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Causality theory for $C^{1,1}$ metrics

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This thesis is dedicated to the loving memory of my father. He is sorely missed but his spirit will always live in my heart.

Contents

Introduction

One of the greatest achievements of the 20th century - general relativity, the geometric theory of physics of gravity, space and time, was obtained by Albert Einstein exactly one hundred years ago. Lorentzian geometry and, in particular, causality theory, the mathematical foundations of general relativity, have since been considered as part of mathematical physics as well as of differential geometry. The development of the theory hence branched into many different areas of mathematics and physics and has continued to inspire researchers even a hundred years after.

Traditionally, the standard references on general relativity and causality theory assume either smoothness of the spacetime metric, see e.g. [BEE96], [Kr99], [MS08], [ON83], [Pen72], or C^2-differentiability, see e.g. [Chr11], [Cl93], [HE73], [Se98], [GaSe05]. Despite the mathematical convenience of the C^2 assumption there are a number of good reasons for considering spacetimes with metrics of lower differentiability. From a physical point of view, one would like to study systems where there is a jump in the material composition or density as for example, when one crosses the surface of a star. However, the issue of regularity of the metric becomes apparent when matching different spacetimes over a common boundary. On matching these regions the matter variables become discontinuous, forcing the differentiability of the metric via the field equations to be less than C^2. The approach presented by Lichnerowicz in [L55] deals with metrics which are piecewise C^3 but globally are only C^1. More extreme cases are seen in the example of impulsive waves (see e.g. [GP09, Ch. 20]) where the metric is still C^3 off the impulse but globally is merely C^0.

In addition, from the point of view of PDE theory, the questions of regularity are essential when solving the initial value problem. In the classical local existence theorem for the vacuum Einstein equations (see [CG69]) the assumption on the regularity of the metric is H^s_{loc} with $s > 5/2$ and more recent studies have significantly lowered the regularity ([KR05, M06, KRS12]).

The arguments presented above have led to an increased interest in causality theory with metrics of low regularity. Various approaches to causal structures and discussions on the problems that can occur when the regularity of the metric is below C^2 have been presented in [Chr11], [CG12], [Cl93], [HE73], [MS08], [Se98]. As emphasised by Senovilla in [Se98], the main problem is the existence of normal coordinates and their regularity, as well as the existence of totally normal (convex) neighborhoods. However, it has recently been demonstrated in [Chr11] that the entire smooth causality theory can be preserved for C^2-metrics.

Moreover, in the early years of general relativity, the appearance of some kind of irregularities of spacetimes or a singular behavior of solutions of the Einstein field equations was thought to be arising from the high degree of symmetry or that it was unphysical in some way. This way of thinking changed in 1965 with the first modern singularity theorem presented by Penrose in his paper [Pen65], where it was shown that deviations from spherical symmetry could not prevent gravitational collapse. He also introduced the notion of closed trapped surface and used geodesic incompleteness to mathematically characterize a singular spacetime. Shortly afterwards, several papers by Hawking, Penrose, Ellis, Geroch and others followed, leading to the development of singularity theorems that are considered to be one of the greatest achievements within general relativity. However, the conclusion of the singularity theorems is their weak point. In fact, they only show the existence of an incomplete causal geodesic but do not say much about the nature of the singularity. In particular, it is not clear whether the curvature blows up (see, however [Cl82, Cl93] as well as [SeGa14, Sec. 5.1.5] and the references therein) or whether the singularity is simply a result of the differentiability dropping below C^2. Nevertheless, in the case that the regularity of the metric simply drops to $C^{1,1}$ (continuously differentiable with locally Lipschitz first order derivatives, often also denoted by C^{2-}), the curvature is still locally bounded which, from the physical point of view, would not be regarded as a singular behavior since it corresponds to a finite jump in the matter variables. Examples of physically realistic systems of this type are given by the Oppenheimer-Snyder model of a collapsing star [OppSny39] and general matched spacetimes, see e.g. [L55, MaSe93].

Therefore a reasonable candidate for the lowest degree of differentiability where one could expect the standard results of causality theory to remain valid is given by the $C^{1,1}$ regularity class. Indeed, it represents the threshold of the unique solvability of the geodesic equation. Also, as already indicated by Senovilla in [Se98], $C^{1,1}$ regularity of the metric is the natural differentiability class from the point of view of the singularity theorems as well. Since the existence of normal coordinates, normal neighborhoods and maximal curves is vastly important when proving the singularity theorems, explicit places where the C^2-assumption enters the proofs of the singularity theorems and the number of technical difficulties a proof in the $C^{1,1}$-case would have to overcome is presented in [Se98, Sec. 6.1].

On the other hand, it is well-known that below $C^{1,1}$ many standard properties cease to be true: explicit counterexamples by Hartman and Wintner [HW51] show that for metrics of Hölder regularity $C^{1,\alpha}$ with $0 < \alpha < 1$, radial geodesics may fail to be minimizing between any two points that they contain. Also a recent study of the causality theory for continuous metrics in [CG12] has proved that many fundamental results of smooth causality are wrong: light cones need no longer be topological hypersurfaces of codimension one and there may exist causal curves that are not everywhere null but for which there is no fixed-endpoint deformation into a timelike curve, i.e., the push-up principle is no longer true in general. In fact, for any $0 < \alpha < 1$ there are metrics of regularity $C^{0,\alpha}$, called 'bubbling metrics', whose light-cones have nonempty interior, and for whom the push-up principle ceases to hold. In addition, for metrics which are merely continuous it is still unknown whether or not the timelike futures remain open. Nevertheless, Chrusciel and Grant established many key results of causality theory for continuous metrics in [CG12] such as the existence of smooth time functions on Cauchy developments, see also [FS12].

One of the main ingredients for studying local causality and therefore, singularity theory is the exponential map. In smooth pseudo-Riemannian geometry, the fact that the exponential map is a diffeomorphism locally around 0 is highly important for many fundamental constructions such as normal coordinates, normal neighborhoods, injectivity radius but also for comparison methods and in the Lorentzian case, for studying local causality theory. However, the standard way of proving this result uses the inverse function theorem which is no longer applicable in the case of $C^{1,1}$ metrics as the exponential map is then only Lipschitz.

The aim of this thesis is to first show that the exponential map of a $C^{1,1}$ pseudo-Riemannian metric retains its maximal possible regularity, namely, that is a bi-Lipschitz homeomorphism locally around 0. Proving in addition the existence of totally normal neighborhoods will allow us to establish one of the main results of local causality theory, that is, relating the causal structure of Minkowski spacetime to that of a manifold in any given point. Once the key elements of causality theory for $C^{1,1}$ metrics are developed, we will prove the Hawking singularity theorem in this regularity.

This work is organized as follows. In Chapter 1 we give a brief introduction to pseudo-Riemannian geometry. Using approximation techniques and methods from comparison geometry, we then prove that the exponential map of a $C^{1,1}$ pseudo-Riemannian metric is locally a bi-Lipschitz homeomorphism. We also establish the existence of totally normal neighborhoods in an appropriate sense.

In Chapter 2, we show that the standard results of local causality theory remain valid for $C^{1,1}$ metrics. In particular, we will establish the push-up principle and the existence of accumulation curves.

Chapter 3 is devoted to the global structure of spacetimes. We review causality conditions and basic notions such as Cauchy developments and Cauchy hypersurfaces and we further develop the causality theory for $C^{1,1}$ metrics.

Finally, in Chapter 4 we provide a detailed proof of Hawking's singularity theorem in the regularity class $C^{1,1}$ using the results on causality theory obtained in previous chapters.

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Chapter 1

Uniqueness and regularity of geodesics

1.1 Introduction to pseudo-Riemannian Geometry

The aim of this section is to introduce basic definitions and notions that will be used throughout. In what follows we give a brief historical overview of the development of pseudo-Riemannian geometry and present its most important results and tools that will be used in this work based on [Chr11] and [ON83]. Our basic reference for differential geometry is [ON83].

In the well known geometry of the Euclidian space $\mathbb{R}^3$ the notions of length and angles are defined by means of its natural inner product, that is, the dot product. The natural isomorphism $T_p\mathbb{R}^3 \approx \mathbb{R}^3$ allows us to deploy the dot product on each tangent space and therefore perform basic geometric operations such as measuring the length of a tangent vector or the angle between two tangent vectors. The study of curves and surfaces, together with their properties, was of great importance even in ancient times, and today these concepts have significant applications to many different fields such as physics. The discovery of calculus in the 17th century enabled further investigations of curves and surfaces in the three-dimensional Euclidean space that built the basis for the development of differential geometry in the 18th and 19th century.

A crucial contribution to the theory of surfaces was made by Carl Friedrich Gauss, who created the notion of Gaussian curvature and clarified the distinction between intrinsic and extrinsic quantities: that is, between geometrical quantities that are observed by the inhabitants of the surface, hence entirely determined by it, and those which depend on how the surface is positioned in the surrounding space. This led to the "Theorema Egregium", where it was established that the Gaussian curvature is an intrinsic invariant, i.e., that it can be determined entirely by measuring distances along paths on the surface itself. Bernhard Riemann extended these two special cases to n-dimensional manifolds, introducing the famous construction central to his geometry, now known as a Riemannian metric. In the 20th century, after Albert Einstein's theory of general relativity was introduced, further generalization appeared: the positive definiteness of the Riemannian metric was weakened to nondegeneracy, leading to the development of pseudo-Riemannian geometry. With the very influential text by Hawking and Ellis in the 70's, [HE73], progress on causality theory, singularity theory and black holes in general relativity has been achieved leading to increased interest in global Lorentzian geometry, see [BEE96].

Pseudo-Riemannian geometry is the study of smooth manifolds equipped with a nondegenerate bilinear symmetric $(0,2)$-tensor field of arbitrary signature called pseudo-Riemannian manifolds. We will denote by (M, g) a smooth pseudo-Riemannian manifold of dimension n with a metric tensor g for which the regularity will be explicitly stated. Assuming M to be a C^∞-manifold is no loss of generality since any C^k-manifold with $k \geq 1$ possesses a unique C^∞-structure that is C^k-compatible with the given C^k-structure on M (see [Hi76, Th. 2.9]). Riemannian manifolds are an important special class of pseudo-Riemannian manifolds, for which the metric tensor is positive definite and thus of signature $(+, ..., +)$. Hence the metric induced on the tangent space of a Riemannian manifold is Euclidean. Another important case of pseudo-Riemannian manifolds which is our primary interest, is the case of Lorentzian manifolds that are equipped with a metric tensor of signature $(-, +, ..., +)$.

If $x^1, ..., x^n$ is a coordinate system on $U \subseteq M$ then the metric components on U are $g_{ij} := g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$, $1 \leq i, j \leq n$. Thus for vector fields $v = \sum v^i \frac{\partial}{\partial x^i}$, $w = \sum v^i \frac{\partial}{\partial x^j}$, $g(v, w) = \sum g_{ij} v^i w^j$. The metric in coordinates is often written as the line element $ds^2 = g_{ij} dx^i dx^j$. We will often use an alternative notation for g and write $g(v, w) = \langle v, w \rangle$ for tangent vectors $v, w \in T_pM$, $p \in M$.

In standard references to general relativity, [BEE96], [HE73], [Kr99], [ON83], [Wa84], the metric is assumed to be smooth. It has recently been demonstrated in [Chr11] that all the results from smooth causality theory hold true if the regularity of the metric is assumed to be C^2 and the curves considered are locally Lipschitz, unlike in usual treatments of causality theory where the corresponding curves are required to be piecewise smooth. The approach presented in [Chr11] appears to be very convenient for several reasons, primarily for studying the global causal structure of spacetimes where the key tool is taking limits of causal curves, see Section 2.5 for more detail. Hence we assume in the following introductory part that the metric g is C^2 and that curves are locally Lipschitz:

Let h be some auxiliary Riemannian metric such that (M, h) is complete- such a metric always exists, cf. [NO61], and denote by d_h the corresponding distance function. A curve $\alpha : I \to M$ is called *locally Lipschitz* if for every compact set K of I there exists a constant $C(K)$ such that

$$\forall t_1, t_2 \in K,\ d_h(\alpha(t_1), \alpha(t_2)) \leq C(K)|t_1 - t_2|.$$

Given a locally Lipschitz curve $\alpha : [a, b] \to M$, its length is defined by:

$$L(\alpha) := \int_a^b \|\alpha'(t)\| dt,$$

where

$$\|\alpha'(t)\| = \sqrt{g(\alpha'(t), \alpha'(t))}.$$

Remark 1.1.1. By Rademacher's theorem any locally Lipschitz curve α is differentiable almost everywhere. Also, this class of curves is independent of the choice of a background Riemannian metric h and these curves can be parametrized by h arc-length, see for more details Chapter 2.

Now let (M, g) be a Lorentzian manifold. A nonzero tangent vector v is said to be timelike, null, causal or spacelike if $g(v, v) < 0$, $= 0$, ≤ 0, or ≥ 0, respectively. A locally Lipschitz curve α is called timelike, null, causal or spacelike if $\alpha'(t)$ has the corresponding property almost everywhere.

If M admits a continuous, nowhere vanishing, timelike vector field X, then M is said to be time oriented by X. This vector is used to separate all causal vectors at each point into two classes called future directed and past directed. A *spacetime* is then a Lorentzian manifold (M, g) together with a choice of time orientation.

1.2 Geodesics

The Levi-Civita connection on a pseudo-Riemannian manifold (M, g) will be denoted by ∇. In a coordinate system $x^1, ..., x^n$ on U one has (cf. [ON83, Prop. 3.13]):

$$\nabla_{\partial_i}(\sum W^j \partial_j) = \sum_k \left(\frac{\partial W^k}{\partial x^i} + \sum_j \Gamma^k_{ij} W^j \right) \partial_k,$$

where Γ^k_{ij} are the Christoffel symbols given by:

$$\Gamma^k_{ij} = \frac{1}{2} \sum_m g^{km} \left(\frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m} \right).$$

The natural generalization of a straight line in Euclidean space is in a pseudo-Riemannian manifold M given by a *geodesic*: a curve $\alpha : I \to M$ whose vector field α' is parallel, i.e., a curve of zero acceleration, $\alpha'' = 0$. Using a local coordinate system $\{x^1, ..., x^n\}$ on $U \subseteq M$, the geodesic equation is then given by

$$\frac{d^2(x^k)}{dt^2} + \sum_{i,j} \Gamma^k_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0, \tag{1.1}$$

for $1 \leq k \leq n$, where x^i is an abbreviation for the coordinate functions $x^i \circ \alpha$ of a curve α. Hence, a curve α in U is a geodesic of M if and only if it satisfies the equation (1.1). Since the geodesic equation can be re-written as a first order system of ordinary differential equations, the standard existence and uniqueness theorem for ODE's implies that there is a unique solution to (1.1).

We summarize the most important properties of geodesics in the following (cf. [ON83, Chapter 3, Section Geodesics]):

Theorem 1.2.1. *Let M be a pseudo-Riemannian manifold. Then:*

1. *For any $p \in M$ and any vector $v \in T_pM$ there exists a unique maximal geodesic $\alpha_v : I_v \to M$ with $0 \in I_v$, $\alpha_v(0) = p$ and $\alpha'_v(0) = v$.*

2. *For $t \in I_v$ and $w := \alpha'_v(t)$, $I_w = I_v - t$ and $\alpha_w(s) = \alpha_v(t + s)$, $\forall s \in I_w$.*

3. *If $0 \neq \lambda \in \mathbb{R}$ then $\alpha_{\lambda v}(t) = \alpha_v(\lambda t)$, $\forall t \in I_{\lambda v} = \lambda^{-1} I_v$.*

4. *$\mathscr{G} := \{(t,v) \mid t \in I_v\}$ is open in $\mathbb{R} \times TM$ and the map $f : \mathscr{G} \to M$, $(t,v) \mapsto \alpha_v(t)$ is smooth on $\mathscr{G}$.*

A central object in pseudo-Riemannian geometry and a very important tool in the theory of general relativity is the *Riemann curvature tensor*. It is the generalization of Gaussian curvature to arbitrary pseudo-Riemannian manifolds and it is a way of measuring the curvature of spacetimes. Following [ON83], we define it to be given by $R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$. This convention differs by a sign from that of [HE73]. Other important curvature tensors in general relativity can then be defined in terms of the Riemann curvature tensor: the *Ricci tensor* by $R_{ab} = R^c{}_{abc}$ (which again differs by a sign from that in [HE73] where $R_{ab} = R^c{}_{acb}$, so overall the two definitions of Ricci curvature agree) and the *scalar curvature* $S = g^{ab}R_{ab} = R^b_b$.

Another way of describing the curvature of pseudo-Riemannian manifolds is in terms of the *sectional curvature*: Let $p \in M$ and let v, w be any basis of a two-dimensional subspace P of the tangent space T_pM. Then

$$K(v,w) = \frac{\langle R_{vw}v, w\rangle}{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle}$$

is called the sectional curvature $K(P)$ of P.

Note that K is defined only for timelike and spacelike planes since the denominator is negative in the first case and positive in the second.

A fundamental property of curvature is its control over the relative behavior of nearby geodesics. The following notion arises naturally when describing the difference between two infinitesimally close geodesics:

Definition 1.2.2. A vector field J along a geodesic α is called a *Jacobi field* if it satisfies the *Jacobi equation*:

$$\frac{D^2}{dt^2}J(t) + R(J(t), \alpha'(t))\alpha'(t) = 0,$$

where D is the covariant derivative with respect to the Levi-Civita connection.

Consider now a geodesic $\alpha : [0,1] \to M$ with $\alpha(0) = p$ and $\alpha(1) = q$. A point q is said to be *conjugate* to p if there exists a non-zero Jacobi field J along α such that $J(0) = 0$ and $J(1) = 0$.

1.3 The exponential map

Now let (M, g) be an n-dimensional pseudo-Riemannian manifold with a smooth metric tensor g. Let $p \in M$. For

$$D_p := \{v \in T_pM \mid \text{inextendible geodesic } \alpha_v \text{ is defined at least on } [0,1]\},$$

the exponential map is given by $\exp_p : D_p \to M$ such that $\exp_p(v) = \alpha_v(1)$ for all $v \in D_p$. Hence intuitively speaking, the exponential map assigns to a tangent vector $v \in T_pM$ a geodesic on the manifold starting at p and going in that direction for a unit time. The actual distance traveled will depend on the vector v since it corresponds to the velocity vector of the geodesic.

Given a point p in a Riemannian manifold (M, g) and $v \in T_pM$, the geodesic α_v will be locally minimizing in the sense that there exists $t > 0$ such that

$$d_g(p, \alpha(t)) = L(\alpha|_{[0,t]}).$$

Generally, a geodesic need not be minimizing. It can happen that there are two different geodesics of the same length between two points on a manifold or that there are two conjugate points along a geodesic. Contrary to the Riemannian case, on a Lorentzian manifold, causal geodesics are locally the longest curves connecting two points.

Now, $D_p \subseteq T_pM$ is obviously the largest set on which the exponential map can be defined. If M is complete, then $D_p = T_pM$, for every point $p \in M$. Thinking about the exponential map with such a property, i.e., being defined on the whole tangent space, justifies the notion of *geodesic completeness*: A manifold is said to be *geodesically complete* if all geodesics can be defined for all real values of the affine parameter, which is clearly equivalent to the requirement that the domain of the exponential map is T_pM, for every point $p \in M$.

The following theorem, due to Hopf and Rinow, in particular shows that compact Riemannian manifolds are geodesically complete:

Theorem 1.3.1. *(Hopf-Rinow Theorem) Let (M, g) be a connected Riemannian manifold. Then the following statements are equivalent:*

1. *M is a complete metric space (with respect to the Riemannian distance d)*
2. *M is geodesically complete*

3. The closed and bounded subsets of M are compact.

There is no Lorentzian analogue of this, see [Chr11, Example 2.2.1].

Even if the exponential map is defined on the whole tangent space, it need not be a global diffeomorphism in general. However, by the fact that its differential at the origin of the tangent space is the identity map and using the inverse function theorem, one can prove that it is a local diffeomorphism (cf. [ON83, Prop. 3.30]):

Theorem 1.3.2. *Let (M, g) be a pseudo-Riemannian manifold with a smooth metric g and let $p \in M$. Then there exist neighborhoods $\tilde{U} \subseteq T_pM$ around 0 and $U \subseteq M$ around p such that*

$$\exp_p : \tilde{U} \to U$$

is a diffeomorphism.

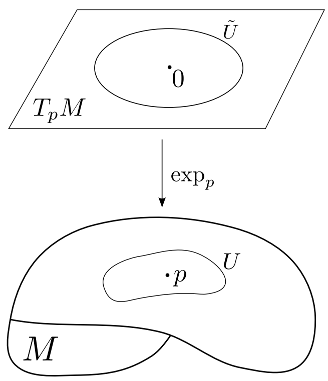

Figure 1.1: The exponential map

A subset A of a vector space is called *starshaped* around 0 if for any vector $v \in A$, it follows that $tv \in A$, for all $t \in [0, 1]$. Now, if $\tilde{U}$ is starshaped around 0, then U is called a normal neighborhood of p. Then a special kind of coordinate system can be defined for any normal neighborhood U, called a *normal coordinate system*: Fix an orthonormal basis $e_1, ..., e_n$ for T_pM and let

$$\exp_p^{-1}(q) =: x^i e_i, \tag{1.2}$$

for $q \in U$, $i = 1, ..., n$. Then $((x^1, ..., x^n), U)$ is called the normal coordinate system around p. We have (cf. [ON83, Lemma 3.14]):

Proposition 1.3.3. *Let $x^1, ..., x^n$ be a normal coordinate system at $p \in M$. Then for all $1 \leq i, j, k \leq n$, it follows 1. $g_{ij}(p) = \delta_{ij}\varepsilon_j$ and 2. $\Gamma^k_{ij}(p) = 0$.*

Now let (M, g) be a Riemannian manifold, $p \in M$ and denote by $B_g(0, r) := \{v \in T_pM \mid \langle v, v \rangle < r\}$ the open ball of radius r. By Theorem 1.3.2, there exists $r > 0$ such that $\exp_p : B_g(0, r) \to B_g(p, r)$ is a diffeomorphism, where $B_g(p, r) := \{q \in M \mid d_g(p, q) < r\}$. Once there exist two distinct geodesics between p and some point q, the exponential map ceases to be injective. On the other hand, if there exist two conjugate points along a geodesic, the exponential map is no longer a local diffeomorphism. This leads to the following definition:

Definition 1.3.4. The injectivity radius $\mathrm{Inj}(p)$ of (M, g) at $p \in M$ is the supremum of values of radii r such that the exponential map defines a global diffeomorphism from $B_g(0, r)$ onto its image in M.

In the Lorentzian case, defining the injectivity radius is not as straightforward. The main obstacle in defining it directly with respect to the Lorentzian metric g is that the Lorentzian norm of a non-zero vector may vanish. Hence, the definition will depend on the background Riemannian metric (cf. [CleF08]).

For a complete pseudo-Riemannian manifold M, the exponential maps $\exp_p : T_pM \to M$ for all $p \in M$, constitute a single mapping $\exp : TM \to M$ of the tangent bundle TM. Define

$$E : TM \to M$$

$$E(v) = (\pi(v), \exp(v)),$$

where π is the natural projection of TM onto M. Explicitly, $E(v) = (p, \exp_p(v))$, for $v \in T_pM \subset TM$.

