\gamma^2 sgn(f_1,f_2),$$

where γ is the fixed winding number of the reference morphism in consideration. For the field operators this then leads to

$$F^{c_1}(f_1)F^{c_2}(f_2) = e^{\pm i\pi\gamma^2 c_1c_2}F^{c_2}(f_2)F^{c_1}(f_1), \tag{IV.105}$$

where the sign depends on which of the operators is localized to the right of the other. A short consideration also shows that the adjoint $F^c(f)^*$ of a field operator is simply given by

$$F^c(f)^* = F^{-c}(f) \tag{IV.106}$$

so they are already included in the set of field operators with arbitrary charge $c \in \mathbb{Z}$.

This construction then leads to a translation-covariant net of field algebras $I \to \mathcal{F}(I)$ localized in intervals on $\mathbb{R}$ where the statistics parameter of the net is determined by the fixed winding number we chose at the beginning $\nu = \gamma^2$.

There are still two things missing in this construction needed to obtain a local covariant massless net on $\mathbb{R}^{1+1}$. Namely, the positivity of the generator of translations and a representation of the dilations. The first issue is addressed by the following proposition.

Proposition IV.4. *The representation U_{qh} of $\mathbb{R}$ defined in* (IV.99) *has a positive generator for every charge $q \in \mathbb{Z}$ and every admissible reference function h.*

Proof. Using the definition of U_{qh} and Y_{qh} the generator turns out to be

$$T_q := i\frac{d}{dw}U_{qh}(w)\Big|_{w=0} = q^2\int dx\, h'(x)^2 + q\, d\hat{\Gamma}(h') + d\hat{\Gamma}(p), \tag{IV.107}$$

where $d\hat{\Gamma}(p)$ is just the generator of the vacuum representation U_0. To show that this is a positive operator for every h with winding number $\gamma \in \mathbb{R}$ we will first consider the case $h = n\,\mathfrak{f}$, where $\mathfrak{f}$ has winding number $q(\mathfrak{f}) = 1$ and $n \in \mathbb{Z}$. In this case we know that $\hat{\Gamma}(e^{in\mathfrak{f}})$

actually exists as an operator on Fock space, so on this larger Hilbert space we can split up the representation U_{qh} to obtain

$$\begin{aligned} U_{qh}(w) &= e^{-iq^2n^2\sigma(\mathfrak{f},\mathfrak{f}_w)}\,\hat{\Gamma}(e^{iqn(\mathfrak{f}_w-\mathfrak{f})})\,U_0(w) \\ &= \hat{\Gamma}(e^{-iqn\,\mathfrak{f}})\,\hat{\Gamma}(e^{iqn\,\mathfrak{f}_w})\,U_0(w) \\ &= \hat{\Gamma}(e^{-iqn\,\mathfrak{f}})\,U_0(w)\,\hat{\Gamma}(e^{iqn\,\mathfrak{f}}). \end{aligned}$$

This shows that on the Fock space U_{qh} is unitarily equivalent to U_0 and hence also the respective generators are connected by the unitary $\hat{\Gamma}(e^{-iqn\,\mathfrak{f}})$ and hence positive. More precisely

$$T_q = q^2n^2\,||\mathfrak{f}'||_2^2 + qn\,d\hat{\Gamma}(\mathfrak{f}') + d\hat{\Gamma}(p) \geq 0, \quad \forall n \in \mathbb{Z}.$$

Now because the terms $q^2n^2\,||\mathfrak{f}'||^2$ and $d\hat{\Gamma}(p)$ are always non-negative we get the inequality

$$nq\,d\hat{\Gamma}(\mathfrak{f}') \geq -n^2q^2||\mathfrak{f}'||^2 - d\hat{\Gamma}(p).$$

Using this estimate in the expression of T_q for arbitrary $h = \gamma\,\mathfrak{f}, \gamma \in \mathbb{R}$ then leads to

$$\begin{aligned} T_q &= q^2\gamma^2||\mathfrak{f}'||^2 + d\hat{\Gamma}(p) + q\gamma\,d\hat{\Gamma}(\mathfrak{f}') \\ &\geq q^2\gamma^2||\mathfrak{f}'||^2 + d\hat{\Gamma}(p) - \frac{\gamma}{n}n^2q^2||\mathfrak{f}'||^2 - \frac{\gamma}{n}\,d\hat{\Gamma}p \\ &= q^2||\mathfrak{f}'||^2\left(\gamma^2 - \gamma n\right) + d\hat{\Gamma}(p)\left(1 - \frac{\gamma}{n}\right). \end{aligned}$$

This holds for every $n \in \mathbb{Z}$ so by choosing $n = -\,\mathrm{sign}(\gamma)$ we see that the generator T_q is indeed positive for every winding number $\gamma \in \mathbb{R}$. □

Moreover, the minimum of the spectrum of T_q lies at zero, because on the vacuum sector $q = 0$ the generator is simply the second quantized generator of the one-particle momentum operator p and the spectrum of $d\hat{\Gamma}(p)$ is $\{p \in \mathbb{R} | p \geq 0\}$. This ensures that the theory we will get on $\mathbb{R}^{1+1}$ is really massless, because the joint spectrum of the generators of space-time translations fills up the whole forward light cone.

In addition to the representation of translations we have also defined a representation of the dilations on the one particle Hilbert space according to

$$(U_1^d(\mu)\tilde{\varphi})(p) := \tilde{\varphi}\,(e^{\mu}p)\,.$$

This representation obviously satisfies $U_1^d(\mu_1)U_1^d(\mu_2) = U_1^d(\mu_1 + \mu_2)$ and on the vacuum Hilbert space $\mathcal{H}_0$ it leads to a second quantized representation $U_0^d(\mu) := \hat{\Gamma}(U_1^d(\mu))$ satisfying

$$\hat{\Gamma}(U_1^d(\mu))\hat{\Gamma}(e^{i\alpha})\hat{\Gamma}(U_1^d(\mu))^* = \hat{\Gamma}(e^{i\alpha^{\mu}})\,, \quad \alpha^{\mu}(x) := \alpha\left(e^{-\mu}x\right).$$

Now because a function h and the "dilated" function h^{μ} obviously have the same winding number the respective morphisms are equivalent and there is an intertwiner $V_{h,h^{\mu}}$ between

ρ_h and ρ_{h^μ}. We can use this to implement the dilations on the charge q Hilbert space in the same way as the translations by defining

$$\begin{aligned} U^d(\mu)(q,\Psi) &= (q, U^d_{qh}\Psi) \\ U^d_{qh}(\mu) &:= e^{-iq^2\sigma(h,h^\mu)}\hat{\Gamma}(e^{iq(h^\mu-h)})U^d_0(\mu). \end{aligned} \tag{IV.108}$$

A short calculation then shows that this definition indeed satisfies property (IV.79) needed to obtain a representation of the Poincaré group on $\mathbb{R}^{1+1}$.

Lemma IV.19. *The representations defined by* (IV.108) *and* (IV.99) *satisfy*

$$U^d_{qh}(\mu)U_{qh}(w)U^d_{qh}(\mu)^* = U_{qh}\left(e^\mu w\right), \qquad \forall w, \mu \in \mathbb{R}.$$

Proof. We first note that

$$\begin{aligned} (h_w)^\mu(x) &= h\left(e^{-\mu}x - w\right) \\ \Rightarrow (h^\mu)_{e^\mu w}(x) &= h\left(e^{-\mu}x - w\right) = (h_w)^\mu(x). \end{aligned}$$

With this in mind we compute

$$\begin{aligned} &U^d_{qh}(\mu)U_{qh}(w)U^d_{qh}(\mu)^* \\ &= \hat{\Gamma}(e^{iq(h^\mu-h)})U^d_0(\mu)\, e^{-iq^2\sigma(h,h_w)}\hat{\Gamma}(e^{iq(h_w-h)})U_0(w)\, U^d_0(-\mu)\hat{\Gamma}(e^{-iq(h^\mu-h)}) \\ &= e^{-iq^2\sigma(h,h_w)}\,\hat{\Gamma}\big(e^{iq(h^\mu-h)}\big)\,\hat{\Gamma}\big(e^{iq((h_w)^\mu-h^\mu)}\big)\,\hat{\Gamma}\big(e^{-iq((h^\mu)_{e^\mu w}-h_{e^\mu w})}\big)\,U_0(e^\mu w) \\ &= e^{-iq^2[\sigma(h,h_w)-\sigma(h^\mu-h,(h_w)^\mu-h^\mu)+\sigma((h_w)^\mu-h,(h_w)^\mu-h_{e^\mu w})]}\,\hat{\Gamma}\big(e^{iq(h_{e^\mu w}-h)}\big)\,U_0(e^\mu w) \\ &= e^{-iq^2\sigma(h,h_{e^\mu w})}\,\hat{\Gamma}\big(e^{iq(h_{e^\mu w}-h)}\big)\,U_0(e^\mu w) = U_{qh}(e^\mu w) \end{aligned}$$

which completes the proof. □

With this last lemma we now have a net of field algebras on the line which satisfies all the requirements needed to use it as a building block for a full massless Poincaré covariant local theory for anyons on the two-dimensional Minkowski space.

