# MASTERARBEIT

Titel der Masterarbeit

„The determinacy of locally uncountable games“

Verfasser

Martin Benedikt Köberl

angestrebter akademischer Grad

Master of Science (MSc)

Wien, 2015

| | |
|---|---|
| Studienkennzahl lt. Studienblatt: | A 066 821 |
| Studienrichtung lt. Studienblatt: | Masterstudium Mathematik |
| Betreuer: | O. Univ.-Prof. Sy-David Friedman |

*in memory of my grandparents*

*Abstract*

In this thesis we try to investigate the notion of games of uncountable length. While much is known about games of length $\omega$, research in the determinacy of uncountable length is sparse and mostly very recent. We try to better understand methods used by Itay Neeman in [13] to prove determinacy of the local game of Chapter 4.

The chapters before are used to introduce the needed notions: In Chapter 1 we review the basic notions of ultrapower via an ultrafilter and ultrapower via an extender. We introduce the large cardinals notions needed, the most important of them is the concept of a Woodin cardinal, and define the notion of an iteration tree, which has come to be an indispensable tool in determinacy studies and inner model theory.

In Chapter 2 we introduce a forcing notion due to Woodin, called Woodin's extender algebra. In the local game introduced in Chapter 4 we fix a name in this forcing notion for a set of sequences of reals of the same length. In this game it is player I and II's goal to produce a sequence of reals so that I wins iff the name can be adequately interpreted in a shift of the ground model so that the generated sequence is an element of it. Using chain conditions for Woodin's extender algebra we get that runs of the game are locally uncountable.

Chapter 3 introduces the so-called branching game. It is an auxiliary game needed to prove the determinacy of the local game. We will reduce a run of the local game to several runs of the branching game and construct a winning strategy for the local game from one for the branching game.

# Contents

# Introduction

Let $A$ be a set of reals and define the game $\mathcal{G}(A)$ for two players as follows: Players I and II alternate in taking turns and play one natural number. Player I starts with playing $a_0$, player II responds with $a_1$ and so on. Player I plays $a_k$ for all even numbers $k \in \omega$, player II plays $a_k$ for odd $k \in \omega$. A situation of

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| I | $a_0$ | | $a_2$ | | $\dots$ | $a_{2n}$ | | $\dots$ |
| II | | $a_1$ | | $a_3$ | $\dots$ | | $a_{2n+1}$ | $\dots$ |

Figure 0.1: A run of the game $\mathcal{G}(A)$

this is depicted in Figure 0.1. We say that the sequence $a = (a_k \mid k \in \omega)$ is a *run* of the game $\mathcal{G}(A)$. Player I wins this run of the game $\mathcal{G}(A)$ just in case that $a \in A$. Otherwise, if $a \notin A$, player II wins the run. A natural question to ask now is whether one player can always win $\mathcal{G}(A)$, more specifically, whether no matter what one player does, the other player has a strategy to ensure to win the run of the game. A *strategy* is a function $\sigma : \omega^{<\omega} \to \omega$ (clearly the idea is that players use the numbers played until now as argument for the strategy to determine their next move). We can use strategies to play against real numbers (the moves by the other player) to define a run of a game:

$$\sigma * x = (\sigma(\emptyset), x_0, \sigma((x_0)), x_1, \sigma((x_0, x_1)), \dots).$$

Analogously we define $x * \sigma$ so that $x$ describes player I's moves and $\sigma$ is used to determine the moves of player II. We say that a strategy $\sigma$ is a *winning strategy for player* I (II) in $\mathcal{G}(A)$ iff for every real $x$, the sequence $\sigma * x$ ($x * \sigma$) does (not) belong to $A$. If one of the two players has a winning strategy (clearly it is not possible for both of them to have one) we call the set *$A$ determined*. The first determinacy result is due to Gale and Stewart [1] where above notions are introduced:

**Theorem.** *Every closed set of reals is determined.*

There seem to be two immediate ways of further research in this area. The first one, proving determinacy of more complicated sets of reals, was soon fruitful in the years after the definition of these games. In ZFC more than determinacy of closed sets can be proved. The following result by Martin ([6]) extends the determinacy far beyond closed sets:

**Theorem.** *Every Borel set of reals is determined.*

A few years earlier, Martin already had proved this and stronger determinacy results using large cardinals (see [7]):

**Theorem.** *If there is a measurable cardinal, then every analytic set of reals is determined.*

That some form of large cardinal notion (precisely, a sharp for every real $x$) was needed was shown by Harrington in [2]. This put a bound to determinacy results provable in ZFC alone and showed that further determinacy proofs had to make use of large cardinal notions.

A breakthrough result introducing the very important technique of *iteration trees* was established by Martin and Steel in [8]:

**Theorem.** *If there are infinitely many Woodin cardinals, projective determinacy holds.*

Since it is inconsistent with the axiom of choice that all sets of reals are determined we have a rough idea on how many large cardinals are needed to get determinacy for several classes of sets of reals.

Another idea is to look at games of longer length. A naïve idea is to fix a set $A \subset \omega^\alpha$ for some $\alpha > \omega$ and define the notions from above for this game. Results in this area are mostly newer, a good survey can be found in Neeman [10]. The goal of this thesis is to have a look at games of this kind (i.e., two players alternate in playing natural numbers) so that a complete run in these games has at least uncountable length in some inner model in which it is interpreted.

In Chapter 1 we review the main tools used in this area of study. We outline the construction of ultrapowers and extender models, define the large cardinal notions needed and introduce iteration trees. This material is mostly standard and can be found in Jech [3] or Kanamori [4] or the appendix of Neeman [13]. An introduction to extenders and iteration trees can also be found in [8]. We will also make use of forcing and generally will assume that the reader is familiar with basic set theory and some forcing such as covered by Kunen [5].

In Chapter 2 we start working towards the main theorem. We present a variant of a forcing notion due to Woodin, called Woodin's extender algebra and denoted by $\mathbb{W}$. We start by introducing so called positions which correspond closely to the objects constructed in the branching game of Chapter 3. The conditions of Woodin's extender algebra correspond to equivalence classes of statements about these positions. A generic object in Woodin's extender algebra will uniquely define a position with additional properties which are known as extender axioms.

Chapter 3 introduces an auxiliary game known as branching game. In this game players I and II start off with a position and a name for a set of longer positions. They extend both the model (via extenders they change to elementary extensions) they are working in but also the position. It is player I's goal that at some point the extended position is part of an interpretation of the forcing name mapped via an elementary embedding into an extension of the starting model. The determinacy proof for this game is not part of this thesis.

In Chapter 4 we establish Theorem 73 which is the soul of many results stating determinacy for some uncountable games. Roughly, it asserts determinacy for games whose runs are uncountable in some inner model. The much stronger assertion of having uncountable runs in V is the topic of Neeman [11].

The material of Chapters 2 to 4 is due to Itay Neeman and completely included in his book [13]. For an introduction into the topic in general and a survey of the first half of his book (this thesis covers the second half), the reader is advised to look at [10].

### Acknowledgements

I thank all the people who acted as mathematical teachers in my life: For helping me to develop my interest, to broaden my knowledge and for answering my questions. I am grateful to my parents and my whole family for discovering and promoting my interest in mathematics. It would never have become such an interesting subject for me, however, without Elfi Paleta whose support as my high school teacher was endless. My high school colleague Markus provided me with a competitor, I am not sure how far and high I would have come without him.

My time of studying mathematics was made pleasant through efforts of Angela, Anastasia, Fabio, Markus and Thomas. I owe the students at the Kurt Gödel Research Center numerous interesting discussions which helped me to determine my interest and understand the field better. First and foremost, these are Fabio, Stefan and Wolfgang. I would also like to thank my advisor, Sy-David Friedman, both for his patience with me in completing this thesis, but also for his demands upon me. I am thankful to Diana and Fabio for commenting on parts of this thesis.

### Notation

Our notation is mostly standard. By reals we mean elements of $\omega^\omega$. Concatenation of sequences is denoted by $^\frown$. The main caveat is the use of $\restriction$. We use it for the restriction of classes of ordinals (where it just means intersection). For a model $M$ we mean by $M \restriction \alpha$ the initial segment $\mathrm{V}_\alpha$ relativized to $M$. For classes or sets of sequences or products of classes and class models we mean it to restrict every coordinate.

# 1 Prerequisites

For reasons which will not become apparent in this thesis we mostly do not work with models of ZFC but of ZFC$^*$: For this, we extend the language by a unary predicate $\mathcal{K}$ (see Definition 5 for its use), and extend the axiom schemes of replacement and comprehension to be also valid for formulas involving $\mathcal{K}$. Further details can be found in [13, p. 301]. In this chapter, although we state everything for models of ZFC$^*$, most constructions transfer directly to models of ZFC.

1 **Definition.** *Let $M$ and $N$ be transitive models of* ZFC$^*$. *We say $M$* and *$N$ agree to* an ordinal $\rho$ *if* $M \upharpoonright \rho = N \upharpoonright \rho$. *$M$ and $N$ are said to* agree past $\rho$ *if they agree to some* $\alpha > \rho$.

## 1.1 Ultrapowers

We recall the ordinary ultrapower construction from model theory fitted to (class) models of ZFC or ZFC$^*$. Let $U$ be an ultrafilter on some cardinal $\kappa$. Furthermore, let us assume that $U$ is $\sigma$-complete.

Then we can construct the *ultrapower* $\mathrm{Ult}(\mathrm{V}, U)$: Its elements are equivalence classes of functions $f : \kappa \to \mathrm{V}$ where we set $f \sim g$ if $f(\alpha) = g(\alpha)$ for all $\alpha \in X$ for some set $X \in U$. In the following if some property $\varphi$ holds for all elements $x$ of a measure-1-set of an ultrafilter $U$, we say that $\varphi$ holds for $U$ almost every $x$. Rather than working with the real equivalence classes $[f] = \{g \in \mathrm{V} \mid g : \kappa \to \mathrm{V},\ g \sim f\}$ which turn out to be proper classes we chop them off and replace them by $[f] \cap \mathrm{V}_\lambda$ where $\lambda$ is chosen least possible so that the resulting set is non-empty (this procedure is known as *Scott's trick*). For $f$, $g$ as above, we set $[f] \in^* [g]$ iff $f(\alpha) \in g(\alpha)$ for $U$ a.e. $\alpha$. We write $\mathcal{K}([f])$ iff $\mathcal{K}(f(\alpha))$ for $U$ a.e. $\alpha$. This is well-defined and using the $\sigma$-completeness of $U$ we see that the relation $\in^*$ also is well-founded. Hence we can build the transitive collapse of $(\mathrm{Ult}(\mathrm{V}, U), \in^*)$, which in the following we denote (by a slight abuse of notation) by $(\mathrm{Ult}(\mathrm{V}, U), \in)$.

The so-constructed model is an elementary extension of V:

**2 Theorem** (Łoś)**.** *Let $\varphi$ be a formula with n free variables and $[f_1], \ldots, [f_n] \in \mathrm{Ult}(\mathrm{V}, U)$. Then*

$$\mathrm{Ult}(\mathrm{V}, U) \models \varphi([f_1], \ldots, [f_n]) \Leftrightarrow \text{for } U \text{ a.e. } \alpha\text{: } \mathrm{V} \models \varphi(f_1(\alpha), \ldots, f_n(\alpha)).$$

*Furthermore, the map $i_U : x \mapsto [c_x]$ where $c_x$ is the constant function on $\kappa$ with value x, is an elementary embedding.*

A proof of the results stated in this section can be found in [3, pp. 159-161] and [3, pp. 285-289].

If $U$ even is $\kappa$-complete, it can be shown that the embedding is non-trivial (in fact, non-triviality implies $\kappa$-completeness, as we see below) and that $\mathrm{Ult}(V, U)$ is properly included in V (It is not too hard to see that the ultrafilter $U$ itself can not be an element of the ultrapower $\mathrm{Ult}(V, U)$). More specifically, $[d]$ where

$$d(\alpha) = \alpha \text{ for } \alpha < \kappa$$

is an element of $\mathrm{Ult}(\mathrm{V}, U)$ which is at least $\kappa$ but strictly below $[c_\kappa] = i_U(\kappa)$. The ultrafilter $U$ itself does not belong to $\mathrm{Ult}(\mathrm{V}, U)$, which means that V and $\mathrm{Ult}(\mathrm{V}, U)$ do not have the same sets of rank $\kappa + 1$. Above construction can of course be relativized to models $M$ of ZFC or ZFC$^*$.

Note also that this construction works when $U \in M$ is an ultrafilter in the sense of $M$ on $\kappa$ and $N$ and $M$ agree up to $\kappa$: We then can build the ultrapower $\mathrm{Ult}(N, U)$ by measuring functions in $N$ rather than in $M$ using $U$. However wellfoundedness of the ultrapower may fail in this case due to $\sigma$-completeness being a second order property ($M$ and $N$ need not have the same $\omega$-sequences of sets in $U$). Furthermore, the ultrapower is not a subclass of $N$ anymore.

Whenever we have a nontrivial elementary embedding $j : \mathrm{V} \to M$, it necessarily follows (using the elementarity) that for some cardinal $\kappa$, we have that $j(\kappa) > \kappa$. The least such $\kappa$ is called the *critical point* of $j$, denoted by $\mathrm{crit}(j)$. We define $U \subset \mathcal{P}(\kappa)$ by

$$X \in U \Leftrightarrow \kappa \in j(X),$$

it can be easily checked that this in fact is a $\kappa$-complete ultrafilter on $\kappa$. Hence, cardinals $\kappa$ carrying $\kappa$-complete ultrafilters are equivalent to the existence of nontrivial embeddings. They deserve their own name:

**3 Definition.** *A cardinal $\kappa$ is called* measurable *if there is a $\kappa$-complete ultrafilter $U$ on $\kappa$.*

It can be easily shown that measurable cardinals are inaccessible (and in fact much more: they do not exist in the constructible universe).

Suppose now we have an elementary embedding $i : \mathrm{V} \to M$, where V and $M$ agree up to $\lambda$ for some $\lambda$ bigger than $\kappa$. For the ultrafilter $U$ derived from $i$ above, we have $U \in \mathrm{V}$ but $U \notin \mathrm{Ult}(\mathrm{V}, U)$ so the embedding $i$ cannot be an ultrapower embedding of the form described above.

**4 Definition.** *Assume $M$ and $N$ are transitive models of* ZFC$^*$. *Let $i : M \to N$ be a nontrivial elementary embedding. If $M$ and $N$ agree up to $\lambda$, then i is called $\lambda$*-strong.

As the ultrafilter $U$ is a set of rank $\kappa + 1$, the derived embedding $i_U$ is not even $\kappa + 1$-strong. If we are in the situation that we have an elementary embedding $i : M \to N$ with $M$ and $N$ agreeing far beyond the critical point of $i$, this means that we need to find other ways to fully describe these embeddings using sets. One possibility of doing this is to use extenders introduced in the next Section.

## 1.2 Extenders

Given an elementary embedding $j : \mathrm{V} \to M$ with $\kappa = \mathrm{crit}(j)$, we can define the $\kappa, \lambda)$*-extender* $E_j = (E_a \mid a \in [\lambda]^{<\omega}$ for $\kappa \leq \lambda < j(\kappa)$ by:

$$X \in E_a \iff a \in j(X) \ \wedge \ X \in \mathcal{P}([\kappa]^{|a|}). \tag{1.1}$$

The idea is to catch the elementary embedding $j$ up to $\lambda$ with $E$, the specific meaning can be found in Theorem 7. It can be checked that the $E_a's$ are ultrafilters which cohere and satisfy the other properties in Definition 5. This means that we can construct the ordinary ultrapowers $\mathrm{Ult}(\mathrm{V}, E_a)$ and using coherence it can be checked that there are embeddings between them which allow us to build the direct limit. This direct limit agrees with $M$ up to $\lambda$. Before we can explain the detailed construction, we need to introduce a few notions to formulate the coherence conditions: Let $a$ and $b$ be two finite sets of ordinals with $a \subset b$. Let $(b_i \mid i < |b|)$ be the increasing enumeration of $b$ and fix $i_k$ such that $a = \{b_{i_k} \mid k < |a|\}$. If $c$ is another finite set of ordinals of the same size as $b$ with increasing enumeration $c_i \mid i < |c|)$, we define the projection

$$c^{b,a} = \{c_{i_k} \mid k < |a|\},$$

so that $c^{b,a}$ has size $|a|$ and is produced from $c$ in the same way as $a$ is produced from $b$, specifically $b^{b,a} = a$. For sets $X \subset [\lambda]^{|a|}$ for some ordinal $\lambda$, we can define the upwards projection:

$$X^{a,b} = \{c \in [\lambda]^{|b|} \mid c^{b,a} \in X\}.$$

If $f$ is a function whose domain is equal to $[\lambda]^{|a|}$, we define the projection $f^{a,b}$: It is a function with domain $[\lambda^{|b|}]$ which simulates $f$:

$$f^{a,b}\colon \{\alpha_i \mid i < |b|\} \mapsto f(\{\alpha_{i_k} \mid k < |a|\}, \quad \text{where } \alpha_0 < \alpha_1 < \cdots < \alpha_{|b|-1}.$$

**5 Definition.** *A system of ultrafilters $E = (E_a \mid a \in [\lambda]^{<\omega})$ for some ordinal $\lambda$ is called a $(\kappa, \lambda)$-*extender *if it satisfies the following conditions:*

*(1) For each $a \in [\lambda]^{<\omega}$, the object $E_a$ is a $\kappa$-complete ultrafilter on $[\kappa]^{|a|}$.*

*(2) Coherence: If $a \subset b \in [\lambda]^{<\omega}$ and $X \in E_a$, then $X^{a,b} \in E_b$.*

*(3) Normality: Whenever $a \in [\lambda]^{<\omega}$ and $f : [\lambda]^{|a|} \to \mathrm{V}$ is such that for some $i < |a| : f(u) < u_i$ for $E_a$ a.e. $u$ and some $i < |a|$, then there is $\xi < a_i$ such that for $\xi$ and $k < i$, being so that $\xi$ is the $k$th element of $a \cup \{\xi\}$:*

$$\text{for } E_{a\cup\{\xi\}} \text{ a.e. } u: \ f^{a,a\cup\{\xi\}}(u) = u_k.$$

*(4) $\omega$-Completeness: Given $a_i \in [\lambda]^{<\omega}$ and $X_i \in E_{a_i}$ for all $i \in \omega$, there is a map $g : \bigcup_{i\in\omega} a_i \to \kappa$ so that $g''a_i \in X_i$ for all $i$.*

*(5) $\mathcal{K}(E)$.*

Given an object $E$ as above, we can construct an inner model of V using $E$, which we call the ultrapower of V via $E$ and denote by $\mathrm{Ult}(\mathrm{V}, E)$. Strictly speaking this is not an ultrapower but a direct limit of a system of ultrapowers of V: We can build the ultrapowers $\mathrm{Ult}(\mathrm{V}, E_a)$ for all $a \in [\lambda]^{<\omega}$ in the usual way as described in the previous Section, using that $E_a$ is an ultrafilter on $[\kappa]^{|a|}$ and using the compatibility requirement define elementary embeddings

$\mathrm{Ult}(\mathrm{V}, E_a) \to \mathrm{Ult}(\mathrm{V}, E_b)$ whenever $a \subset b \in [\lambda]^{<\omega}$. The direct limit of these objects is the ultrapower $\mathrm{Ult}(\mathrm{V}, E)$ which is a model of $\mathsf{ZFC}^*$. The well-foundedness of $\mathrm{Ult}(V, E)$ follows from the $\omega$-completeness of $E$.

The explicit construction of the ultrapower can be described as follows: For $a, b \in [\lambda]^{<\omega}$, $f : [\lambda]^{|a|} \to \mathrm{V}$ and $g : [\lambda]^{|b|} \to \mathrm{V}$ we write $(f, a) \sim (g, b)$ iff for $E_{a\cup b}$ a.e. $u$ it holds that $f^{a,a\cup b}(u) = g^{b,a\cup b}(u)$. This defines an equivalence relation whose equivalence classes are proper classes. For $a$ and $f$ as above we write $[f, a]$ for the equivalence class intersected with $\mathrm{V}_\mu$ where $\mu$ is chosen such that $[f, a]$ is non-empty (this again is an application of *Scott's trick*). Given such objects $[f, a]$ and $[g, b]$ we write $[f, a] \in^* [g, b]$ iff for $E_{a\cup b}$ a.e. $u$ we have $f^{a,a\cup b}(u) \in g^{b,a\cup b}(u)$. This clearly is well-defined and using the $\omega$-completeness of $E$ it follows that $\in^*$ is well-founded. Furthermore we set $\mathrm{Ult}(\mathrm{V}, E) \models \mathcal{K}([f, a])$ if for $E_a$ a.e. $u \in [\lambda]^{|a|}$ we have that $\mathcal{K}(f(u))$. Therefore, we can build the transitive collapse of $(\{[f, a] \mid a \in [\lambda]^{<\omega}, f : [\lambda]^{|a|} \to \mathrm{V}\}, \in^*)$. We denote this object by $\mathrm{Ult}(\mathrm{V}, E)$. Again we have a version of Łoś's Theorem:

**6 Theorem.** *Let $\varphi$ be a formula with $n$ free variables, $a_i \in [\lambda]^{<\omega}$ and $f_i : [\lambda]^{|a_i|} \to \mathrm{V}$ for $i < n \in \omega$. Then*

$$\mathrm{Ult}(\mathrm{V}, E) \models \varphi\left([f_0, a_0], \ldots, [f_{n-1}, a_{n-1}]\right)$$

*holds if and only if*

$$\text{for } E_a \text{ a.e. } u \in [\lambda]^{|a|} : \ \mathrm{V} \models \varphi\left(f_0^{a_0,a}(u), \ldots, f_{n-1}^{a_{n-1},a}(u)\right),$$

*where $a = \bigcup_{i<n} a_i$. As a consequence,* V *and* $\mathrm{Ult}(\mathrm{V}, E)$ *are first-order equivalent.*

From this we get easily that the map

$$i_E : \mathrm{V} \to \mathrm{Ult}(\mathrm{V}, E),\ x \mapsto [c_x, \emptyset], \quad \text{where } c_x \text{ is the constant function with value } x$$

is an elementary embedding from V into its ultrapower.