Whether or not M is complete, exp and E have the same largest domain-the set U of all vectors $v \in TM$ such that the geodesic α_v is defined at least on the interval $[0, 1]$. Again, by the inverse function theorem, one can show the following:

Theorem 1.3.5. *The map E is a diffeomorphism of a neighborhood of $TM_0 := \{0_p | \ p \in M\} \subseteq TM$ onto a neighborhood of $\triangle_M := \{(p, p) | \ p \in M\}$ in $M \times M$.*

1.4 The exponential map of a $C^{1,1}$-metric

In the preceding sections we have seen that in smooth pseudo-Riemannian geometry, the fact that the exponential map is a local diffeomorphism is the key tool for many fundamental constructions such as normal coordinates, normal neighborhoods, injectivity radius but also for comparison methods and in the Lorentzian case, for studying local causality theory, see Chapter 2.

As already indicated in the introduction, the way of proving this result is based on an application of the inverse function theorem. It is hence possible to do such a proof for C^2 pseudo-Riemannian metrics since the exponential map is then C^1. However, lowering the differentiability of the metric below C^2 causes many problems. On the other hand, it is generally held in the literature that $C^{1,1}$ (continuously differentiable with Lipschitz derivatives) delimits the regularity where one can still reasonably expect the 'standard' results to remain valid since it is the lowest regularity of the metric for which the geodesic equation is still uniquely solvable. Nevertheless, the exponential map for $C^{1,1}$-metrics is only Lipschitz hence the inverse function theorem is no longer applicable.

The aim of this section is to show that the exponential map of a $C^{1,1}$ pseudo-Riemannian metric retains its maximal possible regularity, namely that it is a bi-Lipschitz homeomorphism locally around 0, following [KSS14].

Our notation is standard, cf., e.g., [Jo11, ON83]. If K is a compact set in M we write $K \Subset M$.

Our strategy is to regularize the metric locally via convolution with a mollifier to obtain a net g_ε of smooth metrics of the same signature. We then use methods from comparison geometry to obtain sufficiently strong estimates on the exponential maps of the regularized metrics to be able to carry the bi-Lipschitz property through the limit $\varepsilon \to 0$. More precisely, we rely on new comparison methods, developed only recently by B. L. Chen and P. LeFloch in their studies on the injectivity radius of Lorentzian metrics ([CleF08]). In the Riemannian case, one may alternatively use the Rauch comparison theorem, as well as injectivity radius estimates due to Cheeger, Gromov and Taylor, as will be pointed out in Section 3. To show existence of totally normal neighborhoods we use the uniform estimates derived above to adapt the standard proof of the smooth case.

We first recall what is known about the exponential map of $C^{1,1}$ pseudo-Riemannian metrics due to J. H. C. Whitehead's paper [Wh32]. Consider a system of ODEs of the form

$$\frac{d^2c^k}{dt^2} + \Gamma^k_{ij}(c(t))\frac{dc^i}{dt}\frac{dc^j}{dt} = 0, \tag{1.3}$$

where the Γ^k_{ij} are functions symmetric in i, j. Note that they need not be the Christoffel symbols of some metric. A curve that satisfies Equation (1.3) is called a *path* and the theory of these paths is called the geometry of paths, see [Wh32].

A well known fact in Riemannian geometry is that in a normal neighborhood U around p and for any $q \in U$ the radial geodesic segment between p and q is the unique shortest curve connecting these two points and is entirely contained in U. Similar question arose in the geometry of paths: Under which conditions does a neighborhood around some point exist in which any two points can be connected by a unique path that does not leave that neighborhood? In [Wh32, Sec. 3] Whitehead proved that only assuming that $\Gamma^k_{ij}(c(t))$ are Lipschitz and symmetric in their lower indices such a neighborhood, called a *simple region*, exists. In particular, this result holds when Equation (1.3) is considered as the geodesic equation of a $C^{1,1}$ pseudo-Riemannian metric g of arbitrary signature since in this case, the Christoffel symbols $\Gamma^k_{ij}(c(t))$ are Lipschitz. It follows that $\exp^g_p : (\exp^g_p)^{-1}(S) \to S$ is continuous and bijective, hence a homeomorphism by invariance of domain, and one has:

Theorem 1.4.1. *Let M be a smooth manifold with a $C^{1,1}$ pseudo-Riemannian metric g and let $p \in M$. Then there exist open neighborhoods U of $0 \in T_pM$ and V of p in M such that*

$$\exp^g_p : U \to V$$

is a homeomorphism.

We thus strengthen Theorem 1.4.1 by additionally establishing the bi-Lipschitz property of $\exp^g_p$. We note, however, that our proof is self-contained and will not pre-suppose Th. 1.4.1. Rather, it implicitly provides an alternative proof for this result.

Hence we prove the following theorem:

Theorem 1.4.2. *Let M be a smooth manifold with a $C^{1,1}$ pseudo-Riemannian metric g and let $p \in M$. Then there exist open neighborhoods U of $0 \in T_pM$ and V of p in M such that*

$$\exp^g_p : U \to V$$

is a bi-Lipschitz homeomorphism.

As already indicated, our method of proof is to approximate g by a net g_ε of smooth pseudo-Riemannian metrics and then use comparison results to control the relevant geometrical quantities derived from the g_ε uniformly in ε so as to preserve the bi-Lipschitz property as

$\varepsilon \to 0$. By $B_h(p,r)$ we denote the open ball around the point p of radius r with respect to the Riemannian metric h. To distinguish exponential maps stemming from various metrics we will use a superscript, as in $\exp_p^g$.

Since the result is local, we may assume $M = \mathbb{R}^n$ and $p = 0$[1]. The standard Euclidean metric on $\mathbb{R}^n$ will be denoted by g_E or $\langle\, ,\, \rangle_E$ and we write $\|\ \|_E$ for the corresponding standard Euclidean norm, as well as for mapping norms induced by the Euclidean norm.

Now take $\rho \in \mathcal{D}(\mathbb{R}^n)$ with unit integral and define the standard mollifier $\rho_\varepsilon := \varepsilon^{-n}\rho\left(\frac{x}{\varepsilon}\right)$ $(\varepsilon > 0)$. We set $g_\varepsilon := g * \rho_\varepsilon$ (componentwise convolution).

Remark 1.4.3. For later reference, we note the following properties of the approximating net g_ε.

(i) It is well known that $g_\varepsilon \to g$ in $C^1(M)$. Also, the second derivatives of g_ε are bounded, uniformly in ε, on compact sets: For any $k \geq 1$, the norm on $C^{k,1}$ is given by

$$\|f\|_{C^{k,1}(\overline{U})} := \sum_{|s|\leq k} \|\partial^s f\|_{C^0(\overline{U})} + \sum_{|s|=k} \text{Höl}_1(\partial^s f, \overline{U}),$$

where $\text{Höl}_1(f,S) := \sup\{\frac{|f(x)-f(y)|}{|x-y|} |\ x \neq y,\ x,y \in S\}$ for a closed and bounded set S. Consider now $f \in C^{0,1}$. Then we have:

$$\frac{|f*\rho_\epsilon(x) - f*\rho_\epsilon(y)|}{|x-y|} = \left|\int \frac{f(x-z)-f(y-z)}{|x-y|}\rho_\epsilon(z)dz\right| \leq \text{Höl}_1(f,S),$$

and since $\frac{|f*\rho_\epsilon(x)-f*\rho_\epsilon(y)|}{|x-y|} \to |Df * \rho_\epsilon(x)|$ as $y \to x$, it follows that $\forall\epsilon, \forall x \in S$, $|Df * \rho_\epsilon(x)| \leq \text{Höl}_1(f,S)$. Therefore $D^2 g_\epsilon$ is uniformly bounded on compact sets.

(ii) On any compact subset of M, for ε sufficiently small, g_ε is a pseudo-Riemannian metric. Indeed, for the eigenvalues λ^i and λ^i_ε of g and g_ε respectively, by [GKOS01, Lemma 3.2.76], $|\lambda^i - \lambda^i_\varepsilon| \leq \|g - g_\varepsilon\|$ thus $\lambda^i_\varepsilon \to \lambda^i$, i.e., all the eigenvalues are of the same sign for ε small enough. By the above we obtain that the g_ε form a family of pseudo-Riemannian metrics of the same signature as g whose Riemannian curvature tensors R_ε are bounded uniformly in ε.

In order to proceed we need to determine a neighborhood of 0 in T_pM that is a common domain for all $\exp_p^{g_\varepsilon}$ for ε sufficiently small. Here, and in several places later on, we will

[1]Nevertheless we will write p below to distinguish considerations in T_pM from those in M.

make use of the following consequence of a standard result on the comparison of solutions to ODE [Die80, 10.5.6, 10.5.6.1]:

Lemma 1.4.4. *Let $F, G \in C(H, X)$ where H is a convex open subset of a Banach space X. Suppose:*

$$\sup_{x \in H} \|F(x) - G(x)\| \leq \alpha,$$

G is Lipschitz continuous on H with Lipschitz constant $\leq k$, and F is locally Lipschitz on H. For $\mu > 0$, define

$$\varphi(\xi) := \mu e^{\xi k} + \alpha(e^{\xi k} - 1)\frac{1}{k}, \ \xi \geq 0.$$

Let $x_0 \in H$, $t_0 \in \mathbb{R}$ and let u be a solution of $x' = G(x)$ with $u(t_0) = x_0$ defined on $J := (t_0 - b, t_0 + b)$ such that $\forall t \in J$, $\overline{B(u(t), \varphi(|t - t_0|))} \subseteq H$. Then for every $\tilde{x} \in H$ with $\|\tilde{x} - x_0\| \leq \mu$ there exists a unique solution v of $x' = F(x)$ with $v(t_0) = \tilde{x}$ on J with values in H. Moreover, $\|u(t) - v(t)\| \leq \varphi(|t - t_0|)$ for $t \in J$.

We rewrite the geodesic equation for the metric g as a first order system:

$$\begin{aligned} \frac{dc^k}{dt} &= y^k(t) \\ \frac{dy^k}{dt} &= -\Gamma^k_{g,ij}(c(t))y^i(t)y^j(t) \end{aligned} \tag{1.4}$$

and analogously for the metrics g_ε. Hence, $\exp^g_p(v) = c(1)$ where $c(0) = p, \ y(0) = v$. Let $t_0 = 0$ and $x_0 = (p, 0)$. We fix $b > 1$ and set $J = (-b, b)$. Now take u to be the constant solution of (1.4) with initial condition $x_0 = (p, 0)$, let $\delta > 0$ and set $H := B(x_0, 2\delta) \subseteq \mathbb{R}^{2n}$. The Christoffel symbols Γ_g are Lipschitz functions on H, and by Remark 1.4.3 (i) it follows that there is a common Lipschitz constant k for Γ_g and the Γ_{g_ε} on H. Choose $\alpha > 0, \ \mu > 0$ such that

$$\varphi(b) = \mu e^{bk} + \frac{\alpha}{k}(e^{bk} - 1) < \delta.$$

and choose $\varepsilon_0 > 0$ such that $\forall \varepsilon < \varepsilon_0$ we have $\sup_H \|\Gamma_g - \Gamma_{g_\varepsilon}\| \leq \alpha$. Then $\overline{B(u(t), \varphi(|t|))} \subseteq H, \ \forall t \in J$. By Lemma 1.4.4, for all $\tilde{x} = (p, w) \in H$ with $\|\tilde{x} - x_0\| = \|w\| \leq \mu$, there exists a unique solution u_ε on $(-b, b)$ of

$$\begin{aligned} \frac{dc^k}{dt} &= y^k(t) \\ \frac{dy^k}{dt} &= -\Gamma^k_{g_\varepsilon,ij}(c(t))y^i(t)y^j(t), \end{aligned}$$

with values in H and $u_\varepsilon(0) = \tilde{x} = (p, w)$, as well as a unique solution to (1.4) with these initial conditions. Therefore a common domain of $\exp_p^g$ and all $\exp_p^{g_\varepsilon}$ $(\varepsilon < \varepsilon_0)$ is given by $\{w \in \mathbb{R}^n | \ \|w\|_E < \mu\} =: B_E(0, \mu)$.

Remark 1.4.5. From Remark 1.4.3 we obtain that for some $\varepsilon_0 > 0$ we have:

(i) There exists a constant $K_1 > 0$ such that, for $\varepsilon < \varepsilon_0$, $\|R_\varepsilon\|_E \le K_1$ uniformly on $B_E(0, \mu)$.

(ii) For some $K_2 > 0$ and $\varepsilon < \varepsilon_0$,
$$\|\Gamma_{g_\varepsilon}\|_E \le K_2,$$
uniformly on $B_E(0, \mu)$.

Lemma 1.4.6. *Let $r_1 < \min\left(\frac{1}{2K_2}, \frac{1}{2}\mu\right)$. Then for all $\varepsilon < \varepsilon_0$,*
$$\exp_p^{g_\varepsilon}(\overline{B_E(0, r_1)}) \subseteq B_E(p, \mu).$$

Proof. Let $\gamma : [0, r_1] \to M$ be a g_ε-geodesic with $\gamma(0) = p$ and $\|\gamma'(0)\|_E = 1$ and set $s_0 := \sup\{s \in [0, r_1] \mid \gamma|_{[0,s]} \subseteq B_E(p, \mu)\}$. Then $s_0 > 0$ and for $s \in [0, s_0)$ we have
$$\left| \frac{d}{ds} \langle \gamma'(s), \gamma'(s) \rangle_E \right| = 2 |\langle \gamma''(s), \gamma'(s) \rangle_E| = 2 \left| \langle \Gamma_{g_\varepsilon}(\gamma'(s), \gamma'(s)), \gamma'(s) \rangle_E \right| \le 2K_2 \|\gamma'(s)\|_E^3,$$
and therefore $\left| \frac{d}{ds} \|\gamma'(s)\|_E^{-1} \right| \le K_2$. From this, setting $f(s) := \|\gamma'(s)\|_E$, for $s \in [0, s_0)$ we obtain
$$\frac{f(0)}{f(s)} = \int_0^s f(0) \frac{d}{d\tau} \left(\frac{1}{f(\tau)} \right) d\tau + 1 \in [1/2, 3/2],$$
so
$$\frac{1}{2} \|\gamma'(0)\|_E \le \|\gamma'(s)\|_E \le 2 \|\gamma'(0)\|_E \qquad (s \in [0, s_0)). \tag{1.5}$$
Therefore,
$$L_E(\gamma|_{[0,s_0]}) = \int_0^{s_0} \|\gamma'(s)\|_E \, ds \le 2r_1 \|\gamma'(0)\|_E < \mu,$$
implying that $s_0 = r_1$. □

We next want to determine a ball around $0 \in T_pM$ on which each $\exp_p^{g_\varepsilon}$ is a local diffeomorphism. To achieve this, we first need to derive estimates on Jacobi fields along geodesics, based on [CleF08, Sec. 4].

Lemma 1.4.7. *Set* $C_1 := 2K_2$, $C_2 := 4K_1$ *and let*

$$r_2 < \min\left(r_1, \frac{1}{C_1}\log\left(\frac{C_1 + C_2}{C_1/2 + C_2}\right), (2 + C_1)^{-1}\right).$$

Then for $\varepsilon < \varepsilon_0$, *any* g_ε*-geodesic* $\gamma : [0, r_2] \to M$ *with* $\gamma(0) = p$ *and* $\|\gamma'(0)\|_E = 1$ *lies entirely in* $B_E(p, \mu)$. *Moreover, if* J *is a* g_ε*-Jacobi field along* γ *with* $J(0) = 0$ *and* $\|\nabla_{g_\varepsilon,\gamma'} J(0)\|_E = 1$ *then* $\|J(s)\|_E \le 1$ *and* $\frac{1}{2} \le \|\nabla_{g_\varepsilon,\gamma'} J\|_E \le 2$ *for all* $s \in [0, r_2]$.

Proof. By Lemma 1.4.6, γ lies in $B_E(p, \mu)$. Also, (1.5) implies

$$\max_{s\in[0,r_2]} \|\gamma'(s)\|_E \le 2 \max_{s\in[0,r_2]} \|\gamma'(0)\|_E \le 2. \tag{1.6}$$

Suppose that $s_0 := \sup\{s \in [0, r_2] \mid \|J(t)\|_E \le 1\ \forall t \in [0, s]\} < r_2$. By assumption, J satisfies the Jacobi equation

$$\begin{aligned} \nabla_{g_\varepsilon,\gamma'}\nabla_{g_\varepsilon,\gamma'} J(s) &= -R_\varepsilon(J(s), \gamma'(s))\gamma'(s) \\ J(0) = 0, \quad &\|J'(0)\|_E = 1. \end{aligned}$$

Thus by Remark 1.4.5 and (1.6), on $[0, s_0]$ we obtain

$$\begin{aligned} \left|\frac{d}{ds}\langle\nabla_{g_\varepsilon,\gamma'} J, \nabla_{g_\varepsilon,\gamma'} J\rangle_E\right| &= 2|\langle\nabla_{E,\gamma'}\nabla_{g_\varepsilon,\gamma'} J, \nabla_{g_\varepsilon,\gamma'} J\rangle_E| \\ &= 2|\langle\nabla_{g_\varepsilon,\gamma'}\nabla_{g_\varepsilon,\gamma'} J, \nabla_{g_\varepsilon,\gamma'} J\rangle_E + \langle(\nabla_{E,\gamma'} - \nabla_{g_\varepsilon,\gamma'})\nabla_{g_\varepsilon,\gamma'} J, \nabla_{g_\varepsilon,\gamma'} J\rangle_E| \\ &= 2|\langle\nabla_{g_\varepsilon,\gamma'}\nabla_{g_\varepsilon,\gamma'} J, \nabla_{g_\varepsilon,\gamma'} J\rangle_E - \langle\Gamma_{g_\varepsilon}(\nabla_{g_\varepsilon,\gamma'} J, \gamma'), \nabla_{g_\varepsilon,\gamma'} J\rangle_E| \\ &\le 8K_1\|\nabla_{g_\varepsilon,\gamma'} J\|_E + 4K_2\|\nabla_{g_\varepsilon,\gamma'} J\|_E^2, \end{aligned}$$

so that

$$\left|\frac{d}{ds}\|\nabla_{g_\varepsilon,\gamma'} J\|_E\right| \le 4K_1 + 2K_2\|\nabla_{g_\varepsilon,\gamma'} J\|_E = C_1\|\nabla_{g_\varepsilon,\gamma'} J\|_E + C_2. \tag{1.7}$$

Taking into account that $\|\nabla_{g_\varepsilon,\gamma'} J(0)\|_E = 1$ by assumption, integration of (1.7) leads to

$$-\frac{C_2}{C_1} + \left(1 + \frac{C_2}{C_1}\right)e^{-C_1 s} \le \|\nabla_{g_\varepsilon,\gamma'} J(s)\|_E \le -\frac{C_2}{C_1} + \left(1 + \frac{C_2}{C_1}\right)e^{C_1 s}.$$

Due to our choice of r_2, this entails

$$\frac{1}{2} \le \|\nabla_{g_\varepsilon,\gamma'} J\|_E \le 2 \tag{1.8}$$

on $[0, s_0]$. From this, we get

$$\left|\frac{d}{ds}\|J(s)\|_E\right| = \frac{1}{\|J(s)\|_E}\,|\langle\nabla_{g_\varepsilon,\gamma'}J(s), J(s)\rangle_E - \langle\Gamma_{g_\varepsilon}(J(s),\gamma'(s)), J(s)\rangle_E| \le 2 + 2K_2.$$

Therefore,

$$\|J(s)\|_E \le (2+2K_2)s < s/r_2 < 1 \tag{1.9}$$

for $s \in [0, s_0]$. For $s = s_0$, this gives a contradiction to the definition of s_0. □

Lemma 1.4.8. *There exists some $0 < r_3 < r_2$ such that, for all $\varepsilon < \varepsilon_0$, $\exp_p^{g_\varepsilon}$ is a local diffeomorphism on $B_E(0, r_3)$.*

Proof. For any Jacobi field J as in Lemma 1.4.7 we have:

$$\begin{aligned}\frac{d}{ds}\langle\nabla_{g_\varepsilon,\gamma'}J, J\rangle_E &= \langle\nabla_{g_\varepsilon,\gamma'}\nabla_{g_\varepsilon,\gamma'}J, J\rangle_E - \langle\Gamma_{g_\varepsilon}(\nabla_{g_\varepsilon,\gamma'}J,\gamma'), J\rangle_E \\ &\quad + \langle\nabla_{g_\varepsilon,\gamma'}J, \nabla_{g_\varepsilon,\gamma'}J\rangle_E - \langle\nabla_{g_\varepsilon,\gamma'}J, \Gamma_{g_\varepsilon}(J,\gamma')\rangle_E\end{aligned}$$

Of these four terms, the third one is bounded from below by 1/4 due to (1.8). For the others, employing Lemma 1.4.7 (see (1.6), (1.8), (1.9)), we obtain for $s \in [0, r_2]$:

$$\begin{aligned}|\langle\nabla_{g_\varepsilon,\gamma'}\nabla_{g_\varepsilon,\gamma'}J, J\rangle_E(s)| &= |\langle R_\varepsilon(J,\gamma')\gamma', J\rangle_E(s)| \le K_1\|\gamma'(s)\|_E^2\|J(s)\|_E^2 \le 4\frac{K_1}{r_2^2}s^2 \\ |\langle\Gamma_{g_\varepsilon}(\nabla_{g_\varepsilon,\gamma'}J,\gamma'), J\rangle_E(s)| &\le K_2\|\gamma'(s)\|_E\|\nabla_{g_\varepsilon,\gamma'}J(s)\|_E\|J(s)\|_E \le 4\frac{K_2}{r_2}s \\ |\langle\nabla_{g_\varepsilon,\gamma'}J, \Gamma_{g_\varepsilon}(J,\gamma')\rangle_E(s)| &\le K_2\|\gamma'(s)\|_E\|J(s)\|_E\|\nabla_{g_\varepsilon,\gamma'}J(s)\|_E \le 4\frac{K_2}{r_2}s\end{aligned}$$

From this we obtain an $r_3 = r_3(r_2, K_1, K_2) < r_2$ such that on $[0, r_3]$, $\frac{d}{ds}\langle\nabla_{g_\varepsilon,\gamma'}J, J\rangle_E$ is bounded from below by a positive constant. By the same estimates and (1.8) again, it is also bounded from above. Hence for some $c_1 > 0$, any $\varepsilon < \varepsilon_0$ and $s \in [0, r_3]$ we obtain:

$$e^{-c_1} \le \frac{d}{ds}\langle\nabla_{g_\varepsilon,\gamma'}J, J\rangle_E(s) \le e^{c_1},$$

and therefore

$$e^{-c_1}s \le \langle\nabla_{g_\varepsilon,\gamma'}J, J\rangle_E(s) \le e^{c_1}s.$$

Combined with (1.8) and (1.9), this entails:

$$\frac{1}{r_2}s \geq \|J(s)\|_E \geq \frac{\langle \nabla_{g_\varepsilon,\gamma'} J, J\rangle_E(s)}{\|\nabla_{g_\varepsilon,\gamma'} J(s)\|_E} \geq \frac{e^{-c_1}}{2}s.$$

Altogether, we find $c_2 > 0$ such that for all $\varepsilon < \varepsilon_0$ and $s \in [0, r_3]$:

$$e^{-c_2}s \leq \|J(s)\|_E \leq e^{c_2}s.$$

In terms of the exponential map, any Jacobi field as in Lemma 1.4.7 is of the form $J(s) = T_{s\gamma'(0)} \exp_p^{g_\varepsilon}(s \cdot w)$, with $w \in T_pM$, $\|w\|_E = 1$. Thus

$$e^{-c_2} \leq \|T_{s\gamma'(0)} \exp_p^{g_\varepsilon}(w)\|_E \leq e^{c_2} \qquad (s \in [0, r_3]).$$

Since $\|\gamma'(0)\|_E = 1$ we conclude that $\forall \varepsilon < \varepsilon_0$, $\forall v \in B_E(0, r_3)$, $\forall w \in T_pM$:

$$e^{-c_2}\|w\|_E \leq \|T_v \exp_p^{g_\varepsilon}(w)\|_E \leq e^{c_2}\|w\|_E. \tag{1.10}$$

In particular, $\exp_p^{g_\varepsilon}$ is a local diffeomorphism on $B_E(0, r_3)$. □

We note that (1.10) can equivalently be formulated as

$$e^{-2c_2}g_E \leq (\exp_p^{g_\varepsilon})^* g_E \leq e^{2c_2}g_E \tag{1.11}$$

for $\varepsilon < \varepsilon_0$ on $B_E(0, r_3)$.

