universität wien

DISSERTATION

Titel der Dissertation

„Forcing: Larger cardinals in Cichon's diagram, and PFA(S)[S]“

Verfasser

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Wien, 2015

Studienkennzahl lt. Studienblatt:	A 796 605 405
Dissertationsgebiet lt. Studienblatt:	Mathematik
Betreuerin / Betreuer:	Dr. Jakob Kellner

Acknowledgements

I wish to thank Jakob Kellner for supervising the completion of this thesis. Together with Martin Goldstern and Saharon Shelah he is really responsible for the formative work in the first part of this thesis. Saharon Shelah also welcomed me to Jerusalem in 2012, where aspects of this work came to be further clarified, and for which I am grateful.

The second part of this thesis would be impossible without Frank D. Tall and Stevo Todorcevic who brought these models to my attention. Numerous invaluable conversations lead to my understanding of these techniques, and the results given herein.

Special mention must be given to Sy-David Friedman, who has been very welcoming of myself at the Kurt Gödel Research Center from my arrival in Vienna in 2008.

Last, but certainly not least, this thesis could not have been completed without the love and support of my wife, Vera Fischer. Through the ups and downs of this process she has stood by me, and I only hope I am worthy of such affection.

Contents

Introduction

The main theme of this thesis is the application of the method of *forcing* to different mathematical structures. In particular, to the structure of the real line, $\mathbb{R}$, and general topological spaces.

In the first part (joint work with M. Goldstern, J. Kellner, and S. Shelah) we continue the long study of combinatorial aspects of the real line. Our focus is on the so-called "Cichoń's diagram," which provides a graphical representation of known inequalities between many "cardinal characteristics of the continuum," those mainly dealing with the measure-theoretic properties (as encapsulated by the ideal of Lebesgue null subsets of $\mathbb{R}$), and topological properties (as encapsultated by the ideal of meagre subsets of $\mathbb{R}$).

In the second part, we continue the investigation of properties of certain, special models of ZFC which were first examined by P. Larson and S. Todorcevic. These are the models formed after forcing with a Souslin tree S over a model of $\mathsf{PFA}(S)$ (the restriction of the Proper Forcing Axiom to those proper posets which do not add a cofinal branch through S). In particular, we look at properties of the compact (Hausdorff) spaces in such models. The specific property we look at is one step of a programme initiated by F.D. Tall to derive the consistency of the following statement: all hereditarily normal manifolds of dimension >1 are metrizable.

Part I

Creature forcing and Cichon's diagram

Joint work with M. Goldstern, J. Kellner, and S. Shelah

Chapter 1

Introduction

1.1 The result and its history

Let $\mathcal{N}$ denote the ideal of Lebesgue null sets, and $\mathcal{M}$ the ideal of meager sets. We prove (see Theorem 6.2.0.7) that consistently, several cardinal characteristics of Cichoń's Diagram (see Figure 1.1) are (simultaneously) different:

$$\aleph_1 = \mathrm{cov}(\mathcal{N}) = \mathfrak{d} < \mathrm{non}(\mathcal{M}) < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0}.$$

Since our model will satisfy $\mathfrak{d} = \aleph_1$, will also have $\mathrm{non}(\mathcal{M}) = \mathrm{cof}(\mathcal{M})$. The desired values of the

$$\begin{tikzcd}
 & \mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}) \arrow[r] & 2^{\aleph_0} \\
 & & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[r] \arrow[uu] & \mathrm{add}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{cov}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}$$

Figure 1.1: Cichoń's diagram. An arrow between $\mathfrak{x}$ and $\mathfrak{y}$ indicates that $\mathfrak{x} \leq \mathfrak{y}$. Moreover, $\max(\mathfrak{d}, \mathrm{non}(\mathcal{M})) = \mathrm{cof}(\mathcal{M})$ and $\min(\mathfrak{b}, \mathrm{cov}(\mathcal{M})) = \mathrm{add}(\mathcal{M})$.

cardinals $\mathrm{non}(\mathcal{M}), \mathrm{non}(\mathcal{N}), \mathrm{cof}(\mathcal{N}), 2^{\aleph_0}$ can be chosen quite arbitrarily, as long as they are ordered as indicated and each satisfies $\kappa^{\aleph_0} = \kappa$.

A (by now) classical series of theorems [3, 5, 9, 10, 11, 17, 21, 22, 24, 25, 28] proves these (in)equalities in ZFC and shows that they are the only ones provable, more precisely: all assignments of the values $\aleph_1$ and $\aleph_2$ to the characteristics in Cichoń's Diagram are consistent, provided they do not contradict the above (in)equalities. (A complete proof can be found in [4, chapter 7]).

This does not answer the question whether three (or more) characteristics can be made simultaneously different. The general expectation is that this should always be possible, but may require quite complicated forcing methods: We cannot use the two best understood methods, countable support iterations of proper forcings (as it forces $2^{\aleph_0} \leq \aleph_2$) and, at least for the "right hand side" of the diagram, we cannot use finite support iterations of ccc forcings in the straightforward way (as it adds lots of Cohen reals, and thus increases $\mathrm{cov}(\mathcal{M})$ to $2^{\aleph_0}$).

There are ways to overcome this obstacle. One way would be to first increase the continuum in a "long" finite support iteration, resulting in $\mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}$, and then "collapsing" $\mathrm{cov}(\mathcal{M})$ in another, "short" finite support iteration. In a much more sophisticated version of this idea, Mejía [20] recently constructed several models with many simultaneously different cardinal characteristics in Cichoń's Diagram (building on work of Brendle [7], Blass-Shelah [6] and Brendle-Fischer [8]).

We take a different approach, completely avoiding finite support, and use something in between a countable and finite support product (or: a form of iteration with very "restricted memory").

This construction avoids Cohen reals, in fact it is ω^ω-bounding, resulting in $\mathfrak{d} = \aleph_1$. This way we get an independence result "orthogonal" to the ccc/finite-support results of Mejía.

The fact that our construction is ω^ω-bounding is not incidental, but rather a necessary consequence of the two features which, in our construction, are needed to guarantee properness: a "compact" or "finite splitting" version of pure decision, and fusion (which together gives a strong version of Baumgartner's Axiom A and in particular properness and ω^ω-bounding).

We think that our construction can be used for various other independence results with $\mathfrak{d} = \aleph_1$, but the construction would require considerable remodeling if we want to use it for similar results with $\mathfrak{d} > \aleph_1$, even more so for $\mathfrak{b} > \aleph_1$.

1.2 A very informal overview of the construction

The obvious attempt to prove the theorem would be the following: Find a forcing for each cardinal characteristic $\mathfrak{x}$ that increases $\mathfrak{x}$ but leaves the other characteristics unchanged. More specifically: Find the following forcing notions:

- $\mathbb{Q}_{\mathtt{nm}}$, adding a new meager set which will contain all old reals.
  Adding many such sets will tend to make non($\mathcal{M}$) large.
- $\mathbb{Q}_{\mathtt{nn}}$, adding a new measure zero set which will contain all old reals.
  Adding many such sets will tend to make non($\mathcal{N}$) large.
- $\mathbb{Q}_{\mathtt{cn}}$, adding a new measure zero set which is not contained in any old measure zero set.
  Adding many such sets will tend to make cf($\mathcal{N}$) large.
- $\mathbb{Q}_{\mathtt{sk}}$, adding a kind of Sacks real, in the sense that the generic real does not change any other cardinal characteristics: every new real is bounded by an old real, is contained in an old measure zero set, etc.
  Adding many such reals will tend to make the continuum large.

For each $t \in \{\mathtt{nm}, \mathtt{nn}, \mathtt{cn}, \mathtt{sk}\}$, our $\mathbb{Q}_t$ will be a finitely splitting tree forcing; $\mathbb{Q}_{\mathtt{nm}}$ will be "lim-inf" (think of a tree forcing where we require large splitting at every node, not just infinitely many along every branch; i.e., more like Laver or Cohen than Miller or Sacks; however note that in contrast to Laver all our forcings are finitely splitting); the other ones will be "lim-sup" (think of forcings like Sacks or Silver).

We then fix for each t a cardinal κ_t, and take some kind of product (or: iteration) of κ_t many copies of $\mathbb{Q}_t$, and hope for the best. Here we arrive at the obvious problem: Which product or iteration will work? As mentioned above, neither a finite support iteration[1] nor a countable support iteration will work, and it is not clear why a product will not collapse the continuum. So we will introduce a modification of the product construction.

The paper is divided into two parts: In part 1 we describe the "general" forcing construction (let us call it "framework"), in part 2, the "application", we use the framework to construct a specific forcing that proves the main theorem.

[1] To avoid a wrong impression: our specific forcings $\mathbb{Q}_t$ will not be ccc, so a finite support iteration would not work anyway.

Part 1: In Sections 2–5 we present the "framework": Starting with building blocks (so-called "subatoms"), we define the forcing $\mathbb{Q}$. This is an instance of creature forcing. (The standard reference for creature forcing is Rosłanowski and Shelah [25], but our presentation will be self-contained. Our framework is a continuation of [14, 13], where the central requirement to get properness was "decisiveness". In this paper, decisiveness does not appear explicitly, but is implicit in the way that the subatoms are combined to so-called atoms.)

We fix a set Ξ of indices. (For the application, we will partition Ξ into sets Ξ_t of size κ_t for $t \in \{\mathtt{nm}, \mathtt{nn}, \mathtt{cn}, \mathtt{sk}\}$ as above.) The forcing $\mathbb{Q}$ will "live" on the product $\Xi \times \omega$, i.e., a condition $p \in \mathbb{Q}$ will contain for certain (ξ, n) a "creature" $p(\xi, n)$, a finite object that gives some information about the generic filter.

More specifically: There is a countable subset $\mathrm{supp}(p) \subseteq \Xi$, and for each $\xi \in \mathrm{supp}(p)$ the condition up to some level $n_0(\xi)$ consists of a so-called trunk (where the according finite initial segment of the generic real $\underaccent{\tilde}{y}_\xi$ is already completely determined), and for all $n > n_0(\xi)$ there is a creature $p(\xi, n)$, an element of a fixed finite set $\mathsf{K}_{\xi,n}$, which gives several (finitely many) possibilities for the according segment of the generic real $\underaccent{\tilde}{y}_\xi$. We assign a "norm" to the creature, a real number that measures the "number of possibilities" (or: the amount of freedom that the creature leaves for the generic): More possibilities means larger norm.

Moreover, for each m there are only finitely many ξ with $n_0(\xi) \leq m$ (i.e., at each level m there live only finitely many creatures of p). We can then set the norm of p at m to be the minimum of the norms of $p(\xi, n)$ over all ξ "active" at level m.

A requirement for a p to be a valid condition in $\mathbb{Q}$ is that the norms at level m diverge to infinity for $m \to \infty$ (i.e., the lim-inf of the norms is infinite).

So far, $\mathbb{Q}$ seems to be a lim-inf forcing, but recall that we want to use lim-inf as well as lim-sup.

So let us redefine $\mathbb{Q}$: We will "cheat" by allowing "gluing". We declare a subset of Ξ to be the set $\Xi_{\mathtt{ls}}$ of "lim-sup indices" (in the application this will be $\Xi_{\mathtt{nn}} \cup \Xi_{\mathtt{cn}}$). Forget the "norm of p at level m" and the lim-inf condition above. Instead, we partition the set of levels ω into finite intervals $\omega = I_0 \cup I_1 \cup \dots$ (this partition depends on the condition and can be coarsened when we go to a stronger condition). For such an interval I, we declare all creatures whose levels belong to I to constitute a "compound creature" with a "compound norm":

- Basically, for each $\xi \in \Xi_{\mathtt{ls}}$ we set $\mathrm{nor}(p, I, \xi)$ to be the maximum of the norms of $p(\xi, m)$ with $m \in I$;
- for other ξ we take the minimum rather than the maximum;
- and finally we set $\mathrm{nor}(p, I)$ to be the minimum of $\mathrm{nor}(p, I, \xi)$ for all (finitely many) ξ active at some level in I.

The new lim-inf condition is that $\mathrm{nor}(p, I_k)$ diverges to infinity with $k \to \infty$.

While this may give some basic idea about the construction, things really are more complicated: We will require the well-known "halving" property of creature forcing (to prove Axiom A). Moreover the Sacks part, i.e., $\mathbb{Q}_{\mathtt{sk}}$ on the indices $\Xi_{\mathtt{sk}} \subset \Xi$, does not fit well into the framework as presented above and requires special treatment. This will mathematically not be very complicated but unfortunately will make our notation much more awkward and unpleasant.

A central requirement on our building blocks (subatoms) will be another well-known property of creature-forcing: "bigness", a kind of Ramsey property connected to the statement that creatures at a level m are "much bigger" than everything that "happened below m".

Using these requirements, we will show the following:

- (Assuming CH in V) $\mathbb{Q}$ is $\aleph_2$-cc (a standard Δ-system argument).
- We say that p essentially decides a name $\underset{\sim}{\tau}$ of an ordinal, if there is a level m such that: whenever we increase the trunk of p up to m (for this, there are only finitely many possibilities), we know the value of $\underset{\sim}{\tau}$. In other words, knowing the generic up to m (on some finite set of indices), we also know the value of $\underset{\sim}{\tau}$.
- Pure decision and fusion: Given a name $\underset{\sim}{\tau}$ of an ordinal and a condition p, we can strengthen p to a condition q essentially deciding $\underset{\sim}{\tau}$. Moreover, we can do this in a way such that p and q agree below a given level h and the norms above this level do not drop below a given bound. (This is called "pure decision".)

  This in turn implies "fusion": We can iterate this strengthening for infinitely many names $\underset{\sim}{\tau}_\ell$, resulting in a common extension q_∞ which essentially decides all $\underset{\sim}{\tau}_\ell$.

  (While fusion is an obvious property of the framework, pure decision is the central result of part 1, and will use the requirements on bigness and halving).
- The usual standard argument then gives continuous reading (every real is a continuous image of (countably many) generic reals), a strong version of Axiom A and thus ω^ω-bounding and proper. (Recall that we have "finite splitting", i.e., essentially deciding implies that there are only finitely many potential values.)
- We also get a Lipschitz variant of continuous reading, "rapid reading", which implies that the forcing adds no random reals (and which will be essential for many of the proofs in part 2).

Part 2: In Sections 6–10, we define the specific forcings $\mathbb{Q}_t$ (or rather: the building blocks, i.e., the subatoms, for these forcings) for $t \in \{\mathtt{nm}, \mathtt{nn}, \mathtt{cn}\}$ (the Sacks case is already dealt with in part 1).

We prove that these subatoms satisfy the bigness requirements of the framework, and we prove the various parts of the main theorem.

Annotated Contents

definitions of the t-subatoms (other than Sacks) will be given in Sections 7, 8, 10. For each type t the forcing will contain a "t-part" of size κ_t.
We formulate the main theorem: $\mathbb{Q}$ will force each invariant to be the respective κ_t.
We show that the Sacks part satisfies a Sacks property, which implies $\mathrm{cof}(\mathcal{N}) \leq \kappa_{\mathtt{cn}}$ in the generic extension.
And just using the fact that only the nm-indices are "lim-inf", we show that $\mathrm{non}(\mathcal{M}) \leq \kappa_{\mathtt{nm}}$.

Section 7, p. 46 We define the nm-subatoms and prove $\mathrm{non}(\mathcal{M}) \geq \kappa_{\mathtt{nm}}$.

Section 8, p. 48 We define the nn-subatoms and prove $\mathrm{non}(\mathcal{N}) \geq \kappa_{\mathtt{nn}}$.

Section 9, p. 50 We mention some simple facts about counting, and use them to define the counting norm, lognor, for the cn subatoms.

Section 10, p. 53 We define the cn-subatoms and prove $\mathrm{cof}(\mathcal{N}) \geq \kappa_{\mathtt{cn}}$. And finally, we show $\mathrm{non}(\mathcal{N}) \leq \kappa_{\mathtt{nn}}$.

1.3 Acknowledgements

We are grateful to Diego Mejía for pointing out several embarrassing oversights.

Chapter 2

The definition of the forcing $\mathbb{Q}$

2.1 Subatomic creatures

Definition 2.1.0.1. Let POSS be a finite set. A *subatomic family* living on POSS consists of a finite set K (whose elements are called *subatomic creatures*, or *subatoms*, for short), a quasiorder $\leq$ on K and functions poss and nor with domain K, satisfying the following for all $x \in \mathsf{K}$:

- $\mathrm{poss}(x)$ is a nonempty subset of POSS;
- $\mathrm{nor}(x)$ is a nonnegative real number; and
- $y \leq x$ implies $\mathrm{poss}(y) \subseteq \mathrm{poss}(x)$.

To simplify notation, we further assume:

- If $|\,\mathrm{poss}(y)| = 1$, then $\mathrm{nor}(y) < 1$.
- For each $x \in \mathsf{K}$ and $a \in \mathrm{poss}(x)$ there is a $y \leq x$ with $\mathrm{poss}(y) = \{a\}$. (Such a subatom will be called a *singleton*.)

Notation 2.1.0.2. Abusing notation, we will just write K for the subatomic family $(\mathsf{K}, \leq, \mathrm{nor}, \mathrm{poss})$. If $y \leq x$ we will also say that y is "stronger than x" or is "a successor of x".

Remark 2.1.0.3. Our subatomic families will also have the following properties (which might make the picture clearer, but will not be used in any proof):

- x is determined[1] by $\mathrm{poss}(x)$ (i.e., the function $\mathrm{poss} : \mathsf{K} \to 2^{\mathrm{POSS}}$ is injective). So in particular $\mathrm{nor}(x)$ is determined by $\mathrm{poss}(x)$.
- $\mathrm{poss}(y) \subseteq \mathrm{poss}(x)$ implies $\mathrm{nor}(y) \leq \mathrm{nor}(x)$.
- $y \leq x$ iff $\mathrm{poss}(y) \subseteq \mathrm{poss}(x)$.

In the usual way we often identify a natural number n with the set $\{0, \ldots, n-1\}$, and write $m \in n$ for $m < n$; for example in the following definition.

Definition 2.1.0.4. Fix a natural number $B > 0$. We say that a subatom $x \in \mathsf{K}$ has *B-bigness* if for all colorings $c : \mathrm{poss}(x) \to B$ there is a $y \leq x$ such that $c \,|\, \mathrm{poss}(y)$ is constant and $\mathrm{nor}(y) \geq \mathrm{nor}(x) - 1$.[2] We say that the subatomic family K has *B-bigness* if each $x \in \mathsf{K}$ has B-bigness.

[1] The analogous statement will not be true for "compound creatures" (cf. Definition 2.5.0.19) because of the halving parameters.

[2] As only the number of "colors" is of importance, we may consider the codomain of the coloring function to be any set of cardinality B.

Given a subatom x in a fixed subatomic family K, we have the following facts.

- If $\mathrm{nor}(x) \leq 1$, then x has B-bigness for all $B > 0$. (Any coloring $c : \mathrm{poss}(x) \to B$ will be constant on $\mathrm{poss}(y)$ for any singleton $y \leq x$.)
- If $\mathrm{nor}(x) \geq 2$, then x cannot have $|\mathrm{poss}(x)|$-bigness. (The identity function $c : \mathrm{poss}(x) \to \mathrm{poss}(x)$ is only constant on singleton sets, and any singleton subatom has norm $< \mathrm{nor}(x) - 1$.)
- If x has B-bigness, then x has B'-bigness for all $1 \leq B' \leq B$.

Example 2.1.0.5. The basic example of a subatomic family with B-bigness is the following "counting norm": Fix any finite set POSS. A subatom x is a nonempty subset of POSS, $\mathrm{poss}(x) := x$, $y \leq x$ is defined as $y \subseteq x$, and we set

$$\mathrm{nor}(x) := \log_B |x|.$$

We get a stronger variant of bigness if we divide the norm by B:

$$\mathrm{nor}'(x) := \frac{\log_B(|x|)}{B}.$$

Then for each $F : \mathrm{poss}(x) \to B$ there is a $y \leq x$ such that $F \restriction \mathrm{poss}(y)$ is constant and $\mathrm{nor}'(y) \geq \mathrm{nor}'(x) - {}^1\!/_B$.

Remark 2.1.0.6. The above example (in the version nor') is actually used for the $\mathrm{non}(\mathcal{M})$-subatoms (cf. 7.1.0.1). The $\mathrm{cof}(\mathcal{N})$-subatoms (cf. Section 10.1) still use a counting norm, i.e., $\mathrm{nor}(x)$ only depends on the cardinality of $\mathrm{poss}(x)$, but the relation between $|\mathrm{poss}(x)|$ and $\mathrm{nor}(x)$ is more complicated. The $\mathrm{non}(\mathcal{N})$-subatoms (cf. Section 8.1) will use another kind of norm: $\mathrm{nor}(x)$ will not just depend on the cardinality of $\mathrm{poss}(x)$, but also on its structure.

Given a subatomic family with 2-bigness, it is straightforward to construct another subatomic family with arbitrary bigness by only altering the norm.

Lemma 2.1.0.7. *If K is a subatomic family with 2-bigness, then given any $b \geq 1$ replacing the norm of K with nor' defined by $\mathrm{nor}'(x) := {}^{\mathrm{nor}(x)}\!/_b$ results in a subatomic family with 2^b-bigness.*

Proof. Given $x \in \mathsf{K}$, and a coloring $c : \mathrm{poss}(x) \to \mathcal{P}(b)$, use the 2-bigness of the original subatomic family to inductively pick $x = x_0 \geq x_1 \geq \cdots \geq x_b = y$ so that for each $i < b$ we have $\mathrm{nor}(x_{i+1}) \geq \mathrm{nor}(x_i) - 1$ and $c_i \restriction \mathrm{poss}(x_{i+1})$ is constant, where $c_i : \mathrm{poss}(x_i) \to 2$ is defined by $c_i(a) = 1$ iff $i \in c(a)$. Then $c \restriction \mathrm{poss}(y)$ is constant, and $\mathrm{nor}'(y) = {}^{\mathrm{nor}(y)}\!/_b \geq {}^{(\mathrm{nor}(x)-b)}\!/_b = \mathrm{nor}'(x) - 1$. $\square$

2.2 Atomic creatures

We now describe how to combine subatomic families to create so-called atoms. Fix a natural number $J > 0$, and fix a parameter $\ell \in \omega$. We will first define the "measure" of subsets of J with respect to this parameter:

Definition 2.2.0.8. For $A \subseteq J$, we set

$$\mu^\ell(A) := \frac{\log_3(|A|)}{\ell + 1}$$

(or 0, if $A = \emptyset$).[3]

[3]So, technically $\mu^\ell(A)$ is defined to be ${}^{\log_3(\max\{|A|,1\})}\!/_{\ell+1}$.

We will later use the following easy observation about the "measure":

Lemma 2.2.0.9. *Suppose $k \leq \ell$, and $A_0, \dots, A_k$ are subsets of J. Then there are pairwise disjoint sets $B_0, \dots, B_k$ such that $B_i \subseteq A_i$, and $\mu^\ell(B_i) \geq \mu^\ell(A_i) - 1$ for all $i \leq k$.*

Proof. Note that if for some $i \leq k$ we have that $\mu^\ell(A_i) \leq 1$, then simply picking $B_i := \emptyset$ will introduce no obstructions. We may then assume that $\mu^\ell(A_i) > 1$ (meaning that $|A_i| \geq 3^{\ell+1}$) for each $i \leq k$. We now inductively construct $(k+1)$-tuples $(A_0^j, \dots, A_k^j)$ ($j \leq n := {}^{k(k+1)}/_2$ where $A_i^0 = A_i$ for each $i \leq k$, and at stage $j < n$ we handle a distinct pair (i^0, i^1) with $i^0 < i^1 \leq k$ so that

- $A_{i^0}^{j+1} \subseteq A_{i^0}^j$, $|A_{i^0}^{j+1}| \geq {}^{|A_{i^0}^j|}/_3$;
- $A_{i^0}^{j+1} \subseteq A_{i^1}^j$, $|A_{i^1}^{j+1}| \geq {}^{|A_{i^1}^j|}/_3$; and
- $A_{i^0}^{j+1} \cap A_{i^1}^{j+1} = \emptyset$.

(and $A_i^{j+1} = A_i^j$ for all other $i \leq k$). As $|A_{i^0}| \geq 3^{\ell+1}$ it follows by the induction that $|A_{i^0}^j| \geq 3$, and similarly $|A_{i^1}^j| \geq 3$, and so it is possible to partition the intersection $A_{i^0}^j \cap A_{i^1}^j$ into $Y \cup Z$ so that $|A_{i^0}^j \setminus Y| \geq {}^{|A_{i^0}^j|}/_3$ and $|A_{i^1}^j \setminus Z| \geq {}^{|A_{i^1}^j|}/_3$. We may then take $A_{i^0}^{j+1} := A_{i^0}^j \setminus Y$ and $A_{i^1}^{j+1} := A_{i^1}^j \setminus Z$.

After these steps, set $B_i := A_i^n$ for each $i \leq k$. It is clear that the B_i are pairwise disjoint (since if $i^0 < i^1 \leq k$ at some stage j we would have handled this pair, meaning that $A_{i^0}^{j+1} \cap A_{i^1}^{j+1} = \emptyset$, but $B_{i^0} \subseteq A_{i^0}^{j+1}$ and $B_{i_1} \subseteq A_{i^1}^{j+1}$). As each A_i was modified at most k times in the inductive construction it follows that $|B_i| \geq {}^{|A_i|}/_{3^k}$, and so $\mu^\ell(B_i) = {}^{\log_3(|B_i|)}/_{\ell+1} \geq {}^{\log_3(|A_i|/3^k)}/_{\ell+1} \geq {}^{\log_3(|A_i|)-\ell}/_{\ell+1} \geq \mu^\ell(A_i) - 1$. $\square$

Suppose now that for each $j \in J$ we have a subatomic family K_j living on a finite set POSS_j. We can now define the atoms built from the subatoms:

Definition 2.2.0.10.

- An *atomic creature*, or *atom*, $\mathfrak{a}$ consists of a sequence $(x_j)_{j \in J}$ where x_j is a K_j-subatom for all $j \in J$.
- The *norm* of an atom $\mathfrak{a} = (x_j)_{j \in J}$, $\mathrm{nor}(\mathfrak{a})$, is the maximal r with the following property: There is a set $A \subseteq J$ with $\mu^\ell(A) \geq r$ and $\mathrm{nor}(x_j) \geq r$ for all $j \in A$. We say that such an A "witnesses the norm" of $\mathfrak{a}$.

So the norm of an atom is large if there is a "large" subset A of J such that all subatoms in A are "large".

The following easy fact will be useful later:

Fact 2.2.0.11. *Suppose $A \subseteq J$ witnesses the norm of an atom $\mathfrak{a} = (x_j)_{j \in J}$, and let $\mathfrak{b} = (y_j)_{j \in J}$ be any atom which agrees with $\mathfrak{a}$ on all indices in A. Then* $\mathrm{nor}(\mathfrak{b}) \geq \mathrm{nor}(\mathfrak{a})$. *In particular, if* $\mathrm{nor}(y_j) \leq \mathrm{nor}(x_j)$ *for all $j \notin A$, then* $\mathrm{nor}(\mathfrak{b}) = \mathrm{nor}(\mathfrak{a})$.

2.3 Sacks columns

Given a (finite) tree T, its *splitting-size*, $\mathrm{nor}_{\mathrm{split}}(T)$, is defined as the maximal $\ell \in \omega$ such that there is a subset $S \subseteq T$ (with the induced order) which is order isomorphic to the complete binary tree $2^{\leq \ell}$ (of height ℓ with 2^ℓ many leaves). Equivalently: $2^{\leq \ell}$ embeds into T (i.e., there is an injection which preserves order in both directions, i.e., an order isomorphism onto the image).

Given a finite subset I of ω and $F \subseteq 2^I$, we can identify F with the tree of its restrictions $T_F = F \cup \{\eta | n : \eta \in F,\ n \in I\}$ (a tree of partial functions from I to 2, ordered by inclusion). We write $\mathrm{nor}_{\mathrm{split}}(F)$ for $\mathrm{nor}_{\mathrm{split}}(T_F)$.

The following establishes a basic combinatorial fact about this norm:

Definition and Lemma 2.3.0.12. There exists a function f with the following property:

- For each n, c: Whenever $2^{f(1,n,c)}$ is colored with c colors, then there is a homogeneous subset A of $2^{f(1,n,c)}$ such that $\mathrm{nor}_{\mathrm{split}}(A) \geq n$.
- More generally:
  For each j, n, c: Whenever $(2^{f(j,n,c)})^j$ is colored with c colors, then there are subsets $A_1, \dots, A_j$ of $2^{f(j,n,c)}$ such that the set $A_1 \times \cdots \times A_j$ is homogeneous,[4] and $\mathrm{nor}_{\mathrm{split}}(A_i) \geq n$ for all i.
- Moreover, f may be chosen to be monotone in each argument.

Proof. We define $f(j,n,c)$ recursively on j by $f(1,n,c) = n \cdot c$, and $f(j+1,n,c) = f(1,n,c^{2^{j \cdot f(j,n,c)}}) = n \cdot c^{2^{j \cdot f(j,n,c)}}$. Note that $f(j,n,1) = n$, and clearly any coloring $\pi : (2^n)^j \to 1$ is constant. We may then assume that $c > 1$ for the remainder of the proof.

We first show by induction on c that $f(1,n,c)$ is as required. Suppose that $f(1,n,c)$ works for some $c \geq 1$, and let $\pi : 2^{n \cdot (c+1)} \to c+1$ be a coloring. For $\eta \in 2^n$, let $[\eta] := \{\nu \in 2^{n+c \cdot n} : \eta \subseteq \nu\}$. Note that $\mathrm{nor}_{\mathrm{split}}([\eta]) = 2^{c \cdot n}$ for each $\eta \in 2^n$. If there is an $\eta \in 2^n$ such that $\pi \,|\, [\eta]$ omits one of $0, \dots, c$, then $\pi \,|\, [\eta]$ is a coloring with at most c colors, and so there must be an $A \subseteq [\eta] \subseteq 2^{n+c \cdot n}$ such that $\mathrm{nor}_{\mathrm{split}}(A) \geq n$ and $\pi \,|\, A$ is constant.

Otherwise, for each $\eta \in 2^n$ there is an $\nu_\eta \in [\eta]$ such that $\pi(\nu_\eta) = 0$. It follows that $A := \{\nu_\eta : \eta \in 2^n\}$ has splitting size n, and $\pi \,|\, A$ is constantly 0.

Assume that $f(j,n,c)$ satisfies the desired property for some $j \geq 1$. Set $p := f(j,n,c)$ and $q := c^{2^{j \cdot p}}$, so that $f(j+1,n,c) = n \cdot q = f(1,n,q)$. Suppose $\pi : (2^{n \cdot q})^{j+1} \to c$ is a coloring. Define $T := \{\eta \in 2^{n \cdot q} : \eta \,|\, [p, n \cdot q) \text{ is constantly } 0\}$. Since $c \geq 2$ it follows that $p < n \cdot q$, and so $\mathrm{nor}_{\mathrm{split}}(T) = p$. For $\eta \in 2^{n \cdot q}$ define $\pi_\eta : T^j \to c$ by $\pi_\eta(\eta_1, \dots, \eta_j) = \pi(\eta_1, \dots, \eta_j, \eta)$. Note that the mapping $\eta \mapsto \pi_\eta$ is a coloring of $2^{n \cdot q}$ by at most $c^{(2^p)^j} = q$ many colors. By the above it follows that there is an $A_{j+1} \subseteq 2^{n \cdot q}$ and a $\pi^* : T^j \to c$ such that $\mathrm{nor}_{\mathrm{split}}(A_{j+1}) \geq n$ and $\pi_\eta = \pi^*$ for each $\eta \in A_{j+1}$.

Then as π^* is a coloring of T^j by at most c colors, and as $\mathrm{nor}_{\mathrm{split}}(T) = p = f(j,n,c)$ by hypothesis for each $i \leq j$ there are $A_i \subseteq T \subseteq 2^{n \cdot q}$ with $\mathrm{nor}_{\mathrm{split}}(A_i) \geq n$ (for $i \leq j$) such that $A_1 \times \cdots \times A_j$ is homogeneous for π^*. It then follows that $A_1 \times \cdots \times A_j \times A_{j+1}$ is homogeneous for π. ☐

Definition 2.3.0.13. Suppose that I is a nonempty (finite) interval in ω. By a *Sacks column on* I we mean a nonempty $\mathfrak{s} \subseteq 2^I$. We say that another Sacks column $\mathfrak{s}'$ on I is *stronger* than $\mathfrak{s}$, and write $\mathfrak{s}' \leq \mathfrak{s}$, if $\mathfrak{s}' \subseteq \mathfrak{s}$.

We can naturally take products of columns that are stacked above each other:

Definition 2.3.0.14. Let $\mathfrak{s}_1$ be a Sacks column on an interval I_1 and let $\mathfrak{s}_2$ be a Sacks column on an interval I_2. If $\min(I_2) = \max(I_1) + 1$, then the *product* $\mathfrak{s}' = \mathfrak{s}_1 \otimes \mathfrak{s}_2$ is the Sacks column on $I_1 \cup I_2$ defined by $f \in \mathfrak{s}'$ iff $f \,|\, I_1 \in \mathfrak{s}_1$ and $f \,|\, I_2 \in \mathfrak{s}_2$.
Iterating this, we can take products of finitely many properly stacked[5] Sacks columns.

We now define the norm of a Sacks column $\mathfrak{s}$ on an interval I. Actually, we define a family of norms, using two parameters B and m. Later, we will virtually always use values of B and m determined by $\min(I)$; more details will come in Subsection 2.5 and Section 4.

[4] As in the case of the bigness of subatoms, only the number of "colors" of our coloring functions is of importance. Moreover, by the definition of the splitting norm it follows that $T_1, \dots, T_j$ are trees each of splitting size at least $f(j,n,c)$ and $\pi : T_1 \times \cdots \times T_j \to c$ is a coloring, then there are $A_i \subseteq T_i$ ($i \leq j$) such that $\mathrm{nor}_{\mathrm{split}}(A_i) \geq n$ for each i and $\pi \,|\, A_1 \times \cdots \times A_j$ is constant.

[5] Sacks columns $\mathfrak{s}_1, \dots, \mathfrak{s}_n$ on intervals $I_1, \dots, I_n$, respectively, are called *properly stacked* if $\min(I_{i+1}) = \max(I_i) + 1$ for each $i < n$.

Definition 2.3.0.15. $\mathrm{nor}^{B,m}_{\mathrm{Sacks}}(\mathfrak{s}) \geq n$ iff $n = 0$ or $\mathrm{nor}_{\mathrm{split}}(\mathfrak{s}) \geq F^B_m(n)$ where $F^B_m : \omega \to \omega$ is defined as follows: $F^B_m(0) = 1$ and $F^B_m(n+1) = f(m, F^B_m(n), B)$, where we use the function f of Definition 2.3.0.12.

In other words,

$$\mathrm{nor}^{B,m}_{\mathrm{Sacks}}(\mathfrak{s}) = \max(\{n \in \omega : F^B_m(n) \leq \mathrm{nor}_{\mathrm{split}}(\mathfrak{s})\} \cup \{0\}). \tag{2.3.0.16}$$

The exact definition of this norm will not be important in the rest of the paper; we will only require the following properties:

Lemma 2.3.0.17.

1. *If $\mathfrak{s}, \mathfrak{s}'$ have the same splitting-size, then* $\mathrm{nor}^{B,m}_{\mathrm{Sacks}}(\mathfrak{s}') = \mathrm{nor}^{B,m}_{\mathrm{Sacks}}(\mathfrak{s})$.
2. *If $\mathfrak{s}' \leq \mathfrak{s}$, $B' \geq B$ and $m' \geq m$, then* $\mathrm{nor}^{B',m'}_{\mathrm{Sacks}}(\mathfrak{s}') \leq \mathrm{nor}^{B,m}_{\mathrm{Sacks}}(\mathfrak{s})$.
3. $\mathrm{nor}^{B,m}_{\mathrm{Sacks}}(\mathfrak{s}_1 \otimes \cdots \otimes \mathfrak{s}_n) \geq \mathrm{nor}^{B,m}_{\mathrm{Sacks}}(\mathfrak{s}_i)$ *for all $1 \leq i \leq n$.*
4. *If I is large (with respect to B and m), then $\mathrm{nor}^{B,m}_{\mathrm{Sacks}}(2^I)$ will be large. More precisely, given $a \in \omega$, if $|I| > F^B_m(a)$, then $\mathrm{nor}^{B,m}_{\mathrm{Sacks}}(2^I) \geq a$.*
5. *We will later use the following simple (but awkward) consequence: Fix properly stacked intervals I, I' and a Sacks column $\mathfrak{s}$ on $I \cup I'$. Then there is an $\tilde{\mathfrak{s}} \leq \mathfrak{s}$ such that*
$$\mathrm{nor}^{B,m}_{\mathrm{Sacks}}(\tilde{\mathfrak{s}}) \geq \min\left(\mathrm{nor}^{B,m}_{\mathrm{Sacks}}(\mathfrak{s}), \mathrm{nor}^{B,m}_{\mathrm{Sacks}}(2^I)\right)$$
and $|\tilde{\mathfrak{s}}| \leq |2^I|$.
6. *(Bigness) For $i < m$, fix Sacks columns $\mathfrak{s}_i$ such that $\mathrm{nor}^{B,m}_{\mathrm{Sacks}}(\mathfrak{s}_i) \geq n+1$.*
Then for any "coloring" function $\pi : \prod_{i<m} \mathfrak{s}_i \to B$ there are Sacks columns $\mathfrak{s}'_i \leq \mathfrak{s}_i$ with $\mathrm{nor}^{B,m}_{\mathrm{Sacks}}(\mathfrak{s}'_i) \geq n$ such that π is constant on $\prod_{i<m} \mathfrak{s}'_i$.

Proof. For (5), just prune all unnecessary branches. In more detail: Note that $\mathrm{nor}_{\mathrm{split}}(2^I) = |I|$, and that $\mathrm{nor}^{B,m}_{\mathrm{Sacks}}$ is determined by the splitting-size $\mathrm{nor}_{\mathrm{split}}$. So we have to find $\tilde{\mathfrak{s}} \subseteq \mathfrak{s}$ with splitting size $r := \min(\mathrm{nor}_{\mathrm{split}}(\mathfrak{s}), |I|)$: Obviously we can find the binary tree $2^{\leq r}$ inside $\mathfrak{s}$ (as a suborder). Extend each of its maximal elements (uniquely), and take the downwards closure. This gives $\tilde{\mathfrak{s}}$.

(6) follows immediately from Lemma 2.3.0.12: We have $\mathrm{nor}_{\mathrm{split}}(\mathfrak{s}_i) \geq F^B_m(n+1) = f(m, F^B_m(n), B)$; so by the characteristic property of the function f, for any coloring function $\pi : \prod_{i<m} \mathfrak{s}_i \to B$ there are Sacks columns $\mathfrak{s}'_i \leq \mathfrak{s}_i$ with $\mathrm{nor}_{\mathrm{split}}(\mathfrak{s}'_i) \geq F^B_m(n)$ such that π is constant on $\prod_{i<m} \mathfrak{s}'_i$. So $\mathrm{nor}^{B,m}_{\mathrm{Sacks}}(\mathfrak{s}'_i) \geq n$. □

2.4 Setting the stage

We fix for the rest of this paper a nonempty (index) set Ξ. We furthermore assume that Ξ is partitioned into subsets $\Xi_{\mathtt{ls}}, \Xi_{\mathtt{li}}, \Xi_{\mathtt{sk}}$ ($\Xi_{\mathtt{li}}$ is nonempty, but $\Xi_{\mathtt{ls}}$ and $\Xi_{\mathtt{sk}}$ could be empty). For each $\xi \in \Xi$, we say that ξ is of type *lim-sup*, *lim-inf* or *Sacks* if ξ is an element of $\Xi_{\mathtt{ls}}$, $\Xi_{\mathtt{li}}$, or $\Xi_{\mathtt{sk}}$, respectively. We set $\Xi_{\text{non-sk}} := \Xi_{\mathtt{ls}} \cup \Xi_{\mathtt{li}} = \Xi \setminus \Xi_{\mathtt{sk}}$.