After introducing the construction we also can give proper meaning to the introductory words of this section:

**7 Theorem.** *Let $j$ be an elementary embedding $j : \mathrm{V} \to M$. Let $\kappa = \mathrm{crit}(j)$ and suppose that $\kappa < \lambda < j(\kappa)$. Let $E$ be the $(\kappa, \lambda)$-extender derived from $j$ as in*

*Equation* (1.1). *Then there is an elementary embedding* $k : \mathrm{Ult}(V, E) \to M$ *with* $\mathrm{crit}(k) \geq \lambda$ *so that* $j = k \circ i_E$.

This means that the ultrapower $\mathrm{Ult}(V, E)$ and $M$ agree on sets of rank up to $\lambda$. A proof of this and most of the other results in this section can be found in [4, pp. 352-357] (note however our restriction on extenders).

**8 Definition.** *In the case that* V *and* $\mathrm{Ult}(\mathrm{V}, E)$ *agree to* $\alpha$ *(see Definition 1), we say* $E$ *is* $\alpha$-strong. *The largest* $\alpha$ *for which this is valid is called the* strength *of* $E$.

Again, this construction not only applies to the universe V itself but can be used for arbitrary models $M$ of ZFC$^*$. Here again we do not need that $E$ itself is an element in $M$, which is a model of ZFC$^*$, but only that the components of $E$ measure all subsets of $[\kappa]^n$ (for $n \in \omega$). Thus if $E \in M$ is an extender as above and $N$ is a transitive model ZFC$^*$, which agrees with $M$ to $\kappa$, we can also build the ultrapower $\mathrm{Ult}(N, E)$. This also is a model of ZFC$^*$ but may again fail to be wellfounded. This is due to the fact that $\omega$-completeness of extenders is not an absolute property of $E$ as it involves second-order quantification over $E$. If this happens, we use $\mathrm{Ult}(N, E)$ to denote the ultrapower of $N$ via $E$ of which only the wellfounded part is collapsed. The possibility of applying extenders to models different to those in which they originate will play a big role in the definition of iteration trees in the last Section of this chapter..

Given two extenders $E$ and $F$ in $V$, we can compare them by asking whether one is an element of the ultrapower via the other: We set $E \lhd F$ if $E \in \mathrm{Ult}(V, F)$. This relation is intransitive so it is not an order, however since its original definition just involved normal ultrafilters on which it is an order relation it is known under the name *Mitchell order*. Wellfoundedness of the order however also holds for the relation on extenders. For normal ultrafilters, this result is due to Mitchell and can be found in [9]. For extenders we have the result by Steel (see [14]):

**9 Theorem.** *The Mitchell order on extenders is wellfounded.*

Furthermore we are going to need the notion of *countability* which we use for extenders whose canonical embeddings preserve countability (see [13, p. 302]): If $E$ is a $(\kappa, \lambda)$-extender of a model $M$, we say $E$ is countable in V if $\lambda$ and $(2^\kappa)^M$ are both countable in V. An embedding $h : N \to Q$ is called *countability preserving* if for every $N$-inaccessible $\vartheta$ which is countable in

V, we also have $h(\vartheta)$ is countable in V and if for every $\delta \in N$, countability of $(2^\delta)^N$ in V implies countability of $(2^{h(\delta)})^Q$ in V. The following facts will be helpful in the later work:

**10 Lemma.** *Let $M$ and $N$ be models of* ZFC* *and $E$ be an extender of $M$ so that $M$ and $N$ agree past* crit$(E)$. *If $E$ is countable in* V, *then $i_E^N : N \to \mathrm{Ult}(N, E)$ preserves countability, where $i_E^N$ is the canonical ultrapower embedding given by $E$.*

*The family of embeddings which preserve countability is closed under compositions and countable direct limits.*

## 1.3 Woodin cardinals

The main notion of large cardinals we are going to need in this thesis are *Woodin cardinals*. For this we first need the notion of strongness:

**11 Definition.** *A cardinal $\kappa$ is called $\alpha$-strong if there is an elementary embedding $j : \mathrm{V} \to M$ such that*

- $\mathrm{crit}(j) = \kappa$,
- $\mathrm{V}_\alpha = M \restriction \alpha$.

*The cardinal $\kappa$ is called $\alpha$-strong wrt. (with respect to) a set $H$ if there exists an elementary embedding $j : \mathrm{V} \to M$ with the above requirements and furthermore*

- $j(H) \cap \mathrm{V}_\alpha = H \cap \mathrm{V}_\alpha$.

*If $\kappa$ is $\alpha$-strong (wrt. $H$) for all $\alpha < \lambda$ for some limit ordinal $\lambda$, we call it $< \lambda$-strong (wrt. $H$).*

If there are such elementary embeddings, there are also extenders such that the above properties hold for the canonical elementary embeddings $i_E$ described above.

**12 Definition.** *A cardinal $\delta$ is called a* Woodin cardinal *if for every $H \subset \mathrm{V}_\delta$ there exists $\kappa < \delta$ which is $< \delta$-strong wrt. $H$.*

Woodin cardinals thus always have stationarily many measurable cardinals below them. However they need not be measurable themselves. But we have

**13 Lemma.** *Woodin cardinals are inaccessible.*

A proof of this fact can be found in [8, p. 104].

## 1.4 Iteration trees

**14 Definition.** *We say $T$ is a* tree order *on an ordinal $\alpha$ if $T$ is a suborder of $<\upharpoonright\alpha$, it is linear on all initial segments and it respects successors and limits, i.e.,*

*(1)* $\forall\eta < \alpha\ : \{\zeta \mid \zeta T\eta\}$ *is linearly ordered by $T$,*

*(2)* $\forall\xi$ *with* $\xi + 1 < \alpha\ : \xi + 1$ *is a successor in $T$,*

*(3)* $\forall\gamma < \alpha, \gamma$ *limit* $: \{\zeta \mid \zeta T\gamma\}$ *is cofinal in $\gamma$ wrt. to $<$.*

*A subset $b \subseteq \alpha$ is a* branch *of $T$ if it is linearly ordered by $T$ and downwards closed. The branch $b$ is called* cofinal *if the ordinals in $b$ are cofinal in $\alpha$.*

The last condition of the previous Definition implies that such a $\gamma$ is a limit in $T$ (i.e., it does not have a direct predecessor wrt. $T$). Note furthermore that the pair $(\alpha, U)$ is a set-theoretic tree in the usual sense with the unique root 0 where ordinals are successors iff they are successors wrt. the usual order.

The idea of iteration trees is to have models $M_\xi$ for $\xi < \alpha$ which are extender models of the model indexed with their predecessor in the order $T$. The extender which is used to build $M_\xi$ however comes from $M_{\xi-1}$. If $\xi$ is a limit ordinal, the respective model is just a limit. The original definition goes back to [8].

**15 Definition.** *An* iteration tree $\mathcal{T}$ *of length $\alpha$ on a model $M$ is a tuple containing:*

*(1) a tree order $T$ of length $\alpha$,*

*(2) extenders $E_\xi$ whenever $\xi + 1 < \alpha$,*

*(3) models $M_\xi$ for $\xi < \alpha$ with embeddings $j_{\zeta,\xi} : M_\zeta \to M_\xi$ whenever $\zeta T\xi$,*

*which satisfy the following conditions:*

*(4)* $M_0 = M$;

*(5) whenever $\xi + 1 < \alpha$: $M_xi \models$ "$E_\xi$ is an extender" or $E_\xi$ is equal to* "undefined"*;*

*(6) whenever $E_\xi$ =* "undefined"*, the $T$-predecessor of $\xi + 1$ is $\xi$ and $M_{\xi+1} = M_\xi$ with $j_{\xi,\xi+1}$ being the identity mapping;*

*(7) if $E_\xi$ for $\xi + 1 < \alpha$ is not* "undefined"*, we have* $M_{\xi+1} = \mathrm{Ult}(M_\zeta, E_\xi)$ *where $\zeta$ is the $T$-predecessor of $\xi + 1 < \alpha + 1$ and that $j_{\zeta,\xi+1}$ is the canonical ultrapower embedding;*

*(8) if $\lambda < \alpha$ is a limit, $M_\lambda$ is the direct limit of $(M_\xi, j_{\zeta,\xi} \mid \zeta T \xi T \lambda)$ and the mappings $j_{\xi,\lambda}$ are the canonical direct limit embeddings;*

*(9) all other $j_{\zeta,\xi}$ for $\zeta T \xi < \alpha$ are given by compositions of objects defined above.*

Note that the definition is in some form redundant. Given the extenders, the tree order and the start model $M$; the models $M_\xi$ for $\xi \geq 1$ which appear higher in the tree are uniquely determined given the extenders and the tree order. Quite often, we will describe iteration trees just by defining these objects. Furthermore, the use of undefined extenders (by which we mean if extenders are equal to "undefined") gives us more liberty in defining iteration trees. By removing instances of undefined extenders we can shorten the tree to make it into an iteration tree where every extender is defined. Sometimes setting extenders equal to "undefined" allows us less complicated definitions though. Formally, we code "undefined" just by some low-level object appearing in any model.

Of course, to build the ultrapower $\mathrm{Ult}(M_\zeta, E_\xi)$, we need that $M_\zeta$ and $M_\xi$ (of which $E_\xi$ is an element) agree past $\mathrm{lh}(E)^{M_\xi}$. This condition is implicit in the definition of the iteration trees. As mentioned above, when we build external ultrapowers (that is, ultrapowers where the ultrafilter or the extender is not an element of the model to which it is applied), we are no longer guaranteed that these are wellfounded. Another problem turns out to be the wellfoundedness of models $M_\lambda$ for limit ordinals $\lambda$.

This is the place where the notion of *iteration strategy* comes in. This is a winning strategy in the so-called *iteration game on a model M* in which two players start with $M$ and play extenders and a tree order $T$ to build an iteration tree of length $\omega_1^V + 1$ where one player wants to make sure that all models are wellfounded: Given a model $M_0 = M$, starting with round 0, in successor (or zero) rounds $\xi$ player II plays an extender $E_\xi \in M_\xi$ and $\zeta < \xi + 1$. We then set $M_{\xi+1} = \mathrm{Ult}(M_\zeta, E_\xi)$ (of course, we need there that $M_\xi$ and $M_\zeta$ agree enough so that this makes sense), extend the tree order $T$ by setting $\zeta T \xi + 1$ (and taking $T$'s transitive closure) and continue to round $\xi + 1$. In limit rounds $\lambda$, player I chooses a branch $b$ which is cofinal in $\lambda$. We extend the tree order $T$ by setting $\zeta T \lambda$ for all $\zeta \in b$, and continue to round $\lambda + 1$ where $M_\lambda$ is the direct limit model of the $M_\zeta$'s for $\zeta \in b$ with the canonical embeddings. If ever a model $M_\alpha$ with $\alpha \leq \omega_1^V$ is reached which is illfounded, player II wins, if however $M^{\mathrm{V}}_{\omega_1}$ is wellfounded, player I wins.

**16 Definition.** *An* iteration strategy *for $M$ is a winning strategy for player* I *in the iteration game. If an iteration strategy for $M$ exists, we say $M$ is* iterable. *If $\mathcal{T}$ is an iteration strategy on $M$ and $\Gamma$ is an iteration strategy for $M$, we say $\Gamma$ is consistent with $\mathcal{T}$ if $\mathcal{T}$ is a possible intermediate stage of the iteration game played using $\Gamma$ against an opponent* II.

Note that this means that for any $\xi < \omega_1^{\mathrm{V}}$ there is no extender $E$ in any $M_\zeta$ for $\zeta < \xi$ so that $\mathrm{Ult}(M_\xi, E)$ is illfounded.

Furthermore note that in the final move in which player I decides the branch which leads to $\omega_1^{\mathrm{V}}$ she needs to make sure that this set of ordinals is cofinal in $\omega_1^{\mathrm{V}}$, a requirement towards which she needs to work already at earlier stages. If she manages to find such a branch, then the resulting model $M^{\mathrm{V}}_{\omega_1}$ will be wellfounded for obvious reasons.

Under certain assumptions the existence of iteration strategies is guaranteed through the following theorem which we take from [12]. The object $\mathrm{V}^\sharp_\delta$ is some kind of large cardinal assumptions which follows from the existence of a measurable cardinal above $\delta$. For a rough overview on this see [4].

**17 Theorem.** *Suppose that there is a Woodin limit of Woodin cardinals in* V, *say $\delta$, and suppose that $(\mathrm{V}_\delta)^\sharp$ exists. Then there is a class model $M$ of* $\mathsf{ZFC}^*$ *and some $\bar\delta \in M$ so that: $\bar\delta$ is a Woodin limit of Woodin cardinals in $M$, $\bar\delta$ is countable in* V *and $M$ is iterable.*

# 2 Woodin's extender algebra

In this chapter we are going to present Woodin's extender algebra, a forcing notion due to Woodin. This forcing notion depends on a Woodin limit of Woodin cardinals. The generic objects added satisfy certain statements about extender models of the generic extension. In Section 2.1 we are going to discuss the generic objects added which are called positions. In Section 2.2 we introduce a logic to describe these objects. Furthermore we introduce an inference relation for this logic, where we say that a statement $\rho$ implies a statement $\sigma$ if every object satisfying $\rho$ also satisfies $\sigma$. Section 2.3 introduces the forcing notion: Its conditions are equivalence classes of statements of the logic described in the Section before ordered by inference. However, we do not allow every statement in the forcing notion but only these which do neither contradict the basic axioms nor the extender axioms. The former we use to make sure that the generic extension contains an object as described in Section 2.1, the latter to make certain statements about extender models of the generic extension true. In Section 2.4 we investigate in which cases we can and in which cases we cannot extend a given position to a generic object.

The main point about the generic objects is that they uniquely define objects (so-called positions) corresponding to positions of the game defined in Chapter 3. We use this positions to encode long sequences of real numbers as needed for our main theorem presented in Chapter 4. We work in a fixed inner model $M$ of $\mathsf{ZFC}^*$.

Our presentation follows Sections 4A–4B of [13].

## 2.1 Positions

In the following, we use $\mathcal{W}$ to denote the class of Woodin cardinals in $M$ which are not themselves limits of Woodin cardinals. For $\delta \in \mathcal{W}$, we say $\delta$ is a *relative successor* if $\delta$ is the smallest Woodin cardinal or $\mathcal{W} \upharpoonright \delta$ has a greatest element. Otherwise, if $\mathcal{W} \upharpoonright \delta$ does not have a maximum, we call $\delta$ a *relative limit*. Furthermore, for $\delta \in \mathcal{W}$, we define $e(\delta) = \sup\{\kappa \in \mathcal{W} \mid \kappa < \delta\}$. It is easy to check that this is always smaller than $\delta$. If we refer to a cardinal as a relative successor or as a relative limit it is implicitly to be understood that it is an element of $\mathcal{W}$.

Furthermore we let the class $\mathcal{L}$ consist of all ordinals $\kappa$ such that $\mathcal{W}$ is cofinal in $\kappa$ (where we say that every class is cofinal in 0 and for any ordinal $\alpha$ we say that a class is cofinal in $\alpha + 1$ if it contains $\alpha$; for limits we adopt the usual convention). Then $\mathcal{L}$ contains 0, $\delta + 1$ for every $\delta \in \mathcal{W}$ and all limits of Woodin cardinals. Note furthermore that $e(\delta)$ is the biggest element of $\mathcal{L}$ below $\delta$ for every $\delta \in \mathcal{W}$.

**18 Definition.** *If $\lambda \in \mathcal{L}$ we say that $w$ is a* witness *for $\lambda$ if $w$ is a function such that:*

*(1)* $\mathrm{dom}(w) \subset \omega$,

*(2)* $\mathrm{rng}(w)$ *is a subset of* $\mathcal{W} \upharpoonright \lambda$,

*(3)* $\mathrm{rng}(w)$ *is cofinal in* $\lambda$,

*(4) $w$ is injective.*

Note that if $\lambda$ is a Woodin cardinal itself (in which case it is the limit of Woodin cardinals), a witness for $\lambda$ will not exist in $M$ as for this it need to have countable cofinality.

We now are able to phrase the main definition of this section:

**19 Definition.** *A function $t$ in* V *is an $M$-*position *if for some $\kappa \in \mathcal{L}$, $t$ has domain* $(\mathcal{W} \cup \mathcal{L}) \upharpoonright \kappa$ *such that the following conditions are satisfied:*

*(1) If $\delta$ is a relative successor, $t(\delta)$ is a real (i.e., an element of $\omega^\omega$),*

*(2) if $\delta$ is a relative limit, $t(\delta)$ is an element of* $(M \upharpoonright e(\delta) + 1)^\omega$,

*(3) if $\lambda \in \mathcal{L} \upharpoonright \kappa$, $t(\lambda)$ is a witness for $\lambda$.*

*We call $\kappa$ the relative domain of $t$, denoted by* $\mathrm{rdm}(t)$.

To better understand this definition it may be useful to imagine the following examples. Note first that $\emptyset$ is a position of relative domain 0.

Suppose $\delta^\dagger$ is a relative successor and let $\delta$ be the biggest element of $\mathcal{W}$ below it (If $\delta^\dagger$ is the smallest Woodin cardinal, work with $t = \emptyset$ in the following). Given a position $t$ with relative domain $\delta + 1$, we may extend it to a position $t^\dagger$ of relative domain $\delta^\dagger + 1$. For this we need a witness for $\delta + 1$. As this is a successor ordinal any injective function $w$ whose domain is a subset of $\omega$ such that $w(n) = \delta$ for some $n \in \omega$ suffices. However for the forthcoming applications it is more useful to look at $t(e(\delta))$ which is a witness for $e(\delta)$. If there is some $n \in \omega \setminus \mathrm{dom}(e(\delta))$ we may set $w = t(e(\delta)) \cup (n, \delta)$ to get a witness for $\delta + 1$. By taking a real $x$ and defining $t^\dagger = t \cup \{(\delta + 1, w), (\delta^\dagger, x)\}$ we get a position of relative domain $\delta^\dagger + 1$ (If $\delta^\dagger$ is the first Woodin cardinal, this can be safely ignored of course)

If $\kappa$ is a limit of elements in $\mathcal{W}$ the situation is more complex but in usually the situation is as follows: For some $\gamma$ we have positions $(t_\alpha \colon \alpha < \gamma)$ such that:

- The position $t_\beta$ extends $t_\alpha$ whenever $\alpha < \beta < \gamma$;
- the sequence $(\mathrm{rdm}(t_\alpha) \colon \alpha < \gamma)$ tends to $\kappa$.

By taking now $t' = \bigcup_{\alpha<\gamma} t_\alpha$ we get a position of relative domain $\kappa$.

If now $\delta$ is a relative limit and we have a position of relative domain $e(\delta)$, we only need a witness for $e(\delta)$ and an element of $(M \restriction e(\delta) + 1)^\omega$. Motivated by the above paragraph we furthermore ask for the following condition (it is safe to assume that $\mathrm{rdm}(t_\alpha)$ are successors of elements in $\mathcal{W}$):

- The witness $t_\beta(e(\mathrm{rdm}(t_\beta) - 1))$ extends $t_\alpha(e(\mathrm{rdm}(t_\alpha) - 1))$ whenever $\alpha < \beta < \gamma$.

Now by setting $w = \bigcup_{\alpha<\gamma} t_\alpha(e(\mathrm{rdm}(t_\alpha)))$ we get a witness for $e(\delta)$ and together with $x \in (M \restriction e(\delta) + 1)^\omega$ we get $t = t' \cup \{(e(\delta), w), (\delta, x)\}$, a position of relative domain $\delta + 1$.

## 2.2 Talking about positions

As indicated in the outline the forcing conditions of Woodin's extender algebra are statements about positions. To formalize these properly we give in the following a short exposition of the logic's syntax and semantics.

**20 Definition.** *The following statements are called* basic identities*:*

*(1)* $\check{t}(\delta)(n) = m$ *for* $\delta$ *a relative successor and* $m, n \in \omega$*,*

*(2)* $\check{t}(\delta)(n) = a$ *for* $\delta$ *a relative limit and* $n \in \omega, a \in M \restriction e(\delta) + 1$*,*

*(3)* $\check{t}(\lambda)(n) = \alpha$ *for* $\lambda \in \mathcal{L}$ *and* $n \in \omega$*,* $\alpha < \lambda$*.*

Identities *then are all objects generated by the above using negation and transfinite disjunction. For an identity $\sigma$ we recursively define its* height*,* $\mathrm{ht}(\sigma)$*:*

*(1) if $\sigma$ is a basic identity $\check{t}(\kappa)(n) = x$ for $\lambda \in \mathcal{W} \cup \mathcal{L}$ (and the other objects appropriate), then* $\mathrm{ht}(\sigma) = \kappa + 1$*;*

*(2) if $\sigma = \neg\rho$ for some identity $\rho$, then* $\mathrm{ht}(\sigma) = \mathrm{ht}(\rho)$*;*

*(3) if $\sigma = \bigvee_{\xi<\alpha} \sigma_\xi$ for identities $\sigma_\xi$ $(\xi < \alpha)$, we define* $\mathrm{ht}(\sigma) = \sup_{\xi<\alpha} \mathrm{ht}(\sigma_\xi)$.

We use the letter $\check{t}$ since there is a uniform definition of a position $t(G)$ from the generic filter $G$ which will satisfy the above given statements if the generic filter forces these statements to be true. At this point it should be easy come up with a formal representation of identities such that the rank of the formal representation corresponds to the height of the identity (and the length of the disjunctions involved).

The definition provides one with a proper class of identities. However, in an application we will only talk about positions of bounded height (bounded by a Woodin limit of Woodin cardinals) Still, the unbounded length of disjunctions may lead to problems: However, we will see that we can safely replace (i.e., without changing the truth-conditions) an identity with too long disjunctions by another one inside a rank initial segment of $M$ fixed in advance.

**21 Definition.** *For a position $t$ and an identity $\sigma$ with* $\mathrm{ht}(\sigma) \le \mathrm{rdm}(t)$ *we define $t \models \sigma$ as follows:*

*(1) if $\sigma$ is a basic identity $\check{t}(\kappa)(n) = x$ for $\kappa \in \mathcal{W} \cup \mathcal{L}$ (and the other objects appropriate), then $t \models \sigma$ iff $t(\kappa)(n) = x$;*

*(2) if $\sigma = \neg\rho$, then $t \models \sigma$ iff $t \not\models \rho$;*

*(3) if $\sigma = \bigvee_{\xi<\alpha} \sigma_\xi$ for some $\alpha$, then $t \models \sigma$ iff there is a $\xi < \alpha$ for which $t \models \sigma_\xi$.*

If $A$ is a set of identities of height at most $\mathrm{rdm}(t)$, we write $t \models A$ in case that $t \models \sigma$ for all $\sigma \in A$.