Lemma 1.4.9. *For $r_4 < e^{-c_2}r_3$, $r_5 < e^{-c_2}r_4$ and $\tilde{r} := e^{c_2}r_4$ we have, $\forall \varepsilon < \varepsilon_0$:*

$$\exp_p^{g_\varepsilon}(\overline{B_E(0, r_5)}) \subseteq B_E(p, r_4) \subseteq \exp_p^{g_\varepsilon}(\overline{B_E(0, \tilde{r})}) \subseteq \exp_p^{g_\varepsilon}(B_E(0, r_3)).$$

Proof. For $q \in B_E(p, r_4)$, let $\alpha : [0, a] \to M$ be a piecewise smooth curve from p to q in $B_E(p, r_4)$ of Euclidean length less than r_4. Since $\exp_p^{g_\varepsilon}$ is a local diffeomorphism on $B_E(0, r_3)$, for $b > 0$ sufficiently small there exists a unique $\exp_p^{g_\varepsilon}$-lift $\hat{\alpha} : [0, b] \to B_E(0, r_3)$ of $\alpha|_{[0,b]}$ starting at 0. We claim that $a' := \sup\{b < a \mid \hat{\alpha} \text{ exists on } [0, b]\} = a$. Indeed, suppose that $a' < a$. Then

$$L_{(\exp_p^{g_\varepsilon})^* g_E}(\hat{\alpha}|_{[0,a')}) = L_{g_E}(\alpha|_{[0,a')}) = \int_0^{a'} \|\alpha'(t)\|_E dt \leq r_4.$$

Hence by (1.11) we obtain $L_{g_E}(\hat{\alpha}|_{[0,a')}) \le \tilde{r}$. Now let $a_n \nearrow a'$. Then $\hat{\alpha}(a_n) \in \overline{B_E(0,\tilde{r})}$, so some subsequence $(\hat{\alpha}(a_{n_k}))$ converges to a point v in $\overline{B_E(0,\tilde{r})}$. Since $\exp_p^{g_\varepsilon}$ is a diffeomorphism on a neighborhood of v and $\exp_p^{g_\varepsilon}(v) = \lim \alpha(a_{n_k}) = \alpha(a')$, this shows that $\hat{\alpha}$ can be extended past a', a contradiction. Thus $q = \exp_p^{g_\varepsilon}(\hat{\alpha}(a)) \in \exp_p^{g_\varepsilon}(\overline{B_E(0,\tilde{r})})$.

For the first inclusion, take $v \in T_pM$ with $\|v\|_E \le r_5$ and set $q := \exp_p^{g_\varepsilon}(v)$. Then for the radial geodesic $\gamma : [0,1] \to M$, $t \mapsto \exp_p^{g_\varepsilon}(tv)$ from p to q, by (1.10) we obtain for s small:

$$L_E(\gamma|_{[0,s]}) = \int_0^s \|T_{tv} \exp_p^{g_\varepsilon}(v)\|_E \, dt \le e^{c_2}\|v\|_E < r_4$$

From this we conclude that $\sup\{s \in [0,1] \mid \gamma|_{[0,s]} \subseteq B_E(p,r_4)\} = 1$, so $q \in B_E(p,r_4)$. □

Note that $\exp_p^{g_\varepsilon} : \overline{B_E(0,\tilde{r})} \to \exp_p^{g_\varepsilon}(\overline{B_E(0,\tilde{r})})$ is a surjective local homeomorphism between compact Hausdorff spaces, hence is a covering map. Using this, we obtain:

Lemma 1.4.10. *For any $\varepsilon < \varepsilon_0$, $\exp_p^{g_\varepsilon}$ is injective (hence a diffeomorphism) on $B_E(0,r_5)$.*

Proof. Suppose to the contrary that there exist $v_0, v_1 \in B_E(0,r_5)$, $v_0 \ne v_1$, and $\varepsilon < \varepsilon_0$ such that $\exp_p^{g_\varepsilon}(v_0) = q = \exp_p^{g_\varepsilon}(v_1)$. Hence, $\gamma_i(t) := \exp_p^{g_\varepsilon}(tv_i)$, $i = 0,1$, are two distinct geodesics starting at p which intersect at the point q. Then $\gamma_s(t) = s\gamma_1(t) + (1-s)\gamma_0(t)$ is a fixed endpoint homotopy connecting γ_0 and γ_1 in the ball $B_E(p,r_4)$. Since $\exp_p^{g_\varepsilon}$ is a covering map, and using Lemma 1.4.9, we can lift this homotopy to $\overline{B_E(0,\tilde{r})}$. But the lifts of γ_0 and γ_1 are $t \mapsto tv_i$, $i = 0,1$, which obviously are not fixed endpoint homotopic in $\overline{B_E(0,\tilde{r})}$, a contradiction. □

From (1.10) we obtain a uniform Lipschitz constant for all $\exp_p^{g_\varepsilon}$ with $\varepsilon < \varepsilon_0$: $\exists c_3 > 0$ such that $\forall u, v \in B_E(0,r_5)$

$$\|\exp_p^{g_\varepsilon}(u) - \exp_p^{g_\varepsilon}(v)\|_E \le c_3\|u - v\|_E.$$

For the corresponding estimate from below we use the following result that provides a mean value estimate for C^1-functions on not necessarily convex domains (cf. [GKOS01, 3.2.47]).

Lemma 1.4.11. *Let $\Omega \subseteq \mathbb{R}^n$, $\Omega' \subseteq \mathbb{R}^m$ be open, $f \in C^1(\Omega,\Omega')$ and suppose that $K \Subset \Omega$. Then there exists $C > 0$, such that $\|f(x) - f(y)\| \le C\|x - y\|$, $\forall x, y \in K$. C can be chosen as $C_1 \cdot \sup_{x\in L}(\|f(x)\| + \|Df(x)\|)$ for any fixed compact neighborhood L of K in Ω, where C_1 depends only on L.*

Using Lemma 1.4.9 we now pick $r_7 < r_6 =: e^{-c_2}\hat{r} < \hat{r} < r_5$ such that for $\varepsilon < \varepsilon_0$ we have

$$\exp_p^{g_\varepsilon}(\overline{B_E(0,r_7)}) \subseteq B_E(p,r_6) \subseteq \exp_p^{g_\varepsilon}(\overline{B_E(0,\hat{r})}) \Subset \exp_p^{g_\varepsilon}(B_E(0,r_5)).$$

Again by (1.10), we have $\forall\varepsilon < \varepsilon_0$,

$$e^{-c_2}\|\xi\|_E \leq \|T_q(\exp_p^{g_\varepsilon})^{-1}(\xi)\|_E \leq e^{c_2}\|\xi\|_E,$$

$\forall q \in \overline{B_E(p,r_6)}, \forall\xi \in T_qM$. Thus by Lemma 1.4.11 there exists some $c_4 > 0$ such that

$$\|(\exp_p^{g_\varepsilon})^{-1}(q_1) - (\exp_p^{g_\varepsilon})^{-1}(q_2)\|_E \leq c_4^{-1}\|q_1 - q_2\|_E$$

$\forall\varepsilon < \varepsilon_0$, $\forall q_1, q_2 \in \exp_p^{g_\varepsilon}(B_E(0,r_7))$.

Summing up, for all $\varepsilon < \varepsilon_0$ and all $u, v \in B_E(0,r_7)$ we have

$$c_4\|u - v\|_E \leq \|\exp_p^{g_\varepsilon}(u) - \exp_p^{g_\varepsilon}(v)\|_E \leq c_3\|u - v\|_E.$$

Finally, let $\varepsilon \to 0$. Then for all $u, v \in B_E(0,r_7)$ we get

$$c_4\|u - v\|_E \leq \|\exp_p^{g}(u) - \exp_p^{g}(v)\|_E \leq c_3\|u - v\|_E.$$

Thus, $\exp_p^g$ is a bi-Lipschitz homeomorphism on $U := B_E(0,r_7) \subseteq T_pM$. In particular, $V = \exp_p^g(U)$ is open in M (invariance of domain). This concludes the proof of Theorem 1.4.2.

1.5 The Riemannian case

In this section we point out some alternatives to the reasoning presented in the proof of Theorem 1.4.2, when the special case of a $C^{1,1}$ Riemannian metric g is considered. As already shown in the preceding section, our strategy was to approximate g by a net g_ε of smooth pseudo-Riemannian metrics and then use results from comparison geometry, recently obtained by Chen and LeFloch, see [CleF08]. Methods from comparison geometry appear to be one of the most powerful tools in global Riemannian geometry. Many important geometrical conclusions can be drawn just by studying manifolds for which only information about bounds on the sectional curvature is given. One of the fundamental results which relates the sectional curvature of a Riemannian manifold to the rate at which geodesics spread apart is the following version of the Rauch comparison theorem, cf., e.g., [Jo11, Cor. 4.6.1].

Theorem 1.5.1. *Let (M, h) be a smooth Riemannian manifold and suppose that $\exp_p^h$ is defined on a ball $B_{h_p}(0, R)$, for some $R > 0$, and that there exist $\rho \leq 0$, $\kappa > 0$ such that the sectional curvature K of M satisfies $\rho \leq K \leq \kappa$ on some open set which contains $\exp_p^h(B_{h_p}(0, R))$. Then for all $v \in T_pM$ with $\|v\|_{h_p} = 1$, all $w \in T_pM$, and all $0 < t < \min(R, \frac{\pi}{\sqrt{\kappa}})$,*

$$\frac{\mathrm{sn}_\kappa(t)}{t}\|w\| \leq \|(T_{tv}\exp_p^h)(w)\| \leq \frac{\mathrm{sn}_\rho(t)}{t}\|w\|.$$

Here, for $\alpha \in \mathbb{R}$,

$$\mathrm{sn}_\alpha(t) := \begin{cases} \frac{1}{\sqrt{\alpha}}\sin(\sqrt{\alpha}t) & \text{for } \alpha > 0 \\ t & \text{for } \alpha = 0 \\ \frac{1}{\sqrt{-\alpha}}\sinh(\sqrt{-\alpha}t) & \text{for } \alpha < 0 \end{cases}$$

As an immediate consequence, we obtain that for any $0 < r < \min(R, \frac{\pi}{\sqrt{\kappa}})$, there exists some $c > 0$ such that $\forall v \in B_{h_p}(0, r)$, $\forall w \in T_pM$

$$e^{-c}\|w\| \leq \|(T_v \exp_p^h)(w)\| \leq e^c\|w\|. \tag{1.12}$$

Remark 1.5.2. Denote by $G(k, n) := \{k\text{-dimensional subspaces of } \mathbb{R}^n\}$ the Grassmann manifold. $G(k, n)$ is an $(n - k)k$-dimensional manifold diffeomorphic to

$$\mathscr{O}(n) \Big/ (\mathscr{O}(k) \times \mathscr{O}(n-k)),$$

where $\mathscr{O}(n)$ is the orthogonal group of dimension n, hence is compact. For (M, g) a smooth Riemannian manifold, $G(2, TM) = \bigsqcup_{p\in M} G(2, T_pM)$ is a fiber bundle with compact fibers. The sectional curvature can be viewed as a smooth function on the 2-Grassmannian bundle $G(2, TM)$, i.e., $K : G(2, TM) \to \mathbb{R}$ and it is given by:

$$K(v, w) = \frac{\langle R_{vw}v, w\rangle}{\langle v, v\rangle\langle w, w\rangle - \langle v, w\rangle}.$$

Since K is continuous, it is bounded on $G(2, TM)|_U \cong U \times G(2, \mathbb{R}^n)$, for any relatively compact subset U of M. However, an analogous argument is not possible in the Lorentzian (or general pseudo-Riemannian) setting.

Namely, for a smooth Lorentzian manifold (M, g), its sectional curvature K has quite different behavior than in the case of Riemannian manifolds, since it is now only defined on non-degenerate 2-planes, forming an open subbundle of $G(2, TM)$. More precisely, a Lorentzian manifold has bounded sectional curvature K only in the trivial case where K is constant ([K79], cf. also [Harris82]). This is the main reason that only timelike planes are considered in the curvature conditions. However, this was not enough for our considerations since for obtaining that each $\exp_{g\varepsilon}$ is a local diffeomorphism, we had to derive estimates on Jacobi fields along any causal geodesic, see Section 1.4. In [CleF08], several injectivity radius estimates for Lorentzian manifolds were established and formulated in terms of a reference Riemannian metric that is determined by a future directed timelike unit vector field T, referred to as the reference vector field prescribed on M. One of the reasons for using a reference Riemannian metric is the fact that a Lorentzian norm of a non-zero vector may vanish hence the injectivity radius is not supposed to be defined directly for a Lorentzian metric g. Hence, for $p \in M$, the geodesic ball of radius r in T_pM is given by $B_T(0, r) := \{v \in T_pM \mid \langle v, v\rangle_{g_T} < r\}$, where g_T is the reference Riemannian metric determined by T, and one can also define the geodesic ball $\mathscr{B}(p, r) := \exp_p(B_T(0, r))$. Then the injectivity radius $\mathrm{Inj}(p, M)$ is defined as the largest radius r for which the exponential map is a diffeomorphism from $B_T(0, r)$ onto $\mathscr{B}(p, r) := \exp_p(B_T(0, r))$. Analogously, the injectivity radius can be defined for any pseudo-Riemannian metric g. We used similar methods to the ones obtained in [CleF08], only using the Euclidean metric instead of an arbitrary Riemannian one.

Now let g be a $C^{1,1}$ Riemannian metric on M, and let g_ε be approximating smooth metrics. Then we may fix some $r' > 0$ and some $\varepsilon_0 > 0$ such that $\exp_p^g$ and $\exp_p^{g_\varepsilon}$ $(\varepsilon < \varepsilon_0)$ are defined on $B_{g_p}(0, r')$. Since $\exp_p^{g_\varepsilon}$ converges locally uniformly to $\exp_p^g$, there exists an open, relatively compact subset $W \subseteq M$ with $\bigcup_{\varepsilon<\varepsilon_0} \exp_p^{g_\varepsilon}(B_{g_p}(0, r')) \subseteq W$. On W, by Remarks 1.4.3 (ii) and 1.5.2 we obtain uniform bounds on the sectional curvatures K_ε of g_ε, i.e.,

$$\exists \rho \leq 0, \kappa > 0 : \forall \varepsilon < \varepsilon_0 \quad \rho \leq K_\varepsilon \leq \kappa.$$

Thus by (1.12), for any $r < \min(r', \frac{\pi}{\sqrt{\kappa}})$, there exists some $c > 0$ depending only on ρ and κ such that for all $\varepsilon < \varepsilon_0$

$$e^{-c}\|w\|_{g_\varepsilon} \leq \|(T_v \exp_p^{g_\varepsilon})(w)\|_{g_\varepsilon} \leq e^c\|w\|_{g_\varepsilon}, \tag{1.13}$$

$\forall v \in B_{g_p}(0, r), \forall w \in T_pM$. In particular, by the inverse function theorem every $\exp_p^{g_\varepsilon}$ is a local diffeomorphism on $B_{g_p}(0,r)$. Thus we may rewrite (1.13) equivalently as

$$e^{-2c} g_{\varepsilon,p} \le (\exp_p^{g_\varepsilon})^* g_\varepsilon \le e^{2c} g_{\varepsilon,p},$$

on $B_{g_p}(0,r)$. Since $g_\varepsilon \to g$ locally uniformly, by increasing c we obtain (1.11) on a suitable Euclidean ball and can proceed as in Section 1.4.

Finally, we note that to obtain a common domain (and injectivity) of the approximating exponential maps $\exp_p^{g_\varepsilon}$ one may alternatively employ the following result of Cheeger, Gromov and Taylor ([CGT82], the formulation below is taken from [CleF08]), which provides a lower bound on the injectivity radii $\mathrm{Inj}_{g_\varepsilon}(M,p)$.

Theorem 1.5.3. *Let M be a C^∞ n-manifold with a smooth Riemannian metric g. Suppose that $\overline{B_g(p,1)} \Subset M$ for some point p in M. Then for any $K, v > 0$ there exists some $i = i(K,v,n) > 0$ such that if*

$$\|R_g\|_{L^\infty(B(p,1))} \le K, \;\; Vol_g(B(p,1)) \ge v,$$

then the injectivity radius $Inj_g(M,p)$ at p is bounded from below by i,

$$Inj_g(M,p) \ge i.$$

Since the distance function d_g of the $C^{1,1}$-metric g induces the manifold topology, $B_g(p,2r)$ is an open, relatively compact subset of M for $r > 0$ sufficiently small. Thus for ε small, $B_{g_\varepsilon}(p,r) \subseteq B_g(p,2r)$ is relatively compact and

$$\mathrm{Vol}_{g_\varepsilon}(B_{g_\varepsilon}(p,r)) \ge \mathrm{Vol}_{g_\varepsilon}(B_g(p,r/2)) \ge \frac{1}{2}\mathrm{Vol}_g(B_g(p,r/2)) > 0.$$

By Theorem 1.5.3, there exists some r_0 such that

$$\mathrm{Inj}(g_\varepsilon, p) \ge r_0, \;\; \forall \varepsilon \le \varepsilon_0,$$

so $\exp_p^{g_\varepsilon}$ is a diffeomorphism on $B_{g_\varepsilon}(p,r_0)$ $\forall \varepsilon \le \varepsilon_0$. Since $B_g(p,\frac{r_0}{2}) \subseteq B_{g_\varepsilon}(p,r_0)$ for ε small, it follows that $\exp_p^{g_\varepsilon}$ is a diffeomorphism on $B_g(p,\frac{r_0}{2})$. From here, using Theorem 1.5.1, we may proceed as in the argument following Lemma 1.4.10 to conclude that $\exp_p^g$ is a bi-Lipschitz homeomorphism on some neighborhood of $0 \in T_pM$.

1.6 Strong differentiability of the exponential map

An alternative approach to the question of regularity of the exponential map for $C^{1,1}$-metrics has recently been presented in [Min13]. In this paper it is shown that both, the exponential map $\exp_p$ and the map E (see Section 1.3) are not only locally bi-Lipschitz homeomorphisms, but also strongly differentiable at the origin. In this section we give an idea of the proof as done in [Min13]. This approach and ours (see Section 1.4) nicely complement each other: in [Min13], there is no regularization or comparison geometry needed. In fact, the result can be derived without any prior knowledge of pseudo-Riemannian geometry since it is based on the Picard-Lindelöf approximation method and on a version of the inverse function theorem due to Leach. For this theorem, the notion of *strong derivative* and the corresponding notion of *strong differential* is needed:

Definition 1.6.1. Let X and Y be Banach spaces and let $f : X \supseteq S \to Y$, where S is open. *The strong differential of* f at $x \in S$ is a bounded linear transformation $L : X \to Y$ for which it holds: for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $|y_1 - x| < \delta$ and $|y_2 - x| < \delta$, then:

$$|f(y_1) - f(y_2) - L(y_1 - y_2)| \leq \varepsilon |y_1 - y_2|. \tag{1.14}$$

Obviously, if f is strongly differentiable at a point x, it is also classically differentiable and these differentials coincide. The converse is not true: $f(x) = x^2 \sin(\frac{1}{x})$ is differentiable but not strongly differentiable at 0.

The following result is Leach's inverse function theorem, cf.,e.g., [Min13, Theorem 2] (see also [Le61], [Nij74]):

Theorem 1.6.2. *Let $f : S \to \mathbb{R}^n$ be a function defined on an open subset $S \subseteq \mathbb{R}^n$ and let $L : \mathbb{R}^n \to \mathbb{R}^n$ be the strong differential of f at $x \in S$. If L is invertible then there exist an open neighborhood U of x, an open neighborhood $\tilde{U}$ of $f(x)$ and a function $g : \tilde{U} \to \mathbb{R}^n$ such that $f(U) = \tilde{U}$, $g(\tilde{U}) = U$ and $f|_U$ and g are inverse to each other. They are both Lipschitz and g has strong differential L^{-1} at $f(x)$.*

In addition, f is differentiable (resp. strongly differentiable) at $y \in U$ iff g is differentiable (resp. strongly differentiable) at $f(y)$, in which case the differentials (resp. strong differentials) are invertible.

The strong differentiability of the map E is then proved by a local analysis of the geodesic equation using the Picard-Lindelöf approximation method. The main ingredient in this proof is the fact that the right-hand side of the geodesic equation is Lipschitz in both variables and homogeneous of second degree in the second variable. Now, note that if $f(x, y)$ is a strongly differentiable function, keeping x fixed, one easily obtains that the function $f(x, .)$ is strongly differentiable. Moreover, the composition of strongly differentiable functions is strongly differentiable. Since $E(v) = (p, \exp_p(v))$, we obtain that $\exp_p$ is strongly differentiable at the origin. Having proved the strong differentiability of E and $\exp_p$, since they clearly satisfy the conditions of Theorem 1.6.2, one immediately obtains that their inverses are also Lipschitz.

1.7 Totally normal neighborhoods

Recall that for a smooth pseudo-Riemannian metric g on a manifold M, a neighborhood U of $p \in M$ is called a normal neighborhood of p if $\exp_p^g$ is a diffeomorphism from a starshaped open neighborhood $\tilde{U}$ of $0 \in T_pM$ onto U. U is called *totally normal* if it is a normal neighborhood of each of its points. This terminology is in line with [DoCarmo92] while, e.g., in [ON83] such sets are called geodesically convex.

Analogously, if g is a $C^{1,1}$-pseudo-Riemannian metric on a smooth manifold M we call a neighborhood of a point $p \in M$ normal if there exists a starshaped open neighborhood $\tilde{U}$ of $0 \in T_pM$ such that $\exp_p^g$ is a bi-Lipschitz homeomorphism from $\tilde{U}$ onto U. U is called totally normal if it is a normal neighborhood of each of its points.

In what follows we adapt the standard proof for the existence of totally normal neighborhoods, cf., e.g., [ON83, Prop. 5.7] (tracing back to [Wh32, Sec. 4]) to the $C^{1,1}$-situation.

Theorem 1.7.1. *Let M be a smooth manifold with a $C^{1,1}$ pseudo-Riemannian metric g. Then each point $p \in M$ possesses a basis of totally normal neighborhoods.*

Proof. The main point to note is that the explicit bounds derived in Section 1.4 on the radius of the ball in T_pM where $\exp_p^g$ is a bi-Lipschitz homeomorphism depend only on quantities that can be uniformly controlled on compact sets. Therefore, for any $p \in M$ there exists a neighborhood V' of p and some $r > 0$ such that, $\forall q \in V'$,

$$\exp_q^g : B_{h,q}(0, r) \to \exp_q^g(B_{h,q}(0, r)) \tag{1.15}$$

is a bi-Lipschitz homeomorphism. Here, h is any background Riemannian metric.