IV.4. Conclusion

As we have seen in the previous two sections, the theory of loop groups and implementable Bogoliubov transformations allows one to define a net of operator algebras on the two-dimensional Minkowski space which satisfies the following properties.

- *Compact Localization*: Field algebras are localized in (arbitrarily small) double cones: $\mathcal{O} \mapsto \mathcal{F}(\mathcal{O})$. The algebras $\mathcal{F}(\mathcal{O})$ are generated by basic fields $\Phi, \Phi^\dagger$ which raise or lower the charge by one.

- *Covariance* under the Poincaré group: There is a unitary representation U of $\mathcal{P}_+^\uparrow$ such that

$$U(a,\Lambda)\mathcal{F}(\mathcal{O})U(a,\Lambda)^* = \mathcal{F}(\Lambda\mathcal{O}+a), \qquad a \in \mathbb{R}^2, \Lambda \in \mathcal{L}_+^\uparrow.$$

  Spectrum condition: The spectrum of the generators of translations lies in the closure of the forward light cone $\overline{V^+}$.

- The fields have *anyonic statistics*, i.e. for spacelike separated $\mathcal{O}_1$ and $\mathcal{O}_2$ the basic fields from the respective algebras satisfy

$$\Phi_1\Phi_2 = e^{i\pi\nu\, sgn(\mathcal{O}_1,\mathcal{O}_2)}\,\Phi_2\Phi_1,$$

  for some parameter $\nu \in \mathbb{R}$.

Such a construction is possible for *every mass* $m \geq 0$, i.e. for massive and massless theories, and for all statistics parameters $\nu \in \mathbb{R}$. In contrast to the wedge local situation of chapter III the (basic) fields here are *not* polarization free generators. This means that the energy-momentum spectrum of the vectors created from the vacuum by $\Phi, \Phi^\dagger$ does not lie solely on the mass-shell H_m^+ (or the boundary of V^+ in the massless case). Moreover, the whole construction is *non-perturbative* and every step is mathematically well-defined.

It would now be tempting to try this construction in one dimension higher, i.e. on $2+1$ dimensional Minkowski space, but unfortunately multiplication operators on position space do *not* lead to implementable Bogoliubov transformations there. Maybe the more algebraic approach that has been sketched in the massless case could be used to try a similar construction with some explicit vacuum representation of a suitable observable algebra.

In the next chapter we will therefore pursue a more modest strategy and use implementable multiplication operators on the circle to illustrate the connection between non-trivial behavior under 2π rotations and localization of the fields on a kind of covering space (i.e. dependence of the localization regions on some additional information).

V. Anyons and "Winding Number"†

As we have seen in the previous chapter, in $d = 1+1$ dimensions localized quantum fields with anyonic commutation relations can be explicitly constructed using implementable Bogoliubov transformations on the one-particle space. In 2+1 dimensions, explicit constructions of quantum fields with anyonic statistics usually carry electric charge and magnetic flux and they often satisfy formal commutation relations of the form

$$\phi(x_1)\phi(x_2) \sim e^{i\pi\lambda\,\mathrm{sign}[\arg(x_1-x_2)]}\phi(x_2)\phi(x_1),$$

where $\arg(x)$ denotes the angle of the vector $x \in \mathbb{R}^2$. Smearing such ϕ's with compactly localized test functions would then lead to localization of the operators in smeared out "double strings", extending to infinity in two opposite directions. Similar localization regions have already been encountered in chapter II, in the case of deformations with $R = 1$. However, to speak of "proper" anyons in the sense of algebraic quantum field theory one would need field algebras localized in spacelike cones C, extending to infinity only in a connected compact set of spacelike directions. More precisely, the localization regions for anyons can be labelled by paths $\tilde{C}$ of such spacelike cones depending also on a kind of winding number which determines the commutation relations of mutually spacelike separated fields [9, 29, 31, 36, 68](instead of heaving a mere sign-function in the exchange phase). The spin statistics theorem for anyons [36, 69] then forces these fields to transform under a representation of the universal covering of the Poincaré group $\widetilde{\mathcal{P}}_+^\uparrow$ with arbitrary real-valued spin $s \in \mathbb{R}$.

It would be desirable to find an explicit construction for such cone-localized anyon fields for any spin s in $2+1$ dimensions. Unfortunately the no-go theorem by Bros and Mund [7] states that there are no interaction-free fields for anyons. This also means that there are no anyon fields which simply create single particle states from the vacuum [67]. This makes the explicit non-perturbative construction of anyons very complicated.

It is, however, possible to circumvent this no-go theorem by considering fields which are only localized in wedge regions in Minkowski space, as we have seen in chapter II. In this case one can modify the method of multiplicative deformations, established in [42, 57], to obtain one-particle generators localized in so-called "paths of wedges" which have anyonic commutation relations and are covariant w.r.t a representation of $\widetilde{\mathcal{P}}_+$ for spin $s \in \mathbb{R}$ [74].

The no-go theorem now makes it very difficult to explicitly construct anyon fields in $d = 2+1$ with sharper localization. To better understand the concepts of winding number,

† The work in this chapter has been submitted to "Letters in Mathematical Physics" (see also [75]).

anyonic commutation relations and arbitrary spin and their connection, we will therefore consider first the simplified case of anyon quantum fields on the circle. A similar construction using also the current algebra on the circle (see also [12]) can be found in [20, 21] where, however, the phase factor appearing in the commutation relations is again governed by a sign-function. After smearing the point-fields this leads to localization in two disconnected regions lying on opposite sides of the circle. Here we will modify this construction in such a way that one obtains a compactly localized field algebra, covariant under a representation of the (covering group of the) rotations with arbitrary spin and with commutation relations depending on the relative winding number of the respective localization regions.

By taking tensor products of these fields on the circle with local fields on $\mathbb{R}^{2+1}$ one arrives at a rotation- and translation covariant cone-localized quantum field theory with anyonic commutation relations. However, due to the simple tensor product structure, covariance w.r.t boosts is lost in the construction.

V.1. Local Anyons on the Circle

Because the spin for anyons can be an arbitrary real number, rotations around multiples of 2π will act non-trivially on our fields. We will therefore consider the fields to be localized in intervals $\tilde{I}$ on the *universal covering* $\widetilde{S_1}$ of the circle with radius $R = 1$ (the generalization to any $R > 0$ being straightforward). Our aim is to construct a net $\tilde{I} \mapsto \mathcal{F}(\tilde{I})$ of $*$-algebras of operators acting on a Hilbert space $\mathcal{H}$ satisfying the following properties, which define an anyonic field net on the circle:

i) **Charge Sectors:** The Hilbert space splits into a direct sum of Hilbert spaces for fixed charge,

$$\mathcal{H} = \bigoplus_{q\in\mathbb{Z}} \mathcal{H}_q, \tag{V.1}$$

and $\mathcal{H}_0$ contains a unique vacuum vector Ω. Moreover, there exist basic fields Φ in every local algebra $\mathcal{F}(\tilde{I})$ that change the charge of a vector by 1, i.e. $\Phi\mathcal{H}_q \subset \mathcal{H}_{q+1}$ and $\Phi^*\mathcal{H}_q \subset \mathcal{H}_{q-1}$.

ii) **Isotony:** The map $\tilde{I} \mapsto \mathcal{F}(\tilde{I})$ preserves inclusions, i.e. $\mathcal{F}(\tilde{I}_1) \subset \mathcal{F}(\tilde{I}_2)$ if $\tilde{I}_1 \subset \tilde{I}_2$.

iii) **Covariance and Spin:** The field algebras are covariant under a unitary representation U of the universal covering group of the rotations $\widetilde{U(1)} \simeq \mathbb{R}$, i.e.

$$U(\omega)\mathcal{F}(\tilde{I})U(\omega)^* \subset \mathcal{F}(\tilde{I} + \omega), \quad \omega \in \mathbb{R}. \tag{V.2}$$

The vacuum vector is invariant under this representation, i.e. $U(\omega)\Omega = \Omega$. Furthermore 2π rotations act as

$$U(2\pi) = \sum_{q\in\mathbb{Z}} e^{2\pi i S_q} P_q, \tag{V.3}$$

where P_q is the projector onto the charge q subspace and $S_q \in \mathbb{R}$ is the spin of the sector with charge q (defined only modulo 1).

iv) **(Twisted) Locality:** Basic fields Φ_1, Φ_2 localized in intervals $\tilde{I}_1, \tilde{I}_2$, whose projections I_1, I_2 onto the base space S_1 do not intersect, satisfy commutation relations of the form

$$\Phi_1 \ \Phi_2 = \pm e^{2\pi i s(2N(\tilde{I}_1,\tilde{I}_2)+1)} \ \Phi_2 \ \Phi_1. \tag{V.4}$$

Here $s \in \mathbb{R}$ is a real parameter and $N(\tilde{I}_1, \tilde{I}_2) \in \mathbb{Z}$ is the relative winding number of $\tilde{I}_1$ w.r.t. $\tilde{I}_2$, which will be defined more explicitly below.