In the following fix $\vartheta$, a Woodin limit of Woodin cardinals. The purpose of the forcing notion $\mathbb{W}_\vartheta$ is to add a position of relative domain $\vartheta$. We are going to see that this position can be adequately described using identities whose formal representations are in $M \restriction \vartheta$.

Let $\vdash$ be the smallest set of pairs consisting of identities $\sigma \in M \restriction \vartheta$ (which implies that $\mathrm{ht}(\sigma) < \vartheta$) and sets $A$ of identities in $M \restriction \vartheta$ such that the following conditions hold:

(I1) If $\sigma \in A$, then $A \vdash \sigma$;

(I2) if for some identity $\tau \in M \restriction \vartheta$, $A \vdash \tau$ and $A \cup \{\tau\} \vdash \sigma$, then $A \vdash \sigma$;

(I3) if $\sigma = \neg\neg\tau$ for some $\tau$ and $A \vdash \tau$, then $A \vdash \sigma$;

(I4) if $\tau = \neg\neg\sigma$ for some $\tau$ and $A \vdash \tau$, then $A \vdash \sigma$;

(I5) if $\sigma = \bigvee_{\xi<\alpha} \sigma_\xi$ for some $\sigma_\xi$ ($\xi < \alpha$) for some $\alpha < \vartheta$ and there is a $\xi < \alpha$ such that $A \vdash \sigma_\xi$, then $A \vdash \sigma$;

(I6) if $\sigma = \neg \bigvee_{\xi<\alpha} \sigma_\xi$ for some $\sigma_\xi$ ($\xi < \alpha$) for some $\alpha < \vartheta$ and for all $\xi < \alpha$ we have $A \vdash \neg\sigma_\xi$, then $A \vdash \sigma$;

(I7) if there is an identity $\tau \in M \restriction \vartheta$ such that $A \cup \{\neg\sigma\} \vdash \tau$ and $A \cup \{\neg\sigma\} \vdash \neg\tau$, then $A \vdash \sigma$.

If $A$ is a singleton $\{\rho\}$ we omit the brackets and write $\rho \vdash \sigma$ instead of $\{\rho\} \vdash \sigma$. Using (I1) as the start of the recursive construction and keeping adding pairs $(A, \sigma)$ to satisfy the other conditions, one can show that the construction ends after $\vartheta$ steps. The so-defined relation $\vdash$ has the following properties (where $A, B$ are sets of identities in $M \restriction \vartheta$, $\sigma, \tau$ are such identities and $t$ is a position with big enough relative domain, $\tau \wedge \sigma$ is short for $\neg(\neg\sigma \vee \neg\tau)$ and $\tau \to \sigma$ is short for $\neg\tau \vee \sigma$):

(P1) $t \models A$ and $A \vdash \sigma$ imply $t \models \sigma$;

(P2) $A \vdash \sigma$ and $B \supset A$ imply $B \models \sigma$;

(P3) $\tau \vdash \sigma$ implies $\neg\tau \vdash \neg\sigma$;

(P4) $\sigma \wedge \tau \vdash \sigma$ and $\sigma \wedge \tau \vdash \tau$;

(P5) $A \vdash \neg(\tau \wedge \sigma)$ implies $A \cup \{\tau\} \vdash \neg\sigma$;

(P6) $A \vdash \tau \to \sigma$ implies $A \cup \{\tau\} \vdash \sigma$.

**22 Definition.** *If $\tau$ and $\sigma$ are identities in $M \restriction \vartheta$, we define $\tau \leq \sigma$ iff $\tau \vdash \sigma$. If $\tau \leq \sigma$ and $\sigma \leq \tau$ we write $\sigma \cong \tau$.*

It can then be easily checked that $\leq$ is reflexive and transitive and that $\cong$ is an equivalence relation which is compatible with $\leq$. We use $[\sigma]$ to denote the equivalence class of $\sigma$ with respect to $\cong$ and denote by $\mathbb{A}$ (as is common practice and a slight misuse of notation, we use this letter both to denote the ordered structure but also its underlying set. We will continue doing so in the following without explicitly mentioning it.) the poset of equivalence classes together with the ordering $\leq_{\mathbb{A}}$ which we get by taking the quotient of $\preceq$ modulo $\cong$:

$$\mathbb{A} = \{[\sigma] \mid \sigma \text{ is an identity in } M \restriction \vartheta\}$$
$$[\sigma] \leq_{\mathbb{A}} [\tau] \Leftrightarrow \sigma \vdash \tau$$

The forcing poset we want to come up with at the end is a restriction of this poset. One reason to restrict it is to make sure that the generic object added gives rise to a unique position.

## 2.3 Defining Woodin's extender algebra

**23 Definition.** *An identity is called a* basic axiom *if it is of one of the following forms for some $\delta \in \mathcal{W} \restriction \vartheta$ or some $\kappa \in \mathcal{L} \restriction \vartheta$:*

*(1)* $\bigvee_{i<\omega} \check{t}(\delta)(n) = i$ *if $\delta$ is a relative successor and $n \in \omega$;*

*(2)* $\bigvee_{\xi<\alpha} \check{t}(\delta)(n) = a_\xi$ *if $\delta$ is a relative limit and $n \in \omega$, where $(a_\xi \mid \xi < \alpha)$ is an enumeration of $M \restriction e(\delta) + 1$ for $\alpha < \vartheta$;*

*(3)* $\neg\check{t}(\delta)(n) = x \vee \neg\check{t}(\delta)(n) = y$ *for $n \in \omega$ and $x$ and $y$ are distinct elements of $\omega$ if $\delta$ is a relative successor or of $M \restriction e(\delta) + 1$ if $\delta$ is a relative limit;*

*(4)* $\neg\check{t}(\kappa)(n) = \alpha \vee \neg\check{t}(\kappa)(n) = \beta$ *where $n \in \omega$ and $\alpha$ and $\beta$ are distinct ordinals $< \kappa$;*

*(5)* $\neg\check{t}(\kappa)(n) = \alpha \vee \neg\check{t}(\kappa)(m) = \alpha$ *where $\alpha < \kappa$ and $n$ and $m$ are distinct natural numbers;*

*(6)* $\bigvee_{n<\omega} \bigvee_{\alpha\in\mathcal{W}\restriction(\bar{\kappa},\kappa)} \check{t}(\kappa)(n) = \alpha$ *where $\bar{\kappa}$ is some element of $\mathcal{L} \restriction \kappa$.*

**24 Remark.** If $t$ is a position with $\mathrm{rdm}(t) = \vartheta$, then all the basic axioms are true for $t$.

So far, we can make sure that the generic object added by the forcing poset will define a unique position. Furthermore, we ask the identities in the generic filter to satisfy the axioms given by the next Definition:

**25 Definition.** *Let $\delta \in \mathcal{W} \restriction \vartheta$ and $\kappa < \delta$ be not a Woodin cardinal in $M$. Let furthermore be $E$ an ($M$-)extender such that $E \in M \restriction \delta + \omega$,* $\mathrm{crit}(E) = \kappa$ *and such that $M \restriction \delta + 1 \subset \mathrm{Ult}(M, E)$. If additionally $\vec{\sigma} = (\sigma_\xi \mid \xi < \kappa) \in M$ is* *a sequence of identities such that each of them is an element of $M \upharpoonright \kappa$ and the identity $\pi(\vec{\sigma})(\kappa)$, where $\pi$ is the canonical ultrapower embedding, has height at most $\delta + 1$, we define the* extender axiom

$$\chi(\kappa, \delta, E, \vec{\sigma}) = (\pi(\vec{\sigma}))(\kappa) \to \bigvee_{\xi<\kappa} \sigma_\xi \,.$$

Note first that this is a well-defined identity: As $\kappa$ is the critical point of $\pi$, $\pi(\vec{\sigma})$ is a sequence of identities (in the ultrapower) of length $\pi(\kappa) > \kappa$, so it makes sense to talk about $(\pi(\vec{\sigma}))(\kappa)$. Now, if $t$ is a position of high enough relative domain such that $t \models (\pi(\vec{\sigma}))(\kappa)$ it certainly holds that $t \models \bigvee_{\xi<\pi(\kappa)}(\pi(\vec{\sigma}))_\xi$. Now one expects to be able to pull this back to get $t \models \bigvee_{\xi<\kappa} \vec{\sigma}(\xi)$ but in this argumentation one neglects the effects $\pi$ may have on $t$. However as Theorem 31 will show we can get rid of cases where $t$ does not satisfy an extender axiom by passing to certain ultrapowers.

It is now time to define our forcing notion. As we have stated before we restrict $\mathbb{A}$ so that its elements (or rather the members of the equivalence classes $\mathbb{A}$ contains) do not contradict the basic or the extender axioms.

**26 Definition.** *Let $\mathsf{Ax}$ be the set of all basic axioms as defined in Definition 23 and all extender axioms as defined in Definition 25. An identity $\sigma \in M \upharpoonright \vartheta$ is called* good *just in case that $\mathsf{Ax} \nvdash \neg\sigma$. Otherwise $\sigma \in M \upharpoonright \vartheta$ is called* bad. *Now define $\mathbb{W}$ as the subset of $\mathbb{A}$ consisting of all equivalence classes containing good identities, i.e.,*

$$\mathbb{W} = \{[\sigma] \in \mathbb{A} \mid \sigma \text{ is a good identity}\} \,.$$

*We order $\mathbb{W}$ by $\leq_{\mathbb{W}}$ which is the restriction of $\leq_{\mathbb{A}}$ to $\mathbb{W}$.*

This clearly is well-defined as $\sigma \cong \tau$ implies that $\sigma$ is good iff $\tau$ is good using property (P3). At this place the reader should note the following: $\mathbb{W}$ is clearly dependent on $\vartheta$ and on $M$ and in fact it is not the first of the objects defined with these dependencies. The same holds for $\mathbb{A}$, $\mathsf{Ax}$, $\cong$, $\prec$ and other objects defined and used before. Note furthermore that all of the objects are sets in $M$.

Recall that the $\gamma$ chain condition ($\gamma$-cc for a forcing poset $\mathbb{P}$ and a cardinal $\gamma$ means that all antichains in $\mathbb{P}$ have size less than $\gamma$. It is a well-known fact that a forcing with the $\gamma$-cc preserves cardinals bigger or equal to $\gamma$.

**27 Theorem.** *$\mathbb{W}_\vartheta$ has the $\vartheta$ chain condition in $M$.*

*Proof.* We work in $M$ and assume that there is an antichain of length $\vartheta$. From such an antichain we are going to construct objects $\delta, \kappa, E, \vec{\sigma}$ satisfying the assumptions of Definition 25 such that the extender axiom $\chi(\kappa, \delta, E, \vec{\sigma})$ leads to a contradiction (which corresponds to Ax containing bad identities).

Working by induction we may assume that

$$M \models \mathbb{W}_{\bar{\vartheta}} \text{ has the } \bar{\vartheta}\text{-cc for all Woodin limits of Woodin cardinals } \bar{\vartheta} < \vartheta.$$

Let $A \subset \mathbb{W}_\vartheta$ be an antichain of length $\vartheta$ in $M$. Enumerate $A$ as $(\sigma_\xi \mid \xi < \vartheta)$ where each $\sigma_\xi$ is a good identity.

Let $F = \{(\xi, \sigma_\xi) \mid \xi < \vartheta\} \subset M \restriction \vartheta$ and using Woodinness of $\vartheta$ pick a $\kappa$ which is $< \vartheta$-strong with respect to $F$. Let $\nu < \vartheta$ be so that $F \restriction \kappa + 1 \in M \restriction \nu$. As $\vartheta$ is a limit of Woodin cardinals, fix $\delta \in \mathcal{W}$ so that $\delta \geq \nu$ (which certainly implies that $\delta > \kappa$).

As $\kappa$ is $< \vartheta$-strong with respect to $F$ there is an extender $E$ with

(1) critical point $\mathrm{crit}(E) = \kappa$,

(2) so that $M \restriction \delta + 1 \subset \mathrm{Ult}(M, E)$,

(3) $\pi(F) \cap M \restriction \delta = F \cap M$ where $\pi$ is the canonical elementary ultrapower embedding $M \to \mathrm{Ult}(M, E)$

. Furthermore we can demand that

(4) $E$ belongs to $M \restriction \delta + \omega$

by replacing it by a restriction. As $E$ is $\delta + 1$-strong, we have that

(5) $\kappa < \delta < \pi(\kappa)$.

We now want to establish that $\kappa$ is not a Woodin cardinal of $M$.

**Claim.** $\pi(F)) \restriction \kappa + 1 = F \restriction \kappa + 1$, more specifically: for all $\xi \leq \kappa : F(\xi) = \pi(F)(\xi)$.

*Proof.* Observe first that for $\xi \leq \kappa$, $(\xi, \sigma_\xi) \in M \restriction \delta$ hence $(\xi, \sigma_\xi) \in \pi(F)$ by (3). Furthermore $F$ is a function and $\pi$ is elementary, thus $\pi(F)$ is a function too. ⊣

**Claim.** For each $\xi < \kappa$, $\sigma_\xi$ belongs to $M \restriction \kappa$.

*Proof.* By elementarity of $\pi$ we have $\pi(F)(\pi(\xi)) = \pi(\sigma_\xi)$ and as $\xi < \kappa$, $\pi(\xi) = \xi$ and hence $\pi(F))\pi(\xi)) = \pi(F)(\xi) = \sigma_\xi$ by the previous Claim. So have $\pi(\sigma_\xi) = \sigma_\xi$ and as $\sigma_\xi \in M \restriction \delta$ we also have $\pi(\sigma_\xi) \in M \restriction \delta$. Because of (2) we have $M \restriction \delta = \mathrm{Ult}(M, E) \restriction \delta$ and using $\delta < \pi(\kappa)$ (which we know from (5)) this leads to $\pi(\sigma_\xi) \in \mathrm{Ult}(M, E) \restriction \pi(\kappa)$. Using that $\pi$ is elementary we can pull this statement back to $M$ to get $\sigma_\xi \in M \restriction \kappa$. ⊣

**Claim.** $\kappa$ is not a Woodin cardinal in $M$.

*Proof.* First we prove that $\kappa$ is a limit of Woodin cardinals. Assume not: Then there was $\alpha < \kappa$ such that there were no Woodin cardinals between $\alpha$ and $\kappa$ (in $M$). By elementarity of $\pi$ there are no Woodin cardinals between $\alpha$ and $\pi(\kappa)$ in $\mathrm{Ult}(M, E)$ which is a contradiction by (5). If $\kappa$ itself was a Woodin cardinal it would be a Woodin limit of Woodin cardinals.

Assume $\kappa$ is Woodin, so that $\mathbb{W}_\kappa$ is defined. Then it can be proved that $\mathbb{W}_\kappa = \mathbb{W}_\vartheta \cap M \restriction \kappa$. For the basic axioms this is clear. If $\chi$ is an extender axiom of $\mathbb{W}_\kappa$, it is easy to see that $\chi \in \mathbb{W}_\vartheta \cap M \restriction \kappa$. If $\chi$ however is an extender axiom of $\mathbb{W}_\vartheta$ with $\chi \in M \restriction \kappa$, it can be checked that the length of the sequence of identities involved is smaller than $\kappa$ and then using that $\kappa$ is Woodin one may simulate the extender used in $\chi$ by another extender (satisfying the conditions for extender axioms in $\mathbb{W}_\kappa$) to see that $\chi$ is an element of $\mathbb{W}_\kappa$.

As $\sigma_\xi \in \mathcal{W}_\vartheta \cap M \restriction \kappa$ (by the previous claim) for all $\xi < \kappa$, $A \restriction \kappa$ is an antichain of length $\kappa$ in $\mathbb{W}_\kappa$, contradicting the induction hypothesis . ⊣

Now let $\vec{\sigma} = F \restriction \kappa$. Then $(\pi(\vec{\sigma}))_\kappa = \pi(F)(\kappa) = \sigma_\kappa$ by the first claim. As $\delta$ was chosen above $\nu$ which was chosen such that $\sigma_\kappa \in M \restriction \nu$, we clearly have $\mathrm{ht}(\pi(\vec{\sigma}_\kappa)) < \delta$.

By now we have established that $(\kappa, \delta, E, \vec{\sigma})$ satisfies the requirements of Definition 25 so that the extender axiom $\chi(\kappa, \delta, E, \vec{\sigma})$ is an element of $\mathsf{Ax}_\vartheta$. This means (using the above paragraph)

$$\sigma_\kappa \to \bigvee_{\xi<\kappa} \sigma_\xi \ \in \mathsf{Ax}_\vartheta$$

or, put differently using property (P6)

$$\mathsf{Ax} \cup \{\sigma_\kappa\} \vdash \bigvee_{\xi<\kappa} \sigma_\xi \,. \tag{2.1}$$

As $A$ is an antichain, every two elements of $A$ are incompatible, which means that for every $\xi < \vartheta$, the identity $\sigma_\xi \wedge \sigma_\kappa$ is bad since this is a common strengthening of $\sigma_\xi$ and $\sigma_\kappa$ by property (P4):

$$\mathsf{Ax} \vdash \neg(\sigma_\xi \wedge \sigma_\kappa)\,.$$

Thus we get $\mathsf{Ax} \cup \{\sigma_\kappa\} \vdash \sigma_\xi$ for all $\xi < \kappa$ by property (P5) and hence by inference rule (I6) we have

$$\mathsf{Ax} \cup \{\sigma_\kappa\} \vdash \neg \bigvee_{\xi<\kappa} \sigma_\xi\,. \tag{2.2}$$

Applying inference rule (I7) to Equation (2.1) and Equation (2.2) thus establishes $\mathsf{Ax} \vdash \neg\sigma_\kappa$, hence $\sigma_\kappa$ is a bad identity, which clearly is a contradiction to $\sigma_\kappa \in \mathbb{W}_\vartheta$. □

If $G$ is generic for $\mathbb{W}_\vartheta$ (in a forcing extension of $M$) we define $t = t(G)$ as follows:

for all $\delta \in \mathcal{W} \restriction \vartheta$ a relative successor and $n, m \in \omega$

$$t(\delta)(n) = x \;\Leftrightarrow G \vdash \check{t}(\delta)(n) = m,$$

for all $\delta \in \mathcal{W} \restriction \vartheta$ a relative limit, $n \in \omega$ and $x \in M \restriction e(\delta) + 1$

$$t(\delta)(n) = x \;\Leftrightarrow G \vdash \check{t}(\delta)(n) = x,$$

for all $\kappa \in \mathcal{L} \restriction \vartheta$, $n \in \omega$ and $\alpha \in \mathcal{W} \restriction \kappa$

$$t(\kappa)(n) = \alpha \;\Leftrightarrow G \vdash \check{t}(\kappa)(x) = \alpha.$$

Using that $G$ is a generic filter of $\mathbb{W}$ whose elements all satisfy the basic axioms one argues as following to establish Lemma 28 Let $\delta \in \mathcal{W} \restriction \vartheta$ be a relative successor and let $n \in \omega$. Note that for every identity $\sigma$ such that $[\sigma] \in \mathbb{A}$, the set $D = \{[\tau] \in \mathbb{W} \mid \tau \vdash \sigma \vee \tau \vdash \neg\sigma\}$ is dense: Let $[\rho] \in \mathbb{W}$ be a condition and assume that there is no $\nu \leq \rho$ so that $\nu \in D$. Then specifically $\sigma \wedge \rho$ and $\sigma \wedge \neg\rho$ are not elements of $\mathbb{W}$ which means that $\mathsf{Ax} \vdash \neg(\sigma \wedge \rho)$ and $\mathsf{Ax} \vdash \neg(\sigma \wedge \neg\rho)$. Using property (P5) and inference rule (I7) one sees that $\rho$ is bad which is a contradiction to $\rho \in \mathbb{W}$. If now $G \vdash \neg\check{t}(\delta)(n) = m$ for every $m \in \omega$ we get that $G \vdash \neg \bigvee_{m\in\omega} \check{t}(\delta)(n) = m$ which contradicts (1) of Definition 23. However the existence of two distinct $m, l$ such that $G \vdash \check{t}(\delta)(n) = m \wedge \check{t}(\delta)(n) = l$ directly contradicts (3) of Definition 23. Arguing similarly for the other cases one sees that the generic filter defines a unique position.

**28 Lemma.** *If $\vartheta$ is a Woodin limit of Woodin cardinals and $G$ is $\mathbb{W}_\vartheta$-generic over $M$, $t$ is a position of relative domain $\vartheta$.*

## 2.4 Obstructions

Given a $M$-position $t$ of some relative domain $\delta + 1$ for some $\delta \in \mathcal{W}$ we might ask how far $t$ is from being an initial segment of $t(G)$ for some $\mathbb{W}_\vartheta$-generic filter $G$ for some Woodin limit of Woodin cardinals $\vartheta$? In the later applications we want to extend/lengthen $t$ to such a condition: Obviously any problem arising will not be due to the basic axioms. This means then that there is some extender axiom which fails for a possible $G$ (with $t \subset t(G)$) which can only happen if there are objects $(\kappa, \delta, E, \vec{\sigma})$ satisfying the assumptions of Definition 25 such that $t$ does not satisfy the accompanying extender axiom.