Now define $S := \{v \in TM \mid \pi(v) \in V', \|v\|_h < r\}$, with π the natural projection of TM onto M. Let $E : TM \to M \times M$, $E(v) = (\pi(v), \exp^g(v))$. Then by (1.15) $E : S \to E(S) =: W$ is a continuous bijection, hence a homeomorphism by invariance of domain. Let $(\psi = (x^1, ..., x^n), V)$ be a coordinate system centered at p (in the smooth case ψ is usually taken to be a normal coordinate system, which is not available to us, but this is in fact not needed). Define the $(0,2)$-tensor field B on V by

$$B_{ij}(q) := \delta_{ij} - \sum_k \Gamma^k_{ij}(q)x^k(q).$$

Since $\psi(p) = 0$ we may assume V small enough that B is positive definite on V. In addition, we may suppose that $W \subseteq V \times V$. Set $N(q) := \sum_{i=1}^n (x^i(q))^2$, and let $V(\delta) := \{q \in V \mid N(q) < \delta\}$. Then if δ is so small that $V(\delta) \times V(\delta) \subseteq W$, E is a homeomorphism from $U_\delta := E^{-1}(V(\delta) \times V(\delta))$ onto $V(\delta) \times V(\delta)$ and $\exp^g([0,1] \cdot U_\delta) \subseteq \exp^g(S) \subseteq V$.

We will show that $V(\delta)$ is totally normal. For $q \in V(\delta)$ and $U_q := U_\delta \cap T_qM$, $\exp^g_q = E|_{U_q} : U_q \to V(\delta)$ is a homeomorphism, so it is left to show that U_q is starshaped. Let $v \in U_q$. Then $\sigma : [0,1] \to M$, $\sigma(t) = \exp^g_q(tv)$ is a geodesic from q to $\sigma(1) =: \tilde{q} \in V(\delta)$ that lies entirely in V.

If σ is contained in $V(\delta)$ then $tv \in U_q$, $\forall t \in [0,1]$: suppose to the contrary that $\bar{t} := \sup\{t \in [0,1] |\ [0,t] \cdot v \in U_q\} < 1$. Then $\bar{t}v \in \partial U_q$ and since $(\exp^g_q|_{U_q})^{-1}(\sigma([0,1])) \Subset U_q$, there exists some $t_1 < \bar{t}$ such that $U_q \ni t_1 v \notin (\exp^g_q|_{U_q})^{-1}(\sigma([0,1]))$, a contradiction. Hence the entire segment $\{tv |\ t \in [0,1]\}$ lies in U_q, so U_q is starshaped. It remains to show that σ cannot leave $V(\delta)$. If it did, there would exist $t_0 \in [0,1]$ such that $N(\sigma(t_0)) \geq \delta$. Since $N(q), N(\tilde{q}) < \delta$, the function $t \mapsto N \circ \sigma$ has a maximum at some point $\tilde{t} \in (0,1)$. However,

$$\frac{d^2(N \circ \sigma)}{dt^2}(\tilde{t}) = 2B_{\sigma(\tilde{t})}((\psi \circ \sigma)'(\tilde{t}), (\psi \circ \sigma)'(\tilde{t})) > 0\,,$$

a contradiction. □

Chapter 2

Causality theory with $C^{1,1}$ metrics

2.1 Introduction

Motivated by studies of the Einstein equations in low regularity [KR05], [KRS12], [M06], there has recently been an increased interest in determining the minimal degree of regularity of the metric under which the standard results of Lorentzian causality theory remain valid. The standard references on general relativity and in particular, causality theory, assume either smoothness of the spacetime metric, see e.g. [BEE96], [Kr99], [MS08], [ON83], [Pen72], or C^2-differentiability, see e.g. [Chr11], [Cl93], [HE73], [Se98], [GaSe05]. These assumptions on the regularity of the metric may exclude some very important cases from the physical point of view, such as shock waves or the matching of two different spacetimes over a common boundary, see [Se98]. Several approaches to causal structures and discussions on the problems that can occur when the regularity of the metric is below C^2 have been presented in [Chr11], [CG12], [Cl93], [HE73], [MS08], [Se98]. As emphasised by Senovilla in [Se98], the main problem is the existence of normal coordinates and their regularity, as well as the existence of totally normal (convex) neighborhoods. However, it has recently been demonstrated in [Chr11] that the entire smooth causality theory can be preserved for C^2-metrics.

As already indicated, a reasonable candidate for the lowest degree of differentiability where one could expect the standard results of causality theory to remain valid is given by metrics of class $C^{1,1}$ (contin- uously differentiable with locally Lipschitz first order derivatives, often also denoted by C^{2-}) since it is the threshold where one still has unique solvability of the geodesic equation. However, it is well-known that below $C^{1,1}$ many standard properties fail to be true. In [HW51], Hartman and Wintner show that for metrics of Hölder regularity $C^{1,\alpha}$ with $0 < \alpha < 1$, radial geodesics may fail to be minimizing between any two points that they contain. Also, in [CG12], Chrusciel and Grant established the existence of so-called 'bubbling metrics', which are of $C^{0,\alpha}$ regularity, $0 < \alpha < 1$, whose light-cones have nonempty interior, and for whom the push-up principle is no longer true. In addition, for metrics which are merely continuous it is still unknown whether or not the timelike futures remain open.

As already explained in the preceding chapter, the fundamental tool for studying local causality is the exponential map. More precisely, using approximation techniques and methods from comparison geometry, we showed that the exponential map retains its maximal possible regularity (see Section 1.4). Hence our next goal is to develop the key elements of local causality theory for $C^{1,1}$-Lorentzian metrics, essentially based on [KSSV14]. We will then further develop causality theory following the proofs given in [Chr11], [ON83]. However, an alternative approach to causality theory for $C^{1,1}$-Lorentzian metrics by E. Minguzzi has recently appeared in [Min13]. As already indicated in Section 1.6, this paper establishes that $\exp_p$ is a bi-Lipschitz homeomorphism, and in addition shows that the map E (see Section 1.3) is a bi-Lipschitz homeomorphism on a neighborhood of the zero-section in TM and is strongly differentiable over this zero section [Min13, Th. 1.11]. In this work, the required properties of the exponential map are derived from a careful analysis of the corresponding ODE problem based on Picard-Lindelöf approximations, as well as from an inverse function theorem for Lipschitz maps. In [Min13] the author also establishes the Gauss Lemma and develops the essential elements of $C^{1,1}$-causality, thereby obtaining many of the results that are also contained in this chapter, some even in greater generality.

Nevertheless, our approach and the one presented in [Min13] nicely complement each other: Our methods are a direct continuation of the regularization approach of P. Chrusciel and J. Grant ([CG12]) and are completely independent from those employed in [Min13]. The basic idea is to approximate a given metric of low regularity (which may be as low as C^0) by two nets of smooth metrics $\check{g}_\epsilon$ and $\hat{g}_\epsilon$ whose light cones sandwich those of g. We then continue the line of argument of [CG12, KSS14] to establish the key results of causality theory for a $C^{1,1}$-metric. The advantage of these methods is that they quite easily adapt to regularity below $C^{1,1}$, which as far as we can see is the natural lower bound for the applicability of those employed in [Min13]. As an example, we note that the push-up lemmas from [CG12], cf. Prop. 2.3.14 and 2.3.15 below, in fact even hold for $C^{0,1}$-metrics (or, more generally, for causally plain C^0-metrics), whereas the corresponding results in [Min13, Sec. 1.4] require the metric to be $C^{1,1}$. Moreover, the regularization approach adopted here, together with methods from Lorentzian comparison geometry as used in [CleF08] and [KSS14], will allow us to establish the Hawking singularity theorem for $C^{1,1}$-metrics, see Chapter 4.

2.2 Regularization techniques

As already mentioned, a fundamental tool in our approach is approximating a given metric of regularity $C^{1,1}$ by a net g_ε of C^∞-metrics, in the following sense:

Remark 2.2.1. We cover M by a countable and locally finite collection of relatively compact chart neighborhoods and denote the corresponding charts by (U_i, ψ_i) $(i \in \mathbb{N})$. Let $(\zeta_i)_i$ be a subordinate partition of unity with $\mathrm{supp}(\zeta_i) \Subset U_i$ for all i and choose a family of cut-off functions $(\chi_i)_i \in \mathcal{D}(U_i)$ with $\chi_i \equiv 1$ on a neighborhood of $\mathrm{supp}(\zeta_i)$. Finally, let $\rho \in \mathcal{D}(\mathbb{R}^n)$ be a test function with unit integral and define the standard mollifier $\rho_\varepsilon(x) := \varepsilon^{-n}\rho\left(\frac{x}{\varepsilon}\right)$ $(\varepsilon > 0)$. Then denoting by f_* (respectively f^*) push-forward (respectively pullback) under a map f, the following formula defines a family $(g_\varepsilon)_\varepsilon$ of smooth sections of $T^0_2(M)$

$$g_\varepsilon := \sum_i \chi_i\, \psi_i^*\Big(\big(\psi_{i\,*}(\zeta_i\, g)\big) * \rho_\varepsilon\Big)$$

which satisfies

(i) g_ε converges to g in the C^1-topology as $\varepsilon \to 0$, and

(ii) the second derivatives of g_ε are bounded, uniformly in ε, on compact sets.

On any compact subset of M, therefore, for ε sufficiently small the g_ε form a family of pseudo-Riemannian metrics of the same signature as g whose Riemannian curvature tensors R_ε are bounded uniformly in ε.

Observe that the above procedure can be applied even to distributional sections of any vector bundle $E \to M$ (using the corresponding vector bundle charts) and that the usual convergence properties of smoothings via convolution are preserved.

As in the preceding chapter, we will write $\exp_p^{g_\varepsilon}$ to distinguish the exponential maps stemming from metrics g_ε. For brevity we will drop this superscript for the $C^{1,1}$-metric g itself, though. We shall need the following properties of the exponential maps corresponding to an approximating net as above:

Lemma 2.2.2. *Let g be a $C^{1,1}$-pseudo-Riemannian metric on M and let g_ε be a net of smooth pseudo-Riemannian metrics that satisfy conditions (i) and (ii) of Remark 2.2.1. Then any $p \in M$ has a basis of normal neighborhoods U such that, with $\exp_p : \tilde{U} \to U$, all $\exp_p^{g_\varepsilon}$ are diffeomorphisms with domain $\tilde{U}$ for ε sufficiently small. Moreover, the inverse maps $(\exp_p^{g_\varepsilon})^{-1}$ also are defined on a common neighborhood of p for ε small, and converge locally uniformly to $\exp_p^{-1}$.*

Proof. The claims about the common domains of $\exp_p^{g_\varepsilon}$, respectively of $(\exp_p^{g_\varepsilon})^{-1}$ follow from Lemma 1.4.4 and Lemma 1.4.9. To obtain the convergence result, we first note that without loss of generality, given a common domain V of the $(\exp_p^{g_\varepsilon})^{-1}$ for $\varepsilon < \varepsilon_0$, we may assume that $\bigcup_{\varepsilon<\varepsilon_0}(\exp_p^{g_\varepsilon})^{-1}(V)$ is relatively compact in $\tilde{U}$: this follows from the fact that the maps $(\exp_p^{g_\varepsilon})^{-1}$ are Lipschitz, uniformly in ε (see the argument following Lemma 1.4.11).

Now if $(\exp_p^{g_\varepsilon})^{-1}$ did not converge uniformly to $\exp_p^{-1}$ on some compact subset of V then by our compactness assumptions we could find a sequence q_k in V converging to some $q \in V$ and a sequence $\varepsilon_k \searrow 0$ such that $w_k := (\exp_p^{g_{\varepsilon_k}})^{-1}(q_k) \to w \neq \exp_p^{-1}(q)$. But since $(\exp_p^{g_\varepsilon}) \to \exp_p$ locally uniformly, we arrive at $q_k = \exp_p^{g_{\varepsilon_k}}(w_k) \to \exp_p(w) \neq q$, a contradiction. □

In the particular case of g being Lorentzian, a more sophisticated approximation procedure, adapted to the causal structure of g, was given in [CG12, Prop. 1.2].

To formulate this result, we first recall that a space-time is a time-oriented Lorentzian manifold, with time-orientation determined by some continuous time-like vector field. In what follows, all Lorentzian manifolds will be supposed to be time-oriented. Also we recall from [CG12] that for two Lorentzian metrics g, h, we say that h has strictly wider light cones than g, denoted by $g \prec h$, if for any tangent vector $X \neq 0$, $g(X,X) \leq 0$ implies that $h(X,X) < 0$.

We will also need the following technical tools:

Lemma 2.2.3. *Let (K_m) be an exhaustive sequence of compact subsets of a manifold M ($K_m \subseteq K_{m+1}^\circ$, $M = \bigcup_m K_m$), and let $\varepsilon_1 \geq \varepsilon_2 \geq \cdots > 0$ be given. Then there exists some $\psi \in C^\infty(M)$ such that $0 < \psi(p) \leq \varepsilon_m$ for $p \in K_m \setminus K_{m-1}^\circ$ (where $K_{-1} := \emptyset$).*

Proof. See, e.g., [GKOS01, Lemma 2.7.3]. □

Lemma 2.2.4. *Let M, N be manifolds, and set $I := (0, \infty)$. Let $u : I \times M \to N$ be a smooth map and let (P) be a property attributable to values $u(\varepsilon, p)$, satisfying:*

(i) For any $K \Subset M$ there exists some $\varepsilon_K > 0$ such that (P) holds for all $p \in K$ and $\varepsilon < \varepsilon_K$.

(ii) (P) is stable with respect to decreasing K and ε: if $u(\varepsilon, p)$ satisfies (P) for all $p \in K \Subset M$ and all ε less than some $\varepsilon_K > 0$ then for any compact set $K' \subseteq K$ and any $\varepsilon_{K'} \leq \varepsilon_K$, u satisfies (P) on K' for all $\varepsilon \leq \varepsilon_{K'}$.

Then there exists a smooth map $\tilde{u} : I \times M \to N$ such that (P) holds for all $\tilde{u}(\varepsilon, p)$ ($\varepsilon \in I$, $p \in M$) and for each $K \Subset M$ there exists some $\varepsilon_K \in I$ such that $\tilde{u}(\varepsilon, p) = u(\varepsilon, p)$ for all $(\varepsilon, p) \in (0, \varepsilon_K] \times K$.

Proof. See [HKS12, Lemma 4.3]. □

Based on these auxiliary results, we can prove the following refined version of [CG12, Prop. 1.2]:

Proposition 2.2.5. *Let (M, g) be a space-time with a continuous Lorentzian metric, and h some smooth background Riemannian metric on M. Then for any $\varepsilon > 0$, there exist smooth Lorentzian metrics $\check{g}_\varepsilon$ and $\hat{g}_\varepsilon$ on M such that $\check{g}_\varepsilon \prec g \prec \hat{g}_\varepsilon$ and $d_h(\check{g}_\varepsilon, g) + d_h(\hat{g}_\varepsilon, g) < \varepsilon$, where*

$$d_h(g_1, g_2) := \sup_{0 \neq X, Y \in TM} \frac{g_1(X, Y) - g_2(X, Y)}{\|X\|_h \|Y\|_h}.$$

Moreover, $\hat{g}_\varepsilon$ and $\check{g}_\varepsilon$ depend smoothly on ε, and if $g \in C^{1,1}$ then $\check{g}_\varepsilon$ and $\hat{g}_\varepsilon$ additionally satisfy (i) and (ii) from Rem. 2.2.1.

Proof. First we use time-orientation to obtain a continuous timelike one-form $\tilde{\omega}$ (the g-metric equivalent of a continuous timelike vector field). Using the smoothing procedure of Rem. 2.2.1, on each U_i we can pick $\varepsilon_i > 0$ so small that $\tilde{\omega}_{\varepsilon_i}$ is timelike on U_i. Then $\omega := \sum_i \zeta_i \tilde{\omega}_{\varepsilon_i}$ is a smooth timelike one-form on M. By compactness we obtain on every U_i a constant $c_i > 0$ such that

$$|\omega(X)| \geq c_i \quad \text{for all } g\text{-causal vector fields } X \text{ with } \|X\|_h = 1. \tag{2.1}$$

Next we set on each U_i and for $\eta > 0$ and $\lambda < 0$

$$\hat{g}^i_{\eta,\lambda} = g^i_\eta + \lambda\, \omega \otimes \omega, \tag{2.2}$$

where g^i_η is as in Remark 2.2.1 (set $\varepsilon := \eta$ there and $g^i_\eta := g_\eta|_{U_i}$). Let Λ_k ($k \in \mathbb{N}$) be a compact exhaustion of $(-\infty, 0)$. For each k, there exists some $\eta_k > 0$ such that $\eta_k < \min_{\lambda\in\Lambda_k} |\lambda|$, $\eta_k > \eta_{k+1}$ for all k, and

$$|g^i_\eta(X,X) - g(X,X)| \leq |\lambda|\,\frac{c_i^2}{2} \tag{2.3}$$

for all g-causal vector fields X on U_i with $\|X\|_h = 1$, all $\lambda \in \Lambda_k$, and all $0 < \eta \leq \eta_k$. Thus by Lemma 2.2.3 there exists a smooth function $\lambda \mapsto \eta(\lambda, i)$ on $(-\infty, 0)$ with $0 < \eta(\lambda, i) \leq |\lambda|$ and such that (2.3) holds for all g-causal vector fields X on U_i with $\|X\|_h = 1$, all λ, and all $0 < \eta \leq \eta(\lambda, i)$.

Combining (2.1) with (2.3) we obtain

$$\hat{g}^i_{\eta,\lambda}(X,X) = g(X,X) + (g^i_\eta - g)(X,X) + \lambda\,\omega(X)^2 \leq 0 + \Big(|\lambda|\frac{c_i^2}{2} + \lambda c_i^2\Big)\|X\|_h^2 < 0,$$

for all g-causal X and hence $g \prec \hat{g}^i_{\eta,\lambda}$ for all $\lambda < 0$ and $0 < \eta \leq \eta(\lambda, i)$.

Given a compact exhaustion E_k ($k \in \mathbb{N}$) of $(0, \infty)$, for each k there exists some $\lambda_k < 0$ such that $|\lambda_k| < \min_{\varepsilon\in E_k} \varepsilon$, $\lambda_k < \lambda_{k+1}$ for all k, and

$$d_{U_i}(\hat{g}^i_{\eta(\lambda,i),\lambda}, g) := \sup_{0\neq X,Y\in TU_i} \frac{|\hat{g}^i_{\eta(\lambda,i),\lambda}(X,Y) - g(X,Y)|}{\|X\|_h\|Y\|_h} < \frac{\varepsilon}{2^{i+1}}.$$

for all $\varepsilon \in E_k$ and all $\lambda_k \leq \lambda < 0$. Again by Lemma 2.2.3 we obtain a smooth map $(0, \infty) \to (-\infty, 0)$, $\varepsilon \to \lambda_i(\varepsilon)$ such that $|\lambda_i(\varepsilon)| < \varepsilon$ for all ε, and $d_{U_i}(\hat{g}^i_{\eta(\lambda_i(\varepsilon),i),\lambda_i(\varepsilon)}, g) < \frac{\varepsilon}{2^{i+1}}$ for all $\varepsilon > 0$. We now consider the smooth symmetric $(0, 2)$-tensor field on M,

$$g_\varepsilon := \sum_i \chi_i \hat{g}^i_{\eta(\lambda_i(\varepsilon),i),\lambda_i(\varepsilon)}.$$

By construction, $(\varepsilon, p) \mapsto g_\varepsilon(p)$ is smooth, and g_ε converges to g locally uniformly as $\varepsilon \to 0$. Therefore, for any $K \Subset M$ there exists some ε_K such that for all $0 < \varepsilon < \varepsilon_K$, g_ε is of the same signature as g, hence a Lorentzian metric on K, with strictly wider lightcones than g. We are thus in a position to apply Lemma 2.2.4 to obtain a smooth map $(\varepsilon, p) \mapsto \hat{g}_\varepsilon(p)$

such that for each fixed ε, $\hat{g}_\varepsilon$ is a globally defined Lorentzian metric which on any given $K \Subset M$ coincides with g_ε for sufficiently small ε.

Then $d_h(\hat{g}_\varepsilon, g) < \varepsilon/2$, and $\varepsilon \to 0$ implies $\lambda_i(\varepsilon) \to 0$ and a fortiori $\eta(\lambda_i(\varepsilon), i) \to 0$ for each $i \in \mathbb{N}$.

From this, by virtue of (2.2), (i) and (ii) of Remark 2.2.1 hold for $\hat{g}_\varepsilon$ if $g \in C^{1,1}$.

The approximation $\check{g}_\varepsilon$ is constructed analogously choosing $\lambda > 0$. □

Remark 2.2.6.

(i) From Rem. 2.2.1 and the above proof it follows that, given a Lorentzian metric of some prescribed regularity (e.g., Sobolev, Hölder, etc.), the inner and outer regularizations $\check{g}_\varepsilon$ and $\hat{g}_\varepsilon$ have the same convergence to g as regularizations by convolution do locally.

(ii) If g is a metric of general pseudo-Riemannian signature, then since g_ε in Rem. 2.2.1 depends smoothly on ε, also in this case an application of Lemma 2.2.4 allows one to produce regularizations $\tilde{g}_\varepsilon$ that are pseudo-Riemannian metrics on all of M of the same signature as g and satisfy (i) and (ii) from that remark.

Next we wish to prove the Gauss Lemma. In order to proceed, we need to show the following:

Lemma 2.2.7. *Let g be a $C^{1,1}$-pseudo-Riemannian metric on M, $p \in M$, and let $f(t,s) := \exp_p(t(v+sw))$, for $t \in I$, $s \in J$, where I and J are intervals around* 0*. Then $\partial_s \partial_t f(t,s) = \partial_t \partial_s f(t,s)$ in $L^\infty_{loc}(I \times J)$.*

Proof. Without loss of generality, we may assume $M = \mathbb{R}^n$. We start out by rewriting the geodesic equation as a first order system:

$$\begin{aligned} \frac{dx^k}{dt} &= y^k(t) \\ \frac{dy^k}{dt} &= -\Gamma^k_{g,ij}(x(t)) y^i(t) y^j(t) \end{aligned} \tag{2.4}$$

such that $x(0) = x_0$, $y(0) = y_0$ and let $v_0 := (x_0, y_0)$. Then $\|y^k(t, v_0) - y^k(t, v_1)\|_{L^\infty(I)} \leq L\|v_0 - v_1\|$ for some $L > 0$. Hence $\|y^k(., v_0) - y^k(., v_1)\|_{L^\infty(I)} \leq L\|v_0 - v_1\|$ and therefore $v_0 \mapsto y^k(., v_0) \in \mathrm{Lip}(\mathbb{R}^{2n}, C^0(I, \mathbb{R}^n))$. Analogously it follows that $v_0 \mapsto x^k(., v_0) \in \mathrm{Lip}(\mathbb{R}^{2n}, C^0(I, \mathbb{R}^n))$ thus implying $s \mapsto f(., s) \in \mathrm{Lip}(J, C^1(I, \mathbb{R}^n))$.