Our forcing will "live" on $\Xi \times \omega$. For $(\xi, \ell) \in \Xi \times \omega$ we call ξ the *index* and ℓ the *level.*

The "frame" of the forcing will be as follows:

Definition 2.4.0.18. 1. (For the "Sacks part":) We fix a sequence $(I_{\mathtt{sk},\ell})_{\ell\in\omega}$ of properly stacked intervals in ω.[6] For simplicity we further assume that $\min(I_{\mathtt{sk},0}) = 0$. Given natural numbers

[6] I.e., $I_{\mathtt{sk},\ell} = [\min(I_{\mathtt{sk},\ell}), \min(I_{\mathtt{sk},\ell+1}))$ for all $\ell \in \omega$.

$\ell < m$ we set $I_{\mathtt{sk},[\ell,m)} := \bigcup_{\ell \le h < m} I_{\mathtt{sk},h} = [\min(I_{\mathtt{sk},\ell}), \min(I_{\mathtt{sk},m}))$. A Sacks column on $I_{\mathtt{sk},[\ell,m)}$ is also called a "Sacks column between ℓ and m".

2. We fix for each level $\ell \in \omega$ some $J_\ell \in \omega$. A *sublevel* is a pair (ℓ, j) for $\ell \in \omega$ and $j \in J_\ell \cup \{-1\}$. (The sublevel $(\ell, -1)$ will be associated with the Sacks part at level ℓ.) We will usually denote sublevels by $\mathbf{u}$ or $\mathbf{v}$.

3. We say $\mathbf{v}$ is below $\mathbf{u}$, or $\mathbf{v} < \mathbf{u}$, if $\mathbf{v}$ lexicographically precedes $\mathbf{u}$. Note that this order has order type ω.

4. A sublevel $(\ell, -1)$ is called a Sacks sublevel; all other sublevels are called subatomic. Instead of $(\ell, -1)$ we will sometimes just write "the sublevel ℓ", and we sometimes just write "$\mathbf{v}$ is below ℓ" instead of $\mathbf{v} < (\ell, -1)$.

5. (For the "non-Sacks part":) For each subatomic sublevel $\mathbf{u}$ and index $\xi \in \Xi_{\text{non-}\mathtt{sk}}$ we fix a subatomic family $\mathsf{K}_{\xi,\mathbf{u}}$ living on a finite set $\mathrm{POSS}_{\xi,\mathbf{u}}$.

6. For each level $\ell \in \omega$ and index $\xi \in \Xi_{\text{non-}\mathtt{sk}}$, each sequence $(x_j)_{j \in J_\ell}$ with $x_j \in \mathsf{K}_{\xi,\mathbf{u}}$ constitutes (as in 2.2.0.10) an atom $\mathfrak{a}$, where we use ℓ as parameter in μ^ℓ for the definition of the norm of the atom.

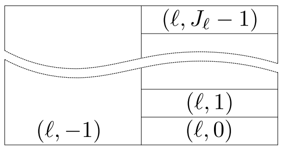

Figure 2.1: Diagram of the sublevels at level ℓ, with the Sacks sublevel $(\ell, -1)$ occurring "before" the subatomic sublevels $(\ell, 0), (\ell, 1), \dots, (\ell, J_\ell - 1)$.

To be able to use this frame to construct a reasonable (in particular: proper) forcing, we will have to add several additional requirements of the following form: The Sacks intervals $I_{\mathtt{sk},\ell}$ (that "appear" at sublevel ℓ) are "large" with respect to everything that was constructed in sublevels $\mathbf{v}$ below ℓ; and the subatoms at a subatomic sublevel $\mathbf{u}$ have "large" bigness with respect to everything that was constructed at sublevels $\mathbf{v} < \mathbf{u}$. The complete construction with all requirements will be given in Section 4.

2.5 Compound creatures

We can now define compound creatures, which are made up from subatomic creatures and Sacks columns.

Definition 2.5.0.19. A *compound creature* $\mathfrak{c}$ consists of:

1. Natural numbers $m^{\mathrm{dn}} < m^{\mathrm{up}}$.

2. supp, a nonempty finite[7] subset of Ξ.

3. For each $\xi \in \mathrm{supp} \cap \Xi_{\mathtt{sk}}$, the object $\mathfrak{c}(\xi)$, which is a Sacks column between m^{dn} and m^{up}.

4. For each $\xi \in \mathrm{supp} \cap \Xi_{\text{non-}\mathtt{sk}}$ and each subatomic sublevel $\mathbf{u} = (\ell, j)$ with $m^{\mathrm{dn}} \le \ell < m^{\mathrm{up}}$ and $j \in J_\ell$, the object $\mathfrak{c}(\xi, \mathbf{u})$ which is a subatom in $\mathsf{K}_{\xi,\mathbf{u}}$.

[7]We could assume without loss of generality that the size of supp is at most m^{dn}. This will be shown in Lemma 3.4.0.11.

5. For each $m^{\mathrm{dn}} \leq \ell < m^{\mathrm{up}}$ a real number $d(\ell)$ (called the *halving parameter* of $\mathfrak{c}$ at level ℓ).[8]

6. We additionally require "modesty":[9] For each subatomic sublevel $\mathbf{u}$ there is at most one $\xi \in \mathrm{supp}(\mathfrak{c})$ such that the subatom $\mathfrak{c}(\xi, \mathbf{u})$ is not a singleton.

7. This defines for each $\xi \in \mathrm{supp} \cap \Xi_{\texttt{non-sk}}$ the *atom* $\mathfrak{c}(\xi, \ell) = (\mathfrak{c}(\xi, (\ell, j)) : j \in J_\ell)$.

We also write $m^{\mathrm{dn}}(\mathfrak{c}), m^{\mathrm{up}}(\mathfrak{c}), \mathrm{supp}(\mathfrak{c}),\ d(\mathfrak{c}, h)$.

We will use the following assumptions (later there will be more, a complete list will be given in Section 4):

Assumption 2.5.0.20. For each $\ell \in \omega$,

- we fix natural numbers $B(\ell)$ and $\mathrm{maxposs}(<\ell)$, such that $k \leq \ell$ implies $B(k) \leq B(\ell)$ and $\mathrm{maxposs}(<k) \leq \mathrm{maxposs}(<\ell)$. (These parameters will be defined in Section 4.)
- we assume that $I_{\texttt{sk},\ell}$ is large enough so that there are Sacks trees of large norm; more concretely: $\mathrm{nor}_{\mathrm{Sacks}}^{B(\ell),\ell}(2^{I_{\texttt{sk},\ell}}) \geq \ell$ for every $\ell \in \omega$.
- we assume that J_ℓ is large enough such that $\mu^\ell(J_\ell)$ is big; more concretely: $\mu^\ell(J_\ell) \geq 2^{\ell \cdot \mathrm{maxposs}(<\ell)}$.
- we assume that for every $\xi \in \Xi_{\texttt{non-sk}}$ and $j \in J_\ell$ there is (at least) one subatom $x \in \mathsf{K}_{\xi,(\ell,j)}$ with $\mathrm{nor}(x) \geq 2^{\ell \cdot \mathrm{maxposs}(<\ell)}$.

Using these assumptions, we can now define the norm of a compound creature:

Definition 2.5.0.21. The *norm* of a compound creature $\mathfrak{c}$, $\mathrm{nor}(\mathfrak{c})$, is defined to be the minimum of the following values:

1. $\mathrm{nor}_{\mathrm{width}}(\mathrm{supp}(\mathfrak{c}))$;
2. $\mathrm{nor}_{\mathrm{Sacks}}(\mathfrak{c}(\xi))$ for all $\xi \in \mathrm{supp} \cap \Xi_{\texttt{sk}}$;
3. $\mathrm{nor}_{\mathrm{limsup}}(\mathfrak{c}, \xi)$ for all $\xi \in \mathrm{supp} \cap \Xi_{\texttt{ls}}$; and
4. $\mathrm{nor}_{\mathrm{liminf}}^{\mathrm{maxposs}(<m^{\mathrm{dn}})}(\mathfrak{c}, h)$ for all $m^{\mathrm{dn}} \leq h < m^{\mathrm{up}}$.

where we use the following:

- The "width norm" $\mathrm{nor}_{\mathrm{width}}(\mathrm{supp}(\mathfrak{c})) = \frac{m^{\mathrm{dn}}(\mathfrak{c})}{|\mathrm{supp}(\mathfrak{c})|}$.
  As $\mathrm{supp}(\mathfrak{c})$ is nonempty, the width norm (and thus $\mathrm{nor}(\mathfrak{c})$ as well) is at most $m^{\mathrm{dn}}(\mathfrak{c})$.
- The Sacks norm $\mathrm{nor}_{\mathrm{Sacks}}(\mathfrak{c}(\xi)) := \mathrm{nor}_{\mathrm{Sacks}}^{B(m^{\mathrm{dn}}),m^{\mathrm{dn}}}(\mathfrak{c}(\xi))$ (with $m^{\mathrm{dn}} := m^{\mathrm{dn}}(\mathfrak{c})$) as defined in (2.3.0.16).
- The "lim sup norm", the maximal norm of the atoms appearing at index ξ at any level h, i.e.,

$$\mathrm{nor}_{\mathrm{limsup}}(\mathfrak{c}, \xi) = \max(\mathrm{nor}(\mathfrak{c}(\xi, h)) : m^{\mathrm{dn}} \leq h < m^{\mathrm{up}}).$$

- The "lim inf norm", the minimal norm of the atoms appearing at level h at any lim-inf index, modified by the halving parameter:[10,11]

$$\mathrm{nor}_{\mathrm{liminf}}^{\mathrm{maxposs}(<m^{\mathrm{dn}})}(\mathfrak{c}, h) = \frac{\log_2(N - d(\mathfrak{c}, h))}{\mathrm{maxposs}(<m^{\mathrm{dn}})} \quad \text{for } N := \min\{\mathrm{nor}(\mathfrak{c}(\xi, h)) : \xi \in \mathrm{supp} \cap \Xi_{\texttt{li}}\}.$$

[8] One could (without loss of generality, in some sense) restrict the halving parameter to a finite subset of the reals; then for fixed $\mathrm{supp}, m^{\mathrm{dn}}, m^{\mathrm{up}}$ there are only finitely many compound creatures.

[9] Again: without this requirement, the resulting forcing poset would be equivalent.

[10] As usual: If the following logarithm results in a negative number, or if we apply the logarithm to a negative number, then we instead define the resulting norm as $\mathrm{nor}(\mathfrak{c}) := 0$. So really we mean $\mathrm{nor}_{\mathrm{liminf}}^{\mathrm{maxposs}(<m^{\mathrm{dn}})}(\mathfrak{c}, h) = \frac{\log_2(\max(1, N - d(\mathfrak{c}, h)))}{\mathrm{maxposs}(<m^{\mathrm{dn}})}$.

[11] The reason for the logarithm, and the use of the halving parameters, will become clear only in Section 5.2.

(So for both $\mathrm{nor}_{\mathrm{limsup}}$ and $\mathrm{nor}_{\mathrm{liminf}}$ we use the norms of atoms $\mathfrak{c}(\xi,h)$; recall that the level h of this atom is used in Definition 2.2.0.10 of $\mathrm{nor}(\mathfrak{c}(\xi,h))$, more specifically: μ^h is used to measure the size of subsets of J_h.)

The assumptions imply the following:

Lemma 2.5.0.22. *Fix $2 < m^{\mathrm{dn}} < m^{\mathrm{up}}$ and $\mathrm{supp} \subseteq \Xi$ with $|\mathrm{supp}| < m^{\mathrm{dn}}$ and $\mathrm{supp}\cap\Xi_{\mathtt{sk}}$, $\mathrm{supp}\cap\Xi_{\mathtt{li}}$, $\mathrm{supp}\cap\Xi_{\mathtt{ls}}$ all nonempty. Then there is a compound creature $\mathfrak{c}$ with $m^{\mathrm{dn}}(\mathfrak{c}) = m^{\mathrm{dn}}$, $m^{\mathrm{up}}(\mathfrak{c}) = m^{\mathrm{up}}$, $\mathrm{supp}(\mathfrak{c}) = \mathrm{supp}$ such that $\mathrm{nor}(\mathfrak{c}) = \mathrm{nor}_{\mathrm{width}}(\mathrm{supp})$.*

Proof. We can first use for all subatoms and Sacks columns the "large" ones guaranteed by the assumptions. However, this will in general not satisfy modesty. So we just apply Lemma 2.2.0.9 at each $m^{\mathrm{dn}} \leq \ell < m^{\mathrm{up}}$, resulting (for each ℓ) in disjoint sets $A^\ell_\xi \subseteq J_\ell$ for $\xi \in \mathrm{supp}\cap\Xi_{\text{non-}\mathtt{sk}}$. We keep the large subatoms at the sublevels in A^ℓ_ξ, and choose arbitrary singleton subatoms at other sublevels. Now we have a compound creature, whose norm is the minimum of the following:

- the width norm;
- the (unchanged) Sacks norms, which are $\geq m^{\mathrm{dn}} > \mathrm{nor}_{\mathrm{width}}(\mathrm{supp})$;
- the lim sup norms; here, all atoms at level ℓ have norm $\geq 2^{\ell\cdot\mathrm{maxposs}(<\ell)} - 1 \geq 2^{m^{\mathrm{dn}}\cdot\mathrm{maxposs}(<m^{\mathrm{dn}})} - 1 > \mathrm{nor}_{\mathrm{width}}(\mathrm{supp})$; so all lim sup norms drop by at most 1.
- the lim inf norms, which drop by an even smaller amount, due to the logarithm.

□

Fact 2.5.0.23. *Let $\mathfrak{c}$ be a compound creature and $u \subseteq \mathrm{supp}(\mathfrak{c})$ such that $u\cap\Xi_{\mathtt{sk}}$, $u\cap\Xi_{\mathtt{li}}$, $u\cap\Xi_{\mathtt{ls}}$ are all nonempty. Then the naturally defined $\mathfrak{c}\,|\,u$ is again a compound creature with norm at least $\mathrm{nor}(\mathfrak{c})$.*

Definition 2.5.0.24. A compound creature $\mathfrak{d}$ is purely stronger than $\mathfrak{c}$, if $\mathfrak{c}$ and $\mathfrak{d}$ have the same m^{dn}, m^{up}, the same halving parameters, the same supp; and if for each $\xi \in \mathrm{supp}\cap\Xi_{\mathtt{sk}}$ the Sacks column $\mathfrak{d}(\xi)$ is stronger than $\mathfrak{c}(\xi)$ and for each subatomic sublevel $\mathbf{u}$ that appears in $\mathfrak{c}$ and $\xi \in \mathrm{supp}\cap\Xi_{\text{non-}\mathtt{sk}}$ the subatom $\mathfrak{d}(\xi,\mathbf{u})$ is stronger than $\mathfrak{c}(\xi,\mathbf{u})$.

In other words, the only difference between $\mathfrak{c}$ and $\mathfrak{d}$ occurs at Sacks columns and subatoms, where they become stronger.

$\mathfrak{d}$ is r-purely stronger than $\mathfrak{c}$, if additionally $\mathrm{nor}(\mathfrak{d}) \geq \mathrm{nor}(\mathfrak{c}) - r$.

To show $\aleph_2$-cc, we will later use the following property:

Lemma 2.5.0.25. *Fix two compound creatures $\mathfrak{c}_1$ and $\mathfrak{c}_2$ with same m^{dn} and m^{up} and the same halving parameters, with disjoint supports, and such that $\mathrm{nor}(\mathfrak{c}_1), \mathrm{nor}(\mathfrak{c}_2) > x$. Then there exists a compound creature $\mathfrak{d}$ with same m^{dn} and m^{up} and support $\mathrm{supp}(\mathfrak{c}_1)\cup\mathrm{supp}(\mathfrak{c}_2)$ such that $\mathrm{nor}(\mathfrak{d}) \geq \frac{x}{2} - 1$ and $\mathfrak{d}\,|\,\mathrm{supp}(\mathfrak{c}_i)$ is purely stronger than $\mathfrak{c}_i$ for $i = 1, 2$.*

More generally, the same is true if $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are not necessarily disjoint, but identical on the intersection $u := \mathrm{supp}(\mathfrak{c}_1)\cap\mathrm{supp}(\mathfrak{c}_2)$, i.e., $\mathfrak{c}_1\,|\,u = \mathfrak{c}_2\,|\,u$.

Proof. Set $\mathfrak{d}'$ be the union of $\mathfrak{c}_1$ and $\mathfrak{c}_2$ (which is defined in the obvious way: $\mathfrak{d}'(\xi,k) = \mathfrak{c}_1(\xi,k)$ if $\xi \in \mathrm{supp}(c_1)$, and $\mathfrak{c}_2(\xi,k)$ otherwise).

It is easy to see that $\mathfrak{d}'$ satisfies all requirements apart from modesty 2.5.0.19(6). As in the proof of Lemma 2.5.0.22, we can make it modest, resulting in a compound creature $\mathfrak{d}$, of norm $\geq \frac{x}{2} - 1$. (The factor $\frac{1}{2}$ comes from doubling the size of the support, which decreases the width norm.) □

2.6 The elements (conditions) of the forcing poset $\mathbb{Q}$

Definition 2.6.0.26. $\emptyset$ is the weakest condition. Any other condition p consists of $\mathtt{w}^p$, $(p(h))_{h\in\mathtt{w}^p}$ and t^p such that:

- $\mathtt{w}^p \subseteq \omega$ is infinite.
- For each $h \in \mathtt{w}^p$, $p(h)$ is a compound creature such that
  - $m^{\mathrm{dn}}(p(h)) = h$,
  - $m^{\mathrm{up}}(p(h))$ is the $\mathtt{w}^p$-successor of h,
  - for $h < h'$ in $\mathtt{w}^p$, $\mathrm{supp}(p(h)) \subseteq \mathrm{supp}(p(h'))$,
  - $\lim_{h\in\mathtt{w}^p}(\mathrm{nor}(p(h))) = \infty$.
- We set $\mathrm{supp}(p) := \bigcup_{h\in\mathtt{w}^p} \mathrm{supp}(p(h))$ (a nonempty subset of Ξ which is finite or countable).
- For $\xi \in \mathrm{supp}(p)$, we define $\mathrm{trklg}^p(\xi)$ (the *trunk length at* ξ) to be the minimal h such that $\xi \in \mathrm{supp}(p(h))$.
- The trunk t^p assigns
  - to each $\xi \in \mathrm{supp}(p) \cap \Xi_{\mathtt{sk}}$ and $\ell < \mathrm{trklg}^p(\xi)$ an element of $2^{I_{\mathtt{sk},\ell}}$;
  - to each $\xi \in \mathrm{supp}(p) \cap \Xi_{\text{non-}\mathtt{sk}}$ and subatomic sublevel $\mathbf{u}$ below $\mathrm{trklg}^p(\xi)$ an element of $\mathrm{POSS}_{\xi,\mathbf{u}}$.

Note that Assumption 2.5.0.20 guarantees that $\mathbb{Q}$ is nonempty (cf. Lemma 2.5.0.22).

Notation 2.6.0.27. Given $p \in P$, $h \in \mathtt{w}^p$ and ℓ which is $\geq h$ and less than the $\mathtt{w}^p$-successor of h, and a sublevel $\mathbf{u} = (\ell, j)$ we set

- $\mathrm{supp}(p, \mathbf{u}) = \mathrm{supp}(p, \ell) := \mathrm{supp}(p(h))$.
- $d(p, \ell) := d(p(h), \ell)$ (the halving parameter of p at level ℓ),
- For $\xi \in \Xi_{\text{non-}\mathtt{sk}} \cap \mathrm{supp}(p, \mathbf{u})$ and $j \neq -1$ we set $p(\xi, \mathbf{u}) := p(h)(\xi, \mathbf{u})$, the subatom located at index ξ and sublevel $\mathbf{u}$.
- For $\xi \in \Xi_{\mathtt{sk}} \cap \mathrm{supp}(p(h))$ we set $p(\xi, h) := p(h)(\xi)$, the Sacks column at index ξ starting at level h (note that we require $h \in \mathtt{w}^p$).

2.7 The set of possibilities

We will now define the "possibilities" of a condition p, which give information about the possible value of the generic objects $\underset{\sim}{y}_\xi$ and which we will use to define the order of the forcing. Informally speaking, a condition p consists of

- the trunk part t^p, where there is a unique possibility,
- subatoms x (each with a set of possibilities $\mathrm{poss}(x)$),
- Sacks columns $\mathfrak{s}$ (which we interpret as a set of possible branches) which "live" between $h \in \mathtt{w}^p$ and the $\mathtt{w}^p$-successor h^+ of h, and this set of possible branches generally cannot be written as a product of possibilities at levels $h \leq l < h^+$, let alone sublevels.

> This property of the Sacks columns will make our notation quite awkward. As a consequence, the following section has the worst ratio of mathematical contents to notational awkwardness. Things will improve later on. We promise.

We first (in 2.7.0.28) describe a way to define the set of possibilities separately for each $\xi \in \mathrm{supp}(p)$; all possibilities then are the product over the ξ-possibilities.

Then (in 2.7.0.29) we will describe a variant: We define the possibilities at a sublevel $\mathbf{u}$, and all possibilities are a product over the $\mathbf{u}$-possibilities.

Both versions result in the same set of possibilities (apart from an awkward but canonical bijection, see 2.7.0.30). The first version is more useful in formulating things such as "a stronger condition has as smaller set of possibilities"; but the second one is the notion that will actually be used later on in proofs.

Definition 2.7.0.28. Fix a condition p and an index $\xi \in \mathrm{supp}(p)$.

- If $\xi \in \Xi_{\text{non-}\mathtt{sk}}$ and $\mathbf{u} = (\ell, j)$ is a subatomic sublevel, then we set $\mathrm{poss}(p, \xi, =\mathbf{u})$ to be $\mathrm{poss}(x)$ for the according subatom $x = p(\xi, \mathbf{u})$. However, if $\ell < \mathrm{trklg}^p(\xi)$ (and so there is no subatom, but instead a part of the trunk), then we let $\mathrm{poss}(p, \xi, =\mathbf{u})$ be the singleton $\{t^p(\xi, \mathbf{u})\}$. (In either case we have $\mathrm{poss}(p, \xi, =\mathbf{u}) \subseteq \mathrm{POSS}_{\xi,\mathbf{u}}$.)
  We set $\mathrm{poss}(p, \xi, <\mathbf{u})$ to be $\prod_{\mathbf{v}<\mathbf{u} \text{ subatomic sublevel}} \mathrm{poss}(p, \xi, =\mathbf{v})$.

- If $\xi \in \Xi_{\mathtt{sk}}$ and $\mathbf{u} = (m, j)$ is a sublevel, then we will define $\mathrm{poss}(p, \xi, <\mathbf{u})$. We first define a number ℓ as follows:

  - If $j = -1$ and $m \in \mathtt{w}^p$, then $\ell := m$.
  - Otherwise: ℓ is the least number $> m$ in $\{0, \dots, \mathrm{trklg}^p(\xi) - 1\} \cup \mathtt{w}^p$.

  We then set $\mathrm{poss}(p, \xi, <\mathbf{u})$ to be the set of all functions $\eta \in 2^{[0,\min(I_{\mathtt{sk},\ell}))}$ compatible[12] with the trunk and the Sacks columns at ξ.

- Set $\mathrm{poss}(p, <\mathbf{u})$ to be $\prod_{\xi \in \mathrm{supp}(p)} \mathrm{poss}(p, \xi, <\mathbf{u})$.

- Recall that we identify ℓ with the sublevel $(\ell, -1)$, so we can write $\mathrm{poss}(p, <\ell)$ instead of $\mathrm{poss}(p, <(\ell, -1))$.

Note that each possibility below $\mathbf{u}$ restricted to the non-Sacks part can be seen as a "rectangle" with width $\mathrm{supp}(p) \cap \Xi_{\text{non-}\mathtt{sk}}$ and height $\mathbf{u}$; whereas the restriction to the Sacks part is a rectangle with height in $\mathtt{w}^p$ (which is generally above $\mathbf{u}$). So together this gives an "L-shaped" domain. Only in case $\mathbf{u} = (\ell, -1)$ for $\ell \in \mathtt{w}^p$ we get a more pleasant overall rectangular shape.

In the following alternative definition we ignore a part of p which is "trivial" because we have no freedom/choice left. More specifically, we ignore the trunk and singleton subatoms (but not, e.g., singleton Sacks columns). Also, we do not first concentrate on some fixed index ξ, but directly define $\mathrm{poss}'(p, =\mathbf{u})$ for certain sublevels $\mathbf{u}$.

Definition 2.7.0.29. We define the set $\mathrm{sblvls}(p)$ of "active" sublevels of p by case distinction; for each $\mathbf{u} \in \mathrm{sblvls}(p)$ we define the object $\mathrm{poss}'(p, =\mathbf{u})$:

- If $\mathbf{u} = (\ell, -1)$ is a Sacks sublevel, then $\mathbf{u} \in \mathrm{sblvls}(p)$ iff $\ell \in \mathtt{w}^p$ and $S := \mathrm{supp}(p, \ell) \cap \Xi_{\mathtt{sk}} \neq \emptyset$.

  We set $p(\mathbf{u})$ to be the sequence $(p(\xi, \ell))_{\xi \in S}$ of these Sacks columns, and set $\mathrm{poss}'(p, =\mathbf{u})$ to be the product of this sequence.

[12] In more detail: Whenever $h < \mathrm{trklg}^p(\xi))$ and $h \leq m$ (or $< m$, if $j = -1$), then $\eta \restriction I_{\mathtt{sk},h} = t^p(\xi, h)$. And: if $h \geq \mathrm{trklg}^p(\xi)$ and $h \leq m$ (or $< m$, if $j = -1$), and $h \in \mathtt{w}^p$ then $\eta \restriction I_{\mathtt{sk},h} \in p(\xi, h)$.

- If $\mathbf{u} = (\ell, j)$ is a subatomic sublevel, then $\mathbf{u} \in \mathrm{sblvls}(p)$ if $\ell \geq \min(\mathtt{w}^p)$ and if there is a non-singleton subatom at sublevel $\mathbf{u}$, at index ξ, say. In that case, according to modesty 2.5.0.19(6), this is the only non-singleton subatom at $\mathbf{u}$. We call ξ the active index at $\mathbf{u}$, and we set $p(\mathbf{u}) := p(\xi, \mathbf{u})$ (the "active subatom") and $\mathrm{poss}'(p, {=}\mathbf{u}) := \mathrm{poss}(p(\mathbf{u}))$.

So $\mathrm{sblvls}(p)$ is a subset (and thus suborder) of the set of all sublevels (again of order type ω). We set $\mathrm{poss}'(p, {<}\mathbf{u}) = \prod_{\mathbf{v}<\mathbf{u},\ \mathbf{v}\in\mathrm{sblvls}(p)} \mathrm{poss}'(p, {=}\mathbf{v})$.

The definition of the following bijection ι is easy to see/understand, but very awkward to formulate precisely, and hence left as an exercise.

Fact 2.7.0.30. *There is a natural/canonical correspondence* $\iota : \mathrm{poss}(p, {<}\mathbf{u}) \to \mathrm{poss}'(p, {<}\mathbf{u})$*: Given an* $\eta \in \mathrm{poss}(p, {<}\mathbf{u})$*, we first omit from* η *all the "trivial" information contained in the trunk and in the singleton subatoms; and then "relabel" the resulting sequence (instead of a sequence indexed by elements of* ξ *we wish to have one indexed by elements of* $\mathrm{sblvls}(p)$*).*

Later in this paper we will not distinguish between poss and poss′; actually we will mostly use poss′, and often use the following trivial observation:

Fact 2.7.0.31. *For* $\mathbf{v} < \mathbf{u}$ *in* $\mathrm{sblvls}(p)$*,*

$$\mathrm{poss}'(p, {<}\mathbf{u}) = \mathrm{poss}'(p, {<}\mathbf{v}) \times \mathrm{poss}'(p, {=}\mathbf{v}) \times \mathrm{poss}'(p, {>}\mathbf{v}),$$

where we set $\mathrm{poss}'(p, {>}\mathbf{v}) := \prod_{\mathbf{v}'\in\mathrm{sblvls}(p), \mathbf{v}<\mathbf{v}'<\mathbf{u}} \mathrm{poss}(p, {=}\mathbf{v}')$*.*
$\mathrm{poss}'(p, {=}\mathbf{v})$ *is a product of Sacks columns if* $\mathbf{v}$ *is Sacks, otherwise it is* $\mathrm{poss}(x)$ *for the active subatom at* $\mathbf{v}$*.*

2.8 The order of the forcing

Definition 2.8.0.32. A condition q is stronger than p (or: $q \leq p$), if

1. $\mathtt{w}^q \subseteq \mathtt{w}^p$.
2. $\mathrm{supp}(p) \cap \mathrm{supp}(q(h)) = \mathrm{supp}(p(h))$ for each $h \in \mathtt{w}^q$. This implies:
   - $\mathrm{supp}(q(h)) \supseteq \mathrm{supp}(p(h))$.
   - For $\xi \in \mathrm{supp}(p)$, $\mathrm{trklg}^q(\xi) = \min\{\ell \in \mathtt{w}^q : \ell \geq \mathrm{trklg}^p(\xi)\}$.
   - So the trunk t^q is defined on a bigger domain than t^p.
3. The trunk of q, i.e., the function t^q, extends the function t^p and is "compatible" with p: The singleton $\mathrm{poss}(q, \xi, {<}\mathrm{trklg}^q(\xi))$ is a subset of $\mathrm{poss}(p, \xi, {<}\mathrm{trklg}^q(\xi))$. I.e., the subatoms and Sacks columns that disappeared have been replaced by a trunk which is compatible with the respective possibilities of p.

   Equivalently, we could also write: For any $\eta \in \mathrm{poss}(q, {<}\min(\mathtt{w}^q))$, the restriction of η to $\mathrm{supp}(p)$ is in $\mathrm{poss}(p, {<}\min(\mathtt{w}^q))$.
4. If $\xi \in \mathrm{supp}(p) \cap \Xi_{\mathtt{non\text{-}sk}}$ and $\mathbf{u}$ is a subatomic sublevel above $\mathrm{trklg}^q(\xi)$, then the subatom $q(\xi, \mathbf{u})$ is stronger than $p(\xi, \mathbf{u})$.
5. If $\xi \in \mathrm{supp}(p) \cap \Xi_{\mathtt{sk}}$ and $h \in \mathtt{w}^q$ such that $h \geq \mathrm{trklg}^q(\xi)$, then the Sacks column $q(\xi, h)$ is stronger than (i.e., a subset of) the product of the Sacks columns $p(\xi, \ell)$ for $\ell \in \mathtt{w}^p$, $h \leq \ell < h^+$, where h^+ is the $\mathtt{w}^q$-successor of h.
6. The halving parameters do not decrease, i.e.: $d(q, \ell) \geq d(p, \ell)$ for all $\ell \in \omega$ with $\ell \geq \min(\mathtt{w}^q)$.

Chapter 3

Some simple properties of $\mathbb{Q}$

3.1 Increasing the trunk

We now introduce an obvious way to strengthen a condition: Increasing the trunk.

Definition 3.1.0.1. Given $\ell \in \mathtt{w}^p$ and $\eta \in \mathrm{poss}(p, <\ell)$, we define $p \wedge \eta$ to be the condition q resulting from replacing the compound creatures below ℓ with the trunk η.

More formally: $\mathtt{w}^q = \mathtt{w}^p \setminus \ell$; for $k \in \mathtt{w}^q$ we have $q(k) = p(k)$; for a subatomic $\mathbf{u}$ below ℓ and $\xi \in \mathrm{supp}(p) \cap \Xi_{\texttt{non-sk}}$, the trunk $t^q(\xi, \mathbf{u})$ is $\eta(\xi, \mathbf{u})$; and for $h < \ell$ and $\xi \in \mathrm{supp}(p) \cap \Xi_{\texttt{sk}}$, the trunk $t^q(\xi, h)$ is $\eta(\xi) \restriction I_{\texttt{sk},h}$.

The definition of the order implies:

Fact 3.1.0.2. *Fix $\ell \in \mathtt{w}^p$.*

- *For $\eta \in \mathrm{poss}(p, <\ell)$, $p \wedge \eta \le p$.*
- *$\{p \wedge \eta : \eta \in \mathrm{poss}(p, <\ell)\}$ is predense below p.*
- *In particular: Assume that p and q are conditions that above some ℓ_1 have the same $\mathtt{w}$ and the same compound creatures,[1] and that $\mathrm{poss}(q, <\ell_1) \subseteq \mathrm{poss}(p, <\ell_1)$. Then $q \le^* p$.*
  (Here, $q \le^* p$ means $q \Vdash p \in G$; equivalently: every $r \le q$ is compatible with p.)

We can define a variant of $\wedge$, which works for any sublevel (not only those sublevels $\mathbf{u} = (\ell, -1)$ with $\ell \in \mathtt{w}^p$):

Definition 3.1.0.3. Given $\eta \in \mathrm{poss}(p, <\mathbf{u})$, we define $q = p \curlywedge \eta$ as the condition obtained by replacing the according parts of p with the singleton subatoms (or: singleton Sacks columns) given by η.

More formally: $\mathtt{w}^q = \mathtt{w}^p$; $\mathrm{supp}(q, n) = \mathrm{supp}(p, n)$ for all $n \in \mathtt{w}^p$; $t^q = t^p$; if $\xi \in \mathrm{supp}(p, \mathbf{v}) \cap \Xi_{\texttt{non-sk}}$ and the subatomic sublevel $\mathbf{v}$ is below $\mathbf{u}$, then $q(\xi, \mathbf{v})$ is the singleton subatom $\{\eta(\xi, \mathbf{v})\}$; if $\ell \in \mathtt{w}^p$ and $\xi \in \mathrm{supp}(p, \ell) \cap \Xi_{\texttt{sk}}$ and $\mathbf{u}$ strictly above ℓ, then the Sacks column $q(\xi, \ell)$ consists of the single branch given by $\eta(\xi)$ restricted to $I_{\texttt{sk},[\ell,\ell^+)}$, where ℓ^+ is the $\mathtt{w}^p$-successor of ℓ.

We can now define the generic sequence added by the forcing (note that the generic filter will generally not be determined by this sequence, due to additional information given by $\mathtt{w}$ and the halving parameters).

[1] more formally: $\ell_1 \in \mathtt{w}^p$, $\mathtt{w}^p \setminus \ell_1 = \mathtt{w}^q \setminus \ell_1$, and $p(h) = q(h)$ for all $h \in \mathtt{w}^p \setminus \ell_1$. Note that this implies $\mathrm{supp}(p) = \mathrm{supp}(q)$.

Definition 3.1.0.4. For $\xi \in \Xi_{\text{non-}\mathtt{sk}}$, let $\underset{\sim}{y}_\xi$ be (the name for)

$$\{(\mathbf{u}, a) : \mathbf{u} \text{ a subatomic sublevel and } (\exists p \in G)\ t^p(\xi, \mathbf{u}) = a\}.$$

For $\xi \in \Xi_{\mathtt{sk}}$, we set $\underset{\sim}{y}_\xi$ to be

$$\bigcup \{t^p(\xi, \ell) : p \in G,\ \ell < \mathrm{trklg}^p(\xi)\}.$$

Fact 3.1.0.5. *Let* $\mathbf{u}$ *be a sublevel.*

- *For* $\eta \in \mathrm{poss}(p, <\mathbf{u})$, $p \curlywedge \eta \le p$.
- *If* $\ell \in \mathtt{w}^p$, $\mathbf{u} = (\ell, -1)$ *and* $\eta \in \mathrm{poss}(p, <\ell)$, *then* $p \curlywedge \eta \le^* p \wedge \eta$ *and* $p \wedge \eta \le p \curlywedge \eta$.
- $\{p \curlywedge \eta : \eta \in \mathrm{poss}(p, <\mathbf{u})\}$ *is predense below* p.
- $p \curlywedge \eta$ *and* $p \curlywedge \eta'$ *are incompatible if* $\eta' \neq \eta$ *in* $\mathrm{poss}(p, <\mathbf{u})$.
- $p \curlywedge \eta$ *forces that* $\underset{\sim}{\bar{y}}$ *extends* η, *i.e., that* $\underset{\sim}{y}_\xi$ *extends* $\eta(\xi)$ *for all* $\xi \in \mathrm{supp}(p)$.
  In particular, p *forces that* $\underset{\sim}{\bar{y}}$ *extends* t^p.
- $\eta \in \mathrm{poss}(p, <\mathbf{u})$ *iff*[2] p *does not force that* η *is incompatible with the generic reals* $\underset{\sim}{\bar{y}}$.
- *For* $\eta \in \mathrm{poss}(p, <\mathbf{u})$, p *forces:* $\underset{\sim}{\bar{y}}$ *extends* η *iff* $p \curlywedge \eta \in G$.
- $\mathbb{Q}$ *forces that* $\underset{\sim}{\bar{y}}$ *is "defined everywhere":*
  - *For* $\xi \in \Xi_{\mathtt{sk}}$ $\underset{\sim}{y}_\xi \in 2^\omega$,
  - *and for* $\xi \in \Xi_{\text{non-}\mathtt{sk}}$ *and* $\mathbf{u}$ *a subatomic sublevel,* $\underset{\sim}{y}_\xi(\mathbf{u}) \in \mathrm{POSS}_{\xi,\mathbf{u}}$ *is defined.*

(Proof of the last item: Given a condition p and $\xi \in \Xi$, we have to show that we can find a $q \le p$ with $\xi \in \mathrm{supp}(q)$. This is shown just as Lemma 2.5.0.25, using at ξ the large Sacks columns/subatoms guaranteed by 2.5.0.20. Then "increasing the trunk" shows that $\underset{\sim}{y}_\xi(n)$ is defined for all n.)

Note that we can use the equivalent poss' (defined in 2.7.0.29) instead of poss. Formally, we could use the bijection ι of 2.7.0.30 and set $p \wedge \eta' := p \wedge \iota^{-1}(\eta')$ for $\eta' \in \mathrm{poss}'(p, <l)$ (and $p \curlywedge \eta' := p \curlywedge \iota^{-1}(\eta')$ for $\eta' \in \mathrm{poss}'(p, <\mathbf{u})$). But what we really mean: For some $\eta' \in \mathrm{poss}'$ we can define $p \wedge \eta'$ (and: $p \curlywedge \eta'$) in the obvious and natural way; and this results in the same object as when using $p \wedge \eta$ (or: $p \curlywedge \eta$) for the $\eta \in \mathrm{poss}$ that corresponds to η' (i.e., for $\eta = \iota^{-1}(\eta')$).

3.2 The set of possibilities of stronger conditions

If $q \le p$, then $\mathrm{poss}(q, <\mathbf{u})$ morally is a subset of $\mathrm{poss}(p, <\mathbf{u})$ for any $\mathbf{u}$.

If we just consider a sublevel $(\ell, -1)$ for $\ell \in \mathtt{w}^q$ then this is literally true:

> Assume that $q \le p$, $\xi \in \mathrm{supp}(p)$ and $\ell \in \mathtt{w}^q$. Then $\mathrm{poss}(q, \xi, <\ell) \subseteq \mathrm{poss}(p, \xi, <\ell)$.

In the general case, it is more cumbersome to make this explicit for the Sacks part. We will only need the following:

Lemma 3.2.0.6. *Given* $q \le p$ *and* $\eta \in \mathrm{poss}(q, <\mathbf{u})$ *there is a unique* $\eta' \in \mathrm{poss}(p, <\mathbf{u})$ *such that* $q \curlywedge \eta \le p \curlywedge \eta'$.

[2] for the direction "right to left", which we will not need in this paper, we of course have to assume that η has the right "format", i.e., $\eta = \prod_{\xi \in \mathrm{supp}(p)} \eta(\xi)$ and each $\eta(\xi)$ has the appropriate length/domain.