Remarks:
i) We allow in general for a $\pm$ sign in equation (V.4) because as we will see in the subsequent explicit construction we will arrive at commutation relations with an additional minus sign in front of the exchange phase factor.
ii) Property iv) shows that the commutation relations will not be governed by a simple two-valued sign-function of the localization points, but depend on the winding number of two disjoint regions which makes sense because we work on the universal covering space $\widetilde{S_1}$ of the circle.
iii) Due to Lemma II.1 from the introduction we know that (V.3) is basically the most general representation of the 2π-rotations possible. It has also been shown that S_q has to be quadratic in q so we have

$$S_q = sq^2, \tag{V.5}$$

where $s \in \mathbb{R}$ is the spin of the model.

It therefore suggests itself to define the representation of $\widetilde{U(1)}$ on the full Hilbert space according to

$$U(\omega) := e^{is\omega Q^2} U_0(\omega), \tag{V.6}$$

where Q is the charge operator, $Q\mathcal{H}_q = q\mathcal{H}_q$, and U_0 is some (2π-periodic) representation of $U(1)$, defined naturally for real ω as $U_0(\omega) = U_0\big(\omega (\mathrm{mod}\, 2\pi)\big)$.

Our field algebra will be constructed on the charged (anti-symmetric) Fock space, i.e. on $\mathcal{F}(\mathcal{H}_1) = \mathcal{F}(\mathcal{H}_1^+ \oplus \mathcal{H}_1^-) \simeq \mathcal{F}(\mathcal{H}_1^+) \otimes \mathcal{F}(\mathcal{H}_1^-)$ with the usual charge operator and corresponding decomposition into charged sectors. In this case the considerations of section (IV.1) thus show that the representation (V.6) is *not* the second quantization of some one-particle representation, where the phase would be linear in q, and we will have to take care of this peculiarity in our construction of the fields.

V.1.1. Construction of the Fields

The idea is to first construct auxiliary fields $\hat{\Phi}$ on the circle $S_1 \simeq [0, 2\pi)$ by second quantization of specific one-particle operators, which are covariant under the 2π-periodic representation U_0 and thus still have the "wrong" commutation relations. They are then lifted to the covering space $\widetilde{S_1}$ by using the full representation U of the rotations. More precisely we define for $\omega \in \mathbb{R}$,

$$\begin{aligned} \Phi_\omega &:= e^{is\omega Q^2}\, \hat{\Phi}_\omega\, e^{-is\omega Q^2} = e^{is\omega(2Q-1)}\hat{\Phi}_\omega, \\ \hat{\Phi}_\omega &:= U_0(\omega)\hat{\Phi}_0 U_0(\omega), \end{aligned} \tag{V.7}$$

where $\hat{\Phi}_0$ is a field localized in an interval around the (arbitrary) reference direction $x = 0$. By using the representation U_0 we get auxiliary fields $\hat{\Phi}_\omega$ localized in intervals around $x = \hat{\omega}$, where $\hat{\omega}$ denotes the projection of $\omega \in \mathbb{R}$ onto the interval $[0, 2\pi)$[1].

One can see that the difference between $\hat{\Phi}_\omega$ and the final field Φ_ω is just the operator $e^{i\omega s(2Q-1)}$ which will lead to an additional phase factor in the commutation relations of the Φ's. The important thing is that these auxiliary fields $\hat{\Phi}$ will be defined in such a way that for non-intersecting localization intervals they satisfy commutation relations of the form

$$\hat{\Phi}_{\omega_1}\hat{\Phi}_{\omega_2} = e^{-2is[(\widehat{\omega_1-\omega_2})-\pi]\pm i\pi}\, \hat{\Phi}_{\omega_2}\hat{\Phi}_{\omega_1}, \tag{V.8}$$

which still depend explicitly on the relative distance of the respective localization regions.

As already stated in section V.1 our basic fields will raise the charge of a vector by one and thus satisfy $\hat{\Phi}Q = (Q-1)\hat{\Phi}$. Using this relations and the definition of the fields Φ_ω one can calculate that their commutation relations turn out to be

$$\Phi_{\omega_1}\Phi_{\omega_2} = -e^{2is[(\omega_1-\omega_2)-(\widehat{\omega_1-\omega_2})+\pi]}\, \Phi_{\omega_2}\Phi_{\omega_1}. \tag{V.9}$$

Now remember that $\hat{\omega}$ has been defined as $\omega \pmod{2\pi}$, which means that there exists an integer $n(\omega) \in \mathbb{Z}$ such that

$$\omega = \hat{\omega} + 2\pi n(\omega). \tag{V.10}$$

We call this $n(\omega)$ the *"winding number"* of ω and using its definition we can rewrite the commutation relations according to

$$\Phi_{\omega_1}\Phi_{\omega_2} = -e^{2\pi is[2n(\omega_1-\omega_2)+1]}\, \Phi_{\omega_2}\Phi_{\omega_1}. \tag{V.11}$$

For intervals $\tilde{I}_1, \tilde{I}_2 \in \tilde{S}_1$ such that $I_1 \cap I_2 = \emptyset$ the number $n(\omega_1 - \omega_2)$ is constant for all $\omega_1 \in \tilde{I}_1, \omega_2 \in \tilde{I}_2$ allowing us to define a relative winding number of $\tilde{I}_1$ w.r.t. $\tilde{I}_2$ according to

$$N(\tilde{I}_1, \tilde{I}_2) := n(\omega_1 - \omega_2), \text{ for } \omega_1 \in \tilde{I}_1, \omega_2 \in \tilde{I}_2. \tag{V.12}$$

Hence for fields localized in non-intersecting intervals we get exactly the desired commutation relations (V.4).

[1] Note that because we chose our standard interval not to be symmetric around 0, $\hat{\omega}$ satisfies $(\widehat{-\omega}) = 2\pi - \hat{\omega}$.

The interesting question that remains is how we can construct the auxiliary fields $\hat{\Phi}_\omega$ such that they satisfy the right commutation relations (V.8). In the following we will describe in more detail the explicit construction of these fields by using implementers of certain Bogoliubov transformations.

Mathematical Preliminaries

Consider the Hilbert space of square integrable functions on the circle $\mathcal{H}_1 = L^2(S_1)$ which can be seen as a kind of auxiliary one-particle space. Fourier transformation leads to the equivalence[2]

$$\mathcal{H}_1 \simeq l^2(\mathbb{Z}) \simeq l^2(\mathbb{N}_0) \oplus l^2(\mathbb{N}) =: \mathcal{H}_1^+ \oplus \mathcal{H}_1^-, \tag{V.13}$$

where, without loss of generality, we have chosen to include the zero-mode into the first summand $\mathcal{H}_1^+$. On this Hilbert space we work with the usual representation of $SO(2) \simeq U(1)$, namely

$$(U_1(\omega)\varphi)(x) := \varphi(x - w), \tag{V.14}$$

which is diagonal in Fourier space, i.e.

$$(U_1(\omega)\tilde{\varphi})_n = e^{-in\omega}\tilde{\varphi}_n. \tag{V.15}$$

The Fourier modes $\tilde{\varphi}_n$ and the inverse transformation are defined according to

$$\tilde{\varphi}_n := \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} dx \varphi(x) e^{-ixn}, \quad \varphi(x) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \tilde{\varphi}_n e^{inx}, \tag{V.16}$$

and to avoid complicating the notation we do not make a notational distinction between the representation U_1 in x-space and in momentum-space. The projections $P_\pm$ onto $\mathcal{H}_1^+$ and $\mathcal{H}_1^-$ are then defined according to

$$\begin{aligned} (P_+\varphi)(x) &\equiv \varphi^+(x) = \frac{1}{\sqrt{2\pi}} \sum_{n\geq 0} \tilde{\varphi}_n e^{inx}, \\ (P_-\varphi)(x) &\equiv \varphi^-(x) = \frac{1}{\sqrt{2\pi}} \sum_{n< 0} \tilde{\varphi}_n e^{inx}. \end{aligned} \tag{V.17}$$

Over this one-particle space one can now take the *anti-symmetrized Fock space*

$$\mathcal{H} := \mathcal{F}_F(\mathcal{H}_1) \simeq \mathcal{F}_F(\mathcal{H}_1^+) \otimes \mathcal{F}_F(\mathcal{H}_1^-)$$

and we will again use the implementers of certain Bogoliubov transformations to construct charge-shifting field operators on this Hilbert space.

[2] Here $\mathbb{N}_0$ denotes the non-negative integers and $\mathbb{N} = \mathbb{N}_0 \setminus \{0\}$.

An important requirement for being able to compute commutation relations on the Fock space is that the one-particle operators commute. To ensure this we will again work only with unitary multiplication operators on $L^2(S_1)$, i.e. operators of the form

$$(V\varphi)(x) = e^{if(x)}\varphi(x) \equiv (e^{if}\varphi)(x),$$

where we will use the same symbol for the operator on Hilbert space and the function with which it multiplies. For such operators the following lemma characterizes a large class of implementable Bogoliubov transformations on the circle [17].

