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"Bridgeland stability conditions and exceptional collections"

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To my son Bogoljub Georgiev Dimitrov, born 21.12.2011, with love.

Да сине, аз съм твоят тате. Нищо, никой, никога не може да угаси обичта ни...

Abstract

The “moduli space” of stability conditions is currently an important ingredient in the framework of Homological Mirror Symmetry (HMS). It was introduced by T. Bridgeland (2002) as an approach to mathematical understanding of certain moduli spaces arising in string theory. He assigned to any triangulated category a complex manifold, whose elements are referred to as Bridgeland stability conditions. HMS predicts a parallel between dynamical systems and categories whereby the space of Bridgeland stability conditions is a candidate to play the role of the Teichmüller space. However, global information for the stability space is known in only a handful of examples.

Long before HMS (1994), Beilinson et. al. observed patterns in the structure of some triangulated categories which they called exceptional collections (Beilinson's paper appeared in 1978).

The main motivation for the present work comes from a procedure generating stability conditions from exceptional collections, described by E. Macrì in his paper from 2007.

This thesis explores some aspects of the interplay between the two notions in the title and unveils novelties for both sides. On the one hand, the findings concerning stability conditions are new evidences supporting the parallel mentioned above. On the other hand, remarkable relations between exceptional representations of quivers appear in the thesis.

The work consists of three parts.

In the first part is defined the notion of a *σ-exceptional collection* so that any full σ-exceptional collection (if such exists) generates σ, where σ denotes a stability condition. The focus here lies on constructing σ-exceptional collections from a given stability condition σ on $D^b(\mathcal{A})$, where $\mathcal{A}$ is a hereditary, hom-finite category, linear over an algebraically closed field. One difficulty is due to the *Ext-nontrivial couples*: exceptional objects $X, Y \in \mathcal{A}$ with non-vanishing $Ext^1(X,Y)$ and $Ext^1(Y,X)$. A new constraint on the category $\mathcal{A}$, called *regularity-preserving*, makes this difficulty manageable. Examples of regularity-preserving categories are demonstrated. Finally, all stability conditions on the acyclic triangular quiver are shown to be generated by exceptional collections.

The central result in the second part of the thesis is a characterization of the Dynkin/Euclidean/all other quivers on the language of Bridgeland stability conditions.

The third part continues with the study of the entire space of stability conditions on the acyclic triangular quiver. The main conclusion here is that this space is contractible. This is the first example of a quiver Q different from Dynkin and Kronecker quivers for which the stability space on the derived category of representations of Q is shown to be contractible. It follows that the stability space on the weighted projective line $\mathbb{P}^1(1,2)$ is contractible.

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I am, first and foremost, indebted to Ludmil Katzarkov. His genuine and generous attitude as an advisor is indispensable to bringing this work into existence. Ludmil's expertise in all the three: classical problems in Geometry, HMS, "Brave New" Mathematics, and his remarkable insight into the interrelations between them are an invaluable source of inspiration, knowledge, and motivation to me. Ludmil is also a great friend of mine and companion for skiing. Thank you Ludmil.

This thesis evolved from questions raised by Ludmil Katzarkov and Maxim Kontsevich. On this account I am also deeply grateful to Maxim Kontsevich, as well as for his valuable comments, and no less for numerous inspiring talks given by him at various conferences.

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The endeavor to produce mathematical research is demanding and it can be daunting. Having had the fortune and the privilege to meet the people mentioned above is a reward enough for me.

The main motivation for this work came from parts of [37]. I am indebted to E. Macrì for this.

I am also grateful to J. Woolf for specifying clearly in the introduction of his paper [49] that the tame representation type case is beyond the scope of the paper. I thank Fabian Haiden for pointing out the reference [49] the day after it appeared in arxiv.

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Contents

0.1 Some general notations and conventions used throughout the dissertation

In these notes k is an algebraically closed field.[1] The letters $\mathcal{T}$ and $\mathcal{A}$ denote always a triangulated category and an abelian category, respectively, linear over k, the shift functor in $\mathcal{T}$ is designated by [1]. I write $\mathrm{Hom}^i(X,Y)$ for $\mathrm{Hom}(X,Y[i])$ and $\hom^i(X,Y)$ for $\dim_k(\mathrm{Hom}(X,Y[i]))$, where $X,Y\in\mathcal{T}$.

By $K_0(\mathcal{T})$, resp. $K_0(\mathcal{A})$, will be denoted the Grothendieck groups of $\mathcal{T}$, resp. $\mathcal{A}$.

For $X,Y\in\mathcal{A}$, writing $\mathrm{Hom}^i(X,Y)$, I consider X,Y as elements in $\mathcal{T}=D^b(\mathcal{A})$, i.e. $\mathrm{Hom}^i(X,Y)=\mathrm{Ext}^i(X,Y)$.

For a subset $S\subset Ob(\mathcal{T})$ the notation $\langle S\rangle\subset\mathcal{T}$ means the triangulated subcategory of $\mathcal{T}$ generated by S, i. e. the minimal triangulated subcategory containing S.

An *exceptional object* is an object $E\in\mathcal{T}$ satisfying $\mathrm{Hom}^i(E,E)=0$ for $i\neq 0$ and $\mathrm{Hom}(E,E)=k$. The set of all exceptional objects of $\mathcal{A}$, resp. of $D^b(\mathcal{A})$, will be denoted by $\mathcal{A}_{exc}$, resp. $D^b(\mathcal{A})_{exc}$.

The property that for two $X,Y\in\mathcal{T}$ hold the vanishings $\hom^l(X,Y)=0$ for any $l\in\mathbb{Z}$ will be denoted by writing just $\hom^*(X,Y)=0$.

An *exceptional collection* is a sequence $\mathcal{E}=(E_1,E_2,\dots,E_n)\subset\mathcal{T}_{exc}$ satisfying $\hom^*(E_i,E_j)=0$ for $n\geq i>j\geq 1$. If in addition we have $\langle\mathcal{E}\rangle=\mathcal{T}$, then $\mathcal{E}$ will be called a full exceptional collection.

An exceptional collection $(E_1,\dots,E_n)$ is said to be *Ext-exceptional* if $\forall i\neq j$ $\mathrm{Hom}^{\leq 0}(E_i,E_j)=0$.

An abelian category $\mathcal{A}$ is said to be *hereditary*, if $\mathrm{Ext}^i(X,Y)=0$ for any $X,Y\in\mathcal{A}$ and $i\geq 2$, it is said to be of finite length, if it is Artinian and Noterian.

For an object $X\in D^b(\mathcal{A})$ of the form $X\cong X'[j]$, where $X'\in\mathcal{A}$ and $j\in\mathbb{Z}$, I write $\deg(X)=j$.

For any quiver Q I write $\Gamma(Q)$ for the underlying graph and $D^b(Q)$ for $D^b(Rep_k(Q))$.

Throughout the dissertation the term *Dynkin quiver* means a quiver Q, s. t. $\Gamma(Q)$ is one of the simply laced Dynkin diagrams $\mathbb{A}_m, m\geq 1$, $\mathbb{D}_m$, $m\geq 4$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$ (see for example [1, p. 32]) and the term *Euclidean quiver* means an acyclic quiver Q, s. t. $\Gamma(Q)$ is one of the extended Dynkin diagrams $\widetilde{\mathbb{A}}_m, m\geq 1$, $\widetilde{\mathbb{D}}_m$, $m\geq 4$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$ (see for example [1, fig. (4.13)]).

The letter $\mathbb{H}$ will denote the upper half plane with the negative real axis included, i. e. $\mathbb{H}=\{r\exp(\mathrm{i}\pi t): r>0 \text{ and } 0<t\leq 1\}$.

[1]in some sections algebraically closedness of k is not important, but overall this feature is necessary.

Chapter 1

Introduction

In my PhD thesis, which is in the papers: [17], [18], [19], [20] are explored some aspects of the interplay between the two notions of the title. This interplay unveils novelties for both: *Bridgeland stability conditions* and *exceptional collections*.

I give first some words about the two notions in question and about the initial goal.

1.1 Motivation

In 1994 Maxim Kontsevich interpreted a duality coming from physics in a powerful mathematical framework called Homological Mirror Symmetry (HMS). HMS is now the foundation of a wide range of contemporary mathematical research. Many authors have demonstrated the interaction of mirror symmetry and HMS with new and subtle mathematical structures. One of these structures is the space of stability conditions associated to a triangulated category.

Motivated by M. Douglas's work [22], [23] in string theory, and especially by the notion of Π-stability, T. Bridgeland defined in [8] a map:

$$\left\{ \begin{array}{c} \text{triangulated} \\ \text{categories} \end{array} \right\} \xrightarrow{\text{Stab}} \left\{ \begin{array}{c} \text{complex} \\ \text{manifolds} \end{array} \right\}. \tag{1.1}$$

For a triangulated cateogry $\mathcal{T}$ the associated complex manifold $\mathrm{Stab}(\mathcal{T})$ is refered to as the space of stability conditions (or the stability space) on $\mathcal{T}$.

Bridgeland's manifolds are expected to provide a rigorous understanding of certain moduli spaces arising in string theory. Furthermore, a parallel between dynamical systems and categories was established (see Subsection 1.3). According to this parallel the stability space plays the role of the Teichmüller space. Thus, the study of the topology of the spaces of stability conditions became a subject of significant importance. These spaces provide a link between: topology, representation theory, dynamical systems, algebraic geometry, category theory.

The map (1.1) is well defined but far from well understood. The problem of describing the entire $\mathrm{Stab}(\mathcal{T})$ is notoriously difficult (from now on $\mathcal{T}$ denotes a triangulated category linear over an algebraically closed field k).

It is an exercise with Bridgeland's axioms to show that $\mathrm{Stab}(\langle E\rangle) = \mathbb{C}$, where E is an exceptional object (see Section 0.1 for definition) in $\mathcal{T}$.

Recall that an *exceptional collection* is a sequence $\mathcal{E} = (E_0, E_1, \ldots, E_n)$ of exceptional objects satisfying $\hom^*(E_i, E_j) = 0$ for $i > j$. Exceptional collections were introduced by Rudakov, Beilinson, Bondal, Kuznetsov et. al. From now on throughout the introduction $\mathcal{E} = (E_0, E_1, \ldots, E_n)$ denotes an exceptional collection. Some triangulated categories have the form $\mathcal{T} = \langle E_0, E_2, \ldots, E_n\rangle$ and then $(E_0, E_1, \ldots, E_n)$ is said to be a full exceptional collection in $\mathcal{T}$. The question about studying $\mathrm{Stab}(\langle E_0, E_1, \ldots, E_n\rangle)$ $(n \geq 1)$ seemed to me as a natural next step after having obtained $\mathrm{Stab}(\langle E\rangle) = \mathbb{C}$. This turns out to be not so easy. This study was initiated by E. Macrì in [37]. We proceed further in [18] and [19] (joint with my advisor Prof. Katzarkov).

Categories which have a decomposition of the form $\langle E_0, E_1, \ldots, E_n\rangle$ arise from representations of quivers. For each acyclic quiver Q the derived category of representations of Q : $D^b(Rep_k(Q))$ has such a decomposition with $n + 1$ equal to the number of the vertices of Q. I will write just $D^b(Q)$ for this category. In 2012 $\mathrm{Stab}(D^b(Q))$ had been studied for some Dynkin quivers and for $Q = K(l)$, where $K(l)$ is the l-Kronecker quiver (two vertices and l parallel arrows between them). Our goal was to give a satisfactory description of $D^b(Q)$ for a non-Dynkin quiver with number of vertices bigger than two and hopefully to prove that $\mathrm{Stab}(D^b(Q))$ is contractible.

In July 2014 appeared the papers [12], [49] which clarified all Dynkin quivers. However the results in [49], [12] do not cover tame representation type quivers, these quivers are beyond the scope of [49], [12]. "*The natural next case is to consider tame representation type quivers. ... The situation here is much more complicated:... - new ideas will be required to study the tame representation type case from the point of view of this paper*"[49].

In Chapter 4 of the dissertation, which is the paper [20], is given a new example of a tame representation type quiver with contractible space of stability conditions.

1.2 Non-semistable exceptional objects in hereditary categories

In this subsection of the introduction I explain some features and the main results of Chapter 2. This chapter consists of the paper [18]. The Appendix 2.B consists of the paper [19].

Assume that $\mathcal{T}$ has a decomposition of the form $\mathcal{T} = \langle E_0, E_1, \ldots, E_n\rangle$. E. Macri constructed in [37] stability conditions on $\mathcal{T}$ via exceptional collections. Thus we obtain a subset:

$$\left\{ \begin{array}{c} \text{stability conditions generated} \\ \text{by exceptional collections} \end{array} \right\} \subset \mathrm{Stab}(\mathcal{T}). \tag{1.2}$$

E. Macrì, studying $\mathrm{Stab}(D^b(K(l))$ in [37], gave an idea for producing an exceptional pair generating a given stability condition σ on $D^b(K(l))$, where $K(l)$ is the l-Kronecker quiver.

The main goal of Chapter 2 is to find non-trivial examples of $\mathcal{T}$ with $rank(K_0(\mathcal{T})) \geq 3$, for which the inclusion (1.2) is equality, where $K_0(\mathcal{T})$ is the Grothendieck group of $\mathcal{T}$. To that end is defined the notion of a *σ-exceptional collection* (Definition 2.33), so that the full σ-exceptional collections are exactly the exceptional collections which generate σ, and then the focus falls on constructing σ-exceptional collections from a given $\sigma \in \mathrm{Stab}(D^b(\mathcal{A}))$, where $\mathcal{A}$ is a hereditary, hom-finite, abelian category. In Chapter 2 are developed some tools for constructing σ-exceptional collections of length at least three in $D^b(\mathcal{A})$. These tools are based on the notion of regularity-preserving hereditary category, introduced in Section 2.6 to avoid difficulties related to the *Ext-nontrivial couples* (couples of exceptional objects in $\mathcal{A}$ with $\mathrm{Ext}^1(X,Y) \neq 0$ and $\mathrm{Ext}^1(Y,X) \neq 0$).

1.2.1 Regularity-preserving hereditary category

I explain now in more detail how appeared the notion of a regularity preserving category. By $\mathcal{A}$ I denote a k-linear hom-finite hereditary abelian category and let $\mathcal{T} = D^b(\mathcal{A})$, $\sigma \in \mathrm{Stab}(\mathcal{T})$.

By definition each stability condition, in particular the chosen $\sigma \in \mathrm{Stab}(\mathcal{T})$, determines a set of non-zero objects in $\mathcal{T}$ (called *semi-stable objects*) labeled by real numbers (called *phases of the semistable objects*). The semi-stable objects correspond to the so called "BPS" branes from string theory. The set of semi-stable objects will be denoted by σ^{ss}.

In Section 2.5 the non-semistable exceptional objects are divided into two types: σ-regular and σ-irregular and a procedure is explained, which produces (at least one) exceptional pair (S,E) with semistable S from any σ-regular object R . This procedure is denoted by $R \dashrightarrow (S,E)$. It is the basic step towards constructing σ-exceptional collections and it can not be performed on σ-irregular objects. The irregular objects appear due to the Ext-nontrivial couples.

The σ-regular objects in turn are divided into final and non-final as follows. In each relation $R \dashrightarrow (S,E)$ the first component S is a semistable exceptional object, and the second is not restricted to be always semistable. If there is such a relation with a non-semistable E, then I refer to R as a *nonfinal* σ-regular object, otherwise - *final* (see Definition 2.46). The name non-final is justified, when the category $\mathcal{A}$ has a specific property, defined as follows:

Definition 1.1. (*Definition 2.47*) *A hereditary abelian category $\mathcal{A}$ will be said to be regularity-preserving, if for each $\sigma \in \mathrm{Stab}(D^b(\mathcal{A}))$ from the the following data:*
$R \in D^b(\mathcal{A})$ is a σ-regular object; $R \dashrightarrow (S,E)$; $E \notin \sigma^{ss}$
it follows that E is a σ-regular object as well.

In a regularity-preserving category $\mathcal{A}$ the relation $\dashrightarrow$ circumvents the σ-irregular objects, and each non-final σ-regular object R generates a long sequence of the form:

$$\begin{array}{ccccccccccc} R & \overset{X_1}{\dashrightarrow} & (S_1,E_1) & \overset{proj_2}{\longmapsto} & E_1 & \overset{X_2}{\dashrightarrow} & (S_2,E_2) & \overset{proj_2}{\longmapsto} & E_2 & \overset{X_3}{\dashrightarrow} & (S_3,E_3) & \overset{proj_2}{\longmapsto} & \dots \\ & & {\scriptstyle proj_1}\downarrow & & & & {\scriptstyle proj_1}\downarrow & & & & {\scriptstyle proj_1}\downarrow & & . \\ & & S_1 & & & & S_2 & & & & S_3 & & \end{array} \qquad (1.3)$$

The sequences of the form (1.3) generated by σ-regular objects are the main tool used in Sections 2.7, 2.8, 2.9 for constructing σ-exceptional collections.

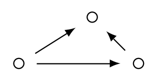
To check the property in Definition 1.1 is not an easy task. If $\mathcal{A}$ has no Ext-nontrivial couples, then σ-irregular objects do not appear for any σ and $\mathcal{A}$ is regularity preserving (Lemma 2.49).

It follows from [37, Lemma 4.1] that there are no Ext-nontrivial couples in $Rep_k(K(l))$ and hence $Rep_k(K(l))$ is an example of regularity preserving category (see Appendix 2.C.1).

It is shown in Appendix 2.B that for any Dynkin quiver Q the category $Rep_k(Q)$ has no Ext-nontrivial couples, hence $Rep_k(Q)$ is regularity preserving as well. Appendix 2.B consists of the paper [19].

1.2.2 In search for the equality:

$$\left\{ \begin{array}{c} \textbf{stab. cond. generated} \\ \textbf{by exceptional collections} \end{array} \right\} = \mathrm{Stab}(\mathcal{T}) \qquad \boldsymbol{with} \qquad rank(K_0(\mathcal{T})) \geq 3.$$

The only affine acyclic quiver with three vertices is $Q_1 =$  and it is natural to try to apply the methods from Sections 2.7, 2.8, 2.9 to it. However $Rep_k(Q_1)$ has one Ext-nontrivial couple.

It is shown in Subsection 2.6.3 that in the case when $\mathcal{A}$ has Ext-nontrivial couples certain conditions on these couples, called *RP property 1 and RP property 2* (see Definition 2.51) imply regularity-preserving. After a detailed study of the exceptional objects of the quiver Q_1 in Section 2.2 it turns out that these relations do hold in $Rep_k(Q_1)$. Analogous procedure is carried out also successfully for the quiver Q_2 shown on figure (2.2) (here are two Ext-nontrivial couples).

Having shown that $Rep_k(Q_1)$ is regularity preserving, the newly obtained methods for constructing σ-triples are applied to the case $\mathcal{A} = Rep_k(Q_1)$ in Section 2.10 leading to the following result:

Theorem 1.2. *For each $\sigma \in \mathrm{Stab}(D^b(Q_1))$ there exists a full σ-exceptional collection.*

Thus, all stability conditions on $D^b(Q_1)$ are generated by exceptional collections and (1.2) is equality for $\mathcal{T} = D^b(Q_1)$. This implies that $\mathrm{Stab}(D^b(Q_1))$ is connected (Corollary 2.82).

1.3 Density of phases

In a joint work [17] with Haiden, Katzarkov, Kontsevich, following results and ideas in [10], [25], [33], we studied questions motivated by the classical theory of dynamical systems in the context of triangulated categories. [17, Section 3] is based on the interplay between exceptional collections and Bridgeland stability conditions. Chapter 3 of the dissertation is a slightly improved version of [17, Section 3]. Here I explain features of Chapter 3.

For any triangulated category $\mathcal{T}$ and $\sigma \in \mathrm{Stab}(\mathcal{T})$ by P_σ will be denoted the subset of the unit circle $\mathbb{S}^1$ obtained by projecting the phases of semi-stable objects to the unit circle $\mathbb{S}^1$ via $\exp(\mathrm{i}\pi\_)$ (see Definition 3.1).

In [17] we searched for a categorical analogue of the density of the set of slopes of closed geodesics on a Riemann surface. This is done in [17, section 3], where the focus falls on constructing stability conditions for which the set P_σ is dense in a non-trivial arc of the circle. As a result was obtained the following characterization of the map (1.1), when restricted to categories of the form $D^b(Q)$:

Dynkin quivers	P_σ is always finite
Euclidean quivers	P_σ is either finite or has exactly two limit points
All other acyclic quivers	P_σ is dense in an arc for a family of stability conditions

(1.4)

where by Euclidean quiver I mean an acyclic quiver, whose underlying graph is an extended Dynkin diagram. When $k = \mathbb{C}$, the table (1.4) holds after removing "acyclic" in the third row.

Table (1.4) is the main result of Chapter 3.

The first and the second row in table (1.4) are Lemma 3.10 and Corollary 3.12, they follow quickly by the axioms of Bridgeland stability conditions.

In Section 3.4 is shown how to construct stability conditions σ for which P_σ satisfies the density property in the third row or the two limit points property on the second raw of the table with the help of Kronecker pairs. An exceptional pair (E, F) in $\mathcal{T}$ is an *l-Kronecker pair* if $\hom^{\leq 0}(E, F) = 0$, and $\hom^1(E, F) = l$ (Definition 3.20). Theorem 3.24 and its corollaries use l-Kronecker pairs with $l \geq 3$ to obtain density and with $l = 2$ to obtain two limit points. Results and ideas in [34], [37], and in Chapter 2 are useful for the proof of Theorem 3.24 and its corollaries in Section 3.4.

Another result which proves the third row in table (1.4) is:

In all acyclic quivers different from Dynkin and Euclidean there exist l-Kronecker pairs with $l \geq 3$ (Proposition 3.31).

On the other hand the first and the second rows of table (1.4) together with Theorem 3.24 imply the following:

For any Dynkin or Euclidean quiver Q, any exceptional pair (E_1, E_2) in $D^b(Q)$ satisfies $\dim_k(\mathrm{Hom}^i(E_1, E_2)) < 3$ *for all i, in particular only 1- and 2-Kronecker pairs can appear in* $D^b(Q)$ *(Corollary 3.28).*

Lemma 3.38 says that for each Euclidean quiver $\widetilde{Q}$ there exists a 2-Kronecker pair in $D^b(Rep_k(\widetilde{Q}))$. This fact with the help of Theorem 3.24 imply that:

Any Euclidean quiver $\widetilde{Q}$ has a family of stability conditions σ on $D^b(Rep_k(\widetilde{Q}))$, s. t. P_σ has exactly two limit points of the type $\{p, -p\}$ (Proposition 3.29).

Further examples of density of phases (see Section 3.6) are on $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^n$, $n \geq 2$ and their blow ups and on any smooth projective variety X, such that $D^b(Coh(X))$ is generated by a strong exceptional collection of length three.

The last Section 3.7 contains several questions related to the content of Chapter 3.

1.4 Bridgeland stability conditions on the acyclic triangular quiver

The last Chapter 4 of the dissertation consists of the paper [20]. In [20] we use Theorem 1.2 to prove: *the space* $\mathrm{Stab}(D^b(Q_1))$ *is a contractible (and in particular connected) manifold (Theorem 4.1).* This is the first example of a quiver Q different from Dynkin and Kronecker quivers for which $\mathrm{Stab}(D^b(Rep_k(Q)))$ is shown to be contractible. I give here some details about the structure of $\mathrm{Stab}(D^b(Rep_k(Q_1)))$. Let us fix $\mathcal{T} = D^b(Rep_k(Q_1))$.

The braid group on two strings $B_2 \cong \mathbb{Z}$ acts on the set of equivalence classes of 2-Kronecker pairs. In Subsection 4.4.1 are described the orbits of this action on the 2-Kronecker pairs. There are two such orbits and in terms of our notations they are $\{(a^m, a^{m+1}[-1])\}_{m\in\mathbb{Z}}$ and $\{(b^m, b^{m+1}[-1])\}_{m\in\mathbb{Z}}$.

It turns out that the exceptional objects of $D^b(Q_1)$ can be grouped as follows $\{a^m\}_{m\in\mathbb{Z}} \cup \{M, M'\} \cup \{b^m\}_{m\in\mathbb{Z}}$, where $\{M, M'\} \subset Rep_k(Q_1)$ is the unique Ext-nontrivial couple of $Rep_k(Q_1)$.

Let $\mathfrak{T}_a^{st}$ and $\mathfrak{T}_b^{st}$ be the stability conditions generated by the exceptional triples containing a subsequence of the from $(a^m[p], a^{m+1}[q])$ and $(b^m[p], b^{m+1}[q])$ for some $m, p, q \in \mathbb{Z}$, respectively. Theorem 1.2 amounts to the equality (see Section 4.5) $\mathrm{Stab}(D^b(Q_1)) = \mathfrak{T}_a^{st} \cup (\_, M, \_) \cup (\_, M', \_) \cup \mathfrak{T}_b^{st}$, where $(\_, M, \_) \cup (\_, M', \_)$ denotes the set of stability conditions generated by triples of the form $(A, M[p], C)$ or $(A, M'[p], C)$ with $p \in \mathbb{Z}$ (these turn out to be the triples (A, B, C) for which $\hom^i(A, B) \le 1$, $\hom^i(A, C) \le 1$, $\hom^i(B, C) \le 1$ for all $i \in \mathbb{Z}$).

The main steps in the proof of Theorem 4.1 are as follows. Section 4.6 contains the proof that $\mathfrak{T}_a^{st} \cap \mathfrak{T}_b^{st} = \emptyset$. It is shown in Section 4.7 that $\mathfrak{T}_a^{st}$ and $\mathfrak{T}_b^{st}$ are contractible. In Section 4.8 the subsets $\mathfrak{T}_a^{st}$ and $\mathfrak{T}_b^{st}$ are connected by $(\_, M, \_) \cup (\_, M', \_)$ and it is shown that in this procedure the contractibility is preserved.

1.5 Weighted projective lines

The quivers Q_1, Q_2 (shown in figure (2.2)) are special cases of the quivers depicted in the beginning of [27, Section 4] (see also [21, Section 4]). Geigle and Lenzing have constructed in [27] equivalences between triangulated categories:

$$D^b(\mathrm{coh}(\mathbb{P}^1(1,2))) \cong D^b(Rep_k(Q_1)) \qquad D^b(\mathrm{coh}(\mathbb{P}^1(2,2))) \cong D^b(Rep_k(Q_2)), \tag{1.5}$$

where $\mathbb{P}^1(1,2)$ and $\mathbb{P}^1(2,2)$ are the weighted projective lines of weight $(1,2)$ and $(2,2)$, respectivley, and $\mathrm{coh}(\mathbb{P}^1(1,2))$, $\mathrm{coh}(\mathbb{P}^1(2,2))$ are the categories of coherent sheaves as defined in [27, Section 1]. The equivalences (1.5) can be found in [27, Subsection 5.4.1].

Thus, the results for $D^b(Rep_k(Q_1))$ and $D^b(Rep_k(Q_2))$ obtained in this thesis are also results for $D^b(\mathrm{coh}(\mathbb{P}^1(1,2)))$ and $D^b(\mathrm{coh}(\mathbb{P}^1(2,2)))$. In particular, Theorem 4.1 implies that

Corollary 1.3. *The space* $\mathrm{Stab}(D^b(\mathrm{coh}(\mathbb{P}^1(1,2))))$ *is a contractible manifold.*

Chapter 2

Non-semistable exceptional objects in hereditary categories

2.1 Introduction

Bridgeland's axioms imply[1] that $\mathrm{Stab}(\langle E \rangle) = \mathbb{C}$ for an exceptional object E in $\mathcal{T}$. The guiding motivation of this chapter is the study of $\mathrm{Stab}(\langle E_1, E_2, \dots, E_n \rangle)$, where $(E_1, \dots, E_n)$ is an exceptional collection in $\mathcal{T}$ and $n \geq 2$. This study was initiated by E. Macrì in [37]. Here, we proceed further.

J. Collins and A. Polishchuk defined and studied in [14] a gluing procedure for Bridgeland stability conditions in the situation when $\mathcal{T}$ has a semiorthogonal decomposition $\mathcal{T} = \langle \mathcal{A}_1, \mathcal{A}_2 \rangle$.

1.1. T. Bridgeland constructed a stability condition $\sigma \in \mathrm{Stab}(\mathcal{T})$ from a bounded t-structure $\mathcal{A} \subset \mathcal{T}$ and a stability function[2] $Z : K_0(\mathcal{A}) \to \mathbb{C}$ satisfying certain restrictions. Keeping $\mathcal{A}$ fixed and varying Z produces a family of stability conditions, which we denote by $\mathbb{H}^{\mathcal{A}} \subset \mathrm{Stab}(\mathcal{T})$. E. Macrì proved in [37, Lemma 3.14], using results of [2], that the extension closure $\mathcal{A}_{\mathcal{E}}$ of a full Ext-exceptional collection[3] $\mathcal{E} = (E_0, E_1, \dots, E_n)$ in $\mathcal{T}$ is a heart of a bounded t-structure, and for each $\sigma \in \mathbb{H}^{\mathcal{A}_{\mathcal{E}}}$ the objects $E_0, E_1, \dots, E_n$ are σ-stable with phases in $(0, 1]$.[4] Motivated by this result, for a given $\sigma \in \mathrm{Stab}(\mathcal{T})$ we define a *σ-exceptional collection*(Definition 2.33) as an Ext-exceptional collection $\mathcal{E} = (E_0, E_1, \dots, E_n)$, s. t. the objects $\{E_i\}_{i=0}^n$ are σ-semistable, and $\{\phi(E_i)\}_{i=0}^n \subset (t, t+1)$ for some $t \in \mathbb{R}$. It follows easily from [37, Lemmas 3.14, 3.16] that for any full Ext-exceptional collection $\mathcal{E}$ the set $\{\sigma \in \mathrm{Stab}(\mathcal{T})$: $\mathcal{E}$ is σ-exceptional$\}$ coincides with[5] $\mathbb{H}^{\mathcal{A}_{\mathcal{E}}} \cdot \widetilde{GL}^+(2, \mathbb{R})$(Corollary 2.34).

E. Macrì, studying $\mathrm{Stab}(D^b(K(l))$ in [37], gave an idea for producing a σ-exceptional pair in

[1]Recall that for a subset $S \subset Ob(\mathcal{T})$ we denote by $\langle S \rangle \subset \mathcal{T}$ the triangulated subcategory of $\mathcal{T}$ generated by S.

[2]I.e. Z is homomorphism $K_0(\mathcal{A}) \xrightarrow{Z} \mathbb{C}$, s. t. $Z(X) \in \mathbb{H} = \{r \exp(\mathrm{i}\pi t) : r > 0 \ \text{ and } \ 0 < t \leq 1\}$ for $X \in \mathcal{A} \setminus \{0\}$.

[3]Recall that an exceptional collection $\mathcal{E} = (E_0, \dots, E_n)$ is said to be Ext-exceptional if $\forall i \neq j$ $\mathrm{Hom}^{\leq 0}(E_i, E_j) = 0$.

[4]Furthermore, $\mathcal{A}_{\mathcal{E}}$ is artinian and noetherian, and its simple objects are $\mathcal{E}$.

[5]Recall that $\mathrm{Stab}(\mathcal{T})$ carries a right action by $\widetilde{GL}^+(2, \mathbb{R})$.

$D^b(K(l))$ from a non-semistable exceptional object, where $K(l)$ is the l-Kronecker quiver.

Throughout sections 2.4, 2.5, 2.6, 2.7, 2.8, 2.9 are developed tools for constructing σ-exceptional collections of length at least three in $D^b(\mathcal{A})$, where $\mathcal{A}$ is a hereditary hom-finite abelian category. Combining them with the findings of Section 2.2 about $Rep_k(Q_1)$ we prove in Section 2.10 the following theorem:

Theorem 2.1. *Let* Q_1 *be the quiver* $\begin{array}{ccc} & \circ & \\ \nearrow & & \nwarrow \\ \circ & \longrightarrow & \circ \end{array}$. *Let* k *be an algebraically closed field. For each* $\sigma \in \mathrm{Stab}(D^b(Rep_k(Q_1)))$ *there exists a full* σ*-exceptional collection.*

Theorem 2.1 is one novelty of this chapter.[6] In particular, it implies that $\mathrm{Stab}(D^b(Rep_k(Q_1)))$ is connected (Corollary 2.82).

The $K(l)$-analogue of Theorem 2.1(Lemma 2.147) is already treated by E. Macrì in [37, Lemma 4.2 on p.10]. For the sake of completeness, we add a proof of this analogue in Appendix 2.C.2.

The proof of Theorem 2.1 is more complicated than of its $K(l)$-analogue not only because the full collections are triples instead of pairs, but also due to the presence of Ext-nontrivial couples[7] in $Rep_k(Q_1)$. We circumvent this difficulty by observing remarkable patterns, which the Ext-nontrivial couples obey. These patterns and the notion of *regularity-preserving hereditary category*, which they imply, are other novelties of the chapter.

1.2. We explain now the organization of the chapter and give details about the intermediate results.

Here, by $\mathcal{A}$ we denote a k-linear hom-finite hereditary abelian category, where k is an algebraically closed field, and we denote $D^b(\mathcal{A})$ by $\mathcal{T}$.

In Section 2.4 we analyze the following data: an exceptional object $E \in D^b(\mathcal{A})$, which is not σ-semistable for a given stability condition $\sigma \in \mathrm{Stab}(\mathcal{T})$. Macrì initiated such an analysis in [37, p.10].
We end up in Section 2.4 with a distinguished triangle, denoted by $\mathfrak{alg}(E)$, which satisfies one of five possible lists of properties, named **C1,C2,C3,B1,B2**. If the resulting list is one of **C1,C2** or **C3**, then we say that the object E is σ*-regular*, otherwise - σ*-irregular*. The triangle $\mathfrak{alg}(R) = \begin{array}{ccc} U & \longrightarrow & R \\ & \nwarrow \quad \swarrow & \\ & V & \end{array}$ of a σ-regular R has the feature that for any indecomposable components S and E of V and U, respectively, the pair (S,E) is exceptional with semistable first element S. We denote this relation between a σ-regular object R and the exceptional pair (S,E) by $R \overset{X}{\dashrightarrow} (S,E)$, where X contains further information as explained in Section 2.5. This feature is not available in the irregular cases **B1** and **B2**, and the obstruction to obtaining it are the Ext-nontrivial couples. Such couples exist in $Rep_k(Q_1)$ and $Rep_k(Q_2)$, as shown in Section 2.2. Essential part of our efforts concerns the Ext-nontrivial couples. It follows from [37, Lemma 4.1] that there are no such couples in $Rep_k(K(l))$ (Appendix 2.C.1).

[6]In other words, this theorem says that all the stability conditions on $D^b(Rep_k(Q_1))$ are generated by exceptional collections.

[7]These are couples of exceptional objects X, Y with $\mathrm{Ext}^1(X,Y) \neq 0$, $\mathrm{Ext}^1(Y,X) \neq 0$(Definition 2.48).

Thus, in Sections 2.4, 2.5 from each σ-regular exceptional object R we obtain at least one exceptional pair (S,E) with $R \overset{X}{\dashrightarrow} (S,E)$. The first component S in such a pair is always semistable. If the second component E is not semistable, which is possible iff R is non-final as defined in Definition 2.46, then it is natural to ask: Is E a σ-regular exceptional object?

Motivated by this question, we introduce in Section 2.6 certain conditions on the Ext-nontrivial couples of $\mathcal{A}$, which we call *RP property 1 and RP property 2* (Subsection 2.6.2), and using them we give a positive answer. We say that $\mathcal{A}$ is a *regularity-preserving category*(Definition 2.47), when the answer is positive. RP properties 1, 2 themselves are not important for the rest of this chapter, but that $\mathcal{A}$ is regularity-preserving, which follows from them.

Whence, in regularity-preserving category $\mathcal{A}$ the relation $\dashrightarrow$ circumvents the irregular objects, and each non-final σ-regular object R generates a long sequence[8] of the form:

$$\begin{array}{ccccccccccc} R & \overset{X_1}{\dashrightarrow} & (S_1,E_1) & \overset{proj_2}{\longmapsto} & E_1 & \overset{X_2}{\dashrightarrow} & (S_2,E_2) & \overset{proj_2}{\longmapsto} & E_2 & \overset{X_3}{\dashrightarrow} & (S_3,E_3) \overset{proj_2}{\longmapsto} \dots \\ & & \downarrow{\scriptstyle proj_1} & & & & \downarrow{\scriptstyle proj_1} & & & & \downarrow{\scriptstyle proj_1} \\ & & S_1 & & & & S_2 & & & & S_3 \end{array} \quad . \tag{2.1}$$

In such a sequence, which we call an *R-sequence*, the exceptional objects $S_1, S_2, \dots$ are all semistable, and furthermore, if E_n is final for some n, then, by the very definition of a final object(Definition 2.46), the pair (S_{n+1}, E_{n+1}) is semistable and exceptional.

In Section 2.7 we proceed further in direction σ-exceptional collections by refining on the phases and the degrees of $\{S_i\}$, and showing various situations, in which the vanishings $\mathrm{Hom}^*(S_i,S_1) = \mathrm{Hom}^*(E_i,S_1) = 0$ hold for $i>1$. However, these vanishings do not hold in each R-sequence. Nevertheless, we show that starting from any σ-regular R through any R-sequence we reach a final σ-regular object E_n for some $n \geq 1$.

After a careful examination of the final σ-regular objects, in Section 2.8, we find that an exceptional pair (S,E) produced from such an object is not only semistable, but also $(S,E[-i])$ is a σ-exceptional pair for some $i \geq 0$ (e.g., a situation as: $\phi(S)=\phi(E)$, $\mathrm{Hom}(S,E)\neq 0$ cannot happen).

The proofs in Sections 2.7 and 2.8 are facilitated by the use of a function $\theta_\sigma : Ob(\mathcal{T}) \to \mathbb{N}^{(\sigma^{ss}_{ind}/\cong)}$, introduced in subsection 2.3.2. For an object $X \in Ob(\mathcal{T})$ the function $\theta_\sigma(X) : \sigma^{ss}_{ind}/\cong \to \mathbb{N}$ indicates (with multiplicities) the indecomposable components of the Harder-Narasimhan factors of X. The relation $R \overset{X}{\dashrightarrow} (S,E)$ implies $\theta_\sigma(E) < \theta_\sigma(R)$ and $\theta_\sigma(R)(S) > 0$. This feature gives an upper bound of the lengths of all R-sequences with a fixed R. It also plays a role in avoiding some situations as the mentioned in the end of the previous paragraph.

In Section 2.2 we obtain tables with dimensions of $\mathrm{Hom}(X,Y)$, $\mathrm{Ext}^1(X,Y)$ for any two exceptional objects X, Y of the categories $Rep_k(Q_1)$, $Rep_k(Q_2)$, and observe that one of these always vanishes. RP property 1 and RP property 2 follow by a careful analysis of these tables. For the

[8] By "long" we mean that it has at least two steps. This sequence is not uniquely determined by R.

Ext-nontrivial couples of the quiver Q_1 we observe an additional pattern: Corollary 2.8, which helps us further to avoid the irregular cases. We refer to it as the additional RP property. It does not hold in Q_2. In the end of Subsection 2.2.2 we obtain the lists of all exceptional pairs and triples in $Rep_k(Q_1)$. In Section 2.B is shown that for each Dynkin quiver Q there are no Ext-nontrivial couples in $Rep_k(Q)$, hence $Rep_k(Q)$ is regularity preserving.

The results before Section 2.9 contain the implications (the first is due to regularity-preserving): σ-regular object $\Rightarrow$ final σ-regular object $\Rightarrow$ σ-exceptional pair (Corollary 2.62 and Remark 2.63).

In Section 2.9 we develop various criteria for existence of σ-exceptional triples in $D^b(\mathcal{A})$, assuming that the exceptional objects of $\mathcal{A}$ obey the global properties observed for $Rep_k(Q_1)$ in Section 2.2.[9] It is shown that any non-final **C2** or **C3** object induces such a triple. Thus, if R is a **C2** or **C3** object, then any R-sequence of length two produces a σ-exceptional triple. If R is a **C1** object, then our results imply that any R-sequence of length three is enough, but for length less or equal to two - only under special circumstances (Lemmas 2.72, 2.77, Corollary 2.75).
If R is a final σ-regular object, then we have no long R-sequences, they are all of length one and each of them induces a σ-exceptional pair. To obtain a σ-triple in this case we apply two ideas. The first is to combine the pairs coming from different R-sequences, which leads to the result that a final σ-regular object R whose Harder-Narasimhan filtration differs from $\mathfrak{alg}(R)$ induces a σ-exceptional triple. The other idea is to utilize the infimum ϕ_{min} and the supremum ϕ_{max} of the set of phases of semistable exceptional objects in $\mathcal{A}$. More precisely, we show that a relation $R \dashrightarrow (S[1], E)$ with a final **C3** object $R \in \mathcal{A}$ and $\phi(S) > \phi_{min}$ induces a σ-triple(Corollary 2.71). There is an analogous criterion using a final **C2** object $R \in \mathcal{A}$ and ϕ_{min}, shown in Corollary 2.74, but there is not an analogue for final **C1** objects (Lemma 2.77 uses a non-final **C1** object and in different setting). When $\phi_{max} - \phi_{min} > 1$, we show that, if (S_{min}, E, S_{max}) is an exceptional triple in $\mathcal{A}$ with $S_{min} \in \mathcal{P}(\phi_{min})$ and $S_{max} \in \mathcal{P}(\phi_{max})$, then non-semistability of E (no matter regular or irregular) implies a σ-exceptional triple. The last is widely used in Subsection 2.10.3.

The criteria obtained in Section 2.9 combined with the lists of the exceptional pairs and the exceptional triples of $Rep_k(Q_1)$ at our disposal (due to Section 2.2) turn out to be enough for the proof of the main Theorem 2.1, which is demonstrated in Section 2.10. The locally finiteness of the stability condition $\sigma \in \mathrm{Stab}(\mathcal{T})$ plays an important role as well. The proof is divided into two steps: $\phi_{max} - \phi_{min} > 1$ and $\phi_{max} - \phi_{min} \le 1$.

1.3. In the next Chapter 3 is shown that any connected quiver Q, which is neither affine nor Dynkin, has a family of stability conditions with phases which are dense in an arc (Proposition 3.29). The proof of this fact relies on extendability, as defined in Definition 3.22, of certain stability conditions on a subcategory of $D^b(Q)$ to the entire $D^b(Q)$ (the precise setting is described right after Theorem 3.24). In Subsection 2.3.3 we comment on the stability conditions constructed by E. Macrì [37] via exceptional collections. By slightly modifying the statement of [37, Proposition 3.17] and refining its proof is obtained Proposition 2.31, which provides the extendability needed in next chapter for

[9]The precise assumptions are specified after Lemma 2.65.

the proof of Proposition 3.29.

1.4. It is known [15] that the Braid group acts transitively on the full exceptional collections of $Rep_k(Q_1)$. This action is not free (Remark 2.13).

1.5. We expect that there is a proof of Theorem 2.1, governed by a general principle related to the notion of a regularity-preserving hereditary category (Definition 2.47). RP property 1 and RP property 2 are our method to prove regualrity-preserving. The fact that they hold not only in $Rep_k(Q_1)$, but also in $Rep_k(Q_2)$ (Corollary 2.7) seems to be a trace of a larger unexplored picture. In Appendix 2.B we show that $Rep_k(Q)$ is regularity preserving for any Dynkin quiver Q. We expect that there are further non-trivial examples of regularity-preserving categories.

We do not give an answer to the question: is there a σ-exceptional quadruple for each $\sigma \in \mathrm{Stab}(D^b(Q_2))$ (the Q_2-analogue of Theorem 2.1). $Rep_k(Q_2)$ is regularity-preserving, and the results of Sections 2.7, 2.8, and Subsection 2.9.1 hold for $Rep_k(Q_2)$ entirely. These are clues for a positive answer(see especially Corollary 2.64). In section 2.2 we give the dimensions of $\mathrm{Hom}(X,Y)$, $\mathrm{Ext}^1(X,Y)$ for any two exceptional objects X,Y in Q_2 as well. This lays a ground for working on the Q_2-analogue of Theorem 2.1.

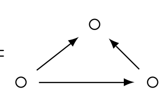
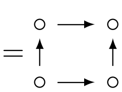
2.2 On the Ext-nontrivial couples of some hereditary categories

In Sections 2.6, 2.7, 2.8, 2.9 we treat hereditary abelian categories whose exceptional objects are supposed to obey specific pairwise relations. In this section we give examples of such categories.

2.2.1 The categories

For any finite quiver Q and an algebraically closed field k we denote the category of k-representations of Q by $Rep_k(Q)$. It is well known that $Rep_k(Q)$ is a hom-finite hereditary k-linear abelian category (see e. g. [16]).

In this section we compute the dimensions of $\mathrm{Hom}(X,Y)$, $\mathrm{Ext}^1(X,Y)$ for any two exceptional objects X,Y in the following quivers:

$Q_1 =$  $Q_2 =$  . (2.2)

The obtained information reveals some patterns, which are of importance for the rest of this chapter.

More precisely, Corollary 2.7 **(a)** claims that $Rep_k(Q_1)$ and $Rep_k(Q_2)$ have RP property 1 and RP property 2(see subsection 2.6.2 for definition). These properties ensure that $Rep_k(Q_1)$ and $Rep_k(Q_2)$ are regularity-preserving(Definition 2.47, Proposition 2.52), which is of primary importance for Sections 2.7, 2.9, 2.10.

In the end of Section 2.7 and in Section 2.9, the property that for any two exceptional objects X,Y at most one of the spaces $\mathrm{Hom}(X,Y)$, $\mathrm{Ext}^1(X,Y)$ is nonzero plays an important role. Corollary 2.7 **(b)** asserts that this property holds for both the quivers Q_1, Q_2.

For Q_1 we observe the additional RP property(see Corollary 2.8), used in Subsection 2.9.2. In the end we obtain the lists of exceptional pairs and exceptional triples in $Rep_k(Q_1)$, which are widely used in Section 2.10.

We give now more details.

2.2.2 The dimensions $\hom(X, Y), \hom^1(X, Y)$ for $X, Y \in Rep_k(Q_i)_{exc}$ and $i \in \{1, 2\}$

For a representation $\rho = \begin{array}{ccc} k^{\alpha_+} & \longrightarrow & k^{\alpha_e} \\ \uparrow & & \uparrow \\ k^{\alpha_b} & \longrightarrow & k^{\alpha_-} \end{array} \in Rep_k(Q_2)$, where $\alpha_b, \alpha_-, \alpha_+, \alpha_e \in \mathbb{N}$, we denote its dimension vector by $\underline{\dim}(\rho) = (\alpha_b, \alpha_-, \alpha_+, \alpha_e)$ and for $\begin{array}{ccccc} & & k^{\alpha_e} & & \\ & \nearrow & & \nwarrow & \\ k^{\alpha_b} & & \longrightarrow & & k^{\alpha_{mid}} \end{array} = \rho \in Rep_k(Q_1)$ we denote $\underline{\dim}(\rho) = (\alpha_b, \alpha_{mid}, \alpha_e)$. The Euler forms (see (3.4) for definition) of Q_1, Q_2 are:

$$\big\langle (\alpha_b, \alpha_{mid}, \alpha_e),\ (\alpha'_b, \alpha'_{mid}, \alpha'_e) \big\rangle = \alpha_b \alpha'_b + \alpha_{mid}\alpha'_{mid} + \alpha_e \alpha'_e - \alpha_b \alpha'_e - \alpha_b \alpha'_{mid} - \alpha_{mid}\alpha'_e,$$

$$\big\langle (\alpha_b, \alpha_-, \alpha_+, \alpha_e),\ (\alpha'_b, \alpha'_-, \alpha'_+, \alpha'_e) \big\rangle = \begin{array}{c} \alpha_+\alpha'_+ + \alpha_-\alpha'_- + \alpha_b\alpha'_b + \alpha_e\alpha'_e \\ -\alpha_b\alpha'_+ - \alpha_b\alpha'_- - \alpha_+\alpha'_e - \alpha_-\alpha'_e \end{array}.$$

Recall(see page 8 in [16]) that for any $\rho, \rho' \in Rep_k(Q)$ we have the formula

$$\hom(\rho, \rho') - \hom^1(\rho, \rho') = \big\langle \underline{\dim}(\rho), \underline{\dim}(\rho') \big\rangle . \tag{2.3}$$

In particular, it follows that if $\rho \in Rep_k(Q)$ is an exceptional object, then $\langle \underline{\dim}(\rho), \underline{\dim}(\rho) \rangle = 1$. The vectors satisfying this equality are called real roots(see [16, p. 17]). For example, one can show that the real roots of Q_1 are $(m+1, m, m)$,$(m, m+1, m+1)$, $(m, m, m+1)$, $(m+1, m+1, m)$, $(m+1, m, m+1)$, $(m, m+1, m)$, $m \geq 0$. The imaginary roots[10] of Q_1, are (m, m, m), $m \geq 1$. Not every real root is a dimension vector of an exceptional representation. More precisely:

Lemma 2.2. *Let $m \geq 1$. If $(\alpha_b, \alpha_{mid}, \alpha_e) \in \{(m+1, m, m+1), (m, m+1, m)\}_{m \in \mathbb{N}}$, then $(\alpha_b, \alpha_{mid}, \alpha_e)$ is not dimension vector of any exceptional representation in $Rep_k(Q_1)$. If $(\alpha_b, \alpha_-, \alpha_+, \alpha_e) \in \{(m, m+1, m, m),\ (m, m, m+1, m),\ (m+1, m, m+1, m+1),\ (m+1, m+1, m, m+1)\}_{m \in \mathbb{N}}$, then $(\alpha_b, \alpha_-, \alpha_+, \alpha_e)$ is not dimension vector of any exceptional representation in $Rep_k(Q_2)$.*

Sketch of proof. For the proof of this lemma one can use (see [16, Lemma 1 on page 13]) that a representation $\rho \in Rep_k(Q_i)$ is without self-extensions iff $\dim(\mathcal{O}_\rho) = \dim(Rep_k(Q_i))$, where $\mathcal{O}_\rho$ is the orbit of ρ in $Rep_k(Q_i)$ as defined in [16, page 11,12]. Using this argument, it can be shown that any representation without self-extensions with dimension vector among the listed in the lemma is decomposable. □

[10] Imaginary root is a vector ρ with $\langle \underline{dim}(\rho), \underline{dim}(\rho) \rangle \leq 0$.

Now we classify the exceptional objects on $Rep_k(Q_1)$, $Rep_k(Q_2)$(Propositions 2.3 and 2.4). In these propositions we use the following notations for any $m \geq 1$:

$$\pi_+^m : k^{m+1} \to k^m, \quad \pi_-^m : k^{m+1} \to k^m, \quad j_+^m : k^m \to k^{m+1}, \quad j_-^m : k^m \to k^{m+1}$$
$$\pi_+^m(a_1, a_2, \dots, a_m, a_{m+1}) = (a_1, a_2, \dots, a_m) \qquad \pi_-^m(a_1, a_2, \dots, a_m, a_{m+1}) = (a_2, \dots, a_m, a_{m+1})$$
$$j_+^m(a_1, a_2, \dots, a_m) = (a_1, a_2, \dots, a_m, 0) \qquad j_-^m(a_1, a_2, \dots, a_m) = (0, a_1, \dots, a_m).$$

Proposition 2.3. *The exceptional objects up to isomorphism in $Rep_k(Q_1)$ are ($m = 0, 1, 2, \dots$)*

$$E_1^m = \begin{array}{ccc} & k^m & \\ {\scriptstyle \pi_+^m}\nearrow & & \nwarrow{\scriptstyle Id} \\ k^{m+1} & \xrightarrow{\pi_-^m} & k^m \end{array} \qquad E_2^m = \begin{array}{ccc} & k^{m+1} & \\ {\scriptstyle j_+^m}\nearrow & & \nwarrow{\scriptstyle Id} \\ k^m & \xrightarrow{j_-^m} & k^{m+1} \end{array} \qquad E_3^m = \begin{array}{ccc} & k^{m+1} & \\ {\scriptstyle j_+^m}\nearrow & & \nwarrow{\scriptstyle j_-^m} \\ k^m & \xrightarrow{Id} & k^m \end{array}$$

$$E_4^m = \begin{array}{ccc} & k^m & \\ {\scriptstyle \pi_+^m}\nearrow & & \nwarrow{\scriptstyle \pi_-^m} \\ k^{m+1} & \xrightarrow{Id} & k^{m+1} \end{array} \qquad M = \begin{array}{ccc} & 0 & \\ \nearrow & & \nwarrow \\ 0 & \longrightarrow & k \end{array} \qquad M' = \begin{array}{ccc} & k & \\ {\scriptstyle Id}\nearrow & & \nwarrow \\ k & \longrightarrow & 0 \end{array}.$$

Sketch of proof. We showed that the dimension vectors of the exceptional representations are real roots. The list of real roots is given before Lemma 2.2 and some of them are excluded in Lemma 2.2. Moreover, there is at most one representation without self-extensions of a given dimension vector up to isomorphism [16, p. 13]. Taking into account these arguments, the proposition follows by showing that the endomorphism space of each of the listed representations is k (recall also (2.3)). The computations, which we skip, are reduced to table (2.109) in Appendix 2.A. □

Proposition 2.4. *The exceptional objects up to isomorphism in $Rep_k(Q_2)$ are($m = 0, 1, 2, \dots$)*

$$E_1^m = \begin{array}{ccc} k^m & \xrightarrow{Id} & k^m \\ {\scriptstyle \pi_+^m}\uparrow & & \uparrow{\scriptstyle Id} \\ k^{m+1} & \xrightarrow{\pi_-^m} & k^m \end{array} \quad E_2^m = \begin{array}{ccc} k^{m+1} & \xrightarrow{Id} & k^{m+1} \\ {\scriptstyle j_+^m}\uparrow & & \uparrow{\scriptstyle Id} \\ k^m & \xrightarrow{j_-^m} & k^{m+1} \end{array} \quad E_3^m = \begin{array}{ccc} k^m & \xrightarrow{j_+^m} & k^{m+1} \\ {\scriptstyle Id}\uparrow & & \uparrow{\scriptstyle j_-^m} \\ k^m & \xrightarrow{Id} & k^m \end{array} \quad E_4^m = \begin{array}{ccc} k^{m+1} & \xrightarrow{\pi_+^m} & k^m \\ {\scriptstyle Id}\uparrow & & \uparrow{\scriptstyle \pi_-^m} \\ k^{m+1} & \xrightarrow{Id} & k^{m+1} \end{array}$$

$$E_5^m = \begin{array}{ccc} k^m & \xrightarrow{j_+^m} & k^{m+1} \\ {\scriptstyle Id}\uparrow & & \uparrow{\scriptstyle Id} \\ k^m & \xrightarrow{j_-^m} & k^{m+1} \end{array} \quad E_6^m = \begin{array}{ccc} k^{m+1} & \xrightarrow{\pi_+^m} & k^m \\ {\scriptstyle Id}\uparrow & & \uparrow{\scriptstyle Id} \\ k^{m+1} & \xrightarrow{\pi_-^m} & k^m \end{array} \quad E_7^m = \begin{array}{ccc} k^m & \xrightarrow{Id} & k^m \\ {\scriptstyle \pi_+^m}\uparrow & & \uparrow{\scriptstyle \pi_-^m} \\ k^{m+1} & \xrightarrow{Id} & k^{m+1} \end{array} \quad E_8^m = \begin{array}{ccc} k^{m+1} & \xrightarrow{Id} & k^{m+1} \\ {\scriptstyle j_+^m}\uparrow & & \uparrow{\scriptstyle j_-^m} \\ k^m & \xrightarrow{Id} & k^m \end{array}$$

$$F_+ = \begin{array}{ccc} k & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array} \quad F_- = \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & k \end{array} \quad G_+ = \begin{array}{ccc} k & \xrightarrow{Id} & k \\ {\scriptstyle Id}\uparrow & & \uparrow \\ k & \longrightarrow & 0 \end{array} \quad G_- = \begin{array}{ccc} 0 & \longrightarrow & k \\ \uparrow & & \uparrow{\scriptstyle Id} \\ k & \xrightarrow{Id} & k \end{array}.$$

Sketch of proof. The same as Proposition 2.3. □

Now we compute $\hom(\rho, \rho')$, $\hom^1(\rho, \rho')$ with ρ, ρ' varying throughout the obtained lists.

Proposition 2.5. *The dimensions of the vector spaces* $\mathrm{Hom}(X, Y)$ *and* $\mathrm{Hom}^1(X, Y)$ *for any pair of exceptional objects $X, Y \in Rep_k(Q_1)$ are contained in the following table:*

		hom	$\hom^1$		hom	$\hom^1$
$0 \le m < n$	(E_1^m, E_1^n)	0	$n-m-1$	(E_1^n, E_1^m)	$1+n-m$	0
$0 \le n < m$	(E_2^m, E_2^n)	0	$m-n-1$	(E_2^n, E_2^m)	$1+m-n$	0
$0 \le n < m$	(E_3^m, E_3^n)	0	$m-n-1$	(E_3^n, E_3^m)	$1+m-n$	0
$0 \le m < n$	(E_4^m, E_4^n)	0	$n-m-1$	(E_4^n, E_4^m)	$1+n-m$	0
$m \ge 0, n \ge 0$	(E_1^m, E_2^n)	0	$n+m+2$	(E_2^n, E_1^m)	$n+m$	0
$m \ge 0, n \ge 0$	(E_1^m, E_3^n)	0	$n+m+1$	(E_3^n, E_1^m)	$n+m$	0
$0 \le m \le n$	(E_1^m, E_4^n)	0	$n-m$	(E_4^n, E_1^m)	$1+n-m$	0
$0 \le n < m$	(E_1^m, E_4^n)	$m-n$	0	(E_4^n, E_1^m)	0	$m-n-1$
$0 \le n \le m$	(E_2^m, E_3^n)	0	$m-n$	(E_3^n, E_2^m)	$1+m-n$	0
$0 \le m < n$	(E_2^m, E_3^n)	$n-m$	0	(E_3^n, E_2^m)	0	$n-m-1$
$m \ge 0, n \ge 0$	(E_2^m, E_4^n)	$1+n+m$	0	(E_4^n, E_2^m)	0	$n+m+2$
$m \ge 0, n \ge 0$	(E_3^m, E_4^n)	$n+m$	0	(E_4^n, E_3^m)	0	$n+m+2$
$m \ge 0$	(M, E_1^m)	0	0	(E_1^m, M)	0	1
$m \ge 0$	(M, E_2^m)	0	0	(E_2^m, M)	1	0
$m \ge 0$	(M, E_3^m)	0	1	(E_3^m, M)	0	0
$m \ge 0$	(M, E_4^m)	1	0	(E_4^m, M)	0	0
$m \ge 0$	(M', E_1^m)	1	0	(E_1^m, M')	0	0
$m \ge 0$	(M', E_2^m)	0	1	(E_2^m, M')	0	0
$m \ge 0$	(M', E_3^m)	0	0	(E_3^m, M')	1	0
$m \ge 0$	(M', E_4^m)	0	0	(E_4^m, M')	0	1
	(M, M')	0	1	(M', M)	0	1

(2.4)

Sketch of proof. Via computations, which we do not write out here, we obtain $\hom(\rho, \rho')$ for any two representations ρ, ρ' taken from Proposition 2.3. The computations are reduced to determining the dimensions of some vector spaces of matrices. These spaces and their dimensions are listed in Appendix 2.A, table (2.109). Having $\hom(\rho, \rho')$, the dimension $\hom^1(\rho, \rho')$ is computed by (2.3). □

Proposition 2.6. *The dimensions* $\hom(X, Y)$ *and* $\hom^1(X, Y)$ *for any pair of exceptional objects* $X, Y \in Rep_k(Q_2)$ *are contained in the following table:*

		hom	$\hom^1$		hom	$\hom^1$
$0 \le n < m$	(E_1^m, E_1^n)	$1+m-n$	0	(E_1^n, E_1^m)	0	$m-n-1$
$0 \le m < n$	(E_2^m, E_2^n)	$1+n-m$	0	(E_2^n, E_2^m)	0	$n-m-1$
$0 \le m < n$	(E_3^m, E_3^n)	$1+n-m$	0	(E_3^n, E_3^m)	0	$n-m-1$
$0 \le n < m$	(E_4^m, E_4^n)	$1+m-n$	0	(E_4^n, E_4^m)	0	$m-n-1$
$0 \le m < n$	(E_5^m, E_5^n)	$1+n-m$	0	(E_5^n, E_5^m)	0	$n-m-1$
$0 \le n < m$	(E_6^m, E_6^n)	$1+m-n$	0	(E_6^n, E_6^m)	0	$m-n-1$
$0 \le n < m$	(E_7^m, E_7^n)	$1+m-n$	0	(E_7^n, E_7^m)	0	$m-n-1$
$0 \le m < n$	(E_8^m, E_8^n)	$1+n-m$	0	(E_8^n, E_8^m)	0	$n-m-1$
$0 \le m, 0 \le n$	(E_1^m, E_2^n)	0	$2+n+m$	(E_2^n, E_1^m)	$m+n$	0
$0 \le m, 0 \le n$	(E_1^m, E_3^n)	0	$n+m$	(E_3^n, E_1^m)	$m+n$	0
$0 \le n < m$	(E_1^m, E_4^n)	$m-n-1$	0	(E_4^n, E_1^m)	0	$m-n-1$
$0 \le m \le n$	(E_1^m, E_4^n)	0	$n-m+1$	(E_4^n, E_1^m)	$n-m+1$	0
$0 \le m, 0 \le n$	(E_1^m, E_5^n)	0	$n+m+1$	(E_5^n, E_1^m)	$n+m$	0
$0 \le n < m$	(E_1^m, E_6^n)	$m-n$	0	(E_6^n, E_1^m)	0	$m-n-1$

		hom	$\hom^1$		hom	$\hom^1$
$0 \leq m \leq n$	(E_1^m, E_6^n)	0	$n-m$	(E_6^n, E_1^m)	$n-m+1$	0
$0 \leq n < m$	(E_1^m, E_7^n)	$m-n$	0	(E_7^n, E_1^m)	0	$m-n-1$
$0 \leq m \leq n$	(E_1^m, E_7^n)	0	$n-m$	(E_7^n, E_1^m)	$n-m+1$	0
$0 \leq m, 0 \leq n$	(E_1^m, E_8^n)	0	$n+m+1$	(E_8^n, E_1^m)	$n+m$	0
$0 \leq n \leq m$	(E_2^m, E_3^n)	0	$m-n+1$	(E_3^n, E_2^m)	$m-n+1$	0
$0 \leq m < n$	(E_2^m, E_3^n)	$n-m-1$	0	(E_3^n, E_2^m)	0	$n-m-1$
$0 \leq m, 0 \leq n$	(E_2^m, E_4^n)	$2+m+n$	0	(E_4^n, E_2^m)	0	$n+m+2$
$0 \leq m < n$	(E_2^m, E_5^n)	$n-m$	0	(E_5^n, E_2^m)	0	$n-m-1$
$0 \leq n \leq m$	(E_2^m, E_5^n)	0	$m-n$	(E_5^n, E_2^m)	$m-n+1$	0
$0 \leq m, 0 \leq n$	(E_2^m, E_6^n)	$1+m+n$	0	(E_6^n, E_2^m)	0	$n+m+2$
$0 \leq m, 0 \leq n$	(E_2^m, E_7^n)	$1+m+n$	0	(E_7^n, E_2^m)	0	$n+m+2$
$0 \leq m < n$	(E_2^m, E_8^n)	$n-m$	0	(E_8^n, E_2^m)	0	$n-m-1$
$0 \leq n \leq m$	(E_2^m, E_8^n)	0	$m-n$	(E_8^n, E_2^m)	$m-n+1$	0
$0 \leq m, 0 \leq n$	(E_3^m, E_4^n)	$m+n$	0	(E_4^n, E_3^m)	0	$n+m+2$
$0 \leq m \leq n$	(E_3^m, E_5^n)	$n-m+1$	0	(E_5^n, E_3^m)	0	$n-m$
$0 \leq n < m$	(E_3^m, E_5^n)	0	$m-n-1$	(E_5^n, E_3^m)	$m-n$	0
$0 \leq m, 0 \leq n$	(E_3^m, E_6^n)	$m+n$	0	(E_6^n, E_3^m)	0	$n+m+1$
$0 \leq m, 0 \leq n$	(E_3^m, E_7^n)	$m+n$	0	(E_7^n, E_3^m)	0	$n+m+1$
$0 \leq m \leq n$	(E_3^m, E_8^n)	$n-m+1$	0	(E_8^n, E_3^m)	0	$n-m$
$0 \leq n < m$	(E_3^m, E_8^n)	0	$m-n-1$	(E_8^n, E_3^m)	$m-n$	0
$0 \leq m, 0 \leq n$	(E_4^m, E_5^n)	0	$2+m+n$	(E_5^n, E_4^m)	$1+m+n$	0
$0 \leq m < n$	(E_4^m, E_6^n)	0	$n-m-1$	(E_6^n, E_4^m)	$n-m$	0
$0 \leq n \leq m$	(E_4^m, E_6^n)	$1+m-n$	0	(E_6^n, E_4^m)	0	$m-n$
$0 \leq m < n$	(E_4^m, E_7^n)	0	$n-m-1$	(E_7^n, E_4^m)	$n-m$	0
$0 \leq n \leq m$	(E_4^m, E_7^n)	$m-n+1$	0	(E_7^n, E_4^m)	0	$m-n$
$0 \leq m, 0 \leq n$	(E_4^m, E_8^n)	0	$2+m+n$	(E_8^n, E_4^m)	$1+m+n$	0
$0 \leq m, 0 \leq n$	(E_5^m, E_6^n)	$m+n$	0	(E_6^n, E_5^m)	0	$2+m+n$
$0 \leq m, 0 \leq n$	(E_5^m, E_7^n)	$1+m+n$	0	(E_7^n, E_5^m)	0	$1+m+n$
$0 \leq m \leq n$	(E_5^m, E_8^n)	$n-m$	0	(E_8^n, E_5^m)	0	$n-m$
$0 \leq n \leq m$	(E_5^m, E_8^n)	0	$m-n$	(E_8^n, E_5^m)	$m-n$	0
$0 \leq n \leq m$	(E_6^m, E_7^n)	$m-n$	0	(E_7^n, E_6^m)	0	$m-n$
$0 \leq m \leq n$	(E_6^m, E_7^n)	0	$n-m$	(E_7^n, E_6^m)	$n-m$	0
$0 \leq m, 0 \leq n$	(E_6^m, E_8^n)	0	$1+m+n$	(E_8^n, E_6^m)	$1+m+n$	0
$0 \leq m, 0 \leq n$	(E_7^m, E_8^n)	0	$2+m+n$	(E_8^n, E_7^m)	$m+n$	0
$0 \leq m$	(F_+, E_1^m)	0	0	(E_1^m, F_+)	0	1
$0 \leq m$	(F_-, E_1^m)	0	0	(E_1^m, F_-)	0	1
$0 \leq m$	(F_+, E_2^m)	0	0	(E_2^m, F_+)	1	0
$0 \leq m$	(F_-, E_2^m)	0	0	(E_2^m, F_-)	1	0
$0 \leq m$	(F_+, E_3^m)	0	1	(E_3^m, F_+)	0	0
$0 \leq m$	(F_-, E_3^m)	0	1	(E_3^m, F_-)	0	0
$0 \leq m$	(F_+, E_4^m)	1	0	(E_4^m, F_+)	0	0
$0 \leq m$	(F_-, E_4^m)	1	0	(E_4^m, F_-)	0	0
$0 \leq m$	(F_+, E_5^m)	0	1	(E_5^m, F_+)	0	0
$0 \leq m$	(F_-, E_5^m)	0	0	(E_5^m, F_-)	1	0
$0 \leq m$	(F_+, E_6^m)	1	0	(E_6^m, F_+)	0	0
$0 \leq m$	(F_-, E_6^m)	0	0	(E_6^m, F_-)	0	1
$0 \leq m$	(F_+, E_7^m)	0	0	(E_7^m, F_+)	0	1
$0 \leq m$	(F_-, E_7^m)	1	0	(E_7^m, F_-)	0	0
$0 \leq m$	(F_+, E_8^m)	0	0	(E_8^m, F_+)	1	0
$0 \leq m$	(F_-, E_8^m)	0	1	(E_8^m, F_-)	0	0
$0 \leq m$	$(G_\pm, E_1^m)$	1	0	$(E_1^m, G_\pm)$	0	0
$0 \leq m$	$(G_\pm, E_2^m)$	0	1	$(E_2^m, G_\pm)$	0	0
$0 \leq m$	$(G_\pm, E_3^m)$	0	0	$(E_3^m, G_\pm)$	1	0
$0 \leq m$	$(G_\pm, E_4^m)$	0	0	$(E_4^m, G_\pm)$	0	1
$0 \leq m$	(G_+, E_5^m)	0	1	(E_5^m, G_+)	0	0
$0 \leq m$	(G_-, E_5^m)	0	0	(E_5^m, G_-)	1	0
$0 \leq m$	(G_+, E_6^m)	1	0	(E_6^m, G_+)	0	0
$0 \leq m$	(G_-, E_6^m)	0	0	(E_6^m, G_-)	0	1
$0 \leq m$	(G_+, E_7^m)	0	0	(E_7^m, G_+)	0	1
$0 \leq m$	(G_-, E_7^m)	1	0	(E_7^m, G_-)	0	0
$0 \leq m$	(G_+, E_8^m)	0	0	(E_8^m, G_+)	1	0
$0 \leq m$	(G_-, E_8^m)	0	1	(E_8^m, G_-)	0	0
	(F_+, F_-)	0	0	(F_-, F_+)	0	0
	(F_+, G_+)	0	0	(G_+, F_+)	0	0
	(F_+, G_-)	0	1	(G_-, F_+)	0	1
	(F_-, G_+)	0	1	(G_+, F_-)	0	1
	(F_-, G_-)	0	0	(G_-, F_-)	0	0
	(G_+, G_-)	0	0	(G_-, G_+)	0	0

Sketch of proof. The table for $Rep_k(Q_2)$ is obtained by the same method as for $Rep_k(Q_1)$. □
The next subsection contains corollaries of the obtained tables.

2.2.3 The Ext-nontrivial couples and their properties

From the table in Proposition 2.5 we see that the only couple $\{X, Y\}$ of exceptional objects in $Rep_k(Q_1)$ satisfying $\hom^1(X, Y) \neq 0$ and $\hom^1(Y, X) \neq 0$ is $\{M, M'\}$. We call such a couple an *Ext-nontrivial couple* (see Definition 2.48). By Proposition 2.6 we see that the Ext-nontrivial couples in $Rep_k(Q_2)$ are $\{F_+, G_-\}$, $\{F_-, G_+\}$.

Corollary 2.7 concerns both $Rep_k(Q_1)$ and $Rep_k(Q_2)$.

Corollary 2.7. *The categories $Rep_k(Q_1)$, $Rep_k(Q_2)$ satisfy the following properties:*

- ***(a)*** *RP property 1, RP property 2 (see Definition 2.51).*
- ***(b)*** *For any two exceptional objects $X, Y \in Rep_k(Q_i)$ at most one degree in $\{\hom^p(X, Y)\}_{p\in\mathbb{Z}}$ is nonzero, where $i \in \{1, 2\}$.*

Proof. It follows by a careful case by case check, using the tables in Propositions 2.5, 2.6. □

The following four corollaries concern only $Rep_k(Q_1)$ and are contained in table (2.4).

Corollary 2.8. *If $\{\Gamma_1, \Gamma_2\}$ is an Ext-nontrivial couple in $Rep_k(Q_1)$(see Definition 2.48), then for each exceptional object $X \in Rep_k(Q_1)$ we have $\hom^p(\Gamma_i, X) \neq 0$ for some $i \in \{1, 2\}$, $p \in \mathbb{Z}$ and* $\hom^q(X, \Gamma_j)$ *for some $j \in \{1, 2\}$, $q \in \mathbb{Z}$.*

Corollary 2.9. *The exceptional pairs (X, Y) in $Rep_k(Q_1)$ are ($m \in \mathbb{N}$):*

$$\begin{array}{c}(E_1^{m+1}, E_1^m)\ \ (E_2^m, E_2^{m+1})\ \ (E_3^m, E_3^{m+1})\ \ (E_4^{m+1}, E_4^m)\ \ (E_1^0, E_2^0)\ \ (E_1^0, E_3^0)\\ (E_4^m, E_1^m)\ \ (E_1^{m+1}, E_4^m)\ \ (E_3^m, E_2^m)\ \ (E_2^m, E_3^{m+1})\ \ (E_4^0, E_3^0)\ \ (E_1^m, M)\\ (E_2^m, M)\ \ (M, E_3^m)\ \ (M, E_4^m)\ \ (M', E_1^m)\ \ (M', E_2^m)\ \ (E_3^m, M')\ \ (E_4^m, M').\end{array} \tag{2.5}$$

Using this corollary we obtain the list of the exceptional triples of $Rep_k(Q_1)$, which by [15] are the full exceptional collections.

Corollary 2.10. *The full exceptional collections in $Rep_k(Q_1)$ up to isomorphism are ($m \in \mathbb{N}$):*

$$\begin{array}{ccc}(E_1^{m+1}, E_1^m, M) & (E_1^{m+1}, E_4^m, E_1^m) & (E_1^{m+1}, M, E_4^m)\\ (E_1^0, E_2^0, M) & (E_1^0, E_3^0, E_2^0) & (E_1^0, M, E_3^0)\\ (E_2^m, E_2^{m+1}, M) & (E_2^m, E_3^{m+1}, E_2^{m+1}) & (E_2^m, M, E_3^{m+1})\\ (E_3^m, E_2^m, E_3^{m+1}) & (E_3^m, E_3^{m+1}, M') & (E_3^m, M', E_2^m)\\ (E_4^{m+1}, E_4^m, M') & (E_4^{m+1}, E_1^{m+1}, E_4^m) & (E_4^{m+1}, M', E_1^{m+1})\\ (E_4^0, E_1^0, E_3^0) & (E_4^0, E_3^0, M') & (E_4^0, M', E_1^0)\\ (M, E_3^m, E_3^{m+1}) & (M, E_4^{m+1}, E_4^m) & (M, E_4^0, E_3^0)\\ (M', E_1^{m+1}, E_1^m) & (M', E_2^m, E_2^{m+1}) & (M', E_1^0, E_2^0).\end{array}$$

The following corollary is a special case of [15, Lemma 2]. It also follows from Corollary 2.10.

Corollary 2.11. *Let (A_0, A_1, A_2),(A'_0, A'_1, A'_2) be two exceptional triples in $Rep_k(Q_1)$. If $A_i \cong A'_i$, $A_j \cong A'_j$ for two different $i, j \in \{0, 1, 2\}$, then $A_k \cong A'_k$ for the third $k \in \{0, 1, 2\}$.*

Remark 2.12. *In [17] is shown that any exceptional pair (A, B) in $D^b(Q)$ for an acyclic affine quiver Q satisfies $\hom^i(A, B) \leq 2$. The pairs of $Rep_k(Q_1)$ are listed in Corollary 2.9. Equality is attained in the following pairs: (E_1^{m+1}, E_1^m), (E_2^m, E_2^{m+1}), (E_3^m, E_3^{m+1}), (E_4^{m+1}, E_4^m), $(E_1^0, E_2^0) =$, (E_4^0, E_3^0).*

Remark 2.13. *From Corollaries 2.10 and 2.11 we see that the action of the Braid group B_3 on the exceptional collections of $Rep_k(Q_1)$ is not free. We give examples below.*

Example of fixed triples by a Braid group element. *For any exceptional triple (A, B, C) we denote here the triple*[11] *$(A, L_B(C), B)$ by $L_1(A, B, C)$. We keep in mind also Corollary 2.11 and that each exceptional object in $D^b(Q_1)$ is a shift of an exceptional object in $Rep_k(Q_1)$.*

The first row in the list of Corollary 2.10 shows that, up to shifts, we have the equalities $L_1(E_1^{m+1}, M, E_4^m) = (E_1^{m+1}, E_1^m, M)$; $L_1(E_1^{m+1}, E_1^m, M) = (E_1^{m+1}, E_4^m, E_1^m)$; $L_1(E_1^{m+1}, E_4^m, E_1^m) = (E_1^{m+1}, M, E_4^m)$. Thus, the triple (E_1^{m+1}, M, E_4^m) is fixed by $(L_1)^3$. The element $(L_1)^3$ is not trivial in the braid group B_3, since B_3 is torsion free.

Acting with L_1 on each of the rest rows, except the last two rows, we find the same behavior.

2.3 Preliminaries

Here we comment on Bridgeland's stability conditions and on Macrì's construction of stability conditions via exceptional collections.

In Subsection 2.3.2 for a a Krull-Schmidt category $\mathcal{T}$, we introduce a function $Ob(\mathcal{T}) \xrightarrow{\theta_\sigma} \mathbb{N}^{(\sigma_{ind}^{ss}/\cong)}$, depending on a stability condition $\sigma \in \mathrm{Stab}(\mathcal{T})$. It helps us later to encode useful features of the relation $R \dashrightarrow (S, E)$ in the simple expressions $\theta_\sigma(R) > \theta_\sigma(E)$, $\theta_\sigma(R)(S) > 0$(see Section 2.5). Lemma 2.17, based on the locally finiteness of the elements in $\mathrm{Stab}(\mathcal{T})$, has an important role in Section 2.10. The simple fact observed in Lemma 2.19, used throughout Sections 2.6,..., 2.10, is helpful in our study of long R-sequences.

After having recalled Macrì's construction in Subsection 2.3.3, we define in the final Subsection 2.3.4 the notion of a σ-exceptional collection.

[11] Recall that for any exceptional pair (A, B) the exceptional objects $L_A(B)$ and $R_B(A)$ are determined by the triangles $L_A(B) \longrightarrow \mathrm{Hom}^*(A, B) \otimes A \xrightarrow{ev^*_{A,B}} B$; $A \xrightarrow{coev^*_{A,B}} \mathrm{Hom}^*(A, B)\check{} \otimes B \longrightarrow R_B(A)$ and that $(L_A(B), A)$, $(B, R_B(A))$ are exceptional pairs.

2.3.1 Krull-Schmidt property. The function $\theta : Ob(\mathcal{C}) \to \mathbb{N}^{(\mathcal{C}_{ind}/\cong)}$

Let $\mathcal{C}$ be an additive category. We denote by $\mathcal{C}_{ind}$ the set of all indecomposable objects in $\mathcal{C}$.[12] We discuss here the well known Krull Schmidt property.

Definition 2.14. *We say that an additive category $\mathcal{C}$ has Krull-Schmidt property if for each $X \in Ob(\mathcal{C}) \setminus \{0\}$ there exists unique up to isomorphism and permutation sequence $\{X_1, X_2, \dots X_n\}$ in $\mathcal{C}_{ind}$ with $X \cong \bigoplus_{i=1}^n X_i$.*

For $X \in Ob(\mathcal{C}) \setminus \{0\}$ with a decomposition $X \cong \bigoplus_{i=1}^n X_i$ as above we denote by $Ind(X)$ the set $\{Y \in Ob(\mathcal{C}) : \ Y \cong X_i \text{ for some } i = 1, 2, \dots, n\}$. If X is a zero object, then $Ind(X) = \emptyset$.

We will use two simple observations related to this property.

Lemma 2.15. *Let $\mathcal{A}$ be a hereditary abelian category. If $\mathcal{A}$ has Krull-Schmidt property, then $D^b(\mathcal{A})$ has Krull-Schmidt property.*

Proof. Recall that any object $X \in D^b(\mathcal{A})$ decomposes as follows $X \cong \bigoplus_i H^i(X)[-i]$ and if $X \cong \bigoplus_i X_i[-i]$ for some collection $\{X_i\} \subset \mathcal{A}$, then $X_i \cong H^i(X)$ for all i. In particular $\mathcal{A}$ is a thick subcategory of $D^b(\mathcal{A})$.[13] Now the lemma follows. □

Lemma 2.16. *Let $\mathcal{C}$ have Krull-Schmidt property. There exists unique function $Ob(\mathcal{C}) \xrightarrow{\theta} \mathbb{N}^{(\mathcal{C}_{ind}/\cong)}$ satisfying:*[14]

- ***(a)*** *If $Y \cong \bigoplus_{i=1}^m Y_i$ in $\mathcal{C}$, then $\theta(Y) = \sum_{i=1}^n \theta(Y_i)$.*
- ***(b)*** *For any $X \in \mathcal{C}_{ind}$ the function $\mathcal{C}_{ind}/\cong \xrightarrow{\theta(X)} \mathbb{N}$ assigns one to the equivalence class containing X, and zero elsewhere.*

Proof. For an object $X \in Ob(\mathcal{C})$ with a decomposition $X \cong \bigoplus_{i=1}^n X_i$ as in Definition 2.14 the function $\mathcal{C}_{ind}/\cong \xrightarrow{\theta(X)} \mathbb{N}$ assigns to each $u \in \mathcal{C}_{ind}/\cong$ the number $\#\{i : X_i \in u\}$. □

2.3.2 Comments on Bridgeland stability conditions. The family $\{\theta_\sigma : Ob(\mathcal{T}) \to \mathbb{N}^{(\sigma^{ss}_{ind}/\cong)}\}_{\sigma \in \mathrm{Stab}(\mathcal{T})}$

T. Bridgeland defined in [8] the notion of a locally finite stability condition on a triangulated category $\mathcal{T}$ and equipped the set of all locally finite stability conditions on a given $\mathcal{T}$ with a structure of a complex manifold, this manifold is denoted by $\mathrm{Stab}(\mathcal{T})$. The manifold $\mathrm{Stab}(\mathcal{T})$ carries a natural right action by the group $\widetilde{GL}^+(2, \mathbb{R})$.

[12]the set $\mathcal{C}_{ind}$ does not contain zero objects.
[13]By "thick" we mean "closed under direct summands"
[14]By $\mathbb{N}^{(\mathcal{C}_{ind}/\cong)}$ we denote the set of functions from $\mathcal{C}_{ind}/\cong$ to $\mathbb{N}$ with finite support.

A stability condition on $\mathcal{T}$ as a pair $(\mathcal{P}, Z)$, where $\{\mathcal{P}(t)\}_{t\in\mathbb{R}}$ is a family of full additive subcategories and $Z : K_0(\mathcal{T}) \to \mathbb{C}$ is a group homomorphism satisfying certain axioms. The homomorphism Z is called *central charge*. If $\sigma = (\mathcal{P}, Z)$ is a locally finite stability condition on a triangulated category $\mathcal{T}$, then for each $t \in \mathbb{R}$ the subcategory $\mathcal{P}(t)$ is an abelian category of finite length (see [9, p. 6]). Furthermore [8], the short exact sequences in $\mathcal{P}(t)$ are exactly these sequences $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ with $A, B, C \in \mathcal{P}(t)$, s. t. for some $\gamma : C \to A[1]$ the sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]$ is a triangle in $\mathcal{T}$. The first lemma in this subsection, used in Section 2.10, follows from **locally finiteness**.

Lemma 2.17. *Let* $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}(\mathcal{T})$*,* $t \in \mathbb{R}$*,* $A \in \mathcal{P}(t)$*. For any object* $X \in \mathcal{T}$ *denote by* $[X] \in K_0(\mathcal{T})$ *the corresponding equivalence class in the Grothendieck group* $K_0(\mathcal{T})$*. Then the set*

$$\{[X] \in K_0(\mathcal{T}) : X \in \mathcal{P}(t) \text{ and there exists a monic arrow } X \to A \text{ in } \mathcal{P}(t)\} \tag{2.6}$$

is finite.

Proof. Since $\mathcal{P}(t)$ is abelian category of finite length, we have a Jordan-Holder filtration for the given $A \in \mathcal{P}(t)$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow \dots \longrightarrow & E_{n-1} & \longrightarrow & E_n = A \\ & & \swarrow & & \swarrow & & & & \swarrow \\ & S_1 & & S_2 & & & & S_n & \end{array}$$

where $E_i \to E_{i+1} \to S_{i+1}$ are short exact sequences in $\mathcal{P}(t)$ and $S_1, S_2, \dots, S_n$ are simple objects in $\mathcal{P}(t)$. We will show that the set (2.6) is finite by showing that it is a subset of:

$$\left\{ \sum_{i=1}^{m} [S_{\xi(i)}] : \{1, 2, \dots, m\} \xrightarrow{\xi} \{1, 2, \dots, n\} \text{ is injective} \right\}.$$

For any monic arrow $X \to A$ in $\mathcal{P}(t)$ we have a Jordan-Holder filtration of X

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E'_1 & \longrightarrow & E'_2 & \longrightarrow \dots \longrightarrow & E'_{n-1} & \longrightarrow & E'_m = X \\ & & \swarrow & & \swarrow & & & & \swarrow \\ & S'_1 & & S'_2 & & & & S'_m & \end{array} \tag{2.7}$$

where $S'_1, S'_2, \dots, S'_m$ are simple objects in $\mathcal{P}(t)$, s. t. $S'_i \cong S_{\xi(i)}, i = 1, \dots, m$ for some injection $\xi : \{1, 2, \dots, m\} \to \{1, 2, \dots, n\}$. Since $E'_i \to E'_{i+1} \to S'_{i+1}$ is a short exact sequences in $\mathcal{P}(t)$, it is also a part of a triangle $E'_i \to E'_{i+1} \to S'_{i+1} \to E'_i[1]$ in $\mathcal{T}$. Hence by (2.7) it follows $[X] = \sum_{i=1}^{m} [S'_i] = \sum_{i=1}^{m} [S_{\xi(i)}]$. □

Recall that one of Bridgeland's axioms [8] is: for any nonzero $X \in Ob(\mathcal{T})$ there exists a diagram of triangles,[15] called **Harder- Narasimhan filtration**:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow \dots \longrightarrow & E_{n-1} & \longrightarrow & E_n = X \\ & \nwarrow & \swarrow & \nwarrow & \swarrow & & & \nwarrow & \swarrow \\ & & A_1 & & A_2 & & & & A_n \end{array} \tag{2.8}$$

[15] Whenever we refer to a collection of three arrows in a triangulated category as to a triangle, we mean a distinguished triangle.

where $\{A_i \in \mathcal{P}(t_i)\}_{i=1}^n$, $t_1 > t_2 > \cdots > t_n$ and A_i is non-zero object for any $i = 1, \dots, n$ (the non-vanishing condition makes the factors $\{A_i \in \mathcal{P}(t_i)\}_{i=1}^n$ unique up to isomorphism). In [8] is used the notation $\phi_-^\sigma(X) := t_n$, $\phi_+^\sigma(X) := t_1$. The objects in the set (2.9) are said to be σ-semistable and for a semistable object A we have a unique t s. t. $A \in \mathcal{P}(t) \setminus \{0\}$, it is called phase of A and denoted by $\phi^\sigma(A) := t$. *The objects* $\{A_i\}_{i=1}^n$ *in* (2.8) *will be called HN factors of* X (HN for Harder-Narasimhan). It is useful to give a name of the minimal HN factor A_n.

Definition 2.18. *For any* $X \in \mathcal{T} \setminus \{0\}$ *we choose*[16] *a Harder-Narasimhan filtration as in* (2.8). *Having this diagram, we denote the semistable HN factor of minimal phase* A_n *by* $\sigma_-(X)$, *and the last triangle* $E_{n-1} \longrightarrow X \longrightarrow A_n \longrightarrow E_{n-1}[1]$ *by* $\mathrm{HN}_-(X)$. *In particular,* $\phi(\sigma_-(X)) = \phi_-(X)$.

In the next Lemma 2.19 we treat $\sigma_-(X)$. We recall first that **from** $\phi(A) > \phi(B)$ **with semistable** A**,** B **it follows** $\hom(A,B) = 0$ (another axiom of Bridgeland [8]). This axiom implies that from $\phi_-(X) > \phi_+(Y)$ it follows $\hom^{\leq 0}(X,Y) = \hom^{\leq 0}(\sigma_-(X),Y) = 0$. We get $\hom^{\leq 1}(\sigma_-(X),Y) = 0$ in the following situation:

Lemma 2.19. *If* $\phi_-(X) \geq \phi_+(Y)$ *and* $\hom^{\leq 1}(X,Y) = 0$, *then* $\hom^{\leq 1}(\sigma_-(X),Y) = 0$.

Proof. Let $\mathrm{HN}_-(X) = Z \longrightarrow X \longrightarrow \sigma_-(X) \longrightarrow Z[1]$. Then $\phi_-(Z) > \phi(\sigma_-(X)) = \phi_-(X) \geq \phi_+(Y)$. Hence $\mathrm{Hom}^{\leq 0}(Z,Y) = 0$. We apply $\mathrm{Hom}(\_, Y[i])$ with $i \leq 1$ to this triangle and obtain:
$0 = \mathrm{Hom}(Z[1], Y[i]) \to \mathrm{Hom}(\sigma_-(X), Y[i]) \to \mathrm{Hom}(X, Y[i]) = 0$. The lemma follows. □

In [8] for a slicing $\mathcal{P}$ of $\mathcal{T}$ and an interval $I \subset \mathbb{R}$ by $\mathcal{P}(I)$ is denoted the extension closure of $\{\mathcal{P}(t)\}_{t\in I}$, and $\mathcal{P}([t,t+1)), \mathcal{P}((t,t+1])$ are shown to be hearts of bounded t-structures for any $t \in \mathbb{R}$. If $\mathcal{P}$ is a part of a stability condition $(\mathcal{P}, Z) \in \mathrm{Stab}(\mathcal{T})$, then $\mathcal{P}(t)$ is shown to be abelian. The nonzero objects in the subcategory $\mathcal{P}(I)$ are exactly those $X \in \mathcal{T} \setminus \{0\}$, which satisfy $\phi_\pm(X) \in I$.

From these facts it follows that $\mathcal{P}(I)$ is a thick[17] subcategory for any interval $I \subset \mathbb{R}$:

Lemma 2.20. *For any slicing* $\mathcal{P}$ *of a triangulated category* $\mathcal{T}$ *and any interval* $I \subset \mathbb{R}$ *the category* $\mathcal{P}(I)$ *is a thick subcategory of* $\mathcal{T}$. *In particular, if* $\mathcal{T}$ *has Krull-Schmidt property, then* $\mathcal{P}(I)$ *has it.*

Proof. In [28] t-structures are defined as pairs of subcategories. For any slicing $\mathcal{P}$ and any $t \in \mathbb{R}$ the hearts $\mathcal{P}((t,t+1]), \mathcal{P}([t,t+1))$ come from the pairs $(\mathcal{P}((t,+\infty)), \mathcal{P}((-\infty,t+1]))$,
$(\mathcal{P}([t,+\infty)), \mathcal{P}((-\infty,t+1)))$, respectively, which are bounded t-structures. Let us consider for example the t-structure $(\mathcal{P}((t,+\infty)), \mathcal{P}((-\infty,t+1]))$. In terms of the notations used in [28] we denote $\mathcal{T}^{\leq 0} = \mathcal{P}((t,+\infty))$, $\mathcal{T}^{\geq 0} = \mathcal{P}((-\infty,t+1])$. From the properties of t-structures we know that

$$X \in \mathcal{T}^{\leq 0} \iff \forall Y \in \mathcal{T}^{\geq 1} \quad \hom(X,Y) = 0; \qquad X \in \mathcal{T}^{\geq 0} \iff \forall Y \in \mathcal{T}^{\leq -1} \quad \hom(Y,X) = 0.$$

Hence $\mathcal{T}^{\leq 0} = \mathcal{P}((t,+\infty))$, $\mathcal{T}^{\geq 0} = \mathcal{P}((-\infty,t+1])$ are thick subcategories. Similarly $\mathcal{P}([t,+\infty))$, $\mathcal{P}((-\infty,t+1))$ are thick. Since for any interval $I \subset \mathbb{R}$ the subcategory $\mathcal{P}(I)$ is an intersection of two subcategories of the considered types, the lemma follows. □

[16]by the axiom of choice

[17]Recall that by "thick" we mean "closed under direct summands".

Corollary 2.21. *Let $X, A, B \in \mathcal{T}$ and $X \cong A \oplus B$, then for any slicing $\mathcal{P}$ of $\mathcal{T}$ we have $\phi_-(X) \leq \phi_-(A) \leq \phi_+(A) \leq \phi_+(X)$.*

Proof. We have $X \in \mathcal{P}([\phi_-(X), \phi_+(X)])$. From the previous lemma $A, B \in \mathcal{P}([\phi_-(X), \phi_+(X)])$ and the statement follows. □

Thus, if $\mathcal{T}$ has Krull-Schmidt property, then all $\{\mathcal{P}(t)\}_{t\in\mathbb{R}}$ have it(Lemma 2.20). From Lemma 2.16 we obtain a family of functions $\{\mathcal{P}(t) \to \mathbb{N}^{(\mathcal{P}(t)_{ind}/\cong)}\}_{t\in\mathbb{R}}$. In Definition 2.22 below we build a single function on $Ob(\mathcal{T})$ from this family of functions, using the HN filtrations. We need first some notations.

For $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}(\mathcal{T})$ we denote by σ^{ss} the set of σ-semistable objects, i. e.

$$\sigma^{ss} = \cup_{t\in\mathbb{R}}\mathcal{P}(t) \setminus \{0\}. \tag{2.9}$$

By σ^{ss}_{ind} we denote the set of all indecomposable semistable objects, i. e.[18]

$$\sigma^{ss}_{ind} = \cup_{t\in\mathbb{R}}\mathcal{P}(t)_{ind} = \sigma^{ss} \cap \mathcal{T}_{ind}. \tag{2.10}$$

In **(a)** of Definition 2.22 we consider $\mathbb{N}^{(\mathcal{P}(t)_{ind}/\cong)}$ as a subset of $\mathbb{N}^{(\sigma^{ss}_{ind}/\cong)}$, which is reasonable since the family $\{\mathcal{P}(t)_{ind}\}_{t\in\mathbb{R}}$ is pairwise disjoint.

Definition 2.22. *Let $\mathcal{T}$ have Krull-Schmidt property. Let $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}(\mathcal{T})$.*

We define $\theta_\sigma : Ob(\mathcal{T}) \to \mathbb{N}^{(\sigma^{ss}_{ind}/\cong)}$ as the unique function satisfying the following:

***(a)** For each $t \in \mathbb{R}$ the restriction of θ_σ to $\mathcal{P}(t)$ coincides with the function $\mathcal{P}(t) \to \mathbb{N}^{(\mathcal{P}(t)_{ind}/\cong)}$, given by Lemmas 2.16, 2.20.*

***(b)** For any non-zero $X \in Ob(\mathcal{T})$ with a HN filtration[19] (2.8) holds the equality $\theta_\sigma(X) = \sum_{i=1}^{n} \theta_\sigma(A_i)$.*

We use freely that $X \cong Y$ implies $\theta_\sigma(X) = \theta_\sigma(Y)$, $X \neq 0$ implies $\theta_\sigma(X) \neq 0$, and $\theta_\sigma(X) \leq \theta_\sigma(Y)$ implies $\phi_-(Y) \leq \phi_-(X) \leq \phi_+(X) \leq \phi_+(Y)$. Another property of θ_σ, to which we refer later, is:

Lemma 2.23. *Let $\phi_-(X_1) > \phi_+(X_2)$. For any triangle $X_1 \to X \to X_2 \to X_1[1]$ we have $\theta_\sigma(X) = \theta_\sigma(X_1) + \theta_\sigma(X_2)$.*

Proof. If the HN factors of X_1 and X_2 are $A_1, A_2, \dots, A_n$ and $B_1, B_2, \dots, B_m$, respectively, then, using the octahedral axiom, one can show that the HN factors of X are $A_1, A_2, \dots, A_n, B_1, B_2, \dots, B_m$. Now the lemma follows from **(b)** in Definition 2.22. □

The property $\theta_\sigma(X \oplus Y) = \theta_\sigma(X) + \theta_\sigma(Y)$ for $X, Y \in \mathcal{P}(t)$ follows from **(a)** in Lemma 2.16. To show this additive property for any two objects X, $Y \in \mathcal{T}$ we note first:

[18]Recall that $\mathcal{P}(t)$ is thick in $\mathcal{T}$ (Lemma 2.20), hence $\mathcal{P}(t)_{ind} = \mathcal{P}(t) \cap \mathcal{T}_{ind}$.

[19]Recall that the collection $\{A_i\}_{i=1}^{n}$ of the HN factors is determined by X up to isomorphism.

Lemma 2.24. *For any diagram of the type (composed of distinguished triangles):*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow \dots \longrightarrow & B_{n-1} & \longrightarrow & B_n = X, \\ & \nwarrow & \swarrow & \nwarrow & \swarrow & & & \nwarrow & \swarrow \\ & & A_1 & & A_2 & & & & A_n & \end{array}$$

where $\{A_i \in \mathcal{P}(t_i)\}_{i=1}^n$, $t_1 > t_2 > \cdots > t_n$, *without the constraint that* $A_1, A_2, \dots, A_n$ *are non-zero objects, we have* $\theta_\sigma(X) = \sum_{i=1}^n \theta_\sigma(A_i)$.

Proof. We can remove all triangles where A_i is zero and in the end we obtain the HN filtration of X, then the equality follows from **(b)** in Definition 2.22 and $\theta_\sigma(A_i) = 0$ if A_i is a zero object. $\square$

Given two non-zero objects $X_1, X_2 \in Ob(\mathcal{T})$, then after inserting triangles of the form $\begin{array}{ccc} E & \xrightarrow{Id} & E \\ \nwarrow & & \swarrow \\ & 0 & \end{array}$ to their HN filtrations we can obtain two($i = 1, 2$) equally long diagrams with distinguished triangles

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & B_1^i & \longrightarrow & B_2^i & \longrightarrow \dots \longrightarrow & B_{n-1}^i & \longrightarrow & B_n^i = X_i, \\ & \nwarrow & \swarrow & \nwarrow & \swarrow & & & \nwarrow & \swarrow \\ & & A_1^i & & A_2^i & & & & A_n^i & \end{array}$$

where $\{A_j^i \in \mathcal{P}(t_j)\}_{j=1}^n$, $i = 1, 2$ and $t_1 > t_2 > \cdots > t_n$. Hence, we get a diagram of triangles:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_1^1 \oplus B_1^2 & \longrightarrow & B_2^1 \oplus B_2^2 & \longrightarrow \dots & B_{n-1}^1 \oplus B_{n-1}^2 & \longrightarrow & X_1 \oplus X_2. \\ & \nwarrow & \swarrow & \nwarrow & \swarrow & & & \nwarrow & \swarrow \\ & & A_1^1 \oplus A_1^2 & & A_2^1 \oplus A_2^2 & & & & A_n^1 \oplus A_n^2 \end{array}$$

We have $\{A_j^1 \oplus A_j^2 \in \mathcal{P}(t_j)\}_{j=1}^n$ by the additivity of $\mathcal{P}(t_j)$. Using Lemma 2.24 we obtain: $\theta_\sigma(X_1 \oplus X_2) = \sum_{j=1}^n \theta_\sigma(A_j^1 \oplus A_j^2) = \sum_{j=1}^n \theta_\sigma(A_j^1) + \sum_{j=1}^n \theta_\sigma(A_j^2) = \theta_\sigma(X_1) + \theta_\sigma(X_2)$, i. e. we proved:

Lemma 2.25. *For any pair of objects* X_1, X_2 *in* $\mathcal{T}$ *we have:* $\theta_\sigma(X_1 \oplus X_2) = \theta_\sigma(X_1) + \theta_\sigma(X_2)$.

The remaining axioms of Bridgeland [8] consist in saying that **a stability condition** $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}(\mathcal{T})$ **has the properties:** $\mathcal{P}(t)[1] = \mathcal{P}(t+1)$ **for each** $t \in \mathbb{R}$**, and**

$$X \in \sigma^{ss} \qquad \Rightarrow \qquad Z(X) = r(X)\ \exp(\mathrm{i}\pi\phi(X)),\ r(X) > 0. \tag{2.11}$$

We end this subsection by recalling one more result of [8]. We recall first the following definition:

Definition 2.26. *Let* $(\mathcal{A}, K_0(\mathcal{A}) \xrightarrow{Z} \mathbb{C})$ *be an abelian category and a stability function on it.*[20] *A non-zero object* $X \in \mathcal{A}$ *is said to be* Z*-semistable of phase* t *if every* $\mathcal{A}$*-monic* $X' \to X$ *satisfies*[21] $\arg Z(X') \leq \arg Z(X) = \pi t$ *(if equality is attained only for* $X' \cong X$ *then* X *is said to be stable).*

[20] I.e. Z is homomorphism, s. t. $Z(X) \in \mathbb{H} = \{r \exp(\mathrm{i}\pi t) | r > 0 \ \text{ and } \ 0 < t \leq 1\}$ for $X \in \mathcal{A}$, $X \neq 0$

[21] For $u \in \mathbb{H}$ we denote by $\arg(u)$ the unique number satisfying $\arg(u) \in (0, 1]$, $u = \exp(\mathrm{i}\pi \arg(u))$. It is convenient to set $\arg(0) = -\infty$.