Similarly, using the fact that the right-hand side of (2.4) is Lipschitz, we obtain:

$$\|\dot{y}^k(t, v_0) - \dot{y}^k(t, v_1)\| \le \tilde{L}\|y^k(t, v_0) - y^k(t, v_1)\| \le L\tilde{L}\|v_0 - v_1\|.$$

From this it follows that $s \mapsto f_t(., s) \in \mathrm{Lip}(J, C^1(I, \mathbb{R}^n))$ and therefore $s \mapsto f(., s) \in \mathrm{Lip}(J, C^2(I, \mathbb{R}^n))$.

We next use the *Radon-Nikodym property* (see e.g. [BeLi00]): A Banach space E has the *Radon-Nikodym property* (RNP) if every Lipschitz map $\alpha : \mathbb{R} \to E$ is differentiable almost everywhere. $C^2(I, \mathbb{R}^n)$ does not have the RNP but every $W^{k,p}$ does since it is reflexive (cf. [Aj13, Section 7], [BePe96]). Since $\forall p$, $C^2(I, \mathbb{R}^n)$ can be continuously embedded in $W^{2,p}(I)$, the map $s \mapsto f(., s) \in \mathrm{Lip}(J, W^{2,p}(I))$ for all p hence $s \mapsto \partial_s f(., s) \in L^\infty(J, W^{2,p}(I))$. For $p > 2$, by the Sobolev embedding of $W^{2,p}(I)$ into $C^1(I, \mathbb{R}^n)$, the map $s \mapsto \partial_s f(., s) \in L^\infty(J, C^1(I, \mathbb{R}^n))$ for almost all s and $\|\partial_s f(., s)\|_{C^1(I,\mathbb{R}^n)} \le L$, $L > 0$. Hence for almost all s and for all t, (thus for almost all (t, s)), $\partial_t\partial_s f(t, s)$ exists thus $\|\partial_t\partial_s f(t, s)\|_{L^\infty(I\times J)} \le L$ so $\partial_t\partial_s f(t, s) \in L^\infty(I \times J)$.

Similarly, since $s \mapsto f(., s) \in \mathrm{Lip}(J, C^2(I, \mathbb{R}^n))$, it follows that $\forall s$, $\forall t$, $\partial_t f(t, s)$ exists and $s \mapsto \partial_t f(., s) \in \mathrm{Lip}(J, C^1(I, \mathbb{R}^n)) \subseteq \mathrm{Lip}(J, W^{1,p}(I))$. Now the space $W^{1,p}$ has the RNP thus for almost all s and for all t, $\partial_s\partial_t f(t, s) \in L^\infty(J, W^{1,p}(I))$. For $p > 2$, $W^{1,p}(I)$ can be embedded in $C^0(I, \mathbb{R}^n)$ thus for almost all (t, s), $s \mapsto \partial_s\partial_t f(t, s)$ exists and $s \mapsto \partial_s\partial_t f(t, s) \in L^\infty(J, C^0(I, \mathbb{R}^n)) \subseteq L^\infty(I \times J)$.

Therefore $\partial_t\partial_s f(t, s)$ and $\partial_s\partial_t f(t, s)$ exist for almost all $(t, s) \in I \times J$ and $\partial_s\partial_t f(t, s)$, $\partial_s\partial_t f(t, s) \in L^\infty(I\times J) \subseteq L^1_{\mathrm{loc}}(I\times J) \subseteq \mathscr{D}'(I\times J)$. Thus $\partial_s\partial_t f = \partial_t\partial_s f$ almost everywhere.
□

Remark 2.2.8. We will define the Levi-Civita connection of a $C^{1,1}$ metric g via local coordinate expressions, namely:

$$\nabla_{X^i\partial_i}(Y^i\partial_j) := X^i \left(\frac{\partial Y^k}{\partial X^i} + \Gamma^k_{ij} Y^j\right) \partial_k,$$

$X, Y \in \mathfrak{X}(M)$. This is well defined hence all the properties of the Levi-Civita connection hold true as they can be expressed in local terms that are identical to the ones in the smooth case and they also remain true for vector fields X, Y, Z that are Lipschitz along a curve since they are still differentiable almost everywhere.

In particular, we have

$$\partial_i g_{jk} = \Gamma^m_{ij} g_{mk} + \Gamma^m_{ik} g_{mj}.$$

Now let $t \mapsto \alpha(t)$ be a Lipschitz curve and let $Z : I \to TM$ be Lipschitz such that $\pi \circ Z = \alpha$. We set

$$Z'(t) := \sum_k \left(\frac{dZ^k}{dt} + \sum_{i,j} \Gamma^k_{ij}(\alpha(t)) \frac{d\alpha^j}{dt} Z^i(t) \right) \partial_k|_{\alpha(t)}$$

hence $Z' \in L^\infty_{\mathrm{loc}}(I, TM)$. Thus the usual properties of $X \mapsto X'$ hold and in particular,

$$\langle X, Y\rangle' = \langle X', Y\rangle + \langle X, Y'\rangle \tag{2.5}$$

holds almost everywhere for $X, Y \in \mathfrak{X}_{\mathrm{Lip}}(\alpha)$.

Theorem 2.2.9. *(The Gauss Lemma) Let g be a $C^{1,1}$-pseudo-Riemannian metric on M, and let $p \in M$. Then p possesses a basis of normal neighborhoods U with the following properties: $\exp_p : \tilde{U} \to U$ is a bi-Lipschitz homeomorphism, where $\tilde{U}$ is an open star-shaped neighborhood of 0 in T_pM. Moreover, for almost all $x \in \tilde{U}$, if v_x, $w_x \in T_x(T_pM)$ and v_x is radial, then*

$$\langle T_x \exp_p(v_x), T_x \exp_p(w_x)\rangle = \langle v_x, w_x\rangle.$$

Proof. Take U, $\tilde{U}$ as in Lemma 2.2.2 and let $x \in \tilde{U}$ be such that $T_x \exp_p$ exists. By bilinearity, we may assume that $x = v_x = v$ and $w_x = w$. Let $f(t, s) := \exp_p(t(v + sw))$. Then $s \mapsto f(., s) \in \mathrm{Lip}([\varepsilon, \varepsilon], C^2([0, 1], M))$. We have that $f(t, 0) = \exp_p(tv)$ thus $f_t(1, 0) = T_v \exp_p(v)$ and $f(1, s) = \exp_p(v + sw)$ so since v is such that $T_v \exp_p$ exists, by the chain rule we have $f_s(1, 0) = T_v \exp_p(w)$. It remains to show that $\langle f_t(1, 0), f_s(1, 0)\rangle = \langle v, w\rangle$. Since $t \mapsto f(t, s)$ is a geodesic with initial velocity $v + sw$, $f_{tt} = 0$ hence for some constant C, using (2.5) we obtain:

$$\langle f_t, f_t\rangle = C = \langle f_t(0, s), f_t(0, s)\rangle = \langle v + sw, v + sw\rangle.$$

By Lemma 2.2.7 and again by (2.5),

$$\begin{aligned}\partial_t \langle f_s, f_t\rangle &= \langle f_{st}, f_t\rangle + \langle f_s, f_{tt}\rangle = \langle f_{ts}, f_t\rangle = \frac{1}{2}\partial_s \langle f_t, f_t\rangle \\ &= \frac{1}{2}\partial_s(\langle v + sw, v + sw\rangle) = \langle v, w\rangle + s\langle w, w\rangle.\end{aligned}$$

Thus $\partial_t \langle f_s, f_t\rangle(t, 0) = \langle v, w\rangle$ for all t.

We have $f(0,s) = \exp_p(0) = p$ for all s hence $f_s(0,0) = 0$ and $\langle f_s, f_t\rangle(0,0) = 0$. Therefore, $\langle f_s, f_t\rangle(t,0) = t\langle v, w\rangle$ and for $t = 1$, $\langle f_t(1,0), f_s(1,0)\rangle = \langle v, w\rangle$. □

2.3 Local causality theory for $C^{1,1}$-metrics

A signal can be sent between two points of a spacetime only if they can be joined by a causal curve. Therefore by causality we refer to the general question of which points in a Lorentz manifold can be joined by causal curves. This relativistically means, which events influence or can be influenced by a given event. As in [Chr11] we will base our approach to causality theory on locally Lipschitz curves: fix a smooth complete Riemannian metric h (such a metric always exists, cf. [NO61]) and denote the corresponding distance function by d_h. Recall that a curve $\alpha : I \to M$ is said to be *locally Lipschitz* if for every $K \Subset I$ there is a constant $C(K)$ such that

$$\forall t_1, t_2 \in K \; d_h(\alpha(t_1), \alpha(t_2)) \leq C(K)|t_1 - t_2|.$$

The class of locally Lipschitz curves is independent on the choice of the background Riemannian metric h (this follows from the proof of [Chr11, Prop. 2.3.1]):

Proposition 2.3.1. *Let h_1 and h_2 be two complete Riemannian metrics on M. Then a curve $\alpha : I \to M$ is locally Lipschitz with respect to h_1 if and only if it is locally Lipschitz with respect to h_2.*

Proof. For $K \Subset I$, $\alpha(K)$ is compact. Let L_i, $i = 1, 2$ denote the h_i-length of α and set

$$K_i := \bigcup_{t\in K} B_{h_i}(\alpha(t), L_i)$$

where $B_{h_i}(p, r)$ is a geodesic ball with respect to the metric h_i. Then the sets K_i are relatively compact. The sets $\tilde{K}_i \subset TM$ of h_i-unit vectors over K_i are relatively compact as well. Hence there exists a constant C_K such that for all $v \in T_pM$, $p \in K_i$, we have

$$C_K^{-1} h_1(v, v) \leq h_2(v, v) \leq C_K h_1(v, v).$$

Let α_i denote any h_i-minimizing geodesic between $\alpha(t_1)$ and $\alpha(t_2)$. Then

$$\forall\ t_1, t_2 \in K,\ \ \alpha_i \subset K_i.$$

(It is contained in $B_{h_i}(\alpha(t), L_i)$ since the h_i-length of α is L_i and α_i is minimizing h_i-geodesic). It follows that

$$\begin{aligned}
d_{h_2}(\alpha(t_1), \alpha(t_2)) &= \int_0^{L_2(\alpha_2)} (h_2(\alpha_2'(t), \alpha_2'(t)))^{\frac{1}{2}} dt \\
&\geq C_k^{-1} \int_0^{L_2(\alpha_2)} (h_1(\alpha_2'(t), \alpha_2'(t)))^{\frac{1}{2}} dt \\
&\geq C_k^{-1} \inf_{\alpha} \int_0^{L_i(\alpha)} (h_1(\alpha'(t), \alpha'(t)))^{\frac{1}{2}} dt \\
&= C_k^{-1} \int_0^{L_1(\alpha_1)} (h_1(\alpha_1'(t), \alpha_1'(t)))^{\frac{1}{2}} dt \\
&= C_k^{-1}\ d_{h_1}(\alpha(t_1), \alpha(t_2)).
\end{aligned}$$

From symmetry with respect to the interchange of h_1 and h_2 we conclude:

$$C_k^{-1} d_{h_1}(\alpha(t_1), \alpha(t_2)) \leq d_{h_2}(\alpha(t_1), \alpha(t_2)) \leq C_k d_{h_1}(\alpha(t_1), \alpha(t_2))$$

$\forall t_1, t_2 \in K$ and the result follows. □

Any locally Lipschitz curve α is differentiable almost everywhere by Rademacher's theorem, cf. [EvGa92]. We call α timelike, causal, spacelike or null, if $\alpha'(t)$ has the corresponding property almost everywhere. If the time-orientation of M is determined by a continuous timelike vector field X then a causal curve α is called future- resp. past-directed if $\langle X(\alpha(t)), \alpha'(t)\rangle < 0$ resp. > 0 almost everywhere.

An important property of locally Lipschitz curves is that they can be parametrized with respect to the h-arclength. Consider a causal curve $\alpha : [a, b) \to M$ and suppose that α' is non-zero almost everywhere. By Rademacher's theorem, the integral

$$s(t) = \int_a^t |\alpha'|_h(u) du$$

is well-defined, $s(t)$ is a continuous strictly increasing function of t so $s : I \to s(I)$ is bijective and $s'(t) = |\gamma'(t)|_h$ almost everywhere. Now define $\tilde{\alpha} := \alpha \circ s^{-1}$. The function

s^{-1} is also strictly increasing hence differentiable almost everywhere so we have:

$$\begin{aligned}\tilde{\alpha}'(t) &= \alpha'(s^{-1}(t))(s^{-1})'(t) \\ &= \alpha'(s^{-1}(t))\frac{1}{s'(s^{-1}(t))} \\ &= \alpha'(s^{-1}(t))\frac{1}{|\alpha'(s^{-1}(t))|_h}\end{aligned}$$

thus $|\tilde{\alpha}'(t)|_h = 1$ almost everywhere. Moreover, $\tilde{\alpha}$ is Lipschitz with Lipschitz constant smaller or equal to 1:

We claim that $d_h(\tilde{\alpha}(t_1), \tilde{\alpha}(t_2)) \leq |t_1 - t_2|$. For $t_1 < t_2$ and taking the infimum over the curves $\hat{\alpha}$ that start at $\alpha(t_1)$ and end at $\alpha(t_2)$, we calculate:

$$\begin{aligned}t_2 - t_1 &= \int_{t_1}^{t_2} dt \\ &= \int_{t_1}^{t_2} \sqrt{\langle \tilde{\alpha}', \tilde{\alpha}' \rangle_h} dt \\ &\geq \inf_{\hat{\alpha}} \int_{\hat{\alpha}} \sqrt{\langle \hat{\alpha}', \hat{\alpha}' \rangle_h} dt = d_h(\tilde{\alpha}(t_1), \tilde{\alpha}(t_2)).\end{aligned}$$

Remark 2.3.2. This approach to causal curves differs from that in [Min13], where the corresponding curves are required to be C^1 (see, however, Cor. 2.3.11 below). As one of the key tools when studying causality theory is taking limits of causal curves, considering piecewise C^1 curves leads to many difficulties. Namely, the limit of such curves will rarely be piecewise C^1, also causing timelike and causal curves to have completely different properties in which case, separate proofs need to be given. The approach based on Lipschitz curves overcomes these problems.

With these notions we have:

Definition 2.3.3. Let g be a C^0-Lorentzian metric on M. For $p \in A \subseteq M$ we define the relative chronological, respectively causal future of p in A by (cf. [Chr11, 2.4]):

$$\begin{aligned}I^+(p, A) &:= \{q \in A|\text{ there exists a future directed timelike curve in } A \text{ from } p \text{ to } q\ \} \\ J^+(p, A) &:= \{q \in A|\text{ there exists a future directed causal curve in } A \text{ from } p \text{ to } q\ \} \cup A.\end{aligned}$$

For $B \subseteq A$ we set $I^+(B, A) := \bigcup_{p \in B} I^+(p, A)$ and analogously for $J^+(B, A)$. We set $I^+(p) := I^+(p, M)$. Replacing 'future directed' by 'past-directed' we obtain the corresponding definitions of the chronological respectively causal pasts I^-, J^-.

These sets are of a very simple structure in the case of Minkowski spacetime R_1^n. Given $p \in R_1^n$, future directed null rays emanating from p constitute the future cone at p, namely, $\partial I^+(p) = J^+(p) \setminus I^+(p)$. The chronological future $I^+(p)$ is the set of all the points inside the cone and the causal future $J^+(p)$ consists of p, the points inside and on the cone, see Figure 2.3. One of the most important results of causality theory, see Theorem 2.3.10, shows that locally, any Lorentzian manifold has the same structure as the Minkowski spacetime.

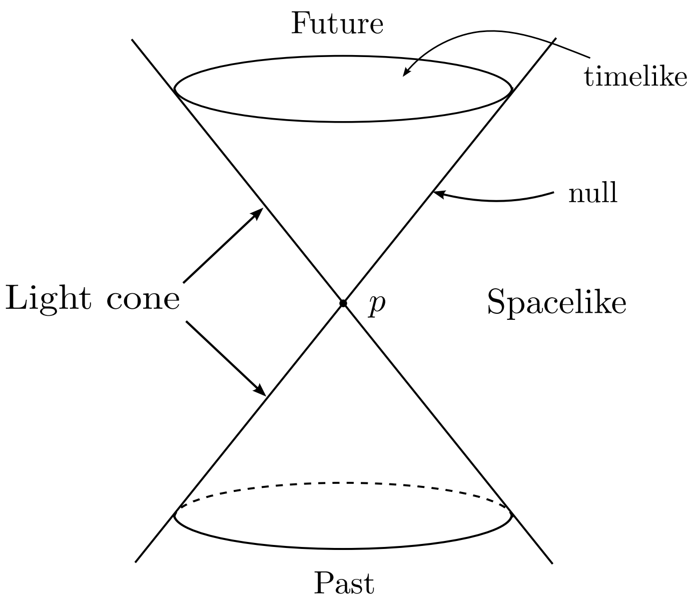

Figure 2.1: The light cone at p

Note that, even though the class of locally Lipschitz timelike curves is considerably wider than the class of piecewise C^1 curves, the resulting sets are the same as the standard ones, cf. 2.3.11. Below we will formulate all results for I^+, J^+. By symmetry, the corresponding claims for chronological or causal pasts follow in the same way.

As usual, for p, $q \in M$ we write $p < q$, respectively $p \ll q$, if there is a future directed causal, respectively timelike, curve from p to q. By $p \leq q$ we mean $p = q$ or $p < q$.

The following result shows some elementary properties of futures and pasts (cf. [Chr11, Prop. 2.4.2]):

Proposition 2.3.4. *Let g be a continuous Lorentzian metric on M. Then:*

1. $I^+(U) \subset J^+(U)$

2. $p \in I^+(q) \Leftrightarrow q \in I^-(p)$

3. $V \subset I^+(U) \Rightarrow I^+(V) \subset I^+(U)$

Similar properties hold when I^+ is replaced with J^+.

Proof. 1. is clear.

2. If $[0,1] \ni t \to \alpha(t)$ is a future directed causal curve from q to p, then $[0,1] \ni t \to \alpha(1-t)$ is a past directed causal curve from p to q.

3. Consider $\alpha_i : [0,1] \to M,\ i = 1,2,$ two causal curves such that $\alpha_1(1) = \alpha_2(0)$. The concatenation $\alpha_1 \cup \alpha_2$ is given by:

$$(\alpha_1 \cup \alpha_2)(t) = \begin{cases} \alpha_1(t), & t \in [0,1], \\ \alpha_2(t-1), & t \in [1,2]. \end{cases}$$

Now let $r \in I^+(V)$. Then there exists $q \in V$ and a future directed timelike curve α_2 from q to r. Since $V \subset I^+(U)$, there exists a future directed timelike curve α_1 from some point $p \in U$ to q. Then the curve $\alpha_1 \cup \alpha_2$ is a future directed timelike curve from U to r so $r \in I^+(U)$. □

We now recall some definitions that were introduced in [CG12] and results there obtained which will be of use in this work.

Definition 2.3.5. A locally Lipschitz curve $\alpha : [0,1] \to M$ is said to be locally uniformly timelike (l.u.-timelike) with respect to the C^0-metric g if there exists a smooth Lorentzian metric $\check{g} \prec g$ such that $\check{g}(\alpha', \alpha') < 0$ almost everywhere. Then for $p \in A \subseteq M$

$$\check{I}^+_g(p, A) := \{q \in A |\ \text{there exists a future directed l.u.-timelike curve in } A \text{ from } p \text{ to } q\}.$$

Thus $\check{I}^+_g(A) = \bigcup_{\check{g} \prec g} I^+_{\check{g}}(A)$, hence it is open ([CG12, Prop. 1.4]). The following definition ([CG12, Def. 1.8]) introduces a highly useful substitute for normal coordinates in the context of metrics of low regularity:

Definition 2.3.6. Let (M, g) be a smooth Lorentzian manifold with continuous metric g and let $p \in M$. A relatively compact open subset U of M is called a cylindrical neighborhood of $p \in U$ if there exists a smooth chart (φ, U), $\varphi = (x^0, ..., x^{n-1})$ with $\varphi(U) = I \times V$, I an interval around 0 in $\mathbb{R}$ and V open in $\mathbb{R}^{n-1}$, such that:

1. $\frac{\partial}{\partial x^0}$ is timelike and $\frac{\partial}{\partial x^i}$, $i = 1, ..., n-1$, are spacelike,
2. For $q \in U,\ v \in T_qM$, if $g_q(v, v) = 0$ then $\frac{|v^0|}{\|\vec{v}\|} \in (\frac{1}{2}, 2)$ (where $T_q\varphi(v) = (v^0, \vec{v})$, and $\|\ \|$ is the Euclidean norm on $\mathbb{R}^{n-1}$),
3. $(\varphi_* g)_{\varphi(p)} = \eta$ (the Minkowski metric).

By [CG12, Prop. 1.10], every point in a spacetime with continuous metric possesses a basis of cylindrical neighborhoods. According to [CG12, Def. 1.16], a Lorentzian manifold M with C^0-metric g is called *causally plain* if for every $p \in M$ there exists a cylindrical neighborhood U of p such that $\partial \check{I}^{\pm}(p, U) = \partial J^{\pm}(p, U)$. This condition excludes causally 'pathological' behaviour (bubbling metrics). By [CG12, Cor. 1.17], we have:

Proposition 2.3.7. *Let g be a $C^{0,1}$-Lorentzian metric on M. Then (M, g) is causally plain.*

The most important property of causally plain Lorentzian manifolds for our purposes is given in the following result ([CG12, Prop. 1.21]).

Proposition 2.3.8. *Let g be a continuous, causally plain Lorentzian metric and let $A \subseteq M$. Then*

$$I^{\pm}(A) = \check{I}^{\pm}(A). \tag{2.6}$$

Returning now to our main object of study, for the remainder of this section g will denote a $C^{1,1}$-Lorentzian metric. Then in particular, g is causally plain by Prop. 2.3.7. To analyze the local causality for g in terms of the exponential map we first introduce some terminology. Let $\tilde{U}$ be a star-shaped neighborhood of $0 \in T_pM$ such that $\exp_p : \tilde{U} \to U$ is a bi-Lipschitz homeomorphism (Th. 1.4.2). On T_pM we define the position vector field $\tilde{P} : v \mapsto v_v$ and the quadratic form $\tilde{Q} : T_pM \to \mathbb{R}$, $v \mapsto g_p(v, v)$. By P, Q we denote the push-forwards of these maps via $\exp_p$, i.e.,

$$P(q) := T_{\exp_p^{-1}(q)} \exp_p(\tilde{P}(\exp_p^{-1}(q)))$$
$$Q(q) := \tilde{Q}(\exp_p^{-1}(q)).$$

As $\exp_p$ is locally Lipschitz, P is an L^∞_{loc}-vector field on U, while Q is locally Lipschitz (see, however, Rem. 2.3.9 below).

Let X be some smooth vector field on U and denote by $\tilde X$ its pullback $\exp_p^* X$ (note that $T_v \exp_p$ is invertible for almost every $v \in \tilde U$). Then by Th. 2.2.9, for almost every $q \in U$ we have, setting $\tilde q := \exp_p^{-1}(q)$:

$$\langle \mathrm{grad} Q(q), X(q)\rangle = X(Q)(q) = \tilde X(\tilde Q)(\tilde q) = \langle \mathrm{grad}\tilde Q, \tilde X\rangle|_{\tilde q} = 2\langle \tilde P, \tilde X\rangle|_{\tilde q} = 2\langle P, X\rangle|_q.$$

It follows that $\mathrm{grad} Q = 2P$.