Proof. Uniqueness follows from the fact that $p \curlywedge \eta'$ and $p \curlywedge \eta''$ are incompatible for different η', η'' in $\mathrm{poss}(p, <\mathbf{u})$.

We define $\eta'(\xi)$ separately for each $\xi \in \mathrm{supp}(p)$. For $\xi \in \Xi_{\text{non-}\mathtt{sk}}$ we just use $\eta'(\xi) := \eta(\xi)$. So assume $\xi \in \Xi_{\mathtt{sk}}$. Let k be the smallest element of $\mathtt{w}^p$ above $\mathbf{u}$.

- If $\mathbf{u}$ is below $\mathrm{trklg}^p(\xi)$ (and therefore also below $\mathrm{trklg}^q(\xi)$), then again we set $\eta'(\xi) := \eta(\xi)$.
- If $\mathbf{u}$ is above $\mathrm{trklg}^p(\xi)$ but below $\mathrm{trklg}^q(\xi)$, then we extend $\eta(\xi)$ up to k with the values given by the trunk t^q. This gives $\eta'(\xi)$.
- If $\mathbf{u}$ is above $\mathrm{trklg}^q(\xi) \geq \mathrm{trklg}^p(\xi)$, then $\eta'(\xi)$ is the restriction of $\eta(\xi)$ to k. □

Remark 3.2.0.7. Note that $q \leq p$ does not imply $\mathrm{sblvls}(q) \subseteq \mathrm{sblvls}(p)$; as a previously "inactive" sublevel can become active (outside of $\mathrm{supp}(p)$, of course). Also, $\mathbf{u}$ can be an active subatomic sublevel in both p and q, and still the active index can change: The "old" active subatom at ξ can shrink to a singleton in q, while q gains a new index with an active subatom (outside of $\mathrm{supp}(p)$). Because of this, it is even more cumbersome to formulate an exact version of "stronger conditions have fewer possibilities" for poss' instead of poss.

3.3 $\aleph_2$ chain condition

Lemma 3.3.0.8. *Assuming CH, $\mathbb{Q}$ is $\aleph_2$-cc.*

Proof. Assume that $A = \{p_i : i \in \aleph_2\}$ is a set of conditions. By thinning out A (only using CH and the Δ-system lemma for families of countable sets), we may assume that there is a countable set $\Delta \subseteq \Xi$ such that for $p \neq q$ in A the following holds:

- $\mathtt{w}^p = \mathtt{w}^q$.
- $d(p, \ell) = d(q, \ell)$ for all $\ell \geq \min(\mathtt{w}^p)$.
- $\Delta = \mathrm{supp}(p) \cap \mathrm{supp}(q)$. Moreover $\mathrm{supp}(p, \ell) \cap \Delta = \mathrm{supp}(q, \ell) \cap \Delta$ for all $\ell \in \mathtt{w}^p$.
- p and q are identical on Δ, i.e., for each $\ell \in \mathtt{w}^p$ the compound creatures $p(\ell)$ and $q(\ell)$ are identical on the intersection, as in Lemma 2.5.0.25; and the trunks agree on Δ, i.e., $t^p(\xi, \ell)$ is the same as $t^q(\xi, \ell)$ for each $\xi \in \Delta \cap \Xi_{\mathtt{sk}}$ and $\ell < h(\xi)$; and analogously for the subatomic sublevels.

As in Lemma 2.5.0.25 we can (for each $p, q \in A$ and $\ell \in \mathtt{w}^p$) find a compound creature $\mathfrak{d}(\ell)$ "stronger than" both $p(\ell)$ and $q(\ell)$. These creatures (together with the union of the trunks) form a condition stronger than both p and q. Hence A is not an antichain. □

3.4 Pruned conditions

Let p be a condition. All compound creatures $p(\ell)$ above some ℓ_0 will have norm at least 1. Note that by the definition of $\mathrm{nor}_{\mathrm{width}}$ this implies that $|\mathrm{supp}(p, \ell)| \leq \ell$.

The norm of a compound creature $\mathfrak{c}$ is at most m^{dn} (where we set $m^{\mathrm{dn}} := m^{\mathrm{dn}}(\mathfrak{c})$). We assumed that $\mathrm{nor}_{\mathrm{Sacks}}^{B(m^{\mathrm{dn}}), m^{\mathrm{dn}}}(2^{I_{\mathtt{sk}, m^{\mathrm{dn}}}})$ is at least m^{dn}. Let $\mathfrak{s}$ be any Sacks column in $\mathfrak{c}$. By Lemma 2.3.0.17(5) (using $I := I_{\mathtt{sk}, m^{\mathrm{dn}}}$ and $I' := I_{\mathtt{sk}, [m^{\mathrm{dn}}+1, m^{\mathrm{up}})}$), there is an $\tilde{\mathfrak{s}} \subseteq \mathfrak{s}$ with $|\tilde{\mathfrak{s}}| \leq 2^{I_{\mathtt{sk}, m^{\mathrm{dn}}}}$ and $\mathrm{nor}_{\mathrm{Sacks}}^{B(m^{\mathrm{dn}}), m^{\mathrm{dn}}}(\tilde{\mathfrak{s}}) \geq \min(m^{\mathrm{dn}}, \mathrm{nor}_{\mathrm{Sacks}}^{B(m^{\mathrm{dn}}), m^{\mathrm{dn}}}(\mathfrak{s}))$. So when we replace $\mathfrak{s}$ by $\tilde{\mathfrak{s}}$ in $\mathfrak{c}$, the norm of the compound creature does not change. The same is true if we replace all Sacks columns with an appropriate strengthening: The resulting compound creature $\mathfrak{d}$ will be 0-purely stronger than $\mathfrak{c}$, cf. Definition 2.5.0.24.

So we get the following:

Definition 3.4.0.9. We call a Sacks-column $\mathfrak{s}$ between ℓ and n Sacks-pruned, if $|\mathfrak{s}| \leq 2^{|I_{\mathtt{sk},\ell}|}$. A compound creature is Sacks-pruned, if all its Sacks columns are. A condition q is Sacks-pruned, if all $q(h)$ are. A condition p is pruned, if it is Sacks pruned and all compound creatures $p(h)$ have norm bigger than 1.

Definition 3.4.0.10. A condition q is purely stronger (or: r-purely stronger) than p, if $\mathtt{w}^q = \mathtt{w}^p$, $q(\ell)$ is purely stronger[3] than $p(\ell)$ (or: r-purely stronger, respectively) for all $\ell \in \mathtt{w}^q$, and $t^q = t^p$. (This implies $q \leq p$.)

For every condition p there is a 0-purely stronger Sacks-pruned q. Given $p \in \mathbb{Q}$ Sacks-pruned, $\ell \in \mathtt{w}^p$ sufficiently large, and $\eta \in \mathrm{poss}(p, {<}\ell)$, the condition $q = p \wedge \eta < p$ is pruned.

In particular, we get:

Fact 3.4.0.11.

- *If p is pruned, then $|\mathrm{supp}(p(h))| < h$ for all $h \in \mathtt{w}^p$.*
- *The set of pruned conditions in $\mathbb{Q}$ is dense.*

3.5 Gluing

So far we have increased trunks to strengthen conditions, as well as taking disjoint unions and pure strengthenings. There are two additional constructions:

Definition 3.5.0.12. A compound creature $\mathfrak{d}$ is the result of increasing the halving parameters in $\mathfrak{c}$, if $\mathfrak{d}$ and $\mathfrak{c}$ are identical except for the halving parameters: $d(\mathfrak{d}, \ell)$ can be bigger than $d(\mathfrak{c}, \ell)$ for each $m^{\mathrm{dn}} \leq \ell \leq m^{\mathrm{up}}$.

Analogously, we define a condition q to be the result of increasing the halving parameters in p. (Again, this implies $q \leq p$.)

Definition 3.5.0.13. We call a finite sequence of compound creatures $\mathfrak{c}_1, \dots, \mathfrak{c}_n$ "properly stacked", if $m^{\mathrm{up}}(\mathfrak{c}_i) = m^{\mathrm{dn}}(\mathfrak{c}_{i+1})$ and $\mathrm{supp}(\mathfrak{c}_i) \subseteq \mathrm{supp}(\mathfrak{c}_{i+1})$. Given such a sequence, we can glue it together to get the new creature $\mathfrak{d} = \mathrm{glue}(\mathfrak{c}_1, \dots, \mathfrak{c}_n)$ in the following way:

- $m^{\mathrm{dn}}(\mathfrak{d}) = m^{\mathrm{dn}}(\mathfrak{c}_1)$ and $m^{\mathrm{up}}(\mathfrak{d}) = m^{\mathrm{up}}(\mathfrak{c}_n)$ (i.e., vertically the creature lives on the union of the levels of the old creatures).
- $\mathrm{supp}(\mathfrak{d}) = \mathrm{supp}(\mathfrak{c}_1)$ (i.e., the rectangle-shape of the new creature is the result of taking the union of the old rectangles and cutting off the stuff that sticks out horizontally beyond the base).
- For $\xi \in \mathrm{supp}(\mathfrak{d}) \cap \Xi_{\mathtt{non\text{-}sk}}$ and subatomic sublevels $\mathbf{u}$ between $m^{\mathrm{dn}}(\mathfrak{d})$ and $m^{\mathrm{up}}(\mathfrak{d})$, the subatom $\mathfrak{d}(\xi, \mathbf{u})$ is $\mathfrak{c}_i(\xi, \mathbf{u})$ for the appropriate i.
- For $\xi \in \mathrm{supp}(\mathfrak{d}) \cap \Xi_{\mathtt{sk}}$, the Sacks column $\mathfrak{d}(\xi)$ is defined as the product $\mathfrak{c}_1(\xi) \otimes \cdots \otimes \mathfrak{c}_n(\xi)$.

[3] see Definition 2.5.0.24

By the definition of the norm (see 2.5.0.21) monotonicity of B and maxposs (Assumption 2.5.0.20) and Lemma 2.3.0.17(2),(3), we get

$$\mathrm{nor}(\mathrm{glue}(\mathfrak{c}_1, \dots, \mathfrak{c}_n)) \geq \min(\mathrm{nor}(\mathfrak{c}_1), \dots, \mathrm{nor}(\mathfrak{c}_n)).$$

This gives another way to strengthen a condition p: shrinking the set $\mathtt{w}$:

Definition 3.5.0.14. Given a condition p and an infinite subset U of $\mathtt{w}^p$ such that $\min(U) = \min(\mathtt{w}^p)$, we say that q results from gluing p along U, if

- $\mathtt{w}^q = U$,
- for $h \in \mathtt{w}^q$, let $h = h_1 < h_2 < \cdots < h_n$ enumerate the elements of $\mathtt{w}^p$ that are $\geq h$ and less than the $\mathtt{w}^q$-successor of h. Then the compound creature $q(h)$ is $\mathrm{glue}(p(h_1), \dots, p(h_n))$,
- The new parts of the trunk are compatible with p.

Note that q is not unique, as there are many choices to increase the trunk (in the last item). Of course the resulting q is stronger than p.

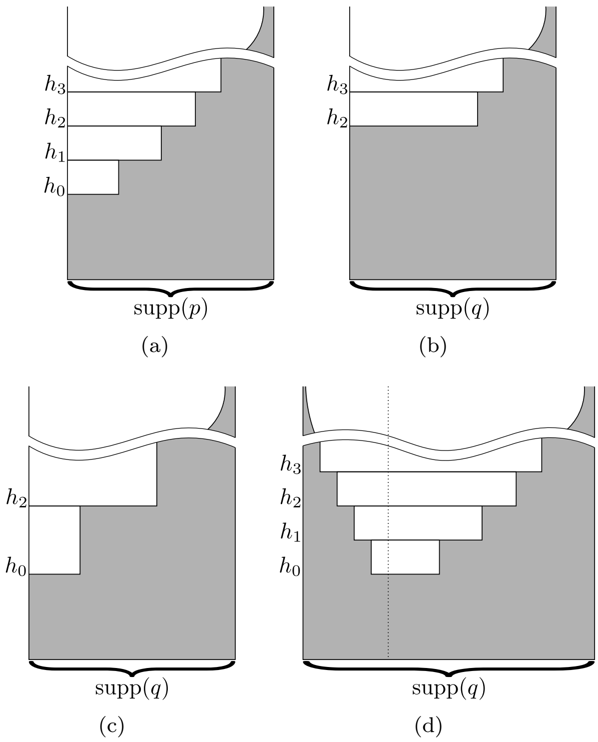

Figure 3.1: (a) A schematic diagram of a condition p of the forcing. The h_i indicate an increasing enumeration of $\mathtt{w}^p$, while the shaded region represents the domain of the trunk function t^p. (b) A condition $q = p \wedge \eta$, where $\eta \in \mathrm{poss}(p, {<}h_2)$. In particular, all of the compound creatures above level h_2 have been left unchanged, and the below level h_2 the condition q consists entirely of trunk, with values determined by η. (c) A condition q obtained from p by gluing the pairs of compound creatures $p(h_0), p(h_1)$ and $p(h_2), p(h_3)$. Note that $\mathrm{trklg}^q(\eta) = h_2$ for any $\eta \in \mathrm{supp}(p)$ with $\mathrm{trklg}^p(\eta) = h_1$ (and similarly if $\mathrm{trklg}^p(\eta) = h_3$). (d) A condition q obtained as the "disjoint union" of p and another condition (represented to the left of the dotted line) with the same $\mathtt{w}$ and the same halving parameters at each level as p.

By now we have seen five specific ways to strengthen a condition. Actually, every $q \leq p$ can be obtained by a combination of these methods. (We will not use the following fact, nor the subsequent remark, in the rest of the paper.)

Fact 3.5.0.15. *For $p, q \in \mathbb{Q}$, $q \leq p$ iff there are p_1, p_2, p_3 and p_4 such that:*

1. *p_1 results from increasing the trunk in p, i.e., $p_1 = p \wedge \eta$ for some $\eta \in \operatorname{poss}(p, < \min(\mathtt{w}^q))$ (in fact: for the (unique) η which is extended by t^q).*
2. *$p_2 \leq p_1$ results from gluing p along $\mathtt{w}^q$, as above.*
3. *$p_3 \leq p_2$ is purely stronger.*
4. *$p_4 \leq p_3$ results from increasing halving parameters.*
5. *q is the disjoint union of p_4 with some condition p'; i.e., the conditions p and p' have the same $\mathtt{w}$, the same halving parameters, disjoint domain, and jointly satisfy "modesty" 2.5.0.19(6); and q is the naturally defined union.*

Remark 3.5.0.16.
- Every q obtained by the construction above is stronger than p, provided it is a condition. Note that constructions (1), (2) and (5) always result in conditions (for (5), this is the same argument as in 2.5.0.25), whereas in constructions (3) and (4) we will generally decrease the norms of the compound creatures in an uncontrolled fashion. So to get a condition, we have to make sure that the norms of the new compound creatures still converge to infinity. Also, to be able to find a suitable p' in (5), we should make enough room for modesty in (3).
- The order is not entirely irrelevant: gluing (2) has to be done before pure strengthening (3), since glued Sacks columns always have the form of products along the old $\mathtt{w}^p$, whereas generally the Sacks columns in q will not be of this form.

We will later use a specific gluing construction:

Lemma 3.5.0.17. *Assume that $\mathfrak{c}_0, \dots, \mathfrak{c}_n$ is a properly stacked sequence of compound creatures, $n > 0$, and $\operatorname{nor}(\mathfrak{c}_i) \geq M$ for all $i \leq n$. Pick for each $i < n$ some compound creatures $\mathfrak{d}_i$, purely stronger than $\mathfrak{c}_i$, such that $\mathfrak{d}_i$ and $\mathfrak{c}_i$ agree on the lim-inf part (but $\mathfrak{d}_i$ could consist of singletons on the lim-sup and the Sacks part). Set $\mathfrak{d}_n = \mathfrak{c}_n$. Then $\operatorname{glue}(\mathfrak{d}_0, \dots, \mathfrak{d}_n)$ has norm $\geq M$ as well.*

Proof. The lim sup norm and the Sacks-norms will be large because $\operatorname{nor}(\mathfrak{d}_n) = \operatorname{nor}(\mathfrak{c}_n) \geq M$; the lim inf norm will be large because we did not change anything on the lim inf part. □

3.6 Projections and complete subforcings

Lemma 3.6.0.18. *Assume that[4] $\Xi_{\mathtt{1i}} \subseteq \Xi' \subseteq \Xi$. Let $\mathbb{Q}_{\Xi'} \subseteq \mathbb{Q}$ consist of all $p \in \mathbb{Q}$ with $\operatorname{supp}(p) \subseteq \Xi'$. Then $\mathbb{Q}_{\Xi'}$ is a complete subforcing, and the restriction map is a projection on an open dense subset. Of course, $\mathbb{Q}_{\Xi'}$ will satisfy all the properties that we will prove generally for $\mathbb{Q}$ (as $\mathbb{Q}_{\Xi'}$ is defined just like $\mathbb{Q}$, only with other sets index sets Ξ).*

Proof. The dense set D is the set of all conditions p with $\operatorname{supp}(p) \cap \Xi_{\mathtt{1i}} \neq \emptyset$. Fix $p \in D$ and set $p' = p \,|\, \Xi'$. Assume that $q' \leq p'$ is in $\mathbb{Q}'$. It is enough to show that q' is compatible with p. We will construct $q \leq p$ such that $q' = q \,|\, \Xi'$ as follows: Fix $p_1 := p \,|(\Xi \setminus \Xi')$. Increase the trunk of p_1 to $\min(\mathtt{w}^{q'})$ and glue along $\mathtt{w}^{q'}$. This gives a condition $q_1 \leq p_1$ with $\mathtt{w}^{q_1} = \mathtt{w}^{q'}$. We let q be the union of q_1 and q' in the obvious way: For each $\ell \in \mathtt{w}^{q_1}$, we take the union (as in the proof of 2.5.0.25) of the compound creatures $q_1(\ell)$ and $q'(\ell)$, using the (potentially bigger) halving parameters of $q'(\ell)$. The compound norms still converges to infinity: The lim-inf norm only uses the information of q', and the width norms of $q'(\ell)$ and $q_1(\ell)$ both converge to infinity, and the width norm of the union is at least one half of the minimum of these two. □

[4]If we do not assume $\Xi' \supseteq \Xi_{\mathtt{1i}}$, then we get problems with the lim-inf norm when we combine the increased halving parameters of q' with the lim-inf creatures in p_1.

Chapter 4

An inductive construction of $\mathbb{Q}$

We will now review the "framework" 2.4.0.18, finally giving all the assumptions (including the previous Assumption 2.5.0.20) that are required to make the forcing proper.

In the following construction, we have the freedom to choose (as long as the assumptions are satisfied).

- Ξ, $\Xi_{\mathtt{ls}}$, $\Xi_{\mathtt{li}}$ as in 2.4.0.18,
- natural numbers $H({<}\mathbf{u})$ (for each sublevel $\mathbf{u}$) such that H is increasing.
  Remark: The function H gives us the possibility to impose additional demands on the bigness B (as given in (4.0.0.1)). It is not needed to get properness and ω^ω-bounding, but will be used later[1] in the constructions that are specific to our cardinal characteristics.
- the subatomic families $\mathsf{K}_{\xi,\mathbf{u}}$ living on some finite set $\mathrm{POSS}_{\xi,\mathbf{u}}$

The other parameters are determined by the construction, namely:

- Natural numbers $\mathrm{maxposs}({<}\mathbf{u})$ for each sublevel $\mathbf{u}$.
  This will turn out to be an upper bound to the cardinality of $\mathrm{poss}(p, {<}\mathbf{u})$ for any pruned p.
- For each sublevel $\mathbf{u}$, we set
  $$B(\mathbf{u}) := 2^{H({<}\mathbf{u})\cdot \mathrm{maxposs}({<}\mathbf{u})}. \tag{4.0.0.1}$$
  (And we set $B((0,-1)) := 2$). $B(\mathbf{u})$ is the bigness required for the subatoms (or: Sacks columns) at $\mathbf{u}$.
- The Sacks intervals $I_{\mathtt{sk},\ell}$ and subatomic index sets J_ℓ, for each $\ell \in \omega$, as in 2.4.0.18.

Note that, as usual, for a Sacks sublevel $\mathbf{u} = (\ell, -1)$ we may write $B(\ell)$ for $B(\mathbf{u})$. Similarly, $\mathrm{maxposs}({<}\ell) := \mathrm{maxposs}({<}(\ell,-1))$ and $H({<}\ell) := H({<}(\ell,-1))$.

We define and require the following, by induction on ℓ, where we set the "initial values" $\mathrm{maxposs}({<}(0,-1)) := 1$ and $I_{\mathtt{sk},-1} = \{-1\}$:

Basic Construction.

($*$1) We require that $H({<}\ell) > \mathrm{maxposs}({<}\ell) + \ell + 2$.

[1] Here is a very informal description of how H will be used. The basic requirement is that at each sublevel $\mathbf{u}$ we have bigness (namely $B(\mathbf{u})$) which is large with respect to everything that happened below. However, the notion of "large with respect to" will slightly depend on the actual construction that increases the relevant cardinal characteristic. The parameter H will allow us to accommodate these different interpretations. The function H will be used as a parameter when defining "rapid reading" in Definition 5.1.0.1.

(∗2) The Sacks sublevel:
We let $I_{\mathtt{sk},\ell}$ be the interval starting at $\max(I_{\mathtt{sk},\ell-1})+1$ and of minimal size such that $\mathrm{nor}^{B(\ell),\ell}_{\mathrm{Sacks}}(2^{I_{\mathtt{sk},\ell}}) \geq \ell$.
The relevant information is: We have "bigness" in the form of Lemma 2.3.0.17(6) for $B := B(\ell)$.

(∗3) We set $\mathrm{maxposs}(<(\ell,0)) := \mathrm{maxposs}(<\ell) \cdot 2^{|I_{\mathtt{sk},\ell}|\cdot\ell}$.

(∗4) We set $J_\ell := 3^{(\ell+1)\cdot 2^{\ell\cdot\mathrm{maxposs}(<\ell)}}$. So $\mu^\ell(J_\ell) = 2^{\ell\cdot\mathrm{maxposs}(<\ell)}$. ($\mu$ is defined in 2.2.0.8.)

(∗5) The subatomic sublevels: By induction on $j \in J_\ell$ we now deal with the sublevel $\mathbf{u} = (\ell, j)$:

(a) For each $\xi \in \Xi_{\text{non-}\mathtt{sk}}$, we require that $\mathsf{K}_{\xi,\mathbf{u}}$ is a subatomic family living on some finite set $\mathrm{POSS}_{\xi,\mathbf{u}}$.

(b) For each $\xi \in \Xi_{\text{non-}\mathtt{sk}}$, we require that there is a subatom $x \in \mathsf{K}_{\xi,\mathbf{u}}$ with norm at least $2^{\ell\cdot\mathrm{maxposs}(<\ell)}$.

(c) For each $\xi \in \Xi_{\text{non-}\mathtt{sk}}$, we require that $\mathsf{K}_{\xi,\mathbf{u}}$ is $B(\mathbf{u})$-big.

(d) We require that there is a uniform bound $M(\mathbf{u}) = \max(\{|\mathrm{POSS}_{\xi,\mathbf{u}}| : \xi \in \Xi_{\text{non-}\mathtt{sk}}\})$. Then we set, for $\mathbf{v}$ the successor sublevel of $\mathbf{u}$,

$$\mathrm{maxposs}(<\mathbf{v}) := \mathrm{maxposs}(<\mathbf{u}) \cdot M(\mathbf{u})^{\ell+1}.$$

In particular this defines $\mathrm{maxposs}(<(\ell+1,-1))$ if $\mathbf{u} = (\ell, J_\ell - 1)$.

The assumptions guarantee that the previous Assumption 2.5.0.20 is satisfied (so in particular that there are compound creatures with norm m^{dn}, and that $\mathbb{Q} \neq \emptyset$).

By induction, we immediately get the following (which is the reason for the name "maxposs"):

Fact 4.0.0.2. *Let p be pruned. Then $|\mathrm{poss}(p,<\mathbf{u})| \leq \mathrm{maxposs}(<\mathbf{u})$ for $\mathbf{u} \in \mathrm{sblvls}(p)$. In particular, $|\mathrm{poss}(p,<h)| \leq \mathrm{maxposs}(<h)$ for $h \in \mathtt{w}^p$.*

Each $p(\mathbf{u})$ is $B(\mathbf{u})$-big:

Fact 4.0.0.3. *Let p be a pruned condition, $\mathbf{u} = (\ell, j)$ be a p-sublevel (which can be Sacks or subatomic), and $\mathbf{v} < \mathbf{u}$ another sublevel.*

Whenever $F : \mathrm{poss}'(p,=\mathbf{u}) \to B(\mathbf{u})$ is a coloring function, then there is a strengthening $q(\mathbf{u})$ of the $p(\mathbf{u})$ (i.e., either $q(\mathbf{u})$ is a subatom stronger than $p(\mathbf{u})$; or $q(\mathbf{u})$ is a sequence of Sacks column such that each one is stronger than the according column in $p(\mathbf{u})$) such that the subatomic norm (or: each Sacks norm) decreases by at most 1 and such that $F\,|\,\mathrm{poss}(y)$ is constant.

As $B(\mathbf{u})$ is much larger than $\mathrm{maxposs}(<\mathbf{u})$, we also get a version of "compound bigness" (we will not directly use the following version, but we will use similar constructions):

A function $G : \mathrm{poss}'(p,\leq\mathbf{u}) \to H(<\mathbf{u})$ can be interpreted as $F : \mathrm{poss}'(p,=\mathbf{u}) \to H(<\mathbf{u})^Y$ for $\mathrm{poss}'(p,<\mathbf{u})$ (cf. 2.7.0.31). As $|\mathrm{poss}'(p,<\mathbf{u})| \leq \mathrm{maxposs}(<\mathbf{u}))$, and $B(\mathbf{u})$ is big with respect to $\mathrm{maxposs}(<\mathbf{u}))$ and $H(<\mathbf{u})$, we can use the previous item and strengthen $p(\mathbf{u})$ to make G independent of the possibilities at $\mathbf{u}$.

Iterating this downwards from the predecessor of $\mathbf{u}$ to $\mathbf{v}$, we get:

Fact 4.0.0.4.
- *If $G : \mathrm{poss}'(p,<\mathbf{u}) \to H(<\mathbf{v})$, then we can increase the $p(\mathbf{u}')$ to $q(\mathbf{u}')$ for $\mathbf{v} \leq \mathbf{u}' < \mathbf{u}$, decreasing all subatomic/Sacks norms (and therefore also all compound norms) by at most 1, such that G restricted to $\mathrm{poss}'(q,<\mathbf{u})$ only depends on $\mathrm{poss}'(q,<\mathbf{v})$.*
- *In particular, if $G : \mathrm{poss}'(p,<\mathbf{u}) \to 2$, then we can strengthen p to q as above such that $G\,|\,\mathrm{poss}'(q,<\mathbf{u})$ is constant.*

Chapter 5

Properness, ω^ω-bounding and rapid reading

5.1 Bigness, rapid reading from continuous reading

(Remark: This section is the straightforward modification of [14, Lemma 1.13].)

Definition 5.1.0.1.

- Let $\underset{\sim}{\tau}$ be the name of an ordinal. We say that $\underset{\sim}{\tau}$ is decided below the sublevel $\mathbf{u}$ (with respect to the condition p), if $p \curlywedge \eta$ decides the value of $\underset{\sim}{\tau}$ for all $\eta \in \text{poss}(p, <\mathbf{u})$; in other words, there is a function $R : \text{poss}(p, <\mathbf{u}) \to \text{Ord}$ such that $p \curlywedge \eta \Vdash \underset{\sim}{\tau} = R(\eta)$ for all $\eta \in \text{poss}(p, <\mathbf{u})$.
- We also write "$\underset{\sim}{\tau}$ is decided $< \mathbf{u}$"; and we write "$\underset{\sim}{\tau}$ is decided $\leq \mathbf{u}$" for the obvious concept (i.e., "$\underset{\sim}{\tau}$ is decided $< \mathbf{v}$", where $\mathbf{v}$ is the successor sublevel of $\mathbf{u}$).
- p *essentially decides* $\underset{\sim}{\tau}$, if there is some sublevel $\mathbf{u}$ such that τ is decided below $\mathbf{u}$.
- Let $\underset{\sim}{r}$ be the name of an ω-sequence of ordinals. We say that a condition p continuously reads $\underset{\sim}{r}$, if all $\underset{\sim}{r}(m)$ are essentially decided by p.
- p rapidly reads $\underset{\sim}{r} \in 2^\omega$, if, for each sublevel $\mathbf{u}$, $\underset{\sim}{r} \mid H(<\mathbf{u})$ is decided below $\mathbf{u}$.
- Let $\Xi_0 \subseteq \Xi$. We say that p "reads $\underset{\sim}{r}$ continuously only using indices in Ξ_0" if p reads $\underset{\sim}{r}$ continuously and moreover (using the relevant functions R mentioned above) the value of $R(\eta)$ depends only on $\eta \mid \Xi_0$.

  In other words: For every n there exists a sublevel $\mathbf{u}$ such that $p \curlywedge \eta$ decides the value of $\underset{\sim}{r}(n)$ for all $\eta \in \text{poss}(p, <\mathbf{u})$, and whenever $\eta \mid \Xi_0 = \eta' \mid \Xi_0$, then $p \curlywedge \eta$ and $p \curlywedge \eta'$ agree on the value of $\underset{\sim}{r}(n)$.
- We define the notion "reads $\underset{\sim}{r}$ rapidly only using indices in Ξ_0" similarly.
- Instead of "only using indices in $\Xi \setminus \Xi_1$" we also write "not using indices in Ξ_1".

Note that for $X \supseteq \Xi_{\mathtt{li}}$, a real $\underset{\sim}{r}$ is read continuously from X iff it exists in the $\mathbb{Q}_X$-extension (cf. 3.6.0.18).

Remark 5.1.0.2. For a fixed condition p, the possibilities (at all sublevels) form an infinite tree in the obvious way. The set of branches T_p of this tree carries a natural topology. p continuously reads

τ iff there is a continuous function F on T_p in the ground model such that p forces $\underset{\sim}{\tau} = \tilde{F}(\underset{\sim}{\bar{y}})$, where $\tilde{F}$ is the canonical extension of F.

In our case, the tree is finitely splitting, so T_p is compact, and continuous is the same as uniformly continuous. (Note that the definition above really uses a uniform notion of continuity.)

Rapid reading corresponds to a form of Lipschitz continuity.

Lemma 5.1.0.3. *1. If p continuously (or: rapidly) reads $\underset{\sim}{r}$ and $q \leq p$ with $\mathrm{supp}(q) \supseteq \mathrm{supp}(p)$, then q continuously (or: rapidly) reads $\underset{\sim}{r}$. The same holds if we add "only using Ξ_0" or: "not using Ξ_1".*

2. If $q \leq^* p$, and $\underset{\sim}{\tau}$ is a name of an ordinal essentially decided by p, then also q essentially decides $\underset{\sim}{\tau}$.

Proof. (1) Intuitively, this is clear: If $q \leq p$ and $\eta \in \mathrm{poss}(q, <\mathbf{u})$ then η morally is an element of $\mathrm{poss}(p, <\mathbf{u})$, and $q \curlywedge \eta \leq p \curlywedge \eta$.

The formal proof uses Lemma 3.2.0.6.

(2) p forces that $\underset{\sim}{\tau}$ is decided by a finite case distinction; so q forces the same. $\square$

Lemma 5.1.0.4. *In V, let κ be $\max(\aleph_0, |\Xi_0|)^{\aleph_0}$. Then in the extension, there are at most κ many reals which are continuously read only using[1] indices in Ξ_0.*

Proof. This is the usual "nice names" argument: Given p continuously reading $\underset{\sim}{r}$. We can define the obvious name $\underset{\sim}{r}'$ continuously read by $p' = p \,|\, \Xi_0$, such that p forces $\underset{\sim}{r} = \underset{\sim}{r}'$. There are at most κ many countable subsets of Ξ_0, and therefore only κ many conditions p' with $\mathrm{supp}(p') \subseteq \Xi_0$. Given such a condition p', there are only $2^{\aleph_0}$ many ways to continuously read a real (with respect to p'). $\square$

We will first show that we can "densely" get from continuous reading to rapid reading. Later we will show that "densely" we can continuously read reals. Both proofs are the obvious modifications of the corresponding proofs in [14].

Lemma 5.1.0.5. *Assume that p continuously reads $\underset{\sim}{r} \in 2^\omega$, then there is a $q \leq p$ rapidly reading $\underset{\sim}{r}$. The same is true if we add "only using Ξ_0".*

Proof. Without loss of generality we can assume that p is pruned (use Lemmas 3.4.0.11 and 5.1.0.3).

For a sublevel $\mathbf{u}$, we set

$$\mathbf{v}^{\mathrm{dec}}(\mathbf{u}) \text{ is the maximal sublevel such that } \underset{\sim}{r} \,|\, H(<\mathbf{v}^{\mathrm{dec}}(\mathbf{u})) \text{ is decided below } \mathbf{u}, \tag{5.1.0.6}$$

The function $\mathbf{v}^{\mathrm{dec}}$ is nondecreasing; and continuous reading implies that $\mathbf{v}^{\mathrm{dec}}$ is an unbounded function on the sublevels; but $\mathbf{v}^{\mathrm{dec}}$ can generally grow very slowly. (p "rapidly reads $\underset{\sim}{r}$" would mean that $\mathbf{v}^{\mathrm{dec}}(\mathbf{u}) \geq \mathbf{u}$ for all $\mathbf{u}$.)

For all sublevels $\mathbf{v} \leq \mathbf{u}$ we set

$$\underset{\sim}{x}^{\mathbf{u}}_{\mathbf{v}} := \underset{\sim}{r} \,|(H(<\min(\mathbf{v}, \mathbf{v}^{\mathrm{dec}}(\mathbf{u})))) \text{ (which is by definition decided below } \mathbf{u}). \tag{5.1.0.7}$$

There are at most

$$2^{H(<\mathbf{v})} \tag{5.1.0.8}$$

many possibilities for $\underset{\sim}{x}^{\mathbf{u}}_{\mathbf{v}}$, as $H((<\min(\mathbf{v}, \mathbf{v}^{\mathrm{dec}}(\mathbf{u})))) \leq H(<\mathbf{v})$.

[1] More formally: reals r such that there is a $p \in G$ and a name $\underset{\sim}{r}$ such that p continuously reads $\underset{\sim}{r}$ only using Ξ_0 and such that G evaluates $\underset{\sim}{r}$ to r.

1 For now, fix a Sacks sublevel $\mathbf{u} = (\ell, -1)$ with $\ell \in \mathtt{w}^p$.

We will define (or rather: pick) by downwards induction on $\mathbf{u}' \in \mathrm{sblvls}(p)$, $\mathbf{u}' \leq \mathbf{u}$, objects $\mathfrak{d}^{\mathbf{u}}_{\mathbf{u}'}$, which are either a sequence of Sacks columns (if $\mathbf{u}'$ is Sacks) or a subatom; and functions $\psi^{\mathbf{u}}_{\mathbf{u}'}$.

1a For $\mathbf{u}' = \mathbf{u}$, we set $\mathfrak{d}^{\mathbf{u}}_{\mathbf{u}} := p(\mathbf{u})$, i.e., the sequence of Sacks columns of level ℓ. We let $\psi^{\mathbf{u}}_{\mathbf{u}}$ be the function with domain $\mathrm{poss}(p, {<}\mathbf{u})$ which assigns to each $\eta \in \mathrm{poss}(p, {<}\mathbf{u})$ the corresponding value of $\underset{\sim}{x}^{\mathbf{u}}_{\mathbf{u}}$.

In other words: $p \curlywedge \eta$ forces that $\underset{\sim}{x}^{\mathbf{u}}_{\mathbf{u}} = \psi^{\mathbf{u}}_{\mathbf{u}}(\eta)$ for each $\eta \in \mathrm{poss}(p, {<}\mathbf{u})$.

1b We continue the induction on $\mathbf{u}'$. For now, we write $\mathfrak{d}' := \mathfrak{d}^{\mathbf{u}}_{\mathbf{u}'}$, $\psi' := \psi^{\mathbf{u}}_{\mathbf{u}'}$, and $x' := x^{\mathbf{u}}_{\mathbf{u}'}$.

- If $\mathbf{u}'$ is subatomic, then we choose for $\mathfrak{d}'$ a subatom stronger than the active subatom $p(\mathbf{u}')$, with $\mathrm{nor}(\mathfrak{d}') \geq \mathrm{nor}(p(\mathbf{u}')) - 1$.
- Otherwise, i.e., if $\mathbf{u}' = (\ell', -1)$ is Sacks with $\ell' \in \mathtt{w}^p$, set $S := \mathrm{supp}(p, \ell') \cap \Xi_{\mathbf{sk}} \neq \emptyset$. Then $\mathfrak{d}'$ is a sequence $(\mathfrak{s}'_\xi)_{\xi \in S}$ of Sacks columns such that $\mathfrak{s}'_\xi \subseteq p(\xi, \ell')$ and $\mathrm{nor}_{\mathrm{Sacks}}(\mathfrak{s}'_\xi) \geq \mathrm{nor}_{\mathrm{Sacks}}(p(\xi, \ell)) - 1$ for each $\xi \in S$.
- ψ' is a function with domain $\mathrm{poss}(p, {<}\mathbf{u}')$ such that

$$\text{modulo } (\mathbf{v} : \mathbf{u}' \leq \mathbf{v} < \mathbf{u}), \text{ each } \eta \in \mathrm{poss}(p, {<}\mathbf{u}') \text{ decides } \underset{\sim}{x}' \text{ to be } \psi'(\eta), \tag{5.1.0.9}$$

  by which we mean:

  > $p \curlywedge \eta$ forces the following: If the generic $\underset{\sim}{\bar{y}}$ is compatible with $\mathfrak{d}^{\mathbf{u}}_{\mathbf{v}}$ for each sublevel $\mathbf{v} \in \mathrm{sblvls}(p)$ with $\mathbf{u}' \leq \mathbf{v} < \mathbf{u}$, then $\underset{\sim}{x}' = \psi'(\eta)$.

How can we find such $\mathfrak{d}'$, ψ'?

Let $\mathbf{u}''$ be the smallest element of $\mathrm{sblvls}(p)$ above $\mathbf{u}'$. By induction we already know that $\psi'' := \psi^{\mathbf{u}}_{\mathbf{u}''}$ is a function with domain $\mathrm{poss}(p, {<}\mathbf{u}'')$ such that modulo $(\mathbf{v} : \mathbf{u}'' \leq \mathbf{v} < \mathbf{u})$ each $\eta \in \mathrm{poss}(p, {<}\mathbf{u}'')$ decides $\underset{\sim}{x}'' := \underset{\sim}{x}^{\mathbf{u}}_{\mathbf{u}''}$ to be $\psi''(\eta)$.

Let $\psi''_0(\eta)$ be the restriction of $\psi''(\eta)$ to $H({<}\min(\mathbf{u}', \mathbf{v}^{\mathrm{dec}}(\mathbf{u})))$, i.e., ψ''_0 maps each $\eta \in \mathrm{poss}(p, {<}\mathbf{u}'')$ to a restriction of $\underset{\sim}{x}''$, which is a potential value for $\underset{\sim}{x}'$.

We can write[2] ψ''_0 as a function $A \times B \to C$, for $A := \mathrm{poss}(p, {<}\mathbf{u}')$, $B = \mathrm{poss}(p, {=}\mathbf{u}')$ and C is the set of possible values of $\underset{\sim}{x}'$, which has, according to (5.1.0.8), size $\leq 2^{H({<}\mathbf{u}')}$. This defines a function from B to C^A, a set of cardinality $\leq 2^{\mathrm{maxposs}({<}\mathbf{u}') \cdot H({<}\mathbf{u}')}$; so according to (4.0.0.1) and Fact 4.0.0.3 we can use bigness at sublevel $\mathbf{u}'$ to find $\mathfrak{d}'$ such that ψ''_0 does not depend on sublevel $\mathbf{u}'$. This naturally defines ψ'.