Proposition 2.27 (Proposition 5.3 in [8]). *Let $\mathcal{A} \subset \mathcal{T}$ be a bounded t-structure in a triangulated category $\mathcal{T}$ and $K_0(\mathcal{A}) \xrightarrow{Z} \mathbb{C}$ be a stability function on $\mathcal{A}$ with HN property.*[22] *Then there exists unique stability condition*[23] *$\sigma = (\mathcal{P}, Z_e)$ on $\mathcal{T}$ satisfying:*

(a) *$Z_e(X) = Z(X)$ for $X \in \mathcal{A}$;*

(b) *For $t \in (0, 1]$ the objects of $\mathcal{P}(t)$ are the Z-semistable objects in $\mathcal{A}$ of phase t (as defined in Definition 2.26).*

*Conversely, for each stability condition $\sigma = (\mathcal{P}, Z_e)$ on $\mathcal{T}$ the subcategory $\mathcal{P}((0, 1]) = \mathcal{A}$ is a heart of a bounded t-structure of $\mathcal{T}$, the restriction $Z = Z_e \circ (K_0(\mathcal{A}) \to K_0(\mathcal{T}))$ of Z_e to $K_0(\mathcal{A})$ is a stability function on $\mathcal{A}$ with HN property and for $t \in (0, 1]$ the set of objects of $\mathcal{P}(t)$ is the same as in **(b)**.*

Definition 2.28. *We denote by $\mathbb{H}^{\mathcal{A}}$ the family of stability conditions on $\mathcal{T}$ obtained by **(a)**, **(b)** above keeping $\mathcal{A}$ fixed and varying Z in the set of all stability functions on $\mathcal{A}$ with HN property. In particular $\mathbb{H}^{\mathcal{A}} \ni (\mathcal{P}, Z) \mapsto Z_{|K_0(\mathcal{A})}$ is a bijection between $\mathbb{H}^{\mathcal{A}}$ and this set.*

Remark 2.29. *Let $\mathcal{A} \subset \mathcal{T}$ be as in the previous definition. If $\mathcal{A}$ is an abelian category of finite length, then any stability function $Z : K_0(\mathcal{A}) \to \mathbb{C}$ satisfies the HN property ([8, Proposition 2.4]). If in addition $\mathcal{A}$ has finitely many, say $s_1, s_2, \ldots, s_n$, simple objects then all stability conditions in $\mathbb{H}^{\mathcal{A}}$ are locally finite. Hence, in this setting we have $\mathbb{H}^{\mathcal{A}} \subset \mathrm{Stab}(\mathcal{T})$ and bijection $\mathbb{H}^{\mathcal{A}} \ni (\mathcal{P}, Z) \mapsto (Z(s_1), \ldots, Z(s_n)) \in \mathbb{H}^n$.*

2.3.3 On the stability conditions constructed by E. Macrì via exceptional collections

E. Macrì proved in [37, Lemma 3.14] that the extension closure $\mathcal{A}_{\mathcal{E}}$ of a full Ext-exceptional collection $\mathcal{E} = (E_0, E_1, \ldots, E_n)$ in $\mathcal{T}$ is a heart of a bounded t-structure. Furthermore, $\mathcal{A}_{\mathcal{E}}$ is of finite length and $E_0, E_1, \ldots, E_n$ are the simple objects in it. Bridgeland's Proposition 2.27 produces a family $\mathbb{H}^{\mathcal{A}_{\mathcal{E}}} \subset \mathrm{Stab}(\mathcal{T})$ (see Definition 2.28).

Definition 2.30. *Let $\mathcal{E}$ be a full Ext-exceptional collection and let $\mathcal{A}_{\mathcal{E}}$ be its extension closure. We write $\mathbb{H}^{\mathcal{E}}$ for $\mathbb{H}^{\mathcal{A}_{\mathcal{E}}}$ and denote by $\Theta'_{\mathcal{E}} \subset \mathrm{Stab}(\mathcal{T})$ the set obtained by acting on $\mathbb{H}^{\mathcal{E}}$ with $\widetilde{GL}^+(2, \mathbb{R})$.*

If $\mathcal{T}$ is of finite type, then starting with any full exceptional collection $\mathcal{E} = (E_0, E_1, \ldots, E_n)$ the collection $\mathcal{E}[p] = (E_0[p_0], E_1[p_1], \ldots, E_n[p_n])$ is Ext for some integer vector $p = (p_0, p_1, \ldots, p_n) \in \mathbb{Z}^{n+1}$ and to each such vector corresponds a subset $\Theta'_{\mathcal{E}[p]} \subset \mathrm{Stab}(\mathcal{T})$. E. Macrì denotes the union of these open subsets by $\Theta_{\mathcal{E}}$, and the union of the subsets $\{\Theta_{\mathcal{M}} : \mathcal{M} \text{ is a mutation of } \mathcal{E}\}$ by $\Sigma_{\mathcal{E}}$, i. e.

$$\Theta_{\mathcal{E}} = \bigcup_{\{p \in \mathbb{Z}^{n+1} : \mathcal{E}[p] \text{ is Ext}\}} \Theta'_{\mathcal{E}[p]} \subset \mathrm{Stab}(\mathcal{T}); \qquad \Sigma_{\mathcal{E}} = \bigcup_{\{\Theta_{\mathcal{M}} : \mathcal{M} \text{ is a mutation of } \mathcal{E}\}} \Theta_{\mathcal{M}}. \quad (2.12)$$

[22] HN property for $K_0(\mathcal{A}) \xrightarrow{Z} \mathbb{C}$ is defined in [8, Definition 2.3].

[23] If $\mathcal{A}$ has finite length and finitely many simple objects, then the obtained stability condition σ is locally finite.

Lemma 3.19 in [37] says that $\Theta_{\mathcal{E}}$ is an open, connected and simply connected subset of $\mathrm{Stab}(\mathcal{T})$, which implies (see [37, Corollary 3.20]) that, if all iterated mutations of $\mathcal{E}$ are regular,[24] then $\Sigma_{\mathcal{E}}$ is an open, connected subset of $\mathrm{Stab}(\mathcal{T})$.

The following proposition ensures extendability of certain stability conditions used in [17, Section 3]. The statement of Proposition 2.31 is a slight modification of the first part of [37, Proposition 3.17]. The difference is that in the statement of [37, Proposition 3.17] is claimed that one must take $\mathcal{E}_{ij} = (E_i, E_j)$, whereas we take $\mathcal{E}_{ij} = (E_i, E_{i+1}, \dots, E_j)$. For the sake of clarity, we give a proof of Proposition 2.31 here.

Proposition 2.31. *Let $\mathcal{E} = (E_0, E_1, \dots, E_n)$ be a full Ext-exceptional collection in $\mathcal{T}$. Let $0 \le i < j \le n$ and denote $\mathcal{E}_{ij} = (E_i, E_{i+1}, \dots, E_j)$, $\mathcal{T}_{ij} = \langle \mathcal{E}_{ij} \rangle \subset \mathcal{T}$. Let $\mathbb{H}^{\mathcal{E}_{ij}} \subset \mathrm{Stab}(\mathcal{T}_{ij})$, $\mathbb{H}^{\mathcal{E}} \subset \mathrm{Stab}(\mathcal{T})$ be the corresponding families as in Definition 2.30.*

Then the map $\pi_{ij} : \mathbb{H}^{\mathcal{E}} \to \mathbb{H}^{\mathcal{E}_{ij}}$, which assigns to $(\mathcal{P}, Z) \in \mathbb{H}^{\mathcal{E}}$ the unique $(\mathcal{P}', Z') \in \mathbb{H}^{\mathcal{E}_{ij}}$ with $\{Z'(E_k) = Z(E_k)\}_{k=i}^{j}$, is surjective. For any $(\mathcal{P}, Z) \in \mathbb{H}^{\mathcal{E}}$ and $(\mathcal{P}', Z') \in \mathbb{H}^{\mathcal{E}_{ij}}$ holds the implication

$$\pi_{ij}(\mathcal{P}, Z) = (\mathcal{P}', Z') \quad \Rightarrow \quad \{\mathcal{P}'(t) = \mathcal{P}(t) \cap \mathcal{T}_{ij}\}_{t \in \mathbb{R}}. \tag{2.13}$$

Proof. Using the definition of $\mathbb{H}^{\mathcal{E}}$, $\mathbb{H}^{\mathcal{E}_{ij}}$ (Definitions 2.28, 2.30), one easily reduces the proof of this proposition to the following lemma (compare with the proof of [37, Proposition 3.17, p.7]). □

Lemma 2.32. *Let $\mathcal{E}$, $\mathcal{E}_{ij}$ be as in Proposition 2.31. Let us denote by $\mathcal{A}$, $\mathcal{A}_{ij}$ the extension closures of $\mathcal{E}$ and $\mathcal{E}_{ij}$ in $\mathcal{T}$. Then $\mathcal{A}_{ij}$ is an exact Serre subcategory of $\mathcal{A}$. In particular the embedding functor induces an embedding $K_0(\mathcal{A}_{ij}) \to K_0(\mathcal{A})$.*

Proof. Since both $\mathcal{A}, \mathcal{A}_{ij}$ are abelian categories ([37, Lemma 3.14]), if $\mathcal{A}_{ij}$ is a Serre subcategory of $\mathcal{A}$ it follows that $\mathcal{A}_{ij}$ is an exact subcategory. Whence, it is enough to show that $\mathcal{A}_{ij}$ is a Serre subcategory. Let $0 \to B_1 \to S \to B_2 \to 0$ be any short exact sequence in $\mathcal{A}$.

Assume that $B_1, B_2 \in \mathcal{A}_{ij}$. Since $\mathcal{A}$ is a heart of bounded t-structure[25] in $\mathcal{T}$, the given short exact sequence is part of a triangle in $\mathcal{T}$. Since $\mathcal{A}_{ij}$ is extension closed in $\mathcal{T}$, it follows $S \in \mathcal{A}_{ij}$.

Next, assume that $S \in \mathcal{A}_{ij}$. We have to show that $B_1, B_2 \in \mathcal{A}_{ij}$. By $B_1, B_2 \in \mathcal{A}$ and the definition of $\mathcal{A}$, we have diagrams of short exact sequences in $\mathcal{A}$ for $l = 1, 2$(the superscript is a power of E_i):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U_{l,n} & \longrightarrow & U_{l,n-1} & \longrightarrow \dots \longrightarrow & U_{l,1} & \longrightarrow & U_{l,0} = B_l \\ & \nearrow & & \swarrow & & & & \swarrow & \\ E_n^{p_{l,n}} & & E_{n-1}^{p_{l,n-1}} & & & & E_0^{p_{l,0}} & & \end{array} \qquad l = 1, 2. \tag{2.14}$$

From $S \in \mathcal{A}_{ij}$ it follows $\mathrm{Hom}^*(S, E_l) = 0$ for $l < i$ and $\mathrm{Hom}^*(E_l, S) = 0$ for $l > j$. Since we have $\mathcal{A}$-epic arrows $S \to B_2$, $B_2 \to E_0^{p_{2,0}}$ and $\mathcal{A}$-monic arrows $E_0^{p_{1,n}} \to B_1$, $B_1 \to S$, it follows that

[24] Here *regular* means that for $0 \le i \le n-1$ at most one degree in $\{\mathrm{Hom}^p(E_i, E_{i+1}) = 0\}_{p \in \mathbb{Z}}$ does not vanish.

[25] Recall that the short exact sequences in a heart of a t-structure $\mathcal{A}$ are exactly those sequences $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ with $A, B, C \in \mathcal{A}$, s. t. for some $\gamma : C \to A[1]$ the triangle $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]$ is distinguished in $\mathcal{T}$.

$p_{2,0} = 0$, if $0 < i$ and $p_{1,n} = 0$, if $n > j$. Now by induction it follows:

$$p_{2,k} = 0 \text{ for } k < i, \qquad p_{1,k} = 0 \text{ for } k > j. \tag{2.15}$$

We show bellow that $\mathrm{Hom}(E_k, B_2) = 0$ for $k > j$ and $\mathrm{Hom}(B_1, E_k) = 0$ for $k < i$. Since there exist $\mathcal{A}$-monic $E_n^{p_{2,n}} \to B_2$ and $\mathcal{A}$-epic $B_1 \to E_0^{p_{1,0}}$, by the diagrams (2.14) and induction we obtain $p_{2,k} = 0$ for $k > j$, $\quad p_{1,k} = 0$ for $k < i$. These vanishings together with (2.15) imply the lemma.

Having (2.14) and (2.15) we can write $B_2 \in \langle E_i, E_{i+1}, \dots, E_n \rangle$ and $B_1 \in \langle E_0, E_1, \dots, E_j \rangle$, hence

$$\mathrm{Hom}^*(B_2, E_k) = \mathrm{Hom}^*(S, E_k) = 0 \text{ for } k < i, \mathrm{Hom}^*(E_k, B_1) = \mathrm{Hom}^*(E_k, S) = 0 \text{ for } k > j. \tag{2.16}$$

From the short exact sequence $0 \to B_1 \to S \to B_2 \to 0$ in $\mathcal{A}$ we get a distinguished triangle $B_1 \to S \to B_2 \to B_1[1]$ in $\mathcal{T}$. Since we have (2.16), applying to this triangle $\mathrm{Hom}(E_k, \_)$ and $\mathrm{Hom}(\_, E_k)$ we obtain the desired $\mathrm{Hom}(E_k, B_2) = 0$ for $k > j$, $\mathrm{Hom}(B_1, E_k) = 0$ for $k < i$. □

2.3.4 σ-exceptional collections

Motivated by the work of E. Macrì, discussed in the introduction and in the previous Subsection 2.3.3, we define:

Definition 2.33. *Let $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}(\mathcal{T})$. We call an exceptional collection $\mathcal{E} = (E_0, E_1, \dots, E_n)$ σ-exceptional collection if the following properties hold:*

- *$\mathcal{E}$ is semistable w. r. to σ (i. e. all E_i are semistable).*
- *$\forall i \neq j$ $\hom^{\leq 0}(E_i, E_j) = 0$ (i. e. this is an Ext-exceptional collection).*
- *There exists $t \in \mathbb{R}$, s. t. $\{\phi(E_i)\}_{i=0}^n \subset (t, t+1]$.*

The set stability conditions for which $\mathcal{E}$ is σ-exceptional coincides with $\Theta'_{\mathcal{E}} = \mathbb{H}^{\mathcal{E}} \cdot \widetilde{GL}^+(2, \mathbb{R})$ (Definition 2.30). More precisely, we have:

Corollary 2.34 (of Lemmas 3.14, 3.16 in [37])**.** *Let $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}(\mathcal{T})$. Let $\mathcal{E}$ be a full Ext-exceptional collection in $\mathcal{T}$. Then we have the equivalences:*

$\sigma \in \Theta'_{\mathcal{E}} \iff \mathcal{E} \subset \mathcal{P}(t, t+1]$ *for some* $t \in \mathbb{R} \iff \mathcal{E}$ *is a σ-exceptional collection.*

Proof. First, note [37, Lemma 3.16] that from $\{E_i\}_{i=0}^n \subset \mathcal{P}((t, t+1])$ it follows $\mathcal{A}_{\mathcal{E}} = \mathcal{P}((t, t+1])$, and then all $\{E_i\}_{i=0}^n$ are stable in σ, because they are simple in $\mathcal{A}_{\mathcal{E}} = \mathcal{P}((t, t+1])$. Indeed, $\mathcal{A}_{\mathcal{E}}$ and $\mathcal{P}((t, t+1])$ are both bounded t-structures, therefore the inclusion $\mathcal{A}_{\mathcal{E}} \subset \mathcal{P}((t, t+1])$ implies equality $\mathcal{A}_{\mathcal{E}} = \mathcal{P}((t, t+1])$. Whence, if $\{E_i\}_{i=0}^n \subset \mathcal{P}((t, t+1])$, then $\mathcal{E}$ is σ-exceptional (see Definition 2.33).

Now the corollary follows from the last part of Bridgeland's Proposition 2.27 and the following comments on the action of $\widetilde{GL}^+(2, \mathbb{R})$. If $(\widetilde{\mathcal{P}}, \widetilde{Z})$ is obtained by the action with $\widetilde{GL}^+(2, \mathbb{R})$ on $(\mathcal{P}, Z)$, then $\{\widetilde{\mathcal{P}}(\psi(t)) = \mathcal{P}(t)\}_{t \in \mathbb{R}}$ for some strictly increasing smooth function $\psi : \mathbb{R} \to \mathbb{R}$ with $\psi(t+1) = \psi(t)+1$, and hence $\mathcal{P}(0, 1] = \widetilde{\mathcal{P}}(\psi(0), \psi(0)+1]$. Conversely, for any $t \in \mathbb{R}$ and any $(\mathcal{P}, Z)$ we can act on it with element in $\widetilde{GL}^+(2, \mathbb{R})$, so that the resulting $(\widetilde{\mathcal{P}}, \widetilde{Z})$ satisfies $\mathcal{P}(t, t+1] = \widetilde{\mathcal{P}}(0, 1]$. □

Since the exceptional collection $\mathcal{E}$ in Definition 2.33 has finite length, we have:

Remark 2.35. *The third condition in Definition 2.33 is equivalent to each of the following three conditions:* $\{\phi(E_i)\}_{i=1}^n \subset (t, t+1)$ *for some* $t \in \mathbb{R}$; $\{\phi(E_i)\}_{i=1}^n \subset [t, t+1)$ *for some* $t \in \mathbb{R}$; $\max(\{\phi(E_i)\}_{i=0}^n) - \min(\{\phi(E_i)\}_{i=0}^n) < 1$.

Furthermore, by Corollary 2.34 we have $\Theta'_{\mathcal{E}} = \left\{\sigma : \max\{\phi_+^\sigma(E_i)\}_{i=0}^n - \min\{\phi_-^\sigma(E_i)\}_{i=0}^n < 1\right\} = \{\sigma : \mathcal{E} \subset \sigma^{ss}$ *and* $|\phi^\sigma(E_i) - \phi^\sigma(E_j)| < 1$ *for* $i < j\}$, *therefore*[26] $\Theta'_{\mathcal{E}}$ *is an open subset of* $\mathrm{Stab}(\mathcal{T})$.

One can now easily show that the assignment:

$$\Theta'_{\mathcal{E}} \ni \sigma = (\mathcal{P}, Z) \quad \mapsto \quad (|Z(E_0)|, \dots, |Z(E_n)|, \phi^\sigma(E_0), \dots, \phi^\sigma(E_n))$$

is well defined, and gives a homeomorphism between $\Theta'_{\mathcal{E}}$ *and the following simply connected set:*

$$\left\{(x_0, \dots, x_n, y_0, \dots, y_n) \in \mathbb{R}^{2(n+1)} \quad : \quad x_i > 0, \ |y_i - y_j| < 1\right\}.$$

From the first part of this remark and Corollary 2.34 we see that for each $\sigma \in \Theta'_{\mathcal{E}}$ we have an open interval, in which $\mathcal{P}(x)$ is trivial (take $t \in \mathbb{R}$ and $\epsilon > 0$ so that $\{\phi(E_i)\}_{i=0}^n \subset (t, t+1] \cap (t+\epsilon, t+\epsilon+1]$, then $(t, t+\epsilon)$ is such an interval). In particular(recall also that $\mathcal{P}(x)[1] = \mathcal{P}(x+1)$), we have:

Remark 2.36. *Let* $\mathcal{E}$ *be as in Corollary 2.34. For each* $\sigma \in \Theta'_{\mathcal{E}}$ *the set*[27] P_σ *is not dense in* $\mathbb{S}^1$.

2.4 Non-semistable exceptional objects in hereditary abelian categories

In this section is written an algorithm, denoted by $\mathfrak{alg}$. In subsection 2.4.1 we define the input data of the algorithm, in subsection 2.4.2 - the data at the output. The rest sections of the text refer mainly to subsections 2.4.1 and 2.4.2.

2.4.1 Presumptions

For the rest of the chapter $\mathcal{A}$ is an abelian hereditary hom-finite category, linear over an algebraically closed field k.[28] It can be shown[29] that such a category has Krull-Schmidt property(Definition 2.14). Hence, by Lemma 2.15, the derived category $D^b(\mathcal{A})$ also satisfies the Krull-Schmidt property. For brevity, we set $\mathcal{T} = D^b(\mathcal{A})$. Let $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}(\mathcal{T})$ be a stability condition. In this setting by Definition 2.22 we obtain the function $\theta_\sigma : Ob(\mathcal{T}) \to \mathbb{N}^{(\sigma^{ss}_{ind}/\cong)}$.

The input data of the algorithm $\mathfrak{alg}$ *is a non-semistable w. r. to* σ *exceptional object* $E \in \mathcal{T}$. *The output data is a triangle, denoted by* $\mathfrak{alg}(E)$. *We distinguish five cases at the output, depending*

[26] For a a fixed nonzero object $X \in \mathcal{T}$ the functions $\sigma \mapsto \phi_\pm^\sigma(X)$ on the manifold $\mathcal{T}$ are continuous

[27] see Definition 3.1 for the notation P_σ

[28] In all the sections 2.4, 2.5, 2.6, 2.7, 2.8, 2.9 the symbol $\mathcal{A}$ denotes such a category.

[29] using some facts for modules over unital associative ring shown around page 302 of [35], see also [36]

on the features of the triangle $\mathfrak{alg}(E)$*, and denote them by* ***C1, C2, C3, B1, B2****. Only one of the five possible cases can occur at the output, i. e.* $\mathfrak{alg}(E)$ *has all the features of exactly one case, say* $X \in \{$***C1, C2, C3, B1, B2***$\}$*, and then* $\mathfrak{alg}(E)$ *is said to be of type* X*.*

We note two facts, which we keep in mind further.

Remark 2.37. *It can be shown*[30] *that, under the given assumptions on* $\mathcal{A}$*, if* $X \in \mathcal{A}_{ind}$ *satisfies* $\mathrm{Ext}^1(X,X) = 0$*, then* $\mathrm{Hom}(X,X) = k$*, and hence* X *is an exceptional object.*

Remark 2.38. *Since* $\mathcal{A}$ *is a hereditary category, for any two indecomposable* $A, B \in D^b(\mathcal{A})$ *with* $\deg(A) = \deg(B)$ *from* $\phi_-(A) > \phi_+(B) + 1$ *it follows that* $\mathrm{Hom}^*(A,B) = 0$.

Another simple observation due to hereditariness, which we will apply throughout, is:

Lemma 2.39. *Let* $\mathcal{A}$ *be a hereditary abelian category and let* $0 \to X \to Y \to Z \to 0$ *be a short exact sequence in* $\mathcal{A}$*. For each* $W \in \mathcal{A}$ *hold the following implications:*

(a) *If* $\hom^1(Y,W) = 0$*, then* $\hom^1(X,W) = 0$

(b) *If* $\hom^1(W,Y) = 0$*, then* $\hom^1(W,Z) = 0$.

Proof. To prove **(a)** we apply $\mathrm{Hom}(\_, W[1])$ to the triangle $X \to Y \to Z \to X[1]$, corresponding to the given exact sequence. It follows $0 = \mathrm{Hom}(Y, W[1]) \to \mathrm{Hom}(X, W[1]) \to \mathrm{Hom}(Z[-1], W[1]) = 0$, where the right vanishing is because $\mathcal{A}$ is hereditary. In **(b)** we apply $\mathrm{Hom}(W, \_)$. □

We could work here with weaker assumptions on $\mathcal{A}$. More precisely:

Remark 2.40. *Given that* $\mathcal{A}$ *is a hereditary* k*-linear abeilan category with Krull Schmidt property as defined in Definition 2.14, without assuming* hom*-finiteness and that* k *is algebraically closed, then everything in Sections 2.4, 2.5, 2.6, 2.7, 2.8, 2.9 remains valid, if we replace "exceptional" by "pre-exceptional".*[31] *Under such seemingly weaker assumptions on* $\mathcal{A}$*, we do not have the statement in Remark 2.37.*

2.4.2 The cases

Here we explain the features of each of the five cases **C1, C2, C3, B1, B2** occurring at the output of $\mathfrak{alg}$. The other subsections of 2.4 contain the algorithm.

Let $E \in \mathfrak{T}$ be a non-semistable w. r. to σ exceptional object. We recall that the meaning of the notation $\deg(E)$, used here, is explained in tSection 0.1 before the introduction. The properties (a),(b),(c) below are common features of $\mathfrak{alg}(E)$ for all the cases, property (d) is common for **C1, C2, C3**:

[30]by adapting the proof of this fact for quivers, given on [16, p. 9,10], to $\mathcal{A}$

[31]By *Pre-exceptional object* we mean an indecomposable object $X \in \mathfrak{T}$ with $\mathrm{Hom}^i(X,X) = 0$ for $i \neq 0$. *Pre-exceptional collection* is a sequence of pre-exceptional objects $(E_1, E_2, \ldots, E_n)$ with $\mathrm{Hom}^*(E_i, E_j) = 0$ for $i > j$.

$$\mathfrak{alg}(E) = \begin{array}{ccc} U & \longrightarrow & E \\ & \nwarrow \quad \swarrow & \\ & V & \end{array} \qquad U \in \mathfrak{T}, V \in \sigma^{ss}, U \neq 0, V \neq 0, \text{ where:} \tag{2.17}$$

(a) V is the degree j component of[32] $\sigma_-(E)$, where $j \in \{\deg(E), \deg(E)+1\}$.

(b) $\theta_\sigma(U) < \theta_\sigma(E) \quad \Rightarrow \quad \phi_-(U) \geq \phi(V) = \phi_-(E)$. [33] [34]

(c) Any $\Gamma \in Ind(V)$ satisfies $\hom(E, \Gamma) \neq 0$ (see Definition 2.14 for the notation $Ind(V)$).

(d) In the cases **C1, C2, C3** hold the vanishings $\hom^*(U,V) = \hom^1(U,U) = \hom^1(V,V) = 0$, in particular for any $S \in Ind(V)$, $E' \in Ind(U)$ the pair (S, E') is exceptional with $S \in \sigma^{ss}$.

We give now the complete lists of properties. For simplicity we assume that $E \in \mathcal{A}$, i. e. $\deg(E) = 0$, for other degrees everything is shifted with the corresponding number.

C1. *The triangle is of the form* $\mathfrak{alg}(E) = \begin{array}{ccc} A & \longrightarrow & E \\ & \nwarrow \quad \swarrow & \\ & B & \end{array}$ *with the properties:*

C1.1 $\{A, B\} \subset \mathcal{A}$, $A \neq 0$, $B \neq 0$, $\hom^1(A,A) = \hom^1(B,B) = \hom^*(A,B) = 0$,

C1.2 *B is the zero degree component of $\sigma_-(E)$, in particular B is semistable of phase $\phi_-(E)$,*

C1.3 $\theta_\sigma(A) < \theta_\sigma(E) \Rightarrow \phi_-(A) \geq \phi_-(E)$,

C1.4 *any $\Gamma \in Ind(A)$ satisfies $\hom^1(B, \Gamma) \neq 0$.*

C2. *The triangle is of the form*

$$\mathfrak{alg}(E) = \begin{array}{ccc} A_1 \oplus A_2[-1] & \longrightarrow & E \\ & \nwarrow \quad \swarrow & \\ & B & \end{array} \tag{2.18}$$

with the properties:

C2.1 $\{A_1, A_2, B\} \subset \mathcal{A}$, $A_2 \neq 0$, $B \neq 0$, *A_1 is a proper sub-object(in $\mathcal{A}$) of E,* $\hom^1(A_2, A_2) = \hom^1(A_1, A_1) = \hom^*(A_1, B) = \hom^*(A_2, B) = \hom^*(A_1, A_2) = 0$,

C2.2 *B is the zero degree component of $\sigma_-(E)$, in particular B is semistable of phase $\phi_-(E)$,*

C2.3 $\theta_\sigma(A_1) + \theta_\sigma(A_2[-1]) < \theta_\sigma(E)$, *in particular* $\phi_-(A_1) \geq \phi_-(E)$ *and* $\phi_-(A_2[-1]) \geq \phi_-(E)$,

[32] $\sigma_-(E)$ is defined in Definition 2.18

[33] We write $f < g$ for two functions $f, g \in \mathbb{N}^{(\sigma^{ss}_{ind}/\cong)}$, if $f(u) < g(u)$ for some $u \in \sigma^{ss}_{ind}/\cong$.

[34] Note below that in cases **C3, B2** we have proper inequality $\phi_-(U) > \phi(V)$.

C2.4 *any* $\Gamma \in Ind(A_1)$ *satisfies* $\hom(B, \Gamma[1]) \neq 0$, *any* $\Gamma \in Ind(A_2)$ *satisfies the three conditions:* $\hom(B, \Gamma) \neq 0$, $\hom(\Gamma, E[1]) \neq 0$, $\hom(E, \Gamma[1]) = 0$.

C3. *The triangle is of the form* $\mathfrak{alg}(E) = \begin{array}{ccc} A & \longrightarrow & E \\ & \nwarrow \quad \swarrow & \\ & B[1] & \end{array}$ *with the properties:*

C3.1 $\{A, B\} \subset \mathcal{A}$, $A \neq 0$, $B \neq 0$, $\hom^1(A, A) = \hom^1(B, B) = \hom^*(A, B) = 0$,

C3.2 $\mathfrak{alg}(E) \cong \mathbb{HN}_-(E)$, *hence* $\theta_\sigma(A) < \theta_\sigma(E)$ *and* $\phi_-(A) > \phi_-(E) = \phi(B) + 1$,

C3.3 *any* $\Gamma \in Ind(B)$ *satisfies* $\hom^1(E, \Gamma) \neq 0$ *and* $\hom^1(\Gamma, E) = 0$, *any* $\Gamma \in Ind(A)$ *satisfies* $\hom(B, \Gamma) \neq 0$ *and* $\hom(\Gamma, E) \neq 0$.

B1. *The triangle is of the form* $\mathfrak{alg}(E) = \begin{array}{ccc} A_1 \oplus A_2[-1] & \longrightarrow & E \\ & \nwarrow \quad \swarrow & \\ & B & \end{array}$ *with the properties:*

B1.1 $\{A_1, A_2, B\} \subset \mathcal{A}$, $A_2 \neq 0$, $B \neq 0$, $\hom^1(A_2, A_2) = \hom^1(A_1, A_1) = \hom^*(A_2, B) = 0$, A_1 *is a proper subobject(in* $\mathcal{A}$*) of* E,

B1.2 B *is the zero degree component of* $\sigma_-(E)$, *in particular* B *is semistable of phase* $\phi_-(E)$,

B1.3 $\theta_\sigma(A_1) + \theta_\sigma(A_2[-1]) < \theta_\sigma(E)$, *in particular* $\phi_-(A_1) \geq \phi_-(E)$ *and* $\phi_-(A_2[-1]) \geq \phi_-(E)$,

B1.4 *there exists* $\Gamma \in Ind(A_2)$ *with* $\hom^1(\Gamma, E) \neq 0$, $\hom^1(E, \Gamma) \neq 0$.[35]

B2. *The triangle is of the form* $\mathfrak{alg}(E) = \begin{array}{ccc} A & \longrightarrow & E \\ & \nwarrow \quad \swarrow & \\ & B[1] & \end{array}$ *with the properties:*

B2.1 $\{A, B\} \subset \mathcal{A}$, $A \neq 0$, $B \neq 0$, $\hom^1(B, B) = \hom^*(A, B) = 0$,

B2.2 $\mathfrak{alg}(E) \cong \mathbb{HN}_-(E)$, *hence* $\theta_\sigma(A) < \theta_\sigma(E)$ *and* $\phi_-(A) > \phi_-(E) = \phi(B) + 1$,

B2.3 *there exists* $\Gamma \in B$ *with* $\hom^1(\Gamma, E) \neq 0$, $\hom^1(E, \Gamma) \neq 0$.[36]

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[35] A comparison with **C2.4** shows that **B1** and **C2** cannot appear together.
[36] A comparison with **C3.3** shows that **B2** and **C3** cannot appear together.

2.4.3 The last HN triangle

Now we start explaining $\mathfrak{alg}$.

Let $E \in \mathcal{A}_{exc}$, $E \not\in \sigma^{ss}$. Macrì initiated in [37, p. 10] an analysis of the last HN triangle of E, when $E \in Rep_k(K(l))$. The arguments on [37, p. 10] are used here in formulas (2.20), (2.21), and in the derivation of the vanishings **C3.1**(Subsection 2.4.4).

Consider the last HN triangle $\mathrm{HN}_-(E)$(see Definition 2.18):

$$\mathrm{HN}_-(E) \;=\; X \longrightarrow E \xrightarrow{f} \sigma_-(E) \longrightarrow X[1], \qquad \phi_-(X) > \phi(\sigma_-(E)) = \phi_-(E). \tag{2.19}$$

Lemma 2.41. *The triangle* $\mathrm{HN}_-(E)$ *is of the form (with* $B_0, B_1 \in \mathcal{A}$*):*

$$X \longrightarrow E \xrightarrow{f} B_0 \oplus B_1[1] \longrightarrow X[1], \quad \phi_-(X) > \phi(B_0) = \phi(B_1) + 1 = \phi_-(E), \tag{2.20}$$

$$\hom^{\leq 0}(X, B_0) = \hom^{\leq 0}(X, B_1[1]) = 0 \tag{2.21}$$

$$\theta_\sigma(E) = \theta_\sigma(X) + \theta_\sigma(B_0) + \theta_\sigma(B_1[1]). \tag{2.22}$$

For any $i \in \{0,1\}$*,* $\Gamma \in Ind(B_i)$ *the component of* f *to* $\Gamma[i]$ *is non-zero and* $\hom(E, \Gamma[i]) \neq 0$*.*
Any $\Gamma \in Ind(X)$ *satisfies* $\hom(\Gamma, E) \neq 0$ *and* $\hom(B_0 \oplus B_1[1], \Gamma[1]) \neq 0$*.*

Proof. We show first that for each $\Gamma \in Ind(\sigma_-(E))$ the component of f from E to Γ is non-zero.

Indeed, suppose that for some $\Gamma \in Ind(\sigma_-(E))$ this component vanishes, then by the Krull-Schmidt property we can write $\sigma_-(E) = U \oplus \Gamma$, and f is of the form: $f = (f' : E \to U) \oplus (0 \to \Gamma)$. After summing the triangles $X' \longrightarrow E \xrightarrow{f'} U \longrightarrow X'[1]$ and $\Gamma[-1] \longrightarrow 0 \longrightarrow \Gamma \longrightarrow \Gamma$ we obtain a triangle $X' \oplus \Gamma[-1] \longrightarrow E \xrightarrow{f} \sigma_-(E) \longrightarrow X'[1] \oplus \Gamma$ (recall that $E \neq \sigma^{ss}$, hence $X' \neq 0$). From (2.19) it follows that $X' \oplus \Gamma[-1] \cong X$. From Corollary 2.21 we see that $\phi_-(X') \geq \phi_-(X) > \phi_-(E) = \phi(U)$. By this inequality and the uniqueness of the HN filtration of E we deduce that $\sigma_-(E) \cong U$, i. e. $U \oplus \Gamma \cong U$, which contradicts the Krull-Schmidt property.

Thus, for each $\Gamma \in Ind(\sigma_-(E))$ the component of f to Γ is non-zero and $\hom(E, \Gamma) \neq 0$. Now the triangle (2.19) reduces to (2.20), since $\mathcal{A}$ is hereditary. From $\phi_-(X) > \phi(B_i[i])$ $(i = 0, 1)$ it follows (2.21). Applying Lemmas 2.23, 2.25 to (2.20) we obtain (2.22). It remains to prove the last property.

Suppose that $\hom(\Gamma, E) = 0$ for some $\Gamma \in Ind(X)$. Then we can represent $X \to E$ as a direct sum $(U \to E) \oplus (\Gamma \to 0)$. By the triangle (2.19) we get $Y' \oplus \Gamma[1] \cong \sigma_-(E)$, where Y' is the cone of $U \to E$. From Corollary 2.21 we see $\phi_-(U) \geq \phi_-(X) > \phi_-(E) = \phi(Y')$. Since $\phi_-(U) > \phi_-(E)$, we have $U \not\cong E$ and $Y' \neq 0$. It follows that $\mathrm{HN}_-(E) = U \longrightarrow E \longrightarrow Y' \longrightarrow U[1]$. Therefore $X \cong U$, i. e. $U \oplus \Gamma \cong U$, which contradicts the Krull-Schmidt property.

Suppose that for some $\Gamma \in Ind(X)$ we have $\hom(B_0 \oplus B_1[1], \Gamma[1]) = 0$, then by similar arguments we get $E \cong E' \oplus \Gamma$, and hence $\Gamma \cong E$ (since E is indecomposable), which contradicts $\phi_-(\Gamma) \geq \phi_-(X) > \phi_-(E)$. The lemma is proved. □

By f_i will be denoted the component of f to $B_i[i]$(see (2.20)), i. e. we have commutative diagrams (the right arrow is the projection)

$$\begin{array}{ccc} E & \xrightarrow{f} & B_0 \oplus B_1[1] \\ {\scriptstyle Id}\downarrow & & \downarrow \\ E & \xrightarrow{f_i} & B_i[i] \end{array} \qquad i \in \{0,1\}. \tag{2.23}$$

The algorithm $\mathfrak{alg}$ tests now the condition $B_0 = 0$.

2.4.4 If $B_0 = 0$

This condition leads to one of the cases **C3, B2** depending on the outcome of one test. Since $B_0 = 0$, the triangle (2.20) is reduced to a short exact sequence $0 \longrightarrow B_1 \longrightarrow X \longrightarrow E \longrightarrow 0$, and $X \in Ob(\mathcal{A})$. Hence (2.21) is now the same as $\hom^*(X, B_1) = 0$, which by Lemma 2.39 **(a)** and the given exact sequence implies $\hom^1(B_1, B_1) = 0$. By Lemma 2.41 any $\Gamma \in Ind(B_1)$ satisfies $\hom(E, \Gamma[1]) \neq 0$. Therefore, if $\hom(\Gamma, E[1]) \neq 0$ for some $\Gamma \in Ind(B_1)$, then the triangle:

$$\mathrm{HN}_-(E) = \begin{array}{ccc} X & \longrightarrow & E \\ & \nwarrow \quad \swarrow & \\ & B_1[1] & \end{array} \tag{2.24}$$

satisfies **B2.1, B2.2, B2.3** (with $A = X, B = B_1$). By setting $\mathfrak{alg}(E)$ to (2.24) we get **B2**.

It remains to consider the case when $\hom(\Gamma, E[1]) = 0$ for each $\Gamma \in Ind(B_1)$, i. e.

$$\hom(B_1, E[1]) = 0. \tag{2.25}$$

Setting again $\mathfrak{alg}(E)$ to (2.24)(with X replaced by A, B_1 replaced by B) we obtain the property **C3.2** immediately. The property **C3.3** follows from Lemma 2.41. We have already all the features of **C3.1** except the vanishing $\hom^1(X, X) = 0$.

The vanishing $\hom^1(X, X) = 0$ follows from (2.25), since the triangle (2.24) and $\mathrm{Hom}(X, \_)$ give an exact sequence $\mathrm{Hom}^1(X, B_1) \to \mathrm{Hom}^1(X, X) \to \mathrm{Hom}^1(X, E)$, where the left and the right terms vanish. The vanishing $\mathrm{Hom}^1(X, B_1) = 0$ is already shown (before (2.24)). The other vanishing $\hom^1(X, E) = 0$ follows from (2.25), $\hom^1(E, E) = 0$, and $\mathrm{Hom}(\_, E[1])$ applied to the same triangle.

Thus, $\mathfrak{alg}(E)$ is of type **C3**.

2.4.5 If $B_0 \neq 0$

Under this condition we obtain one of the cases **C1, C2, B1** at the output depending on the outcomes of additional tests.

By Lemma 2.41 we have $f_0 \neq 0$. Let us take kernel and cokernel of f_0 in $\mathcal{A}$:

$$A_1 \xrightarrow{ker(f_0)} E \xrightarrow{f_0} B_0 \xrightarrow{coker(f_0)} A_2. \tag{2.26}$$

Since $f_0 \neq 0$, $ker(f_0)$ is a proper subobject of E. Let $E \xrightarrow{e_0} B_0' \xrightarrow{im(f_0)} B_0$ be a decomposition of f_0 in $\mathcal{A}$, where e_0 is $\mathcal{A}$-epic and $im(f_0)$ is $\mathcal{A}$-monic. In particular, we have an exact sequence in $\mathcal{A}$

$$0 \longrightarrow A_1 \xrightarrow{ker(f_0)} E \xrightarrow{e_0} B_0' \longrightarrow 0. \tag{2.27}$$

The next step of the algorithm $\mathfrak{alg}$ is to test the condition $A_2 = 0$. We show first some preliminary facts, which do not depend on the vanishing of A_2.

Preliminary facts

These facts are (2.28),(2.29),(2.30), (2.31), and Lemma 2.42.

The equalities below will help us later to obtain **C1.1**, when $A_2 = 0$, and **C2.1**, when $A_2 \neq 0$:

$$\hom^1(A_1, A_1) = \hom^1(A_2, A_2) = 0 \tag{2.28}$$
$$\hom(A_1, B_0) = \hom^*(A_2, B_0) = 0 \tag{2.29}$$
$$\hom^1(A_1, E) = \hom(A_1, A_2) = 0 \tag{2.30}$$

The inequality (2.31) ensures **C1.3** and **C2.3**, and Lemma 2.42 ensures **C1.4** and half of **C2.4**.

$$\theta_\sigma(A_1) + \theta_\sigma(A_2[-1]) < \theta_\sigma(E). \tag{2.31}$$

To show these facts we start by recalling that the triangle in $\mathfrak{T}$ containing f_0 is

$$E \xrightarrow{f_0} B_0 \longrightarrow C(f_0) \longrightarrow E[1] \tag{2.32}$$

where $C(f_0)$ is the cochain complex (B_0 is in degree 0)

$$\dots \longrightarrow 0 \longrightarrow E \xrightarrow{f_0} B_0 \longrightarrow 0 \longrightarrow \dots \tag{2.33}$$

and the non-trivial part of the cochain maps $B_0 \to C(f_0) \to E[1]$ is $\begin{array}{ccccc} 0 & \longrightarrow & E & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ B_0 & \longrightarrow & B_0 & \longrightarrow & 0 \end{array}$.

Since $\mathcal{A}$ is hereditary, we have $C(f_0) \cong \bigoplus_i H^i(C(f_0))[-i]$, which we can reduce by (2.26) and (2.33) to

$$C(f_0) \cong A_1[1] \oplus A_2. \tag{2.34}$$

Since we have the commutative diagram (2.23) with $i = 0$, by the 3×3 lemma in triangulated categories [2, Proposition 1.1.11] we can put the triangles (2.20), (2.32) in a diagram

$$
\begin{array}{ccccccc}
E & \xrightarrow{f} & B_0 \oplus B_1[1] & \longrightarrow & X[1] & \longrightarrow & E[1] \\
{\scriptstyle Id}\downarrow & & \downarrow & & \downarrow & & {\scriptstyle Id}\downarrow \\
E & \xrightarrow{f_0} & B_0 & \longrightarrow & C(f_0) & \longrightarrow & E[1] \\
\downarrow & & {\scriptstyle 0}\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & B_1[2] & \longrightarrow & Y & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E[1] & \xrightarrow{f_0} & B_0[1] \oplus B_1[2] & \longrightarrow & X[2] & \longrightarrow & E[1]
\end{array}
$$

where $X[1] \longrightarrow C(f_0) \longrightarrow Y \longrightarrow X[2]$; $0 \longrightarrow B_1[2] \longrightarrow Y \longrightarrow 0$ are distinguished triangles. Hence $Y \cong B_1[2]$ and we obtain a distinguished triangle

$$X \longrightarrow C(f_0)[-1] \longrightarrow B_1[1] \longrightarrow X[1]. \tag{2.35}$$

The vanishings (2.28),(2.29),(2.30) will be obtained from triangles (2.35),(2.32), and the exact sequence (2.27).

We apply $\mathrm{Hom}(-, B_0)$ and $\mathrm{Hom}(-, B_0[-1])$ to (2.35) and by (2.21) the result is: $\mathrm{Hom}(C(f_0), B_0[1]) = \mathrm{Hom}(C(f_0), B_0) = 0$. These vanishings and (2.34) imply (2.29). The vanishing $\hom^1(A_1, E) = 0$ (the first part of (2.30)) follows from $\hom^1(E, E) = 0$, the exact sequence (2.27), and Lemma 2.39 **(a)**. Now we can write $\hom(C(f_0), E[2]) = \hom(A_1[1] \oplus A_2, E[2]) = \hom^1(A_1, E) = 0$. Having $0 = \hom(C(f_0), B_0[1]) = \hom(C(f_0), E[2])$, we apply $\mathrm{Hom}(C(f_0), \_)$ to (2.32) and obtain

$$0 = \mathrm{Hom}(C(f_0), B_0[1]) \to \mathrm{Hom}(C(f_0), C(f_0)[1]) \to \mathrm{Hom}(C(f_0), E[2]) = 0.$$

Hence $\hom(A_1[1] \oplus A_2, A_1[2] \oplus A_2[1]) = 0$, which contains (2.28) and the second vanishing in (2.30).

The next step is to show (2.31). From Lemma 2.23 and the triangle (2.35) we get $\theta_\sigma(C(f_0)[-1]) = \theta_\sigma(X) + \theta_\sigma(B_1[1])$. From $B_0 \neq 0$ it follows $\theta_\sigma(B_0) > 0$, and hence:

$$\theta_\sigma(C(f_0)[-1]) = \theta_\sigma(X) + \theta_\sigma(B_1[1]) < \theta_\sigma(X) + \theta_\sigma(B_1[1]) + \theta_\sigma(B_0) = \theta_\sigma(E),$$

where the last equality is taken from (2.22). Now (2.31) follows from (2.34).

Since $\mathfrak{alg}(E)$ in both the cases $A_2 = 0$ and $A_2 \neq 0$ will be set to (2.32), the following corollary ensures **C1.4**, and part of **C2.4**.

Lemma 2.42. *Each* $\Gamma \in Ind(C(f_0)) = Ind(A_1[1] \oplus A_2)$ *satisfies* $\hom(B_0, \Gamma) \neq 0$*, and each* $\Gamma \in Ind(A_2)$ *satisfies* $\hom(\Gamma, E[1]) \neq 0$.

Proof. Suppose that $\hom(B_0, \Gamma) = 0$ for some $\Gamma \in Ind(C(f_0))$ and split $C(f_0) = U \oplus \Gamma$, then the arrow $B_0 \to C(f_0)$ in (2.32) can be represented as $(B_0 \to U) \oplus (0 \to \Gamma)$. The sum of the triangle $E' \longrightarrow B_0 \longrightarrow U \longrightarrow E'[1]$ extending $B_0 \to U$ and the triangle $\Gamma[-1] \longrightarrow 0 \longrightarrow \Gamma \longrightarrow \Gamma$ is isomorphic to (2.32), hence $E \cong E' \oplus \Gamma[-1]$. Since E is exceptional and $\Gamma \neq 0$, it follows $E' = 0$ and $E \cong \Gamma[-1]$, hence $\theta_\sigma(E) = \theta_\sigma(\Gamma[-1]) \leq \theta_\sigma(C(f_0)[-1]) < \theta_\sigma(E)$, where we used $C(f_0) = U \oplus \Gamma$ and the inequality derived before this corollary. Thus, we get a contradiction.

If $\hom(\Gamma, E[1]) = 0$ for some $\Gamma \in Ind(A_2)$, then we can split $C(f_0) \cong A_1[1] \oplus A_2 \cong V \oplus \Gamma$, and the last arrow in (2.32) is of the form $(V \to E[1]) \oplus (\Gamma \to 0)$. It follows by similar arguments as above that $B_0 \cong U \oplus \Gamma$ for some U. Therefore $\hom(A_2, B_0) \neq 0$, which contradicts (2.29) □

If $A_2 = 0$

Under this condition we get here a triangle or type **C1**.

Now f_0 is epic(see (2.26)) and (2.32) becomes a short exact sequence

$$0 \longrightarrow A_1 \xrightarrow{ker(f_0)} E \xrightarrow{f_0} B_0 \longrightarrow 0. \tag{2.36}$$

The triangle $\mathfrak{alg}(E)$ is set to (2.36), so $A = A_1$, and $B = B_0$. From (2.31) we get $\theta_\sigma(A_1) < \theta_\sigma(E)$, which is the same as **C1.3**. In Lemma 2.42 we have **C1.4**, and in (2.41) - **C1.2**. It remains to show **C1.1**. We have $A_1 \neq 0$, for otherwise E would be semistable. We have also (2.28) and (2.29), therefore we have to show only $\hom^1(A_1, B_0) = 0 = \mathrm{Hom}^1(B_0, B_0)$.

By $\hom^1(A_1, E) = 0$ (see (2.30)), the sequence (2.36), and Lemma 2.39 **(b)** we obtain $\hom^1(A_1, B_0) = 0$. The same lemma and $\hom^1(E, E) = 0$ imply $\hom^1(E, B_0) = 0$, hence $\mathrm{Hom}(\_, B_0[1])$ applied to (2.36) gives: $0 = \mathrm{Hom}(A_1[1], B_0[1]) \to \mathrm{Hom}(B_0, B_0[1]) \to \mathrm{Hom}(E, B_0[1]) = 0$, i. e. $\hom^1(B_0, B_0) = 0$.

If $A_2 \neq 0$**.**

Under this condition we will obtain either the case **C2** or the case **B1** depending on the outcome of one additional test. The triangle $\mathfrak{alg}(E)$ is set to (2.32), which by $C(f_0) \cong A_1[1] \oplus A_2$ can be rewritten as:

$$\mathfrak{alg}(E) = \begin{array}{ccc} A_1 \oplus A_2[-1] & \longrightarrow & E \\ & \nwarrow & \swarrow \\ & B_0 & \end{array} \tag{2.37}$$

From Lemma 2.42 we have $\hom^1(\Gamma, E) \neq 0$ for each $\Gamma \in Ind(A_2)$. If $\hom^1(E, \Gamma) \neq 0$ for some $\Gamma \in Ind(A_2)$, then the triangle (2.37) has all the features of the case **B1** due to (2.27), (2.28), (2.29), (2.31).

Thus, it remains to show that if each $\Gamma \in Ind(A_2)$ satisfies $\hom(E, \Gamma[1]) = 0$, in particular

$$\mathrm{Hom}(E, A_2[1]) = 0 \quad \Rightarrow \quad \mathrm{Hom}(E, C(f_0)[1]) = \mathrm{Hom}(E, A_1[2] \oplus A_2[1]) = 0, \tag{2.38}$$

then the triangle (2.37) satisfies **C2.1**, **C2.2**, **C2.3**, **C2.4** (with $B = B_0$).

C2.2 is in (2.20), **C2.3** is (2.31), and **C2.4** is contained in (2.38), Lemma 2.42. It remains to obtain the vanishings in **C2.1**, that are not claimed in (2.28), (2.29), (2.30). These vanishings are $\hom^1(A_1, B_0) = \hom^1(A_1, A_2) = \hom^1(B_0, B_0) = 0$. We obtain them in this order below.

The equality (2.38) together with $\hom^1(E, E) = 0$ and the triangle (2.32) imply

$$\hom(E, B_0[1]) = 0,$$

hence by the sequence (2.27) and Lemma 2.39 we get $\hom^1(A_1, B_0) = 0$. From this vanishing it follows $\hom(C(f_0), B_0[2]) = 0$ and applying $\mathrm{Hom}(C(f_0), \_)$ to (2.32) we obtain

$$0 = \mathrm{Hom}(C(f_0), B_0[2]) \to \mathrm{Hom}(C(f_0), C(f_0)[2]) \to \mathrm{Hom}(C(f_0), E[3]) = 0$$
$$\Rightarrow 0 = \hom(C(f_0), C(f_0)[2]) = \hom(A_1[1] \oplus A_2, A_1[3] \oplus A_2[2]) \quad \Rightarrow \quad \hom(A_1, A_2[1]) = 0.$$

Finally, we apply $\mathrm{Hom}(\_, B_0[1])$ to (2.32): $0 = \mathrm{Hom}(C(f_0), B_0[1]) \to \mathrm{Hom}(B_0, B_0[1]) \to \mathrm{Hom}(E, B_0[1]) = 0$, where the left vanishing is contained in (2.29), and the right vanishing is above.

Now we have already the complete list **C2** for $\mathfrak{alg}(E)$.

2.5 Some terminology. The relation $R \dashrightarrow (S, E)$

The terminology introduced here is important for the rest of the dissertation. All definitions in this section assume a given stability condition on[37] $D^b(\mathcal{A})$, which we denote by σ. We divide the non-semistable exceptional objects into two types: σ-regular and σ-irregular (Definition 2.43 and Remark 2.44). In turn the σ-regular objects are divided into final and non-final (Definition 2.46).

We refer to **C1, C2, C3** as regular cases and to **B1, B2** as irregular cases. More precisely:

Definition 2.43. *Let $E \in D^b(\mathcal{A})_{exc}$ and $E \notin \sigma^{ss}$. If the triangle $\mathfrak{alg}(E)$ given by section 2.4 is of type X, where X is one of **C1,C2,C3,B1,B2**, then E is said to be an X object w. r. to σ.*

*The **Ci** objects(for $i = 1, 2, 3$) will be called <u>σ-regular exceptional objects</u> and the **Bi** objects(for $i = 1, 2$) will be called <u>σ-irregular exceptional objects</u>.*[38]

Remark 2.44. *In cases **C1, C2, C3** the triangle $\mathfrak{alg}(E)$ satisfies the vanishings in* (d) *after* (2.17). *It can be shown that some of these vanishings fails in cases **B1, B2**. Thus, an object $E \in \mathcal{T}_{exc} \setminus \sigma^{ss}$ is σ-regular iff $\mathfrak{alg}(E)$ satisfies these vanishings and σ-irregular iff some of these vanishings fails.*

We introduce now the relation $R \overset{X}{\dashrightarrow} (S, E)$. It facilitates the next steps of the exposition.

[37] Here and in all sections that follow $\mathcal{A}$ is as in subsection (2.4.1).

[38] In this text the adjectives "σ-regular", "σ-irregular" regard either exceptional objects or the cases at the output of $\mathfrak{alg}$. We often omit "exceptional object" after these adjectives, when this is by default. We sometimes omit "σ-", which is akin to writing semistable instead of σ-semistable.

Definition 2.45. *Let $R, S, E \in D^b(\mathcal{A})$ and let X be one of the symbols* ***C1,C2a,C2b,C3***. *By the notation $R \overset{X}{\dashrightarrow} (S,E)$ we mean the following data:*

- *R is a σ-regular exceptional object, in particular $\mathfrak{alg}(R)$ is of type* ***Ci***$(i \in \{1,2,3\})$,
- *$S \in Ind(V)$, $E \in Ind(U)$, where (V,U) are the lower and the left vertices of $\mathfrak{alg}(R)$ in* (2.17),
- *if $i \in \{1,3\}$ and R is a* ***Ci*** *object, then we set $X =$* ***Ci***,
- *if R is a* ***C2*** *object and E is a component of $A_2[-1]$ in diagram* (2.18), *then we set $X =$* ***C2a***,
- *if R is a* ***C2*** *object and E is a component of A_1 in* (2.18), *then we set $X =$* ***C2b***.

In the next sections we refer mainly to the following features(explained below) of the pair (S,E):

$$R \overset{X}{\dashrightarrow} (S,E) \quad X \in \{\mathbf{C1},\ \mathbf{C2a},\ \mathbf{C2b},\ \mathbf{C3}\}$$

$$\{S,E\} \subset D^b(\mathcal{A})_{exc}, \quad \hom^*(E,S) = 0, \quad \deg(E)+1 \geq \deg(S) \geq \deg(R) \geq \deg(E) \tag{2.39}$$

$$\theta_\sigma(E) < \theta_\sigma(R), \quad S \in \sigma^{ss}, \quad \theta_\sigma(R)(S) > 0, \quad \phi_-(E) \geq \phi(S) = \phi_-(R). \tag{2.40}$$

The first two statements in (2.39) amount to saying that (S,E) is an exceptional pair,[39] which is the same as: S, E are indecomposable and $\hom^*(E,S) = \hom^1(S,S) = \hom^1(E,E) = 0$. This follows from (d) right after (2.17) and $S \in Ind(V)$, $E \in Ind(U)$. In (a) right after (2.17) is specified that V is a direct summand of $\sigma_-(R)$, hence by $S \in Ind(V)$ and the definition of θ_σ(Definition 2.22) it follows that $\theta_\sigma(R)(S) > 0$ and $S \in \sigma^{ss}$. In (b) right after (2.17) we have specified $\theta_\sigma(U) < \theta_\sigma(R)$, $\phi_-(U) \geq \phi(V) = \phi_-(R)$, which by $E \in Ind(U)$, $S \in Ind(V)$ implies $\theta_\sigma(E) < \theta_\sigma(R)$, $\phi_-(E) \geq \phi(S) = \phi_-(R)$. Thus we obtain (2.40). The degrees of R, S, E are interrelated as shown in the following table,[40] which follows from the very definition of **C1,C2a,C2b,C3**:[41]

X	$\deg(S) - \deg(R)$	$\deg(R) - \deg(E)$	
C1, C2b	0	0	$\phi_-(E) \geq \phi(S)$
C2a	0	+1	$\phi_-(E) \geq \phi(S)$
C3	+1	0	$\phi_-(E) > \phi(S)$

(2.41)

The inequalities $\deg(E)+1 \geq \deg(S) \geq \deg(R) \geq \deg(E)$ follow, so (2.39) is shown completely.

We divide the σ-regular objects into final and non-final as follows:

Definition 2.46. *If R is a σ-regular object and all the indecomposable components of U (in diagram* (2.17)*) are semistable, then R is said to be <u>final</u>, otherwise - <u>non-final</u>.*

[39] In general, this pair is not uniquely determined by R, because we make choices among $Ind(U)$ and $Ind(V)$.
[40] Recall that for $X \in \mathcal{A}$ and $j \in \mathbb{Z}$ we write $\deg(X[j]) = j$.
[41] the description of **C1, C2, C3** is in subsection 2.4.2

If R is a non-final regular object then some indecomposable component of U is not semistable. By regularity this component is also an exceptional object and then we can apply to it $\mathfrak{alg}$. Now we cannot exclude the occurrence of the irregular cases **B1, B2**, i. e. we cannot exclude the occurrence of an irregular component of U.

2.6 Regularity-preserving categories. RP prpoerties 1,2

Recall that $\mathfrak{alg}$ can be applied to any non-semistable exceptional object. Using the terminology from Section 2.5, we can say that if R is σ-regular and non-final, then from the output data $\mathfrak{alg}(R)$ we can extract some number of non-semistable exceptional objects (the non-semistable components of U in diagram (2.17)). The algorithm $\mathfrak{alg}$ can be applied to any of them again. If the category $\mathcal{A}$ has the property that the cases **B1, B2** cannot occur after this second iteration of $\mathfrak{alg}$ we say that $\mathcal{A}$ is regularity-preserving. More precisely:

Definition 2.47. *A hereditary abelian category $\mathcal{A}$ will be said to be regularity-preserving, if for each $\sigma \in \mathrm{Stab}(D^b(\mathcal{A}))$ from the the following data:*

*$R \in D^b(\mathcal{A})$ is a σ-regular object; $R \overset{X}{\dashrightarrow} (S,E)$, where $X \in \{$**C1, C2a, C2b, C3**$\}$; $E \notin \sigma^{ss}$*

it follows that E is a σ-regular object as well.

In this section 2.6 we show two restrictions on the exceptional objects, called RP property 1 and RP property 2, which ensure that $\mathcal{A}$ is regularity-preserving.

2.6.1 Ext-nontrivial couples

Looking at the description of **B1, B2** (see **B1.4, B2.3**) we see that in any of these cases occur couples $\{L,\Gamma\} \subset \mathcal{A}$ of exceptional objects with $\hom^1(L,\Gamma) \neq 0, \hom^1(\Gamma,L) \neq 0$. It is useful to give a name to such a couple:

Definition 2.48. *An Ext-nontrivial couple is a couple of exceptional objects $\{L,\Gamma\} \subset \mathcal{A}_{exc}$, s. t. $\hom^1(L,\Gamma) \neq 0$ and $\hom^1(\Gamma,L) \neq 0$.*

Trivially coupling object is an exceptional object $E \in \mathcal{A}_{exc}$, s. t. for each $\Gamma \in \mathcal{A}_{exc}$ we have $\hom^1(E,\Gamma) = 0$ or $\hom^1(\Gamma,E) = 0$, i. e. for each $\Gamma \in \mathcal{A}_{exc}$ the couple $\{E,\Gamma\}$ is not Ext-nontrivial.

From **B1.4, B2.3** it follows

Lemma 2.49. *If $E \in \mathcal{A}_{exc}$ is a trivially coupling object, then for each stability condition $\sigma \in \mathrm{Stab}(D^b(\mathcal{A}))$ it is either σ-semistable or σ-regular.*

Thus, an object can be σ-irregular only if it is an element of an Ext-nontrivial couple. The following lemma gives some information about the other element of the couple.

Lemma 2.50. *Let each $X \in \mathcal{A}_{exc}$ satisfy the dichotomy that it is either trivially coupling or there exists unique up to isomorphism another object $Y \in \mathcal{A}_{exc}$ such that $\{X, Y\}$ is an Ext-nontrivial couple. Then for each Ext-nontrivial couple $\{E, \Gamma\} \subset \mathcal{A}_{exc}$ and each $\sigma \in \mathrm{Stab}(D^b(\mathcal{A}))$ we have:*

(a) *If E is a **B2** object, then Γ is semistable of phase $\phi_-(E) - 1$.*

(b) *If E is a **B1** object, then $\phi_-(\Gamma) \geq \phi_-(E) + 1$.*

(c) *At most one of the objects $\{E, \Gamma\}$ can be σ-irregular.*

Proof. **(a)** By **B2.3** there exists a semistable $X \in \mathcal{A}_{exc}$ of phase $\phi_-(E) - 1$, s. t. $\{E, X\}$ is an Ext-nontrivial couple. From the assumption of the lemma it follows $X \cong \Gamma$.

(b) By **B1.3** and **B1.4** there exists $X \in \mathcal{A}_{exc}$ with $\phi_-(X) \geq \phi_-(E) + 1$, s. t. $\{E, X\}$ is an Ext-nontrivial couple. From the assumption of the lemma we have $X \cong \Gamma$, hence $\phi_-(\Gamma) \geq \phi_-(E)+1$.

(c) It is enough to prove that if E is σ-irregular then Γ is not σ-irregular. If E is **B2**, then by **(a)** Γ is semistable, i. e. it is not σ-irregular. By **(a)** applied to Γ it follows also that if E is **B1** then Γ is not **B2**. Whence, it remains to show that E and Γ cannot both be **B1**. By **(b)** we see that if both are **B1** then $\phi_-(\Gamma) \geq \phi_-(E) + 1$ and $\phi_-(E) \geq \phi_-(\Gamma) + 1$ which is impossible. □

The next step is to show that even with the presence of Ext-nontrivial couples $\mathcal{A}$ could be regularity-preserving.

2.6.2 RP property 1 and RP property 2

Our key to regularity-preserving of $\mathcal{A}$ are the following patterns of the Ext-nontrivial couples of $\mathcal{A}$.

Definition 2.51. *Let $\mathcal{A}$ be a hereditary category. We say that $\mathcal{A}$ has*

RP Property 1*: if for each Ext-nontrivial couple $\{\Gamma, \Gamma'\} \subset \mathcal{A}$ and for each $X \in \mathcal{A}_{exc}$ from* $\hom^*(\Gamma, X) = 0$ *it follows* $\hom^*(X, \Gamma') = 0$*;*

RP Property 2*: if for each Ext-nontrivial couple $\{\Gamma, \Gamma'\} \subset \mathcal{A}$ and for any two $X, Y \in \mathcal{A}_{exc}$ from* $\hom(\Gamma, X) \neq 0, \hom(X, Y) \neq 0, \hom^*(\Gamma, Y) = 0$ *it follows* $\hom(\Gamma', Y) \neq 0$.[42]

The main result of Section 2.6 is:

Proposition 2.52. *If $\mathcal{A}$ has RP Property 1 and RP Property 2,*[43] *then $\mathcal{A}$ is regularity-preserving.*

2.6.3 Proof of Proposition 2.52

We can assume that $R \in \mathcal{A}$. We split the proof in two lemmas. The first lemma uses RP property 1, but does not use RP property 2.

[42] note that $\hom(\Gamma, X) \neq 0, \hom(X, Y) \neq 0, \hom^*(\Gamma, Y) = 0$ imply $X \neq \Gamma$, $X \neq Y$

[43] $\mathcal{A}$ is as in Subsection 2.4.1.

Lemma 2.53. *Let R be a **C3** object with* $\mathfrak{alg}(R) = \begin{array}{ccc} A & \longrightarrow & R \\ & \nwarrow \;\; \swarrow & \\ & B[1] & \end{array}$. *Then each non-semistable $E \in Ind(A)$ is σ-regular.*

Proof. Recall that in **C3.1, C3.2** we have $A, B \neq 0$, $\hom^*(A,B) = \hom^1(A,A) = \hom^1(B,B) = 0$, and $\phi_-(A) > \phi(B) + 1$. The last inequality, together with Corollary 2.21 and Remark 2.38, implies:

$$\phi_-(E) > \phi(B) + 1 \quad \Rightarrow \quad \hom^*(E,B) = 0. \tag{2.42}$$

If E is a **B1** object, then we get $\mathfrak{alg}(E) = \begin{array}{ccc} A_1 \oplus A_2[-1] & \longrightarrow & E, \\ & \nwarrow \;\; \swarrow & \\ & B' & \end{array}$ where $B' \in \mathcal{A}$ is a direct summand of $\sigma_-(E)$ (see **B1.2**). By (2.42) we can apply Lemma 2.19 to E, B and obtain $\hom^{\leq 1}(B', B) = 0$, hence $\hom^*(B', B) = 0$. From the triangle $\mathfrak{alg}(E)$ it follows $\hom^*(A_2, B) = 0$. By **B1.4** there exists $E' \in Ind(A_2)$ s. t. $\{E, E'\}$ is an Ext-nontrivial couple. So, we obtained $\hom^*(E', B) = 0$. Since $\hom^1(B,B) = 0$, RP property 1 in subsection 2.6.2 implies $\hom^*(B,E) = 0$, which contradicts **C3.3**.

If E is **B2** object, then we get $\mathfrak{alg}(E) = \begin{array}{ccc} A' & \longrightarrow & E \\ & \nwarrow \;\; \swarrow & \\ & B'[1] & \end{array}$, where $B'[1] = \sigma_-(E)$ (see **B2.2**). By (2.42) we can apply Lemma 2.19 to $E, B[1]$ and obtain $\hom^{\leq 1}(B'[1], B[1]) = 0$, hence $\hom^*(B', B) = 0$. By **B2.3** there exists $E' \in Ind(B')$, s. t. $\{E, E'\}$ is an Ext-nontrivial couple. So, we obtained $\hom^*(E', B) = 0$ which by RP property 1 implies $\hom^*(B, E) = 0$. This contradicts **C3.3**. $\square$

The second lemma uses both RP property 1 and RP property 2.

Lemma 2.54. *Let R, $E \in \mathcal{A}_{exc}$, $R \notin \sigma^{ss}$, $E \notin \sigma^{ss}$. If R, E fit into any of the following two situations:*

(a) *R is a **C1** object,* $\mathfrak{alg}(R) = \begin{array}{ccc} A & \longrightarrow & R \\ & \nwarrow \;\; \swarrow & \\ & B & \end{array}$, *$E \in Ind(A)$;*

(b) *R is a **C2** object,* $\mathfrak{alg}(R) = \begin{array}{ccc} A_1 \oplus A_2[-1] & \longrightarrow & R \\ & \nwarrow \;\; \swarrow & \\ & B & \end{array}$, *$E \in Ind(A_1)$ or $E \in Ind(A_2)$;*

then E is σ-regular.

Proof. The arguments for $E \in Ind(A)$, R is **C1** and $E \in Ind(A_1)$, R is **C2** are similar. We give them first. Recall that in **C1.3** and **C2.3** we have $\phi_-(A) \geq \phi(B)$ and $\phi_-(A_1) \geq \phi(B)$, respectively.

By Corollary 2.21, in **C1** case we have $\phi_-(E) \geq \phi_-(A) \geq \phi(B)$, and in **C2** case we have $\phi_-(E) \geq \phi_-(A_1) \geq \phi(B)$. In both the cases (see **C2.1**, **C1.1**) we have $\hom^*(E,B) = 0$. In both the cases we have also $\hom(E,R) \neq 0$ (recall that in **C2** case A_2 is a subobject of R), so we can write

$$\phi_-(E) \geq \phi(B), \quad \hom^*(E,B) = 0, \quad \hom(E,R) \neq 0 \qquad E, B \in \mathcal{A}. \tag{2.43}$$

If we take any $X \in Ind(B)$, then $\hom(R, X) \neq 0$ (this is valid in all the five cases[44]). Since R is σ-regular, we have $X, E \in \mathcal{A}_{exc}$ and combining with (2.43) we can write:

$$\hom(E, R) \neq 0, \hom(R, X) \neq 0, \hom^*(E, X) = 0, \qquad X, E, R \in \mathcal{A}_{exc}. \tag{2.44}$$

If E is a **B2** object, then $\mathfrak{alg}(E)$ is of the form

$$\mathfrak{alg}(E) = \begin{array}{ccc} A' & \longrightarrow & E \\ & \nwarrow \quad \swarrow & \\ & B'[1] & \end{array}. \tag{2.45}$$

From (2.43) we see that Lemma 2.19 can be applied, which implies $\hom(B', B) = 0$. By **B2.3**, there exists $E' \in Ind(B')$, s. t. $\{E, E'\}$ is an Ext-nontrivial couple. Then by (2.44) and RP property 2 we obtain $\hom(E', X) \neq 0$, which contradicts $\hom(B', B) = 0$.

If E is **B1** object, then $\mathfrak{alg}(E)$ is of the form

$$\mathfrak{alg}(E) = \begin{array}{ccc} A'_1 \oplus A'_2[-1] & \longrightarrow & E \\ & \nwarrow \quad \swarrow & \\ & B' & \end{array} \tag{2.46}$$

with $B' \in \mathcal{A}$ and for some $E' \in Ind(A'_2)$ the couple $\{E, E'\}$ is Ext-nontrivial. From (2.43) and Lemma 2.19 it follows $\hom^{\leq 1}(B', B) = 0$, hence $\hom^*(B', B) = 0$, which combined with $\hom^*(E, B) = 0$ and the triangle (2.46), implies $\hom^*(A'_2, B) = 0$. Whence, we obtain $\hom^*(E', B) = 0$, which by RP property 1 and $\hom^1(B, B) = 0$ implies $\hom^*(B, E) = 0$. The last contradicts **C1.4, C2.4**.

Suppose now that we are in the situation **(b)** and $E \in Ind(A_2)$ is a **B2** object. Then we again have (2.45) and some $E' \in Ind(B')$, s. t. $\{E, E'\}$ is an Ext-nontrivial couple. However, now in addition to $\hom^*(E, B) = 0$ we have $\phi_-(E) \geq \phi_-(A_2) = \phi_-(A_2[-1]) + 1 \geq \phi(B) + 1 = \phi(B[1])$. Now Lemma 2.19 gives $\hom^{\leq 1}(B'[1], B[1]) = 0$, i. e. $\hom^*(B', B) = 0$. Thus, we obtain $\hom^*(E', B) = 0$, hence $\hom^*(B, E) = 0$ by RP property 1, which contradicts **C2.4**.

Finally, suppose that $E \in Ind(A_2)$ is a **B1** object. Then we can use again (2.46) and take some $E' \in Ind(A'_2)$, s. t. $\{E, E'\}$ is an Ext-nontrivial couple. As in the preceding paragraph, in addition to $\hom^*(E, B) = 0$, we have again $\phi_-(E) \geq \phi(B[1])$. Now Lemma 2.19 gives $\hom^{\leq 1}(B', B[1]) = 0$, i. e. $\hom^*(B', B) = 0$. Combining with $\hom^*(E, B) = 0$ and the triangle (2.46) we obtain $\hom^*(E', B) = 0$. As in the previous paragraph, the last vanishing gives a contradiction. □

2.7 Sequence of regular cases

In this section we assume that $\mathcal{A}$ is regularity-preserving. If we are given a non-final σ-regular object R, then we can apply $\mathfrak{alg}$ iteratively(Definition 2.47). As a result we obtain a sequences of

[44]by the last part of Lemma 2.41 and since X is a direct summand of $\sigma_-(E)$

exceptional pairs(between the subsequent iterations we make a choice, whence the resulting sequence is not uniquely determined by R in general):

$$\begin{array}{ccccccccccc} R & \overset{X_1}{\dashrightarrow} & (S_1,E_1) & \overset{proj_2}{\longmapsto} & E_1 & \overset{X_2}{\dashrightarrow} & (S_2,E_2) & \overset{proj_2}{\longmapsto} & E_2 & \overset{X_3}{\dashrightarrow} & (S_3,E_3) \overset{proj_2}{\longmapsto} \dots \\ & & {\scriptstyle proj_1}\downarrow & & & & {\scriptstyle proj_1}\downarrow & & & & {\scriptstyle proj_1}\downarrow \\ & & S_1 & & & & S_2 & & & & S_3 \end{array} \tag{2.47}$$

where $X_i \in \{\mathbf{C1, C2a, C2b, C3}\}$. Such a sequence will be called an *R-sequence*. The number of the objects $\{S_i\}$ will be called length of the R-sequence.[45] We study here R-sequences.

The sequence (2.47) can be extended after E_i iff $E_i \notin \sigma^{ss}$, which is possible only if E_{i-1} is not final (Definition 2.46). From (2.40) it follows (recall that $\theta_\sigma(R)$ is an $\mathbb{N}$-valued function with finite support)

$$\theta_\sigma(R) > \theta_\sigma(E_1) > \theta_\sigma(E_2) > \dots . \tag{2.48}$$

Hence we see that *after finitely many steps we reach a final σ-regular object.* More precisely:

Lemma 2.55. *Let R be σ-regular. There does not exist an infinite R-sequence. The lengths of all R-sequence are bounded above by $\sum_{u \in \sigma^{ss}_{ind}/\cong} \theta_\sigma(R)(u)$.*

Some features of the individual steps in any R-sequence, specified in (2.39), (2.40), and Lemma 2.52, are readily integrated to the following basic features of the whole R-sequence:

Lemma 2.56. *Let R be σ-regular. Let an R-sequence as* (2.47) *have length n. Then $\{(S_i,E_i)\}_{i=1}^n$ is a sequence of exceptional pairs, which, in addition to* (2.48)*, satisfies the following monotonicities:* [46]

$$\phi_-(R) = \phi(S_1) \leq \phi_-(E_1) = \phi(S_2) \leq \phi_-(E_2) = \phi(S_3) \leq \dots \tag{2.49}$$

$$\deg(R) \geq \deg(E_1) \geq \deg(E_2) \geq \deg(E_3) \geq \dots \tag{2.50}$$

where $\{S_i\}_{i=1}^n$ are semistable, $\{E_i\}_{i=1}^{n-1}$ are σ-regular, and the last object E_n is either semistable or again σ-regular (and then the sequence can be extended).

In the rest of this section we make various refinements of Lemma 2.56. Whence, in the rest of this section the objects R, $\{(S_i,E_i)\}_{i=1}^n$, and the integer $n \in \mathbb{N}$ will be as in Lemma 2.56, in particular these objects fit in an R-sequence (2.47), which ends at E_n. Assuming this data, we will show that under additional conditions some of the inequalities in (2.49) are strict, and vanishings, other than the already known $\{\hom^*(E_i,S_i)=0\}_{i=1}^n$, appear. The basic lemma is:

[45] R is the exceptional object, which is the origin of the sequence, so for example if the length is ≥ 2, then after removing the first step X_1 we get an E_1-sequence.

[46] Recall that the notation $\deg(X)$ is explained in Section 0.1 before the introduction.

Lemma 2.57. *Let $1 \leq i < n$. Then the following implications hold:*

- ***(a)*** *If* $\deg(S_i) \geq \deg(S_{i+1})$, *then* $\hom^*(S_{i+1}, S_i) = 0$.
- ***(b)*** *If* $\deg(S_i) = \deg(S_{i+1})$, *then* $\hom^*(S_{i+1}, S_i) = 0$ *and* $\phi(S_{i+1}) > \phi(S_i)$.
- ***(c)*** *If* $\deg(S_i) + 1 = \deg(S_{i+1})$, *then* $\hom^1(S_{i+1}, S_i) = 0$.

Proof. Since E_i and E_{i-1} are regular, all the four features specified right after (2.17) hold for $\mathfrak{alg}(E_{i-1})$ and $\mathfrak{alg}(E_i)$. Now we unfold the definitions and use these features to write:

$$\mathfrak{alg}(E_{i-1}) = \begin{array}{ccc} U & \longrightarrow & E_{i-1} \\ & \nwarrow \; \swarrow & \\ & V & \end{array} \quad \mathfrak{alg}(E_i) = \begin{array}{ccc} U' & \longrightarrow & E_i \\ & \nwarrow \; \swarrow & \\ & V' & \end{array} \qquad \begin{array}{c} S_i \in Ind(V) \\ E_i \in Ind(U) \\ S_{i+1} \in Ind(V') \end{array} \quad \begin{array}{c} \deg(S_i) = \deg(V) \\ \deg(S_{i+1}) = \deg(V') \\ \phi(S_i) = \phi(V) \\ \phi(S_{i+1}) = \phi(V') \end{array}$$

$$\hom^*(E_i, V) = 0, \qquad \phi(V') = \phi_-(E_i) \geq \phi(V), \qquad \theta_\sigma(E_i) < \theta_\sigma(E_{i-1}). \tag{2.51}$$

The first two expressions in (2.51) show that we can apply Lemma 2.19 to E_i and V. Since V' is a direct summand of $\sigma_-(E_i)$ and $\deg(S_{i+1}) = \deg(V'), \deg(V) = \deg(S_i)$, this lemma gives us: $\hom^*(V', V) = 0$, if $\deg(S_{i+1}) \leq \deg(S_i)$; $\hom(V', V[1]) = 0$, if $\deg(S_{i+1}) = \deg(S_i) + 1$.

So far we proved **(a)**, **(c)**. It remains to show that the inequality $\phi(S_{i+1}) \geq \phi(S_i)$ given by (2.49) is strict inequality $\phi(S_{i+1}) > \phi(S_i)$ in **(b)**. We first observe the following implication:

$$\phi(S_{i+1}) = \phi(S_i) \qquad \Rightarrow \qquad S_{i+1} \in Ind(\sigma_-(E_{i-1})) \cap Ind(\sigma_-(E_i)). \tag{2.52}$$

Indeed, by (2.40) we have $\theta_\sigma(E_i)(S_{i+1}) \neq 0$. From (2.51) it follows that $\theta_\sigma(E_{i-1})(S_{i+1}) \neq 0$, hence S_{i+1} is an indecomposable component of some HN factor of E_{i-1}. This must be $\sigma_-(E_{i-1})$, because the assumption $\phi(S_{i+1}) = \phi(S_i)$ implies $\phi_-(E_{i-1}) = \phi(S_{i+1})$, so we obtain (2.52).

Suppose that $\phi(S_i) = \phi(S_{i+1})$ and $\deg(S_i) = \deg(S_{i+1})$, then $\phi(V) = \phi(V')$ and $\deg(V) = \deg(V') = j$ for some $j \in \mathbb{Z}$. Hence V and V' are the degree j terms of $\sigma_-(E_{i-1})$ and $\sigma_-(E_i)$, respectively. Now (2.52) and Krull-Schmidt property imply $S_{i+1} \in Ind(V) \cap Ind(V')$, which contradicts the already proven $\hom^*(V', V) = 0$. Hence **(b)** and the lemma follow. □

Corollary 2.58. *If for each $i \in \{1, 2, \ldots, n\}$ we have* $\deg(S_1) \geq \deg(S_i)$, *then:*

- ***(a)*** *the vanishings* $\hom^*(S_i, S_1) = \hom^*(E_i, S_1) = 0$ *hold for each integer i with $2 \leq i \leq n$,*
- ***(b)*** *furthermore, if* $\deg(S_i) = \deg(S_1)$ *for some $i \geq 2$ then $\phi(S_1) < \phi(S_i)$.*

The inequalities $\{\deg(S_1) \geq \deg(S_i)\}_{i=1}^n$ *hold in any of the following cases:*

- $X_1 =$ ***C2a***

- $X_1 =$ ***C3***
- ***C3*** *does not occur in the sequence* $\{X_1, X_2, X_3, \dots, X_n\}$.

Proof. From Lemma 2.56 we have $\{\phi_-(E_i) \geq \phi(S_1), \phi(S_i) \geq \phi(S_1)\}_{i=1}^n$ and $\hom^*(E_1, S_1) = 0$.

Suppose that for some i with $1 \leq i < n$ we are given $\hom^*(E_i, S_1) = 0$ (here we make an induction assumption). We use the triangle $\mathfrak{alg}(E_i)$(it must be of type **C1, C2, C3**):

$$\mathfrak{alg}(E_i) = \begin{array}{ccc} U & \longrightarrow & E_i \\ & \nwarrow \quad \swarrow & \\ & V & \end{array} \qquad U, V \in \mathfrak{T}, U \neq 0, V \neq 0, \qquad \begin{array}{l} S_{i+1} \in Ind(V) \\ E_{i+1} \in Ind(U) \end{array}$$

where V is a direct summand of $\sigma_-(E_i)$ and V is of pure degree.

By $\hom^*(E_i, S_1) = 0$, $\phi_-(E_i) \geq \phi(S_1)$ we can apply Lemma 2.19 and we obtain

$$\hom^{\leq 1}(V, S_1) = 0. \tag{2.53}$$

Therefore, if $\deg(S_{i+1}) \leq \deg(S_1)$, then $\hom^*(V, S_1) = 0$, since $\deg(V) = \deg(S_{i+1})$. Now $\hom^*(V, S_1) = 0$ together with the induction assumption $\hom^*(E_i, S_1) = 0$ and the triangle $\mathfrak{alg}(E_i)$ give $\hom^*(U, S_1) = 0$. Hence $\hom^*(E_{i+1}, S_1) = 0$ and $\hom^*(S_{i+1}, S_1) = 0$. Part **(a)** follows.

We prove part **(b)** by contradiction. Suppose that $\deg(S_i) = \deg(S_1)$ and $\phi(S_i) = \phi(S_1)$. From (2.48) and (2.40) it follows $\theta_\sigma(R)(S_i) > \theta_\sigma(E_{i-1})(S_i) > 0$, therefore S_i is a direct summand of some HN factor of R. On the other hand by $\phi(S_1) = \phi_-(R)$, $\phi(S_i) = \phi(S_1)$, and $\deg(S_i) = \deg(S_1)$ it follows $S_1, S_i \in Ind(V)$, where V is the degree $\deg(S_i) = \deg(S_1)$ term of $\sigma_-(R)$. Therefore (recall also **C1.2, C2.2, C3.2**), we can write $\mathfrak{alg}(R) = \begin{array}{ccc} U & \longrightarrow & R \\ & \nwarrow \quad \swarrow & \\ & V & \end{array}$ and $S_i \in Ind(V)$. The definition of $\dashrightarrow$ (Definition 2.45) implies that we can replace S_1 by S_i in the R-sequence which we consider. However now part **(a)** of the corollary says that $\hom^*(S_i, S_i) = 0$, which contradicts $S_i \neq 0$. Hence $\phi(S_i) > \phi(S_1)$, if $\deg(S_i) = \deg(S_1)$ and part **(b)** is shown.

To prove the rest of the corollary, we use table (2.41) for comparing degrees.

If we are given $X_1 =$ **C2a** or $X_1 =$ **C3**, then $\deg(E_1) = \deg(S_1) - 1$. From (2.50) in Lemma 2.56 we can write that $\deg(E_i) \leq \deg(E_1) = \deg(S_1) - 1$ for $i = 1, 2, \dots, n-1$, hence $\deg(E_i) + 1 \leq \deg(S_1)$. By $E_i \dashrightarrow (S_{i+1}, E_{i+1})$ and the last expression in (2.39) we have also $\deg(S_{i+1}) \leq \deg(E_i) + 1$. Hence, we obtain $\deg(S_{i+1}) \leq \deg(S_1)$ for $i = 1, 2, \dots, n-1$.

Finally, assume that the sequence $\{X_1, X_2, X_3, \dots, X_n\}$ does not contain **C3**. By the already proven, we can assume that $X_1 =$ **C2b** or $X_1 =$ **C1**, which implies $\deg(E_1) = \deg(S_1)$. Since **C3** is forbidden, it follows $\{\deg(S_{i+1}) = \deg(E_i)\}_{i=1}^{n-1}$, hence by (2.50) we obtain $\{\deg(S_{i+1}) \leq \deg(S_1)\}_{i=1}^{n-1}$. The corollary is completely proved. □

Corollary 2.58 does not ensure the vanishings $\{\hom^*(S_i, S_1) = \hom^*(E_i, S_1) = 0\}_{i \geq 2}$ for R-sequences with first step **C1** or **C2b** and containing a **C3** step. The obstacle to obtain these vanishings for each R-sequence is that the data $\hom^*(X, S) = 0$, $S \in \sigma^{ss}$, $\phi_-(X) \geq \phi(S)$ gives $\hom^{\leq 1}(\sigma_-(X), S) = 0$, but not $\hom^*(\sigma_-(X), S) = 0$ (see Lemma 2.19).

For certain R-sequences starting with a **C1** step and ending with a **C3** step we obtain these vanishings in the next lemma, but here we use the property in Corollary 2.7 **(b)** for the first time.

Lemma 2.59. *Assume that, besides being regularity-preserving, the category $\mathcal{A}$ satisfies the following: for any two $X, Y \in \mathcal{A}_{exc}$ at most one degree in $\{\hom^p(X,Y)\}_{p\in\mathbb{Z}}$ is nonzero.*

*If an R-sequence (as in Lemma 2.56) obeys the following restrictions (all the three): $X_1 =$ **C1**; in the sequence $\{X_2, X_3, \dots, X_{n-1}\}$ do not occur **C2a** and **C3**; $X_n =$ **C3**, then it satisfies* $\hom^*(S_i, S_1) = \hom^*(E_i, S_1) = 0$ *for $i = 2, \dots, n$.*

Proof. Applying the previous lemma to the sequence obtained by truncating the last step X_n, we obtain the given vanishings for $i < n$. We have to prove only $\hom^*(S_n, S_1) = \hom^*(E_n, S_1) = 0$.

We first observe that from $B \overset{X}{\dashrightarrow} (S,E)$, $X \in \{\mathbf{C2b}, \mathbf{C1}\}$ it follows by Definition 2.45 that $\deg(B) = \deg(E)$ and there exists a monic $E \to B$ in $\mathcal{A}[\deg(B)]$. Therefore we can assume that $0 = \deg(R) = \deg(E_1) = \dots = \deg(E_{n-1})$ and $E_1, E_2, \dots, E_{n-1}$ are $\mathcal{A}$-subobjects of R. Since $X_n =$ **C3**, we have, by **C3.2**, that $\mathfrak{alg}(E_{n-1}) \cong \mathrm{HN}_-(E_{n-1})$, and we can write:

$$\mathfrak{alg}(E_{n-1}) = \begin{array}{ccc} A & \longrightarrow & E_{n-1} \\ & \nwarrow \quad \swarrow & \\ & B[1] & \end{array}, \qquad \begin{array}{c} A, B \in \mathcal{A} \\ \phi_-(A) > \phi(B[1]) = \phi_-(E_{n-1}) \end{array} \quad \begin{array}{c} E_n \in Ind(A) \\ S_n \in Ind(B[1]). \end{array} \tag{2.54}$$

Let us take now any $\Gamma \in Ind(A)$. From Lemma 2.41 we have $\hom(\Gamma, E_{n-1}) \neq 0$. Since E_{n-1} is an $\mathcal{A}$-subobject of R and $\Gamma \in \mathcal{A}$, it follows that $\hom(\Gamma, R) \neq 0$. By the given property of $\mathcal{A}$ it follows that $\hom^1(\Gamma, R) = 0$ (any $\Gamma \in Ind(A)$ is exceptional object). Therefore we obtain $\hom^1(A, R) = 0$. Since $X_1 =$ **C1**, we have a diagram $\mathfrak{alg}(R) = \begin{array}{ccc} A' & \longrightarrow & R, \\ & \nwarrow \quad \swarrow & \\ & B' & \end{array}$ and $S_1 \in Ind(B')$, $E_1 \in Ind(A')$. By Lemma 2.39 **(b)** it follows $\hom^1(A, B') = 0$. We have also $\phi_-(A) > \phi_-(E_{n-1}) \geq \phi_-(R) = \phi(B')$, therefore $\hom(A, B') = 0$. Thus, we obtain $\hom^*(A, B') = 0$, and hence $\hom^*(A, S_1) = 0$. The triangle (2.54) and $\hom^*(E_{n-1}, S_1) = 0$ imply $\hom^*(B, S_1) = 0$. The lemma follows. □

2.8 Final regular cases

Let R be a final σ-regular object and (S,E) be any exceptional pair satisfying $R \overset{X}{\dashrightarrow} (S,E)$, $X \in \{$**C1, C2a, C2b, C3**$\}$. We have that $E \in \sigma^{ss}$ from the very definition of final(Definition 2.46). We show here that, besides being semistable, the exceptional pair (S,E) satisfies $\phi(S) < \phi(E)$(Corollary 2.61). Furthermore, if R is the middle term of an exceptional triple (S_{min}, R, S_{max})(see Corollary 2.64), then the quadruple (S_{min}, S, E, S_{max}) is also exceptional.

All results here, except the second part of Corollary 2.62, hold without regularity-preserving.

The first lemma ensures some strict inequalities. In this respect it is similar to Lemma 2.57 **(b)** and Corollary 2.58 **(b)**. As in their proofs, the function θ_σ will be useful again here.

Lemma 2.60. *Let R be a σ-regular object with $\mathfrak{alg}(R) = \begin{array}{ccc} U & \longrightarrow & R \\ & \nwarrow \swarrow & \\ & V & \end{array}$. For each $\Gamma \in Ind(U)$ from $\Gamma \in \sigma^{ss}$ it follows that $\phi(V) < \phi(\Gamma)$. In particular, if R is a final, then $\phi_-(U) > \phi(V)$.*

Proof. For simplicity, let $R \in \mathcal{A}$. If R is a **C3** object, then the lemma is true by **C3.2**, so we can assume that R is a **C1** or a **C2** object. Then the triangle $\mathfrak{alg}(R)$ is of the form (if R is **C1**, then $A_2 = 0$, otherwise $A_2 \neq 0$)

$$\begin{array}{ccc} A_1 \oplus A_2[-1] & \longrightarrow & R \\ & \nwarrow \swarrow & \\ & B & \end{array} \qquad \begin{array}{c} \hom^*(A_1, B) = \hom^*(A_2, B) = 0 \\ A_1, A_2, B \in \mathcal{A} \\ \theta_\sigma(A_1 \oplus A_2[-1]) < \theta_\sigma(R). \end{array} \tag{2.55}$$

We consider first the case $\Gamma \in \sigma^{ss} \cap Ind(A_1)$. Then $\theta_\sigma(\Gamma) \leq \theta_\sigma(A_1 \oplus A_2[-1]) < \theta_\sigma(R)$. Since Γ is semistable, the last inequality implies $\theta_\sigma(R)(\Gamma) \neq 0$, hence Γ is an indecomposable component of some HN factor of R. If $\phi(\Gamma) = \phi(B)$ then this must be the minimal HN factor $\sigma_-(R)$. On the other hand $\deg(\Gamma) = 0$ and B is the zero degree of $\sigma_-(R)$. Therefore, we see that if $\phi(\Gamma) = \phi(B)$, then $\Gamma \in Ind(B)$, which contradicts $\hom^*(A_1, B) = 0$.

Now let $\Gamma \in \sigma^{ss} \cap Ind(A_2)$. Then $\theta_\sigma(\Gamma[-1]) \leq \theta_\sigma(A_1 \oplus A_2[-1]) < \theta_\sigma(R)$ and as in the previous case we deduce that $\Gamma[-1]$ is an indecomposable component of an HN factor of R. If $\phi(\Gamma[-1]) = \phi(B)$ then this must be $\sigma_-(R)$, but $\deg(\Gamma[-1]) = -1$, which contradicts Lemma 2.41 **(a)**.

If R is final, then each $\Gamma \in Ind(U)$ is semistable and the lemma follows. □

By this lemma and Definition 2.45 we obtain:

Corollary 2.61. *Let R be final σ-regular. Let $R \overset{X}{\dashrightarrow} (S, E)$. Then $S, E \in \sigma^{ss}$ and $\phi(E) > \phi(S)$.*

Having $\phi(S) < \phi(E)$, it follows that $(S, E[-i])$ is a σ-pair(Definition 2.33) for some $i \geq 1$. Indeed, we have $\phi(S) - 1 < \phi(E[-i]) \leq \phi(S)$ for some $i \geq 1$. Since $\deg(S) \geq \deg(E)$(recall (2.39)), the pair $(S, E[-i])$ has all the features of a σ-pair. Thus, we obtain the first part of the following corollary:

Corollary 2.62. *Each final σ-regular object implies the existence of a σ-exceptional pair.*

In particular, if $\mathcal{A}$ is regularity-preserving, then each σ-regular object induces such a pair.

Proof. If there exists a σ-regular object, then by preserving of regularity and Lemma 2.55 we get a final σ-regular object. Hence, by the first part, we obtain a σ-exceptional pair. □

If $\mathcal{A}$ has not Ext-nontrivial couples, then each non-semistable exceptional object is σ-regular for each stability condition, hence:

Remark 2.63. *If there are no Ext-nontrivial couples in $\mathcal{A}$, as in $\mathcal{A} = Rep_k(K(l))$, then each non-semistable exceptional object induces a σ-exceptional pair.*

The origin of our main σ-triples criterion(Proposition 2.80) is in the next corollary.

Corollary 2.64. *If we are given the following data:*

- $S_{min}, S_{max} \in \sigma^{ss} \cap \mathcal{A}_{exc}$ *with* $\phi(S_{min}) \leq \phi(A) \leq \phi(S_{max})$ *for each* $A \in \sigma^{ss} \cap \mathcal{A}_{exc}$
- (S_{min}, R, S_{max}) *is an exceptional triple, s. t.* $R \in \mathcal{A}_{exc}$ *is final and* σ*-regular*
- $R \overset{X}{\dashrightarrow} (S, E)$, $X \in$ ***C1,C2a,C2b,C3,***

then (S_{min}, S, E, S_{max}) *is a semistable exceptional quadruple (and no two of* R, S, E *are isomorphic).*

Proof. We have $\hom^*(E, S) = 0$ (in particular $S \not\cong E$) and we must show that $\hom^*(S_{max}, S) = \hom^*(S_{max}, E) = \hom^*(S, S_{min}) = \hom^*(E, S_{min}) = 0$. By assumption R is final and then both S, E are semistable. Since R is not semistable, it cannot be isomorphic to S or to E.

Let us assume first that R is a **C3** object. Then we have a triangle $\mathfrak{alg}(R) = \begin{array}{ccc} A & \longrightarrow & R \\ & \nwarrow \; \swarrow & \\ & B[1] & \end{array}$ with $\hom^*(A, B) = 0$ and $E \in Ind(A)$, $S \in Ind(B[1])$. The assumptions on S_{min},S_{max} and **C3.2** imply

$$\phi(S_{max}) \geq \phi(E) > \phi(B) + 1 = \phi(S) + 1 \geq \phi(S_{min}) + 1.$$

Hence $\hom^*(S_{max}, B) = 0$, which, combined with $\hom^*(S_{max}, R) = 0$ and the triangle $\mathfrak{alg}(R)$, implies $\hom^*(S_{max}, A) = 0$. Thus, we get $\hom^*(S_{max}, S) = \hom^*(S_{max}, E) = 0$. Since each $\Gamma \in Ind(A)$ satisfies $\phi(\Gamma) > \phi(B)+1 \geq \phi(S_{min})+1$, we have $\hom^*(A, S_{min}) = 0$. Now $\hom^*(R, S_{min}) = 0$ and $\mathfrak{alg}(R)$ imply $\hom^*(B, S_{min}) = 0$. Thus, we get $\hom^*(S, S_{min}) = \hom^*(E, S_{min}) = 0$ as well.

Let us assume now that R is a **C1** or **C2** object. Then the triangle $\mathfrak{alg}(R)$ is of the form (if R is **C1**, then $A_2 = 0$, otherwise $A_2 \neq 0$):

$$\mathfrak{alg}(R) = \begin{array}{ccc} A_1 \oplus A_2[-1] & \longrightarrow & R \\ & \nwarrow \; \swarrow & \\ & B & \end{array} \qquad \begin{array}{c} A_1, A_2, B \in \mathcal{A} \\ \hom^*(A_1, B) = \hom^*(A_2, B) = 0 \\ E \in Ind(A_1 \oplus A_2[-1]), \; S \in Ind(B). \end{array} \qquad (2.56)$$

Since $B \neq 0$ is semistable and $\hom^1(B, B) = 0$, it follows $\phi(S_{max}) \geq \phi(B) \geq \phi(S_{min})$. On the other hand we have $\hom^*(R, S_{min}) = 0$ and $\phi_-(R) = \phi(B)$. From Lemma 2.19 it follows $\hom^*(B, S_{min}) = 0$, which, combined with $\hom^*(R, S_{min}) = 0$ and the triangle $\mathfrak{alg}(R)$, implies $\hom^*(A_1 \oplus A_2[-1], S_{min}) = 0$. So, we obtained $\hom^*(S, S_{min}) = \hom^*(E, S_{min}) = 0$ and it remains to show $\hom^*(S_{max}, S) = \hom^*(S_{max}, E) = 0$. From Lemma 2.60 it follows that for each indecomposable component Γ of A_1, resp A_2, we have $\phi(\Gamma) > \phi(B)$, resp. $\phi(\Gamma[-1]) > \phi(B)$, and combining with $\phi(S_{max}) \geq \phi(\Gamma)$ we see that $\phi(S_{max}) > \phi(B)$, hence $\hom(S_{max}, B) = 0$.

Furthermore, if R is **C2**, then $A_2 \neq 0$ and $\phi(S_{max}) \geq \phi(\Gamma)$, $\phi(\Gamma[-1]) > \phi(B)$ for each $\Gamma \in Ind(A_2)$. Therefore $\phi(S_{max}) > \phi(B) + 1$ and $\hom^*(S_{max}, B) = 0$. The latter together with $\hom^*(S_{max}, R) = 0$ imply $\hom^*(S_{max}, A_1 \oplus A_2[-1]) = 0$, and the corollary follows.

Finally, if R is **C1**, then $A_2 = 0$ in the triangle (2.56) and we have a short exact sequence $0 \to A_1 \to R \to B \to 0$. Hence, by Lemma 2.39 and $\hom(S_{max}, R[1]) = 0$ we get $\hom(S_{max}, B[1]) = 0$. We showed already that $\hom(S_{max}, B) = 0$, therefore $\hom^*(S_{max}, B) = 0$. Using again the triangle (2.56) and $\hom^*(S_{max}, R) = 0$ we obtain $\hom^*(S_{max}, A_1) = 0$. The corollary follows. □

2.9 Constructing σ-exceptional triples

So far, the property of Corollary 2.7 **(b)** was used only in Lemma 2.59. In this section it is used throughout. We start with a simple observation:

Lemma 2.65. *Let $\mathcal{A}$ be as in Subsection 2.4.1. Let Corollary 2.7* **(b)** *hold for $\mathcal{A}$. Then for any two non-isomorphic exceptional objects $A, B \in \mathcal{A}$ we have* $\hom(A, B) = 0$ *or* $\hom(B, A) = 0$.

In particular, if $C \in \mathcal{A}$ satisfies $\hom^1(C, C) = 0$, *then for any two non-isomorphic $A, B \in Ind(C)$ one of the pairs (A, B), (B, A) is exceptional.*

Proof. Let $\hom(A, B) \neq 0$. Take a nonzero $u : A \to B$. By Corollary 2.7 **(b)** it follows $\hom^1(A, B) = 0$. One can show that [16, Lemma 1, page 9] holds for $\mathcal{A}$, so $\hom^1(A, B) = 0$ implies that every nonzero $f \in \hom(B, A)$ is either monic or epic. Suppose that $f \in \hom(B, A)$ is epic, then $u \circ f \in \hom(B, B) = k$ is nonzero, hence f is also monic. Therefore f is invertible, which contradicts the assumptions. If f is monic, then we consider $f \circ u$ and again get a contradiction.

The second part follows from Remark 2.37. □

Besides the restrictions of Subsection 2.4.1, we assume throughout Section 2.9 that Corollary 2.7 **(b)** holds for $\mathcal{A}$ and that $\mathcal{A}$ is regularity-preserving. In Subsection 2.9.2, besides these features, we assume that $\mathcal{A}$ has the additional RP property (Corollary 2.8) and that Corollary 2.11 holds for it. In particular, all results hold for $\mathcal{A} = Rep_k(Q_1)$ (the preserving of regularity follows from Corollary 2.7 **(a)** and Proposition 2.52).

We denote $D^b(\mathcal{A})$ by $\mathcal{T}$, and choose any $\sigma \in \mathrm{Stab}(\mathcal{T})$. In Corollary 2.62 is shown that any σ-regular object R induces a σ-pair. If R is final, then this pair is of the form $(S, E[-j])$ with $j \geq 0$, for any $R \dashrightarrow (S, E)$. Using a σ-regular object R, we will obtain in this section various criteria for existence of σ-exceptional triples in $\mathcal{T}$. To obtain a σ-triple we utilize three approaches: using long R-sequences(of length greater than one); combining the σ-pairs induced by several single step R-sequences with a final R; combining a σ-pair induced from R with a semistable $S \in \mathcal{A}_{exc} \cap \sigma^{ss}$ of phase close to the minimal/maximal phase. The minimal and maximal phases are defined by[47]

$$\phi_{min} = \inf(\{\phi(S) : S \in \sigma^{ss} \cap \mathcal{A}_{exc}\}) \qquad \phi_{max} = \sup(\{\phi(S) : S \in \sigma^{ss} \cap \mathcal{A}_{exc}\}). \tag{2.57}$$

Note that if Corollary 2.11 holds for $\mathcal{A}$, which is assumed in Subsection 2.9.2, then we have $-\infty < \phi_{min} \leq \phi_{max} < \infty$. Indeed, if some of the strict inequalities fails, then we can construct

[47] For the notation σ^{ss} see (2.9) and recall that by $\mathcal{A}_{exc}$ we denote the set of exceptional objects of $\mathcal{A}$

a sequence $S_1, S_2, S_3, S_4, \dots, S_n$ (as long as we want) of semistable exceptional objects in $\mathcal{A}$, s. t. $\{\phi(S_i) + 1 < \phi(S_{i+1})\}_{i=1}^{n-1}$, which contradicts Corollary 2.11.

We denote by S_{min}/S_{max} objects in $\mathcal{A}_{exc} \cap \sigma^{ss}$ satisfying $\phi(S_{min}) = \phi_{min}/\phi(S_{max}) = \phi_{max}$, this can be expressed by writing $S_{min/max} \in \mathcal{P}(\phi_{min/max}) \cap \mathcal{A}_{exc}$.

We note in advance that by replacing "**C3**" with "**C2**" and "$> \phi_{min}$" with "$< \phi_{max}$" we obtain the criteria in which R is a **C2** object from those in which R is a **C3** object. However, the proof of the **C2** versions demands more efforts and more assumptions on $\mathcal{A}$(the additional RP property and Corollary 2.11).