Remark 2.3.9. It is proved in [Min13] that the regularity of both P and Q is better than would be expected from the above definitions. Indeed, [Min13, Prop. 2.3] even shows that P, as a function of (p, q) is strongly differentiable on a neighborhood of the diagonal in $M \times M$, and by [Min13, Th. 1.18], Q is in fact $C^{1,1}$ as a function of (p, q). We will however not make use of these results in what follows and only remark that slightly weaker regularity properties of P and Q (as functions of q only) can also be obtained directly from standard ODE-theory. In fact, setting $\alpha_v(t) := \exp_p(tv)$ for $v \in T_pM$, it follows that $P(q) = \alpha'_{v_q}(1)$, where $v_q := \exp_p^{-1}(q)$. Since $t \mapsto (\alpha_v(t), \alpha'_v(t))$ is the solution of the first-order system corresponding to the geodesic equation with initial value (p, v), and since the right-hand side of this system is Lipschitz-continuous, [Amann90, Th. 8.4] shows that $v \mapsto \alpha'_v(1)$ is Lipschitz-continuous. Since also $q \mapsto v_q$ is Lipschitz, we conclude that P is Lipschitz-continuous. From this, by the above calculation, it follows that Q is $C^{1,1}$.

As in the smooth case, we may use $\exp_p$ to introduce normal coordinates. To this end, let $e_0, \ldots, e_n$ be an orthonormal basis of T_pM and for $q \in U$ set $x^i(q)e_i := \exp_p^{-1}(q)$. The coordinates x^i then are of the same regularity as $\exp_p^{-1}$, i.e., locally Lipschitz. The coordinate vector fields $\frac{\partial}{\partial x^i}\big|_q = T_{\exp_p^{-1}(q)} \exp_p(e^i)$ themselves are in L^∞_{loc}. Note, however, that in the $C^{1,1}$-setting we can no longer use the relation $g_p = \eta$ (the Minkowski-metric in the x^i-coordinates), since it is not clear a priori that $\exp_p$ is differentiable at 0 with $T_0 \exp_p = \mathrm{id}_{T_pM}$[1]. Due to the additional loss in regularity it is also usually not advisable to write the metric in terms of the exponential chart (the metric coefficients in these coordinates would only be L^∞_{loc}).

The following is the main result on the local causality in normal neighborhoods.

[1]See, however, [Min13] where it is shown that indeed $\exp_p$ is even strongly differentiable at 0 with derivative id_{T_pM}.

Theorem 2.3.10. *Let g be a $C^{1,1}$-Lorentzian metric, and let $p \in M$. Then p has a basis of normal neighborhoods U, $\exp_p : \tilde{U} \to U$ a bi-Lipschitz homeomorphism, such that:*

$$
\begin{aligned}
I^+(p,U) &= \exp_p(I^+(0) \cap \tilde{U}) \\
J^+(p,U) &= \exp_p(J^+(0) \cap \tilde{U}) \\
\partial I^+(p,U) = \partial J^+(p,U) &= \exp_p(\partial I^+(0) \cap \tilde{U})
\end{aligned}
$$

Here, $I^+(0) = \{v \in T_pM \mid \tilde{Q}(v) < 0\}$, and $J^+(0) = \{v \in T_pM \mid \tilde{Q}(v) \leq 0\}$. In particular, $I^+(p,U)$ (respectively $J^+(p,U)$) is open (respectively closed) in U.

Proof. We first note that the third claim follows from the first two and the fact that $\exp_p$ is a homeomorphism on U. For the proof of the first two claims we take a normal neighborhood U that is contained in a cylindrical neighborhood of p. In addition, we pick a regularizing net $\hat{g}_\varepsilon$ as in Prop. 2.2.5 and let U, $\tilde{U}$ as in Lemma 2.2.2 (fixing a suitable $\varepsilon_0 > 0$).

($\supseteq$) Let $v \in \tilde{U}$ and let $\alpha := t \mapsto \exp_p(tv)$, $t \in [0,1]$. Set $\alpha_\varepsilon(t) := \exp_p^{\hat{g}_\varepsilon}(tv)$. Then by continuous dependence on initial data we have that $\alpha_\varepsilon \to \alpha$ in C^1 (cf. Lemma 1.4.4). Hence applying the smooth Gauss lemma for each ε it follows that for each $t \in [0,1]$ we have

$$g(\alpha'(t), \alpha'(t)) = \lim_{\varepsilon \to 0} \hat{g}_\varepsilon(\alpha'_\varepsilon(t), \alpha'_\varepsilon(t)) = \lim_{\varepsilon \to 0} (\hat{g}_\varepsilon)_p(v,v) = g_p(v,v).$$

Also, time-orientation is respected by $\exp_p$ since both $I(0) \cap \tilde{U}$ and $I(p,U)$ (by [CG12, Prop. 1.10]) have two connected components, and the positive x^0-axis in $\tilde{U}$ is mapped to $I^+(p,U)$.

($\subseteq$): We denote the position vector fields and quadratic forms corresponding to $\hat{g}_\varepsilon$ by $\tilde{P}_\varepsilon$, P_ε and $\tilde{Q}_\varepsilon$, Q_ε, respectively.

If $\alpha : [0,1] \to U$ is a future-directed causal curve in U emanating from p then α is timelike with respect to each $\hat{g}_\varepsilon$. Set $\beta := (\exp_p)^{-1} \circ \alpha$ and $\beta_\varepsilon := (\exp_p^{\hat{g}_\varepsilon})^{-1} \circ \alpha$. By [Chr11, Prop. 2.4.5], $\beta_\varepsilon([0,1]) \subseteq I^+_{\hat{g}_\varepsilon(p)}(0)$ for all $\varepsilon < \varepsilon_0$. Then by Lemma 2.2.2 we have that $\beta_\varepsilon \to \beta$ uniformly, and that $\tilde{Q}_\varepsilon \to \tilde{Q}$ locally uniformly, so $\tilde{Q}(\beta(t)) = \lim \tilde{Q}_\varepsilon(\beta_\varepsilon(t)) \leq 0$ for all $t \in [0,1]$, and therefore $\beta((0,1]) \subseteq J^+(0) \cap \tilde{U}$. Together with the first part of the proof it follows that $\exp_p(J^+(0) \cap \tilde{U}) = J^+(p,U)$. Now assume that α is timelike. Then by Prop. 2.3.8, $\alpha((0,1]) \subseteq \check{I}^+(p,U)$. This means that there exists a smooth metric $\check{g} \prec g$ such that α is $\check{g}$-timelike. Let $f_{\check{g}}, f_g$ denote the graphing functions of $\partial I^+_{\check{g}}(p,U)$ and $\partial J^+(p,U)$, respectively (in a cylindrical chart, see [CG12, Prop. 1.10]). Then by [CG12, Prop. 1.10], since α lies in $I^+_{\check{g}}(p,U)$, it has to lie strictly above $f_{\check{g}}$, hence also strictly above f_g, and so $\alpha((0,1]) \cap \partial J^+(p,U) = \emptyset$. But then, since $\exp_p$ is a homeomorphism on U, we have that

$$\beta((0,1]) \cap (\partial J^+(0) \cap \tilde{U}) = \beta((0,1]) \cap \exp_p^{-1}(\partial J^+(p,U)) = \exp_p^{-1}(\alpha((0,1]) \cap \partial J^+(p,U)) = \emptyset$$

Hence β lies entirely in $I^+(0) \cap \tilde{U}$, as claimed. □

Corollary 2.3.11. *Let $U \subseteq M$ be open, $p \in U$. Then the sets $I^+(p,U)$, $J^+(p,U)$ remain unchanged if, in Def. 2.3.3, Lipschitz curves are replaced by piecewise C^1 curves, or in fact by broken geodesics.*

Proof. Let $\alpha : [0,1] \to U$ be a, say, future directed timelike Lipschitz curve in U. By Th. 1.7.1 and Th. 2.3.10 we may cover $\alpha([0,1])$ by finitely many totally normal open sets $U_i \subseteq U$, such that there exist $0 = t_0 < \cdots < t_N = 1$ with $\alpha([t_i,t_{i+1}]) \subseteq U_{i+1}$ and $I^+(\alpha(t_i),U_i) = \exp_{\alpha(t_i)}(I^+(0) \cap \tilde{U}_i)$ for $0 \le i < N$. Then the concatenation of the radial geodesics in U_i connecting $\alpha(t_i)$ with $\alpha(t_{i+1})$ gives a timelike broken geodesic from $\alpha(0)$ to $\alpha(1)$ in U. □

The existence of totally normal neighborhoods allows us to prove the following:

Corollary 2.3.12. *If $q \in I^+(p)$ (resp. $q \in J^+(p)$), then there exists a future directed piecewise broken timelike (causal) geodesic from p to q.*

Proof. Let $q \in J^+(p)$ and let $\alpha : [0,1] \to M$ be a future directed causal curve such that $\alpha(0) = p$ and $\alpha(1) = q$. Cover α by totally normal sets U_i, $i \in \mathbb{N}$ and let $p_1 \in U_1$. Since U_1 is totally normal, it is a normal neighborhood of each of its points, therefore a normal neighborhood of p and p_1. By definition, $\tilde{U}_1$ is a starshaped neighborhood of $0 \in T_pM$, such that $\exp_p|_{\tilde{U}_1}$ is a Lipschitz homeomorphism onto U_1. Let $q_1 = \exp_p^{-1}(p_1) \in \tilde{U}_1$. The ray $\rho(t) = tq_1$, $(0 \le t \le 1)$ lies in $\tilde{U}_1$ since $\tilde{U}_1$ is starshaped. Thus the geodesic segment $\exp_p \circ \rho$ lies in U_1 and runs from p to p_1, so we have connected those points with a radial geodesic. We repeat this procedure for points p_1 and $p_2 \in U_1 \cap U_2$ and later on, for p_i and $p_{i+1} \in U_i \cap U_{i+1}$, so we get radial geodesics between each two of these points.

If α is timelike, all the segments are also timelike. Concatenating these segments, we obtain the desired future directed piecewise broken geodesic from p to q. □

The following analogue of [Chr11, Cor. 2.4.10] provides more information about causal curves intersecting the boundary of $J^+(p,U)$:

Corollary 2.3.13. *Under the assumptions of Th. 2.3.10, suppose that $\alpha : [0,1] \to U$ is causal and $\alpha(1) \in \partial J^+(p,U)$. Then α lies entirely in $\partial J^+(p,U)$ and there exists a reparametrization of α as a null-geodesic segment.*

Proof. Suppose to the contrary that there exists $t_0 \in (0,1)$ such that $\alpha(t_0) \in I^+(p,U)$. Then there exists a future directed timelike curve γ from p to $\alpha(t_0)$. Applying Prop. 2.3.15 to the concatenation $\gamma \cup \alpha|_{[t_0,1]}$ it follows that there exists a future directed timelike curve from p to $\alpha(1)$. But then $\alpha(1) \in I^+(p,U)$, a contradiction. Thus $\alpha(t) \in \partial J^+(p,U)$, $\forall t \in [0,1]$, implying that $\beta(t) = \exp_p^{-1} \circ \alpha(t) \in \partial J^+(0)$, $\forall t \in [0,1]$, so $\tilde{Q}(\beta(t)) = 0$, $\forall t \in [0,1]$ and for almost all t we have

$$0 = \frac{d}{dt}\tilde{Q}(\beta(t)) = g_p(\mathrm{grad}\tilde{Q}(\beta(t)), \beta'(t)) = g_p(2\tilde{P}(\beta(t)), \beta'(t)).$$

Hence $\beta(t)$ is collinear with $\beta'(t)$ almost everywhere, and it is easily seen that this implies the existence of some $v \neq 0, v \in \partial J^+(0)$, and of some $h : \mathbb{R} \to \mathbb{R}$ such that $\beta(t) = h(t)v$. The function h is locally Lipschitz since β is, and injective since α is (on every cylindrical neighborhood there is a natural time function). Thus h is strictly monotonous, and in fact strictly increasing since otherwise β would enter $J^-(0)$. Thus $\beta'(t) = f(t)\beta(t)$ where $f(t) := \frac{h'(t)}{h(t)} \in L^\infty_{\mathrm{loc}}$. From here we may argue exactly as in [Chr11, Cor. 2.4.10]: the function $r(s) := \int_0^s f(\tau)\, d\tau$ is locally Lipschitz and strictly increasing, hence a bijection from $[0,1]$ to some interval $[0,r_0]$. Thus so is its inverse $r \to s(r)$, and we obtain $\beta(s(r))' = \beta'(s(r))/f(s(r)) = \beta(s(r))$ a.e., where the right hand side is even continuous. It follows that in this parametrization, β is C^1 and in fact is a straight line in the null cone, hence α can be parametrized as a null-geodesic segment, as claimed. □

The following result, called the push-up principle, is one of the essential tools in many arguments of causality theory. In fact, together with the fact that accumulation curves of causal curves are again causal (see Section 2.5), it is the main ingredient for developing causality theory. Note that this result has recently been proved in [CG12] in greater generality, namely for causally plain metrics (cf. [CG12, Lemma 1.22]) but for consistency reasons we will include the proof for $C^{1,1}$-metrics:

Proposition 2.3.14. *Let p, q, $r \in M$ with $p \leq q$ and $q \ll r$ or $p \ll q$ and $q \leq r$. Then $p \ll r$.*

Proof. It suffices to consider the case $p < q \ll r$. Let $\alpha : [0,1] \to M$ be causal and future-directed from p to q and $\beta : [0,1] \to M$ timelike future-directed from q to r. Following [Chr11, Lemma 2.4.14], the strategy of proof is to 'push up' the concatenation of α and β slightly to obtain a timelike connection of p and r. To this end, as in the proof of Cor. 2.3.11 we cover $\alpha([0,1])$ by finitely many totally normal open sets $U_i \subseteq U$ $(1 \le i \le N)$, such that there exist $0 = t_0 < \cdots < t_N = 1$ with $\alpha([t_i, t_{i+1}]) \subseteq U_{i+1}$ and $I^+(\alpha(t_i), U) = \exp_{\alpha(t_i)}(I^+(0) \cap \tilde{U}_i)$ for $0 \le i < N$.

Since β is non-null and intersects U_N, by Th. 2.3.10 and Cor. 2.3.13 there exists a timelike geodesic segment β_1 from $\alpha(t_{N-1})$ to some point in $\beta([0,1]) \cap U_N$. Next, we apply the same reasoning to β_1 to obtain a timelike geodesic segment from $\alpha(t_{N-2})$ to an element of U_{N-1} lying on β_1. Continuing in this way, after $N-1$ steps we obtain by concatenation a timelike future-directed curve connecting p and r. □

Since the U_i in the proof of Prop. 2.3.14 can be chosen in any given neighborhood of the concatenation of α and β, we obtain (cf. [CG12, Prop. 1.23] for causally plain metrics):

Corollary 2.3.15. *Let $\alpha : [0,1] \to M$ be causal and future-directed from p to q. If there exist $t_1 < t_2$ in $[0,1]$ such that $\alpha|_{[t_1,t_2]}$ is timelike, then in any neighborhood of α there exists a timelike future-directed curve from p to q.*

Proposition 2.3.16. *Let α be causal curve from p to q in (M,g) which is not a null geodesic. Then there exists a timelike curve from p to q.*

Proof. By Corollary 2.3.12 we may assume without loss of generality that α is a piecewise broken geodesic. If one of the geodesics forming α is timelike, the result follows from the previous Corollary. Consider now a piecewise broken null geodesic with a break point, say $\tilde{p}$. Let a point q on α be in $J^-(\tilde{p})$ and close enough to $\tilde{p}$ so that $\tilde{p}$ belongs to a domain of normal coordinates U centered at q. Then points on α lying to the causal future of $\tilde{p}$ are not in $\partial J^+(q,U)$ by Cor. 2.3.13 (since if they were, α would entirely lie in the boundary and there would be a reparametrization of it so that it is a null geodesic segment through q. Therefore, there would be no break points). Hence, they are in $I^+(q,U)$ and α can be deformed within U to a timelike curve. The result follows now from the previous Corollary. □

Corollary 2.3.17. *The relation $\ll$ is open: if $p \ll q$ then there exist neighborhoods V of p and W of q such that $p' \ll q'$ for all $p' \in V$ and $q' \in W$. In particular, for any $p \in M$, $I^+(p)$ is open in M.*

Proof. Let α be a future-directed timelike curve from p to q and pick totally normal neighborhoods N_p, N_q of p, q as in Th. 2.3.10. Now let $p' \in N_p$ and $q' \in N_q$ be points on α. Then $V := I^-(p', N_p)$ and $W := I^+(q', N_q)$ have the required property. □

From this we immediately conclude:

Corollary 2.3.18. *Let $A \subseteq U \subseteq M$, where U is open. Then*

$$I^+(A,U) = I^+(I^+(A,U)) = I^+(J^+(A,U)) = J^+(I^+(A,U)) \subseteq J^+(J^+(A,U)) = J^+(A,U)$$

We have already established that timelike futures are open. In Minkowski spacetime the sets $J^\pm(p)$ are closed, with

$$\overline{I^\pm(p)} = J^\pm(p). \tag{2.7}$$

This need not be true in general:

Example 2.3.19. Let (M,g) be the two-dimensional Minkowski space-time $\mathbb{R}^{1,1}$ from which a point, say $(1,1)$, has been removed. Taking p as the origin, it is easily seen that no causal curve reaches the region after the point that has been deleted, which is represented by the dashed line in Figure 2.3. Hence $J^+(p,M)$ is neither open nor closed and since $I(p)$ stays unchanged, (2.7) does not hold, i.e., $J^+(p) \subsetneq \overline{I^+(p)}$.

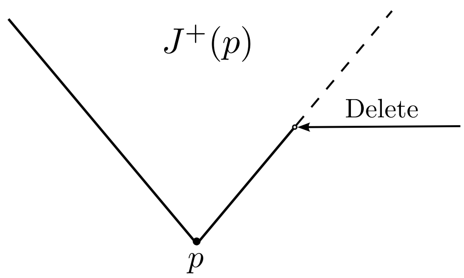

Figure 2.2: $J^+(p)$ not closed

Another straightforward consequence of Lemma 2.3.14 is the following property of J:

Corollary 2.3.20. *For any $p \in M$ we have*

$$J^+(p) \subset \overline{I^+(p)}.$$

Proof. Let $q \in J^+(p)$ and consider a sequence of points $r_i \in I^+(q)$ which accumulate at q. Using the push-up principle (see Prop. 2.3.14), we obtain that $r_i \in I^+(J^+(p)) = I^+(p)$ hence $q \in \overline{I^+(p)}$. □

A consequence of Prop. 2.3.15 is that the causal future of any $A \subseteq M$ consists (at most) of A, $I^+(A)$ and of null-geodesics emanating from A:

Corollary 2.3.21. *Let $A \subseteq M$ and let α be a causal curve from some $p \in A$ to some $q \in J^+(A) \setminus I^+(A)$. Then α is a null-geodesic that does not meet $I^+(A)$.*

Proof. By Prop. 2.3.15, α has to be a null curve. Moreover, if $\alpha(t) \in I^+(A)$ for some t then for some $a \in A$ we would have $a \ll \alpha(t) \leq q$, so $q \in I^+(A)$ by Prop. 2.3.14, a contradiction. Covering α by totally normal neighborhoods as in Cor. 2.3.11 and applying Cor. 2.3.13 gives the claim. □

Following [ON83, Lemma 14.2] we next give a more refined description of causality for totally normal sets. For this, recall from the proof of Th. 1.7.1 that the map $E : v \mapsto (\pi(v), \exp_{\pi(v)}(v))$ is a homeomorphism from some open neighborhood S of the zero section in TM onto an open neighborhood W of the diagonal in $M \times M$. If U is totally normal as in Th. 2.3.10 and such that $U \times U \subseteq W$ then the map $U \times U \to TM$, $(p, q) \mapsto \overrightarrow{pq} := \exp_p^{-1}(q) = E^{-1}(p, q)$ is continuous.

Proposition 2.3.22. *Let $U \subseteq M$ be totally normal as in Th. 2.3.10.*

(i) Let p, $q \in U$. Then $q \in I^+(p, U)$ (resp. $\in J^+(p, U)$) if and only if $\overrightarrow{pq}$ is future-directed timelike (resp. causal).

(ii) $J^+(p, U)$ is the closure of $I^+(p, U)$ relative to U.

(iii) The relation $\leq$ is closed in $U \times U$.

(iv) If K is a compact subset of U and $\alpha : [0, b) \to K$ is causal, then α can be continuously extended to $[0, b]$.

Proof. (i) and (ii) are immediate from Th. 2.3.10.

(iii) Let $p_n \leq q_n$, $p_n \to p$, $q_n \to q$. By (i), $\overrightarrow{p_n q_n}$ is future-directed causal for all n. By continuity (see Th. 1.7.1), therefore, $\langle \overrightarrow{pq}, \overrightarrow{pq} \rangle \leq 0$, so $\overrightarrow{pq}$ is future-directed causal as well.

(iv) Let $0 < t_1 < t_2 < \cdots \to b$. Since K is compact, $\alpha(t_i)$ has an accumulation point p and it remains to show that p is the only accumulation point. Suppose that $q \neq p$ is also an accumulation point. Choose a subsequence t_{i_k} such that $\alpha(t_{i_{2k}}) \to p$ and $\alpha(t_{i_{2k+1}}) \to q$. Then since $\alpha(t_{i_{2k}}) \leq \alpha(t_{i_{2k+1}}) \leq \alpha(t_{i_{2k+2}})$, (iii) implies that $p \leq q \leq p$. By (i), then, $\overrightarrow{pq}$ would be both future- and past-directed, which is impossible. □

From this, with the same proof as in [ON83, Lemma 14.6] we obtain:

Corollary 2.3.23. *Let $A \subseteq M$. Then*

(i) $J^+(A)^\circ = I^+(A)$.

(ii) $J^+(A) \subseteq \overline{I^+(A)}$.

(iii) $J^+(A) = \overline{I^+(A)}$ *if and only if* $J^+(A)$ *is closed.*

Finally, as in the smooth case, one may introduce a notion of causality also for general continuous curves (cf. [HE73, p. 184], [Kr99, Def. 8.2.1]):

Definition 2.3.24. A continuous curve $\alpha : I \to M$ is called future-directed causal (resp. timelike) if for every $t \in I$ there exists a totally normal neighborhood U of $\alpha(t)$ such that for any $s \in I$ with $\alpha(s) \in U$ and $s > t$, $\alpha(s) \in J^+(\alpha(t))\setminus\{\alpha(t)\}$ (resp. $\alpha(s) \in I^+(\alpha(t))\setminus\{\alpha(t)\}$), and analogously for $s < t$ with J^- resp. I^-.

The proof of [Kr99, Lemma 8.2.1]) carries over to the $C^{1,1}$-setting, showing that any continuous causal (resp. timelike) curve is locally Lipschitz.

Remark 2.3.25. While a continuous causal curve α need not be a causal Lipschitz curve in the sense of our definition (cf. [Min13, Rem. 1.28]), it still follows that $\langle \alpha'(t), \alpha'(t)\rangle \leq 0$ almost everywhere (however, $\alpha'(t)$ might be 0).

To see this, consider first the case where g is smooth. Set $p := \alpha(t)$, pick a normal neighborhood U around p and set $\beta := \exp_p^{-1} \circ \alpha$. Then $\beta'(t) = \alpha'(t)$ and by Def. 2.3.24 and Th. 2.3.10, $\beta(s) \in J^+(0)$ for $s > t$ small. Therefore, $\beta'(t) \in J^+(0)$, so $\langle \alpha'(t), \alpha'(t)\rangle \leq 0$. In the general case, where g is only supposed to be $C^{1,1}$, pick a regularization $\hat{g}_\varepsilon$ as in Prop. 2.2.5. Then $\hat{g}_\varepsilon(\alpha'(t), \alpha'(t)) \leq 0$ for all ε by the above and letting $\varepsilon \to 0$ gives the claim.