2 We perform this downwards induction from each Sacks sublevel $\mathbf{u}$ of p. So this defines for each $\mathbf{v} < \mathbf{u}$ in $\mathrm{sblvls}(p)$ the objects $\mathfrak{d}^{\mathbf{u}}_{\mathbf{v}}$ and $\psi^{\mathbf{u}}_{\mathbf{v}}$, satisfying (which is just 5.1.0.9):

$$\text{modulo } (\mathbf{v}' : \mathbf{v} \leq \mathbf{v}' \leq \mathbf{u}), \text{ each } \eta \in \mathrm{poss}(p, {<}\mathbf{v}) \text{ decides } \underset{\sim}{x}^{\mathbf{u}}_{\mathbf{v}} \text{ to be } \psi^{\mathbf{u}}_{\mathbf{v}}(\eta). \tag{5.1.0.10}$$

Also, the norms of each Sacks column and subatom drop by at most 1.

3 Note that for a given $\mathbf{v}$, there are only finitely many possibilities for $\mathfrak{d}^{\mathbf{u}}_{\mathbf{v}}$ and $\psi^{\mathbf{u}}_{\mathbf{v}}$. So by König's Lemma there is a sequence $(\mathfrak{d}^*_{\mathbf{v}}, \psi^*_{\mathbf{v}})_{\mathbf{v} \in \mathrm{sblvls}(p)}$ such that

$$\text{for each sublevel } \mathbf{v}' \text{ there is an } \mathbf{u} > \mathbf{v}' \text{ such that } \mathfrak{d}^{\mathbf{u}}_{\mathbf{v}''} = \mathfrak{d}^*_{\mathbf{v}''} \text{ and } \psi^{\mathbf{u}}_{\mathbf{v}''} = \psi^*_{\mathbf{v}''} \text{ for all } \mathbf{v}'' \leq \mathbf{v}'. \tag{5.1.0.11}$$

[2] cf. 2.7.0.31

4 We now construct q by replacing the subatoms and Sacks columns in p at sublevel $\mathbf{v}$ with $\mathfrak{d}^*_{\mathbf{v}}$ (for each $\mathbf{v} \in \mathrm{sblvls}(p)$). So q has the same $\mathtt{w}$ as p, the same supports, the same halving parameters and the same trunk; and all norms decrease by at most 1. We claim that q rapidly reads $\underset{\sim}{r}$, i.e., we claim that each $\eta \in \mathrm{poss}(q, {<}\mathbf{v})$ decides $\underset{\sim}{r} \,|\, H({<}\mathbf{v})$.

5 Pick a $\mathbf{v}' > \mathbf{v}$ such that $\mathbf{v}^{\mathrm{dec}}(\mathbf{v}') \geq \mathbf{v}$. According to the definition (5.1.0.6), this means that $\underset{\sim}{r} \,|\, H({<}\mathbf{v})$ is decided below $\mathbf{v}'$. Then pick $\mathbf{u} > \mathbf{v}'$ as in (5.1.0.11). Recall (from (5.1.0.10)) that $\underset{\sim}{x}^{\mathbf{u}}_{\mathbf{v}}$ is decided below $\mathbf{v}$ by $\psi^{\mathbf{u}}_{\mathbf{v}}$ modulo the sequence $(\mathfrak{d}^{\mathbf{u}}_{\mathbf{v}''} : \mathbf{v} \leq \mathbf{v}'' < \mathbf{u})$. Recall that $\mathbf{v}^{\mathrm{dec}}(\mathbf{v}') \geq \mathbf{v}$ and $\mathbf{u} \geq \mathbf{v}'$. So $\min(\mathbf{v}^{\mathrm{dec}}(\mathbf{u}), \mathbf{v}) = \mathbf{v}$, therefore $\underset{\sim}{x}^{\mathbf{u}}_{\mathbf{v}} = \underset{\sim}{r} \,|\, H({<}\mathbf{v})$. And, since $\mathbf{v}^{\mathrm{dec}}(\mathbf{v}') \geq \mathbf{v}$, $\underset{\sim}{x}^{\mathbf{u}}_{\mathbf{v}}$ is decided already (by the original condition p) below $\mathbf{v}'$. So we can omit the assumption that the generic is compatible with $\mathfrak{d}^{\mathbf{u}}_{\mathbf{u}''}$ for any $\mathbf{v}' \leq \mathbf{u}'' < \mathbf{u}$ and still correctly compute $\underset{\sim}{x}^{\mathbf{u}}_{\mathbf{v}}$ with $\psi^{\mathbf{u}}_{\mathbf{v}}$ modulo $(\mathfrak{d}^{\mathbf{u}}_{\mathbf{u}''} : \mathbf{v} \leq \mathbf{u}'' < \mathbf{v}')$.

In particular, $\psi^{\mathbf{u}}_{\mathbf{v}} = \psi^*_{\mathbf{v}}$ correctly computes $\underset{\sim}{x}^{\mathbf{u}}_{\mathbf{v}} = \underset{\sim}{r} \,|\, H({<}\mathbf{v})$ modulo q (since q contains $\mathfrak{d}^{\mathbf{u}}_{\mathbf{u}''} = \mathfrak{d}^*_{\mathbf{u}''}$ for each $\mathbf{u}'' < \mathbf{v}'$.) $\square$

5.2 Halving and unhalving

We will now, for the first and only time in this paper, make use of the halving parameter. We will show how to "halve" a condition q to $\mathrm{half}(q)$, and then "unhalve" any $r \leq \mathrm{half}(q)$ with "positive norms" to some $s \leq^* q$ with "large norms". This fact will only be used in the next section, to show pure decision.

We repeat the definition of the lim-inf norm from 2.5.0.21:

$$\mathrm{nor}^{\mathrm{maxposs}({<}m^{\mathrm{dn}})}_{\mathrm{liminf}}(\mathfrak{c}, h) = \frac{\log_2(N^{\mathfrak{c}}_h - d(\mathfrak{c}, h))}{\mathrm{maxposs}({<}m^{\mathrm{dn}})} \quad \text{for } N^{\mathfrak{c}}_h := \min\{\mathrm{nor}(\mathfrak{c}(\xi, h)) : \xi \in \mathrm{supp} \cap \Xi_{\mathtt{1i}}\}.$$

If we increase $d := d(\mathfrak{c}, h)$ to

$$d' := d + \frac{N^{\mathfrak{c}}_h - d}{2} = \frac{N^{\mathfrak{c}}_h + d}{2}, \tag{5.2.0.12}$$

then the resulting lim-inf norm (hence also the compound norm) decreases by at most $1/\mathrm{maxposs}({<}m^{\mathrm{dn}})$.

Definition 5.2.0.13. Given a compound creature $\mathfrak{c}$, we set $\mathrm{half}(\mathfrak{c})$ to be the same compound creature as $\mathfrak{c}$, except that we replace each halving parameter $d(h)$ by the $d'(h)$ described above.
So $\mathrm{nor}(\mathrm{half}(\mathfrak{c})) \geq \mathrm{nor}(\mathfrak{c}) - 1/\mathrm{maxposs}({<}m^{\mathrm{dn}})$.
Similarly, given a condition p and a level $h \in \mathtt{w}^p$, we set $\mathrm{half}(p, {\geq}h)$ to be the same as p, except that all compound creatures $p(\ell)$ for $\ell \geq h$ are halved (and nothing changes below h).

The point of halving is the following: Assume that the norms in q are "large" and that $r \leq \mathrm{half}(q)$ has norms that are just > 0. Then there is an "unhalved version" of r, an $s \leq q$, such that the norms in s are "large" and still $s \leq^* r$.

In more detail:

Lemma 5.2.0.14 (Unhalving). *Fix*

- $M \in \mathbb{R}$,
- *a condition* q,
- $h \in \mathtt{w}^q$ *such that* $\mathrm{nor}(q(\ell)) \geq M$ *for all* $\ell \geq h$ *in* $\mathtt{w}^q$,
- *a condition* $r \leq \mathrm{half}(q, {\geq}h)$ *such that* $\min(\mathtt{w}^r) = h$ *and* $\mathrm{nor}(r(\ell)) > 0$ *for all* ℓ *in* $\mathtt{w}^r$.

Then there is an s such that

1. $s \leq q$.

2. $h = \min(\mathtt{w}^s)$.

3. *Writing h_1 for the successor of h in $\mathtt{w}^s$, we have $\operatorname{nor}(s,\ell) \geq M$ for all $\ell \geq h_1$ in $\mathtt{w}^s$.*

4. $\operatorname{supp}(s,h) = \operatorname{supp}(q,h)$.

5. *Above h_1, s is the same as r, i.e.:*

   - *For $\ell \geq h_1$: $\ell \in \mathtt{w}^s$ iff $\ell \in \mathtt{w}^r$, and for such ℓ we have $s(\ell) = r(\ell)$.*
   - *The trunks agree above h_1.*
   - *So in particular, $\operatorname{supp}(s) = \operatorname{supp}(r)$, and the norms do not change above h_1 (hence are $\geq M$).*

6. $\operatorname{nor}(s,h) \geq M - 1/\operatorname{maxposs}(<h)$.

7. $\operatorname{poss}(s,<h_1) \subseteq \operatorname{poss}(r,<h_1)$.

Note that (5) and (7) implies $s \leq^* r$ (by 3.1.0.2). So (by 5.1.0.3), if r essentially decides a name $\underset{\sim}{\tau}$, then so does s.

Proof. First fix $h_0 \in \mathtt{w}^r$ bigger than h such that $\operatorname{nor}(r(\ell)) > M$ for all $\ell \geq h_0$. Let h_1 be the $\mathtt{w}^r$-successor of h_0.

We set $\mathtt{w}^s := \{h\} \cup \mathtt{w}^r \setminus h_1$. The trunk t^s will extend t^r (and will contain some additional in the "area" $[h,h_1) \times (\operatorname{supp}(r,h_0) \setminus \operatorname{supp}(q,h))$).

For $\ell \geq h_1$ in $\mathtt{w}^s$, we set $s(\ell) := r(\ell)$.

We set $\mathfrak{d}_0 := \operatorname{glue}(r(h),\dots,r(h_0))$, and choose arbitrary r-compatible elements for the new parts of the trunk t^s. We then let $\mathfrak{d}_1$ be the restriction of $\mathfrak{d}_0$ to $\operatorname{supp}(q,h)$ (again, choosing r-compatible elements for the new parts of the trunk t^s).

Now we construct $\mathfrak{d}$ from $\mathfrak{d}_1$ by replacing each halving parameter $d^{\mathfrak{d}_1}(k)$ by $d^q(k)$ (for all $h \leq k < h_1$). We set $s(h) = \mathfrak{d}$. This completes the construction of the condition s.

It is straightforward to check that the requirements are satisfied. We will show $\operatorname{nor}(s(h)) = \operatorname{nor}(\mathfrak{d}) \geq M - 1/\operatorname{maxposs}(<h)$:

The norm of $\mathfrak{d}$ is the minimum of several subnorms:

- The width norm, which is $\geq M$, as $\operatorname{supp}(\mathfrak{d}) = \operatorname{supp}(q,h)$ and $\operatorname{nor}(q(h)) \geq M$.
- The Sacks norms of the Sacks columns $\mathfrak{d}(\xi) = r(\xi,h) \otimes \cdots \otimes r(\xi,h_0)$ for $\xi \in \operatorname{supp}(\mathfrak{d}) \cap \Xi_{\mathtt{sk}}$:

$$\begin{aligned}\operatorname{nor}_{\text{Sacks}}(\mathfrak{d}(\xi)) = \operatorname{nor}_{\text{Sacks}}^{B(h),h}(\mathfrak{d}(\xi)) \geq \operatorname{nor}_{\text{Sacks}}^{B(h),h}(r(\xi,h_0)) \geq \\ \geq \operatorname{nor}_{\text{Sacks}}^{B(h_0),h_0}(r(\xi,h_0)) = \operatorname{nor}_{\text{Sacks}}(r(\xi,h_0)) \geq M,\end{aligned}$$

  by 2.3.0.17.

- The lim-sup norms: $\operatorname{nor}_{\text{limsup}}(\mathfrak{d},\xi) \geq \operatorname{nor}_{\text{limsup}}(r(h_0),\xi) \geq M$.
- So it remains to deal with the lim-inf norm.

So we have to show that for $h \le \ell < h_1$,

$$\mathrm{nor}^{\mathrm{maxposs}(<h)}_{\mathrm{liminf}}(\mathfrak{d},\ell) = \frac{\log_2(N^{\mathfrak{d}}_\ell - d(\mathfrak{c},\ell))}{\mathrm{maxposs}(<h)} \ge M - \frac{1}{\mathrm{maxposs}(<h)}, \tag{5.2.0.15}$$

where $N^{\mathfrak{d}}_\ell := \min\{\mathrm{nor}(\mathfrak{d}(\xi,\ell)) : \xi \in \mathrm{supp}(\mathfrak{d}) \cap \Xi_{\mathtt{1i}}\}$.

Recall $d'(\ell)$ as defined in (5.2.0.12). These are the halving parameters used in half(q), and since $r \le \mathrm{half}(q)$ we know that $d^r(\ell) \ge d'(\ell)$ (where d^r are the halving parameters used in r).

Let $m \in \mathtt{w}^r$ correspond to ℓ (i.e., $m \le \ell$ and ℓ less than the $\mathtt{w}^r$-successor of m). As $\mathrm{nor}(r(m)) > 0$, we know that

$$0 < \mathrm{nor}^{\mathrm{maxposs}(<m)}_{\mathrm{liminf}}(r(m),\ell) \le \mathrm{nor}^{\mathrm{maxposs}(<h)}_{\mathrm{liminf}}(r(m),\ell) \le \frac{\log_2(N^{\mathfrak{d}}_\ell - d^r(\ell))}{\mathrm{maxposs}(<h)}$$

for $N^{\mathfrak{d}}_\ell$ as above.[3]

Fix any $\xi \in \mathrm{supp}(q,h) \cap \Xi_{\mathtt{1i}}$. Let $k \in \mathtt{w}^q$ correspond to ℓ (as above), and set $\mathfrak{c} = q(k)$. The inequality above gives $0 < \log_2(\mathrm{nor}(\mathfrak{d}(\ell,\xi)) - d^r(\ell))$, which implies

$$\mathrm{nor}(\mathfrak{d}(\xi,\ell)) > d^r(\ell) \ge d'(\ell) = d^q(\ell) + \frac{N^{\mathfrak{c}}_\ell - d^q(\ell)}{2}.$$

So $\mathrm{nor}(\mathfrak{d}(\ell,\xi)) - d^q(\ell) > \frac{N^{\mathfrak{c}}_\ell - d^q(\ell)}{2}$ for all ξ, and so

$$\begin{aligned}\mathrm{nor}^{\mathrm{maxposs}(<h)}_{\mathrm{liminf}}(\mathfrak{d},\ell) &\ge \mathrm{nor}^{\mathrm{maxposs}(<h)}_{\mathrm{liminf}}(\mathfrak{c},\ell) - \frac{1}{\mathrm{maxposs}(<h)}\\ &\ge \mathrm{nor}^{\mathrm{maxposs}(<k)}_{\mathrm{liminf}}(\mathfrak{c},\ell) - \frac{1}{\mathrm{maxposs}(<h)}\\ &\ge M - \frac{1}{\mathrm{maxposs}(<h)}.\end{aligned}$$

□

5.3 Halving and pure decision

(Remark: This section is the straightforward modification of [14, Lemma 1.17].)

Lemma 5.3.0.16. *Suppose that $\underset{\sim}{\tau}$ is a name for an element of V, that $p_0 \in \mathbb{Q}$, that $M_0 \in \mathtt{w}^{p_0}$ and $n_0 \ge 1$ are such that* $\mathrm{nor}(p_0(h)) \ge n_0 + 2$ *for all $h \in \mathtt{w}^{p_0} \setminus M_0$. Then there is a condition q such that:*

- *$q \le p_0$.*
- *q essentially decides $\underset{\sim}{\tau}$.*
- *Below M_0, q and p_0 are identical,[4] i.e.: $\mathtt{w}^q \cap M_0 = \mathtt{w}^{p_0} \cap M_0$ and $q(h) = p_0(h)$ for all $h \in \mathtt{w}^q \cap M_0$.*
- $\mathrm{nor}(q(h)) \ge n_0$ *for all $h \in \mathtt{w}^q \setminus M_0$.*

Proof. We may assume that p_0 is pruned. Our proof will consist of several steps:

1. Using halving; the mini-steps.

Suppose that we are given $p \in \mathbb{Q}$, $M \in \mathtt{w}^p$, and $n \ge 1$ such that $\mathrm{nor}(p(h)) > n$ for all $h \in \mathtt{w}^p \setminus M$. We show how to construct an extension of p, denoted $r(p, M, n)$.

First enumerate $\mathrm{poss}(p, <M)$ as $(\eta^1, \dots, \eta^m)$. Note that $m \le \mathrm{maxposs}(<M)$. Setting $p^0 = p$, we inductively construct conditions $p^1, \dots, p^m$ and the auxiliary conditions $\tilde{p}^1, \dots, \tilde{p}^m$ so that for each $k < m$ the following holds:

[3] The last $\le$ holds since $r(m)$ contains the same subatoms as $\mathfrak{d}$ (on the common support; however the support of $r(m)$ may be larger, therefore the last inequality is not necessarily an equality).

[4] $\mathrm{supp}(q)$ can be larger than $\mathrm{supp}(p)$, so below M_0 there will be new parts of the trunk t^q.

1. $\tilde{p}^{k+1}$ is p^k where we replace everything below M (and in $\mathrm{supp}(p)$) with η^{k+1}.
   Remarks:
   - By (3) below, we will get $\min(\mathtt{w}^{\tilde{p}^{k+1}}) = M$.
   - If $k = 0$, then $\tilde{p}^1$ is just $p \wedge \eta^1$. But for $k > 0$, η^{k+1} will not be in $\mathrm{poss}(p^k, {<}M)$, so we cannot use the notation $\tilde{p}^{k+1} = p^k \wedge \eta^{k+1}$.
   - Note that generally $\mathrm{supp}(p^k)$ will be larger than $\mathrm{supp}(p)$, so we do not replace the whole trunk below M by η^{k+1}, but just the part in $\mathrm{supp}(p)$.
2. $p^{k+1} \leq \tilde{p}^{k+1}$. Note that we do not have $p_{k+1} \leq p_k$, for trivial reasons: their trunks are incompatible.
3. $\min(\mathtt{w}^{p^{k+1}}) = M$.
   Remarks:
   - So by strengthening $\tilde{p}^{k+1}$ to p^{k+1}, we do not increase the overall trunk-length $\min(\mathtt{w})$.
   - Note that we do not assume that $\mathtt{w}^{p^{k+1}} = \mathtt{w}^{p^k} \setminus M$, i.e., generally the $\mathtt{w}$-sets will become thinner due to gluing.
4. $\mathrm{supp}(p^{k+1}, M) = \mathrm{supp}(p, M)$.
   - Remark: This only holds at level M: Generally, $\mathrm{supp}(p^{k+1})$ will be larger than $\mathrm{supp}(p^k)$.
5. $\mathrm{nor}(p^{k+1}, h) > n - \frac{k+1}{\mathrm{maxposs}({<}M)}$ for all $h \in \mathtt{w}^{p^{k+1}} \setminus M$.
6. One of the following two cases holds:
   - **(decide)** p^{k+1} essentially decides $\underset{\sim}{\tau}$.
   - **(halve)** $p^{k+1} = \mathrm{half}(\tilde{p}^{k+1}, {\geq}M)$.

   More explicitly: If the deciding case is possible, then we use it. Only if it is not possible, we halve.

We then define $r = r(p, M, n)$ as follows: Below M, r is identical to p; and above (including) M, r is identical to p^m (the last one of the p^k constructed above). In more detail:

- $\mathtt{w}^r = (\mathtt{w}^p \cap M) \cup (\mathtt{w}^{p^m} \setminus M)$; i.e., below M the levels of r are the ones of p; and above (including) M the levels of r are the ones of p^m.
- $r(h) = p(h)$ for all $h \in \mathtt{w}^r \cap M$;
- $r(h) = p^m(h)$ for all $h \in \mathtt{w}^r \setminus M$;
- This determines the domain of t^r; and we set t^r to be t^{p^m} restricted to this domain.

$r = r(p, M, n)$ has the following properties:

- $r \in \mathbb{Q}$, $r \leq p$.
- $\mathrm{nor}(r(\ell)) > n - 1$ for all $\ell \geq M$ in $\mathtt{w}^r$.
- If $\eta \in \mathrm{poss}(r, {<}M)$ and if there is a $s \leq r \wedge \eta$ such that s essentially decides $\underset{\sim}{\tau}$, $\min(\mathtt{w}^s) = M$ and $\mathrm{nor}(s(\ell)) > 0$ for all $\ell \geq M$ in $\mathtt{w}^s$, then $r \wedge \eta$ essentially decides $\underset{\sim}{\tau}$. (5.3.0.17)

Proof of (5.3.0.17). η extends some $\eta^{k+1} \in \text{poss}(p, {<}M)$; so $s \leq r \wedge \eta \leq p^{k+1} \leq \tilde{p}^{k+1}$. All we have to show is that p^{k+1} was constructed using the "decide" case. Assume towards a contradiction that the "halve" case was used. Then s is stronger than $\text{half}(\tilde{p}^{k+1}, {\geq}M)$, so we can unhalve it (using Lemma 5.2.0.14) to get some $s' \leq \tilde{p}^{k+1}$ with large norm such that $s' \leq^* s$, showing that we could have used the "decide" case after all. This ends the proof of (5.3.0.17). $\square$

2. Iterations of the mini-steps; the condition q.

Given p_0, M_0, n_0 as in the statement of the Lemma, we inductively construct conditions p_k and natural numbers M_k for $k \geq 1$. Given p_k and M_k, our construction of p_{k+1} and M_{k+1} is as follows: Choose $M_{k+1} \in \mathtt{w}^{p_k}$ bigger than M_k such that

$$\text{nor}(p_k(h)) > k + n_0 + 3 \text{ for all } h \in \mathtt{w}^{p_k} \setminus M_{k+1}.$$

Then set $p'_{k+1} = r(p_k, M_{k+1}, k + n_0 + 3)$, and construct p_{k+1} by gluing together everything between (including) M_k and (excluding) M_{k+1}.

The sequence of conditions $(p_k)_{k\in\omega}$ converges to a condition of $\mathbb{Q}$, which we will denote by q. Note that $r \leq q$ implies that $\mathtt{w}^r$ is a subset of $(\mathtt{w}^{p_0} \cap M_0) \cup \{M_0, M_1, M_2, \dots\}$ (as we have glued everything between each M_i and M_{i+1}).

It is clear that $q \leq p_0$, and that $\text{nor}(q, h) > n_0 + 1$ for all $h \in \mathtt{w}^q \setminus M_0$.

We will later show that q essentially decides $\underset{\sim}{\tau}$ (thus proving the lemma).

The following property will be central:

$$\begin{gathered}\text{Assume that } \eta \in \text{poss}(q, {<}M_\ell) \text{ for some } \ell \in \omega, \text{ and } r \leq q \wedge \eta \text{ essentially decides} \\ \underset{\sim}{\tau} \text{ and } \min(\mathtt{w}^r) = M_\ell \text{ and each } r(m) \text{ has norm} > 1 \text{ for each } m \in \mathtt{w}^r. \\ \text{Then } q \wedge \eta \text{ essentially decides } \underset{\sim}{\tau}.\end{gathered} \tag{5.3.0.18}$$

Proof of (5.3.0.18): η (or rather: a restriction of η to $\text{supp}(p)$) was considered as a possible trunk η^{k+1} in the "mini-step" when constructing $r(p_{\ell-1}, M_\ell, \ell + n_0 + 2)$. So we can use (5.3.0.17). This ends the proof of (5.3.0.18).

3. Using bigness to thin out q to prove essentially deciding.

We now repeat the construction of the proof of Lemma 5.1.0.5, but this time we do not homogenize on the potential values of some $\underset{\sim}{x}$, but rather on whether $q \curlywedge \eta$ essentially decides $\underset{\sim}{\tau}$ or not.

For now, fix a sublevel $\mathbf{u} = (\ell, -1)$ above $(M_0, -1)$ with $\ell \in \mathtt{w}^q$.

- We set $\mathfrak{d}^{\mathbf{u}}_{\mathbf{u}}$ to be the collection of Sacks columns $q(\mathbf{u})$. We set $B^{\mathbf{u}}_{\mathbf{u}}$ to be the set of $\eta \in \text{poss}(q, {<}\mathbf{u})$ such that $q \curlywedge \eta$ essentially decides $\underset{\sim}{\tau}$.
- By downwards induction on $\mathbf{u}' \in \text{sblvls}(q)$, $(M_0, -1) \leq \mathbf{u}' < \mathbf{u}$, we construct $\mathfrak{d}^{\mathbf{u}}_{\mathbf{u}'}$ and $B^{\mathbf{u}}_{\mathbf{u}'}$ such that the following is satisfied:
  - $\mathfrak{d}^{\mathbf{u}}_{\mathbf{u}'}$ is a strengthening of the subatom (or: collection of Sacks columns) $q(\mathbf{u}')$, the norm decreases by at most 1.
  - (Homogeneity) $B^{\mathbf{u}}_{\mathbf{u}'}$ is a subset of $\text{poss}(q, {<}\mathbf{u}')$, such that for each $\eta \in B^{\mathbf{u}}_{\mathbf{u}'}$ and each $\nu \in \text{poss}(\mathfrak{d}^{\mathbf{u}}_{\mathbf{u}'})$ $\eta^\frown\nu \in B^{\mathbf{u}}_{\mathbf{u}'+1}$; and analogously for each $\eta \in \text{poss}(q, {<}\mathbf{u}') \setminus B^{\mathbf{u}}_{\mathbf{u}'}$ and each $\nu \in \text{poss}(\mathfrak{d}^{\mathbf{u}}_{\mathbf{u}'})$, $\eta^\frown\nu \notin B^{\mathbf{u}}_{\mathbf{u}'+1}$.

(Just as in the case of rapid reading, we can find these objects using bigness: Assume that $\mathbf{u}''$ is the sblvls(q)-successor of $\mathbf{u}'$; by induction there is a function F which maps each $\eta \in \text{poss}(q, {<}\mathbf{u}''$ to $\{\in B, \notin B\}$; we thin out $q(\mathbf{u}')$ to $\mathfrak{d}^{\mathbf{u}}_{\mathbf{u}'}$ such that for each $\nu \in \text{poss}(q, {<}\mathbf{u}')$ each extension of ν compatible with $\mathfrak{d}^{\mathbf{u}}_{\mathbf{u}'}$ has the same F-value $F^*(\nu)$; this in turn defines $B^{\mathbf{u}}_{\mathbf{u}'}$.)

- Assume that $\mathbf{v} < \mathbf{u}$ as above, that $\eta \in \text{poss}(q, <\mathbf{v})$, that $q \curlywedge \eta$ essentially decides $\underset{\sim}{\tau}$ and that $\eta' \in \text{poss}(q, <\mathbf{u})$ extends η. Then trivially $q \curlywedge \eta'$ also essentially decides $\underset{\sim}{\tau}$. So we get:

$$\text{If } q \curlywedge \eta \text{ essentially decides } \underset{\sim}{\tau} \text{ for } \eta \in \text{poss}(q, <\mathbf{v}), \text{ then } \eta \in B^{\mathbf{u}}_{\mathbf{v}} \text{ for any } \mathbf{u} > \mathbf{v}. \tag{5.3.0.19}$$

- We now show the converse:

$$\text{Whenever } \eta \in B^{\mathbf{u}}_{\mathbf{u}'} \text{ for some sublevel } \mathbf{u}' \text{ of the form } (M_{\ell'}, -1) \leq \mathbf{u} \text{ for some } \ell', \text{ then } q \wedge \eta \text{ essentially decides } \underset{\sim}{\tau}. \tag{5.3.0.20}$$

  (Equivalently: $q \curlywedge \eta$ essentially decides $\underset{\sim}{\tau}$, as $q \curlywedge \eta =^* q \wedge \eta$.) Proof: We can modify q to a stronger condition r using η as trunk and using $\mathfrak{d}^{\mathbf{u}}_{\mathbf{u}''}$ for all $\mathbf{u}' \leq \mathbf{u}'' \leq \mathbf{u}$. Any $\eta' \in \text{poss}(r, <\mathbf{u})$ is in $B^{\mathbf{u}}_{\mathbf{u}}$, so $q \curlywedge \eta' =^* r \curlywedge \eta'$ essentially decides $\underset{\sim}{\tau}$. So r essentially decides $\underset{\sim}{\tau}$. Also, each compound creature in r has norm > 1, so we can use (5.3.0.18). This ends the proof of (5.3.0.20).

- So to show that q essentially decides $\underset{\sim}{\tau}$, it is enough to show that for all $\eta \in \text{poss}(q, <(M_0, -1))$ there is a $\mathbf{u}$ such that $\eta \in B^{\mathbf{u}}_{(M_0,-1)}$.

- As in the rapid reading case, we choose an "infinite branch" $(\mathfrak{d}^*_{\mathbf{v}}, B^*_{\mathbf{v}})$. I.e.: for each $\mathbf{v}'$ there is a $\mathbf{u} > \mathbf{v}'$ such that $(\mathfrak{d}^{\mathbf{u}}_{\mathbf{v}}, B^{\mathbf{u}}_{\mathbf{v}}) = (\mathfrak{d}^*_{\mathbf{v}}, B^*_{\mathbf{v}})$ for each $\mathbf{v} \leq \mathbf{v}'$. This defines a condition $q_1 \leq q$.

- To show that q essentially decides $\underset{\sim}{\tau}$, it is enough to show $\eta \in B^*_{(M_0,-1)}$ for all $\eta \in \text{poss}(q, <M_0) = \text{poss}(q_1, <(M_0, -1))$.

  So fix such an η. Find any $r \leq q_1 \wedge \eta$ deciding $\underset{\sim}{\tau}$. Without loss of generality, $\min(\mathtt{w}^r) = M_\ell$ for some ℓ, and each compound creature in r has norm at least 1. Let $\eta' > \eta$ be the trunk of r (restricted to $\text{supp}(q)$ and M_ℓ). According to (5.3.0.18), $q \wedge \eta'$ essentially decides $\underset{\sim}{\tau}$.

  Pick some $\mathbf{u} > (M_\ell, -1)$ such that $(\mathfrak{d}^{\mathbf{u}}_{\mathbf{v}}, B^{\mathbf{u}}_{\mathbf{v}}) = (\mathfrak{d}^*_{\mathbf{v}}, B^*_{\mathbf{v}})$ for each $\mathbf{v} \leq (M_\ell, -1)$. According to (5.3.0.19), $\eta' \in B^*_{\mathbf{v}}$. By homogeneity, $\eta \in B^*_{(M_0,-1)}$. So according to (5.3.0.20), $q \wedge \eta$ essentially decides $\underset{\sim}{\tau}$. □

5.4 Properness, ω^ω-bounding, rapid reading, no randoms

A standard argument now gives the following:

Theorem 5.4.0.21. *$\mathbb{Q}$ satisfies (the finite/ω^ω-bounding version of) Baumgartner's Axiom A, in particular it is proper and ω^ω-bounding and (assuming CH in the ground model) preserves all cofinalities. Also, $\mathbb{Q}$ rapidly reads every $\underset{\sim}{r} \in 2^\omega$.*

Proof. We already know that we can rapidly read each real if we can continuously read it.

We define $q \leq_n p$ as: $q \leq p$ and there is an $h \in \mathtt{w}^q$, $h \geq n$, such that q and p are identical below h and $\text{nor}(q(\ell)) > n$ for all $\ell \geq h$.

It is clear that any sequence $p_0 \geq_0 p_1 \geq_1 p_2 \geq_2 \dots$ has a limit; and Lemma 5.3.0.16 shows that for any name $\underset{\sim}{\tau}$ of an ordinal, $n \in \omega$ and $p \in \mathbb{Q}$, there is a $q \leq_n p$ such that modulo q there are only finitely many possibilities for $\underset{\sim}{\tau}$. □

Rapid reading gives us:

Lemma 5.4.0.22. *Every new real is contained in a ground model null set, i.e., no random reals are added. So assuming CH in the ground model, we will have* $\text{cov}(\mathcal{N}) = \aleph_1$ *in the extension.*

Proof. Let $\underset{\sim}{r}$ be the name of an element of 2^ω and p a condition. Let $q \leq p$ rapidly read $\underset{\sim}{r}$. So for all $\ell \in \mathtt{w}^q$, $\underset{\sim}{r} \,|\, H(<\ell)$ is determined by each $\eta \in \mathrm{poss}(q, <\ell)$. Hence, the set A^q_ℓ of possibilities for $\underset{\sim}{r} \,|\, H(<\ell)$ has size at most $\mathrm{maxposs}(<\ell) < H(<\ell) < 2^{H(<\ell)}/\ell$. So A^q_ℓ has "relative size" $< 1/\ell$, and the sequence $(A^q_\ell)_{\ell \in \omega}$ defines (in the ground model) the null set

$$N = \{s \in 2^\omega : (\forall \ell \in \mathtt{w}^q)\ s \,|\, H(<\ell) \in A^q_\ell\}.$$

And q forces that $\underset{\sim}{r} \in N$. $\square$

Chapter 6

The specific forcing and the main theorem

6.1 The forcing

Recall that Ξ is partitioned into $\Xi_{\texttt{sk}}$, $\Xi_{\texttt{li}}$ and $\Xi_{\texttt{ls}}$. We now further partition $\Xi_{\texttt{ls}}$ into $\Xi_{\texttt{nn}}$ and $\Xi_{\texttt{cn}}$. So every $\xi \in \Xi$ has one of the following four types:

- type `sk` (Sacks) for $\xi \in \Xi_{\texttt{sk}}$,
- type `cn` (cofinality null) for $\xi \in \Xi_{\texttt{cn}}$,
- type `nn` (non null) for $\xi \in \Xi_{\texttt{nn}}$, and
- type `nm` (non meager) for $\xi \in \Xi_{\texttt{li}}$. So `nm` is the only lim-inf type.

Let κ_t be the size of Ξ_t.

In the inductive construction of $\mathbb{Q}$ in Section 4, several assumptions are made in the subatom stages $\mathbf{u}$. We will satisfy those assumptions in the following way:

For each type $t \in \{\texttt{cn}, \texttt{nn}, \texttt{nm}\}$ we assume that we have a family of subatomic families $\mathsf{K}'_{t,b}$ indexed by a parameter b, such that for each $b \in \omega$, $\mathsf{K}'_{t,b}$ is a subatomic family living on some $\mathrm{POSS}'_{t,b}$ satisfying b-bigness. Actually, we will require a stronger variant of b-bigness such that we can find an homogeneous successor subatom while decreasing the norm not by 1 but by at most $1/b$. I.e., we require:

$$\text{For } x \in \mathsf{K}'_{t,b} \text{ and } F : \mathrm{poss}(x) \to b \text{ there is a } y \leq x \text{ such that } \mathrm{nor}(y) \geq \mathrm{nor}(x) - {}^1\!/b \text{ and } F \restriction \mathrm{poss}(y) \text{ is constant.} \tag{6.1.0.1}$$

Additionally we require that

$$\text{there is at least one subatom in } \mathsf{K}'_{t,b} \text{ with norm} \geq b \tag{6.1.0.2}$$

.

Then we set for each subatomic sublevel $\mathbf{u} = (\ell, j)$

$$b(\mathbf{u}) := B(\mathbf{u}) \cdot (b(\mathbf{v}) + 1) + 1, \tag{6.1.0.3}$$

where $\mathbf{v}$ is the largest[1] subatomic sublevel smaller than $\mathbf{u}$. So the sequence $b(\mathbf{u})$ is strictly (actually: very quickly) increasing. According to the definition 4.0.0.1 of $B(\mathbf{u})$, we also get:

Lemma 6.1.0.4. *XX where do we need this? XX*
$b(\mathbf{u}) \geq 2 \cdot \text{maxposs}(<\mathbf{u})$*, and even* $b(\mathbf{u}) \geq 2^{(\textit{number of sublevels below } \mathbf{u}) \cdot \text{maxposs}(<\mathbf{u})}$.

Then we set (for all $\xi \in \Xi_t$)

$$\mathsf{K}_{\xi,\mathbf{u}} := \mathsf{K}'_{t,b(\mathbf{u})}.$$

This way we automatically satisfy requirements (b) and (c) of item ($*5$) on page 28. And since there are only four, i.e., finitely many, types, there is automatically a bound M on $|\,\text{POSS}_{\xi,\mathbf{u}}\,|$ as required in (d).

Strong bigness gives us the following property:

Lemma 6.1.0.5. *Let I be a finite set of subatomic sublevels (and thus I is naturally ordered). Let $\mathbf{v}$ be the minimum of I. For each $\mathbf{u} \in I$ let $\xi_\mathbf{u} \in$ non-$\mathtt{sk}$ and $x_\mathbf{u}$ a subatom in $\mathsf{K}_{\xi_\mathbf{u},\mathbf{u}}$. Let $F : \prod_{\mathbf{u}\in I} \text{poss}(x_\mathbf{u}) \to b(\mathbf{v})$. Then there are $y_\mathbf{u} < x_\mathbf{u}$ with $\text{nor}(y_\mathbf{u}) \geq \text{nor}(x_\mathbf{u}) - 1/b(\mathbf{u})$ and such that $F \restriction \prod_{\mathbf{u}\in I} \text{poss}(y_\mathbf{u})$ is constant.*

Proof. We construct $y_\mathbf{u}$ by downwards induction on $\mathbf{u} \in I$: Let $\mathbf{u}'$ be the maximum of I, then F can be written as function from $\text{poss}(x_{\mathbf{u}'})$ to $b(\mathbf{v})^P$, where $P = \prod_{\mathbf{u}\in I\setminus\{\mathbf{u}'\}} \text{poss}(x_\mathbf{u})$. As $|P|$ is less than the number of sublevels below $\mathbf{u}'$ times $\text{maxposs}(<\mathbf{u}')$, we get $|P| < b(\mathbf{u}')$, and thus can use strong bigness to get $y_{\mathbf{u}'} < x_{\mathbf{u}'}$.

Now continue by induction. $\square$

The families $\mathsf{K}'_{t,r}$ that we will actually use are described in Section 10 for $t = \mathtt{cn}$, Section 8 for $t = \mathtt{nn}$, and Section 7 for $t = \mathtt{nm}$.

In addition, we will define there for each $\mathsf{K}'_{t,b}$ a number $H'(t,{=}b)$, and in the inductive construction, we define H as follows:

Definition. $H(<(0,-1)) := 3$. If $\mathbf{u} = (\ell, j)$ is a sublevel with immediate predecessor $\mathbf{u}'$, we define $H(<\mathbf{u}) = H(\leq\mathbf{u}')$ in cases by:

- For a Sacks sublevel $\mathbf{u}$ (i.e., $j = -1$), $H(<\ell) = H(<\mathbf{u}) := 2 + \ell + \text{maxposs}(<\ell) + H(<\mathbf{u}') + \max(\{H'(t,{=}b(\mathbf{u}')) : t \in \{\mathtt{nm}, \mathtt{nn}, \mathtt{cn}\}\})$.
- For $j = 0$: $H(<\mathbf{u}) := 1 + H(<\mathbf{u}') + \max(I_{\mathtt{sk},\ell})$.
- For $j > 0$, $H(<\mathbf{u}) := 1 + H(<\mathbf{u}') + \max\{H'(t,{=}b(\mathbf{u}')) : t \in \{\mathtt{nm}, \mathtt{nn}, \mathtt{cn}\}\})$.