The criteria using long R-sequences with a **C1** object R are weaker than those with **C2**/**C3**.

The distinction between **C1, C2, C3** is not essential in Lemma 2.70 (based on the second approach, where R is final) and in Proposition 2.80. Furthermore, Proposition 2.80 asserts that if $\phi_{min} - \phi_{max} > 1$, then any non-semistable $E \in \mathcal{A}_{exc}$, which is a middle term of an exceptional triple (S_{min}, E, S_{max}) induces a σ-exceptional triple (the regularity of E follows).

2.9.1 Constructions without assuming the additional RP property

Recall(Definition 2.3.4) that an exceptional triple (S_0, S_1, S_2) is said to be σ-exceptional under three conditions: it must be semistable, it must satisfy $\hom^{\leq 0}(S_0, S_1) = \hom^{\leq 0}(S_0, S_2) = \hom^{\leq 0}(S_1, S_2) = 0$, and the phases of its elements must be in $(t, t+1]$ for some $t \in \mathbb{R}$. If we are given only that (S_0, S_1, S_2) is semistable, then we can always ensure the second or the third condition by applying the shift functor to S_1, S_2, but both together - not always. For example if $\phi(S_i) = \phi(S_{i+1})$, $\hom(S_i, S_{i+1}) \neq 0$ $(i = 0, 1)$, then this cannot be achieved (similarly, if $\phi(S_i) = \phi(S_{i+1}) + 1$, $\hom^1(S_i, S_{i+1}) \neq 0$). In the following lemma are given some cases in which this can be achieved. We give the arguments for one of them. The rest are also easy. Keeping in mind Remark 2.35 is useful, when checking these implications.

Lemma 2.66. *Let (S_0, S_1, S_2) be a semistable exceptional triple, where $S_0, S_1, S_2 \in \mathcal{A}$. If any of the following conditions holds:*

(a) $\phi(S_0) < \phi(S_1) < \phi(S_2)$, $1 + \phi(S_0) < \phi(S_2)$

(b) $\phi(S_0) \leq \phi(S_1) < \phi(S_2)$, $\hom(S_0, S_1) = 0$

(c) $\phi(S_0) < \phi(S_1) \leq \phi(S_2)$, $\hom(S_1, S_2) = 0$

(d) $\phi(S_0) < \phi(S_2) \leq \phi(S_1) < \phi(S_2) + 1$, $\hom(S_1, S_2) = 0$

(e) $\phi(S_0) < \phi(S_1) + 1$, $\phi(S_1) < \phi(S_2)$, $\phi(S_0) < \phi(S_2)$, $\hom(S_0, S_1) = 0$

(f) $\phi(S_0) < \phi(S_1) + 1$, $\phi(S_0) < \phi(S_2) + 1$, $\phi(S_1) < \phi(S_2) + 1$, $\hom(S_0, S_1) = \hom(S_0, S_2) = \hom(S_1, S_2) = 0$

(g) $\phi(S_0) < \phi(S_2)$,$\phi(S_0) + 1 < \phi(S_1)$,$\phi(S_2) \neq \phi(S_1[-1])$, $\hom(S_0, S_2) = \hom(S_1, S_2) = 0$, *then for some integers $0 \leq i$, $0 \leq j$ the triple $(S_0, S_1[-i], S_2[-j])$ is σ-exceptional.*

Proof. **(d)** From $\phi(S_0) < \phi(S_2)$ it follows that $\phi(S_2[-j]) \leq \phi(S_0) < \phi(S_2[-j]) + 1$ for some $j \geq 1$. From $\phi(S_2) \leq \phi(S_1) < \phi(S_2) + 1$ it follows $\phi(S_2[-j]) \leq \phi(S_1[-j]) < \phi(S_2[-j]) + 1$. Now $\hom(S_1, S_2) = 0$ implies that $(S_0, S_1[-j], S_2[-j])$ is a σ-exceptional triple. □

The next lemma is a step in the proof of our basic long R-sequences criterion Proposition 2.68.

Lemma 2.67. *Let $R \overset{\boldsymbol{X}}{\dashrightarrow} (S, E)$, where $X \in \{$**C1, C2b**$\}$. Then there exists S', such that*

$$R \overset{X}{\dashrightarrow} (S', E),\ \hom(S', E) = 0,\ \hom(R, S') \neq 0,\ \hom(E, R) \neq 0.$$

Proof. By Definition 2.45 with $X \in \{$**C1, C2b**$\}$, there is a triangle of the form[48] $A_1 \oplus A_2[-1] \to R \to B \to A_1[1] \oplus A_2$ and $S \in Ind(B)$, $E \in Ind(A_1)$. Furthermore, any $A' \in Ind(A_1)$, $B' \in Ind(B)$ satisfy $\hom^1(B, A') \neq 0$, $\hom(R, B') \neq 0$, $\hom(A', R) \neq 0$(see **C1, C2** and Lemma 2.41). In particular, there exists $S' \in Ind(B)$, with $\hom^1(S', E) \neq 0$. By Corollary 2.7 (b) it follows $\hom(S', E) = 0$. The lemma follows. □

Now we obtain σ-triples from certain, but not all, long[49] R-sequences.

Proposition 2.68. *If there exists an R-sequence*

$$\begin{array}{ccccccccccccc} R \overset{X_1}{\dashrightarrow} & (S_1, E_1) & \overset{proj_2}{\longmapsto} E_1 \overset{X_2}{\dashrightarrow} & (S_2, E_2) & \overset{proj_2}{\longmapsto} E_2 \overset{X_3}{\dashrightarrow} \dots \overset{proj_2}{\longmapsto} E_{n-1} \overset{X_n}{\dashrightarrow} & (S_n, E_n) & \overset{proj_2}{\longmapsto} E_n \\ & {\scriptstyle proj_1}\downarrow & & {\scriptstyle proj_1}\downarrow & & {\scriptstyle proj_1}\downarrow & \\ & S_1 & & S_2 & \dots & S_n & \end{array} \tag{2.58}$$

with $n \geq 2$, E_{n-1} is final, and $\{\deg(S_1) \geq \deg(S_i)\}_{i=1}^n$, then there exists a σ-exceptional triple.

Proof. Assume that such a sequence exists. Since E_{n-1} is final, Corollary 2.61 implies that S_n and E_n are both semistable and $\phi(E_n) > \phi(S_n)$. Since $\deg(S_1) \geq \deg(S_i)$ for each $i = \{1, 2, \dots, n\}$, by Corollary 2.58 and table (2.41) we obtain

$$\hom^*(S_n, S_1) = \hom^*(E_n, S_1) = 0$$
$$\deg(S_1) \geq \deg(S_n) \geq \deg(E_n), \qquad \phi(S_1) \leq \phi(S_n) < \phi(E_n).$$

In particular, the exceptional triple (S_1, S_n, E_n) is semistable and after shifting we obtain a triple of the form $(A, B[-i], C[-i-j])$ with $0 \leq i, 0 \leq j$, $\phi(A) \leq \phi(B[-i]) < \phi(C[-j-i])$, $A, B, C \in \mathcal{A}$. If $i \neq 0$, then Lemma 2.66, **(a)** can be applied to the triple (A, B, C) and the proposition follows.

If $i = 0$, then $\deg(S_1) = \deg(S_n)$. By Corollary 2.58 (b) it follows $\phi(S_1) < \phi(S_n)$. Whence, we obtain a semistable triple $(A, B, C[-j])$ with $0 \leq j$, $\phi(A) < \phi(B) < \phi(C[-j])$. If $j \neq 0$, then the

[48] If $X =$ **C1**, then $A_2 = 0$. If $X =$ **C2b**, then $A_2 \neq 0$.
[49] by "long" we mean of length greater than one

triple (A, B, C) satisfies the conditions in Lemma 2.66, **(a)**. If $j = 0$, then $X_n \in \{\mathbf{C2b,C1}\}$ and due to Lemma 2.67 we can assume that $\hom(S_n, E_n) = \hom(B, C) = 0$. Now the triple (A, B, C) satisfies the conditions in Lemma 2.66, **(c)**. The proposition follows. □

It follows that any long R-sequence starting with a **C3** or a **C2a** step induces a σ-triple:

Corollary 2.69. *From the data:* $R \overset{X}{\dashrightarrow} (S, E)$, $X \in \{\textit{C3, C2a}\}$, $E \notin \sigma^{ss}$ *it follows that there exists a σ-exceptional triple. In particular each non-final **C3** object implies such a triple.*

Proof. Since $E \notin \sigma^{ss}$, by Lemma 2.55 we obtain an R-sequence with maximal length $n \geq 2$ and with first step the given $R \overset{X}{\dashrightarrow} (S, E)$. This sequences is of the form (2.58) with $X_1 = X$, $n \geq 2$. As far as the sequence is of maximal length, the object E_{n-1} must be final and σ-regular. Since $X_1 = X \in \{\mathbf{C3,\ C2a}\}$, Corollary 2.58 gives $\{\deg(S_1) \geq \deg(S_i)\}_{i=1}^{n}$. Thus, we constructed an R-sequence (2.58) with the three properties used in Proposition 2.68. The corollary follows. □

The next lemma uses a final regular object R, so we do not have long R-sequences here.

Lemma 2.70. *Let R be a final σ-regular object with* $\mathfrak{alg}(R) = \begin{array}{ccc} U & \longrightarrow & R \\ & \nwarrow \ \swarrow & \\ & V & \end{array}$. *Then we have:*

- ***(a)*** *If $\mathfrak{alg}(R)$ is not the HN filtration of R, then U is not semistable.*
- ***(b)*** *If U is not semistable, then there exists a σ-exceptional triple.*

Proof. Without loss of generality we can assume that $R \in \mathcal{A}$. Since R is a final σ-regular object, any $\Gamma \in Ind(U)$ is a semistable exceptional object, and hence by Lemma 2.60 it satisfies $\phi(\Gamma) > \phi(V)$. Now part **(a)** is clear and it remains to prove **(b)**.

If U is not semistable, then there exists a pair of non-isomorphic $\Gamma_1, \Gamma_2 \in Ind(U)$ with different phases. We can assume $\phi(\Gamma_2) > \phi(\Gamma_1)$. In particular, for the rest of the proof we can use

$$\hom(\Gamma_2, \Gamma_1) = 0 \qquad \phi(\Gamma_2) > \phi(\Gamma_1) > \phi(V). \tag{2.59}$$

First, assume that R is a **C1** object. Then the triangle $\mathfrak{alg}(R)$ and some of its properties are

$$\mathfrak{alg}(R) = \begin{array}{ccc} A & \longrightarrow & R \\ & \nwarrow \ \swarrow & \\ & B & \end{array} \qquad A, B \in \mathcal{A}, \hom^1(A, A) = \hom^1(B, B) = \hom^*(A, B) = 0.$$

By $\hom^1(A, A) = 0$ we have $\hom^1(\Gamma_2, \Gamma_1) = 0$, which, combined with $\hom(\Gamma_2, \Gamma_1) = 0$, implies $\hom^*(\Gamma_2, \Gamma_1) = 0$. By $\hom^*(A, B) = 0$ it follows that for each $\Gamma \in Ind(B)$ we have $\hom^*(\Gamma_i, \Gamma) = 0$, $i = 1, 2$. Hence for each $\Gamma \in Ind(B)$ the triple $(\Gamma, \Gamma_1, \Gamma_2)$ is exceptional and $\phi(V) = \phi(\Gamma) < \phi(\Gamma_1) < \phi(\Gamma_2)$. By **C1.4** we have $\hom^1(B, \Gamma_1) \neq 0$, and hence we can choose Γ so that $\hom^1(\Gamma, \Gamma_1) \neq 0$, which by Corollary 2.7 **(b)** implies $\hom(\Gamma, \Gamma_1) = 0$. Thus, we constructed an exceptional triple

$(\Gamma, \Gamma_1, \Gamma_2)$ with $\hom(\Gamma, \Gamma_1) = 0$, $\phi(\Gamma) < \phi(\Gamma_1) < \phi(\Gamma_2)$. By Lemma 2.66 **(b)**, after shifting this triple becomes σ-exceptional.

In **C3** case:

$$\mathfrak{alg}(R) = \begin{array}{ccc} A & \longrightarrow & R \\ & \nwarrow \quad \swarrow & \\ & B[1] & \end{array} \qquad A, B \in \mathcal{A} \setminus \{0\}, \hom^1(A,A) = \hom^1(B,B) = \hom^*(A,B) = 0.$$

As in the previous case we obtain that for each $\Gamma \in Ind(B)$ the triple $(\Gamma, \Gamma_1, \Gamma_2)$ is exceptional. Now (2.59) becomes $\phi(V) = \phi(\Gamma) + 1 < \phi(\Gamma_1) < \phi(\Gamma_2)$ and Lemma 2.66, **(a)** gives a σ-triple.

In **C2** case the triangle $\mathfrak{alg}(R)$ and some of its properties are:

$$\begin{array}{ccc} A_1 \oplus A_2[-1] & \longrightarrow & R \\ & \nwarrow \quad \swarrow & \\ & B & \end{array} \qquad \begin{array}{c} A_2, B \in \mathcal{A} \setminus \{0\} \\ \hom^1(A_1, A_1) = \hom^1(A_2, A_2) = \hom^1(B, B) = 0 \\ \hom^*(A_1, A_2) = \hom^*(A_1, B) = \hom^*(A_2, B) = 0 \end{array} . \qquad (2.60)$$

If both $\Gamma_1, \Gamma_2 \in Ind(A_1)$, then the arguments are the same as in **C1** case.

If both $\Gamma_1, \Gamma_2 \in Ind(A_2[-1])$, then $\hom^1(B,B) = \hom^*(A_2, B) = 0$ imply that for each $\Gamma \in Ind(B)$ the triple $(\Gamma, \Gamma_1, \Gamma_2)$ is exceptional and now $\Gamma_i[1] \in \mathcal{A}$, $\phi(\Gamma_i[1]) > \phi(B) + 1 = \phi(\Gamma) + 1$, i. e. $\phi(\Gamma) + 1 < \phi(\Gamma_1[1]) < \phi(\Gamma_2[1])$. From this data Lemma 2.66 **(a)** produces a σ-exceptional triple.

Before we continue with the other possibility, we note that

$$\hom(A_2, A_1) = 0. \qquad (2.61)$$

Indeed, by **C2.4** for each $\Gamma \in Ind(A_2)$ we have $\hom(\Gamma, R[1]) \neq 0$, then by Corollary 2.7 **(b)** it follows $\hom(\Gamma, R) = 0$, i. e. $\hom(A_2, R) = 0$. Now $\hom(A_2, A_1) = 0$ follows from the fact that A_1 is a proper subobject of R in $\mathcal{A}$.

If $\Gamma_1 \in Ind(A_1)$, $\Gamma_2 \in Ind(A_2[-1])$, then(see (2.60)) for each $\Gamma \in Ind(B)$ the triple $(\Gamma, \Gamma_2, \Gamma_1)$ is exceptional. We will show that $\Gamma \in Ind(B)$ can be chosen so that the conditions of Lemma 2.66 **(g)** hold with the triple $(\Gamma, \Gamma_2[1], \Gamma_1)$. These conditions are: $\phi(\Gamma) < \phi(\Gamma_1)$, $\phi(\Gamma) + 1 < \phi(\Gamma_2[1])$, $\phi(\Gamma_2) \neq \phi(\Gamma_1)$, $\hom(\Gamma, \Gamma_1) = \hom(\Gamma_2[1], \Gamma_1) = 0$.

By **C2.4** we see that Γ can be chosen so that $\hom^1(\Gamma, \Gamma_1) \neq 0$ and then by Corollary 2.7 **(b)** $\hom(\Gamma, \Gamma_1) = 0$. We have the vanishing $\hom(\Gamma_2[1], \Gamma_1) = 0$ by $\hom(A_2, A_1) = 0$. The inequalities $\phi(\Gamma_1) > \phi(\Gamma)$, $\phi(\Gamma_2[1]) > \phi(\Gamma) + 1$ hold because Γ_1, Γ_2 are components of $U = A_1 \oplus A_2[-1]$. Finally, we have $\phi(\Gamma_2) \neq \phi(\Gamma_1)$ by assumption and the conditions of Lemma 2.66 (g) are verified. The lemma follows.[50] □

Corollary 2.71. *Let $R \in \mathcal{A}_{exc}$ be a **C3** object with* $\mathfrak{alg}(R) = \begin{array}{ccc} A & \longrightarrow & R \\ & \nwarrow \quad \swarrow & \\ & B[1] & \end{array}$. *If $\mathfrak{alg}(R)$ differs from the HN filtration of R or they coincide and $\phi_{min} < \phi(B)$, then there exists a σ-exceptional triple.*

[50] We do not need to consider separately the case: $\Gamma_1 \in Ind(A_2)$, $\Gamma_2 \in Ind(A_1)$, for the relation $\phi(\Gamma_2) \neq \phi(\Gamma_1)$ is symmetric.

Proof. By the previous lemma and Corollary 2.69 we can assume that $\mathfrak{alg}(R)$ is the HN filtration, hence A is semistable and $\phi(A) > \phi(B) + 1$. If $\phi_{min} < \phi(B)$, then $\phi(B) > \phi(S)$ for some $S \in \mathcal{A}_{exc} \cap \sigma^{ss}$, and by $\phi(A) > \phi(B) + 1$ we obtain $\hom^*(A, S) = 0$. Since we have $\phi_-(R) = \phi(B) + 1 > \phi(S) + 1$, it follows $\hom^*(R, S) = 0$, which due to $\mathfrak{alg}(R)$ gives $\hom^*(B, S) = 0$. Thus, we see that for any $A' \in Ind(A)$, $B' \in Ind(B)$ the triple (S, B', A') is semistable and exceptional with $\phi(S) < \phi(B') < \phi(A')$, $\phi(S) + 1 < \phi(A')$. Now the corollary follows from Lemma 2.66, **(a)**. □

We obtain now σ-triples from some R-sequences starting with a **C1** object R.

Lemma 2.72. *Let $R \in \mathcal{A}$ be a **C1** object. Let $R \overset{\textbf{C1}}{\dashrightarrow} (S_1, E_1) \overset{proj_2}{\longmapsto} E_1 \overset{\textbf{C3}}{\dashrightarrow} (S_2[1], E_2)$ be an R-sequence. Then (S_1, S_2, E_2) is an exceptional triple with $\phi(S_2) + 1 < \phi_-(E_2)$ and $\hom(S_1, S_2) = 0$.*

Furthermore, any of the three conditions $E_2 \notin \sigma^{ss}$; $\phi(S_2) > \phi_{min}$; $\phi(S_1) \neq \phi(S_2) + 1$ implies an existence of a σ-exceptional triple.

Proof. By Lemma 2.59 we see that (S_1, S_2, E_2) is an exceptional triple. Since E_1 is a **C3** object, we can write $\mathfrak{alg}(E_1) = \begin{array}{ccc} A' & \longrightarrow & E_1 \\ & \nwarrow \quad \swarrow & \\ & B'[1] & \end{array}$ and(see **C3.2**) $\phi_-(A') > \phi(B') + 1$. From $E_2 \in Ind(A'), S_2 \in Ind(B')$ we obtain the first property $\phi(S_2) + 1 < \phi_-(E_2)$.

Next, we consider the vanishing $\hom(S_1, S_2) = 0$. From **C3.3** it follows $\hom(E_2, E_1) \neq 0$. As far as R is a **C1** object, we can write $\mathfrak{alg}(R) = \begin{array}{ccc} A & \longrightarrow & R \\ & \nwarrow \quad \swarrow & \\ & B & \end{array}$ and $E_1 \in Ind(A)$, $S_1 \in Ind(B)$. In particular E_1 is a subobject of R in $\mathcal{A}$. Now by $E_1, E_2, R \in \mathcal{A}$ and $\hom(E_2, E_1) \neq 0$ it follows that $\hom(E_2, R) \neq 0$, and hence Corollary 2.65 implies $\hom(R, E_2) = 0$. These arguments hold for each element in $Ind(A')$, hence $\hom(R, A') = 0$. By the exact sequence $\mathfrak{alg}(E_1)$ we get $\hom(R, B') = 0$, and by the exact sequence $\mathfrak{alg}(R)$ we get $\hom(B, B') = 0$, hence $\hom(S_1, S_2) = 0$.

If $E_2 \notin \sigma^{ss}$, then we get a σ-triple from Corollary 2.69, so let $E_2 \in \sigma^{ss}$. If $\phi(S_2) > \phi_{min}$, then by Corollary 2.71 the lemma follows.

Finally, consider the condition $\phi(S_1) \neq \phi(S_2) + 1$. Since we have also $\phi(B'[1]) = \phi_-(E_1) \geq \phi(S_1)$, we can write $\phi(S_1) < \phi(S_2) + 1$. We already obtained $\phi(S_2) + 1 < \phi(E_2)$ in the beginning of the proof. Thus, the triple (S_1, S_2, E_2) satisfies $\phi(S_1) < \phi(S_2) + 1$, $\phi(S_2) < \phi(E_2)$, $\phi(S_1) < \phi(E_2)$, $\hom(S_1, S_2) = 0$ and by Lemma 2.66 **(e)** it produces a σ-exceptional triple. □

2.9.2 Constructions assuming the additional RP property

In this subsection we restrict $\mathcal{A}$ further by assuming that the properties in Corollary 2.8 and Corollary 2.11 hold.[51]

In the previous subsection we obtained a σ-triple (without using the additional RP property) from any long R-sequence with a **C3** object R. One difficulty to obtain analogous criterion when

[51] We refer to the property in Corollary 2.8 as the additional RP property. If $\mathrm{rank}(K_0(\mathcal{A})) = 3$ and each exceptional triple generates $D^b(\mathcal{A})$, then Corollary 2.11 holds.

R is a **C2** or a **C1** object is mentioned before Lemma 2.59. It makes it difficult to obtain the vanishings $\{\hom^*(S_1, S_i) = \hom^*(S_1, E_i)\}_{i\geq 2}$ and so to obtain an exceptional triple. Nevertheless, when R is **C2**, with some extra efforts and utilizing the additional RP property and the property in Corollary 2.11 we obtain exceptional triples in Proposition 2.73. Furthermore, we show that these exceptional triples can be shifted to σ-triples. We have not an analogous criterion with a **C1** object. [52]

Proposition 2.73. *Each non-final* ***C2*** *object produces a σ-exceptional triple.*

Proof. Let $R \in \mathcal{A}$ be a non-final **C2** object. Consider the triangle $\mathfrak{alg}(R)$:

$$\begin{array}{ccc} A_1 \oplus A_2[-1] & \longrightarrow & R \\ & \nwarrow \quad \swarrow & \\ & B & \end{array} \qquad \begin{array}{c} A_2, B \in \mathcal{A} \setminus \{0\} \\ \hom^1(A_1, A_1) = \hom^1(A_2, A_2) = \hom^1(B, B) = 0 \\ \hom^*(A_1, A_2) = \hom^*(A_1, B) = \hom^*(A_2, B) = 0 \end{array} . \qquad (2.62)$$

For any $\Gamma_0 \in Ind(B)$, $\Gamma \in Ind(A_2[-1])$ we have $R \overset{\mathbf{C2a}}{\dashrightarrow} (\Gamma_0, \Gamma)$, hence by Corollary 2.69 if $\Gamma \notin \sigma^{ss}$, the proposition follows. Thus, we can assume that all components of A_2 are semistable and $A_1 \neq 0$.

For any $\Gamma_0 \in Ind(B), \Gamma_1 \in Ind(A_2), \Gamma_2 \in Ind(A_1)$ the triple $(\Gamma_0, \Gamma_1, \Gamma_2)$ is exceptional, hence by Corollary 2.11 we see that each of $Ind(A_1), Ind(A_2), Ind(B)$ has up to isomorphism unique element. Whence we can write

$$A_1 = \Gamma_2^p, \; A_2 = \Gamma_1^q, \; B = \Gamma_0^r \qquad (\Gamma_0, \Gamma_1, \Gamma_2) \text{ is exceptional triple.} \qquad (2.63)$$

We explained that $\Gamma_1 \in \sigma^{ss}$, furthermore by Lemma 2.60 it follows $\phi(\Gamma_1[-1]) > \phi(\Gamma_0)$:

$$\Gamma_0, \Gamma_1 \in \sigma^{ss}, \qquad \phi(\Gamma_1) > \phi(\Gamma_0) + 1. \qquad (2.64)$$

By Proposition 2.52, we know that Γ_2 is σ-regular, so $\mathfrak{alg}(\Gamma_2)$ is of type $X \in \{$**C1**, **C2**, **C3**$\}$. We will construct a σ-exceptional triple in each case.

If Γ_2 is a **C3** object, then by Corollary 2.69 we can assume that Γ_2 is final. For the triangle

$$\mathfrak{alg}(\Gamma_2) = \begin{array}{ccc} A' & \longrightarrow & \Gamma_2 \\ & \nwarrow \quad \swarrow & \\ & B'[1] & \end{array} \qquad \begin{array}{c} A', B' \in \mathcal{A} \setminus \{0\} \\ \hom^1(A', A') = \hom^1(B', B') = 0 \\ \hom^*(A', B') = 0 \end{array} \qquad (2.65)$$

due to Lemma 2.70 **(b)** and Corollary 2.71, we can assume also that A' is semistable with $\phi(A') > \phi(B') + 1$ and $\phi(B') = \phi_{min}$. We have also $\phi(B') + 1 = \phi_-(\Gamma_2) \geq \phi_-(A_1) \geq \phi(B) = \phi(\Gamma_0) \geq \phi_{min} = \phi(B')$. Therefore we can write

$$\phi(A') > \phi(B') + 1 \geq \phi(\Gamma_0) = \phi(B) \geq \phi(B'). \qquad (2.66)$$

[52] Lemma 2.72 and Corollary 2.75 cover all R-sequences with a **C1** object R and of length greater than two.

For any $A'' \in Ind(A'), B'' \in Ind(B')$ we have $R \overset{\mathbf{C2b}}{\dashrightarrow} (\Gamma_0, \Gamma_2) \overset{proj_2}{\longmapsto} \Gamma_2 \overset{\mathbf{C3}}{\dashrightarrow} (B''[1], A'')$, hence by $\deg(\Gamma_0) + 1 = \deg(B''[1])$ and Lemma 2.57 **(c)** we get $\hom(B'', \Gamma_0) = 0$, hence

$$\hom(B', B) = 0. \tag{2.67}$$

We show now an implication, which will be used twice later:

$$\text{If} \quad \hom(B, B') = 0 \ \text{ and } \ A'' \in Ind(A'), \quad \text{then} \quad A'' \not\cong \Gamma_1. \tag{2.68}$$

Indeed, if $A'' \cong \Gamma_1$, then by **C2.4** applied to (2.62) and recalling (2.63) we obtain $\hom(B, A') \neq 0$, and then by the short exact sequence (2.65) and $\hom(B, B') = 0$ we get $\hom(B, \Gamma_2) \neq 0$. Now from Corollary 2.7 **(b)** it follows $\hom^1(\Gamma_0, \Gamma_2) = \hom^1(B, A_1) = 0$, which contradicts **C2.4**.

Keeping (2.66) in mind, we consider two options $\phi(A') > \phi(B) + 1$ and $\phi(A') \leq \phi(B) + 1$.

If $\phi(A') > \phi(B) + 1$, then $\hom^*(A', B) = 0$, which, together with $\hom^*(\Gamma_2, B) = 0$, implies $\hom^*(B', B) = 0$. Therefore (see (2.63)) $\hom^*(A', \Gamma_0) = \hom^*(B', \Gamma_0) = \hom^*(A', B') = 0$, which by Corollary 2.11 imply that $Ind(A')/\cong$, $Ind(B')/\cong$ have unique elements,say A'', B'', and (Γ_0, B'', A'') is a semistable exceptional triple with $\phi(B'') = \phi(B')$, $\phi(A'') = \phi(A')$.
Next, we show that the inequality $\phi(\Gamma_0) \leq \phi(B'') + 1$ in (2.66) must be an equality. Indeed, if $\phi(\Gamma_0) < \phi(B'') + 1$, then we have $\phi(\Gamma_0) < \phi(B'') + 1$, $\phi(B'') < \phi(A'')$, $\phi(\Gamma_0) < \phi(A'')$ and by Lemma 2.66 **(e)** we can assume $\hom(\Gamma_0, B'') \neq 0$, so $\hom(\Gamma_0, B') \neq 0$. Hence, the triangle $\mathfrak{alg}(\Gamma_2)$ implies $\hom(\Gamma_0, A') \neq 0, \hom(\Gamma_0, A'') \neq 0$. Now Corollary 2.7 **(b)** implies $\hom^1(\Gamma_0, A'') = \hom^1(\Gamma_0, A') = 0$. From the exact sequence $0 \to B' \to A' \to \Gamma_2 \to 0$ and Lemma 2.39 it follows $\hom^1(\Gamma_0, \Gamma_2) = 0$. The latter is the same as $\hom^1(B, A_1) = 0$, which contradicts **C2.4**. So, we obtained $\phi(\Gamma_0) = \phi(B'') + 1$ and (2.66) becomes:

$$\phi(B) = \phi(\Gamma_0) = \phi(B'') + 1 = \phi(B') + 1 \quad \Rightarrow \quad \hom(B, B') = 0. \tag{2.69}$$

Now we utilize the semistable Γ_1 in (2.64). If $\phi(\Gamma_1) > \phi(B'') + 1$, then $\hom^*(\Gamma_1, B'') = 0$ as well as $\hom^*(\Gamma_1, \Gamma_0) = 0$, hence the triple $(\Gamma_0, B'', \Gamma_1)$ is exceptional. From Corollary 2.11 and the triple (Γ_0, B'', A'') it follows $\Gamma_1 \cong A''$, which contradicts (2.68). Therefore $\phi(\Gamma_1) \leq \phi(B'') + 1$. Now (2.69) implies $\phi(\Gamma_1) \leq \phi(B)$. Since we consider the subcase $\phi(A') > \phi(B) + 1$, therefore $\phi(A') = \phi(A'') > \phi(\Gamma_1) + 1$. Hence, in addition to $\hom^*(A'', \Gamma_0) = \hom^*(\Gamma_1, \Gamma_0) = 0$, we get $\hom^*(A'', \Gamma_1) = 0$. Whence, the assumption $\phi(A') > \phi(B) + 1$ leads us to an exceptional triple $(\Gamma_0, \Gamma_1, A'')$. However, the triple $(\Gamma_0, \Gamma_1, \Gamma_2)$ implies $\Gamma_2 \cong A''$, which contradicts $\Gamma_2 \notin \sigma^{ss}$, $A'' \in \sigma^{ss}$.

Therefore, it remains to consider the subcase $\phi(A') \leq \phi(B) + 1$. The latter together with $\phi(B') + 1 < \phi(A')$, taken from (2.66), imply $\phi(B') < \phi(B)$. Combining with (2.64) and (2.66) we get

$$\phi(B') < \phi(B) \leq \phi(B') + 1 < \phi(A') \leq \phi(B) + 1 < \phi(\Gamma_1). \tag{2.70}$$

These inequalities show that, in addition to $\hom(B', B) = 0$ (equality (2.67)) and $\hom^*(\Gamma_1, \Gamma_0) = 0$, we get $\hom(B, B') = 0$ and $\hom^*(\Gamma_1, B') = 0$. For clarity, we put together these vanishings:

$$\hom(B', \Gamma_0) = \hom(\Gamma_0, B') = 0, \qquad \hom^*(\Gamma_1, \Gamma_0) = \hom^*(\Gamma_1, B') = 0. \tag{2.71}$$

The vanishings $\hom^*(\Gamma_1,\Gamma_0) = \hom^*(\Gamma_1, B') = 0$ and the additional RP property (Corollary 2.8) show that for each $B'' \in Ind(B')$ the couple $\{\Gamma_0, B''\}$ is not Ext-nontrivial, i. e. we have $\hom^1(\Gamma_0, B'') = 0$ or $\hom^1(B'',\Gamma_0) = 0$. Therefore, for each $B'' \in Ind(B')$ we have $\hom^*(\Gamma_0, B'') = 0$ or $\hom^*(B'',\Gamma_0) = 0$. If $\hom^*(\Gamma_0, B'') = 0$ for some $B'' \in Ind(B')$, then (B'',Γ_0,Γ_1) is a semistable exceptional triple with $\phi(B'') < \phi(\Gamma_0) < \phi(\Gamma_1)$, $\hom(B'',\Gamma_0) = 0$ and we can apply Lemma 2.66 **(b)**. Hence, we can assume that for each $B'' \in Ind(B')$ we have $\hom^*(B'',\Gamma_0) = 0$ and (Γ_0, B'',Γ_1) is an exceptional triple. Therefore the set $Ind(B')/\cong$ has unique element, say B''. Thus, we arrive at an exceptional triple

$$(\Gamma_0, B'',\Gamma_1),\quad \hom(\Gamma_0, B'') = 0, \qquad B' \cong (B'')^s. \tag{2.72}$$

On the other hand, the vanishings $\hom^*(B',\Gamma_0) = \hom^*(\Gamma_2,\Gamma_0) = 0$ and the triangle (2.65) imply $\hom^*(A',\Gamma_0) = 0$. The last vanishing and $\hom^*(A', B') = 0$ give rise to a triple (Γ_0, B'', A'') with $(A'')^u \cong A'$. Both the triples (Γ_0, B'', A''), (Γ_0, B'',Γ_1) imply $A'' \cong \Gamma_1$, which contradicts (2.68). Thus, the proposition follows, when Γ_2 is a **C3** object.

If Γ_2 is a **C2** object, then $\mathfrak{alg}(\Gamma_2)$ and some of its features are

$$\begin{array}{ccc} A'_1 \oplus A'_2[-1] & \longrightarrow & \Gamma_2 \\ & \nwarrow \swarrow & \\ & B' & \end{array} \qquad \begin{array}{c} A'_2, B' \in \mathcal{A} \setminus \{0\} \\ \hom^1(A'_1, A'_1) = \hom^1(A'_2, A'_2) = \hom^1(B', B') = 0 \\ \hom^*(A'_1, A'_2) = \hom^*(A'_1, B') = \hom^*(A'_2, B') = 0. \end{array}$$

For any $A'' \in Ind(A'_1 \oplus A'_2[-1]), B'' \in Ind(B')$ we have an R-sequence $R \xrightarrow{\mathbf{C2b}} (\Gamma_0,\Gamma_2) \xmapsto{proj_2} \Gamma_2 \xrightarrow{\mathbf{C2a/b}} (B'', A'')$ without a **C3**-step in it. From Corollary 2.58 (the last case) it follows $\hom^*(B',\Gamma_0) = \hom^*(A'_1,\Gamma_0) = \hom^*(A'_2,\Gamma_0) = 0$. Combining these vanishings with $\hom^*(A'_1, B') = \hom^*(A'_2, B') = 0$, $A'_2 \neq 0$ we conclude by Corollary (2.11) that

$$A'_2 \cong (A'')^s;\ B' \cong (B'')^t;\ (\Gamma_0, B'', A'') \text{ is exceptional; if } A'_1 \neq 0 \text{ then } A'_1 \cong (A'')^u \tag{2.73}$$

for some $A'', B'' \in \mathcal{A}_{exc}$. By Corollary 2.69 and $\Gamma_2 \xrightarrow{\mathbf{C2a}} (B'', A''[-1])$ we reduce to the case $A'' \in \sigma^{ss}$. Thus, Γ_2 becomes final. Furthermore, by $\deg(B) = \deg(B')$ we have $\deg(\Gamma_0) = \deg(B'')$ and we see that the R-sequence $R \xrightarrow{\mathbf{C2b}} (\Gamma_0,\Gamma_2) \xmapsto{proj_2} \Gamma_2 \xrightarrow{\mathbf{C2a}} (B'', A''[-1])$ satisfies the three conditions of Proposition 2.68. This proposition ensures a σ-exceptional triple. It remains to consider:

Γ_2 is a **C1** object. Denote the corresponding triangle as follows:

$$\mathfrak{alg}(\Gamma_2) = \begin{array}{ccc} A' & \longrightarrow & \Gamma_2 \\ & \nwarrow \swarrow & \\ & B' & \end{array} \qquad \begin{array}{c} A', B' \in \mathcal{A} \setminus \{0\} \\ \hom^1(A', A') = \hom^1(B', B') = 0 \\ \hom^*(A', B') = 0. \end{array} \tag{2.74}$$

Now we have again $\deg(B') = \deg(\Gamma_0)$. It follows from Corollaries 2.58, 2.11 that

$$A' \cong (A'')^s,\quad B' \cong (B'')^t,\quad (\Gamma_0, B'', A'') \text{ is exceptional,} \tag{2.75}$$

$$\phi(B'') > \phi(\Gamma_0). \tag{2.76}$$

for some $A'', B'' \in \mathcal{A}_{exc}$. The arguments which give (2.75) are as those giving (2.73), and (2.76) follows from Corollary 2.58 **(b)**. If $A'' \in \sigma^{ss}$, then Γ_2 is final, and Proposition 2.68 produces a σ-sequence from the R-sequence $R \xrightarrow{\mathbf{C2b}} (\Gamma_0, \Gamma_2) \xmapsto{proj_2} \Gamma_2 \xrightarrow{\mathbf{C1}} (B'', A'')$. Therefore, we can assume that $A'' \notin \sigma^{ss}$.

If A'' is **C1** or **C2**, then we get an R-sequence, in which a **C3** step does not appear as follows:

$$\begin{array}{ccccccccccc} R & \xrightarrow{\mathbf{C2b}} & (\Gamma_0, \Gamma_2) & \xmapsto{proj_2} & \Gamma_2 & \xrightarrow{\mathbf{C1}} & (B'', A'') & \xmapsto{proj_2} & A'' & \xrightarrow{X_3} & (S, E) \xmapsto{proj_2} E \\ & & \downarrow{\scriptstyle proj_1} & & & & \downarrow{\scriptstyle proj_1} & & & & \downarrow{\scriptstyle proj_1} \\ & & \Gamma_0 & & & & B'' & & & & S \end{array} \qquad X_3 \in \{\mathbf{C1}, \mathbf{C2a}, \mathbf{C2b}\}.$$

From Corollary 2.58 it follows that the sequence (Γ_0, B'', S, E) is exceptional, which contradicts Corollary 2.11.

Therefore A'' must be a **C3** object, which ensures a Γ_2-sequence of the form $\Gamma_2 \xrightarrow{\mathbf{C1}} (B'', A'') \xmapsto{proj_2} A'' \xrightarrow{\mathbf{C3}} (S[1], E)$. In Lemma 2.72 is shown that the triple (B'', S, E) is exceptional. The criteria given there show that $E \in \sigma^{ss}$ and reduce the phases of (B'', S, E) to

$$\phi(B'') = \phi(S) + 1 = \phi_{min} + 1 < \phi(E); \qquad (B'', S, E) \text{ is semistable and exceptional.} \tag{2.77}$$

From Corollary 2.11 it follows that $\mathfrak{alg}(A'') = \begin{array}{ccc} E^i & \longrightarrow & A'' \\ & \nwarrow \quad \swarrow & \\ & S[1]^j & \end{array}$ for some integers $i, j \in \mathbb{N}$.

If $\phi(E) > \phi(\Gamma_0)+1$, then $\hom^*(E, \Gamma_0) = 0$, which, combined with $\hom^*(A'', \Gamma_0) = 0$ (see (2.75)), implies $\hom^*(S, \Gamma_0) = 0$. These vanishings and the exceptional triples (Γ_0, B'', A''), (B'', S, E) imply that (Γ_0, B'', S, E) is an exceptional sequence, which is impossible.

Thus, $\phi(E) \leq \phi(\Gamma_0) + 1$ and we can write (see also (2.64))

$$\phi(S) + 1 < \phi(E) \leq \phi(\Gamma_0) + 1 < \phi(\Gamma_1) \qquad \Rightarrow \qquad \hom^*(\Gamma_1, S) = 0. \tag{2.78}$$

Since $\hom^*(\Gamma_1, \Gamma_0) = 0$ as well, the additional RP property(Corollary 2.8) ensures that the couple $\{S, \Gamma_0\}$ is not Ext-nontrivial, therefore $\hom^1(\Gamma_0, S) = 0$ or $\hom^1(S, \Gamma_0) = 0$. We show below that $\hom(\Gamma_0, S) = \hom(S, \Gamma_0) = 0$, hence $\hom^*(\Gamma_0, S) = 0$ or $\hom^*(S, \Gamma_0) = 0$. It follows that some of the triples $(S, \Gamma_0, \Gamma_1), (\Gamma_0, S, \Gamma_1)$ is exceptional.

If (S, Γ_0, Γ_1) is exceptional, then Lemma 2.66, **(a)** produces σ-exceptional triple, due to the inequalities $\phi(S) < \phi(\Gamma_0)$, $\phi(\Gamma_0) + 1 < \phi(\Gamma_1)$ (see (2.78)).

If (Γ_0, S, Γ_1) is exceptional, then due to the inequalities $\phi(S) < \phi(\Gamma_1)$, $\phi(\Gamma_0) < \phi(\Gamma_1)$, $\phi(\Gamma_0) < \phi(S) + 1$ (the last comes from (2.76), (2.77)) and $\hom(\Gamma_0, S) = 0$ we can apply Lemma 2.66 **(e)**.

The used in advance $\hom(\Gamma_0, S) = 0$ follows from $\phi(S) < \phi(\Gamma_0)$ (see (2.78)). The other vanishing $\hom(S, \Gamma_0) = 0$ follows from $\phi_-(A'') \geq \phi(\Gamma_0)$, $\hom^*(A'', \Gamma_0) = 0$ (see (2.75)), and Lemma 2.19.

Now the proposition is completely proved. □

It follows now the **C2**-analogue of Corollary 2.71. After a proper reformulation,[53] Corollary 2.71 is transformed to Corollary 2.74 by replacing "**C3**" with "**C2**" and "$> \phi_{min}$" with "$< \phi_{max}$".

Corollary 2.74. *Let $R \in \mathcal{A}$ be a **C2** object with* $\mathfrak{alg}(R) = \begin{array}{ccc} A_1 \oplus A_2[-1] & \longrightarrow & R \\ & \nwarrow \; B \; \swarrow & \end{array}$. *If either $\mathfrak{alg}(R)$ differs from the HN filtration of R or they coincide and $\phi(A_2) < \phi_{max}$, then there exists a σ-triple.*

Proof. Due to the criteria given in Proposition 2.73 and Lemma 2.70, we reduce to the case: R is final and $\mathfrak{alg}(R)$ is the HN filtration of R. In particular $A_1 \oplus A_2[-1] \in \sigma^{ss}$.

If $\phi(A_2) < \phi_{max}$, then $\phi(S) > \phi(A_2)$ for some $S \in \mathcal{A}_{exc} \cap \sigma^{ss}$. Since $\mathfrak{alg}(R)$ is the HN filtration of R, it follows that $\phi_+(R) = \phi(A_2) - 1$. Therefore $\phi(S) > \phi_+(R) + 1 > \phi(B) + 1$, which implies $\hom^*(S, R) = \hom^*(S, B) = 0$. From the triangle $\mathfrak{alg}(R)$ we obtain also $\hom^*(S, A_2) = 0$. Therefore, for any $A' \in Ind(A_2)$, $B' \in Ind(B)$ the semistable triple (B', A', S) is exceptional and it satisfies $\phi(B') < \phi(A') < \phi(S)$, $\phi(B') + 1 < \phi(S)$. Now Lemma 2.66 **(a)** produces a σ-triple. □

In the next corollary we obtain σ-triples from some, but not all, long R-sequences with a **C1** object R.

Corollary 2.75. *Let $R \overset{\textbf{C1}}{\dashrightarrow} (S_1, E_1)$. If E_1 is either a **C2** or a **C1** object, then there exists a σ-exceptional triple.*

Proof. If E_1 is **C2**, then we have an R-sequence $R \overset{\textbf{C1}}{\dashrightarrow} (S_1, E_1) \overset{proj_2}{\longmapsto} E_1 \overset{\textbf{C2a}}{\dashrightarrow} (S_2, E_2[-1])$. By Proposition 2.73, we can assume that E_1 is final, and then Proposition 2.68 ensures a σ-triple.

If E_1 is **C1**, then we get a second step $E_1 \overset{\textbf{C1}}{\dashrightarrow} (S_2, E_2)$, for some (S_2, E_2), and then we go on further until a final object occurs, which will certainly happen by Lemma 2.55. We can assume that in this process a **C2** step does not occur (otherwise the corollary follows by the proven case). By Corollary 2.69 we can assume that all **C3** objects are final. Hence, if a **C3** step occurs, then this is the last step. The other possibility is to reach a final **C1** case and then Proposition 2.68 gives a σ-triple. Whence, we reduce to an R-sequence with $n \geq 3$ of the form:

$$\begin{array}{ccccccccccccc} R & \overset{\textbf{C1}}{\dashrightarrow} & (S_1, E_1) & \overset{proj_2}{\longmapsto} & E_1 & \overset{\textbf{C1}}{\dashrightarrow} & (S_2, E_2) & \overset{proj_2}{\longmapsto} & E_2 & \overset{\textbf{C1}}{\dashrightarrow} \dots \overset{proj_2}{\longmapsto} & E_{n-1} & \overset{\textbf{C3}}{\dashrightarrow} (S_n, E_n) & \overset{proj_2}{\longmapsto} E_n \\ & & \downarrow{\scriptstyle proj_1} & & & & \downarrow{\scriptstyle proj_1} & & & & & \downarrow{\scriptstyle proj_1} & \\ & & S_1 & & & & S_2 & & & \dots & & S_n & \end{array}.$$

We apply Lemma 2.59 to the R-sequence above and to the E_1-sequence in it, and obtain: $\hom^*(S_n, S_1) = \hom^*(E_n, S_1) = \hom^*(S_n, S_2) = \hom^*(E_n, S_2) = 0$. Furthermore, by Lemma 2.57 **(b)** and $\deg(S_2) = \deg(S_1) = 0$ (see table (2.41)) it follows $\hom^*(S_2, S_1) = 0$. These vanishings imply that (S_1, S_2, S_n, E_n) is a semistable exceptional sequence, which is a contradiction. □

[53] The part of Corollary 2.71 using ϕ_{min} can be reformulated as saying that the data: a final **C3** object $R \in \mathcal{A}_{exc}$, $R \dashrightarrow (S, F)$, $X \in \{S, F\}$, $\deg(X) \neq 0$, and $\phi(X) - \deg(X) > \phi_{min}$ implies a σ-triple.

We summarize now the results concerning R-sequences with a **C1** object R.

Corollary 2.76. *Let there be no a σ-exceptional triple. If $R \overset{\textbf{C1}}{\dashrightarrow} (S_1, E_1)$, then the object E_1 is either semistable or a **C3** object. If E_1 is a **C3** object, then for each R-sequence $R \overset{\textbf{C1}}{\dashrightarrow} (S_1, E_1) \overset{proj_2}{\longmapsto} E_1 \overset{\textbf{C3}}{\dashrightarrow} (S_2[1], E_2)$ the triple (S_1, S_2, E_2) is exceptional, semistable, and it satisfies: $\phi(S_2) = \phi_{min}$, $\phi(S_1) = \phi(S_2) + 1 < \phi(E_2)$, $\hom(S_1, S_2) = 0$, $\hom^1(S_1, S_2) \neq 0$.*

Proof. Follows from Corollary 2.75 and Lemma 2.72. □

A next step to the proof of Proposition 2.80 is to show that, given a **C1**-object R, each long R-sequence induces a σ-triple, when R is part of an exceptional pair (R, S_{max}) or (S_{min}, R).

Lemma 2.77. *Let $R \in \mathcal{A}$ be a non-final **C1** object. If we are given one of the following:*

(a) *$S_{min} \in \mathcal{A}_{exc}$ with $\phi(S_{min}) = \phi_{min}$ and $\hom^*(R, S_{min}) = 0$,*

(b) *$S_{max} \in \mathcal{A}_{exc}$ with $\phi(S_{max}) = \phi_{max}$ and $\hom^*(S_{max}, R) = 0$,*

then there exists a σ-exceptional triple.

Proof. By the criterion given in Corollary 2.75 we can assume that there exists an R-sequence of the form $R \overset{\textbf{C1}}{\dashrightarrow} (S_1, E_1) \overset{proj_2}{\longmapsto} E_1 \overset{\textbf{C3}}{\dashrightarrow} (S_2[1], E_2)$. The triple (S_1, S_2, E_2) is exceptional by Lemma 2.72 and using the criteria given there we can assume that it is semistable and:

$$\phi(S_1) = \phi(S_2) + 1 = \phi_{min} + 1 < \phi(E_2), \qquad \phi(S_2) = \phi_{min}.$$

In part (a) we are given that $\hom^*(R, S_{min}) = 0$. We claim that the triple (S_{min}, S_1, E_2) is exceptional. Indeed, we have: $\hom^*(E_2, S_{min}) = 0$ by $\phi(E_2) > \phi(S_{min}) + 1$, and $\hom^*(E_2, S_1) = 0$ by the exceptional triple (S_1, S_2, E_2). Finally $\hom^*(S_1, S_{min}) = 0$ by $\hom^*(R, S_{min}) = 0$, $\phi_-(R) = \phi(S_1) \geq \phi(S_{min})$ and Lemma 2.19. Thus, we constructed a semistable exceptional triple (S_{min}, S_1, E_2) with $\phi(S_{min}) < \phi(S_1) = \phi(S_{min}) + 1 < \phi(E_2)$. Now Lemma 2.66 **(a)** produces a σ-triple.

Let $\hom^*(S_{max}, R) = 0$ for some $S_{max} \in \mathcal{A}_{exc}$ with maximal phase. Unfolding the definition of **C1** we get a short exact sequence $0 \to E \to R \to S \to 0$ with $E_1 \in Ind(E)$, $S_1 \in Ind(S)$, $\phi(S) = \phi(S_1)$. Since S_{max} is of maximal phase, we have $\phi(S_{max}) \geq \phi(E_2) > \phi(S_2) + 1 = \phi(S_1) = \phi(S)$, which implies $\hom^*(S_{max}, S_2) = 0$, $\hom(S_{max}, S) = 0$. By Lemma 2.39 and $\hom^*(S_{max}, R) = 0$ we get also $\hom(S_{max}, S[1]) = 0$, hence $\hom^*(S_{max}, S) = 0$, which in turn implies $\hom^*(S_{max}, E) = 0$. So far, using the conditions of **(b)**, we obtained

$$\hom^*(S_{max}, S_1) = \hom^*(S_{max}, E_1) = \hom^*(S_{max}, S_2) = 0. \tag{2.79}$$

We show below that $\hom^*(S_{max}, E_2)$ also vanishes, and then the sequence (S_1, S_2, E_2, S_{max}) becomes exceptional, which is a contradiction. Then the corollary follows.

Since any relation of the form $E_1 \overset{\mathbf{C3}}{\dashrightarrow} (X[1], Y)$ gives by Lemma 2.72 an exceptional triple (S_1, X, Y), it follows from Corollary 2.11 that $\mathfrak{alg}(E_1) = \begin{array}{ccc} E_2^i & \longrightarrow & E_1 \\ & \nwarrow \quad \swarrow & \\ & S_2[1]^j & \end{array}$. This triangle and the already shown $\hom^*(S_{max}, E_1) = \hom^*(S_{max}, S_2) = 0$ give the desired $\hom^*(S_{max}, E_2) = 0$. □

The additional RP property gives us another situation, where the irregular cases **B1** and **B2** cannot occur. This is shown in Lemmas 2.78, 2.79 below. In this respect these lemmas are similar to Proposition 2.52, but the latter uses RP properties 1,2.

Lemma 2.78. *If (S_{min}, E) is an exceptional pair in $\mathcal{A}$ with $S_{min} \in \mathcal{P}(\phi_{min})$, then E is not **B2**.*

Proof. If E is a **B2** object, then $\mathfrak{alg}(E) = \begin{array}{ccc} A & \longrightarrow & E \\ & \nwarrow \quad \swarrow & \\ & B[1] & \end{array}$ with $B \in \sigma^{ss}$, $\phi(B) + 1 = \phi_-(E)$, $\phi_-(A) > \phi(B)+1$, and for some $\Gamma \in Ind(B)$ the couple $\{E, \Gamma\}$ is Ext-nontrivial. From $\Gamma \in \mathcal{A}_{exc} \cap \sigma^{ss}$ it follows that $\phi(\Gamma) = \phi(B) \geq \phi_{min}$, therefore $\phi_-(A) > \phi_{min} + 1$ and $\hom^*(A, S_{min}) = 0$. The vanishings $\hom^*(A, S_{min}) = 0$, $\hom^*(E, S_{min}) = 0$ imply $\hom^*(B, S_{min}) = 0$. Thus, we obtain an Ext-nontrivial couple $\{\Gamma, E\}$ and $S_{min} \in \mathcal{A}_{exc}$ with $\hom^*(E, S_{min}) = \hom^*(\Gamma, S_{min}) = 0$, which contradicts the additional RP property (Corollary 2.8). □

Lemma 2.79. *Let $\phi_{max} > \phi_{min} + 1$. If (S_{min}, E, S_{max}) is an exceptional triple in $\mathcal{A}$ with $S_{min} \in \mathcal{P}(\phi_{min})$, $S_{max} \in \mathcal{P}(\phi_{max})$, then E is not σ-irregular.*

Proof. In the previous lemma we showed that E is not a **B2** object. Suppose that E is a **B1** object. Then $\mathfrak{alg}(E) = \begin{array}{ccc} A_1 \oplus A_2[-1] & \longrightarrow & E \\ & \nwarrow \quad \swarrow & \\ & B & \end{array}$ with $B \in \sigma^{ss}$, $\phi_-(A_1 \oplus A_2[-1]) \geq \phi(B)$, $\phi(B) = \phi_-(E)$, and for some $\Gamma \in Ind(A_2)$ the couple $\{E, \Gamma\}$ is Ext-nontrivial.

If $\phi(B) > \phi(S_{min})$, then we have $\phi_-(\Gamma[-1]) \geq \phi_-(A_1 \oplus A_2[-1]) \geq \phi(B) > \phi(S_{min})$, hence $\phi_-(\Gamma) > \phi(S_{min})+1$. However, this implies $\hom^*(\Gamma, S_{min}) = 0$ and we have also $\hom^*(E, S_{min}) = 0$, which contradicts the additional RP property(Corollary 2.8).

If $\phi(B) \leq \phi(S_{min})$, then by $\phi_{max} > \phi_{min} + 1$ we have $\hom^*(S_{max}, B) = 0$, which, combined with $\hom^*(S_{max}, E) = 0$ and the triangle $\mathfrak{alg}(E)$, implies $\hom^*(S_{max}, A_2) = 0$. Thus, we have $\hom^*(S_{max}, \Gamma) = \hom^*(S_{max}, E) = 0$, which contradicts Corollary 2.8. □

We can prove now easily:

Proposition 2.80. *Let $\phi_{max} - \phi_{min} > 1$. Let (S_{min}, E, S_{max}) be an exceptional triple in $\mathcal{A}$ with $S_{min} \in \mathcal{P}(\phi_{min})$, $S_{max} \in \mathcal{P}(\phi_{max})$. If $E \notin \sigma^{ss}$, then there exists a σ-exceptional triple.*

Proof. From Lemma 2.79 and $E \notin \sigma^{ss}$ it follows that E is regular. From Corollary 2.64 it follows that E cannot be final (due to Corollary 2.11 there are no exceptional sequences of length 4). Now the existence of a σ-exceptional triple follows from Corollary 2.69, Proposition 2.73, and Lemma 2.77. □

2.10 Application to $\mathrm{Stab}(D^b(Q_1))$

The criteria of Section 2.9 hold for $\mathcal{A} = Rep_k(Q_1)$, due to Section 2.2. In this section we apply these criteria to $Rep_k(Q_1)$. The result is the following theorem:

Theorem 2.81. *Let k be an algebraically closed field. For each $\sigma \in \mathrm{Stab}(D^b(Rep_k(Q_1)))$ there exists a σ-exceptional triple.*

In Remark 2.40 we pointed out a variant of Sections 2.4, 2.5, 2.6, 2.7, 2.8, 2.9 in which k is any field. We cannot point out a variant of Theorem 2.81 without the restriction that k is algebraically closed.

Corollary 2.82. *The manifold $\mathrm{Stab}(D^b(Rep_k(Q_1)))$ is connected.*

Proof. Let $\mathcal{E} = (E_1^0, M, E_3^0)$. Let $\Sigma_{\mathcal{E}}$ be as in (2.12). From Corollary 2.7 **(b)** we see that all triples in $D^b(Q_1)$ are regular. Therefore $\Sigma_{\mathcal{E}}$ is connected [37, Corollary 3.20]. From [15] it follows that all exceptional triples in $D^b(Q_1)$ are obtained by shifts and mutations of $\mathcal{E}$. Recalling Corollary 2.34 we see that Theorem 2.1 is the same as the equality $\mathrm{Stab}(D^b(Q_1)) = \Sigma_{\mathcal{E}}$. The corollary follows. □

Throughout the proof of Theorem 2.81(the entire Section 2.10) we fix the notations $\mathcal{A} = Rep_k(Q_1)$ and $\mathcal{T} = D^b(\mathcal{A})$. We prove the theorem by contradiction.

Let $\sigma \in \mathrm{Stab}(D^b(\mathcal{A}))$. In all subsections of Section 2.10, except subsection 2.10.1, we assume that there does not exist a σ-exceptional triple.

Loosely speaking, this assumption leads to certain "non-generic" situations (see (2.86)). However, using the locally finiteness of σ, we show that these situations cannot occur (Corollaries 2.85, 2.86) and so we get a contradiction.

The notations $M, M', E_1^m, E_2^m, E_3^m, E_4^m$ are explained in Proposition 2.3. We will refer often to table (2.4) and Corollary 2.10. Whenever we claim that a triple (A_0, A_1, A_2) is an exceptional triple(with A_0, A_1, A_2 one of the symbols $M, M', E_1^m, E_2^m, E_3^m, E_4^m$), then we refer implicitly to Corollary 2.10, and whenever we discuss $\hom^*(A, B)$ with A, B varying in these symbols, we refer to table (2.4).

Remark 2.83. *Recall that(see right after Definition 2.18) $\hom(A, B) \neq 0$ implies $\phi_-(A) \leq \phi_+(B)$. Using table (2.4) we can write for any $n \in \mathbb{N}$*

- $\hom(E_1^{n+1}, E_1^n) \neq 0$ *hence* $\phi_-(E_1^{n+1}) \leq \phi_+(E_1^n)$
- $\hom(E_2^n, E_1^{n+1}) \neq 0$ *hence* $\phi_-(E_2^n) \leq \phi_+(E_2^{n+1})$
- $\hom(E_3^n, E_3^{n+1}) \neq 0$ *hence* $\phi_-(E_3^n) \leq \phi_+(E_3^{n+1})$
- $\hom(E_4^{n+1}, E_4^n) \neq 0$ *hence* $\phi_-(E_4^{n+1}) \leq \phi_+(E_4^n)$.

2.10.1 Basic lemmas

The facts explained here are basic tools used in the following subsections. These facts are individual for Q_1. The reader may skip this subsection on a first reading and return to it only when we refer to these tools.

In this subsection we do not put any restrictions on $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}(\mathcal{T})$. In all the rest subsections σ is assumed not to admit a σ-exceptional triple.

We note first that the values of Z on $\mathcal{A}_{exc}$ (see Proposition 2.3) are: $Z(M)$, $Z(M') = Z(E_1^0) + Z(E_3^0)$, and

$$Z(E_j^m) = m\delta_Z + Z(E_j^0), \quad m \in \mathbb{N}, j = 1,2,3,4, \quad \text{where} \quad \delta_Z = Z(E_1^0) + Z(E_3^0) + Z(M). \qquad (2.80)$$

By Kac's theorem,[54] proved in [31], and the description of the roots before Lemma 2.2 it follows that the values of Z on all indecomposable objects are the already given above and the following:

$$\{m\delta_Z\}_{m\geq 1}, \quad \{m\delta_Z + Z(M), \quad m\delta_Z + Z(E_1^0) + Z(E_3^0)\}_{m\in\mathbb{N}}. \qquad (2.81)$$

Useful short exact sequences in $\mathcal{A}$ and two corollaries based on locally finiteness.

It is easy to check:

Lemma 2.84. *There exist arrows in $\mathcal{A}$ as shown below, so that the resulting sequences are exact:*

$$0 \longrightarrow E_2^{m-1} \longrightarrow E_1^m \longrightarrow (E_1^0)^2 \longrightarrow 0 \qquad (2.82)$$
$$0 \longrightarrow E_3^m \longrightarrow E_2^m \longrightarrow M \longrightarrow 0 \qquad (2.83)$$
$$0 \longrightarrow E_3^{m-1} \longrightarrow E_4^m \longrightarrow (E_4^0)^2 \longrightarrow 0 \qquad (2.84)$$
$$0 \longrightarrow M \longrightarrow E_4^m \longrightarrow E_1^m \longrightarrow 0 \qquad (2.85)$$

These short exact sequences combined with the locally finiteness of σ result in Corollaries 2.85, 2.86. These corollaries exclude the following two situations:

$$\{E_2^m\}_{m\in\mathbb{N}} \subset \mathcal{P}(t), \{E_1^m\}_{m\in\mathbb{N}} \subset \mathcal{P}(t+1) \quad \text{or} \quad \{E_3^m\}_{m\in\mathbb{N}} \subset \mathcal{P}(t), \{E_4^m\}_{m\in\mathbb{N}} \subset \mathcal{P}(t+1). \qquad (2.86)$$

We will sometimes refer to these two cases as non-locally finite cases.

Corollary 2.85. *Assume that $\{E_1^m, E_2^m\}_{m\in\mathbb{N}} \subset \sigma^{ss}$ and $\{E_2^m\}_{m\in\mathbb{N}} \subset \mathcal{P}(t)$ for some $t \in \mathbb{R}$. Then for each $m \in \mathbb{N}$ we have $t \leq \phi(E_1^m) \leq t+1$, and there exists $n \in \mathbb{N}$ with $t \leq \phi(E_1^n) < t+1$.*

Proof. By table (2.4) we have $\hom(E_2^m, E_1^n) \neq 0$ and $\hom(E_1^n, E_2^m[1]) \neq 0$ for $m \geq 1$, hence $t = \phi(E_2^m) \leq \phi(E_1^n) \leq \phi(E_2^m) + 1 = t+1$. It remains to show the last claim.

[54] saying that the dimension vectors of the indecomposables are the same as the roots

The short exact sequence (2.82) gives a distinguished triangle $E_1^m \longrightarrow (E_1^0)^2 \longrightarrow E_2^{m-1}[1] \longrightarrow E_1^m[1]$. Suppose that $\phi(E_1^m) = t+1$ for each m. Then $\{E_1^m, (E_1^0)^2, E_2^{m-1}[1]\}_{m\in\mathbb{N}} \subset \mathcal{P}(t+1)$. It follows that $0 \longrightarrow E_1^m \longrightarrow (E_1^0)^2 \longrightarrow E_2^{m-1}[1] \longrightarrow 0$ is a short exact sequence in the abelian category $\mathcal{P}(t+1)$ for each $m \in \mathbb{N}$ (see the beginning of subsection 2.3.2). Hence $E_1^m \longrightarrow (E_1^0)^2$ is a monic arrow in $\mathcal{P}(t+1)$ for each $m \in \mathbb{N}$. It follows by Lemma 2.17 that the set $\{[E_1^m]\}_{m\in\mathbb{N}}$ is a finite subset of $K_0(D^b(\mathcal{A}))$. On the other hand (see Lemma 2.3) we can write $\{\ [E_1^m] = (m+1)[E_1^0] + m[M] + m[E_3^0]\ \}_{m\in\mathbb{N}}$, which is infinite in $K_0(D^b(\mathcal{A}))$. Thus, the assumption that $\phi(E_1^n) = t+1$ for each n leads to a contradiction. □

Corollary 2.86. *Assume that $\{E_3^m, E_4^m\}_{m\in\mathbb{N}} \subset \sigma^{ss}$ and $\{E_3^m\}_{m\in\mathbb{N}} \subset \mathcal{P}(t)$ for some $t \in \mathbb{R}$. Then for each $m \in \mathbb{N}$ we have $t \leq \phi(E_4^m) \leq t+1$, and there exists $l \in \mathbb{N}$ with $t \leq \phi(E_4^l) < t+1$.*

Proof. By table (2.4) we have $\hom(E_3^m, E_4^n) \neq 0$ and $\hom(E_4^n, E_3^m[1]) \neq 0$ for $m \geq 1$, hence $t \leq \phi(E_4^n) \leq t+1$. The rest of the proof is the same as the proof of Corollary 2.85, but one must use the short exact sequence (2.84) instead of (2.82). □

The short exact sequences with middle terms E_2^0, E_4^0, M' are unique:

Lemma 2.87. *If $0 \to A \to C \to B \to 0$ is a short exact sequence in $\mathcal{A}$ with $A \neq 0$ and $B \neq 0$, then we have the following implications:*

- *if $C \cong E_2^0$, then $A \cong E_3^0$ and $B \cong M$;*
- *if $C \cong E_4^0$, then $A \cong M$ and $B \cong E_1^0$;*
- *if $C \cong M'$, then $A \cong E_3^0$ and $B \cong E_1^0$.*

Proof. See the representations $E_1^0, E_2^0, E_3^0, E_4^0, M, M'$ in Proposition 2.3. □

Comments on C1 objects

Recall(see Lemma 2.67) that for any **C1** object $R \in \mathcal{A}$ there exists an exceptional pair (X, Y) in $\mathcal{A}$ satisfying $R \overset{\mathbf{C1}}{\cdots\!\!\rightarrow} (X, Y)$, $\hom(X, Y) = 0$, $\hom(R, X) \neq 0$, $\hom(Y, R) \neq 0$. A list of the exceptional pairs in $\mathcal{A}$ is given in Lemma 2.9. Using table (2.4) we see that the exceptional pairs (X, Y) in $\mathcal{A}$ with $\hom(X, Y) = 0$ are

$$(E_1^0, E_2^0), (E_1^0, E_3^0), (E_4^0, E_3^0), (E_1^m, M), (M, E_3^m), (M', E_2^m), (E_4^m, M') \qquad m \in \mathbb{N}. \qquad (2.87)$$

By setting R to specific objects in $\mathcal{A}_{exc}$ we can shorten this list further as follows:

Lemma 2.88. *Let $R \in \{E_i^m : m \in \mathbb{N}, 1 \leq i \leq 4\}$ and let R be a **C1** object. Then there exists a pair $(X, Y) \in P_R$ which satisfies $R \overset{\textbf{C1}}{\dashrightarrow} (X, Y)$, where P_R is a set of pairs depending on R as shown in the table:*

R	P_R
$E_1^m, m \geq 1$	$\{(E_1^0, E_2^0),\ (E_4^0, E_3^0),\ (E_1^0, E_3^0)\} \cup \{(E_4^n, M') : n < m\}$
$E_2^m, m \geq 0$	$\{(E_1^0, E_2^0),\ (E_4^0, E_3^0),\ (E_1^0, E_3^0)\} \cup \{(M, E_3^n) : n \leq m\}$
$E_3^m, m \geq 1$	$\{(E_1^0, E_2^0),\ (E_4^0, E_3^0),\ (E_1^0, E_3^0)\} \cup \{(M', E_2^n) : n < m\}$
$E_4^m, m \geq 0$	$\{(E_1^0, E_2^0),\ (E_4^0, E_3^0),\ (E_1^0, E_3^0)\} \cup \{(E_1^n, M) : n \leq m\}$

(2.88)

Proof. We shorten the list (2.87) using $\hom(R, X) \neq 0, \hom(Y, R) \neq 0$ and table (2.4). $\square$

Recall that for each **C1** object $C \in \mathcal{A}$ we have a short exact sequence $0 \to A \to C \to B \to 0$ with $A \neq 0$, $B \neq 0$. It follows the first part of:

Lemma 2.89. *The simple objects E_1^0, E_3^0, M cannot be **C1** objects. Furthermore: If $E_2^0 \overset{\textbf{C1}}{\dashrightarrow} (X, Y)$, then $(X, Y) \cong (M, E_3^0)$. If $E_4^0 \overset{\textbf{C1}}{\dashrightarrow} (X, Y)$, then $(X, Y) \cong (E_1^0, M)$. If $M' \overset{\textbf{C1}}{\dashrightarrow} (X, Y)$, then $(X, Y) \cong (E_1^0, E_3^0)$.*

Proof. The rest of the lemma follows from Lemma 2.87. $\square$

σ-exceptional triples from the low dimensional exceptional objects $\{E_i^0\}_{i=1}^4$, M, M'

We have the following corollaries of Lemma 2.66

Corollary 2.90. *Let $\{E_1^0,\ E_2^0,\ E_3^0,\ M\} \subset \sigma^{ss}$. If $\phi(E_2^0) > \phi(E_1^0)$ or $\phi(E_3^0) > \phi(E_1^0)$, then there exists a σ-exceptional triple.*

Proof. If $\phi(E_3^0) > \phi(E_1^0)$, then by $\phi(E_3^0) \leq \phi(E_2^0)$ (since $\hom(E_3^0, E_2^0) \neq 0$) we have $\phi(E_2^0) > \phi(E_1^0)$. Therefore, it is enough to construct a σ-exceptional triple assuming $\phi(E_2^0) > \phi(E_1^0)$.

By $\hom(E_2^0, M) \neq 0$ we have $\phi(E_2^0) \leq \phi(M)$. If $\phi(E_2^0) < \phi(M)$, then we obtain a σ-exceptional triple from the triple (E_1^0, E_2^0, M) with $\hom(E_1^0, E_2^0) = 0$ and Lemma 2.66 **(b)**. Hence, we reduce to the case $\phi(E_2^0) = \phi(M) > \phi(E_1^0)$.

Next, we consider the triple (E_1^0, M, E_3^0) with $\hom(E_1^0, M) = \hom(E_1^0, E_3^0) = \hom(M, E_3^0) = 0$. By $\hom^1(M, E_3^0) \neq 0$ it follows $\phi(M) \leq \phi(E_3^0) + 1$. If $\phi(M) < \phi(E_3^0) + 1$, then we obtain a σ-triple from Lemma 2.66 (f), due to the inequalities $\phi(E_1^0) < \phi(M) < \phi(E_3^0) + 1$. Thus, it remains to consider the case $\phi(E_1^0) < \phi(E_3^0) + 1 = \phi(E_2^0) = \phi(M)$. In this case we apply Lemma 2.66 (e) to the triple (E_1^0, E_3^0, E_2^0) with $\hom(E_1^0, E_3^0) = 0$ and obtain a σ-triple. $\square$

Corollary 2.91. *Let $\{E_1^0,\ E_4^0,\ E_3^0,\ M'\} \subset \sigma^{ss}$. If $\phi(E_3^0) > \phi(E_4^0)$ or $\phi(E_3^0) > \phi(E_1^0)$, then there exists a σ-exceptional triple.*

Proof. By $\hom(E_4^0, E_1^0) \neq 0$, we see that $\phi(E_3^0) > \phi(E_1^0)$ implies $\phi(E_3^0) > \phi(E_4^0)$. Hence, it is enough to show that the inequality $\phi(E_3^0) > \phi(E_4^0)$ induces a σ-triple.

The triple (E_4^0, E_3^0, M') has $\hom(E_4^0, E_3^0) = 0$ and $\hom(E_3^0, M') \neq 0$, therefore $\phi(E_4^0) < \phi(E_3^0) \leq \phi(M')$. By Lemma 2.66 **(b)** we reduce to the case $\phi(E_3^0) = \phi(M') > \phi(E_4^0)$.

Now, the triple (E_4^0, M', E_1^0) has $\hom(E_4^0, M') = 0$, $\hom(M', E_1^0) \neq 0$ and $\phi(E_4^0) < \phi(M') \leq \phi(E_1^0)$. Therefore, by Lemma 2.66 **(b)** we can reduce the phases to $\phi(E_4^0) < \phi(E_1^0) = \phi(E_3^0) = \phi(M')$.

Due to the obtained setting of the phases and $\hom(E_1^0, E_3^0) = 0$, Lemma 2.66 **(c)** produces a σ-triple from the exceptional triple (E_4^0, E_1^0, E_3^0). The corollary follows. □

2.10.2 On the existence of S_{min}, S_{max}

For the rest of section 2.10 we assume that $\sigma \in \mathrm{Stab}(D^b(Q_1))$ does not admit a σ-exceptional triple. Hence, Corollaries 2.71, 2.74 imply:

Corollary 2.92. *If R is a **C2** or a **C3** object, then the HN filtration of R is $\mathfrak{alg}(R)$ and R is final.*

Moreover, by Corollary 2.71/2.74, any **C3**/**C2** object induces a semistable $S_{min/max} \in \mathcal{A}_{exc}$ with $\phi(S_{min/max}) = \phi_{min/max}$, i. e. each **C3**/**C2** object ensures that $\mathcal{P}(\phi_{min/max}) \cap \mathcal{A}_{exc} \neq \emptyset$. In this subsection we generalize these implications. The main proposition here is in terms of the numbers ϕ_{min}, ϕ_{max} defined in (2.57). The following lemma gives some information about these numbers.

Lemma 2.93. *If there exists $R \in \mathcal{A}_{exc}$ which is either **C2** or **C3** object, then $\phi_{max} - \phi_{min} > 1$.*

Proof. We use that R is final and apply Corollary 2.61. Therefore, we have either $R \xrightarrow{\mathbf{C2}} (S, E[-1])$ with $\phi(S) < \phi(E[-1])$ or $R \xrightarrow{\mathbf{C3}} (S[1], E)$ with $\phi(S[1]) < \phi(E)$, where $S, E \in \sigma^{ss} \cap \mathcal{A}_{exc}$. Hence there exist $S, E \in \sigma^{ss} \cap \mathcal{A}_{exc}$ with $\phi(E) > \phi(S) + 1$, therefore $\phi_{max} - \phi_{min} > 1$. □

The main proposition of this subsection is:

Proposition 2.94. *If $\phi_{max} - \phi_{min} > 1$, then $\mathcal{P}(\phi_{min}) \cap \mathcal{A}_{exc} \neq \emptyset$ and $\mathcal{P}(\phi_{max}) \cap \mathcal{A}_{exc} \neq \emptyset$.*

In the proof of Proposition 2.94 we use Corollaries 2.98, 2.100, proved later independently.

Proof. [of Proposition 2.94] Suppose first that $\mathcal{P}(\phi_{max}) \cap \mathcal{A}_{exc} = \emptyset$. It follows that there exists a sequence $\{S_i\}_{i\in\mathbb{N}} \subset \sigma^{ss} \cap \mathcal{A}_{exc}$ such that

$$\phi_{min} + 1 < \phi(S_0) < \phi(S_1) < \cdots < \phi(S_i) < \phi(S_{i+1}) < \cdots < \phi_{max} \tag{2.89}$$

$$\lim_{i\to\infty} \phi(S_i) = \phi_{max}. \tag{2.90}$$

The objects $\{S_i\}_{i\in\mathbb{N}}$ are pairwise non-isomorphic. Since $\phi(S_0) - 1 > \phi_{min}$, there exists $S \in \sigma^{ss} \cap \mathcal{A}_{exc}$ with $\phi(S_0) - 1 > \phi(S) \geq \phi_{min}$. In particular, for each $i \in \mathbb{N}$ holds $\hom^*(S_i, S) = 0$. From table

(2.4) it follows that either $S = M$ or $S = M'$, i. e. there can be at most two elements in $\sigma^{ss} \cap \mathcal{A}_{exc}$ with phase strictly smaller than $\phi(S_0) - 1$ and such an element exists. Whence, there exists $S_{min} \in \sigma^{ss} \cap \mathcal{A}_{exc}$ of minimal phase, i. e. $\phi(S_{min}) = \phi_{min}$. Furthermore $S_{min} \in \{M, M'\}$.

If $S_{min} = M$. Now, due to $\hom^*(S_i, M) = 0$, table (2.4) shows that $\{S_i\}_{i\in\mathbb{N}} \subset \{E_3^m, E_4^m\}_{m\in\mathbb{N}}$. From Remark 2.83 and the monotone behavior (2.89) it follows that $S_i = E_3^{m_i}$ and $m_i < m_{i+1}$ for big enough $i \in \mathbb{N}$. Later in Corollary 2.98 **(a)** we show that such a sequence $\{S_i\}_{i\in\mathbb{N}}$ with (2.90) and the equality $\phi(M) = \phi_{min}$ imply that all elements of $\{E_3^j\}_{j\in\mathbb{N}}$ are semistable. Therefore, from

$$\phi(M) + 1 < \phi(E_3^{m_i}) \leq \phi(E_3^{m_i+1}) \leq \phi(E_3^{m_i+2}) \leq .. \leq \phi(E_3^{m_{i+1}-1}) \leq \phi(E_3^{m_{i+1}});$$
$$\phi(E_3^{m_i}) < \phi(E_3^{m_{i+1}})$$

it follows that for some $j \in \{m_i, m_i + 1, \dots, m_{i+1}\}$ we have $\phi(M) + 1 < \phi(E_3^j) < \phi(E_3^{j+1})$, hence we can apply Lemma 2.66 **(a)** to the triple (M, E_3^j, E_3^{j+1}), which contradicts our assumption on σ.

If $S_{min} = M'$. Now table (2.4) shows that $\{S_i\}_{i\in\mathbb{N}} \subset \{E_1^m, E_2^m\}_{m\in\mathbb{N}}$ and Remark 2.83 shows that for big enough $i \in \mathbb{N}$ we have $S_i = E_2^{m_i}, m_i < m_{i+1}$. By Corollary 2.100 (a) we obtain $\{E_2^j\}_{j\in\mathbb{N}} \subset \sigma^{ss}$. Now similar arguments as in the previous case (with an exceptional triple (M', E_2^j, E_2^{j+1}) for some $j \in \mathbb{N}$) lead us to a contradiction.

So far, we derived that there exists $S_{max} \in \mathcal{P}(\phi_{max}) \cap \mathcal{A}_{exc}$. Next, suppose that $\mathcal{P}(\phi_{min}) \cap \mathcal{A}_{exc} = \emptyset$. Then we have a sequence $\{S_i\}_{i\in\mathbb{N}} \subset \sigma^{ss} \cap \mathcal{A}_{exc}$ with

$$\phi_{max} - 1 > \phi(S_i) > \phi(S_{i+1}) > \phi_{min} \qquad \lim_{i\to\infty} \phi(S_i) = \phi_{min}. \tag{2.91}$$

It is clear that $\hom^*(S_{max}, S_i) = 0$ for each $i \in \mathbb{N}$, hence (by table (2.4)) we see that $S_{max} \in \{M, M'\}$.

If $S_{max} = M'$. In this case from table (2.4) it follows that $\{S_i\}_{i\in\mathbb{N}} \subset \{E_3^m, E_4^m\}_{m\in\mathbb{N}}$. By Remark 2.83 and the monotone behavior (2.91) we can construct the sequence so that $S_i = E_4^{m_i}, m_i < m_{i+1}$ for $i \in \mathbb{N}$. Now Corollary 2.98 **(b)** shows that $\{E_4^j\}_{j\in\mathbb{N}} \subset \sigma^{ss}$. Hence, for some $j \in \mathbb{N}$ we can apply Lemma 2.66 **(a)** to the triple (E_4^{j+1}, E_4^j, M'), which is a contradiction.

If $S_{max} = M$. Since we have $\{\hom^*(M, S_i) = 0\}_{i\in\mathbb{N}}$, table (2.4) shows that $\{S_i\}_{i\in\mathbb{N}} \subset \{E_1^m, E_2^m\}_{m\in\mathbb{N}}$. From Remark 2.83 we get $S_i = E_1^{m_i}$, $m_i < m_{i+1}$ for $i \in \mathbb{N}$. Corollary 2.100 **(b)** shows that $\{E_1^j\}_{j\in\mathbb{N}} \subset \sigma^{ss}$, hence for some $j \in \mathbb{N}$ we can use Lemma 2.66 **(a)** with the triple (E_1^{j+1}, E_1^j, M), which gives us a contradiction. The proposition is proved. □

We divide the proof of Corollaries 2.98, 2.100 in several lemmas.

Lemma 2.95. *Let $S_{min} \in \mathcal{P}(\phi_{min}) \cap \mathcal{A}_{exc}$. Let $R \in \mathcal{A}_{exc}$ be either a **C2** object or a **C3** object. If $\hom^*(R, S_{min}) = 0$, then there exists $S \in \sigma^{ss} \cap \mathcal{A}_{exc}$ with $\hom^*(S, S_{min}) = 0$ and $\phi(S) + 1 < \phi_{max}$.*

Proof. Presenting the arguments below we keep in mind Corollary 2.92.

If R is **C2**, then we have $\mathfrak{alg}(R) = \begin{array}{ccc} A_1 \oplus A_2[-1] & \longrightarrow & R \\ & \nwarrow \quad \swarrow & \\ & B & \end{array}$, $A_2, B \in \mathcal{A} \setminus \{0\}$, $\phi_{max} \geq \phi(A_2) > \phi(B) + 1$. From $\phi_-(R) = \phi(B) \geq \phi(S_{min})$, $\hom^*(R, S_{min}) = 0$, and Lemma 2.19 it follows, that $\hom^*(B, S_{min})$. Any $S \in Ind(B)$ satisfies the desired properties and the lemma follows.

If R is **C3**, then $\mathfrak{alg}(R) = \begin{array}{ccc} A & \longrightarrow & R \\ & \nwarrow \quad \swarrow & \\ & B[1] & \end{array}$, $A, B \in \mathcal{A} \setminus \{0\}$, $\phi_{max} \geq \phi(A) > \phi(B) + 1 \geq \phi(S_{min}) + 1$, hence $\hom^*(A, S_{min}) = 0$, which, together with $\hom^*(R, S_{min}) = 0$, implies $\hom^*(B, S_{min}) = 0$. Now the lemma follows with any $S \in Ind(B)$. □

Lemma 2.96. *Let $S_{max} \in \mathcal{A}_{exc}$ satisfy $\phi(S_{max}) = \phi_{max}$, and let $R \in \mathcal{A}_{exc}$ be either a **C2** or a **C3** object. If* $\hom^*(S_{max}, R) = 0$*, then there exists $S \in \sigma^{ss} \cap \mathcal{A}_{exc}$ with* $\hom^*(S_{max}, S) = 0$ *and* $\phi(S) > \phi_{min} + 1$.

Proof. If R is **C2**, then we can write $\mathfrak{alg}(R) = \begin{array}{ccc} A_1 \oplus A_2[-1] & \longrightarrow & R \\ & \nwarrow \quad \swarrow & \\ & B & \end{array}$, $A_2, B \in \mathcal{A} \setminus \{0\}$, $\phi_{max} = \phi(S_{max}) \geq \phi(A_2) > \phi(B) + 1 \geq \phi_{min} + 1$. Hence $\hom^*(S_{max}, B) = 0$, which, together with $\hom^*(S_{max}, R) = 0$, implies $\hom^*(S_{max}, A_2) = 0$. Now the lemma follows with $S \in Ind(A_2)$.

If R is **C3**, then $\mathfrak{alg}(R) = \begin{array}{ccc} A & \longrightarrow & R \\ & \nwarrow \quad \swarrow & \\ & B[1] & \end{array}$, $A, B \in \mathcal{A} \setminus \{0\}$, $\phi(S_{max}) \geq \phi(A) > \phi(B) + 1 \geq \phi_{min} + 1$, hence $\hom^*(S_{max}, B) = \hom^*(S_{max}, A) = 0$. Now any $S \in Ind(A)$ has the desired properties. □

Lemma 2.97. *Let $M \in \mathcal{P}(\phi_{min})$ or $M' \in \mathcal{P}(\phi_{max})$. If for some $m > 0$ we have $E_3^m \in \sigma^{ss}$ or $E_4^m \in \sigma^{ss}$, then there is not a **C1** object in the set $\{E_3^j, E_4^j\}_{j \in \mathbb{N}}$.*

Proof. Suppose that some $R \in \{E_3^j, E_4^j\}_{j \in \mathbb{N}}$ is a **C1** object. From Lemma 2.77 and $\hom^*(E_{3/4}^j, M) = \hom^*(M', E_{3/4}^j) = 0$ for each $j \in \mathbb{N}$ we see that R must be final,[55] hence $\mathfrak{alg}(R)$ is the HN filtration of R. In particular, from $R \overset{\mathbf{C1}}{\dashrightarrow} (X, Y)$ it follows that X, Y are semistable and $\phi(Y) > \phi(X)$. Now Lemma 2.88 (look at the last two rows in the table) contradicts the following negations:[56]

$\neg \left(E_3^0, E_1^0 \in \sigma^{ss} \text{ and } \phi(E_3^0) > \phi(E_1^0)\right)$. *Proof:* If $E_3^0, E_1^0 \in \sigma^{ss}$, then from $E_3^m \in \sigma^{ss}$ or $E_4^m \in \sigma^{ss}$,$m > 0$ and $\hom(E_3^0, E_{3/4}^m) \neq 0$, $\hom(E_{3/4}^m, E_1^0) \neq 0$ it follows $\phi(E_3^0) \leq \phi(E_1^0)$.

$\neg \left(E_3^0, E_4^0 \in \sigma^{ss} \text{ and } \phi(E_3^0) > \phi(E_4^0)\right)$. *Proof:* We are given $m > 0$ with $E_3^m \in \sigma^{ss}$ or $E_4^m \in \sigma^{ss}$, hence $\hom(E_3^0, E_{3/4}^m) \neq 0$, $\hom(E_{3/4}^m, E_4^0) \neq 0$ imply $\phi(E_3^0) \leq \phi(E_4^0)$).

$\neg \left(E_2^0, E_1^0 \in \sigma^{ss} \text{ and } \phi(E_2^0) > \phi(E_1^0)\right)$. *Proof:* If $\phi(M) = \phi_{min}$, then from $\hom(E_2^0, M) \neq 0$ it follows $\phi(E_2^0) = \phi(M) = \phi_{min} \leq \phi(E_1^0)$. If $\phi(M') = \phi_{max}$, then $\hom(M', E_1^0) \neq 0$ implies $\phi(E_2^0) \leq \phi(M') = \phi_{max} = \phi(E_1^0)$.

$\neg \left(E_2^n, M' \in \sigma^{ss} \text{ and } \phi(E_2^n) > \phi(M')\right)$. *Proof:* If $\phi(M) = \phi_{min}$, then by $\hom(E_2^n, M) \neq 0$ we get $\phi(E_2^n) = \phi_{min}$. If $\phi(M') = \phi_{max}$, then $\phi(E_2^n) \leq \phi(M')$.

[55]Recall that we have Corollary 2.92 at our disposal, due to our assumption on σ.

[56]For a statement p, when we write $\neg p$ we mean: "p is not true".

$\neg\ (M, E_1^n \in \sigma^{ss}$ and $\phi(M) > \phi(E_1^n))$. *Proof:* If $\phi(M) = \phi_{min}$, then from $E_1^n \in \sigma^{ss}$ it follows $\phi(M) \leq \phi(E_1^n)$. If $\phi(M') = \phi_{max}$, then $\hom(M', E_1^n) \neq 0$ implies $\phi(M) \leq \phi(M') = \phi_{max} = \phi(E_1^n)$.

The lemma follows. □

Corollary 2.98. *Let $\{S_i\}_{i\in\mathbb{N}}$ be a sequence of pairwise non-isomorphic, semistable objects with $\{S_i\}_{i\in\mathbb{N}} \subset \{E_3^j,\ E_4^j\}_{j\in\mathbb{N}}$. If any of the two conditions below is satisfied*

- ***(a)*** *$M \in \mathcal{P}(\phi_{min})$, $\lim_{i\to\infty} \phi(S_i) = \phi_{max}$,*
- ***(b)*** *$M' \in \mathcal{P}(\phi_{max})$, $\lim_{i\to\infty} \phi(S_i) = \phi_{min}$,*

then all the exceptional objects in the set $\{E_3^j,\ E_4^j\}_{j\in\mathbb{N}}$ are semistable.

Proof. Since each $E \in \{E_3^j,\ E_4^j\}_{j\in\mathbb{N}}$ is a trivially coupling object, it is neither **B1** nor **B2** (Corollary 2.49). From Lemma 2.97 we know that any E is either semistable or **Ci**(i=2,3). However, if it is **Ci**(i=2,3), then:

(a) By $\hom^*(E, M) = 0$ (see table (2.4)), $\phi(M) = \phi_{min}$, and Lemma 2.95 there exists $S \in \sigma^{ss} \cap \mathcal{A}_{exc}$ with $\hom^*(S, M) = 0$ and $\phi(M) + 1 \leq \phi(S) + 1 < \phi_{max}$, which by $\lim_{i\to\infty} \phi(S_i) = \phi_{max}$ implies that (M, S, S_i) is an exceptional triple for big enough i. By Corollary 2.11 this cannot happen, since $\{S_i\}_{i\in\mathbb{N}}$ are pairwise non-isomorphic.

(b) By $\hom^*(M', E) = 0$(see table (2.4)), $\phi(M') = \phi_{max}$, and Lemma 2.96 there exists $S \in \sigma^{ss} \cap \mathcal{A}_{exc}$ with $\hom^*(M', S) = 0$ and $\phi(M') \geq \phi(S) > \phi_{min} + 1$, which by $\lim_{i\to\infty} \phi(S_i) = \phi_{min}$ implies that (S_i, S, M') is an exceptional triple for big enough i. This contradicts Corollary 2.11. □

The arguments for the proof of Corollary 2.100 are the same, but the role of Lemma 2.97 is played by the following Lemma 2.99.

Lemma 2.99. *Let $M' \in \mathcal{P}(\phi_{min})$ or $M \in \mathcal{P}(\phi_{max})$. If for some $m > 0$ we have $E_1^m \in \sigma^{ss}$ or $E_2^m \in \sigma^{ss}$, then there is not a **C1** object in the set $\{E_1^j,\ E_2^j\}_{j\in\mathbb{N}}$.*

Proof. Using that for each $j \in \mathbb{N}$ we have $\hom^*(E_{1/2}^j, M') = 0$, $\hom^*(M, E_{1/2}^j) = 0$ and Lemma 2.88 (this time the first two rows in the table) by the same arguments as in Lemma 2.97 we reduce the proof to the negations:

$\neg\ \big(E_2^0, E_1^0 \in \sigma^{ss}$ and $\phi(E_2^0) > \phi(E_1^0)\big)$. *Proof:* If $E_2^0, E_1^0 \in \sigma^{ss}$, then by $\hom(E_2^0, E_{1/2}^m) \neq 0$, $\hom(E_{1/2}^m, E_1^0) \neq 0$, $m > 0$ it follows that $\phi(E_2^0) \leq \phi(E_1^0)$.

$\neg\ \big(E_3^0, E_1^0 \in \sigma^{ss}$ and $\phi(E_3^0) > \phi(E_1^0)\big)$. *Proof:* Follows from $\hom(E_3^0, E_{1/2}^m) \neq 0$ and $\hom(E_{1/2}^m, E_1^0) \neq 0$.

$\neg\ (E_3^0, E_4^0 \in \sigma^{ss}$ and $\phi(E_3^0) > \phi(E_4^0))$. *Proof:* If $\phi(M') = \phi_{min}$, then from $\hom(E_3^0, M') \neq 0$ it follows $\phi_{min} = \phi(E_3^0) \leq \phi(E_4^0)$. If $\phi(M) = \phi_{max}$, then $\hom(M, E_4^0) \neq 0$ implies $\phi_{max} = \phi(E_4^0) \geq \phi(E_3^0)$.

$\neg\ (M, E_3^n \in \sigma^{ss}$ and $\phi(E_3^n) > \phi(M))$. *Proof:* If $\phi(M') = \phi_{min}$, then we use $\hom(E_3^n, M') \neq 0$. If $\phi(M) = \phi_{max}$, then $E_3^n \in \sigma^{ss}$ implies $\phi(E_3^n) \leq \phi(M)$.

$\neg\ (E_4^n, M' \in \sigma^{ss}$ and $\phi(M') > \phi(E_4^n))$. *Proof:* If $\phi(M') = \phi_{min}$, then $E_4^n \in \sigma^{ss}$ implies $\phi(M') \leq \phi(E_4^n)$. If $\phi(M) = \phi_{max}$, then the negation follows from $\hom(M, E_4^n) \neq 0$.

The lemma follows. □

Corollary 2.100. *Let $\{S_i\}_{i\in\mathbb{N}} \subset \{E_1^j,\ E_2^j\}_{j\in\mathbb{N}}$ be a sequence of pairwise non-isomorphic, semistable objects. Any of the following two settings:*

(a) $M' \in \mathcal{P}(\phi_{min})$, $\lim_{i\to\infty} \phi(S_i) = \phi_{max}$,

(b) $M \in \mathcal{P}(\phi_{max})$, $\lim_{i\to\infty} \phi(S_i) = \phi_{min}$,

implies that $\{E_1^j,\ E_2^j\}_{j\in\mathbb{N}} \subset \sigma^{ss}$.

Proof. The arguments are the same as those used in the proof of Corollary 2.98, but we use Lemma 2.99 instead of Lemma 2.97. □

Note that the conclusions of Corollaries 2.100 and 2.98, namely that $\{E_2^m, E_1^m\} \subset \sigma^{ss}$ and $\{E_3^m, E_4^m\} \subset \sigma^{ss}$, are components of the data in the two non-locally finite cases (2.86). In the next subsection we derive (2.86) from the assumption $\phi_{max} - \phi_{min} > 1$, and Corollaries 2.100, 2.98 will be helpful at some points.

The implications given below are further minor steps towards derivation of the non-locally finite cases (2.86). These implications will be used in both Subsection 2.10.3 and Subsection 2.10.4.

Lemma 2.101.

(a) *If $\phi_{max} = \phi(M')$ and $\{E_4^m : m \in \mathbb{N}\} \subset \sigma^{ss}$, then $\{E_4^m : m \in \mathbb{N}\} \subset \mathcal{P}(t)$ for some $t \leq \phi_{max}$.*

(b) *If $\phi_{max} = \phi(M)$ and $\{E_1^m : m \in \mathbb{N}\} \subset \sigma^{ss}$, then $\{E_1^m : m \in \mathbb{N}\} \subset \mathcal{P}(t)$ for some $t \leq \phi_{max}$.*

(c) *If $\phi_{min} = \phi(M')$ and $\{E_2^m : m \in \mathbb{N}\} \subset \sigma^{ss}$, then $\{E_2^m : m \in \mathbb{N}\} \subset \mathcal{P}(t)$ for some $t \leq \phi_{max}$.*

(d) *If $\phi_{min} = \phi(M)$ and $\{E_3^m : m \in \mathbb{N}\} \subset \sigma^{ss}$, then $\{E_3^m : m \in \mathbb{N}\} \subset \mathcal{P}(t)$ for some $t \leq \phi_{max}$.*

Proof. Presenting the proof we keep in mind Remark 2.83:

(a) For any $m \in \mathbb{N}$ we have $\phi(E_4^{m+1}) \leq \phi(E_4^m) \leq \phi(M')$. The triple (E_4^{m+1}, E_4^m, M') has $\hom(E_4^m, M') = 0$. Hence from Lemma 2.66 **(c)** it follows $\phi(E_4^{m+1}) = \phi(E_4^m)$ for each $m \in \mathbb{N}$.

(b) We apply the same arguments as in **(a)** to the triple (E_1^{n+1}, E_1^n, M) with $\hom(E_1^n, M) = 0$.

(c) Now $\phi(M') \leq \phi(E_2^n) \leq \phi(E_2^{n+1})$, $\hom(M', E_2^n) = 0$ and we can apply Lemma 2.66 **(b)** to the triple (M', E_2^n, E_2^{n+1}), which implies $\phi(E_2^n) = \phi(E_2^{n+1})$ for each $n \geq 0$.

(d) We apply the same arguments as in **(c)** to the triple (M, E_3^n, E_3^{n+1}) with $\hom(M, E_3^n) = 0$. □

2.10.3 The case $\phi_{max} - \phi_{min} > 1$

In this subsection we show that the inequality $\phi_{max} - \phi_{min} > 1$ is inconsistent with the assumption that there is not a σ-exceptional triple. The inequality $\phi_{max} - \phi_{min} > 1$ implies by Proposition 2.94 that (for brevity we denote this product by Φ):

$$\Phi = (\mathcal{P}(\phi_{min}) \cap \mathcal{A}_{exc}) \times (\mathcal{P}(\phi_{max}) \cap \mathcal{A}_{exc}) \neq \emptyset. \tag{2.92}$$

If $(S_{min}, S_{max}) \in \Phi$, then (S_{min}, S_{max}) is an exceptional pair, since $\phi_{max} - \phi_{min} > 1$. Hence there exists unique $E \in \mathcal{A}_{exc}$, s. t. (S_{min}, E, S_{max}) is an exceptional triple. It is essential that E must be necessarily semistable, which follows from 2.80.

F the rest of this subsection we assume that $\phi_{max} - \phi_{min} > 1$. In the end we conclude that $\Phi \neq \emptyset$ contradicts the non-existence of a σ-exceptional triple.

Since any $(S_{min}, S_{max}) \in \Phi$ is an exceptional pair in $\mathcal{A}$, it must be some of the pairs listed in Corollary 2.9. We show case-by-case (in a properly chosen order) that for each pair (A, B) in this list the incidence $(A, B) \in \Phi$ leads to a contradiction. We show first that $(E_1^0, E_3^0) \notin \Phi$.

Lemma 2.102. $(E_1^0, E_3^0) \notin \Phi$.

Proof. Suppose that $(E_1^0, E_3^0) \in \Phi$. We consider the triple (E_1^0, M, E_3^0). From Proposition 2.80 it follows that $M \in \sigma^{ss}$, hence $\phi_{min} = \phi(E_1^0) \leq \phi(M) \leq \phi(E_3^0) = \phi_{max}$. One of these inequalities must be proper. However, by $\hom(E_1^0, M) = \hom(M, E_3^0) = 0$ and Lemma 2.66 **(b)**, **(c)** we obtain a σ-exceptional triple, which is a contradiction. □

We introduce the following formal rules, which facilitate the exposition:

$$(A, C) \in \Phi \xrightarrow{(A,B,C)} \text{ either } (B, C) \in \Phi \text{ or } (A, B) \in \Phi \tag{2.93}$$

$$(A, C) \in \Phi \xrightarrow{(A,B,C),\ \hom(A,B)=0} (A, B) \in \Phi \tag{2.94}$$

$$(A, C) \in \Phi \xrightarrow{(A,B,C),\ \hom(B,C)=0} (B, C) \in \Phi. \tag{2.95}$$

In (2.93), (2.94), and (2.95) the triple (A, B, C) is the unique exceptional triple(taken from Lemma 2.10) with first element A and last element C. In all the three rules we implicitly use Proposition 2.80, from which it follows $B \in \sigma^{ss}$, and hence $\phi_{min} = \phi(A) \leq \phi(B) \leq \phi(C) = \phi_{max}$. The specific arguments assigned to each individual rule are:

(2.93) from Lemma 2.66 **(a)** and $\phi_{max} - \phi_{min} > 1$ it follows that either $\phi(A) = \phi(B) = \phi_{min}$ or $\phi(B) = \phi(C) = \phi_{max}$, whence we reduce to either $(B, C) \in \Phi$ or $(A, B) \in \Phi$;

(2.94) by Lemma 2.66 **(b)** and $\hom(A, B) = 0$ we get $\phi(B) = \phi(C) = \phi_{max}$, whence $(A, B) \in \Phi$;

(2.95) by Lemma 2.66 **(c)** and $\hom(B, C) = 0$ we get $\phi(A) = \phi(B) = \phi_{min}$, whence $(B, C) \in \Phi$.

Now we eliminate some pairs (X, Y) by showing that $(X, Y) \in \Phi$ implies $(E_1^0, E_3^0) \in \Phi$.

Corollary 2.103. *For each $n \in \mathbb{N}$ any of the pairs $(E_4^0, E_3^0), (E_1^0, E_2^0), (M, E_3^n), (E_4^n, M'), (E_1^n, M)$, (M', E_2^n), (E_1^{n+1}, E_4^n), (E_4^n, E_1^n), (E_4^{n+1}, E_4^n), (E_1^{n+1}, E_1^n), (E_2^n, E_3^{n+1}), (E_3^n, E_2^n), (E_3^n, E_3^{n+1}), (E_2^n, E_2^{n+1}) is not in Φ.*

Proof. We keep in mind the formal rules (2.93), (2.94), (2.95). The following expressions and Lemma 2.102 show that each of the listed pairs is not in Φ.