2.4 Extendible and inextendible curves

When studying causality and singularity theory, a very important notion is that of an inextendible causal curve. As an object travels through spacetime it follows a future directed causal curve. Intuitively, an object would travel for "eternity" if the corresponding causal curve associated to it "goes on forever" into the future, i.e., is future inextendible. Note that in the case of a spacetime having a "hole", for example $M = \mathbb{R}^2_1 \setminus \{0, 0\}$, one can construct a future directed inextendible causal curve which comes arbitrarily close to the hole as the time runs to infinity. For a curve that "stops", a natural question is if by concatenation with another curve, it could be made inextendible. This section explains that concept in more detail following [Chr11].

Definition 2.4.1. A future directed causal curve $\alpha : [a, b) \to M$ is future extendible provided there exists $b < c \in \mathbb{R} \cup \{\infty\}$ and a causal curve $\tilde{\alpha} : [a, c) \to M$ such that

$$\tilde{\alpha}|_{[a,b)} = \alpha. \tag{2.8}$$

Then $\tilde{\alpha}$ is called an *extension* of α.

Definition 2.4.2. Let $\alpha : [a, b) \to M$ be a future directed causal curve. A point p is called a future end point of α if $\lim_{s\to b} \alpha(s) = p$.

The curve α is said to be *future inextendible* if it is not future extendible. Notions of past inextendibility, extendibility and past end points are defined analogously.

For a given future directed causal curve $\alpha : [a, b) \to M$ with an end point p, the question arises under which conditions it can be extended to a causal curve $\tilde{\alpha} : [a, b] \to M$, such that $\tilde{\alpha}|_{[a,b)} = \alpha$ and $\tilde{\alpha}(b) = p$. One of the problems that can occur is that $\tilde{\alpha}$ need not be locally Lipschitz in general: $[0, 1) \ni \alpha(t) = (-\sqrt{1-t}, 0) \in \mathbb{R}^1_1$ is locally Lipschitz on $[0, 1)$ but is not on $[0, 1]$. This is why we now fix an auxiliary complete Riemannian metric h and denote by d_h the corresponding distance function. Let α be a causal curve. We know that α can be parametrized with respect to the h-arc-length, leading to a Lipschitz curve with Lipschitz constant 1 (see the argument following Prop. 2.3.1). Then we have the following result:

Lemma 2.4.3. *Let $\alpha : [a, b) \to M,\ b < \infty$, be a Lipschitz curve with an end point p. Then α can be extended to a Lipschitz curve $\hat{\alpha} : [a, b] \to M$, with $\hat{\alpha} = p$.*

Proof. From the condition that α is Lipschitz,

$$d_h(\alpha(t), \alpha(t')) \leq L|t - t'|, \ \forall s \in [a, b), \tag{2.9}$$

passing from t' to b in that equation we obtain

$$d_h(\alpha(t), p) \leq L|t - b|$$

hence $\hat{\alpha}$ is also Lipschitz. □

Consider now $\alpha : [a, b) \to M, \ b \in \mathbb{R} \cup \{\infty\}$. Then p is said to be an ω-limit point if there exists a sequence $t_k \to b$ such that $\alpha(t_k) \to p$. An end point is always an ω-limit point, but the converse need not be true in general. For example, consider $\alpha(s) = \exp(is) \in \mathbb{C}$, then by setting $s_k = x + 2\pi k$, we see that every point $\exp(ix) \in S^1 \subset \mathbb{C}^1$ is an ω-limit point of α. For Lipschitz curves and $b < \infty$, the notions of an end point and of an ω-limit point coincide:

Lemma 2.4.4. *Let $\alpha : [a, b) \to M, \ b < \infty$ be a Lipschitz curve. Then every ω-limit point of α is an end point of α. In particular, α has at most one ω-limit point.*

Proof. Let p be an ω-limit point of α. By definition there exists a sequence $t_i \to b$ such that $\alpha(t_i) \to p$. Using (2.9) we obtain that $d_h(\alpha(t_i), \alpha(t)) \leq L|t_i - t|$. Since d_h is a continuous function of its arguments, letting $i \to \infty$, we have

$$d_h(p, \alpha(t)) \leq L|b - t|,$$

so p is an end point of α. If exists, the end point is unique, hence the result. □

Note that, since a locally Lipschitz curve is Lipschitz on a compact set, any extension $\tilde{\alpha}$ is Lipschitz on the compact subset $[a, b]$ of its domain of definition. But then by (2.8), $\tilde{\alpha}|_{[a,b)}$ is also Lipschitz. Hence, extendibility in the class of locally Lipschitz causal curves forces a causal curve $\alpha : [a, b) \to M$ to be Lipschitz. This differs from the classical theory where causality is based on piecewise smooth (C^1) curves, since in that case, the resulting extension need not be a piecewise smooth curve.

So far we have seen how a Lipschitz curve can be extended by adding an end point. But when this is the case, it easily follows that it can also be extended as a strictly longer curve:

Lemma 2.4.5. *A Lipschitz curve $\alpha : [a, b) \to M,\ b < \infty$, is extendible if and only if it has an end point.*

Proof. Let $\hat{\alpha} : [a, b] \to M,\ \hat{\alpha}(b) = p$, and let $\tilde{\alpha} : [0, d) \to M$ be a future directed causal geodesic starting at p that can be maximally extended to the future, for an appropriate $d \in (0, \infty)$. Since the concatenation of two locally Lipschitz causal curves is again locally Lipschitz and causal, it follows that $\hat{\alpha} \cup \tilde{\alpha}$ is an extension of α. □

It fact, the curves considered in the previous lemma are always extendible:

Theorem 2.4.6. *Let (M, g) be a spacetime with a continuous metric g. Let $\alpha : [0, b) \to M,\ b \in \mathbb{R} \cup \{\infty\}$, be a future directed causal curve parameterized by h-arc-length. Then α is future inextendible if and only if $b = \infty$.*

Proof. Suppose to the contrary that $b < \infty$. By the Hopf-Rinow theorem, (M, d_h) is complete and $\alpha([0, b)) \subseteq (M, d_h)$. Since $[0, b)$ is dense in $[0, b]$ and α is Lipschitz on $[0, b)$, it follows that there exists a unique extension $\hat{\alpha} : [0, b] \to (M, d_h)$. □

Consider again the geodesic equation as a first order system:

$$\begin{aligned} \frac{dx^k}{dt} &= y^k(t) \\ \frac{dy^k}{dt} &= -\Gamma^k_{g,ij}(x(t))y^i(t)y^j(t), \end{aligned} \tag{2.10}$$

where $x(0) = p$ and $y(0) = v$. We have seen in the previous chapter that for $C^{1,1}$ metrics, (2.10) is uniquely solvable. Let the solution be defined on a maximal interval $I \ni 0$. I is called maximal if for another interval I' that contains 0 and on which a solution to the geodesic equation is defined, one has that $I' \subset I$. If I is maximal, the geodesic is said to be *maximally extended.* Now the question arises if a maximally extended geodesic is inextendible in the sense we have defined earlier, see Def. 2.4.1. This is not immediate since even though affinely parameterized geodesics are locally Lipschitz, they need not be Lipschitz when maximally extended and the inextendibility was defined for Lipschitz curves. The following result resolves the problem in question (cf. [Chr11, Prop. 2.5.6]):

Proposition 2.4.7. *A causal geodesic $\alpha : I \to M$ is maximally extended as a geodesic if and only if it is inextendible as a causal curve.*

Proof. If α is inextendible as a causal curve then clearly, it cannot be extended as a geodesic.

Now suppose to the contrary, that α is a maximally extended geodesic which is extendible as a curve. Let $\hat{\alpha} : [0, b] \to M$ be a continuous extension of α and let U be a totally normal neighborhood of $\hat{\alpha}(b)$. Then there exists a, $0 \leq a < b$, such that $\hat{\alpha}[a, b] \subset U$. Since U is totally normal, it is a normal neighborhood of $\alpha(a)$ and $\alpha|_{[a,b)}$ is a radial geodesic, hence it can be geodesically extended until it approaches ∂U or until its domain is $[a, \infty)$. Since $\hat{\alpha}(b) \in U$ and not in ∂U, α can be extended past b, which gives a contradiction with maximality of α as a geodesic. □

The following result holds true even for continuous metrics (see also [Chr11, Prop. 2.5.7]):

Theorem 2.4.8. *Let (M, g) be a spacetime with a continuous metric g. Let α be a future directed causal, respectively timelike, curve. Then there exists an inextendible causal, respectively timelike, extension of α.*

Proof. A simpler case is when g is assumed to be $C^{1,1}$: If $\alpha : [a, b) \to M$ is inextendible, there is nothing to prove. Otherwise, for $\hat{\alpha} : [a, b] \to M$, $\hat{\alpha}(b) = p$, and for any maximally extended future directed causal geodesic $\tilde{\alpha}$ starting at p, the path $\tilde{\alpha} \cup \hat{\alpha}$ provides an extension which is, by Prop. 2.4.7, inextendible.

Now assume that g is continuous. Suppose that α is extendible with an end point p. Let Ω_p be the collection of all future directed timelike curves starting at p which are parameterized by h-arc-length. An example of such a curve is an integral curve of a timelike vector field (which certainly exists since we have a time-orientable manifold hence Ω_p is non-empty). Ω_p can be directed using the property of "being an extension": by $\alpha_1 < \alpha_2$ we mean that α_2 is an extension of α_1. Since every chain of curves has an upper bound in Ω_p given by the concatenation of those curves, from the Zorn lemma it follows that Ω_p contains at least one maximal element which provides the existence of inextendible curves in Ω_p. If α_1 is any maximal element of Ω_p and $\hat{\alpha} : [a, b] \to M$, $\hat{\alpha}(b) = p$, then $\hat{\alpha} \cup \alpha_1$ is an inextendible future directed extension of α. □

2.5 Accumulation curves

The aim of this section is to establish the existence of accumulation curves. They are one of the main tools when it comes to studying the global properties of spacetimes. Recall that we have fixed an auxiliary complete Riemannian metric h. We will consider causal curves that are parametrized with respect to the h-arc-length. By the argument following Prop. 2.3.1, we obtain that all causal curves are Lipschitz with Lipschitz constant 1.

Definition 2.5.1. Let g be a continuous metric on M and let $\alpha_n : I \to M$ be a sequence of curves in (M, g). Then $\alpha : I \to M$ is an accumulation curve of the α_n's if there exists a subsequence α_{n_k} that converges to α uniformly on compact subsets of I.

Definition 2.5.2. An elementary neighborhood is an open ball B within the domain of a normal coordinate neighborhood U around $p \in M$ such that B is relatively compact in U and ∇x^0 and ∂x^0 are timelike on $\bar{U}$, where x^0 is the time coordinate on U.

Remark 2.5.3. Recall that by $g \prec \hat{g}$ we mean that the metric $\hat{g}$ has strictly wider lightcones than g. Note that on compact sets, one can always obtain a sequence smaller than a given metric. Namely, let g_n be a sequence of smooth metrics that converges locally uniformly to a continuous metric g such that $g \prec g_{n+1} \prec g_n$ and let $K \Subset M$. Then for a given metric $g' \succ g$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $g_n \prec g'$ on K, see [Sae15, Lemma 1.4].

The following result holds true also for continuous metrics, see [CG12, Prop. 1.5]:

Proposition 2.5.4. *Let (M, g) be a spacetime with continuous metric g. A curve α is causal for g if and only if it is causal for every smooth metric $\hat{g} \succ g$.*

Proof. If $\alpha : I \to M$ is causal for g, it is clear that it is causal (even timelike) for each metric $\hat{g}$.

Now assume α is causal for every metric $\hat{g} \succ g$ and assume to the contrary, that α is not g-causal. Then there exists a set $A \subseteq I$ of non-zero measure such that the weak derivative α' is g-spacelike for all $p \in A$. Since α is locally Lipschitz, by Rademacher's theorem it is differentiable almost everywhere hence the set of points B at which α has classical derivatives has full measure in I. Thus A and $A \cap B$ have the same measure and in particular, $A \cap B$ is not empty.

For $\tilde{t} \in A \cap B$, there is a metric $\hat{g} \succ g$ such that $\alpha'(\tilde{t})$ is spacelike for $\hat{g}$. Since $\hat{g}$ is smooth, there is a normal neighborhood U of $\alpha(\tilde{t})$ such that, for any $q \in U$, there exists a radial $\hat{g}$-geodesic $\hat{\alpha}_q : [0, 1] \to U$ starting at $\alpha(\tilde{t}) = \hat{\alpha}_q(0)$ and ending at $q = \hat{\alpha}_q(1)$. Define the function $\sigma : U \to \mathbb{R}$ (as in [Chr11, Prop. 2.2.3]) by $\sigma(q) := \hat{g}_{\alpha(\tilde{t})}\left(\frac{d\hat{\alpha}_q}{dt}(0), \frac{d\hat{\alpha}_q}{dt}(0)\right)$. By

Taylor expansion of α we obtain $\alpha(t) = \alpha(\tilde{t}) + \alpha'(\tilde{t})(t - \tilde{t}) + o(t - \tilde{t})$. Then

$$\sigma(\alpha(t)) = \hat{g}_{\alpha(\tilde{t})}(\exp^{-1}_{\alpha(\tilde{t})}(\alpha(t)), \exp^{-1}_{\alpha(\tilde{t})}(\alpha(t))).$$

Let $f(\alpha(t)) := \exp^{-1}_{\alpha(\tilde{t})}(\alpha(t))$. Taylor expanding f, we have

$$\begin{aligned} f(\alpha(t)) &= f(\alpha(\tilde{t}) + \alpha'(\tilde{t})(t - \tilde{t}) + o(t - \tilde{t})) \\ &= f(\alpha(\tilde{t})) + Df(\alpha(\tilde{t}))(\alpha(t) - \alpha(\tilde{t})) + o(|\alpha(t) - \tilde{\alpha}(t)|^2) \\ &= f(\alpha(\tilde{t})) + Df(\alpha(\tilde{t}))(\alpha'(\tilde{t})(t - \tilde{t}) + o(t - \tilde{t})) + o((t - \tilde{t})^2), \end{aligned}$$

and therefore, since $f(\alpha(\tilde{t})) = 0$ and $Df(\alpha(\tilde{t})) = \mathrm{id}$, $\sigma(\alpha(t)) = \hat{g}_{\alpha(\tilde{t})}(f(\alpha(t)), f(\alpha(t))) = \hat{g}(\alpha'(\tilde{t}), \alpha'(\tilde{t}))(t-\tilde{t})^2 + o((t-\tilde{t})^2)$. For t sufficiently close to $\tilde{t}$, the right-hand side is positive since $\hat{g}(\alpha'(\tilde{t}), \alpha'(\tilde{t})) > 0$. Then $\alpha'(t)$ is $\hat{g}$-spacelike hence the set of points for which α' is $\hat{g}$-spacelike is also of non-zero measure. This is a contradiction since α was assumed to be causal for $\hat{g}$. Hence $A \cap B$ is empty, i.e., α' is causal almost everywhere. □

As a consequence, we obtain the existence of accumulation curves (cf. [CG12, Thm. 1.6], [Sae15, Thm. 1.5]):

Proposition 2.5.5. *Let (M, g) be a smooth manifold with a continuous metric g. Let $\alpha_n : I \to M$ be a sequence of causal curves that accumulates at $p \in M$. Then there exists a causal curve $\alpha : I \to M$ through p which is an accumulation curve of the α_n's.*

Proof. Let $\hat{g} \succ g$ be any smooth metric. Then the curves α_n are causal for $\hat{g}$ and by the standard smooth result (see [Chr11, Thm. 2.6.7]), there exists a $\hat{g}$-causal accumulation curve α through p and a subsequence α_{n_k} of α_n such that $\alpha_{n_k} \to \alpha_n$ locally uniformly. Now let g' be any smooth metric such that $g \prec g' \prec \hat{g}$ (such a metric exists by uniform convergence and Remark 2.5.3). Then there exists a subsequence $\alpha_{n_{k_i}}$ of α_{n_k} such that the $\alpha_{n_{k_i}}$ are g'-causal. Hence they converge to a g'-causal curve which has to be α since it is a subsequence of a converging sequence. Hence the notion of accumulation curve does not depend on the metric, i.e., α is $\hat{g}$-causal for every $\hat{g} \succ g$ thus the result by Prop. 2.5.4. □

Next we address the question of inextendibility of accumulation curves. The following results turn out to be useful (see [Chr11, Lemma 2.6.6] and [Chr11, Lemma 2.6.5]):

Lemma 2.5.6. *Let X be a continuous timelike vector field defined on a compact set K. Then there exists a constant $C > 0$ such that for all $q \in K$ and for all causal vectors*

$Y \in T_qM$ we have

$$|g(X,Y)| \geq C|Y|_h. \tag{2.11}$$

Proof. By homogeneity it suffices to show (2.11) for causal $Y \in T_qM$ such that $|Y|_h = 1$. Denote by U_q the set of all such vectors. Since the function

$$\bigcup_{q\in K} U_q \ni Y \longrightarrow |g(X,Y)|$$

is strictly positive and continuous on the compact set $\bigcup_{q\in K} U_q$, it will have a minimum C hence the result. □

Lemma 2.5.7. *Let U be an elementary neighborhood. There exists a constant L such that for any past-directed causal curve $\alpha : I \to U$ the h-length $|\alpha|_h$ of α is bounded by L.*

Proof. Let x^0 be the local time coordinate on U. Since $X := \nabla x^0$ is timelike, by Lemma 2.5, for $K = \overline{U}$, there exists a constant C such that for any causal curve α in U, we have

$$|g(X,\alpha')| \geq C|\alpha'|_h > 0$$

at all points at which α is differentiable. Then, for $t_2 \geq t_1$,

$$\begin{aligned} |x^0(\alpha(t_2)) - x^0(\alpha(t_1))| &= \int_{t_1}^{t_2} \frac{d}{dt}(x^0\circ\alpha)dt \\ &= \int_{t_1}^{t_2} |g(\nabla x^0,\alpha')|dt \\ &\geq C\int_{t_1}^{t_2} |\alpha'|_h ds = CL_{t_1}^{t_2}(\alpha). \end{aligned}$$

Then it follows that

$$\begin{aligned} L_{t_1}^{t_2}(\alpha) &\leq \frac{1}{C}\sup |x^0(\alpha(t_2)) - x^0(\alpha(t_1))| \\ &\leq \frac{2}{C}\sup_{\overline{U}} |x^0| =: L < \infty. \end{aligned}$$

□

Finally, we have (see [Chr11, Lemma 2.6.4]):

Lemma 2.5.8. *Let α_n be a sequence of d_h-parameterized inextendible causal curves converging to α uniformly on compact subsets of $\mathbb{R}$. Then α is inextendible.*

Remark 2.5.9. The parameter range of α is $\mathbb{R}$ so the result would follow from Theorem 2.4.8 if α were d_h-parameterized, but this might not hold: consider $\mathbb{R}^1_1$ with the background Riemannian metric $h = dt^2 + dx^2$ and a sequence of null geodesics in $\mathbb{R}^1_1$ threading back and forth around the $\{x = 0\}$ axis up to a distance $1/n$, namely

$$\alpha_n(s) = \begin{cases} (s, s) & s \in [0, \frac{1}{n}] \\ (s, -s) & s \in [0, \frac{1}{n}]. \end{cases}$$

$\|\alpha_n'\| = \sqrt{2}$ so let $\hat{\alpha}_n(s) = \alpha_n(s/\sqrt{2}),\ s \in [0, \sqrt{2}/n]$ Then the limit curve $\alpha(s) = (s/\sqrt{2}, 0)$ is not d_h-parameterized.

Proof. We need to show that both $\alpha|_{[0,\infty)}$ and $\alpha|_{(-\infty,0]}$ have infinite length. It suffices to consider $\alpha|_{[0,\infty)}$, for which we will only write α. Assume to the contrary that there exists $a < \infty$ so that the reparametrization $\tilde{\alpha} = \alpha \circ f$ of α with respect to the h-arc length is defined on $[0, a)$. By Theorem 2.4.8, it can be extended to a causal curve defined on $[0, a]$, still denoted by $\tilde{\alpha}$.

Now let U be an elementary neighborhood around $\tilde{\alpha}(a)$ and let $0 < b < a$ be such that $\tilde{\alpha}(b) \in U$. By definition of an accumulation curve, there exists a sequence $n_i \in \mathbb{N}$ such that:

$$\tilde{\alpha}(b) = \alpha(f(b)) = \lim \alpha_{n_i}(f(b)),\ (f(b) = t_i).$$

In particular, $\alpha_{n_i}(f(b)) \in U$ for i large enough. From Lemma 2.5.7 applied to α_{n_i} it follows that $\alpha_{n_i}|_{[f(b),f(b)+L]}$ must leave U. Since $\alpha_{n_i}(f(b)) \to \tilde{\alpha}(b)$, then $\tilde{\alpha}_{n_i}(f(b) + L)$ should converge to some point between $\tilde{\alpha}(b)$ and $\tilde{\alpha}(a)$ since $\tilde{\alpha}(a)$ is an endpoint, but this cannot be because they are not in U. Hence, $\alpha_{n_i}|_{[0,f(b)+L]}$ cannot accumulate at a curve which has an endpoint $\alpha(a) \in U$ hence the result. □

It follows from Lemma 2.5.8 together with Proposition 2.5.5 that:

Proposition 2.5.10. *Let (M, g) be a Lorentzian manifold with a $C^{1,1}$ metric. Every sequence of future directed, inextendible causal curves which accumulates at a point $p \in M$ accumulates at some future directed inextendible causal curve through p.*

2.6 Achronal causal curves

A curve $\alpha : I \to M$ is called *achronal* if $\forall\ s, s' \in I,\ \alpha(t) \notin I^+(\alpha(t'))$, i.e., there are no two points on α which are timelike related. An example of such curves are spacelike and null geodesics in Minkowski spacetime. However, there exist spacetimes where null geodesics need not be achronal. An example is given by the spacetime $\mathbb{R} \times S^1$ with the flat metric $g = -dt^2 + dx^2$ where x. If we assume that x is an angle-type coordinate along S^1 with periodicity 2π, then the points $(0,0)$ and $(2\pi, 0)$ lie on the null geodesic $s \to (s, s \bmod 2\pi)$ and are timelike related to each other.

The following result shows that any achronal causal curve is necessarily a null geodesic (cf. [Chr11, Prop. 2.6.9]):

Proposition 2.6.1. *Let α be an achronal causal curve. Then α is a null geodesic.*

Proof. Let $\alpha : [0,1] \to M$ be an achronal causal curve and let U be a totally normal neighborhood around $\alpha(0) = p$. Then α in U is a causal curve from p to a point q lying in $\partial J^+(p, U)$. Otherwise, we would get a contradiction to the achronality of α. Hence by Corollary 2.3.13, it lies entirely in $\partial J^+(p, U)$ and there exists a reparametrization of α such that it is a null geodesic segment. Covering α by totally normal neighborhoods U_i, $i \in \mathbb{N}$, the claim follows. □

The proof of the following theorem only uses the fact that timelike futures and pasts are open, cf. Theorem 2.3.10. Hence the result also holds for continuous metrics with this property and in particular for causally plain metrics (see [Chr11, Thm. 2.6.10] for C^2 metrics):

Theorem 2.6.2. *Let $\alpha_n : I \to M$ be a sequence of future directed achronal causal curves accumulating at α. Then α is achronal.*

Remark 2.6.3. Note that by Theorem 2.5.10 it follows that α is inextendible if the α_n's are.