So in particular, if p rapidly reads $\underset{\sim}{r}$, then for all $t \in \{\mathtt{nm}, \mathtt{nn}, \mathtt{cn}\}$ and all subatomic sublevels $\mathbf{u}$

$$\underset{\sim}{r} \restriction H'(t,{=}\mathbf{u}) \text{ is decided } \leq b(\mathbf{u}). \tag{6.1.0.6}$$

Note that once we fix the parametrized subatomic families $\mathsf{K}'_{t,b}$ and $H'(t,{=}b)$ (and the cardinalities κ_t), we have specified everything required to construct $\mathbb{Q}$, and $\mathbb{Q}$ will satisfy Baumgartner's Axiom A, will be ω^ω-bounding, and, assuming CH, will have the $\aleph_2$-cc. We also get rapid reading.

[1]If $\mathbf{u}$ is $(0,0)$, the smallest of all subatomic sublevels, we just set $b(\mathbf{u}) := B(\mathbf{u})$. By the way, it would be enough to set $b(\mathbf{u}) := B(\mathbf{u})$, as this sequence would be increasing sufficiently fast, but this would require two extra lines of calculations.

6.2 The main theorem

We will show:

Theorem 6.2.0.7. *Assume (in V) CH, $\kappa_{\mathtt{nm}} \leq \kappa_{\mathtt{nn}} \leq \kappa_{\mathtt{cn}} \leq \kappa_{\mathtt{sk}}$ and $\kappa_t^{\aleph_0} = \kappa_t$ for $t \in \{\mathtt{nm}, \mathtt{nn}, \mathtt{cn}, \mathtt{sk}\}$. Then there is a forcing $\mathbb{Q}$ which forces*

1. $\mathrm{cov}(\mathcal{N}) = \mathfrak{d} = \aleph_1$,

2. $\mathrm{non}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) = \kappa_{\mathtt{nm}}$,

3. $\mathrm{non}(\mathcal{N}) = \kappa_{\mathtt{nn}}$,

4. $\mathrm{cof}(\mathcal{N}) = \kappa_{\mathtt{cn}}$,

5. $2^{\aleph_0} = \kappa_{\mathtt{sk}}$.

Moreover, $\mathbb{Q}$ preserves all cardinals and all cofinalities.

As mentioned above, we fix disjoint index sets Ξ_t ($t \in \{\mathtt{sk}, \mathtt{cn}, \mathtt{nn}, \mathtt{nm}\}$) of respective sizes κ_t, and we construct $\mathbb{Q}$ as described above. Then the following points are obvious or have already been shown:

(1) $\mathfrak{d} = \aleph_1$, since $\mathbb{Q}$ is ω^ω-bounding. And it was already shown in Lemma 5.4.0.22 that no random reals are added, so $\mathrm{cov}(\mathcal{N}) = \aleph_1$.

(5) If $\alpha \neq \beta \in \Xi_{\mathtt{sk}}$, then the generic reals at α and β are forced to be different, so we have at least $\kappa_{\mathtt{sk}}$ many reals. Every real in the extension is read continuously, so by Lemma 5.1.0.4 there are at most $\kappa_{\mathtt{sk}}^{\aleph_0} = \kappa_{\mathtt{sk}}$ many reals.

(•) The "moreover" part is clear because $\mathbb{Q}$ satisfies Baumgartner's Axiom A and has the $\aleph_2$-cc.

In the rest of the paper, we will describe the families $K'_{t,b}$ and $H'(t,{=}b)$ and prove the remaining parts of the main theorem:

(2) In ZFC, $\max(\mathfrak{d}, \mathrm{non}(\mathcal{M})) = \mathrm{cof}(\mathcal{M})$. And $\mathrm{non}(\mathcal{M}) \leq \kappa_{\mathtt{nm}}$ is shown in 6.4.0.12, and $\geq$ in 7.3.0.7.

(3) $\mathrm{non}(\mathcal{N}) \leq \kappa_{\mathtt{nn}}$ is shown in 10.5.0.10; and $\geq$ in 8.3.0.5.

(4) $\mathrm{cof}(\mathcal{N}) \leq \kappa_{\mathtt{cn}}$ is shown in 6.3.0.11; and $\geq$ in 10.4.0.8.

6.3 The Sacks part: $\mathrm{cof}(\mathcal{N}) \leq \kappa_{\mathtt{cn}}$

We will show that every null set added by $\mathbb{Q}$ is contained in a null set which is already added by the non-Sacks part.

We will first show that the quotient $\mathbb{Q}/\mathbb{Q}_{\Xi_{\text{non-sk}}}$ (in other words: the extension from the universe obtained not using the sacks coordinates to the full generic extension) has the Sacks property.

Recall that the Sacks property states (or, depending on the definition, is equivalent to): Every function in ω^ω in the extension is caught by an $(n+2)$-slalom from the ground model. (I.e., there is a function $S : \omega \to [\omega]^{<\omega}$ in the ground model with $|S(n)| \leq n+2$, and $f(n) \in S(n)$ for all n.)

The Laver property is similar, but only applies to functions f in the extension which are bounded by a ground model function.

We get

Lemma 6.3.0.8. *1. Laver property is equivalent to:*

Whenever $\utilde{r} \in 2^\omega$ is in the extension and $G : \omega \to \omega$ in the ground model, then there is in the ground model a tree T (without terminal nodes) such that $\utilde{r} \in [T]$ and $|T \restriction 2^{G(n)}| < n + 2$ for all n.

2. *The Sacks property is equivalent to the conjunction of Laver property and ω^ω-bounding.*

3. *If an extension has the Sacks property, then any new null set is contained in an old null set.*

Proof. For the well known (2) and (3) see, e.g., [4, Theorem 2.3.12]. For (1), we only show how to get the Laver property (which is enough for this paper, and the other direction is similarly easy).

Suppose that $g : \omega \to \omega$ is given. Enumerate $\{(n, m) : m \le g(n)\}$ in lexicographic order as (n_i, m_i). Define a function $G : \omega \to \omega$ by

$$G(n) = \min\{i : n_i > n\} = n + 1 + \textstyle\sum_{k \le n} g(k).$$

(For convenience we will think of $G(-1) = 0$.) Note that according to the enumeration given above, every function $r : \omega \to 2$ determines a subset of $\prod_{n<\omega}(g(n)+1)$ by $\{(n_i, m_i) : r(i) = 1\}$. Accordingly, certain functions r induce a function bounded by g: those functions r such that given any n there is a unique $m \le g(n)$ such that (n, m) is in the subset determined by r as described above. (Equivalently, for each n there is a unique $G(n-1) \le i < G(n)$ such that $r(i) = 1$.) Given such an r, by $\mathrm{val}(r, n)$ we denote m_i where $G(n-1) \le i < G(n)$ is such that $r(i) = 1$.

Note that given any function f bounded by g there is a unique function $r_f : \omega \to 2$ (which determines a function bounded by g as described above) such that $\mathrm{val}(r_f, n) = f(n)$ for all n.

Suppose that $\utilde{f}$ is a name for a function bounded by the ground model function g. Let $\utilde{r}_f$ be a name for the function $\omega \to 2$ as described above, and let T be the tree guaranteed to exist by the assumption (using the function G defined from g above). We may assume that all branches x of T determine a function bounded by g as described above. Now define a slalom S by $S(n) = \{\mathrm{val}(x, n) : x \in [T]\}$. It is clear that S catches $\utilde{f}$. □

We now prove our version of the Laver property for the quotient. As the whole forcing is ω^ω-bounding, this implies the Sacks property.

Lemma 6.3.0.9. 1. *Assume that p is a condition, $\utilde{r} \in 2^\omega$ a name and $G : \omega \to \omega$ is in V. Then there is a $q \le p$ and a name $\utilde{T} \subseteq 2^{<\omega}$ (of a tree without terminal nodes) such that: q continuously reads $\utilde{T}$ not using any Sacks indices; q forces $r \in [\utilde{T}]$; and $|\utilde{T} \restriction 2^{G(n)}| < n + 2$ for all n.*

2. *Therefore the quotient $\mathbb{Q}/\mathbb{Q}_{\Xi_{\text{non-sk}}}$ has the Laver property (and thus the Sacks property).*

Proof. If $G_1(n) \le G_2(n)$ for all n, and $\utilde{T}$ witnesses the conclusion of the lemma for G_2, then $\utilde{T}$ also witnesses the lemma for G_1. So we may without loss of generality increase the function G whenever this is convenient.

We can assume that p rapidly reads $\utilde{r}$, i.e., $\mathrm{poss}(p, {<}n)$ determines $\utilde{r} \restriction H({<}n)$ for all $n \in \mathtt{w}^p$.

We can then assume that there is a strictly increasing function G' such that $G'(n) \in \mathtt{w}^p$ and $G(n) = H({<}G'(n))$ for all n (as we can increase G).

Also, to simplify notation, we can assume that $\mathtt{w}^p = \{G'(0), G'(1), \dots\}$. (Otherwise, just glue.)

So each $\eta \in \mathrm{poss}(p, {<}G'(n))$ determines a value for $\utilde{r} \restriction G(n)$, which we call $R^n(\eta)$. We view η as a pair $(\eta_{\mathtt{sk}}, \eta_{\text{non-}\mathtt{sk}})$ for $\eta_t := \eta \restriction \Xi_t$ for $t \in \{\text{non-}\mathtt{sk}, \mathtt{sk}\}$. Accordingly we write $R^n(\eta_{\mathtt{sk}}, \eta_{\text{non-}\mathtt{sk}})$. If we fix $\eta_{\mathtt{sk}}$, then $R^n(-, \eta_{\mathtt{sk}})$ can be viewed as a name (for an element of $2^{G(n)}$) which does not depend on the Sacks part, in the following way: If there is a $\eta_{\text{non-}\mathtt{sk}}$ compatible with the generic filter such that $(\eta_{\text{non-}\mathtt{sk}}, \eta_{\mathtt{sk}}) = \eta \in \mathrm{poss}(p, {<}G'(n))$, then the value is $R^n(\eta)$ (and otherwise $\emptyset$, say).

Below we will construct $q \le p$ by gluing and by strengthening Sacks columns (and we will leave the support, the subatoms and the halving parameters unchanged).

Assume we have such a q, and assume that $G'(m_0) < G'(m_1)$ are consecutive elements of $\mathtt{w}^q$. Note that $G'(m_0) < G'(m_0+1) < \cdots < G'(m_1-1) < G'(m_1)$ are consecutive elements of $\mathtt{w}^p$. Fix $\eta \in \mathrm{poss}(q, {<}G'(m_1))$ and $m_0 \le \ell \le m_1$. Then η extends a unique element of $\mathrm{poss}(q, {<}G'(\ell))$, which we call η^ℓ. We can then restrict η^ℓ to the Sacks part: $\eta^\ell_{\mathtt{sk}} := \eta^\ell \restriction \Xi_{\mathtt{sk}}$.

Note:

- $\eta^\ell_{\mathtt{sk}}$ is η restricted to the Sacks part and to "height $G'(\ell)$", i.e.,

$$\eta^\ell_{\mathtt{sk}} := \eta \restriction \Xi_{\mathtt{sk}} \times (1 + \max(I_{\mathtt{sk},G'(\ell)})).$$

- $q \wedge \eta$ forces that the name $R^\ell(-, \eta^\ell_{\mathtt{sk}})$ (which does not depend on the Sacks part) is evaluated to $\underset{\sim}{r} \restriction G(\ell)$.
- So q forces that $\underset{\sim}{r} \restriction G(\ell)$ is an element of

$$\underset{\sim}{T}^\ell := \{R^\ell(-, \eta^\ell_{\mathtt{sk}}) : \eta \in \mathrm{poss}(q, {<}G'(m_1))\},$$

  a name not depending on the Sacks part.

So it is enough to show that there are few $\eta^\ell_{\mathtt{sk}}$, i.e.,

$$|S_\ell| < \ell + 2 \text{ for } S_\ell := \{\eta^\ell_{\mathtt{sk}} : \eta \in \mathrm{poss}(q, {<}G'(m_1))\}. \tag{$\star_\ell$}$$

We will now by induction on n:

1. construct h_n, where $\mathtt{w}^q$ will be the set $\{G'(h_0), G'(h_1), \dots\}$;
2. construct q below $G'(h_n)$,
3. and show that $(\star_\ell)$ holds for all $\ell \le h_n$.

We set $h_0 = 0$; so $G'(h_0) = \min(\mathtt{w}^p)$ and q below $G'(h_0)$ has to be identical to p. And $(\star_0)$ holds as S_0 is a singleton.

Assume we have already constructed h_n and q below $G'(h_n)$, satisfying $(\star_\ell)$ for $\ell \le h_n$.

1. For any I and $\mathfrak{s} \subseteq 2^I$, we write $\mathrm{nor}^*_{\mathrm{Sacks}}(\mathfrak{s})$ for $\mathrm{nor}^{B(G'(h_n)),G'(h_n)}_{\mathrm{Sacks}}(\mathfrak{s})$, see 2.3.0.16. (I.e., the Sacks norm that would be assigned to a Sacks column starting at $G'(h_n)$ which has the same $\mathrm{nor}_{\mathrm{split}}$ as $\mathfrak{s}$.) Let $\Sigma := \mathrm{supp}(p, G'(h_n)) \cap \Xi_{\mathtt{sk}}$, the set of Sacks indices active at the current level. Let s be minimal such that $\mathrm{nor}^*_{\mathrm{Sacks}}(2^s) \ge n$, and define h' by

$$h' := (h_n + 1) \cdot 2^{s \cdot |\Sigma|}. \tag{6.3.0.10}$$

   Finally, let h_{n+1} be minimal such that for all $\xi \in \Sigma$ there is an $\ell(\xi)$ with $h' \le \ell(\xi) < h_{n+1}$ and $\mathrm{nor}^*_{\mathrm{Sacks}}(p(\xi, G'(\ell(\xi)))) \ge n$. (We can find such $\ell(\xi)$, as even $\mathrm{nor}_{\mathrm{Sacks}}(p(\xi, G'(\ell)))$ diverges to infinity.)

2. $G'(h_n) < G'(h_n+1) < \cdots < G'(h_{n+1}-1) < G'(h_{n+1})$ are consecutive elements of $\mathtt{w}^p$. We glue p between $G'(h_n)$ and $G'(h_{n+1}-1))$, so $G'(h_n)$ and $G'(h_{n+1})$ will be consecutive elements of $\mathtt{w}^q$.

   We now define the compound creature $q(G'(h_n))$, a pure strengthening of the compound creature $\mathrm{glue}(p(G'(h_n)), \dots, p(G'(h_{n+1}-1))))$: The subatoms are unchanged. So we just have to specify for each $\xi \in \text{supp}(p, h_n) \cap \Xi_{\mathtt{sk}}$ the new Sacks column $q(\xi, h_n) \leq p(\xi, G'(h_n)) \otimes \cdots \otimes p(\xi, G'(h_{n+1} - 1))$ as follows: Recall that there is one $\ell(\xi)$ such that $h' \leq \ell(\xi) < h_{n+1}$ and $\text{nor}^*_{\text{Sacks}}(p(\xi, G'(\ell(\xi)))) \geq n$. Choose a singleton subset of $p(\xi, G'(m))$ for all $m \neq \ell(\xi)$, and at $m = \ell(\xi)$ pick a subtree of $p(\xi, G'(m))$ which is isomorphic to 2^s (in the sense that each branch has s splitting points).

By the definition of s, we have $\text{nor}_{\text{Sacks}}(q(\xi, h_n)) \geq n$, and therefore

$$\text{nor}(q(h_n)) \geq \min(n, \text{nor}(p(h_n), \dots, \text{nor}(p(h_{n+1} - 1)))).$$

So in particular the q we get after the induction will be an element of $\mathbb{Q}$.

3. As we choose singletons below $G'(h')$, $|S_{h_n}| = |S_{h_n+1}| = \cdots = |S_{h'-1}|$. By induction, $|S_{h_n}| < h_n + 2$; so $(*_\ell)$ holds for $\ell \leq h'$. For each $h' \leq \ell \leq h_{n+1}$, we added at each $\xi \in \Sigma$ at most once at most 2^s many possibilities. So $|S_\ell| \leq (h_n + 1) \cdot 2^{s \cdot |\Sigma|} < \ell + 2$, by (6.3.0.10). □

By Lemma 6.3.0.8(3), we conclude:

Corollary 6.3.0.11. *1. If $\underset{\sim}{N}$ is the name of a null set and p a condition, then there is a $q \leq p$ and some name of a null set $\underset{\sim}{N}'$ not depending on any Sacks indices such that q forces $\underset{\sim}{N} \subseteq \underset{\sim}{N}'$.*

2. $\mathbb{Q}$ forces $\text{cof}(\mathcal{N}) \leq \kappa_{\mathtt{cn}}$.

6.4 Lim inf and lim sup: $\text{non}(\mathcal{M}) \leq \kappa_{\mathtt{nm}}$

The following does not require any knowledge about the particular subatoms used in the forcing construction, the only relevant fact is that the nm indices are the only ones that a lim-inf construction.

Lemma 6.4.0.12. *$\mathbb{Q}$ forces $\text{non}(\mathcal{M}) \leq \kappa_{\mathtt{nm}}$.*

Proof. We claim that the set of all reals that can be read continuously from nm-indices is not meager. This set has size $\leq \kappa_{\mathtt{nm}}$ by Lemma 5.1.0.4.

Let $\underset{\sim}{M}$ be a name for a meager set. We can find names $\underset{\sim}{T}_n \subseteq 2^{<\omega}$ for nowhere dense trees such that $\underset{\sim}{M} = \bigcup_{n \in \omega} [\underset{\sim}{T}_n]$ is forced. We want to show that we can continuously read a real $\underset{\sim}{r} \notin \underset{\sim}{M}$ using only the nm-indices.

As $\mathbb{Q}$ is ω^ω-bounding and $\underset{\sim}{T}_n$ is nowhere dense, there is in V a function $f_n : \omega \to \omega$ such that for each $\nu \in 2^k$ there is a $\nu' \in 2^{f_n(k)}$ extending ν and not in $\underset{\sim}{T}_n$.

We fix some $p \in \mathbb{Q}$ forcing the above, and assume that p is pruned and continuously reads $\underset{\sim}{T}_n$ for each n. We will construct (in V) a $q \leq p$ and an $\underset{\sim}{r}$ continuously read by q only using nm indices, such that q forces $\underset{\sim}{r} \notin \underset{\sim}{M}$.

Assume we have already constructed q below some $k_n \in \mathtt{w}^q$, and that we already have some $h_n \in \omega$ and a name $\underset{\sim}{\ell}_n$ for an element of 2^{h_n} that is decided by $\text{poss}(q, {<}k_n) \,|\, \Xi_{\mathtt{nm}}$. (The real $\underset{\sim}{r}$ will be the union of the $\underset{\sim}{\ell}_n$.) We also assume that is already guaranteed that $\underset{\sim}{\ell}_n$ is not in $\underset{\sim}{T}_0 \cup \cdots \cup \underset{\sim}{T}_{n-1}$).

Enumerate $\text{poss}(q, {<}k_n)$ as $\eta_0, \dots, \eta_{K-1}$.

Set $k^0 := k_n$, $h^0 := h_n$, $\underset{\sim}{\ell}^0 := \underset{\sim}{\ell}_n$, and we define q' below k^0 to be q. By induction on $r \in K$ we now deal with η_r: Assume we are given a name $\underset{\sim}{\ell}^r$ for an element of 2^{h^r} that is decided by $\text{poss}(q', {<}k^r) \,|\, \Xi_{\mathtt{nm}}$, and that we have constructed q' below $k^r \in \mathtt{w}^p$, in a way that between k^0 and k^r on the non-nm indices, all subatoms and Sacks-columns in q' are singletons.

Set $h^{r+1} := f_n(h^r)$. Choose $k^{r+1} \in \mathtt{w}^p$ bigger than k^r and large enough to determine $\underset{\sim}{X} := \underset{\sim}{T}_n \,|\, h^{r+1}$. I.e., there is a function F from $\text{poss}(p, {<}k^{r+1})$ to potential values of $\underset{\sim}{X}$. We now define q' between k^r and k^{r+1}: The nm-subatoms are unchanged (i.e., the ones of p), for the other subatoms and Sacks columns, we choose arbitrary singletons. A $\nu \in \mathrm{poss}(p, {<}k^{r+1})$ consists of: the part below k^r called A, then non-nm-part above k^r called B, and the nm-part above k^r called C. So we can write $\underset{\sim}{X} = F(A, B, C)$. If we assume that the generic chooses η_r (i.e., $A = \eta_r$) and then follows the singleton values of q on the non-nm-part (which determines B to be some B_q), then $\underset{\sim}{X}$ can be written as nm-name. More formally: We can define $\underset{\sim}{X}'$ as $F(\eta_r, B_q, -)$, which is a nm-name and forced by q to be $\underset{\sim}{X}$.

Also, we know that p forces that there is an element $\ell' \in 2^{h^{r+1}}$ which extends $\underset{\sim}{\ell}^r$ (which by induction is already determined by the nm-part of η_r) and which is not in $\underset{\sim}{X}$. So (in V) we can pick for all choices of C an $\ell'(C) \in 2^{h^{r+1}} \setminus F(\eta_r, B_q, C)$ extending $\underset{\sim}{\ell}^r$. Then $\underset{\sim}{\ell}^{r+1} = \ell(-)$ is a nm-name determined below k^{r+1}, and q forces that $\underset{\sim}{\ell}^{n+1}$ extends $\underset{\sim}{\ell}^n$, and $q \wedge \eta_r$ forces that $\underset{\sim}{\ell}^{n+1} \notin \underset{\sim}{T}_n$.

We repeat the construction for all $r \in K$, and set $\ell_{n+1} := \ell^K$, $h_{n+1} := h^K$ and set k_{n+1} to be the $\mathtt{w}^p$-successor of k^K, where we use the Sacks columns and subatoms of p between k^K and k_{n+1}. We now glue the condition between k_n and k_{n+1}. This results in a condition that still has "large" norm, as described in Lemma 3.5.0.17. $\square$

Chapter 7

The nm part

7.1 The subatomic creatures for type nm

We now describe the subatomic family $\mathsf{K}'_{\mathtt{nm},b}$ used at nm-indices (depending on the parameter b).

Definition 7.1.0.1. 1. Fix a finite index set $I \subseteq \omega$ which is large enough so that item (4) below is satisfied. For notational simplicity, we assume that I is disjoint to all intervals already used.[1]

2. $\mathrm{POSS}_{\mathtt{nm},b} := 2^I$.

3. A subatomic creature x is just a nonempty subset of 2^I, where we set $\mathrm{poss}(x) := x$ and

$$\mathrm{nor}(x) := \frac{1}{b}\log_b(|\,\mathrm{poss}(x)|).$$

4. We require $\mathrm{nor}(\mathrm{POSS}) > b$ (thus satisfying (6.1.0.2)).

5. We set $H'(\mathtt{nm}, b) := 2^{\max(I)+1}$.

Clearly, the norm satisfies strong b-bigness (i.e., satisfies the requirement (6.1.0.1)).

Note 7.1.0.2. We just used the simplest possible norm here. It turns out that the details of the definition of this norm are not relevant, as long as the norm has bigness. Later in section 10.6 we will use a different norm to get a different constellation of cardinal characteristics.

7.2 The generic object

Recall that (according to Section 6.1) when constructing the forcing at subatomic sublevels $\mathbf{u}$, we use for all $\xi \in \Xi_{\mathtt{nm}}$ the subatomic family $\mathsf{K}_{\xi,\mathbf{u}} = \mathsf{K}'_{\mathtt{nm},b(\mathbf{u})}$ living on some interval I, which we will call $I_{\mathtt{nm},\mathbf{u}}$.

Fix α of type nm. Recall that the generic object $\underset{\sim}{y}_\alpha$ assigns to each subatomic sublevel $\mathbf{u}$ the element of $\mathrm{POSS}_{\alpha,\mathbf{u}}$ chosen by the generic filter.

We define the name $\underset{\sim}{M}_\alpha$ of a meager set as follows:

$$\text{A real } r \in 2^\omega \text{ is in } \underset{\sim}{M}_\alpha \text{ iff for all but finitely many levels } \ell \text{ there is a subatomic sublevel } \mathbf{u} = (\ell, j) \text{ such that } r \restriction I_{\mathtt{nm},\mathbf{u}} \neq \underset{\sim}{y}_\alpha(\mathbf{u}). \tag{7.2.0.3}$$

[1] This is a bit fuzzy, but it does not matter how we interpret it. More specifically, we could use any of the following: "disjoint to all I that are associated to smaller parameter values $b' < b$", or: "disjoint to all I that have actually been used in type nm for some $\mathsf{K}_{\xi,\mathbf{v}}$"; and since $H'(\mathtt{nm}, {=}b)$ is larger than $\max(I)$, it would also follow from: "the minimum of I is bigger than $H({<}\mathbf{u}')$, where $\mathbf{u}'$ is the predecessor of the current sublevel".

If p rapidly reads $\underset{\sim}{r}$, then according to (6.1.0.6) and 7.1.0.1(5),

$$\underset{\sim}{r} \restriction I_{\mathtt{nm},\mathbf{u}} \text{ is decided } \leq \mathbf{u}. \tag{7.2.0.4}$$

Also, since $b(\mathbf{u}) > \mathrm{maxposs}(<\mathbf{u})$, we get:

If the norm of a $\mathtt{nm}$-subatom x at sublevel $\mathbf{u}$ is at least 1, then $|\mathrm{poss}(x)| > \mathrm{maxposs}(<\mathbf{u})$. (7.2.0.5)

(Recall Note 7.1.0.2: This is true whenever the norm has bigness.)

7.3 $\mathrm{non}(\mathcal{M}) \geq \kappa_{\mathtt{nm}}$

Lemma 7.3.0.6. *Let $\underset{\sim}{r}$ be a name of a real, p a condition that rapidly reads $\underset{\sim}{r}$ not using*[2] $\alpha \in \Xi_{\mathtt{nm}}$. *Then p forces that $\underset{\sim}{r} \in \underset{\sim}{M}_\alpha$.*

Proof. It is enough to prove that some $q \leq p$ forces that $\underset{\sim}{r} \in \underset{\sim}{M}_\alpha$: Assume that p does not force $\underset{\sim}{r} \in \underset{\sim}{M}_\alpha$, then some $p' \leq p$ forces the negation; p' still rapidly reads $\underset{\sim}{r}$ not using α, so if we know that there is a $q \leq p'$ as claimed, we get a contradiction.

We can assume that p is pruned and that $\alpha \in \mathrm{supp}(p)$. We will construct a q purely stronger than p (in particular with the same $\mathtt{w}$, halving parameters, and trunk). Actually, we will only strengthen one subatom at index α for each level $h \geq \min(\mathtt{w}^p)$.

For all $h \geq \min(\mathtt{w}^p)$ (not necessarily in $\mathtt{w}^p$), there are several $j \in J_h$ such that $\mathrm{nor}(x) > 1$ for the subatom $x = p(\alpha, (h, j))$. For each such h we pick exactly one subatomic sublevel $\mathbf{u}(h) = (h, j)$, with $x(h)$ the according subatom.

According to (7.2.0.4), $\underset{\sim}{r} \restriction I_{\mathtt{nm},\mathbf{u}}$ is decided $\leq \mathbf{u}$ and therefore even below $\mathbf{u}$ (since α is the active index at sublevel $\mathbf{u}$; according to modesty no other index can be active; and $\underset{\sim}{r}$ does not depend on α). Therefore there are at most $\mathrm{maxposs}(<\mathbf{u})$ many possibilities for $\underset{\sim}{r} \restriction I_{\mathtt{nm},\mathbf{u}}$. According to (7.2.0.5) there has to be at least one element s of $\mathrm{poss}(x(h))$ which differs from all of these possibilities. So we can in q replace the subatom $x(h)$ with the singleton $\{s\}$. Then the norms in q will still be large. (If $A \subseteq J_h$ witnesses the large $\mathrm{nor}_{\mathrm{liminf}}$ of p, then $A \setminus \{j\}$ for $\mathbf{u}(h) = (h, j)$ witnesses that the $\mathrm{nor}_{\mathrm{liminf}}$ of q decreases only slightly.)

So q is constructed by strengthening each $x(h)$ in this way. Clearly $q \leq p$ is still a valid condition, and forces $\underset{\sim}{r} \in \underset{\sim}{M}_\alpha$, as $\underset{\sim}{r} \restriction I_{\mathtt{nm},(h,\mathbf{u}(h))}$ disagrees with $\underset{\sim}{y}_\alpha$ for all $h \geq \min(\mathtt{w}^p)$. □

Corollary 7.3.0.7. $\mathbb{Q}$ *forces* $\mathrm{non}(\mathcal{M}) \geq \kappa_{\mathtt{nm}}$.

Proof. Assume that $\kappa_{\mathtt{nm}} > \aleph_1$ (otherwise there is nothing to show). Fix a condition p and $\kappa < \kappa_{\mathtt{nm}}$ and names $(\underset{\sim}{r}_i)_{i \in \kappa}$ of reals. It is enough to show that there is an $\alpha \in \Xi_{\mathtt{nm}}$ such that p forces that $\{\underset{\sim}{r}_i : i \in \kappa\}$ is a subset of the meager set $\underset{\sim}{M}_\alpha$.

For each i fix a maximal antichain A_i below p such that each $a \in A_i$ rapidly reads $\underset{\sim}{r}_i$. Due to $\aleph_2$-cc, and since $\kappa_{\mathtt{nm}} > \aleph_1$ and $\kappa_{\mathtt{nm}} > \kappa$, we can find an index $\alpha \in \Xi_{\mathtt{nm}}$ not appearing in the support of any condition in any A_i. According to the previous lemma, every element $a \in A_i$ (and hence also p itself) forces that $\underset{\sim}{r}_i \in \underset{\sim}{M}_\alpha$. □

[2] cf. 5.1.0.1

Chapter 8

The nn part

8.1 The subatomic creatures for type $\mathtt{nn}$

We describe the subatomic families $\mathsf{K}'_{\mathtt{nn},b}$, depending on a parameter b.

Definition 8.1.0.1. 1. Fix an interval I large enough such that (4) is satisfied (and in particular $|I| > b$). As in the $\mathtt{nm}$ subatoms, we assume that this interval I is disjoint to all intervals previously chosen.

2. The basic set of all possibilities, POSS, consists of all subsets X of 2^I with relative size $1-1/2^b$:

$$\mathrm{POSS} := \{X \subseteq 2^I : |X| = (1 - 1/2^b)|2^I|\}.$$

3. A subatom $C = \mathrm{poss}(C)$ is a subset of POSS, where we set

$$\mathrm{nor}(C) := \frac{1}{b}\log_b(\mathrm{nor}_0(C)), \text{ where}$$
$$\mathrm{nor}_0(C) := \min\{|Y| : Y \subseteq 2^I, (\forall X \in \mathrm{poss}(C))\, X \cap Y \neq \emptyset\}.$$

4. We require $\mathrm{nor}(\mathrm{POSS}) > b$ (thus satisfying (6.1.0.2)).

5. We set $H'(\mathtt{nn}, {=}b) := \max(I) + 1$.

Note that nor_0 of the subatom with full possibility set is approximately $2^{|I|}/2^b$. In particular, for large I the norm gets large, i.e., we can satisfy (4).

Lemma 8.1.0.2. *1. The subatomic family has strong b-bigness (i.e., satisfies the requirement* (6.1.0.1)*).*

2. Given $E \subseteq 2^I$ and a subatom C, then the subatom C' with possibilities $\{H \in \mathrm{poss}(C) : H \cap E = \emptyset\}$ satisfies $\mathrm{nor}_0(C') \geq \mathrm{nor}_0(C) - |E|$.

3. From the above it follows that: If $|E| \leq b^{\mathrm{nor}(C)}/2$, then $\mathrm{nor}(C') \geq \mathrm{nor}(C) - \log_b(2)$.

Proof. (1): Fix $F : \mathrm{poss}(C) \to b$. Let C_i be the subatom with $F \restriction \mathrm{poss}(C_i) = i$ for all $i \in b$. Assume that all C_i have nor_0 at most r, witnesses by $X_i \subseteq 2^I$. Then $\bigcup X_i$ witnesses that $\mathrm{nor}_0(C) \leq b \cdot r$. So $\mathrm{nor}(C) \leq \log_b(b\cdot r)/b \leq 1/b + \max(\mathrm{nor}(C_i))$. So there is at least one i with $\mathrm{nor}(C_i) \geq \mathrm{nor}(C) - 1/b$, as required.

(2): Assume Y witnesses $\mathrm{nor}_0(C')$, then $Y \cup E$ witnesses $\mathrm{nor}_0(C)$.

(3):

$$\frac{b^{\mathrm{nor}(C)}}{2} = \frac{\mathrm{nor}_0(C)^{1/b}}{2} = \left(\frac{\mathrm{nor}_0(C)}{2^b}\right)^{1/b} \leq$$
$$\leq \left[(1-\frac{1}{2^b})^{1/b} \cdot \mathrm{nor}_0(C)^b\right]^{1/b} = (1-\frac{1}{2^b}) \cdot \mathrm{nor}_0(C) \qquad \square$$

8.2 The generic object

The following paragraph is just as in the nm case 7.2:

According to Section 6.1, when constructing the forcing at subatomic sublevels $\mathbf{u}$, we use for all $\xi \in \Xi_{\mathtt{nn}}$ the subatomic family $\mathsf{K}_{\xi,\mathbf{u}} = \mathsf{K}'_{\mathtt{nn},b(\mathbf{u})}$ living on some interval I, which we temporarily call $I_{\mathtt{nn},\mathbf{u}}$. Also, if p rapidly reads $\underset{\sim}{r}$, then $\underset{\sim}{r} \restriction I_{\mathtt{nn},\mathbf{u}}$ is decided below $\leq \mathbf{u}$.

Fix α of type nn. Recall that the generic object $\underset{\sim}{y}_\alpha$ assigns to each subatomic sublevel $\mathbf{u}$ the element $\underset{\sim}{R}_{\alpha,\mathbf{u}}$ of $\mathrm{POSS}_{\alpha,\mathbf{u}}$ chosen by the generic filter. So $\underset{\sim}{R}_{\alpha,\mathbf{u}}$ is a subset of $2^{I_{\mathtt{nn},\mathbf{u}}}$ of relative size $(1 - 1/2^{b(\mathbf{u})})$.

Note that $b(\mathbf{u})$ is strictly monotone (cf. (6.1.0.3)), and hence $\prod_{\mathbf{u}\text{ subatomic sublevel}}(1-1/2^{b(\mathbf{u})}) > 0$. Therefore

$$\{x \in 2^\omega : \forall \mathbf{u} : x \restriction I_{\mathtt{nn},\mathbf{u}} \in \underset{\sim}{R}_{\alpha,\mathbf{u}}\}$$

is positive, and

$$\{x \in 2^\omega : \forall^\infty \mathbf{u} : x \restriction I_{\mathtt{nn},\mathbf{u}} \in \underset{\sim}{R}_{\alpha,\mathbf{u}}\}$$

has measure one. Therefore

$$\underset{\sim}{N}_\alpha := \{x \in 2^\omega : \exists^\infty \mathbf{u} : x \restriction I_{\mathtt{nn},\mathbf{u}} \notin \underset{\sim}{R}_{\alpha,\mathbf{u}}\} \tag{8.2.0.3}$$

is a null set. (Here, $\mathbf{u}$ ranges over all subatomic sublevels.)

8.3 $\mathrm{non}(\mathcal{N}) \geq \kappa_{\mathtt{nn}}$

Lemma 8.3.0.4. *Let $p \in \mathbb{Q}$ rapidly read $\underset{\sim}{r} \in 2^\omega$ not using $\alpha \in \Xi_{\mathtt{nn}}$. Then p forces $r \in N_\alpha$.*

Proof. As in 7.3.0.6, it is enough to find a $q \leq p$ forcing $r \in N_\alpha$; and we assume that p is pruned and that $\alpha \in \mathrm{supp}(p)$.

We construct q purely stronger than p by induction, only modifying subatoms at index α (and decreasing their subatom norms by at most 1):

Pick a subatomic sublevel $\mathbf{u}$ (higher than any sublevel previously considered) where α is active with the subatom C "living" on $I := I_{\mathtt{nn},\mathbf{u}}$.

$\underset{\sim}{r} \restriction I$ is decided $\leq \mathbf{u}$ and therefore even below $\mathbf{u}$ (as $\underset{\sim}{r}$ is read from p not using α; and due to modesty α is the only index active at sublevel $\mathbf{u}$). So the set E of possibilities for $\underset{\sim}{r} \restriction I$ has size at most $\mathrm{maxposs}(<\mathbf{u})$, and we can remove them all from the subatom at C while decreasing the norm by at most 1, according to Lemma 8.1.0.2(2) and (6.1.0.4).

Repeat this for infinitely many sublevels $\mathbf{u}$. $\square$

Just as in 7.3.0.7, this implies:

Corollary 8.3.0.5. $\mathbb{Q}$ *forces* $\mathrm{non}(\mathcal{N}) \geq \kappa_{\mathtt{nn}}$.

Chapter 9

Some simple facts about counting

We now list some simple combinatorial properties that will be used for the definitions and proofs in the `cn`-part.

9.1 Large families of positive sets have positive intersection, $\mathrm{nor}^{\cap}$

Lemma 9.1.0.1. *For $\delta \in (0,1)$ and $\ell \in \omega$ there are $M(\delta,\ell) \in \omega$ and $\varepsilon^{\cap}(\delta,\ell) > 0$ such that: Whenever we have a probability space Ω and a family $(A_i : i < M)$ of sets of measure $\geq \delta$, we can find a subfamily of ℓ many sets whose intersection has measure at least $\varepsilon^{\cap}(\delta,\ell)$.*

Proof. By straightforward counting.[1]

We write χ_B for the characteristic function of B. Assume we have M many sets A_i, and set $X \subset \Omega$ to contain all points that lie in at least ℓ many of the A_i. Then

$$\delta \cdot M \leq \int \sum_{i \in M} \chi_{A_i} \leq \mu(X) \cdot M + \mu(\Omega \setminus X) \cdot (\ell - 1) \leq \mu(X) \cdot M + \ell,$$

and $\mu(X) \geq \delta - \ell/M$. So if we set

$$M > 2\frac{\ell}{\delta},$$

then there are at least $\delta/2$ "many" points in X. We can assign to each point $x \in X$ a subset M_x of M (of size at least ℓ) by

$$i \in M_x \text{ iff } x \in A_i.$$

This partitions X into at most 2^M many sets; and at least one of the pieces has to have size at least

$$\varepsilon^{\cap}(\delta,\ell) := \frac{\delta}{2 \cdot 2^M}. \qquad \square$$

Let us set $F^0_b := 1$ and $F^{n+1}_b = M(1/b, F^n_b)$. We can use this notion to define a norm on natural numbers:

Definition 9.1.0.2. For $m > 0$: $\mathrm{nor}^{\cap}_b(m) \geq n$ iff $m \geq F^n_b$.

[1] Originally we used a stronger statement for which we only had a more complicated proof. We are grateful to William B. Johnson for pointing out in `http://mathoverflow.net/q/108380` that the statement in the current form has the obvious straightforward proof.

So we get the following:

> Fix a measure space Ω and a sequence $(T_i)_{i\in A}$ of sets of measure $\geq {}^1\!/\!_b$. Then there is a subset $B \subseteq A$ such that $\mathrm{nor}^{\cap}_b(|B|) \geq \mathrm{nor}^{\cap}_b(|A|) - 1$ and $\bigcap_{i\in B} T_i$ has measure $\geq \varepsilon^{\cap}({}^1\!/\!_b, |A|)$. (9.1.0.3)

Note that without loss of generality the function $\varepsilon^{\cap}$ satisfies: $\varepsilon^{\cap}(\delta, \ell_1) \geq \varepsilon^{\cap}(\delta, \ell_2)$ whenever $\ell_2 > \ell_1 > 0$. We write down the following trivial consequence of (9.1.0.3) for later reference:

> Assume that A is a subset of some finite set POSS. Fix a measure space Ω and a sequence $(T_i)_{i\in A}$ of sets of measure $\geq {}^1\!/\!_b$. Then there is a subset $B \subseteq A$ such that $\mathrm{nor}^{\cap}_b(|B|) \geq \mathrm{nor}^{\cap}_b(|A|) - 1$ and $\bigcap_{i\in B} T_i$ has measure $\geq \varepsilon^{\cap}({}^1\!/\!_b, |\,\mathrm{POSS}\,|)$. (9.1.0.4)

9.2 Most large subsets do not cover a half-sized set

Let Ω be the set of subsets of some finite set $A \in \omega$ of relative size $1 - \epsilon$ (for $0 < \epsilon < {}^1\!/\!_4$). (Since $A \in \omega$, we can write A for the cardinality $|A|$.) I.e.: $x \in \Omega$ implies $x \subseteq A$ and $|x| = A \cdot (1 - \epsilon)$. We can assume $A \gg {}^1\!/\!_\epsilon$ and that $A \cdot \epsilon$ is an integer.