$(E_4^0, E_3^0) \in \Phi \xrightarrow{(E_4^0,E_1^0,E_3^0),\ \hom(E_1^0,E_3^0)=0} (E_1^0, E_3^0) \in \Phi.$

$(E_1^0, E_2^0) \in \Phi \xrightarrow{(E_1^0,E_3^0,E_2^0),\ \hom(E_1^0,E_3^0)=0} (E_1^0, E_3^0) \in \Phi.$

$(M, E_3^0) \in \Phi \xrightarrow{(M,E_4^0,E_3^0),\ \hom(E_4^0,E_3^0)=0} (E_4^0, E_3^0) \in \Phi.$

$(M, E_3^n) \in \Phi, n \geq 1 \xrightarrow{(M,E_3^{n-1},E_3^n),\ \hom(M,E_3^{n-1})=0} (M, E_3^{n-1}) \in \Phi \xrightarrow{\text{induction}} (M, E_3^0).$

$(E_4^0, M') \in \Phi \xrightarrow{(E_4^0,E_3^0,M'),\ \hom(E_4^0,E_3^0)=0} (E_4^0, E_3^0) \in \Phi.$

$(E_4^n, M') \in \Phi, n \geq 1 \xrightarrow{(E_4^n,E_4^{n-1},M'),\ \hom(E_4^{n-1},M')=0} (E_4^{n-1}, M') \in \Phi \xrightarrow{\text{induction}} (E_4^0, M').$

$(E_1^0, M) \in \Phi \xrightarrow{(E_1^0,E_2^0,M),\ \hom(E_1^0,E_2^0)=0} (E_1^0, E_2^0) \in \Phi.$

$(E_1^n, M) \in \Phi, n \geq 1 \xrightarrow{(E_1^n,E_1^{n-1},M),\ \hom(E_1^{n-1},M)=0} (E_1^{n-1}, M) \in \Phi \xrightarrow{\text{induction}} (E_1^0, M).$

$(M', E_2^0) \in \Phi \xrightarrow{(M',E_1^0,E_2^0),\ \hom(E_1^0,E_2^0)=0} (E_1^0, E_2^0) \in \Phi.$

$(M', E_2^n) \in \Phi, n \geq 1 \xrightarrow{(M',E_2^{n-1},E_2^n),\ \hom(M',E_2^{n-1})=0} (M', E_2^{n-1}) \in \Phi \xrightarrow{\text{induction}} (M', E_2^0).$

$(E_1^{n+1}, E_4^n) \in \Phi, n \geq 0 \xrightarrow{(E_1^{n+1},M,E_4^n),\ \hom(E_1^{n+1},M)=0} (E_1^{n+1}, M) \in \Phi.$

$(E_4^n, E_1^n) \in \Phi, n \geq 0 \xrightarrow{(E_4^n,M',E_1^n),\ \hom(E_4^n,M')=0} (E_4^n, M') \in \Phi.$

$(E_4^{n+1}, E_4^n) \in \Phi, n \geq 0 \xrightarrow{(E_4^{n+1},E_1^{n+1},E_4^n)}$ either $(E_1^{n+1}, E_4^n) \in \Phi$ or $(E_4^{n+1}, E_1^{n+1}) \in \Phi.$

$(E_1^{n+1}, E_1^n) \in \Phi, n \geq 0 \xrightarrow{(E_1^{n+1},E_4^n,E_1^n)}$ either $(E_4^n, E_1^n) \in \Phi$ or $(E_1^{n+1}, E_4^n) \in \Phi.$

$(E_2^n, E_3^{n+1}) \in \Phi, n \geq 0 \xrightarrow{(E_2^n,M,E_3^{n+1}),\ \hom(M,E_3^{n+1})=0} (M, E_3^{n+1}) \in \Phi.$

$(E_3^n, E_2^n) \in \Phi, n \geq 0 \xrightarrow{(E_3^n,M',E_2^n),\ \hom(M',E_2^n)=0} (M', E_2^n) \in \Phi.$

$(E_3^n, E_3^{n+1}) \in \Phi, n \geq 0 \xrightarrow{(E_3^n,E_2^n,E_3^{n+1})}$ either $(E_2^n, E_3^{n+1}) \in \Phi$ or $(E_3^n, E_2^n) \in \Phi.$

$(E_2^n, E_2^{n+1}) \in \Phi, n \geq 0 \xrightarrow{(E_2^n,E_3^{n+1},E_2^{n+1})}$ either $(E_3^{n+1}, E_2^{n+1}) \in \Phi$ or $(E_2^n, E_3^{n+1}) \in \Phi.$ □

We eliminated many pairs by using only Section 2.2, Proposition 2.80, and Lemma 2.66. It remains to consider the incidences: (M, E_4^n), (E_3^n, M'), (M', E_1^n), $(E_2^n, M) \in \Phi$ for $n \geq 0$. From any of these incidences, with the help of Corollaries 2.98, 2.100 and Lemma 2.101, we will derive some of the non-locally finite cases (2.86), which are excluded by Corollaries 2.86, 2.85. We start with (M, E_4^n).

Lemma 2.104. *For each $n \geq 0$ we have $(M, E_4^n) \notin \Phi$.*

Proof. Suppose that $(M, E_4^n) \in \Phi$. In the previous corollary we showed that $(E_4^{n+1}, E_4^n) \notin \Phi$. Now from the implication $(M, E_4^n) \in \Phi, n \geq 0 \xrightarrow{(M,E_4^{n+1},E_4^n)}$ either $(E_4^{n+1}, E_4^n) \in \Phi$ or $(M, E_4^{n+1}) \in \Phi$ we deduce that $(M, E_4^{n+1}) \in \Phi$, and by induction we obtain $\phi(E_4^i) = \phi_{max}$ for $i \geq n$. We are given also $\phi(M) = \phi_{min}$. Hence, by Corollary 2.98 **(a)**, we obtain $\{E_3^j, E_4^j\}_{j\in\mathbb{N}} \subset \sigma^{ss}$. Now Remark 2.83 implies that $\{E_4^j\}_{j\in\mathbb{N}} \subset \mathcal{P}(\phi_{max})$. We will obtain a contradiction to Corollary 2.86 by deriving

$$\forall i \geq 0 \quad \phi(E_3^i) = \phi_{max} - 1. \tag{2.96}$$

Since $\hom^1(E_4^0, E_3^0) \neq 0$, we have $\phi(M) = \phi_{min} < \phi_{max} - 1 = \phi(E_4^0) - 1 \leq \phi(E_3^0) \leq \phi_{max}$. Whence:

$$\phi(M) < \phi(E_3^0) \leq \phi(E_4^0) \leq \phi(E_3^0) + 1.$$

If $\phi(E_4^0) < \phi(E_3^0) + 1$, then we have $\phi(M) < \phi(E_3^0) \leq \phi(E_4^0) < \phi(E_3^0) + 1$ and Lemma 2.66 **(d)** applied to the triple (M, E_4^0, E_3^0) gives us a σ-exceptional triple. Therefore $\phi(E_4^0) = \phi(E_3^0) + 1$. We showed above that E_3^j is semistable for each $j \in \mathbb{N}$. From Lemma 2.101 **(d)** we get $\phi(E_3^0) = \phi(E_3^i)$ for any $i \geq 0$, thus we get (2.96). □

Lemma 2.105. *For each $n \geq 0$ we have $(E_3^n, M') \notin \Phi$.*

Proof. Suppose that $(E_3^n, M') \in \Phi$. We obtain a contradiction of Corollary 2.86 as follows:
$(E_3^n, M') \in \Phi, n \geq 0 \xrightarrow{(E_3^n, E_3^{n+1}, M')}$ either $(E_3^{n+1}, M') \in \Phi$ or $(E_3^n, E_3^{n+1}) \in \Phi \xrightarrow{\text{Corollary 2.103}}$ $(E_3^{n+1}, M') \in \Phi \xrightarrow{\text{ind.}} \forall i \geq n \;\; \phi(E_3^i) = \phi_{min} \xrightarrow{\text{Corollary 2.98 (b)}} \{E_3^j, E_4^j\}_{j\in\mathbb{N}} \subset \sigma^{ss}$. By Remark 2.83 we see that $\phi(E_3^i) = \phi_{min}$ for $i \geq 0$. We show below that $(E_3^0, M') \in \Phi$ implies that $\phi(E_4^i) = \phi_{min} + 1$ for each $i \geq 0$, which contradicts Corollary 2.86.

Indeed, by $\hom^1(E_4^0, E_3^0) \neq 0$ we can write $\phi_{min} = \phi(E_3^0) \leq \phi(E_4^0) \leq \phi(E_3^0)+1 < \phi_{max} = \phi(M')$. The triple (E_4^0, E_3^0, M') has $\hom(E_4^0, E_3^0) = 0$, therefore from $\phi(E_4^0) < \phi(E_3^0) + 1$ it follows that for some $j \geq 1$ the triple $(E_4^0, E_3^0, M'[-j])$ is σ-exceptional. Therefore $\phi(E_4^0) = \phi(E_3^0) + 1$. We showed above that $\{E_4^j\} \subset \sigma^{ss}$. By Lemma 2.101 **(a)** we conclude that $\phi(E_4^n) = \phi_{min} + 1$ for each $n \geq 0$. □

Lemma 2.106. *For each $n \geq 0$ we have $(M', E_1^n) \notin \Phi$.*

Proof. Suppose that $(M', E_1^n) \in \Phi$. We show that this contradicts Corollary 2.85 as follows:
$(M', E_1^n) \in \Phi, n \geq 0 \xrightarrow{(M', E_1^{n+1}, E_1^n)}$ either $(E_1^{n+1}, E_1^n) \in \Phi$ or $(M', E_1^{n+1}) \in \Phi \xrightarrow{\text{Corollary 2.103}}$ $(M', E_1^{n+1}) \in \Phi \xrightarrow{\text{ind.}} \forall i \geq n \;\; \phi(E_1^i) = \phi_{max} \xrightarrow{\text{Corollary 2.100 (a)}} \{E_1^j, E_2^j\}_{j\in\mathbb{N}} \subset \sigma^{ss}$.
By Remark 2.83 we see that $\phi(E_1^i) = \phi_{max}$ for each $i \geq 0$. Furthermore, using $(M', E_1^0) \in \Phi$ we show below that $\phi(E_2^i) = \phi_{max} - 1$ must hold for $i \geq 0$, which contradicts Corollary 2.85.

Indeed, it follows from $\hom^1(E_1^0, E_2^0) \neq 0$ that $\phi_{min} = \phi(M') < \phi(E_1^0) - 1 \leq \phi(E_2^0) \leq \phi_{max} = \phi(E_1^0)$. If $\phi(E_1^0) < \phi(E_2^0)+1$, then $\phi(M') < \phi(E_2^0) \leq \phi(E_1^0) < \phi(E_2^0)+1$, and the triple (M', E_1^0, E_2^0) with $\hom(E_1^0, E_2^0) = 0$ gives rise to a σ-triple by Lemma 2.66 **(d)**. Therefore $\phi(E_1^0) = \phi(E_2^0) + 1$. Since $\{E_2^j\} \subset \sigma^{ss}$, Lemma 2.101 **(c)** implies that $\phi(E_2^n) = \phi(E_2^0)$ for $n \geq 0$. The lemma is proved. □

Lemma 2.107. *For each $n \geq 0$ we have $(E_2^n, M) \notin \Phi$.*

Proof. Suppose that $(E_2^n, M) \in \Phi$. We will obtain a contradiction of Corollary 2.85 as follows: $(E_2^n, M) \in \Phi, n \geq 0 \xrightarrow{(E_2^n, E_2^{n+1}, M)}$ either $(E_2^{n+1}, M) \in \Phi$ or $(E_2^n, E_2^{n+1}) \in \Phi \xrightarrow{\text{Corollary 2.103}}$ $(E_2^{n+1}, M) \in \Phi \xrightarrow{\text{ind.}} \forall i \geq n \;\; \phi(E_2^i) = \phi_{min} \xrightarrow{\text{Corollary 2.100 } \mathbf{(b)}} \{E_1^j, E_2^j\}_{j \in \mathbb{N}} \subset \sigma^{ss}$.
By Remark 2.83 we conclude that $\phi(E_2^i) = \phi_{min}$ for $i \geq 0$. We show below that $(E_2^0, M) \in \Phi$ implies that $\phi(E_1^i) = \phi_{min} + 1$ for each $i \geq 0$, which contradicts Corollary 2.85.

Indeed, it follows from $\hom^1(E_1^0, E_2^0) \neq 0$ and $(E_1^0, M) \notin \Phi$(see Corollary 2.103) that $\phi_{min} = \phi(E_2^0) < \phi(E_1^0) \leq \phi(E_2^0) + 1 < \phi_{max} = \phi(M)$. If $\phi(E_1^0) < \phi(E_2^0) + 1$, then for some $j \geq 1$ the triple $(E_1^0, E_2^0, M[-j])$ is σ-exceptional, since (E_1^0, E_2^0, M) is exceptional and $\hom(E_1^0, E_2^0) = 0$. Therefore $\phi(E_1^0) = \phi(E_2^0) + 1 < \phi(M)$. Lemma 2.101 **(b)** gives us $\phi(E_1^n) = \phi(E_1^0) = \phi_{min} + 1$ for $n \geq 0$. □

Therefore, we reduce to $\phi_{max} - \phi_{min} \leq 1$, which will be assumed until the end of the proof.

2.10.4 The case $\phi_{max} - \phi_{min} \leq 1$

From this inequality we obtain a contradiction here again, by deriving the non-locally finite cases (2.86). We show first in a series of lemmas that $\mathcal{A}_{exc} \subset \mathcal{P}(\phi_{min}) \cup \mathcal{P}(\phi_{max})$, $\phi_{\max} - \phi_{min} = 1$. Lemma 2.93 and Corollary 2.76 imply immediately

Lemma 2.108. *Any $E \in \mathcal{A}_{exc}$ is either semistable or irregular or a final **C1** object.*

Any $X \in \{E_i^j : j \in \mathbb{N}, 1 \leq i \leq 4\}$ is a trivially coupling object, hence by Lemma 2.49 we have only two possibilities: X is semistable or X is a final **C1** object (cannot be irregular).

Corollary 2.109. *The objects E_1^0, E_3^0 are semistable, and M is either irregular or semistable.*

Proof. The objects E_1^0, E_3^0, M cannot be **C1** by Lemma 2.89. □

Lemma 2.110. *The object E_2^0 is semistable.*

Proof. Suppose that E_2^0 is not semistable.

Therefore E_2^0 must be **C1**, and we have $E_2^0 \overset{\mathbf{C1}}{\dashrightarrow} (X, Y)$ for some exceptional pair (X, Y). By Lemma 2.89, we see that $(X, Y) = (M, E_3^0)$. Since E_2^0 is final, we can write

$$M, E_3^0 \in \sigma^{ss} \qquad \phi(M) < \phi(E_3^0).$$

From the triple (E_1^0, M, E_3^0), which satisfies $\hom(E_1^0, M) = \hom(E_1^0, E_3^0) = \hom(M, E_3^0) = 0$, and Lemma 2.66 (f) we see that $\phi(E_1^0) = \phi(M) + 1$ (recall that E_1^0 is semistable). From $\phi_{max} - \phi_{min} \leq 1$ it is clear that

$$\phi(M) = \phi_{min}, \qquad \phi(E_1^0) = \phi_{max} = \phi(M) + 1.$$

The obtained relations imply that E_4^0 is semistable. Indeed, if E_4^0 is not semistable, then it must be final **C1**, hence by Lemma 2.89 we have $E_4^0 \overset{\mathbf{C1}}{\dashrightarrow} (E_1^0, M)$, which in turn implies $\phi(M) > \phi(E_1^0)$

contradicting $\phi(E_1^0) = \phi(M) + 1$. Therefore E_4^0 is semistable. Now consider the triple (M, E_4^0, E_3^0) with $\hom(E_4^0, E_3^0) = 0$. We have $\phi(M) \leq \phi(E_4^0) \leq \phi(M) + 1$, $\phi(M) < \phi(E_3^0) \leq \phi(M) + 1$. If $\phi(M) < \phi(E_4^0)$, then $\phi(M) - 1 < \phi(E_4^0[-1]) \leq \phi(M)$, $\phi(M) - 1 < \phi(E_3^0[-1]) \leq \phi(M)$ and $(M, E_4^0[-1], E_3^0[-1])$ is a σ-exceptional triple. So far, assuming that E_2^0 is not semistable we get:

$$\phi_{min} = \phi(M) = \phi(E_4^0) < \phi(E_3^0) \leq \phi(E_1^0) = \phi(E_4^0) + 1.$$

Therefore $\phi(E_4^0) - 1 < \phi(E_3^0[-1]) \leq \phi(E_1^0[-1]) = \phi(E_4^0)$ and then the triple $(E_4^0, E_1^0[-1], E_3^0[-1])$ is a σ-exceptional triple (since $\hom(E_1^0, E_3^0) = 0$). This triple contradicts our assumption on σ. □

Lemma 2.111. *The object E_4^0 is semistable.*

Proof. Suppose that E_4^0 is not semistable. Hence it is final **C1**, and by Lemma 2.89 we have $E_4^0 \overset{\mathbf{C1}}{\dashrightarrow} (E_1^0, M)$. Since E_4^0 is final, it follows:

$$M, E_1^0 \in \sigma^{ss} \qquad \phi(M) > \phi(E_1^0).$$

From the simple objects triple (E_1^0, M, E_3^0) and Lemma 2.66 **(f)** it follows that $\phi(M) = \phi(E_3^0) + 1$ (recall that E_3^0 is semistable), hence by $\phi_{max} - \phi_{min} \leq 1$:

$$\phi(E_3^0) = \phi_{min}, \qquad \phi(M) = \phi_{max} = \phi_{min} + 1.$$

Now we have $\{E_1^0, E_2^0, E_3^0, M\} \subset \sigma^{ss}$. From Corollary 2.90 it follows $\phi(E_2^0) \leq \phi(E_1^0)$. Whence, we have $\phi_{min} = \phi(M) - 1 \leq \phi(E_2^0) \leq \phi(E_1^0) < \phi(M)$. The triple (E_1^0, E_2^0, M) has $\hom(E_1^0, E_2^0) = 0$. We rewrite the last inequalities as follows $\phi(M[-1]) \leq \phi(E_2^0) \leq \phi(E_1^0) < \phi(M[-1]) + 1$ and obtain a σ-exceptional triple $(E_1^0, E_2^0, M[-1])$, which is a contradiction. The lemma follows. □

Now, using that $\{E_i^0\}_{i=1}^4 \subset \sigma^{ss}$, we show that M, M' cannot be irregular.

Corollary 2.112. *There does not exist a **B2** object.*

Proof. Suppose that $E \in \mathcal{A}$ is a **B2** object. Since the only Ext-nontrivial couple is $\{M, M'\}$, we have $E \in \{M, M'\}$ and we can write

$$\mathfrak{alg}(E) = \begin{array}{ccc} A & \longrightarrow & E \\ & \nwarrow \; \swarrow & \\ & B[1] & \end{array} \qquad \begin{array}{c} \{E, \Gamma\} = \{M, M'\} \text{ for some } \Gamma \in Ind(B), \\ \phi_-(A) > \phi(B) + 1 = \phi(\Gamma) + 1. \end{array} \tag{2.97}$$

From $\hom(M, E_4^0) \neq 0$, $\hom(M', E_1^0) \neq 0$, and $\{E_1^0, E_4^0\} \subset \sigma^{ss}$ (shown in the preceding lemmas) it follows that there exists $X \in \sigma^{ss} \cap \mathcal{A}_{exc}$ with $\hom(E, X) \neq 0$, hence $\hom(A, X) \neq 0$ and $\phi_-(A) \leq \phi(X)$. Whence, we obtain $\phi(X) \geq \phi_-(A) > \phi(\Gamma) + 1$ with $X, \Gamma \in \sigma^{ss} \cap \mathcal{A}_{exc}$, which contradicts the inequality $\phi_{max} - \phi_{min} \leq 1$. □

Lemma 2.113. *There does not exist a σ-irregular object.*

Proof. By Corollary 2.112 we have to show that neither M nor M' can be **B1**.

Suppose that $E \in \{M, M'\}$ is **B1**, then we can write:

$$\mathfrak{alg}(E) = \begin{array}{ccc} A_1 \oplus A_2[-1] & \longrightarrow & E \\ & \nwarrow \, \swarrow & \\ & B & \end{array} \qquad \begin{array}{c} \{E, \Gamma\} = \{M, M'\} \text{ for some } \Gamma \in Ind(A_2), \\ \hom^1(A_1, A_1) = \hom^1(A_2, A_2) = 0 \\ \phi_-(A_1 \oplus A_2[-1]) \geq \phi(B) = \phi_-(E). \end{array}$$

We show first that each $Y \in Ind(A_2)$ must be semistable with $\phi(Y) = \phi(B) + 1$, which implies

$$A_2 \in \sigma^{ss}, \quad \phi(A_2[-1]) = \phi(B). \tag{2.98}$$

To that end we observe that there exists $X \in \sigma^{ss} \cap \mathcal{A}_{exc}$ with $\hom(X, B) \neq 0$, and hence

$$\phi(X) \leq \phi(B) \qquad X \in \sigma^{ss} \cap \mathcal{A}_{exc}. \tag{2.99}$$

Indeed, if we find $X \in \sigma^{ss} \cap \mathcal{A}_{exc}$ with $\hom^*(X, E) = 0$ and $\hom(X, \Gamma) \neq 0$, then from the triangle $\mathfrak{alg}(E)$ it follows that $\hom(X, B) \cong \hom(X, A_1[1] \oplus A_2) \neq 0$ (the latter does not vanish by $\Gamma \in Ind(A_2)$ and $\hom(X, \Gamma) \neq 0$). Looking at table (2.4) we see that $\hom^*(E_2^0, M') = 0, \hom(E_2^0, M) \neq 0$, $\hom^*(E_3^0, M) = 0, \hom(E_3^0, M') \neq 0$, therefore

$$\begin{array}{lll} X = E_2^0 & \text{if} & E = M' \\ X = E_3^0 & \text{if} & E = M. \end{array} \tag{2.100}$$

Let us take any $Y \in Ind(A_2)$. From Lemma 2.50 **(c)** it follows that Y cannot be σ-irregular. Hence it is either semistable or a final **C1** object. If Y is **C1**, then $Y \overset{\mathbf{C1}}{\dashrightarrow} (Z, W)$ for some $Z, W \in \sigma^{ss} \cap \mathcal{A}_{exc}$, and we can write $\phi(W) > \phi(Z) = \phi_-(Y) \geq \phi_-(A_2) \geq \phi(B) + 1 \geq \phi(X) + 1$, which contradicts $\phi_{max} - \phi_{min} \leq 1$. If Y is semistable, then by $\phi_{max} - \phi_{min} \leq 1$ it follows $\phi(Y) \leq \phi(X) + 1$, which, together with $\phi(Y) \geq \phi(B) + 1 \geq \phi(X) + 1$, implies $\phi(Y) = \phi(B) + 1 = \phi(X) + 1$. Whence, we proved (2.98). Furthermore, we see that (2.99) must be equality.

Being a **B1** object, E is not semistable. From the triangle $\mathfrak{alg}(E)$, the equality $\phi(A_2[-1]) = \phi(B)$ and the fact that $\mathcal{P}(t)$ is an extension closed subcategory of $\mathcal{T}$ it follows that $A_1 \neq 0$ and $\phi_+(A_1) > \phi(B)$. From **B1.1** we know that A_1 is a proper $\mathcal{A}$-subobject of E. Since M is simple in $\mathcal{A}$, it follows that E cannot be M. Whence, E must be M' and then $X = E_2^0$ (see (2.100)). The only proper subobject of M' in $\mathcal{A}$ up to isomorphism is E_3^0 and we know that it is semistable. Whence, we arrive at $\phi(E_3^0) > \phi(B) = \phi(X) = \phi(E_2^0)$, It follows that $\hom(E_3^0, E_2^0) = 0$, which contradicts table (2.4). $\square$

Corollary 2.114. *The objects M, M' are semistable and*

$$\phi(E_2^0) \leq \phi(E_1^0) \qquad \phi(E_3^0) \leq \phi(E_1^0) \qquad \phi(E_3^0) \leq \phi(E_4^0). \tag{2.101}$$

Proof. The semistability of M follows from Corollary 2.109 and Lemma 2.113. Then from Corollary 2.90 we get $\phi(E_2^0) \leq \phi(E_1^0)$ and $\phi(E_3^0) \leq \phi(E_1^0)$.

Using $\phi_{max} - \phi_{min} \leq 1$ we showed so far that the cases **C2, C3, B1, B2** can not appear. Therefore we have only two options for M': either semistable or final **C1**.

Suppose that M' is final **C1**. Lemma 2.89 implies that $M' \overset{\mathbf{C1}}{\dashrightarrow} (E_1^0, E_3^0)$. Therefore $\phi(E_3^0) > \phi(E_1^0) = \phi_-(M')$. However we showed already that $\phi(E_3^0) \leq \phi(E_1^0)$. Hence, M' must be also semistable. Now Corollary 2.91 implies $\phi(E_3^0) \leq \phi(E_4^0)$. □

So far, we showed that the low dimensional exceptional objects $\{E_i^0\}_{i=1}^4$, M, M' are semistable. The following implications, due to table (2.88) in Lemma 2.88, will help us to show that $\mathcal{A}_{exc} \subset \sigma^{ss}$.

Corollary 2.115. *Let $R \in \{E_i^m : m \geq 1, 1 \leq i \leq 4\}$ and let R be non-semistable.*

(a) *If $R = E_1^m$, then $\phi(E_4^n) < \phi(M')$ for some $n < m$, and hence $\phi(M) < \phi(M')$*

(b) *If $R = E_2^m$, then $\phi(M) < \phi(E_3^n)$ for some $n \leq m$, and hence $\phi(M) < \phi(M')$*

(c) *If $R = E_3^m$, then $\phi(M') < \phi(E_2^n)$ for some $n < m$, and hence $\phi(M') < \phi(M)$*

(d) *If $R = E_4^m$, then $\phi(E_1^n) < \phi(M)$ for some $n \leq m$, and hence $\phi(M') < \phi(M)$.*

Proof. Now we have $M, M' \in \sigma^{ss}$ and any non-semistable $R \in \mathcal{A}_{exc}$ is a final **C1** object. Note also that for each $n \in \mathbb{N}$ we have $\hom(M, E_4^n) \neq 0$, $\hom(E_3^n, M') \neq 0$, $\hom(E_2^n, M) \neq 0$, $\hom(M', E_1^n) \neq 0$, which implies $\phi(M) \leq \phi_+(E_4^n)$, $\phi_-(E_3^n) \leq \phi(M')$, $\phi_-(E_2^n) \leq \phi(M)$, $\phi(M') \leq \phi_+(E_1^n)$. Due to Lemma 2.61 and the inequalities (2.101) in Corollory 2.114, we can remove the pairs (E_1^0, E_2^0), (E_4^0, E_3^0), (E_1^0, E_3^0) from table (2.88) in Lemma 2.88. If $E_i^m \notin \sigma^{ss}$ for some $m \geq 1$, $1 \leq i \leq 4$, then E_i^m is a final **C1** object and the corollary follows from table (2.88) in Lemma 2.88. □

Knowing that the triple (E_1^0, M, E_3^0) of the simple objects is semistable, we obtain that one of three equalities below must hold, which implies $\phi_{max} - \phi_{min} = 1$.

Lemma 2.116. *There is an equality $\phi_{max} - \phi_{min} = 1$. One of the following equalities must hold:*

$$\phi(E_1^0) = \phi(M) + 1, \quad \phi(E_1^0) = \phi(E_3^0) + 1, \quad \phi(M) = \phi(E_3^0) + 1. \tag{2.102}$$

Proof. From $\hom(E_1^0, M[1]) \neq 0$, $\hom(E_1^0, E_3^0[1]) \neq 0$, $\hom(M, E_3^0[1]) \neq 0$ we have $\phi(E_1^0) \leq \phi(M) + 1$, $\phi(E_1^0) \leq \phi(E_3^0) + 1$, $\phi(M) \leq \phi(E_3^0) + 1$. Applying **(f)** of Lemma 2.66 to the triple (E_1^0, M, E_3^0), we see that one of the equalities (2.102) holds. Hence $\phi_{max} - \phi_{min} \geq 1$ and the lemma follows. □

Corollary 2.117. $\phi(M) \in \{\phi_{min}, \phi_{max}\}$.

Proof. Suppose that $\phi_{min} < \phi(M) < \phi_{max}$. By Lemma 2.116 we get $\phi_{min} = \phi(E_3^0)$ and $\phi(E_3^0)+1 = \phi(E_1^0) = \phi_{max}$. Therefore, we can write $\phi(E_3^0) < \phi(M) < \phi(E_3^0)+1$ and $\phi(E_3^0) \leq \phi(E_2^0) \leq \phi(E_3^0)+1$. Now by combining (2.11) and the equality $Z(E_2^0) = Z(E_3^0) + Z(M)$ (see Lemma (2.3)) we obtain:

$$\phi_{min} = \phi(E_3^0) < \phi(E_2^0) < \phi(E_3^0) + 1 = \phi(E_1^0) = \phi_{max}. \tag{2.103}$$

By semistability of M' we have either $\phi_{min} = \phi(E_3^0) < \phi(M')$ or $\phi_{min} = \phi(E_3^0) = \phi(M')$. We aim at a contradiction[57] by using either the triple (E_3^0, M', E_2^0) with $\hom(M', E_2^0) = 0$ or the triple (M', E_1^0, E_2^0) with $\hom(E_1^0, E_2^0) = 0$. If $\phi_{min} = \phi(E_3^0) < \phi(M')$, then we have $\phi(E_3^0) < \phi(M') \leq \phi(E_3^0) + 1$, $\phi(E_3^0) < \phi(E_2^0) < \phi(E_3^0) + 1$, hence the triple $(E_3^0, M'[-1], E_2^0[-1])$ is σ-exceptional. If $\phi_{min} = \phi(E_3^0) = \phi(M')$, then we have $\phi(E_1^0) = \phi(M') + 1$, $\phi(M') < \phi(E_2^0) < \phi(M') + 1$, hence the triple $(M', E_1^0[-1], E_2^0[-1])$ is σ-exceptional. □

Corollary 2.118. *We have* $\{\phi(M), \phi(M'), \phi(E_j^0)\} \subset \{\phi_{min}, \phi_{max}\}$ *for* $j = 1, 2, 3, 4$.

Proof. Now we have $\{\phi(M), \phi(M'), \phi(E_j^0)\} \subset \sigma^{ss}$ and $\phi(M) \in \{\phi_{min}, \phi_{max}\}$. It is enough to show $\{\phi(E_1^0), \phi(E_3^0)\} \subset \{\phi_{min}, \phi_{max}\}$, because then by formula (2.11), the equalities $Z(M') = Z(E_1^0) + Z(E_3^0)$, $Z(E_2^0) = Z(M) + Z(E_3^0)$, $Z(E_4^0) = Z(M) + Z(E_1^0)$, and the inequalities $\phi_{min} \leq \phi(M'), \phi(E_2^0), \phi(E_4^0) \leq \phi_{max}$ it follows that $\{\phi(M'), \phi(E_2^0), \phi(E_4^0)\} \subset \{\phi_{min}, \phi_{max}\}$.

If $\phi(M) = \phi_{min}$, then by $\hom(E_2^0, M) \neq 0$ it follows that $\phi(E_2^0) = \phi_{min}$, and by Lemma 2.116 it follows that $\phi(E_1^0) = \phi_{max}$. Expanding the equality $Z(E_2^0) = Z(M) + Z(E_3^0)$ by formula (2.11), and using $\phi(M) = \phi(E_2^0) = \phi_{min}$, $\phi_{min} \leq \phi(E_3^0) \leq \phi_{max}$, we conclude $\phi(E_3^0) \in \{\phi_{min}, \phi_{max}\}$.

If $\phi(M) = \phi_{max}$, then by $\hom(M, E_4^0) \neq 0$ it follows $\phi(E_4^0) = \phi_{max}$, and by Lemma 2.116 it follows $\phi(E_3^0) = \phi_{min}$. Finally, $\phi(E_1^0) \in \{\phi_{min}, \phi_{max}\}$ follows from $\phi(M) = \phi(E_4^0) = \phi_{max}$, $Z(E_4^0) = Z(M) + Z(E_1^0)$, and formula (2.11). The corollary is proved. □

The proofs of semistability for E_1^m and E_2^m share some steps because the non-semistability of any of them implies $\phi(M) < \phi(M')$(Corollary 2.115 **(a)**, **(b)**). Similarly, the starting argument in the proof of Lemma 2.120 is that the non-semistability of E_3^m or E_4^m implies $\phi(M') < \phi(M)$.

Lemma 2.119. *All objects in* $\{E_1^m, E_2^m\}_{m \in \mathbb{N}}$ *are semistable.*

Proof. Suppose that E_1^m is not semistable for some $m \in \mathbb{N}$. Corollary 2.115 **(a)** shows that $E_4^n \in \sigma^{ss}$, $\phi(E_4^n) < \phi(M')$ for some $n \in \mathbb{N}$, and $\phi(M) < \phi(M')$. The latter inequality implies, due to Corollary 2.115 **(c)** and **(d)**, that $\{E_4^m, E_3^m\}_{m \in \mathbb{N}} \subset \sigma^{ss}$, and, due to Corollary 2.118, it implies

$$\phi_{min} = \phi(M), \quad \phi(M') = \phi_{max} = \phi_{min} + 1. \tag{2.104}$$

By Lemma 2.101 **(a)** we can write $\phi(E_4^0) = \phi(E_4^n) < \phi(M')$ and combining with Corollary 2.114 we arrive at $\phi_{min} = \phi(M') - 1 \leq \phi(E_3^0) \leq \phi(E_4^0) < \phi(M')$, hence the triple $(E_4^0, E_3^0, M'[-1])$ with $\hom(E_4^0, E_3^0) = 0$ is a σ-exceptional triple. Therefore $\{E_1^m\}_{m \in \mathbb{N}} \subset \sigma^{ss}$.

[57] of the assumption that there is not a σ-exceptional triple

Next, suppose that E_2^m is not semistable for some $m \in \mathbb{N}$. Then by Corollary 2.115 **(b)** we have $E_3^n \in \sigma^{ss}$, $\phi(M) < \phi(E_3^n)$ for some $n \in \mathbb{N}$, and $\phi(M) < \phi(M')$. Now by the same arguments as above we get (2.104) and $\{E_4^m, E_3^m\}_{m\in\mathbb{N}} \subset \sigma^{ss}$. By Lemma 2.101 **(d)** we can write $\phi(E_3^0) = \phi(E_3^n) > \phi(M)$. Combining with Corollary 2.114 we arrive at $\phi_{min} = \phi(M) < \phi(E_3^0) \leq \phi(E_4^0) \leq \phi(M)+1$. These inequalities and the exceptional triple (M, E_4^0, E_3^0) with $\hom(E_4^0, E_3^0) = 0$ provide a σ-exceptional triple $(M, E_4^0[-1], E_3^0[-1])$. The lemma follows. □

Lemma 2.120. *All objects in* $\{E_3^m, E_4^m\}_{m\in\mathbb{N}}$ *are semistable.*

Proof. Suppose that E_4^m or E_3^m is not semistable for some $m \in \mathbb{N}$. By Lemma 2.115 we get $\phi(M') < \phi(M)$. Since $\{\phi(M), \phi(M')\} \subset \{\phi_{min}, \phi_{max}\}$ (Corollary 2.118), we find that:

$$\phi_{min} = \phi(M'), \quad \phi(M) = \phi_{max} = \phi_{min} + 1.$$

We have also $\{E_1^m, E_2^m\}_{m\in\mathbb{N}} \subset \sigma^{ss}$. Thus, **(b)** and **(c)** in Lemma 2.101 can be used to obtain:

$$\forall m \in \mathbb{N} \qquad \phi(E_1^m) = \phi(E_1^0), \quad \phi(E_2^m) = \phi(E_2^0). \tag{2.105}$$

From $\hom(M, E_4^0) \neq 0$, and $\hom(E_4^0, E_1^0) \neq 0$ (note that $\hom(E_4^0, E_1^m) = 0$ for $m \geq 1$) it follows $\phi(M) = \phi_{max} = \phi(E_1^0)$. On the other hand, from the triple (M', E_1^0, E_2^0) with $\hom(E_1^0, E_2^0) = 0$ it follows that $\phi(E_2^0) = \phi(M') = \phi_{min}$ (otherwise $(M', E_1^0[-1], E_2^0[-1])$ would be a σ-exceptional triple). Using (2.105) we obtain

$$\forall m \in \mathbb{N} \qquad \phi_{max} = \phi(M) = \phi(E_1^m), \quad \phi_{min} = \phi(M') = \phi(E_2^m).$$

However, due to **(c)** and **(d)** in Corollary 2.115, these equalities contradict the assumption that E_3^m or E_4^m is not semistable for some m. The lemma follows. □

Corollary 2.121. *All exceptional objects are semistable and their phases are in* $\{\phi_{min}, \phi_{max}\}$.

Proof. We have already proved that the exceptional objects are semistable. Recall that (see (2.80)) we denote $\delta_Z = Z(M) + Z(E_1^0) + Z(E_3^0)$. By Bridgleand's axiom (2.11) we can rewrite (2.80) as follows:

$$r(E_j^m)\exp(\mathrm{i}\pi\phi(E_j^m)) = m\delta_Z + r(E_j^0)\exp(\mathrm{i}\pi\phi(E_j^0)) \qquad m \in \mathbb{N}, j = 1,2,3,4. \tag{2.106}$$

In Corollary 2.118 we have $\{\phi(M), \phi(E_1^0), \phi(E_3^0)\} \subset \{\phi_{min}, \phi_{max}\}$, therefore we can write $\delta_Z = \Delta\exp(\mathrm{i}\pi\gamma)$ with $\Delta \geq 0$ and $\gamma \in \{\phi_{min}, \phi_{max}\}$.

Now (2.106) restricts all the phases in the set $\{\phi_{min}, \phi_{max}\}$, since $\phi_{min} \leq \phi(E_j^m) \leq \phi_{max} = \phi_{min} + 1$ for any $1 \leq j \leq 4$, $m \in \mathbb{N}$. □

We are already close to (2.86). To derive completely some of the non-locally finite cases in (2.86) we consider each of the three equalities (2.102). We showed that one of them holds.

If $\phi(E_1^0) = \phi(M) + 1$

Then $\phi_{min} = \phi(M)$ and $\phi_{max} = \phi(E_1^0)$.

Since $\hom(E_2^m, M) \neq 0$, we have $\phi(E_2^m) \leq \phi(M) = \phi_{min}$ for $m \in \mathbb{N}$. Hence $\{E_2^m\} \subset \mathcal{P}(\phi_{min})$. We will show below that $\{E_1^m\} \subset \mathcal{P}(\phi_{max})$ and so we obtain the first case in (2.86).

The sequence $\{\phi(E_1^m)\}_{m\in\mathbb{N}}$ is non-increasing (see Remark 2.83) and has at most two values. The first value is $\phi(E_1^0) = \phi_{max} = \phi(M)+1$. Suppose that $\phi(E_1^l) = \phi(M)$ for some $l > 0$. We can assume that l is minimal, so $\phi(E_1^{l-1}) = \phi(M)+1$. In table (2.4) we see that $\hom(M', E_1^l) \neq 0$, hence $\phi(M') \leq \phi(M) = \phi_{min}$, i. e. $\phi(M') = \phi(M) = \phi_{min}$. We have the triple (E_1^l, M, E_4^{l-1}) with $\hom(E_1^l, M) = 0$ and $\phi(E_1^l) = \phi(M)$. It follows that $\phi(E_4^{l-1}) = \phi(M)$, otherwise Lemma 2.66 **(b)** produces a σ-triple. However, now the exceptional triple $(E_4^{l-1}, M', E_1^{l-1})$ with $\hom(E_4^{l-1}, M') = 0$ satisfies $\phi(M) = \phi(E_4^{l-1}) = \phi(M') < \phi(E_1^{l-1}) = \phi(M) + 1$ and Lemma 2.66 **(b)** gives a contradiction.

Whence, the equality $\phi(E_1^0) = \phi(M) + 1$ implies the first case in (2.86), which contradicts Corollary 2.85. Therefore, for the rest of the proof we can use the strict inequality:

$$\phi(E_1^0) < \phi(M) + 1. \tag{2.107}$$

If $\phi(E_1^0) = \phi(E_3^0) + 1$ **or** $\phi(M) = \phi(E_3^0) + 1$**.**

In both cases $\phi_{min} = \phi(E_3^0)$, $\phi_{max} = \phi(E_3^0) + 1$.

We note first that $\hom(M, E_4^m) \neq 0$ and $\hom(E_4^m, E_1^m) \neq 0$ for each integer m, hence

$$\phi(M) \leq \phi(E_4^m) \leq \phi(E_1^m) \leq \phi_{max} = \phi(E_3^0) + 1 \qquad m \in \mathbb{N}. \tag{2.108}$$

Threfore, it is enough to consider the case $\phi(E_1^0) = \phi(E_3^0) + 1$. The latter equality and (2.107) imply $\phi_{max} = \phi(E_1^0)$ and $\phi(E_3^0) = \phi_{min} < \phi(M)$. It follows that $\phi(M) = \phi_{max}$. Now (2.108) implies $\{E_4^m\}_{m\in\mathbb{N}} \subset \mathcal{P}(\phi_{max})$. We will show that $\{E_3^m\}_{m\in\mathbb{N}} \subset \mathcal{P}(\phi_{min})$ and so we obtain the second case in (2.86).

Now we have $\phi(E_3^0) = \phi_{min}$. Suppose that $\phi(E_3^l) = \phi_{max}$ for some $l > 0$. Choosing the minimal l with this property, we have $\phi(E_3^{l-1}) = \phi_{min}$. By $\hom(E_3^l, M') \neq 0$ we get $\phi(M') = \phi_{max} = \phi(M)$. It follows that $\phi(E_2^{l-1}) = \phi_{min}$, because otherwise $(E_3^{l-1}, M'[-1], E_2^{l-1}[-1])$ is a σ-triple, due to $\hom(M', E_2^{l-1}) = 0$. However, now $(E_2^{l-1}, M[-1], E_3^l[-1])$ is a σ-exceptional triple, due to $\hom(M, E_3^l) = 0$.

Whence, any of the equalities $\phi(E_1^0) = \phi(E_3^0) + 1$ and $\phi(M) = \phi(E_3^0) + 1$ implies (2.86), which is the desired contradiction. Theorem 2.81 is proved.

Appendix

2.A Table

In the table below we present the dimensions of some vector spaces of matrices. We skip the computations. For $m, n \geq 1$ we denote by $\mathcal{M}_k(m,n)$ the vector space of $m \times n$ matrices over the field k. The notations $\pi_{\pm}^m, j_{\pm}^m$ for $m \in \mathbb{N}$ are explained before Proposition 2.3.

	V	$\dim_k(V)$
$1 \leq n < m$	$\{(X,Y) \in \mathcal{M}_k(n+1,m+1) \times \mathcal{M}_k(n,m) : X \circ j_+^m = j_+^n \circ Y, X \circ j_-^m = j_-^n \circ Y\}$	0
$1 \leq m \leq n$	$- \cdot\cdot -$	$1+n-m$
$1 \leq m < n$	$\{(X,Y) \in \mathcal{M}_k(n,m) \times \mathcal{M}_k(n+1,m+1) : X \circ \pi_+^m = \pi_+^n \circ Y, X \circ \pi_-^m = \pi_-^n \circ Y\}$	0
$1 \leq n \leq m$	$- \cdot\cdot -$	$1+m-n$
$1 \leq m, 1 \leq n$	$\{(X,Y) \in \mathcal{M}_k(n+1,m) \times \mathcal{M}_k(n,m+1) : X \circ \pi_+^m = j_+^n \circ Y, X \circ \pi_-^m = j_-^n \circ Y\}$	0
$1 \leq m, 1 \leq n$	$\{X \in \mathcal{M}_k(n,m) : j_+^n \circ X \circ \pi_-^m = j_-^n \circ X \circ \pi_+^m\}$	0
$1 \leq m \leq n$	$\{X \in \mathcal{M}_k(n+1,m) : \pi_-^n \circ X \circ \pi_+^m = \pi_+^n \circ X \circ \pi_-^m\}$	0
$0 \leq n < m$	$- \cdot\cdot -$	$m-n$
$1 \leq n \leq m$	$\{X \in \mathcal{M}_k(n,m+1) : j_-^n \circ X \circ j_+^m = j_+^n \circ X \circ j_-^m\}$	0
$0 \leq m < n$	$- \cdot\cdot -$	$n-m$
$1 \leq m, 1 \leq n$	$\{(X,Y) \in \mathcal{M}_k(m,n+1) \times \mathcal{M}_k(m+1,n) : X \circ j_+^n = \pi_+^m \circ Y, X \circ j_-^n = \pi_-^m \circ Y\}$	$m+n$
$0 \leq n < m$	$\{(X,Y) \in \mathcal{M}_k(n+1,m)^2 : \pi_+^n \circ X = \pi_-^n \circ Y, \quad X \circ \pi_+^m = Y \circ \pi_-^m\}$	$m-n-1$
$0 \leq m < n$	$\{(X,Y) \in \mathcal{M}_k(n,m+1)^2 : j_+^n \circ X = j_-^n \circ Y, \quad X \circ j_+^m = Y \circ j_-^m\}$	$n-m-1$
$0 \leq m, 0 \leq n$	$\{(X,Y) \in \mathcal{M}_k(n+1,m+1)^2 : \pi_+^n \circ X = \pi_-^n \circ Y, \quad X \circ j_+^m = Y \circ j_-^m\}$	$n+m+2$
$0 \leq m, 0 \leq n$	$\{X \in \mathcal{M}_k(n+1,m+1) : \pi_-^n \circ X \circ j_+^m = \pi_+^n \circ X \circ j_-^m\}$	$m+n+1$

(2.109)

2.B $Rep_k(Q)$ is regularity preserving for Dynkin quiver Q

So far in the present chapter 2 were studied non-semistable exceptional objects in hereditary categories and the notion of regularity preserving category was introduced, but quite a few examples of such categories were given. Certain conditions on the Ext-nontrivial couples (exceptional objects $X, Y \in \mathcal{A}$ with $\mathrm{Ext}^1(X,Y) \neq 0$ and $\mathrm{Ext}^1(Y,X) \neq 0$) were shown to imply regularity-preserving.

The study of exceptional objects in quivers goes back to [45], [46], [15], and to [42] for more general hereditary categories. However, to the best of our knowledge, no attention to the Ext-nontrivial couples has been focused.

It is known that in Dynkin quivers $\mathrm{Hom}(\rho,\rho')=0$ or $\mathrm{Ext}^1(\rho,\rho')=0$ for any two exceptional representations. On one hand, the present Section proves this fact by a new method, which allows to extend it to representation infinite cases: quivers with graphs the extended Dynkin diagrams $\widetilde{\mathbb{E}}_6,\widetilde{\mathbb{E}}_7,\widetilde{\mathbb{E}}_8$. On the other hand, we use it to show that for any Dynkin quiver Q there are no Ext-nontrivial couples in $Rep_k(Q)$, which implies regularity preserving of $Rep_k(Q)$, where k is an algebraically closed field. The present Section contains the paper [19] (joint with Ludmil Katzarkov).

2.B.1 Introduction

A brief description of the content of the present section is as follwos.

The basic observation is that if a quiver Q satisfies $\mathrm{Hom}_Q(\rho,\rho')=0$ or $\mathrm{Ext}^1_Q(\rho,\rho')=0$ for any two exceptional representations ρ,ρ', then the dimension vectors of any Ext-nontrivial couple $\{\rho,\rho'\}$ satisfy $\langle\underline{\dim}(\rho)+\underline{\dim}(\rho'),\underline{\dim}(\rho)+\underline{\dim}(\rho')\rangle\leq 0$ (Lemma 2.122). This motivates us to study in more detail the property that $\mathrm{Hom}(\rho,\rho')=0$ or $\mathrm{Ext}^1(\rho,\rho')=0$ for given exceptional representations ρ,ρ' $\in Rep_k(Q)$. In Section 2.2 is shown that this property holds in $Rep_k(Q_1)$ and $Rep_k(Q_2)$ for any two exceptional representations (Corollary 2.7 (b)). An example with an acyclic quiver where this fails is obtained by changing the orientation of the quiver Q_2 (see (2.123)). In particular, the category of representations changes by changing the orientation of the arrows and keeping the quiver acyclic (Lemma 2.128).

In Subsection 2.B.2 is recalled the definition of the standard differential in the 2-term complex computing $\mathbb{R}\mathrm{Hom}_Q(\rho,\rho')$ for any two representations ρ, $\rho'\in Rep_k(Q)$, which we denote by $F^Q_{\rho,\rho'}$. We utilize this linear map because the condition that one of the two spaces $\mathrm{Hom}(\rho,\rho')$ or $\mathrm{Ext}^1(\rho,\rho')$ vanishes is the same as the condition that $F^Q_{\rho,\rho'}$ has maximal rank. In Subsections 2.B.3, 2.B.4 we find conditions which ensure maximality of the rank of $F^Q_{\rho,\rho'}$. The strategy is to expand the simple linear-algebraic observations: Lemma 2.135 (a),(b) and Lemma 2.133 to big enough quivers by using Corollary 2.126. [58] The obtained conditions, which ensure maximality of the rank, are as follows.

Let ρ, ρ' be exceptional representations, α, α' be their dimension vectors and let A, A' be the supports of α, α'. When Q has no edges loops and α or α' has only one nontrivial value, i. e. A or A' is a single element set, then $F^Q_{\rho,\rho'}$ has maximal rank (Lemma 2.127).

In Subsection 2.B.3 we consider quivers without loops and exceptional representations whose dimension vectors are thin, i. e. the componenets of these vectors take values in $\{0,1\}$ (see Definition 2.130). The main result of this Subsection (Lemma 2.132) is that, when the graph of Q has no loops, for any two thin exceptional representations ρ, ρ' the linear map $F^Q_{\rho,\rho'}$ has maximal rank. The last

[58] Corollary 2.126 is based on the algebro-geometric fact (see e.g. [16, p. 13]) that the orbit $\mathcal{O}_\rho$ of an exceptional representation ρ is Zariski open in a certain affine space.

Lemma 2.134 of this subsection considers some cases in which $A \cap A'$ is a single element set and ρ, ρ' are not restricted to be with thin dimension vectors.

In Subsection 2.B.4 we restrict Q further. We consider star shaped quivers with any orientation of the arrows (see Figure 2.135). We allow here the exceptional representations to have hill dimension vector (Definition 2.136) in addition to thin dimension vectors. It is shown that for any two exceptional representations $\rho, \rho' \in Rep_k(Q)$, whose dimension vectors are hill or thin, the map $F^Q_{\rho,\rho'}$ has maximal rank (Proposition 2.140).

A natural question is whether the dimension vectors of all exceptional representations in a given star shaped quiver are either hill or thin ?

Since the answer of this question is positive for any Dynkin quiver Q (Remark 2.142) and for the extended Dynkin quivers $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$ (Lemma 2.143), it follows that for any two exceptional representations ρ, ρ' in a Dynkin quiver Q or in an extended Dynkin quiver of the type $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$ the linear map $F^Q_{\rho,\rho'}$ has maximal rank(Corollary 2.144). Thus, in these quivers we have $\mathrm{Hom}(\rho, \rho') = 0$ or $\mathrm{Ext}^1(\rho, \rho') = 0$. This property for Dynkin quivers follows easily from the fact that $Rep_k(Q)$ is representation directed for Dynkin Q (see [43, p. 59] for the argument and [3], [24] for the fact that Dynkin quivers are representation directed). [59] However $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$ are not representation directed, since they are representation-infinite (see [47, 5.5, p. 307]), so this argument can not be applied to them. We expect that the arguments in Subsections 2.B.2, 2.B.3, 2.B.4 can be extended further to show that $\mathrm{Hom}(\rho, \rho') = 0$ or $\mathrm{Ext}^1(\rho, \rho') = 0$ for any two exceptional representations ρ, ρ' in $\widetilde{\mathbb{D}}_n$ for $n \geq 4$.

Star-shaped quivers have been extensively studied (going back to [29], [3] and recently e.g. [30]), but to the best of our knowledge Proposition 2.140 is new.

From here till the end of the introduction Q is a Dynkin quiver Q (i. e. the graph of Q is A_n with $n \geq 1$ or D_n with $n \geq 4$ or E_n with $n = 6, 7, 8$). Lemma 2.122 combined with Corollary 2.144 and the positivity of the Euler form imply that there are no Ext-nontrivial couples in $Rep_k(Q)$ (Corollary 2.145). This in turn implies that $Rep_k(Q)$ is regularity preserving. Furthermore, there are no σ-irregular objects for any $\sigma \in \mathrm{Stab}(D^b(Rep_k(Q)))$.

Corollary 2.145 means that for any two exceptional representations $\rho, \rho' \in Rep_k(Q)$ we have $\mathrm{Ext}^1(\rho, \rho') = 0$ or $\mathrm{Ext}^1(\rho', \rho) = 0$. Analogous property in degree zero, which says that $\mathrm{Hom}(\rho, \rho') = 0$ or $\mathrm{Hom}(\rho', \rho) = 0$ for any two non-equivalent exceptional representations $\rho, \rho' \in Rep_k(Q)$, is well known for Dynkin quivers. These facts about any Dynkin quiver Q can be summarized by saying that for any two non-equivalent exceptional representations $\rho, \rho' \in Rep_k(Q)$ the product of the two numbers in each row and in each column of the table below vanishes (see Section 0.1 for the notations $\mathrm{hom}(\rho, \rho'), \mathrm{hom}^1(\rho, \rho')$):

$\mathrm{hom}(\rho, \rho')$	$\mathrm{hom}^1(\rho, \rho')$
$\mathrm{hom}(\rho', \rho)$	$\mathrm{hom}^1(\rho', \rho)$

.

[59] We thank Pranav Pandit for pointing out these references.

2.B.2 The differential $F^Q_{\rho,\rho'}$

In this Subsection Q is any connected quiver. We denote the set of vertices by $V(Q)$, the set of arrows by $Arr(Q)$, and the underlying non-oriented graph by $\Gamma(Q)$. Let

$$Arr(Q) \to V(Q) \times V(Q) \quad a \mapsto (s(a), t(a)) \in V(Q) \times V(Q) \tag{2.110}$$

be the function assigning to an arrow $a \in Arr(Q)$ its origin $s(a) \in V(Q)$ and its end $t(a) \in V(Q)$. By $\langle,\rangle$ we denote the Euler form of Q (see (3.4)). The dual quiver $Q^\vee$ has $V(Q^\vee) = V(Q)$, $Arr(Q^\vee) = Arr(Q)$, but $(s^\vee, t^\vee) = (t, s)$. By transposing matrices we obtain an equivalence

$$Rep_k(Q)^{op} \xrightarrow{\vee} Rep_k(Q^\vee). \tag{2.111}$$

Th following properties hold

$$\forall \rho, \rho' \in Rep_k(Q) \qquad \underline{\dim}(\rho) = \underline{\dim}(\rho^\vee) \quad \hom^i_Q(\rho, \rho') = \hom^i_{Q^\vee}(\rho'^\vee, \rho^\vee) \tag{2.112}$$

$$\forall \alpha, \beta \in \mathbb{N}^{V(Q)} \qquad \langle \alpha, \beta \rangle_{Q^\vee} = \langle \beta, \alpha \rangle_Q \tag{2.113}$$

The basic observation of this Section 2.B is:

Lemma 2.122. *If any two exceptional objects $\rho, \rho' \in Rep_k(Q)$ satisfy $\hom(\rho, \rho') = 0$ or $\hom^1(\rho, \rho') = 0$, then any Ext-nontrivial couple $\{\rho, \rho'\}$ satisfies $\langle \underline{\dim}(\rho) + \underline{\dim}(\rho'), \underline{\dim}(\rho) + \underline{\dim}(\rho') \rangle \leq 0$.*

Proof. Since ρ, ρ' are exceptional representations, we have $\langle \underline{\dim}(\rho), \underline{\dim}(\rho) \rangle = \langle \underline{\dim}(\rho'), \underline{\dim}(\rho') \rangle = 1$, therefore

$$\langle \underline{\dim}(\rho) + \underline{\dim}(\rho'), \underline{\dim}(\rho) + \underline{\dim}(\rho') \rangle = 2 + \langle \underline{\dim}(\rho), \underline{\dim}(\rho') \rangle + \langle \underline{\dim}(\rho'), \underline{\dim}(\rho) \rangle. \tag{2.114}$$

Since $\hom^1(\rho, \rho') \neq 0$, $\hom^1(\rho', \rho) \neq 0$, by the given property of the exceptional objects we obtain $\hom(\rho, \rho') = \hom(\rho', \rho) = 0$, hence by (2.3) we obtain $\langle \underline{\dim}(\rho), \underline{\dim}(\rho') \rangle = -\hom^1(\rho, \rho') < 0$, $\langle \underline{\dim}(\rho'), \underline{\dim}(\rho) \rangle = -\hom^1(\rho', \rho) < 0$. Now the lemma follows from (2.114). □

In Corollary 2.7 (b) we see that the condition of the lemma above is satisfied in $Rep_k(Q_1)$, $Rep_k(Q_2)$. The condition of Lemma 2.122 is related to the standard differential in the 2-term complex computing $\mathbb{R}\mathrm{Hom}_Q(\rho, \rho')$. We recall this definition:

Definition 2.123. *For any two representations $\rho, \rho' \in Rep_k(Q)$ we denote a by $F^Q_{\rho,\rho'}$ (we omit the superscript Q, when it is clear which is the quiver in question) the standard differential in the 2-term complex computing $\mathbb{R}\mathrm{Hom}_Q(\rho, \rho')$. Recall that:*

$$F^Q_{\rho,\rho'} : \prod_{i \in V(Q)} \mathrm{Hom}(k^{\alpha_i}, k^{\alpha'_i}) \to \prod_{a \in Arr(Q)} \mathrm{Hom}(k^{\alpha_{s(a)}}, k^{\alpha'_{t(a)}}) \tag{2.115}$$

where $\alpha = \underline{\dim}(\rho)$, $\alpha' = \underline{\dim}(\rho') \in \mathbb{N}^{V(Q)}$, as follows:

$$F^Q_{\rho,\rho'}\left(\{f_i\}_{i \in V(Q)}\right) = \{f_{t(a)} \circ \rho_a - \rho'_a \circ f_{s(a)}\}_{a \in Arr(Q)}. \tag{2.116}$$

This differential will be used to obtain the condition in Lemma 2.122. More precisely, we have the following standard facts:

Lemma 2.124. *Let Q be a quiver and $\rho, \rho' \in Rep_k(Q)$ be two representations. The following hold:*

(a) $\mathrm{Hom}_{Rep_k(Q)}(\rho, \rho') = \mathfrak{ker}\left(F^Q_{\rho,\rho'}\right)$

(b) $\langle \underline{\dim}(\rho), \underline{\dim}(\rho')\rangle = \dim\left(\mathrm{dom}\left(F^Q_{\rho,\rho'}\right)\right) - \dim\left(\mathrm{cod}\left(F^Q_{\rho,\rho'}\right)\right)$, *where* $\mathrm{dom}\left(F^Q_{\rho,\rho'}\right)$ *and* $\mathrm{cod}\left(F^Q_{\rho,\rho'}\right)$ *denote the domain and codomain of* $F^Q_{\rho,\rho'}$.

(c) *Let* $\langle \underline{\dim}(\rho), \underline{\dim}(\rho')\rangle \geq 0$. *Then* $F^Q_{\rho,\rho'}$ *has maximal rank iff* $\hom(\rho, \rho') = \langle \underline{\dim}(\rho), \underline{\dim}(\rho')\rangle$ *and* $\hom^1(\rho, \rho') = 0$.

(d) *Let* $\langle \underline{\dim}(\rho), \underline{\dim}(\rho')\rangle < 0$. *Then* $F^Q_{\rho,\rho'}$ *has maximal rank iff* $\hom(\rho, \rho') = 0$ *and* $\hom^1(\rho, \rho') = -\langle \underline{\dim}(\rho), \underline{\dim}(\rho')\rangle$.

(e) *Let* $Q^\vee$ *be the dual quiver and* $\vee$ *be the equivalence in* (2.111), *then* $F^Q_{\rho,\rho'}$ *has maximal rank iff* $F^{Q^\vee}_{\rho'^\vee,\rho^\vee}$ *has maximal rank*

Proof. (a) and (b) follow from the definitions, (c) and (d) follow from (a), (b) and (2.3). Finally, (e) follows from (c), (d), (2.112), and (2.113). □

For any $\alpha \in \mathbb{N}^{V(Q)}$ we denote

$$GL(\alpha) = \prod_{i\in V(Q)} GL(\alpha_i, k); Rep(\alpha) = \{\rho \in Rep_k(Q) : \underline{\dim}(\rho) = \alpha\} = \prod_{a\in Arr(Q)} \mathrm{Hom}(k^{\alpha_{s(a)}}, k^{\alpha_{t(a)}}).$$

For any $\alpha \in \mathbb{N}^{V(Q)}$ the isomorphism classes of representations with dimension vector α are the orbits of the left action:

$$GL(\alpha) \times Rep(\alpha) \to Rep(\alpha) \qquad g.\rho = \{g(t(a)) \circ \rho_a \circ g(s(a))^{-1}\}_{a\in Arr(Q)}. \tag{2.117}$$

For $\rho \in Rep(\alpha)$ the orbit containing ρ is denoted by $\mathcal{O}_\rho$.

Let $\alpha, \alpha' \in \mathbb{N}^{V(Q)}$, $g \in GL(\alpha)$, $g' \in GL(\alpha')$. It is easy to show that for any $\rho \in Rep(\alpha)$, $\rho' \in Rep(\alpha')$ we have

$$F_{g.\rho,\ \rho'} = R_{g^{-1}} \circ F_{\rho,\rho'} \circ R_g \qquad F_{\rho,\ g'.\rho'} = L_{g'} \circ F_{\rho,\rho'} \circ L_{g'^{-1}}, \tag{2.118}$$

where:

$$\begin{gathered} L_{g'}, R_g : \prod_{i\in V(Q)} \mathrm{Hom}(k^{\alpha_i}, k^{\alpha'_i}) \to \prod_{i\in V(Q)} \mathrm{Hom}(k^{\alpha_i}, k^{\alpha'_i}) \\ L_{g'}\left(\{f_i\}_{i\in V(Q)}\right) = \{g'_i \circ f_i\}_{i\in V(Q)}, \quad R_g\left(\{f_i\}_{i\in V(Q)}\right) = \{f_i \circ g_i\}_{i\in V(Q)}; \end{gathered} \tag{2.119}$$

$$\begin{gathered} L_{g'}, R_g : \prod_{a\in Arr(Q)} \mathrm{Hom}(k^{\alpha_{s(a)}}, k^{\alpha'_{t(a)}}) \to \prod_{a\in Arr(Q)} \mathrm{Hom}(k^{\alpha_{s(a)}}, k^{\alpha'_{t(a)}}) \\ L_{g'}\left(\{u_a\}_{a\in Arr(Q)}\right) = \{g'_{t(a)} \circ u_a\}_{a\in Arr(Q)} \quad R_g\left(\{u_a\}_{a\in Arr(Q)}\right) = \{u_a \circ g_{s(a)}\}_{a\in Arr(Q)}. \end{gathered} \tag{2.120}$$

In particular, we see immediately that

Lemma 2.125. *Let $\alpha, \alpha' \in \mathbb{N}^{V(Q)}$, $(\rho, \rho') \in Rep(\alpha) \times Rep(\alpha')$. If $F_{\rho,\rho'}$ is not of maximal rank, then $F_{x,y}$ is not of maximal rank for any $(x, y) \in \mathcal{O}_\rho \times \mathcal{O}_{\rho'}$.*

The following corollary will be widely used.

Corollary 2.126. *Let $\alpha, \alpha' \in \mathbb{N}^{V(Q)}$ be real roots of Q. Let $\rho \in Rep(\alpha)$, $\rho' \in Rep(\alpha')$ be exceptional representations. If $F_{x,y}$ has maximal rank for some $(x, y) \in Rep(\alpha) \times Rep(\alpha')$, then $F_{\rho,y}$ and $F_{x,\rho'}$ have maximal rank. For each $a \in Arr(Q)$ the linear maps ρ_a, ρ'_a have maximal rank.*

Proof. First recall that, since ρ, ρ' are exceptional, the orbits $\mathcal{O}_\rho$ and $\mathcal{O}_{\rho'}$ are Zariski open in $Rep(\alpha)$ and $Rep(\alpha')$, respectively (see [16, p. 13]). For a given $x \in Rep(\alpha)$ the condition on $y \in Rep(\alpha')$ to be such that $F_{x,y}$ is not of maximal rank is expressed by vanishing of certain family of polynomials on $Rep(\alpha')$. If there is $y \in Rep(\alpha')$ such that $F_{x,y}$ is of maximal rank, then the zero set of this family of polynomials is a proper Zariski closed subset of $Rep(\alpha')$, hence, by the previous lemma, not maximality of the rank of $F_{x,\rho'}$ implies that the orbit $\mathcal{O}_{\rho'}$ is contained in this proper zariski closed subset, and then $\mathcal{O}_{\rho'}$ can not be an open subset of $Rep(\alpha')$. Thus, we showed that if $F_{x,y}$ is of maximal rank for some $y \in Rep(\alpha')$, then $F_{x,\rho'}$ is of maximal rank. The claim about $F_{\rho,y}$ is proved by the same arguments applied to ρ.

Finally, the property that ρ_a is not of maximal rank is invariant under the action of $GL(\alpha)$, for any $a \in Arr(Q)$. It follows that non-maximality of the rank of ρ_a implies that $\mathcal{O}_\rho$ is contained in a proper Zariski closed subset of $Rep(\alpha)$. If ρ is an exceptional representation, then $\mathcal{O}_\rho$ is Zariski open in $Rep(\alpha)$, therefore ρ_a is of maximal rank for each $a \in Arr(Q)$. □

It is useful to give a more precise description of the map defined in Definition 2.123. For any $(\rho, \rho') \in Rep(\alpha) \times Rep(\alpha')$ we denote $A = \{i \in V(Q) : \alpha_i \neq 0\}$, $A' = \{i \in V(Q) : \alpha'_i \neq 0\}$. We denote also $Arr(A, A') = \{a \in Arr(Q) : s(a) \in A, t(a) \in A'\}$. Then for $F_{\rho,\rho'}$ we can write

$$F_{\rho,\rho'} : \prod_{i\in A\cap A'} \mathrm{Hom}(k^{\alpha_i}, k^{\alpha'_i}) \to \prod_{a\in Arr(A,A')} \mathrm{Hom}(k^{\alpha_{s(a)}}, k^{\alpha'_{t(a)}}) \tag{2.121}$$

$$F_{\rho,\rho'}\left(\{f_i\}_{i\in A\cap A'}\right) = \left\{ \begin{array}{cc} f_{t(a)} \circ \rho_a - \rho'_a \circ f_{s(a)} & a \in Arr(A\cap A', A\cap A') \\ -\rho'_a \circ f_{s(a)} & a \in Arr(A\cap A', A'\setminus A) \\ f_{t(a)} \circ \rho_a & a \in Arr(A\setminus A', A\cap A') \\ 0 & a \in Arr(A\setminus A', A'\setminus A) \end{array} \right\}. \tag{2.122}$$

In the rest Subsections we will study the question about maximality of the rank of $F_{\rho,\rho'}$, where ρ, ρ' are exceptional representations. We prove first the following lemma:

Lemma 2.127. *Let Q have no edges loops. Let $\rho, \rho' \in Rep_k(Q)$ be exceptional representations. If A or A' is a single element set, then $F_{\rho,\rho'}$ has maximal rank.*

Proof. Recall that we denote $\alpha = \underline{\dim}(\rho)$, $\alpha' = \underline{\dim}(\rho')$.

Assume that $A = \{j\}$. If $A \cap A' = \emptyset$, then $F_{\rho,\rho'}$ is injective. Let $A \cap A' = \{j\}$.

Now $A \setminus A' = \emptyset$ and, since Q has no edges loops, we have $Arr(A \cap A', A \cap A') = \emptyset$, hence for any $y \in Rep(\alpha')$ the map $F_{\rho,y}$ has the form (we use (2.121), (2.122) and that now $\alpha_j = 1$):

$$F_{\rho,y} : \operatorname{Hom}(k, k^{\alpha'_j}) \to \prod_{a \in Arr(\{j\}, A' \setminus \{j\})} \operatorname{Hom}(k, k^{\alpha'_{t(a)}})$$
$$F_{\rho,y}(f) = \{-y_a \circ f\}_{a \in Arr(\{j\}, A' \setminus \{j\})}$$

Obviously, we can choose $y \in Rep(\alpha')$ so that $F_{\rho,y}$ has maximal rank. Therefore by Corollary 2.126 $F_{\rho,\rho'}$ has maximal rank as well.

If $A' = \{j\}$, then by the already proved $F^{Q^\vee}_{\rho'^\vee,\rho^\vee}$ has a maximal rank. Now we apply Lemma 2.124, (e) and obtain that $F_{\rho,\rho'}$ has maximal rank. □

An example of a quiver Q and exceptional representations ρ, ρ', s. t. $F^{Q}_{\rho,\rho'}$ is not of maximal rank is as follows:

$$Q = \begin{array}{ccc} \circ & \longrightarrow & \circ \\ \uparrow & & \downarrow \\ \circ & \longrightarrow & \circ \end{array} \quad \rho = \begin{array}{ccc} k & \longrightarrow & 0 \\ \uparrow & & \downarrow \\ k & \longrightarrow & k \end{array} \quad \rho' = \begin{array}{ccc} 0 & \longrightarrow & k \\ \uparrow & & \downarrow \\ k & \longrightarrow & k \end{array}. \tag{2.123}$$

One easily computes $\hom(\rho,\rho') = 1$, $\langle \underline{\dim}(\rho), \underline{\dim}(\rho') \rangle = 0$, and hence $\hom^1(\rho,\rho') = 1$. Furthermore, ρ, ρ' are exceptional representations. Now from Lemma 2.124 (c) it follows that $F_{\rho,\rho'}$ is not of maximal rank. Comparing with Corollary 2.7 (b) we obtain

Lemma 2.128. *The categories of representations of the quivers* $\begin{array}{ccc} \circ & \longrightarrow & \circ \\ \uparrow & & \downarrow \\ \circ & \longrightarrow & \circ \end{array}$, $\begin{array}{ccc} \circ & \longrightarrow & \circ \\ \uparrow & & \uparrow \\ \circ & \longrightarrow & \circ \end{array}$ *are not equivalent.*

In the next Subsection we restrict our considerations to a quiver Q without loops.

2.B.3 Remarks about $F_{\rho,\rho'}$ in quivers without loops

Throughout this subsection Q is quiver without loops (i. e. the underlying graph $\Gamma(Q)$ is simply connected), in particular there is at most one edge between any two vertices of Q. Here we consider exceptional representations whose dimension vectors take values in $\{0,1\}$. These exceptional representations are said to have thin dimension vector (Definition 2.130). The main result of this

Subsection is that, when the graph of Q has no loops, then for any two exceptional representations ρ, ρ' with thin dimension vectors the linear map $F^{Q}_{\rho,\rho'}$ has maximal rank. The last Lemma 2.134 of this Subsection considers some cases in which $A \cap A'$ is a single element set and where ρ, ρ' are not restricted to be with thin dimension vectors.

For any subset $X \subset V(Q)$ we denote by Q_X the quiver with $V(Q_X) = X$ and $Arr(Q_X) = Arr(X, X)$. We denote by ρ, ρ' two representations of Q. We denote by $\alpha, \alpha' \in \mathbb{N}^{V(Q)}$ their dimension vectors, and by $A = supp(\alpha) \subset V(Q)$, $A' = supp(\alpha') \subset V(Q)$ the supports of α, α'. If $Arr(A \setminus A', A' \setminus A) \neq \emptyset$, then by the simply-connectivity of Q it follows that $A \cap A' = \emptyset$ and then $F_{\rho,\rho'}$ is trivially injective(see (2.121)). Thus, we see

Lemma 2.129. *If $Arr(A \setminus A', A' \setminus A) \neq \emptyset$, then $F_{\rho,\rho'}$ has maximal rank.*

From now on we assume that $Arr(A \setminus A', A' \setminus A) = \emptyset$, and then the last row in (2.122) can be erased, and we have a disjoint union:

$$Arr(A, A') = Arr(A \cap A', A \cap A') \cup Arr(A \cap A', A' \setminus A) \cup Arr(A \setminus A', A \cap A'). \tag{2.124}$$

Now we consider exceptional representations ρ, ρ' whose dimension vectors contain only units and zeroes. More precisely:

Definition 2.130. *A vector $\alpha \in \mathbb{N}^{V(Q)}$ is said to be thin if for any $i \in A$ we have $\alpha_i = 1$, where $A = supp(\alpha) \subset V(Q)$ is the support of α.*

Remark 2.131. *If $\rho \in Rep_k(Q)$ is an exceptional representation with a thin dimension vector (thin exceptional representation), then the sub-quiver Q_A mist be connected and one can assume that $\forall a \in Arr(A, A) \quad \rho_a = \mathrm{Id}_k$.*

Lemma 2.132. *Let ρ and ρ' be exceptional representations with thin dimension vectors. Then $F_{\rho,\rho'}$ has maximal rank.*

Proof. Due to the given conditions we can write:

$$\langle \underline{\dim}(\rho), \underline{\dim}(\rho') \rangle = \#(A \cap A') - \#(Arr(A, A')) \tag{2.125}$$

$$\prod_{i \in A \cap A'} k \xrightarrow{F_{\rho,\rho'}} \prod_{a \in Arr(A,A')} k \qquad F_{\rho,\rho'}\left(\{f_i\}_{i \in A \cap A'}\right) = \left\{ \begin{array}{cc} f_{t(a)} - f_{s(a)} & a \in Arr(A \cap A', A \cap A') \\ -f_{s(a)} & a \in Arr(A \cap A', A' \setminus A) \\ f_{t(a)} & a \in Arr(A \setminus A', A \cap A') \end{array} \right\}.$$

We can assume that $A \cap A' \neq \emptyset$. Since Q_A and $Q_{A'}$ are connected, it follows that $Q_{A \cap A'} = Q_A \cap Q_{A'}$ is connected. Since there are no loops in $\Gamma(Q)$, the graph of $Q_{A \cap A'}$ is simply connected, therefore $\#(A \cap A') = \#(Arr(A \cap A', A \cap A')) + 1$. Putting (2.124) and the latter equality in (2.125) we obtain

$$\langle \underline{\dim}(\rho), \underline{\dim}(\rho') \rangle = 1 - \#(Arr(A \cap A', A' \setminus A)) - \#(Arr(A \setminus A', A \cap A')). \tag{2.126}$$

The following lemma will be helpful for the rest of the proof

Lemma 2.133. *Let T be a quiver, s. t. $\Gamma(T)$ is simply-connected. Consider the linear map*

$$F:\prod_{i\in V(T)}k\to\prod_{a\in Arr(T)}k \qquad F\left(\{f_i\}_{i\in V(T)}\right)=\left\{f_{t(a)}-f_{s(a)}\right\}_{a\in Arr(T)} \tag{2.127}$$

For each $j\in V(T)$, each $x\in k$, and each $y\in\prod_{a\in Arr(T)}k$ there exists unique $\{f_i\}_{i\in V(T)}\in\prod_{i\in V(T)}k$ with $f_j=x$ and $F(\{f_i\}_{i\in V(T)})=y$. In particular, for each $j\in V(T)$ the linear map

$$\prod_{i\in V(T)}k\to k\oplus\prod_{a\in Arr(T)}k \qquad \{f_i\}_{i\in V(T)}\mapsto\left(f_j,\left\{f_{t(a)}-f_{s(a)}\right\}_{a\in Arr(T)}\right) \tag{2.128}$$

is isomorphism.

Proof. Easy induction on the number of vertices. □

We will apply this lemma to $Q_{A\cap A'}$.

Consider first the case $\langle\underline{\dim}(\rho),\underline{\dim}(\rho')\rangle\geq 0$. We need to show that $F_{\rho,\rho'}$ is surjective. Now by (2.126) we have $1\geq Arr(A\cap A',A'\setminus A)+Arr(A\setminus A',A\cap A')$, and then the map $F_{\rho,\rho'}$ is the same as one of the maps (2.127) or (2.128) corresponding to $T=Q_{A\cap A'}$, hence $F_{\rho,\rho'}$ is surjective.

In the case $\langle\underline{\dim}(\rho),\underline{\dim}(\rho')\rangle<0$, we have $1<Arr(A\cap A',A'\setminus A)+Arr(A\setminus A',A\cap A')$. Hence, for some projection π the map $\pi\circ F_{\rho,\rho'}$ is the same as the map (2.128) corresponding to $T=Q_{A\cap A'}$, hence $F_{\rho,\rho'}$ is injective. □

In the end we consider the map $F_{\rho,\rho'}$ in the case, when $A\cap A'$ has a single element.

Lemma 2.134. *Let ρ,ρ' be exceptional representations, s. t. $A\cap A'=\{j\}$. Let $\Gamma(Q)$ does not split at j, i. e. the edges adjacent to j can be represented as follows x — j — y . Finally, assume that α is constant on $A\cap\{x,y,j\}$ or α' is constant on $A'\cap\{x,y,j\}$. Then $F_{\rho,\rho'}$ has maximal rank.*

Proof. Since there are no loops in $\Gamma(Q)$, we have $Arr(A\cap A',A\cap A')=\emptyset$ and $F_{\rho,\rho'}$ has the form

$$\langle\underline{\dim}(\rho),\underline{\dim}(\rho')\rangle=\alpha_j\alpha'_j-\sum_{a\in Arr(\{j\},A'\setminus A)}\alpha_j\alpha'_{t(a)}-\sum_{a\in Arr(A\setminus A',\{j\})}\alpha_{s(a)}\alpha'_j \tag{2.129}$$

$$F_{\rho,\rho'}:\mathrm{Hom}(k^{\alpha_j},k^{\alpha'_j})\to\prod_{a\in Arr(\{j\},A'\setminus A)}\mathrm{Hom}(k^{\alpha_j},k^{\alpha'_{t(a)}})\oplus\prod_{a\in Arr(A\setminus A',\{j\})}\mathrm{Hom}(k^{\alpha_{s(a)}},k^{\alpha'_j}) \tag{2.130}$$

$$F_{\rho,\rho'}(\{f\})=\left\{\begin{array}{ll}-\rho'_a\circ f & a\in Arr(\{j\},A'\setminus A)\\ f\circ\rho_a & a\in Arr(A\setminus A',\{j\})\end{array}\right\}. \tag{2.131}$$

From Lemma 2.127 we can assume that $\#(A)\geq 2$, $\#(A')\geq 2$. Since ρ,ρ' are exceptional representations, Q_A and $Q_{A'}$ are connected. Then the edges adjacent to j can be represented as follows $A\setminus A'\ni x$ —— j —— $y\in A'\setminus A$. We consider three cases.

If $Arr(\{j\}, A' \setminus A) \neq \emptyset$, $Arr(A \setminus A', \{j\}) = \emptyset$, then we can represent the arrows adjacent to j as follows

$$A \setminus A' \ni x \longleftarrow j \xrightarrow{a} y \in A' \setminus A \tag{2.132}$$

and $\langle \underline{\dim}(\rho), \underline{\dim}(\rho') \rangle = \alpha_j \alpha'_j - \alpha_j \alpha'_y = \alpha_j(\alpha'_j - \alpha'_y)$, $F_{\rho,\rho'}(\{f\}) = -\rho'_a \circ f$. Since ρ' is an exceptional representation, the map ρ'_a has maximal rank (see the last part of Corollary 2.126). Therefore $F_{\rho,\rho'}$ has maximal rank.

If $Arr(\{j\}, A' \setminus A) = \emptyset$, $Arr(A \setminus A', \{j\}) \neq \emptyset$, then we can represent the arrows adjacent to j as follows

$$A \setminus A' \ni x \xrightarrow{a} j \longleftarrow y \in A' \setminus A \tag{2.133}$$

and $\langle \underline{\dim}(\rho), \underline{\dim}(\rho') \rangle = \alpha_j \alpha'_j - \alpha_x \alpha'_j = (\alpha_j - \alpha_x)\alpha'_j$, $F_{\rho,\rho'}(\{f\}) = f \circ \rho_a$. Since ρ is an exceptional representation, then ρ_a has maximal rank. Therefore $F_{\rho,\rho'}$ has maximal rank.

If $Arr(\{j\}, A' \setminus A) \neq \emptyset$, $Arr(A \setminus A', \{j\}) \neq \emptyset$, then we can represent the arrows adjacent to j as follows

$$A \setminus A' \ni x \xrightarrow{a} j \xrightarrow{b} y \in A' \setminus A \tag{2.134}$$

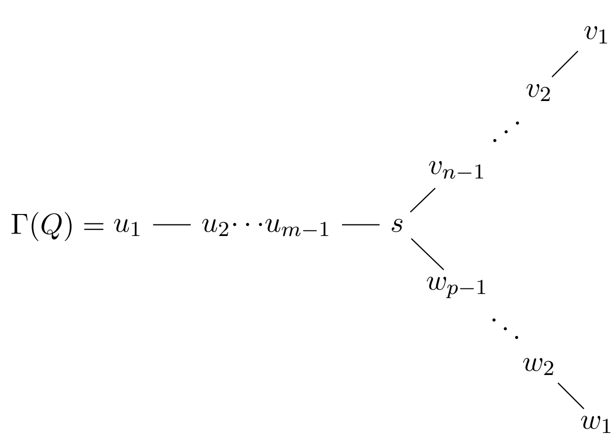
and $\langle \underline{\dim}(\rho), \underline{\dim}(\rho') \rangle = \alpha_j \alpha'_j - \alpha_x \alpha'_j - \alpha_j \alpha'_y$, $F_{\rho,\rho'}(\{f\}) = (-\rho'_b \circ f, f \circ \rho_a)$.

If α is constant on $A \cap \{x, y, j\}$, then $\alpha_x = \alpha_j$ and $\langle \underline{\dim}(\rho), \underline{\dim}(\rho') \rangle = -\alpha_j \alpha'_y < 0$. Since $\alpha_x = \alpha_j$ and ρ_a has maximal rank, it follows that ρ_a is isomorphism, hence $F_{\rho,\rho'}$ is injective. Therefore $F_{\rho,\rho'}$ has maximal rank.

If α' is constant on $A' \cap \{x, y, j\}$, then $\alpha'_y = \alpha'_j$ and $\langle \underline{\dim}(\rho), \underline{\dim}(\rho') \rangle = -\alpha'_j \alpha_x < 0$. Since $\alpha'_y = \alpha'_j$ and ρ'_b has maximal rank, it follows that ρ'_b is isomorphism, hence $F_{\rho,\rho'}$ is injective. Therefore $F_{\rho,\rho'}$ has maximal rank. The lemma is completely proved. □

2.B.4 Remarks about $F_{\rho,\rho'}$ in star shaped quivers

In this Subsection 2.B.4 we restrict Q further. We assume that its graph is of the type$(m, n, p \geq 2)$:



(2.135)

Everything in this Subsection holds for star shaped quivers with more than three rays. For simplicity of the notations we work with three rays. For such quivers we consider exceptional representations with hill dimension vector (Definition 2.136) in addition to the already considered thin dimension vectors. We show that for any two exceptional representations $\rho, \rho' \in Rep_k(Q)$, whose dimension vectors are hill or thin, the map $F^Q_{\rho,\rho'}$ has maximal rank.

The following lemma (more precisely parts (a) and (b) in this lemma) is basic for this section.

Lemma 2.135. *Let L be a quiver whose vertices are the numbers $\{1, 2, \dots, n\}(n \geq 2)$, whose graph is $\Gamma(L) = 1 — 2 — \dots — (n-1) — n$, and with any orientation of the arrows. Let $\rho, \rho' \in Rep_k(L)$ be two representations with dimension vectors $\alpha = \underline{\dim}(\rho)$, $\alpha' = \underline{\dim}(\rho')$, s. t.*

$$0 < \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_{n-1} \leq \alpha_n \qquad 0 < \alpha'_1 \leq \alpha'_2 \leq \dots \leq \alpha'_{n-1} \leq \alpha'_n,$$

and s. t. for each $a \in Arr(L)$ the linear maps ρ_a, ρ'_a have maximal rank. Then the linear map

$$\prod_{i=1}^{n} \mathrm{Hom}(k^{\alpha_i}, k^{\alpha'_i}) \xrightarrow{F^L_{\rho,\rho'}} \prod_{a \in Arr(L)} \mathrm{Hom}(k^{\alpha_{s(a)}}, k^{\alpha'_{t(a)}}) \quad \{f_i\}_{i=1}^n \mapsto \left\{f_{t(a)} \circ \rho_a - \rho'_a \circ f_{s(a)}\right\}_{a \in Arr(L)} \tag{2.136}$$

has the following properties:

(a) *The map $\mathfrak{ker}(F^L_{\rho,\rho'}) \to \mathrm{Hom}(k^{\alpha_n}, k^{\alpha'_n})$, defined by $\mathfrak{ker}(F^L_{\rho,\rho'}) \ni \{f_i\}_{i=1}^n \mapsto f_n$, is injective.*

(b) *For any $x \in \mathrm{Hom}(k^{\alpha_1}, k^{\alpha'_1})$ and any $y \in \prod_{a \in Arr(L)} \mathrm{Hom}(k^{\alpha_{s(a)}}, k^{\alpha'_{t(a)}})$ there exists $\{f_i\}_{i=1}^n \in \prod_{i=1}^n \mathrm{Hom}(k^{\alpha_i}, k^{\alpha'_i})$, s. t. $F^L_{\rho,\rho'}(\{f_i\}_{i=1}^n) = y$ and $f_1 = x$.*

(c) *In particular, the map $F^L_{\rho,\rho'}$ is surjective and $\dim(\mathfrak{ker}(F^L_{\rho,\rho'})) = \sum_{i=1}^n \alpha_i \alpha'_i - \sum_{a \in Arr(L)} \alpha_{s(a)} \alpha'_{t(a)}$.*

(d) *If we are given a surjective map $k^{\alpha'_0} \xleftarrow{x} k^{\alpha'_1}$, then the linear map*

$$\prod_{i=1}^{n} \mathrm{Hom}(k^{\alpha_i}, k^{\alpha'_i}) \xrightarrow{G^L_{\rho,\rho',x}} \mathrm{Hom}(k^{\alpha_1}, k^{\alpha'_0}) \oplus \prod_{a \in Arr(L)} \mathrm{Hom}(k^{\alpha_{s(a)}}, k^{\alpha'_{t(a)}})$$
$$\{f_i\}_{i=1}^n \mapsto (-x \circ f_1, F^L_{\rho,\rho'}(\{f_i\}_{i=1}^n)) \tag{2.137}$$

is surjective and the dimension of its kernel is $\sum_{i=1}^n \alpha_i \alpha'_i - \sum_{a \in Arr(L)} \alpha_{s(a)} \alpha'_{t(a)} - \alpha_1 \alpha'_0$.

(e) *If we are given an injective map $k^{\alpha_0} \xrightarrow{x} k^{\alpha_1}$, then the linear map*

$$\prod_{i=1}^{n} \mathrm{Hom}(k^{\alpha_i}, k^{\alpha'_i}) \xrightarrow{H^L_{\rho_r,\rho'_r,x}} \mathrm{Hom}(k^{\alpha_0}, k^{\alpha'_1}) \oplus \prod_{a \in Arr(L)} \mathrm{Hom}(k^{\alpha_{s(a)}}, k^{\alpha'_{t(a)}})$$
$$\{f_i\}_{i=1}^n \mapsto (f_1 \circ x, F^L_{\rho,\rho'}(\{f_i\}_{i=1}^n)) \tag{2.138}$$

is surjective and the dimension of its kernel is $\sum_{i=1}^n \alpha_i \alpha'_i - \sum_{a \in Arr(L)} \alpha_{s(a)} \alpha'_{t(a)} - \alpha_0 \alpha'_1$.

(f) *If we are given an injective map $k^{\alpha'_n} \xrightarrow{x} k^{\alpha'_{n+1}}$, then the linear map*

$$\prod_{i=1}^{n} \mathrm{Hom}(k^{\alpha_i}, k^{\alpha'_i}) \longrightarrow \mathrm{Hom}(k^{\alpha_n}, k^{\alpha'_{n+1}}) \oplus \prod_{a\in Arr(L)} \mathrm{Hom}(k^{\alpha_{s(a)}}, k^{\alpha'_{t(a)}})$$
$$\{f_i\}_{i=1}^{n} \mapsto (-x \circ f_n, F^L_{\rho,\rho'}(\{f_i\}_{i=1}^{n}))$$

is injective.

Proof. We prove first (a) and (b). Let $n = 2$. We consider the two possible orientations of the arrow 1 ——— 2.

If the arrow starts at 1, then consider the diagram

$$\begin{array}{ccc} k^{\alpha_1} & \xrightarrow{\rho} & k^{\alpha_2} \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow \\ k^{\alpha'_1} & \xrightarrow{\rho'} & k^{\alpha'_2} \end{array}$$

Now the map $F^L_{\rho,\rho'}$ is $F^L_{\rho,\rho'}(f_1, f_2) = f_2 \circ \rho - \rho' \circ f_1$ and ρ, ρ' are injective. If $F^L_{\rho,\rho'}(f_1, f_2) = 0$ and $f_2 = 0$, then $\rho' \circ f_1 = 0$, and by the injectivity of ρ' we obtain $f_1 = 0$. Thus, we obtain (a).

To show (b) we have to find $f_2 \in \mathrm{Hom}(k^{\alpha_2}, k^{\alpha'_2})$, s. t. $f_2 \circ \rho - \rho' \circ x = y$ for any $x \in \mathrm{Hom}(k^{\alpha_1}, k^{\alpha'_1})$ and any $y \in \mathrm{Hom}(k^{\alpha_1}, k^{\alpha'_2})$. Since ρ is injective, then it has left inverse $\pi : k^{\alpha_2} \to k^{\alpha_2}$, and then we can choose $f_2 = (y + \rho' \circ x) \circ \pi$.

If the arrow starts at 2, then consider the diagram

$$\begin{array}{ccc} k^{\alpha_1} & \xleftarrow{\rho} & k^{\alpha_2} \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow \\ k^{\alpha'_1} & \xleftarrow{\rho'} & k^{\alpha'_2} \end{array}$$

Now the map $F^L_{\rho,\rho'}$ is $F^L_{\rho,\rho'}(f_1, f_2) = f_1 \circ \rho - \rho' \circ f_2$ and ρ, ρ' are surjective. If $F^L_{\rho,\rho'}(f_1, f_2) = 0$ and $f_2 = 0$, then $f_1 \circ \rho = 0$, and by the surjectivity of ρ we obtain $f_1 = 0$. Thus, we obtain (a).

To show (b) we have to find $f_2 \in \mathrm{Hom}(k^{\alpha_2}, k^{\alpha'_2})$, s. t. $x \circ \rho - \rho' \circ f_2 = y$ for any $x \in \mathrm{Hom}(k^{\alpha_1}, k^{\alpha'_1})$ and any $y \in \mathrm{Hom}(k^{\alpha_2}, k^{\alpha'_1})$. Since ρ' is surjective, then it has right inverse $in : k^{\alpha'_1} \to k^{\alpha'_2}$, and then we can choose $f_2 = in \circ (x \circ \rho - y)$.

So far, we proved the lemma, when $n = 2$. Now by using induction and the already proved case $n = 2$ one can easily prove (a), (b) for each $n \geq 2$. The statements in (c), (d), (e), and (f) follow from (a) and (b). ☐

In Lemma 2.138 we allow one of the components ρ, ρ' to be of a type different from thin. More precisely:

Definition 2.136. *Let Q be a star shaped quiver(as in Figure (2.135)). We say that $\alpha \in \mathbb{N}^{V(Q)}$ is a hill vector if*

$$\begin{array}{ll} \alpha(u_1) \leq \alpha(u_2) \leq \cdots \leq \alpha(u_{m-1}) \leq \alpha(s), & \alpha(u_{m-1}) > 0 \\ \alpha(v_1) \leq \alpha(v_2) \leq \cdots \leq \alpha(v_{n-1}) \leq \alpha(s), & \alpha(v_{n-1}) > 0 \\ \alpha(w_1) \leq \alpha(w_2) \leq \cdots \leq \alpha(w_{p-1}) \leq \alpha(s), & \alpha(w_{p-1}) > 0. \end{array} \quad (2.139)$$

The non-vanishing condition for $\alpha(u_{n-1})$, $\alpha(v_{n-1})$, $\alpha(w_{n-1})$ in this definition simplifies our considerations, but we suspect that this condition can be relaxed.

In the proof of Lemma 2.138 and Lemma 2.139 we use the following simple observation:

Lemma 2.137. *Let $Y \subset V$ be a vector subspace in a vector space V and $\dim(Y) = y$, $\dim(V) = n$. Let $\{x_1, \ldots, x_m\}$ be integers in $\{0, 1, \ldots, n\}$.*

(a) If $y + \sum_{i=1}^{m} x_i - mn \geq 0$, then one can choose vector supspaces $\{X_i \subset V\}_{i=1}^{m}$ so that $\dim(X_i) = x_i$ and

$$\dim\left(Y \cap \bigcap_{i=1}^{m} X_i\right) = y + \sum_{i=1}^{m} x_i - mn \quad (2.140)$$

(b) If $y + \sum_{i=1}^{m} x_i - mn < 0$, then one can choose vector supspaces $\{X_i \subset V\}_{i=1}^{m}$ so that $\dim(X_i) = x_i$ and

$$Y \cap \bigcap_{i=1}^{m} X_i = \{0\}. \quad (2.141)$$

Proof. **(a)** If $m = 1$, then we have $y + x_1 \geq n$. Therefore we can choose $X_1 \subset V$, so that $\dim(X_1) = x_1$ and $X_1 + Y = V$. Therefore by a well known formula, we have $n = \dim(X_1 + Y) = \dim(X_1) + \dim(Y) - \dim(X_1 \cap Y) = x_1 + y - \dim(X_1 \cap Y)$. Hence $\dim(X_1 \cap Y) = x_1 + y - n$.

Suppose that (a) holds for some $m \geq 1$ and take any collection of integers $\{x_1, \ldots, x_m, x_{m+1}\}$ in $\{0, 1, \ldots, n\}$, s. t. $y + \sum_{i=1}^{m+1} x_i - (m+1)n \geq 0$. We can rewrite the last inequality as follows $n \leq y + \sum_{i=1}^{m} x_i + x_{m+1} - mn$. On the other hand $x_{m+1} \leq n$, therefore

$$n \leq y + \sum_{i=1}^{m} x_i + x_{m+1} - mn \leq y + \sum_{i=1}^{m} x_i + n - mn \quad \Rightarrow \quad 0 \leq y + \sum_{i=1}^{m} x_i - mn. \quad (2.142)$$

Now by the induction assumption we obtain vector subspaces $\{X_i \subset V\}_{i=1}^{m}$ with $\{\dim(X_i) = x_i\}_{i=1}^{m}$ and $\dim\left(Y \cap \bigcap_{i=1}^{m} X_i\right) = y + \sum_{i=1}^{m} x_i - mn$. Now we have $\dim\left(Y \cap \bigcap_{i=1}^{m} X_i\right) + x_{m+1} \geq n$ and as in the case $m = 1$ we find a vector subspace $X_{m+1} \subset V$ with $\dim(X_{m+1}) = x_{m+1}$ and $\dim\left(Y \cap \bigcap_{i=1}^{m+1} X_i\right) = \dim\left(Y \cap \bigcap_{i=1}^{m} X_i\right) + x_{m+1} - n = y + \sum_{i=1}^{m+1} x_i - (m+1)n$. Thus, we proved (a).

(b) If $m = 1$, then the statement is obvious. Now we assume that we have (b) for some $m \geq 1$. Let $\{x_1, \ldots, x_m, x_{m+1}\}$ be any collection of integers in $\{0, 1, \ldots, n\}$, s. t. $y+\sum_{i=1}^{m+1} x_i-(m+1)n < 0$. If $y + \sum_{i=1}^{m} x_i - mn < 0$, then we use the induction assumption. If $y + \sum_{i=1}^{m} x_i - mn \geq 0$, then we use (a) to obtain vector subspaces $\{X_i \subset V\}_{i=1}^{m}$ with $\{\dim(X_i) = x_i\}_{i=1}^{m}$ and $\dim\left(Y \cap \bigcap_{i=1}^{m} X_i\right) = y + \sum_{i=1}^{m} x_i - mn$. Now we have $\dim\left(Y \cap \bigcap_{i=1}^{m} X_i\right) + x_{m+1} = y + \sum_{i=1}^{m+1} x_i - mn < n$, therefore we can choose $X_{m+1} \subset V$ with $\dim(X_{m+1}) = x_{m+1}$ and $Y \cap \bigcap_{i=1}^{m+1} X_i = \{0\}$. The lemma is proved. $\square$

Lemma 2.138. *Let $\rho, \rho' \in Rep_k(Q)$ be two exceptional representations with thin and hill dimension vectors, respectively. Then $F_{\rho,\rho'}$ and $F_{\rho',\rho}$ have maximal rank.*

Proof. We show first that $F_{\rho,\rho'}$ has maximal rank. Due to Lemma 2.127 we can assume that $\#(A) \geq 2$, hence $A \cap A' \neq \{s\}$(we are given also $\alpha'(u_{m-1}) > 0$, $\alpha'(v_{n-1}) > 0$, $\alpha'(w_{p-1}) > 0$). Due to Lemma 2.134 we can assume that $\#(A \cap A') \geq 2$. Lemma 2.129 considers the case $Arr(A \setminus A', A' \setminus A) \neq \emptyset$, hence we can assume that $Arr(A \setminus A', A' \setminus A) = \emptyset$ and we can write

$$\langle \alpha, \alpha' \rangle = \sum_{i \in A \cap A'} \alpha'_i - \sum_{a \in Arr(A \cap A', A \cap A')} \alpha'_{t(a)} - \sum_{a \in Arr(A \cap A', A' \setminus A)} \alpha'_{t(a)} - \sum_{a \in Arr(A \setminus A', A \cap A')} \alpha'_{t(a)} \quad (2.143)$$

$$F_{\rho,\rho'} : \prod_{i \in A \cap A'} \operatorname{Hom}\left(k, k^{\alpha'_i}\right) \to \prod_{a \in Arr(A,A')} \operatorname{Hom}\left(k, k^{\alpha'_{t(a)}}\right) \quad (2.144)$$

$$F_{\rho,\rho'}\left(\{f_i\}_{i \in A \cap A'}\right) = \left\{ \begin{array}{cc} f_{t(a)} - \rho'_a \circ f_{s(a)} & a \in Arr(A \cap A', A \cap A') \\ -\rho'_a \circ f_{s(a)} & a \in Arr(A \cap A', A' \setminus A) \\ f_{t(a)} & a \in Arr(A \setminus A', A \cap A') \end{array} \right\}. \quad (2.145)$$

We consider first the case(see (2.135)) $A \cap A' \subset \{u_1, u_2, \ldots, u_{m-1}\}$. Now $\Gamma(Q_{A\cap A'})$ has the form u_i —— u_{i+1} —— $\ldots$ —— u_{i+k-1} —— u_{i+k}. Let us denote by ρ_r, ρ'_r the representations ρ, ρ' restricted to $Q_{A\cap A'}$. Then $Q_{A\cap A'}$, ρ_r, ρ'_r satisfy the conditions in Lemma 2.135 (recall also the last statement in Corollary 2.126). Let us denote by a the arrow adjacent to $Q_{A\cap A'}$ at u_{i+k}. If a starts at u_{i+k}, i. e. it points towards the splitting point s, then, due to (2.139), $a \in Arr(A \cap A', A' \setminus A)$ and ρ'_a is injective. In this case $\pi \circ F_{\rho,\rho'}$, where π is some projection, is the same as the linear map in Lemma 2.135 (f) with $x = \rho'_a$, hence $F_{\rho,\rho'}$ is injective. Let the arrow a ends at u_{i+k}, then it is neither in $Arr(A \cap A', A' \setminus A)$ nor in $Arr(A \setminus A', A' \cap A)$. Let us denote by b the arrow adjacent to $Q_{A\cap A'}$ at u_i. Now if b starts at u_i and $u_{i-1} \in A'$, then $b \in Arr(A \cap A', A' \setminus A)$ and $F_{\rho,\rho'}$ is the same as the linear map in Lemma 2.135 (d) with $x = \rho'_b$. If b starts at u_i and $u_{i-1} \notin A'$, then $Arr(A \cap A', A' \setminus A) = Arr(A \setminus A', A' \cap A) = \emptyset$ and $F_{\rho,\rho'}$ is the same as $F^{Q_{A\cap A'}}_{\rho_r,\rho'_r}$ from Lemma 2.135. If b ends at u_i and $u_{i-1} \in A$, then $b \in Arr(A \setminus A', A' \cap A)$ and $F_{\rho,\rho'}$ is the same as the linear map in Lemma 2.135 (e) with $x = \mathrm{Id}_k$. If b ends at u_i and $u_{i-1} \notin A$, then $Arr(A \cap A', A' \setminus A) = Arr(A \setminus A', A' \cap A) = \emptyset$ and $F_{\rho,\rho'}$ is the same as $F^{Q_{A\cap A'}}_{\rho_r,\rho'_r}$ from Lemma 2.135. Thus, we see that $F_{\rho,\rho'}$ has maximal rank, when $A \cap A' \subset \{u_1, u_2, \ldots, u_{m-1}\}$.