**29 Definition.** *A pair* $(E, \vec{\sigma}) \in M$ *is called an* $M$-obstruction *for the position* $t$ *of relative domain* $\delta + 1$ *if the following conditions hold:*

*(1)* $E$ *is an extender such that its critical point* $\kappa = \mathrm{crit}(E)$ *is not a Woodin cardinal in* $M$*,* $E$ *is* $\delta + 1$*-strong and* $E \in M \upharpoonright \delta + \omega$*;*

*(2)* $\vec{\sigma} \in M$ *is a sequence of identities of length* $\kappa$ *such that each of them is an element of* $M \upharpoonright \kappa$*;*

*(3) for the canonical ultrapower embedding* $\pi$*, the identity* $(\pi(\vec{\sigma}))(\kappa)$ *has height at most* $\delta + 1$*;*

*(4) the position* $t$ *does not satisfy the extender axiom* $\chi(\kappa, \delta, E, \vec{\sigma})$*, which means:*

$$t \models (\pi(\vec{\sigma}))(\kappa),$$
$$t \models \neg\sigma_\xi \qquad \text{for all } \xi < \kappa.$$

*If there are no* $M$*-obstructions for* $t$*, we call* $t$ obstruction free *over* $M$*.*

Note that it is enough here to restrict ourselves to positions of relative domain $\delta + 1$ for some $\delta \in \mathcal{W}$ as the height of extender axioms is restricted to this value. If a position of relative domain $\kappa$ for some limit of Woodin cardinals $\kappa \in \mathcal{L}$ does not satisfy some extender axiom, this is witnessed by an initial segment of $t$. Furthermore notice that the critical point of the extender $E$ of an obstruction $(E, \vec{\sigma})$ for a position $t$ is a limit of Woodin cardinals which is not Woodin: This follows from the strongness condition on $E$ and can be seen using a representation of $\delta = \mathrm{rdm}(t) - 1$ in the ultrapower by $E$.

As stated before there are ways to remove obstructions by passing to certain ultrapowers:

**30 Definition.** *An $M$-obstruction $(E, \vec{\sigma})$ for $t$ is called* minimal *if it satisfies (additionally to the conditions in Definition 29) the following condition:*

$$t \text{ is obstruction free over } \mathrm{Ult}(M, E).$$

Minimal obstructions exist always:

**31 Theorem.** *Let $\delta \in \mathcal{W}$ and suppose $t \in \mathrm{V}$ is a $M$-position of relative domain $\delta + 1$. If $t$ is not obstruction free over $M$, there is a minimal $M$-obstruction for $t$.*

*Proof.* Start by choosing an obstruction $(E_0, \vec{\sigma}_0)$ for $t$. If $t$ is not obstruction free in the ultrapower $\mathrm{Ult}(M, E_1)$ pick an obstruction $(E_1, \vec{\sigma}_1)$ and notice that this also is an $M$-obstruction for $t$ using the previous Definition. If $t$ is not obstruction free in the ultrapower $\mathrm{Ult}(M, E_2)$, pick an obstruction $(E_2, \vec{\sigma}_2)$ witnessing that and notice that this also is an $M$-obstruction for $t$. Continuing in the same fashion we get a chain of extenders in $M$ which is descending in the Mitchell order. By Theorem 9 this chain has to be finite. The lowest element of this chain corresponds to a minimal $M$-obstruction. □

**32 Definition.** *$t$ is called $M$-*clear *if $t \restriction \delta + 1$ is obstruction free over $M$ for all $\delta \in \mathcal{W} \restriction \mathrm{rdm}(t)$.*

Now if we have a $M$-position $t$ of relative domain $\vartheta$ let us define $G(t)$:

$$G(t) = \{[\sigma] \in \mathbb{W}^M \mid t \models \sigma\}$$

$G = G(t)$ clearly satisfies all basic axioms (which just make sure that $t(G)$ is a position) and it is easy to check that being $M$-clear means that all extender axioms are satisfied (as a counterexample would provide an obstruction). Using that $t(G(t)) = t$ one gets:

**33 Lemma.** *Let $t$ be a $M$-position of relative domain $\vartheta$ where $\vartheta$ is a Woodin limit of Woodin cardinals in $M$. If $t$ is $M$-clear then $G(t)$ is $\mathbb{W}$-generic over $M$.*

The main statements for which we will use $\mathbb{W}$ are of the form $\check{t} \in \dot{Y}$ for some $\mathbb{W}$-name $\dot{Y}$. The next lemma then tells us that the Truth Lemma for the forcing relation can be strengthened in certain cases to the following:

**34 Lemma.** *Suppose that $\dot{Y} \in M$ is a $\mathbb{W}$-name for a set of positions of relative domain $\vartheta$. Then there is some ordinal $\alpha$, strictly smaller than $\vartheta$, so that:*

*For any $G$ which is $\mathbb{W}$-generic/$M$, $\check{t}_G \in \dot{Y}_G$ iff this is forced by some condition in $G \cap (M \restriction \alpha)$.*

*Proof.* Let $K \subset \mathbb{W}$ be the set of conditions forcing $\check{t} \in \dot{Y}$. Let $A \subset K$ be a maximal antichain in $K$. By the $\vartheta$-cc of $\mathbb{W}$ (see Theorem 27), we have that $| A | < \vartheta$. As $\vartheta$ is inaccessible and $A \subset M \restriction \vartheta$, we therefore have that $A \subset M \restriction \alpha$ for some $\alpha < \vartheta$. This $\alpha$ witnesses the statement of the theorem. □

## 2.5 Coding obstructions

Here we intend to shed some more light on the definition of positions given in Definition 19. Given a position $t$ of relative domain $\delta + 1$ for $\delta \in \mathcal{W}$ in some model $M$ which exists in some generic extension of $M$. We intend to code somehow certain obstructions of $t$ inside of $t$.

Recall that $t(\tau)$ for $\tau \in \mathcal{W}$ a relative limit is not a real but rather an element of $(M \restriction e(\tau) + 1)^\omega$ where $e(\tau) = \sup\{\kappa \in \mathcal{W} \mid \kappa < \tau\}$. We try to code information at these places which helps us to tell which extenders to choose when we want to pass to an ultrapower. For this we need the following Definition.

**35 Definition.** *Let $\kappa$ be an element of $\mathcal{L}$. We say a function $c$ is a $\kappa$-*functor *if it satisfies the following conditions:*

*(1) $c$ has domain $J = \{(\bar{\kappa}, \bar{\delta}) \in (\mathcal{L} \times \mathcal{W}) \restriction \kappa \mid \bar{\kappa} < \bar{\delta}\}$,*

*(2)* $\mathrm{rng}(c) \subset M \restriction \kappa$*,*

*(3) $c$ is injective,*

*(4) whenever $(\bar{\kappa}, \bar{\delta}) \in J$, the image $c(\bar{\kappa}, \bar{\delta})$ is a $col(\omega, \bar{\delta})$-name.*

We split $t(\tau)$ into its even and its odd part, denoted by $t_{\mathrm{I}}(\tau)$ and $t_{\mathrm{II}}(\tau)$:

$$t_{\mathrm{I}}(\tau) = (t(\tau)(2n) \mid n \in \omega) \quad t_{\mathrm{II}}(\tau) = (t(\tau)(2n+1) \mid n \in \omega).$$

As the names may suggest, the intuition behind these should be that player I is going to use the even part to code information important for him, while player II uses the odd part.

**36 Definition.** *For $t$ and $\tau$ as above, we say $t$ is* I-suitable *at $\tau$ if for every $n \in \omega$, the object $t_{\mathrm{I}}(\tau)(n)$ is an $e(\tau)$-functor.*

This Definition indeed makes sense: An $e(\tau)$-functor is a subset of $(\mathcal{L} \times \mathcal{W} \times M) \upharpoonright e(\tau)$ and thus an element of $M \upharpoonright e(\tau) + 1$. It thus says that if $t$ satisfies the definition, $t_{\mathrm{I}}(\tau)(n)$ is not just any element of $M \upharpoonright e(\tau) + 1$ but a special one.

For the main definition of this section we furthermore need the following:

**37 Definition.** *For a forcing notion $\mathbb{Q} \in M$ and a $\mathbb{Q}$-name $\dot{u} \in M$ we write $a \in \dot{u}[*]$ for a set $a$ (we do not require that $a \in M$) if there is some $\mathbb{Q}$-generic filter $G$ over $M$ such that $a \in \dot{u}[G]$. By $a \notin \dot{u}[*]$ we mean that $a \notin u[G]$ for any $\mathbb{Q}$-generic filter $G$ over $M$.*

**38 Definition.** *Let $t$ be as above. The pair $(E, \vec{\sigma})$ is a* I-*acceptable obstruction for $t$ if it satisfies the following conditions:*

*(1) $(E, \vec{\sigma})$ is a minimal $M$-obstruction for $t$,*

*(2) $t$ is* I-*suitable at $\tau(E)$,*

*(3) there is $n \in \omega$ for which $t \in \pi(c)(\kappa, \delta))(n))[*]$ where $\pi$ denotes the ultrapower embedding $M \to \mathrm{Ult}(M, E)$, $c = t_{\mathrm{I}}(\tau(E))(n)$ and $\kappa = \mathrm{crit}(E)$.*

It requires some thought to see that this Definition makes sense: Recall that $\tau(E)$ is the first Woodin cardinal above $\mathrm{crit}(E)$ and thus a relative limit since $\mathrm{crit}(E)$ is a limit of Woodin cardinals. Hence condition (2) makes sense. This implies that $c = t_{\mathrm{I}}(\tau(E))(n)$ is a $\mathrm{crit}(E)$-functor. Furthermore we have to check that condition (3) is satisfiable. The extender $E$ is $\delta + 1$-strong by definition and hence $\pi(\kappa) > \delta$ and $\delta$ is still a Woodin cardinal in the ultrapower. It follows that the pair $(\kappa, \delta)$ is in the domain of $\pi(c)$. By assumption $\pi(c)(\kappa, \delta))(n)$ is a name in $\mathrm{col}(\omega, \delta)$ and so condition (3) follows.

# 3 The branching game

This chapter is devoted to introduce the auxiliary game which is called the branching game. We will use the determinacy of this game to prove the determinacy of the local game in Chapter 4. In the branching game players I and II start within a model $M$ with an $M$-position $t$ and work to extend this position to have a certain relative domain fixed in advance. However, every time the position is extended they shift the model (and by this, also the length which has to be reached is shifted). If in the end the position is created is inside the interpretation of some shift of a set $\dot{C}$ which was fixed in advance or is I-acceptably obstructed, player I wins.

The content of this chapter can mostly be found in Section 6A and Section 6G of [13].

**39 Definition.** *For $\delta \in \mathcal{W}$, we say that a $\delta$-*sequence *is an $M$-clear annotated position of relative domain $\delta + 1$. By a $\delta$-*name *we mean a name $\dot{C}$ in the standard forcing collapsing $\delta$ to $\omega$ for a set of $\delta$-sequences. If $\delta$ is a Woodin limit of Woodin cardinals in $M$, an $\delta$-*sequence *is an $M$-clear position of relative domain $\delta$. A $\delta$-*name *is a $\mathbb{W}_\delta$-name $\dot{C}$ for a set of $\delta$-sequences.*

## 3.1 The rules of the branching game

Given a transitive model $M$ of $\mathsf{ZFC}^*$, a $\delta^*$-name $\dot{C}^*$ for some Woodin cardinal $\delta^*$ and a $\delta$-sequence $t$ for some Woodin cardinal $\delta < \delta^*$ we define the game $G_{\text{br}}(M, t, \delta^*)(\dot{C}^*)$. In this game, players I and II collaborate to extend $t$. While doing that they shift the model they are working in via branches of iteration trees. In the end they will have constructed a model $M^*$ together with an elementary embedding $j^* : M \to M^*$ and a $j^*(\delta^*)$-sequence $t^*$ over $M^*$ which extends $t$.

It is I's goal to make sure that in the end the constructed position $t^*$ is inside of $j^*(\dot{C})[g]$ where $g$ is a generic filter for some forcing poset suitable to interpret $j^*(\dot{C})$ or it is I-acceptably obstructed. Player I has to take care that at no point the current position is obstructed by an obstruction which is not I-acceptable. Furthermore the players have to ensure that the models appearing in the construction are wellfounded.

Let $M_0 = M$, $j_{0,0} = \mathrm{id}_M$, $t_0 = t$. We give separate rules for successor rounds and limit rounds. Furthermore we have another distinction for limit rounds $\beta$ depending on whether $\mathrm{rdm}(t_\beta)$, the relative domain of the position constructed in previous rounds, is a Woodin cardinal or not.

During the construction players I and II have to avoid certain conditions which put an early end on the game: These are labelled by E and referred to as snags. These snags generally prevent the construction of the objects as stated in the introduction of this chapter.

### 3.1.1 Successor round

At the beginning of round 0 or a successor round $\beta$ we are given a model $M_\beta$, an elementary embedding $j_{0,\beta} : M_0 \to M_\beta$ and an $M_\beta$-position $t_\beta$ which is $M_\beta$-clear and whose relative domain is not too big: $\mathrm{rdm}(t_\beta) < j_{0,\beta}(\delta^*)$.

First the position is extended: I picks a witness and then both players, I and II, collaborate to produce a real, so that the current position can be extended to a longer one. After that I plays an iteration tree of which II picks a branch. Then all objects are shifted to the direct limit along the branch in which certain properties are checked to make sure that when the next round starts off, the initial assumptions are satisfied:

(S1) I picks a witness $w_\beta$ for $\mathrm{rdm}(t_\beta)$ over $M_\beta$;

(S2) I and II alternate in playing natural numbers to produce a real $y_\beta$;

(S3) I plays a length-$\omega$-iteration tree $\mathcal{T}_\beta$ on $M_\beta$ such that all extenders of $\mathcal{T}_\beta$ have critical points above $\mathrm{rdm}(t_\beta)$ and are countable in V;

(S4) II picks a cofinal branch $b_\beta$ in $\mathcal{T}_\beta$.

We then let $Q_\beta$ be the direct limit of $\mathcal{T}_\beta$ along the branch $b_\beta$.

(E1) If $Q_\beta$ is illfounded, the game ends and player I wins.

If not, we can assume that $Q_\beta$ is wellfounded and let $k_\beta : M_\beta \to Q_\beta$ be the canonical elementary embedding given by the direct limit construction.

Due to the restrictions on $\mathcal{T}_\beta$, we have $\mathrm{crit}(k_\beta) > \mathrm{rdm}(t_\beta)$. This means that $t_\beta$ is a position of relative domain $\mathrm{rdm}^{Q_\beta}(t_\beta) = \mathrm{rdm}^{M_\beta}(t_\beta)$ over $Q_\beta$ and $w_\beta$ is a witness for $\mathrm{rdm}(t_\beta)$. Set $t^\dagger_\beta = t_\beta{}^\frown(w_\beta, y_\beta)$ (by this we mean that we extend the function $t_\beta$ in a way so that it still is a position and $w_\beta$) is the image of $\mathrm{rdm}(t_\beta)$ and $y_\beta$ is the image of the smallest element of $\mathcal{W}$ above $\mathrm{rdm}(t_\beta)$) which then is a position of relative domain $k_\beta(\delta^\dagger_\beta) + 1$ where $\delta^\dagger_\beta$ is the smallest Woodin cardinal above $\mathrm{rdm}(t_\beta)$. Note that $\delta^\dagger_\beta$ exists since $\mathrm{rdm}(t_\beta) < j_{0,\beta}(\delta^*)$ which is a Woodin cardinal.

(E2) If $t^\dagger_\beta$ is $Q_\beta$-obstructed, the game ends. Player I wins if and only if $t^\dagger_\beta$ is I-acceptably $Q_\beta$-obstructed.

Otherwise, we let $M_{\beta+1} = Q_\beta$, $j_{\beta,\beta+1} = k_\beta$ and $t_{\beta+1} = t^\dagger_\beta$. Since $t_{\beta+1}$ is not obstructed by (E2) and $t_{\beta+1} \restriction j_{\beta,\beta+1}(\mathrm{rdm}(t_\beta) + 1)$ is $M_{\beta+1}$-clear by the initial assumptions on $t_\beta$ and the elementarity of $j_{\beta,\beta+1}$, we have that $t_{\beta+1}$ is $M_{\beta+1}$-clear. We furthermore have that $\mathrm{rdm}(t_{\beta+1}) = \delta_{\beta+1} + 1$ by the definition of $\delta_{\beta+1}$. From the initial assumption $\delta^\dagger_\beta < j_{0,\beta}(\delta^*)$ we get $\delta_{\beta+1} \leq j_{0,\beta+1}(\delta^*)$ where $j_{0,\beta+1} = j_{\beta,\beta+1} \circ j_{0,\beta}$.

(P1) If $\delta_{\beta+1} = j_{0,\beta+1}(\delta^*)$ the game ends. Player I wins if and only if there exists a $g$ such that

- $g$ is $\mathrm{col}(\omega, j_{0,\beta+1}(\delta^*))$-generic over $M_{\beta+1}$ and
- $t_{\beta+1} \in j_{0,\beta+1}(\dot{C}^*)[g]$.

Otherwise we may assume that $\delta_{\beta+1} < j_{0,\beta+1}(\delta^*)$. We then let $j_{\alpha,\beta+1} = j_{\beta,\beta+1} \circ j_{\alpha,\beta}$ for $\alpha < \beta$. It can be seen then that the initial assumptions of the successor mega-round $\beta + 1$ are satisfied. We transition to this round.

### 3.1.2 The limit round

In a limit round $\beta$ we have wellfounded models $M_\xi$ for $\xi < \beta$ with elementary embeddings $j_{\zeta,\xi} : M_\zeta \to M_\xi$ for $\zeta < \xi$ and $M_\xi$-positions $t_\xi$ such that the following conditions hold for all $\xi < \beta$:

(a) $t_\xi$ is $M_\xi$-clear,

(b) $\mathrm{rdm}(t_\xi) < j_{0,\xi}(\delta^*)$,

(c) $(t_\xi \mid \xi < \beta)$ is a strictly increasing sequence of positions and

(d) $\mathrm{crit}(j_{\zeta,\xi}) \geq \mathrm{rdm}(t_\zeta)$ for each $\zeta < \xi$.

We then let $M_\beta$ be the direct limit of the above chain of models

$$M_\beta = \operatorname{dir\,lim}(M_\zeta, j_{\zeta,\xi} \mid \zeta < \xi < \beta) \tag{3.1}$$

with elementary embeddings $j_{\xi,\beta} : M_\xi \to M_\beta$ for $\xi < \beta$.

(E3) If $M_\beta$ is illfounded, then the run of $G_{\mathrm{br}}$ ends and player I wins.

(E4) If $\beta = \omega_1^{\mathrm{V}}$, then the run of $G_{\mathrm{br}}$ ends and player I loses.

Notice that (E3) and (E4) exclude each other as limits of chains of wellfounded models of length $\omega_1$ are wellfounded.

We can conclude from the conditions above:

(e) $\operatorname{crit}(j_{\xi,\beta}) \geq \operatorname{rdm}(t_\beta)$ for $\xi < \beta$. Hence the $t_\xi$'s are positions over $M_\beta$.

We let $t_\beta = \bigcup_{\xi<\beta} t_\xi$. This $t_\beta$ is a $M_\beta$-position for which the following conditions hold:

(f) $t_\beta$ is $M_\beta$-clear since all the $t_\xi$'s are,

(g) $\operatorname{rdm}(t_\beta) \leq j_{0,\beta}(\delta^*)$, and

(h) $\operatorname{rdm}(t_\beta)$ is a limit of Woodin cardinals in $M_\beta$.

Then, one of the following two cases occurs:

A) If $\operatorname{rdm}(t_\beta)$ is a Woodin cardinal in $M_\beta$ itself (so it is a Woodin limit of Woodin cardinals), we continue with the phantom limit case.

B) If $\operatorname{rdm}(t_\beta)$ is not a Woodin cardinal in $M_\beta$, we follow the rules for the standard limit case.

#### The phantom limit case

In this case we just immediately transition to the next round with the objects produced up to this point, unless our position has reached the required relative domain. Let $M_{\beta+1} = M_\beta$, $j_{\beta,\beta+1} = \mathrm{id}_{M_\beta}$, $t_{\beta+1} = t_\beta$ and $\delta_{\beta+1} = \delta_\beta = \operatorname{rdm}(t_\beta)$. All the not yet defined embeddings $j_{\xi,\beta}$ for $\xi < \beta$ are built by composition. By condition (g) above we have $\delta_{\beta+1} \leq j_{0,\beta+1}(\delta^*)$.

(P2) If $\delta_{\beta+1} = j_{0,\beta+1}(\delta^*)$, the game $G_{\mathrm{br}}(M, t, \delta^*)(\dot{C}^*)$ ends. Player I wins if and only if there exists a $G$ such that

- $G$ is $j_{0,\beta+1}(\mathbb{W}_{\delta^*})$-generic over $M_{\beta+1}$ and
- $t_{\beta+1} \in j_{0,\beta+1}(\dot{C}^*)[G]$.

In the other case, if $\delta_{\beta+1} < j_{0,\beta+1}(\delta^*)$, we pass to the next round $\beta + 1$. It is easy to see that the initial conditions of this successor round are satisfied.

### The standard limit case

If $\mathrm{rdm}(t_\beta)$ is not Woodin, we know that $\lambda_\beta = \mathrm{rdm}(t_\beta) < j_{0,\beta}(\delta^*)$ since $j_{0,\beta}(\delta^*)$ is Woodin. For $\tau_\beta$ the least Woodin cardinal above $\lambda_\beta$ we have $\tau_\beta \leq j_{0,\beta}(\delta^*)$. First I and II continue similar to the successor case. Note however that not a real is played as here $t(\kappa)$ is produced for $\kappa$ a relative limit.

(L1) I plays a witness $w_\beta$ for $\lambda_\beta$ over $M_\beta$,

(L2) I plays a length $\omega$ iteration tree $\mathcal{T}_\beta$ on $M_\beta$ all of whose extenders have critical points above $\lambda_\beta$ and are countable in V,

(L3) II plays a cofinal branch $b_\beta$ in $\mathcal{T}_\beta$.

We let $Q_\beta$ be the direct limit of the models in $\mathcal{T}_\beta$ along the branch $b_\beta$ and $k_\beta : M_\beta \to Q_\beta$ be the elementary embedding given by this construction. Again the game ends with player I winning if player II chooses a bad branch:

(E5) If $Q_\beta$ is illfounded, the run of $G_{\mathrm{br}}$ ends and player I wins the game.

Otherwise we continue building the position.

(L4) I picks $y_\beta$ in $(Q_\beta \upharpoonright \lambda_\beta + 1)^\omega$.

Note that in this case, not I and II construct this $\omega$-sequence together but rather player I constructs it on his own. Due to the restrictions on the extenders used in the iteration tree by (L2), we have that $t_\beta$ is a position in $Q_\beta$ of relative domain $\lambda_\beta$. Thus, $s_\beta = t_\beta{}^\frown(w_\beta, y_\beta)$ is a $Q_\beta$-position with $\mathrm{rdm}(s_\beta) = k_\beta(\tau_\beta) + 1$.