Let $T \subseteq A$ be of relative size $\geq 1/2$, i.e., $|T| \geq {}^A\!/\!_2$. Let Ω_T be the elements of Ω that cover T, i.e., $x \in \Omega^T$ iff $x \in \Omega$ and $T \subseteq x$.

We will use the following easy fact from combinatorics:

Fact 9.2.0.5. *For any natural number $k \geq 2$, the quotient*

$$\frac{\binom{2Nk}{N}}{\binom{Nk}{N}}$$

tends to infinity with $N \to \infty$.

Proof. This can be checked with Stirling's approximation formula, or with the following elementary estimate: From

$$\forall a, b : \frac{(a-b)^b}{b!} \leq \binom{a}{b} \leq \frac{a^b}{b!}$$

we get

$$N! \cdot \binom{2Nk}{N} \geq (2Nk - N)^N \quad \text{and} \quad N! \cdot \binom{Nk}{N} \leq (Nk)^N,$$

and hence

$$\frac{\binom{2Nk}{N}}{\binom{Nk}{N}} \geq \frac{(2Nk-N)^N}{(Nk)^N} \geq (2 - \frac{1}{k})^N \to \infty.$$

□

Lemma 9.2.0.6. *Fix $b > 2$ and a finite set I with $|I| > b$. Let* POSS *be the family of subsets of 2^I of relative size $1 - {}^1\!/\!_{2^b}$. For $m \in \omega$ we define $\mathrm{nor}^{\div}_{I,b}(m) := \lfloor {}^m\!/\!_{\binom{2^{|I|-1}}{2^{|I|-b}}} \rfloor$.*

Then:

1. *For any $T \subseteq 2^I$ of at least relative size ${}^1\!/\!_2$ and for any $C \subseteq$ POSS there is a subset $D \subseteq C$ with $\mathrm{nor}^{\div}_{I,b}(|D|) \geq \mathrm{nor}^{\div}_{I,b}(|C|) - 1$ and $T \not\subseteq x$ for all $x \in D$.*

2. *If I is chosen sufficiently large (with respect to b), then $\mathrm{nor}^{\div}_{I,b}(\mathrm{POSS})$ is large.*

Proof. 1. It is enough to show this in case T has exactly size $2^{|I|-1}$. If $x \in C \setminus D$, then the set $2^I \setminus x$ has size $2^{|I|-b}$ and is a subset of $2^I \setminus T$. So there are at most $\binom{2^{|I|-1}}{2^{|I|-b}}$ possibilities for

$2^I \setminus x$, hence (by definition of $\mathrm{nor}^{\div}_{I,b}$) we get $\mathrm{nor}^{\div}(C \setminus D) \leq 1$. From the implication

$$x \leq y \text{ and } \lfloor x - y \rfloor \leq 1 \;\Rightarrow\; \lfloor x \rfloor - \lfloor y \rfloor \leq 1$$

we get $\mathrm{nor}^{\div}_{I,b}(C) - \mathrm{nor}^{\div}_{I,b}(D) \leq 1$.

2. Note that the cardinality of POSS is equal to $\binom{2^{|I|}}{2^{|I|-b}}$. Using Fact 9.2.0.5 with $N := 2^{|I|-b}$ and $k := 2^{b-1}$ we get that $\binom{2^{|I|}}{2^{|I|-b}} / \binom{2^{|I|-1}}{2^{|I|-b}}$ is large for large I. □

9.3 Providing bigness

In this section, we write log to denote $\log_2$.

Apart from unimportant rounding effects, log of $\mathrm{nor}^{\div}$ satisfies 2-bigness (and the same for $\mathrm{nor}^{\cap}$). Instead of thinking about such effects, we just define for any norm a 2-big version. Actually, we define a 2-big version of the combinations of two norms (of course, any finite number of norms can be combined in this way):

Definition 9.3.0.7. Assume that $\mathrm{nor}_1, \mathrm{nor}_2 : \omega \to \omega$ are weakly increasing and converge to infinity.

Then we define $\mathrm{lognor} = \mathrm{lognor}(\mathrm{nor}_1, \mathrm{nor}_2) : \omega \to \omega$ as follows: By induction on m, we define $\mathrm{lognor}(x) \geq m$ by the conjunction of the following clauses:

- $\mathrm{nor}_1(x) \geq m$ and $\mathrm{nor}_2(x) \geq m$.
- $\mathrm{lognor}(\lfloor \frac{x}{2} \rfloor) \geq m - 1$.
- If $y \in \omega$ and $i \in \{1, 2\}$ satisfies $\mathrm{nor}_i(y) \geq \mathrm{nor}_i(x) - 1$, then $\mathrm{lognor}(y) \geq m - 1$.

We set $\mathrm{lognor}(x) := \mathrm{lognor}(\mathrm{nor}^{\cap}, \mathrm{nor}^{\div})$.

Lemma 9.3.0.8. *Let* $\mathrm{lognor} = \mathrm{lognor}(\mathrm{nor}_1, \mathrm{nor}_2)$.

- $\mathrm{lognor}(x)$ *is a well-defined natural number for all* x*, i.e., there is a maximal* m *such that* $\mathrm{lognor}(x) \geq m$ *holds.*
- lognor *is weakly increasing and diverges to infinity.*
- lognor *has* 2*-bigness: If* $F : m \to 2$ *is a coloring function and* $\mathrm{lognor}(m) = n$*, then there is some* $c \in 2$ *such that* $\mathrm{lognor}(F^{-1}(c)) \geq n - 1$.
- *So if we define* $\mathrm{nor}_b(x)$ *as* $\frac{\mathrm{lognor}(x)}{\lceil \log(b) \rceil}$*, then* nor_b *will be* b*-big.*
- *If* $\mathrm{nor}_i(y) \geq \mathrm{nor}_i(x) - 1$ *for some* $i \in \{1, 2\}$*, then* $\mathrm{lognor}(y) \geq \mathrm{lognor}(x) - 1$.

Proof. "Well-defined" follows from $\mathrm{lognor}(x) \leq \mathrm{nor}_i(x)$.

Monotonicity follows from the monotonicity of nor_1 and nor_2.

We now prove that by induction on m that there are only finitely many x with $\mathrm{lognor}(x) < m$. For $m = 0$ this is obvious, as all x satisfy $\mathrm{lognor}(x) \geq 0$. For $m > 0$: $\mathrm{lognor}(x) < m$ iff either $\mathrm{nor}_1(x) < m$ or $\mathrm{nor}_2(x) < m$ or $\mathrm{lognor}(\lfloor \frac{x}{2} \rfloor) < m - 1$ or there is some y and some $i \in \{1, 2\}$ with $\mathrm{nor}_i(y) \geq \mathrm{nor}_i(x) - 1$ and $\mathrm{lognor}(y) < m - 1$; for each case there are only finitely many possibilities.

2-bigness and the last item follow directly from the definition. b-bigness is Lemma 2.1.0.7. □

Chapter 10

The $\mathtt{cn}$ part

10.1 The subatomic creatures for type $\mathtt{cn}$

We now describe the subatomic families $\mathsf{K}'_{\mathtt{cn},b}$ used for the $\mathtt{cn}$-indices.

Definition 10.1.0.1. 1. Fix an interval I which is large enough to satisfy (4). In particular, $|I| > b$. Again, we assume that this interval is disjoint to all intervals previously chosen.

2. The basic set of all possibilities and the set of subatoms is the same as in the $\mathtt{nn}$-case 8.1.0.1 (but the norm will be different). So POSS consists of all subsets X of 2^I with relative size $1 - 1/2^b$:
$$\mathrm{POSS} = \{X \subseteq 2^I : |X| = (1 - 1/2^b)|2^I|\}.$$

3. A subatom C is a subset of POSS, with $\mathrm{poss}(C) := C$, and
$$\mathrm{nor}(C) := \frac{\mathrm{lognor}(\mathrm{nor}^{\cap}_b, \mathrm{nor}^{\dot{-}}_{I,b})(|C|)}{2^{\min(I)} \cdot b^2}.$$

4. We require $\mathrm{nor}(\mathrm{POSS}) > b$ (thus satisfying (6.1.0.2)).

5. We set $H'(\mathtt{cn}, {=}b) := \max(H'_0, H'_1)$ for $H'_0 := 2^{\binom{2^{|I|}}{2^{|I|-b}}}$ and $H'_1 := 1/\varepsilon^{\cap}(1/b, |\mathrm{POSS}|)$, where $\varepsilon^{\cap}$ is defined in 9.1.0.1.

Note that $H'(\mathtt{cn}, {=}b) > |\mathsf{K}'_{\mathtt{cn},b}|$ (this is what we need H'_0 for).

Recall that lognor satisfies 2-bigness, so after dividing by b (actually, $\lceil \log_2(b) \rceil \cdot b$ would be sufficient) we get strong b-bigness (i.e., the norm satisfies the requirement (6.1.0.1)).

Note that (in contrast to the $\mathtt{nn}$ case) this norm is a counting norm, i.e., $\mathrm{nor}(C)$ only depends on $|C|$, not on the "structure" of C.

10.2 The generic object

Just as in the $\mathtt{nn}$-case, we set $I_{\mathtt{nn},\mathbf{u}}$ to be the I used for $\mathsf{K}'_{\mathtt{nn},b(\mathbf{u})}$; and we define $\underset{\sim}{N}_\alpha$ analogously to the $\mathtt{nn}$-case.[1]

As before, $\underset{\sim}{N}_\alpha$ is a name for a null set, and a real r is in $\underset{\sim}{N}_\alpha$ iff there are infinitely many sublevels $\mathbf{u}$ such that $r \restriction I_{\mathtt{cn},\mathbf{u}}$ is not in the possibility X of $\mathsf{K}'_{\mathtt{cn},\mathbf{u}} = \mathsf{K}_{\alpha,\mathbf{u}}$ that is chosen by the generic filter.

This time, the purpose of $\underset{\sim}{N}_\alpha$ is not to cover all reals not depending on α, but rather to avoid being covered by any null set not depending on α.

[1] Of course, generally $I_{\mathtt{cn},\mathbf{u}} \neq I_{\mathtt{nn},\mathbf{u}}$, so $\underset{\sim}{N}_\alpha$ for $\alpha \in \Xi_{\mathtt{nn}}$ lives on a different domain than $\underset{\sim}{N}_\beta$ for $\beta \in \Xi_{\mathtt{cn}}$.

Lemma 10.2.0.2. *Fix a subatomic sublevel* $\mathbf{u}$*, an index* $\alpha \in \Xi_{\mathtt{cn}}$ *and a subatom* $C \in \mathsf{K}'_{\mathtt{cn},\mathbf{u}} = \mathsf{K}_{\alpha,\mathbf{u}}$*.*

1. *Given* $T \subseteq 2^{I_{\mathtt{cn},\mathbf{u}}}$ *of relative size* $\geq 1/2$ *we can strengthen* C *to* D*, decreasing the norm by at most* $1/2^{\min(I)} \cdot b(\mathbf{u})$ *such that* $T \not\subseteq X$ *for all* $X \in \mathrm{POSS}(D)$*.*
2. *Fix a probability space* Ω *and a function* F *that maps every* $X \in \mathrm{poss}(C)$ *to* $F(X) \subseteq \Omega$ *of measure* $\geq 1/b(\mathbf{u})$*. Then we can strengthen* C *to* D*, decreasing the norm by at most* $1/2^{\min I} \cdot b(\mathbf{u})$ *such that* $\bigcap_{X \in \mathrm{poss}(D)} F(X)$ *has measure at least* $1/b(\mathbf{u}+1)$*. Here,* $\mathbf{u}+1$ *denotes the smallest subatomic sublevel above* $\mathbf{u}$*.*

Proof. This is an immediate consequences of (9.1.0.4), 9.2.0.6 and 9.3.0.8, just note that

$$b(\mathbf{u}+1) > H'(\mathtt{cn}, {=}b(\mathbf{u})) \geq 1/\varepsilon^{\cap}(1/b(\mathbf{u}), |\mathrm{POSS}|). \qquad \square$$

Again, let $\mathbf{u}+1$ denote the smallest subatomic sublevel above $\mathbf{u}$. Then

$$b(\mathbf{u}+1) > H'(\mathtt{cn}, {=}b(\mathbf{u})) > |\mathsf{K}_{\mathtt{cn},b(\mathbf{u})}|.$$

In other words,

The cardinality of $\mathsf{K}_{\mathtt{cn},b(\mathbf{u})}$ is less than $b(\mathbf{u}+1)$. (10.2.0.3)

10.3 Names for null sets

Let $T \subseteq 2^{<\omega}$ be a tree (without terminal nodes) of measure $1/2$. (Such trees correspond bijectively to closed sets of measure $1/2$.) Then the set

$$N_T := 2^\omega \setminus \bigcup \{r + [T] : r \in \mathbb{Q}\}. \tag{10.3.0.4}$$

is a null set (closed under rational translations). Conversely, for every null set N there is such a T with $N \subseteq N_T$.

The relative measure of s in T (for $s \in 2^n$, $n \in \omega$) is defined as $\mu([T] \cap [s]) \cdot 2^n$. For completeness, we say that the relative measure of s is 0 if $s \notin T$. (Analogously, we can define the relative measure of a node s in a finite tree $T \subseteq 2^{\leq m}$ with no terminal nodes of height $< m$.) Note the following easy consequence of the Lebesgue density theorem:

Fact 10.3.0.5. *If* T *is a tree without terminal nodes,* $s \in T$ *has positive relative measure, and* $\delta < 1$*, then there is a* $t > s$ *with relative measure* $> \delta$*. (And for all levels above the level of* t*, there is an extension* $t' > t$ *which also has relative measure* $> \delta$*.)*

By removing nodes with relative measure 0, the measure of T does not change. We give such trees a name:

Definition 10.3.0.6. T is a pruned-$1/2$ tree, if $T \subseteq 2^{<\omega}$ has measure $1/2$ and has no nodes of relative measure zero (and in particular no terminal nodes).

Note that each null set is contained in N_T for some pruned-$1/2$ T. So instead of investigating arbitrary names for null sets, we will consider names $\underaccent{\sim}{T}$ for pruned-$1/2$ trees.

Note that there are fewer than 2^{2^h} many possibilities for the level h of $\underaccent{\sim}{T}$. So we can "code" $\underaccent{\sim}{T}$ by a real $\underaccent{\sim}{r} \in 2^\omega$ such that $\underaccent{\sim}{T} \restriction h$ is determined by $\underaccent{\sim}{r} \restriction 2^{2^{(h+1)}}$.

Assume that p rapidly reads this $\underaccent{\sim}{r}$. Then $\underaccent{\sim}{T} \restriction (\max(I_{\mathtt{cn},\mathbf{u}}) + 1)$ is determined $\leq \mathbf{u}$ (according to (6.1.0.6) and 10.1.0.1(5)).

We will describe this situation by "p rapidly reads $\underaccent{\sim}{T}$".

10.4 $\mathrm{cof}(\mathcal{N}) \geq \kappa_{\mathtt{cn}}$

Lemma 10.4.0.7. *Let $p \in Q$ rapidly read the pruned-1/2 tree $\underset{\sim}{T}$ not using the index $\alpha \in \Xi_{\mathtt{cn}}$. Then p forces that $\underset{\sim}{N}_\alpha$ is not a subset of $N_{\underset{\sim}{T}}$, i.e.,[2] there is some $s \in \underset{\sim}{N}_\alpha \cap [\underset{\sim}{T}]$.*

Proof. We can assume that p is pruned and that $\alpha \in \mathrm{supp}(p)$. It is enough to find a name $\underset{\sim}{r} \in 2^\omega$ and a $q \leq p$ forcing $\underset{\sim}{r} \in \underset{\sim}{N}_\alpha \cap [T]$. For this, we will inductively modify p at infinitely many sublevels $\mathbf{u}$ (resulting in the 1-purely stronger q):

Let $\mathbf{u}$ be a subatomic sublevel (above all the sublevels that we have already modified), where α is the active index with subatom C of norm at least 10, living on the interval $I := I_{\mathtt{cn},\mathbf{u}}$.

The finite tree $\underset{\sim}{T}' := \underset{\sim}{T} \,|\, \max(I)+1$ is determined $\leq \mathbf{u}$, and even $< \mathbf{u}$, as $\underset{\sim}{T}$ does not depend on α (as usual, note that due to modesty α is the only active index at sublevel $\mathbf{u}$). In particular the set Y of potential values of $\underset{\sim}{T}'$ has size $\leq \mathrm{maxposs}(<\mathbf{u})$.

We now enumerate all $T^* \in Y$ and $t \in T^* \cap 2^{\min(I)}$ with relative measure (in T^*) at least 1/2. There are at most $\mathrm{maxposs}(<\mathbf{u}) \times 2^{\min(I)}$ many such pairs (T^*, t).

Starting with $C^0 := C$, we iteratively use Lemma 10.2.0.2(1) to strengthen the subatom C^n to some C^{n+1} such that for the current (T^*, t) and all $X \in \mathrm{poss}(C^{n+1})$ there is some $t' \in 2^I \setminus X$ such that $t^\frown t' \in T^*$.

So in the end we get a subatom $D \leq C$ of norm $\geq \mathrm{nor}(C) - 1$ such that for all (T^*, t) and $X \in \mathrm{poss}(D)$ there is some $t' \in 2^I \setminus X$ with $t^\frown t' \in T^*$.

In this way, we modify infinitely many sublevels $\mathbf{u}$, resulting in a condition $q \leq p$.

Now work in the forcing extension, where q is in the generic filter. We can now construct by induction an element r of $\underset{\sim}{N}_\alpha \cap [\underset{\sim}{T}]$ (i.e., $r \,|\, I_{\mathtt{cn},\mathbf{u}}$ is not in the generically chosen X at index α and sublevel $\mathbf{u}$, for infinitely many sublevels $\mathbf{u}$.)

Assume we already have $r \,|\, n \in \underset{\sim}{T}$ for some n. Since $\underset{\sim}{T}$ has no nodes of relative norm 0, there is a $h' > n$ and an $t' \in T \cap 2^{h'}$ extending $r \,|\, n$ with relative measure $\geq$ 1/2 (see 10.3.0.5). Pick a sublevel $\mathbf{u}$ such that: $\min(I) =: h > h'$ for $I := I_{\mathtt{cn},\mathbf{u}}$, and $\mathbf{u}$ was considered in our construction of q. There is still some $t \in 2^{h'}$ extending $\underset{\sim}{r} \,|\, n$ of relative measure 1/2. Set $T^* := \underset{\sim}{T} \,|\, \max(I)+1$. Note that in our construction of q, when considering $\mathbf{u}$, we dealt with the pair (T^*, t), and thus made sure for all $X \in \mathrm{poss}(q(\alpha, \mathbf{u}))$ (so in particular for the one actually chosen by the generic filter) there is some $t' \in 2^I$ such that $t^\frown t' \in T^*$ and $t' \notin X$. So we can just set $r \,|\, \max I := t^\frown t'$. $\square$

Corollary 10.4.0.8. *$\mathbb{Q}$ forces that $\mathrm{cof}(\mathcal{N}) \geq \kappa_{\mathtt{cn}}$.*

Proof. This is very similar to the proof of 7.3.0.7: Assume that there is a $\aleph_1 \leq \kappa < \kappa_{\mathtt{cn}}$ and a p forcing that $(\underset{\sim}{N}^*_i)_{i\in\kappa}$ is a basis of null sets. As described above, we can assume that each $\underset{\sim}{N}^*_i = N_{\underset{\sim}{T}_i}$ for some pruned-1/2 tree $\underset{\sim}{T}_i$ of measure 1/2. For each i, fix a maximal antichain A_i below p of conditions rapidly reading $\underset{\sim}{T}_i$. $X := \bigcup_{i\in\kappa, q\in A_i} \mathrm{supp}(q)$ has size κ, so there is an $\alpha \in \Xi_{\mathtt{cn}} \setminus X$. Each $a \in A_i$ rapidly reads $\underset{\sim}{T}_i$ not using α. So by the preceding lemma, $\underset{\sim}{N}_\alpha \not\subseteq N_{\underset{\sim}{T}_i}$ is forced by a (and therefore by p, as A_i is predense below p). $\square$

10.5 $\mathrm{non}(\mathcal{N}) \leq \kappa_{\mathtt{nn}}$

We want to show that the set X of reals reals that are added by (or more precisely: rapidly read from) the `nm` and `nn` parts (i.e., not depending on the `cn` and Sacks parts) is not null.

Let $\mathbb{Q}_{\Xi_{\text{non-sk}}}$ be the set of conditions p with $\mathrm{supp}(p) \cap \Xi_{\mathtt{sk}} = \emptyset$. Recall that according to Lemma 3.6.0.18, $\mathbb{Q}_{\Xi_{\text{non-sk}}}$ is a complete subforcing of $\mathbb{Q}$ (and satisfies ω^ω-bounding, rapid reading,

[2] as $\underset{\sim}{N}_\alpha$ is closed under rational translates

etc). We have seen in 6.3 that the quotient of $\mathbb{Q}$ and $\mathbb{Q}_{\Xi_{\text{non-sk}}}$ satisfies the Sacks property, and in particular that every null set N in the Q-extension is contained in a null-set $N' \supseteq N$ in the intermediate $\mathbb{Q}_{\Xi_{\text{non-sk}}}$-extension.

So it is enough to show that X is still non-null in the $\mathbb{Q}_{\Xi_{\text{non-sk}}}$-extension; in other words, we can in the rest of the paper ignore the Sacks indices altogether (i.e., work in $\mathbb{Q}_{\Xi_{\text{non-sk}}}$, or in other words assume that $\Xi_{\mathtt{sk}} = \emptyset$).

We have seen that the sets of the form N_T for pruned-$1/2$ trees T form a basis of null sets; so we just have to show the following:

Lemma 10.5.0.9. *Let $\underset{\sim}{T}^*$ be a pruned-$1/2$ tree rapidly read by p. Then there is a $q \leq p$ continuously reading some $\underset{\sim}{r} \in 2^\omega$ not using the* cn *part, such that q forces $\underset{\sim}{r} \in [\underset{\sim}{T}^*]$. (As described above, the Sacks part is not used at all.)*

As $\underset{\sim}{r} \in [\underset{\sim}{T}^*]$ implies $\underset{\sim}{r} \notin N_{\underset{\sim}{T}^*}$, and $\underset{\sim}{r}$ only depends on the nm and nn parts, we get:

Corollary 10.5.0.10. $\mathbb{Q}$ *forces* $\mathrm{non}(\mathcal{N}) \leq \kappa_{\mathtt{nn}}$.

To prove Lemma 10.5.0.9 we will use:

Lemma 10.5.0.11. *Let T be a tree of positive measure and fix $\epsilon > 0$. Then for all sufficiently large $m \in \omega$ there are many fat nodes in $T \cap 2^m$, by which we mean:*

$$\mu([T^{[s]}]) \geq 2^{-m}(1-\epsilon) \text{ for at least } |[T] \cap 2^m| \cdot (1-\epsilon) \text{ many } s \in T \cap 2^m.$$

Proof. Write μ for the measure of $[T]$. Note that $|T \cap 2^m| \cdot 2^{-m}$ decreases and converges to μ. Hence from some m on, we have

$$|T \cap 2^m| \cdot 2^{-m} - \mu\epsilon^2 \leq \mu. \tag{10.5.0.12}$$

Let l be the number of fat nodes at level m, and $s = |T \cap 2^m| - l$ the number of non-fat nodes. We want to show $l \geq 2^m \mu \cdot (1-\epsilon)$.

Clearly,

$$\mu < l \cdot 2^{-m} + s \cdot 2^{-m}(1-\epsilon) = |T \cap 2^m| \cdot 2^{-m} - 2^{-m} s\epsilon. \tag{10.5.0.13}$$

Combining (10.5.0.12) and (10.5.0.13), we get $|T \cap 2^m| \cdot 2^{-m} - \mu\epsilon^2 \leq |T \cap 2^m| \cdot 2^{-m} - 2^{-m} s\epsilon$, and hence $s \leq 2^m \mu \cdot \epsilon$. As $l + s = |T \cap 2^m| \geq 2^m \mu$, we get $l \geq 2^m \mu \cdot (1-\epsilon)$, as required. □

Proof of Lemma 10.5.0.9. We can assume that p is pruned. By induction on $n \in \omega$, we construct:

(a) $k_n \in \omega$.

(b) A condition $q_n \leq p$ with $k_n \in \mathtt{w}^{q_n}$ such that $\mathrm{nor}(q_n, k') \geq n + 6$ for all $k' \geq k_n$ in $\mathtt{w}^{q_n}$.

(c) We will additionally require: $q_{n+1} \leq q_n$; q_{n+1} is identical to q_n below k_n, and has norms $\geq n$ between k_n and k_{n+1}.

(Therefore there is a limit condition q_ω stronger than each q_n.)

(d) $i_n \in \omega$ and a name $\underset{\sim}{s}_n$ for an element of $\underset{\sim}{T}^* \cap 2^{i_n}$ such that q_n decides $\underset{\sim}{s}_n$ below k_n not using any cn-indices.

(e) We additionally require that i_n is "not too large" with respect to k_n, more particularly:

$$2^{i_n+2} < b((k_n, 0)).$$

($(k_n, 0)$ is the the smallest subatomic sublevel above k_n.) (As b is strictly monotone, it suffices to have $k_n > 2^{i_n+2}$.)

(f) We additionally require: $i_{n+1} > i_n$, and $\underset{\sim}{s}_{n+1}$ is forced (by q_{n+1}) to extend $\underset{\sim}{s}_n$.

So q_∞ will force that the union of the $\underset{\sim}{s}_n$ will be the required branch through $\underset{\sim}{T}^*$, proving the Lemma.

(g) We will also construct a name $\underset{\sim}{T}_n$, which is (forced by q_n to be) a subtree of $\underset{\sim}{T}^*$ with stem $\underset{\sim}{s}_n$ and relative measure $> 1/2$ (i.e., $\mu([\underset{\sim}{T}_n]) > 1/2 \cdot 2^{-i_n}$), which is read continuously by q_n not using any $\mathtt{cn}$-indices below k_n.[3]

We set $i_0 := 0$, $\underset{\sim}{s}_0 := \langle\rangle$ and $\underset{\sim}{T}_0 = \underset{\sim}{T}^*$. We choose k_0 such that the norms of the compound creatures in p are ≥ 6 above k_0 and set q_0 to be p where we increase the trunk to k_0. So $\underset{\sim}{T}_0$ does not depend on any $\mathtt{cn}$-indices below k_0 (as below k_0 there is only trunk and thus a unique possibility).

So assume we already have the objects mentioned above for some n (i.e., k_n, q_n, i_n, $\underset{\sim}{s}_n$ and $\underset{\sim}{T}_n$). For notational simplicity we refer to them without the subscript n, i.e., we set $k := k_n$ etc. We will now construct the objects for $n+1$.

1. We choose k^* so large that for each $\xi \in \mathrm{supp}(q(k)) \cap \Xi_{\mathtt{ls}}$ there is an atom $q_n(\xi, \ell)$ of norm $> n+2$ for some ℓ between k and k^*.

2. It is forced that Lemma 10.5.0.11 holds for $\underset{\sim}{T}$ and for $\epsilon := 1/\mathrm{maxposs}(<k^*)\cdot\mathrm{maxposs}(<k)$. So we get a name $\underset{\sim}{m}$ for a level where there are many fat nodes. Using Lemma 5.3.0.16, we strengthen q to q^1, not changing anything below k^* and keeping all norms $\geq n+4$, such that we can find (in V) some $m > i$ which is forced by q^1 to be $\geq \underset{\sim}{m}$. Note that Lemma 10.5.0.11 is forced to hold for this $m \geq \underset{\sim}{m}$ as well, i.e., there is a name of a "large" set $\underset{\sim}{L} \subseteq 2^m$ of "fat" nodes.

   This m will be our i_{n+1}. So $i_{n+1} > i_n$ is satisfied.

3. So can further strengthen q^1 to q^2 not changing anything below k^* and keeping all norms $\geq n+2$ such that $\underset{\sim}{L} \subseteq 2^m$ is essentially decided, i.e., decided below some level $k^{**} > k^*$. Since we already assumed that $\underset{\sim}{T}$ is read continuously, we can assume that q^2 also decides $\underset{\sim}{T} \cap 2^m$ below k^{**}. Also, we can assume that all norms of compound creatures in q^2 above (including) k^{**} are $> n+7$, and that $k^{**} > 2^{m+2}$.

   This k^{**} will be k_{n+1}. Note that this ensures item (e) for $n+1$.

4. $\underset{\sim}{L}$ is forced to be a subset of $\underset{\sim}{T} \cap 2^m$ of relative size $\geq (1-\epsilon)$, and both $\underset{\sim}{L}$ and $\underset{\sim}{T} \cap 2^m$ are decided below k^{**}. Also, $\underset{\sim}{T} \cap 2^m$ does not depend on the $\mathtt{cn}$-part below k. Therefore, we can construct a name $\underset{\sim}{L}' \subseteq \underset{\sim}{L}$ that also does not depend on such coordinates, and such that $\underset{\sim}{L}' \subseteq \underset{\sim}{T} \cap 2^m$ has relative size $\geq (1 - \epsilon \cdot \mathrm{maxposs}(<k)) \geq 1/2$.

   Proof: Each $\eta \in \mathrm{poss}(q_2, k^{**})$ determines objects $L_\eta \subseteq S_\eta$ (where $q^2 \wedge \eta$ forces "$L_\eta = \underset{\sim}{L}$ and $S_\eta = \underset{\sim}{T} \cap 2^m$"). We call η_1, η_2 equivalent if they differ only on the $\mathtt{cn}$-part below k (which implies $S_{\eta_1} = S_{\eta_2}$). Clearly, each equivalence class has size at most $\mathrm{maxposs}(<k)$. For an equivalence class $[\eta]$, we set $L'_{[\eta]} := \bigcap_{\eta' \in [\eta]} L_{\eta'}$. So the map assigning η to $L'_{[\eta]}$ defines a name (not depending on the $\mathtt{cn}$-part below k) of a subset of S_η of relative size $\geq 1/2$.

   Recall that $\underset{\sim}{T}$ is forced to have stem $s \in 2^i$ and measure $> 1/2 \cdot 2^{-i}$, so the cardinality of $\underset{\sim}{T} \cap 2^m$ is forced to be $> 2^{m-i-1}$, and thus the cardinality of $\underset{\sim}{L}'$ is forced to be $> 2^{m-i-1}(1/2) = 2^{m-i-2} > 2^m/b((k,0))$, according to item (e).

   To summarize:

[3] I.e.: For all ℓ there is a k and a function defined on $\mathrm{poss}(q_n, <k)$ giving the value of $\underset{\sim}{T}_n \cap 2^\ell$ such that the value is the same for $\eta, \eta' \in \mathrm{poss}(q_n, <k)$ that differ only on the $\mathtt{cn}$-part below k_n.

- $\underset{\sim}{T} \cap 2^m$ and its subset $\underset{\sim}{L}'$ are decided by q^2 below k^{**}, not using the cn-part below k.
- We set $\Omega = 2^m$. (As a finite set, it carries the uniform probability measure.) $\underset{\sim}{L}'$ as subset of Ω is forced to have measure $> 1/b((k,0))$.
- q^2 forces that each $s \in \underset{\sim}{L}'$ satisfies $\mu([\underset{\sim}{T}^{[s]}]) \geq 2^{-m}(1-\epsilon)$.

5. Now we glue q^2 between k and k^{**}, and replace all lim-sup subatoms between k^* and k^{**} with singletons (not changing the lim-inf subatoms, nor anything between k and k^*), resulting in q^* and the compound creature $\mathfrak{d}^* = q^*(k)$ (with $m^{\mathrm{dn}}(\mathfrak{d}^*) = k$, $m^{\mathrm{up}}(\mathfrak{d}^*) = k^{**}$ and $\mathrm{supp}(\mathfrak{d}^*) = \mathrm{supp}(q,k)$). So above k^{**}, q^* is identical to q^2, and below k^* it is identical to q.

   Note that $\mathrm{nor}(\mathfrak{d}^*) \geq n+2$: Gluing results in a norm at least the minimum of the norms of the glued creatures; and replacing lim-sup subatoms above k^* with singletons does not drop the norm below $n+2$ as we made sure that there are large subatoms between k and k^*.

   We will in the following find a strengthening $\mathfrak{d}^{**}$ of $\mathfrak{d}^*$ with $\mathrm{nor}(\mathfrak{d}^{**}) \geq \mathrm{nor}(\mathfrak{d}^*) - 2 \geq n$ and we will set q_{n+1} to be q^* where we replace $\mathfrak{d}^*$ with $\mathfrak{d}^{**}$. Then items (b) and (c) will be satisfied for $n+1$.

6. Recall that q^* decides both $\underset{\sim}{L}'$ and $\underset{\sim}{T} \cap 2^m$ below k^{**}, not using the cn-part below k. Note that $\mathrm{poss}(q^*, {<}k^{**})$ is isomorphic to $X \times Y \times Z$, for

   - $X := \mathrm{poss}(q^*, {<}k) = \mathrm{poss}(q, {<}k)$,
   - Y are the possibilities of $\mathfrak{d}^*$ between k and k^*, and
   - Z are the possibilities of $\mathfrak{d}^*$ between k^* and k^{**} (which we can restrict to the lim-inf part, as there are only singletons in the lim-sup-part).

7. Fix a $\nu \in Z$. We will now perform an induction on the (subatomic) sublevels $\mathbf{u}$ between k and k^*, starting with the lowest one, $(k,0)$. We assume that we have arrived in this construction at sublevel $\mathbf{u}$ with the active subatom C, and that we already have constructed the following:

   - The (final) subatoms for all sublevels $\mathbf{v}$ below $\mathbf{u}$ (and above k), with subatom-norm at most 2 smaller than the norms of the original subatoms (i.e., those in $\mathfrak{d}^*$).
   - (Preliminary) subatoms for all sublevels $\mathbf{u}'$ above (including) $\mathbf{u}$ (and below k^*), where the norm of the subatom at $\mathbf{u}'$ has been reduced from the original one by at most $K/b(\mathbf{u}')$, where K is the number of steps already performed in the current induction (i.e., K is the number of subatomic sublevels between k and $\mathbf{u}$). So our current C is one of these "preliminary subatoms".
   - A function $F^{\mathbf{u}}$ that maps each possibility $\eta \in X \times Y$ to a subsets $F^{\mathbf{u}}(\eta)$ of 2^m; such that for all η
     - $F^{\mathbf{u}}(\eta)$ is forced to be a subset of $\underset{\sim}{L}'$ by the condition q^* modulo the fixed $\nu \in Z$, modulo η and modulo the already constructed subatoms (the final ones as well as the preliminary ones).[4]
     - $F^{\mathbf{u}}(\eta) \subseteq 2^m$ is of relative size $\geq 1/b(\mathbf{u})$.
     - $F^{\mathbf{u}}(\eta)$ does not depend on any cn-indices below $\mathbf{u}$.

   The first sublevel, $(k,0)$, is clear: there are no sublevels below where we have to define final subatoms, the preliminary subatoms above are just the original ones, and $F^{(k,0)}$ is just given by the name $\underset{\sim}{L}'$.

[4] See (5.1.0.9) for a definition of "modulo". If η is not a compatible with the currently constructed (final and preliminary) subatoms, then $F^{\mathbf{u}}(\eta)$ is irrelevant.

Now we perform the inductive step. If our subatom C is not of $\mathtt{cn}$-type, we do nothing[5] and go to the next step. So let us assume that the current (preliminary) C is of $\mathtt{cn}$-type.

Let Y^- be Y restricted to the sublevels below $\mathbf{u}$, and Y^+ to the ones above. Every[6] $\eta \in X \times Y$ can be written as $(\eta^-, \eta^{\mathbf{u}}, \eta^+)$ for $\eta^- \in X \times Y^-$, $\eta^{\mathbf{u}} \in \mathrm{poss}(C)$ and $\eta^+ \in Y^+$.

When we fix some $\eta^- \in X \times Y^-$ and $\eta^+ \in Y^+$, the function $F^{\mathbf{u}}$ reduces to a function F^{η^-,η^+} that maps $\mathrm{poss}(C)$ to subsets of 2^m of relative size $\geq 1/b(\mathbf{u})$. So we can use Lemma 10.2.0.2(2) and strengthen C to $D(\eta^-, \eta^+)$ decreasing the norm by at most $1/b(\mathbf{u})$ such that

$$F'(\eta^-, \eta^+) := \bigcap_{\mu \in \mathrm{poss}(D(\eta^-,\eta^+))} F^{\eta^-,\eta^+}(\mu)$$

is a set of measure $\geq 1/b(\mathbf{u}+1)$.

For fixed $\eta^+ \in Y^+$, we can iterate this strengthening for all $\eta^- \in X \times Y^-$: From D to some $\tilde{D} := D(\eta^-, \eta^+)$, then from $\tilde{D}$ to $D(\eta'^-, \eta^+)$ for the next η'^-, etc., resulting in a $D(\eta^+)$ with norm reduced by at most $\mathrm{maxposs}(<\mathbf{u})/b(\mathbf{u}) < 1$.

Note that there are less than $b(\mathbf{u}+1)$ many possibilities for $D(\eta^+)$, cf (10.2.0.3). Finally we can use bigness of the Y^+-part, as stated in Lemma 6.1.0.5, to find successor subatoms at all sublevels above $\mathbf{u}$, resulting in a new set of possibilities $\tilde{Y}^+ \subseteq Y^+$ such that for each $\eta^+ \in \tilde{Y}^+$ we get the same $D := D(\eta^+)$. This D will be the (final) subatom at our current level $\mathbf{u}$.

We can now define

$$F^{\mathbf{u}+1}(\eta) := \bigcap_{\mu \in \mathrm{poss}(D)} F^{\mathbf{u}}(\eta^-, \mu, \eta^+).$$

As above, this is a set of measure $\geq 1/b(\mathbf{u}+1)$, does not depend on the $\mathtt{cn}$-part $\leq \mathbf{u}$, and it is forced (modulo D) to be a subset of $\underset{\sim}{L}'$.

We have now chosen the new final subatom D, the new preliminary subatoms and $F^{\mathbf{u}+1}$ in a way that we can perform the next step of the iteration.

8. We perform the whole inductive construction of (7) for every $\nu \in Z$ independently (i.e., we start at the original $\mathfrak{d}^*$ for each $\nu \in Z$).

   So for every ν we get a different sequence $\bar{D}(\nu)$ of subatoms between k and k^*. Using bigness (again as in Lemma 6.1.0.5), we can thin out the subatoms between k^* and k^{**}, resulting in $Z' \subseteq Z$, such that for each $\nu \in Z'$ we get the same sequence $\bar{D}(\nu) =: \bar{D}$ which finally defines the compound creature $\mathfrak{d}^{**}$ stronger than $\mathfrak{d}^*$.

   We set q_{n+1} to be q^* with $\mathfrak{d}^*$ strengthened to $\mathfrak{d}^{**}$, and we set $i_{n+1} := m$ and $k_{n+1} := k^{**}$.