Next, we consider the case $A \cap A' \subset \{u_1, u_2, \ldots, u_{m-1}, s\}$ and $s \in A \cap A'$. Now $\Gamma(Q_{A\cap A'})$ has the form u_i —— u_{i+1} —— $\ldots$ —— u_{m-1} —— s and v_{n-1}, w_{p-1} are not elements of A. We denote by ρ_r, ρ'_r

the restrictions of ρ, ρ' to $Q_{A\cap A'}$. Let us denote the arrow between s and v_{n-1} by b and the arrow between s and w_{p-1} by c. If b and c both end at s, then $F_{\rho,\rho'}$ is one of the following three linear maps: $F^{Q_{A\cap A'}}_{\rho_r,\rho'_r}$ (see (2.136)), $G^{Q_{A\cap A'}}_{\rho_r,\rho'_r,x}$ (see (2.137)), $H^{Q_{A\cap A'}}_{\rho_r,\rho'_r,\mathrm{Id}_k}$ (see (2.138)) considered in Lemma 2.135, hence $F_{\rho,\rho'}$ is surjective. It is useful to denote

$$S = \{b,c\} \cap Arr(A\cap A', A'\setminus A). \tag{2.146}$$

For $a \in S$ the linear map ρ'_a is surjective and

$$\forall a\in S \qquad \dim(\mathfrak{ker}(\rho'_a)) = \alpha'_s - \alpha'_{t(a)} \tag{2.147}$$

Looking at (2.145) we see that $F_{\rho,\rho'}$ has the form

$$F_{\rho,\rho'}\left(\{f_i\}_{i\in A\cap A'}\right) = \left(T\left(\{f_i\}_{i\in A\cap A'}\right), \left\{-\rho'_a\circ f_s\right\}_{a\in S}\right), \tag{2.148}$$

where T is one of $F^{Q_{A\cap A'}}_{\rho_r,\rho'_r}$, $G^{Q_{A\cap A'}}_{\rho_r,\rho'_r,x}$, $H^{Q_{A\cap A'}}_{\rho_r,\rho'_r,\mathrm{Id}_k}$. In the tree cases $\mathfrak{ker}(T)\subset \mathfrak{ker}\left(F^{Q_{A\cap A'}}_{\rho_r,\rho'_r}\right)$ and by Lemma 2.135 (a) the linear map $\mathfrak{ker}(T)\xrightarrow{\kappa}\mathrm{Hom}(k,k^{\alpha_s})$ defined by projecting to the $\mathrm{Hom}(k,k^{\alpha_s})$-component is injective(here and in (2.148) the notation s is the splitting vertex in figure (2.135)). Now from (2.148) we see that

$$\dim(\mathfrak{ker}(F_{\rho,\rho'})) = \dim\left(\kappa(\mathfrak{ker}T)\cap\bigcap_{a\in S}\mathfrak{ker}(\rho'_a)\right) \tag{2.149}$$

On the other hand by and (c), (d), (e) in Lemma 2.135, (2.147), and the formula (2.143) one easily shows that

$$\langle\alpha,\alpha'\rangle = \langle\underline{\dim}(\rho),\underline{\dim}(\rho')\rangle = \dim(\kappa(\mathfrak{ker}T)) + \sum_{a\in S}\dim(\mathfrak{ker}(\rho'_a)) - \#(S)\ \alpha'_s. \tag{2.150}$$

The feature of ρ', due to the fact that it is an exceptional representation, used so far is that ρ'_a is of maximal rank for any $a\in Arr(Q)$. All considerations hold for any $\widetilde{\rho'}\in Rep(\alpha')$ s. t. $\widetilde{\rho'}_a$ is of maximal rank for $a\in Arr(Q)$. For any such $\widetilde{\rho'}$ we have $\kappa(\mathfrak{ker}(\widetilde{T}))\subset k^{\alpha'_s}$, $\mathfrak{ker}(\widetilde{\rho'}_a)\subset k^{\alpha'_s}$ for $a\in S$ and (2.149), (2.150) hold. If $\langle\alpha,\alpha'\rangle\geq 0$, then by Lemma 2.137 (a) we can choose $\{\widetilde{\rho'}_a\}_{a\in S}$ (without changing the rest elements of ρ') so that

$$\langle\alpha,\alpha'\rangle = \dim(\kappa(\mathfrak{ker}T)) + \sum_{a\in S}\dim(\mathfrak{ker}(\widetilde{\rho'_a})) - \#(S)\ \alpha'_s = \dim\left(\kappa(\mathfrak{ker}T)\cap\bigcap_{a\in S}\mathfrak{ker}(\widetilde{\rho'}_a)\right). \tag{2.151}$$

Therefore, by (2.149) we get $\dim(\mathfrak{ker}(F_{\rho,\widetilde{\rho'}})) = \langle\underline{\dim}(\rho),\underline{\dim}(\widetilde{\rho'})\rangle$, which implies that $F_{\rho,\widetilde{\rho'}}$ is surjective(see Lemma 2.124 (b)). Now Corollary 2.126 shows that $F_{\rho,\rho'}$ has maximal rank. If $\langle\alpha,\alpha'\rangle<0$, then by Lemma 2.137 (b) we can choose $\{\widetilde{\rho'}_a\}_{a\in S}$ (without changing the rest elements of ρ') so that $\{0\} = \kappa(\mathfrak{ker}\widetilde{T}) \cap \bigcap_{a\in S} \mathfrak{ker}(\widetilde{\rho'}_a)$. Hence (2.149) implies that $F_{\rho,\widetilde{\rho'}}$ is injective. Now Corollary 2.126 shows that $F_{\rho,\rho'}$ has maximal rank.

Finally, we consider the case $\{u_{m-1}, s\} \subset A \cap A' \not\subset \{u_1, u_2, \ldots, u_{m-1}, s\}$. Now $v_{n-1} \in A\cap A'$ or $w_{p-1} \in A \cap A'$. We will give details about the case when $v_{n-1} \in A \cap A'$ and $w_{p-1} \in A \cap A'$. The steps for the other cases are the same. Now the quiver $Q_{A\cap A'}$ has the form

$$Q_{A\cap A'} \quad = \quad u_i \ \cdots\ u_{m-1} \ — \ s \begin{array}{l} \quad v_j \\ \ \ \cdot^{\cdot^{\cdot}} \\ v_{n-1} \\ / \\ \\ \backslash \\ w_{p-1} \\ \ \ \ddots \\ \quad w_k \end{array} \tag{2.152}$$

Let us denote $L_u = Q_{A\cap A'\cap\{u_1,u_2,\ldots,u_{m-1},s\}}$, and let ρ_u, ρ'_u be the restrictions of ρ, ρ' to L_u. Similarly we obtain L_v, ρ_v, ρ'_v and L_w, ρ_w, ρ'_w. Then we can apply Lemma 2.135 to L_i, ρ_i, ρ'_i for $i \in \{u, v, w\}$. Furthermore, we can express $F_{\rho,\rho'}$ as follows:

$$F_{\rho,\rho'}\left(\{f_i\}_{i\in A\cap A'}\right) = \Big(\ T_u\left(\{f_i\}_{i\in V(L_u)}\right), T_v\left(\{f_i\}_{i\in V(L_v)}\right), T_w\left(\{f_i\}_{i\in V(L_w)}\right)\ \Big), \tag{2.153}$$

where for $i \in \{u, v, w\}$ the linear map T_i is one of $F^{L_i}_{\rho_i,\rho'_i}$(see (2.136)), $G^{L_i}_{\rho_i,\rho'_i,x_i}$(see (2.137)), $H^{L_i}_{\rho_i,\rho'_i,\mathrm{Id}_k}$(see (2.138)). Using (c), (d), (e) in Lemma 2.135 and (2.143) one easily shows that

$$\langle\alpha, \alpha'\rangle = \dim(\mathfrak{ker}(T_u)) + \dim(\mathfrak{ker}(T_v)) + \dim(\mathfrak{ker}(T_w)) - 2\alpha'_s. \tag{2.154}$$

By Lemma 2.135 (a) the linear map $\mathfrak{ker}(T_i) \xrightarrow{\kappa_i} \mathrm{Hom}(k, k^{\alpha_s})$ defined by projecting to the $\mathrm{Hom}(k, k^{\alpha_s})$-component is injective for $i \in \{u, v, w\}$. From (2.153) one easily shows that

$$\dim(\mathfrak{ker}(F_{\rho,\rho'})) = \dim\big(\ \ \kappa_u(\mathfrak{ker}T_u) \cap \kappa_v(\mathfrak{ker}T_v) \cap \kappa_w(\mathfrak{ker}T_w)\ \ \big). \tag{2.155}$$

The obtained formulas hold for any $\widetilde{\rho'} \in Rep(\alpha')$ s. t. $\widetilde{\rho'}_a$ is of maximal rank for $a \in Arr(Q)$ (we denote the corresponding linear maps be $\widetilde{T}_i$). Due to (2.118) and having that $\widetilde{T}_i$ is $F^{L_i}_{\rho_i,\widetilde{\rho'}_i}$ or $G^{L_i}_{\rho_i,\widetilde{\rho'}_i,x_i}$ or $H^{L_i}_{\rho_i,\widetilde{\rho'}_i,\mathrm{Id}_k}$ we can move $\kappa_i(\mathfrak{ker}(\widetilde{T}_i))$ inside k^{α_s} by varying $\widetilde{\rho'} \in Rep(\alpha')$. Therefore, if $\langle\alpha, \alpha'\rangle \geq 0$, using (2.154) and Lemma 2.137 (a), we can ensure

$$\dim\Big(\ \ \kappa_u(\mathfrak{ker}\widetilde{T}_u) \cap \kappa_v(\mathfrak{ker}\widetilde{T}_v) \cap \kappa_w(\mathfrak{ker}\widetilde{T}_w)\ \ \Big) = \langle\alpha, \alpha'\rangle. \tag{2.156}$$

Therefore $\dim(\mathfrak{ker}(F_{\rho,\widetilde{\rho'}})) = \langle\underline{\dim}(\rho), \underline{\dim}(\widetilde{\rho'})\rangle$, which implies that $F_{\rho,\widetilde{\rho'}}$ is surjective(see Lemma 2.124 (b)). Now Corollary 2.126 shows that $F_{\rho,\rho'}$ has maximal rank. If $\langle\alpha, \alpha'\rangle < 0$, then, due to Lemma

2.137 (b) and (2.154), we can vary $\widetilde{\rho'}$ so that $\{0\} = \kappa_u(\mathfrak{ker}\widetilde{T}_u) \cap \kappa_v(\mathfrak{ker}\widetilde{T}_v) \cap \kappa_w(\mathfrak{ker}\widetilde{T}_w)$. Hence (2.155) implies that $F_{\rho,\widetilde{\rho'}}$ is injective. Now Corollary 2.126 shows that $F_{\rho,\rho'}$ has maximal rank.

So far we proved that $F^{Q}_{\rho,\rho'}$ has a maximal rank. The quiver $Q^{\vee}$ and the representations $\rho^{\vee}$, $\rho'^{\vee}$ satisfy the same conditions as Q, ρ, ρ', respectively. Therefore $F^{Q^{\vee}}_{\rho^{\vee},\rho'^{\vee}}$ has maximal rank. Now Lemma 2.124 (e) shows that $F^{Q}_{\rho',\rho}$ has maximal rank. The lemma is proved. □

Lemma 2.139. *Let ρ, $\rho' \in Rep_k(Q)$ be exceptional representations with hill dimension vectors. Then $F_{\rho,\rho'}$ has maximal rank.*

Proof. From Definition 2.136 we see that $Q_{A\cap A'}$ is in Figure (2.152). Now the arguments are the same as the arguments after Figure (2.152) in the proof of Lemma 2.138. In this case we use Lemma 2.135 in its full generality, when both representations have non-decreasing dimension vectors, so far we used it with one constant dimension vector and one non-decreasing dimension vector. □

Combining Lemmas 2.139, 2.138 and 2.132 we obtain:

Proposition 2.140. *Let Q be a star shaped quiver. For any two exceptional representations $\rho, \rho' \in Rep_k(Q)$, whose dimension vectors are hill or thin, the linear map $F_{\rho,\rho'}$ has maximal rank. In particular, for any two such ρ, ρ' we have* $\hom(\rho,\rho') = 0$ *or* $\hom^1(\rho,\rho') = 0$.

In the end we discuss hill dimension vectors:

Lemma 2.141. *Let Q be a star-shaped quiver. The sum of any two hill vectors in $\mathbb{N}^{V(Q)}$ is hill.*

Let s be the splitting vertex of Q. Let $\delta \in \mathbb{N}^{V(Q)}$ be a hill vector with $\delta(s) \geq 3$ such that in each ray of the form x_1 —— x_2 $\cdots$ x_{k-1} —— s we have $\delta(x_i) = \frac{i}{k}\delta(s)$ for $i = 1, 2, \dots, k-1$.[60] *Then*

(a) *For each thin vector $\alpha \in \mathbb{N}^{V(Q)}$ the vectors $\delta \pm \alpha$ are hill.*

(b) *For each hill vector $\alpha \in \mathbb{N}^{V(Q)}$, s. t. in each ray of the form x_1 —— x_2 $\cdots$ x_{k-1} —— s we have $\alpha(x_1) \leq \frac{\delta(s)}{k}$, $\{\alpha(x_{i+1}) - \alpha(x_i) \leq \frac{\delta(s)}{k}\}_{i=1}^{k-1}$ and $\alpha(x_{k-1}) < \frac{k-1}{k}\delta(s)$ the vector $\delta - \alpha$ is hill.*

Proof. Recalling the definition of hill vectors (Definition 2.136), it is clear that the sum of two hill vectors is hill. Let x_1 —— x_2 $\cdots$ x_{k-1} —— s be any ray of Q ($k \geq 2$) and let us denote $x_k = s$.

(a) If α is thin, then by $\{\delta(x_{i+1}) - \delta(x_i) = \frac{\delta(s)}{k} \geq 1\}_{i=1}^{k-1}$ one easily shows that $\alpha + \delta$ and $\alpha - \delta$ are non-decreasing along the ray towards the splitting vertex. By $\delta(s) \geq 3$ and the given properties of δ it follows that $\delta(x_{k-1}) > 1$ (otherwise $k = 2$ and $1 = \delta(s)/2$, which is a contradiction) now it is clear that $\delta \pm \alpha$ are hill.

(b) By the given properties of α we can represent it as follows $\alpha(x_1) = a_1$, $\alpha(x_2) = a_1 + a_2, \dots,$ $\alpha(x_s) = \sum_{l=1}^{k} a_l$, where $0 \leq a_l \leq \frac{\delta(s)}{k}$ for $l = 1, 2, \dots, k$. Therefore $\delta(x_i) - \alpha(x_i) = \sum_{l=1}^{i}\left(\frac{\delta(s)}{k} - a_l\right)$ for $i = 1, 2, \dots, k$, so we see that $\delta - \alpha$ is non-decreasing along the ray towards the splitting vertex. Now $\alpha(x_{k-1}) < \frac{k-1}{k}\delta(s)$ implies that $\delta - \alpha$ is a hill vector. □

[60] In particular $\frac{\delta(s)}{k}$ is a positive integer and $\delta(s) \geq 2$.

2.B.5 Remarks about $F_{\rho,\rho'}$ in Dynkin and in some extended Dynkin quivers.

A natural question motivated by Proposition 2.140 is whether the dimension vectors of all exceptional representations in a given star shaped quiver are either hill or thin ?

Recalling that the dimension vectors of the exceptional representations in any quiver are real roots (see the paragraph after (2.3)) and that for Dynkin quivers all roots are real, finding the answer of this question for Dynkin quivers is easy:

Remark 2.142. *Tables with the dimension vectors of roots of Dynkin quivers can be found in Patrick Browne's webpage [13]. In these tables one verifies that if Q is a Dynkin quiver (i. e. the graph of Q is $\mathbb{A}_n$ with $n \geq 1$ or $\mathbb{D}_n$ with $n \geq 4$ or $\mathbb{E}_n$ with $n = 6, 7, 8$), then all the roots of Q are either thin or hill. For the sake of clarity we give an explanation with pictures below.*

If $\Gamma(Q) = \mathbb{A}_n$, then all roots are thin.

The $n(n-1)$ roots of a quiver with graph $D_n (n \geq 4)$ are the roots with thin dimension vectors together with the roots of the following type:

$$0 - 0 \cdots 0 - 1 - 1 \cdots 1 - 2 - 2 - \cdots 2 \begin{array}{l} \nearrow 1 \\ \searrow 1 \end{array}$$

there are k zeroes, l ones, and m twos in the horizontal ray, where $m > 0$, $l > 0$, $k + l + m = n - 2$, $k < n - 3$. Thus, we see that when $\Gamma(Q) = \mathbb{D}_n$, then all the roots are thin or hill.

The 36 roots of a quiver with graph $\mathbb{E}_6$ are the roots with thin dimension vectors together with the roots coming from both the $\mathbb{D}_5$ subgraphs, and the following 6 roots

$$\begin{array}{c} 1 \\ | \\ 1 - 1 - 2 - 2 - 1 \end{array} \quad \begin{array}{c} 1 \\ | \\ 1 - 2 - 2 - 2 - 1 \end{array} \quad \begin{array}{c} 1 \\ | \\ 1 - 1 - 2 - 1 - 1 \end{array}$$

$$\begin{array}{c} 1 \\ | \\ 1 - 2 - 3 - 2 - 1 \end{array} \quad \begin{array}{c} 1 \\ | \\ 1 - 2 - 2 - 1 - 1 \end{array} \quad \begin{array}{c} 2 \\ | \\ 1 - 2 - 3 - 2 - 1 \end{array}.$$

Therefore all roots are thin or hill.

Note that $\mathbb{E}_7$ is obtained from $\mathbb{E}_6$ by adding one vertex. From any root of $\mathbb{E}_6$ we can obtain a root of $\mathbb{E}_7$ by inserting a number which is the same as an adjacent number. For example from an $\mathbb{E}_6$-root $\begin{array}{c} 2 \\ | \\ 1 - 2 - 3 - 2 - 1 \end{array}$ we obtain E_7-roots: $\begin{array}{c} 2 \\ | \\ 1 - 2 - 3 - 2 - 2 - 1 \end{array}$; $\begin{array}{c} 2 \\ | \\ 1 - 2 - 3 - 3 - 2 - 1 \end{array}$; $\begin{array}{c} 2 \\ | \\ 1 - 2 - 3 - 2 - 1 - 1 \end{array}$. We say that these roots are obtained by inserting from the corresponding $\mathbb{E}_6$ root. Note that a root obtained by inserting from a hill $\mathbb{E}_6$-root is a hill $\mathbb{E}_7$-root.

The 63 roots of a quiver with graph $\mathbb{E}_7$ are the roots with thin dimension vectors, together with the roots coming from the $\mathbb{D}_6$, $\mathbb{E}_6$ subgraphs, the roots obtained by inserting from the $\mathbb{E}_6$ roots, and the following 3 roots

$$\begin{array}{c} 2 \\ | \\ 1 - 2 - 4 - 3 - 2 - 1 \end{array} \quad \begin{array}{c} 2 \\ | \\ 1 - 3 - 4 - 3 - 2 - 1 \end{array} \quad \begin{array}{c} 2 \\ | \\ 2 - 3 - 4 - 3 - 2 - 1 \end{array}$$

which are also hill dimension vectors.

Finally, the 120 *roots of a quiver with graph* $\mathbb{E}_8$ *are the roots with thin dimension vectors, together with the roots coming from the* $\mathbb{D}_7$, $\mathbb{E}_7$ *subgraphs, the roots obtained by inserting from the* $\mathbb{E}_7$ *roots, and the following* 11 *roots*

$$1 — 3 — \overset{\displaystyle 2}{\overset{|}{5}} — 4 — 3 — 2 — 1 \qquad 2 — 3 — \overset{\displaystyle 2}{\overset{|}{5}} — 4 — 3 — 2 — 1$$

$$1 — 3 — \overset{\displaystyle 3}{\overset{|}{5}} — 4 — 3 — 2 — 1 \qquad 2 — 3 — \overset{\displaystyle 3}{\overset{|}{5}} — 4 — 3 — 2 — 1$$

$$2 — 4 — \overset{\displaystyle 2}{\overset{|}{5}} — 4 — 3 — 2 — 1 \qquad 2 — 4 — \overset{\displaystyle 3}{\overset{|}{5}} — 4 — 3 — 2 — 1$$

$$2 — 4 — \overset{\displaystyle 3}{\overset{|}{6}} — 4 — 3 — 2 — 1 \qquad 2 — 4 — \overset{\displaystyle 3}{\overset{|}{6}} — 5 — 3 — 2 — 1$$

$$2 — 4 — \overset{\displaystyle 3}{\overset{|}{6}} — 5 — 4 — 2 — 1 \qquad 2 — 4 — \overset{\displaystyle 3}{\overset{|}{6}} — 5 — 4 — 3 — 1$$

$$2 — 4 — \overset{\displaystyle 3}{\overset{|}{6}} — 5 — 4 — 3 — 2,$$

which are hill dimension vectors as well.

For those extended Dynkin diagrams, which are star-shaped with three rays, the answer is also positive (which is probably known):

Lemma 2.143. *All the real roots of the extended Dynkin diagrams* $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$ *(see for example [1, fig. (4.13)]) are either hill or thin.*

Proof. Let $\widetilde{Q}$ be some of the listed extended Dynkin diagrams. There is a vertex $\star \in V(\widetilde{Q})$ in the end of one of the rays, called extended vertex, s. t. after removing it and the connecting edge we obtain a corresponding embedded Dynkin diagram, which we denote by Q. Using the inclusion $V(Q) \subset V(\widetilde{Q})$, one can consider the roots of Q as a subset of the real roots of $\widetilde{Q}$.

Let $(\alpha, \beta) = \frac{1}{2}(\langle \alpha, \beta\rangle_{\widetilde{Q}} + \langle \beta, \alpha\rangle_{\widetilde{Q}})$ be the symmetrization of $\langle , \rangle_{\widetilde{Q}}$. Then by definition (see [16, p. 15,17]) we have $\Delta^{re}(\widetilde{Q}) = \{\alpha \in \mathbb{Z}^{V(\widetilde{Q})} : (\alpha, \alpha) = 1\}$ and $\Delta(Q) = \{\alpha \in \Delta^{re}(\widetilde{Q}) : \alpha(e) = 0\}$. Let $\delta \in \mathbb{N}_{\geq 1}^{V(\widetilde{Q})}$ be the minimal imaginary root of $\Delta_+(\widetilde{Q})$. In [1, fig. (4.13)] are given the coordinates of δ for all extended Dynkin diagrams, and one sees that the component of δ at $\star$ is $\delta(\star) = 1$. Furthermore, δ is a hill dimension vector satisfying the conditions in Lemma 2.141 (applied to $\widetilde{Q}$).

Let us take any $\alpha \in \Delta_+^{re}(\widetilde{Q})$. We will show that α is hill or thin. We can assume that $\alpha \notin \mathbb{Z}\delta$. If $\alpha(e) = 0$, then $\alpha \in \Delta_+(Q)$ and the lemma follows from Remark 2.142, so we can asume that

$\alpha(e) = a \geq 1$. Since $(\delta, \_) = 0$, we see that $\alpha - a\delta \in \Delta(Q)$. It follows that either $\alpha = a\delta + x$ or $\alpha = a\delta - x$ for some $x \in \Delta_+(Q)$. If $\alpha = a\delta + x$, then we immediately see that α is hill by Lemma 2.141 (a) and Remark 2.142. In the case $\alpha = a\delta - x$ we write $\alpha = (a-1)\delta + (\delta - x)$. Using Lemma 2.141 and the explanation given in Remark 2.142, one can show that for any $x \in \Delta_+(Q)$ the vector $\delta - x$ is thin or hill (Lemma 2.141 is helpful in showing this, but also some case by case checks are necessary). Applying again Lemma 2.141 we see that $\alpha = (a-1)\delta + (\delta - x)$ is thin or hill and the lemma follows. □

Due to Remark 2.142 and Lemma 2.143, Proposition 2.140 implies immediately

Corollary 2.144. *Let Q be a Dynkin quiver or an extended Dynkin quiver of type $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$. Then for any two exceptional representations $\rho, \rho' \in Rep_k(Q)$ the linear map $F_{\rho,\rho'}$ has maximal rank.*

In particular, for any two exceptional representations $\rho, \rho' \in Rep_k(Q)$ we have $\hom(\rho, \rho') = 0$ or $\hom^1(\rho, \rho') = 0$.

The part of Corollary 2.144 concerning Dynkin quivers follows easily from the fact that $Rep_k(Q)$ is representation directed for Dynkin Q (see [43, p. 59] for the argument and [3], [24] for the fact that Dynkin quivers are representation directed).[61]

Corollary 2.145. *If Q is a Dynkin quiver, then there are no Ext-nontrivial couples in $Rep_k(Q)$, i. e. for any two exceptional representations $\rho, \rho' \in Rep_k(Q)$ we have $\hom^1(\rho, \rho') = 0$ or $\hom^1(\rho', \rho) = 0$.*

Proof. Recall that for such a quiver we have $\langle \alpha, \alpha \rangle > 0$ for each $\alpha \in \mathbb{N}^{V(Q)} \setminus \{0\}$. Since for any two exceptional representations $\rho, \rho' \in Rep_k(Q)$ we have $\hom(\rho, \rho') = 0$ or $\hom^1(\rho, \rho') = 0$, we can apply Lemma 2.122. The corollary follows. □

Corollary 2.146. *If Q is a Dynkin quiver, then $Rep_k(Q)$ is regularity preserving category. Furthermore, there are no σ-irregular objects for any $\sigma \in \mathrm{Stab}(D^b(Rep_k(Q)))$.*

Proof. Since there are no Ext-nontrivial couples, RP properties 1,2(Definition 2.51) are tautologically satisfied. Then by Proposition 2.52 $Rep_k(Q)$ is regularity preserving. Actually, due to Lemma 2.49, there are no σ-irregular objects for any $\sigma \in \mathrm{Stab}(D^b(Rep_k(Q)))$. □

2.C The Kronecker quiver

2.C.1 There are no Ext-nontrivial couples in $Rep_k(K(l))$

The quiver with two vertices and $l \geq 2$ parallel arrows will be denoted by $K(l)$. Here we revisit [37, Lemma 4.1]. This lemma implies the title of this subsection.

[61] We thank Pranav Pandit for pointing out this fact.

Following the notations of [37], let s_0 and s_1 be the exceptional objects in $D^b(K(l))$, such that $s_0[1]$ is the simple representation with k at the source, and s_1 is the simple representation with k at the sink, and then define s_i for each $i \in \mathbb{Z}$ as follows:

$$s_{-i} = L_{s_{-i+1}}(s_{-i+2}), \quad s_{i+1} = R_{s_i}(s_{i-1}) \qquad i \geq 1. \tag{2.157}$$

The Braid group B_2 is isomorphic to $\mathbb{Z}$. By the transitivity of the action of B_2 on the set of full exceptional collections, shown in [15], it follows that, up to shifts, the complete list of the exceptional pairs in $Rep_k(K(l))$ is $\{(s_i, s_{i+1})\}_{i\in\mathbb{Z}}$. *Lemma 4.1 in [37] says that* $s_{\leq 0}[1], s_{\geq 1} \in Rep_k(K(l))$, *and:*

$$p \neq 0 \Rightarrow \hom^p(s_i, s_j) = 0; \qquad p \neq 1 \Rightarrow \hom^p(s_j, s_i) = 0; \qquad i < j. \tag{2.158}$$

Now $\{s_{-i}[1]\}_{i\geq 0} \cup \{s_i\}_{i\geq 1}$ is the complete list of exceptional objects of $Rep_k(K(l))$, and from the vanishings (2.158) it follows that for any couple $\{X, Y\}$ in this list $\hom^1(X, Y) \neq 0$ implies $\hom^1(Y, X) = 0$. Thus, there are no Ext-nontrivial couples in $Rep_k(K(l))$.

One can show that the following inequalities hold for each $i \in \mathbb{Z}$:

$$l = \hom(s_i, s_{i+1}) < \hom(s_i, s_{i+2}) < \dots; \quad 0 = \hom^1(s_i, s_{i-1}) < \hom^1(s_i, s_{i-2}) < \dots, \tag{2.159}$$

$$\dim_k(s_1) = \dim_k(s_0[1]) < \dim_k(s_2) = \dim_k(s_{-1}[1]) < \dots. \tag{2.160}$$

which implies that $\{s_{-i}[1]\}_{i\geq 0} \cup \{s_i\}_{i\geq 1}$ are pairwise non-isomorphic. Whence, in this case the action of the Braid group is free (compare with Remark 2.13).

2.C.2 σ-exceptional pairs in $D^b(K(l))$

The full exceptional collections in $D^b(K(l))$ have length two, so the analogue of Theorem 2.81 is:

Lemma 2.147. *For each* $\sigma \in \operatorname{Stab}(D^b(K(l)))$ *there exists a* σ*-exceptional pair.*

The statement of [37, Lemma 4.2] is equivalent to the statement of Lemma 2.147. For the sake of completeness we give a proof of Lemma 2.147 here.

Denote, for brevity $\mathcal{A} = Rep_k(K(l))$, and take any $\sigma = (\mathcal{P}, Z) \in \operatorname{Stab}(D^b(\mathcal{A}))$. There are no Ext-nontrivial couples in $\mathcal{A}$ and the exceptional pairs of $D^b(\mathcal{A})$, up to shifts, are a sequence $\{(s_i, s_{i+1})\}_{i\in\mathbb{Z}}$, where $\{s_{-i}[1]\}_{i\geq 0} \cup \{s_i\}_{i\geq 1} \subset \mathcal{A}$(see Appendix 2.C.1). By Remark 2.63 we reduce the proof immediately to the case, where all the exceptional objects are semistable. In (2.159) we have $\{\hom(s_i, s_{i+1}) \neq 0\}_{i\in\mathbb{Z}}$, hence $\{\phi(s_i) \leq \phi(s_{i+1})\}_{i\in\mathbb{Z}}$. If $\phi(s_i) < \phi(s_{i+1})$ for some $i \in \mathbb{Z}$, then there exists $j \geq 1$ with $\phi(s_{i+1}[-j]) \leq \phi(s_i) < \phi(s_{i+1}[-j]) + 1$, and hence, due to (2.158), the pair $(s_i, s_{i+1}[-j])$ is σ-exceptional. Thus, we reduce to the case, where all $\{s_i\}_{i\in\mathbb{Z}}$ have the same phase,[62] say $t \in \mathbb{R}$:

$$\{s_i\}_{i\in\mathbb{Z}} \subset \mathcal{P}(t). \tag{2.161}$$

[62] In the end of Appendix 2.C.1 we pointed out that $\{s_i\}_{i\geq 1}$ are pairwise non-isomorphic.

We show now that the obtained inclusion contradicts the locally finiteness of σ, i. e. (2.161) is a non-locally finite case.

Since all the exceptional pairs in $\mathcal{A}$ are $\{(s_{i-1}[1], s_i[1])\}_{i\leq -1} \cup \{(s_0[1], s_1)\} \cup \{(s_i, s_{i+1})\}_{i\geq 1}$, it follows from (2.161) that:

$$\text{\textit{For each exceptional pair} } (S,E) \text{ \textit{with} } S, E \in \mathcal{A} \text{ \textit{we have} } \phi(S) \geq \phi(E). \tag{2.162}$$

We will obtain a contradiction by constructing an exceptional pair (S, E) in $\mathcal{A}$ with $\phi(S) < \phi(E)$. Recall that Z is the central charge of σ. By (2.161) and (2.11) we have $\{Z(s_1), Z(s_0[1]) = -Z(s_0)\} \subset \mathbb{R}\exp(\mathrm{i}\pi t)$. Since[63] $K_0(D^b(\mathcal{A})) \cong \mathbb{Z}^2$ and the simple objects $s_0[1]$, s_1 form a basis of $K_0(D^b(\mathcal{A}))$, it follows that $\mathfrak{im}(Z) \subset \mathbb{R}\exp(\mathrm{i}\pi t)$. Now using (2.11) again, we concude that $\mathcal{P}(x)$ is trivial for $x \in (t-1, t)$, therefore $\mathcal{P}(t-1, t] = \mathcal{P}(t)$. From the very foundation [8] given by T. Bridgeland, we know that $\mathcal{P}(t-1, t]$ is a heart of a bounded t-structure of $D^b(\mathcal{A})$, so $\mathcal{P}(t)$ is a heart as well. Due to this property of $\mathcal{P}(t)$, it is also well known that $K_0(\mathcal{P}(t)) \xrightarrow{K_0(\mathcal{P}(t)\subset D^b(\mathcal{A}))} K_0(D^b(\mathcal{A}))$ is an isomorphism, so $K_0(\mathcal{P}(t)) \cong \mathbb{Z}^2$. The locally finiteness of σ implies that $\mathcal{P}(t)$ is an abelian category of finite length, which in turn, combined with $K_0(\mathcal{P}(t)) \cong \mathbb{Z}^2$, implies that $\mathcal{P}(t)$ has exactly two simple objects, say $X, Y \in \mathcal{P}(t)$. It follows by Lemma 2.20, that $\{X, Y\}$ are indecomposable in $D^b(\mathcal{A})$, therefore $X = X'[i]$, $Y = Y'[j]$ for some $i, j \in \mathbb{Z}$ and $X', Y' \in \mathcal{A}$. Viewing $\mathcal{A}$ as the extension closure of $s_0[1]$, s_1, we see that $X', Y' \in \mathcal{A} \subset \mathcal{P}[t, t+1]$. Now from $\{X'[i], Y'[j]\} \subset \mathcal{P}(t)$ it follows that either $\phi(X') = t$, $i = 0$ or $\phi(X') = t+1$, $i = -1$, and the same holds for Y', j. If either $i = i' = -1$ or $i = i' = 0$, then $\hom(s_1, X) = \hom(s_1, Y) = 0$ or $\hom(X, s_0) = \hom(Y, s_0) = 0$, which contradicts the existence of a Jordan-Hölder filtration of $s_0, s_1 \in \mathcal{P}(t)$ via the simples X, Y of $\mathcal{P}(t)$. Thus, we arrive at:

$$X = X', \qquad Y = Y'[-1], \qquad X', Y' \in \mathcal{A} \qquad \phi(X') = t,\ \phi(Y') = t+1. \tag{2.163}$$

By $\phi(Y') > \phi(X')$ it follows $\hom(Y', X') = 0$. Since $Y'[-1]$, X' are non-isomorphic simple objects in the abelian category $\mathcal{P}(t)$, it follows that $\hom(Y'[-1], X') = 0$ as well, hence $\hom^*(Y', X') = 0$.

The pair (X', Y') in $\mathcal{A}$ has $\phi(X') < \phi(Y')$ and $\hom^*(Y', X') = 0$, and it almost contradicts (2.162), but we have no arguments for the vanishings $\mathrm{Ext}^1(X', X') = 0$ and $\mathrm{Ext}^1(Y', Y') = 0$.

Keeping in mind the comments in the beginning of Subsection 2.3.2, we can view $\mathcal{P}(t)$ as the extension closure in $D^b(\mathcal{A})$ of the set $\{Y'[-1], X'\}$. Denoting the extension closures of X' and Y' by $\mathcal{X}$ and $\mathcal{Y}$, respectively, it is clear that $\mathcal{P}(t)$ *is the extension closure of* $\mathcal{Y}[-1] \cup \mathcal{X}$ *and*

$$[\mathcal{X}] = \mathbb{N}[X'], \quad [\mathcal{Y}] = \mathbb{N}[Y'], \qquad \hom^*(\mathcal{Y}, \mathcal{X}) = 0, \qquad \mathcal{X} \subset \mathcal{A} \cap \mathcal{P}(t), \quad \mathcal{Y} \subset \mathcal{A} \cap \mathcal{P}(t+1), \tag{2.164}$$

where the first two equalities are between subsets of $K_0(D^b(\mathcal{A}))$. Using $\hom^*(\mathcal{Y}, \mathcal{X}) = 0$ and that $\mathcal{P}(t)$ is the extension closure of $\mathcal{Y}[-1] \cup \mathcal{X}$, as in the case of semi-orthogonal decompositions, one can show

[63] This isomorphism is determined by assigning to $[X] \in K_0(D^b(\mathcal{A}))$, for $X \in \mathcal{A}$, the dimension vector $\underline{\dim}(X) \in \mathbb{Z}^2$.

that for each $X \in \mathcal{P}(t)$ there exists a triangle $A[-1] \longrightarrow X \longrightarrow B \longrightarrow A$ with $A \in \mathcal{Y}, B \in \mathcal{X}$ and $\hom^*(A, B) = 0$. Since $s_j \in \mathcal{A}_{exc} \cap \mathcal{P}(t)$ for $j \geq 1$, the corresponding triangle for s_j is:

$$s_j \longrightarrow B \longrightarrow A \longrightarrow s_j[1], \qquad \hom^*(A, B) = 0, \; A \in \mathcal{Y}, B \in \mathcal{X}, s_j \in \mathcal{A}_{exc}. \tag{2.165}$$

To prove Lemma 2.147, we show first that we can assume $A \neq 0$. After that we recall some of the arguments used in Subection 2.4.5 for obtaining the properties **C2.1** in the triangle (2.32). These arguments lead to the vanishings $\hom^1(B, B) = \hom^1(A, A) = 0$. Taking any $S \in Ind(B)$, $E \in Ind(A)$, we obtain an exceptional pair (S, E) in $\mathcal{A}$ with $\phi(S) < \phi(E)$, which contradicts (2.162).

Suppose that $A = 0$. Then $s_j \cong B \in \mathcal{X}$ and by (2.164) we have $\underline{\dim}(s_j) = p \; \underline{\dim}(X')$ for some $p \in \mathbb{N}$. Since s_j is exceptional and X' is indecomposable, then $\langle \underline{\dim}(s_j), \underline{\dim}(s_j) \rangle = 1$(see (2.3)) and $\langle \underline{\dim}(X'), \underline{\dim}(X') \rangle \leq 1$ (see [31, p. 58]).[64] It follows that $\underline{\dim}(s_j) = \underline{\dim}(X')$, $\langle \underline{\dim}(X'), \underline{\dim}(X') \rangle = 1$. Recall that X' is simple in $\mathcal{P}(t)$, which implies $\hom(X', X') = 1$. Now formula (2.3) shows that X' is an exceptional object, and hence $\underline{\dim}(X') = \underline{\dim}(s_j)$ implies that $X' \cong s_j$.[65] Thus, $A = 0$ implies $X' \cong s_j$. It follows, since $\{s_i\}_{i \geq 1}$ are pairwise non-isomorphic, that in (2.165) the object A can vanish for at most one integer $j \geq 1$. Hence, we can take $j \geq 1$ so that $A \neq 0$.

Since $\mathrm{Hom}^1(A, B) = \mathrm{Hom}^2(A, s_j) = 0$, by applying $\mathrm{Hom}(A, \_)$ to (2.165) we obtain $\mathrm{Hom}^1(A, A) = 0$. Because we have $\hom^*(A, B)$, it follows that $\{\hom^1(\Gamma, s_j) \neq 0\}_{\Gamma \in Ind(A)}$.[66] Since there are no Ext-nontrivial couples in $\mathcal{A}$, we obtain $\{\hom^1(s_j, \Gamma) = 0\}_{\Gamma \in Ind(A)}$, hence $\hom^1(s_j, A) = 0$. Now the triangle (2.165) and $\mathrm{Hom}(s_j, \_)$ imply $\hom^1(s_j, B) = 0$. Finally, the same triangle and $\mathrm{Hom}(\_, B[1])$ imply $\mathrm{Hom}(B, B[1]) = 0$. Lemma 2.147 is proved.

[64] where $\langle , \rangle$ is the Euler form of $K(l)$.

[65] There is at most one representation without self-extensions of a given dimension vector([16, p. 13]).

[66] see the last paragraph of the proof of Lemma 2.42 with E replaced by s_j, A_2 by A, B_0 by B, and letting $A_1 = 0$

Chapter 3

Density of phases

3.1 Introduction

In a series of papers Gaiotto-Moore-Neitzke [25], Kontzevich-Soibelman [33], and Bridgeland-Smith [10] have established a connection between Teichmüller theory and the theory of stability conditions on triangulated categories. One of the results is a correspondence between geodesics of finite length and stable objects, with slopes of the former giving the phases of the latter. In a joint work [17] with Haiden, Katzarkov, Kontsevich we develop further this parallel between dynamical systems and categories. The density of the set of slopes of closed geodesics on a Riemann surface is a motivation to investigate in [17, Section 3] the question whether a given triangulated category admits a stability condition such that the set of phases of stable objects is dense somewhere in the circle. This Chapter of the Dissertation is a slight improvement of [17, Section 3].

In case of stability conditions non-dense behavior is possible (Lemma 3.10 and Corollary 3.12):

Theorem. *The phases are never dense in an arc for Dynkin and Euclidean quivers.*[1]

Similarly as in the case of geodesics density property is expected to hold in general.

To obtain density property we prove the following (Theorem 3.24):

Theorem. *If a k-linear triangulated category*[2] $\mathcal{T}$ *contains an l-Kronecker pair with $l \geq 3$, s. t. a certain family of stability conditions on it is extendable to the entire category, then the extended stability conditions have phases dense in some arc.*

where an *l-Kronecker pair* in $\mathcal{T}$ is an exceptional pair (E, F) with $\hom^{\leq 0}(E, F) = 0$ and $\hom^1(E, F) = l$ (Definition 3.20), and by *extending of a stability condition* we mean Definition 3.22.

Using this theorem we obtain (Proposition 3.29):

[1] i. e. acyclic quivers with underlying graph a Dynkin or an extended Dynkin diagram

[2] Recall that k is an algebraically closed field.

Theorem. *Any connected acyclic quiver Q, which is neither Euclidean nor Dynkin has a family of stability conditions with phases which are dense in an arc.*

The main findings of this chapter are collected in the following table:

Dynkin quivers	P_σ is always finite	
Euclidean quivers	P_σ is either finite or has exactly two limit points	(3.1)
All other acyclic quivers	P_σ is dense in an arc for a family of stability conditions	

where P_σ denotes the set of stable phases (see Definition 3.1). Furthermore, the first part of Proposition 3.29 claims that *for each Euclidean quiver Q there exists a family of stability conditions on Q for which P_σ has exactly two limit points.* When $k = \mathbb{C}$, the table (3.1) holds after removing "acyclic" in the third row (Remark 3.30).

By the non-dense behavior of stability conditions it follows that *on Dynkin and Euclidean quivers the dimensions of Hom spaces of exceptional pairs are strictly smaller than* 3 (Corollary 3.28).

Further examples of density of phases (blow ups of projective spaces) are given in subsection 3.6.

3.2 Preliminaries

In this chapter we study the behavior of the set of phases of semi-stable objects, we denote this set by P_σ. More precisely:

Definition 3.1. *Let $\mathcal{T}$ be a triangulated category and $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}(\mathcal{T})$ a stability condition on it. We denote:*[3]

$$P_\sigma^{\mathcal{T}} = \exp(\mathrm{i}\pi\{t \in \mathbb{R} | \mathcal{P}(t) \neq \{0\}\}) \subset S^1. \tag{3.2}$$

By $\mathcal{P}(t+1) = \mathcal{P}(t)[1]$[4] *it follows $-P_\sigma^{\mathcal{T}} = P_\sigma^{\mathcal{T}}$.*

3.2.1 On θ-stability and a theorem by A. King

In the next Section we use a result by King. We recall first

Definition 3.2 (θ-stability)**.** *Let $\theta : K_0(\mathcal{A}) \to \mathbb{R}$ be a non-trivial group homomorphism, where $\mathcal{A}$ is an abelian category. Then $X \in \mathcal{A}$ is called θ-semistable if $\theta(X) = 0$ and for each monic arrow $X' \to X$ in $\mathcal{A}$ we have $\theta(X') \geq 0$ (if $\theta(X') = 0$ only for the sub-objects* 0 *and X then it is called θ-stable).*

Remark 3.3. *Z-semistable of phase t (as defined in Definition 2.26) is the same as θ-semistable with $\theta = -Im(\mathrm{e}^{-\mathrm{i}\pi t} Z)$.*

[3]When the triangulated category $\mathcal{T}$ is fixed in advance we write just P_σ.

[4]which is one of Bridgeland's axioms

From Proposition 4.4 in [34] it follows

Proposition 3.4 (A. King). *Let A be a finite dimensional, hereditary k-algebra (recall that k is an algebraically closed field throughout the entire dissertation). Let $\alpha \in K_0(A\text{-Mod})$. Then the following conditions are equivalent:*

1. *There exist $X \in A\text{-Mod}$ and a non-trivial $\theta : K_0(A\text{-Mod}) \to \mathbb{R}$, s. t. $[X] = \alpha$ and X is θ-stable.*

2. *α is a Schur root, which by definition means that some $Y \in A\text{-Mod}$ with $[Y] = \alpha$ satisfies $\mathrm{End}_{A\text{-Mod}}(Y) = k$.*

This Proposition will be used in the proof of Corollary 3.16.

3.3 Dynkin, Euclidean quivers and Kronecker quiver

In this Section we comment on the set P_σ as σ varies in the set of stability conditions on Dynkin, Euclidean quivers and on the Kronecker quiver. The main results here are Lemma 3.10 , Corollary 3.12 and Corollary 3.17.

3.3.1 Kac's theorem

For any quiver Q the notations $V(Q)$, $Arr(Q)$, $\Gamma(Q)$ are explained in the beginning of Subsection 2.B.2. Recall also that we denote by s, t the functions assigning to an arrow $a \in Arr(Q)$ its origin $s(a) \in V(Q)$ and its end $t(a) \in V(Q)$ (see (2.110)). A vertex $v \in V(Q)$ is called *source/sink* if all arrows touching it start/end at it (more precisely $v \neq t(a)/v \neq s(a)$ for each $a \in Arr(Q)$).

The terms *Dynkin quiver* and *Euclidean quiver* are explained in Section 0.1. By $K(l)$, $l \geq 1$ will be denoted the quiver, which consists of two vertices with l parallel arrows between them, and will be referred to as *l-Kronecker quiver.* Note that $K(1)$ is Dynkin, $K(2)$ is Euclidean.

Recall the Kac's Theorem.

Remark 3.5 (On Kac's Theorem). *Let Q be a connected quiver without edges-loops. In [31] is defined the positive root system of Q. We denote this root system by $\Delta_+(Q) \subset \mathbb{N}^{V(Q)}$. For $X \in Rep_k(Q)$ we denote by $\underline{\dim}(X) \in \mathbb{N}^{V(Q)}$ its dimension vector. The main result of [31] is:*

$$\{\underline{\dim}(X) | X \in Rep_k(Q), X \ \textit{is indecomposable}\} = \Delta_+(Q). \tag{3.3}$$

The Euler form of any quiver Q is defined by

$$\langle \alpha, \beta \rangle_Q = \sum_{j \in V(Q)} \alpha_j \beta_j - \sum_{j \in Q_1} \alpha_{s(j)} \beta_{t(j)}, \qquad \alpha, \beta \in \mathbb{N}^{V(Q)}. \tag{3.4}$$

The set $\Delta_+(Q)$ has a simple description for Dynkin, extended Dynkin or hyperbolic quivers ($K(l)$, $l \geq 3$ are hyperbolic quivers) as shown by Kac in [31]. It is determined by the Euler form as follows

$$\Delta(Q) = \{r \in \mathbb{Z}^{V(Q)} \setminus \{0\} | \langle r, r \rangle_Q \leq 1\}, \qquad \Delta_+(Q) = \Delta(Q) \cap \mathbb{N}^{V(Q)}. \tag{3.5}$$

If Q is acyclic,[5] then the path algebra kQ is finite dimensional and we have an isomorphism $K_0(Rep_k(Q)) \cong \mathbb{Z}^{V(Q)}$ determined by $K_0(Rep_k(Q)) \ni [X] \mapsto \underline{\dim}(X) \in \mathbb{Z}^{V(Q)}$ for $X \in Rep_k(Q)$. In particular, for any homomorphism $Z : K_0(Rep_k(Q)) \to \mathbb{C}$ and any $X \in Rep_k(Q)$ we have

$$Z(X) = \sum_{i \in V(Q)} \underline{\dim}_i(X)\ Z(s_i) = (v, \underline{\dim}(X)), \qquad \{v_i = Z(s_i)\}_{i \in V(Q)}, \tag{3.6}$$

where s_i is the simple representation with k in the vertex $i \in V(Q)$ and 0 in the other vertices. *Throughout this chapter 3* $(,)$ *denotes the bilinear form on* $\mathbb{C}^{V(Q)} \times \mathbb{C}^{V(Q)}$ *defined by* $(\alpha, \beta) = \sum_{i \in V(Q)} \alpha_i \beta_i$, $\alpha, \beta \in \mathbb{C}^{V(Q)}$, *NOT the symmetrization* $\langle \alpha, \beta \rangle_Q + \langle \beta, \alpha \rangle_Q$ *of* $\langle , \rangle_Q$. *We mention once this symmetrization and denote it by* $(,)_Q$.

3.3.2 The inclusion $P_\sigma \subseteq R_{v,\Delta_+}$.

Lemma 3.6. *Let* $\mathcal{T}$ *be any triangulated category. Then for each* $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}(\mathcal{T})$ *we have:* $\{t \in \mathbb{R} | \mathcal{P}(t) \neq 0\} = \{\phi_\sigma(I) | I \text{ is } \mathcal{T}\text{-indecomposable and } \sigma\text{-semistable}\}$.

Proof. Let $X \in \mathcal{P}(t)$ be non-zero. Since $\mathcal{P}(t)$ is of finite length , we have a decomposition in $\mathcal{P}(t)$ of the form $X \cong \bigoplus_{i=1}^n X_i$, where X_i are indecomposable in $\mathcal{P}(t)$, therefore (here we use that $\mathcal{P}(t)$ is abelian) there are no non-trivial idempotents in $\mathrm{End}_{\mathcal{P}(t)}(X_i) = \mathrm{End}_{\mathcal{T}}(X_i)$, hence X_i is indecomposable in $\mathcal{T}$. Thus, we see that $t = \phi_\sigma(X_i)$, where X_i is an indecomposable in $\mathcal{T}$ and σ-semistable. The lemma follows. □

Corollary 3.7. *Let* $\mathcal{A}$ *be a hereditary abelian category. For each* $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}(D^b(\mathcal{A}))$ *holds the inclusion:*

$$P_\sigma \subseteq \left\{ \pm \frac{Z(X)}{|Z(X)|} \,|X \text{ is indecomposable in } \mathcal{A}, Z(X) \neq 0 \right\}. \tag{3.7}$$

Proof. Take any $t \in \mathbb{R}$ with $\mathcal{P}(t) \neq \{0\}$. From the previous lemma there is a semi-stable, indecomposable $X \in D^b(\mathcal{A})$, s. t. $\phi_\sigma(X) = t$. Since $\mathcal{A}$ is hereditary, it follows that $X = X'[i]$ for some indecomposable $X' \in \mathcal{A}$, $i \in \mathbb{Z}$. Now we can write

$$(-1)^i Z(X') = Z(X) = m(X) \exp(\mathrm{i}\pi\phi_\sigma(X)) = m(X) \exp(\mathrm{i}\pi t) \qquad m(X) > 0,$$

where we use that X is σ-semistable and one of the Bridgeland's axioms ([8, Definition 1.1 a)]). The corollary is proved. □

[5] i. e. it has no oriented cycles.

When $\mathcal{A} = Rep_k(Q)$ with Q-acyclic we can rewrite this corollary in a useful form. Putting (3.3) and (3.6) in the righthand side of (3.7) with $\mathcal{A} = Rep_k(Q)$ we get a set $\left\{ \frac{\pm(v,r)}{|(v,r)|} \,|r \in \Delta_+(Q), (v,r) \neq 0 \right\}$, where $v \in \mathbb{C}^{V(Q)}$ is a non-zero vector. It is useful to define

Definition 3.8. *For any finite set F, any subset $A \subset \mathbb{N}^F \setminus \{0\}$ and any non-zero vector $v \in \mathbb{C}^F$ we denote* [6]

$$R^F_{v,A} = \left\{ \pm \frac{(v,r)}{|(v,r)|} \,|r \in A, (v,r) \neq 0 \right\} \subset S^1, \text{ where } (v,r) = \sum_{i \in F} v_i \; r_i. \tag{3.8}$$

Then we can rewrite (3.7) as follows (we assume that Q is an acyclic, because we used (3.6), which holds only for acyclic quivers) :

Corollary 3.9. *Let Q be an acyclic quiver. For any $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}(D^b(Rep_k(Q)))$ holds the inclusion*

$$P_\sigma \subseteq R_{v,\Delta_+(Q)} \qquad v = \{v_i = Z(s_i)\}_{i \in V(Q)}. \tag{3.9}$$

3.3.3 On the set $R_{v,\Delta_+(Q)}$

Lemma 3.10. *Let Q be a Dynkin quiver. For any stability condition $\sigma \in \mathrm{Stab}(D^b(Rep_k(Q)))$ the set of semi-stable phases P_σ is finite.*

Proof. It is well known that for a Dynkin quiver Q the positive root system $\Delta_+(Q)$ is finite. Hence for any non-zero $v \in \mathbb{C}^{V(Q)}$ the set $R_{v,\Delta_+(Q)}$ is finite. Now the lemma follows from Corollary 3.9. □

Lemma 3.11. *Let Q be an Euclidean quiver (see subsubsection 3.3.1 for definition). For any non-zero $v \in \mathbb{C}^{V(Q)}$ the set $R_{v,\Delta_+(Q)}$ is either finite or there exist $m \in \mathbb{N}$, $p \in S^1$ and sequences $\{p^i_j \subset S^1\}_{i=1,\dots,m; j \in \mathbb{N}}$, s. t. $\{\lim_{j\to\infty} p^i_j = p\}_{i=1}^m$ and $R_{v,\Delta_+} = \cup_{i=1}^m \{\pm p^i_j\}_{j \in \mathbb{N}}$.*

Proof. The root system Δ of an Euclidean quiver Q (as described in the first equality of (3.5)) has an element $\delta \in \mathbb{N}^{V(Q)}_{\geq 1}$ with the properties $\Delta \cup \{0\} + \mathbb{Z}\delta \subset \Delta \cup \{0\}$ and $\Delta \cup \{0\}/\mathbb{Z}\delta$ is finite (see [16, p. 18]). Hence there is a finite set $\{\alpha_1, \alpha_2 \dots, \alpha_m\} \subset \Delta$, s. t. $\Delta \cup \{0\} = \bigcup_{i=1}^m (\alpha_i + \mathbb{Z}\delta)$. If for any $i \in \{1, 2, \dots, m\}$ we choose the minimal $n_i \in \mathbb{Z}$, s. t. $\alpha_i + n_i\delta \in \Delta_+$ and denote $\beta_i = \alpha_i + n_i\delta$, then $\Delta_+ = \bigcup_{i=1}^m (\beta_i + \mathbb{N}\delta)$. From the definition (3.8) of R_{v,Δ_+} we see that

$$R_{v,\Delta_+} = \bigcup_{i=1}^m \left\{ \pm \frac{(v,\beta_i) + n(v,\delta)}{|(v,\beta_i) + n(v,\delta)|} |n \in \mathbb{N}, i = 1, 2, \dots, m, (v,\beta_i) + n(v,\delta) \neq 0 \right\}. \tag{3.10}$$

If $(v,\delta) = 0$, then the set is finite. Otherwise for $i = \{1, 2, \dots, m\}$ we have $\lim_{n\to\infty} \frac{(v,\beta_i)+n(v,\delta)}{|(v,\beta_i)+n(v,\delta)|} = \frac{(v,\delta)}{|(v,\delta)|}$. □

[6] When the set F is clear we write just $R_{v,A}$.

From this lemma and Corollary 3.9 it follows:

Corollary 3.12. *Let Q be an Euclidean quiver. Then for any $\sigma \in D^b(Rep_k(Q))$ the set P_σ is either finite or has exactly two limit points of the type $\{p, -p\}$.*[7]

Proof. If P_σ is infinite, then by the previous lemma and $P_\sigma \subset R_{v,\Delta_+}$, $\{v_i = Z(s_i)\}_{i\in V(Q)}$ it follows that $R_{v,\Delta_+} = \cup_{i=1}^m \{\pm p_j^i\}_{j\in\mathbb{N}}$ with $\lim_{j\to\infty} p_j^i = p$ for $i = 1, 2, \dots, m$. In particular P_σ can not have more than two limit points. Since P_σ is infinite, the sets $P_\sigma \cap \{p_j^i\}_{j\in\mathbb{N}}$, $P_\sigma \cap \{-p_j^i\}_{j\in\mathbb{N}}$ are infinite for some i (recall that $-P_\sigma = P_\sigma$). Hence $\{p, -p\}$ are limit points of P_σ. The corollary follows. □

Next we discuss the set R_{v,Δ_+} for the l-Kronecker quiver $K(l)$ (two vertices with l parallel arrows between them), when $l \geq 2$. In this case the vertices are two, so v has two complex coordinates. I.e. R_{v,Δ_+} consists of fractions like $\frac{nz_1+mz_2}{|nz_1+mz_2|}$, where $z_1, z_2 \in \mathbb{C}$, $n, m \in \mathbb{N}$. It is useful to note

Remark 3.13. *Let $z_i = r_i \exp(\mathrm{i}\phi_i)$, $r_i > 0$, $i = 1, 2$, $0 < \phi_2 < \phi_1 \leq \pi$. Then*

$$\frac{\alpha z_1 + \beta z_2}{|\alpha z_1 + \beta z_2|} = \begin{cases} \exp\left(\mathrm{i} f\left(\frac{\alpha}{\beta}\right)\right) & \alpha \geq 0, \beta > 0, \\ \exp(\mathrm{i}\phi_1) & \alpha > 0, \beta = 0, \end{cases} \tag{3.11}$$

where $f : [0, \infty) \to [\phi_2, \phi_1) \subset (0, \pi)$ is the strictly increasing smooth function:

$$f(x) = \arccos\left(\frac{x r_1 \cos(\phi_1) + r_2 \cos(\phi_2)}{\sqrt{x^2 r_1^2 + r_2^2 + 2x r_1 r_2 \cos(\phi_1 - \phi_2)}}\right), \quad f(0) = \phi_2, \ \lim_{x\to\infty} f(x) = \phi_1. \tag{3.12}$$

From (3.4) we see that the Euler form for the quiver $K(l)$ is $\langle(\alpha_1, \alpha_2), (\beta_1, \beta_2)\rangle_{K(l)} = \alpha_1\beta_1 + \alpha_2\beta_2 - l\alpha_1\beta_2$. Hence the positive roots are

$$\Delta_{l+} = \Delta_+(K(l)) = \{(n, m) \in \mathbb{N}^2 | n^2 + m^2 - lmn \leq 1\} \setminus \{(0, 0)\}. \tag{3.13}$$

Remark 3.14. *Since the root systems $\Delta(K(l))$ with $l \geq 2$ will play an important role, we reserve for them the notation $\Delta_l = \Delta(K(l))$, respectively $\Delta_{l+} = \Delta_+(K(l))$.*

The roots with $n^2 + m^2 - lmn = 1$ are called real roots and with $n^2 + m^2 - lmn \leq 0$ - imaginary roots. We can represent the real and the imaginary roots as follows:

$$\Delta_{l+}^{re} = \{(1, 0)\} \cup \{(0, 1)\} \cup \left\{\frac{n}{m} = \frac{1}{2}\left(l \pm \sqrt{l^2 - 4 + \frac{4}{m^2}}\right) | n, m \in \mathbb{N}_{\geq 1}, (n, m) = 1\right\} \tag{3.14}$$

$$\Delta_{l+}^{im} = \left\{\frac{1}{2}\left(l - \sqrt{l^2 - 4}\right) \leq \frac{n}{m} \leq \frac{1}{2}\left(l - \sqrt{l^2 - 4}\right) | n \in \mathbb{N}_{\geq 0}, m \in \mathbb{N}_{\geq 1}\right\}. \tag{3.15}$$

[7]Later we show that one can always find σ s. t. P_σ is with two limit points.

Lemma 3.15. *Let $v = (z_1, z_2)$, $z_i = r_i \exp(\mathrm{i}\phi_i)$, $r_i > 0$, $0 < \phi_2 < \phi_1 \leq \pi$, $l \geq 2$. Let us denote $u = f\left(\frac{1}{2}\left(l - \sqrt{l^2-4}\right)\right)$, $v = f\left(\frac{1}{2}\left(l + \sqrt{l^2-4}\right)\right)$, where f is defined in Remark 3.13. Then*

$$R_{v,\Delta_{l+}} = \{\pm c_j\}_{j\in\mathbb{N}} \cup \pm D \cup \{\pm a_j\}_{j\in\mathbb{N}},$$

where D is a dense subset in the arc $\exp(\mathrm{i}[u,v]) \subset S^1$, $\{a_j\}_{j\in\mathbb{N}}$ is a sequence with $a_0 = \exp(\mathrm{i}\phi_2)$ and anti-clockwise monotonically converges to $\exp(\mathrm{i}u)$, $\{c_j\}_{j\in\mathbb{N}}$ is a sequence with $c_0 = \exp(\mathrm{i}\phi_1)$ and clockwise monotonically converging to $\exp(\mathrm{i}v)$ (note that, if $l = 2$ then $u = v = f(1)$, if $l > 2$ then $u < v$).

Proof. Now $R_{v,\Delta_{l+}} = \left\{\pm\frac{nz_1+mz_2}{|nz_1+mz_2|} | (n,m) \in \Delta_{l+}\right\}$. We have a disjoint union $\Delta_{l+} = \Delta_{l+}^{re} \cup \Delta_{l+}^{im}$, where Δ_{l+}^{re}, Δ_{l+}^{im} are taken from (3.14), (3.15). Recall also that if $m \geq 1$ then $\frac{nz_1+mz_2}{|nz_1+mz_2|} = \exp(\mathrm{i}f\,(n/m))$ (see Remark 3.13). Therefore we can write for $R_{v,\Delta_{l+}}$:

$$\left\{\pm\frac{nz_1+mz_2}{|nz_1+mz_2|} | (n,m) \in \Delta_{l+}^{re}\right\} \cup \left\{\pm\frac{nz_1+mz_2}{|nz_1+mz_2|} | (n,m) \in \Delta_{l+}^{im}\right\} =$$
$$\{\pm\exp(\mathrm{i}\phi_1)\} \cup \{\pm\exp(\mathrm{i}\phi_2)\} \cup \left\{\pm\exp\left(\mathrm{i}f\left(\frac{1}{2}\left(l \pm \sqrt{l^2-4+\frac{4}{m^2}}\right)\right)\right) | m \in \mathbb{N}_{\geq 1}\right\}$$
$$\cup \left\{\pm\exp\left(\mathrm{i}f\,(n/m)\right) | n/m \in \left[\frac{1}{2}\left(l - \sqrt{l^2-4}\right), \frac{1}{2}\left(l - \sqrt{l^2-4}\right)\right]\right\}.$$

Now the lemma follows from the properties of f given in Remark 3.13 and the fact that $\mathbb{Q} \cap \left[\frac{1}{2}\left(l - \sqrt{l^2-4}\right), \frac{1}{2}\left(l - \sqrt{l^2-4}\right)\right]$ is dense in $\left[\frac{1}{2}\left(l - \sqrt{l^2-4}\right), \frac{1}{2}\left(l - \sqrt{l^2-4}\right)\right]$. □

3.3.4 Stability conditions σ on $K(l)$ with $P_\sigma = R_{v,\Delta_{l+}}$, $l \geq 2$

In this Subsection $l \geq 2$ is fixed and $Q = K(l)$, Δ_{l+} is the positive root system of $K(l)$, $\mathcal{A} = Rep_k(Q)$, $\mathcal{T} = D^b(Rep_k(Q))$.

For a representation $X = k^n \overset{\vdots}{\rightrightarrows} k^m \in \mathcal{A}$ we write $\underline{\dim}(X) = (n,m)$, $\underline{\dim}_1(X) = n$, $\underline{\dim}_2(X) = m$. The simple objects of the standard t-structure $\mathcal{A} = Rep_k(Q) \subset D^b(Rep_k(Q))$ are:

$$s_1 = k \overset{\vdots}{\rightrightarrows} 0, \quad s_2 = 0 \overset{\vdots}{\rightrightarrows} k. \tag{3.16}$$

To $\mathcal{A} \subset \mathcal{T}$ we can apply Remark 2.29 and then we have $\mathbb{H}^{\mathcal{A}} \subset \mathrm{Stab}(\mathcal{T})$ and bijection $\mathbb{H}^{\mathcal{A}} \ni (\mathcal{P}, Z) \mapsto (Z(s_1), Z(s_2)) \in \mathbb{H}^2$.

For any $(\mathcal{P}, Z) \in \mathbb{H}^{\mathcal{A}}$, $t \in (0,1]$ $\mathcal{P}(t)$ consists of the objects in $\mathcal{A}$ satisfying the condition in Definition 2.26. If we denote $v = (Z(s_1), Z(s_2)) \in \mathbb{H}^2$, then by $Z(X) = (v, \underline{\dim}(X))$:

$$\begin{aligned} X \in \mathcal{P}(t), t \in (0,1] \iff \\ \text{for any } \mathcal{A}\text{-monic } \ X' \to X \qquad \arg(v, \underline{\dim}(X')) \leq \arg(v, \underline{\dim}(X)) = \pi t. \end{aligned} \tag{3.17}$$

Lemma 3.16. *Let* $\sigma = (\mathcal{P}, Z) \in \mathbb{H}^{\mathcal{A}}$ *and* $\arg(Z(s_1)) > \arg(Z(s_2))$. *Let* $(n,m) \in \Delta_{l+}$ *be a Schur root. Then* $\frac{nz_1+mz_2}{|nz_1+mz_2|} \in P_\sigma$, *where* $z_i = Z(s_i)$, $i = 1,2$.

Proof. So, let $(n,m) \in \Delta_{l+}$ be a Schur root. We show that there exists a σ-semistable X with $\underline{\dim}(X) = (n,m)$. Then the lemma follows because $X \in \mathcal{P}(t) \neq \{0\}$ for some $t \in (0,1]$ and by the formula $m(X)\exp(\mathrm{i}\pi t) = Z(X) = nz_1 + mz_2$.

If $m = 0$, then $n = 1$ (recall (3.13)) and then $X = s_1$ is the semistable, which we need (it is even stable in σ, since it is a simple object in $\mathcal{A}$). Hence we can assume that $m \geq 1$. Similarly, we can assume that $n \geq 1$.

Denote $\arg(z_i) = \phi_i$, $i = 1,2$, $v = (z_1, z_2)$. Then $0 < \phi_2 < \phi_1 \leq \pi$. By (3.11) for any X with $\underline{\dim}(X) = (n,m)$ we have $\arg(v, \underline{\dim}(X)) = \arg(nz_1 + mz_2) = f(n/m)$. Then by (3.17) such a X is semi-stable in σ iff any $\mathcal{A}$-monic $X' \to X$ satisfies

$$\arg(\underline{\dim}_1(X')z_1 + \underline{\dim}_2(X')z_2) \leq f\left(\frac{n}{m}\right).$$

Recall that $f(n/m) < \phi_1$ (see Remark (3.13)). From the last inequality we get $\underline{\dim}_2(X') \neq 0$ and then by (3.11) this inequality can be rewritten as $f(\underline{\dim}_1(X')/\underline{\dim}_2(X')) \leq f(n/m)$.

So, we see that $X \in \mathcal{A}$ with $\underline{\dim}(X) = (n,m)$ is σ-semistable iff any $\mathcal{A}$-monic arrow $X' \to X$ satisfies:

$$\underline{\dim}_2(X') \neq 0, \quad \frac{\underline{\dim}_1(X')}{\underline{\dim}_2(X')} \leq \frac{n}{m}. \tag{3.18}$$

Now since (n,m) is a Schur root, by Proposition 3.4 there exists $X \in \mathcal{A}$ with $\underline{\dim}(X) = (n,m)$ and a non-zero $\theta : K_0(\mathcal{A}) \to \mathbb{R}$, s. t. X is θ-semistable (see definition 3.2). We will show that this X is the σ-semistable, which we need.

By θ-semistability of X we have $\theta(1,0)n + \theta(0,1)m = 0$. By $m \neq 0$ we have a monic map[8] $s_2 \to X$ and then again by θ-semistability $\theta(0,1) \geq 0$, which together with $\theta(1,0)n + \theta(0,1)m = 0$, $\theta \neq 0$ implies

$$\theta(1,0) < 0, \quad \theta(0,1) > 0, \quad \theta(1,0)\frac{n}{m} + \theta(0,1) = 0.$$

Let us take now any monic arrow $X' \to X$ in $\mathcal{A}$ with $X' \neq 0$. By θ-semistability $0 \leq \theta(X') = \theta(1,0)\underline{\dim}_1(X') + \theta(0,1)\underline{\dim}_2(X')$. Hence by $\theta(1,0) < 0$ we obtain $\underline{\dim}_2(X') \neq 0$. Therefore we can

[8] Since the vertex corresponding to s_2 is a sink, s_2 is a subobject of any $X \in \mathit{Rep}_k(K(l))$ with $\underline{\dim}_2(X) \neq 0$.

write

$$\theta(1,0)\frac{\underline{\dim}_1(X')}{\underline{\dim}_2(X')}+\theta(0,1)\geq 0=\theta(1,0)\frac{n}{m}+\theta(0,1).$$

By $\theta(1,0)<0$ it follows $\frac{\underline{\dim}_1(X')}{\underline{\dim}_2(X')}\leq\frac{n}{m}$. Hence, we verified (3.18) and the lemma follows. □

Corollary 3.17. *Let $\sigma=(\mathcal{P},Z)\in\mathbb{H}^{Rep_k(K(l))}\subset \mathrm{Stab}D^b(Rep_k(K(l)))$ and $\arg(Z(s_1))>\arg(Z(s_2))$. Then $P_\sigma=R_{v,\Delta_{l+}}$, where $v=(Z(s_1),Z(s_2))$.*

Proof. In [44] one can read that the indecomposable representations of $K(l)$ with dimension vectors real roots have no self-extensions and hence by Remark 2.37 they are Schur.

For $l\geq 3$, [31, Theorem 4 a)] says that all imaginary roots Δ_{l+}^{im} are Schur roots as well, hence any $(n,m)\in\Delta_{l+}$ is Schur and we can apply the previous lemma to it.

When $l=2$, then the imaginary roots are $\Delta_{l+}^{im}=\{(n,n)\}_{n\geq 1}$ and, using the previous Lemma, it is enough to show that $(1,1)$ is a Schur root. It is easy to show that the representation $k \underset{Id}{\overset{Id}{\rightrightarrows}} k\in$ *Rep*$(K(2))$ is a Schur representation. The corollary follows. □

Remark 3.18. *Recently it was noted in [26] that there is a connection between [50], [51], [40] and the density in an arc for the Kronecker quiver.*

3.4 Kronecker pairs

In this Section we generalize Corollary 3.17. The most general statement is Theorem 3.24, but we use further only its Corollary 3.26 (corollary 3.25 is intermediate). The first step is: [9]

Lemma 3.19. *Let $\mathcal{T}$ be a k-linear triangulated category (here k can be any field). Let (E_1,E_2) be a full exceptional pair, s. t. $\mathrm{Hom}^{\leq 0}(E_1,E_2)=0$, $0<\dim_k(\mathrm{Hom}^1(E_1,E_2))=l<\infty$. Let $\mathcal{A}$ be the extension closure of (E_1,E_2) in $\mathcal{T}$.*

Then $\mathcal{A}$ is a heart of a bounded t-structure in $\mathcal{T}$ and there exists an equivalence of abelian categories: $F:\mathcal{A}\to Rep_k(K(l))$, s. t. $F(E_1)=s_1$, $F(E_2)=s_2$ (s_1, s_2 are as in (3.16)).

Proof. In [14, p. 6] or [41, section 3] it is shown that by $\hom^{\leq 0}(E_1,E_2)=0$ and[10] $\mathcal{T}=\langle E_1,E_2\rangle$ it follows that $\mathcal{A}$ is a heart of a bounded t-structure of $\mathcal{T}$ (see also [32, section 8]). In particular $\mathcal{A}$ is an abelian category.

[9]It is motivated by Bondal's result in [4] for equivalence between triangulated category generated by a strong exceptional collection and the derived category of modules over an algebra of homomorphisms of this collection and by a note on this equivalence in [37]. Observe however that we do not have restriction on (E_1,E_2) to be a strong pair and we construct equivlanece between t-structures.

[10]If S is a subset of objects in a triangulated category $\mathcal{T}$ we denote by $\langle S\rangle$ the triangulated subcategory generated by S.

Let $DT(\mathcal{T})$ be the category of distinguished triangles in $\mathcal{T}$ (objects are the distinguished triangles and morphisms are triple of arrows between triangles making commutative the corresponding diagram). Using the semi orthogonal decomposition $\mathcal{T} = \langle \langle E_1 \rangle , \langle E_2 \rangle \rangle$ one can construct three functors:

$$G : \mathcal{T} \to DT(\mathcal{T}), \quad \lambda_1 : \mathcal{T} \to \langle E_1 \rangle , \quad \lambda_2 : \mathcal{T} \to \langle E_2 \rangle \tag{3.19}$$

s. t. the triangle $G(X) \in DT(\mathcal{T})$ for any $X \in \mathcal{T}$ is:

$$G(X) \;=\; \lambda_2(X) \xrightarrow{u_X} X \xrightarrow{v_X} \lambda_1(X) \xrightarrow{w_X} \lambda_2(X)[1], \quad \lambda_1(X) \in \langle E_1 \rangle , \lambda_2(X) \in \langle E_2 \rangle . \tag{3.20}$$

It is well known that λ_2 is right adjoint to the embedding functor $\langle E_2 \rangle \to \mathcal{T}$ and λ_1 is left adjoint to $\langle E_1 \rangle \to \mathcal{T}$ (see for example [28, p. 279]). The adjoint functor (left or right) to an exact functor is also an exact functor ([7, Proposition 1.4]). Therefore λ_1 and λ_2 are exact functors. If we restrict λ_1, λ_2 to $\mathcal{A}$ then we obtain exact functors between abelian categories

$$\lambda_i^{\mathcal{A}} : \mathcal{A} \to \mathcal{A}_i \qquad i = 1, 2,$$

where $\mathcal{A}_i \cong k\text{-}Vect$ is the additive closure of E_i.

We define the functor $F : \mathcal{A} \to Rep_k(K(l))$ as follows. First choose a basis of $\mathrm{Hom}^1(E_1, E_2)$ and a decomposition of any $Y \in \mathcal{A}_i$ into $\dim(\mathrm{Hom}(E_i, Y))$ number of copies of E_i, $i = 1, 2$. Take any $X \in \mathcal{A}$, then we get a distinguished triangle $G(X)$ as in (3.20) with $\lambda_i(X) = \lambda_i^{\mathcal{A}}(X)$, in particular we get an arrow $\lambda_1^{\mathcal{A}}(X) \xrightarrow{w_X} \lambda_2^{\mathcal{A}}(X)[1]$. This arrow, using the chosen decompositions and the basis of $\mathrm{Hom}^1(E_1, E_2)$, can be expressed by l $a_2 \times a_1$ matrices over k, where $a_i = \dim(E_i, \lambda_i(X))$, $i = 1, 2$. In particular these l matrices are a representation of $K(l)$ with dimension vector (a_1, a_2) and we define $F(X)$ to be this representation.

Let $f : X \to Y$ be an arrow in $\mathcal{A}$ then, as far as $G : \mathcal{T} \to DT(\mathcal{T})$ is a functor, $G(f)$ is a morphism of triangles, hence the diagram:

$$\begin{array}{ccc} \lambda_1^{\mathcal{A}}(X) & \xrightarrow{w_X} & \lambda_2^{\mathcal{A}}(X)[1] \\ {\scriptstyle \lambda_1^{\mathcal{A}}(f)}\downarrow & & \downarrow{\scriptstyle \lambda_2^{\mathcal{A}}(f)[1]} \\ \lambda_1^{\mathcal{A}}(Y) & \xrightarrow{w_Y} & \lambda_2^{\mathcal{A}}(Y)[1] \end{array}$$

is commutative. Let M_1, M_2 be the matrices of $\lambda_1^{\mathcal{A}}(f)$, $\lambda_2^{\mathcal{A}}(f)$. The commutativity of the diagram above implies that $(M_1, M_2) : F(X) \to F(Y)$ is an arrow in $Rep_k(K(l))$ and our definition of $F(f)$ is $F(f) = (M_1, M_2)$.

By the exactness of $\lambda_i^{\mathcal{A}}$, $i = 1, 2$ it follows that F is an exact functor between abelian categories. Now, by straightforward computations one can show that F is an equivalence. $\square$

This lemma prompts the following definition

Definition 3.20. *A pair of objects (E_1, E_2) in a k-linear triangulated category $\mathcal{T}$ is called* <u>*l-Kronecker pair*</u> *if:*

- *(E_1, E_2) is an exceptional pair*

- $\mathrm{Hom}^{\leq 0}(E_1, E_2) = 0$
- $1 \leq l = \dim_k(\mathrm{Hom}^1(E_1, E_2)) < \infty$.

Corollary 3.21. *Let (E_1, E_2) be an l-Kronecker pair in a k-linear triangulated category $\mathcal{D}$ with $l \geq 2$. Denote $\mathcal{T} = \langle E_1, E_2 \rangle \subset \mathcal{D}$ and $\mathcal{A}$ - the extension closure of (E_1, E_2).*

Then any $\sigma = (\mathcal{P}, Z) \in \mathbb{H}^{\mathcal{A}} \subset \mathrm{Stab}\mathcal{T}$ with $\arg(Z(E_1)) > \arg(Z(E_2))$ satisfies $P_\sigma = R_{v,\Delta_{l+}}$, where $v = (Z(E_1), Z(E_2))$. In particular, if $l \geq 3$ then P_σ is dense in an arc of non-zero length, if $l = 2$ then P_σ has exactly two limit points.

Proof. We take the equivalence $F : \mathcal{A} \to Rep_k(K(l))$ constructed in Lemma 3.19, $\mathcal{A} \subset \mathcal{T}$, $Rep_k(K(l)) \subset D^b(Rep_k(K(l)))$. This equivalence induces a natural bijection $F^* : \mathbb{H}^{Rep_k(K(l))} \to \mathbb{H}^{\mathcal{A}}$. For $\sigma = (\mathcal{P}, Z) \in \mathbb{H}^{\mathcal{A}}$, $\sigma' = (\mathcal{P}', Z') \in \mathbb{H}^{Rep_k(K(l))}$, from $F^*(\sigma') = \sigma$ it follows $Z(E_i) = Z'(s_i)$ (because $F(E_i) = s_i$) and $P_\sigma = P_{\sigma'}$. Then the corollary follows from Corollary 3.17. □

Thus, in this Corollary we obtained $\sigma \in \mathrm{Stab}(\langle E_1, E_2 \rangle)$ with P_σ dense in a nontrivial arc. To obtain $\sigma' \in \mathrm{Stab}(\mathcal{D})$ with such a property, we want to extend the given $\sigma \in \mathrm{Stab}(\langle E_1, E_2 \rangle)$ to a stability condition on $\mathcal{D} \supset \mathcal{T}$ in the following sense:

Definition 3.22. *Let $\mathcal{T} \subset \mathcal{D}$ be a triangulated subcategory in a triangulated category $\mathcal{D}$. We say that $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}(\mathcal{T})$ can be extended to $\mathcal{D}$ (or extendable to $\mathcal{D}$) if there exists $\sigma_e = (\mathcal{P}_e, Z_e) \in \mathrm{Stab}(\mathcal{D})$, s. t. $Z_e \circ K_0(\mathcal{T} \subset \mathcal{D}) = Z$ and $\{\mathcal{P}(t) \subset \mathcal{P}_e(t)\}_{t \in \mathbb{R}}$. In this case σ_e is called extension of σ.*

Remark 3.23. *From Definition 3.1 it follows that if σ_e is an extension of σ, then $P_{\sigma_e} \supset P_\sigma$.*

By Corollary 3.21 it follows:

Theorem 3.24. *Let (E_1, E_2) be an l-Kronecker pair with $l \geq 2$ in a k-linear triangulated category $\mathcal{D}$. Denote $\mathcal{T} = \langle E_1, E_2 \rangle \subset \mathcal{D}$ and $\mathcal{A}$ - the extension closure of (E_1, E_2).*

Then any $\sigma \in \mathrm{Stab}(\mathcal{D})$, which is an extension of a stability condition $(\mathcal{P}, Z) \in \mathbb{H}^{\mathcal{A}} \subset \mathrm{Stab}(\mathcal{T})$ with $\arg(Z(E_1)) > \arg(Z(E_2))$ satisfies $P_\sigma \supset R_{v,\Delta_{l+}}$, where $v = (Z(E_1), Z(E_2))$. In particular, if $l \geq 3$ then P_σ is dense in an arc of non-zero length, if $l = 2$ then P_σ contains at least two limit points.

One setting, where we can extend these stability conditions, is as follows.

Assume that $(E_0, E_1, \dots, E_n)$ is a full Ext-exceptional collection[11] in $\mathcal{D}$. Then for any $0 \leq i < j \leq n$ the extension closure $\mathcal{A}_{ij}$ of $E_i, E_{i+1}, \dots, E_j$ is a heart of a bounded t-structure in $\mathcal{T}_{ij} = \langle E_i, E_{i+1}, \dots, E_j \rangle \subset \mathcal{D}$ (see [37, Lemma 3.14],[14]), hence we have a corresponding family $\mathbb{H}^{\mathcal{A}_{ij}} \subset \mathrm{Stab}(\mathcal{T}_{ij})$. In this setting all stability conditions in $\mathbb{H}^{\mathcal{A}_{ij}}$ are extendable to $\mathcal{D}$. The precise statement is (see Proposition 2.31 and [37, Proposition 3.17]) that there is a surjective map[12] $\pi_{ij} : \mathbb{H}^{\mathcal{A}} \to \mathbb{H}^{\mathcal{A}_{ij}}$, s. t. for any $\sigma \in \mathbb{H}^{\mathcal{A}_{ij}}$, $\sigma_e \in \mathbb{H}^{\mathcal{A}}$ from $\pi_{ij}(\sigma_e) = \sigma$ it follows that σ_e is an extension of σ. Having the desired extensions we obtain by Corollary 3.24:

[11] The "Ext-" means $\mathrm{Hom}^{\leq 0}(E_i, E_j) = 0$ for $0 \leq i < j \leq n$.
[12] We denote here $\mathcal{A} = \mathcal{A}_{0n}$.

Corollary 3.25. *Let $(E_0, E_1, \dots, E_n)$ be a full Ext-exceptional collection in a k-linear triangulated category $\mathcal{D}$. Let (E_i, E_{i+1}) be an l-Kronecker pair with $l \geq 2$ for some $0 \leq i \leq n-1$. Denote the extension closure of $(E_0, E_1, \dots, E_n)$ by $\mathcal{A}$.*

Then any $\sigma = (\mathcal{P}, Z) \in \mathbb{H}^{\mathcal{A}}$ with $\arg(Z(E_i)) > \arg(Z(E_{i+1}))$ satisfies $P_\sigma \supset R_{v,\Delta_{l+}}$, where $v = (Z(E_i), Z(E_{i+1}))$. In particular, if $l \geq 3$ then P_σ is dense in an arc of non-zero length, if $l = 2$ then P_σ contains at least two limit points.

Corollary 3.26. *Let $(E_0, E_1, \dots, E_n)$ be any full exceptional collection in a k-linear triangulated category $\mathcal{D}$ of finite type.[13] Let (E_i, E_j) be a l-Kronecker pair with $l \geq 2$ for some $0 \leq i < j \leq n$. Then there exists a family of stability conditions σ on $\mathcal{T}$ for which*

(a) *P_σ is dense in an arc of non-zero length, when $l \geq 3$,*

(b) *P_σ contains at least two limit points, when $l = 2$.*

Proof. First by mutations of the exceptional collection $(E_0, E_1, \dots, E_n)$ we can obtain a full exceptional collection $(E_i, E_j, C_2, \dots, C_n)$. Then, because $\mathcal{T}$ is of finite type, after shifts of $C_2, C_3, \dots, C_n$ we can obtain a full exceptional collection $\mathcal{B} = \{B_0, B_2, \dots, B_n\}$, which is Ext and $B_0 = E_i$, $B_1 = E_j$. So we get a full Ext-exceptional collection $\mathcal{B}$ for which (B_0, B_1) is an l-Kronecker pair with $l \geq 2$. Now if we denote by $\mathcal{A}$ the extension closure of $\mathcal{B}$ by Corollary 3.25 it follows that any $\sigma = (\mathcal{P}, Z) \in \mathbb{H}^{\mathcal{A}}$ with $\arg(Z(B_0)) > \arg(Z(B_1))$ satisfies $P_\sigma \supset R_{v,\Delta_{l+}}$, where $v = (Z(B_0), Z(B_1))$ and $l = \dim_k(\mathrm{Hom}^1(B_0, B_1))$. □

Remark 3.27. *A more general setting, where the stability conditions $\mathbb{H}^{\mathcal{A}}$ in Theorem 3.24 can be extended, is that there exists a semi-orthogonal decomposition $(\mathcal{D}', \langle E_1, E_2\rangle)$ of $\mathcal{D}$ with additional assumptions, specified in [14, Theorem 3.6] and [14, Proposition 3.5].*

3.5 Application to quivers

In this Section we apply the results of the previous Section 3.4 to quivers and obtain Corollary 3.28, Proposition 3.29. Table (3.1) contains Proposition 3.29 and the results of Section 3.3.

Let Q be an acyclic quiver. The notations $V(Q)$, $Arr(Q)$, $\Gamma(Q)$ are explained in Subsection 2.B.2. It is shown in [15] that any exceptional collection $(E_1, E_2, \dots, E_n)$ in $Rep_k(Q)$ of length $n = \#(V(Q))$ is a full exceptional collection of $D^b(Rep_k(Q))$. Furthermore, any exceptional collection $(E_1, E_2, \dots, E_i)$ in $Rep_k(Q)$ with $i < n$ can be completed to a full $(E_1, E_2, \dots, E_i, E_{i+1}, \dots, E_n)$ exceptional collection. In particular if we are given an l-Kronecker pair in $Rep_k(Q)$ we can complete it to a full exceptional collection and, since $D^b(Rep_k(Q))$ is of finite type, then we can apply Corollary 3.26. Therefore, only existence of an l-Kronecker pair with $l \geq 2$ in $Rep_k(Q)$ is enough to apply Corollary 3.26, hence if $l \geq 3$ then we obtain σ with P_σ dense in an arc, if $l = 2$ then we obtain σ with P_σ having at least two limit points. Now using Corollaries 3.10 , 3.12, we can easily prove:

[13]by finite type we mean that for any pair $X, Y \in \mathcal{T}$ we have $\sum_{k\in\mathbb{Z}} \dim(\mathrm{Hom}(X, Y[k])) < \infty$.

Corollary 3.28. *Let Q be either an Euclidean or a Dynkin quiver. Then any exceptional pair (E_1, E_2) in $Rep_k(Q)$ satisfies* $\dim_k(\mathrm{Hom}(E_1, E_2)) < 3$,$\dim_k(\mathrm{Ext}^1(E_1, E_2)) < 3$.

Proof. Since $Rep_k(Q)$ is hereditary, the exceptional objects in $D^b(Rep_k(Q))$ are just shifts of exceptional objects in $Rep_k(Q)$ and then from the arguments above and Corollaries 3.10 , 3.12 it follows that there does not exists a l-Kronecker pair with $l \geq 3$ in $D^b(Rep_k(Q))$. In other words for any exceptional pair (E_1, E_2) in $D^b(Rep_k(Q))$ the minimal nonzero degree $\mathrm{Hom}^{min}(E_1, E_2) \neq 0$, $\mathrm{Hom}^{<min}(E_1, E_2) = 0$ has dimension $\dim_k(\mathrm{Hom}^{min}(E_1, E_2)) \leq 2$. Since $Rep_k(Q)$ is hereditary, there are at most two nonzero degrees in $\mathrm{Hom}^*(E_1, E_2)$ and it remains to show that the maximal nonzero degree $\mathrm{Hom}^{max}(E_1, E_2) \neq 0$, $\mathrm{Hom}^{>max}(E_1, E_2) = 0$ has dimension $\dim_k(\mathrm{Hom}^{max}(E_1, E_2)) \leq 2$.

For any exceptional pair (E_1, E_2) it is well known that $(L_{E_1}(E_2), E_1)$ is also an exceptional pair, where $L_{E_1}(E_2)$ is determined by the distinguished triangle:

$$L_{E_1}(E_2) \longrightarrow \mathrm{Hom}^*(E_1, E_2) \otimes E_1 \xrightarrow{ev_{E_1,E_2}} E_2 \longrightarrow L_{E_1}(E_2)[1]. \qquad (3.21)$$

Take any $i \in \mathbb{Z}$. We show below that $\mathrm{Hom}^i(L_{E_1}(E_2), E_1) \cong \mathrm{Hom}^{-i}(E_1, E_2)$, which means that the maximal non-zero degree of $\mathrm{Hom}^*(E_1, E_2)$ is the minimal non-zero degree of $\mathrm{Hom}^*(L_{E_1}(E_2), E_1)$ and they are isomorphic. Then the corollary follows by the proved inequality for the minimal non-vanishing degrees.

We apply $\mathrm{Hom}^i(\_, E_1)$ to the triangle above and by $\mathrm{Hom}^*(E_2, E_1) = 0$ it follows

$$\mathrm{Hom}^i(\mathrm{Hom}^*(E_1, E_2) \otimes E_1, E_1) \cong \mathrm{Hom}^i(L_{E_1}(E_2), E_1). \qquad (3.22)$$

On the other hand (recall that E_1 is an exceptional object)

$$\mathrm{Hom}^*(E_1, E_2) \otimes E_1 \cong \bigoplus_j E_1[-j]^{\dim(\mathrm{Hom}^j(E_1,E_2))} \qquad \Rightarrow$$

$$\mathrm{Hom}^i(\mathrm{Hom}^*(E_1, E_2) \otimes E_1, E_1) \cong \mathrm{Hom}(\oplus_j E_1[-j]^{\dim(\mathrm{Hom}^j(E_1,E_2))}, E_1[i]) \cong k^{\dim(\mathrm{Hom}^{-i}(E_1,E_2))},$$

which together with (3.22) give $\mathrm{Hom}^i(L_{E_1}(E_2), E_1) \cong \mathrm{Hom}^{-i}(E_1, E_2)$ and the corollary is proved. □

Next we want to prove:

Proposition 3.29. *Any Euclidean quiver $\widetilde{Q}$ has a family of stability conditions σ on $D^b(Rep_k(\widetilde{Q}))$, s. t. P_σ has exactly two limit points of the type $\{p, -p\}$.*

Any acyclic connected quiver Q, which is neither Euclidean nor Dynkin has a family of stability conditions σ on $D^b(Rep_k(Q))$, s. t. P_σ is dense in an arc of non-zero length.

Remark 3.30. *If there are oriented cycles in Q and $k = \mathbb{C}$, then one can show that there is a family*[14] $\{s_\lambda\}_{\lambda\in\mathbb{C}}$ *of non isomorphic simple objects in $\mathcal{A} = Rep_{\mathbb{C}}(Q)$. Then if we define for simple object $s \in \mathcal{A}$*

$$Z(s) = \begin{cases} \frac{\lambda}{|\lambda|} & \text{if } s = s_\lambda, \lambda \in \mathbb{H} \\ \mathrm{i} & \text{otherwise} \end{cases} \tag{3.23}$$

we obtain a stability function $Z : K_0(\mathcal{A}) \to \mathbb{C}$, which has HN property, since $\mathcal{A}$ is of finite length. One can show that the corresponding stability condition $\sigma = (\mathcal{P}, Z) \in \mathbb{H}^{\mathcal{A}}$ is locally finite. Since all $\{s_\lambda\}_{\lambda\in\mathbb{C}}$ are simple in $\mathcal{A}$, they are σ-semistable. Hence $P_\sigma = S^1$.

By the arguments given in the beginning of this Section (before Corollary 3.28) we reduce the proof of Proposition 3.29 to finding an l-Kronecker pair with $l \geq 3$ in $D^b(Rep_k(Q))$ for non-euclidean quiver Q and finding a 2-Kronecker pair in $D^b(Rep_k(\widetilde{Q}))$ for euclidean quiver $\widetilde{Q}$ (recall also Corollary 3.12). From here till the end of this subsection we present the proof of the following:

Proposition 3.31. *Any acyclic connected quiver Q, which is neither Euclidean nor Dynkin has an l-Kronecker pair with $l \geq 3$ in $Rep_k(Q)$ (i. e. a pair or representations (ρ, ρ') in $Rep_k(Q)$ with $\mathrm{Hom}^*_{\mathcal{T}}(\rho', \rho) = \mathrm{Hom}^{\leq 0}_{\mathcal{T}}(\rho, \rho') = 0$, $\dim_k(\mathrm{Hom}^1_{\mathcal{T}}(\rho, \rho')) \geq 3$, where $\mathcal{T} = D^b(Rep_k(Q))$).*

Along the proof of this proposition we obtain also Lemma 3.38, which implies the first part of Proposition 3.29. For any acyclic connected quiver Q and for any $\rho \in Rep_k(Q)$ we will denote:

$$supp(\rho) = supp(\underline{\dim}(\rho)) = \{i \in V(Q) | \underline{\dim}_i(\rho) \neq 0\}. \tag{3.24}$$

For $i \in V(Q)$ the simple representation s_i is characterized by $supp(s_i) = \{i\}$, $\underline{\dim}_i(s_i) = 1$. Obviously $\{s_i | i \in V(Q)\}$ are exceptional objects in $D^b(Rep_k(Q))$. One of the representations in the Kronecker pair (ρ, ρ'), which we shall obtain, is among the exceptional objects $\{s_i | i \in V(Q)\}$.

It is useful to denote

$$\begin{gathered} A, B \subset V(Q) \quad Arr(A, B) = \{a \in Arr(Q) | s(a) \in A, t(a) \in B\}, \\ Ed(A, B) = Ed(B, A) = Arr(A, B) \cup Arr(B, A). \end{gathered} \tag{3.25}$$

To find Kronecker pairs in $D^b(Rep_k(Q))$ we observe first, that for $\rho, \rho' \in Rep_k(Q)$ we have $\mathrm{Hom}^{\leq -1}(\rho, \rho') = 0$ in $D^b(Rep_k(Q))$ and

$$\begin{gathered} supp(\rho) \cap supp(\rho') = \emptyset \quad \Rightarrow \quad \mathrm{Hom}(\rho, \rho') = \mathrm{Hom}(\rho', \rho) = 0, \\ \dim_k(\mathrm{Hom}^1(\rho, \rho')) = \sum_{a \in Arr(supp(\rho), supp(\rho'))} \underline{\dim}_{s(a)}(\rho) \underline{\dim}_{t(a)}(\rho') \end{gathered} \tag{3.26}$$

which follows by (2.3). Another useful statement is

[14] For example the representations $\{\mathbb{C} \xrightarrow{\lambda} \mathbb{C}\}_{\lambda\in\mathbb{C}}$ of the quiver with one vertex and one loop are all simple and mutually non isomorphic

Lemma 3.32. *Let Q be an Euclidean quiver. Then for each $n \in \mathbb{N}$ there exists an exceptional representation $\rho \in Rep_k(Q)$, s. t. $\underline{\dim}_v(\rho) \geq n$ for each $v \in V(Q)$.*

Proof. Let $\delta \in \mathbb{N}_{\geq 1}^{V(Q)}$ be the minimal imaginary root of $\Delta_+(Q)$, used in the proof of Lemma 3.11.[15] One property of δ is that $(\_, \delta)_Q = 0$ on $\mathbb{N}^{V(Q)}$, where $(\alpha, \beta)_Q = \langle \alpha, \beta \rangle_Q + \langle \beta, \alpha \rangle_Q$ is the symmetrization of $\langle , \rangle_Q$. We find below a vertex $v \in V(Q)$, s. t. $\langle 1_v, 1_v \rangle_Q = \frac{1}{2}(1_v, 1_v)_Q = 1$, $\langle 1_v, \delta \rangle_Q \neq 0$. Then for any $m \in \mathbb{N}$ we have $1 = \frac{1}{2}(1_v + m\delta, 1_v + m\delta)_Q = \langle 1_v + m\delta, 1_v + m\delta \rangle_Q = 1$, $\langle 1_v + m\delta, \delta \rangle_Q \neq 0$, hence, for big enough m, $r = 1_v + m\delta$ is a real positive root $r \in \Delta_+(Q)$ with $\langle r, \delta \rangle_Q \neq 0$, $\{r_v \geq n\}_{v \in V(Q)}$. Hence by [16, p. 27] there is an exceptional representation ρ with $\underline{\dim}(\rho) = r$ and the lemma follows.

If[16] $\Gamma(Q) = \widetilde{\mathbb{A}}_m$, $m \geq 1$: As far as Q is not an oriented cycle then there is a sink $s \in V(Q)$ (i. e. both the arrows touching s end at it). Hence by (3.4) $\langle 1_s, 1_s \rangle_Q = - \langle 1_s, \delta \rangle_Q = 1$.

If $\Gamma(Q) = \widetilde{\mathbb{D}}_m$, $m \geq 4$, $\widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8$. In [1, fig. (4.13)] are given the coordinates of δ for all these options for $\Gamma(Q)$. We take $v \in V(Q)$ to be the extending vertex, in [1, fig. (4.13)] this vertex is denoted by $\star$, i.e. $v = \star$. Then by (3.4) and the given in [1, fig. (4.13)] coordinates of δ one computes $\langle 1_v, 1_v \rangle_Q = 1$, $\langle 1_v, \delta \rangle_Q = \pm 1$, depending on whether $\star$ is a sink/source in Q. □

Definition 3.33. *Let $A \subset V(Q)$, $A \neq \emptyset$. By Q_A we denote the quiver with $V(Q_A) = A$ and $Arr(Q_A) = Arr(A, A) = \{a \in Arr(Q) | s(a) \in A, t(a) \in A\}$. For any $\rho \in Rep_k(Q_A)$ we denote by the same letter ρ the representation in $Rep_k(Q)$, which on A, $Arr(A, A)$ coincides with ρ and is zero elsewhere.*

We say that a vertex $v \in V(Q)$ is adjacent to Q_A if $v \notin A$ and $Ed(A, \{v\}) \neq \emptyset$.

Remark 3.34. *If $\rho \in Rep_k(Q_A)$ is an exceptional representation then the corresponding extended representation in $Rep_k(Q)$ is also exceptional.*

Remark 3.35. *If $v \in V(Q)$ is adjacent to Q_A then $Arr(Q_{A \cup \{v\}}) = Arr(Q_A) \cup Ed(A, \{v\})$.*

In the following two corollaries we consider a configuration of a subset $A \subset V(Q)$ and an adjacent to it vertex $v \in V(Q)$, s. t. the arrows connecting v and A are all directed either from v to A or from A to v, which means that v is either a source or a sink in $Q_{A \cup \{v\}}$.

In Corollary 3.36 we show that if Q_A is an Euclidean quiver then we get a l-Kronecker pair (E_1, E_2) in $Rep_k(Q)$ with $l = \dim_k(\mathrm{Hom}^1(E_1, E_2))$ as big as we want (without additional assumption on the quiver Q).

In Corollary 3.37 we show that if $\Gamma(Q_A)$ is either $\mathbb{A}_n (n \geq 1)$ or $\mathbb{D}_n (n \geq 4)$ then, under the additional assumption that there are at least three edges between v and A, we get a Kronecker pair (E_1, E_2) in $Rep_k(Q)$ with $\dim_k(\mathrm{Hom}^1(E_1, E_2)) = l$ equal to this number of edges.

[15] In [1, fig. (4.13)] are given the coordinates of δ for all euclidean graphs

[16] Recall that by $\Gamma(Q)$ we denote the underlying non-oriented graph and that $\widetilde{\mathbb{A}}_1$ is graph with two vertices and two parallel edges connecting them, $\widetilde{\mathbb{A}}_m$ with $m \geq 2$ is a loop with $m+1$ vertices and m edges connecting them forming a simple loop

Corollary 3.36. *Let $A \subset V(Q)$ be such that Q_A is Euclidean. Let $v \in V(Q)$ be a vertex, which is adjacent to Q_A and either a sink or a source in $Q_{A\cup\{v\}}$. Then for any $n \geq 3$ there exists an l-Kronecker pair (E_1, E_2) in $Rep_k(Q)$ with $l = \dim(\mathrm{Hom}^1(E_1, E_2)) \geq n$.*

Proof. From Lemma 3.32 and Remark 3.34 we get an exceptional representation $\rho \in Rep_k(Q)$, s. t. $supp(\rho) = A$ and $\{\underline{\dim}_i(\rho) \geq n\}_{i\in A}$.

If v is a sink in $Q_{A\cup\{v\}}$ then $Arr(\{v\}, A) = \emptyset$, and, since v is adjacent to Q_A, $Arr(A, \{v\}) \neq \emptyset$. From $\{v\} \cap A = \emptyset$ and (3.26) we get $\mathrm{Hom}^*(s_v, \rho) = 0$, $\mathrm{Hom}^{\leq 0}(\rho, s_v) = 0$, $\dim_k(\mathrm{Hom}^1(\rho, s_v)) \geq n$. Therefore (ρ, s_v) is the l-Kronecker pair with $l \geq 3$ we need.

If v is a source in $Q_{A\cup\{v\}}$ then the same arguments show that (s_v, ρ) is such a Kronecker pair. □

Corollary 3.37. *Let $A \subset V(Q)$ be such that $\Gamma(Q_A)$ is either $\mathbb{A}_n (n \geq 1)$ or $\mathbb{D}_n (n \geq 4)$. Let $v \in V(Q)$ be adjacent to Q_A and either a sink or a source in $Q_{A\cup\{v\}}$. Let $\#(Ed(\{v\}, A)) = n \geq 3$. Then there exists a Kronecker pair (E_1, E_2) in $Rep_k(Q)$ with $\dim(\mathrm{Hom}^1(E_1, E_2)) = n$.*

Proof. Using that $\Gamma(Q_A)$ is either $\mathbb{A}_n (n \geq 1)$ or $\mathbb{D}_n (n \geq 4)$ we see that the representation ρ, with $A \ni i \mapsto k$, $Arr(A, A) \ni a \mapsto Id_k$ and zero elsewhere is an exceptional representation in $Rep_k(Q)$ with $supp(\rho) = A$ and $\{\underline{\dim}_i(\rho) = 1\}_{i\in A}$.

If v is a sink in $Q_{A\cup\{v\}}$ then $Arr(\{v\}, A) = \emptyset$ and $\#(Arr(A, \{v\})) = \#(Ed(\{v\}, A)) = n$. From $\{v\} \cap A = \emptyset$ and (3.26) we get $\mathrm{Hom}^*(s_v, \rho) = 0$, $\mathrm{Hom}^{\leq 0}(\rho, s_v) = 0$, $\dim_k(\mathrm{Hom}^1(\rho, s_v)) = n$. So that (ρ, s_v) is the l-Kronecker pair with $l \geq 3$ we need.

If v is a source in $Q_{A\cup\{v\}}$ then the same arguments show that (s_v, ρ) is such a Kronecker pair. □

By similar arguments we prove now the following lemma which together with Corollary 3.12 implies the first part of Proposition 3.29:

Lemma 3.38. *For each Euclidean quiver $\widetilde{Q}$ there exists a 2-Kronecker pair in $D^b(Rep_k(\widetilde{Q}))$.*

Proof. The underlying graph $\Gamma(\widetilde{Q})$ is some of the listed in [1, fig. (4.13)] extended Dynkin diagrams. Let $\star \in V(\widetilde{Q})$ be an extended vertex and Q be the corresponding embedded Dynkin diagram (as in the beginning of the proof of Lemma 2.143). When $\Gamma(\widetilde{Q}) = \widetilde{\mathbb{A}}_n$, $n \geq 2$ we choose $\star$ to be a sink or a source in $\widetilde{Q}$. When $\Gamma(\widetilde{Q}) \neq \widetilde{\mathbb{A}}_n$, then $Ed(\{\star\}, V(Q))$ (see (3.25) for the notation) has only one element and $\star$ is necessarily a sink or a source in $\widetilde{Q}$. Let $\rho \in Rep_k(Q)$ be the exceptional representation s. t. $\underline{\dim}(\rho)$ is the maximal root in $\Delta_+(Q)$. When $\Gamma(Q)$ is $\mathbb{A}_n$ then $\underline{\dim}(\rho)$ assigns 1 to all vertices of Q, otherwise one can easily determine $\underline{\dim}(\rho)$ using Remark 2.142. We denote by the same letter $\rho \in Rep_k(\widetilde{Q})$ the extended representation, it is exceptional in $Rep_k(\widetilde{Q})$ (see Remark 3.34). Let $s_\star \in Rep_k(\widetilde{Q})$ be the simple representation supported at $\star$. Now we have $supp(\rho) \cap supp(s_\star) = \emptyset$ and either $Arr(\star, supp(\rho)) = \emptyset$ or $Arr(supp(\rho), \star) = \emptyset$. From (3.26) we see that either $(s_\star, \rho)$ or $(\rho, s_\star)$ is a 2-Kronecker pair: when $\Gamma(\widetilde{Q}) = \widetilde{\mathbb{A}}_n$,$n \geq 2$ then $Ed(\star, supp(\rho))$ has two elements, otherwise $Ed(\star, supp(\rho))$ has one element but $\underline{\dim}_v(\rho) = 2$ for the vertex $v \in supp(\rho)$ touching this edge. □

So let us fix from here till the end of this Section a quiver Q, satisfying the conditions of Proposition 3.31. We are searching for an l-Kronecker pair with $l \geq 3$ in $Rep_k(Q)$.