Lemma V.1. *For a smooth real-valued function $\alpha \in C^\infty(S_1, \mathbb{R})$ the multiplication operator $(\alpha\varphi)(x) := \alpha(x)\varphi(x)$ has Hilbert Schmidt off-diagonal elements and therefore $e^{it\alpha} \in \mathcal{G}_F(\mathcal{H}_1), \forall t \in \mathbb{R}$ and $q(e^{it\alpha}) = 0$.*

Proof. The off-diagonal elements of α by definition act according to

$$(P_+\alpha P_-\varphi)(x) \equiv (\alpha_{+-}\varphi^-)(x) = \frac{1}{2\pi}\sum_{n\geq 0}\sum_{k<0}\tilde{\alpha}_{n-k}\tilde{\varphi}_k\, e^{inx},$$

so the Hilbert-Schmidt condition reads

$$\begin{aligned}\mathrm{Tr}[(\alpha_{+-})^*\alpha_{+-}] = \mathrm{Tr}[\alpha_{-+}\alpha_{+-}] &\propto \sum_{n<0}\sum_{k\geq 0}\tilde{\alpha}_{n-k}\tilde{\alpha}_{k-n} = \sum_{n<0}\sum_{k\geq 0}|\tilde{\alpha}_{n-k}|^2 \\ &= \sum_{k\geq 0}\sum_{n<k}|\tilde{\alpha}_n|^2 = \sum_{n=1}^{\infty} n\,|\tilde{\alpha}_n|^2 < \infty\end{aligned}$$

which is fulfilled for smooth functions α. □

Just like in the massless case in the previous chapter using multiplication operators in position space now allows us to explicitly compute a simple expression for the Schwinger term [17].

Lemma V.2. *Consider self-adjoint operators acting as multiplication with the smooth (real-valued) functions α, β on $L^2(S_1)$. Then the Schwinger term $S(\alpha,\beta)$ turns out to be*

$$S(\alpha,\beta) = \frac{1}{2\pi}\int_0^{2\pi} dx\ \alpha(x)\beta'(x) \ = \frac{1}{4\pi}\int_0^{2\pi} dx\,\big(\alpha(x)\beta'(x) - \alpha'(x)\beta(x)\big) \tag{V.18}$$

Proof. Similar to the proof of lemma V.1 one calculates

$$\begin{aligned} iTr(\alpha_{-+}\beta_{+-} - \beta_{-+}\alpha_{+-}) &= \frac{i}{2\pi}\sum_{n<0}\sum_{l\geq 0}\Big(\tilde{\alpha}_{n-l}\,\tilde{\beta}_{l-n} - \tilde{\beta}_{n-l}\,\tilde{\alpha}_{l-n}\Big) \\ = \frac{i}{2\pi}\sum_{l\geq 0}\sum_{n<-l}\Big(\tilde{\alpha}_n\tilde{\beta}_{-n} - \tilde{\beta}_n\tilde{\alpha}_{-n}\Big) &= -\frac{i}{2\pi}\sum_{n=1}^{\infty} n\Big(\tilde{\alpha}_n\tilde{\beta}_{-n} - \tilde{\beta}_n\tilde{\alpha}_{-n}\Big) \\ = -\frac{i}{2\pi}\sum_{n\in\mathbb{Z}} n\,\tilde{\alpha}_n\tilde{\beta}_{-n} &= -\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\widetilde{(\alpha')}_n\tilde{\beta}_{-n}.\end{aligned}$$

Inverse Fourier transform then leads to the simple expression in position space

$$\begin{aligned}
-\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\widetilde{(\alpha')}_n\tilde{\beta}_{-n} &= -\frac{1}{2\pi}\int_0^{2\pi} dxdy\,\alpha'(x)\beta(y)\frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{-in(x-y)} \\
&= -\frac{1}{2\pi}\int_0^{2\pi} dx\,\alpha'(x)\beta(x) = \frac{1}{4\pi}\int_0^{2\pi} dx\,\left(\alpha(x)\beta'(x)-\alpha'(x)\beta(x)\right),
\end{aligned}$$

where potential boundary terms cancel because of the continuity of α and β. □

Construction of the Auxiliary Field

Apart from exponentials of smooth multiplication operators we also need unitaries with non-vanishing Fredholm index. We therefore consider first of all the operator

$$(V\varphi)(x) := e^{ix}\varphi(x), \tag{V.19}$$

which is obviously unitary, but the function $[0,2\pi)\ni x\mapsto x$ is *not* smooth on S_1 so Lemma V.1 is not applicable here. However we still have the following result.

Proposition V.1. *The unitary multiplication operator $(V\varphi)(x) = e^{ix}\varphi(x)$ has the following properties:*

i) Its off-diagonal elements $V_{\pm\mp}$ are Hilbert-Schmidt.

ii) The diagonal elements satisfy

$$\ker V_{--} = \{\lambda e_-|\lambda\in\mathbb{R}\},\quad \ker V_{--}^* = \ker V_{++} = \emptyset,$$

with $e_-(x) = \frac{1}{\sqrt{2\pi}}e^{-ix}$ and $(Ve_-)(x) = e_0(x) = \frac{1}{\sqrt{2\pi}}$.

Proof. Consider the basis $\{e_n\in\mathcal{H}_1|n\in\mathbb{Z}\}$ for $\mathcal{H}_1 = L^2(S_1)$ where $e_n(x) := \frac{1}{\sqrt{2\pi}}e^{inx}$. Then the operator V acts as a *shift operator* with respect to this basis, i.e. it satisfies

$$Ve_n = e_{n+1}.$$

Now because $\mathcal{H}_1^+$ is spanned by $\{e_n|n\geq 0\}$ and $\mathcal{H}_1^-$ by $\{e_n|n<0\}$ it is thus obvious that $e_-\equiv e_{-1}$ spans $\ker V_{--}$ and that $\ker V_{++} = \emptyset$.
This also immediately leads to $V_{-+} = 0$ and $ImV_{+-} = \{\lambda e_0|\lambda\in\mathbb{R}\}$ so $V_{\pm\mp}$ are evidently Hilbert-Schmidt operators because they have finite range. □

This shows that such a V is a suitable operator to obtain a simple charge shift on the Fock space. One can also define rotated V's according to

$$V_\omega = U_1(\omega)VU_1(\omega)^* = e^{-i\omega}V. \tag{V.20}$$

For the implementers this leads to

$$U_0(\omega)\hat{\Gamma}(V)U_0(\omega)^* = \hat{\Gamma}(U_1(\omega)VU_1(\omega)^*) = \hat{\Gamma}(e^{-i\omega}V) = \hat{\Gamma}(V)e^{-i\omega Q}. \tag{V.21}$$

(The definition $e^{-i\omega Q}\hat{\Gamma}(V)$ would also be possible, which simply amounts to another choice of phase factor for the implementer.) This simple relation now allows us to compute the commutation relations of different charge shifting operators $\hat{\Gamma}(V_{\omega_1}), \hat{\Gamma}(V_{\omega_2})$, which can in general be a more involved task as we have seen in section IV.3. Using the charge shifting property of V, i.e. $VQ = (Q-1)V$, one immediately gets

$$\hat{\Gamma}(V_{\omega_1})\hat{\Gamma}(V_{\omega_2}) = e^{-(\omega_1-\omega_2)}\,\hat{\Gamma}(V_{\omega_2})\hat{\Gamma}(V_{\omega_1}). \tag{V.22}$$

These are not yet the right commutation relations we need for our auxiliary field so we consider in addition an operator of the form

$$(e^{i\lambda\alpha}\varphi)(x) := e^{i\lambda\alpha(x)}\varphi(x), \tag{V.23}$$

where $\alpha \in C^\infty(S_1, \mathbb{R})$ is a smooth real-valued function on the circle and λ is some real parameter. Lemma V.1 then tells us that this is an implementable unitary operator allowing us to define the auxiliary field $\hat{\Phi}$ according to

$$\hat{\Phi}_\omega := \hat{\Gamma}(V_\omega)\hat{\Gamma}(e^{i\lambda\alpha_\omega}) = U_0(\omega)\,\hat{\Gamma}(V)\hat{\Gamma}(e^{i\lambda\alpha})\,U_0(\omega)^*, \tag{V.24}$$

where α_ω is again defined as $\alpha_\omega(x) := \alpha(x-\omega)$.

The simple form of the operator V and lemma IV.2 then allow us to compute the relative commutation relations between $V_{\omega'}$ and $e^{i\lambda\alpha_\omega}$.