(E6) If the position $s_\beta$ is obstructed but not I-acceptably obstructed, the game $G_{\mathrm{br}}(M, t, \delta^*)(\dot{C}^*)$ ends and player I loses.

Then, if $s_\beta$ is obstruction free, player II decides whether the round $\beta$ ends now. If player II decides to end the round, we let $M_{\beta+1} = Q_\beta$, $j_{\beta,\beta+1} = k_\beta$, $t_{\beta+1} = s_\beta$, $\delta_{\beta+1} = k_\beta(\tau_\beta)$. One can see then that $t_{\beta+1}$ is $M_{\beta+1}$-clear and has relative domain $\delta_{\beta+1} + 1$ which is such that $\delta_{\beta+1} \leq j_{0,\beta+1}(\delta^*)$. If the two are equal, the game ends with payoff condition (P1); otherwise we continue with round $\beta + 1$. If $s_\beta$ is I-acceptably obstructed this round continues in every case.

### Leaps

If II does not choose to end the round early or if $s_\beta$ is I-acceptably obstructed, player II has extra moves to which we refer as leap. In this moves she may leap forward from and extend the current position $s_\beta$ on her own. Thereby she may skip rounds which would have otherwise been played. She does this by choosing a larger Woodin cardinal $\delta_{\beta+1}$ and extending the currently produced position to have relative domain $\delta_{\beta+1} + 1$.

(L5) II plays an extender $E^*_\beta$ such that

(1) $E^*_\beta$ is $k_\beta(\tau_\beta) + 1$-strong in some model $Q^*_\beta$ such that $Q^*_\beta$ and $Q_\beta$ agree to $k_\beta(\tau_\beta) + 1$,

(2) $E^*_\beta$ is countable in V and

(3) $\operatorname{crit}(E^*_\beta) = \lambda_\beta$.

Obviously, $M_\beta$ and $Q^*_\beta$ agree past $\lambda_\beta$ because of rule (L2). We define $M_{\beta+1} = \operatorname{Ult}(M_\beta, E^*_\beta)$ and let $j_{\beta,\beta+1} \colon M_\beta \to M_{\beta+1}$ be the canonical elementary embedding.

(E7) If $M_{\beta+1}$ is illfounded, then the game ends and player I wins.

Now, one should notice that $s_\beta$ is a position over $M_{\beta+1}$ too because of the strongness condition (1) on $E^*_\beta$ in (L5). Player II can now extend $s_\beta$ to another position provided this is in line with what player I has coded at an earlier stage:

(L6) II plays $\delta_{\beta+1} \in \mathcal{W}^{M_{\beta+1}} \restriction j_{\beta,\beta+1}(\lambda_\beta)$ and an $M_{\beta+1}$-position $t_{\beta+1}$ with relative domain $\delta_{\beta+1} + 1$ such that the following conditions are satisfied:

(1) $t_{\beta+1}$ extends $s_\beta$,

(2) $t_{\beta+1}$ is $M_{\beta+1}$-clear,

(3) there exists some $n^*_\beta \in \omega$ so that $u_\beta(n^*_\beta)$ is a $\lambda_\beta$-functor as in Definition 35 in the sense of $M_\beta$, and so that $t_{\beta+1} \in c_{\beta+1}(\lambda_\beta, \delta_{\beta+1})[*]$ where $c_{\beta+1} = j_{\beta,\beta+1}(u_\beta(n^*_\beta))$.

Since $\lambda_\beta < j_{0,\beta}(\delta^*)$ and $\delta_{\beta+1} < j_{\beta,\beta+1}(\lambda_\beta)$ by definition of these objects, we thus have $\operatorname{rdm}(t_{\beta+1}) < j_{0,\beta+1}(\delta^*)$. Hence, the requirements of payoff condition (P1) cannot be possibly satisfied and we directly continue with round $\beta + 1$. It is easily checked that the assumptions on the objects $M_{\beta+1}$, $t_{\beta+1}$ and $j_{0,\beta+1}$ at the beginning of successor mega-round $\beta + 1$ are satisfied.

This finishes the description of the rules of the game. We refer to the sequence of objects constructed by two players at any stage of the game as position. Its length is the number of rounds played:

**40 Definition.** *For a* position *$P$ of length $\beta$ in the branching game of the form*

$$(\mathcal{T}_\xi, b_\xi, E^*_\xi, t_{\xi+1} \mid \xi < \beta)$$

*(not all of these objects are defined for all $\xi$ (but we are going to neglect this) and not all objects constructed in a run of the branching game but they can be reconstructed from the objects above), we say*

- $(M_\xi, j_{\zeta,\xi}, t_\xi \mid \zeta < \xi \leq \beta)$ *is the* history *of $P$, and*
- $(M_\beta, j_{0,\beta}, t_\beta)$ *is the* outcome *of $P$,*

*where the $M_\xi$ and $j_{\zeta,\xi}$ are constructed from $P$ in the obvious way.*

## 3.2 Leaping in the branching game

Here we want to describe legal moves for player II when it comes to a leap. We do this not only to give further insight into the rules. Also to have it at hand at a later point to use it in defining our winning strategy for the local game in Chapter 4.

Suppose that $P$ is a non-terminal position in $G_{\mathrm{br}}(M, t, \delta^*)(\dot{C})$ and suppose $(M_\beta, j_{0,\beta}, t_\beta)$ is the outcome of $P$, i.e., suppose $\beta$ is a limit ordinal, $M_\beta, j_{0,\beta}, t_\beta$ are as in the limit case and $P$ is non-terminal through the snags (E3) to (E4). Let $\lambda_\beta = \mathrm{rdm}(t_\beta)$. If $\lambda_\beta$ itself is not a Woodin cardinal in $M_\beta$, the mega-round $\beta$ of the game following $P$ is played according to the standard limit round case. Let $\tau_\beta$ be the least Woodin cardinal in $M_\beta$ above $\lambda_\beta$. For this, note that $\lambda_\beta < j_{0,\beta}(\delta^*)$. Assume that $w_\beta, \mathcal{T}_\beta, b_\beta, y_\beta$ are legal moves for (L1) to (L4). Suppose that these do not lead to an end of the game through one of the snags (E5) or (E6). Then let $Q_\beta$ be the direct limit of the branch $b_\beta$ in $\mathcal{T}_\beta$ and let $k_\beta : M_\beta \to Q_\beta$ be the direct limit embedding. Let $s_\beta = t_\beta{}^\frown(w_\beta, y_\beta)$ which is a $Q_\beta$-position of relative domain $k_\beta(\tau_\beta)+1$. This can be summarized by saying that players I and II have played $G_{\mathrm{br}}(M, t, \delta^*)(\dot{C})$ up to the leap in the limit round $\beta$.

**41 Lemma.** *Suppose $Q^*_\beta$ is transitive and agrees with $Q_\beta$ to $k_\beta(\tau_\beta)+1$. Denote by $W^*_\beta$ the class $\mathcal{W}$ computed in $Q^*_\beta$. Let $\delta^*_\beta \in \mathcal{W}^*_\beta$ and $t^*_\beta$ be a $Q^*_\beta$-position of relative domain $\delta^*_\beta + 1$. Assume furthermore that $t^*_\beta$ is* I-*acceptably obstructed over $Q^*_\beta$ with witnessing obstruction $(E^*_\beta, \vec{\sigma}^*_\beta)$. If*

*(1) $Q^*_\beta \restriction \delta^*_\beta + \omega$ is countable in* V*,*

*(2) $t^*_\beta$ extends $s_\beta$,*

*(3) every strict initial segment of $t^*_\beta$ is $Q^*_\beta$-clear,*

*(4)* $\mathrm{crit}(E^*_\beta)$ *is equal to $\lambda_\beta$,*

*(5)* $\mathrm{Ult}(M_\beta, E^*_\beta)$ *is wellfounded.*

*Then $E^*_\beta, \delta_{\beta+1} = \delta^*_\beta, t_{\beta+1} = t^*_\beta$ are legal, non-terminal moves for the leap rules (L5) to (L6).*

*Proof.* To see that $E^*_\beta$ satisfies the conditions in (L5), first use that $Q_\beta$ and $Q^*_\beta$ agree to $k_\beta(\tau_\beta) + 1$ by assumption. $E^*_\beta$ is part of an obstruction for the $Q^*_\beta$-position $t^*_\beta$ with relative domain $\delta^*_\beta + 1$ and thus, by Definition 29, is $\delta^*_\beta + 1$-strong over $Q^*_\beta$. By assumption, $t^*_\beta$ extends $s_\beta$, which implies that $\delta^*_\beta$ is bigger than or equal to $k_\beta(\tau_\beta)$, and so $E^*_\beta$ clearly also is $k_\beta(\tau_\beta) + 1$-strong over $Q^*_\beta$. $Q^*_\beta \restriction \delta^*_\beta + \omega$ is countable in V and thus, also $E^*_\beta$, which by Definition 29 is an element of it, is countable. The third claim of (L5) is part of our assumptions.

As in the description of the leaps, it follows that $M_\beta$ and $Q^*_\beta$ agree past $\lambda_\beta = \mathrm{crit}(E^*_\beta)$, so $M_{\beta+1} = \mathrm{Ult}(M_\beta, E^*_\beta)$ makes sense. Denote by $j_{\beta,\beta+1} : M_\beta \to M_{\beta+1}$ the canonical ultrapower embedding and let $\mathcal{W}_{\beta+1}$ be the class $\mathcal{W}$ computed in $M_{\beta+1}$. Because of the strongness condition on $E^*_\beta$ established in the first paragraph, also $M_{\beta+1}$ and $Q^*_\beta$ agree to $\delta^*_\beta + 1$ and hence $\mathcal{W}^*_\beta \restriction \delta^*_\beta + 1 = \mathcal{W}_{\beta+1} \restriction \delta^*_\beta + 1$. Using that $\mathrm{crit}(E^*_\beta)$ gets sent above its strength, we see that $j_{\beta,\beta+1}(\lambda_\beta) > \delta^*_\beta + 1$, and so we have $\delta^*_\beta \in \mathcal{W}_{\beta+1} \restriction j_{\beta,\beta+1}(\lambda_\beta)$.

It remains to prove that $t^*_\beta$ satisfies the conditions of (L6): The first one is clear from the assumptions of the Lemma. All strict initial segments of the $Q^*_\beta$-position $t^*_\beta$ are $Q^*_\beta$-clear by assumption. Using this, together with the agreement between $Q^*_\beta$ and $M_{\beta+1}$, it follows that $t^*_\beta$ is an $M_{\beta+1}$-position all of whose strict initial segments are $M_{\beta+1}$-clear. To see that $t^*_\beta$ itself is obstruction free over $M_{\beta+1}$, use that $(E^*_\beta, \vec{\sigma}^*_\beta)$ is a minimal obstruction for $t^*_\beta$ over $Q^*_\beta$: This means that it is obstruction free over $\mathrm{Ult}(Q^*_\beta, E^*_\beta)$ which definitely agrees with $M_{\beta+1}$ to $\delta^*_\beta + \omega$, making $t^*_\beta$ also obstruction free over $M_{\beta+1}$ by Definition 29.

For the last property let $u_\beta = (y(2n) \mid n \in \omega)$. Using the definition of an obstruction, Definition 38 and the agreement of $Q^*_\beta$ and $M_\beta$, it follows that $u_\beta(n)$ is a $\lambda_\beta$-functor over $M_\beta$ for every $n \in \omega$. Since $\mathrm{Ult}(Q^*_\beta, E^*_\beta)$ and $M_{\beta+1}$ are built by using the same extender, they agree on subsets of its critical point $\lambda_\beta$, such as $\lambda_\beta$-functors. Hence, again by the definition of I-acceptable obstructions, there is $n^*_\beta \in \omega$ for which $t^*_\beta \in j_{\beta,\beta+1}(c^*_\beta)(\lambda_\beta, \delta^*_\beta)[*]$ where $c^*_\beta = c_\beta(n^*_\beta)$.

Furthermore, by assumption (5) the snag (E7) does not apply and so these moves are non-terminal in the game. □

## 3.3 Determinacy of the branching game

The determinacy we want for the branching game is not what we normally mean by determinacy. It is not the case here that either player I or II have a winning strategy in the game $G_{\mathrm{br}}(M, \emptyset, \delta^*)(\dot{C})$ but rather that we can redefine everything above in such a way that we get a branching game for player II where the roles of the players are changed: It is now II's task to work in such a way that at the end of the run of the branching game the produced position lies within some predefined set or reaches a II-acceptable obstruction. We get this game by exchanging the terms I and II in the above description except for these parts where the two players construct real numbers together. This is for the simple reason that the branching game is really just an auxiliary game in which another game (the local game from Chapter 4) is embedded whose runs precisely consist of these reals.

It can be shown now that there are formulas $\varphi_{\mathrm{ini}}$ and $\psi_{\mathrm{ini}}$ such that whenever $\varphi_{\mathrm{ini}}$ ($\psi_{\mathrm{ini}}$) holds of a Woodin limit of Woodin cardinals $\delta$ (we only will need these cases in the next chapter) and a $\delta$-name $\dot{C}$ ($\dot{D}$), then player I (II) has a strategy for the branching game $G_{\mathrm{br}}(M, \emptyset, \delta)(\dot{C})$ ($H_{\mathrm{br}}(M, \emptyset, \delta)(\dot{D})$.

**42 Theorem.** *Let $M$ be a model of* $\mathsf{ZFC}^*$ *and $\delta$ be a Woodin cardinal in $M$. If $\dot{C}$ is a $\delta$-name in $M$ such that*

*(1) $M \upharpoonright \delta + 1$ is countable in* V*, and*

*(2) $\varphi_{ini}(\delta, \dot{C})$ holds in $M$,*

*then player* I *has a winning strategy in $G_{br}(M, \emptyset, \delta)(\dot{C})$.*

Of course the previous Theorem applies also with $\psi_{\mathrm{ini}}$ for the branching game of player II.

It can however happen that none of these formulas holds. In this case the other players win their opponents' branching game. Putting these things together we get a position which does not lie in any of the interpreted shifts of $\dot{C}$, $\dot{D}$. The following Theorem mentions two properties of $\delta$-sequences (niceness and saturation) which we did not define in this thesis. We will not need their exact meanings.

**43 Theorem.** *Let $\delta$ be a Woodin limit of Woodin cardinals. Let $\dot{C}$ and $\dot{D}$ be $\delta$-names such that both formulas $\varphi_{ini}(\delta, \dot{C})$ and $\psi_{ini}(\delta, \dot{D})$ fail. Then there exists a supernice, saturated $\delta$-sequence avoiding both $\dot{C}$ and $\dot{D}$.*

# 4 Games of uncountable length (in inner models)

In this chapter we work towards proving Theorem 44. It asserts determinacy for the local game in which players I and II alternate playing natural numbers until they have produced a sequence of reals. In advance we fix a name in Woodin's extender algebra for a set of sequences of a certain length and if this set can be adequately shifted into an elementary superstructure of the model we are working in and interpreted there (via a generic object) so that the sequence produced by players I and II lies in this set, player I wins. The main point of this is that because of the chain condition of Woodin's extender algebra we get that the runs of this game are uncountable in the inner models we are working in. We will prove the determinacy of the game using the results of the previous Chapter by using a winning strategy in the branching game to define a winning strategy for the local game.

The material of this Chapter can be found in Sections 7A–7E of [13].

## 4.1 Definition of the local game

For the following, we let $M$ be a transitive model of $\mathsf{ZFC}^*$ and $\Gamma$ be an iteration strategy for $M$. Moreover, let $\vartheta$ be a Woodin limit of Woodin cardinals in $M$.

If $\dot{A} \in M$ is a $\mathbb{W}_\vartheta$-name for a set of length-$\vartheta$-sequences of reals we define the game $G_{\text{loc}}(M, \Gamma, \vartheta, \dot{A})$. In mega-round $\xi$ of this game, players I and II alternate (with player I starting) playing natural numbers producing a real $z_\xi$. Player I wins iff there exists an $\alpha < \omega_1^V$ so that the following condition is satisfied:

(P) There exists an iteration tree $\mathcal{U}$ on $M$ of countable length leading to a final model $M^*$ such that for $M^*$, the elementary embedding

$j^* : M \to M^*$ given by $\mathcal{U}$ and some set $H$, the following conditions hold:

1) The iteration tree $\mathcal{U}$ is consistent with the iteration strategy $\Gamma$;
2) $H$ is $j^*(\mathbb{W}_\vartheta)$-generic over $M^*$;
3) $(z_\xi \mid \xi < \alpha) \in j^*(\dot{A})[H]$.

Note that 3) implies that $(z_\xi \mid \xi < \alpha) \in M^*[H]$.

As $\dot{A}$ is a name for a set of sequences of length $\vartheta$, the interpreted shift $j^*(\dot{A})[H]$ consists of sequences of length $j^*(\vartheta)$. Hence, if player I wins $G_{\mathrm{loc}}(M, \Gamma, \vartheta, \dot{A})$, we have for $\alpha$ of 3) that $\alpha = j^*(\vartheta)$.

Since $\mathbb{W}_\vartheta$ has the $\vartheta$-cc, $j^*(\mathbb{W}_\vartheta)$ has the $j^*(\vartheta)$-cc in $M^*$. This means that $j^*(\vartheta)$ remains a cardinal in an extension of $M^*$ by a generic filter of $j^*(\mathbb{W}_\vartheta)$ as in payoff condition (P). Thus, if player I wins the game after $\alpha$ many rounds, the produced sequence $(z_\xi \mid \xi < \alpha)$ is of length at least $\omega_1^{M^*}$ (by the previous paragraph).

Here we see the importance of the $\vartheta$-cc of Woodin's extender algebra. This guarantees the uncountable length of the run of the game (in the model which it is interpreted in) and gives rise to many applications.

Furthermore we define the mirrored game $H_{\mathrm{loc}}(M, \Gamma, \vartheta, \dot{B})$ for the objects as above and $\dot{B}$ a $\mathbb{W}_\vartheta$-name for sequences of reals of length $\vartheta$. Players I and II play as in $G_{\mathrm{loc}}$ but here it is II's task to get into a shifted interpretation of $\dot{B}$. Player II wins iff there is an $\alpha$ countable in V so that the payoff condition (P) holds with $\dot{A}$ replaced by $\dot{B}$. It is then obvious that our observations from above carry over to the mirrored game. In particular runs of this game winning for player II have uncountable length in the top model of the iteration tree witnessing the winning condition.

We work towards establishing the following theorem assuming the determinacy result Theorem 42 for the branching game. It says that given names $\dot{A}, \dot{B}$ as above; either one of the players has a strategy to enter interpretations of these names in some elementary extensions or that there is a generic filter so that the two interpretations do not cover all sequences:

**44 Theorem.** *Let $M, \Gamma, \vartheta, \dot{A}, \dot{B}$ be as above and suppose $\vartheta$ is countable in* V. *Then one of the following cases holds:*

*(1)* I *has a winning strategy in $G_{loc}(M, \Gamma, \vartheta, \dot{A})$;*

*(2)* II *has a winning strategy in* $H_{loc}(M, \Gamma, \vartheta, \dot{B})$*;*

*(3) there is* $G \in \mathrm{V}$ *which is* $\mathbb{W}_\vartheta$*-generic over* $M$ *and a sequence* $\vec{z} = (z_\xi \mid \xi < \vartheta) \in M[G]$ *so that* $\vec{z}$ *is neither in* $\dot{A}[G]$ *nor in* $\dot{B}[G]$.

*Additionally, there are formulas* $\varphi$ *and* $\psi$ *so that* $M \models \varphi(\vartheta, \dot{A})$ *implies (1); and* $M \models \psi(\vartheta, \dot{B})$ *implies (2). Otherwise (if neither* $\varphi$ *nor* $\psi$ *hold in* $M$*) case (3) holds.*

Note the following special case of the previous Theorem. If we define $\dot{B}$ as the canonical name for the complement of $\dot{A}$ (relative to the set of all sequences of reals of length $\vartheta$), case (3) is not possible and one of the first two cases must hold.

## 4.2 Proving determinacy of the local game

In this section we work under various assumptions to prove determinacy of the local game. The main tool in the proof will be the determinacy of the branching game, stated in Theorem 42. More specifically we will reduce a run of the local game to several runs of the branching game (which are played along branches of the witnessing iteration tree).