An immediate consequence of Corollary 3.37 is

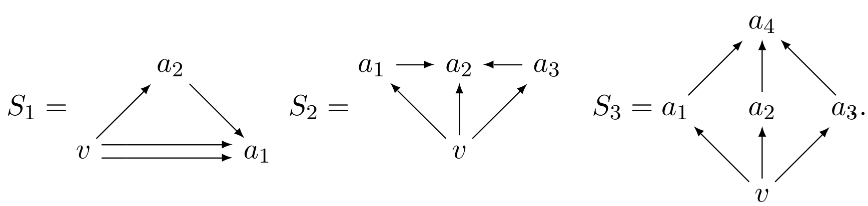
Corollary 3.39. *If there are $n \geq 3$ parallel arrows in $Arr(Q)$, then there exists an l-Kronecker pair (E_1, E_2) in $Rep_k(Q)$ with $l = \dim(\mathrm{Hom}^1(E_1, E_2)) = n$.*

Proof. Let these arrows start at a vertex i and end at a vertex j, then $\#Arr(\{i\},\{j\}) = n \geq 3$, $Arr(\{j\},\{i\}) = \emptyset$ and we apply Corollary 3.37 to $A = \{j\}$, $v = \{i\}$. □

Hence to prove Proposition 3.31, we can assume that there are no more than two parallel arrows in $Arr(Q)$. We consider next the case that two parallel arrows do occur.

Remark 3.40. *In the considerations, that follow, we refer most often to Corollary 3.36 (i. e. then we get l-Kronecker pairs with arbitrary big l), but there are three situations, where we need Corollary 3.37*[17] *with the minimal admissible number of edges connecting v and A, namely 3, and then the produced l-Kronecker pair is with minimal possible $l = 3$ ensuring density.*

The quiver $Q_{A\cup\{v\}}$ observed in these three special situations[18]*, in which we use Corollary 3.37, is as follows (we denote the set A by $A = \{a_1, a_2, \dots, a_n\}$):*



In S_1 $\Gamma(Q_A) = \mathbb{A}_2$, in S_2 $\Gamma(Q_A) = \mathbb{A}_3$, in S_3 $\Gamma(Q_A) = \mathbb{D}_4$.

3.5.1 If there are 2 parallel arrows in $Arr(Q)$

Let these two arrows start at a vertex i and end at a vertex j. Thus, throughout this subsubsection we have $\#(Arr(\{i\},\{j\})) = 2$, $Arr(\{j\},\{i\}) = \emptyset$. Then (recall Definition 3.33):

$$Q_{\{i,j\}} = i \rightrightarrows j, \quad \Gamma(Q_{\{i,j\}}) = \widetilde{\mathbb{A}}_1. \tag{3.27}$$

By our assumption that Q is connected and not Euclidean there is a vertex $k \in V(Q)$ which is adjacent to $Q_{\{i,j\}}$. If either $Arr(\{k\},\{i,j\}) = \emptyset$ or $Arr(\{i,j\},\{k\}) = \emptyset$ then by Corollary 3.36 we get an l-Kronecker pair with $l \geq 3$. Hence we can assume $Arr(\{k\},\{i,j\}) \neq \emptyset$ and $Arr(\{i,j\},\{k\}) \neq \emptyset$. From the condition that there are no oriented cycles we reduce to

$$Arr(\{k\},\{j\}) \neq \emptyset \quad Arr(\{i\},\{k\}) \neq \emptyset \quad \Rightarrow \quad Arr(\{j\},\{k\}) = Arr(\{k\},\{i\}) = \emptyset. \tag{3.28}$$

[17] we already used it once in Corollary 3.39

[18] in Corollary 3.39 the quiver $Q_{A\cup\{v\}}$ is the Kronecker quiver, i. e. $Q_{A\cup\{v\}} = K(n)$, $n \geq 3$

If $Arr(\{k\},\{j\})$ has two elements then $Q_{\{k,j\}} = k \rightrightarrows j$, $\Gamma(Q_{\{k,j\}}) = \widetilde{\mathbb{A}}_1$, i is adjacent to $Q_{\{k,j\}}$ and $Arr(\{k,j\},\{i\}) = Arr(\{k\},\{i\}) \cup Arr(\{j\},\{i\}) = \emptyset$, hence Corollary 3.36 produces a l-Kronecker pair with $l \geq 3$. Therefore we can assume that $Arr(\{k\},\{j\})$ has only one element.

If $Arr(\{i\},\{k\})$ has two elements, then $Q_{\{i,k\}} = i \rightrightarrows k$, $\Gamma(Q_{\{i,k\}}) = \widetilde{\mathbb{A}}_1$, j is adjacent to $Q_{\{i,k\}}$ and $Arr(\{j\},\{i,k\}) = Arr(\{j\},\{i\}) \cup Arr(\{j\},\{k\}) = \emptyset$, hence Corollary 3.36 produces a l-Kronecker pair with $l \geq 3$. Hence we can assume that $Arr(\{i\},\{k\})$, $Arr(\{k\},\{j\})$ are single element sets. Using that $Q_{\{i,j\}} = i \rightrightarrows j$ we obtain that $Q_{\{i,j,k\}}$ is the same as the quiver S_1 in Remark 3.40 with $v = i$, $a_1 = j$, $a_2 = k$ and then by Corollary 3.37 we obtain a l-Kronecker pair with $l \geq 3$.

We reduce to the case

3.5.2 If there are no parallel arrows in $Arr(Q)$

In this case for any pair $i, j \in V(Q)$, $i \neq j$ we have $\#(Arr(i,j)) = \#(Ed(i,j)) \leq 1$. In this subsubsection the term *loop with m vertices, $m \geq 1$ in $\Gamma(Q)$* means a sequence $a_1, a_2, \ldots, a_m$ in $V(Q)$, s. t. $\#\{a_1, a_2, \ldots, a_m\} = m$ and $\{Ed(a_i, a_{i+1}) \neq \emptyset\}_{i=1}^{m-1}$, $Ed(a_m, a_1) \neq \emptyset$. Since there are no edges-loops and there are no parallel arrows in Q, any loop in $\Gamma(Q)$ (if there is such) must be with $m \geq 3$ vertices.

First we show quickly how to get a l-Kronecker pair with $l \geq 3$ if there are no loops.

If there are no loops in $\Gamma(Q)$ (recall also that by assumption Q is neither Dynkin nor Euclidean) then one can show that for some proper subset $A \subset V(Q)$ Q_A is an Euclidean quiver of the type $\widetilde{\mathbb{D}}_m$, $m \geq 4$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$. Take an adjacent to Q_A vertex $v \in V(Q)$. The assumption that there are no loops in $\Gamma(Q)$ imply that there is a unique edge between v and Q_A, i. e. either $Arr(\{v\}, A) = \emptyset$ or $Arr(A, \{v\}) = \emptyset$, and then we can apply Corollary 3.36 to obtain a l-Kronecker pair with $l \geq 3$.

Till the end of this subsubsection we assume that there is a loop in $\Gamma(Q)$. Let us fix a loop with minimal number of vertices $a_1, a_2, \ldots, a_m$, i. e. m is the minimal possible number of vertices in a loop. Denote $A = \{a_1, a_2, \ldots, a_m\}$, $\#(A) = m$. From the minimality of m it follows that $Ed(a_i, a_j) = \emptyset$, if $1 \leq i < j \leq m$, $2 \leq j - i \leq m - 2$, hence Q_A (recall Definition 3.33) is a quiver with $\Gamma(Q_A) = \widetilde{\mathbb{A}}_{m-1}$. As above, there exists an adjacent to Q_A vertex $v \in V(Q)$. From Corollary 3.36 it follows that we can assume $Arr(\{v\}, A) \neq \emptyset$, $Arr(A, \{v\}) \neq \emptyset$. In particular $\#(Ed(A, \{v\})) \geq 2$. Let us summarize

$$\begin{gathered} \Gamma(Q_A) = \widetilde{\mathbb{A}}_{m-1}, \quad \{v\} \cap A = \emptyset, \quad Arr(\{v\}, A) \neq \emptyset, \ Arr(A, \{v\}) \neq \emptyset \\ m \geq \#(Ed(A, \{v\})) \geq 2, m \geq 3. \end{gathered} \tag{3.29}$$

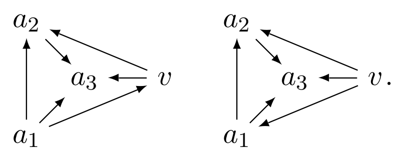
We consider several cases depending on the numbers m, $\#(Ed(A, \{v\}))$.

The case $\#(Ed(A, \{v\})) = m = 3$.

We can order $A = \{a_1, a_2, a_3\}$ so that $Q_A = \begin{array}{l} a_2 \\ \uparrow \ \searrow \\ \ \ \ \ \ \ \ a_3 \\ \mid \ \nearrow \\ a_1 \end{array}$. By $\#(Ed(A, \{v\})) = 3$ it follows that $v \in V(Q)$

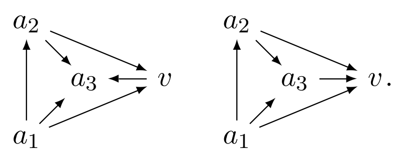
must be connected to all the three vertices $\{a_1, a_2, a_3\}$ and by (3.29) one of the arrows must start at v and another must end at it. We have either $Arr(\{v\}, \{a_2\}) \neq \emptyset$ or $Arr(\{a_2\}, \{v\}) \neq \emptyset$.

If $Arr(\{v\}, \{a_2\}) \neq \phi$, then by the assumption that there are no oriented cycles we have $Arr(\{a_3\}, \{v\}) = \phi$, $Arr(\{v\}, \{a_3\}) \neq \phi$, so we can only choose the direction of $Ed(v, a_1)$, i. e. we have two options for $Q_{\{a_1,a_2,a_3,v\}}$:



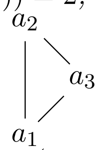
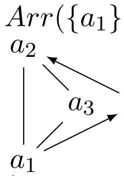
In both the cases a_3 is a sink in $Q_{\{a_1,a_2,a_3,v\}}$, hence we can apply Corollary 3.36 to $Q_{\{a_1,a_2,v\}}$ and $a_3 \in V(Q)$ and get l-Kronecker pair with $l \geq 3$.

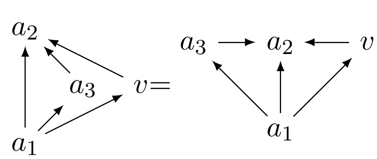
If $Arr(\{a_2\}, \{v\}) \neq \phi$, now the direction of $Ed(v, a_1)$ is fixed and both the options for $Q_{\{a_1,a_2,a_3,v\}}$ are:



In the first case we apply Corollary 3.36 to $Q_{\{a_1,a_2,v\}}$ and $a_3 \in V(Q)$ and in the second we apply it to $Q_{\{a_1,a_2,a_3\}}$ and $v \in V(Q)$.

The case $\#(Ed(A, \{v\})) = 2$, $m = 3$.

Let us consider $\Gamma(Q_A) =$  without fixing the orientation. Now there are only two edges between A and v. Hence by (3.29) there are exactly two arrows between A and v and one of them must start at v and the other must end at it. As long as we have not fixed the orientations of the arrows in Q_A, we can assume that $Arr(\{a_1\}, \{v\}) \neq \emptyset$, $Arr(\{v\}, \{a_2\}) \neq \emptyset$. So that $Q_{\{a_1,a_2,a_3,v\}}$, up to a choice of orientation in Q_A, is  v. We consider now the possible choices of directions of the arrows in Q_A. If a_3 is a source/sink in Q_A, then it is a source/sink in $Q_{A\cup\{v\}}$ and we can apply Corollary 3.36 to $Q_{\{a_1,a_2,v\}}$ and $a_3 \in V(Q)$. Hence, by the condition that Q is acyclic, we reduce to



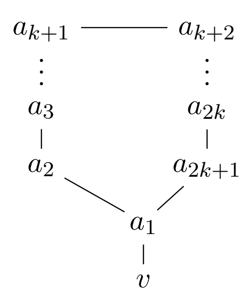
and this is a permutation of the special case S_2 of Remark 3.40. In this case we obtain a l-Kronecker pair with $l \geq 3$ by Corollary 3.37 applied to $Q_{\{v,a_2,a_3\}}$ and $a_1 \in V(Q)$.

The case $m = 4$.

In this case $\Gamma(Q_A) = \begin{array}{ccc} a_4 & — & a_3 \\ | & & | \\ a_1 & — & a_2 \end{array}$ and by the minimality of $m = 4$ it follows $\#(Ed(A, \{v\})) = 2$ (recall that we have reduced to (3.29)). In other words the adjacent vertex v must be connected to two of the vertices of the quadrilateral $\Gamma(Q_A)$. Again by the minimality of $m = 4$ these two vertices must be diagonal and, as long as we have not fixed the orientations of the arrows, we can assume that $Arr(\{a_1\}, \{v\}) \neq \emptyset$, $Arr(\{v\}, \{a_3\}) \neq \emptyset$. So that $Q_{\{a_1,a_2,a_3,a_4,v\}}$, up to a choice of orientation in Q_A, is $\begin{array}{ccccc} & & a_3 & & \\ & / & | & \nwarrow & \\ a_4 & & a_2 & & v \\ & \diagdown & | & \nearrow & \\ & & a_1 & & \end{array}$. It follows to assign directions of the arrows in Q_A. If a_4 or a_2 is a source/sink in Q_A then we can apply Corollary 3.36 to $Q_{\{a_1,a_2,a_3,v\}}$ and $a_4 \in V(Q)$ or to $Q_{\{a_1,a_4,a_3,v\}}$ and $a_2 \in V(Q)$, respectively. Therefore, we reduce to $Q_{\{a_1,a_2,a_3,a_4,v\}} = \begin{array}{ccccc} & & a_3 & & \\ & \nearrow & \uparrow & \nwarrow & \\ a_4 & & a_2 & & v \\ & \nwarrow & \uparrow & \nearrow & \\ & & a_1 & & \end{array}$,which is a permutation of the quiver S_3 in Remark 3.40.

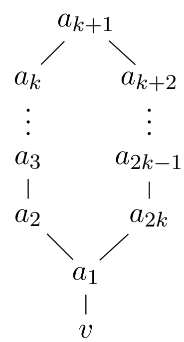
The case $m \geq 5$.

If $m = 2k+1$ is odd, $k \geq 2$, then we can depict $\Gamma(Q_A)$, $v \in V(Q)$ and one edge in $Ed(\{v\}, Q_A)$ as follows:



We have reduced to the case $\#(Ed(\{v\}, Q_A)) \geq 2$ (see (3.29)). If we add another edge between v and A then we obtain another loop with number of vertices less or equal to $k + 2$. By $k \geq 2$ we have $k + 2 < 2k + 1$, which contradicts the minimality of $m = 2k + 1$.

If $m = 2k$ is even, $k \geq 3$, then we can depict $\Gamma(Q_A)$, $v \in V(Q)$ and one edge in $Ed(\{v\}, Q_A)$ as follows:



Again, another edge between v and A produces another loop with number of vertices less or equal to $k+2$. By $k \geq 3$ we have $k+2 < 2k$, which contradicts the minimality of $m = 2k$.

Proposition 3.31 is completely proved and it implies Proposition 3.29.

Having Proposition 3.29, Remark 3.30, Corollary 3.12 and Lemma 3.10 we obtain table 3.1.

3.6 Further examples of Kronecker pairs

Here we give some more examples of l-Kronecker pairs with $l \geq 3$. In all the cases Corollary 3.26 can be applied, hence we obtain a family of stability conditions as in the third row of table 3.1.

3.6.1 Markov triples

It is shown in [6, Example 3.2] that if X is a smooth projective variety (we assume over $\mathbb{C}$), such that $D^b(Coh(X))$ is generated by a strong exceptional collection of three elements (for example $X = \mathbb{P}^2$) then for any such collection (E_0, E_1, E_2) the dimensions $a = \dim(\mathrm{Hom}(E_0, E_1))$, $b = \dim(\mathrm{Hom}(E_0, E_2))$, $c = \dim(\mathrm{Hom}(E_1, E_2))$ satisfy Markov's equation $a^2+b^2+c^2 = abc$. If $(a, b, c) \neq (0, 0, 0)$ and $a, b, c \leq 3$ then (a, b, c) satisfy Markov's equation iff $a = b = c = 3$, i. e. the "minimal" such triple is $(3, 3, 3)$. Hence for any strong collection (E_0, E_1, E_2) on $D^b(Coh(X))$ for some $i < j$ the pair $(E_i, E_j[-1])$ is an l-Kronecker pair with $l \geq 3$. Corollary 3.26 can be applied, since $D^b(Coh(X))$ is of finite type.

3.6.2 $\mathbb{P}^1 \times \mathbb{P}^1$

In [39, p. 3] a full exceptional collection consisting of sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ is described. The matrix given there contains the dimensions of $\mathrm{Hom}(E_i, E_j)$, where E_i, E_j are pairs in the exceptional collection. The number 4 in this matrix corresponds to a 4-Kronecker pair.

3.6.3 $\mathbb{P}^n$, $n \geq 2$ and their blow ups

Another example, where Corollary 3.26 can be applied, is the standard strong exceptional collection $(\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n))$ on $\mathbb{P}^n$, $n \geq 2$. For $n \geq 2$ we have $\{\dim(\mathrm{Hom}(\mathcal{O}(i-1), \mathcal{O}(i))) \geq 3\}_{i=1}^{n}$, hence

$\{(\mathcal{O}(i-1), \mathcal{O}(i)[-1])\}_{i=1}^{n}$ are all l-Kronecker pairs with $l \geq 3$.

Take now $\mathbb{P}^n$, $n \geq 2$ and blow it up in finite number of points and let the obtained variety be X. By [5, Theorem 4.2] we know that $D^b(X)$ has a semiorthogonal decomposition $\langle E_1, E_2, \dots, E_l, D^b(\mathbb{P}^n)\rangle$, where $E_1, E_2, \dots, E_l$ are exceptional objects. The Kronecker pairs of $D^b(\mathbb{P}^n)$ are also l-Kronecker pairs with $l \geq 3$ in $D^b(X)$ and Corollary 3.26 can be applied. In particular these arguments hold for all del Pezzo surfaces.

After blowing up in a more general subvarieties $Y \subset \mathbb{P}^n$ we still get l-Kronecker pairs with $l \geq 3$ but in this case one must check the extendability condition in Theorem 3.24 (see remark 3.27).

3.7 Questions

We end this Chapter with a few questions related to its content.

Recall that by Corollary 3.28 $\{\hom^i(E_1, E_2) \leq 2\}_{i\in\mathbb{Z}}$ hold for any exceptional pair (E_1, E_2) in $D^b(Q)$ and for any Euclidean or a Dynkin quiver Q. For a big class of quivers (where Corollary 3.36 can be applied) it was shown that these dimensions are not bounded above. However Corollary 3.36 can not be applied to all non-Dynkin and non-Euclidean quivers, for example to the quivers listed in Remark 3.40. Nevertheless, in any of the quivers listed in Remark 3.40 we found Kronecker pairs (E_1, E_2) with $\hom^i(E_1, E_2) = 3$ for $i = 1$ and 0 for $i \neq 1$. We expect that the following question has a positive answer:

Do the inequality $\hom^i(E_1, E_2) \leq 3$ hold for any $i \in \mathbb{Z}$, any exceptional pair (E_1, E_2), in any of the quivers listed in Remark 3.40 ?

After answering this question it would be interesting to determine all the quivers Q, s. t. the dimensions $\{\hom^i(E_1, E_2)\}_{i\in\mathbb{Z}}$, where (E_1, E_2) vary through all exceptional pairs in $D^b(Q)$, are bounded above.

Finally, we come to some vague reflections related to the phases. We expect that the triangulated categories with non-dense behaviour of phases form a "thin" set of categories. Among these, the categories $D^b(Rep_k(Q))$ with Q an Euclidean quiver have somehow remarkable behaviour of P_σ (see the second row of Table 3.1). It is interesting to classify the categories with such a behavior of P_σ.

Chapter 4

Bridgeland stability conditions on the acyclic triangular quiver

4.1 Introduction

The previous chapters of the dissertation are based on results and ideas by T. Bridgeland [8], A. King [34], E. Macrí [37], J. Collins and A. Polishchuck [14].

Recently J. Woolf in [49] and N. Broomhead, D. Pauksztello, D. Ploog in [12] showed classes of categories with contractible components in the space of stability conditions. These papers generalize and unify various known results (e.g. results in [11], [48]) for stability spaces of specific categories, and settle some conjectures about the stability spaces associated to Dynkin quivers, and to their Calabi-Yau-N Ginzburg algebras (the latter are not in the scope of [12]). However the results in [49], [12] do not cover tame representation type quivers, these quivers are beyond the scope of [49], [12].

In the present chapter we give a new example of a tame representation type quiver with contractible space of stability conditions.

After a parallel between dynamical systems and categories was established ([17], [10], [25], [33]) the study of the topology of the spaces of stability conditions became a subject of significant importance. According to this analogy the stability space plays the role of the Teichmüller space. In such a way the moduli space of stability conditions provides a link between: topology, representation theory, dynamical systems, algebraic geometry, category theory.

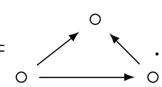
1.1. We outlined in Subsection 2.3.3 how E. Macri constructed [37] stability conditions using a a full Ext-exceptional collection $\mathcal{E} = (E_0, E_1, \dots, E_n)$ and the action of $\widetilde{GL}^+(2,\mathbb{R})$ on $\mathrm{Stab}(\mathcal{T})$. These stability conditions are $\mathbb{H}^{\mathcal{E}} \cdot \widetilde{GL}^+(2,\mathbb{R})$ (see Definition 2.30) and they will be referred to as *generated by* $\mathcal{E}$.

E. Macrì, studying $\mathrm{Stab}(D^b(K(l))$ in [37], gave an idea for producing an exceptional pair generating a given stability condition σ on $D^b(K(l))$, where $K(l)$ is the l-Kronecker quiver.

We defined in the Chapter 2 the notion of a *σ-exceptional collection*, so that the full σ-exceptional collections are exactly the exceptional collections which generate σ (Corollary 2.34), and we focused there on constructing σ-exceptional collections from a given $\sigma \in \mathrm{Stab}(D^b(\mathcal{A}))$, where $\mathcal{A}$ is a hereditary, hom-finite, abelian category. We developed tools for constructing σ-exceptional collections of length at least three in $D^b(\mathcal{A})$. These tools are based on the notion of regularity-preserving hereditary category, introduced in Chapter 2 to avoid difficulties related to the Ext-nontrivial couples (couples of exceptional objects in $\mathcal{A}$ with $\mathrm{Ext}^1(X,Y) \neq 0$ and $\mathrm{Ext}^1(Y,X) \neq 0$).

In this chapter for simplicity we will denote the triangular acyclic quiver just by Q (it was denoted in the previous chapters and in figure (2.2) by Q_1) . It was shown in Chapter 2 that $Rep_k(Q)$ is regularity preserving and the newly obtained methods for constructing σ-triples were applied to the case $\mathcal{A} = Rep_k(Q)$. As a result we obtained Theorem 2.1. In other words, all stability conditions on $D^b(Q)$ are generated by exceptional collections (in this case exceptional triples). This theorem implies that $\mathrm{Stab}(D^b(Q))$ is connected (Corollary 2.82). Using Theorem 2.1 and the data about the exceptional collections given in Section 2.2, we prove in the present chapter the following:

Theorem 4.1. *Let k be an algebraically closed field. Let Q be the following quiver:*

$Q =$  . (4.1)

The space of Bridgeland stability conditions $\mathrm{Stab}(D^b(Rep_k(Q)))$ *is a contractible (and connected) manifold, where* $D^b(Rep_k(Q))$ *is the derived category of representations of Q.*

1.2. We give now more details about the structure of $\mathrm{Stab}(D^b(Rep_k(Q)))$ and about the proof of Theorem 4.1.

Recall that we call an exceptional pair (E,F) in $D^b(Rep_k(Q))$ an l-Kronecker pair if $\hom^{\leq 0}(E,F) = 0$, and $\hom^1(E,F) = l \neq 0$ (Definition 3.20). In Corollary 3.28 was shown that for any affine acyclic quiver A (like the quiver Q) only 1- and 2-Kronecker pairs can appear in $D^b(A)$.

Recall that the Braid group on two strings $B_2 \cong \mathbb{Z}$ acts on the set of equivalence classes of exceptional pairs in $\mathcal{T}$. [1] The set of equivalence classes of 2-Kronecker pairs is invariant under the action of B_2. In Subsection 4.4.1 are described the orbits of this action on the 2-Kronecker pairs (using Corollary 2.10). There are two such orbits and in terms of our notations they are $\{(a^m, a^{m+1}[-1])\}_{m\in\mathbb{Z}}$ and $\{(b^m, b^{m+1}[-1])\}_{m\in\mathbb{Z}}$ (see Remark 4.25).

It turns out that the exceptional objects of $D^b(Rep_k(Q))$ can be grouped as follows $\{a^m\}_{m\in\mathbb{Z}} \cup \{M, M'\} \cup \{b^m\}_{m\in\mathbb{Z}}$, where $\{M, M'\} \subset Rep_k(Q)$ is the unique Ext-nontrivial couple of $D^b(Rep_k(Q))$.

Let $\mathfrak{T}_a^{st}$ and $\mathfrak{T}_b^{st}$ be the stability conditions generated by the exceptional triples containing a subsequence of the form $(a^m[p], a^{m+1}[q])$ and $(b^m[p], b^{m+1}[q])$ for some $m, p, q \in \mathbb{Z}$, respectively. Using Theorem 2.1 we show in Section 4.5 that $\mathrm{Stab}(D^b(Rep_k(Q))) = \mathfrak{T}_a^{st} \cup (\_, M, \_) \cup (\_, M', \_) \cup$

[1] Here we take the equivalence $\sim$ explained in Section 4.2 and it is clear when a given equivalence class w.r. $\sim$ will be called a 2-Kronecker pair

$\mathfrak{T}_b^{st}$, where $(\_, M, \_) \cup (\_, M', \_)$ denotes the set of stability conditions generated by triples of the form $(A, M[p], C)$ or $(A, M'[p], C)$ with $p \in \mathbb{Z}$ (these turn out to be the triples (A, B, C) for which $\dim(\mathrm{Hom}^i(A, B)) \leq 1$, $\dim(\mathrm{Hom}^i(A, C)) \leq 1$, $\dim(\mathrm{Hom}^i(B, C)) \leq 1$ for all $i \in \mathbb{Z}$).

The main steps are as follows. In Section 4.6 is shown that $\mathfrak{T}_a^{st} \cap \mathfrak{T}_b^{st} = \emptyset$. In Section 4.7 the subsets $\mathfrak{T}_a^{st}$ and $\mathfrak{T}_b^{st}$ are shown to be contractible. In Section 4.8 we connect $\mathfrak{T}_a^{st}$ and $\mathfrak{T}_b^{st}$ by $(\_, M, \_) \cup (\_, M', \_)$ and show that in this procedure the contractibility is preserved.

The theorem from topology which we use to glue stability conditions generated by different exceptional triples is the Seifert-van Kampen theorem, modified about contractile subsets in manifolds (see Remark 4.67). In Section 4.3 are given several important tools used throughout to analyze the intersection of the sets of stability conditions generated by different exceptional collections. These tools are extensions of results and ideas in [34], [37], Chapters 3 and 2. In the final step (Section 4.8) we utilize as such a tool also the relation $R \dashrightarrow (S, E)$ between a σ-regular object R and an exceptional pair generated by it (introduced in Chapter 2).

The obtained in Section 2.2 data about $\mathrm{Hom}(X, Y)$ and $\mathrm{Ext}^1(X, Y)$, where X, Y vary throughout the exceptional objects of $Rep_k(Q)$, is organized in a better way in Section 4.4. We add there also some observations about the behavior of the central charges (see the beginning of subsection 2.3.2 for the notion central charge) of the exceptional objects, which are very essential for the proof of Theorem 4.1 as well.

1.4. We expect that with the picture about $\mathrm{Stab}(D^b(Q_1))$ obtained here one can explicitly show what is $\mathrm{Stab}(D^b(Q_1))$. We hope that a proper interpretation of the relation between the Kronecker pairs and the structure of $\mathrm{Stab}(D^b(Rep_k(Q)))$, observed in this chapter, will open a way to better conceptual understanding and a successful analysis of other cases will follow.

4.2 Some more notations.

In addition to the notations fixed in Section 0.1 the following conventions will hold throughout this last chapter of the dissertation.

First, we recall that *in this chapter for simplicity we will denote the triangular acyclic quiver just by Q (it was denoted in the previous chapters and in figure* (2.2) *by Q_1)* .

For a vector $\mathbf{p} = (p_0, p_1, \ldots, p_n) \in \mathbb{Z}^{n+1}$ and an exceptional collection $\mathcal{E} = (E_0, E_1, \ldots, E_n) \subset \mathcal{T}_{exc}$ we denote $\mathcal{E}[\mathbf{p}] = (E_0[p_0], E_1[p_1], \ldots, E_n[p_n])$. Obviously $\mathcal{E}[\mathbf{p}]$ is also an exceptional collection. The exceptional collections of the form $\{\mathcal{E}[\mathbf{p}] : \mathbf{p} \in \mathbb{Z}^{n+1}\}$ will be said to be shifts of $\mathcal{E}$.

For two exceptional collections $\mathcal{E}_1$, $\mathcal{E}_2$ of equal length we write $\mathcal{E}_1 \sim \mathcal{E}_2$ if $\mathcal{E}_2 \cong \mathcal{E}_1[\mathbf{p}]$ for some $\mathbf{p} \in \mathbb{Z}^{n+1}$.

For any $a \in \mathbb{R}$ and any complex number $z \in \mathrm{e}^{\mathrm{i}\pi a} \cdot (\mathbb{R} + \mathrm{i}\mathbb{R}_{>0})$, respectively $z \in \mathrm{e}^{\mathrm{i}\pi a} \cdot (\mathbb{R}_{<0} \cup (\mathbb{R} + \mathrm{i}\mathbb{R}_{>0}))$, we denote by $\arg_{(a,a+1)}(z)$, resp. $\arg_{(a,a+1]}(z)$, the unique $\phi \in (a, a+1)$, resp. $\phi \in (a, a+1]$, satisfying $z = |z| \exp(\mathrm{i}\pi\phi)$.

For a non-zero complex number $v \in \mathbb{C}$ we denote the two connected components of $\mathbb{C} \setminus \mathbb{R}v$ by:

$$v^c_+ = v \cdot (\mathbb{R} + \mathrm{i}\mathbb{R}_{>0}) \qquad v^c_- = v \cdot (\mathbb{R} - \mathrm{i}\mathbb{R}_{>0}) \qquad\qquad v \in \mathbb{C} \setminus \{0\}. \tag{4.2}$$

For $b \in (a, a+1)$, $c \in (a-1, a)$ $r_1 > 0$, $r_2 > 0$ we have

$$\begin{aligned} \arg_{(a,a+1)}(r_1 \exp(\mathrm{i}\pi a) + r_2 \exp(\mathrm{i}\pi b)) = a + \arg_{(0,1)}(r_1 + r_2 \exp(\mathrm{i}\pi(b-a))) \\ \arg_{(a-1,a)}(r_1 \exp(\mathrm{i}\pi a) + r_2 \exp(\mathrm{i}\pi c)) = a + \arg_{(-1,0)}(r_1 + r_2 \exp(\mathrm{i}\pi(c-a))). \end{aligned} \tag{4.3}$$

These formulas imply that for $c \in (a-1, a)$, $r_1 > 0$, $r_2 > 0$ we have

$$\arg_{(a-1,a)}\left(r_1 \exp(\mathrm{i}\pi a) + r_2 \exp(\mathrm{i}\pi c)\right) = -\arg_{(-a,-a+1)}\left(r_1 \exp(-\mathrm{i}\pi a) + r_2 \exp(-\mathrm{i}\pi c)\right). \tag{4.4}$$

4.3 General remarks

This section provides several tools which will be used throughout to analyze the intersection of the sets of stability conditions generated by different exceptional collections (Propositions 4.3, 4.10, 4.11 and Lemmas 4.12, 4.13). A description of the set of stability conditions generated by all shifts of a fixed exceptional triple is given in Proposition 4.8.

4.3.1 One remark related to Bridgeland stability conditions

We use freely the axioms and notations on stability conditions introduced by Bridgeland in [8] and some additional notations used in Subsection 2.3.2. In particular, for $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}(\mathcal{T})$ we denote by σ^{ss} the set of σ-semistable objects, see (2.9).

In particular, for any interval $I \subset \mathbb{R}$ the extension closure of the slices $\{\mathcal{P}(x)\}_{x \in I}$ is denoted by $\mathcal{P}(I)$. The nonzero objects in the subcategory $\mathcal{P}(I)$ are exactly those $X \in \mathcal{T} \setminus \{0\}$, which satisfy $\phi_\pm(X) \in I$, i. e. whose HN factors have phases in I. In particular, if $X \in \mathcal{P}(a-1, a] \setminus \{0\}$ then $Z(X) \in \exp(\mathrm{i}\pi a)^c_- \cup \mathbb{R}_{>0} \exp(\mathrm{i}\pi a)$.

From [8] we know that for any $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}(\mathcal{T})$ and any $t \in \mathbb{R}$ the subcategory $\mathcal{P}(t, t+1]$ is a heart of a bounded t-structure. In particular $\mathcal{P}(t, t+1]$ is an abelian category, whose short exact sequences are exactly these sequences $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ with $A, B, C \in \mathcal{P}(t, t+1]$, s. t. for some $\gamma : C \to A[1]$ the sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]$ is a triangle in $\mathcal{T}$. Using these remarks, the HN filtration and by drawing pictures one easily shows the following properties:

Remark 4.2. *Let $t \in \mathbb{R}$ and $X \in \mathcal{P}(a-1, a]$. Then:*

(a) If $X \notin \sigma^{ss}$ then $\phi_-(X) < \arg_{(a-1,a]}(Z(X)) < \phi_+(X)$.

(b) $X \notin \sigma^{ss}$ iff there exists a monic arrow $X' \to X$ in the abelian category $P(a-1, a]$ satisfying $\arg_{(a-1,a]}(Z(X')) > \arg_{(a-1,a]}(Z(X))$.

(c) *If* $Z(X) \in v_+^c$ *for some* $v \in \mathbb{C}^*$ *with* $v = |v| \exp(\mathrm{i}\pi t)$ *and* $a - 1 \in (t, t+1)$ *or* $a \in (t, t+1)$, *then* $\arg_{(a-1,a]}(Z(X)) = \arg_{(t,t+1)}(Z(X))$. *In particular, when* $X \in \sigma^{ss}$, *we have:* $\phi(X) = \arg_{(t,t+1)}(Z(X))$.

4.3.2 Remarks on σ-exceptional collections

E. Macrì proved in [37, Lemma 3.14] that the extension closure $\mathcal{A}_{\mathcal{E}}$ of a full Ext-exceptional collection $\mathcal{E} = (E_0, E_1, \dots, E_n)$ in $\mathcal{T}$ is a heart of a bounded t-structure. Furthermore, $\mathcal{A}_{\mathcal{E}}$ is of finite length and $E_0, E_1, \dots, E_n$ are the simple objects in it. By Bridgeland's Proposition 2.27 from the bounded t-structure $\mathcal{A}_{\mathcal{E}}$ is produced a family of stability conditions, which we denote by $\mathbb{H}^{\mathcal{A}_{\mathcal{E}}} \subset \mathrm{Stab}(\mathcal{T})$ (see Definition 2.28) or sometimes just $\mathbb{H}^{\mathcal{E}} \subset \mathrm{Stab}(\mathcal{T})$. For a given $\sigma \in \mathrm{Stab}(\mathcal{T})$ we defined a *σ-exceptional collection* in Definition 2.33.

The following Proposition is basic for this chapter:

Proposition 4.3. *Let* $\mathcal{T}$ *be a* k*-linear triangulated category and* $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}(\mathcal{T})$. *Let* $\mathcal{E} = (E_0, E_1, \dots, E_n)$ *be a full* σ*-exceptional collection such that* $\phi(E_i) \geq \phi(E_{i+1})$ *and* $\hom^1(E_i, E_{i+1}) \neq 0$ *for some* $i \in \{0, 1, \dots, n-1\}$. *Let* $\mathcal{A}_{i,i+1}$ *be the extension closure of* E_i, E_{i+1} *in* $\mathcal{T}$. *Then each element in* $\mathcal{T}_{exc} \cap \mathcal{A}_{i,i+1}$ *is semistable.*[2]

Proof. If $\phi(E_i) = \phi(E_{i+1}) = t$, then $\mathcal{A}_{i,i+1} \subset \mathcal{P}(t)$ and hence all non-zero objects in $\mathcal{A}_{i,i+1}$ are semistable, therefore we can assume that $\phi(E_i) > \phi(E_{i+1})$.

By Corollary 2.34 we have $\sigma \in \Theta'_{\mathcal{E}} = \mathbb{H}^{\mathcal{E}} \cdot \widetilde{GL}^+(2, \mathbb{R})$. Since the action of $\widetilde{GL}^+(2, \mathbb{R})$ does not change the order of the phases, we can assume that $\sigma = (\mathcal{P}, Z) \in \mathbb{H}^{\mathcal{E}}$, which means that the extension closure of $\mathcal{E}$ is the t-structure $\mathcal{P}(0, 1]$ and

$$\phi(E_j) = \arg_{(0,1]}(Z(E_j)) \qquad j = 1, \dots, n. \tag{4.5}$$

Let us denote $\mathcal{T}_{i,i+1} = \langle E_i, E_{i+1} \rangle$. From Proposition 2.31 we have a projection map $\mathbb{H}^{\mathcal{E}} \to \mathbb{H}^{\mathcal{A}_{i,i+1}} \subset \mathrm{Stab}(\mathcal{T}_{i,i+1})$ and it maps $\sigma = (\mathcal{P}, Z)$ to a stability condition $\sigma' = (\mathcal{P}', Z') \in \mathbb{H}^{\mathcal{A}_{i,i+1}}$ with $Z'(E_i) = Z(E_i)$, $Z'(E_{i+1}) = Z(E_{i+1})$ and $\{\mathcal{P}'(t) = \mathcal{P}(t) \cap \mathcal{T}_{i,i+1}\}_{t \in \mathbb{R}}$. Therefore it remains to show that the objects in $\mathcal{T}_{exc} \cap \mathcal{A}_{i,i+1}$ are σ'-semistable.

From Lemma 3.19 we have that $\mathcal{A}_{i,i+1}$ is a bounded t-structure in $\mathcal{T}_{i,i+1}$ and an equivalence of abelian categories $F : \mathcal{A}_{i,i+1} \to Rep_k(K(l))$ with $F(E_i) = s_1$, $F(E_{i+1}) = s_2$, where $l = \hom^1(E_i, E_{i+1})$ and s_1, s_2 are the simple representations of $K(l)$ with k at the source, sink, respectively. This equivalence maps $\sigma' \in \mathbb{H}^{\mathcal{A}_{i,i+1}}$ to a stability condition

$$\sigma'' = (\mathcal{P}'', Z'') \in \mathbb{H}^{Rep_k(K(l))} \subset \mathrm{Stab}(D^b(K(l))) \quad Z(E_i) = Z''(s_1), Z(E_{i+1}) = Z''(s_2).$$

If $E \in \mathcal{T}_{exc} \cap \mathcal{A}_{i,i+1}$, then by the fact that F is an equivalence of abelian categories it follows that $F(E) \in Rep_k(K(l))$ is an exceptional representation.[3] Since $\{F(\mathcal{P}'(t)) = \mathcal{P}''(t)\}_{t \in (0,1]}$, it remains to prove that each exceptional representation of $Rep_k(K(l))$ is σ''-semistable.

[2] Recall that by $\mathcal{T}_{exc}$ we denote the set of exceptional objects in $\mathcal{T}$

[3] To show this one uses also the fact that $\mathcal{A}_{i,i+1}$ is a bounded t-structure in $\mathcal{T}_{i,i+1}$.

Let $\rho \in Rep_k(K(l))_{exc}$. Then the dimension vector $\underline{\dim}(\rho) = (n,m) \in (n,m)$ is a real root of $K(l)$, furthermore it is a Schur root. From (4.5) we have $\arg(Z''(s_1)) > \arg(Z''(s_2))$. By the arguments in the proof of Lemma 3.16 using a theorem by King ([34, Proposition 4.4]) and $\arg(Z''(s_1)) > \arg(Z''(s_2))$ we obtain a σ''-stable representation $X \in Rep_k(K(l))$ with $\underline{\dim}(X) = (n,m)$. Since X is stable, it is simple in $\mathcal{P}''(t)$, where $t = \phi''(X)$, in particular it is indecomposable in $\mathcal{P}''(t)$. Since $\mathcal{P}''(t)$ is a thick subcategory (see Lemma 2.20), it follows that X is indecomposable in $Rep_k(K(l))$. Since $\underline{\dim}(\rho)$ is a real root and both X, ρ are indecompsable representations, the equality $\underline{\dim}(\rho) = \underline{\dim}(X)$ implies $\rho \cong X$(see [31, Theorem 2, c)]). The proposition follows. □

Other statements, which will be widely used in the next sections are Propositions 4.8, 4.10 and 4.11. For the proof of Proposition 4.8 it is useful to define:

Definition 4.4. *Let $n \geq 1$ be an integer. Let $\mathcal{I} = \{I_{ij} = (l_{ij}, r_{ij}) \subset \mathbb{R}\}_{0\leq i<j\leq n}$ be a family of non-empty open intervals, and let $\mathfrak{l} = \{l_{ij} \in \{-\infty\} \cup \mathbb{R}\}_{0\leq i<j\leq n}$, $\mathfrak{r} = \{r_{ij} \in \mathbb{R} \cup \{+\infty\}\}_{0\leq i<j\leq n}$ be the corresponding families of left and right endpoints.*

We will denote the following open convex set $\{(y_0, y_1, \dots, y_n) \in \mathbb{R}^{n+1} : y_i - y_j \in I_{ij}\ i<j\} \subset \mathbb{R}^{n+1}$ by $S^n(\mathcal{I})$ or $S^n(\mathfrak{l}, \mathfrak{r})$.

For a full Ext-exceptional collection $\mathcal{E} = (E_0, E_1, \dots, E_n)$ in $\mathcal{T}$ we denote $\Theta'_{\mathcal{E}} = \mathbb{H}^{\mathcal{E}} \cdot \widetilde{GL}^+(2,\mathbb{R})$. If $\mathcal{E}$ is a full Ext-exceptional collection, then we have (see Remark 2.35):

$$\Theta'_{\mathcal{E}} = \mathbb{H}^{\mathcal{E}} \cdot \widetilde{GL}^+(2,\mathbb{R}) = \{\sigma : \mathcal{E} \subset \sigma^{ss} \text{ and } |\phi^\sigma(E_i) - \phi^\sigma(E_j)| < 1 \text{ for } i<j\} \tag{4.6}$$

and the assignment:

$$\{\sigma \in \mathrm{Stab}(\mathcal{T}) : \mathcal{E} \subset \sigma^{ss}\} \ni (\mathcal{P}, Z) \xmapsto{f_{\mathcal{E}}} (\{|Z(E_i)|\}_{i=0}^n, \{\phi^\sigma(E_i)\}_{i=0}^n) \in \mathbb{R}^{2(n+1)} \tag{4.7}$$

restricted to $\Theta'_{\mathcal{E}}$ defines a homeomorphism between $\Theta'_{\mathcal{E}}$ and $\mathbb{R}^{n+1}_{>0} \times S^n(-\mathbf{1}, +\mathbf{1})$ (as defined in Definition 4.4).

Assume now that $\mathcal{E} = (E_0, E_1, \dots, E_n)$ is any full exceptional collection in $\mathcal{T}$ (not restricted to be Ext). If $\mathcal{T}$ is a triangulated category of finite type, then there are infinitely many choices of $\mathbf{p} \in \mathbb{Z}^{n+1}$ such that $\mathcal{E}[\mathbf{p}] = (E_0[p_0], E_1[p_1], \dots, E_n[p_n])$ is an Ext-exceptional collection. [37, Lemma 3.19] says that the following open subset of stability conditions is connected and simply conected:

$$\Theta_{\mathcal{E}} = \bigcup_{\{\mathbf{p}\in\mathbb{Z}^{n+1}:\mathcal{E}[\mathbf{p}] \text{ is Ext}\}} \Theta'_{\mathcal{E}[\mathbf{p}]} \subset \mathrm{Stab}(\mathcal{T}). \tag{4.8}$$

For the sake of completeness we will comment on this set as well (compare with [37, proof of Lemma 3.19]).

By Corollary 2.34 $\Theta_{\mathcal{E}}$ is the set of stability conditions $\sigma \in \mathrm{Stab}(\mathcal{T})$ for which a shift of $\mathcal{E}$ is a σ-exceptional collection, in particular for each $\sigma \in \Theta_{\mathcal{E}}$ we have $\mathcal{E} \subset \sigma^{ss}$. Hence the assignment (4.7) is well defined on $\Theta_{\mathcal{E}}$. Furthermore, this defines a homeomorphism between $\Theta_{\mathcal{E}}$ and $f_{\mathcal{E}}(\Theta_{\mathcal{E}})$. Indeed, if $\mathcal{E}[\mathbf{p}]$ is an Ext-collection for some $\mathbf{p} \in \mathbb{Z}^{n+1}$, then $f_{\mathcal{E}[\mathbf{p}]}$ maps $\Theta'_{\mathcal{E}[\mathbf{p}]}$ homeomorphically to $\mathbb{R}^{n+1}_{>0} \times S^n(-\mathbf{1},+\mathbf{1})$ (see after (4.7)) and due to $f_{\mathcal{E}[\mathbf{p}]} - (0,\mathbf{p}) = f_{\mathcal{E}}$ we see that $f_{\mathcal{E}|\Theta'_{\mathcal{E}[\mathbf{p}]}}$ is homeomorphism onto its image $\mathbb{R}^{n+1}_{>0} \times (S^n(-\mathbf{1},+\mathbf{1}) - \mathbf{p})$. Therefore, provided that $f_{\mathcal{E}}$ is injective on $\Theta_{\mathcal{E}}$, **the following restriction is a homeomorphism:**

$$f_{\mathcal{E}|\Theta_{\mathcal{E}}} : \Theta_{\mathcal{E}} \to \mathbb{R}^{n+1}_{>0} \times \left(\bigcup_{\mathbf{p}\in A} S^n(-\mathbf{1},+\mathbf{1}) - \mathbf{p} \right), \quad \text{where} \quad A = \left\{ \mathbf{p} \in \mathbb{Z}^{n+1} : \mathcal{E}[\mathbf{p}] \text{ is Ext} \right\}. \quad (4.9)$$

To prove that the obtained function is injective, assume that $\sigma_i = (\mathcal{P}_i, Z_i) \in \Theta_{\mathcal{E}}$, $i = 1,2$ and $f_{\mathcal{E}}(\sigma_1) = f_{\mathcal{E}}(\sigma_2)$, i. e. $|Z_1(E_j)| = |Z_2(E_j)|, \phi^{\sigma_1}(E_j) = \phi^{\sigma_2}(E_j)$ for all j, then by (4.6) and the axiom $\phi^{\sigma}(E_j[p_j]) = \phi^{\sigma}(E_j) + p_j$ we see that for any $\mathbf{p}$ the incidence $\sigma_1 \in \Theta'_{\mathcal{E}[\mathbf{p}]}$ is equivaaent to $\sigma_2 \in \Theta'_{\mathcal{E}[\mathbf{p}]}$, hence by the injectivity of $f_{\mathcal{E}[\mathbf{p}]}$ and $f_{\mathcal{E}[\mathbf{p}]} - (0,\mathbf{p}) = f_{\mathcal{E}}$ we obtain $\sigma_1 = \sigma_2$. Thus, we see that (4.9) is a homeomorphism.

Finally, note that by (4.6), (4.7), (4.8), and $f_{\mathcal{E}[\mathbf{p}]} = f_{\mathcal{E}} + (0,\mathbf{p})$ it follows that

$$\Theta_{\mathcal{E}} = \left\{ \sigma \in \mathrm{Stab}(\mathcal{T}) : \mathcal{E} \subset \sigma^{ss} \text{ and } \phi^{\sigma}(\mathcal{E}) \in \bigcup_{\mathbf{p}\in A} S^n(-\mathbf{1},+\mathbf{1}) - \mathbf{p} \right\}. \quad (4.10)$$

4.3.3 The set $f_{\mathcal{E}}(\Theta_{\mathcal{E}})$ when $n = 2$

We give an explicit representation of $f_{\mathcal{E}}(\Theta_{\mathcal{E}})$, when $n = 2$. Remark 4.6 shows that the case $n \geq 3$ is not completely analogous. The only statement, which will be used later, is Proposition 4.8. This subsection is dedicated to the proof of Proposition 4.8.

Let us denote:

$$B^n = \{(0, q_1, q_2, \dots, q_n) \subset \mathbb{N}^{n+1} : 0 \leq q_1 \leq q_2 \leq \dots \leq q_n\}. \quad (4.11)$$

The following properties are clear from the definitions of $S^n(\mathcal{J})$ (Definition 4.4) and of $A \subset \mathbb{Z}^{n+1}$ (formula (4.9))

$$\forall \mathbf{v} \in diag(\mathbb{R}^{n+1}) \quad S^n(\mathcal{J}) - \mathbf{v} = S^n(\mathcal{J}) \quad (4.12)$$

$$\forall \mathbf{v} \in diag(\mathbb{Z}^{n+1}) \quad A - \mathbf{v} = A \quad (4.13)$$

$$\forall \mathbf{v} \in B^n \quad A - \mathbf{v} \subset A. \quad (4.14)$$

Any $\mathbf{p} = (p_0, p_1, \dots, p_n) \in A$ can be represented as $\mathbf{p} - (p_0, p_0, \dots, p_0) + (p_0, p_0, \dots, p_0)$, hence if we denote

$$A_0 = \left\{ \mathbf{p} \in \mathbb{Z}^{n+1} : p_0 = 0, \mathcal{E}[\mathbf{p}] \text{ is Ext} \right\} \quad (4.15)$$

by the properties above we can write

$$\bigcup_{\mathbf{p}\in A} S^n(-\mathbf{1},+\mathbf{1})-\mathbf{p}=\bigcup_{\mathbf{p}\in A_0} S^n(-\mathbf{1},+\mathbf{1})-\mathbf{p}=\bigcup_{\mathbf{p}\in A_0}\left(\bigcup_{\mathbf{v}\in B^n} S^n(-\mathbf{1},+\mathbf{1})+\mathbf{v}\right)-\mathbf{p}. \tag{4.16}$$

For the cases $n=1,2$ we have the following simple form of the expression in the brackets:

Lemma 4.5. *The following equalities hold:*

$$\bigcup_{\boldsymbol{v}\in B^1} S^1(-1,+1)+\boldsymbol{v}=S^1(-\infty,1) \qquad \bigcup_{\boldsymbol{v}\in B^2} S^2(-\boldsymbol{1},+\boldsymbol{1})+\boldsymbol{v}=S^2(-\infty,\boldsymbol{1}). \tag{4.17}$$

Recall that $S^n(-\infty,\boldsymbol{1})=\{(y_0,y_1,\dots,y_n)\in\mathbb{R}^{n+1}: y_i-y_j<1, i<j\}$ (see Definition 4.4).

Remark 4.6. *For $n\geq 3$ we have not such an equality. For example, we have $(0,-\frac{1}{2},\frac{1}{2},0,\dots,0)\in S^n(-\infty,\boldsymbol{1})$ but $(0,-\frac{1}{2},\frac{1}{2},0,\dots,0)\notin\bigcup_{\boldsymbol{v}\in B^n} S^n(-\boldsymbol{1},+\boldsymbol{1})+\boldsymbol{v}$ for $n\geq 3$.*

More precisely, it holds $\bigcup_{\boldsymbol{v}\in B^n} S^n(-\boldsymbol{1},+\boldsymbol{1})+\boldsymbol{v}\subset\neq S^n(-\infty,\boldsymbol{1})$ for $n\geq 3$.

Proof. (of Lemma 4.5) Note first that for any $\mathcal{I}=\{I_{ij}: i<j\}$ as in Definition 4.4 and any $\mathbf{p}\in\mathbb{Z}^{n+1}$ we have

$$S^n(\{I_{ij}: i<j\})-\mathbf{p}=S^n(\{I_{ij}-(p_i-p_j): i<j\}). \tag{4.18}$$

In particular for $n=1$ we have (now the index set of $\mathcal{I}$ has only one element: $(0,1)$):

$$\bigcup_{\mathbf{v}\in B^1} S^1(-1,+1)+\mathbf{v}=\bigcup_{(0,k)\in\mathbb{N}^2} S^1(-1,+1)+(0,k)=\bigcup_{k\in\mathbb{N}} S^1(-1-k,1-k)$$
$$=\bigcup_{k\in\mathbb{N}}\{-1-k<y_0-y_1<1-k\}=\{y_0-y_1<1\}=S^1(-\infty,+1)$$

Using (4.12) and (4.18) one easily shows that:

$$\bigcup_{\mathbf{v}\in B^2} S^2(-\mathbf{1},+\mathbf{1})+\mathbf{v}=diag(\mathbb{R}^{n+1})\oplus\{y_2=0\}\cap\left(\bigcup_{\mathbf{v}\in B^2} S^2(-\mathbf{1},+\mathbf{1})+\mathbf{v}\right)$$
$$S^2(-\infty,\mathbf{1})=diag(\mathbb{R}^{n+1})\oplus\{y_2=0\}\cap S^2(-\infty,\mathbf{1}).$$

Obviously we have

$$\{y_2=0\}\cap S^2(-\infty,\mathbf{1})=\{y_2=0\}\cap\left\{\begin{array}{c} y_0-y_1<1\\ y_0-y_2<1\\ y_1-y_2<1\end{array}\right\}=\left\{\begin{array}{c} y_0-y_1<1\\ y_0<1\\ y_1<1\end{array}\right\}.$$

We will prove the second equality in (4.17) by showing that:

$$\{y_2 = 0\} \cap \left(\bigcup_{\mathbf{v} \in B^2} S^2(-\mathbf{1}, +\mathbf{1}) + \mathbf{v} \right) = \left\{ \begin{array}{c} y_0 - y_1 < 1 \\ y_0 < 1 \\ y_1 < 1 \end{array} \right\}. \tag{4.19}$$

Let $(0, k, k+l)$ be any vector in B^2, where $k,\, l \in \mathbb{N}$. By (4.18) we have:

$$S^2(-\mathbf{1}, \mathbf{1}) + (0, k, k+l) = \left\{ \begin{array}{c} -1 - k < y_0 - y_1 < 1 - k \\ -1 - k - l < y_0 - y_2 < 1 - k - l \\ -1 - l < y_1 - y_2 < 1 - l \end{array} \right\} \subset \left\{ \begin{array}{c} y_0 - y_1 < 1 \\ y_0 - y_2 < 1 \\ y_1 - y_2 < 1 \end{array} \right\}. \tag{4.20}$$

Denoting the unit open square by $C(-1, +1) = \{|y_i| < 1; i = 0, 1\} \subset \mathbb{R}^2$, we can write:

$$\begin{aligned} \{y_2 = 0\} \cap \big(S^2(-\mathbf{1}, \mathbf{1}) + (0, k, k+l)\big) &= \left\{ \begin{array}{c} -1 - k < y_0 - y_1 < 1 - k \\ -1 - k - l < y_0 < 1 - k - l \\ -1 - l < y_1 < 1 - l \end{array} \right\} \\ &= S^1(-1 - k, +1 - k) \cap (C(-1, +1) - (k + l, l)) \\ &= \big(S^1(-1, +1) + (0, k)\big) \cap (C(-1, +1) - (k + l, l)) \\ &= \big(S^1(-1, +1) - (k + l, k + l) + (0, k)\big) \cap (C(-1, +1) - (k + l, l)) \\ &= \big(S^1(-1, +1) \cap C(-1, +1)\big) - (k + l, l). \end{aligned}$$

Therefore:

$$\{y_2 = 0\} \cap \left(\bigcup_{\mathbf{v} \in B^2} S^2(-\mathbf{1}, \mathbf{1}) + \mathbf{v} \right) = \bigcup_{k \in \mathbb{N}} \left(\bigcup_{l \in \mathbb{N}} \big(S^1(-1, 1) \cap C(-1, 1)\big) - (l, l) \right) - (k, 0). \tag{4.21}$$

Before we continue with the proof of Lemma 4.5, we prove:

Lemma 4.7. *For any $k \in \mathbb{Z} \cup \{+\infty\}$ we have the following equality:*

$$\bigcup_{l \leq k} S^1(-1, +1) \cap C(-1, +1) + (l, l) = S^1(-1, +1) \cap \left\{ \begin{array}{c} y_0 < 1 + k \\ y_1 < 1 + k \end{array} \right\}. \tag{4.22}$$

Proof. We prove first the equality for $k = +\infty$. Let $(a_0, a_1) \in S^1(-1, +1)$, i. e. $|a_0 - a_1| < 1$. Since $\mathbb{R} = \bigcup_{l \in \mathbb{Z}} [2l - 1, 2l + 1)$, there exists $l \in \mathbb{Z}$ such that $a_0 + a_1 \in [2l - 1, 2l + 1)$, i. e. $-1 \leq a_0 + a_1 - 2l \leq 1$. We have also $-1 < a_0 - a_1 < +1$ and due to the equalities:

$$a_0 - l = \frac{a_0 + a_1 - 2l}{2} + \frac{a_0 - a_1}{2}; \quad a_1 - l = \frac{a_0 + a_1 - 2l}{2} + \frac{a_1 - a_0}{2}$$

we obtain $-1 = -\frac{1}{2} - \frac{1}{2} < a_i - l < \frac{1}{2} + \frac{1}{2} = 1$ for $i = 0,1$. Hence $(a_0, a_1) - (l,l) \in C(-1,+1) \cap S(-1,+1)$, and we proved the equality (4.22) with $k = +\infty$. By (4.12) and since the translation in $\mathbb{R}^2$ is bijective we rewrite this equality as follows $S^1(-1,+1) = \bigcup_{l\in\mathbb{Z}} S^1(-1,+1) \cap C(-1,+1) + (l,l) = \bigcup \left(S^1(-1,+1) + (l,l)\right) \cap (C(-1,+1) + (l,l)) = S^1(-1,+1) \cap \left(\bigcup_{l\in\mathbb{Z}} C(-1,+1) + (l,l)\right)$. Hence

$$S^1(-1,1) \cap \left\{ \begin{array}{c} y_0 < 1+k \\ y_1 < 1+k \end{array} \right\} = S^1(-1,1) \cap \left(\bigcup_{l\in\mathbb{Z}} C(-1,1) + (l,l) \right) \cap \left\{ \begin{array}{c} y_0 < 1+k \\ y_1 < 1+k \end{array} \right\}. \tag{4.23}$$

Due to the equalities

$$\left\{ \begin{array}{c} y_0 < 1+k \\ y_1 < 1+k \end{array} \right\} \cap (C(-1,1) + (l,l)) = \left\{ \begin{array}{ll} \emptyset & \text{if } \ l \geq k+2 \\ \left\{ \begin{array}{c} k < y_0 < 1+k \\ k < y_1 < 1+k \end{array} \right\} \subset C(-1,1) + (k,k) & \text{if } \ l = k+1 \\ C(-1,+1) + (l,l) & \text{if } \ l \leq k \end{array} \right.$$

we obtain $\left(\bigcup_{l\in\mathbb{Z}} C(-1,1) + (l,l)\right) \cap \left\{ \begin{array}{c} y_0 < 1+k \\ y_1 < 1+k \end{array} \right\} = \bigcup_{l\leq k} C(-1,1) + (l,l)$. By (4.23) and applying again (4.12) we obtain the equality (4.22) for $k \in \mathbb{Z}$. □

Now we put (4.22) with $k = 0$ in (4.21) and obtain

$$\{y_2 = 0\} \cap \left(\bigcup_{\mathbf{v}\in B^2} S^2(-\mathbf{1}, \mathbf{1}) + \mathbf{v} \right) = \bigcup_{k\in\mathbb{N}} \left(S^1(-1,+1) \cap \left\{ \begin{array}{c} y_0 < 1 \\ y_1 < 1 \end{array} \right\} \right) - (k,0). \tag{4.24}$$

The next step is to show that

$$\bigcup_{k\in\mathbb{N}} \left(S^1(-1,+1) \cap \left\{ \begin{array}{c} y_0 < 1 \\ y_1 < 1 \end{array} \right\} \right) - (k,0) = \bigcup_{k\in\mathbb{N}} \left(S^1(-1,+1) - (k,0)\right) \cap \left\{ \begin{array}{c} y_0 < 1 \\ y_1 < 1 \end{array} \right\}. \tag{4.25}$$

The inclusion $\subset$ is clear. Assume now that $a_0, a_1 \in \mathbb{R}$, $k \in \mathbb{N}$ and $|a_0 - a_1| < 1$ and $a_0 - k < 1$, $a_1 < 1$. We have to find $a_0' \in \mathbb{R}$, and $k' \in \mathbb{N}$ such that

$$\left|a_0' - a_1\right| < 1 \qquad a_0' < 1 \qquad a_0' - k' = a_0 - k. \tag{4.26}$$

First note that $a_0 = a_0 - a_1 + a_1 < |a_0 - a_1| + a_1 < 2$. If $k = 0$ or $a_0 < 1$, then we put $a_0' = a_0$, $k' = k$. Thus, we can assume that $k \geq 1$ and $1 \leq a_0 < 2$. Now $a_1 < 1$ and $|a_0 - a_1| < 1$ imply $0 \leq a_1 < 1$. It follows that $-1 < -a_1 \leq a_0 - 1 - a_1 < 1$, therefore we can put $a_0' = a_0 - 1$, $k' = k - 1$. Hence we obtain (4.25).

On the other hand by (4.12) and the already proven first equality in (4.17) we have

$$\bigcup_{k\in\mathbb{N}} S^1(-1,1) - (k,0) = \bigcup_{k\in\mathbb{N}} S^1(-1,1) + (0,k) = S^1(-\infty,1).$$

The latter equality and equalities (4.24), (4.25) imply (4.19) and the lemma follows. □

Putting (4.17) in (4.16) and then using (4.18) we obtain for the case $n = 2$:

$$\bigcup_{\mathbf{p}\in A} S^2(-\mathbf{1},+\mathbf{1}) - \mathbf{p} = \bigcup_{\mathbf{p}\in A_0} S^2(-\infty,\mathbf{1}) - \mathbf{p} = \bigcup_{(0,p_1,p_2)\in A_0} \left\{ \begin{array}{c} y_0 - y_1 < 1 + p_1 \\ y_0 - y_2 < 1 + p_2 \\ y_1 - y_2 < 1 + p_2 - p_1 \end{array} \right\}. \tag{4.27}$$

Using the equality (4.27), the homeomorphism (4.9), and (4.10) we will prove the main result of this subsection:

Proposition 4.8. *Let $\mathcal{T}$ be a k-linear triangulated category. Let $\mathcal{E} = (A_0, A_1, A_2)$ be a full exceptional collection, such that:*

$$\begin{aligned} 1+\alpha &= \min\{i : \hom^i(A_0, A_1) \neq 0\} \in \mathbb{Z} \\ 1+\beta &= \min\{i : \hom^i(A_0, A_2) \neq 0\} \in \mathbb{Z} \\ 1+\gamma &= \min\{i : \hom^i(A_1, A_2) \neq 0\} \in \mathbb{Z}. \end{aligned} \tag{4.28}$$

Then the subset $\Theta_{\mathcal{E}} \subset \mathrm{Stab}(\mathcal{T})$ defined in (4.8) has the following description:

$$\Theta_{\mathcal{E}} = \left\{ \sigma \in \mathrm{Stab}(\mathcal{T}) : \mathcal{E} \subset \sigma^{ss} \ \mathit{and} \ \begin{array}{c} \phi^\sigma(A_0) - \phi^\sigma(A_1) < 1 + \alpha \\ \phi^\sigma(A_0) - \phi^\sigma(A_2) < 1 + \min\{\beta, \alpha+\gamma\} \\ \phi^\sigma(A_1) - \phi^\sigma(A_2) < 1 + \gamma \end{array} \right\} \tag{4.29}$$

and $\Theta_{\mathcal{E}}$ is homeomorphic with the set $\mathbb{R}^3_{>0} \times \left\{ \begin{array}{c} y_0 - y_1 < 1 + \alpha \\ y_0 - y_2 < 1 + \min\{\beta, \alpha+\gamma\} \\ y_1 - y_2 < 1 + \gamma \end{array} \right\}$ *by the map $f_{\mathcal{E}}$ in (4.7) restricted to $\Theta_{\mathcal{E}}$. In particular $\Theta_{\mathcal{E}}$ is contractible.*

Proof. Given a family $\mathcal{I}$ of the form: $\mathcal{I} = \{I_{01} = (-\infty, u), I_{02} = (-\infty, v), I_{12} = (-\infty, w)\}$, we write $S\begin{pmatrix} -\infty, u \\ -\infty, v \\ -\infty, w \end{pmatrix}$ for $S^2(\mathcal{I})$ throughout the proof. By (4.27), (4.10), and (4.9) the proof is reduced to showing that:

$$\bigcup_{(0,p_1,p_2)\in A_0} S\begin{pmatrix} -\infty, 1+p_1 \\ -\infty, 1+p_2 \\ -\infty, 1+p_2-p_1 \end{pmatrix} = S\begin{pmatrix} -\infty, 1+\alpha \\ -\infty, 1+\min\{\beta,\alpha+\gamma\} \\ -\infty, 1+\gamma \end{pmatrix}. \tag{4.30}$$

From the definition of A_0 in (4.15) and the definition of α, β, γ one easily obtains:

$$(0, p_1, p_2) \in A_0 \quad \Rightarrow \quad p_1 \le \alpha, p_2 \le \min\{\beta, \alpha+\gamma\}; \qquad (0, \alpha, \min\{\beta, \alpha+\gamma\}) \in A_0. \tag{4.31}$$

If $u \leq u'$, $v \leq v'$, $w \leq w'$, then $S(-\infty,(u,v,w)) \subset S(-\infty,(u',v',w'))$, hence by (4.31) we have:

$$\bigcup_{(0,p_1,p_2)\in A_0} S\begin{pmatrix} -\infty, 1+p_1 \\ -\infty, 1+p_2 \\ -\infty, 1+p_2-p_1 \end{pmatrix} = S\begin{pmatrix} -\infty, 1+\alpha \\ -\infty, 1+\min\{\beta,\alpha+\gamma\} \\ -\infty, 1+\min\{\beta,\alpha+\gamma\}-\alpha \end{pmatrix} \cup \bigcup_{\left\{\begin{array}{c}(0,p_1,p_2)\in A_0 : \\ p_2-p_1 > \min\{\beta,\alpha+\gamma\}-\alpha\end{array}\right\}} S\begin{pmatrix} -\infty, 1+p_1 \\ -\infty, 1+p_2 \\ -\infty, 1+p_2-p_1 \end{pmatrix}. \tag{4.32}$$

Now we consider two cases.

If $\min\{\beta,\alpha+\gamma\} = \alpha+\gamma$, then $\min\{\beta,\alpha+\gamma\}-\alpha = \gamma$ and $(A_0, A_1[p_1], A_2[p_2])$ is not an Ext-collection for $p_2-p_1 > \gamma$ (since $\hom^{p_1+\gamma+1-p_2}(A_1[p_1], A_2[p_2]) \neq 0$, $p_2-p_1-\gamma-1 \geq 0$), hence the equality (4.32) reduces to (4.30).

If $\min\{\beta,\alpha+\gamma\} = \beta < \alpha+\gamma$, then $\beta \leq \alpha - i + \gamma$ for $i \leq \alpha+\gamma-\beta$ and hence

$$\{(0,\alpha-i,\beta) : 0 \leq i \leq \alpha+\gamma-\beta\} \subset A_0. \tag{4.33}$$

Furthermore, we claim that the equality (4.32) reduces to

$$\bigcup_{(0,p_1,p_2)\in A_0} S\begin{pmatrix} -\infty, 1+p_1 \\ -\infty, 1+p_2 \\ -\infty, 1+p_2-p_1 \end{pmatrix} = \bigcup_{i=0}^{\alpha+\gamma-\beta} S\begin{pmatrix} -\infty, 1+\alpha-i \\ -\infty, 1+\beta \\ -\infty, 1+\beta-\alpha+i \end{pmatrix}. \tag{4.34}$$

Indeed, the first set of the union in (4.32) is the same as the first set of the union (4.34). Now assume that $(0,p_1,p_2) \in A_0$ and $p_2-p_1 > \beta-\alpha$, then $\beta-\alpha < p_2-p_1 \leq \gamma$. Therefore for some $1 \leq i \leq \gamma+\alpha-\beta$ we have $p_2-p_1 = \beta-\alpha+i$. From (4.31) we have also $p_2 \leq \beta$, therefore $p_1 = p_2-\beta+\alpha-i \leq \alpha-i$, and then $S\begin{pmatrix} -\infty, 1+p_1 \\ -\infty, 1+p_2 \\ -\infty, 1+p_2-p_1 \end{pmatrix} \subset S\begin{pmatrix} -\infty, 1+\alpha-i \\ -\infty, 1+\beta \\ -\infty, 1+\beta-\alpha+i \end{pmatrix}$ and we obtain (4.34). The last step of the proof is to show that

$$\bigcup_{i=0}^{\alpha+\gamma-\beta} S\begin{pmatrix} -\infty, 1+\alpha-i \\ -\infty, 1+\beta \\ -\infty, 1+\beta-\alpha+i \end{pmatrix} = S\begin{pmatrix} -\infty, 1+\alpha \\ -\infty, 1+\beta \\ -\infty, 1+\gamma \end{pmatrix}. \tag{4.35}$$

The inclusion $\subset$ is clear. To prove the inclusion $\supset$, assume that $(a_0,a_1,a_2) \in \mathbb{R}^3$ and $a_0-a_1 < 1+\alpha$, $a_0-a_2 < 1+\beta$, $a_1-a_2 < 1+\gamma$.

If $a_0-a_1 < 1+\alpha-(\alpha+\gamma-\beta) = 1+\beta-\gamma$, then by $a_1-a_2 < 1+\gamma$ it follows that (a_0,a_1,a_2) is in the set with index $i = \alpha+\gamma-\beta$ on the right-hand side.

It remains to consider the case, when $1+\alpha-i > a_0-a_1 \geq 1+\alpha-i-1$ for some $0 \leq i < \alpha+\gamma-\beta$. Now (a_0,a_1,a_2) is in the set indexed by the given i. Indeed, now $a_1-a_0 \leq i-\alpha$ and by $a_0-a_2 < 1+\beta$ we have $a_1-a_2 = a_1-a_0+a_0-a_2 < 1+\beta+i-\alpha$. □

4.3.4 More propositions used for gluing

Since we will often use the notion of a σ-triple, for the sake of completeness we rewrite here Definition 2.33 for triples (see also Remark 2.35):

Definition 4.9. *An exceptional triple* (A_0, A_1, A_2) *is a* σ*-triple iff the following conditions hold:* **(a)** $\hom^{\leq 0}(A_i, A_j) = 0$ *for* $i \neq j$*;* **(b)** $\{A_i\}_{i=0}^2 \subset \sigma^{ss}$ *;* **(c)** $\{\phi(A_i)\}_{i=0}^2 \subset (t, t+1)$ *for some* $t \in \mathbb{R}$.

We enhance now Proposition 4.3 for the case $n = 2$:

Proposition 4.10. *Let* $\mathcal{T}$ *be a* k*-linear triangulated category. Let* $\mathcal{E} = (A_0, A_1, A_2)$*,* α*,* β*,* γ *be as in Proposition 4.8. Let* $\sigma \in \Theta_{\mathcal{E}}$ *(hence we have the inequalities in* (4.29)*).*

(a) *If* $\phi^\sigma(A_0) \geq \phi^\sigma(A_1[\alpha])$*, then* $\mathcal{A} \cap \mathcal{T}_{exc} \subset \sigma^{ss}$*, where* $\mathcal{A}$ *is the extension closure of* $(A_0, A_1[\alpha])$.

(b) *If* $\phi^\sigma(A_1) \geq \phi^\sigma(A_2[\gamma])$*, then* $\mathcal{A} \cap \mathcal{T}_{exc} \subset \sigma^{ss}$*, where* $\mathcal{A}$ *is the extension closure of* $(A_1, A_2[\gamma])$.

Proof. If an equality holds in (a) or (b), then we have $\mathcal{A} \subset \mathcal{P}(t)$ for some $t \in \mathbb{R}$ and the Proposition follows. Hence we can assume that we have a proper inequality in both the cases.

(a) By the definition of $\Theta_{\mathcal{E}}$ in (4.8) and Corollary 2.34 we see that $(A_0[l], A_1[i], A_2[j])$ is a σ-triple for some $l, i, j \in \mathbb{Z}$. We can assume[4] $l = 0$ and then $\hom^{\leq 0}(A_0, A_1[i]) = 0$ and $|\phi(A_0) - \phi(A_1[i])| < 1$. From the definition of α we see that $i \leq \alpha$. Actually we must have $i = \alpha$, otherwise the given inequality $\phi(A_0) - \phi(A_1[\alpha]) > 0$ implies $\phi(A_0) - \phi(A_1[i]) > 1$, which is a contradiction. Thus $(A_0, A_1[\alpha], A_2[j])$ is a σ-triple for some $j \in \mathbb{Z}$. Now we apply Proposition 4.3.

(b) In this case we shift the given triple to a σ-triple of the form $(A_0[l], A_1, A_2[j])$ for some $l, j \in \mathbb{Z}$, in particular we have $\hom^{\leq 0}(A_1, A_2[j]) = 0$ and $|\phi(A_1) - \phi(A_2[j])| < 1$. From the definition of γ and the given inequality $\phi(A_1) - \phi(A_2[\gamma]) > 0$ it follows that $j = \gamma$. Thus $(A_0[l], A_1, A_2[\gamma])$ is a σ-triple for some $l \in \mathbb{Z}$. Now we apply Proposition 4.3. ☐

Proposition 4.11. *Let* $\mathcal{T}$ *has the property that for each exceptional triple* (A_0, A_1, A_2) *and for any two* $0 \leq i < j \leq 2$ *there exists unique* $k \in \mathbb{Z}$ *satisfying* $\hom^k(A_i, A_j) \neq 0$*. Let* $\mathcal{E} = (A_0, A_1, A_2)$ *be a full exceptional collection in* $\mathcal{T}$.

Let $R_0(\mathcal{E}) = (A_1, R_{A_1}(A_0), A_2)$*,* $L_0(\mathcal{E}) = (L_{A_0}(A_1), A_0, A_2)$*,* $R_1(\mathcal{E}) = (A_0, A_2, R_{A_2}(A_1))$*,* $L_1(\mathcal{E}) = (A_0, L_{A_1}(A_2), A_1)$ *be the triples obtained by a single mutation applied to* $\mathcal{E}$.[5] *Then the four intersections* $\Theta_{\mathcal{E}} \cap \Theta_{R_0(\mathcal{E})}$*,* $\Theta_{\mathcal{E}} \cap \Theta_{L_0(\mathcal{E})}$*,* $\Theta_{\mathcal{E}} \cap \Theta_{R_1(\mathcal{E})}$*,* $\Theta_{\mathcal{E}} \cap \Theta_{L_1(\mathcal{E})}$ *are all contractible and non-empty.*

Proof. Since $\mathcal{E} \sim \mathcal{E}'$ implies $\Theta_{\mathcal{E}} = \Theta_{\mathcal{E}'}$, $R_i(\mathcal{E}) \sim R_i(\mathcal{E}')$, $L_i(\mathcal{E}) \sim L_i(\mathcal{E}')$, we can assume that $l = \hom^1(A_0, A_1) > 0$, $p = \hom^1(A_1, A_2) > 0$. By the assumptions on $\mathcal{T}$ the other degrees are zero

[4] note that $(A_0, A_1[i], A_2[j])$ is a σ-triple iff $(A_0[k], A_1[i+k], A_2[j+k])$ is a σ-triple

[5] Recall that for any exceptional pair (A, B) the exceptional objects $L_A(B)$ and $R_B(A)$ are determined by the triangles $L_A(B) \rightarrow \mathrm{Hom}^*(A, B) \otimes A \xrightarrow{ev^*_{A,B}} B$; $A \xrightarrow{coev^*_{A,B}} \mathrm{Hom}^*(A, B)^{\check{}} \otimes B \rightarrow R_B(A)$ and that $(L_A(B), A)$, $(B, R_B(A))$ are exceptional pairs.

and it follows that the integers α, γ defined in (4.28) vanish and from Proposition 4.8 we get:

$$\Theta_{\mathcal{E}} = \left\{ \sigma \in \mathrm{Stab}(\mathcal{T}) : \mathcal{E} \subset \sigma^{ss} \text{ and } \begin{array}{l} \phi^\sigma(A_0) - \phi^\sigma(A_1) < 1 \\ \phi^\sigma(A_0) - \phi^\sigma(A_2) < 1 + \min\{\beta, 0\} \\ \phi^\sigma(A_1) - \phi^\sigma(A_2) < 1 \end{array} \right\}. \tag{4.36}$$

We start with the intersection $\Theta_{\mathcal{E}} \cap \Theta_{R_0(\mathcal{E})}$. Let us denote $X = R_{A_1}(A_0)[-1]$. Let α', β', γ' be the integers corresponding to the triple (A_1, X, A_2) used in Proposition 4.8. We have $1 + \beta' = \min\{k : \hom^k(A_1, A_2) \neq 0\} = 1 + \gamma = 1$, hence $\beta' = 0$. On the other hand from the definition of $R_{A_1}(A_0)$ we have a triangle

$$A_1^{\oplus l} \to X \to A_0 \to A_1^{\oplus l}[1] \tag{4.37}$$

and it follows that $\hom(A_1, X) \neq 0$, hence $\alpha' = -1$. We apply Proposition 4.8 to the triple (A_1, X, A_2) and obtain (note that $1 + \min\{\beta', \alpha' + \gamma'\} = 1 + \min\{0, \gamma' - 1\} = \min\{1, \gamma'\}$)

$$\Theta_{R_0(\mathcal{E})} = \Theta_{(A_1,X,A_2)} = \left\{ \sigma \in \mathrm{Stab}(\mathcal{T}) : \begin{array}{l} A_1 \in \sigma^{ss} \\ X \in \sigma^{ss} \\ A_2 \in \sigma^{ss} \end{array} \text{ and } \begin{array}{l} \phi^\sigma(A_1) - \phi^\sigma(X) < 0 \\ \phi^\sigma(A_1) - \phi^\sigma(A_2) < \min\{1, \gamma'\} \\ \phi^\sigma(X) - \phi^\sigma(A_2) < 1 + \gamma' \end{array} \right\}. \tag{4.38}$$

From the defintion of β, γ we have $0 = \hom^{\leq \min\{\beta,\gamma\}}(A_0, A_2) = \hom^{\leq \min\{\beta,\gamma\}}(A_1, A_2)$, and then the triangle (4.37) implies that $\hom^{\leq \min\{\beta,\gamma\}}(X, A_2) = 0$, it follows that

$$\min\{\beta, \gamma\} = \min\{\beta, 0\} \leq \gamma'. \tag{4.39}$$

Assume that $\sigma \in \Theta_{(A_1,X,A_2)} \cap \Theta_{\mathcal{E}}$. Then A_0, A_1, A_2, X are all semi-stable and $\phi(A_1) < \phi(X)$.[6] It is easy to show that $\hom(X, A_0) \neq 0$ (using the triangle (4.37)), hence $\phi(X) \leq \phi(A_0)$ and therefore $\phi(A_1) < \phi(A_0)$, and we obtain the inclusion $\subset$ in the following formula (the third inequality in this formula is the second in (4.38), the other inequalities are in (4.36) together with $\phi(A_1) < \phi(A_0)$)

$$\Theta_{\mathcal{E}} \cap \Theta_{R_0(\mathcal{E})} = \left\{ \sigma \in \mathrm{Stab}(\mathcal{T}) : \mathcal{E} \subset \sigma^{ss} \text{ and } \begin{array}{l} 0 < \phi^\sigma(A_0) - \phi^\sigma(A_1) < 1 \\ \phi^\sigma(A_0) - \phi^\sigma(A_2) < 1 + \min\{\beta, 0\} \\ \phi^\sigma(A_1) - \phi^\sigma(A_2) < \min\{\gamma', 1\} \end{array} \right\}. \tag{4.40}$$

We prove now the inclusion $\supset$. Assume that $\mathcal{E} \subset \sigma^{ss}$ and that the inequalities on the right hand side of (4.40) hold. In particular the inequalities in (4.36) hold, hence we have $\sigma \in \Theta_{\mathcal{E}}$ and $\phi^\sigma(A_0) > \phi^\sigma(A_1)$. Proposition 4.10 (a) ensures $X \in \sigma^{ss}$ and by (4.37) we get $\hom(A_1, X) \neq 0$, $\hom(X, A_2) \neq 0$, hence

$$X \in \sigma^{ss} \quad \phi(A_1) \leq \phi(X) \leq \phi(A_0) \quad Z(X) = lZ(A_1) + Z(A_0). \tag{4.41}$$

[6] We omit sometimes the superscript σ in expressions like $\phi^\sigma(X)$ and write just $\phi(X)$.

Using (2.11) and $0 < \phi^\sigma(A_0) - \phi^\sigma(A_1) < 1$ we see that $Z(A_1), Z(A_0)$ are not collinear(see Definition 4.27), therefore $Z(X) = lZ(A_1) + Z(A_0)$ is collinear neither with $Z(A_1)$ nor with $Z(A_0)$. Now we apply (2.11) again and by (4.41) we obtain $\phi(A_1) < \phi(X) < \phi(A_0)$. In particular, we obtain the first inequality in (4.38). The second inequality in (4.38) is the same as the third inequality of (4.40). From $\phi^\sigma(A_0) - \phi^\sigma(A_2) < 1 + \min\{\beta, 0\}$ and (4.39) we get $\phi(X) - \phi^\sigma(A_2) < \phi^\sigma(A_0) - \phi^\sigma(A_2) < 1 + \gamma'$, hence the third inequality in (4.38) is verified also. Thus, we obtain (4.40). This equality implies that the set $\Theta_{\mathcal{E}} \cap \Theta_{R_0(\mathcal{E})}$ is contractible. Indeed,we have a homeomorphism $f_{\mathcal{E}|\Theta_{\mathcal{E}}} : \Theta_{\mathcal{E}} \to f_{\mathcal{E}}(\Theta_{\mathcal{E}})$ (see (4.9), (4.7)). The proved equality (4.40) gives rise to: $f_{\mathcal{E}}\left(\Theta_{\mathcal{E}} \cap \Theta_{R_0(\mathcal{E})}\right) = \mathbb{R}^3_{>0} \times \left\{ \begin{array}{c} 0 < \phi_0 - \phi_1 < 1 \\ \phi_0 - \phi_2 < 1 + \min\{\beta, 0\} \\ \phi_1 - \phi_2 < \min\{\gamma', 1\} \end{array} \right\}$, hence $\Theta_{\mathcal{E}} \cap \Theta_{R_0(\mathcal{E})}$ is contractible.

Next, we consider the intersection $\Theta_{\mathcal{E}} \cap \Theta_{L_1(\mathcal{E})}$, where $L_1(\mathcal{E}) = (A_0, L_{A_1}(A_2), A_1)$.

Let us denote $Y = L_{A_1}(A_2)[1]$. Let α', β', γ' be the integers corresponding to the triple (A_0, Y, A_1). Obviously $\beta' = \alpha = 0$. From the definition of $L_{A_1}(A_2)$ we have a triangle

$$A_2 \to Y \to A_1^{\oplus p} \to A_2[1] \tag{4.42}$$

and it follows that $\hom(Y, A_1) \neq 0$, hence $\gamma' = -1$. Proposition 4.8 applied to the triple (A_0, Y, A_1) results in the equality(note that $1 + \min\{0', \alpha' - 1\} = \min\{1, \alpha'\}$)

$$\Theta_{L_1(\mathcal{E})} = \Theta_{(A_0,Y,A_1)} = \left\{ \sigma \in \mathrm{Stab}(\mathcal{T}) : \begin{array}{c} A_0 \in \sigma^{ss} \\ Y \in \sigma^{ss} \\ A_1 \in \sigma^{ss} \end{array} \text{ and } \begin{array}{l} \phi^\sigma(A_0) - \phi^\sigma(Y) < 1 + \alpha' \\ \phi^\sigma(A_0) - \phi^\sigma(A_1) < \min\{1, \alpha'\} \\ \phi^\sigma(Y) - \phi^\sigma(A_1) < 0 \end{array} \right\}. \tag{4.43}$$

From the defintion of α, β for the initial sequence $\mathcal{E}$ we have $0 = \hom^{\leq \min\{\alpha,\beta\}}(A_0, A_1)$, $0 = \hom^{\leq \min\{\alpha,\beta\}}(A_0, A_2)$, and then the triangle (4.42) implies that $\hom^{\leq \min\{\alpha,\beta\}}(A_0, Y) = 0$, it follows that

$$\min\{\alpha, \beta\} = \min\{0, \beta\} \leq \alpha'. \tag{4.44}$$

Assume that $\sigma \in \Theta_{(A_0,Y,A_1)} \cap \Theta_{\mathcal{E}}$. Then A_0, A_1, A_2, Y are all semi-stable and by (4.43) $\phi(Y) < \phi(A_1)$. The triangle (4.42) implies $\hom(A_2, Y) \neq 0$, hence $\phi(A_2) \leq \phi(Y) < \phi(A_1)$. Combining this inequality with the inequalities in (4.43), (4.36) we obtain the inclusion $\subset$ in the following formula:

$$\Theta_{\mathcal{E}} \cap \Theta_{L_1(\mathcal{E})} = \left\{ \sigma \in \mathrm{Stab}(\mathcal{T}) : \mathcal{E} \subset \sigma^{ss} \text{ and } \begin{array}{l} \phi^\sigma(A_0) - \phi^\sigma(A_1) < \min\{1, \alpha'\} \\ \phi^\sigma(A_0) - \phi^\sigma(A_2) < 1 + \min\{\beta, 0\} \\ 0 < \phi^\sigma(A_1) - \phi^\sigma(A_2) < 1 \end{array} \right\}. \tag{4.45}$$

To prove the inclusion $\supset$, assume that $\mathcal{E} \subset \sigma^{ss}$ and that the inequalities on the right hand side of (4.45) hold. In particular, we have $\sigma \in \Theta_{\mathcal{E}}$(see (4.36)). It remains to show that $Y \in \sigma^{ss}$ and that the inequalities in (4.43) hold. From Proposition 4.10 (b) and $\sigma \in \Theta_{\mathcal{E}}$, $\phi^\sigma(A_1) > \phi^\sigma(A_2)$ we obtain $Y \in \sigma^{ss}$. The triangle (4.42) implies

$$Y \in \sigma^{ss} \quad \phi(A_2) \leq \phi(Y) \leq \phi(A_1) \quad Z(Y) = pZ(A_1) + Z(A_2). \tag{4.46}$$

By similar arguments as in the previous case, using (2.11), $0 < \phi^\sigma(A_1) - \phi^\sigma(A_2) < 1$ and (4.46) one derives the inequalities $\phi(A_2) < \phi(Y) < \phi(A_1)$. In particular, we obtain the third inequality in (4.43). The second inequality in (4.43) is the same as the first inequality of (4.45). From $\phi^\sigma(A_0) - \phi^\sigma(A_2) < 1 + \min\{\beta, 0\}$ and (4.44) we get $\phi(A_0) - \phi^\sigma(Y) < \phi^\sigma(A_0) - \phi^\sigma(A_2) < 1 + \alpha'$ and the first inequality in (4.45) is verified also. Thus, we proved (4.45). As in the previous case this implies that $\Theta_{\mathcal{E}} \cap \Theta_{L_1(\mathcal{E})}$ is contractible.

Finally, recall that $\mathcal{E} \sim R_0(L_0(\mathcal{E}))$, therefore $\Theta_{\mathcal{E}} \cap \Theta_{L_0(\mathcal{E})} = \Theta_{R_0(L_0(\mathcal{E}))} \cap \Theta_{L_0(\mathcal{E})}$ and by the already proved first case we see that $\Theta_{\mathcal{E}} \cap \Theta_{L_0(\mathcal{E})}$ is contractible. For the case $\Theta_{\mathcal{E}} \cap \Theta_{R_1(\mathcal{E})}$ we have $\Theta_{\mathcal{E}} \cap \Theta_{R_1(\mathcal{E})} = \Theta_{L_1(R_1\mathcal{E})} \cap \Theta_{R_1(\mathcal{E})}$ and contractibillity follows from a previous case. The Proposition is proved. $\square$

Propositions 4.10 and 4.3 ensure semi-stability of certain exceptional objects. The following two lemmas are similar in that respect and will be used later, when we analyze the intersections of the form $\Theta_{\mathcal{E}_1} \cap \Theta_{\mathcal{E}_2}$, when $\mathcal{E}_2$ is obtained from $\mathcal{E}_1$ by more than one and different mutations.

Lemma 4.12. *Let $\mathcal{T} = D^b(\mathcal{A})$, where $\mathcal{A}$ is a hereditary abelian hom-finite category, and let for any two exceptional objects $E, F \in \mathcal{T}_{exc}$ there exists at most one $k \in \mathbb{Z}$ satisfying $\hom^k(E, F) \neq 0$.*

Let (A_0, A_1, A_2) be a full Ext-exceptional ("Ext-" means that it satisfies (a) in Definition 4.9) collection in $\mathcal{T}$, such that $\hom^1(A_0, A_2) = 0$ and A_0, A_1, A_2 are semistable. Let X, Y be exceptional objects in $\mathcal{T}$ for which we have a diagram of distinguished triangles, where all arrows are non-zero:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_2 & \longrightarrow & X & \longrightarrow & Y \\ & \nwarrow \swarrow & & \nwarrow \swarrow & & \nwarrow \swarrow & \\ & A_2 & & A_1 & & A_0 & \end{array} . \tag{4.47}$$

(a) *If we have the following system of inequalities:*

$$\begin{array}{c} \phi(A_0) - 1 < \phi(A_1) < \phi(A_0), \quad \phi(A_0) - 1 < \phi(A_2) < \phi(A_0) \\ \arg_{(\phi(A_0)-1, \phi(A_0))}(Z(A_0) + Z(A_1)) > \phi(A_2) \end{array},$$

then $Y \in \sigma^{ss}$ and $\phi(Y) < \phi(A_0)$.

(b) *If we have $\phi(A_2) < \phi(A_1) \leq \phi(A_0) < \phi(A_2) + 1$, then $Y \in \sigma^{ss}$ and $\phi(Y) < \phi(A_0)$.*

Proof. We note first some vanishings. From the given diagram it follows that $\hom(Y, A_0) \neq 0$ and $\hom(X, A_1) \neq 0$. Since X, Y are also exceptional objects, from Lemma 2.65 and the hereditariness of $\mathcal{A}$ it follows that $\hom(A_0, Y) = \hom(A_1, X) = 0$. On the other hand $\hom(A_1, Y) = \hom(A_1, X)$ (follows by applying $\hom(A_1, \_)$ to the last triangle and using $\hom^*(A_1, A_0) = 0$). Thus, we obtain

$$\hom(A_0, Y) = \hom(A_1, Y) = 0. \tag{4.48}$$

Since (A_0, A_1, A_2) is an Ext-exceptional collection, its extension closure is a heart of a bounded t-structure([37, Lemma 3.14]), furthermore this heart is of finite length and (A_0, A_1, A_2) are the simple objects in it. Let us denote for simplicity $t = \phi(A_0)$(in case (a)) or $t - 1 = \phi(A_2)$(in case (b)). In both the cases from the given inequalities and since $\mathcal{P}(t-1,t]$ is also a heart, it follows that the extension closure of (A_0, A_1, A_2) is exactly $\mathcal{P}(t-1,t]$. Now (4.47) can be considered as the Jordan-Hölder filtration of Y in the abelian category $\mathcal{P}(t-1,t]$ and the composition factors of Y are $\{A_0, A_1, A_2\}$.

Suppose that $Y \notin \sigma^{ss}$. From Remark 4.2 (b) there exists $Y' \in \mathcal{P}(t-1,t]$ and a non-trivial monic arrow $Y' \to Y$, s. t. $\arg_{(t-1,t]}(Z(Y')) > \arg_{(t-1,t]}(Z(Y))$.

We have $Z(Y) = Z(A_0) + Z(A_1) + Z(A_2)$ and one can show that the given inequalities in either case (a) or (b) imply that

$$\arg_{(t-1,t]}(Z(Y)) > \arg_{(t-1,t]}(Z(A_2)), \quad \arg_{(t-1,t]}(Z(Y)) > \arg_{(t-1,t]}(Z(A_2) + Z(A_1)). \tag{4.49}$$

Since Y' is a subobject of Y, the composition factors of Y' in $\mathcal{P}(t-1,t]$ are subset of $\{A_0, A_1, A_2\}$. The cases $Y' \cong A_0$, $Y' \cong A_1$ are excluded by (4.48). The case $Y' \cong A_2$ is excluded by the first inequality in (4.49) and the condition $\arg_{(t-1,t]}(Z(Y')) > \arg_{(t-1,t]}(Z(Y))$. Since Y' is a proper subobject of Y we reduce to the case when Y' has two composition factors (two different elements of the set $\{A_0, A_1, A_2\}$). Using (4.48) again we reduce to the following options for a Jordan Hölder filtration

$$0 \longrightarrow A_2 \longrightarrow Y' \longrightarrow A_1 \longrightarrow 0 \qquad 0 \longrightarrow A_2 \longrightarrow Y' \longrightarrow A_0 \longrightarrow 0. \tag{4.50}$$

In the first case we have $Z(Y') = Z(A_2) + Z(A_1)$ which contradicts the second inequality on (4.49). In the second case we have a distinguished triangle$A_2 \longrightarrow Y' \longrightarrow A_0 \longrightarrow A_2[1]$ in $\mathcal{T}$, and form the given vanishing $\hom^1(A_0, A_2) = 0$ it follows $Y' \cong A_0 \oplus A_2$, which contradicts (4.48). So we proved $Y \in \sigma^{ss}$. The inequality $\phi(Y) < \phi(A_0)$(in either case (a) or (b)) follows from the given inequalities and $Z(Y) = Z(A_0) + Z(A_1) + Z(A_2)$. The lemma is proved. □

Lemma 4.13. *Let $\mathcal{T} = D^b(\mathcal{A})$ be as in Lemma 4.12.*

Let (A_0, A_1, A_2) be a full Ext-exceptional collection[7] *in $\mathcal{T}$ such that A_0, A_1, A_2 are semistable. Let Y be an exceptional object in $\mathcal{T}$ for which we have a triangle, where all arrows are non-zero:*

$$\begin{array}{ccc} A_2 & \longrightarrow & Y \\ & \nwarrow \quad \swarrow & \\ & A_0 & \end{array} . \tag{4.51}$$

If one of the two systems: $\begin{array}{c} \phi(A_2) < \phi(A_0) < \phi(A_2) + 1 \\ \phi(A_2) < \phi(A_1) < \phi(A_2) + 1 \end{array}$ *or* $\begin{array}{c} \phi(A_0) - 1 < \phi(A_1) < \phi(A_0) \\ \phi(A_0) - 1 < \phi(A_2) < \phi(A_0) \end{array}$ *holds, then we have: $Y \in \sigma^{ss}$, $\phi(Y) = \arg_{(\phi(A_2),\phi(A_2)+1)}(Z(A_0) + Z(A_2)) = \arg_{(\phi(A_0)-1,\phi(A_0))}(Z(A_0) + Z(A_2))$ and $\phi(A_2) < \phi(Y) < \phi(A_0)$.*

Proof. Since Y, A_0 are exceptional objects and $\hom(Y, A_0) \neq 0$, from Lemma 2.65 and the hereditarines of $\mathcal{A}$ it follows that $\hom(A_0, Y) = 0$. Due to the given inequalities, in both the cases we can

[7] "Ext-" means that it satisfies (a) in Definition 4.9

choose t so that $\phi(A_0), \phi(A_1), \phi(A_2) \in (t-1,t]$. By the same arguments as in the previous lemma, one sees that the extension closure of (A_0, A_1, A_2) is $\mathcal{P}(t-1,t]$ and that this is an abelian category of finite length with simple objects A_0, A_1, A_2. Now (4.51) can be considered as the Jordan-Hölder filtration of Y in the abelian category $\mathcal{P}(t-1,t]$ and the composition factors of Y are $\{A_0, A_2\}$.

We have $Z(Y) = Z(A_0) + Z(A_2)$ and the given inequalities (in either case) imply that:

$$\begin{aligned} \phi(A_2) = \arg_{(t-1,t]}(Z(A_2)) < \arg_{(t-1,t]}(Z(Y)) = \arg_{(\phi(A_0)-1,\phi(A_0))}(Z(A_0) + Z(A_2)) \\ = \arg_{(\phi(A_2),\phi(A_2)+1)}(Z(A_0) + Z(A_2)) < \arg_{(t-1,t]}(Z(A_0)) = \phi(A_0). \end{aligned} \tag{4.52}$$

Suppose that $Y \notin \sigma^{ss}$. From Remark 4.2 (b) it follows that there exists $Y' \in \mathcal{P}(t-1,t]$ and a non-trivial monic arrow $Y' \to Y$ in $\mathcal{P}(t-1,t]$, s. t. $\arg_{(t-1,t]}(Z(Y')) > \arg_{(t-1,t]}(Z(Y))$. Since the composition factors of Y are $\{A_0, A_2\}$ and Y' is a non-zero proper sub-object of Y, then we have $Y' \cong A_2$ or $Y' \cong A_0$. The case $Y' \cong A_0$, is excluded by $\hom(A_0, Y) = 0$. The case $Y' \cong A_2$ is excluded by (4.52) and the condition $\arg_{(t-1,t]}(Z(Y')) > \arg_{(t-1,t]}(Z(Y))$. So we proved $Y \in \sigma^{ss}$.

By $Y \in \mathcal{P}(t, t+1]$ it follows that $\phi(Y) = \arg_{(t-1,t]}(Z(Y))$. Now the lemma follows from (4.52). □

4.4 The exceptional objects in $D^b(Q)$

From now on we fix $\mathcal{T} = D^b(Rep_k(Q))$, where Q is the affine quiver in figure (2.2).

In this Section we organize in a better way the data about $\{\mathrm{Hom}(X,Y), \mathrm{Ext}^1(X,Y)\}_{X\in\mathcal{T}_{exc}}$ obtained in Section 2.2. Subsection 4.4.3 contains some observations about the behavior of the vectors $\{Z(X)\}_{X\in\mathcal{T}_{exc}} \subset \mathbb{C}$, which will be helpful when we analyze the intersections of the form $\Theta_{\mathcal{E}_1} \cap \Theta_{\mathcal{E}_2}$ in the next sections.

In Proposition 2.3 were classified the exceptional objects of $Rep_k(Q)$:

Recall that we denote by $K_0(\mathcal{T})$ the Grothendieck group of $\mathcal{T}$ and for $X \in \mathcal{T}$ we denote by $[X] \in K(T)$ the corresponding equivalence class in $K_0(\mathcal{T})$. From Proposition 2.3 it follows:

Corollary 4.14. *Let us denote $\delta = [E_1^0] + [E_3^0] + [M] \in K_0(\mathcal{T})$. We have the following equalities in $K_0(\mathcal{T})$:*

$$\delta = [E_1^0] + [E_3^0] + [M] = [E_1^0] + [E_2^0] = [E_3^0] + [E_4^0] = [M] + [M'] \tag{4.53}$$

$$[E_1^m] = m\delta + [E_1^0] = (m+1)\delta - [E_2^0] \qquad [E_2^m] = m\delta + [E_2^0] = (m+1)\delta - [E_1^0] \tag{4.54}$$

$$[E_3^m] = m\delta + [E_3^0] = (m+1)\delta - [E_4^0] \qquad [E_4^m] = m\delta + [E_4^0] = (m+1)\delta - [E_3^0] \tag{4.55}$$

$$[E_1^m] + [M] = [E_4^m] \quad [E_3^m] + [M] = [E_2^m] \quad [E_4^m] + [M'] = [E_1^{m+1}] \quad [E_2^m] + [M'] = [E_3^{m+1}]. \tag{4.56}$$

4.4.1 The two orbits of 2-Kronecker pairs in $D^b(Q)$

In Propositions 4.3 and 4.10 were discussed exceptional pairs (E,F) with $\hom^{\leq 0}(E,F)=0$ and $\hom^1(E,F)=l\neq 0$, and their extension closures. We call such a pair *l-Kronecker pair.* Kronecker pairs were used in Capter 3 for studying the density of the set of phases of Bridgeland stability conditions. In Corollary 3.28 was shown that for any affine acyclic quiver A (like the quiver Q in figure (2.2)) only 1- and 2-Kronecker pairs can appear in $D^b(A)$. In this subsection we give some comments on the 1- and 2-Kronecker pairs in $D^b(Q)$, which will be useful later when we apply Propositions 4.3, 4.10 and Lemmas 4.12, 4.13.