9. Now work modulo q_{n+1}. So the final function F of the induction in (7) gives us a name for a subset $\underset{\sim}{L}'' \subseteq \underset{\sim}{L} \subseteq 2^m$ of positive relative size (in 2^m), and the name $\underset{\sim}{L}''$ does not depend on any $\mathtt{cn}$ indices: Not on any below k, since we started with the name $\underset{\sim}{L}'$ which did not depend on such subatoms; not on any between k and k^*, as we removed this dependence sublevel by sublevel during the induction; and not on any $\mathtt{cn}$ subatoms between k^* and k^{**}, as $\mathtt{cn}$ indices are of lim-sup type, and we have only singleton subatoms for the lim sup part between k^* and k^{**}.

   So we can pick a non-$\mathtt{cn}$-name s_{n+1} for an arbitrary (the leftmost, say) element of $\underset{\sim}{L}$.

[5] slightly more formally: we make the current preliminary subatom final, and set $F^{\mathbf{u}+1} := F^{\mathbf{u}}$

[6] We are concerned only about the η still are compatible with the currently constructed preliminary/final subatoms.

10. q_{n+1} forces that s_{n+1} is in $\underset{\sim}{L}$, i.e., a "fat" node, more specifically: $\underset{\sim}{T}' := \underset{\sim}{T}_n^{[s_{n+1}]}$ has a measure greater than $\frac{1-\epsilon}{2^m}$.

    The tree $\underset{\sim}{T}'$ is read continuously by q_n and therefore also by q_{n+1}. In particular, for each $\ell > m$ the finite tree $\underset{\sim}{T}' \cap 2^\ell$ is decided below some ℓ'. For $\eta \in \mathrm{poss}(q_{n+1}, <\ell')$ let $T^{\ell,\eta}$ be the according value of $\underset{\sim}{T}' \cap 2^\ell$ (a subset of 2^ℓ with at least $2^\ell \cdot \frac{1-\epsilon}{2^m}$ elements). We call η and η' equivalent if they differ only on the $\mathtt{cn}$ part below k^{**}. Each equivalence class has size $\leq \mathrm{maxposs}(<k^*)$, as there are only singleton values in the lim-sup part between k^* and k^{**}. We assign to each equivalence class $[\eta]$ the tree $T^{\ell,[\eta]} := \bigcap_{\eta' \in [\eta]} T^{\ell,\eta'}$. Then $T^{\ell,[\eta]}$ has size at least $2^\ell \cdot \frac{1-\mathrm{maxposs}(<k^*)\cdot\epsilon}{2^m}$ (and of course does not depend on the $\mathtt{cn}$-part below k^{**}). So the family $T^{\ell,[\eta]}$ defines a continuous name for a tree $\underset{\sim}{T}_{n+1}$ not depending on the $\mathtt{cn}$-part below k^{**} with root s_{n+1} and measure $> 1/2^{m+1}$, as required. $\square$

10.6 Switching non

It turns out that the same proof can be used for the following variant of Theorem 6.2.0.7, where the order of $\kappa_{\mathtt{nn}}$ and $\kappa_{\mathtt{cn}}$ is reversed:

Theorem 10.6.0.14. *In Theorem 6.2.0.7, we can also use $\kappa_{\mathtt{cn}} \leq \kappa_{\mathtt{nn}}$. In more detail: Assume (in V) CH, $\kappa_{\mathtt{nm}} \leq \kappa_{\mathtt{cn}} \leq \kappa_{\mathtt{nn}} \leq \kappa_{\mathtt{sk}}$ and $\kappa_t^{\aleph_0} = \kappa_t$ for $t \in \{\mathtt{nm}, \mathtt{nn}, \mathtt{cn}, \mathtt{sk}\}$. Then there is a forcing $\mathbb{Q}$ which forces*

1. $\mathrm{cov}(\mathcal{N}) = \mathfrak{d} = \aleph_1$,
2. $\mathrm{non}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) = \kappa_{\mathtt{nm}}$,
3. $\mathrm{cof}(\mathcal{N}) = \kappa_{\mathtt{cn}}$,
4. $\mathrm{non}(\mathcal{N}) = \kappa_{\mathtt{nn}}$,
5. $2^{\aleph_0} = \kappa_{\mathtt{sk}}$.

Moreover, $\mathbb{Q}$ preserves all cardinals and all cofinalities.

Proof. We now use the $\mathtt{cn}$-norm for the $\mathtt{nm}$ part as well. (Recall 7.1.0.2: We can use any $\mathtt{nm}$-norm, as long as bigness is satisfied.) The proofs above do not change, apart the one of $\mathrm{non}(\mathcal{N}) \leq \kappa_{\mathtt{nn}}$: In the inductive construction, we only had to do something at the $\mathtt{cn}$-indices, and we could ignore the $\mathtt{nm}$-indices (as there were only few). In the new version, we have to include the $\mathtt{nm}$-indices as well. But this is no problem: We now do exactly the same at $\mathtt{nm}$-indices as at $\mathtt{cn}$-indices (which we can, as the $\mathtt{nm}$-norm is the same as the $\mathtt{cn}$-norm). $\square$

Part II

Internal forcing: $\mathsf{PFA}(S)$ and an application to topology

Chapter 11

Introduction

The main result of this part is that Balogh's Σ holds in "models of the form $\mathsf{PFA}(S)[S]$" for coherent S. In the following we briefly describe the notions used above.

Σ is the following statement:

> All locally countable subspaces of size $<\mathfrak{c}$ in countably tight compact spaces are σ-discrete.

This statement was excised by Z. Balogh in ([2]) from the work of Z. Szentmiklossy [29]. Specific instances of Σ were central to Szentmiklossy's proof under $\mathsf{MA} + \neg\mathsf{CH}$ that every hereditarily separable subspace of a countably tight space is Lindelöf. For more information about this and similar variations of Σ one can read Balogh's ([2]), in which he also demonstrates that the general statement Σ holds under $\mathsf{MA} + \neg\mathsf{CH}$.

Given a Souslin tree S, $\mathsf{PFA}(S)$ is the weakening of PFA to those proper forcings which keep S Souslin. By "models of the form $\mathsf{PFA}(S)[S]$" we mean models obtained by forcing with the Souslin tree S over a model of $\mathsf{PFA}(S)$. These models were first investigated by S. Todorcevic and P. Larson in [19], where they showed that for suitably chosen S they yield a positive solution to Katetov's problem ([12]): it is consistent that every Hausdorff space with hereditarily normal square is metrizable. Note that the usual method of obtaining any model of PFA uses a supercompact cardinal, so consistency results using models of PFA show consistency relative to a supercompact. In many cases this large cardinal is unnecessary.

It has been known since the 1970s that consistently, there are non-metrizable hereditarily normal manifolds of dimension > 1: for example CH ([27]) and $\mathsf{MA} + \neg\mathsf{CH}$ ([26]) both yield such manifolds.

Soon after the Larson-Todorcevic result, F. Tall initiated a programme to demonstrate that it is consistent that all hereditarily normal manifolds of dimension > 1 are metrizable. He saw that this would following from the follwing four statements [30]:

- Σ.
- All first-countable perfect pre-images of ω_1 include a homeomorphic copy of ω_1.
- Normal first-countable spaces are $\aleph_1$-collectionwise Hausdorff.
- Given any countable covering ideal $\mathcal{I}$ of countable subsets of an stationary subset S of ω_1, either there is a stationary $A \subseteq S$ such that $[A]^\omega \subseteq \mathcal{I}$, or there is a stationary $B \subseteq S$ such that $B \cap I$ is finite for all $I \in \mathcal{I}$.

The programme was then to demonstrate that these four statements each hold in models of the form $\mathsf{PFA}(S)[S]$ for suitable S.

So our contribution to the programme is the demonstration of Σ.

Chapter 12

PFA(S)

12.1 ω_1 -trees and coherent Souslin trees

We will be taking ω_1-trees to be downward-closed normal subtrees of ${}^{<\omega_1}\omega$, with the inherited ordering. In particular, all Souslin trees will be of this type. Note that if T is an ω_1-tree, and $t \in T$, then the *height* of t in T, and the *length* of t as a sequence of natural numbers agree. We denote this common value by $\mathrm{ht}(t)$. We follow common practice and denote for each countable ordinal δ the family of all nodes of T of height δ by $T(\delta)$. We will use non-standardised notation and define for each node $s \in T$ the *cone of* T *above* s by

$$T^{(s)} = \{t \in T : s \leq_T t\}.$$

Given some $C \subseteq \omega_1$, we denote by $T \,|\, C$ the family of all nodes $t \in T$ with $\mathrm{ht}(t) \in C$. Also, if $X \subseteq T$ consists entirely of nodes of height at least $\delta \in \omega_1$, we define the *projection of* X *onto the* δ^{th} *level of* T by

$$X{\downarrow}\delta = \{s \in T(\delta) : (\exists t \in X)(s \leq_T t)\} = \{t \,|\, \delta : t \in X\}.$$

We will also make use of interval notation on trees, so that given $s, t \in T$ with $s <_T t$ the notations $[s,t]$, $[s,t)$, etc., have their obvious intended meanings.

We follow [19] and call a Souslin tree $S \subseteq {}^{<\omega_1}\omega$ *coherent* if it is generated by a coherent family of functions. Recall that a family of functions $\{f_\alpha\}_{\alpha<\omega_1}$, where $f_\alpha : \alpha \to \omega$ for each $\alpha < \omega_1$, is called *coherent* if given any two $\alpha < \beta < \omega_1$ the family

$$f_\alpha \triangle f_\beta = \{\xi \in \alpha = \mathrm{dom}(f_\alpha) \cap \mathrm{dom}(f_\beta) : f_\alpha(\xi) \neq f_\beta(\xi)\}$$

is finite. The tree generated by such a family is given by

$$T = \bigcup_{\alpha<\omega_1} \{g \in {}^\alpha\omega : \{\delta \in \alpha : g(\delta) \neq f_\alpha(\delta)\} \text{ is finite}\}.$$

The existence of coherent Souslin trees follows from $\diamondsuit$ (see [18]).

The strength of coherent Souslin trees comes from fact that they are *strongly homogeneous*, as defined in [15] and [18]. In particular, if S is a coherent Souslin tree, then given any $\delta < \omega_1$ and any two $s, t \in S(\delta)$, there is a *canonical isomorphism* $\sigma_{s,t} : S^{(s)} \to S^{(t)}$. Furthermore, the family of these canonical isomorphisms enjoys several nice coherence properties, although these are irrelevant to the discussion at hand. An important consequence of this fact is that all generic branches through a coherent Souslin tree are "isomorphic," leading to isomorphic generic extensions. When dealing with these isomorphisms, we will generally follow bad form and think of them as automorphisms of the entire Souslin tree S. This will allow us to keep notational obfuscation down, especially when considering forcing statements. In particular, given two nodes s, t on the same level of a coherent Souslin tree S and an S-name τ, by $\sigma_{s,t}(\tau)$ we will mean the $S^{(t)}$-name $\sigma_{s,t}(\tau^{(s)})$, where $\tau^{(s)}$ is a canonically chosen $S^{(s)}$-name satisfying $s \Vdash_S \tau = \tau^{(s)}$. More dramatically, given any forcing statement $\phi(\tau_1, \dots, \tau_n)$, and a node $t' \geq_S t$, the statement

$$t' \Vdash_S \phi(\sigma_{s,t}(\tau_1), \dots, \sigma_{s,t}(\tau_n))$$

will mean

$$t' \Vdash_{S^{(t)}} \phi(\sigma_{s,t}(\tau_1^{(s)}), \dots, \sigma_{s,t}(\tau_n^{(s)})).$$

where the $S^{(s)}$-names $\tau_1^{(s)}, \dots, \tau_n^{(s)}$ are chosen as above.

12.2 PFA(S)

Given a (coherent) Souslin tree S, by *PFA(S)* we mean the restriction of the Proper Forcing Axiom to those partial orders which do not destroy (the Souslinity of) S. To be precise, PFA(S) is the following statement: Given any proper partial order $\mathcal{P}$ such that

$$\Vdash_{\mathcal{P}} \text{“}\check{S} \text{ is a Souslin tree”}$$

and any family $\{D_\alpha\}_{\alpha<\omega_1}$ of dense-open subsets of $\mathcal{P}$, there is a filter $G \subseteq \mathcal{P}$ meeting each D_α. This concept has been used in [33], and others, while a form of it, which may be phrased $MA_{\omega_1}(S)$ was used to achieve the result in [2].

Our interest does not lie in models of PFA(S) themselves, but rather their generic extensions by the Souslin tree S. This stems from the fact that certain consequences of $V = L$ hold after forcing with a Souslin tree, while at the same time certain consequences of PFA can be shown to be forced by S over a model of PFA(S).

The method of showing that a PFA consequence holds in such a generic extension then comes down to using PFA(S) to construct an S-name for the desired object. In cases that have been investigated so far, one takes a proper partial order $\mathcal{P}$ using Todorcevic's method of "using elementary submodels as side conditions" (see [31] and [16]) that gives the desired object in models of PFA, and then alters them to produce an S-name instead. We are then left to show that this altered partial order has two properties: it is proper and it preserves S. To do so we apply the following lemma.

Lemma 1 (Miyamoto, [23]). *A partial order $\mathcal{P}$ is proper and preserves a Souslin tree S iff the following condition is satisfied: Given an elementary submodel M of some H_θ (for θ sufficiently large, regular) containing $S, \mathcal{P}$, every condition $p \in \mathcal{P} \cap M$ has an extension $\bar{p}$ such that the* condition-pair *$\langle \bar{s}, \bar{p}\rangle$ is $(M,\, S \times \mathcal{P})$-generic for every $\bar{s} \in S(\delta)$, where $\delta = \omega_1 \cap M$.*

We shall call the condition mentioned in the above lemma *Miyamoto's condition.*

Chapter 13

An Example – The P-Ideal Dichotomy

By an *ideal of countable subsets* of a set X we mean a nonempty $\mathcal{I} \subseteq [X]^{\leq\omega}$ which is closed under finite unions and subsets. In particular, $[X]^{<\omega}$ is a subset of every ideal of countable subsets of X. We call an ideal $\mathcal{I}$ of countable subsets of X a *P-ideal* if, in addition, given any countable $\{I_n : n \in \omega\} \subseteq \mathcal{I}$ there is a $J \in \mathcal{I}$ such that $I_n \subseteq^* J$ for all n.

Given an ideal $\mathcal{I}$ of countable subsets of a set X, by $\mathcal{I}^\perp$ we mean the famaily of all countable $A \subseteq X$ with the property that $A \cap I$ is finite for all $I \in \mathcal{I}$. $B \subseteq X$ is called $\mathcal{I}^\perp$*-homogeneous* if $[B]^{\leq\omega} \subseteq \mathcal{I}^\perp$; all countable subsets of B have finite intersection with all sets in $\mathcal{I}$.

In [1] S. Todorcevic and U. Abraham developed the following statement, called the *P-Ideal Dichotomy.* Given any P-ideal $\mathcal{I}$ of countable subsets of ω_1, one of the following conditions holds:

1. There is an uncountable $A \subseteq \omega_1$ such that $[A]^\omega \subseteq \mathcal{I}$; or

2. ω_1 can be decomposed as $\omega_1 = \bigcup_{i\in\omega} A_i$ where each A_i is $\mathcal{I}^\perp$-homogeneous.

13.1 PFA implies the P-Ideal Dichotomy

In this short section we will outline the proof that PFA implies the P-Ideal Dichotomy. This proof in based largely on that in [34], and not the proof in the Todorcevic–Abraham paper [1].

Theorem 1 (Todorcevic–Abraham)**.** *PFA implies the P-Ideal Dichotomy.*

Proof. Let $\mathcal{I}$ be a P-ideal on ω_1 for which condition (2) in the statement of the P-Ideal Dichotomy fails. For each countable elementary submodel $N \prec H(\aleph_2)$ fix some $J_N \in \mathcal{I}$ such that $I \subseteq^* J_N$ for each $I \in \mathcal{I} \cap N$, and also let ξ_N denote the least countable ordinal not belonging to any $\mathcal{I}^\perp$-homogeneous set belonging to N. We define $\mathbb{P}$ to be the family of all pairs $p = \langle \mathcal{N}_p, \mathcal{M}_p \rangle$ where

1. $\mathcal{M}_p$ is a finite $\in$-chain of countable elementary submodels of $H(\aleph_2)$ containing $\mathcal{I}$ and the mapping $N \mapsto \langle J_N, \xi_N \rangle$;

2. $\mathcal{N}_p \subseteq \mathcal{M}_p$ (the models in $\mathcal{N}_p$ are called *active* in p).

We order $\mathbb{P}$ by declaring $q \leq p$ iff

1. $\mathcal{M}_q \supseteq \mathcal{M}_p$;

2. $\mathcal{N}_q \cap \mathcal{M}_p = \mathcal{N}_p$; and

3. Given $N \in \mathcal{N}_q \setminus \mathcal{N}_p$ and $N' \in \mathcal{N}_p$, if $N \in N'$, then $\xi_N \in J_{N'}$.

To prove that $\mathbb{P}$ is proper, let $\theta \gg \aleph_2$ be a sufficiently large regular cardinal, and let M be a countable elementary submodel of $H(\theta)$ containing $\mathbb{P}$, $\mathcal{I}$, and the mapping $N \mapsto \langle J_N, \xi_N \rangle$. Given $p \in \mathbb{P} \cap M$ define $\bar{p} = \langle \mathcal{N}_p, \mathcal{M}_p \cup \{M \cap H(\aleph_2)\}\rangle$. It is clear that $\bar{p}$ is a condition of $\mathcal{P}$ extending p. We now show that $\bar{p}$ is $(M, \mathcal{P})$-generic.

Let $D \in M$ be a dense open subset of $\mathbb{P}$ and let q be an extension $\bar{p}$ belonging to D.

Consider $q_M = \langle \mathcal{N}_q \cap M, \mathcal{M}_q \cap M \rangle$. Note that $q_M \in M$ and is an "initial part" of q in the sense that $\mathcal{M}_{q_M} \subseteq \mathcal{M}_q$ and $N \in N'$ for all $N \in \mathcal{M}_{q_M}$ and $N' \in \mathcal{M}_q \setminus \mathcal{M}_{q_M}$. Enumerate $\mathcal{N}_q \setminus M = \mathcal{N}_q \setminus \mathcal{N}_{q_M}$ increasingly according to $\in$ as $N_1 \in \cdots \in N_n$. (Note that $M \cap H(\aleph_2) \in N_1$, also.)

Define F to be the family of all n-tuples $\langle \zeta_1, \ldots, \zeta_n \rangle$ in ω_1 such that there is a condition $r \in \mathcal{P}$ satisfying the following conditions:

- $r \in D$;
- r has q_M as an initial part (in the sense defined above);
- $\mathcal{N}_r \setminus \mathcal{N}_{q_M}$ has size n, and $\langle \xi_{N_1^r}, \ldots, \xi_{N_n^r} \rangle = \langle \zeta_1, \ldots, \zeta_n \rangle$ where $N_1^r \in \cdots \in N_n^r$ is the increasing enumeration of $\mathcal{N}_r \setminus \mathcal{N}_{q_M}$.

Clearly $F \in M$, and contains the n-tuple $\langle \xi_1, \ldots, \xi_n \rangle$. Viewing F as the set of maximal nodes of a subtree $T \subseteq \omega_1^{\leq n}$, we may use elementarity arguments to thin T to a subtree T', also in M and containing $\langle \xi_1^q, \ldots, \xi_n^q \rangle$, such that whenever $\langle \zeta_1, \ldots, \zeta_k \rangle$ $(k < n)$ belongs to T' then the set

$$T'(\zeta_1, \ldots, \zeta_k) = \{\zeta : \langle \zeta_1, \ldots, \zeta_k, \zeta \rangle \in T'\}$$

is not $\mathcal{I}^{\perp}$-homogeneous.

We now inductively pick an n-tuple $\langle \eta_1, \ldots, \eta_n \rangle \in T' \cap M$ so that $\eta_i \in J_{N_1^q} \cap \cdots \cap J_{N_n^q}$ for each $i \leq n$. If $\eta_1, \ldots \eta_k$ $(k < n)$ have been appropriately chosen, then as $T'(\eta_1, \ldots \eta_k) \in M$ and is not $\mathcal{I}^{\perp}$-homogeneous, there is an $A \in [T'(\eta_1, \ldots \eta_k)]^{\omega} \cap M$ which belongs to $\mathcal{I}$. By our choice of the J_Ns, it follows that $A \subseteq^* J_{N_j^q}$ for each $j \leq n$. When may then pick $\eta_{k+1} \in A \cap \bigcap_{j \leq n} J_{N_j^q}$.

Then as $\langle \eta_1, \ldots, \eta_n \rangle \in T' \cap M$ by elementarity there is an $r \in \mathcal{P} \cap M$ witnessing that $\langle \zeta_1, \ldots, \zeta_n \rangle$ belongs to F. In particular this means that $r \in D$, and it is straightforward to verify that r and q are compatible (with common extension $\langle \mathcal{N}_r \cup \mathcal{N}_q, \mathcal{M}_r \cup \mathcal{M}_q \rangle$).

It is easy to show that the sets $D_\alpha = \{p \in \mathcal{P} : \xi_N \geq \alpha \text{ for some } N \in \mathcal{N}_p\}$ are dense, and given a $\{D_\alpha\}_{\alpha \in \omega_1}$-generic filter G for $\mathcal{P}$, it follows that $Z_G = \bigcup_{p \in G} Z_p$ is an uncountable set, and $[Z_G]^{\omega} \subseteq \mathcal{I}$, as desired. □

13.2 The P-Ideal Dichotomy holds in models of the form PFA(S)[S]

We now move to proving that the P-Ideal Dichotomy holds in "models of the form PFA(S)[S]." This result was first shown by S. Todorcevic in [33], though the present proof differs from this one. The basic idea of this proof is to alter the above partial order in order to produce an S-name for the desired object: in this case an uncountable subset of ω_1 every countable subset of which is forced to belong to $\dot{\mathcal{I}}$, a fixed S-name for a P-ideal of countable subsets of ω_1.

Theorem 2 (PFA(S)). *The coherent Souslin tree S forces that the P-Ideal Dichotomy holds.*

Proof. Fix an S-name $\dot{\mathcal{I}}$ for a P-idea of countable subsets of ω_1, and assume that S forces that ω_1 is not the countable union of $\dot{\mathcal{I}}^{\perp}$-homogeneous sets. For each countable elementary submodel N of $H(\aleph_2)$ containing S and $\dot{\mathcal{I}}$ fix the following objects:

- an S-name $\dot{J}_N$ such that $\Vdash_S$ "$\dot{J}_N \in \dot{\mathcal{I}}$ and $I \subseteq^* \dot{J}_N$ for all $I \in \dot{\mathcal{I}} \cap N[\dot{B}]$" where $\dot{B}$ denotes the canonical S-name for the generic branch through S;

- an S-name $\dot{\xi}_N$ for the least countable ordinal not belonging to any $\dot{\mathcal{I}}^\perp$-homogeneous set in $N[\dot{B}]$; and

- a countable ordinal ρ_N such that every node in S_{ρ_N} decides $\dot{J}_N$ and the countable ordinal $\dot{\xi}_N$.

We will be using $\mathsf{PFA}(S)$ to construct an S-name for an uncountable subset $\dot{Z}$ of ω_1 in the following manner. The conditions of the forcing will consist of a finte $\in$-chain of countable elementary submodels of $H(\aleph_2)$, and to some of these models N (the *active* models) a node x_N of S of height ρ_N. Our interpretation will be that $x_N \Vdash_S \dot{\xi}_N \in \dot{Z}$. We will ensure that if $q \leq p$ and N is an active model of q which is not among the models in p and N' is an active model of p and additionally $x_N \leq_S x_{N'}$, then $x_{N'} \Vdash_S$ "$\dot{\eta}_N \in \dot{J}_{N'}$". This will help ensure that that the name so constructed is forced to have the property that all its countable subsets belong to $\dot{\mathcal{I}}$.

We then define $\mathcal{P}$ to be the family of all pairs $p = \langle f_p, \mathcal{M}_p \rangle$ where

1. $\mathcal{M}_p$ is a finite $\in$-chain of countable elemntary submodels of $H(\aleph_2)$ containing S, $\dot{\mathcal{I}}$ and the mapping $N \mapsto \langle \dot{J}_N, \dot{\xi}_N, \rho_N \rangle$;

2. f_p is a function with domain $\mathcal{N}_p \subseteq \mathcal{M}_p$ such that $f_p(N) = x^p_N$ is a node of height ρ_N in S for each $N \in \mathcal{N}_p$.

We order $\mathcal{P}$ by declaring $q \leq p$ iff

1. $\mathcal{M}_q \supseteq \mathcal{M}_p$;

2. $f_q \supseteq f_p$ and $\mathcal{N}_q \cap \mathcal{M}_p = \mathcal{N}_p$;

3. given $N \in \mathcal{N}_q \setminus \mathcal{N}_p$ and $N' \in \mathcal{N}_p$, if $x^q_N \leq_S x^p_{N'}$ then $x^p_{N'} \Vdash$ "$\dot{\xi}_N \in \dot{J}_{N'}$".

We now show that $\mathcal{P}$ is proper and preserves S. Given a sufficiently large regular cardinal $\theta \gg \aleph_2$, let M be a countable elementary submodel of $H(\theta)$ containing S, $\dot{\mathcal{I}}$, the mapping $N \mapsto \langle \dot{J}_N, \dot{\xi}_N, \rho_N \rangle$ and $\mathcal{P}$. For $p \in \mathcal{P} \in M$, set

$$\bar{p} = \langle f_p, \mathcal{M}_p \cup \{M \cap H(\aleph_2)\} \rangle.$$

Then $\bar{p}$ is a condition of $\mathcal{P}$ extending p, and we now show that it it satisfies Miyamoto's condition.

Setting $\delta = \omega_1 \cap M$, fix $\bar{s} \in S_\delta$. We will show that the condition-pair $\langle \bar{p}, \bar{s} \rangle$ is $(M, \mathcal{P} \times S)$-generic.

Fix $D \in M$ a dense-open subset of $\mathcal{P} \times S$, and let $\langle q, t \rangle$ be an extension of $\langle \bar{p}, \bar{s} \rangle$. Without loss of generality we may assume that $\langle q, t \rangle$ satisfies the following:

- $\langle q, t \rangle \in D$;

- the maximum model N_q of $\mathcal{M}_q$ is inactive;

- $t \notin N_q$;

- the maximum model N_M of $\mathcal{M}_q \cap M$ is inactive;

- setting $\delta_M = \omega_1 \cap N_M$, the projections of the set $\{x^q_N : N \in \mathcal{N}_q \setminus M\} \cup \{t\}$ onto the δ_M-th and δ-th levels of S have the same size and, furthermore, $x \,|[\delta_M, \delta) = y \,|[\delta_M, \delta)$ for all $x, y \in \{x^q_N : N \in \mathcal{N}_q \setminus M\} \cup \{t\}$.

We may enumerate the projections of $\{x^q_N : N \in \mathcal{N}_q \setminus M\}$ onto the δ_M-th and δ-th levels of S, respectively, as $\{v_1, \dots, v_m\}$, $\{w_1, \dots, w_m\}$ so that $v_j \leq_S w_j$ for all $j \leq m$. For each $j \leq m$, let σ_j denote the canonical isomorphism $S^{(t_M)} \to S^{(v_i)}$ where $t_M = t \mid \delta_M = \bar{s} \mid \delta_M$.

We must find a condition-pair $\langle r, u\rangle \in D \cap M$ with the property that $u \leq_S t$ and r, q are compatible. For the latter it will be necessary to ensure that if $N \in \mathcal{N}_r \setminus \mathcal{N}_q$ and $N' \in \mathcal{N}_q$ are such that $x^r_N \leq_S x^q_{N'}$ then $x^q_{N'} \Vdash$ "$\dot{\xi}_N \in \dot{J}_{N'}$". As $x^r_N \in M$, in order for $x^r_N \leq_S x^q_{N'}$ to hold it must be that $\sigma_j(t_M) \leq_S x^r_N \leq \sigma_j(\bar{s})$ where $j \leq m$ is such that $u_j \leq_S v_j \leq x^q_{N'}$. Much of our work will be to gain some control over when $\sigma_j(t_M) \leq_S x^r_N \leq \sigma_j(\bar{s})$ can happen.

Let $N_1 \in \cdots \in N_n$ be the $\in$-increasing enumeration of those $N \in \mathcal{N}_q$ with the property that there is a $j \leq M$ such that $u_j \leq_S v_j \leq_S x^q_N \leq_S \sigma_j(t)$. For each $i \leq n$ let j_i denote the unique $j \leq m$ such that $u_j \leq_S x^q_{N_i}$.

Setting $q_M = \langle f_q \mid M, \mathcal{M}_q \cap M\rangle$, we have that $q_M \in M$ is an "initial part" of q in the sense that $N \in N'$ for all $N \in \mathcal{M}_{q_M}$ and $N' \in \mathcal{N}_q \setminus \mathcal{N}_{q_M}$.

Define D' to be the family of all condition-pairs $\langle r, u\rangle \in D$ satisfying the following:

- r has q_M as an initial part;
- $t_M \leq_S u$;
- the maximum model of $\mathcal{M}_q$ is inactive and does not contain u;
- there are exactly n models $N \in \mathcal{N}_r \setminus \mathcal{N}_{q_M}$ with the property that $u_j \leq_S x^r_N \leq_S \sigma_j(u)$ for some $j \leq m$, and by enumerating these increasingly according to $\in$ as $N_1^{\langle r,u\rangle} \in \cdots \in N_n^{\langle r,u\rangle}$ we have that $u_{j_i} \leq x^r_{N_i^{\langle r,u\rangle}}$ for each $i \leq n$.

To somewhat ease the indexing, given $\langle r, u\rangle \in D'$ we will denote $x^r_{N_i^{\langle r,u\rangle}}$, $\rho_{N_i^{\langle r,u\rangle}}$, $\dot{\xi}_{N_i^{\langle r,u\rangle}}$ and $\dot{J}_{N_i^{\langle r,u\rangle}}$ by $x_i^{\langle r,u\rangle}$, $\rho_i^{\langle r,u\rangle}$, $\dot{\xi}_i^{\langle r,u\rangle}$ and $\dot{J}_i^{\langle r,u\rangle}$, respectively. As $x_i^{\langle r,u\rangle}$ decides the values of $\dot{\xi}_i^{\langle r,u\rangle}$ and $\dot{J}_i^{\langle r,u\rangle}$, we will remove the dots from the latter two to denote $x_i^{\langle r,u\rangle}$'s decision of these objects.

Note that $D' \in M$ and contains $\langle q, t\rangle$.

Our goal is now to find a $\langle r, u\rangle \in D' \cap M$ with the property that $u \leq t$. Given such a condition-pair, if $N \in \mathcal{N}_r \setminus \mathcal{N}_q$ is such that $u_j \leq x^r_N \leq \sigma_j(v_j)$ for some j, then as $u \leq_S v_j \leq_S t$ it must be that $x^r_N \leq \sigma_j(u)$, and so $N = N_i^{\langle r,u\rangle}$ for some $i \leq n$ with $j_i = j$. In this way we will gain control over when $\sigma_j(t_M) \leq_S x^r_N \leq_S \sigma_j(\bar{s})$ occurs.

Let F denote the family of all $(n+1)$-tuples $\langle\langle x_1, \xi_1\rangle, \dots, \langle x_n, \xi_n\rangle, u\rangle$ for which there is a $\langle r, u\rangle \in D'$ such that $x_i^{\langle r,u\rangle} = x_1$ and $\xi_1 = \xi_i^{\langle r,u\rangle}$ for each $i \leq n$. Note that F belongs to $M \cap H(\aleph_2)$.

For each initial segment $\langle\langle x_1, \xi_1\rangle, \dots, \langle x_n, \xi_n\rangle\rangle$ of an element of F, let $F(\langle x_1, \xi_1\rangle, \dots, \langle x_n, \xi_n\rangle)$ be the family of all $u \in S^{(t_M)}$ such that $\langle\langle x_1, \xi_1\rangle, \dots, \langle x_n, \xi_n\rangle, u\rangle \in F$. In particular, $F(\langle x_1^{\langle q,t\rangle}, \xi_1^{\langle q,t\rangle}\rangle, \dots, \langle x_n^{\langle q,t\rangle}, \xi_n^{\langle q,t\rangle}\rangle)$ belongs to N_M and contains t, so there must be a $z_{n+1} \in N_M$, with $x_n^{\langle q,t\rangle} \leq_S \sigma_{j_n}(z_{n+1})$ and $z_{n+1} \leq_S t$ such that $F(\langle x_1^{\langle q,t\rangle}, \xi_1^{\langle q,t\rangle}\rangle, \dots, \langle x_n^{\langle q,t\rangle}, \xi_n^{\langle q,t\rangle}\rangle)$ is dense above z_{n+1}.

For each initial segment $\langle\langle x_1, \xi_1\rangle, \dots, \langle x_{n-1}, \xi_{n-1}\rangle\rangle$ of an element of F, let $F(\langle x_1, \xi_1\rangle, \dots, \langle x_{n-1}, \xi_{n-1}\rangle)$ denote the family of all triples $\langle z, y, \zeta\rangle$ such that $y <_S \sigma_{j_n}(z)$ and $F(\langle x_1, \xi_1\rangle, \dots, \langle x_{n-1}, \xi_{n-1}\rangle, \langle y, \zeta\rangle)$ is dense above z. In particular, note that $F(\langle x_1^{\langle q,t\rangle}, \xi_1^{\langle q,t\rangle}\rangle, \dots, \langle x_{n-1}^{\langle q,t\rangle}, \xi_{n-1}^{\langle q,t\rangle}\rangle \in N_n$ and contains the triple $\langle z_{n+1}, x_n^{\langle q,t\rangle}, \zeta_n^{\langle q,t\rangle}\rangle$. It follows that there is a $z_n \in S \cap N_n$ with $x_{n-1}^{\langle q,t\rangle} \leq_S \sigma_{j_{n-1}}(z_n)$ and $\sigma_{j_n}(z_n) \leq x_n^{\langle q,t\rangle}$ such that the family of all second-coordinates of elements of $F(\langle x_1^{\langle q,t\rangle}, \xi_1^{\langle q,t\rangle}\rangle, \dots, \langle x_{n-1}^{\langle q,t\rangle}, \xi_{n-1}^{\langle q,t\rangle}\rangle)$ is dense above $\sigma_{j_n}(z_n)$.

Continuing in this fashion, we eventually define $F()$ to be the family of all triples $\langle z, y, \zeta\rangle$ such that $t_M \leq_S z$ and $\sigma_{j_1}(z) \leq_S y$ and the family of all second coordinates of elements of $F(\langle y, \zeta\rangle)$ is dense above $\sigma_{j_1}(z)$. Note that $F() \in M$ and contains the triple $\langle z_2, z_1^{\langle q,t\rangle}, \zeta_1^{\langle q,t\rangle}\rangle$, from which we may conclude that there is a $z_1 \in S \cap M$ with $t_M \leq_S z_1$ and $\sigma_{j_1}(z_1) \leq_S x_1^{\langle q,t \rangle}$ such that the family of all second coordinates of elements of $F()$ is dense above $\sigma_{j_1}(z_1)$.

Let $\dot{X}_1$ be an S-name (in M) for the family of all $\zeta < \omega_1$ for which there are z, y such that $\langle z, y, \zeta \rangle \in F()$ and $\sigma_{j_1}(z) \in \dot{B}$, where $\dot{B}$ denotes the canonical S-name for the generic branch through S. It follows that $\sigma_{j_1}(z_1) \Vdash_S$ "$\dot{X}_1$ is not $\dot{\mathcal{I}}^{\perp}$-homoegenous", and so by extending z_1 to some z_1' (still in M and below t) we can find an countably infinite $A \subseteq \omega_1$ in M such that $\sigma_{j_1}(z_1') \Vdash_S$ "$A \subseteq \dot{X}_1, A \in \dot{\mathcal{I}}$". Given any $N \in \mathcal{N}_q \setminus \mathcal{N}_{q_M}$ with $v_{j_1} \leq_S x_N^q$ it follows that $A \subseteq^* J_N$ where J_N denotes x_N^q's decision of $\dot{J}_N$. It follows that there must be a $\zeta_1 \in A$ such that $x_N^q \Vdash$ "$\zeta_1 \in \dot{J}_N$" for all such N. Then there must be a triple $\langle w_2, y_1, \zeta_1 \rangle \in F() \cap M$ such that $\sigma_{j_1}(z_1') \Vdash$ "$\sigma_{j_1}(w_2) \in \dot{B}$", which means that $w_2 \leq_S z_1' \leq t$.

Letting $\dot{X}_2$ be an S-name (in M) for the family of all $\zeta < \omega_1$ for which there are z, y such that $\langle z, y, \zeta \rangle \in F(\langle y_1, \zeta_1 \rangle)$ and $\sigma_{j_2}(z) \in \dot{B}$, we may follow a similar argument to the above to find a $\beta_2 \in \omega_1 \cap M$ such that $\sigma_{j_2}(w_2') \Vdash$ "$\beta_2 \in \dot{X}_2$" for a suitable extension w_2' of w_2 in M and below t, and such that $x_N^q \Vdash$ "$\beta \in \dot{J}_N$" for all $N \in \mathcal{N}_q \setminus \mathcal{N}_{q_M}$ with $v_{j_2} \leq x_N^q$. Again, there is a triple $\langle w_3, y_2, \zeta_2 \rangle \in F(\langle y_1, \zeta_1 \rangle) \cap M$ such that $w_3 \leq w_2' \leq t$.

Continuing this way we eventually find triples $\langle w_2, y_1, \zeta_1 \rangle, \ldots, \langle w_{n+1}, y_n, \zeta_1 \rangle$, such that for each $i \leq N$ we have that $x_N^q \Vdash_S$ "$\zeta_i \in \dot{J}_N$" for each $N \in \mathcal{N}_q \setminus \mathcal{N}_{q_M}$ with $v_{j_i} \leq_S x_N^q$.

We also have that $F(\langle y_1, \zeta_1 \rangle, \ldots, \langle y_n, \zeta_n \rangle)$ is dense-open above w_n, and so there must be a $u \in F(\langle y_1, \zeta_1 \rangle, \ldots, \langle y_n, \zeta_n \rangle) \cap M$ with $u \leq_S t$.

Then $\langle \langle y_1, \zeta_1 \rangle, \ldots, \langle y_n, \zeta_n \rangle, u \rangle \in D' \cap M$ and so there is a condition pair $\langle r, u \rangle \in D' \cap M$ witnessing this, and it follows that r, q are compatible conditions of $\mathcal{P}$, and clearly $u \leq_S t$. $\square$

Chapter 14

Balogh's Σ in models of the form PFA$(S)[S]$

In [2], Balogh extracted the following result, which he denoted Σ, from Szentmiklóssy's famous [29].

Theorem 3 (MA + $\neg$CH)**.** *Every locally countable subspace of cardinality $<\mathfrak{c}$ in a countably tight compact space is σ-discrete.*

The purpose of this chapter is to demonstrate that this consequence holds in models of the form $\mathsf{PFA}(S)[S]$ for suitable S.

14.1 Combinatorial results

This section will present some of the necessary preliminary results required in the proof of the main theorem.

Lemma 2. *Suppose that S is a Souslin tree contained in a countable elementary submodel M of some sufficiently large H_λ. Suppose further that $F \in M$ is a subset of S containing some node $t \in S \setminus t$. Then there is some $s \leq t$ in M such that F is dense above s.*

Proof. Take A to be the family of all minimal nodes $u \in T$ such that $F \cap S^{(u)} = \emptyset$. Clearly A is an antichain of S, and therefore countable, but also $A \in M$. Therefore it follows that $A \subseteq M$. Note also that since $t \in F$ it follows that $A \cap \{s \in S : s \leq t\} = \emptyset$.