Lemma V.3. *For all smooth real-valued functions α the commutation relations between $V_{\omega'}$ and $e^{i\lambda\alpha_\omega}$ are independent of ω', ω and are of the form*

$$\hat{\Gamma}(e^{i\lambda\alpha_\omega})\hat{\Gamma}(V_{\omega'}) = e^{i\lambda\,const.}\hat{\Gamma}(V_{\omega'})\hat{\Gamma}(e^{i\lambda\alpha_\omega}), \tag{V.25}$$

where const. *only depends on the integral over α.*

Proof. Since the vector e_0, which is created from the vacuum by $\hat{\Gamma}(V)$, is invariant under rotations we get

$$\langle\hat{\Gamma}(V_{\omega'})\Omega, d\hat{\Gamma}(\alpha_\omega)\hat{\Gamma}(V_{\omega'})\Omega\rangle = \langle e_0, \alpha_\omega e_0\rangle = \langle e_0, \alpha\, e_0\rangle = \text{const.}$$

□

These constant phase factors will therefore cancel out in the total commutation relations between two $\hat{\Phi}$'s which then turn out to be

$$\hat{\Phi}_{\omega_1}\hat{\Phi}_{\omega_2} = e^{-i(\omega_1-\omega_2)}e^{i\lambda^2 S(\alpha_{\omega_1},\alpha_{\omega_2})}\,\hat{\Phi}_{\omega_2}\hat{\Phi}_{\omega_1}. \tag{V.26}$$

Comparing this with the relations of section V.1.1, which the auxiliary field has to satisfy, we are now faced with the following problem:

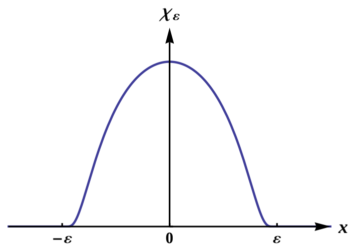
Find a smooth real-valued function α on the circle and a parameter $\lambda \in \mathbb{R}$ for which the Schwinger term $S(\alpha_{\omega_1}, \alpha_{\omega_2})$ satisfies

$$\lambda^2 S(\alpha_{\omega_1}, \alpha_{\omega_2}) - (\widehat{\omega_1 - \omega_2}) = -2s[(\widehat{\omega_1 - \omega_2}) - \pi] \pm \pi, \tag{V.27}$$

for suitable $(\widehat{\omega_1 - \omega_2})$.

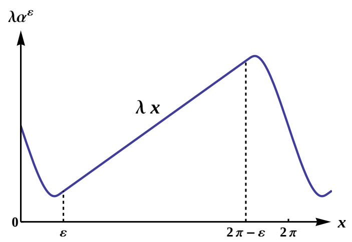
For this purpose consider first the function $x \mapsto \lambda\hat{x}$ on $\mathbb{R}$ for an abitrary real parameter λ. Similar to the shift operator V, multiplication with $e^{i\lambda\hat{x}}$ would lead to commutation relations with exchange phase of the form $e^{-i\lambda(\omega_1-\omega_2)}$. But the function $e^{i\lambda\hat{x}}$ is in general not continuous on the circle and therefore it does not lead to an implementable transformation. This problem can be solved by smearing it with a function $\chi_\varepsilon \in C_0^\infty(\mathbb{R})$ with the following properties:

- $\operatorname{supp} \chi_\varepsilon = [-\varepsilon, \varepsilon]$ with $0 < \varepsilon < \frac{\pi}{2}$
- $\chi_\varepsilon(-x) = \chi_\varepsilon(x) = \overline{\chi_\varepsilon(x)}$
- $\int dx \chi_\varepsilon(x) = 1$



Considering the 2π-periodic function $\hat{x} \equiv x \pmod{2\pi}$ on $\mathbb{R}$ one can use such a χ_ε to define a smooth function according to

$$\alpha^\varepsilon(x) := \int_{\mathbb{R}} dy\, (\widehat{x - y})\chi_\varepsilon(y), \tag{V.28}$$



which is still 2π-periodic and thus defines a smooth function on the circle. An important feature of this function is that it is still linear in the interval $(\varepsilon, 2\pi - \varepsilon)$, because for our choice of a symmetric χ_ε there holds

$$\int dy\, (x - y)\chi_e(y) = x,$$

which means that most of the linear part of $\hat{x}$ is unchanged and only the jump discontinuities get smeared.

For every real parameter $\lambda \in \mathbb{R}$ this α^ε then defines an implementable unitary multiplication operator according to

$$(e^{i\lambda\alpha^\varepsilon}\varphi)(x) = e^{i\lambda\alpha^\varepsilon(x)}\varphi(x). \tag{V.29}$$

It turns out that the Schwinger term for such operators has exactly the right form, namely we have the following proposition.

Proposition V.2. *For functions* $\alpha^{\varepsilon_1}, \alpha^{\varepsilon_2}$ *with* $0 < \varepsilon_i < \frac{\pi}{2}, i = 1, 2$ *as defined in equation* (V.28) *and* $\omega_1, \omega_2 \in \mathbb{R}$ *such that* $\varepsilon_1 + \varepsilon_2 < (\widehat{\omega_1 - \omega_2}) < 2\pi - \varepsilon_1 - \varepsilon_2$ *the Schwinger term satisfies*

$$S(\alpha^{\varepsilon_1}_{\omega_1}, \alpha^{\varepsilon_2}_{\omega_2}) = (\widehat{\omega_1 - \omega_2}) - \pi. \tag{V.30}$$

Proof. Evidently the Schwinger term is invariant under simultaneous rotation of $\alpha^{\varepsilon_1}_{\omega_1}$ and $\alpha^{\varepsilon_2}_{\omega_2}$ so for simplicity we write $\omega \equiv \omega_1 - \omega_2$ and calculate

$$S(\alpha^{\varepsilon_1}_{\omega}, \alpha^{\varepsilon_2}) = \frac{1}{2\pi} \int_0^{2\pi} dx\, \alpha^{\varepsilon_1}_{\omega}(x)(\alpha^{\varepsilon_2})'(x).$$

For this we first need $(\alpha^{\varepsilon})'$ which turns out to be

$$\begin{aligned}
(\alpha^{\varepsilon})'(x) &= \int dy\, \frac{d}{dx}(\widehat{x-y})\chi_{\varepsilon}(y) \\
&= \int dy \left(1 - 2\pi \sum_{k\in\mathbb{Z}} \delta(x - y - 2\pi k)\right) \chi_{\varepsilon}(y) \\
&= 1 - 2\pi \sum_{k\in\mathbb{Z}} \chi_{\varepsilon}(x - 2\pi k).
\end{aligned}$$

Note that, because of the support properties of χ_{ε}, for every $x \in \mathbb{R}$ only one term in the sum $\sum_{k\in\mathbb{Z}} \chi_{\varepsilon}(x - 2\pi k)$ is nonzero. Hence one gets

$$\begin{aligned}
S(\alpha^{\varepsilon_1}_{\omega_1}, \alpha^{\varepsilon_2}_{\omega_2}) &= \int_0^{2\pi} dx\, \alpha^{\varepsilon_1}_{\omega}(x)\big(\frac{1}{2\pi} - \chi_{\varepsilon_2}(x) - \chi_{\varepsilon_2}(x - 2\pi)\big) \\
&= \pi - \int_{-\varepsilon_2}^{\varepsilon_2} dx \int_{-\varepsilon_1}^{\varepsilon_1} dy\, (\widehat{x - y - \omega})\chi_{\varepsilon_1}(y)\chi_{\varepsilon_2}(x),
\end{aligned}$$

by using the periodicity of α^{ε_1} and the fact that $\int dx\, \alpha^{\varepsilon_1}_{\omega}(x) = 2\pi^2$ is independent of ω and χ_{ε_1}. To compute the remaining term we need to calculate integrals of the form

$$\int_{-\varepsilon}^{\varepsilon} dx\, (\widehat{c - x})\chi_{\varepsilon}(x), \qquad \text{for } c \in \mathbb{R} \text{ and } \varepsilon < \hat{c} < 2\pi - \varepsilon.$$

For this purpose remember that we can write $(\widehat{c - x}) = (c - x) - 2\pi n(c - x)$ and that $n(c - x) = n(c)$ for $\varepsilon < \hat{c} < (2\pi - \varepsilon)$ and $-\varepsilon < x < \varepsilon$. Inserting this into the integral we get

$$\int_{-e}^{\varepsilon} dx\, (\widehat{c - x})\chi_{\varepsilon}(x) = \int_{-\varepsilon}^{\varepsilon} dx\, \big((c - x) - 2\pi n(c)\big)\chi_{\varepsilon}(x) = c - 2\pi n(c) = \hat{c}.$$

Using this in the expression for $S(\alpha^{\varepsilon_1}_{\omega_1}, \alpha^{\varepsilon_2}_{\omega_2})$ one immediately arrives at

$$S(\alpha^{\varepsilon_1}_{\omega_1}, \alpha^{\varepsilon_2}_{\omega_2}) = \pi - (\widehat{-\omega}) = \hat{\omega} - \pi.$$

□

With this proposition equation (V.27) leads to the constraint

$$\lambda^2(\hat{\omega} - \pi) - \hat{\omega} \stackrel{!}{=} -2s(\hat{\omega} - \pi) \pm \pi, \tag{V.31}$$

for the parameters λ and s. In order to get a solution for λ we have to choose the minus sign on the right side and for a fixed "spin" s this equation then restricts the parameter λ to

$$\lambda^2 = 1 - 2s. \tag{V.32}$$

Together with the requirement that the operator $e^{i\lambda\alpha^\varepsilon}$ has to be unitary this also shows that the parameter s has to satisfy $s \in (-\infty, \frac{1}{2})$.