From Remark 2.2 we see that the 2-Kronecker pairs in $D^b(Q)$ up to shifts are:

$$\mathfrak{P}_{12}=\{(E_1^{m+1},E_1^m[-1]),(E_1^0,E_2^0),(E_2^m,E_2^{m+1}[-1]) : m\in\mathbb{N}\} \tag{4.57}$$

$$\mathfrak{P}_{43}=\{(E_4^{m+1},E_4^m[-1]),(E_4^0,E_3^0),(E_3^m,E_3^{m+1}[-1]) : m\in\mathbb{N}\}. \tag{4.58}$$

Recall that the Braid group on two strings $B_2\cong\mathbb{Z}$ acts on the set of equivalence classes of exceptional pairs in $\mathcal{T}$ (here we take the equivalence $\sim$ explained in Section 4.2 and it is clear when a given equivalence class w.r. $\sim$ will be called a 2-Kronecker pair). UsingCorollary 2.11 and the list of triples inCorollary 2.10 one can verify that the set of 2-Kronecker pairs is invariant under this action of B_2 and this action on the 2-Kronecker pairs has two orbits. They are (4.57) and (4.58).

We will describe now the sets $\mathcal{T}_{exc}\cap\mathcal{A}$, up to isomorphism, where $\mathcal{A}$ is the extension closure in $\mathcal{T}$ of a 2-Kronecker pair. This will be helpful later (e. g. when we apply Propositions 4.3 and 4.10). We note first a simple lemma (in which $Rep_k(Q)$ can be any hereditary category):

Lemma 4.15. *Let A, $B\in Rep_k(Q)$, let $\mathcal{C}$ be the extension closure of $A, B[-1]$ in $D^b(Rep_k(Q))=\mathcal{T}$. Then any $X\in\mathcal{C}$, which is $\mathcal{T}$-indecomposable, has the form $X'[i]$, where $X'\in Rep_k(Q)$ and $i\in\{0,-1\}$. In particular any $X\in\mathcal{C}\cap\mathcal{T}_{exc}$ has the form $X'[i]$ with $X'\in Rep_k(Q)_{exc}$ and $i\in\{0,-1\}$.*

Proof. Since $Rep_k(Q)$ is hereditary, any object $X\in D^b(Rep_k(Q))$ decomposes as follows $X\cong\bigoplus_{i\in\mathbb{Z}}H^i(X)[-i]$, where $H^i:\mathcal{T}\to Rep_k(Q)$ are the cohomology functors.

Since $A,B\in Rep_k(Q)$, it follows that $H^i(A)=H^i(B[-1])=0$ for each $i\neq\{0,1\}$. The functors $H^i:\mathcal{T}\to Rep_k(Q)$ map triangles to short exact sequences (see e.g. [28]), therefore $H^i(X)=0$ for any $X\in\mathcal{C}$ and any $i\neq\{0,1\}$. By the first paragraph of the proof we see that each $X\in\mathcal{C}$ has the form $X'\oplus X''[-1]$ with X', $X''\in Rep_k(Q)$. If $X\in\mathcal{C}$ is indecomposable in $\mathcal{T}$, then obviously either $X\cong X'$ or $X\cong X'[-1]$ for some $X'\in Rep_k(Q)$. Finally, if $X\in\mathcal{C}\cap\mathcal{T}_{exc}$, then X is indecomposable in $\mathcal{T}$, hence $X\cong X'[i]$, $i\in\{0,-1\}$ and obviously X' is also exceptional, i. e. $X'\in Rep_k(Q)_{exc}$. □

Lemma 4.16. *Let (U,V) be one of the 2-Kronecker pairs given in* (4.57) *or* (4.58)*. Let $\mathcal{A}$ be its extension closure in $\mathcal{T}$. Then representatives of the iso-classes of objects in $\mathcal{A}\cap\mathcal{T}_{exc}$ are:*

$(U,V)=$	$(E_{1/4}^{m+1},E_{1/4}^m[-1])$	$(E_{1/4}^0,E_{2/3}^0)$	$(E_{2/3}^m,E_{2/3}^{m+1}[-1])$
$\mathcal{A}\cap\mathcal{T}_{exc}=$	$\left\{\begin{array}{ll} E_{1/4}^n[-1] & 0\leq n\leq m \\ E_{1/4}^n & n\geq m+1 \\ E_{2/3}^n & n\in\mathbb{N}\end{array}\right\}$	$\left\{\begin{array}{ll} E_{1/4}^n & n\in\mathbb{N} \\ E_{2/3}^n & n\in\mathbb{N}\end{array}\right\}$	$\left\{\begin{array}{ll} E_{2/3}^n & 0\leq n\leq m \\ E_{2/3}^n[-1] & n\geq m+1 \\ E_{1/4}^n[-1] & n\in\mathbb{N}\end{array}\right\}$

where the subscript in the table is either everywhere the first or everywhere the second.

Proof. We consider the case when the subscript is everywhere the first (i. e. we pick out pairs from (4.57)), the other case is analogous. From Lemma 3.19 we have that $\mathcal{A}$ is a bounded t-structure in $\langle U, V\rangle$ and we have also an equivalence of abelian categories

$$F : \mathcal{A} \to Rep_k(K(2)) \qquad F(U) = k \rightrightarrows 0, \quad F(V) = 0 \rightrightarrows k. \tag{4.59}$$

Using the facts that $\mathcal{A}$ is a bounded t-structure in $\langle U, V\rangle$ and that F is equivalence, one can show that if $X \in \mathcal{A} \cap \mathcal{T}_{exc}$, then $F(X) \in Rep_k(K(2))_{exc}$. Furthermore, since $\mathcal{T} = D^b(Rep_k(Q))$ and $Rep_k(Q)$ is a hereditary category, it is easy to prove that (see also Lemma 4.15 and its proof):

$$X \in \mathcal{A} \cap \mathcal{T}_{exc} \quad \Leftrightarrow \quad F(X) \in Rep_k(K(2))_{exc}. \tag{4.60}$$

As in the proof of Proposition 2.3 one can classify $Rep_k(K(2))_{exc}$ and the result is:

$$\forall X \in Rep_k(K(2))_{exc} \qquad X \cong k^{n+1} \underset{\pi_-^n}{\overset{\pi_+^n}{\rightrightarrows}} k^n \;\text{ or }\; X \cong k^n \underset{j_-^n}{\overset{j_+^n}{\rightrightarrows}} k^{n+1} \;\text{ for some }\; n \in \mathbb{N}. \tag{4.61}$$

Since $\mathcal{A}$ is a bounded t-structure in $\langle U, V\rangle$, the inclusion functor $\mathcal{A} \to \mathcal{T}$ induces an embedding of groups $K_0(\mathcal{A}) \to K_0(\mathcal{T})$. Now from (4.59), (4.60), (4.61) it follows that:

$$\{[X] \in K_0(\mathcal{T}) : X \in \mathcal{A} \cap \mathcal{T}_{exc}\} = \{(n+1)[U] + n[V], \quad n[U] + (n+1)[V] : n \in \mathbb{N}\}. \tag{4.62}$$

If $(U, V) = (E_1^{m+1}, E_1^m[-1])$, then using (4.54) we obtain:

$$\begin{aligned}(n+1)[U] + n[V] &= (n+1)\left[E_1^{m+1}\right] - n\left[E_1^m\right] = (n+1)\left((m+1)\delta + \left[E_1^0\right]\right) - n\left(m\delta + \left[E_1^0\right]\right)\\ &= (n+m+1)\delta + \left[E_1^0\right] = \left[E_1^{n+m+1}\right]\\ n[U] + (n+1)[V] &= n\left((m+1)\delta + \left[E_1^0\right]\right) - (n+1)\left(m\delta + \left[E_1^0\right]\right) = (n-m)\delta - \left[E_1^0\right]\\ &= \begin{cases} \left[E_2^{n-m-1}\right] & n \geq m+1 \\ -\left[E_1^{m-n}\right] = \left[E_1^{m-n}[-1]\right] & n \leq m \end{cases}.\end{aligned}$$

Hence (4.62) in this case is $\{[X] \in K_0(\mathcal{T}) : X \in \mathcal{A} \cap \mathcal{T}_{exc}\} = \left\{ \begin{array}{cc} [E_1^n[-1]] & 0 \leq n \leq m \\ [E_1^n] & n \geq m+1 \\ [E_2^n] & n \in \mathbb{N} \end{array} \right\}$. Now the second column in the table follows easily from Lemma 4.15 and the fact that there is at most one, up to isomorphism, exceptional representation in $Rep_k(Q)$ of a given dimension vector ([16, p. 13]).

If $(U, V) = (E_1^0, E_2^0)$, then using (4.54) and (4.53) we reduce (4.62) to $\{[X] \in K_0(\mathcal{T}) : X \in \mathcal{A} \cap \mathcal{T}_{exc}\} = \{[E_1^n], [E_2^n] : n \in \mathbb{N}\}$ and the third column of the table follows.

If $(U, V) = (E_2^m, E_2^{m+1}[-1])$, then using (4.54) we obtain:

$$(n+1)[U] + n[V] = (n+1)\left(m\delta + \left[E_2^0\right]\right) - n\left((m+1)\delta + \left[E_2^0\right]\right) = (m-n)\delta + \left[E_2^0\right]$$
$$= \begin{cases} \left[E_2^{m-n}\right] & n \leq m \\ -\left[E_1^{n-m-1}\right] = \left[E_1^{n-m-1}[-1]\right] & n \geq m+1 \end{cases}$$

$$n[U] + (n+1)[V] = n\left(m\delta + \left[E_2^0\right]\right) - (n+1)\left((m+1)\delta + \left[E_2^0\right]\right)$$
$$= -(n+m+1)\delta - \left[E_2^0\right] = \left[E_2^{n+m+1}[-1]\right]$$

and now (4.62) and similar arguments as in the first case give the fourth column of the table.

The case when the subscript is everywhere the second (i. e. the pairs in (4.58)) is obtained by substituting E_1 with E_4, E_2 with E_3, and using (4.55) instead of (4.54). □

Some 1-Kronecker pairs in $D^b(Q)$ are (see table (2.4)):

$$(M', E_1^m[-1]), (M', E_2^m), (M, E_3^m), (M, E_4^m[-1]). \tag{4.63}$$

In the following lemma are listed several short exact sequences in $Rep_k(Q)$. On one hand, these sequences determine the set $\mathcal{A} \cap \mathcal{T}_{exc}$, where $\mathcal{A}$ is the extension closure of some of the 1-Kronecker pairs in (4.63), so they will be helpful when we apply Propositions 4.3 and 4.10. On the other hand, they (and their combinations) will play the role of the triangles (4.47) and (4.51) when we apply Lemmas 4.12 and 4.13.

Lemma 4.17. *There exist short exact sequences in $Rep_k(Q)$ of the form ($m \in \mathbb{N}$):*

$$0 \longrightarrow E_3^m \longrightarrow E_2^m \longrightarrow M \longrightarrow 0 \tag{4.64}$$
$$0 \longrightarrow M \longrightarrow E_4^m \longrightarrow E_1^m \longrightarrow 0 \tag{4.65}$$
$$0 \longrightarrow M' \longrightarrow E_1^{m+1} \longrightarrow E_4^m \longrightarrow 0 \tag{4.66}$$
$$0 \longrightarrow E_2^m \longrightarrow E_3^{m+1} \longrightarrow M' \longrightarrow 0 \tag{4.67}$$
$$0 \longrightarrow E_3^0 \longrightarrow M' \longrightarrow E_1^0 \longrightarrow 0. \tag{4.68}$$

Proof. The proof is an exercise using Proposition 2.3 . □

4.4.2 Reformulation of some results of Section 2.2 with new notations

It is useful to introduce some notations (see Proposition 2.3 for the notations E_i^j, M, M'):

$$a^m = \begin{cases} E_1^{-m} & m \leq 0 \\ E_2^{m-1}[1] & m \geq 1 \end{cases}; \qquad b^m = \begin{cases} E_4^{-m} & m \leq 0 \\ E_3^{m-1}[1] & m \geq 1 \end{cases}. \tag{4.69}$$

Remark 4.18. *The objects in $\mathcal{T}_{exc}$ up to isomorphism are $\{a^j[k], b^j[k], M[k], M'[k] : j \in \mathbb{Z}, k \in \mathbb{Z}\}$.*

Using table (2.4), one verifies that:

Corollary 4.19. *(of Proposition 2.5) For each $m \in \mathbb{Z}$ we have:*

$$\hom(M', a^m) \neq 0; \quad \hom(M, b^m) \neq 0; \quad \hom^*(a^m, M') = 0; \tag{4.70}$$

$$\hom^1(a^m, M) \neq 0; \quad \hom^1(b^m, M') \neq 0; \quad \hom^*(b^m, M) = 0 \tag{4.71}$$

$$\hom^1(b^{m+1}, a^n) \neq 0 \text{ for } m > n; \quad \hom(b^m, a^n) \neq 0 \text{ for } m \leq n; \quad \hom^*(b^{m+1}, a^m) = 0 \tag{4.72}$$

$$\hom^1(a^m, b^n) \neq 0 \text{ for } m > n; \quad \hom(a^m, b^{n+1}) \neq 0 \text{ for } m \leq n; \quad \hom^*(a^m, b^m) = 0; \tag{4.73}$$

$$\hom(a^m, a^n) \neq 0 \text{ for } m \leq n; \quad \hom^1(a^m, a^n) \neq 0 \text{ for } m > n+1; \quad \hom^*(a^m, a^{m-1}) = 0 \tag{4.74}$$

$$\hom(b^m, b^n) \neq 0 \text{ for } m \leq n; \quad \hom^1(b^m, b^n) \neq 0 \text{ for } m > n+1; \quad \hom^*(b^m, b^{m-1}) = 0 \tag{4.75}$$

$$\hom^1(M, M') \neq 0 \qquad \hom^1(M', M) \neq 0. \tag{4.76}$$

It is useful to keep in mind the following remarks:

Remark 4.20. *Recall that $\phi_-(A) > \phi_+(B)$ implies $\hom(A,B) = 0$ and in particular $\hom(A,B) \neq 0$ implies $\phi_-(A) \leq \phi_+(B)$ (for each stability condition).*

Let $\{x^i\}_{i\in\mathbb{Z}}$ be either $\{a^i\}_{i\in\mathbb{Z}}$ or $\{b^i\}_{i\in\mathbb{Z}}$. From (4.74) and (4.75) it follows that:

(a) *For $m \leq n$ we have $\hom(x^m, x^n) \neq 0$. In particular, if $x^m, x^n \in \sigma^{ss}$ and $m \leq n$ then $\phi(x^m) \leq \phi(x^n)$.*

(b) *For $m+1 < n$ we have $\hom^1(x^n, x^m) \neq 0$. In particular, if $x^m, x^n \in \sigma^{ss}$ and $m+1 < n$ then $\phi(x^n) \leq \phi(x^m) + 1$.*

Remark 4.21. *Let $\{x^i\}_{i\in\mathbb{Z}}$ be either $\{a^i\}_{i\in\mathbb{Z}}$ or $\{b^i\}_{i\in\mathbb{Z}}$. Lemma 4.16 in terms of the notations (4.69) is equivalent to saying that for any three integers $i \leq p$, $p+1 \leq j$ we have that x^i and $x^j[-1]$ are in the extension closure of $\{x^p, x^{p+1}[-1]\}$.*

Keeping these remarks in mind one proves:

Lemma 4.22. *Let $\{x^i\}_{i\in\mathbb{Z}}$ be either $\{a^i\}_{i\in\mathbb{Z}}$ or $\{b^i\}_{i\in\mathbb{Z}}$. If there exists $m \in \mathbb{Z}$ such that $\{x^m, x^{m+1}\} \subset \sigma^{ss}$ and $\phi(x^m) + 1 < \phi(x^{m+1})$, then for $i \notin \{m, m+1\}$ we have $x^i \notin \sigma^{ss}$.*

Proof. Suppose $x^i \in \sigma^{ss}$ with $i < m$, then by Remark (a) we have 4.20 $\phi(x^i) + 1 \leq \phi(x^m) + 1 < \phi(x^{m+1})$, hence $\hom^1(x^{m+1}, x^i) = 0$, which contradicts the second part of Remark 4.20 (b). If $x^i \in \sigma^{ss}$ with $i > m+1$, then by Remark 4.20 (a) we obtain $\hom^1(x^i, x^m) = 0$, which again contradicts Remark 4.20 (b). □

Due to Corollary 2.7 (b) we can apply Lemmas 4.12, 4.13 to $D^b(Q)$. Furthermore, we have:

Corollary 4.23. *(Corollary 2.10) The full exceptional collections in $D^b(Q)$ up to isomorphism and schifts are in the set of triples $\mathfrak{T}$ given below. Propositions 4.8, 4.10, 4.11 can be applied to any of these triples.*

$$\mathfrak{T} = \left\{ \begin{array}{ccc} (M', a^m, a^{m+1}) & (a^m, b^{m+1}, a^{m+1}) & (a^m, a^{m+1}, M) \\ (M, b^m, b^{m+1}) & (b^m, a^m, b^{m+1}) & (b^m, b^{m+1}, M') \\ (b^m, M', a^m) & (a^m, M, b^{m+1}) & . \end{array} : m \in \mathbb{N} \right\}.$$

Proof. The list $\mathfrak{T}$ follows straightforwardly from Corollary 2.10. By Corollary 4.19 $\hom^*(X,Y) \neq 0$ for any exceptional pair (X,Y), therefore Propositions 4.8, 4.10 can be applied to any of the triples. Proposition 4.11 can be applied due to Corollary 2.7 (b). □

Remark 4.24. *It is known [15] that the Braid group on three strings B_3 acts transitively on the exceptional triples of $Rep_k(Q)$. This action is not free (see Remark 2.13).*

Remark 4.25. *With the notations* (4.69) *the two orbits of 2-Kronecker pairs (see* (4.57) *and* (4.58)*) are $\{(a^m, a^{m+1}[-1])\}_{m\in\mathbb{Z}}$ and $\{(b^m, b^{m+1}[-1])\}_{m\in\mathbb{Z}}$. Each of these pairs can be extended to three non-equivalent triples, so we obtain two sets of triples. Having the list $\mathfrak{T}$ above, it follows that these two sets of triples are:*

$$\mathfrak{T}_a = \{(M', a^m, a^{m+1}), (a^m, b^{m+1}, a^{m+1}), (a^m, a^{m+1}, M) : m \in \mathbb{Z}\} \tag{4.77}$$

$$\mathfrak{T}_b = \{(M, b^m, b^{m+1}), (b^m, a^m, b^{m+1}), (b^m, b^{m+1}, M') : m \in \mathbb{Z}\}. \tag{4.78}$$

Furthermore we have:

$$\mathfrak{T} = \mathfrak{T}_a \cup \{(b^m, M', a^m), (a^m, M, b^{m+1}) : m \in \mathbb{Z}\} \cup \mathfrak{T}_b \qquad \mathfrak{T}_a \cap \mathfrak{T}_b = \emptyset. \tag{4.79}$$

Remark 4.26. *The short exact sequences* (4.66)*,* (4.67)*,* (4.68) *in terms of the notations* (4.69) *become a sequence of distinguished triangles (for each p):*

$$\begin{array}{ccc} b^{p+1}[-1] & \longrightarrow & M' \\ & \nwarrow \quad \swarrow & \\ & a^p & \end{array} \tag{4.80}$$

The short exact sequences (4.64) *and* (4.65) *become the following distinguished triangles ($q \in \mathbb{Z}$):*

$$\begin{array}{ccc} a^q[-1] & \longrightarrow & M \\ & \nwarrow \quad \swarrow & \\ & b^q & \end{array} . \tag{4.81}$$

4.4.3 Comments on the vectors $\{Z(X) : X \in \mathcal{T}_{exc}\}$.

The formulas in Corollary 4.14 give rise to the equalities $Z(\delta) = Z(E_1^0) + Z(E_2^0) = Z(E_4^0) + Z(E_3^0)$ and $Z(E_k^m) = mZ(\delta) + Z(E_k^0)$ for each $m \in \mathbb{N}$ each $k = 1, 2, 3, 4$ and for each $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}(\mathcal{T})$. Due to these equalities, with the notations (4.69) we can write (recall that $Z(X[j]) = (-1)^j Z(X)$ for any $j \in \mathbb{Z}$, $X \in \mathcal{T}$):

$$\forall j \in \mathbb{Z} \qquad Z(a^{j+1}) = Z(a^j) - Z(\delta) \ \text{ and } \ Z(b^{j+1}) = Z(b^j) - Z(\delta). \tag{4.82}$$

Therefore for any two integers m, n we have:

$$Z(a^m) = Z(a^n) - (m-n)Z(\delta) \qquad \text{and} \qquad Z(b^m) = Z(b^n) - (m-n)Z(\delta). \tag{4.83}$$

Next we discuss collinear vectors among $\{Z(a^j)\}_{j\in\mathbb{Z}}$ and $\{Z(b^j)\}_{j\in\mathbb{Z}}$. We fix first the meaning of "collinear":

Definition 4.27. *We say that a family $\{A_i\}_{i\in I}$ of complex numbers is collinear if $\{A_i\}_{i\in I} \subset \mathbb{R}c$ for some $c \in \mathbb{C}\setminus\{0\}$. In particular, $0 \in \mathbb{C}$ is collinear to any $a \in \mathbb{C}$.*

With this definition we have:

Lemma 4.28. *Let $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}(\mathcal{T})$. Let $\{x^i\}_{i\in\mathbb{Z}}$ be either $\{a^i\}_{i\in\mathbb{Z}}$ or $\{b^i\}_{i\in\mathbb{Z}}$. Recall that δ is defined in* (4.53) *and consider a sequence in $\mathbb{C}$(infinite in both directions) of the form:*

$$\dots, Z(x^{-i}), \dots, Z(x^{-2}), Z(x^{-1}), Z(x^0), Z(\delta), Z(x^1), Z(x^2), Z(x^3), \dots, Z(x^j), \dots. \tag{4.84}$$

Then the following conditions are equivalent: (a) Two of the vectors in this sequence are collinear; (b) The entire sequence is collinear.

Proof. Recall that formula (4.83) holds for any $m, n \in \mathbb{Z}$.

If $Z(x^i)$ and $Z(\delta)$ are collinear for some $i \in \mathbb{Z}$, then $Z(\delta)$ and $Z(x^0)$ are collinear by $Z(x^0) = Z(x^i) + iZ(\delta)$ and (b) follows from the equalities $Z(x^j) = Z(x^0) - jZ(\delta)$, $j \in \mathbb{Z}$.

If $Z(x^i)$ and $Z(x^j)$ are collinear for some $i \neq j$, then by the equality$Z(\delta) = \frac{1}{j-i}(Z(x^i) - Z(x^j))$ we see that $Z(\delta)$ and $Z(x^i)$ are collinear and (b) follows from the considered above case. □

Corollary 4.29. *Let $\{x^i\}_{i\in\mathbb{Z}}$ be either $\{a^i\}_{i\in\mathbb{Z}}$ or $\{b^i\}_{i\in\mathbb{Z}}$. Let two of the vectors in the sequence* (4.84) *be non-collinear. Then:*

(a) *All the vectors in this sequence are non-zero and no two of them are collinear.*

(b) *If for two integers $n \neq m$ holds $\{x^n, x^m\} \subset \sigma^{ss}$, then we have $\phi(x^n) \notin \phi(x^n) + \mathbb{Z}$.*

(c) *The numbers $\{Z(x^j)\}_{j\in\mathbb{Z}}$ are contained in a common connected component of $\mathbb{C}\setminus\mathbb{R}Z(\delta)$.*

(d) *If for two integers $n < m$ we have $\{x^n, x^m\} \subset \sigma^{ss}$ and $\phi(x^m) < \phi(x^n) + 1$, then:*[8]

$$\{Z(x^j)\}_{j\in\mathbb{N}} \subset Z(\delta)^c_+.$$

Proof. (a) and (b) follow from Lemma 4.28, and the axiom (2.11) in [8].

Since $Z(x^0)$ and $Z(\delta)$ are non-colinear, it follows that either $Z(x^0) \in Z(\delta)^c_+$ or $Z(x^0) \in Z(\delta)^c_-$. From formula (4.83) we have $Z(x^j) = Z(x^0) - jZ(\delta)$ for any $j \in \mathbb{Z}$ therefore either $\{Z(x^j)\}_{j\in\mathbb{Z}} \subset Z(\delta)^c_+$ or $\{Z(x^j)\}_{j\in\mathbb{Z}} \subset Z(\delta)^c_-$. Therefore we obtain (c). Now to prove (d), it is enough to show that $Z(x^m) \in Z(\delta)^c_+$. From (b) and Remark 4.20 (a) we get the inequalities $\phi(x^n) < \phi(x^m) < \phi(x^n)+1$. By drawing a picture and taking into account formula (2.11) and the equality $Z(x^n) = Z(x^m) + (m-n)Z(\delta)$, one sees that $\phi(x^n) < \phi(x^m) < \phi(x^n)+1$ is impossible if $Z(x^m) \in Z(\delta)^c_-$. □

Corollary 4.30. *Let $\{x^i\}_{i\in\mathbb{Z}}$ be either the sequence $\{a^i\}_{i\in\mathbb{Z}}$ or the sequence $\{b^i\}_{i\in\mathbb{Z}}$.*

If $Z(\delta) \neq 0$ and $Z(x^q) \in Z(\delta)^c_+$ for some $q \in \mathbb{Z}$, then $\{Z(x^i)\}_{i\in\mathbb{Z}} \subset Z(\delta)^c_+$ and for any $t \in \mathbb{R}$ with $Z(\delta) = |Z(\delta)|\exp(\mathrm{i}\pi t)$ we have:

$$\forall p \in \mathbb{Z} \quad \arg_{(t,t+1)}(Z(x^p)) < \arg_{(t,t+1)}(Z(x^{p+1})) \tag{4.85}$$

$$\lim_{p\to-\infty} \arg_{(t,t+1)}(Z(x^p)) = t; \quad \lim_{p\to+\infty} \arg_{(t,t+1)}(Z(x^p)) = t+1. \tag{4.86}$$

[8] See (4.2) for the notations $Z(\delta)^c_\pm$.

Proof. Since $Z(\delta)$ and $Z(x^q)$ are not collinear by Corollary 4.29 (c) and $Z(x^q) \in Z(\delta)_+^c$ it follows that $\{Z(x^i)\}_{i\in\mathbb{Z}} \subset Z(\delta)_+^c$. The inequalities (4.85) follow from $Z(x^{p+1}), Z(x^p) \in Z(\delta)_+^c$ and $Z(x^{p+1}) = Z(x^p) - Z(\delta)$ (see (4.83)). The formulas in (4.86) follow also from (4.83) and $\{Z(x^i)\}_{i\in\mathbb{Z}} \subset Z(\delta)_+^c$. □

Corollary 4.31. *Let $Z(M)$ and $Z(M')$ be non-zero and have the same direction.*[9] *Let $Z(a^q), Z(b^p) \in Z(\delta)_+^c$ for some $p, q \in \mathbb{Z}$.*

Then $\{Z(a^j), Z(b^j)\}_{j\in\mathbb{Z}} \subset Z(\delta)_+^c$ and for any $t \in \mathbb{R}$ with $Z(\delta) = |Z(\delta)|\exp(\mathrm{i}\pi t)$ the formulas (4.86), (4.85) *hold for both the sequences $\{Z(a^j)\}_{j\in\mathbb{Z}}$ and $\{Z(b^j)\}_{j\in\mathbb{Z}}$.*

Furthermore, for any three integers i, j, m we have:

$$j < m \leq i \quad \Rightarrow \quad \arg_{(t,t+1)}(Z(a^j)) < \arg_{(t,t+1)}(Z(b^m)) < \arg_{(t,t+1)}(Z(a^i)). \tag{4.87}$$

Proof. Corollary 4.30 implies the first part of the conclusion. To show (4.87) we note first that the equalities (4.56) with the notations (4.69) become the following (for any $m \in \mathbb{Z}$):

$$Z(b^m) - Z(M) = Z(a^m) \qquad Z(b^m) + Z(M') = Z(a^{m-1}). \tag{4.88}$$

Since $Z(a^{m-1}), Z(a^m), Z(b^m) \in Z(\delta)_+^c$ for any $m \in \mathbb{Z}$ and $Z(M)$, $Z(M')$ have the same direction as $Z(\delta)$ (recall (4.53)) the equalities (4.88) imply that $\arg_{(t,t+1)}(Z(a^{m-1})) < \arg_{(t,t+1)}(Z(b^m)) < \arg_{(t,t+1)}(Z(a^m))$ for any $m \in \mathbb{Z}$. Now (4.87) follows from (4.85) (applied to the case $\{x^i\}_{i\in\mathbb{Z}} = \{a^i\}_{i\in\mathbb{Z}}$). □

4.5 The union $\mathrm{Stab}(D^b(Q)) = \mathfrak{T}_a^{st} \cup (\_, M, \_) \cup (\_, M', \_) \cup \mathfrak{T}_b^{st}$

In this Section we distinguish some building blocks of $\mathrm{Stab}(D^b(Q))$ and organize them in a manner consistent with the order in which we will glue these blocks in the next sections.

Theorem 2.1 says that for each $\sigma \in \mathrm{Stab}(D^b(Q))$ there exists a σ-triple. This means that (see Corollary 2.34) for each σ there exists an Ext-exceptional triple $\mathcal{E}$ with $\sigma \in \Theta'_{\mathcal{E}}$. From Corollary 4.23 we see that $\mathcal{E}$ is a shift of some of the triples in $\mathfrak{T}$. Recalling the notation (4.8) we get $\mathrm{Stab}(D^b(Q)) = \bigcup_{\mathcal{E}\in\mathfrak{T}} \Theta_{\mathcal{E}}$. Our basic building blocks are $\{\Theta_{\mathcal{E}}\}_{\mathcal{E}\in\mathfrak{T}}$ and by Proposition 4.8 they are contractible.

For a given triple $(A, B, C) \in \mathfrak{T}$ we will denote the open subset $\Theta_{(A,B,C)} \subset \mathrm{Stab}(D^b(Q))$ by (A, B, C), when (we believe that) no confusion may arise. With this convention we can write

$$\mathrm{Stab}(D^b(Q)) = \bigcup_{(A,B,C)\in\mathfrak{T}} (A, B, C). \tag{4.89}$$

[9] We mean that $Z(M) = yZ(M')$ for some $y \in \mathbb{R}_{>0}$. In particular, by $Z(\delta) = Z(M) + Z(M')$ (recall (4.53)) it follows that $Z(\delta)$ is non-zero.

For a given $X \in \{M, M'\}$ *we denote by* $(X, \_, \_)$ *the following open subset of* $\mathrm{Stab}(D^b(Q))$*:*

$$\mathrm{Stab}(D^b(Q)) \supset (X, \_, \_) = \bigcup_{\{(B_0,B_1,B_2)\in\mathfrak{T}: B_0=X\}} (X, B_1, B_2). \tag{4.90}$$

Similarly we define $(\_, X, \_)$ *and* $(\_, \_, X)$. Looking at the list $\mathfrak{T}$ and denoting (see (4.77), (4.78)):

$$\mathfrak{T}_a^{st} = \bigcup_{(A,B,C)\in\mathfrak{T}_a} (A, B, C) \subset \mathrm{Stab}(D^b(Q)); \qquad \mathfrak{T}_b^{st} = \bigcup_{(A,B,C)\in\mathfrak{T}_b} (A, B, C) \subset \mathrm{Stab}(D^b(Q)) \tag{4.91}$$

we can regroup the union (4.89) using (4.79) as follows:

$$\mathrm{Stab}(D^b(Q)) = \mathfrak{T}_a^{st} \cup (\_, M, \_) \cup (\_, M', \_) \cup \mathfrak{T}_b^{st}. \tag{4.92}$$

Remark 4.32. *From the very definition* (4.8) *of* $\Theta_{\mathcal{E}}$ *it is clear that* $\Theta_{\mathcal{E}[\boldsymbol{p}]} = \Theta_{\mathcal{E}}$ *for any triple* $\mathcal{E} = (A, B, C) \in \mathfrak{T}$ *and any* $\boldsymbol{p} \in \mathbb{Z}^3$. *Using the notations explained here, we have* $(A, B, C) = (A[p_0], B[p_1], C[p_2]) \subset \mathrm{Stab}(D^b(Q))$ *for any* $p_0, p_1, p_2 \in \mathbb{Z}$.

4.6 Some contractible subsets of $\mathfrak{T}_a^{st}$ and $\mathfrak{T}_b^{st}$. Proof that $\mathfrak{T}_a^{st} \cap \mathfrak{T}_b^{st} = \emptyset$

This section is devoted to proving that $(X, \_, \_)$ and $(\_, \_, X)$ are contractible subsets of $\mathrm{Stab}(\mathcal{T})$ for any $X \in \{M, M'\}$ and that $\mathfrak{T}_a^{st} \cap \mathfrak{T}_b^{st} = \emptyset$.

We will refer often to some of the formulas in Corollary 4.19. Whenever we discuss $\hom(A, B)$ or $\hom^1(A, B)$ with A, B varying in the symbols M, M', a^m, b^m, $m \in \mathbb{Z}$, we refer to Corollary 4.19.

Putting (4.77), (4.78) in (4.91) we obtain:

$$\mathfrak{T}_a^{st} = (M', \_, \_) \cup (\_, \_, M) \cup \bigcup_{p\in\mathbb{Z}} (a^p, b^{p+1}, a^{p+1}) \tag{4.93}$$

$$\mathfrak{T}_b^{st} = (M, \_, \_) \cup (\_, \_, M') \cup \bigcup_{q\in\mathbb{Z}} (b^q, a^q, b^{q+1}) \tag{4.94}$$

$$(M', \_, \_) = \bigcup_{m\in\mathbb{Z}} (M', a^m, a^{m+1}); \qquad (M, \_, \_) = \bigcup_{m\in\mathbb{Z}} (M, b^m, b^{m+1}) \tag{4.95}$$

$$(\_, \_, M) = \bigcup_{m\in\mathbb{Z}} (a^m, a^{m+1}, M); \qquad (\_, \_, M') = \bigcup_{m\in\mathbb{Z}} (b^m, b^{m+1}, M'). \tag{4.96}$$

We apply Proposition 4.8 to the triples (a^p, b^{p+1}, a^{p+1}) and (b^q, a^q, b^{q+1}). Using Corollary 2.7 (b) and the formulas in Corollary 4.19 we see that in both the cases the coefficients α, β, γ defined in (4.28) are $\alpha = \beta = \gamma = -1$. Thus, we obtain the following formulas for the sets $(a^p, b^{p+1}, a^{p+1}) \subset$

$\mathrm{Stab}(D^b(\mathfrak{T}))$ and $(b^q, a^q, b^{q+1}) \subset \mathrm{Stab}(D^b(\mathfrak{T}))$ in the first and the second column, respectively:

(a^p, b^{p+1}, a^{p+1})	(b^q, a^q, b^{q+1})
$\left\{ a^p, b^{p+1}, a^{p+1} \in \sigma^{ss} : \begin{array}{r} \phi(a^p) < \phi(b^{p+1}) \\ \phi(a^p) + 1 < \phi(a^{p+1}) \\ \phi(b^{p+1}) < \phi(a^{p+1}) \end{array} \right\}$	$\left\{ b^q, a^q, b^{q+1} \in \sigma^{ss} : \begin{array}{r} \phi(b^q) < \phi(a^q) \\ \phi(b^q) + 1 < \phi(b^{q+1}) \\ \phi(a^q) < \phi(b^{q+1}) \end{array} \right\}$

(4.97)

Similarly, applying Proposition 4.8 to the triples in the unions (4.96), (4.95) (with the help of Corollary 4.19 and Corollary 2.7 (b)) we see that $(M', \_, \_) \cup (\_, \_, M)$ and $(M, \_, \_) \cup (\_, \_, M')$ are the unions of the sets in the first and the second column of the following table, respectively (where m, n, i, j vary throughout $\mathbb{Z}$):

$(M', \_, \_) \cup (\_, \_, M)$	$(M, \_, \_) \cup (\_, \_, M')$
$\left\{ M', a^j, a^{j+1} \in \sigma^{ss} : \begin{array}{r} \phi(M') < \phi(a^j) \\ \phi(M') + 1 < \phi(a^{j+1}) \\ \phi(a^j) < \phi(a^{j+1}) \end{array} \right\}$	$\left\{ M, b^n, b^{n+1} \in \sigma^{ss} : \begin{array}{r} \phi(M) < \phi(b^n) \\ \phi(M) + 1 < \phi(b^{n+1}) \\ \phi(b^n) < \phi(b^{n+1}) \end{array} \right\}$
$\left\{ a^m, a^{m+1}, M \in \sigma^{ss} : \begin{array}{r} \phi(a^m) < \phi(a^{m+1}) \\ \phi(a^m) < \phi(M) \\ \phi(a^{m+1}) < \phi(M) + 1 \end{array} \right\}$	$\left\{ b^i, b^{i+1}, M' \in \sigma^{ss} : \begin{array}{r} \phi(b^i) < \phi(b^{i+1}) \\ \phi(b^i) < \phi(M') \\ \phi(b^{i+1}) < \phi(M') + 1 \end{array} \right\}$

(4.98)

For the triples on the first row of table (4.98) we have $\alpha = \beta = \gamma - 1$ and for the triples on the second row we have $\alpha = -1$, $\beta = \gamma = 0$ (one shows this using Corollaries 4.19 and 2.7 (b)).

4.6.1 Proof that $\mathfrak{T}_a^{st} \cap \mathfrak{T}_b^{st} = \emptyset$

Recall that from the axioms of Bridgeland [8] we have $\phi(A[1]) = \phi(A) + 1$ for any $A \in \sigma^{ss}$, and that $A, B \in \sigma^{ss}$ and $\phi(A) > \phi(B)$ imply $\hom(A, B) = 0$. We will use these axioms often implicitely. We start with:

Lemma 4.33. $((M', \_, \_) \cup (\_, \_, M)) \cap ((M, \_, \_) \cup (\_, \_, M')) = \emptyset$.

Proof. Suppose $\sigma \in (a^m, a^{m+1}, M) \cap (M, b^n, b^{n+1})$, then by the table (4.98) we obtain $\hom^1(b^{n+1}, a^m) = 0$ and $\hom(b^{n+1}, a^{m+1}) = 0$, which contradicts (4.72).

Suppose $\sigma \in (a^m, a^{m+1}, M) \cap (b^i, b^{i+1}, M')$, then by $\hom(M', a^m) \neq 0$ (see (4.70)) and table (4.98) we obtain $\phi(b^i) < \phi(M') \leq \phi(a^m) < \phi(M)$, which contradicts $\hom(M, b^i) \neq 0$ (see (4.70)).

Suppose $\sigma \in (M', a^j, a^{j+1}) \cap (M, b^n, b^{n+1})$, then by $\hom^1(a^{j+1}, M) \neq 0$ (see (4.71)) and table (4.98) we obtain $\phi(M') + 1 < \phi(a^{j+1}) \leq \phi(M) + 1 < \phi(b^{n+1})$, which contradicts $\hom^1(b^{n+1}, M') \neq 0$.

Suppose $\sigma \in (M', a^j, a^{j+1}) \cap (b^i, b^{i+1}, M')$, then by the table we have $\hom^1(a^{j+1}, b^i) = 0$, $\hom(a^{j+1}, b^{i+1}) = 0$, which contradicts (4.73). The lemma is proved. □

Lemma 4.34. *For any $p, q \in \mathbb{Z}$ we have $(a^p, b^{p+1}, a^{p+1}) \cap (b^q, a^q, b^{q+1}) = \emptyset$.*

Proof. Let $\sigma \in (a^p, b^{p+1}, a^{p+1})$, then in table (4.97) we see that $a^p, a^{p+1} \in \sigma^{ss}$ and $\phi(a^p) + 1 < \phi\left(a^{p+1}\right)$. Now by Lemma 4.22 we get $a^q \notin \sigma^{ss}$ for $q \notin \{p, p+1\}$, and therefore $\sigma \notin (b^q, a^q, b^{q+1})$ for $q \notin \{p, p+1\}$.

Suppose that $\sigma \in (b^p, a^p, b^{p+1})$, then from table (4.97) we obtain $\phi(b^p) + 1 < \phi(a^p) + 1 < \phi\left(a^{p+1}\right)$, hence $\hom^1(a^{p+1}, b^p) = 0$, which contradicts (4.73).

Suppose that $\sigma \in (b^{p+1}, a^{p+1}, b^{p+2})$, then from table (4.97) we obtain $\phi(a^p) + 1 < \phi\left(a^{p+1}\right) < \phi\left(b^{p+2}\right)$, hence $\hom^1(b^{p+2}, a^p) = 0$, which contradicts (4.72). The lemma is proved. □

Lemma 4.35. *For any $p, q \in \mathbb{Z}$ we have: $((M', \_, \_) \cup (\_, \_, M)) \cap (b^q, a^q, b^{q+1}) = \emptyset$ and $((M, \_, \_) \cup (\_, \_, M')) \cap (a^p, b^{p+1}, a^{p+1}) = \emptyset$.*

Proof. Assume first that $\sigma \in (b^q, a^q, b^{q+1})$, then we can pick out from table (4.97) the inequality:

$$\phi(b^q) + 1 < \phi(b^{q+1}). \tag{4.99}$$

Suppose that $\sigma \in (\_, \_, M)$, then using (4.99), $\hom(M, b^q) \neq 0$ and table (4.98) we see that $\phi(a^m) + 1 < \phi(b^{q+1})$ and $\phi(a^{m+1}) < \phi(b^{q+1})$ for some $m \in \mathbb{Z}$, hence $\hom^1(b^{q+1}, a^m) = \hom(b^{q+1}, a^{m+1}) = 0$, which contradicts (4.72).

Suppose that $\sigma \in (M', \_, \_)$, then using (4.99), $\hom^1(b^{q+1}, M') \neq 0$ and table (4.98) we see that $\phi(b^q) + 1 < \phi(a^{j+1})$ and $\phi(b^q) < \phi(a^j)$ for some $j \in \mathbb{Z}$, hence $\hom^1(a^{j+1}, b^q) = \hom(a^j, b^q) = 0$, which contradicts (4.73). So far we proved that $((\_, \_, M) \cup (M', \_, \_)) \cap (b^q, a^q, b^{q+1}) = \emptyset$.

Assume now that $\sigma \in (a^p, b^{p+1}, a^{p+1})$, then we pick out from table (4.97) the following inequality:

$$\phi(a^p) + 1 < \phi(a^{p+1}). \tag{4.100}$$

Suppose that $\sigma \in (\_, \_, M')$, then using (4.100), $\hom(M', a^p) \neq 0$ and table (4.98) we deduce that $\phi(b^i) + 1 < \phi(a^{p+1})$ and $\phi(b^{i+1}) < \phi(a^{p+1})$ for some $i \in \mathbb{Z}$, hence $\hom^1(a^{p+1}, b^i) = \hom(a^{p+1}, b^{i+1}) = 0$, which contradicts (4.73).

Suppose that $\sigma \in (M, \_, \_)$, then using (4.100), $\hom^1(a^{p+1}, M) \neq 0$ and table (4.98) we get $\phi(a^p) + 1 < \phi(b^{n+1})$ and $\phi(a^p) < \phi(b^n)$ for some $n \in \mathbb{Z}$, hence $\hom^1(b^{n+1}, a^p) = \hom(b^n, a^p) = 0$, which contradicts (4.72). Thus, we proved the second equality as well. □

Lemmas 4.33, 4.34, 4.35, and formulas (4.93), (4.94) imply that $\mathfrak{T}_a^{st} \cap \mathfrak{T}_b^{st} = \emptyset$.

4.6.2 The subsets $(\_, \_, M), (\_, \_, M')$, $(M, \_, \_)$ and $(M', \_, \_)$ are contractible

We start with:

Lemma 4.36. *Let $\{x^i\}_{i\in\mathbb{Z}}$ be either the sequence $\{a^i\}_{i\in\mathbb{Z}}$ or the sequence $\{b^i\}_{i\in\mathbb{Z}}$. If $m > j$ then:*

$$(x^m, x^{m+1}, X) \cap (x^j, x^{j+1}, X) = \left\{ \sigma : \begin{array}{c} x^m \in \sigma^{ss} \\ x^{m+1} \in \sigma^{ss} \\ X \in \sigma^{ss} \end{array}, \begin{array}{c} 0 < \phi(x^{m+1}) - \phi(x^m) < 1 \\ \phi(x^m) < \phi(X) \\ \phi(x^{m+1}) < \phi(X) + 1 \end{array} \right\},$$

where $X = M$ if $\{x^i\}_{i\in\mathbb{Z}} = \{a^i\}_{i\in\mathbb{Z}}$, and $X = M'$ if $\{x^i\}_{i\in\mathbb{Z}} = \{b^i\}_{i\in\mathbb{Z}}$.

In particular, $(x^m, x^{m+1}, X) \cap (x^j, x^{j+1}, X)$ and $(x^m, x^{m+1}, X) \cup (x^j, x^{j+1}, X)$ are contractible.

Proof. We prove first the inclusion $\subset$. Assume that $\sigma \in (x^m, x^{m+1}, X) \cap (x^j, x^{j+1}, X)$ and $m > j$. Then $X, x^{m+1}, x^m, x^{j+1}, x^j$ are all semistable and by table (4.98) we have

$$\begin{array}{cc} \phi(x^m) < \phi(x^{m+1}) & \phi(x^j) < \phi(x^{j+1}) \\ \phi(x^m) < \phi(X) & \phi(x^j) < \phi(X) \\ \phi(x^{m+1}) < \phi(X) + 1 & \phi(x^{j+1}) < \phi(X) + 1 \end{array} . \tag{4.101}$$

By $m > j$ it follows $\hom^1(x^{m+1}, x^j) \neq 0$, hence $\phi(x^{m+1}) \leq \phi(x^j) + 1$ (see Remark 4.20 (b)). On the other hand from the inequalities above we have $\phi(x^j) + 1 < \phi(x^{j+1}) + 1$ and by Remark 4.20 (a) we obtain $\phi(x^{j+1}) + 1 \leq \phi(x^m) + 1$. Thus we obtain $\phi(x^{m+1}) < \phi(x^m) + 1$ and $\subset$ follows.

Next we consider the converse $\supset$. The condition defining the set on the right-hand side is the same as $\sigma \in (x^m, x^{m+1}, X)$ and $\phi(x^m) > \phi(x^{m+1}[-1])$ (see table (4.98)). From Proposition 4.10 (a) it follows that $\mathcal{A} \cap \mathcal{T}_{exc} \subset \sigma^{ss}$, where $\mathcal{A}$ is the extension closure of $(x^m, x^{m+1}[-1])$, hence By Remark 4.21 we have $\{x^{j+1}, x^j\} \subset \sigma^{ss}$. The inequality $0 < \phi(x^{m+1}) - \phi(x^m) < 1$ and (2.11) show that $Z(x^{m+1})$, $Z(x^m)$ are not collinear, hence by Corollary 4.29 (b) we get $\phi(x^{j+1}) \neq \phi(x^j)$. Now by Ramark 4.20 (a) and the incidence $\sigma \in (x^m, x^{m+1}, X)$ we get:
$$\begin{array}{c} \phi(x^j) < \phi(x^{j+1}) \\ \phi(x^j) \leq \phi(x^m) < \phi(X) \\ \phi(x^{j+1}) \leq \phi(x^{m+1}) < \phi(X) + 1 \end{array} .$$
In table (4.98) we see that $\sigma \in (x^j, x^{j+1}, X)$ and the inclusion $\supset$ is proved.

The proved equality implies that $(x^m, x^{m+1}, X) \cap (x^j, x^{j+1}, X)$ is contractible (see the arguments for the proof that (4.40) is contractible in Proposition 4.11). Since (x^m, x^{m+1}, X) and (x^j, x^{j+1}, X) are contractible, by Remark 4.67 it follows that $(x^m, x^{m+1}, X) \cup (x^j, x^{j+1}, X)$ is contractible as well. □

Corollary 4.37. *The subsets $(\_, \_, M)$ and $(\_, \_, M')$ of $\mathrm{Stab}(D^b(Q))$ are contractible.*

Proof. Recalling (4.96) and using the notations of the previous lemma, we reduce to proving that $\bigcup_{j\in\mathbb{Z}}(x^j, x^{j+1}, X)$ is contractible. For a given $m \in \mathbb{Z}$ Lemma 4.36 says that the intersection $(x^m, x^{m+1}, X) \cap (x^j, x^{j+1}, X)$ is contractible for each $j < m$, furthermore this intersection is the same for all $j < m$. Now by induction and using Remark 4.67 one shows that $\bigcup_{k=0}^{n}(x^{m-k}, x^{m-k+1}, X)$ is contractible for any $n \in \mathbb{N}$ and any $m \in \mathbb{Z}$. Using again Remark 4.67 we deduce that $\bigcup_{j\in\mathbb{Z}}(x^j, x^{j+1}, X)$ is contractible. The corollary follows. □

The proof that $(M, \_, \_)$ and $(M', \_, \_)$ are contractible is analogous. We start with:

Lemma 4.38. *Let $\{x^i\}_{i\in\mathbb{Z}}$ be either the sequence $\{a^i\}_{i\in\mathbb{Z}}$ or the sequence $\{b^i\}_{i\in\mathbb{Z}}$. If $m < j$, then:*

$$(X, x^m, x^{m+1}) \cap (X, x^j, x^{j+1}) = \left\{ \sigma : \begin{array}{c} X \in \sigma^{ss} \\ x^m \in \sigma^{ss} \\ x^{m+1} \in \sigma^{ss} \end{array} , \begin{array}{c} \phi(X) < \phi(x^m) \\ \phi(X) + 1 < \phi(x^{m+1}) \\ 0 < \phi(x^{m+1}) - \phi(x^m) < 1 \end{array} \right\} \tag{4.102}$$

where $X = M'$ *if* $\{x^i\}_{i\in\mathbb{Z}} = \{a^i\}_{i\in\mathbb{Z}}$, *and* $X = M$ *if* $\{x^i\}_{i\in\mathbb{Z}} = \{b^i\}_{i\in\mathbb{Z}}$.

In particular, $(X, x^m, x^{m+1}) \cap (X, x^j, x^{j+1})$ *and* $(X, x^m, x^{m+1}) \cup (X, x^j, x^{j+1})$ *are contractible.*

Proof. By table (4.98) we see that the condition defining the set on the right-hand side of (4.102) is the same as $\sigma \in (X, x^m, x^{m+1})$ and $\phi(x^m) > \phi(x^{m+1}[-1])$.

The inclusion $\subset$ follows from table (4.98), $\hom^1(x^{j+1}, x^m) \neq 0$ and Remark 4.20 (a) as follows $\phi(x^{m+1}) \leq \phi(x^j) < \phi(x^{j+1}) \leq \phi(x^m) + 1$.

To prove the converse inclusion $\supset$ in (4.102), assume that $\sigma \in (X, x^m, x^{m+1})$ and $\phi(x^m) > \phi(x^{m+1}[-1])$. From Proposition 4.10 (b) and Remark 4.21 it follows that $x^j, x^{j+1} \in \sigma^{ss}$. Since we have $0 < \phi(x^{m+1}) - \phi(x^m) < 1$, it follows that $Z(x^m)$, $Z(x^{m+1})$ are not collinear, therefore by Corollary 4.29 (b) and Remark 4.20 (a) we obtain $\phi(x^j) < \phi(x^{j+1})$. Since $j > m$, by Remark 4.20 (a) we obtain also $\phi(X) < \phi(x^m) \leq \phi(x^j)$, $\phi(X) + 1 < \phi(x^{m+1}) \leq \phi(x^{j+1})$, hence $\sigma \in (X, x^j, x^{j+1})$.

The proved equality implies that $(X, x^m, x^{m+1}) \cap (X, x^j, x^{j+1})$ is contractible (see the arguments for the proof that (4.40) is contractible in Proposition 4.11). Since (X, x^m, x^{m+1}) and (X, x^j, x^{j+1}) are contractible, by Remark 4.67 it follows that $(X, x^m, x^{m+1}) \cup (X, x^j, x^{j+1})$ is contractible as well. □

Corollary 4.39. *The subsets* $(M, \_, \_), (M', \_, \_) \subset \mathrm{Stab}(D^b(Q))$ *are contractible.*

Proof. Recalling (4.95) and using the notations of the previous lemma, we reduce to proving that $\bigcup_{j\in\mathbb{Z}} (X, x^j, x^{j+1})$ is contractible. From Lemma 4.38 we know that for a given $m \in \mathbb{Z}$ the intersection $(X, x^m, x^{m+1}) \cap (X, x^j, x^{j+1})$ is contractible and it is the same for all $j > m$. Now by induction and using Remark 4.67 one shows that $\bigcup_{k=0}^{n} (X, x^{m+k}, x^{m+k+1})$ is contractible for any $n \in \mathbb{N}$ and any $m \in \mathbb{Z}$. Using again Remark 4.67 we deduce that $\bigcup_{j\in\mathbb{Z}} (X, x^j, x^{j+1})$ is contractible. The corollary follows. □

4.7 The subsets $\mathfrak{T}_a^{st}$ and $\mathfrak{T}_b^{st}$ are contractible

We start by distinguishing some non-intersecting pairs of sets in the union (4.89):

Lemma 4.40. *The unions* $\bigcup_{p\in\mathbb{Z}} (a^p, b^{p+1}, a^{p+1})$ *and* $\bigcup_{p\in\mathbb{Z}} (b^p, a^p, b^{p+1})$ *are disjoint. Furthermore, we have:*

$$p \neq q \;\Rightarrow\; (a^p, b^{p+1}, a^{p+1}) \cap (a^q, a^{q+1}, M) = (a^p, b^{p+1}, a^{p+1}) \cap (M', a^q, a^{q+1}) = \emptyset \tag{4.103}$$

$$p \neq q \;\Rightarrow\; (b^p, a^p, b^{p+1}) \cap (b^q, b^{q+1}, M') = (b^p, a^p, b^{p+1}) \cap (M, b^q, b^{q+1}) = \emptyset. \tag{4.104}$$

Proof. If $\sigma \in (a^p, b^{p+1}, a^{p+1})$, then these exceptional objects are semistable and by table (4.97) we have $\phi(a^p) + 1 < \phi(a^{p+1})$. Now by Lemma 4.22 we see that a^j with $j \notin \{p, p+1\}$ can not be semistable, therefore $\sigma \notin (a^q, b^{q+1}, a^{q+1})$, $\sigma \notin (a^q, a^{q+1}, M)$, and $\sigma \notin (M', a^q, a^{q+1})$ for $q \neq p$.

If $\sigma \in (b^p, a^p, b^{p+1})$, then b^p, a^p, b^{p+1} are semistable and by table (4.97) we have $\phi(b^p) + 1 < \phi(b^{p+1})$. Now by Lemma 4.22 it follows that b^j with $j \notin \{p, p+1\}$ can not be semistable, therefore $\sigma \notin (b^q, a^q, b^{q+1})$, $\sigma \notin (b^q, b^{q+1}, M')$, and $\sigma \notin (M, b^q, b^{q+1})$ for $q \neq p$. □

Now we attach the pairwise non-intersecting contractible blocks $\{(a^p, b^{p+1}, a^{p+1})\}_{p\in\mathbb{Z}}$ to $(\_, \_, M)$ and to $(M', \_, \_)$

Lemma 4.41. *For any $p \in \mathbb{Z}$ the sets $(a^p, b^{p+1}, a^{p+1}) \cap (\_, \_, M)$; $(a^p, b^{p+1}, a^{p+1}) \cap (M', \_, \_)$; $(b^p, a^p, b^{p+1}) \cap (\_, \_, M')$; and $(b^p, a^p, b^{p+1}) \cap (M, \_, \_)$ are non-empty and contractible.*

Proof. From (4.95), (4.96) and (4.103) it follows that: $(a^p, b^{p+1}, a^{p+1}) \cap (\_, \_, M) = (a^p, b^{p+1}, a^{p+1}) \cap (a^p, a^{p+1}, M)$ and $(a^p, b^{p+1}, a^{p+1}) \cap (M', \_, \_) = (a^p, b^{p+1}, a^{p+1}) \cap (M', a^p, a^{p+1})$.

From (4.95), (4.96) and (4.104) it follows that: $(b^p, a^p, b^{p+1}) \cap (\_, \_, M') = (b^p, a^p, b^{p+1}) \cap (b^p, b^{p+1}, M')$ and $(b^p, a^p, b^{p+1}) \cap (M, \_, \_) = (b^p, a^p, b^{p+1}) \cap (M, b^p, b^{p+1})$.

From Proposition 4.11 it follows that $(a^p, b^{p+1}, a^{p+1}) \cap (a^p, a^{p+1}, M)$, $(a^p, b^{p+1}, a^{p+1}) \cap (M', a^p, a^{p+1})$, $(b^p, a^p, b^{p+1}) \cap (b^p, b^{p+1}, M')$, and $(b^p, a^p, b^{p+1}) \cap (M, b^p, b^{p+1})$ are contractible.

The lemma follows. □

Let us denote:

$$Z = (M', \_, \_) \cup \bigcup_{p\in\mathbb{Z}} (a^p, b^{p+1}, a^{p+1}). \tag{4.105}$$

Corollary 4.39 and Lemmas 4.40, 4.41 imply (recall Remark 4.67) that Z is contractible. From (4.93) and (4.96) we see that:

$$\mathfrak{T}_a^{st} = Z \cup (\_, \_, M) = Z \cup \bigcup_{m\in\mathbb{Z}} (a^m, a^{m+1}, M). \tag{4.106}$$

We start to glue the contractible summands in formula (4.106). The first step is:

Lemma 4.42. *The set $(a^m, a^{m+1}, M) \cap Z$ consists of the stability conditions σ for which a^m, a^{m+1}, M are semistable and:*

$$\begin{array}{c} \phi(a^m) - 1 < \phi(a^{m+1}[-1]) < \phi(a^m) \\ \phi(a^m) - 1 < \phi(M[-1]) < \phi(a^m) \\ \arg_{(\phi(a^m)-1, \phi(a^m))}(Z(a^m) - Z(a^{m+1})) > \phi(M) - 1 \end{array} \quad or \quad \begin{array}{c} \phi(M) < \phi(a^{m+1}) \\ \phi\left(a^m\right) < \phi\left(a^{m+1}\right) \\ \phi\left(a^m\right) < \phi\left(M\right) \\ \phi\left(a^{m+1}\right) < \phi\left(M\right) + 1 \end{array}. \tag{4.107}$$

It follows that $(a^m, a^{m+1}, M) \cap Z$ and $(a^m, a^{m+1}, M) \cup Z$ are contractible.

Proof. In (4.103) we have that $(a^m, a^{m+1}, M) \cap (a^j, b^{j+1}, a^{j+1}) = \emptyset$ for $j \neq m$. Therefore (recall (4.105))

$$(a^m, a^{m+1}, M) \cap Z = (a^m, a^{m+1}, M) \cap ((a^m, b^{m+1}, a^{m+1}) \cup (M', \_, \_)). \tag{4.108}$$

We consider first the inclusion $\subset$. Assume that $\sigma \in (a^m, a^{m+1}, M) \cap Z$. Then a^m, a^{m+1}, M are semistable and from table (4.98) we see that

$$\begin{array}{c} \phi\left(a^m\right) < \phi\left(a^{m+1}\right) \\ \phi\left(a^m\right) < \phi\left(M\right) \\ \phi\left(a^{m+1}\right) < \phi\left(M\right) + 1 \end{array} . \tag{4.109}$$

Taking into account (4.108) we consider two cases.

If $\sigma \in (a^m, b^{m+1}, a^{m+1})$, then $b^{m+1} \in \sigma^{ss}$ and in table (4.97) we see that $\phi(b^{m+1}) < \phi(a^{m+1})$. From $\hom(M, b^{m+1}) \neq 0$ (see (4.70)) it follows that $\phi(M) \leq \phi(b^{m+1}) < \phi(a^{m+1})$ and we obtain the second system of inequalities in (4.107).

If $\sigma \in (M', a^j, a^{j+1})$, then $M', a^j, a^{j+1} \in \sigma^{ss}$ and in table (4.98) we see that $\phi(M')+1 < \phi(a^{j+1})$ and $\phi(a^j) < \phi(a^{j+1})$. From $\hom^1(M, M') \neq 0$ it follows that $\phi(M) \leq \phi(M') + 1 < \phi(a^{j+1})$. Since $\phi(a^m) < \phi(M)$, it follows from Remark 4.20 (a) that $m \leq j$.

If $m = j$, then $\phi(M) < \phi(a^{m+1})$ and we obtain the second system of inequalities.

If $m < j$, then we will derive the first system of inequalities in (4.107). Now $\phi(M) < \phi(a^{j+1})$ and $\hom^1(a^{j+1}, a^m) \neq 0$, hence $\phi(a^{m+1}) \leq \phi(a^j) < \phi(a^{j+1}) \leq \phi(a^m) + 1$ and $\phi(M) < \phi(a^{j+1}) \leq \phi(a^m) + 1$. We have also $M' \in \sigma^{ss}$ and by $\hom^1(M, M') \neq 0$ it follows that $\phi(M) \leq \phi(M') + 1$. From $\hom(M', a^m) \neq 0$ it follows $\phi(M') \leq a^m$. These arguments together with (4.109) imply:

$$\begin{array}{c} \phi(a^m) - 1 < \phi(a^{m+1}[-1]) < \phi(a^m); \\ \phi(a^m) - 1 < \phi(M[-1]) \leq \phi(M') \leq \phi(a^m); \qquad \phi(M[-1]) < \phi(a^m). \end{array} \tag{4.110}$$

In (4.82) we have $Z(a^m) - Z(a^{m+1}) = Z(\delta)$, therefore it remains to show that:

$$\arg_{(\phi(a^m)-1, \phi(a^m))}(Z(\delta)) > \phi(M) - 1. \tag{4.111}$$

From the second row of (4.110) and (2.11) we see that $Z(\delta)$ and $Z(M[-1])$ both lie in the half-plane[10] $Z(a^m)^{c}_{-}$. In (4.53) we have aslo $Z(M') = Z(\delta) + Z(M[-1])$, therefore the vector $Z(M')$ is in $Z(a^m)^{c}_{-}$ as well, hence by $Z(M') = Z(\delta) + Z(M[-1])$ it follows that the inequality (4.111) is equivalent to $\phi(M') > \phi(M[-1])$. Therefore it remains to prove that $\phi(M') \neq \phi(M[-1])$. Indeed, on one hand $\phi(M[-1]) = \phi(M')$ implies $\arg_{(\phi(a^m)-1, \phi(a^m))}(Z(\delta)) = \phi(M')$. On the other hand, $\sigma \in (M', a^j, a^{j+1})$, $m < j$ and (4.109) imply $\phi(M') + 1 < \phi(a^{j+1}) \leq \phi(a^m) + 1 < \phi(M) + 1 \leq \phi(M') + 2$. Thus, we see that $\phi(M[-1]) = \phi(M')$ implies $Z(a^{j+1}) \in Z(\delta)^{c}_{-}$. However, from the first inequality in (4.110) and Corollary 4.29 (d) it follows that $Z(a^{j+1}) \in Z(\delta)^{c}_{+}$, which is a contradiction, and (4.111) follows.

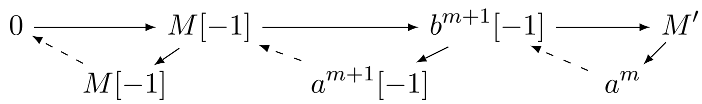
So far we proved that $\sigma \in (a^m, a^{m+1}, M) \cap Z$ implies (4.107). We consider now the converse inclusion.

We assume first that the second system of inequalities in (4.107) holds. In particular $\sigma \in (a^m, a^{m+1}, M)$. By the inequality $\phi(M) < \phi(a^{m+1})$ we can apply Proposition 4.10 (b), hence the

[10] The notation $Z(a^m)^{c}_{-}$ is explained in (4.2).

triangle (4.81) implies that $b^{m+1} \in \sigma^{ss}$, $\phi(M) \leq \phi(b^{m+1}) \leq \phi(a^{m+1})$, and $Z(M) + Z(a^{m+1}) = Z(b^{m+1})$. We have in (4.107) also $\phi(a^{m+1}) - 1 < \phi(M) < \phi(a^{m+1})$ and it follows that $\phi(M) < \phi(b^{m+1}) < \phi(a^{m+1})$. If the inequality $\phi(a^{m+1}) > \phi(a^m) + 1$ holds, then due to $\phi(M) < \phi(b^{m+1}) < \phi(a^{m+1})$ and $\phi(a^m) < \phi(M)$ hold we obtain that $\sigma \in (a^m, b^{m+1}, a^{m+1}) \subset Z$ (see table (4.97)).

Thus, we can assume that $\phi(a^{m+1}[-1]) \leq \phi(a^m)$ and combining with the inequalities $\phi(M[-1]) < \phi(a^{m+1}[-1]) < \phi(M)$, $\phi(a^m) < \phi(M)$ (given in (4.107)) we get $\phi(M[-1]) < \phi(a^{m+1}[-1]) \leq \phi(a^m) < \phi(M)$. It is easy to show now, with the help of Corollaries 4.19 and 2.7 (b), that $(a^m, a^{m+1}[-1], M[-1])$ is a σ-triple (see Definition 4.9). Combining the triangles (4.80) and (4.81) we get the following sequence:


(4.112)

The conditions of Lemma 4.12 (b) are satisfied with the triple $(a^m, a^{m+1}[-1], M[-1])$ and the diagram above. Therefore $M' \in \sigma^{ss}$ and $\phi(M') < \phi(a^m)$.

If $\phi(a^{m+1}[-1]) = \phi(a^m)$, then it follows that $\phi(M') + 1 < \phi(a^{m+1})$, and recalling that we have also $\phi(a^m) < \phi(a^{m+1})$ we see that $\sigma \in (M', a^m, a^{m+1}) \subset Z$ (see table (4.98)).

Therefore we can assume that $\phi(M[-1]) < \phi(a^{m+1}[-1]) < \phi(a^m) < \phi(M)$. We will show in this case that $\sigma \in (M', a^j, a^{j+1})$ for some big enough j. From Proposition 4.3 and Remark 4.21 it follows that $\{a^{j+1}\}_{j\in\mathbb{Z}} \subset \sigma^{ss}$. From $\phi(a^m) < \phi(a^{m+1}) < \phi(a^m) + 1$ and Corollary 4.29 (b) and (d) we deduce that $\phi(a^j) < \phi(a^{j+1})$ and $Z(a^j) \in Z(\delta)_+^c$ for each j (recall also Remark 4.20 (a)). We will prove that for big enough j we have $\phi(M') + 1 < \phi(a^j)$ and then from table (4.98) we obtain $\sigma \in (M', a^j, a^{j+1}) \subset Z$.

Now we have $\phi(a^m[-1]) < \phi(M[-1]) < \phi(a^{m+1}[-1]) < \phi(a^m)$. Since we have $Z(\delta) = Z(a^m) + Z(a^{m+1}[-1])$, we can choose $t \in \mathbb{R}$ so that $t < \phi(a^m) < \phi(M) < \phi(a^{m+1}) < t + 1$ and $Z(\delta) = |Z(\delta)| \exp(\mathrm{i}\pi t)$. Since $\hom^1(a^j, a^m) \neq 0$, $\hom(a^m, a^j) \neq 0$ for $j > m + 1$ and by Corollary 4.29 (b), we have $\phi(a^m) < \phi(a^j) < \phi(a^m) + 1$ for $j > m + 1$. These inequalities together with the incidence $Z(a^j) \in Z(\delta)_+^c$ imply that $\arg_{(t,t+1)}(Z(a^j)) = \phi(a^j)$ for $j > m + 1$ (see Remark 4.2 (c)). Now the formula (4.86) in Corollary 4.30 gives us the following equality:

$$\lim_{j\to\infty} \phi(a^j) = t + 1. \tag{4.113}$$

We showed that $\phi(M') < \phi(a^m)$ (see below (4.112)) and we have also $\phi(a^m) < \phi(M)$. Using $\hom^1(M, M') \neq 0$ we get $\phi(a^m[-1]) < \phi(M') < \phi(a^m)$. We showed also that $t < \phi(a^m) < \phi(M) < \phi(a^{m+1}) < t + 1$. Since $Z(M) + Z(M') = Z(\delta) = |Z(\delta)| \exp(\mathrm{i}\pi t)$, it follows that $Z(M') \in Z(\delta)_-^c$ and $\phi(M') < t$. By (4.113) we get the desired $\phi(a^j) > \phi(M') + 1$ for big j.

So, we proved that the second system of inequalities in (4.107) implies that $\sigma \in (a^m, a^{m+1}, M) \cap Z$. We show now that the first system in (4.107) implies $\sigma \in (a^m, a^{m+1}, M) \cap Z$ as well. Assume that $a^m, a^{m+1}, M \in \sigma^{ss}$ and that these inequalities hold. They contain the inequalities defining (a^m, a^{m+1}, M) (see table (4.98)), therefore we obtain $\sigma \in (a^m, a^{m+1}, M)$ immediately. Furthermore, due to the first two inequalities, the Ext-triple $(a^m, a^{m+1}[-1], M[-1])$ is actually a σ-triple. The conditions of Lemma 4.12 (a) are satisfied with the triple $(a^m, a^{m+1}[-1], M[-1])$ and the diagram (4.112). Therefore $M' \in \sigma^{ss}$ and $\phi(M') < \phi(a^m)$. By $\hom^1(M, M') \neq 0$ we can write $\phi(a^m) - 1 < \phi(M[-1]) \leq \phi(M') < \phi(a^m)$, hence by (2.11) we get $Z(\delta), Z(M[-1]), Z(M') \in Z(a^m)^c_-$. Let us denote $t = \arg_{(\phi(a^m)-1,\phi(a^m))}(Z(\delta))$. The third inequality in (4.107) is the same as $t > \phi(M) - 1$. Combining these arguments with the equality $Z(M') = Z(\delta) + Z(M[-1])$ we write:

$$\phi(a^m[-1]) < \phi(M[-1]) < \phi(M') < t < \phi(a^m). \tag{4.114}$$

We will show that $\sigma \in (M', a^j, a^{j+1})$ for some big enough j. We have $\phi(a^m) < \phi(a^{m+1}) < \phi(a^m)+1$ (the first inequality in (4.107)), which by Proposition 4.3 and Remark 4.21 implies that $\{a^{j+1}\}_{j\in\mathbb{Z}} \subset \sigma^{ss}$, and by Corollary 4.29 (b), (d) implies that $\phi(a^j) < \phi(a^{j+1})$ and $Z(a^j) \in Z(\delta)^c_+$ for each j. Since $\hom^1(a^j, a^m) \neq 0$, $\hom(a^m, a^j) \neq 0$ for $j > m+1$, we have $\phi(a^m) < \phi(a^j) < \phi(a^m) + 1$ for $j > m+1$. These inequalities together with the incidences $Z(a^j) \in Z(\delta)^c_+$, $\phi(a^m) \in (t, t+1)$ imply that $\arg_{(t,t+1)}(Z(a^j)) = \phi(a^j)$ for $j > m+1$ (see Remark 4.2 (c)). Now the formula (4.86) in Corollary 4.30 leads to (4.113) again. Therefore by (4.114) we get $\phi(a^j) > \phi(M')+1$ for big enough j. It follows that $\sigma \in (M', a^j, a^{j+1}) \subset Z$ (see table (4.98)) .

The first part of the lemma is proved. We deduce now that the intersection is contractible. The intersection in question is the same as $(a^m, a^{m+1}[-1], M[-1]) \cap Z$. Let us denote $\mathcal{E} = (a^m, a^{m+1}[-1], M[-1])$. We have a homeomorphism $f_{\mathcal{E}|\Theta_{\mathcal{E}}} : \Theta_{\mathcal{E}} \to f_{\mathcal{E}}(\Theta_{\mathcal{E}})$ (see (4.9), (4.7)). The proved description of $Z \cap \Theta_{\mathcal{E}}$ by the inequalities (4.107) shows that $f_{\mathcal{E}}(Z \cap \Theta_{\mathcal{E}})$ is union of two sets. The first set after permutation of the coordinates in $\mathbb{R}^6$ is the same as the set considered in Corollary 4.64, hence it is also contractible. The second is obviously contractible. Furthermore, one easily verifies that the intersection of these two sets is $\mathbb{R}^3_{>0} \times \{\phi_0 - 1 < \phi_2 < \phi_1 < \phi_0\}$, which is contractible as well. Now by Remark 4.67 it follows that $f_{\mathcal{E}}(Z \cap \Theta_{\mathcal{E}})$ is contractible, therefore $Z \cap \Theta_{\mathcal{E}}$ is contractible as well. Recalling that Z and (a^m, a^{m+1}, M) are contractible and applying again Remark 4.67 we deduce that $(a^m, a^{m+1}, M) \cup Z$ is contractible. The lemma is proved. ☐

Corollary 4.43. *The set $\mathfrak{T}^{st}_a$ is contractible.*

Proof. Recall that $\mathfrak{T}^{st}_a = Z \cup \bigcup_{j\in\mathbb{N}} (a^j, a^{j+1}, M)$ (see (4.106)). We will prove that the set $Z \cup \bigcup_{j=0}^{n} (a^{m-j+1}, a^{m-j}, M)$ is contractible for each $m \in \mathbb{Z}$ and each $n \in \mathbb{N}$. Then the corollary follows from Remark 4.67.

Assume that for some $n \in \mathbb{N}$ the set $Z \cup \bigcup_{j=0}^{n} (a^{m-j+1}, a^{m-j}, M)$ is contractible for each $m \in \mathbb{Z}$. We have proved this statement for $n = 0$ in Lemma 4.42, and now we make induction assumption. Take any $m \in \mathbb{N}$ and consider $Z \cup \bigcup_{j=0}^{n+1} (a^{m-j+1}, a^{m-j}, M) = \left(Z \cup \bigcup_{j=1}^{n+1} (a^{m-j+1}, a^{m-j}, M)\right) \cup (a^m, a^{m+1}, M)$. By the induction assumption $Z \cup \bigcup_{j=1}^{n+1} (a^{m-j+1}, a^{m-j}, M)$ and (a^m, a^{m+1}, M) are contractible. We will show now that the intersection of these sets is contractible as well and then by Remark 4.67 we obtain that the union $Z \cup \bigcup_{j=0}^{n+1} (a^{m-j+1}, a^{m-j}, M)$ is contractible. Indeed, we have

$$\begin{array}{c}\left(Z\cup \bigcup_{j=1}^{n+1}(a^{m-j+1},a^{m-j},M)\right)\cap (a^m,a^{m+1},M) = \\ \big((a^m,a^{m+1},M)\cap Z\big)\cup \left((a^m,a^{m+1},M)\cap \bigcup_{j=1}^{n+1}(a^{m-j+1},a^{m-j},M)\right).\end{array} \tag{4.115}$$

Using Lemmas 4.42 and 4.36 we deduce that the considered intersection consists of the stability conditions for which a^m, a^{m+1}, M are semi-stable and some of the two systems of inequalities in (4.107) or the system $\begin{array}{c}\phi(a^m)<\phi(a^{m+1})<\phi(a^m)+1\\ \phi(a^m)<\phi(M)\\ \phi(a^{m+1})<\phi(M)+1\end{array}$ holds. Since the first system in (4.107) implies the last system we deduce that the intersection (4.115) is described by the inequalities:

$$\begin{array}{c}\phi(a^m)<\phi(a^{m+1})<\phi(a^m)+1\\ \phi(a^m)<\phi(M)\\ \phi(a^{m+1})<\phi(M)+1\end{array} \quad \text{or} \quad \begin{array}{c}\phi\left(a^m\right)<\phi\left(a^{m+1}\right)\\ \phi\left(a^m\right)<\phi\left(M\right)\\ \phi(M)<\phi\left(a^{m+1}\right)<\phi\left(M\right)+1\end{array}. \tag{4.116}$$

Now analogous arguments as in the last paragraph of the proof of Lemma 4.42 clarify that the intersection (4.115) is contractible. The corollary follows. □

We pass to the proof that $\mathfrak{T}_b^{st}$ is contractible. Let us denote

$$W=(M,\_,\_)\cup \bigcup_{p\in\mathbb{Z}}(b^p,a^p,a^{p+1}). \tag{4.117}$$

Corollary 4.39 and Lemmas 4.40, 4.41 imply (recall Remark 4.67) that W is contractible. From (4.94) and (4.96) we see that:

$$\mathfrak{T}_b^{st}=W\cup(\_,\_,M')=W\cup\bigcup_{m\in\mathbb{Z}}(b^m,b^{m+1},M'). \tag{4.118}$$

The proof of the next Lemma 4.44 is analogous to the proof of Lemma 4.42):

Lemma 4.44. *The set $(b^m,b^{m+1},M')\cap W$ consists of the stability conditions σ for which b^m,b^{m+1},M' are semistable and*

$$\begin{array}{c}\phi(b^m)-1<\phi(b^{m+1}[-1])<\phi(b^m)\\ \phi(b^m)-1<\phi(M'[-1])<\phi(b^m)\\ \arg_{(\phi(b^m)-1,\phi(a^m))}(Z(b^m)-Z(b^{m+1}))>\phi(M')-1\end{array} \quad \textit{or} \quad \begin{array}{c}\phi(M')<\phi(b^{m+1})\\ \phi\left(b^m\right)<\phi\left(b^{m+1}\right)\\ \phi\left(b^m\right)<\phi\left(M'\right)\\ \phi\left(b^{m+1}\right)<\phi\left(M'\right)+1\end{array}. \tag{4.119}$$

It follows that $(b^m,b^{m+1},M')\cap W$ and $(b^m,b^{m+1},M')\cup W$ are contractible.

Proof. In (4.104) we have that $(b^m, b^{m+1}, M') \cap (b^j, a^j, b^{j+1}) = \emptyset$ for $j \neq m$. Therefore (recall (4.117))

$$(b^m, b^{m+1}, M') \cap W = (b^m, b^{m+1}, M') \cap ((b^m, a^m, b^{m+1}) \cup (M, \_, \_)). \tag{4.120}$$

We consider first the inclusion $\subset$. Assume that $\sigma \in (b^m, b^{m+1}, M') \cap W$. Then b^m, b^{m+1}, M' are semistable and from table (4.98) we see that

$$\begin{array}{c} \phi\left(b^m\right) < \phi\left(b^{m+1}\right) \\ \phi\left(b^m\right) < \phi\left(M'\right) \\ \phi\left(b^{m+1}\right) < \phi\left(M'\right) + 1 \end{array}. \tag{4.121}$$

Taking into account (4.120) we consider two cases.

If $\sigma \in (b^m, a^m, b^{m+1})$, then $a^m \in \sigma^{ss}$ and $\phi(a^m) < \phi(b^{m+1})$ (see table (4.97)). From $\hom(M', a^m) \neq 0$ (see (4.70)) it follows $\phi(M') \leq \phi(a^m) < \phi(b^{m+1})$ and we get the second system in (4.119).

If $\sigma \in (M, b^j, b^{j+1})$, then $M, b^j, b^{j+1} \in \sigma^{ss}$ and in table (4.98) we see that $\phi(M) + 1 < \phi(b^{j+1})$ and $\phi(b^j) < \phi(b^{j+1})$. From $\hom^1(M', M) \neq 0$ it follows that $\phi(M') \leq \phi(M) + 1 < \phi(b^{j+1})$. Since we have $\phi(b^m) < \phi(M')$, Remark 4.20 (a) implies that $m \leq j$.

If $m = j$, then $\phi(M') < \phi(b^{m+1})$ and we obtain the second system of inequalities.

If $m < j$, then we will derive the first system of inequalities in (4.119). Now $\phi(M') < \phi(b^{j+1})$ and $\hom^1(b^{j+1}, b^m) \neq 0$, hence $\phi(b^{m+1}) \leq \phi(b^j) < \phi(b^{j+1}) \leq \phi(b^m) + 1$, $\phi(M') < \phi(b^{j+1}) \leq \phi(b^m) + 1$. We have also $M \in \sigma^{ss}$ and by $\hom^1(M', M) \neq 0$ and $\hom(M, b^m) \neq 0$ it follows that $\phi(M') \leq \phi(M) + 1$ and $\phi(M) \leq b^m$. These arguments together with (4.121) imply

$$\begin{array}{c} \phi(b^m) - 1 < \phi(b^{m+1}[-1]) < \phi(b^m) \\ \phi(b^m) - 1 < \phi(M'[-1]) \leq \phi(M) \leq \phi(b^m); \qquad \phi(M'[-1]) < \phi(b^m) \end{array}. \tag{4.122}$$

Due to (4.82), to show the first system in (4.119) it remains to derive the following inequality:

$$\arg_{(\phi(b^m)-1, \phi(b^m))}(Z(\delta)) > \phi(M') - 1. \tag{4.123}$$

From (4.122) it follows that $Z(\delta)$ and $Z(M'[-1])$ both lie in the half-plane[11] $Z(b^m)_{-}^{c}$. In (4.53) we have aslo $Z(M) = Z(\delta) + Z(M'[-1])$, therefore the vector $Z(M)$ is in $Z(b^m)_{-}^{c}$ as well. Now the equality $Z(M) = Z(\delta) + Z(M'[-1])$ implies that (4.123) is equivalent to $\phi(M) > \phi(M'[-1])$. Hence we have to prove that $\phi(M) \neq \phi(M'[-1])$. Indeed, on one hand $\phi(M'[-1]) = \phi(M)$ implies $\arg_{(\phi(b^m)-1, \phi(b^m))}(Z(\delta)) = \phi(M)$. On the other hand, $\sigma \in (M, b^j, b^{j+1})$, $m < j$ and (4.121) imply $\phi(M) + 1 < \phi(b^{j+1}) \leq \phi(b^m) + 1 < \phi(M') + 1 \leq \phi(M) + 2$. Thus, we see that $\phi(M'[-1]) = \phi(M)$ implies $Z(b^{j+1}) \in Z(\delta)_{-}^{c}$. However, from the first inequality in (4.122) and Corollary 4.29 (d) it follows that $Z(b^{j+1}) \in Z(\delta)_{+}^{c}$, which is a contradiction, and (4.123) follows.

So far we proved the inclusion $\subset$. We consider now the converse inclusion $\supset$.

[11] The notation $Z(b^m)_{-}^{c}$ is explained in (4.2).

We assume first that the second system of inequalities in (4.119) holds. In particular $\sigma \in (b^m, b^{m+1}, M')$. By the inequality $\phi(M') < \phi(b^{m+1})$ we can apply Proposition 4.10 (b), hence the short exact sequence (4.80) implies that $a^m \in \sigma^{ss}$ and $Z(M') + Z(b^{m+1}) = Z(a^m)$. We have also $\phi(b^{m+1}) - 1 < \phi(M') < \phi(b^{m+1})$, and it follows that $\phi(M') < \phi(a^m) < \phi(b^{m+1})$. If the inequality $\phi(b^{m+1}) > \phi(b^m) + 1$ holds, then recalling that $\phi(b^m) < \phi(M')$ we obtain that $\sigma \in (b^m, a^m, b^{m+1}) \subset W$ (see table (4.97)).

Therefore we reduce to the inequality $\phi(b^{m+1}[-1]) \leq \phi(b^m)$. Combining with $\phi(M'[-1]) < \phi(b^{m+1}[-1]) < \phi(M')$ and $\phi(b^m) < \phi(M')$, we can write $\phi(M'[-1]) < \phi(b^{m+1}[-1]) \leq \phi(b^m) < \phi(M')$ and then $(b^m, b^{m+1}[-1], M'[-1])$ is a σ-triple (see Definition 4.9). Combining the triangles (4.80) and (4.81) we obtain the following sequence of triangles in $\mathcal{T}$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M'[-1] & \longrightarrow & a^m[-1] & \longrightarrow & M \\ & \nwarrow \swarrow & & \nwarrow \swarrow & & \nwarrow \swarrow & \\ & M'[-1] & & b^{m+1}[-1] & & b^m & \end{array} . \qquad (4.124)$$

The conditions of Lemma 4.12 (b) are satisfied with the triple $(b^m, b^{m+1}[-1], M'[-1])$ and the diagram above. Therefore $M \in \sigma^{ss}$ and $\phi(M) < \phi(b^m)$.

If $\phi(b^{m+1}[-1]) = \phi(b^m)$, then we have also $\phi(M) + 1 < \phi(b^{m+1})$, and recalling that we have also $\phi(b^m) < \phi(b^{m+1})$ we get $\sigma \in (M, b^m, b^{m+1}) \subset W$ (see table (4.98)).

Therefore we can assume that $\phi(M'[-1]) < \phi(b^{m+1}[-1]) < \phi(b^m) < \phi(M')$. In this case we will obtain $\sigma \in (M, b^j, b^{j+1})$ for some big j and then by $(M, b^j, b^{j+1}) \subset W$ we get $\sigma \in W$. Proposition 4.3 and Remark 4.21 ensure that $\{b^{j+1}\}_{j\in\mathbb{Z}} \subset \sigma^{ss}$. From $\phi(b^m) < \phi(b^{m+1}) < \phi(b^m) + 1$ and Corollary 4.29 (b) and (d) we see that $\phi(b^j) < \phi(b^{j+1})$ and $Z(b^j) \in Z(\delta)_+^{\mathsf{c}}$ for each j. Now to show that $\sigma \in (M, b^j, b^{j+1})$ it is enough to derive $\phi(M) + 1 < \phi(b^j)$ for big enough j (see table (4.98)).

Since we have $\phi(b^m[-1]) < \phi(M'[-1]) < \phi(b^{m+1}[-1]) < \phi(b^m)$ and $Z(\delta) = Z(b^m) + Z(b^{m+1}[-1])$, we can choose $t \in \mathbb{R}$ so that $t < \phi(b^m) < \phi(M') < \phi(b^{m+1}) < t+1$ and $Z(\delta) = |Z(\delta)|\exp(\mathrm{i}\pi t)$. Since $\hom^1(b^j, b^m) \neq 0$, $\hom(b^m, b^j) \neq 0$ for $j > m+1$ and by Corollary 4.29 (b), we have $\phi(b^m) < \phi(b^j) < \phi(b^m) + 1$ for $j > m+1$. These inequalities together with the incidence $Z(b^j) \in Z(\delta)_+^{\mathsf{c}}$ imply that $\arg_{(t,t+1)}(Z(b^j)) = \phi(b^j)$ for $j > m+1$ (see Remark 4.2 (c)). The formula (4.86) in Corollary 4.30 gives us the following:

$$\lim_{j\to\infty} \phi(b^j) = t + 1. \qquad (4.125)$$

We showed that $\phi(M) < \phi(b^m)$ (see below (4.124)) and we have also $\phi(b^m) < \phi(M')$. From $\hom^1(M', M) \neq 0$ we derive $\phi(b^m[-1]) < \phi(M) < \phi(b^m)$. We showed also that $t < \phi(b^m) < \phi(M') < \phi(b^{m+1}) < t+1$. Since $Z(M) + Z(M') = Z(\delta) = |Z(\delta)|\exp(\mathrm{i}\pi t)$, it follows that $Z(M) \in Z(\delta)_-^{\mathsf{c}}$ and $\phi(M) < t$. Now (4.125) ensures that $\phi(b^j) > \phi(M) + 1$ for big enough j. So far we proved that the second system of inequalities in (4.119) implies $\sigma \in (b^m, b^{m+1}, M') \cap W$.

We pass to the first system of inequalities in (4.119). So assume that $b^m, b^{m+1}, M' \in \sigma^{ss}$ and that these inequalities hold. They contain the inequalities defining (b^m, b^{m+1}, M') (see table (4.98)), hence $\sigma \in (b^m, b^{m+1}, M')$. Furthermore, due to the first two inequalities , the triple

$(b^m, b^{m+1}[-1], M'[-1])$ is a σ-triple and the conditions of Lemma 4.12 (a) are satisfied with this triple and the diagram (4.124). Therefore $M \in \sigma^{ss}$ and $\phi(M) < \phi(b^m)$. By $\hom^1(M', M) \neq 0$ we can write $\phi(b^m) - 1 < \phi(M'[-1]) \leq \phi(M) < \phi(b^m)$ (we use also (4.119)), hence $Z(\delta), Z(M'[-1]), Z(M) \in Z(b^m)_-^c$. Let us denote $t = \arg_{(\phi(b^m)-1,\phi(b^m))}(Z(\delta))$. The third inequality in (4.119) is the same as $t > \phi(M') - 1$. Combining these arguments with the equality $Z(M) = Z(\delta) + Z(M'[-1])$ we deduce that:

$$\phi(b^m[-1]) < \phi(M'[-1]) < \phi(M) < t < \phi(b^m). \tag{4.126}$$

We will deduce that $\sigma \in (M, b^j, b^{j+1})$ for some big enough j. We have $\phi(b^m) < \phi(b^{m+1}) < \phi(b^m)+1$ (the first inequality in (4.119)), which by Proposition 4.3 and Remark 4.21 implies that $\{b^{j+1}\}_{j\in\mathbb{Z}} \subset \sigma^{ss}$, and by Corollary 4.29 (b), (d) implies that $\phi(b^j) < \phi(b^{j+1})$ and $Z(b^j) \in Z(\delta)_+^c$ for each j. Using Remark 4.20 one easily shows that $\phi(b^m) < \phi(b^j) < \phi(b^m) + 1$ for $j > m + 1$. These inequalities together with the incidences $Z(b^j) \in Z(\delta)_+^c$, $\phi(b^m) \in (t, t+1)$ imply that $\arg_{(t,t+1)}(Z(b^j)) = \phi(b^j)$ for $j > m + 1$ (see Remark 4.2 (c)). The formula (4.86) in Corollary 4.30 leads to (4.125) again. Now (4.126) implies that $\phi(b^j) > \phi(M)+1$ for big j, hence $\sigma \in (M, b^j, b^{j+1}) \subset W$ (see table (4.98)).

The arguments showing that $(b^m, b^{m+1}, M') \cap W$ and $(b^m, b^{m+1}, M') \cup W$ are contractible are as in the last paragraph of the proof of Lemma 4.42. The lemma is proved. □

Corollary 4.45. *The set $\mathfrak{T}_b^{st}$ is contractible.*

Proof. Recall that $\mathfrak{T}_b^{st} = W \cup \bigcup_{j\in\mathbb{N}}(b^j, b^{j+1}, M')$ (see (4.118)). Using Lemmas 4.44 and 4.36 one deduces by induction that $W \cup \bigcup_{j=0}^{n}(b^{m-j+1}, b^{m-j}, M')$ is contractible for each $m \in \mathbb{Z}$ and each $n \in \mathbb{N}$ (see the proof of Corollary 4.43 for details). Then the corollary follows from Remark 4.67. □

4.8 Connecting $\mathfrak{T}_a^{st}$ and $\mathfrak{T}_b^{st}$ by $(\_, M, \_)$ and $(\_, M', \_)$

As it follows from the union (4.92), in order to prove Theorem 4.1 it remains to connect the contractible non-intersecting pieces $\mathfrak{T}_a^{st}$, $\mathfrak{T}_b^{st}$ by $(\_, M, \_) \cup (\_, M', \_)$, and to show that in this procedure the contractibility is preserved. We describe first the building blocks of $(\_, M, \_)$ and $(\_, M', \_)$ by Proposition 4.8:

From the list of triples $\mathfrak{T}$ given in Corollary 4.23 we get (see also (4.90)):

$$(\_, M\,\_) = \bigcup_{q\in\mathbb{Z}}(a^q, M, b^{q+1}) \qquad (\_, M'\,\_) = \bigcup_{q\in\mathbb{Z}}(b^q, M', a^q). \tag{4.127}$$

We apply Proposition 4.8 to the triples (a^p, M, b^{p+1}) and (b^q, M', a^q). Using Corollaries 4.19 and 2.7 (b) one computes the coefficients α, β, γ defined in (4.28), which results in $\alpha = 0$, $\beta = \gamma = -1$ in both the cases. Thus we obtain the formulas in the first and the second column of table (4.128) for the contractible subsets $(a^p, M, b^{p+1}) \subset \mathrm{Stab}(D^b(\mathcal{T}))$ and $(b^q, M', a^q) \subset \mathrm{Stab}(D^b(\mathcal{T}))$, respectively:

(a^p, M, b^{p+1})	(b^q, M', a^q)
$\left\{ a^p, M, b^{p+1} \in \sigma^{ss} : \begin{array}{r} \phi(a^p) < \phi(M) + 1 \\ \phi(a^p) < \phi(b^{p+1}) \\ \phi(M) < \phi(b^{p+1}) \end{array} \right\}$	$\left\{ b^q, M', a^q \in \sigma^{ss} : \begin{array}{r} \phi(b^q) < \phi(M') + 1 \\ \phi(b^q) < \phi(a^q) \\ \phi(M') < \phi(a^q) \end{array} \right\}$

. (4.128)

Remark 4.46. $(a^p, M, b^{p+1}[-1])$, $(b^q, M', a^q[-1])$ *are Ext-exceptional triples (satisfy* (a) *in Def. 4.9).*

In some steps of this section, when we need to show that certain exceptional objects are semistable, the tools in Section 4.3 are not efficient enough. For these cases we prove Lemmas 4.47 and 4.48 below. The relation $R \dashrightarrow (S, E)$ between a σ-regular object R and an exceptional pair generated by it (introduced in Chapter 2) is utilized in the proof of these lemmas.

Lemma 4.47. *Let $a^m \notin \sigma^{ss}$ and $t = \phi_-(a^m)$, then one of the following holds:*
(a) *$a^j \in \sigma^{ss}$ for some $j < m-1$ and $t = \phi(a^j)+1$;* **(b)** *$a^j \in \sigma^{ss}$ for some $m < j$ and $t = \phi(a^j)$;*
(c) *$b^j \in \sigma^{ss}$ for some $j < m$ and $t = \phi(b^j) + 1$;* **(d)** *$b^j \in \sigma^{ss}$ for some $m < j$ and $t = \phi(b^j)$;*
(e) *$M \in \sigma^{ss}$ and $t = \phi(M) + 1$.*

Proof. Recall that any $X \in \{E_i^j : j \in \mathbb{N}, 1 \leq i \leq 4\}$ is a trivially coupling object (see after Lemma 2.108). Since $a^m[k] \in \{E_i^j : j \in \mathbb{N}, 1 \leq i \leq 4\}$, where $k \in \{0, -1\}$, from $a^m \notin \sigma^{ss}$ and Lemma 2.49 it follows that $a^m[k]$ is a σ-regular object, hence a^m is a σ-regular object. Therefore we have $R \dashrightarrow (S, E)$ for some exceptional pair (S, E) (see Section 2.5). We will need the following two properties of the exceptional object S. The first is $S \in \sigma^{ss}$, $\phi(S) = \phi_-(a^m)$ (see formula (2.39)). The second property is $\hom(a^m, S) \neq 0$, which follows from (c) after formula (2.17) and the way S was chosen (see Definition 2.45). Recall that there exists at most one nonzero element in the family $\{\hom^k(a^m, X)\}_{k\in\mathbb{Z}}$ for any $X \in \mathcal{T}_{exc}$ (Corollary 2.7 (b)). By Remark 4.18 we have $S \in \{a^j[k], b^j[k] : j \in \mathbb{Z}, k \in \mathbb{Z}\} \cup \{M[k], M'[k]; k \in \mathbb{Z}\}$. Obviously $S \neq a^m[k]$ (since $a^m \notin \sigma^{ss}$ and $S \in \sigma^{ss}$).

Now we will use the property $\hom(a^m, S) \neq 0$ and Corollary 4.19 to prove the lemma. By $\hom^*(a^m, M') = 0$ (see (4.70)) we exclude also the case $S = M'[k]$. It remains to consider the following cases (one of them must appear):

If $S = a^j[k]$ for some $j \neq m$ and $k \in \mathbb{Z}$, then by (4.74) we see that either $j < m-1$ and $k = 1$, or $m < j$ and $k = 0$.

If $S = b^j[k]$ for some $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$, then by (4.73) it follows that either $j < m$ and $k = 1$, or $m < j$ and $k = 0$.

If $S = M[k]$ for some $k \in \mathbb{Z}$, then by (4.71) we get $k = 1$. The lemma follows. □

Lemma 4.48. *Let $b^m \notin \sigma^{ss}$ and $t = \phi_-(b^m)$, then one of the following holds:*

(a) $a^j \in \sigma^{ss}$ *for some* $j < m-1$ *and* $t = \phi(a^j)+1$; **(b)** $a^j \in \sigma^{ss}$ *for some* $m \leq j$ *and* $t = \phi(a^j)$;
(c) $b^j \in \sigma^{ss}$ *for some* $j < m-1$ *and* $t = \phi(b^j)+1$; **(d)** $b^j \in \sigma^{ss}$ *for some* $m < j$ *and* $t = \phi(b^j)$;
(e) $M' \in \sigma^{ss}$ *and* $t = \phi(M')+1$.

Proof. By the same arguments as in the proof of Lemma 4.47 one shows that $\hom(b^m, S) \neq 0$ and $\phi(S) = t$ for some $S \in \sigma^{ss} \cap \left(\{a^j[k], M, M' : j \in \mathbb{Z}, k \in \mathbb{Z}\} \cup \{b^j[k]; k \in \mathbb{Z}, j \in \mathbb{Z}, j \neq m\}\right)$. Now we will use Corollaries 4.19 and 2.7 (b). By $\hom^*(b^m, M) = 0$ (see (4.71)) we exclude the case $S = M[k]$. It remains to consider the following cases (one of them must appear):

If $S = a^j[k]$ for some $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$, then by (4.72) we see that either $j < m-1$ and $k = 1$, or $m \leq j$ and $k = 0$.

If $S = b^j[k]$ for some $j \neq m$ and $k \in \mathbb{Z}$, then by (4.75) it follows that either $j < m-1$ and $k = 1$, or $m < j$ and $k = 0$.

If $S = M'[k]$ for some $k \in \mathbb{Z}$, then by (4.71) we get $k = 1$. The lemma follows. □

Lemmas 4.49 and 4.50 put together the arguments which ensure the semi-stability necessary for the analysis of the intersections $(a^p, M, b^{p+1}) \cap \mathfrak{T}^{st}_{a/b}$ and $(a^p, M, b^{p+1}) \cap (\mathfrak{T}^{st}_a \cup (a^q, M, b^{q+1}) \cup \mathfrak{T}^{st}_b)$.

Lemma 4.49. *Let* $\sigma \in (a^p, M, b^{p+1})$ *and let the following inequality hold:*

$$\phi\left(b^{p+1}\right) - 1 < \phi\left(M\right) < \phi\left(b^{p+1}\right) \ . \tag{4.129}$$

Then we have the following:

(a) $a^{p+1} \in \sigma^{ss}$ *and* $\phi(b^{p+1}) - 1 < \phi(a^{p+1}) - 1 < \phi(M)$.

(b) *If in addition to* (4.129) *we have* $\phi(a^p) < \phi(M)$, *then* $\sigma \in (a^p, a^{p+1}, M)$.

(c) *If in addition to* (4.129) *we have*

$$\phi\left(b^{p+1}\right) - 1 < \phi\left(a^p\right) < \phi\left(b^{p+1}\right), \tag{4.130}$$

then $M' \in \sigma^{ss}$ *and* $\phi(b^{p+1}) - 1 < \phi(M') = \arg_{(\phi(b^{p+1})-1, \phi(b^{p+1}))}(Z(a^p) - Z(b^{p+1})) < \phi(a^p)$.

(d) *If* (4.129), (4.130) *hold and* $\phi(M') < \phi(M)$, *then* $\sigma \in (a^j, a^{j+1}, M)$ *for some* $j \in \mathbb{Z}$.

Proof. (a) We apply Proposition 4.10 (b) to the triple (a^p, M, b^{p+1}) and since $a^{p+1}[-1]$ is in the extension closure of $M, b^{p+1}[-1]$ (by (4.81)) it follows that $a^{p+1} \in \sigma^{ss}$. The inequality $\phi(b^{p+1}) - 1 < \phi(a^{p+1}) - 1 < \phi(M)$ follows from the given inequality (4.129) and $Z(a^{p+1}[-1]) = Z(M) + Z(b^{p+1}[-1])$.

(b) From the given inequalities it follows that $\phi(a^p) < \phi(b^{p+1})$. We have also $\phi(b^{p+1}) < \phi(a^{p+1}) < \phi(M)+1$ from (a). Therefore we obtain the inequalities $\phi(a^p) < \phi(a^{p+1})$, $\phi(a^p) < \phi(M)$, $\phi(a^{p+1}) < \phi(M)+1$, which means that $\sigma \in (a^p, a^{p+1}, M)$ (see table (4.98)).

(c) Follows from Lemma 4.13 applied to the Ext-triple $(a^p, M, b^{p+1}[-1])$ and the triangle (4.80).

(d) Now by the given inequalities and (c) we have $\phi(b^{p+1}) - 1 < \phi(M') < \phi(M) < \phi(b^{p+1})$. Since we have $Z(\delta) = Z(M') + Z(M)$, we can choose $t \in \mathbb{R}$ with $Z(\delta) = |Z(\delta)| \exp(\mathrm{i}\pi t)$ and $\phi(M') < t < \phi(M) < \phi(b^{p+1}) < t+1$. If $\phi(a^p) < \phi(M)$, then (d) follows from (b).

So let $\phi(M) \leq \phi(a^p)$. Since we have also $\phi(a^p) < \phi(b^{p+1})$, we obtain $t < \phi(M) \leq \phi(a^p) < \phi(b^{p+1}) < t+1$. Now Corollary 4.30 ensures that $\{Z(a^j), Z(b^j)\}_{j\in\mathbb{Z}} \subset Z(\delta)_+^c$ and that (4.86), (4.85) hold for both the sequences $\{Z(a^j)\}_{j\in\mathbb{Z}}$ and $\{Z(b^j)\}_{j\in\mathbb{Z}}$. From (a) we see that $\phi(b^{p+1}) < \phi(a^{p+1}) < \phi(M)+1$, hence $t < \phi(a^{p+1}) < t+2$, which combined with $Z(a^{p+1}) \in Z(\delta)_+^c$ implies that $\phi(a^{p+1}) < t+1$. Thus we obtain the inequalities $t < \phi(M) \leq \phi(a^p) < \phi(b^{p+1}) < \phi(a^{p+1}) < t+1$.

From (4.86) we see that there exists $N \in \mathbb{Z}$, $N < p$ such that $t < \arg_{(t,t+1)}(Z(a^j)) < \phi(M)$ for $j < N$. We will prove below that $a^j \in \sigma^{ss}$ for $j < N$. Then (d) follows. Indeed, assume that $a^j \in \sigma^{ss}$ for each $j < N$. Then by (4.74) and Corollary 4.29 (a) it follows that $\phi(a^{p+1}) - 1 < \phi(a^j) < \phi(a^{p+1})$ for $j < N$, therefore $t - 1 < \phi(a^j) < t+1$, which combined with $Z(a^j) \in Z(\delta)_+^c$ implies that $\arg_{(t,t+1)}(Z(a^j)) = \phi(a^j)$. Putting the last equality in (4.85) and in $\arg_{(t,t+1)}(Z(a^j)) < \phi(M)$ we get $\phi(a^{j-1}) < \phi(a^j) < \phi(M)$ which by table (4.98) implies that $\sigma \in (a^{j-1}, a^j, M)$.