### 4.2.1 The general setup and shifting $\mathcal{W}$ and $\mathcal{L}$

Recall the definition of a tree order from Definition 14. In the following, assume that $U$ is a tree order:

**45 Definition.** *A* tree of trees $\mathfrak{U}$ *is an iteration tree with underlying tree order* $U$ *of length* $\alpha$ *for some ordinal* $\alpha$ *where every node is replaced by an iteration tree of length* $\omega + 1$*. More specifically, a tree of trees* $\mathfrak{U}$ *consists of*

- *a tree order* $U$ *on* $\alpha$*,*
- *models* $M_\xi$ *for* $\xi < \alpha$*,*
- *elementary embeddings* $j_{\zeta,\xi} : M_\zeta \to M_\xi$ *whenever* $\zeta U \xi < \alpha$ *which commute,*
- *a length-*$\omega$*-iteration tree* $\mathcal{T}_\xi$ *on* $M_\xi$ *for every* $\xi < \alpha$*,*
- *an infinite branch* $b_\xi$ *through* $\mathcal{T}_\xi$ *whenever* $\xi < \alpha$*,*

*and letting $Q_\xi$ being the direct limit of $\mathcal{T}_\xi$ along the branch $b_\xi$ with $k_\xi : M_\xi \to Q_\xi$ being the canonical elementary direct limit embedding for $\xi < \alpha$, we furthermore have*

- *whenever $\xi + 1 < \alpha$, possibly an extender $E_\xi \in Q_\xi$.*

*These objects have to satisfy the following conditions:*

*(N1)* $E_\xi$ = "undefined" *implies* $\xi U \xi + 1$, $M_{\xi+1} = Q_\xi$ *and* $j_{\xi,\xi+1} = k_\xi$,

*(N2) if $E_\xi$ is defined, $M_{\xi+1} = \mathrm{Ult}(M_\zeta, E_\xi)$ where $\zeta$ is the direct $U$-predecessor of $\xi + 1$,*

*(N3) if $\gamma < \alpha$ is a limit, $M_\gamma$ is the direct limit of the $M_\xi$'s with $\xi U \gamma$ with $j_{\xi,\gamma} : M_\xi \to M_\gamma$ being the canonical elementary direct limit embeddings.*

*The remaining embeddings $j_{\zeta,\xi}$ for $\zeta U \xi$ have to be in such a way that all the embeddings commute.*

*By re-indexing it is possible to see $\mathfrak{U}$ as one big iteration tree of length $\alpha \cdot \omega + 1$ (or $\alpha \cdot \omega$ if $\alpha$ is a limit), an object which we refer to as* $\mathrm{merge}(\mathfrak{U})$. *We say a tree of trees $\mathfrak{U}$ is consistent with an iteration strategy $\Gamma$ if* $\mathrm{merge}(\mathfrak{U})$ *is.*

The first goal of this section is to understand the structure of the classes $\mathcal{W}$ and $\mathcal{L}$ introduced in Section 2.1 within the models of a tree of trees $\mathfrak{U}$ so that we understand how the branching game in these models works. For this we need to assume that Woodin cardinals in these models are present, moreover we demand certain strongness properties of the extenders used so that the models $M_\xi$'s agree up to higher and higher points:

**46 Definition.** *A tree of trees $\mathfrak{U}$ as above is called* regular *if the following conditions hold:*

*(R1) The set $\{\delta \mid \delta$ is Woodin in $M_\xi\}$ has order type $\geq \xi + 1$ in $M_\xi$.*

*Let $\delta_\xi^\dagger$ be the unique $M_\xi$-Woodin cardinal $\delta$ of $M_\xi$ such that the order type of the set of $M_\xi$-Woodin cardinals below $\delta$ is $\xi$. Define $\delta_{\xi+1} = k_\xi(\delta_\xi^\dagger)$.*

*(R2) The iteration tree $\mathcal{T}_\eta$ only uses extenders with critical points greater than $\sup\{\delta_{\xi+1} \mid \xi < \eta\}$ whenever $\eta < \alpha$,*

*(R3) $\mathcal{T}_\eta$ is countable in* V *whenever $\eta < \alpha$.*

*(R4) For all $\eta$ so that $\eta + 1 < \alpha$, the extender $E_\eta$ (if defined) is $\delta_{\eta+1} + 1$-strong in $Q_\eta$.*

For the rest of this section fix a regular tree of trees $\mathfrak{U}$ on $M \models \mathsf{ZFC}^*$ with underlying tree order $U$ of length $\alpha$.

The following lemma establishes the agreement between various $M_\xi$'s and $Q_\xi$'s and is indispensable for the results which follow it:

**47 Lemma.** *(for the regular tree of trees $\mathfrak{U}$) For $\eta < \alpha$, the following conditions hold:*

*(R5) For $\xi < \eta$, $\delta_{\xi+1}$ has (in order type) $\xi$ $M_\eta$-Woodin cardinals below it in $M_\eta$,*

*(R6) $M_\eta$ and all subsequent models in $\mathfrak{U}$ agree past $\sup\{\delta_{\xi+1} \mid \xi < \eta\}$,*

*(R7) $\delta_{\eta+1}$ has (in order type) $\eta$ $Q_\eta$-Woodin cardinals below it in $Q_\eta$,*

*(R8) $Q_\eta$ and all subsequent models in $\mathfrak{U}$ agree past $\delta_{\eta+1}$.*

*Proof.* We show this by induction simultaneously for all of the conditions above. Condition (R6) is only established for the agreement of $M_\eta$ with $Q_\eta$. Similarly, condition (R8) is only established for the agreement of $Q_\eta$ with $M_{\eta+1}$. The general claims then follow from the proof. The reader is advised to check that this poses no problem.

(R5) is vacuous for $\eta = 0$. If $\eta$ is a successor ordinal $\eta' + 1$, we have $M_\eta \restriction \delta_{\eta'} = M_{\eta'} \restriction \delta_{\eta'}$ by (R6) for $\eta'$ and so for $\xi < \eta'$ the claim follows from the induction hypothesis (R5) for $\eta'$. For $\xi = \eta'$ use (R7) for $\eta'$ and the agreement given by (R8). If $\eta$ is a limit ordinal, the claim is clear from the induction hypotheses (R5) and (R6).

It is clear from (R2) that $M_\eta$ and $Q_\eta$ agree past $\sup\{\delta_{\xi+1} \mid \xi < \eta\}$. The claim of (R6) follows from this.

For (R7) note that $\delta_{\eta+1} = k_\eta(\delta_\eta^\dagger)$ and $\delta_\eta^\dagger$ has in $M_\eta$ (in order type) $\eta$ many $M_\eta$-Woodin cardinals below it. The claim follows from this, noting that $\mathrm{crit}(k_\eta) > \eta$.

The claim of (R8) is of course clear for $E_\eta$ = "undefined". So assume that $E_\eta$ is defined and let $\zeta$ be the $U$-predecessor of $\xi + 1$ such that $M_{\eta+1} = \mathrm{Ult}(M_\zeta, E_\eta)$. We have that $M_\zeta$ and $Q_\eta$ agree past $\mathrm{crit}(E_\eta)$. By (R4) it also holds that $Q_\eta$ and $\mathrm{Ult}(Q_\eta, E_\eta)$ agree past $\delta_{\eta+1} + 1$. Noting that $\mathrm{Ult}(Q_\eta, E_\eta) \restriction \delta_{\eta+1} + 1$ only depends on $Q_\eta \restriction \mathrm{crit}(E_\eta)$ (as the critical point gets necessarily mapped above the strength) (which is equal to $M_\zeta \restriction \mathrm{crit}(E_\eta)$), it follows that $Q_\eta$ and $M_{\eta+1} = \mathrm{Ult}(M_\zeta, E_\eta)$ agree past $\delta_{\eta+1}$. □

We now work to establish what $\mathcal{W}$ and $\mathcal{L}$ look like in the models of $\mathfrak{U}$: For this define $\lambda_\gamma$ for $\gamma$ a limit ordinal by

$$\lambda_\gamma = \sup\{\delta_{\xi+1} \mid \xi < \gamma\}.$$

This is a limit of Woodin cardinals in $M_\gamma$ (and all the models following it). The index $\gamma$ is called a *phantom limit* if $\lambda_\gamma$ itself is Woodin in $M_\gamma$; if the cardinal $\lambda_\gamma$ is not Woodin in $M_\gamma$, $\gamma$ is called a *standard limit*. These notions should be compared to the standard limit and the phantom limit case of the branching game in Section 3.1.2.

**48 Lemma.** *For $\eta < \alpha$ the sequence $(\lambda_\gamma \mid \gamma$ is a standard limit, $\gamma \leq \eta)$ enumerates the set*

$$\{\lambda < \delta_{\eta+1} \mid Q_{\eta+1} \models \lambda \textit{ is a non-Woodin limit of Woodin cardinals}\}$$

*in increasing order.*

*Proof.* Using (R7) and (R8) of the previous Lemma, we see that $(\delta_{\xi+1} \mid \xi < \eta)$ enumerates in increasing order the Woodin cardinals of $Q_\eta$ below $\delta_{\eta+1}$. Using the definition of standard limit above, the claim follows. □

Our plan is that the $\lambda_\xi$'s enumerate initial segments of the class $\mathcal{L}$ in the various models. Note that this is in line with the above definition of $\lambda_\gamma$ for limit $\gamma$. We furthermore set $\lambda_0 = 0$ and for $\xi + 1 < \alpha$

$$\lambda_{\xi+1} = \begin{cases} \delta_{\xi+1} & \text{if } \xi \text{ is a phantom limit,} \\ \delta_{\xi+1} + 1 & \text{otherwise.} \end{cases}$$

Furthermore we let $K^{\mathfrak{U}}$ be the set of all ordinals $\eta < \alpha$ which are not phantom limits. The classes $\mathcal{W}_\eta$, $\mathcal{L}_\eta$, denote the classes $\mathcal{W}$, $\mathcal{L}$, respectively, of Section 2.1 computed in $M_\eta$. For $\eta < \alpha$ we also define the function $e_\eta$ as the shift of the function $e$ to the model $M_\eta$: For $\delta \in \mathcal{W}_\eta$ we let $e_\eta(\delta) = \sup\{\tau + 1 \mid \tau \in \mathcal{W}_\eta \upharpoonright \delta\}$.

**49 Lemma.** *For $\eta < \alpha$ the sequence $(\delta_{\xi+1} \mid \xi \in K^{\mathfrak{U}} \upharpoonright \eta)$ enumerates in increasing order $\mathcal{W}_\eta \upharpoonright \lambda_\eta$.*

*Proof.* Note first that there are no elements of $\mathcal{W}_\eta \upharpoonright \lambda_\eta$ between $\lambda_\eta$ and $\delta^\dagger_\eta$ by definition of $\delta^\dagger_\eta$ and (R5) of Lemma 47 which could be enumerated by the sequence without being present in $\mathcal{W}_\eta \restriction \lambda_\eta$. So we can replace the right hand side of the statement by $\mathcal{W}_\eta \restriction \delta^\dagger_\eta$. Now the claim follows immediately from (R5) to (R6) and the definitions of $\delta^\dagger_\eta$ and $K^{\mathfrak{U}}$. Note for this that if $\delta_{\xi+1}$ is a Woodin limit of Woodin cardinals, then $\delta_{\xi+1} = \lambda_\xi$ and $\xi$ is a phantom limit. □

**50 Lemma.** *For $\eta \in K^{\mathfrak{U}}$ we have that $e_\eta(\delta^\dagger_\eta) = \lambda_\eta$.*

*Proof.* The case $\eta = 0$ is clear from the definitions.

If $\eta$ is a limit, $K^{\mathfrak{U}} \restriction \eta$ is cofinal in $\eta$ as it contains all successors. From this and Lemma 49, it follows that the right hand side of the equality is equal to $\sup\{\delta_{\xi+1} + 1 \mid \xi < \eta\}$ which is equal to $\lambda_\eta$ by definition.

If $\eta = \xi + 1$ is a successor ordinal, we have that $\delta_{\xi+1}$ is the largest Woodin cardinal in $M_{\xi+1}$ below $\delta^\dagger_{\xi+1}$ by Lemma 47. If $\xi$ is not a phantom limit, we have from Lemma 49 that $\delta_{\xi+1} \in \mathcal{W}_{\xi+1}$ and the claim is clear. If $\xi$ however is a phantom limit, we do not get $\delta_{\xi+1} \in \mathcal{W}_{\xi+1}$ but arbitrarily large elements of $\mathcal{W}_{\xi+1}$ are enumerated by $\tau + 1$ for $\tau \in \mathcal{W}_{\xi+1} \restriction \delta^\dagger_{\xi+1}$ and the claim follows again from the definition of $e_\eta$. □

**51 Lemma.** *If $\eta < \alpha$ and $\xi < \eta, \xi \in K^{\mathfrak{U}}$, then $e_\eta(\delta_{\xi+1}) = \lambda_\eta$.*

*Proof.* This follows from the agreement between $M_\xi$ and $M_\eta$. Since $\mathrm{crit}(k_\xi) > \lambda_\eta$ by (R2) we have $e^{Q_\eta}(\delta_{\xi+1}) = e^{Q_\eta}(k_\xi(\delta^\dagger_\xi)) = k_\xi(\lambda_\xi) = \lambda_\xi$ by elementarity of $k_\xi$. Since $Q_\xi$ and $M_\eta$ agree past $\delta_{\xi+1}$ by (R8), we get $e_\eta(\delta_{\xi+1}) = \lambda_\xi$. □

Using that $e$ just gives elements of $\mathcal{L}$ when applied to elements of $\mathcal{W}$ we get

**52 Lemma.** *The sequence $(\lambda_\xi \mid \xi \in K^{\mathfrak{U}} \restriction \eta)$ enumerates in increasing order $L_\eta \restriction \lambda_\eta$.*

*Proof.* Using that $\mathcal{L}_\eta = \{e_\eta(\tau) \mid \tau \in \mathcal{W}_\eta \restriction \lambda_\eta\}$, knowledge of an enumeration of $\mathcal{W}_\eta$ by Lemma 49 and the outcome of application of $e_\eta$ by Lemma 51, the claim follows. □

In parallel to the branching game, where objects $w_\xi$ and $y_\xi$ are produced in every round $\xi$ which is not a phantom limit, it is natural to stipulate the following definition.

**53 Definition.** *We say* $(w_\xi, y_\xi \mid \xi \in K^{\mathfrak{U}})$ *is a* $\mathfrak{U}$-sequence *if for each* $\xi \in K^{\mathfrak{U}}$ *it holds that*

*(1)* $w_\xi$ *is a witness for* $\lambda_\xi$ *over* $M_\xi$*,*

*(2) if* $\xi$ *is a successor ordinal or zero,* $y_\xi$ *is a real, and*

*(3) if* $\xi \in K^{\mathfrak{U}}$ *is a standard limit,* $y_\xi \in (Q_\xi \restriction \lambda_\xi + 1)^\omega$.

We fix such a $\mathfrak{U}$-sequence. By enumerating its objects in the right fashion, we get $M_\eta$-positions $t_\eta$ for $\eta < \alpha$:

$$\begin{aligned} \operatorname{dom}(t_\eta) &= \{\delta_{\xi+1}, \lambda_\xi \mid \xi \in K^{\mathfrak{U}} \restriction \eta\}, \\ t_\eta(\delta_\xi) &= y_\xi, \\ t_\eta(\lambda_\xi) &= w_\xi. \end{aligned} \tag{4.1}$$

**54 Lemma.** $t_\eta$ *is an* $M_\xi$*-position of relative domain* $\lambda_\eta$.

*Proof.* Using Lemma 49 and Lemma 52, one sees that $t_\eta$ is a function with domain $(\mathcal{W}_\eta \cup \mathcal{L}_\eta) \restriction \lambda_\eta$. Using the definition of a $\mathfrak{U}$-sequence and the regularity properties (R8) and (R6), the claim follows. □

The reals in this $\mathfrak{U}$-sequence will form a run of the local game. We isolate them:

**55 Definition.** *Let* $t$ *be a position of relative domain* $\lambda$*. Then* $\vec{z}(t) = (t(\delta_\xi) \mid \xi < \alpha)$*, where* $\alpha$ *is the order type of the relative successors below* $\lambda$ *and* $(\delta_\xi \mid \xi < \alpha)$ *enumerates this set in increasing order, is the* sequence of reals *of* $t$.

The reader should go back at this place to the definition of position (Definition 19) to check that this really enumerates all the reals in the range of a position.

**56 Lemma.** *(assuming a regular tree of trees* $\mathfrak{U}$ *as above, a* $\mathfrak{U}$*-sequence and a position* $t_\xi$ *as above)*

$$\vec{z}(t_\eta) = (y_{-1+\xi+1} \mid \xi + 1 < 1 + \eta).$$

*Proof.* The relative successors below $\lambda_\eta = \operatorname{rdm}(t_\eta)$ are enumerated by $\delta_{\zeta+1}$ for $\zeta$ a successor ordinal or zero and being smaller than $\eta$ by definition of $\lambda_\eta$, $\delta_\xi$ and Lemma 49. By the definition of $t_\eta$ above we have $t_\eta(\delta_\xi + 1) = y_\xi$ and thus $\vec{z}(t_\eta) = (y_\zeta \mid \zeta$ is zero or a successor$)$ which in turn equals $(y_{-1+\xi+1} \mid \xi + 1 < 1 + \eta)$. □

Recall that we plan to prove determinacy of local games by building a tree of trees $\mathfrak{U}$ to which is associated a $\mathfrak{U}$-sequence such that the reals of its positions $\vec{z}(t_\xi)$ correspond to the moves by players I and II in the local game. We try to produce the tree of trees $\mathfrak{U}$ in such a way that along its branches we have runs of the branching game, more formally: fix $\eta < \alpha$ and let $r = \{\zeta \mid \zeta U \eta\}$ be the branch of $U$ leading to $\eta$. Furthermore, let $\beta$ be the order type of $(r, U)$ and $f : \beta + 1 \to r \cup \{\eta\}$ be the unique order preserving map. The reader should recall that $U$ is a subset of the usual order on the ordinals. We have for $\xi < \beta$ that $f(\xi + 1)$ is a successor ordinal and hence the definition $E^*_\xi = E_{f(\xi+1)-1}$ (or "undefined") makes sense.

**57 Definition.** *For $\eta$ and $f$ as above, we define the* strand of $\mathfrak{U}$ and of the $\mathfrak{U}$-sequence $(w_\xi, y_\xi \mid \xi \in K^{\mathfrak{U}})$ leading to $\eta$*:*

$$P_\eta = (\mathcal{T}_{f(\xi)}, b_{f(\xi)}, E^*_\xi, t_{f(\xi+1)} \mid \xi < \beta).$$

**58 Lemma.** *The length of the strand $P_\eta$,* $\mathrm{lh}(P_\eta)$ *is a successor iff $\eta$ is a successor ordinal, a limit, if $\eta$ is a limit ordinal and zero if $\eta = 0$.*

*Proof.* First notice that $\mathrm{lh}(P_\eta)$ is the order type of $(r, U)$. Using this and $0 U \xi$ for every $\xi \neq 0$, the claim follows. □

### Extending $\mathfrak{U}$

During the construction we will have to make our tree of trees bigger. Rather than extending $\mathfrak{U}$ to a tree of trees which is longer by 1, we describe an intermediate stage.

**59 Definition.** *Given the tree of trees $\mathfrak{U}$ of length $\alpha$, we say $\mathfrak{U}^+$ is an* extension of length $\alpha + 0.2$ *if $\mathfrak{U}^+$ consists of*

- *a tree order $U^+$ on $\alpha + 1$ which extends $U$, the underlying tree order of $\mathfrak{U}$,*
- *all parts of $\mathfrak{U}$,*
- *(if $\alpha$ is a successor) an extender $E_{\alpha-1} \in Q_{\alpha-1}$ or $E_{\alpha-1}$ =* "undefined",
- *a model $M_\alpha$ which satisfies one of the conditions (N1), (N2), (N3) depending on $\alpha$ and $E_{\alpha-1}$ if $\alpha$ is a successor ordinal,*
- *elementary embeddings $j_{\xi,\alpha}$ for each $\xi$ with $\xi U^+ \alpha$ subject to (N1), (N2), (N3) which commute (where the same condition has to be chosen as in the previous item).*

Note that if $\alpha$ is a limit ordinal, all which is needed for $\mathfrak{U}^+$ apart from $\mathfrak{U}$ is the cofinal branch of $U$ in $\alpha$ leading to it in $U^+$. If $\alpha$ is a successor ordinal and $E_{\alpha-1}$ = "undefined", the extension $\mathfrak{U}^+$ is uniquely defined too. The notion of regularity for a tree of trees of length $\alpha + 0.2$ is defined in the obvious way and it can be checked that many of the above lemmas extend to hold for $M_\alpha$. We will work in the next session to extend an iteration tree of a certain length $\alpha$ to one of length $\alpha + 0.2$.

If we have such an extension $\mathfrak{U}^+$ we need to choose an iteration tree of length $\omega$ on $M_\alpha$ and a branch through it to get a tree of trees of length $\alpha + 1$. These moves however will be handled by a winning strategy of I in the branching game and an iteration strategy, respectively.

### 4.2.2 Extending a tree of trees

We work here to extend a regular tree of trees as it appears in the proof of determinacy of the local game. We have to make sure that the extension is regular too and still satisfies certain assumptions we introduce during this Section.

In this section we work with a transitive inner model $M \models \mathsf{ZFC}^*$. We assume furthermore that there exists an iteration strategy $\Gamma$ for $M$. Let $\eta$ be an ordinal in $M$, $\mathfrak{U}$ a regular tree of trees of length $\eta + 1$ such that $\mathfrak{U}$ is consistent with $\Gamma$ and that there is a $\mathfrak{U}$-sequence $(\vec{w}, \vec{y})$. Assume furthermore that we have $\vartheta$, a Woodin limit of Woodin cardinals in $M$, which is countable in V and let $\dot{Y}$ be a $\vartheta$-name in $M$. Recall from above that if $\mathfrak{U}$ had length $\gamma$ for a limit ordinal $\gamma$, extending it to length $\gamma + 0.2$ means to choose a cofinal $U$-branch in $\gamma$; something which will be done by the iteration tree $\Gamma$.

Assume that

(A1) for each $\bar{\eta} \leq \eta$ the strand $P_{\bar{\eta}}$ is a legal position in $G_{\mathrm{br}}(M, \emptyset, \vartheta)(\dot{Y})$.

Recall Definition 57 for strands. This assumption, of course, is in line with our plans to have a run of the branching game along each branch of the tree of trees. In order to extend the tree of trees we will have to identify the right predecessor for $\eta + 1$ so that also along this branch, the strand is a legal position of the branching game.

**60 Lemma.** *For each $\bar{\eta} \leq \eta$ the following conditions hold:*

- *the outcome of $P_{\bar{\eta}}$ is equal to $(M_{\bar{\eta}}, j_{0,\bar{\eta}}, t_{\bar{\eta}})$, and*

- *the history of $P_{\bar\eta}$ is equal to $(M_\xi, j_{\zeta,\xi}, t_\xi \mid \zeta U \xi U_= \bar\eta)$,*

*where $U$ is the underlying tree order and $U_=$ its reflexive closure.*

*Proof.* To see this one needs to compare Definition 40 of outcome and history to the definition of strand and its properties given by (N1), (N2) and (N3). □

**61 Lemma.** *Suppose that for $\bar\eta \leq \eta$, the strand $P_{\bar\eta}$ is a non-terminal position in $G_{br}(M, \emptyset, \vartheta)(\dot Y)$. Then mega-round $\bar\beta = \mathrm{lh}(P_{\bar\eta})$ of $G_{br}(M, \emptyset, \vartheta)(\dot Y)$ following $P_{\bar\eta}$ is played according to*

*(i) the phantom limit case of the branching game if $\bar\eta$ is a phantom limit in $\mathfrak{U}$,*

*(ii) the standard limit case of the branching game if $\bar\eta$ is a standard limit in $\mathfrak{U}$,*

*(iii) the successor case of the branching game if $\bar\eta$ is $0$ or a successor ordinal.*

*Proof.* From Lemma 58 it follows that $\bar\eta$ is a successor or zero iff $\bar\beta$ is and hence the third case in the statement of the Lemma holds. If $\bar\eta$ is a limit ordinal, we conclude the claim from Lemma 54 and the definitions of the phantom limit case and the standard limit case of the branching game. □

Since no moves are made in phantom limit rounds of the branching game we neglect this possibility in the following. The following lemmata establish that moves defined from the tree of trees $\mathfrak{U}$ and the $\mathfrak{U}$-sequence $(\vec w, \vec y)$ are legal in the branching game. For nodes with successors in $U$ this follows from (A1), so these are most important for nodes which do not have any successors in the tree order.