We can now summarize the construction in the following way: For every $\omega \in \mathbb{R}$, $0 < \varepsilon < \pi$ and symmetric real-valued smearing function χ_ε with $\operatorname{supp} \chi_\varepsilon \subset (-\varepsilon, \varepsilon)$ one can construct field operators

$$\Phi_\omega[\chi_\varepsilon] = U(\omega)\hat{\Gamma}(V)\hat{\Gamma}(e^{i\lambda\alpha^\varepsilon})U(\omega)^* = e^{is\omega(2Q-1)}\,\hat{\Gamma}(V_\omega)\,\hat{\Gamma}(e^{i\lambda\alpha^\varepsilon_\omega}), \tag{V.33}$$

which raise the charge by one and are localized in the interval

$$\widetilde{I^\varepsilon_\omega} := \{x \in \widetilde{S_1} |\, \omega - \varepsilon < x < \omega + \varepsilon\} \subset \widetilde{S_1} \tag{V.34}$$

with width 2ε centered around the point $\omega \in \mathbb{R}$. Together with the adjoint field $\Phi_\omega[\chi_\varepsilon]^*$, lowering the charge by one, these operators then generate a net of algebras on the space $\widetilde{S_1}$ with anyonic statistics.

Lemma V.4. *Consider field operators $\Phi_i = \Phi_{\omega_i}[\chi_{\varepsilon_i}]$, $i = 1, 2$, which are localized in intervals $\tilde{I}_1$ and $\tilde{I}_2$ respectively. If the intervals are non-intersecting, i.e. $I_1 \cap I_2 = \emptyset$, the fields satisfy anyonic commutation relations*

$$\begin{aligned} \Phi_1\,\Phi_2 &= -e^{2\pi is(2N(\tilde{I}_1,\tilde{I}_2)+1)}\,\Phi_2\,\Phi_1, \\ \Phi_1\,\Phi_2^* &= -e^{-2\pi is(2N(\tilde{I}_1,\tilde{I}_2)+1)}\,\Phi_2^*\,\Phi_1, \end{aligned} \tag{V.35}$$

where $N(\tilde{I}_1, \tilde{I}_2)$ is the relative winding number of $\tilde{I}_1$ with respect to $\tilde{I}_2$, defined in (V.12).

In addition the fields are covariant with respect to the unitary representation (V.6) of the universal covering of the rotations $\widetilde{U(1)}$ with real-valued spin $s \subset (-\infty, \frac{1}{2})$. Taking the polynomial algebra over such localized fields then defines the local field algebras $\mathcal{F}(\tilde{I})$ and we have therefore constructed a local, covariant quantum field net for anyons on the circle, satisfying all the requirements defined at the beginning of the chapter.

Special Cases

s=1/2:

For the maximal value $s = \frac{1}{2}$ the relation $\lambda^2 = 1 - 2s$ leads to $\lambda = 0$, which means the operator $e^{i\lambda\alpha^\varepsilon}$ is the identity operator in this case. We are therefore left with only the shift operator V and the field is

$$\Phi_\omega = e^{i\frac{\omega}{2}(2Q-1)}\hat{\Gamma}(V)e^{-i\omega Q} = e^{i\frac{\omega}{2}}\hat{\Gamma}(V), \tag{V.36}$$

which is just $\hat{\Gamma}(V)$ times a constant phase factor which could also be omitted. This shows that the fields for different ω's commute independently of ω which is in accordance with the fact that the exchange phase turns out to be

$$-e^{2\pi is(2n+1)} = -e^{\pi i(2n+1)} = 1.$$

So we see that for $s = \frac{1}{2}$ we get a *bosonic field* "$\Phi \sim \hat{\Gamma}(V) \sim \hat{\Gamma}(e^{ix})$" and because of its simple form it creates one-particle vectors from the vacuum, namely

$$\Phi_\omega\Omega = e^{i\frac{\omega}{2}}\hat{\Gamma}(V)\Omega = e^{i\frac{\omega}{2}}e_0.$$

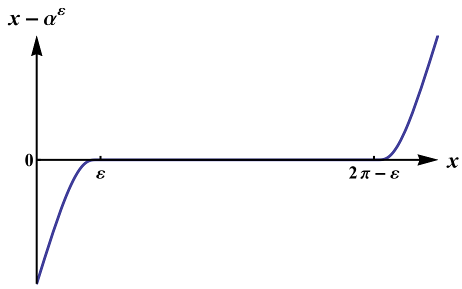
Moreover, taking into account that $Z_{\pm\mp} = 0$ for $V_{-+} = 0$, the explicit form of $\hat{\Gamma}(V)$ (with an appropriate choice of phase) turns out to be

$$\hat{\Gamma}(V) = a^*(e_0)\Gamma_+(-V_{++})\Gamma_-(-\overline{V_{--}}) + \Gamma_+(-V_{++})\Gamma_-(-\overline{V_{--}})b(\overline{e_-}), \tag{V.37}$$

which looks like a free field modified by the unitary operator $\Gamma_+(-V_{++})\Gamma_-(-\overline{V_{--}})$.

s=0:

The case $s = 0$ leads via $\lambda^2 = 1 - 2s$ to $\lambda = \pm 1$ and we will only consider the case $\lambda = -1$. As was already shown the function α^ε is linear in the interval $(\varepsilon, 2\pi - \varepsilon)$ so for $\lambda = -1$ the function $x \mapsto \hat{x} + \lambda\alpha^\varepsilon(x) = \hat{x} - \alpha^\varepsilon(x)$ vanishes for $\varepsilon < x < 2\pi - \varepsilon$. Therefore $e^{ix}e^{-i\alpha^\varepsilon(x)}$ is simply the identity apart from a small interval of length 2ε where its phase changes by the value 2π.



A similar so-called "blip function" was used in [86] and [20, 21] to approximate a step-function on the circle. There it was shown that after taking an appropriate limit the implementer of such a function converges to the free fermi field on the circle. Because of

$$-e^{2\pi is(2n+1)} = -1, \text{ for } s = 0,$$

we will also get an anti-commuting field in our case, but it is unclear if the field converges in some sense to a free fermi field if the smearing function χ_ε tends to a delta function.

Remark: As we have seen the construction leads to a commuting field for $s = \frac{1}{2}$ and an anti-commuting field for $s = 0$. However, we are working on a one-dimensional space — the circle S_1 or its universal covering $\widetilde{S_1}$ respectively — so although we called the parameter s the "spin" it really just labels a representation of the translations on S_1 or $\widetilde{S_1}$. So one would not expect the usual kind of spin-statistics theorem to hold for s in our case.

V.2. Cone-Localized Fields on $\mathbb{R}^2$

It would now be tempting to try the same construction in higher dimensions, e.g. on the Hilbert space $L^2(\mathbb{R}^2)$, to construct a string-local quantum field in two (space-)dimensions, covariant under the Euclidean group $E(2)$ or its universal cover $\widetilde{E(2)}$ respectively. However, such direct attempts are facing serious difficulties concerning either covariance or the Hilbert-Schmidt condition of the occurring operators, because the method of implementing multiplication operators as Bogoliubov transformations is basically restricted to one dimension. We will therefore try to circumvent these problems by considering simply a tensor product of a local field on $\mathbb{R}^2$ with the previously constructed circle-fields.

For this purpose consider for $f \in L^2(\mathbb{R}^2)$ the free field

$$\Psi(f) := c^*(f) + c(\overline{f}), \quad \Psi(\overline{f})^* = \Psi(f), \tag{V.38}$$

on the anti-symmetric Fock space $\mathcal{F}_a(L^2(\mathbb{R}^2))$, where c and c^* are the usual annihilation and creation operators on $\mathcal{F}_a(L^2(\mathbb{R}^2))$. From the anti-commutation relations of c and c^*,

$$\{c(f), c(g)\} = 0, \quad \{c(f), c^*(g)\} = \langle f, g\rangle,$$

it follows that this field satisfies

$$\Psi(f)\Psi(g) = -\Psi(g)\Psi(f),$$

if the test functions f and g are such that $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$. In addition Ψ is covariant with respect to to the second quantization of the pullback representation $\mathcal{U}$ of $E(2)$ defined on $L^2(\mathbb{R}^2)$ according to

$$(\mathcal{U}(\vec{a}, \omega)f)(\vec{x}) := f(R(-\omega)(\vec{x} - \vec{a})), \tag{V.39}$$

where $R(\omega)$ is the usual rotation matrix acting on vectors in $\mathbb{R}^2$.