Letting $\delta = \sup\{\mathrm{ht}(u) : u \in A\} + 1$, set $s = t \,|\, \delta$. As $A \subseteq M$ it follows that $\delta < \omega_1 \cap M$, and therefore $s \in M$. If F is not dense above s, then there must be some minimal extension u of s such that $S^{(u)} \cap F = \emptyset$, but then by definition it would follow that $u \in A$, contradicting our choice of δ, and hence s. $\square$

We will also be interested in finite chains of nodes in a S. We will be taking finite chains to be finite sequences $\langle t_1, \ldots, t_n\rangle$ of nodes of S where $t_1 \leq_S \cdots \leq_S t_n$. In particular, we will say that a family $\mathcal{F}$ of n-chains in S is *dense-above* some $s \in S$ if the family $\{t_1 : \langle t_1, \ldots, t_n\rangle \in \mathcal{F}\}$ is dense-above s.

Corollary 1. *Let S be a Souslin tree contained in a countable elementary submodel M of some sufficiently large H_λ. Suppose further that $\mathcal{F} \in M$ is a family of n-chains in S, containing some $\{t_1, \ldots, t_n\}$ where each t_i is not in M. Then there is some $s \leq t_1$ in M such that $\mathcal{F}$ is dense above s.*

Let $\mathcal{U}$ be a non-principal ultrafilter on ω. The following definitions come from [32]. We recursively define an ultrafilter $\mathcal{U}_\alpha$ on ω for each $\alpha < \omega_1$ by

- $\mathcal{U}_0 = \mathcal{U}_0$,
- $\mathcal{U}_{\alpha+1} = \{A \subseteq \omega : (\mathcal{U}i)(\mathcal{U}_\alpha j)(2^i(2j+1) \in A)\}$,
- $\mathcal{U}_\lambda = \{A \subseteq \omega : (\mathcal{U}i)(\mathcal{U}_{\lambda_i} j)(2^i(2j+1) \in A)\}$ where $\{\lambda_i\}_{i\in\omega}$ is a cofinal sequence in λ.

Given a subset A of a topological space X, we recursively define the *α-sequential closure* $A^{(\alpha)}$ of A in X by

- $A^{(0)} = A$,
- $A^{(\alpha+1)} = \{x \in X : (\exists \{x_i\}_{i\in\omega} \subseteq A^{(\alpha)})(x = \lim_{i\to\infty} x_i)\}$,
- $A^{(\lambda)} = \bigcup_{\alpha<\lambda} A^{(\alpha)}$.

The set $A^{(\omega_1)}$ is called the *sequential closure* of A in X.

Lemma 3. *Let A be a subset of a topological space X, and let $\alpha < \omega_1$. The a point $x \in X$ belongs to $A^{(\alpha)}$ iff there is a sequence $\{x_i\}_{i\in\omega}$ of points of A such that $x = \lim_{i\to\mathcal{U}_\alpha} x_i$.*

Lemma 4. *Let $\mathcal{X}$ be a family on non-decreasing n-sequences of countable ordinals such that the family of all first-coordinates of elements of $\mathcal{X}$ is uncountable. Then there is an uncountable $\mathcal{Y} \subseteq \mathcal{X}$ such that for any $\beta < \omega_1$ and any $i \leq n$ the family*

$$\{\boldsymbol{\alpha} = \langle \alpha_1, \ldots, \alpha_n \rangle \in \mathcal{Y} : \alpha_i = \beta\}$$

is countable.

Proof. By induction on n. The base case is trivial, and so we assume that the result holds for families of non-decreasing n-sequences of countable ordinals, and let $\mathcal{X}$ be a family of non-decreasing $(n+1)$-sequences of countable ordinals such that the family of all of its first-coordinates is uncountable. Letting

$$\mathcal{X}' = \{\langle \alpha_1, \ldots, \alpha_n \rangle : (\exists \alpha)(\langle \alpha_1, \ldots, \alpha_n, \alpha \rangle \in \mathcal{X}\},$$

it is clear that the family of all first-coordinates of elements of $\mathcal{X}'$ is uncountable. Then by the induction hypothesis there is an uncountable $\mathcal{Y}' \subseteq \mathcal{X}'$ such that for each $\beta < \omega_1$ and each $i \leq n$ the family

$$\{\boldsymbol{\alpha} = \langle \alpha_1, \ldots, \alpha_n \rangle \in \mathcal{Y} : \alpha_i = \beta\}$$

is countable. For each $\boldsymbol{\beta} = \langle \beta_1, \ldots, \beta_n \rangle \in \mathcal{Y}'$, choose a countable

$$\mathcal{Y}(\boldsymbol{\beta}) \subseteq \{\langle \alpha_1, \ldots, \alpha_n, \alpha_{n+1} \rangle \in \mathcal{X} : \alpha_i = \beta_i \text{ for each } i \leq n\}$$

and let $\mathcal{Y} = \bigcup\{\mathcal{Y}(\boldsymbol{\beta}) : \boldsymbol{\beta} \in \mathcal{Y}'\}$. It is easy to show that $\mathcal{Y}$ is as required. $\square$

14.2 The proof

Suppose that S is a coherent Souslin tree, and that $\dot{\mathcal{O}}$ is an S-name for a compact sequential topology on some cardinal κ. For notational ease, we will denote by $\dot{K}$ an S-name for the space $\langle \kappa, \dot{\mathcal{O}} \rangle$. Assume that S forces $\{\alpha : \alpha < \omega_1\}$ to be a locally-countable subspace of $\dot{K}$. Furthermore we will fix a non-principal ultrafilter $\mathcal{U}$ in the ground model.

Theorem 4 (PFA(S)). *The coherent Souslin tree S forces that ω_1 is a σ-discrete subspace of $\dot{K}$.*

Proof. For each $\alpha < \omega_1$ fix S-names $\dot{U}_\alpha, \dot{V}_\alpha$ such that for each $\alpha < \omega_1$ the Souslin tree S forces:

1. $\dot{U}_\alpha, \dot{V}_\alpha$ are neighbourhoods of α in $\dot{K}$;
2. $\overline{\dot{V}}_\alpha \subseteq \dot{U}_\alpha$; and
3. $\dot{U}_\alpha \cap \omega_1$ is countable.

We will be using PFA(S) to construct an S-name $\dot{F}$ for a function $\omega_1 \to \omega$ such that for each $n \in \omega$ the tree S forces that $\dot{F}^{-1}[\{n\}]$ is a discrete subset of $\dot{K}$.

To aid in this, we recursively construct a strictly increasing sequence $\{\nu_\alpha\}_{\alpha<\omega_1}$ of countable ordinal such that for each $\alpha < \omega_1$ every node s on the ν_αth level of S decides the value of the countable sets $\dot{V}_\alpha \cap \omega_1, \overline{\dot{V}}_\alpha \cap \omega_1$.

The basic idea is that every node on the ν_αth level of S knows enough about the ordinal α in regards to our situation that it may be used as a representative for α itself.

Letting $C = \{\nu_\alpha : \alpha < \omega_1\}$, we will use PFA($S$) to construct a function $f : S \,|\, C \to \omega$.

The function f will be interpreted as an S-name $\dot{F}$ for a function $\omega_1 \to \omega$ by the following contrivance: given $t \in S(\nu_\alpha)$,

$$t \Vdash \dot{F}(\alpha) = f(t).$$

Our work will then be to construct f so that under this interpretation $\dot{F}$ is forced by S to have discrete fibers.

For later convenience, for each $\nu \in C$ it will be useful to denote by ν^- the unique $\alpha < \omega_1$ such that $\nu_\alpha = \nu$.

Define $\mathcal{P}$ to be the family of all pairs $p = \langle f_p, \mathfrak{N}_p \rangle$ where:

1. f_p is a finite partial function $S \,|\, C \to \omega$; for each $l \in \omega$ by $\mathrm{dom}_l(f_p)$ we denote $f_p^{-1}[\{l\}]$;
2. $\mathfrak{N}_p$ is a finite $\in$-chain of countable elementary submodels of H_{κ^+} containing $S, \dot{\mathcal{O}}, \{\langle \dot{U}_\alpha, \dot{V}_\alpha \rangle : \alpha < \omega_1\}, C$;
3. for each $l \in \omega$ the chain $\mathfrak{N}_p$ separates $\mathrm{dom}_l(f_p)$ in the sense that given any two distinct $s, s' \in \mathrm{dom}_l(f_p)$ there is an $N \in \mathfrak{N}_p$ containing exactly one of these.

For $p, q \in \mathcal{P}$, we let $p \leq q$ iff

1. $f_p \supseteq f_q$;
2. $\mathfrak{N}_p \supseteq \mathfrak{N}_q$; and
3. given $l \in \omega$, if $s \in \mathrm{dom}_l(f_p) \setminus \mathrm{dom}(f_q)$ and $s' \in \mathrm{dom}_l(f_q)$ are compatible nodes of S, then

   $$s' \Vdash \alpha \notin \dot{V}_\beta$$

   where $\alpha = (\mathrm{ht}(s))'$ and $\beta = (\mathrm{ht}(s'))^-$.

Before proving that our partial order $\mathcal{P}$ satisfies Miyamoto's condition, we will show that a sufficiently generic filter in $\mathcal{P}$ gives our required object.

Lemma 5. *For every $s \in S \,|\, C$ the set*

$$D_s = \{p \in \mathcal{P} : s \in \mathrm{dom}(f_p)\}$$

is a dense-open subset of $\mathcal{P}$.

Proof of Lemma 5. If q is a condition of $\mathcal{P}$ not in D_s, pick any natural number n not in the range of f_q. Then the pair $p = \langle f_q \cup \{\langle s, n\rangle\}, \mathcal{N}_q\rangle$ is clearly an extension of q in D_s. $\square$

Lemma 6. *Let $G \subseteq \mathcal{P}$ be a filter generic for $\{D_s : s \in S \,|\, C\}$, and let*

$$f_G = \bigcup_{p \in G} f_p.$$

Then f_G is a function from $S \,|\, C$ to ω. Furthermore, if $\dot{F}_G$ is the S-name derived from f_G as above, then S forces that the fibers of $\dot{F}_G$ are discrete in $\dot{K}$.

Proof of Lemma 6. It clearly suffices to show that each $s' \in S \,|\, C$ forces that $(\dot{F}^{-1}[f_G(s')] \cap \omega_1) \cap \dot{V}_\beta$ is finite, where $\beta = (\mathrm{ht}(s'))^-$. Pick any $q \in D_{s'} \cap G$.

Taking any $b \in S \,|\, C$, pick some $q \in D_b \cap G$, and let $\beta = (\mathrm{ht}(b))^-$. If $a \in f_G^{-1}\{f_G(b)\}$ is comparable with b, let $p \in D_a \cap G$ be an extension of q, and let $\alpha = (\mathrm{ht}(a))^-$. By our extension relation, it follows that either $a \in D_q$, or

$$b \Vdash_S \text{“}\alpha \notin \dot{V}_\beta.\text{”}$$

From this and the definition of $\dot{F}_G$ it follows that

$$b \Vdash_S \text{“}\dot{V}_\beta \cap \dot{F}_G^{-1}\{f_q(b)\} \subseteq \{(\mathrm{ht}(a))^- : a \in D_q, f_q(a) = f_q(b)\}.\text{”}$$

We may now conclude that for any $\beta < \omega_1$ that

$$\Vdash_S \text{“}\dot{V}_\beta \cap \dot{F}_G^{-1}\{\dot{F}_G(\beta)\} \text{ is finite,”}$$

and therefore $\dot{F}_G$ is forced to have discrete fibers. $\square$

We now proceed to prove that $\mathcal{P}$ satisfies Miyamoto's condition. Let $\theta > \kappa^+$ be a sufficiently large regular cardinal, and let M be a countable elementary submodel of H_θ containing $S, \mathcal{P}$, and everything else needed. Let $\delta = \omega_1 \cap M$, and pick any $p \in \mathcal{P} \cap M$. Defining

$$p^M = \langle f_p, \mathcal{N}_p \cup \{M \cap H_{\kappa^+}\}\rangle,$$

by a standard argument it follows that $p^M \in \mathcal{P}$, and $p^M \leq p$. We now show that $\langle p^M t_M\rangle$ is $(M, \mathcal{P} \times S)$-generic for any $t_M \in S_\delta$.

Let $\mathcal{D}$ be a dense open subset of $\mathcal{P} \times S$, and let $\langle q, t\rangle$ be a given extension of $\langle p^M, t_M\rangle$.

Our first step will be to provide a description of this condition-pair $\langle t, q\rangle$ from M's perspective so that by elementarity M contains a condition-pair $\langle u, r\rangle$ with the same description. Further work will be required to allow us to choose such a $\langle u, r\rangle$ compatible with $\langle t, q\rangle$.

Extending $\langle q, t\rangle$, we may assume that $\langle q, t\rangle \in \mathcal{D}$. Since $\mathcal{D}$ is open, by the definition of the extension relationship on $\mathcal{P}$ we may assume that t is not an element of the largest model in $\mathfrak{N}_q$, and that this model contains f_q.

Let $q_M = \langle f_{q_M}, \mathcal{N}_{q_M}\rangle = \langle f_q \,|\, M, \mathcal{N}_q \cap M\rangle = \langle f_q \cap M, \mathcal{N}_q \cap M\rangle$. Note that q_M is an element of the submodel M, and that it is an “initial part” of the condition q. (The exact sense in which we will view q_M as an “initial part” of q will made precise momentarily.)

Note that $\langle q, t\rangle$ could have been chosen so that the maximal model N_{q_M} of $\mathcal{N}_{q_M}$ is so large that, letting $\delta_M = N_{q_M} \cap \omega_1$,

1. $\mathrm{dom}(f_{q_M}) \subseteq N_{-1}$,

2. $\mathrm{dom}(f_q) \setminus N_{-1} = \mathrm{dom}(f_q) \setminus M$, and

3. given any two $a, b \in (\mathrm{dom}(f_q) \setminus M) \cup \{t\}$ we have $a \,|[\delta_M, \delta) = b \,|[\delta_M, \delta)$.

A consequence of (2) and (3) is that the projection of $(\mathrm{dom}(f_q) \setminus M) \cup \{t\}$ on the δ_Mth and δth levels of S have the same size. We may then enumerate the projections of $(\mathrm{dom}(f_q) \setminus M) \cup \{t\}$ on the δ_Mth and δth levels of S, respectively, as $\{u_1, \ldots, u_n\}, \{v_1, \ldots, v_n\}$ so that $u_n = v_n \,|\, \delta_M$ for each $n \leq \tilde{n}$, and $v_1 = t \,|\, \delta = t_M$ (so that $u_1 = t \,|\, \delta_M = t_M \,|\, \delta_M$). For $1 \leq i, j \leq n$, let σ_{ij} be the canonical isomorphism which moves u_i to u_j. Note that $\sigma_{ij}^{-1} = \sigma_{ji}$ and σ_{ii} is the identity function.

Define $F = \{x \in (\mathrm{dom}(f_q) \cup \{t\} \setminus M : (\exists i \leq n)(u_i \leq_S x \leq_S \sigma_{1i}(t))\}$. Note that by our assumptions above, it follows that $x \in F$ iff $x \in (\mathrm{dom}(f_q) \cup \{t\} \setminus N_{q_M}$ and $v_i \leq_S x \leq_S \sigma_{1i}(t)$ for some $1 \leq i \leq n$.

For $x \in F$, let $\hat{x} = \sigma_{i_x 1}(x)$, where i_x is the unique $1 \leq i \leq n$ such that $u_i \leq_S x$.

For $n \leq \tilde{n}$ let σ_n denote the canonical isomorphism $S^{(u_n)} \to S^{(t_M)}$, where $t_M = t \,|\, \delta_M = \bar{s} \,|\, \delta_M$. Note that for each $n \leq \tilde{n}$ the set

$$\{a \in \mathrm{dom}_q \cap S^{(u_n)} : \sigma_n(a) \leq_S t\}$$

is a chain in S. Consider then the family

$$Z = \bigcup_{n \leq \tilde{n}} \{a \in \mathrm{dom}_q \cap S^{(u_n)} : \sigma_n(a) \leq_S t\}.$$

We may enumerate $Z = \{a_1, \ldots, a_{\tilde{m}}\}$ non-decreasing by height. For each $m \leq \tilde{m}$ let $n_m \leq \tilde{n}$ be such that $u_{n_m} \leq_S a_m$, and let $i_m = f_q(a_m)$.

As $\mathcal{N}_q$ separates the fibers of f_q, there are models

$$M \cap H_{\kappa^+} = N_0 \in N_1 \in \cdots \in N_{\tilde{\ell}} = N_t$$

in $\mathcal{N}_q$ such that for all $m < m' \leq \tilde{m}$, if $i_m = i_{m'}$, then there is an $\ell \leq \tilde{\ell}$ such that $a_m \in N_\ell$ and $a_{m'} \notin N_\ell$. For each $\ell \leq \tilde{\ell}$, let $\tilde{k}_\ell = |Z \cap (N_\ell \setminus N_{\ell-1})|$. We may re-enumerate Z as

$$\{a_{1,1}, \ldots, a_{1,\tilde{k}_1}, \ldots, a_{\tilde{\ell},1}, \ldots, a_{\tilde{\ell},\tilde{k}_{\tilde{\ell}}}\},$$

and make similar notational changes to the n_m's and the i_m's. For each $\ell \leq \tilde{\ell}$ let

$$\mathbf{a}_\ell = \langle \sigma_{n_{\ell,1}}(a_{\ell,1}), \ldots, \sigma_{n_{\ell,\tilde{k}_\ell}}(a_{\ell,\tilde{k}_\ell}) \rangle = \langle x_{\ell,1}, \ldots, x_{\ell,\tilde{k}_\ell} \rangle.$$

Clearly each $\mathbf{a}_\ell$ is a chain in S included in the interval $[t_M, t)$, and that $\max \mathbf{a}_\ell <_S \min \mathbf{a}_{\ell+1}$ for all $\ell < \tilde{\ell}$. For completeness we let $\mathbf{a}_{\tilde{\ell}+1} = \langle t \rangle$.

We now consider the family Δ^* of all condition-pairs $\langle u, r \rangle \in \Delta$ satisfying the following conditions.

4. $t_M \leq_S u$,

5. r has q_M as an "initial part" in the following sense: $f_r \supseteq f_{q_M}$, $\mathcal{N}_r \supseteq \mathcal{N}_{q_M}$ and $\mathrm{dom}_r \cap N_{-1} = \mathrm{dom}_{q_M}$,

6. the maximal model $N_u^{\langle u,r \rangle}$ of $\mathcal{N}_r$ is so large that $\mathrm{dom}_r \subseteq N_u^{\langle u,r \rangle}$,

7. $u \notin N_u^{\langle u,r \rangle}$,

8. the family $Z^{\langle u,r \rangle} = \bigcup_{n \leq \tilde{n}} \{a \in \mathrm{dom}_r \cap S^{(u_n)} : \sigma_n(a) \leq_S u\}$ has $\tilde{m}$ elements,

9. there is an enumeration $Z^{\langle u,r\rangle} = \{a_1^{\langle u,r\rangle}, \ldots, a_{\tilde{m}}^{\langle u,r\rangle}\}$ non-decreasing by height such that for each $m \leq \tilde{m}$ $u_{n_n} \leq_S a_m$ and $f_r(a_m) = i_m$, and

10. there are models

$$\min(\mathcal{N}_r \setminus \mathcal{N}_{q_M}) = N_0^{\langle u,r\rangle} \in N_1^{\langle u,r\rangle} \in \cdots \in N_{\tilde{\ell}}^{\langle u,r\rangle} = N_u^{\langle u,r\rangle}$$

in $\mathcal{N}_r$ such that $|Z^{\langle u,r\rangle} \cap (N_\ell^{\langle u,r\rangle} \setminus N_{\ell-1}^{\langle u,r\rangle})| = \tilde{k}_\ell$ for each $\ell \leq \tilde{\ell}$.

It is clear that $\Delta^* \in M$, and that $\langle t, q\rangle \in \Delta^*$.

Note that for each $\langle u, r\rangle \in \Delta^*$ we may re-enumerate

$$Z^{\langle u,r\rangle} = \{a_{1,1}^{\langle u,r\rangle}, \ldots, a_{1,\tilde{k}_1}^{\langle u,r\rangle}, \ldots, a_{\tilde{\ell},1}^{\langle u,r\rangle}, \ldots, a_{\tilde{\ell},\tilde{k}_{\tilde{\ell}}}^{\langle u,r\rangle}\}$$

as we have re-enumerated Z. Then for each $\ell \leq \tilde{\ell}$ we let

$$\mathbf{a}_\ell^{\langle u,r\rangle} = \langle \sigma_{n_{\ell,1}}(a_{\ell,1}^{\langle u,r\rangle}), \ldots, \sigma_{n_{\ell,\tilde{k}_\ell}}(a_{\ell,\tilde{k}_\ell}^{\langle u,r\rangle})\rangle.$$

As with the $\mathbf{a}_\ell$'s, each $\mathbf{a}_\ell^{\langle u,r\rangle}$ is a chain in S in the interval $[t_M, u)$ and $\min \mathbf{a}_\ell^{\langle u,r\rangle} <_S \max \mathbf{a}_{\ell+1}^{\langle u,r\rangle}$ for each $\ell < \tilde{\ell}$. Again for completeness we let $\mathbf{a}_{\tilde{\ell}+1}^{\langle u,r\rangle} = \langle u\rangle$ for each $\langle u, r\rangle \in \Delta^*$.

Define

$$\mathcal{F} = \{\langle \mathbf{a}_1^{\langle u,r\rangle}, \ldots, \mathbf{a}_{\tilde{\ell}}^{\langle u,r\rangle}, \mathbf{a}_{\tilde{\ell}+1}^{\langle u,r\rangle}\rangle : \langle u, r\rangle \in \Delta^*\}$$

Note that $\mathcal{F} \in M \cap H_{\kappa^+}$ and $\langle \mathbf{a}_1, \ldots, \mathbf{a}_{\tilde{\ell}}, \mathbf{a}_{\tilde{\ell}+1}\rangle \in \mathcal{F}$.

For each initial segment $\langle \mathbf{z}_1, \ldots, \mathbf{z}_{\tilde{\ell}}\rangle$ of an element of $\mathcal{F}$, let

$$\mathcal{F}(\mathbf{z}_1, \ldots, \mathbf{z}_{\tilde{\ell}}) = \{\mathbf{z} \in S^1 : \langle \mathbf{z}_1, \ldots, \mathbf{z}_{\tilde{\ell}}, \mathbf{z}\rangle \in \mathcal{F}\}.$$

In particular, note that $\mathbf{a}_{\tilde{\ell}+1} \in \mathcal{F}(\mathbf{a}_1, \ldots, \mathbf{a}_{\tilde{\ell}}) \in N_{\tilde{\ell}}$, however no coordinate of $\mathbf{a}_{\tilde{\ell}+1}$ is in $N_{\tilde{\ell}}$ (that is, $t \notin N_{\tilde{\ell}}$). By Corollary 1 there is a $y_{\tilde{\ell}} \in S \cap N_{\tilde{\ell}}$ with $x_{\tilde{\ell},\tilde{k}_{\tilde{\ell}}} <_S y_{\tilde{\ell}} <_S t$ such that $\mathcal{F}(\mathbf{a}_1, \ldots, \mathbf{a}_{\tilde{\ell}})$ is dense above $y_{\tilde{\ell}}$.

For each initial segment $\langle \mathbf{z}_1, \ldots, \mathbf{z}_{\tilde{\ell}-1}\rangle$ of an element of $\mathcal{F}$, let $\mathcal{F}(\mathbf{z}_1, \ldots, \mathbf{z}_{\tilde{\ell}-1})$ be the family

$$\left\{ \langle \mathbf{z}, y\rangle \in S^{\tilde{k}_{\tilde{\ell}}} \times S : \begin{array}{l} \mathbf{z} \text{ is a chain in } S, \max \mathbf{z}_{\tilde{\ell}-1} <_S \min \mathbf{z}, \\ \max \mathbf{z} <_S y \ , \mathcal{F}(\mathbf{z}_1, \ldots, \mathbf{z}_{\tilde{\ell}-1}, \mathbf{z}) \text{ is dense above } y. \end{array} \right\}.$$

In particular $\langle \mathbf{a}_{\tilde{\ell}}, y_{\tilde{\ell}}\rangle \in \mathcal{F}(\mathbf{a}_1, \ldots, \mathbf{a}_{\tilde{\ell}-1}) \in N_{\tilde{\ell}-1}$ however no coordinate of $\mathbf{a}_{\tilde{\ell}}$ is in $N_{\tilde{\ell}-1}$ and neither is $y_{\tilde{\ell}}$. By Corollary 1 it again follows that there is a $y_{\tilde{\ell}-1} \in S \cap N_{\tilde{\ell}_1}$ with $x_{\tilde{\ell}-1,\tilde{k}_{\tilde{\ell}-1}} <_S y_{\tilde{\ell}_1} <_S t$ such that $\mathcal{F}(\mathbf{a}_1, \ldots, \mathbf{a}_{\tilde{\ell}-1})$ is dense above $y_{\tilde{\ell}_1}$.

Continuing in this fashion, we eventually define $\mathcal{F}(\emptyset)$ to be the family

$$\left\{ \langle \mathbf{z}, y\rangle \in S^{\tilde{k}_1} \times S : \begin{array}{l} \mathbf{z} \text{ is a chain in } S, \, t_M <_S \min \mathbf{z}, \\ \max \mathbf{z} <_S y \ , \mathcal{F}(\mathbf{z}) \text{ is dense above } y. \end{array} \right\}.$$

Again $\langle \mathbf{a}_1, y_1\rangle \in \mathcal{F}(\emptyset) \in N_0$ while no coordinate of $\mathbf{a}_1$ is in N_0, and neither is y_1. Therefore by Corollary 1 there is a $y_0 \in S \cap N_0$ with $x_{1,\tilde{k}_1} <_S y_0 <_S t$ such that $\mathcal{F}(\emptyset)$ is dense above y_0.

For each $n \leq \tilde{n}$, let $\dot{\mathcal{O}}_n$ denote $\sigma_n(\dot{\mathcal{O}})$. Note that each $\dot{\mathcal{O}}_n$ is an S-name (actually, it is an $S^{(t_M)}$-name) for a compact sequential topology on κ. We are now prepared to begin constructing a copy of $\langle \mathbf{a}_1, \ldots, \mathbf{a}_{\tilde{\ell}}, \mathbf{a}_{\tilde{\ell}+1}\rangle$ in $\mathcal{F} \cap M$ that will be instantiated by a condition-pair $\langle u, r\rangle \in \Delta^* \cap M$ compatible with $\langle t, q\rangle$. This will be accomplished in a layer-by-layer argument, where $\mathbf{a}_\ell$ is called the ℓ^{th} *layer* *of $\langle t, q\rangle$ above $\langle t_M, q_M\rangle$.*

Let $\dot{\mathcal{X}}_1$ be an S-name for the family

$$\left\{ \boldsymbol{\alpha} = \langle \alpha_1, \ldots, \alpha_{\tilde{k}_1}\rangle \in \omega_1^{\tilde{k}_1} : \begin{array}{l} \text{there is an } \langle \mathbf{z} = \langle z_1, \ldots, z_{\tilde{k}_1}\rangle, y\rangle \in \mathcal{F}(\emptyset) \\ \text{such that } \{z_1, \ldots, z_{\tilde{k}_1}, y\} \subseteq \dot{B} \text{ and} \\ \alpha_k = (\mathrm{ht}(z_k))^- \text{ for each } k \leq \tilde{k}_1 \end{array} \right\}.$$

As $\mathcal{F}(\emptyset)$ is dense above y_0, it follows that

$$y_0 \Vdash_S \text{“}\{\alpha_1 : \boldsymbol{\alpha} \in \dot{\mathcal{X}}_0\} \text{ is uncountable.”}$$

By Lemma 4 there is an S-name $\dot{\mathcal{Y}}_1$ for a subset of $\dot{\mathcal{X}}_0$ such that

$$y_0 \Vdash_S \text{“}\dot{\mathcal{Y}}_1 \text{ is uncountable with all projections uncountable.”}$$

Note that all of these names may be found (and therefore assumed to be) in M. As $\prod_{k\leq\tilde{k}_1} \dot{\mathcal{O}}_{n_{1,k}}$ is an S-name for a compact sequential topology on $\kappa^{\tilde{k}_1}$, there is an $S^{(t_M)}$-name $\dot{\mathbf{x}}_1 = \langle \dot{x}_{1,1}, \ldots, \dot{x}_{1,\tilde{k}_1}\rangle$ such that

$$y_0 \Vdash_S \text{“}\dot{\mathbf{x}}_1 \text{ is a complete accumulation point for } \dot{\mathcal{Y}}_1 \text{ in } \textstyle\prod_{k\leq\tilde{k}_1} \dot{\mathcal{O}}_{n_{1,k}}\text{”}$$

Then there is an S-names $\langle \dot{\boldsymbol{\xi}}_j\rangle_{j\in\omega}$ for a sequence of points in $\dot{\mathcal{Y}}_0$, and an S-name $\dot{\gamma}_1$ for a countable ordinal such that

$$y_0 \Vdash_S \text{“} \lim_{j\to\mathcal{U}_{\dot{\gamma}_1}} \dot{\boldsymbol{\xi}}_j = \dot{\mathbf{x}}_1.\text{”}$$

As each of these S-names may be assumed to be in M, we may move to an extension $\bar{y}_0$ of y_0, still in M and below t, which decides these names. Let $\langle \boldsymbol{\xi}_j\rangle_{j\in\omega}$ and γ_1 denote $\bar{y}_0$'s decisions, and for each $j \in \omega$ let $\boldsymbol{\xi}_j = \langle \xi_j^1, \ldots, \xi_j^{\tilde{k}_1}\rangle$. Note that for each $k \leq \tilde{k}_0$ we have that

$$y_0 \Vdash_S \text{“} \lim_{j\to\mathcal{U}_{\gamma_1}} \xi_j^k = \dot{x}_{1,k} \text{ in } \dot{\mathcal{O}}_{n_{1,k}}\text{”}$$

Given any $k \leq \tilde{k}_1$, let $b \in \mathrm{dom}_q \cap S^{(u_{n_{1,k}})}$ be such that $f_q(b) = i_{1,k}$, and let $\beta = (\mathrm{ht}(b))^-$. As $\sigma_{n_{1,k}}^{-1}(\bar{y}_0) \leq_S b$ it follows that

$$b \Vdash_S \text{“} \lim_{j\to\mathcal{U}_{\gamma_1}} \xi_j^k = \sigma_{n_{1,k}}^{-1}(\dot{x}_{1,k}) \text{ (in } \mathcal{O}\text{).”}$$

Since $\Vdash_S$ “$\dot{U}_\beta \cap \omega_1$ is countable” and b forces that $\dot{x}_{1,k}$ is a complete accumulation point for the uncountable family of k^{th}-coordinates of elements of $\sigma_{n_{1,k}}^{-1}(\dot{\mathcal{Y}}_1)$, it follows that

$$b \Vdash_S \text{“}\sigma_{n_{1,k}}^{-1}(\dot{x}_{1,k}) \notin \dot{U}_\beta \supseteq \dot{\overline{V}}_\beta.\text{”}$$

As b decides the countable set $\dot{\overline{V}}_\beta \cap \omega_1$, it follows that

$$A_b = \{j \in \omega : b \Vdash_S \text{“}\xi_j^k \notin \dot{\overline{V}}_\beta\text{”}\} \in \mathcal{U}_{\gamma_1}.$$

Taking any $j \in \bigcap_{k\leq\tilde{k}_1} \bigcap\{A_b : b \in \mathrm{dom}_q \cap S^{(u_{1,k})}, f_q(b) = i_{1,k}\}$, it follows that for each $k \leq \tilde{k}_1$ and each $b \in f_q^{-1}\{i_{1,k}\} \cap S^{(u_{1,k})}$ we have

$$b \Vdash_S \text{“}\xi_k^j \notin \dot{V}_\beta\text{”}$$

where $\beta = (\mathrm{ht}(b))^-$. Since $\bar{y}_0 \Vdash_S$ "$\boldsymbol{\xi}_j \in \dot{\mathcal{Y}}_1 \subseteq \dot{\mathcal{X}}_1$" there is a

$$\langle \mathbf{z}_1 = \langle z_{1,1}, \dots, z_{1,\tilde{k}_1} \rangle, w_1 \rangle \in \mathcal{F}(\emptyset)$$

such that

$$\bar{y}_0 \Vdash_S \text{"}\{z_{1,1}, \dots, z_{1,\tilde{k}_1}, w_1\} \in \dot{B}\text{"}$$

and $\xi_j^k = (\mathrm{ht}(z_{1,k}))^-$ for each $k \leq \tilde{k}_1$. By elementarity we may take $\langle \mathbf{z}_1, w_1 \rangle \in M$. Note that by (9) and the definition of $\mathcal{F}(\emptyset)$ it follows that

$$z_{1,1} \leq_S \cdots \leq_S z_{1,\tilde{k}_1} \leq_S w_1 \leq_S \bar{y}_0 \leq_S t.$$

Now let $\dot{\mathcal{X}}_2$ be an S-name for the family

$$\left\{ \boldsymbol{\alpha} = \langle \alpha_1, \dots, \alpha_{\tilde{k}_2} \rangle \in \omega_1^{\tilde{k}_2} : \begin{array}{l} \text{there is an } \langle \mathbf{z} = \langle z_1, \dots, z_{\tilde{k}_2} \rangle, y \rangle \in \mathcal{F}(\mathbf{z}_1) \\ \text{such that } \{z_1, \dots, z_{\tilde{k}_2}, y\} \subseteq \dot{B} \text{ and} \\ \alpha_k = (\mathrm{ht}(z_k))^- \text{ for each } k \leq \tilde{k}_2 \end{array} \right\}.$$

Following the method above (noting that since $\mathcal{F}(\mathbf{z}_1)$ is dense above w_1, then w_1 forces that the family of 1[st]-coordinates of elements of $\dot{\mathcal{X}}_2$ is uncountable), we find a node $\bar{y}_1 \in S \cap M$ above w_1 and below t and a $\tilde{k}_2$-tuple $\boldsymbol{\xi}_2 = \langle \xi_{2,1}, \dots, \xi_{2,\tilde{k}_2} \rangle \in M$ of countable ordinals such that

$$\bar{y}_1 \Vdash_S \text{"}\boldsymbol{\xi}_2 \in \dot{\mathcal{X}}_2\text{"}$$

and for each $k \leq \tilde{k}_2$ and each $b \in f_q^{-1}\{i_{2,k}\} \cap S^{u_{n_{2,k}}}$

$$b \Vdash_S \xi_{2,k} \notin \dot{V}_\beta$$

where $\beta = (\mathrm{ht}(b))^-$. Then by elementarity and definition of $\dot{\mathcal{X}}_2$ there is a pair $\langle \mathbf{z}_2 = \langle z_{2,1}, \dots, z_{2,\tilde{k}_2} \rangle, w_2 \rangle \in \mathcal{F}(\mathbf{z}_1) \cap M$ such that $\bar{y}_1 \Vdash_S$ "$\{z_{2,1}, \dots, z_{2,\tilde{k}_2}, w_2\} \subseteq \dot{B}$" and $\xi_{2,k} = (\mathrm{ht}(z_{2,k}))^-$ for each $k \leq \tilde{k}_2$. It then follows that $w_2 \leq_s \bar{y}_1 \leq_S t$.

Continuing in this fashion, we find

$$\begin{gathered} \langle \mathbf{z}_3 = \langle z_{3,1}, \dots, z_{3,\tilde{k}_3} \rangle, w_3 \rangle \in \mathcal{F}(\mathbf{z}_1, \mathbf{z}_2) \cap M \\ \vdots \\ \langle \mathbf{z}_{\tilde{\ell}} = \langle z_{\tilde{\ell},1}, \dots, z_{\tilde{\ell},\tilde{k}_{\tilde{\ell}}} \rangle, w_{\tilde{\ell}} \rangle \in \mathcal{F}(\mathbf{z}_1, \dots, \mathbf{z}_{\tilde{\ell}-1}) \cap M \end{gathered}$$

such that for each $\ell \leq \tilde{\ell}$ we have

11. $z_{\ell,1} \leq_S \cdots \leq_S z_{\ell,\tilde{k}_\ell} \leq_S w_\ell \leq_S t$, and
12. for each $k \leq \tilde{k}_\ell$ and every $b \in f_q^{-1}\{i_{\ell,k}\}$, if $\beta = (\mathrm{ht}(b))^-$, then $b \Vdash_S$ "$\alpha \notin \dot{V}_\beta$", where $\alpha = (\mathrm{ht}(z_{\ell,k}))^-$.

Since $\mathcal{F}(\mathbf{z}_1, \dots, \mathbf{z}_{\tilde{\ell}})$ is dense above $w_{\tilde{\ell}} \in M$, it follows that there is a $\langle u \rangle \in \mathcal{F}(\mathbf{z}_1, \dots, \mathbf{z}_{\tilde{\ell}}) \cap M$ such that $w_{\tilde{\ell}} \leq_S u \leq_S t$. Then the $(\tilde{\ell}+1)$-tuple $\langle \mathbf{z}_1, \dots, \mathbf{z}_{\tilde{\ell}}, \langle u \rangle \rangle$ is an element of $\mathcal{F} \cap M$, and therefore by definition and elementarity there is a condition-pair $\langle u, r \rangle \in \Delta^* \cap M$ such that

$$\langle \mathbf{a}_1^{\langle u,r \rangle}, \dots, \mathbf{a}_{\tilde{\ell}}^{\langle u,r \rangle}, \mathbf{a}_{\tilde{\ell}+1}^{\langle u,r \rangle} \rangle = \langle \mathbf{z}_1, \dots, \mathbf{z}_{\tilde{\ell}}, \langle u \rangle \rangle.$$

By our choices of the $\mathbf{z}_\ell$ (for $\ell \leq \tilde{\ell}$) it follows that u and q are compatible, and clearly $u \leq_S t$.

Therefore the partial order $\mathcal{P}$ satisfies Miyamoto's condition, and is thus proper and preserves S. This completes our proof. $\square$

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Abstract

This thesis applies the method of forcing to the investigation of the real line, and general topological spaces.

In the first part of the thesis (joint work with M. Goldstern, J. Kellner and S. Shelah), a creature forcing construction is used to construct models of ZFC in which

$$\aleph_1 = \mathfrak{d} = \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < \mathfrak{c}.$$

In the second part of the thesis, we continue the investigation of topological consequenced of S. Todorcevic's PFA(S) (the fragment of the Proper Forcing Axiom consistent with keeping a fixed (coherent) Souslin tree S Souslin). In particular, we show that in the generic extension by S, every locally countable subspace of cardinality $<\mathfrak{c}$ in a compact Hausdorff space is σ-discrete. This is related to Z. Szentmiklossy's investigation of S-spaces under MA, the MA-counterpart of which was extracted by Z. Balogh.

Zusammenfassung

Die vorliegende Dissertation verwendet die *forcing* Methode für Untersuchungen an den reellen Zahlen und allgemeinen topologischen Räumen.

Im ersten Teil der Arbeit, die gemeinschaftlich mit M. Goldstern, J. Kellner und S. Shelah entstanden ist, wird ein *creature forcing* verwendet um ein Modell der Mengenlehre zu konstruieren in dem

$$\aleph_1 = \mathfrak{d} = \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < \mathfrak{c}.$$

Im zweiten Teil der Dissertation setzen wir die Untersuchung der topologischen Konsequenzen von $\mathsf{PFA}(S)$ fort. Das Axiom $\mathsf{PFA}(S)$ wurde von S. Todorcevics eingeführt und ist das Fragment von PFA das konsistent bleibt mit der Aussage, einen vorher fixierten Suslin Baum S-Suslin zu belassen. Wir zeigen insbesondere: In der von S erzeugten generischen Erweiterung ist jeder lokal abzählbare topologische Teilraum von Größe $<\mathfrak{c}$ eines kompakten Hausdorff Raums σ-diskret. Das Resultat bezieht sich auf Z. Szentmiklossys Untersuchungen der S-Räume unter MA. Das MA Gegenstück dazu wurde von Z. Balogh erarbeitet.

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