**62 Lemma.** *Suppose $\bar\eta \leq \eta$ is a successor or zero and $\bar\beta = \mathrm{lh}(P_{\bar\eta})$. Furthermore assume that $P_{\bar\eta}$ is a non-terminal position in $G_{br}(M, \emptyset, \vartheta)(\dot Y)$. Then the objects $w_{\bar\eta}, y_{\bar\eta}, \mathcal{T}_{\bar\eta}, b_{\bar\eta}$ (given by $\mathfrak{U}$ and $\vec w, \vec y$, the $\mathfrak{U}$-sequence) are legal moves in mega-round $\bar\beta$ of $G_{br}(M, \emptyset, \vartheta)\dot Y)$ following $P_{\bar\eta}$.*

*Proof.* We know from the previous Lemma that mega-round $\bar\beta$ proceeds according to the successor case of the rules of the branching game. Assuming that $P_{\bar\eta}$ is a legal position, we have a model $M_{\bar\eta}$ which is wellfounded, an elementary embedding $j_{0,\bar\eta} : M \to M_{\bar\eta}$ and an $M_{\bar\eta}$-position $t_{\bar\eta}$. Copying further from the rules in Section 3.1.1 we have to check that:

- $w_{\bar\eta}$ is a witness for $\mathrm{rdm}(t_{\bar\eta})$ over $M_{\bar\eta}$,
- $y_{\bar\eta}$ is a real,
- $\mathcal{T}_{\bar\eta}$ is a length-$\omega$-iteration tree on $M_{\bar\eta}$ which is countable in V and with critical point above $\mathrm{rdm}(t_{\bar\eta})$,
- $b_{\bar\eta}$ is a cofinal branch through $\mathcal{T}_{\bar\eta}$.

Using the definition of a $\mathfrak{U}$-sequence (cf. Definition 53) and Lemma 54 we see that $w_{\bar\eta}$ is a witness for $\lambda_\eta$ as it should be. From the same lemma and regularity of $\mathfrak{U}$ it follows that the extenders of the iteration tree $\mathcal{T}_{\bar\eta}$ have high enough critical points. The rest of the requirements are clear. □

**63 Lemma.** *Suppose $\bar\eta \leq \eta$ is a standard limit in $\mathfrak{U}$ and let $\bar\beta = \mathrm{lh}(P_{\bar\eta})$. Assume furthermore that $P_{\bar\eta}$ is non-terminal in $G_{br}(M, \emptyset, \vartheta)(\dot{Y})$. Then $w_{\bar\eta}, t_{\bar\eta}, b_{\bar\eta}, y_{\bar\eta}$ (given by $\mathfrak{U}$ and the $\mathfrak{U}$-sequence) are legal moves in the standard limit round $\bar\beta$ up to a (possible) leap.*

*Proof.* This is very similar to the previous proof. It is clear that mega-round $\bar\beta$ is played according to the rules of a standard limit round by Lemma 58. The claim can be checked again by using the relevant definitions. □

Assume now furthermore that I has a winning strategy $\Sigma_{\mathrm{br}}$ for the game $G_{\mathrm{br}}(M, \emptyset, \vartheta)(\dot{Y})$. We want to use $\Sigma_{\mathrm{br}}$ in our construction of the extension of $\mathfrak{U}$. For this to make sense we have to assume that the strands along each branch are consistent with the winning strategy (here $\Sigma_{\mathrm{br}}[P]$ stands for the restriction of $\Sigma_{\mathrm{br}}$ to the round following a non-terminal position $P$ in the branching game):

(A2) for each $\bar\eta \leq \eta$, the strand $P_{\bar\eta}$ is non-terminal in $G_{\mathrm{br}}(M, \emptyset, \vartheta)(\dot{Y})$ and played according to $\Sigma_{\mathrm{br}}$.

(A3) for each $\bar\eta \leq \eta$ which is a standard limit in $\mathfrak{U}$: $w_{\bar\eta}, y_{\bar\eta}, \mathcal{T}_{\bar\eta}, b_{\bar\eta}$ are consistent with $\Sigma_{\mathrm{br}}[P_{\bar\eta}]$,

(A4) for each $\bar\eta \leq \eta$ which is a successor ordinal or zero: $w_{\bar\eta}, \mathcal{T}_{\bar\eta}, b_{\bar\eta}, y_{\bar\eta}$ are consistent with $\Sigma_{\mathrm{br}}[P_{\bar\eta}]$,

(A5) $\eta$ is not a phantom limit in $\mathfrak{U}$.

Since no moves are made in phantom limit rounds of the branching game, it is safe to make the last assumption in order to avoid to have to look at three cases in the following. The other assumptions formalize the motivation that we want to use $\Sigma_{\mathrm{br}}$ to extend our tree of trees $\mathfrak{U}$. (A2) says that every run of the branching game along some branch of $\mathfrak{U}$ up to $\bar{\eta}$ is consistent with the winning strategy $\Sigma_{\mathrm{br}}$. (A3) to (A4) say that furthermore the moves given by $\mathfrak{U}$ and the $\mathfrak{U}$-sequence following some $P_{\bar{\eta}}$ are according to $\Sigma_{\mathrm{br}}$. The previous lemmata established that they are legal. Of course these assumptions are most important in the case where $\bar{\eta}$ does not have any successors in $U$. It remains to establish that these are non-terminal.

Using assumptions (A1) to (A5) (and other assumptions made in this section) we now describe how to extend $\mathfrak{U}$ to a tree of trees $\mathfrak{U}^+$ of length $\eta + 1.2$ (recall that $\mathfrak{U}$ has length $\eta + 1$) such that all of these assumptions are still satisfied for the extension and so that $P_{\eta+1}$ is not a terminal position via any of the snags.

Since $\vartheta$ is a Woodin limit of Woodin cardinals, terminal positions via one of the payoff conditions (P1) to (P2) can only occur after phantom limits. To see this note that if in a mega-round $\alpha$ a non-terminal position $P$ is played, any position extending $P$ in mega-round $\alpha + 1$ cannot be terminal either as the relative domain of the position constructed in mega-round $\alpha + 1$ is $\delta + 1$ for some $\delta \in \mathcal{W}$ which cannot possibly be equal to some shift of $\vartheta$ which is not in $\mathcal{W}$.

Note that the final model of $\mathfrak{U}$ is $Q_\eta$. We define

$$t_\eta^\dagger = t_\eta{}^\frown(w_\eta, y_\eta). \tag{4.2}$$

This is a position of relative domain $k_\eta(\delta_\eta^\dagger) + 1$ over $Q_\eta$.

**64 Lemma.** *Every strict initial segment of $t_\eta^\dagger$ is $Q_\eta$-clear.*

*Proof.* By definition, every strict initial segment of $t_\eta^\dagger$ is an initial segment of $t_\eta$. The strand $P_\eta$ is a legal position in $G_{\mathrm{br}}(M, \emptyset, \vartheta)(\dot{Y})$ by assumption with outcome $(M_\eta, j_{0,\eta}, t_\eta)$. Furthermore we assumed that $P_\eta$ is non-terminal, so $t_\eta$ is $M_\eta$-clear by the rules of the branching game. By regularity, $M_\eta$ and $Q_\eta$ agree past $\lambda_\eta = \mathrm{rdm}(t_\eta)$ which is enough to guarantee that there are no $Q_\eta$-obstructions for any initial segment of $t_\eta$. □

However, there could be an obstruction for $t_\eta^\dagger$ itself:

**65 Lemma.** *The position $t_\eta^\dagger$ is either obstruction free over $Q_\eta$ or else it is* I-*acceptably obstructed over $Q_\eta$.*

*Proof.* By one of the assumptions (A3) to (A4), $t^\dagger_\eta$ can be constructed in a non-terminal position of the branching game. Most importantly this means that in this round the game did not end via (E2) or (E6). □

We therefore need to distinguish two cases depending on whether $t^\dagger_\eta$ is $Q_\eta$-obstructed or not.

### Obstruction free

If $t^\dagger_\eta$ is obstruction free over $Q_\eta$, we let $\mathfrak{U}^+$ be the length-$\eta + 1.2$-extension of $\mathfrak{U}$ with $E_\eta$ = "undefined" (and therefore necessarily $\eta U^+ \eta + 1$). We let $P_{\eta+1}$ be the strand leading to $M_{\eta+1}$ and $t_{\eta+1}$ be the position generated in the usual way from $\mathfrak{U}$ and the $\mathfrak{U}$-sequence. We show that this leads to an extension where the strand leading to $\eta + 1$ is not terminal in the branching game.

**66 Lemma.** *Under the above-stated assumptions $P_{\eta+1}$ is a legal position in $G_{br}(M, \emptyset, \vartheta)(\dot{Y})$. It is non-terminal and played according to the winning strategy $\Sigma_{br}$.*

*Proof.* Assume first that $\eta$ is a successor ordinal or zero. Hence, also $\beta = \mathrm{lh}(P_\eta)$ is a successor ordinal or zero by Lemma 58. Since $\eta U^+ \eta + 1$,

$$P_{\eta+1} = P_\eta{}^\frown(\mathcal{T}_\eta, b_\eta, E_\eta, t_{\eta+1}),$$

where $E_\eta$ = "undefined" and $t_{\eta+1} = t_\eta{}^\frown(w_\eta, y_\eta)$. This is a position in $G_{\mathrm{br}}(M, \emptyset, \vartheta)(\dot{Y})$, where in mega-round $\beta$ moves $\mathcal{T}_\eta, b_\eta, E_\eta, t_{\eta+1}$ were made following $P_\eta$. From (A1) we conclude that $P_\eta$ is a legal position and Lemma 62 says that also $P_{\eta+1}$ is legal. That these moves are according to player I's winning strategy $\Sigma_{\mathrm{br}}$ follows from (A2) for $P_\eta$ and using furthermore (A4) we get the claim for $P_{\eta+1}$.

It is left to check that $P_{\eta+1}$ is non-terminal in $G_{\mathrm{br}}(M, \emptyset, \vartheta)(\dot{Y})$. This is equivalent to the claim that in mega-round $\beta$ the game did not end via any of the snags (E1), (E2) or the payoff condition (P1). Since $Q_\eta$ is a model on a tree of trees which is consistent with the iteration strategy $\Gamma$ it is wellfounded, so the game does not end via the illfoundedness condition (E1). Furthermore $t_{\eta+1} = t^\dagger_\eta$ is obstruction free over $M_{\eta+1} = Q_\eta$, so (E2) does not hold either. The relative domain of this $M_{\eta+1}$-position $\mathrm{rdm}(t_{\eta+1})$ is equal to $\lambda_{\eta+1}$ by Lemma 54, which by definition is $\delta_{\eta+1} + 1$. $\delta_{\eta+1}$, which is a relative successor, however cannot possibly be equal to $j_{0,\eta+1}(\vartheta)$, which is a Woodin limit of Woodin cardinals, thus the game also does not end via the payoff condition (P1).

If $\eta$ is a standard limit, the proof works similar. That $P_\eta$ is legal and played according to $\Sigma_{\mathrm{br}}$ follows using (A1) and (A2). If we let

$$P_{\eta+1} = P_\eta{}^\frown(w_\eta, \mathcal{T}_\eta, b_\eta, y_\eta),$$

we have moves for (L1) to (L4) of mega-round $\beta = \mathrm{lh}(P_\eta)$ which is played according to the standard limit case. We stipulate furthermore that an early end is chosen in round $\beta$ and it is abstained from a leap. By Lemma 63 and (A3), also the moves in round $\beta$ are legal and according to I's winning strategy.

To check that the position is non-terminal it has to be checked that none of the assumptions in the snags (E5), (E6) or in the payoff condition (P1) is satisfied. This follows as above. (The game did not end via one of the snags (E3) or (E4) by (A1) for $\bar{\eta} = \eta$). □

### I-acceptably obstructed

If $t^\dagger_\eta$ defined in Equation (4.2) is I-acceptably obstructed over $Q_\eta$, fix such an obstruction $(E, \vec{\sigma})$ by Definition 29. For $\delta_{\eta+1} = k_\eta(\delta^\dagger_\eta)$, we then have by the definition:

(1) $E$ is $\delta_{\eta+1} + 1$-strong in $Q_\eta$),

(2) $\mathrm{crit}(E) < \delta_{\eta+1}$ and is not Woodin in $Q_\eta$,

which implies that $\mathrm{crit}(E)$ is a non-Woodin limit of Woodin cardinals. By Lemma 48 there exists $\gamma$ such that furthermore

(3) $\gamma \leq \eta$ and is a standard limit in $\mathfrak{U}$,

(4) $\mathrm{crit}(E) = \lambda_\gamma$.

Hence, using Lemma 47, we get that $Q_\eta$ and $M_\gamma$ agree past $\lambda_\gamma$, which means that $E$ can be applied to $M_\gamma$. Let $\mathfrak{U}^+$ be the extension of $\mathfrak{U}$ we get by making this stipulation to a tree of trees of length $\eta + 1.2$: We let $E_\eta = E$ and $\gamma U^+ \eta + 1$. This completely determines $\mathfrak{U}^+$.

**67 Lemma.** *The extension $\mathfrak{U}^+$ satisfies the following conditions:*

- $\mathfrak{U}^+$ *is regular,*

- $M_{\eta+1} = \mathrm{Ult}(M_\gamma, E_\eta)$ *is wellfounded,*
- *Let* $s_\gamma = t_\gamma{}^\frown(w_\gamma, y_\gamma)$. *Then* $t^\dagger_\eta$ *extends* $s_\gamma$.

*Proof.* That $\mathfrak{U}^+$ is regular follows directly from the strongness condition (1) on $E$ at the beginning of this section. Since $\mathfrak{U}$ is consistent with $\Gamma$, the second claim follows. For the third claim, note that if $\gamma = \eta$ the two positions are equal. If $\gamma < \eta$ however, then $s_\gamma = t_{\gamma+1}$ and hence $t^\dagger_\eta$ extends this strictly by definition. □

Setting $\beta = \mathrm{lh}(P_\gamma)$ we see that mega-round $\beta$ of $G_{\mathrm{br}}(M, \emptyset, \vartheta)(\dot{Y})$ following the position $P_\gamma$ is played according to the rules for the standard limit case by Lemma 61 and condition (3) above. By Lemma 63 the objects $w_\gamma, \mathcal{T}_\gamma, b_\gamma, y_\gamma$ are legal moves for rules (L1) to (L4). In the following we want to show that we furthermore can simulate a legal leap in this round. Precisely, we will show that the assumptions of Lemma 41 which constructs legal moves for a leap are satisfied.

**68 Lemma.** *Let* $P^*_\gamma$ *denote the position* $P_\gamma{}^\frown(w_\gamma, \mathcal{T}_\gamma, b_\gamma, y_\gamma)$ *in* $G_{br}(M, \emptyset, \vartheta)(\dot{Y})$. *Then* $P^*_\gamma$ *is*

- *played according to* I*'s winning strategy* $\Sigma_{br}$*, and*
- *non-terminal in* $G_{br}(M, \emptyset, \vartheta)(\dot{Y})$.

*Proof.* From the assumptions (A2) and (A3) for $\bar{\eta} = \gamma$ it follows that $P^*_\gamma$ is played in line with $\Sigma_{\mathrm{br}}$. Moreover we have to check that this position is not terminal through the snags (E5) or (E6): This holds since $Q_\eta$ is wellfounded since it appears on the tree of trees $\mathfrak{U}$ which is consistent with the iteration strategy $\Gamma$ (which picks the branch leading to through $\mathcal{T}_\eta$ leading to $Q_\eta$). Furthermore $t^\dagger_\eta$ is I-acceptably obstructed by the case assumptions. □

**69 Lemma.** $Q_\eta \restriction \delta_{\eta+1} + \omega$ *is countable in* V.

*Proof.* Using that $P_\gamma$ is non-terminal in $G_{\mathrm{br}}(M, \emptyset, \vartheta)(\dot{Y})$ by (A2), it follows that $\mathrm{rdm}(t_\eta) < j_{0,\eta}(\vartheta)$ since neither the payoff condition (P1) nor (P2) was satisfied back then. For $\delta^\dagger_\eta$, the first Woodin cardinal in $M_\eta$ above $\mathrm{rdm}(t_\eta)$, we get $\delta^\dagger_\eta < j_{0,\eta}(\vartheta)$ since the latter is a Woodin limit of Woodin cardinals. Thus $\delta_{\eta+1} = k_\eta(\delta^\dagger_\eta) < (k_\eta \circ j_{0,\eta})(\vartheta)$.

Since $\vartheta$ is Woodin in $M$, it is inaccessible in $M$ and thus the cardinality in $Q_\eta$ of $Q_\eta \restriction \delta_{\eta+1} + \omega$ is smaller than $(k_\eta \circ j_{0,\eta})(\vartheta)$ so it suffices to show that the latter is countable in V to prove the claim.

By the assumptions of this section $\vartheta$ itself is countable in V. Both, $j_{0,\eta}$ and $k_\eta$, preserve this countability since they are embeddings in $\mathfrak{U}$ and $\mathfrak{U}$ is regular and only uses extenders which are countable in V. □

**70 Lemma.** *(assuming that $t^\dagger_\eta$ is* I*-acceptably obstructed over $Q_\eta$) Let $P_{\eta+1}$ be as above wrt. $\mathfrak{U}^+$ as above. Then $P_{\eta+1}$ is legal in $G_{br}(M, \emptyset, \vartheta)(\dot{Y})$, non-terminal and played according to $\Sigma_{br}$.*

*Proof.* Recall that $\gamma$ is the direct $U^+$-predecessor of $\eta + 1$, $\beta = \mathrm{lh}(P_\gamma)$ and that $\gamma$ is a standard limit. Recall from Definition 57 that

$$P_{\eta+1} = P_\gamma{}^\frown(\mathcal{T}_\gamma, b_\gamma, E_\eta, t_{\eta+1}), \tag{4.3}$$

and using Lemma 61 it follows that mega-round $\beta$ of $G_{\mathrm{br}}(M, \emptyset, \vartheta)(\dot{Y})$ is played subject to the rules of the standard limit case.

From this, the equation above and the fact that $E_\eta \neq$ "undefined", we get that the position $P_{\eta+1}$ in the branching game consists of:

(O1) $P_\gamma$, as the moves up to this mega-round,

(O2) $w_\gamma, \mathcal{T}_\gamma, b_\gamma, y_\gamma$, given by the fixed $\mathfrak{U}, \vec{w}, \vec{y}$ and played for rules (L1) to (L4),

(O3) $E_\eta, \delta_{\eta+1}, t_{\eta+1}$, as a leap for rules (L5) and (L6).

It is clear from the assumptions of this section, Lemma 63 and Lemma 68 that (O1) to (O2) give legal non-terminal moves which are consistent with I's winning strategy $\Sigma_{\mathrm{br}}$. Furthermore, also the leap in (O3) gives moves which are consistent with $\Sigma_{\mathrm{br}}$ since they are made by player II.

To check that these moves are legal at this point, we want to appeal to Lemma 41. We use this Lemma with $Q^*_\beta = Q_\eta, E^*_\beta = E_\eta, \delta^*_\beta = \delta^\dagger_\eta, t^*_\beta = t^\dagger_\eta$. The assumptions there follow from Lemma 69, Lemma 67, Lemma 64 and (4) asserting that $\mathrm{crit}(E) = \lambda_\gamma$.

The only reason through which $P_{\eta+1}$ could be terminal in the branching game is snag (E7). This says that the game ends if $M_{\eta+1}$ is illfounded which however is wrong by Lemma 67. □

Putting the results from the previous two subsections together we can conclude:

**71 Lemma.** *If $\mathfrak{U}$ is a regular tree of trees of length $\eta + 1$ and $\vec{w}, \vec{y}$ is a $\mathfrak{U}$-sequence such that assumptions (A1) to (A5) hold, then there is an extension $\mathfrak{U}^+$ of $\mathfrak{U}$ having length $\eta + 1.2$ such that the strand $P_{\eta+1}$ is legal, non-terminal and consistent with $\Sigma_{br}$ in $G_{br}(M, \emptyset, \vartheta)(\dot{Y})$.*

### 4.2.3 Reducing the local game to the branching game

We work with a transitive model $M \models \mathsf{ZFC}^*$ and fix an iteration strategy $\Gamma$ for $M$. Furthermore we let $\vartheta$ be a Woodin limit of Woodin cardinals in $M$ and let $\dot{A}$ be a $\mathbb{W}_\vartheta$-name for a set of sequences of reals of length $\vartheta$.

For illustrational purposes, we fix a $\mathbb{W}_\vartheta$-generic filter $G$ over $M$. We let $\dot{Y}$ be the canonical $\mathbb{W}_\vartheta$-name for the set of $\vartheta$-sequences (see Definition 39) $t \in M[G]$ so that $\vec{z}(t) \in \dot{A}[G]$. Recall Definition 55 where this notion is defined. Note that $\vartheta$ is a Woodin limit of Woodin cardinals and it can be easily checked then that there are $\vartheta$ many relative successors below $\vartheta$. Thus $\vec{z}(t)$ for $t$ a $\vartheta$-sequence is a sequence of reals of length $\vartheta$ and could possibly be an element of $\dot{A}[G]$. Furthermore we use the formula $\varphi_{\text{ini}}$ of Theorem 42 which says that I has a winning strategy in the branching game .

**72 Lemma.** *Assume that $\vartheta$ is countable in* V *and $M \models \varphi_{ini}[\vartheta, \dot{Y}]$. Then player* I *has a winning strategy for $G_{loc}(M, \Gamma, \vartheta, \dot{A})$ in* V.