Now take as Hilbert space the tensor product

$$\mathcal{H} = \mathcal{F}_a(L^2(\mathbb{R}^2)) \otimes \mathcal{F}_a(L^2(S_1)), \tag{V.40}$$

with a representation of $\widetilde{E(2)}$ of the form

$$\hat{\Gamma}(\mathcal{U}(\vec{a},\omega)) \otimes e^{is\omega Q^2}\hat{\Gamma}(U_1(\omega)). \tag{V.41}$$

For every $f \in L^2(\mathbb{R}^2)$, admissible χ_ε and $\omega \in \mathbb{R}$ one can then define on this space the new fields

$$F_\omega[f,\chi_\varepsilon] := \Psi(f) \otimes \Phi_\omega[\chi_\varepsilon]. \tag{V.42}$$

They of course inherit the anyonic commutation relations and are covariant under the representation (V.41), where the translations only act on the first tensor factor (shifting the support of the test function) and the rotations act on both. The motivation behind this definition is that these field operators can be interpreted as being localized in *conelike regions* on the two-dimensional plane. More specifically consider the following subset of $\mathbb{R}^2$,

$$C[f, I^\varepsilon_\omega] := \operatorname{supp} f + \mathbb{R}_+ \bigcup_{\mu \in I^\varepsilon_\omega} \vec{n}_\mu \subset \mathbb{R}^2, {}^3 \tag{V.43}$$

where $\vec{n}_\mu \in \mathbb{R}^2$ is a unit vector in the direction μ, i.e.

$$\vec{n}_\mu := R(-\mu)\vec{n}_0,$$

with a standard unit-vector $\vec{n}_0$, e.g. $\vec{n}_0 = \binom{0}{1}$. This defines a cone-shaped region which "starts" in $\operatorname{supp} f$ and extends to infinity in the set of directions given by I^ε_ω. Since a translation only acts on the test function f it shifts the whole cone, whereas a rotation also changes the interval I corresponding to the asymptotic directions of the cone $C[f, I]$. The interpretation of such cones as localization regions for the fields (V.42) is then possible because of the following commutation relations, which result from the anti-commutativity of the local field Ψ and the anyonic commutation relations of the circle field Φ.

Proposition V.3. *For compactly localized test functions f_1, f_2 and intervals $\tilde{I}_1 = \tilde{I}^{\varepsilon_1}_{\omega_1}$ and $\tilde{I}_2 = \tilde{I}^{\varepsilon_2}_{\omega_2}$ such that the corresponding cones do not intersect, i.e.*

$$C[f_1, I_1] \cap C[f_2, I_2] = \emptyset,$$

the fields $F_1 = F_{\omega_1}[f_1, \chi_{\varepsilon_1}]$ and $F_2 = F_{\omega_2}[f_2, \chi_{\varepsilon_2}]$ satisfy

$$\begin{aligned} F_1\, F_2 &= e^{2\pi i s(2N(\tilde{I}_1,\tilde{I}_2)+1)}\; F_2\, F_1, \\ F_1\, F_2^* &= e^{-2\pi i s(2N(\tilde{I}_1,\tilde{I}_2)+1)}\; F_2^*\, F_1. \end{aligned} \tag{V.44}$$

Proof. From the definition (V.43) it is clear that if the cones $C[f_1, I_1]$ and $C[f_2, I_2]$ are disjoint then also the test function supports and intervals have to satisfy[4]

$$\operatorname{supp} f_1 \cap \operatorname{supp} f_2 = \emptyset, \text{ and } I_1 \cap I_2 = \emptyset.$$

[3]Expressions of the form $\mathbb{R}_+\mathcal{O}$ for $\mathcal{O} \subset \mathbb{R}^2, \vec{0} \notin \mathcal{O}$ are shorthand for $\{\nu\, \mathcal{O} |\ \nu \in \mathbb{R}, \nu > 0\} \subset \mathbb{R}^2$.

[4]Note that the converse is not true, namely there can be non-overlapping test functions and intervals such that the corresponding cones actually have a (finite) overlap.

But under this conditions we get that

$$\Psi(f_1)\Psi(f_2) = (-1)\Psi(f_2)\Psi(f_1), \text{ and}$$
$$\Phi_{\omega_1}[\chi_{\varepsilon_1}]\,\Phi_{\omega_2}[\chi_{\varepsilon_2}] = (-1)e^{2\pi i s(2N(\tilde{I}_1,\tilde{I}_2)+1)}\;\Phi_{\omega_2}[\chi_{\varepsilon_2}]\,\Phi_{\omega_1}[\chi_{\varepsilon_1}].$$

The simple tensor product structure of the fields $F_\omega[f,\chi_\varepsilon]$ then leads to the asserted commutation relations. □

Remark: In addition to the conelike localization regions $C[f, I^\varepsilon_\omega]$ (V.43) (which are obviously invariant under $\omega \mapsto \omega + 2\pi$), the fields also depend on the winding number $n(\omega)$ of ω, which determines the commutation relations. To account for this fact the fields can be interpreted to be localized in *"generalized cones"* (or "paths of cones" as defined in the introduction, see e.g. [68]) which are defined in the following way. A usual cone C in two dimensions is determined by a point $\vec{x} \in \mathbb{R}^2$ (its apex) and an interval I on the circle, specifying the asymptotic directions contained in C. Hence one can denote a cone by the pair $C = (\vec{x}, I)$, where the width of I determining the opening angle of C should be smaller than π.

If we now allow for generalized intervals $\tilde{I}$ on the universal covering $\widetilde{S_1}$ of the circle one can define generalized cones $\tilde{C}$ as pairs

$$\tilde{C} = (\vec{x}, \tilde{I}), \quad \vec{x} \in \mathbb{R}^2, \tilde{I} \subset \widetilde{S_1}. \tag{V.45}$$

After smearing in $\vec{x}$ with a test function f we can obviously also define "smeared" generalized cones as $\tilde{C} = (f, \tilde{I}) \equiv (\operatorname{supp} f, \tilde{I})$.

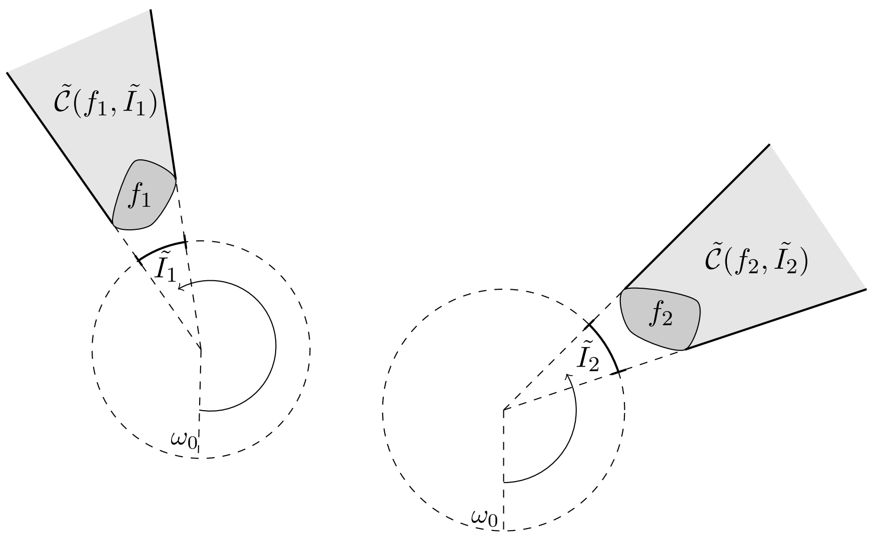


Figure V.1.: *Non-intersecting generalized cones $\tilde{\mathcal{C}}(f_1, \tilde{I}_1)$ and $\tilde{\mathcal{C}}(f_2, \tilde{I}_2)$*

We can therefore construct again, for every generalized cone $\tilde{C}$, the polynomial algebra $\mathcal{F}(\tilde{C})$ of fields localized in $\tilde{C}$. This net $\tilde{C} \mapsto \mathcal{F}(\tilde{C})$ then satisfies the same requirements for an anyonic field net as defined at the beginning of section V.1, with the obvious generalizations that we replace intervals $\tilde{I}$ with cones $\tilde{C}$ and covariance holds w.r.t. a representation of $\widetilde{E(2)}$ instead of just $\widetilde{U(1)}$.

Now this construction has an evident generalization to an arbitrary (interacting) field algebra in the following way. Consider a local and covariant (bosonic or fermionic) field net $\mathcal{O} \mapsto \mathcal{F}(\mathcal{O})$ on $\mathbb{R}^{1+2}$, indexed by double cones $\mathcal{O} \subset \mathbb{R}^{1+2}$. Defining a generalized cone on $\mathbb{R}^3$ by $\tilde{C} = (\mathcal{O}, \tilde{I})$ one can then define the corresponding algebra $\mathcal{F}(\tilde{C})$ as

$$\mathcal{F}(\tilde{C}) = \mathcal{F}(\mathcal{O}) \otimes \mathcal{F}(\tilde{I}),$$

where $\mathcal{F}(\tilde{I})$ is just the algebra of fields on the (covering of the) circle localized in $\tilde{I}$. This leads again to a $\widetilde{E(2)}$-covariant and twisted-local field net, localized in generalized cones $\tilde{C}$ (defined by a double cone $\mathcal{O}$ and an interval $\tilde{I}$) with winding-number dependent commutation relations.