Suppose $a^j \not\in \sigma^{ss}$ for some $j < N$. From Remark 4.21 we know that a^j is in the extension closure of a^p, $a^{p+1}[-1]$. It follows that $a^j \in \mathcal{P}[\phi(a^{p+1}) - 1, \phi(a^p)]$ and then $\phi(a^{p+1}) - 1 \leq \phi_-(a^j)$ (recall the paragraph after (2.9)). We will use Lemma 4.47 and demonstrate that each case (of the five cases given there) leads to a contradiction. We fist derive (4.131). The inequalities $\phi(a^p) - 1 < \phi(a^{p+1}) - 1 < \phi(M) \leq \phi(a^p)$ can be used due to the previous steps. Therefore we have $a^j \in \mathcal{P}[\phi(a^{p+1}) - 1, \phi(a^p)] \subset \mathcal{P}(\phi(a^p) - 1, \phi(a^p)]$. Using $\phi(a^p) \in (t, t+1)$, $Z(a^j) \in Z(\delta)_+^c$, and Remark 4.2 (c) we get: $\arg_{(\phi(a^p)-1,\phi(a^p)]}(Z(a^j)) = \arg_{(t,t+1)}(Z(a^j))$. Now by Remark 4.2 (a) we get $\phi_-(a^j) < \arg_{(t,t+1)}(Z(a^j))$ and by our choice of N we have $\arg_{(t,t+1)}(Z(a^j)) < \phi(M)$. We combine these facts in the following inequalities:

$$\phi(a^{p+1}) - 1 \leq \phi_-(a^j) < \arg_{(t,t+1)}(Z(a^j)) < \phi(M) \leq \phi(a^p) < \phi(a^{p+1}). \tag{4.131}$$

One of the cases in Lemma 4.47 must appear. In case (a) we have $\phi_-(a^j) = \phi(a^k) + 1$ for some $k < j-1$, hence by (4.131) it follows $\hom^1(a^p, a^k) = 0$, which contradicts (4.74) and $j < N < p$.

In case (b): $\phi_-(a^j) = \phi(a^k)$ for some $k > j$. It follows that $\phi(a^k) = \arg_{(t,t+1)}(Z(a^k))$ (see Remark 4.2 (c)), hence by (4.131) and (2.11) we get $\arg_{(t,t+1)}(Z(a^k)) < \arg_{(t,t+1)}(Z(a^j))$, which contradicts (4.85).

In cases (c) and (d) we have $\phi_-(a^j) = \phi(b^k)$ of $\phi(b^k) + 1$ for some $k \in \mathbb{Z}$, and then (4.131) implies $\hom(M, b^k) = 0$, which contradicts (4.70).

Case (e) in Lemma 4.47 and (4.131) imply that $\phi(M)+1 < \phi(M)$ and we proved the lemma. $\square$

Lemma 4.50. *Let $\sigma \in (a^p, M, b^{p+1})$ and let the following inequality hold:*

$$\phi\left(a^p\right) - 1 < \phi\left(M\right) < \phi\left(a^p\right) \ . \tag{4.132}$$

Then we have the following:

(a) *$b^p \in \sigma^{ss}$ and $\phi(M) < \phi(b^p) < \phi(a^p)$.*

(b) *If in addition to (4.132) we have $\phi(M) + 1 < \phi(b^{p+1})$, then $\sigma \in (M, b^p, b^{p+1})$.*

(c) *If in addition to (4.132) we have*

$$\phi\left(a^p\right) - 1 < \phi\left(b^{p+1}\right) - 1 < \phi\left(a^p\right), \tag{4.133}$$

then $M' \in \sigma^{ss}$ *and* $\phi(b^{p+1}) - 1 < \phi(M') = \arg_{(\phi(a^p)-1,\phi(a^p))}(Z(a^p) - Z(b^{p+1})) < \phi(a^p)$.

(d) *If* (4.132), (4.133) *hold and* $\phi(M) < \phi(M')$, *then* $\sigma \in (b^j, b^{j+1}, M')$ *for some* $j \in \mathbb{Z}$ *or* $\sigma \in (M, b^p, b^{p+1})$.

(e) *If* (4.132), (4.133) *hold and* $\phi(M) = \phi(M')$, *then* $\sigma \in (a^j, M, b^{j+1})$ *for each* $j < p$.

Proof. **(a)** We apply Proposition 4.10 (a) to the triple (a^p, M, b^{p+1}) and since b^p is in the extension closure of M, a^p (by (4.81)) it follows that $b^p \in \sigma^{ss}$ and $\phi(M) \leq \phi(b^p) \leq \phi(a^p)$. The inequality $\phi(M) < \phi(b^p) < \phi(a^p)$ follows from the given inequality (4.132) and $Z(b^p) = Z(M) + Z(a^p)$.

(b) From the given inequalities we have $\phi(a^p) < \phi(M) + 1 < \phi(b^{p+1})$. In (a) we proved that $\phi(M) < \phi(b^p) < \phi(a^p)$. Therefore we obtain the inequalities $\phi(M) < \phi(b^p)$, $\phi(M) + 1 < \phi(b^{p+1})$, $\phi(b^p) < \phi(b^{p+1})$, which means that $\sigma \in (M, b^p, b^{p+1})$ (see table (4.98)).

(c) Follows from Lemma 4.13 applied to the Ext-triple $(a^p, M, b^{p+1}[-1])$ and the triangle (4.80).

(d) Now by the given inequalities and (c) we have $\phi(a^p) - 1 < \phi(M) < \phi(M') < \phi(a^p)$. Since $Z(\delta) = Z(M') + Z(M)$, we can choose $t \in \mathbb{R}$ with $Z(\delta) = |Z(\delta)| \exp(i\pi t)$ and $\phi(M) < t < \phi(M') < \phi(a^p) < \phi(M) + 1$. If $\phi(M) + 1 < \phi(b^{p+1})$, then we apply (b).

So, let $\phi(b^{p+1}) \leq \phi(M) + 1$. Since we have also $\phi(a^p) < \phi(b^{p+1})$, we obtain $t < \phi(M') < \phi(a^p) < \phi(b^{p+1}) \leq \phi(M) + 1 < t + 1$. Now Corollary 4.30 ensures that $\{Z(a^j), Z(b^j)\}_{j \in \mathbb{Z}} \subset Z(\delta)_+^c$ and that (4.86), (4.85) hold for both the sequences $\{Z(a^j)\}_{j \in \mathbb{Z}}$ and $\{Z(b^j)\}_{j \in \mathbb{Z}}$. From (a) we see that $\phi(M) < \phi(b^p) < \phi(a^p)$, hence $t - 1 < \phi(b^p) < t + 1$, which combined with $Z(b^p) \in Z(\delta)_+^c$ implies that $t < \phi(b^p)$. Hence we obtain the inequalities

$$t < \phi(b^p) < \phi(a^p) < \phi(b^{p+1}) < t + 1; \quad t < \phi(M') < \phi(a^p) < \phi(b^{p+1}) < t + 1. \tag{4.134}$$

From (4.86) and $t < \phi(M')$ it follows that there exists $N \in \mathbb{Z}$, $N < p$ such that $t < \arg_{(t,t+1)}(Z(b^j)) < \phi(M')$ for $j < N$. We will show below that $b^j \in \sigma^{ss}$ for $j < N$. Then (d) follows. Indeed, assume that $b^j \in \sigma^{ss}$ for each $j < N$. Then by (4.75) and Corollary 4.29 (a) it follows that $\phi(b^{p+1}) - 1 < \phi(b^j) < \phi(b^{p+1})$ for $j < N$, and by (4.134) we get $t - 1 < \phi(b^j) < t + 1$, which combined with $Z(b^j) \in Z(\delta)_+^c$ implies that $\arg_{(t,t+1)}(Z(b^j)) = \phi(b^j)$. Putting the last equality in (4.85) and in $\arg_{(t,t+1)}(Z(b^j)) < \phi(M')$ we obtain $\phi(b^{j-1}) < \phi(b^j) < \phi(M')$, which implies $\sigma \in (b^{j-1}, b^j, M')$.

Suppose $b^j \notin \sigma^{ss}$ for some $j < N$. We apply Lemma 4.48 and demonstrate that each of the five cases given there leads to a contradiction. We show first (4.135). From Remark 4.21 we know that b^j is in the extension closure of b^p, $b^{p+1}[-1]$ (recall that $N < p$) and we have $\phi(b^p) - 1 < \phi(b^{p+1}) - 1 < t < \phi(b^p)$ in (4.134). It follows that $b^j \in \mathcal{P}[\phi(b^{p+1}) - 1, \phi(b^p)] \subset \mathcal{P}(\phi(b^p) - 1, \phi(b^p)]$. Using $\phi(b^p) \in (t, t+1)$, $Z(b^j) \in Z(\delta)_+^c$, Remark 4.2 (c) and (a), we deduce that $\arg_{(\phi(b^p)-1,\phi(b^p)]}(Z(b^j)) = \arg_{(t,t+1)}(Z(b^j)) > \phi_-(b^j)$. The incidence $b^j \in \mathcal{P}[\phi(b^{p+1}) - 1, \phi(b^p)]$ implies $\phi(b^{p+1}) - 1 \leq \phi_-(b^j)$, and we get:

$$\phi(b^{p+1}) - 1 \leq \phi_-(b^j) < \arg_{(t,t+1)}(Z(b^j)) < \phi(M') < \phi(b^{p+1}). \tag{4.135}$$

One of the cases in Lemma 4.48 must appear. In cases (a) and (b) we have $\phi_-(b^j) = \phi(a^k)$ of $\phi(a^k) + 1$ for some $k \in \mathbb{Z}$, and then (4.135) implies $\hom(M', a^k) = 0$, which contradicts (4.70).

In case (c) we have $\phi_-(b^j) = \phi(b^k)+1$ for some $k < j-1$, and (4.135) implies that $\hom^1(b^{p+1}, b^k) = 0$, which contradicts (4.75) and $k < j-1 < p-1$.

In case (d) we have $\phi_-(b^j) = \phi(b^k)$ for some $k > j$. From $Z(b^k) \in Z(\delta)_+^c$, (4.135), and $\phi(b^{p+1}) \in (t, t+1)$ it follows that $\phi(b^k) = \arg_{(t,t+1)}(Z(b^k))$. Hence (4.135) and (2.11) imply $\arg_{(t,t+1)}(Z(b^k)) < \arg_{(t,t+1)}(Z(b^j))$, which contradicts $k > j$ and (4.85).

Case (e) in Lemma 4.48 and (4.135) imply that $\phi(M')+1 < \phi(M')$. We proved completely part (d) of the lemma.

(e) Now by the given inequalities we have $\phi(a^p) - 1 < \phi(M) = \phi(M') < \phi(a^p)$. Recalling that $Z(\delta) = Z(M') + Z(M)$, we see that $t = \phi(M) = \phi(M')$ satisfies $Z(\delta) = |Z(\delta)| \exp(i\pi t)$ and $t < \phi(a^p) < t+1$. From (a) we get $t < \phi(b^p) < \phi(a^p) < t+1$. Now we can apply Corollary 4.31, which besides $\{Z(a^j), Z(b^j)\}_{j\in\mathbb{Z}} \subset Z(\delta)_+^c$ and formulas (4.85), (4.86) gives us the inequalities (4.87).

We extend the inequality $t < \phi(b^p) < \phi(a^p) < t+1$ to (4.136) as follows. We already have that $a^p, b^p, b^{p+1} \in \sigma^{ss}$. In (4.133) is given that $\phi(a^p) < \phi(b^{p+1})$. From $\hom^1(b^{p+1}, M')$ (see (4.71)) it follows $\phi(b^{p+1}) \leq t+1$ and from $Z(b^{p+1}) \in Z(\delta)_+^c$ we get $\phi(b^{p+1}) < t+1 = \phi(M)+1$. We have also $\phi(M) < \phi(b^{p+1})$ (due to $\sigma \in (a^p, M, b^{p+1})$). Therefore $\phi(b^{p+1})-1 < \phi(M) < \phi(b^{p+1})$ and from Lemma 4.49 (a) we get $a^{p+1} \in \sigma^{ss}$ and $\phi(b^{p+1}) - 1 < \phi(a^{p+1}) - 1 < \phi(M)$. Thus, we derive:

$$\phi(a^p)-1 < \phi(b^{p+1})-1 < \phi(a^{p+1})-1 < t < \phi(b^p) < \phi(a^p) < \phi(b^{p+1}) < \phi(a^{p+1}) < t+1. \quad (4.136)$$

We will prove below that a^j and b^j are semi-stable for each $j < p$. We claim that this implies $\sigma \in (a^j, M, b^{j+1})$ for $j < p$. Indeed, assume that $a^j, b^j \in \sigma^{ss}$ for each $j < p$. Then by (4.74), (4.75) we get $\phi(a^{p+1}) - 1 \leq \phi(a^j) \leq \phi(a^{p+1})$ and $\phi(b^{p+1}) - 1 \leq \phi(b^j) \leq \phi(b^{p+1})$, which combined with $t < \phi(b^{p+1}) < \phi(a^{p+1}) < t+1$ and $Z(a^j), Z(b^j) \in Z(\delta)_+^c$ implies that $\phi(a^j), \phi(b^j) \in (t, t+1)$, in particular $\phi(a^j) = \arg_{(t,t+1)}(Z(a^j))$ and $\phi(b^j) = \arg_{(t,t+1)}(Z(b^j))$ for each $j < p$. The last two equalities hold also for $j = p$ by (4.136). Putting these equalities in (4.87) we get that $\phi(a^j) < \phi(b^{j+1})$ for each $j < p$. Thus, we obtain $\phi(M) < \phi(a^j) < \phi(b^{j+1}) < \phi(M)+1$ for each $j < p$, which by table (4.128) gives $\sigma \in (a^j, M, b^{j+1})$.

Suppose that $b^j \not\in \sigma^{ss}$ for some $j < p$. Remark 4.21 asserts that b^j is in the extension closure of b^p, $b^{p+1}[-1]$, therefore $b^j \in \mathcal{P}[\phi(b^{p+1})-1, \phi(b^p)]$, and hence $\phi(b^{p+1}) - 1 \leq \phi_-(b^j)$, $\phi_+(b^j) \leq \phi(b^p)$. Due to (4.136) we can write $b^j \in \mathcal{P}[\phi(b^{p+1})-1, \phi(b^p)] \subset \mathcal{P}(\phi(a^p)-1, \phi(a^p)]$. Using $\phi(a^p) \in (t, t+1)$, $Z(b^j) \in Z(\delta)_+^c$ and Remark 4.2 (c) we conclude that $\arg_{(\phi(a^p)-1,\phi(a^p)]}(Z(b^j)) = \arg_{(t,t+1)}(Z(b^j))$. Now using Remark 4.2 (a), we obtain:

$$\phi(b^{p+1}) - 1 \leq \phi_-(b^j) < \arg_{(t,t+1)}(Z(b^j)) < \phi_+(b^j) \leq \phi(b^p) < \phi(b^{p+1}). \quad (4.137)$$

We use Lemma 4.48 to obtain a contradiction. Some of the five cases given there must appear.

Case (a) ensures $\phi_-(b^j) = \phi(a^k)+1$ for some $k < j-1$ and (4.137) implies that $\hom^1(b^{p+1}, a^k) = 0$, which contradicts (4.72) (now $k < p$).

Case (b) ensures $\phi_-(b^j) = \phi(a^k)$ for some $k \geq j$, and then (4.137) and $Z(a^k) \in Z(\delta)_+^c$ imply $\arg_{(t,t+1)}(Z(a^k)) = \phi(a^k)$, hence by (4.137) and (2.11) we get $\arg_{(t,t+1)}(Z(a^k)) < \arg_{(t,t+1)}(Z(b^j))$, which contradicts (4.87) and $k \geq j$.

Case (c) ensures $\phi_-(b^j) = \phi(b^k) + 1$ for some $k < j - 1 < p - 1$, and (4.137) implies that $\hom^1(b^{p+1}, b^k) = 0$, which contradicts (4.75).

In case (d) we have $\phi_-(b^j) = \phi(b^k)$ for some $k > j$. It follows by $Z(b^k) \in Z(\delta)_+^c$ and (4.137) that $\phi(b^k) = \arg_{(t,t+1)}(Z(b^k))$, and then (4.137) gives $\arg_{(t,t+1)}(Z(b^k)) < \arg_{(t,t+1)}(Z(b^j))$, which contradicts (4.85).

In case (e) using (4.137) we obtain $\phi(M') + 1 < \phi(b^{p+1})$, which contradicts (4.71).

Suppose that $a^j \notin \sigma^{ss}$ for some $j < p$. Since a^j is in the extension closure of a^p, $a^{p+1}[-1]$ (see Remark 4.21), therefore $a^j \in \mathcal{P}[\phi(a^{p+1}) - 1, \phi(a^p)]$, and hence $\phi_\pm(a^j) \in [\phi(a^{p+1}) - 1, \phi(a^p)]$. Due to (4.136) we have $a^j \in \mathcal{P}[\phi(a^{p+1}) - 1, \phi(a^p)] \subset \mathcal{P}(\phi(b^{p+1}) - 1, \phi(b^{p+1})]$ and Remark 4.2 (c) shows that $\arg_{(\phi(b^{p+1})-1,\phi(b^{p+1})]}(Z(a^j)) = \arg_{(t,t+1)}(Z(a^j))$. Combining with Remark 4.2 (a) we put together:

$$\phi(a^{p+1}) - 1 \leq \phi_-(a^j) < \arg_{(t,t+1)}(Z(a^j)) < \phi_+(a^j) \leq \phi(a^p) < \phi(a^{p+1}). \tag{4.138}$$

We use Lemma 4.47 to get a contradiction. One of the five cases given there must appear.

In case (a) of Lemma 4.47 we have $\phi_-(a^j) = \phi(a^k) + 1$ for some $k < j - 1 < p - 1$, and (4.138) implies $\hom^1(a^{p+1}, a^k) = 0$, which contradicts (4.74).

Case (b) ensures $\phi_-(a^j) = \phi(a^k)$ for some $k > j$. It follows that $\phi(a^k) = \arg_{(t,t+1)}(Z(a^k))$ (see Remark 4.2 (c)), hence by (4.138) we get $\arg_{(t,t+1)}(Z(a^k)) < \arg_{(t,t+1)}(Z(a^j))$, which contradicts (4.85).

In case (c) we have $\phi_-(a^j) = \phi(b^k) + 1$ for some $k < j$ and (4.138) implies that $\hom^1(a^{p+1}, b^k) = 0$, which contradicts (4.73) (now $k < p$).

Case (d) ensures $\phi_-(a^j) = \phi(b^k)$ for some $j < k$, and then $\arg_{(t,t+1)}(Z(b^k)) = \phi(b^k)$ (see Remark 4.2 (c)), hence by (4.138) we get $\arg_{(t,t+1)}(Z(b^k)) < \arg_{(t,t+1)}(Z(a^j))$, which contradicts (4.87).

In case (e) we have $\phi_-(a^j) = \phi(M) + 1$, and (4.138) implies $\hom^1(a^{p+1}, M) = 0$, which contradicts (4.71). The lemma is proved. □

Next we glue (a^p, M, b^{p+1}) and $\mathfrak{T}_a^{st}$.

Lemma 4.51. *For any $p \in \mathbb{Z}$ the set $(a^p, M, b^{p+1}) \cap \mathfrak{T}_a^{st}$ consists of the stability conditions σ for which a^p, M, b^{p+1} are semistable and:*

$$\begin{array}{c} \phi\left(b^{p+1}\right) - 1 < \phi\left(M\right) < \phi\left(b^{p+1}\right) \\ \phi\left(b^{p+1}\right) - 1 < \phi\left(a^p\right) < \phi\left(b^{p+1}\right) \\ \arg_{(\phi(b^{p+1})-1,\phi(b^{p+1}))}(Z(a^p) - Z(b^{p+1})) < \phi(M) \end{array} \quad or \quad \begin{array}{c} \phi\left(a^p\right) < \phi\left(M\right) \\ \phi\left(b^{p+1}\right) - 1 < \phi\left(M\right) < \phi\left(b^{p+1}\right) \end{array}. \tag{4.139}$$

It follows that $(a^p, M, b^{p+1}) \cap \mathfrak{T}_a^{st}$ and $(a^p, M, b^{p+1}) \cup \mathfrak{T}_a^{st}$ are contractible.

Proof. We start with the inclusion $\subset$. Assume that $\sigma \in (a^p, M, b^{p+1})$. Then a^p, M, b^{p+1} are semistable and by table (4.128) we get

$$\begin{array}{c} \phi\left(a^p\right) < \phi\left(M\right) + 1 \\ \phi\left(a^p\right) < \phi\left(b^{p+1}\right) \\ \phi\left(M\right) < \phi\left(b^{p+1}\right) \end{array} \tag{4.140}$$

Recalling (4.93), we see that we have to consider three cases.

If $\sigma \in (M', a^j, a^{j+1})$, then M', a^j, a^{j+1} are semi-stable and from table (4.98) we see that $\phi(M')+1 < \phi(a^{j+1})$. Since we have also $\hom^1(b^{p+1}, M')$, $\hom^1(a^{j+1}, M) \neq 0$(see Corollary 4.19), we obtain $\phi\left(b^{p+1}\right) \leq \phi(M') + 1 < \phi(a^{j+1}) \leq \phi(M) + 1$, which combined with (4.140) implies

$$\phi\left(b^{p+1}\right) - 1 < \phi\left(M\right) < \phi\left(b^{p+1}\right) \qquad \phi(M') < \phi(M). \tag{4.141}$$

These non-vanishings and inequalities give also $\phi\left(a^p\right) < \phi\left(b^{p+1}\right) \leq \phi(M') + 1 < \phi(a^{j+1})$. Using Remark 4.20 (a) we deduce that $p \leq j$.

We verify now that $\phi(b^{p+1}) < \phi(a^p) + 1$. If $j = p$, then we immediately obtain this by $\hom^1(b^{p+1}, M') \neq 0$ and $\phi(M') < \phi(a^p)$(see table (4.98)). If $j > p$, then $\hom(b^{p+1}, a^j) \neq 0$ and $\hom^1(a^{j+1}, a^p) \neq 0$ (see Corollary 4.19) and we can write $\phi(b^{p+1}) \leq \phi(a^j) < \phi(a^{j+1}) \leq \phi(a^p) + 1$.

To obtain the first system of inequalities in (4.139) it remains to show the third inequality. From the triangle (4.80) it follows that $\phi(b^{p+1}) - 1 \leq \phi(M') \leq \phi(a^p)$ and $Z(M') = Z(a^p) - Z(b^{p+1})$, now $\phi(M') = \arg_{(\phi(b^{p+1})-1,\phi(b^{p+1}))}(Z(a^p) - Z(b^{p+1})) < \phi(M)$ follows from the already proved $\phi\left(b^{p+1}\right) - 1 < \phi\left(a^p\right) < \phi\left(b^{p+1}\right)$ and (4.141).

If $\sigma \in (a^m, a^{m+1}, M)$, then a^m, a^{m+1} are semistable as well and in table (4.98) we see that $\phi(a^m) < \phi(M)$, which together with the third inequality in (4.140) imply that $\phi(a^m) < \phi(b^{p+1})$ and hence $\hom(b^{p+1}, a^m) = 0$. By (4.72) we deduce that $p \geq m$.

If $p = m$, then we get immediately $\phi(a^p) < \phi(M)$. In table (4.98) we have $\phi(a^{p+1}) < \phi(M) + 1$ and in Corollary 4.19 we have $\hom(b^{p+1}, a^{p+1}) \neq 0$, hence $\phi(b^{p+1}) < \phi(M) + 1$ and we obtain the second system of inequalities in (4.139).

If $p > m$, then $\hom^1(b^{p+1}, a^m) \neq 0$ and from the inequalities $\phi(a^m) < \phi(M)$, $\phi(a^m) < \phi(a^{m+1})$ (due to $\sigma \in (a^m, a^{m+1}, M)$) it follows $\phi(b^{p+1}) < \phi(M)+1$ and $\phi(b^{p+1}) \leq \phi(a^m)+1 < \phi(a^{m+1})+1 \leq \phi(a^p)+1$. Recalling (4.140) we see that the first two equalities in (4.139) hold. Hence by Lemma 4.49 (c) we get $M' \in \sigma^{ss}$ and $\phi(M') = \arg_{(\phi(b^{p+1})-1,\phi(b^{p+1}))}(Z(a^p) - Z(b^{p+1}))$. From $\hom(M', a^m) \neq 0$ it follows $\phi(M') \leq \phi(a^m) < \phi(M)$ and we obtain the complete first system of inequalitites in (4.139).

If $\sigma \in (a^m, b^{m+1}, a^{m+1})$, then $a^m, b^{m+1}, a^{m+1} \in \sigma^{ss}$ and in table (4.97) we see that $\phi(a^m) + 1 < \phi(a^{m+1})$, hence Lemma 4.22 and $a^p \in \sigma^{ss}$ imply that $p = m$ or $p = m + 1$. If $p = m + 1$, then by (4.140) we obtain $\phi(a^m) + 1 < \phi(a^{m+1}) < \phi(b^{m+2})$, and hence $\hom^1(b^{m+2}, a^m) = 0$, which contradicts (4.72). Thus, it remains to consider the case $m = p$. Now we have $\phi(a^p) + 1 < \phi(a^{p+1})$ and $\phi(b^{p+1}) < \phi(a^{p+1})$(see table (4.97)), which together with $\hom^1(a^{p+1}, M) \neq 0$ imply $\phi(a^p) < \phi(M)$ and $\phi(b^{p+1}) < \phi(M) + 1$, hence we obtain the second system in (4.139). The inclusion $\subset$ is proved.

We consider now the converse inclusion $\supset$. Assume that a^p, M, b^{p+1} are semi-stable and that one of the two systems of inequalities in (4.139) holds. The inequalities in each of the two systems imply (4.140), therefore $\sigma \in (a^p, M, b^{p+1})$. If the second system in (4.139) holds, then by Lemma 4.49 (b) we get $\sigma \in (a^p, a^{p+1}, M) \subset \mathfrak{T}_a^{st}$. If the first system in (4.139) holds, then by Lemma 4.49 (c) and (d) we get $\sigma \in (a^j, a^{j+1}, M) \subset \mathfrak{T}_a^{st}$ for some $j \in \mathbb{Z}$, and the inclusion $\supset$ is proved as well.

As in the last paragraph of the proof of Lemma 4.42 one shows that the two systems of inequalities in (4.139) correspond to two contractible sets (the first is contractible by Corollary 4.63), and that their intersection is homeomorphic to $\mathbb{R}^3_{>0} \times \{\phi_2 - 1 < \phi_0 < \phi_1 < \phi_2\}$, which is also contractible. Remark 4.67 implies that $(a^p, M, b^{p+1}) \cap \mathfrak{T}_a^{st}$ is contractible. Since (a^p, M, b^{p+1}) and $\mathfrak{T}_a^{st}$ are both contractible (Proposition 4.8 and Corollary 4.43), Remark 4.67 implies that $(a^p, M, b^{p+1}) \cup \mathfrak{T}_a^{st}$ is contractible as well. □

Lemma 4.52. *For any $p \in \mathbb{Z}$ the set $(a^p, M, b^{p+1}) \cap \mathfrak{T}_b^{st}$ consists of the stability conditions σ for which a^p, M, b^{p+1} are semistable and:*

$$\begin{array}{c} \phi\left(a^p\right) - 1 < \phi\left(M\right) < \phi\left(a^p\right) \\ \phi\left(a^p\right) - 1 < \phi\left(b^{p+1}\right) - 1 < \phi\left(a^p\right) \\ \arg_{(\phi(a^p)-1, \phi(a^p))}(Z(a^p) - Z(b^{p+1})) > \phi(M) \end{array} \quad or \quad \begin{array}{c} \phi\left(a^p\right) - 1 < \phi\left(M\right) < \phi\left(a^p\right) \\ \phi(M) + 1 < \phi(b^{p+1}) \end{array}. \tag{4.142}$$

It follows that $(a^p, M, b^{p+1}) \cap \mathfrak{T}_b^{st}$ and $(a^p, M, b^{p+1}) \cup \mathfrak{T}_b^{st}$ are contractible.

Proof. We start with the inclusion $\subset$. Assume that $\sigma \in (a^p, M, b^{p+1})$. Then a^p, M, b^{p+1} are semistable and by table (4.128) we get

$$\begin{array}{c} \phi\left(a^p\right) < \phi\left(M\right) + 1 \\ \phi\left(a^p\right) < \phi\left(b^{p+1}\right) \\ \phi\left(M\right) < \phi\left(b^{p+1}\right) \end{array} \tag{4.143}$$

Recalling (4.94), we see that we have to consider three cases.

If $\underline{\sigma \in (M, b^j, b^{j+1})}$, then M, b^j, b^{j+1} are semi-stable and from table (4.98) we see that $\phi(M) < \phi(b^j)$ and $\phi(M) + 1 < \phi(b^{j+1})$, hence $\phi(a^p) < \phi(b^{j+1})$ and $\hom(b^{j+1}, a^p) = 0$. From (4.72) it follows that $p \leq j$. If $j = p$, then $\phi(M) + 1 < \phi(b^{p+1})$ and by $\hom(b^p, a^p) \neq 0$ (see (4.72)) we get $\phi(M) < \phi(a^p)$, which implies the second system in (4.142). It remains to consider the case $p < j$.

In this case $\hom^1(b^{j+1}, a^p) \neq 0$ (see (4.72)) and we obtain $\phi(M)+1 < \phi(b^{j+1}) \leq \phi(a^p)+1$, which combined with (4.143) implies $\phi\left(a^p\right) - 1 < \phi\left(M\right) < \phi\left(a^p\right)$. On the other hand, we have $\phi(b^j) < \phi(b^{j+1})$ (see table (4.98)), and by $p < j$ we can write $\phi(b^{p+1}) \leq \phi(b^j) < \phi(b^{j+1}) \leq \phi(a^p) + 1$, which combined with (4.143) implies $\phi\left(a^p\right) - 1 < \phi(b^{p+1}) - 1 < \phi\left(a^p\right)$. Now we can use Lemma 4.50 (c) to deduce that $M' \in \sigma^{ss}$ and $\phi(M') = \arg_{(\phi(a^p)-1, \phi(a^p))}(Z(a^p) - Z(b^{p+1}))$. From $\hom^1(b^{j+1}, M') \neq 0$ and $\phi(M) + 1 < \phi(b^{j+1})$ it follows that $\phi(M) < \phi(M')$ and the first system in (4.142) follows.

If $\underline{\sigma \in (b^m, b^{m+1}, M')}$, then b^m, b^{m+1}, M' are semistable and in table (4.98) we see that $\phi(b^m) < \phi(M')$. By $\hom(M', a^p) \neq 0$ and $\hom(M, b^m) \neq 0$ (see (4.70)) we get:

$$\phi(M) \leq \phi(b^m) < \phi(M') \leq \phi(a^p). \tag{4.144}$$

Whence $\phi(M) < \phi(a^p)$ and combining with (4.143) we derive $\phi\left(a^p\right) - 1 < \phi(M) < \phi\left(a^p\right)$. On the other hand, in (4.144) we have also $\phi(b^m) < \phi(a^p)$, and hence $\hom(a^p, b^m) = 0$, threfore by (4.73) we see that $p \geq m$. In (4.144) we have also $\phi(M) < \phi(M')$. Taking into account Lemma 4.50 (c), we see that if we show $\phi\left(a^p\right) - 1 < \phi\left(b^{p+1}\right) - 1 < \phi\left(a^p\right)$, then the first system in (4.142) follows. Since we have $\phi\left(a^p\right) < \phi\left(b^{p+1}\right)$ (see (4.143)), it remains to verify that $\phi\left(b^{p+1}\right) < \phi\left(a^p\right) + 1$. If $p = m$, then from table (4.98) we obtain $\phi(b^{p+1}) < \phi(M') + 1$ and the inequality in question follows from $\phi(M') \leq \phi(a^p)$. If $m < p$, then $\hom^1(b^{p+1}, b^m) \neq 0$ and we get $\phi(b^{p+1}) \leq \phi(b^m) + 1 < \phi\left(a^p\right) + 1$ (see (4.144)).

If $\sigma \in (b^m, a^m, b^{m+1})$, then $b^m, a^m, b^{m+1} \in \sigma^{ss}$ and in table (4.97) we see that $\phi(b^m) + 1 < \phi(b^{m+1})$, hence Lemma 4.22 and $b^{p+1} \in \sigma^{ss}$ imply that $p = m$ or $p = m - 1$. If $p = m - 1$, then by (4.143) we obtain $\phi(a^{m-1}) + 1 < \phi(b^m) + 1 < \phi(b^{m+1})$, and hence $\hom^1(b^{m+1}, a^{m-1}) = 0$, which contradicts (4.72). Therefore we have $m = p$. Now we have $\phi(b^p) + 1 < \phi(b^{p+1})$ and $\phi(b^p) < \phi(a^p)$ (see table (4.97)), which together with $\hom(M, b^p) \neq 0$ imply $\phi(M) + 1 < \phi(b^{p+1})$ and $\phi(M) < \phi(a^p)$, hence the second system in (4.142) follows. Thus we proved the inclusion $\subset$.

Next we consider the converse inclusion $\supset$. Assume that a^p, M, b^{p+1} are semi-stable and that one of the two systems of inequalities in (4.142) holds. The inequalities in each of the two systems imply (4.143), therefore $\sigma \in (a^p, M, b^{p+1})$. If the second system in (4.142) holds, then by Lemma (4.50) (b) we get $\sigma \in (M, b^p, b^{p+1}) \subset \mathfrak{T}^{st}_b$. If the first system in (4.142) holds, then the desired $\sigma \in \mathfrak{T}^{st}_b$ follows from Lemma (4.50) (c) and (d). The inclusion $\supset$ is proved as well.

In Corollary 4.45 was proved that $\mathfrak{T}^{st}_b$ is contractible. The proof that $(a^p, M, b^{p+1}) \cap \mathfrak{T}^{st}_b$ and $(a^p, M, b^{p+1}) \cup \mathfrak{T}^{st}_b$ are contractible is as in the last paragraph of Lemma 4.51. The two systems in (4.142) correspond to contractible subsets of $(a^p, M, b^{p+1}) \cap \mathfrak{T}^{st}_b$ (the first is contractible by Corollary 4.64). The intersection of these subsets is homeomorphic to $\mathbb{R}^3_{>0} \times \{\phi_0 - 1 < \phi_1 < \phi_2 - 1 < \phi_0\}$, which is also contractible. Now we apply Remark 4.67 twice and the lemma follows. □

Corollary 4.53. *For any $p \in \mathbb{Z}$ the set $\mathfrak{T}^{st}_a \cup (a^p, M, b^{p+1}) \cup \mathfrak{T}^{st}_b$ is contractible.*

Proof. In Lemma 4.51 we proved that $\mathfrak{T}^{st}_a \cup (a^p, M, b^{p+1})$ is contractible. Since $\mathfrak{T}^{st}_a \cap \mathfrak{T}^{st}_b = \emptyset$ (see Subsection 4.6.1), it follows that $(\mathfrak{T}^{st}_a \cup (a^p, M, b^{p+1})) \cap \mathfrak{T}^{st}_b = (a^p, M, b^{p+1}) \cap \mathfrak{T}^{st}_b$, which is contractible by Lemma 4.52. Now we apply Remark 4.67. □

Lemma 4.54. *For any $q < p$ the set $(a^p, M, b^{p+1}) \cap (\mathfrak{T}^{st}_a \cup (a^q, M, b^{q+1}) \cup \mathfrak{T}^{st}_b)$ consists of the stability conditions σ for which a^p, M, b^{p+1} are semistable and:*

$$\begin{array}{ccc} \begin{array}{c}\phi\left(a^p\right) - 1 < \phi\left(M\right) < \phi\left(a^p\right) \\ \phi\left(a^p\right) - 1 < \phi\left(b^{p+1}\right) - 1 < \phi\left(a^p\right)\end{array} & or & \begin{array}{c}\phi\left(a^p\right) - 1 < \phi\left(M\right) < \phi\left(a^p\right) \\ \phi(M) + 1 < \phi(b^{p+1})\end{array} \\ or \quad \begin{array}{c}\phi\left(a^p\right) < \phi\left(M\right) \\ \phi\left(b^{p+1}\right) - 1 < \phi\left(M\right) < \phi\left(b^{p+1}\right)\end{array} & or & \begin{array}{c}\phi\left(b^{p+1}\right) - 1 < \phi\left(M\right) < \phi\left(b^{p+1}\right) \\ \phi\left(b^{p+1}\right) - 1 < \phi\left(a^p\right) < \phi\left(b^{p+1}\right) \\ \arg_{(\phi(b^{p+1})-1, \phi(b^{p+1}))}(Z(a^p) - Z(b^{p+1})) < \phi(M)\end{array} \end{array} . \tag{4.145}$$

It follows that $(a^p, M, b^{p+1}) \cap (\mathfrak{T}^{st}_a \cup (a^q, M, b^{q+1}) \cup \mathfrak{T}^{st}_b)$ and $(a^p, M, b^{p+1}) \cup (\mathfrak{T}^{st}_a \cup (a^q, M, b^{q+1}) \cup \mathfrak{T}^{st}_b)$ are contractible.

Proof. We start with the inclusion $\subset$. Assume that $\sigma \in (a^p, M, b^{p+1})$. Then a^p, M, b^{p+1} are semi-stable and by table (4.128) we get

$$\begin{array}{c} \phi\left(a^p\right) < \phi\left(M\right) + 1 \\ \phi\left(a^p\right) < \phi\left(b^{p+1}\right) \\ \phi\left(M\right) < \phi\left(b^{p+1}\right) \end{array} . \tag{4.146}$$

If $\sigma \in (a^q, M, b^{q+1})$ and $q < p$, then $a^q, b^{q+1} \in \sigma^{ss}$ and $\phi(M) < \phi(b^{q+1})$, $\phi\left(a^q\right) < \phi\left(b^{q+1}\right)$ a well. By (4.72) we have $\hom(b^{q+1}, a^p) \neq 0$ and $\hom^1(b^{p+1}, a^q) \neq 0$, therefore $\phi(M) < \phi(b^{q+1}) \leq \phi(a^p)$ and $\phi(b^{p+1}) \leq \phi(a^q) + 1 < \phi(b^{q+1}) + 1 \leq \phi\left(a^p\right) + 1$. Combining with (4.146) we obtain the system in the first row and first column in (4.145).

If $\sigma \in \mathfrak{T}_a^{st}$, then by Lemma 4.51 some of the systems on the second row of (4.145) follows.

If $\sigma \in \mathfrak{T}_b^{st}$, then by Lemma 4.52 some of the systems on the first row of (4.145) follows ((4.142) implies (4.145)). So, the inclusion $\subset$ is proved.

We show now the inclusion $\supset$. So let a^p, M, b^{p+1} be semi-stable. If some of the systems on the second row of (4.145) holds, then by 4.51 it follows that $\sigma \in (a^p, M, b^{p+1}) \cap \mathfrak{T}_a^{st}$. If the system in the first row and second column of (4.145) holds, then Lemma 4.52 ensures that $\sigma \in (a^p, M, b^{p+1}) \cap \mathfrak{T}_b^{st}$.

Thus, it remains to consider the first system in (4.145). We assume till the end of the proof that

$$\begin{array}{c} \phi\left(a^p\right) - 1 < \phi\left(M\right) < \phi\left(a^p\right) \\ \phi\left(a^p\right) - 1 < \phi\left(b^{p+1}\right) - 1 < \phi\left(a^p\right) \end{array} . \tag{4.147}$$

Lemma 4.50 (c) ensures that

$$M' \in \sigma^{ss}; \qquad \phi(b^{p+1}) - 1 < \phi(M') = \arg_{(\phi(a^p)-1, \phi(a^p))}(Z(a^p) - Z(b^{p+1})) < \phi(a^p). \tag{4.148}$$

Now we consider three cases.

If $\phi(M') > \phi(M)$, then (4.147) and (4.148) yield the first system in (4.142) is satisfied and then Lemma 4.52 says that $\sigma \in (a^p, M, b^{p+1}) \cap \mathfrak{T}_b^{st}$.

If $\phi(M') < \phi(M)$, then by $\hom^1(b^{p+1}, M') \neq 0$ it follows that $\phi(b^{p+1}) - 1 < \phi(M)$. Combining this inequality with (4.147) one easily deduces that:

$$\begin{array}{c} \phi\left(b^{p+1}\right) - 1 < \phi\left(M\right) < \phi\left(b^{p+1}\right) \\ \phi\left(b^{p+1}\right) - 1 < \phi\left(a^p\right) < \phi\left(b^{p+1}\right) \end{array} . \tag{4.149}$$

Having obtained (4.149) we can use Lemma 4.49 (c) and due to $\phi(M') < \phi(M)$ we derive the first system in (4.139). Thus Lemma 4.51 ensures that $\sigma \in (a^p, M, b^{p+1}) \cap \mathfrak{T}_a^{st}$.

Finally, if $\phi(M) = \phi(M')$, then due to (4.147) we can apply Lemma 4.50 **(e)**, which says that $\sigma \in (a^p, M, b^{p+1}) \cap (a^q, M, b^{q+1})$ (recall that $q < p$). So far we proved the first part of the lemma.

We explain now, using the obtained representation through the systems of inequalities (4.145), that $(a^p, M, b^{p+1}) \cap (\mathfrak{T}_a^{st} \cup (a^q, M, b^{q+1}) \cup \mathfrak{T}_b^{st})$ is contractible. The four systems correspond to four open subsets of $(a^p, M, b^{p+1}) \cap (\mathfrak{T}_a^{st} \cup (a^q, M, b^{q+1}) \cup \mathfrak{T}_b^{st})$ (see the last paragraph of the proof of Lemma 4.42). We denote these subsets by $S_{11}, S_{12}, S_{21}, S_{22}$, where S_{ij} corresponds to the system in the i-th row and j-th column of (4.145). The proved part of the lemma is the equality $(a^p, M, b^{p+1}) \cap (\mathfrak{T}_a^{st} \cup (a^q, M, b^{q+1}) \cup \mathfrak{T}_b^{st}) = \bigcup_{1 \leq i,j \leq 2} S_{ij}$. The subset S_{22} is contractible by Corollary 4.63. The subsets S_{11}, S_{12}, S_{21} are contractible since they are homeomorphic to convex subsets of $\mathbb{R}^6$. For example S_{11} is homeomorphic to

$$\mathbb{R}^3_{>0} \times \left\{ (\phi_0, \phi_1, \phi_2) \in \mathbb{R}^3 : \begin{array}{c} \phi_0 - 1 < \phi_1 < \phi_0 \\ \phi_0 - 1 < \phi_2 - 1 < \phi_0 \end{array} \right\}.$$

It is not difficult to check that $S_{11} \cap S_{12}$ is homeomorphic to $\mathbb{R}^3_{>0} \times \{\phi_0 - 1 < \phi_1 < \phi_2 - 1 < \phi_0\}$, hence it is contractible, and by Remark 4.67 we deduce that $S_{11} \cup S_{12}$ is contractible. Note that in S_{12} we have $\phi(M) + 1 < \phi(b^{p+1})$ and in S_{22} we have $\phi(M) + 1 > \phi(b^{p+1})$, therefore $S_{12} \cap S_{22} = \emptyset$. Hence $S_{22} \cap (S_{11} \cup S_{12}) = S_{22} \cap S_{11}$. Furthermore, the intersection $S_{22} \cap S_{11}$ is homeomorphic to:

$$\mathbb{R}^3_{>0} \times \left\{ (\phi_0, \phi_1, \phi_2) \in \mathbb{R}^3 : \begin{array}{c} r_i > 0 \\ \phi_2 - 1 < \phi_1 < \phi_0 < \phi_2 \\ \arg_{(\phi_2 - 1, \phi_2)}(r_0 \exp(\mathrm{i}\pi\phi_0) - r_2 \exp(\mathrm{i}\pi\phi_2)) < \phi_1 \end{array} \right\}, \tag{4.150}$$

which by Corollary 4.66 is contractible as well. Thus, the union $S_{22} \cap (S_{11} \cup S_{12})$ is contractible, therefore by Remark 4.67 it follows that $S_{22} \cup S_{11} \cup S_{12}$ is contractible. In S_{11} and S_{12} we have $\phi(M) < \phi(a^p)$ and in S_{21} we have $\phi(M) > \phi(a^p)$, therefore $S_{21} \cap (S_{22} \cup S_{11} \cup S_{12}) = S_{21} \cap S_{22}$. On the other hand, one easily shows (by drawing a picture) that the intersection $S_{21} \cap S_{22}$ is homeomorphic to $\mathbb{R}^3_{>0} \times \{\phi_2 - 1 < \phi_0 < \phi_1 < \phi_2\}$, which is contractible as well, and hence $S_{21} \cap (S_{22} \cup S_{11} \cup S_{12})$ is contractible. Applying Remark 4.67 again ensures that $S_{21} \cup S_{22} \cup S_{11} \cup S_{12} = (a^p, M, b^{p+1}) \cap (\mathfrak{T}_a^{st} \cup (a^q, M, b^{q+1}) \cup \mathfrak{T}_b^{st})$ is contractible. In Corollary 4.53 is proved that $\mathfrak{T}_a^{st} \cup (a^q, M, b^{q+1}) \cup \mathfrak{T}_b^{st}$ is contractible and with one more reference to Remark 4.67 we prove the lemma. □

Corollary 4.55. *The set* $(\_, M, \_) \cup \mathfrak{T}_a^{st} \cup \mathfrak{T}_b^{st}$ *is contractible.*

Proof. Recall that $(\_, M\_) = \bigcup_{q \in \mathbb{Z}} (a^q, M, b^{q+1})$ (see (4.127)). We will prove that for each $p \in \mathbb{Z}$ and for each $k \geq 1$ the set (4.151) below is contractible, and the corollary follows from Remark 4.67:

$$\bigcup_{i=0}^{k} (a^{p-i}, M, b^{p+1-i}) \cup (\mathfrak{T}_a^{st} \cup \mathfrak{T}_b^{st}). \tag{4.151}$$

In the previous lemma was shown that for $k = 1$ and any $p \in \mathbb{Z}$ the set (4.151) is contractible. Assume that for some $k \geq 1$ this set is contractible for each $p \in \mathbb{Z}$. Take now any $p \in \mathbb{Z}$. We have

$$\bigcup_{i=0}^{k+1} (a^{p-i}, M, b^{p+1-i}) \cup (\mathfrak{T}_a^{st} \cup \mathfrak{T}_b^{st}) = (a^p, M, b^{p+1}) \cup \left(\bigcup_{i=1}^{k+1} (a^{p-i}, M, b^{p+1-i}) \cup (\mathfrak{T}_a^{st} \cup \mathfrak{T}_b^{st}) \right). \tag{4.152}$$

Proposition 4.8 and the induction assumption say that the two components on RHS of (4.152) are contractible. Since the intersection analyzed in Lemma 4.54 does not depend on q, we can write:

$$(a^p, M, b^{p+1}) \cap \left(\bigcup_{i=1}^{k+1} (a^{p-i}, M, b^{p+1-i}) \cup (\mathfrak{T}_a^{st} \cup \mathfrak{T}_b^{st}) \right) = (a^p, M, b^{p+1}) \cap \left((a^{p-1}, M, b^p) \cup (\mathfrak{T}_a^{st} \cup \mathfrak{T}_b^{st}) \right),$$

which by Lemma 4.54 is contractible. Now Remark 4.67 ensures that (4.152) is contractible. □

The next step is to glue $(\_, M, \_) \cup \mathfrak{T}_a^{st} \cup \mathfrak{T}_b^{st}$ and (b^p, M', a^p). This is done in several substeps: Lemmas 4.56, 4.57, 4.58, 4.59, which lead to Corollary 4.60. In the next two lemmas we prove inclusions in only one direction not equality of sets.

Lemma 4.56. *Let $p \in \mathbb{Z}$. If $\sigma \in (b^p, M', a^p) \cap \mathfrak{T}_b^{st}$, then b^p, M', a^p are semistable and:*

$$\begin{array}{c} \phi(a^p) - 1 < \phi(M') < \phi(a^p) \\ \phi(a^p) - 1 < \phi(b^p) < \phi(a^p) \end{array} \quad or \quad \begin{array}{c} \phi(a^p) - 1 < \phi(M') < \phi(a^p) \\ \phi(b^p) < \phi(M') \end{array} . \tag{4.153}$$

Proof. In table (4.128) we see that b^p, M', a^p are semi-stable and:

$$\begin{array}{c} \phi(b^p) < \phi(M') + 1 \\ \phi(b^p) < \phi(a^p) \\ \phi(M') < \phi(a^p) \end{array} \tag{4.154}$$

Recalling (4.94), we see that we have to consider three cases.

If $\sigma \in (M, b^j, b^{j+1})$, then M, b^j, b^{j+1} are semi-stable and from table (4.98) we see that $\phi(M) < \phi(b^j)$ and $\phi(M) + 1 < \phi(b^{j+1})$. By $\hom^1(a^p, M) \neq 0$ and $\hom^1(b^{j+1}, M') \neq 0$ (see (4.71)) we can write $\phi(a^p) \leq \phi(M) + 1 < \phi(b^{j+1}) \leq \phi(M') + 1$, therefore (see also (4.154)) we get

$$\phi(a^p) - 1 < \phi(M') < \phi(a^p). \tag{4.155}$$

Since $\phi(b^p) < \phi(a^p) \leq \phi(M) + 1 < \phi(b^{j+1})$, due to (4.75) the inequality $p \leq j$ must hold.

If $j = p$, then the inequality $\phi(M) < \phi(b^p)$ (coming from $\sigma \in (M, b^j, b^{j+1})$) implies $\phi(a^p) - 1 \leq \phi(M) < \phi(b^p)$ and combining with (4.154) and (4.155) we obtain the first system in (4.153).

If $p < j$, then we have $\hom(a^p, b^j) \neq 0$ (see (4.73)) and $\hom^1(b^{j+1}, b^p) \neq 0$, hence $\phi(a^p) \leq \phi(b^j) < \phi(b^{j+1}) \leq \phi(b^p) + 1$ and again the first system in (4.153) follows.

If $\sigma \in (b^m, b^{m+1}, M')$, then b^m, b^{m+1}, M' are semistable and in table (4.98) we see that $\phi(b^m) < \phi(M')$, therefore $\phi(b^m) < \phi(M') < \phi(a^p)$ and $\hom(a^p, b^m) = 0$. From (4.73) we deduce that $m \leq p$.

If $m = p$, then the incidence $\sigma \in (b^m, b^{m+1}, M')$ gives $\phi(b^p) < \phi(M')$ and $\phi(b^{p+1}) - 1 < \phi(M')$ (see table (4.98)), and from $\hom(a^p, b^{p+1}) \neq 0$ we obtain $\phi(a^p) - 1 < \phi(M')$, therefore the second system in (4.153) holds.

Let $m < p$. Then we have $\phi(b^m) < \phi(b^{m+1})$ and $\phi(b^m) < \phi(M')$ (see table (4.98)). Using $\hom^1(a^p, b^m) \neq 0$ (see (4.73)) we deduce $\phi(a^p) \leq \phi(b^m) + 1 < \phi(b^{m+1}) + 1 \leq \phi(b^p) + 1$ and $\phi(a^p) \leq \phi(b^m) + 1 < \phi(M') + 1$, which combined with (4.154) produces the first system in (4.153).

If $\sigma \in (b^m, a^m, b^{m+1})$, then $b^m, a^m, b^{m+1} \in \sigma^{ss}$ and in table (4.97) we see that $\phi(b^m) + 1 < \phi(b^{m+1})$, hence Lemma 4.22 and $b^p \in \sigma^{ss}$ imply $p = m$ or $p = m + 1$. If $p = m + 1$, then by (4.154) we obtain $\phi(b^m) + 1 < \phi(b^{m+1}) < \phi(a^{m+1})$, and hence $\hom^1(a^{m+1}, b^m) = 0$, which contradicts (4.73). Therefore we have $m = p$. In table (4.97) we see that $\phi(b^p) + 1 < \phi(b^{p+1})$ and $\phi(a^p) < \phi(b^{p+1})$. From $\hom^1(b^{p+1}, M') \neq 0$ it follows that $\phi(b^p) + 1 < \phi(b^{p+1}) \leq \phi(M') + 1$ and $\phi(a^p) < \phi(b^{p+1}) \leq \phi(M') + 1$. These inequalities together with (4.154) produce the second system in (4.153). □

Lemma 4.57. *Let $p \in \mathbb{Z}$. If $\sigma \in (b^p, M', a^p) \cap \mathfrak{T}_a^{st}$, then b^p, M', a^p are semistable and:*

$$\begin{array}{c} \phi(b^p) - 1 < \phi(M') < \phi(b^p) \\ \phi(b^p) - 1 < \phi(a^p) - 1 < \phi(b^p) \end{array} \quad or \quad \begin{array}{c} \phi(b^p) - 1 < \phi(M') < \phi(b^p) \\ \phi(M') + 1 < \phi(a^p) \end{array}. \tag{4.156}$$

Proof. In table (4.128) we see that b^p, M', a^p are semi-stable and:

$$\begin{array}{c} \phi(b^p) < \phi(M') + 1 \\ \phi(b^p) < \phi(a^p) \\ \phi(M') < \phi(a^p) \end{array} \tag{4.157}$$

As it follows from (4.93), we have to consider the following three cases.

If $\sigma \in (M', a^j, a^{j+1})$, then M', a^j, a^{j+1} are semi-stable and $\phi(M') < \phi(a^j)$, $\phi(M')+1 < \phi(a^{j+1})$, $\phi(a^j) < \phi(a^{j+1})$ (see table (4.98)). On the other hand $\phi(b^p) < \phi(M') + 1$, hence $\hom(a^{j+1}, b^p) = 0$ and (4.73) implies that $p - 1 \leq j$. If $p - 1 = j$, then we have $\phi(M') + 1 < \phi(a^p)$ and $\phi(M') < \phi(a^{p-1}) \leq \phi(b^p)$ (see also (4.73)) and combining with (4.157) we derive the second system in (4.156). Let $p \leq j$. Then by (4.73) we have $\hom^1(a^{j+1}, b^p) \neq 0$ and we can write $\phi(M') + 1 < \phi(a^{j+1}) \leq \phi(b^p) + 1$ and $\phi(a^p) \leq \phi(a^j) < \phi(a^{j+1}) \leq \phi(b^p) + 1$, therefore $\phi(M') < \phi(b^p)$ and $\phi(a^p) < \phi(b^p) + 1$, which combined with (4.157) amounts to the first system in (4.156).

If $\sigma \in (a^m, a^{m+1}, M)$, then a^m, a^{m+1}, M are semistable as well and in table (4.98) we see that $\phi(a^m) < \phi(M)$, $\phi(a^{m+1}) < \phi(M)+1$, $\phi(a^m) < \phi(a^{m+1})$. Since $\hom(M', a^m) \neq 0$ and $\hom(M, b^p) \neq 0$, it follows that $\phi(M') \leq \phi(a^m) < \phi(M) \leq \phi(b^p)$ and hence (see also (4.157)):

$$\phi(b^p) - 1 < \phi\left(M'\right) < \phi(b^p) \tag{4.158}$$

On the other hand, $\phi(a^m) < \phi(M)$ and $\hom(M, b^p) \neq 0$ imply that $\phi(a^m) < \phi(b^p)$ and $\hom(b^p, a^m) = 0$. Now from (4.72) we deduce that $m < p$. If $m = p - 1$, then we have $\phi(a^p) < \phi(M) + 1 \leq \phi(b^p) + 1$, which together with (4.158) and (4.157) amounts to the first system in (4.156).

If $m < p - 1$, then $\hom^1(a^p, a^m) \neq 0$ and $\hom(a^{m+1}, b^p) \neq 0$ (see (4.73)). Therefore we have $\phi(a^p) \leq \phi(a^m) + 1 < \phi(a^{m+1}) + 1 \leq \phi(b^p) + 1$ and the first system in (4.156) follows again.

If $\sigma \in (a^m, b^{m+1}, a^{m+1})$, then $a^m, b^{m+1}, a^{m+1} \in \sigma^{ss}$ and in table (4.97) we see that $\phi(a^m) + 1 < \phi(a^{m+1})$, hence Lemma 4.22 and $a^p \in \sigma^{ss}$ imply $p = m$ or $p = m + 1$. If $p = m$, then by (4.157) we obtain $\phi(b^m)+1<\phi(a^m)+1<\phi(a^{m+1})$, and hence $\hom^1(a^{m+1},b^m)=0$, which contradicts (4.73). Thus, it remains to consider the case $m=p-1$. Now we have $\phi(a^{p-1})+1<\phi(a^p)$ and $\phi(a^{p-1})<\phi(b^p)$(see table (4.97)), which together with $\hom(M',a^{p-1})\neq 0$ imply $\phi(M')+1<\phi(a^p)$ and $\phi(M')<\phi(b^p)$, hence we obtain the second system of inequalities in (4.156). □

Lemma 4.58. *Let $\sigma\in(b^p,M',a^p)$ and let the following inequality hold:*

$$\phi\left(b^p\right)-1<\phi\left(M'\right)<\phi\left(b^p\right). \tag{4.159}$$

Then we have the following:

(a) *$a^{p-1}\in\sigma^{ss}$ and $\phi(M')<\phi(a^{p-1})<\phi(b^p)<\phi(a^p)$.*

(b) *If in addition to (4.159) we have $\phi\left(M'\right)+1<\phi\left(a^p\right)$, then $\sigma\in(M',a^{p-1},a^p)$.*

(c) *If in addition to (4.159) we have $\phi\left(b^p\right)-1<\phi\left(a^p\right)-1<\phi\left(b^p\right)$, then $\sigma\in(a^{p-1},M,b^p)$.*

Proof. (a) We apply Proposition 4.10 (a) to the triple (b^p,M',a^p) and since a^{p-1} is in the extension closure of M',b^p (by (4.80)) it follows that $a^{p-1}\in\sigma^{ss}$, $\phi(M')\leq\phi(a^{p-1})\leq\phi(b^p)$. The inequality $\phi(M')<\phi(a^{p-1})<\phi(b^p)$ follows from the given inequality (4.159), formula (2.11) and $Z(a^{p-1})=Z(M')+Z(b^p)$. The inequality $\phi(b^p)<\phi(a^p)$ follows from $\sigma\in(b^p,M',a^p)$ (see table (4.128)).

(b) From the given inequalities and (a) we have $\phi(M')<\phi(a^{p-1})$, $\phi\left(M'\right)+1<\phi\left(a^p\right)$, and $\phi(a^{p-1})<\phi(a^p)$, then table (4.98) ensures that $\sigma\in(M',a^{p-1},a^p)$.

(c) From Lemma 4.13 applied to the Ext-triple $(b^p,M',a^p[-1])$ and the triangle (4.81) we obtain $M\in\sigma^{ss}$ and $\phi(a^p)-1<\phi(M)<\phi(b^p)$. In (a) we got $a^{p-1}\in\sigma^{ss}$ and $\phi(a^{p-1})<\phi(a^p)$, therefore $\phi(a^{p-1})<\phi(M)+1$. In (a) we have also $\phi(a^{p-1})<\phi(b^p)$. Looking at table (4.128) we see that $\sigma\in(a^{p-1},M,b^p)$. □

Lemma 4.59. *Let $\sigma\in(b^p,M',a^p)$ and let the following inequality hold:*

$$\phi\left(a^p\right)-1<\phi\left(M'\right)<\phi\left(a^p\right). \tag{4.160}$$

Then we have the following:

(a) *$b^{p+1}\in\sigma^{ss}$ and $\phi\left(b^p\right)-1<\phi\left(a^p\right)-1<\phi\left(b^{p+1}\right)-1<\phi\left(M'\right)$.*

(b) *If in addition to (4.160) we have $\phi(b^p)<\phi(M')$, then $\sigma\in(b^p,b^{p+1},M')$.*

(c) *If in addition to (4.160) we have $\phi\left(a^p\right)-1<\phi\left(b^p\right)<\phi\left(a^p\right)$, then $\sigma\in(a^p,M,b^{p+1})$*

Proof. (a) We apply Proposition 4.10 (b) to the triple (b^p,M',a^p) and since $b^{p+1}[-1]$ is in the extension closure of $M',a^p[-1]$ (by (4.80)) it follows that $b^{p+1}\in\sigma^{ss}$, $\phi\left(a^p\right)-1\leq\phi\left(b^{p+1}\right)-1\leq\phi\left(M'\right)$. The inequality $\phi\left(a^p\right)-1<\phi\left(b^{p+1}\right)-1<\phi\left(M'\right)$ follows from the given inequality (4.160), formula (2.11), and $Z(b^{p+1}[-1])=Z(M')+Z(a^p[-1])$. The inequality $\phi(b^p)<\phi(a^p)$ follows from $\sigma\in(b^p,M',a^p)$.

(b) From the given inequalities and (a) we have $\phi(b^p)<\phi(b^{p+1})$, $\phi(b^p)<\phi(M')$ and $\phi(b^{p+1})<\phi(M')+1$. Now in table (4.98) we see that $\sigma\in(b^p,b^{p+1},M')$.

(c) From Lemma 4.13 applied to the Ext-triple $(b^p,M',a^p[-1])$ and the triangle (4.81) we obtain $M\in\sigma^{ss}$ and $\phi(a^p)-1<\phi(M)<\phi(b^p)$. In (a) we proved that $b^{p+1}\in\sigma^{ss}$ and $\phi(a^p)<\phi(b^{p+1})$,

$\phi(b^p) < \phi(b^{p+1})$. Now all the conditions determining (a^p, M, b^{p+1}) (given in table (4.128)) are satisfied, hence $\sigma \in (a^p, M, b^{p+1})$. □

Corollary 4.60. *For any $p \in \mathbb{Z}$ the set $(b^p, M', a^p) \cap (\mathfrak{T}_a^{st} \cup (\_, M, \_) \cup \mathfrak{T}_b^{st})$ consists of the stability conditions σ for which b^p, M', a^p are semistable and:*

$$\begin{array}{ccc} \begin{array}{c}\phi(a^p) - 1 < \phi(M') < \phi(a^p) \\ \phi(a^p) - 1 < \phi(b^p) < \phi(a^p)\end{array} & or & \begin{array}{c}\phi(a^p) - 1 < \phi(M') < \phi(a^p) \\ \phi(b^p) < \phi(M')\end{array} \\ or \ \begin{array}{c}\phi(b^p) - 1 < \phi(M') < \phi(b^p) \\ \phi(b^p) - 1 < \phi(a^p) - 1 < \phi(b^p)\end{array} & or & \begin{array}{c}\phi(b^p) - 1 < \phi(M') < \phi(b^p) \\ \phi(M') + 1 < \phi(a^p)\end{array} \end{array}. \tag{4.161}$$

It follows that $(b^p, M', a^p) \cap (\mathfrak{T}_a^{st} \cup (\_, M, \_) \cup \mathfrak{T}_b^{st})$ and $(b^p, M', a^p) \cup (\mathfrak{T}_a^{st} \cup (\_, M, \_) \cup \mathfrak{T}_b^{st})$ are contractible.

Proof. Due to Lemmas 4.56 and 4.57, to prove the inclusion $\subset$ it remains only to show that the incidence $\sigma \in (b^p, M', a^p) \cap (\_, M, \_)$ implies some of the systems in (4.161). Assume that $\sigma \in (b^p, M', a^p) \cap (a^q, M, b^{q+1})$ for some $q \in \mathbb{Z}$. From table (4.128) we see that $b^p, M', a^p, a^q, M, b^{q+1}$ are semi-stable and:

$$\begin{array}{ccc} \phi(b^p) < \phi(M') + 1 & & \phi(a^q) < \phi(M) + 1 \\ \phi(b^p) < \phi(a^p) & \text{and} & \phi(a^q) < \phi(b^{q+1}) \\ \phi(M') < \phi(a^p) & & \phi(M) < \phi(b^{q+1}) \end{array}. \tag{4.162}$$

If $p \leq q$, then the non-vanishings $\hom(a^p, a^q) \neq 0$, $\hom^1(b^{q+1}, M') \neq 0$, and $\hom(M, b^p) \neq 0$ (see Corollary 4.19) together with (4.162) imply the following inequalities $\phi(a^p) \leq \phi(a^q) < \phi(M) + 1 \leq \phi(b^p) + 1$ and $\phi(a^p) \leq \phi(a^q) < \phi(b^{q+1}) \leq \phi(M') + 1$, which combined with (4.162) amount to the system in the first row and the first column of (4.161).

If $q < p$, then the non-vanishings $\hom(M', a^q) \neq 0$, $\hom(b^{q+1}, b^p) \neq 0$, and $\hom^1(a^p, M) \neq 0$ together with (4.162) imply the inequalities $\phi(M') \leq \phi(a^q) < \phi(b^{q+1}) \leq \phi(b^p)$ and $\phi(a^p) \leq \phi(M) + 1 < \phi(b^{q+1}) + 1 \leq \phi(b^p) + 1$. The system in the second row and the first column in (4.161) follows. So far we proved the incusion $\subset$.

Assume that $b^p, M', a^p \subset \sigma^{ss}$ and that (4.161) holds. Each of the systems in (4.161) contains in it the inequalities of (b^p, M', a^p) from table (4.128), hence $\sigma \in (b^p, M', a^p)$. Lemmas 4.58 and 4.59 ensure that $\sigma \in (\mathfrak{T}_a^{st} \cup (\_, M, \_) \cup \mathfrak{T}_b^{st})$ as well and the first part of the corollary follows.

Now the arguments are analogous to those given in the end of the proof of Lemma 4.54.

The four systems in (4.161) correspond to four open subsets of $(b^p, M', a^p) \cap (\mathfrak{T}_a^{st} \cup (\_, M, \_) \cup \mathfrak{T}_b^{st})$. We denote these subsets by $S_{11}, S_{12}, S_{21}, S_{22}$, where S_{ij} corresponds to the system in the i-th row and j-th. The first part of the corollary and Remark 4.67 reduce the proof of the last statement to proving that $\bigcup_{1 \leq i,j \leq 2} S_{ij}$ is contractible .

All of $S_{11}, S_{12}, S_{21}, S_{22}$ are contractible since they are homeomorphic to convex subsets of $\mathbb{R}^6$.

One can show that:

- $S_{11} \cap S_{12}$ is homeomorphic to $\mathbb{R}^3_{>0} \times \{\phi_2 - 1 < \phi_0 < \phi_1 < \phi_2\}$
- $S_{21} \cap (S_{11} \cup S_{12}) = S_{21} \cap S_{11}$ is homeomorphic to $\mathbb{R}^3_{>0} \times \{\phi_2 - 1 < \phi_1 < \phi_0 < \phi_2\}$
- $S_{22} \cap (S_{11} \cup S_{12} \cup S_{21}) = S_{22} \cap S_{21}$ is homeomorphic to $\mathbb{R}^3_{>0} \times \{\phi_0 - 1 < \phi_1 < \phi_2 - 1 < \phi_0\}$.

Since the obtained subsets of $\mathbb{R}^6$ are convex, in particular contractible, it follows by Remark 4.67 that $\bigcup_{1\leq i,j\leq 2} S_{ij}$ is contractible. The corollary follows. □

We can prove now Theorem 4.1:

Theorem 4.61. $\mathrm{Stab}(D^b(Q))$ *is contractible.*

Proof. Recall that $\mathrm{Stab}(\mathcal{T}) = \mathfrak{T}^{st}_a \cup (\_, M', \_) \cup (\_, M, \_) \cup \mathfrak{T}^{st}_b$ (see (4.92)). Recalling (4.127) we get:

$$\mathrm{Stab}(D^b(Q)) = \mathfrak{T}^{st}_a \cup (\_, M, \_) \cup \mathfrak{T}^{st}_b \cup \bigcup_{k\in\mathbb{Z}} (b^k, M', a^k). \tag{4.163}$$

Corollary 4.55 says that $\mathfrak{T}^{st}_a \cup (\_, M, \_) \cup \mathfrak{T}^{st}_b$ is contractible and it remains to show that after adding $\bigcup_{k\in\mathbb{Z}}(b^k, M', a^k)$ the result is still contractible.

We first prove that for any two integers $q > p$ we have:

$$(b^p, M', a^p) \cap (b^q, M', a^q) \subset (b^p, M', a^p) \cap (\mathfrak{T}^{st}_a \cup (\_, M, \_) \cup \mathfrak{T}^{st}_b). \tag{4.164}$$

Assume that $\sigma \in (b^p, M', a^p) \cap (b^q, M', a^q)$. Then in table (4.128) we see that

$$\begin{array}{ccc} \phi(b^p) < \phi(M') + 1 & & \phi(b^q) < \phi(M') + 1 \\ \phi(b^p) < \phi(a^p) & \text{and} & \phi(b^q) < \phi(a^q) \\ \phi(M') < \phi(a^p) & & \phi(M') < \phi(a^q) \end{array} . \tag{4.165}$$

Since $p < q$, we have the non-vanishings $\hom(a^p, b^q) \neq 0$ and $\hom^1(a^q, b^p) \neq 0$ (see (4.73)). We combine with (4.165) as follows $\phi(a^p) \leq \phi(b^q) < \phi(a^q) \leq \phi(b^p) + 1$ and $\phi(a^p) \leq \phi(b^q) < \phi(M') + 1$, hence $\phi(a^p) - 1 < \phi(b^p)$ and $\phi(a^p) - 1 < \phi(M')$. In (4.165) we have also $\phi(b^p) < \phi(a^p)$ and $\phi(M') < \phi(a^p)$ and the system in the first row and the first column of (4.161) follows. Therefore by Corollary 4.60 we get $\sigma \in (b^p, M', a^p) \cap (\mathfrak{T}^{st}_a \cup (\_, M, \_) \cup \mathfrak{T}^{st}_b)$ and we obtain the inclusion (4.164). This implies that for any $p \in \mathbb{Z}$ and any $n \geq 1$ holds the following equality:

$$(b^p, M', a^p) \cap \left(\mathfrak{T}^{st}_a \cup (\_, M, \_) \cup \mathfrak{T}^{st}_b \bigcup_{k=1}^{n} (b^{p+k}, M', a^{p+k}) \right) = (b^p, M', a^p) \cap \left(\mathfrak{T}^{st}_a \cup (\_, M, \_) \cup \mathfrak{T}^{st}_b \right).$$

In Corollary 4.60 we proved that $(b^p, M', a^p) \cap (\mathfrak{T}^{st}_a \cup (\_, M, \_) \cup \mathfrak{T}^{st}_b)$ and $(b^p, M', a^p) \cup (\mathfrak{T}^{st}_a \cup (\_, M, \_) \cup \mathfrak{T}^{st}_b)$ are contractible (for any $p \in \mathbb{Z}$). Now using the equality above and Remark 4.67 one easily shows by induction that $\mathfrak{T}^{st}_a \cup (\_, M, \_) \cup \mathfrak{T}^{st}_b \cup \bigcup_{k=0}^{n} (b^{p+k}, M', a^{p+k})$ is contractible for any $p \in \mathbb{Z}$ and any $n \geq 1$. Applying Remark 4.67 again we deduce that the right-hand side of (4.163) is contractible as well. Therefore $\mathrm{Stab}(D^b(Q))$ is contractible. □

Appendix

4.A Some contractible subsets of $\mathbb{R}^6$

We prove here that some subsets of $\mathbb{R}^6$, which we meet in the proof of Theorem 4.1, are contractible. We start by the following subset

Lemma 4.62. *The set $U_>$, given below, is contractible:*

$$U_> = \left\{ (r_0, r_1, r_2, \phi_0, \phi_1, \phi_2) \in \mathbb{R}^6 : \begin{array}{c} r_i > 0 \\ \phi_0 < \phi_1 < \phi_0 + 1 \\ \phi_0 < \phi_2 < \phi_0 + 1 \\ \arg_{(\phi_0,\phi_0+1)}(r_0 \exp(\mathrm{i}\pi\phi_0) + r_1 \exp(\mathrm{i}\pi\phi_1)) > \phi_2 \end{array} \right\}. \tag{4.166}$$

The set $U_<$ defined by the same inequalities, except the last, where we take $\arg_{(\phi_0,\phi_0+1)}(r_0 \exp(\mathrm{i}\pi\phi_0) + r_1 \exp(\mathrm{i}\pi\phi_1)) < \phi_2$ is contractible as well.

Proof. By drawing a picture one easily shows that:

$$\forall (r_0, r_1, r_2, \phi_0, \phi_1, \phi_2) \in U_> \qquad \begin{array}{c} r'_1 \geq r_1 \\ r'_2 > 0 < r'_0 \leq r_0 \\ \phi_0 < \phi_1 \leq \phi'_1 < \phi_0 + 1 \\ \phi_0 < \phi'_2 \leq \phi_2 < \phi_0 + 1 \end{array} \Rightarrow (r'_0, r'_1, r'_2, \phi_0, \phi'_1, \phi'_2) \in U_>. \tag{4.167}$$

Let $\gamma : \mathbb{S}^n \to U$ be a continuous map with $n \geq 1$. Denote

$$0 < r_0^{min} = \min\{r_0(t) : t \in \mathbb{S}^n\}; \quad 0 < r_1^{max} = \max\{r_1(t) : t \in \mathbb{S}^n\};$$
$$0 < u = \max\{\phi_1(t) - \phi_0(t) : t \in \mathbb{S}^n\} < 1; \quad 0 < v = \min\{\phi_2(t) - \phi_0(t) : t \in \mathbb{S}^n\} < 1;$$

then by (4.167) for any $\delta > 0$ and any $t \in \mathbb{S}^n$, $s \in [0,1]$ the vector given below lies in $U_>$:

$$F(t,s) = \left(\begin{array}{c} r_0(t)(1-s) + s r_0^{min}, r_1(t)(1-s) + s r_1^{max}, r_2(t)(1-s) + s\delta, \\ \phi_0(t), \phi_0(t) + (1-s)(\phi_1(t) - \phi_0(t)) + su, \phi_0(t) + (1-s)(\phi_2(t) - \phi_0(t)) + sv \end{array} \right).$$

Hence we obtain a map $F : \mathbb{S}^n \times [0,1] \to U_>$, whose continuity is obvious. This gives a homotopy from the map γ to the following continuous map:

$$\gamma' : \mathbb{S}^n \to U_> \qquad \gamma'(t) = (r_0^{min}, r_1^{max}, \delta, \phi_0(t), \phi_0(t)+u, \phi_0(t)+v) \tag{4.168}$$

Now we note that:

$$\forall (r_0, r_1, r_2, \phi_0, \phi_1, \phi_2) \in U \quad \forall \delta \in \mathbb{R} \quad (r_0, r_1, r_2, \phi_0+\delta, \phi_1+\delta, \phi_2+\delta) \in U_> \tag{4.169}$$

Therefore for $t \in \mathbb{S}^n$, $s \in [0,1]$ we have

$$G(t,s) = \left(\begin{array}{c} r_0^{min}, r_1^{max}, \delta, \phi_0(t) + s(\phi_0(0) - \phi_0(t)), \\ \phi_0(t) + u + s(\phi_0(0) - \phi_0(t)), \phi_0(t) + v + s(\phi_0(0) - \phi_0(t)) \end{array} \right) \in U_{>0}$$

which gives a homotopy from γ' to the constant map from $\mathbb{S}^n$ to the point $(r_0^{min}, r_1^{max}, \delta, \phi_0(0), \phi_0(0)+u, \phi_0(0)+v) \in U_{>0}$. Thus, we proved that each continuous map $\gamma : \mathbb{S}^n \to U_>$ with $n \geq 1$ is homotopic to a constant map. If we show that $U_>$ is connected, then Whitehead theorem ensures that $U_>$ is contractible. Let $x = (r_0, r_1, r_2, \phi_0, \phi_1, \phi_2) \in U_>$ and $x' = (r'_0, r'_1, r'_2, \phi'_0, \phi'_1, \phi'_2) \in U_>$. By (4.169) we can move continuously x' in $U_>$ to $x'' = (r''_0, r''_1, r''_2, \phi_0, \phi''_1, \phi''_2)$ and now by (4.167) we can connect x, x'' by a continuous path in $U_>$.

The same idea shows that $U_<$ is contractible, one must permute $\leq \leftrightarrow \geq$, min $\leftrightarrow$ max. The lemma is proved. $\square$

Corollary 4.63. *The set V, given below, is contractible:*

$$V = \left\{ (r_0, r_1, r_2, \phi_0, \phi_1, \phi_2) \in \mathbb{R}^6 : \begin{array}{c} r_i > 0 \\ \phi_2 - 1 < \phi_0 < \phi_2 \\ \phi_2 - 1 < \phi_1 < \phi_2 \\ \arg_{(\phi_2-1,\phi_2)}(r_0 \exp(\mathrm{i}\pi\phi_0) - r_2 \exp(\mathrm{i}\pi\phi_2)) > \phi_1 \end{array} \right\}. \tag{4.170}$$

After changing the last inequality to $\arg_{(\phi_2-1,\phi_2)}(r_0 \exp(\mathrm{i}\pi\phi_0) - r_2 \exp(\mathrm{i}\pi\phi_2)) < \phi_1$ the set remains contractible.

Proof. The assignment $(a_0, a_1, a_2, b_0, b_1, b_2) \mapsto (a_2, a_0, a_1, b_2 - 1, b_0, b_1)$ maps homeomorphically the set V to the set U in Lemma 4.62. $\square$

Corollary 4.64. *The set V, given below, is contractible:*

$$V = \left\{ (r_0, r_1, r_2, \phi_0, \phi_1, \phi_2) \in \mathbb{R}^6 : \begin{array}{c} r_i > 0 \\ \phi_0 - 1 < \phi_1 < \phi_0 \\ \phi_0 - 1 < \phi_2 < \phi_0 \\ \arg_{(\phi_0-1,\phi_0)}(r_0 \exp(\mathrm{i}\pi\phi_0) + r_2 \exp(\mathrm{i}\pi\phi_2)) > \phi_1 \end{array} \right\}. \tag{4.171}$$

Proof. The assignment $(a_0, a_1, a_2, b_0, b_1, b_2) \mapsto (a_0, a_2, a_1, -b_0, -b_2, -b_1)$ maps homeomorphically the set V to the set $U_<$ in Lemma 4.62(see (4.4)). □

Lemma 4.65. *The set U, given below, is contractible:*

$$U = \left\{ (r_0, r_1, r_2, \phi_0, \phi_1, \phi_2) \in \mathbb{R}^6 : \begin{array}{c} r_i > 0 \\ \phi_2 < \phi_1 < \phi_0 < \phi_2 + 1 \\ \arg_{(\phi_2, \phi_2+1)}(r_0 \exp(\mathrm{i}\pi\phi_0) + r_2 \exp(\mathrm{i}\pi\phi_2)) < \phi_1 \end{array} \right\}. \quad (4.172)$$

Proof. By drawing a picture one checks that:

$$\forall (r_0, r_1, r_2, \phi_0, \phi_1, \phi_2) \in U \quad \begin{array}{c} r_2' \geq r_2 \\ r_1' > 0 < r_0' \leq r_0 \end{array} \Rightarrow (r_0', r_1', r_2', \phi_0, \phi_1, \phi_2) \in U. \quad (4.173)$$

Let $\gamma : \mathbb{S}^n \to U$ be any continuous map with $n \geq 1$. Denote

$$\begin{gathered} 0 < r_0^{min} = \min\{r_0(t) : t \in \mathbb{S}^n\}; \quad 0 < r_2^{max} = \max\{r_2(t) : t \in \mathbb{S}^n\}; \\ 0 < u = \min\{\phi_1(t) - \phi_2(t) : t \in \mathbb{S}^n\} < 1. \end{gathered} \quad (4.174)$$

By drawing a picture one sees that for big enough $A > r_2^{max}$ we have

$$\forall \phi_2 \forall \phi_0 \quad \phi_2 < \phi_0 < \phi_2 + 1 \Rightarrow \arg_{(\phi_2, \phi_2+1)}(r_0^{min} \exp(\mathrm{i}\pi\phi_0) + A \exp(\mathrm{i}\pi\phi_2)) - \phi_2 < u. \quad (4.175)$$

This implication means that for any $\delta > 0$ the set U', given below, is contained in U:

$$U' = \left\{ (r_0^{min}, \delta, A, \phi_0, \phi_1, \phi_2) \in \mathbb{R}^6 : \begin{array}{c} \phi_2 < \phi_1 < \phi_0 < \phi_2 + 1 \\ u \leq \phi_1 - \phi_2 \end{array} \right\} \subset U \quad (4.176)$$

where A, r_0^{min}, u are fixed in (4.174), (4.175) and we chose any $\delta > 0$. By (4.173) we see that for any $t \in \mathbb{S}^n$, $s \in [0, 1]$ we have:

$$F(t, s) = \big(r_0(t)(1 - s) + s r_0^{min}, r_1(t)(1 - s) + s\delta, r_2(t)(1 - s) + sA, \phi_0(t), \phi_1(t), \phi_2(t)\big) \in U.$$

Hence we obtain a continuous map $F : \mathbb{S}^n \times [0, 1] \to U$, which is a homotopy in U from the map γ to the following continuous map:

$$\gamma' : \mathbb{S}^n \to U \qquad \gamma'(t) = (r_0^{min}, \delta, A, \phi_0(t), \phi_1(t), \phi_2(t)).$$

Furthermore, by (4.174) we have $u \leq \phi_1(t) - \phi_2(t)$ for $t \in \mathbb{S}^n$, which means that $\mathfrak{im}(\gamma') \subset U'$. Since U' is contractible, there exists a homotopy in U' from γ' to a constant map. Since $U' \subset U$(see (4.176)), there exists a homotopy in U from γ to a constant map.

We show below that U is connected, and then by Whitehead theorem U is contractible.

Let $x = (a_0, a_1, a_2, b_0, b_1, b_2) \in U$ and $x' = (a'_0, a'_1, a'_2, b'_0, b'_1, b'_2) \in U$. The formula (4.169) holds again, and by using it we can move continuously x' in U to a point $x'' = (a'_0, a'_1, a'_2, b''_0, b''_1, b_2)$. If we denote

$$0 < r_0^{min} = \min\{a_0, a'_0\}, 0 < r_2^{max} = \max\{a_2, a'_2\}, 0 < u = \min\{b_1 - b_2, b''_1 - b_2\} < 1$$

then choose $A > r_2^{max}$ so that (4.175) holds with the chosen u, r_0^{min}, r_2^{max}, in particular for any $\delta > 0$ the corresponding set U' defined by (4.176) is a subset of U. By the properties (4.173) and by the choice of u, r_0^{min}, A, δ we can move the points x and x'', by changing only $a_0, a_1, a_2, a'_0, a'_1, a'_2$, continuously in U to points y, y' in U', respectively. Now the connectivity of U follows from the connectivity of U'. The lemma is proved. □

Corollary 4.66. *The set V, given below, is contractible:*

$$V = \left\{ (r_0, r_1, r_2, \phi_0, \phi_1, \phi_2) \in \mathbb{R}^6 : \begin{array}{c} r_i > 0 \\ \phi_2 - 1 < \phi_1 < \phi_0 < \phi_2 \\ \arg_{(\phi_2 - 1, \phi_2)}(r_0 \exp(\mathrm{i}\pi\phi_0) - r_2 \exp(\mathrm{i}\pi\phi_2)) < \phi_1 \end{array} \right\}. \quad (4.177)$$

Proof. The assignment $(a_0, a_1, a_2, b_0, b_1, b_2) \mapsto (a_0, a_1, a_2, b_0, b_1, b_2 - 1)$ maps homeomorphically the set V to the set U in Lemma 4.65. □

Remark 4.67. *If we have two contractible open subsets U, V in a f.d. manifold M and the interesection $U \cap V$ is contractible, then by Seifert-van Kampen theorem, Mayer-Vietoris sequence, Hurewicz theorem and Whitehead theorem it follows that $U \cup V$ is contractible.*

If $U = \bigcup_{i \in A} U_i$ is an union of open subsets in a f.d. manifold M and for any finite subset $F \subset A$ we have that $\bigcup_{i \in F} U_i$ is contractible, then using Witehead theorem one can easily deduce that U is contractible as well.

Bibliography

[1] M. Barot, *Representations of quivers*, `http://www.matem.unam.mx/barot/articles/notes_ictp.pdf` 1, 98, 117, 118

[2] A.A. Beilinson, J. Bernstein and P. Deligne, *Faisceaux Pervers*, Astérisque 100, Soc. Math. de France (1982). 8, 34

[3] I. N. Bernšteĭn, I. M. Gelfand, and V. A. Ponomarev, *Coxeter functors, and Gabriel's theorem*, Uspehi Mat. Nauk 28 (1973), no. 2(170), 19-33. iv, 82, 99

[4] A. I. Bondal,*Representations of associative algebras and coherent sheaves*, Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989), no. 1, 25-44 (Russian); English transl., *Math. USSR-Izv.* 34 (1990), no. 1, 23-42. 111

[5] A. I. Bondal and D. Orlov, *Derived categories of coherent sheaves*, Proc. Internat. Congress of Mathematicians vol. II 46-54 (Beijing, 2002) 124

[6] A. I. Bondal and A. E. Polishchuk, *Homological properties of associative algebras: the method of helices*, Izv. Ross. Akad. Nauk. Ser. Mat. 57 (1993), no. 2, 3–50; English transl. in *Russian Acad. Sci. Izv. Math.* 42 (1994), no. 2, 219–260. 123

[7] A. Bondal and M. Kapranov, *Representable functors, Serre functors and mutations*, Math. USSR Izvestija, 35 (1990), 519–541 112

[8] T. Bridgeland, *Stability conditions on triangulated categories*, Annals of Math. 166 no. 2, 317-345 (2007). 2, 19, 20, 21, 23, 24, 101, 106, 125, 128, 148, 151

[9] T. Bridgeland, *Stability conditions on K3 surfaces*, Duke Math. J. 141, no. 2, 241-291 (2008). 20

[10] T. Bridgeland and I. Smith, *Quadratic differentials as stability conditions*, arXiv:1302.7030 5, 103, 125

[11] T. Bridgeland, Y. Qiu, and T. Sutherland. *Stability conditions and the A2 quiver*, to appear in J. Reine Angew. Math. (Crelles' journal), arXiv:1406.2566v1, June 2014. 125

[12] N. Broomhead, D. Pauksztello, and D. Ploog. *Discrete derived categories II: The silting pairs CW complex and the stability manifold.* arXiv:1407.5944 2014 3, 125

[13] P. Browne, http://www.maths.nuigalway.ie/ pbrowne/personal/research.php 97

[14] J. Collins and A. Polishchuck, *Gluing Stability Conditions*, Adv. Theor. Math. Phys. Volume 14, Number 2 (2010), 563-608. 8, 111, 113, 114, 125

[15] W. Crawley-Boevey, *Exceptional sequences of representations of quivers*, in 'Representations of algebras', Proc. Ottawa 1992, eds V. Dlab and H. Lenzing, Canadian Math. Soc. Conf. Proc. 14 (Amer. Math. Soc., 1993), 117-124. 12, 17, 18, 61, 80, 100, 114, 147

[16] W. Crawley-Boevey, *Lectures on Representations of Quivers*, `http://www1.maths.leeds.ac.uk/~pmtwc/quivlecs.pdf` 12, 13, 14, 28, 48, 81, 85, 98, 102, 107, 117, 144

[17] G. Dimitrov, F. Haiden, L. Katzarkov, and M. Kontsevich, *Dynamical systems and categories*, In The Influence of Solomon Lefschetz in Geometry and Topology: 50 Years of Mathematics at CINVESTAV, volume 621 of Con. Math., pages 133-170. American Mathematical Society (AMS), Providence, RI, 2014. 2, 5, 6, 18, 25, 103, 125

[18] G. Dimitrov, L. Katzarkov, *Non-semistable exceptional objects in hereditary categories*, arXiv:1311.7125v1, submitted to International Mathematics Research Notices 2, 3

[19] G. Dimitrov and L. Katzarkov, *Non-semistable exceptional objects in hereditary categories: some remarks and conjectures*, to appear in the proceedings of CATS4 Higher Categorical Structures and their Interactions with Algebraic Geometry, Algebraic Topology and Algebra; Cont. Math., American Mathematical Society (AMS). arXiv:1405.2943 2, 3, 5, 81

[20] G. Dimitrov and L. Katzarkov, *Bridgeland stability conditions on the acyclic triangular quiver*, arxiv: 1410.0904, submitted to Advances in Mathematics 2, 3, 7

[21] Mátyás Domokos and Helmut Lenzing, *Invariant theory of canonical algebras*, J. Algebra 228 (2000), no. 2, 738-762. 7

[22] M. Douglas, *D-branes, categories and N = 1 supersymmetry. Strings, branes, and M-theory*, J. Math. Phys. 42 (2001), no. 7, 2818-2843. 2

[23] M Douglas, *Dirichlet branes, homological mirror symmetry, and stability*, Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002), 395?408, Higher Ed. Press, Beijing, 2002. 2

[24] Vlastimil Dlab and Claus Michael Ringel, *Indecomposable representations of graphs and algebras*, Mem. Amer. Math. Soc. **6** (1976), no. 173, v+57. iv, 82, 99

[25] D. Gaiotto, G. Moore, and A. Neitzke, *Wall-crossing, Hitchin systems, and the WKB approximation*, Adv. Math 234 (2013) 5, 103, 125

[26] Galakhov D, Longhi P, Mainiero T, Moore GW, Neitzke A (2013) *Wild wall crossing and BPS giants*. J High Energ Phys 1311:46, arxiv:1305.5454v1 111

[27] W. Geigle and H. Lenzing, *A class of weighted projective curves arising in representation theory of finite dimensional algebras*, in "Singularities, Representations of Algebras, and Vector Bundles," Lecture Notes in Math., Vol. 1273, pp. 265-297, Springer-Verlag, New YorkrBerlin, 1987 iv, 7

[28] I. Gelfand, Y. Manin, *Methods of Homological algebra*, 2nd edition, Springer. 21, 112, 143

[29] I. M. Gelfand and V. A. Ponomarev, *Problems of linear algebra and classification of quadruples of subspaces in a finite-dimensional vector space*, Coll. Math. Soc. Bolyai, Tihany(Hungary), **5** (1970), 163-237. 82

[30] A. Holtmann, *The s-tame dimension vectors for stars*, Journal of algebra, Volume 289, Issue 1, 1 July 2005, Pages 277-311 82

[31] V.G. Kac: *Infinite root systems, representations of graphs and invariant theory.* Inventiones mathematicae 56, 57-92 (1980). 62, 102, 105, 106, 111, 130

[32] B. Keller and P. Nicolás, *Weight structures and simple dg modules for positive dg algebras.* Int. Math. Res. Not. IMNR 2013, no. 5, 1028–1078. 111

[33] M. Kontsevich and Y. Soibelman, *Motivic Donaldson-Thomas invariants: summary of results, Mirror symmetry and tropical geometry*, 55–89, Contemp. Math., 527, Amer. Math. Soc., Providence, RI, 2010. 5, 103, 125

[34] A. King, *Moduli of representations of finite dimensional algebras*, Quart. J. Math. Oxford Ser. 45 (1994), 515–530. 6, 105, 125, 127, 130

[35] T. Y. Lam, *A first course in non-commutative rings*, Springer-Verlag. 27

[36] H. Lenzing, *Hereditary Categories Lectures 1,2, Advanced ICTP-school on Representation Theory and Related Topics (9-27 January 2006)*, `http://www.math.jussieu.fr/~keller/ictp2006/lecturenotes/lenzing1.pdf` 27

[37] E. Macrì, S*tability conditions on curves*, Math. Res. Lett. 14 (2007) 657-672. Also arXiv:0705.3794. iv, 3, 5, 6, 8, 9, 11, 24, 25, 26, 31, 61, 99, 100, 111, 113, 125, 127, 129, 130, 140

[38] S. Okada, *Stability manifold of* $\mathbb{P}^1$, J. Algebraic Geom., 15(3):487-505, 2006.

[39] M. Perling, *Some quivers describing the derived categories of the toric del Pezzos*, `http://www.mathematik.uni-kl.de/~perling/quivers.pdf` 123

[40] M. Reineke, *Moduli of Representations of Quivers*, Trends in representation theory of algebras and related topics, 589–637, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008. 111

[41] J. Rickard and R. Rouqier, *Stable categories and reconstruction*, arXiv:1008.1976 111

[42] C.M. Ringel, *Exceptional objects in hereditary categories*, Proceedings Constantza Conference, An. St. Univ. Ovidius Constantza Vol. 4 (1996), f. 2, 150-158 80

[43] C.M. Ringel, *PBW-bases of quantum groups*, J. Reine Angew. Math. 470 (1996),51-88 iv, 82, 99

[44] C. M. Ringel, *Indecomposable representations of the Kronecker quivers*, Proc. Amer. Math. Soc. 141 (2013), no. 1, 115–121. 111

[45] A. Schofield, *Semi-invariants of quivers.* J. London Math. Soc. 43 (1991), 383-395. 80

[46] A. Schofield, *The internal structure of real Schur representations*, preprint. 80

[47] A. Skowroński, *Module categories over tame algebras*, Canadian mathematical society, Conference Proceedings, Volume 19 (1996), 281-312 82

[48] Yu Qiu, *Stability conditions and quantum dilogarithm identities for Dynkin quivers*, Adv. Math., 269 (2015), pp 220-264, arXiv:1111.1010. 125

[49] J. Woolf, *Algebraic stability conditions and contractible stability spaces*, arXiv:1407.5986. iv, 3, 125

[50] T. Weist. *On the Euler characteristic of Kronecker moduli spaces* - 2013. J. Algebraic Combin 38 no. 3, 567-583., arXiv:1203.2740 111

[51] T. Weist, *Localization in quiver moduli spaces*, Represent. Theory 17 (2013), 382–425. 111

Zusammenfassung auf Deutsch

Der "Moduliraum" der Stabilitätsbedingungen ist derzeit ein wichtiger Bestandteil der homologischen Spiegelsymmetrie (HMS). Er wurde von T. Bridgeland (2002) als ein Zugang zu einem matematischen Verständnis von bestimmten in der Stringtheorie auftretenden Moduliräumen eingeführt. Er ordnete jeder triangulierten Kategorie eine komplexe Mannigfaltigkeit zu, deren Elemente als Bridgeland Stabilitätsbedingungen bezeichnet werden. HMS sagt eine Parallele zwischen dynamischer Systeme und Kategorien voraus, wobei der Raum der Bridgelandstabilitätsbedingungen ein Kandidat für die Rolle des Teichmüller Raum spielen soll. Jedoch sind globale Informationen über den Stabilitätsraum nur in einer Hand voll Beispielen bekannt.

Lange vor HMS (1994), erkannten Beilinson et. al. Strukturen in einigen triangulierten Kategorien, die sie außergewöhnliche Sammlungen (exceptionall collections) nannten (die Abhandlung von Beilinson erschien im Jahr 1978).

Die Hauptmotivation für die vorliegende Arbeit kommt aus einer Prozedur für Erzeugung von Stabilitätsbedingungen durch außergewöhnlichen Sammlungen, die von E. Macrì in seiner Arbeit aus dem Jahr 2007 beschrieben wurde.

Die vorliegende Dissertation untersucht einige Aspekte des Zusammenspiels zwischen den beiden Begriffen im Titel und präsentiert Neuheiten für beide Seiten. Auf der einen Seite unterstützen die Erkentnisse und Beweise über Stabilitätsbedingungen die oben genannte Parallele. Auf der anderen Seite treten bemerkenswerte Beziehungen zwischen außergewöhnlichen Darstellungen von Köcher in der Dissertation auf.

Die Arbeit besteht aus drei Teilen.

Im ersten Teil wird der Begriff der *σ-außergewöhnlichen Sammlung (σ-exceptional collection)* definiert, so dass jede σ-außergewöhnliche Sammlung (falls vorhanden) σ erzeugt, wobei σ eine Stabilitätsbedingung bezeichnet. Der Fokus liegt hier auf dem Konstruieren von σ-außergewöhnlichen Sammlungen aus einer gegebenen Stabilitätsbedingung σ auf $D^b(\mathcal{A})$, wobei $\mathcal{A}$ eine erbliche, hom-finite-Kategorie, linear über einem algebraisch abgeschlossenen Körper ist. Eine Schwierigkeit kommt von den *Ext-nicht-triviale Paare (Ext-nontrivial couples)*: dies sind außergewöhnliche Objekte $X, Y \in \mathcal{A}$ mit nicht verschwindenden $Ext^1(X,Y)$ und $Ext^1(Y,X)$. Eine neue Einschränkung für die Kategorie $\mathcal{A}$, genannt *Regularitätserhalt (regularity-preserving)*, macht dieses Problem kontrollierbar. Beispiele für Kategorien mit Regularitätserhalt werden demonstriert. Schließlich wird bewiesen, dass alle Stabilitätsbedingungen auf dem azyklische Dreiecksköcher aus außergewöhnlichen Sammlungen erzeugt werden.

Das zentrale Ergebnis im zweiten Teil der Arbeit ist eine Charakterisierung der Dynkin / Euklidischen / alle anderen Köcher mit der Sprache der Bridgelandstabilitätsbedingungen.

Der dritte Teil setzt das Studium des gesamten Raumes der Stabilitätsbedingungen auf dem azyklischen Dreiecksköcher fort. Die wichtigste Schlussfolgerung hier ist, dass dieser Raum zusammenziehbar ist. Dies ist das erste Beispiel eines Köchers Q, verschieden von Dynkin und Kronecker Köcher, für den bewiesen wurde, dass der Stabilitätsraum auf $D^b(Q)$ zusammenziehbar ist. Daraus folgt, dass der Stabilitätsraum auf der gewichteten projektiven Gerade $\mathbb{P}^1(1,2)$ zusammenziehbar ist.

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Brief CV

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Degrees

Expected in May of 2015 — PhD in Mathematics, University of Vienna, Austria,
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July of 2005 — MSc in Mathematics, Sofia University "St. Kliment Ohridski", Sofia, Bulgaria,
Master thesis *"On a class of 8-dimensional complex homogeneous spaces",*
Advisor: Prof. Vasil Tsanov .
Average grade: Excellent (6.00) [1]

July of 2003 — BSc in Communication Technique and Technologies, Technical University of Sofia, Bulgaria,
Bachelor thesis: *"Five-Layer Symmetric Dielectric Slab Waveguide",*
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Publications and preprints

- G. Dimitrov and L. Katzarkov, *Bridgeland stability conditions on the acyclic triangular quiver*, arxiv: 1410.0904, submitted
- G. Dimitrov and L. Katzarkov, *Non-semistable exceptional objects in hereditary categories: some remarks and conjectures*, arXiv:1405.2943, to appear in the proceedings of CATS4, Higher Categorical Structures and their Interactions with Algebraic Geometry, Algebraic Topology and Algebra
- G. Dimitrov and L. Katzarkov, *Non-semistable exceptional objects in hereditary categories*, arxiv: 1311.7125, submitted
- G. Dimitrov, F. Haiden, L. Katzarkov, and M. Kontsevich, *Dynamical systems and categories*, In The Influence of Solomon Lefschetz in Geometry and Topology: 50 Years of Mathematics at CINVESTAV, volume 621 of Con. Math., pages 133-170. American Mathematical Society, (AMS), Providence, RI, 2014. arXiv:1307.8418
- G. Dimitrov and V. Tsanov, *Homogeneous hypercomplex structures II - Coset Spaces of compact Lie Groups*, arxiv: 1204.5222
- G. Dimitrov and V. Tsanov, *Homogeneous hypercomplex structures I - the compact Lie groups*, arxiv: 1005.0172, submitted
- G. Dimitrov and I. Mladenov, *A new formula for the exponents of the generators of the Lorentz group*, In: Proceedings of the Seventh International Conference on Geometry, Integrability and Quantization, SOFTEX, Sofia 2006, pp 98 - 115.
- G. Dimitrov and I. Mladenov, *On the exponents of some 4x4 matrices*, In: Proceedings of the Thirty Fifth Spring Conference of the Union of Bulgarian Mathematicians, Borovets, April 5-8, 2006 pp 152-158.

[1] Grading system: the minimum grade is 2.00, the minimum passing grade is Satisfactory 3.00, and the highest grade is Excellent 6.00.

Selected talks

November 2014 *Bridgeland stability conditions on the acyclic triangular quiver*
Workshop on Wall Crossing, Quantum Integrable Systems, and TQFT; Simons Center for Geometry and Physics (Stony Brook)

February 2014 *Bridgeland stability conditions and exceptional collections*
Conference on Homological Mirror Symmetry, University of Miami (Florida)

November 2013 *Property D*
Workshop on Categories and Complexity, University of Vienna (Austria)

November 2013 *Homogeneous hypercomplex structures - Compact Lie groups and their coset spaces*
Seminar on Complex geometry, Ruhr-Universität Bochum (Germany)

July 2013 *Density of phases via Kronecker pairs*
A meeting dedicated to Vasil Tsanov on his 65-th birthday (Sofia, Bulgaria)

June 2013 *Kronecker pairs II*
Workshop on Higher Categories and Topological Quantum Field Theories, University of Vienna (Austria)

May 2013 *Kronecker pairs*
Workshop on Birational Geometry and Geometric Invariant Theory, University of Vienna (Austria)

February 2012 *Homogeneous hypercomplex structures*
Winter school on TQFT, Langlands and Mirror Symmetry (Huatulco, Mexico)

August 2009 *Hypercomplex structures on compact Lie groups*
Workshop on Homological Mirror Symmetry and Hodge Theory, University of Vienna (Austria)

August 2008 *Complex structures related to SL(3)*
9th International Workshop on Complex Structures, Integrability and Vector Fields (Sofia, Bulgaria)

May 2008 *Moduli spaces of complex structures on the real forms of SL(3,C)*
6th Spring School and Workshop on QFT & HAMILTONIAN SYSTEMS (Calimanesti-Caciulata, Romania)

June 2007 *Complex structures on SU(2,1)*
Fourth Advanced Research Workshop on Gravity, Astrophysics, and strings at the Black sea (Kiten, Bulgaria)

April 2006 *On the exponents of some 4x4 matrices*
35th Spring Conference of the Union of Bulgarian Mathematicians (Borovets, Bulgaria)