*Proof.* We fix an imaginary opponent II in the game $G_{\text{loc}}(M, \Gamma, \vartheta, \dot{A})$ and describe player I's moves. We plan to construct witnessing $\mathcal{U}$ and $\alpha$ for the payoff condition (P). This $\mathcal{U}$ will be the merge of a regular tree of trees $\mathfrak{U}$. We will construct

(D1) a regular tree of trees $\mathfrak{U}^+$ on $M$ of length $\alpha^* + 0.2$ for some $\alpha^*$, so that $\mathfrak{U}^+$ is regular,

(D2) a $\mathfrak{U}$-sequence $(w_\xi, y_\xi \mid \xi \in K^{\mathfrak{U}})$ where $\mathfrak{U} = \mathfrak{U}^+ \restriction \alpha^*$,

so that the reals of this $\mathfrak{U}$-sequence

$$\vec{z} = (y_\xi \mid \xi < \alpha^* \text{ is zero or a successor}) \\ \text{form a complete run of } G_{\text{loc}}(M, \Gamma, \vartheta, \dot{A}) \text{ won by player I.} \quad (4.4)$$

These are the only moves where player II comes in. A major part of the construction of $\mathfrak{U}^+$ will be done using I's winning strategy $\Sigma_{\text{br}}$ in the branching game $G_{\text{br}}(M, \emptyset, \vartheta)(\dot{Y})$. Following the previous section we will construct the tree of trees in such a way that along the branches we have runs of this game.

During the construction we want to make sure that

(1) for each $\eta < \alpha^*$, $P_\eta$ is a legal position in $G_{\text{br}}(M, \emptyset, \vartheta)(\dot{Y})$ which is non-terminal and played according to player I's winning strategy $\Sigma_{\text{br}}$,

(2) the tree of trees $\mathfrak{U}^+$ is consistent with the iteration strategy $\Gamma$,

(3) whenever $\eta < \alpha^*$ is a successor or zero: the objects $w_\eta, y_\eta, \mathcal{T}_\eta, b_\eta$ are consistent with $\Sigma_{\text{br}}[P_\eta]$,

(4) whenever $\eta < \alpha^*$ is a standard limit in $\mathfrak{U}$: the objects $w_\eta, \mathcal{T}_\eta, b_\eta, y_\eta$ are consistent with $\Sigma_{\text{br}}[P_\eta]$,

(5) whenever $\eta < \alpha^*$ is not a phantom limit in $\mathfrak{U}$, the extension $\mathfrak{U} \upharpoonright \eta + 1.2$ is obtained from $\mathfrak{U} \upharpoonright \eta + 1$ through Lemma 71,

(6) whenever $\eta < \alpha^*$ is a phantom limit in $\mathfrak{U}$: $\mathcal{T}_\eta$ is the tree consisting entirely of undefined extenders, $b_\eta$ is the unique branch through it and $E_\eta$ = "undefined".

The reader should recall that $\Sigma_{\text{br}}[P_\eta]$ is I's winning strategy $\Sigma_{\text{br}}$ restricted to moves following $P_\eta$. One should note here the similarity of our construction restriction to the assumptions of the previous sections. We need this in order to be able to use our results from the last section in (5). The conditions described above completely define the tree of trees $\mathfrak{U}^+$ and the $\mathfrak{U}$-sequence and are uniquely determined by the following:

- If $\eta < \alpha^*$ is zero or a successor, the objects $w_\eta, \mathcal{T}_\eta$ are constructed solely by I's winning strategy $\Sigma_{\text{br}}$. The real $y_\eta$ is formed through a collaborative effort of the imaginary opponent II and $\Sigma_{\text{br}}$. The branch $b_\eta$ through $\mathcal{T}_\eta$ is chosen by the iteration strategy $\Gamma$. The extender $E_\eta$ and the $U^+$-predecessor of $\eta + 1$ are given by Lemma 71.
- If $\eta < \alpha^*$ is a standard limit in $\mathfrak{U}$, the witness $w_\eta$, the iteration tree $\mathcal{T}_\eta$ and the $\omega$-sequence $y_\eta$ are constructed using $\Sigma_{\text{br}}$. The branch $b_\eta$ is chosen by the iteration strategy $\Gamma$. Again, the extender $E_\eta$ and the $U^+$-predecessor of $\eta + 1$ are determined by Lemma 71.
- For phantom limits $\eta$ in $\mathfrak{U}$, we use (6).

- It remains to choose the cofinal branches in $\mathfrak{U}^+$ functioning as the $U^+$-branches leading to limit $\eta \leq \alpha^*$ determining $M_\gamma$. Again, these are chosen by the iteration strategy $\Gamma$.

We continue with this construction as long as (1) holds. It is certainly true for $\eta = 0$. As soon as (1) is wrong for some $\eta$, we stop the construction and set $\alpha^* = \eta$. In the following we try to find out more about $\alpha^*$, most importantly we need to check that it is less than $\omega_1^V$ so that the payoff condition (P) can hold.

**Claim.** If $\eta < \omega_1^V$ is a limit ordinal and (1) holds for all $\bar{\eta} < \eta$, then it also holds for $\eta$.

*Proof.* By definition, we have $P_\eta = \bigcup_{\zeta U \eta} P_\zeta$. Since each $P_\zeta$ is a legal non-terminal position in $G_{\mathrm{br}}(M, \emptyset, \vartheta)(\dot{Y})$, which is played according to $\Sigma_{\mathrm{br}}$, it follows that also $P_\eta$ is legal and according to $\Sigma_{\mathrm{br}}$. It is left to check that $P_\eta$ is non-terminal: This follows since $M_\eta$ is a model on the tree of trees $\mathfrak{U}$ which is consistent with $\Gamma$ and is therefore wellfounded which means that (E3) does not apply. Furthermore (E4) does not lead to an end since from $\eta < \omega_1^V$ it follows that the order type of $\{\zeta \mid \zeta U \eta\}$ is less than $\omega_1^V$. ⊣

**Claim.** If $\eta$ is zero or a successor or a standard limit in $\mathfrak{U}$ and condition (1) holds for $\eta$, it also holds for $\eta + 1$.

*Proof.* Note that in this case, the regular tree of trees $\mathfrak{U} \restriction \eta + 1.2$ is constructed from $\mathfrak{U} \restriction \eta + 1$ by an application of Lemma 71. The conclusion there guarantees that the claim holds. ⊣

So if the construction ends with $\alpha^*$ smaller than $\omega_1^V$, by the previous two Lemmata it holds that $\alpha^* = \alpha + 1$ where $\alpha$ is a phantom limit in $\mathfrak{U}$. That this really happens is mostly a consequence of $\Sigma_{\mathrm{br}}$ being a winning strategy for player I in the branching game which has to avoid the snag (E4).

**Claim.** The construction ends before reaching $\omega_1^V$.

*Proof.* Suppose not and let $r = \{\zeta \mid \zeta U \omega_1^V\}$ be the branch leading to $\omega_1^V$. Then each of $P_\zeta$ for $\zeta \in r$ is a legal non-terminal position in $G_{\mathrm{br}}(M, \emptyset, \vartheta)(\dot{Y})$ which is played in line with $\Sigma_{\mathrm{br}}$. It follows that also $P_{\omega_1^V} = \bigcup_{\zeta \in r} P_\zeta$ is legal in this game and according to $\Sigma_{\mathrm{br}}$. Since $r$ is cofinal in $\omega_1^V$ (which is ensured by $\Gamma$), also its order type is $\omega_1^V$ which means that $P_{\omega_1^V}$ is lost by player I through the snag (E4). This however is a contradiction to $\Sigma_{\mathrm{br}}$ being a winning strategy for player I. ⊣

Note here how we make use of the full iteration strategy $\Gamma$ (in order to find a cofinal branch in $\omega_1^V$): We use it to rule out that these big iteration trees actually exist in our construction. Although iteration trees of length $\omega_1^V$ actually turn up in our construction we still need an iteration strategy for iteration games of this length to rule them out.

We now know that $\alpha^*$, the least ordinal so that condition (1) fails, exists and is a successor ordinal following a phantom limit in $\mathfrak{U}$ less than $\omega_1^V$. Since the construction up to this point follows conditions (2) to (6), we get the objects described in (D1) to (D2).

It remains to show that Equation (4.4) holds which is equivalent to saying that the conditions above describe a winning strategy for player I in $G_{\text{loc}}(M, \Gamma, \vartheta, \dot{A})$.

**Claim.** The strand $P_{\alpha^*}$ is a terminal position in $G_{\text{br}}(M, \emptyset, \vartheta)(\dot{Y})$ which is won by player I through payoff condition (P2).

*Proof.* Since $\alpha = \alpha^* - 1$ is a phantom limit in $\mathfrak{U}$, we have from condition (6) that $E_\alpha$ = "undefined". Hence $\alpha U^+ \alpha^*$ which in turn implies $P_\alpha \subset P_{\alpha^*}$. Furthermore mega-round $\beta = \text{lh}(P_\alpha)$ is played subject to the phantom limit case of the branching game and hence $P^*_\alpha$ has to extend $P_\alpha$ by the trivial moves played in this round. These moves are legal in $G_{\text{br}}(M, \emptyset, \vartheta)(\dot{Y})$ following $P_\alpha$ and of course consistent with $\Sigma_{\text{br}}[P_\alpha]$, thus $P_{\alpha^*}$ is a legal position in $G_{\text{br}}(M, \emptyset, \vartheta)(\dot{Y})$ played according to $\Sigma_{\text{br}}$.

Since condition (1) fails for $\alpha^*$, $P_{\alpha^*}$ must be terminal in the branching game. The only way that it is terminal is through condition (P2). Since it was constructed in accordance with I's winning strategy $\Sigma_{\text{br}}$ it is won by player I. ⊣

We then let $M_\xi$ and the embeddings $j_{\zeta,\xi} : M_\zeta \to M_\xi$ for $\zeta U \xi < \alpha^* + 1$ as given by the tree of trees $\mathfrak{U}^+$. The model $M^*_\alpha$ is the final model with the elementary embedding $j_{0,\alpha^*} : M \to M_{\alpha^*}$. We also have $t_{\alpha^*}$ (constructed from $\mathfrak{U}^+$ and the $\mathfrak{U}$-sequence using Equation (4.1) extended to $\mathfrak{U}^+$ ) as an $M_{\alpha^*}$-position.

**Claim.** There is a $G$ such that

- $G$ is $j_{0,\alpha^*}(\mathbb{W}_\vartheta)$-generic over $M$,
- $t_{\alpha^*} \in j_{0,\alpha^*}(\dot{Y})[G]$.

*Proof.* This follows directly from the payoff condition (P2) which holds with the right objects in place. ⊣

Fix such a $G$ as needed for the previous claim.

**Claim.** $\vec{z}$ belongs to $j_{0,\alpha^*}(\dot{A})[G]$.

*Proof.* Note first that Lemma 56 guarantees that $\vec{z}$, defined in Equation (4.4), is equal to $\vec{z}(t_{\alpha^*})$ defined in Definition 55. Now use $t_{\alpha^*} \in j_{0,\alpha^*}(\dot{Y})[G]$ established in the previous Claim and the way $\dot{Y}$ is defined from $\dot{A}$ at the beginning of this section. ⊣

This proves that the sequence of reals of length $\vartheta$, $\vec{z}$, produced using $\Sigma_{\rm br}$ and the imaginary opponent II is in fact a complete run of $G_{\rm loc}(M, \Gamma, \vartheta, \dot{A})$ which is won by player I. The witness for that are the merge of the extended tree of trees merge$(\mathfrak{U}^+)$ and the ordinal $\alpha^*$. □

We quickly note here that the assumptions of the previous Lemma that $\vartheta$ is countable in V was used to use the results from the previous Subsection in the form of Lemma 71. The other assumption, namely that $\varphi_{\rm ini}$ of certain objects holds, was of course needed to have the winning strategy $\Sigma_{\rm br}$ in the branching game at hand.

We are now in a position to prove our main theorem. Because of its importance we introduce all of the objects which are needed for its proof again.

**73 Theorem.** *Let $M$ be a transitive model of* ZFC$^*$ *and $\Gamma$ be an iteration strategy for $M$. Let $\vartheta$ be a Woodin limit of Woodin cardinals in $M$ and $\dot{A}$, $\dot{B}$ be $\mathbb{W}_\vartheta$-names for sequences of reals with length $\vartheta$. Suppose furthermore that in* V*, $\vartheta$ is countable. Then one of the following cases holds:*

*(1) Player* I *has a winning strategy in $G_{loc}(M, \Gamma, \vartheta, \dot{A})$;*

*(2) player* II *has a winning strategy in $H_{loc}(M, \Gamma, \vartheta, \dot{B})$;*

*(3) there exists a $\mathbb{W}_\vartheta$-generic filter $G$ over $M$ and a sequence $(z_\xi \mid \xi < \vartheta \in M[G]$ of reals $z_\xi$ which neither belongs to $\dot{A}[G]$ nor to $\dot{B}[G]$.*

*Moreover there are formulas $\varphi$ and $\psi$ such that their truth values in $M$ correspond to the above cases: if $M \models \varphi(\vartheta, \dot{A})$, then case (1) holds, if $M \models \psi(\vartheta, \dot{B})$, then case (2) holds, and otherwise case (3) holds.*

*Proof.* We fix a generic filter $G$ for $\mathbb{W}_\delta$ over $M$. Let $\dot{Y}$ and $\dot{Z}$ be the canonical names for the set of $\vartheta$-sequences $t \in M[G]$ so that $\vec{z}(t) \in \dot{A}[G]$ (or $\vec{z}(t) \in \dot{B}[G]$). If $\varphi_{\rm ini}(\vartheta, \dot{Y})$ holds in $M$, then applying Lemma 72 produces a winning strategy for player I in $G_{\rm loc}(M, \Gamma, \vartheta, \dot{A})$ and case (1) holds. If

$\psi_{\text{ini}}(\vartheta, \dot{Z})$ holds in $M$, analogous methods for producing Lemma 72 imply that case (2) holds. For this, one needs to repeat the whole development of Chapter 3 and Chapter 4 with the roles of players I and II switched.

Suppose now, that neither of these formulas holds in $M$. Then, using Theorem 43 we get a $\vartheta$-sequence $t$ in an extension $M[G]$ of $M$, where $G$ is $\mathbb{W}_\vartheta$-generic over $M$ and

$$t \notin \dot{Y}[G], \quad t \notin \dot{Z}[G]. \tag{4.5}$$

We now want to produce such objects $G$ and $t$ in V: This is possible if we establish that there are only countably many maximal antichains of $\mathbb{W}_\vartheta$ in $M$. These are subsets of $M \upharpoonright \vartheta$ and from the $\vartheta$-cc of $\mathbb{W}$ it follows that they are actually elements of $M \upharpoonright \vartheta$. Using the inaccessibility of $\vartheta$ in $M$ we get that there are at most $\vartheta$ such objects. From the countability of $\vartheta$ in V it follows now that we can assume that $t$ and $G$ are elements of $M$.

$t \in M[G]$ of course implies $\vec{z}(t) \in M[G]$ and using the inaccessibility of $\vartheta$ in $M$ it follows that this actually is a sequence of reals with length $\vartheta$. From the definition of $\dot{Y}$ and $\dot{Z}$ and Equation (4.5), case (3) follows.

It is clear that these are all possibilities. The formulas $\varphi$ and $\psi$ of the claim are produced in the obvious way using the formulas $\varphi_{\text{ini}}$, $\psi_{\text{ini}}$ and the definitions of $\dot{Y}$ and $\dot{Z}$. □

# Bibliography


[1] David Gale and F. M. Stewart. "Infinite games with perfect information." In: *Contributions to the theory of games, vol. 2*. Annals of Mathematics Studies, no. 28. Princeton University Press, Princeton, N. J., 1953, pp. 245–266.

[2] Leo Harrington. "Analytic determinacy and $0^\sharp$." In: *J. Symbolic Logic* 43.4 (1978), pp. 685–693. ISSN: 0022-4812. DOI: 10.2307/2273508. URL: http://dx.doi.org/10.2307/2273508.

[3] Thomas Jech. *Set theory*. Springer Monographs in Mathematics. The third millennium edition, revised and expanded. Springer-Verlag, Berlin, 2003, pp. xiv+769. ISBN: 3-540-44085-2.

[4] Akihiro Kanamori. *The higher infinite*. Second. Springer Monographs in Mathematics. Large cardinals in set theory from their beginnings. Springer-Verlag, Berlin, 2003, pp. xxii+536. ISBN: 3-540-00384-3.

[5] Kenneth Kunen. *Set theory*. Vol. 34. Studies in Logic (London). College Publications, London, 2011, pp. viii+401. ISBN: 978-1-84890-050-9.

[6] Donald A. Martin. "Borel determinacy." In: *Ann. of Math. (2)* 102.2 (1975), pp. 363–371. ISSN: 0003-486X.

[7] Donald A. Martin. "Measurable cardinals and analytic games." In: *Fund. Math.* 66 (1969/1970), pp. 287–291. ISSN: 0016-2736.

[8] Donald A. Martin and John R. Steel. "A proof of projective determinacy." In: *J. Amer. Math. Soc.* 2.1 (1989), pp. 71–125. ISSN: 0894-0347. DOI: 10.2307/1990913. URL: http://dx.doi.org/10.2307/1990913.

[9] William J. Mitchell. "Sets constructible from sequences of ultrafilters." In: *J. Symbolic Logic* 39 (1974), pp. 57–66. ISSN: 0022-4812.

[10] Itay Neeman. "An introduction to proofs of determinacy of long games." In: *Logic Colloquium '01*. Vol. 20. Lect. Notes Log. Assoc. Symbol. Logic, Urbana, IL, 2005, pp. 43–86.

[11] Itay Neeman. "Games of length $\omega_1$." In: *J. Math. Log.* 7.1 (2007), pp. 83–124. ISSN: 0219-0613. DOI: 10.1142/S0219061307000627. URL: http://dx.doi.org/10.1142/S0219061307000627.

[12] Itay Neeman. "Inner models in the region of a Woodin limit of Woodin cardinals." In: *Ann. Pure Appl. Logic* 116.1-3 (2002), pp. 67–155. ISSN: 0168-0072. DOI: 10.1016/S0168-0072(01)00103-8. URL: http://dx.doi.org/10.1016/S0168-0072(01)00103-8.

[13] Itay Neeman. *The determinacy of long games*. Vol. 7. de Gruyter Series in Logic and its Applications. Walter de Gruyter GmbH & Co. KG, Berlin, 2004, pp. xii+317. ISBN: 3-11-018341-2. DOI: 10.1515/9783110200065. URL: http://dx.doi.org/10.1515/9783110200065.

[14] J. R. Steel. "The well-foundedness of the Mitchell order." In: *J. Symbolic Logic* 58.3 (1993), pp. 931–940. ISSN: 0022-4812. DOI: 10.2307/2275105. URL: http://dx.doi.org/10.2307/2275105.

# Appendix

*Zusammenfassung*

In dieser Arbeit wird versucht Einblicke in die Determiniertheit von überabzählbar langen Spielen zu geben. Während die Untersuchung von Determiniertheit von Spielen abzählbarer Länge bereits seit einigen Jahrzehnten betrieben wird, sind Resultate zur Determiniertheit überabzählbar langer Spiele erst in den letzten Jahren erzielt worden und noch immer relativ dünn gesät. Es wird probiert, die von Itay Neeman in [13] benützten Methoden besser zu verstehen und zu präsentieren. Dadurch soll vermittelt werden mit welchem Aufwand Beweise in diesem Gebiet verbunden sind.

Das Hauptresultat dieser Arbeit ist die Determiniertheit des in Kapitel 4 vorgestellten Spiels. Für einen Forcingnamen für Folgen von reellen Zahlen spielen zwei Personen (I und II genannt) abwechselnd natürliche Zahlen. Kann an irgendeinem Punkt der Forcingname so interpretiert werden, dass die produzierte Folge ein Element ist, gewinnt Spieler I; ist das nicht möglich, gewinnt II.

In Kapitel 1 werden die grundlegenden Konzepte eingeführt. Es wird die Ultrapowerkonstruktion sowohl für Ultrafilter als auch für Extender vorgestellt, die nötigen großen Kardinalzahlen werden erklärt und das Konzept des Iteration Trees, das sich in Determiniertheitstheorie und Inner model theory als unerlässliches Werkzeug erwiesen hat, wird erläutert.

In Kapitel 2 stellen wir das oben erwähnte Forcingposet vor. Seine wichtigste Eigenschaft ist, dass es bestimmte Kardinalzahlen erhält, was garantiert, dass jeder Lauf des Spiels in Kapitel 4 lokal überabzählbar ist. Um die Determiniertheit dieses Spiels zu beweisen, werden wir in Kapitel 3 ein Hilfsspiel vorstellen. Um unser Hauptresultat zu erzielen benützen wir eine Gewinnstrategie für dieses Spiel. Mittels dieser werden wir eine Gewinnstrategie für das lokale Spiel konstruieren.

# Martin Köberl

*Lebenslauf*

## Bildungsweg

| | |
|---|---|
| seit 10/2012 | **Masterstudium der Allgemeinen Sprachwissenschaft**, *Universität Wien*, Schwerpunkt: Grammatiktheorie. |
| seit 10/2012 | **Masterstudium der Mathematik**, *Universität Wien*, Schwerpunkt: mathematische Logik.<br>voraussichtlicher Studienabschluss: 05/2015 |
| 09/2011 – 12/2011 | **Erasmusaufenthalt im Rahmen des Mathematikstudiums**, *University of Edinburgh*. |
| 10/2009 – 09/2012 | **Bachelorstudium der Sprachwissenschaft**, *Universität Wien*, nicht abgeschlossen. |
| 10/2008 – 06/2012 | **Bachelorstudium der Mathematik**, *Universität Wien*, abgeschlossen am 27.06.2012, *ausgezeichneter Erfolg*.<br>Abschlussgrad: Bachelor of Science |
| 10/2000 – 06/2008 | **Höhere Schule**, *Bundesgymnasium Wenzgasse*, Wien.<br>*ausgezeichneter Erfolg* |

## Berufserfahrung

| | |
|---|---|
| seit 2010 | **Tutor**, *Technische Universität Wien*.<br>für Mathematik I und II für Maschinenbau, Verfahrenstechnik und Wirtschaftsingenieurwesen sowie für Mathematik I und II für Bauingenieurwesen |
| 2012, 2013 | **Tutor**, *Universität Wien*.<br>für Analysis II für Physik und für formale Semantik |

## Weitere Aktivitäten

| | |
|---|---|
| seit 07/2013 | **Studierendenvertretung Sprachwissenschaft**, *Mandatar der Interessensvertretung für Studierende*. |
| 2012 – 2013 | **Kolloquium für Doktoratsstudierende**, *Universität Wien*, Organisation von Vorträgen durch Studierende. |