If the original net was interacting then also the new composite net has a nontrivial scattering matrix. However, in the construction of the scattering states one has to take into account the extended localization regions of the fields and the new S-matrix should also depend on the relative winding number of the occurring fields (see e.g. [7, 29, 36]).

V.3. Conclusion

We have seen that it is possible to construct for every interval $\tilde{I}$ on the universal covering of the circle $\widetilde{S_1}$ a field algebra $\mathcal{F}(\tilde{I})$, such that the resulting net $\tilde{I} \mapsto \mathcal{F}(\tilde{I})$ is covariant under a representation of $\widetilde{U(1)}$ for arbitrary real-valued spin. For non-intersecting intervals the corresponding field algebras satisfy twisted commutation relations which only depend on the relative winding number of the intervals. This is achieved by considering unitary implementers of certain one-particle multiplication operators where the occurring Schwinger term is used to change the commutation relations of the operators in the desired way. In contrast to previous similar constructions this leads to a *local* quantum field theory for anyons on the circle. Moreover, everything is defined explicitly and non-perturbatively on the well known anti-symmetric Fock space and no taking of limits (or thereby leaving the Fock space) is needed.

By taking tensor products of these “circle anyons” with any local covariant quantum field theory one can obtain a cone-local theory with anyonic commutation relations. However, potential boost-covariance of the original field theory is lost so this only leads to a non-relativistic theory (covariant under $\widetilde{E(2)}$ plus potential time translations). A construction of a cone-local anyonic field net, which is covariant under the full Poincaré group in $1+2$ dimensions would be desirable but unfortunately this cannot be achieved by the methods described here. The first problem is that the "trick" using an auxiliary (2π-periodic) field, which then gets lifted to a covering space, only works in this simple way for the pure rotation group. For the covering of the full Lorentz group $\tilde{\mathcal{L}}_+^\uparrow$ one would get a representation on Fock space of the form $U(\tilde{\Lambda}) \sim e^{isQ\Omega(\tilde{\Lambda})}U_0(\Lambda)$ where $\Omega(\tilde{\Lambda})$ is now an operator on the Hilbert space instead of a mere constant ω as for the group $\widetilde{U(1)}$ (see e.g. [68] or section III.3.2 for a possible representation of $\tilde{\mathcal{L}}_+^\uparrow$ on the mass shell). Another problem is that the method of considering multiplication operators on the one-particle Hilbert space as implementable Bogoliubov transformations leads to problems concerning the Hilbert-Schmidt property for theories in more than one dimension (see e.g. [50]).

Nevertheless, the above construction provides a simple example of a (cone-)local covariant – and possibly interacting – quantum field net which exhibits the main features expected in a full $2 + 1$ dimensional theory of anyons, namely the relation between nontrivial behavior under 2π-rotations and the dependence on some winding number of the cone-shaped localization regions.

VI. Summary and Outlook

As we briefly sketched in the introduction, it follows from the principles of algebraic quantum field theory that charged particles in low dimensions do not have to be bosons or fermions, but their statistics is governed by the braid group. It has also been shown that in 2+1 dimensions the fields creating them from the vacuum have to be localized in so-called paths of spacelike cones (or wedges) and the phase factor in their commutation relations is determined by their relative winding number and a statistics parameter. Furthermore, they are covariant w.r.t a representation of the Poincaré group with a real-valued spin which is connected to the statistics parameter. However, there are no explicit well-defined examples of such field operators which comply with all the conditions for an anyonic field net in the sense of AQFT. The reason why such a construction is much more difficult than for bosons or fermions is the no-go theorem by Bros and Mund stating that a theory containing cone-localized anyons has to be fully interacting.

In this work we approached this problem from various directions by leaving out some of the underlying assumptions one makes in the abstract algebraic theory. First of all – despite a similar no-go theorem for free local anyons in $d = 1 + 1$ – on two-dimensional Minkowski space it is possible to construct compactly localized quantum field nets with anyonic commutation relations by using the theory of loop groups and implementable Bogoliubov transformations. This works for every mass $m \geq 0$ and every statistics parameter $\lambda \in \mathbb{R}$ and the resulting field operators create vectors with arbitrary high particle number from the vacuum. It has to be pointed out that several important features of 2+1 dimensional anyons are missing in 1+1 dimensional models. There are no localization regions extending to infinity, the localization on some kind of covering space is absent and there is no rotational covariance with real-valued (non half-integer) spin.

In 2+1 dimensions one way to circumvent the no-go theorem is to consider field operators which are only localizable in wedge-shaped regions (or more precisely "paths of wedges"). A relatively recent and convenient method to construct such wedge local generators is based on multiplicative deformations on the Fock space developed by G. Lechner in [57]. Generalizing this procedure to the charged case it is possible to choose the deformation functions in such a way that the resulting polarization free generators satisfy anyonic commutation relations and are covariant under a Poincaré group representation with arbitrary real-valued spin. By taking intersections of such wedge algebras one could in principle obtain algebras localized in spacelike cones (or smaller regions) but from the no-go theorem it follows that these algebras are too small in the sense that they cannot satisfy the Reeh-Schlieder condition. So this approach does not lead to cone-localized anyon field algebras in the sense of algebraic quantum field theory but one obtains wedge-local polarization free generators. Furthermore, it is possible to enlarge the class of admissible deformations in the case of charged fields. In the special case where the standard deformation function R is taken to be the identity the resulting polarization free generators can be interpreted as being localized in tube-like regions extending to infinity in two opposite spacelike directions. Although this is an improvement in localization compared to the wedges, the set of possible tubes which are spacelike separated to a given tube is still rather restricted.

To further demonstrate the connection between localization, statistics and spin of quantum field nets, in the last chapter we focused first only on the rotational degrees of freedom by constructing anyonic field operators on the circle. Using again the method of implementing multiplication operators as Bogoliubov transformations on the Fock space one ends up with a compactly localized field net, which has anyonic commutation relations and is covariant under a representation of $U(1) \simeq SO(2)$ (the "rotations") for arbitrary real-valued spin. More precisely, the field algebras are localized in intervals on the universal covering space of the circle, which allows for the concept of relative winding number of two non-intersecting intervals. By taking the tensor product with a bosonic or fermionic local covariant quantum field theory on $\mathbb{R}^{2+1}$ one ends up with a field net on $\mathbb{R}^3$ which is localized in generalized spatial cones, has real-valued spin and still satisfies anyonic commutation relations. However, the covariance under boosts is lost by taking the tensor product so the resulting field net is not fully Poincaré covariant.

The various constructions mentioned in this work still leave a lot of open questions to be analyzed further. It would for example be desirable to classify the *most general deformation functions* possible in the charged case in 2+1 and 1+1 dimensions, just like for the neutral bosonic case in [55]. Although the functions in equation (III.32) are believed to generate the most general class of deformations in 1+1 dimensions a conclusive proof for this claim could not yet be found. In 2+1 dimensions the same function R has been used on the particle and the anti-particle subspace for the sake of simplicity but it seems a priori also possible to choose different functions in this case. Another interesting question that could be asked is if it is possible to implement *non-abelian statistics* within any of the above approaches. This would involve multi-component fields and the simple commutation relations governed by a mere phase factor would have to be replaced by more complicated exchange algebra relations (see e.g. [31]). Also the charge structure of the theory would have to be more complicated and one cannot have a field algebra acting on a Hilbert space of the form $\oplus \mathcal{H}_q$, with charge quantum numbers $q \in \mathbb{Z}$.

Yet another possible generalization concerns the potential localization regions. Since we restricted ourselves to convex (and causally complete) regions in the beginning the only relevant localization regions are basically double cones, spacelike cones and wedges. If the condition of convexity is dropped one could also consider regions which are not simply connected, like causally completed rings in $d = 2+1$, i.e. regions with a hole. A net of field algebras indexed by such regions could then have a statistics operator that depends on which of two regions is localized "inside" the other. This would be similar to considering localization regions in $d = 1 + 1$ which are not connected allowing for more possibilities of relative localization than just being to the left or right of another region. The phase factor in the commutation relations of such field operators could then take more than two different values.

A still unsolved problem is the construction of a fully Poincaré covariant cone-localized anyonic field net on three dimensional Minkowski space satisfying ideally also the Reeh-Schlieder condition. One would be tempted to generalize the method of using multiplication operators and the current algebra to higher dimensions but unfortunately they don't lead to implementable Bogoliubov transformations in $d \geq 3$. There are, however, works concerning the current algebra and its cocycles in higher dimensions (see e.g. [50, 51, 66] and references therein) and together with a more fundamental algebraic viewpoint this could lead to a similar construction as for the massless case in $d = 1 + 1$. A possibility would be to use ideas similar to the simple kinematic model in [9,10], where states containing gauge charges are defined which can only be localized in spatial cones due to Gauss's theorem. One would need to define an observable algebra, a suitable state determining the vacuum representation and then find physically relevant transportable endomorphisms of this algebra which are localized in spacelike cones and possibly lead to braid group statistics. To make the construction more explicit a concrete realization of the Hilbert space and the action of the algebras on it would then also be desirable. It is, however, unclear how the dependence on a path of cones (needed for proper anyonic statistics) can arise in such an approach.

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