Proof. Suppose to the contrary that α is not achronal. Then there exist $t_1, t_2 \in I$ such that $\alpha(t_2) \in I^+(\alpha(t_1))$ hence, a timelike curve $\hat{\alpha} : [t_1, t_2] \to M$ from $\alpha(t_1)$ to $\alpha(t_2)$. For some $\hat{t} \in (t_1, t_2)$, $\alpha(t_2) \in I^+(\hat{\alpha}(\hat{t}))$ and since $I^+(\hat{\alpha}(\hat{t}))$ is open, there exists an open neighborhood $\mathscr{U}_2$ of $\alpha(t_2)$ such that $\mathscr{U}_2 \subset I^+(\hat{\alpha}(\hat{t}))$. Similarly, there is an open neighborhood $\mathscr{U}_1$ of $\alpha(t_1)$ such that $\mathscr{U}_1 \subset I^-(\hat{\alpha}(\hat{t}))$. For any $p_1 \in \mathscr{U}_1,\ p_2 \in \mathscr{U}_2$ one can go from p_1 along a timelike curve to $\hat{\alpha}(\hat{t})$ and continue along another timelike curve from $\hat{\alpha}(\hat{t})$ to p_2 so p_2 lies in the timelike future of p_1. Passing to a subsequence if necessary, there exist sequences $t_{1,n}$ and $t_{2,n}$ such that $\alpha_n(t_{1,n})$ converges to $\alpha(t_1)$ and $\alpha_n(t_{2,n})$ converges to $\alpha(t_2)$. Then, for n large enough, $\alpha_n(t_{1,n}) \in \mathscr{U}_1$, $\alpha_n(t_{2,n}) \in \mathscr{U}_2$ implying that $\alpha_n(t_{2,n}) \in I^+(\alpha_n(t_{1,n}))$, which is a contradiction to the achronality of α_n. $\square$

Chapter 3

Global structure of spacetimes

As it has already been demonstrated by P. Chruściel in [Chr11], the essential tools needed to obtain a consistent causality theory are the push-up principle and the existence of accumulation curves. Using these results and the local causality theory established in the previous chapter, our next goal will be to further develop causality theory for $C^{1,1}$ metrics following [Chr11] and [ON83].

3.1 Causality conditions

In this section we will be concerned with the causal hierarchy of spacetimes. Different causality conditions can be imposed on a spacetime, each of them having its unique position in the so-called causal ladder. The higher on the causal ladder a spacetime is, the more realistic it is considered to be. Indeed, it is reasonable to think that any realistic spacetime will be on the very top of the causal ladder satisfying the strongest causality condition-global hyperbolicity.

We start out with a definition of the weakest causality condition. A spacetime is called *totally vicious* if $I^+(p) \cap I^-(p) = M$ for all points $p \in M$. If there are no closed timelike curves, a spacetime is said to be *chronological*. Recall that in general relativity each point of a spacetime corresponds to an event. Thus the existence of closed timelike curves may lead to the possibility of time travel and hence the occurrence of many paradoxes such as the grandfather paradox. Having the ability to travel back in time, one could change the past thus leading to the future being different from the one where a time traveller had actually started his journey. Such paradoxes are said to "violate causality".

An example of a space-time which is not chronological is provided by $S^1 \times \mathbb{R}$ with the flat metric $-dt^2 + dx^2$, where t is a local coordinate defined modulo 2π on S^1. Then every circle $x = \text{const}$ is a closed timelike curve.

Compact manifolds are pathological from a Lorentzian perspective:

Proposition 3.1.1. *(Geroch) Let (M, g) be a compact spacetime with a causally plain metric g. Then M is not chronological.*

Proof. Consider the open covering $\{I^+(p) : p \in M\}$ of M. By compactness, it has a finite subcover $I^+(p_1), ..., I^+(p_k)$. We can assume that $I^+(p_1)$ is not contained in any later $I^+(p_i)$, otherwise discard $I^+(p_1)$. But then $p_1 \in I^+(p_1)$, that is, there exists a closed timelike curve through p_1. If p_1 is in some other $I^+(p_i)$, then $I^+(p_1) \subset I^+(p_i)$. □

Thus the spacetimes usually considered in general relativity are noncompact.

The next on the causal ladder is the causality condition. A spacetime is said to be *causal* if there are no closed causal curves.

Consider now a spacetime which contains a family of causal curves α_n with both $\alpha_n(0)$ and $\alpha_n(1)$ converging to p. Such curves can be thought of as being "almost closed". One can produce an arbitrarily small deformation of the metric which will allow one to obtain a closed causal curve in the deformed spacetime. We want to exclude this behaviour. Hence it is said that the strong causality condition holds at $p \in M$ if for any given neighborhood U of p there exists a neighborhood $V \subseteq U$ of p such that every causal curve with endpoints in V is entirely contained in U. A strongly causal spacetime is necessarily causal. However, the converse fails to be true. Delete from $S^1_1 \times \mathbb{R}^1$ two spacelike half-lines whose endpoints were the endpoints of a short null geodesic. The causality condition holds on M, but the strong causality condition fails at each point of the null geodesic.

A result often used in causality theory that describes an important property of time functions is the following (cf. [Chr11, Lemma 2.4.8]):

Lemma 3.1.2. *Let f be a time function, i.e., a differentiable function with past directed timelike gradient. For any f_0, a future directed causal curve α cannot leave the set $\{q : f(q) > f_0\}$ (analogously, the result holds for sets $\{q : f(q) \geq f_0\}$). In fact, f is strictly increasing along α.*

Proof. Let $\alpha : I \to M$ be a future directed causal curve. Then $f \circ \alpha$ is locally Lipschitz since α is, hence:

$$\begin{aligned} f(\alpha(t_2)) - f(\alpha(t_1)) &= \int_{t_1}^{t_2} \frac{d(f \circ \alpha)}{du}(u)du \\ &= \int_{t_1}^{t_2} \langle df, \alpha' \rangle (u)du \\ &= \int_{t_1}^{t_2} g(\nabla f, \alpha')(u)du > 0 \end{aligned}$$

since ∇f is timelike past directed and α' is causal future directed (it cannot be zero because when $g(X, Y) = 0$ for X, Y causal that means that both X and Y are null and collinear). Hence the function $t \to f(\alpha(t))$ is strictly increasing when α is timelike or causal in general since the integrand is strictly positive almost everywhere. □

A spacetime is said to be *stably causal* if there exists a time function t globally defined on M. It then easily follows that stable causality implies strong causality, cf. [Chr11, Prop. 2.7.4].

Lemma 3.1.3. *Suppose the strong causality condition holds on $K \Subset M$ and let $\alpha : [0, b) \to M$ be a future directed inextendible causal curve that starts in K. Then α eventually leaves K never to return.*

Proof. Assume to the contrary, that $\alpha : [0, b) \to M$ is either entirely contained in K or that each time it leaves K, has to return. Then there exists a sequence $s_i \nearrow b$ with $\alpha(s_i) \in K$, $\forall i$. We can assume without loss of generality that $\alpha(s_i) \to p \in K$. Since α is future inextendible, there exists another sequence $t_i \to b$ such that $\alpha(t_i)$ does not converge to p. Choosing a subsequence if necessary, one can additionally assume that there exists a neighborhood U of p that contains no $\alpha(t_i)$. Since both, s_i and t_i, converge to b, they have subsequences that alternate, i.e., $s_1 < t_1 < s_2 < t_2 < ....$ For k large enough, the causal curves $\alpha|_{[s_k, s_{k+1}]}$ start and end arbitrarily close to p, contradicting strong causality of K. □

3.2 Time separation

We are now ready to define the time separation function of an arbitrary spacetime, in parts of the literature also referred to as the Lorentzian distance function, see [BEE96].

Definition 3.2.1. Let $p, q \in M$. The *time separation* $d(p, q)$ from p to q is given by

$$d(p,q) := \sup\{L(\alpha) : \alpha \text{ is a future directed causal curve from } p \text{ to } q\}. \tag{3.1}$$

In particular, $d(p, p) = 0$ and $d(p, q) = 0$ for $q \notin J^+(p)$. Otherwise, we calculate $d(p, q)$ for $q \in J^+(p)$ as the supremum of the Lorentzian arc length of all future directed causal curves from p to q. Hence if α is any future directed causal curve from p to q, $L(\alpha) \leq d(p, q)$.

The time separation function can be thought of as the proper time of the slowest trip between two points of a spacetime. Comparing it to the Riemannian distance function, we conclude that it is maximizing rather than minimizing. Also, unlike the Riemannian distance function, the time separation function need not be finite-valued. If the set of lengths is unbounded, $d(p, q) = \infty$.

The following lemma gives a useful analogue of the triangle inequality:

Lemma 3.2.2. *Let $p, q, r \in M$. Then:*

(i) $d(p, q) > 0$ if and only if $p \ll q$

(ii) Reverse triangle inequality: If $p \leq q \leq r$, then $d(p, q) + d(q, r) \leq d(p, r)$.

Proof. (i) If $d(p, q) > 0$ then by the definition of d there is a future directed causal curve α from p to q with $L(\alpha) > 0$. Then there must be an interval $[t_1, t_2]$ on which α is timelike. Let $p_1, q_1 \in \alpha([t_1, t_2])$, $p_1 \neq q_1$. Then $p \leq p_1 \ll q_1 \leq q$ hence by Proposition 2.3.14, $p \ll q$.

(ii) We distinguish two cases:

1.) There is a future directed timelike curve from p to q and from q to r. Then either both $d(p, q)$ and $d(q, r)$ are finite or at least one of them is infinite. In the first case, let $\varepsilon > 0$, then there exist a future directed causal curve α_1 from p to q and a future directed curve α_2 from q to r such that $L(\alpha_1) \geq d(p, q) - \varepsilon$ and $L(\alpha_2) \geq d(q, r) - \varepsilon$. Hence $d(p, r) \geq L(\alpha_1 \cup \alpha_2) = L(\alpha_1) + L(\alpha_2) \geq d(p, q) + d(q, r) - 2\varepsilon$, thus the result.

Assume now $d(p,q) = \infty$. Then there exists an arbitrarily long future directed timelike curve from p to q. Concatenating it with a timelike curve from q to r, we obtain an arbitrarily long future directed timelike curve from p to r hence $d(p,r) = \infty$.

2.) There is no future directed timelike curve from p to q (or analogously from q to r). Since $p \leq q$ it follows that $p = q$ thus the inequality holds trivially. □

There is the reversed sign in the inequality since causal geodesics in Lorentzian manifolds locally maximize arc length.

Example 3.2.3. (i) Let $M = \mathbb{R}^n_1$ be Minkowskian spacetime. Then for $p < q$, $d(p,q) = \|p - q\|$: $t \mapsto tq + (1-t)p$, $0 \leq t \leq 1$, is a future directed causal curve from p to q of length $\|p - q\|$ thus $d(p,q) \geq \|p - q\|$. For the other inclusion, we consider two cases:

1. $p - q$ is timelike. Then without loss of generality, $p = (0, ..., 0)$ and $q = (T, ..., 0)$ with $T = \|p - q\|$. Now let α be a future directed causal curve from p to q. We can reparametrize it so that $\alpha(t) = (t, x(t))$, $x : [0,T] \to \mathbb{R}^{n-1}$. Then

$$\begin{aligned} L(\alpha) &= \int_0^T \|\alpha'(t)\| dt = \int_0^T \sqrt{|-1 + \|x'(t)^2\||} dt \\ &= \int_0^T \sqrt{1 - \|x'(t)^2\|} dt \leq \int_0^T 1 dt = T = \|p - q\|, \end{aligned}$$

hence $d(p,q) \leq \|p - q\|$.

2. $p - q$ is null. In this case, every causal curve connecting p and q is null thus of length 0. Then $d(p,q) = 0 = \|p - q\|$.

(ii) Let $M = S^1_1 \times \mathbb{R}^1$ be the Lorentz cylinder. Then $d(p,q) = \infty$, $\forall p, q \in M$.

Lemma 3.2.4. *d is lower semi-continuous.*

Proof. Let $p, q \in M$. If $d(p,q) = 0$, there is nothing to prove. Assume $0 < d(p,q) < \infty$ and let $\delta > 0$. By the definition of d, there exists a future directed timelike curve $\alpha : [0,1] \to M$ from p to q with $L(\alpha) \geq d(p,q) - \delta/2$. Consider now $0 < t_1 < t_2 < 1$ such that $0 < L(\alpha|_{[0,t_1]}) < \delta/4$ and $0 < L(\alpha|_{[t_2,1]}) < \delta/4$. For $p_1 := \alpha(t_1)$ and $q_1 := \alpha(t_2)$, let $U := I^-(p_1)$ and $V := I^+(q_1)$. Since $L(\alpha|_{[0,t_1]}) > 0$ we have $d(p, p_1) > 0$ so by Lemma 3.2.2 (i), $p \ll p_1$. Also, by Theorem 2.3.10, U is an open neighborhood of p. Analogously, V is an open neighborhood of q. Now let $(p', q') \in U \times V$. Then by Lemma 3.2.2 (ii):

$$\begin{aligned} d(p', q') &\geq d(p', p_1) + d(p_1, q_1) + d(q_1, q') \geq L(\alpha|_{[t_1,t_2]}) \\ &= L(\alpha) - L(\alpha|_{[0,t_1]}) - L(\alpha|_{[t_2,1]}) \geq d(p, q) - \delta. \end{aligned}$$

Now assume $d(p, q) = \infty$. Then there is an arbitrarily long future directed causal curve connecting p and q. As above, there are neighborhoods U of p and V of q such that any two points from U and V can be connected by arbitrarily long curves. □

Let $A, B \subseteq M$. The time separation $d(A, B)$ is defined as $\sup\{d(a, b) \mid a \in A,\ b \in B\}$. Analogously to the preceding lemma, one can show that the functions $x \mapsto d(x, B)$ and $y \mapsto d(A, y)$ are lower semi-continuous.

3.3 Global hyperbolicity

The strongest causality condition is that of global hyperbolicity. A spacetime (M, g) is said to be *globally hyperbolic* if it is strongly causal and if for every $p, q \in M$ the sets $J^+(p) \cap J^-(q)$ are compact. These sets are called causal diamonds and they are denoted by $J(p, q) := J^+(p) \cap J^-(q)$.

Example 3.3.1.

1. Let $M = \mathbb{R}^n_1$. The Minkowski time x^0 provides a time function on M implying the strong causality condition. Moreover, $J(p, q)$ is compact for all $p, q \in M$ hence M is globally hyperbolic.

2. Let $M := \mathbb{R}^n_1 \setminus \{0\}$. Then M is not globally hyperbolic since any causal diamond $J(p, q)$ containing 0 is not compact.

The following result establishes the existence of accumulation curves in globally hyperbolic spacetimes (cf. [Chr11, Prop. 2.8.1]):

Proposition 3.3.2. *Let (M, g) be a globally hyperbolic spacetime and let α_n be a family of causal curves accumulating both at p and q. Then there exists a causal curve α which is an accumulation curve of the α_n's and which passes through both p and q.*

Proof. We can extend the α_n's to inextendible curves and reparametrize them if necessary, hence assume that they are parametrized by h-arc length, with common domain of definition $I = \mathbb{R}$ and with $\alpha_n(0)$ converging to p. If $p = q$ the result follows from Proposition

2.5.5, hence assume that $p \neq q$. Define a compact set $\tilde{K}$ by:

$$\tilde{K} := (J^+(p) \cap J^-(q)) \cup (J^+(q) \cap J^-(p)) \tag{3.2}$$

One of those sets is necessarily empty since a globally hyperbolic spacetime is causal. The curves α_n are causal and by assumption there exists a sequence t_n such that $\alpha_n(0) \to p$ and $\alpha_n(t_n) \to q$. Choose $p_-, q_+ \in M$ such that $p, q \in K = J^+(p_-) \cap J^-(q_+)$. Then for n large enough, $\alpha_n(0), \alpha_n(t_n) \in K$. Now, K can be covered by a finite number of elementary neighborhoods $U_i,\ i = 1, ..., N$. By strong causality, we can choose the sets U_i so that for every n, α_n does not come back to U_i once it leaves it. By Lemma 2.5.7 we know that there exists a constant L_i, independent of n, such that the h-length $|\alpha_n \cap U_i|_h$ is bounded by L_i. The h-length $|\alpha_n \cap K|_h$ is bounded by:

$$|\alpha_n \cap K|_h \leq L := L_1 + L_2 + ... + L_N. \tag{3.3}$$

Equation (3.3) shows that the sequence t_n is bounded hence, perhaps passing to a subsequence, we have $t_n \to t_*$, for some $t_* \in \mathbb{R}$. Since the α_n's are parameterized with respect to h-arc length, they are Lipschitz with Lipschitz constant smaller or equal to 1: we have:

$$d_h(\alpha_n(t), \alpha_n(t')) \leq |t - t'|. \tag{3.4}$$

Thus the family $\{\alpha_n\}$ is equicontinuous. By equation (3.4) and the Arzela-Ascoli theorem, there exists a curve $\alpha : [0, L] \to M$ and a subsequence α_{n_i} which converges uniformly to α on $[0, L]$. $\alpha_{n_i}(t_{n_i})$ converges both to q and $\alpha(t_*)$:

$$\begin{aligned} d_h(\alpha_{n_i}(t_{n_i}), \alpha(t_*)) \leq\ & d_h(\alpha_{n_i}(t_{n_i}), \alpha_{n_0}(t_n)) \\ & + d_h(\alpha_{n_0}(t_n), \alpha_{n_0}(t_*)) \\ & + d_h(\alpha_{n_0}(t_*), \alpha(t_*)). \end{aligned}$$

Now let $\epsilon > 0$ and let i be such that $\sup_{t \in [0,L]} d_h(\alpha_{n_i}(t), \alpha_{n_0}(t)) < \frac{\epsilon}{3}$. Then it follows that $d_h(\alpha_{n_i}(t_{n_i}), \alpha_{n_0}(t_n)) \leq \frac{\epsilon}{3}$. For n big enough, by continuity of α_{n_0} we have

$$d_h(\alpha_{n_0}(t_n), \alpha_{n_0}(t_*)) \leq \frac{\epsilon}{3}.$$

And for n_0 big enough,

$$d_h(\alpha_{n_0}(t_*), \alpha(t_*)) \leq \frac{\epsilon}{3},$$

hence

$$d_h(\alpha_{n_i}(t_{n_i}), \alpha(t_*)) \leq \epsilon.$$

Hence $\alpha(t_*) = q$ and α is our desired curve joining p and q. □

Remark 3.3.3. The result is wrong if one only assumes stable causality: let (M, g) be the two-dimensional Minkowski spacetime with the origin removed. Let α_n be obtained by following a timelike geodesic from $p = (-1, 0)$ to $(0, \frac{1}{n})$ and then another timelike geodesic to $q = (1, 0)$. Then α_n has two accumulation curves: $s \to (s, 0)$ with $s \in [-1, 0)$ and $s \to (s, 0)$ with $s \in (0, 1]$, none of which passes through both p and q, because $(0, 0)$ has been removed.

The following result shows that the time separation function has better regularity in case of globally hyperbolic spacetimes (cf. [BEE96, Lemma 4.5.])

Proposition 3.3.4. *Let (M, g) be a globally hyperbolic spacetime. Then the time separation function d is finite and continuous on $M \times M$.*

Proof. Let $p, q \in M$. Since $J^+(p) \cap J^-(q)$ is compact, we can cover it by a finite number of totally normal neighborhoods U_i, $1 \leq i \leq n$, such that causal curves which leave U_i never return and such that every causal curve in U_i has length bounded by some $L > 0$. Since any causal curve α from p to q can enter each U_i at most once, it follows that $L(\alpha) \leq Lm$. Hence $d(p, q) \leq Lm$ thus is finite.

Assume now that d fails to be upper semi-continuous at $(p, q) \in M \times M$. Then one could find $\delta > 0$ and sequences $\{p_n\}$ and $\{q_n\}$ such that $p_n \to p$, $q_n \to q$ and

$$d(p_n, q_n) \geq d(p, q) + 2\delta, \ \forall n. \tag{3.5}$$

By definition of the time separation function, one may find a future directed causal curve α_n from p_n to q_n for each n. By Proposition 3.3.2, for the curves α_n, there exists an accumulation curve α from p to q. In case of strongly causal spacetimes (by our assumption M is even globally hyperbolic), it was shown in [Pen72, Thm. 7.5] that the length functional is upper semi-continuous and since $\alpha_n \to \alpha$, we obtain that $L(\alpha) \geq \limsup L(\alpha_n)$. Therefore $L(\alpha) \geq d(p, q) + \delta$ (see also [BEE96, Prop. 3.34] and [BEE96, Remark 3.35]). This is a contradiction to the definition of the time separation function. □

Proposition 3.3.5. *Let $U \subseteq M$ be open and globally hyperbolic. Then the causality relation $\leq$ of M is closed on U.*

Proof. Let $p_n, p, q_n, q \in U$ such that $p_n \leq q_n$, $\forall n$, and $p_n \to p$, $q_n \to q$. We want to show that $p \leq q$. If $p = q$, the result holds trivially. If $p_n = q_n$ for infinitely many n, then $p = q$. Hence we may assume that $p \neq q$ and $p_n < q_n$ for all n. Let α_n be a future directed causal curve from p_n to q_n. As in Proposition 3.3.2 it follows that $\alpha_n([0,1]) \subseteq J(p^-, q^+)$ for suitable $p^-, q^+ \in U$ and there exists a future directed causal curve α from p to q. Hence $p < q$. □

As an immediate consequence, we obtain:

Corollary 3.3.6. *Let M be a globally hyperbolic manifold. Then all sets $J^+(p)$, $J^-(q)$, and $J(p,q)$ are closed.*

3.4 Achronal sets

A set $S \subset M$ is said to be *achronal* if $I^+(S) \cap I^-(S) = \emptyset$. Analogously, a set is called *acausal* if $J^+(S) \cap J^-(S) = \emptyset$. Every acausal set is necessarily achronal but the converse need not be true: consider a null geodesic α in $\mathbb{R}^n_1$. Then $B = \alpha([0,b])$ is achronal but not acausal.

Example 3.4.1. Let $M = \mathbb{R}^n_1$. Then any spacelike hyperplane $t = \text{const}$ and the future light cone in $\mathbb{R}^n_1$ are achronal sets.

Definition 3.4.2. Let S be an achronal set. The *edge* of S is the set of all points $p \in \overline{S}$ such that every neighborhood U of p contains a timelike curve from $I^-(p,U)$ to $I^+(p,U)$ that does not intersect S.

Example 3.4.3.
1. Let $M = \mathbb{R}^n_1$. The achronal sets from the preceding example have empty edge.
2. Let $X = \{(0,x) \mid 0 \leq x \leq 1\} \subseteq \mathbb{R}^2_1$. Then $\text{edge}(X) = \{(0,0),(0,1)\}$. But considered as a subset of $\mathbb{R}^3_1$, $\text{edge}(X) = \overline{X}$.

Next we wish to show that every achronal set without edge is a hypersurface. It need not be a smooth hypersurface as is clearly seen in the example of the nullcone in $\mathbb{R}^n_1$ that is achronal and edgeless.

A Hausdorff space T for which every point has a neighborhood homeomorphic to an open set in $\mathbb{R}^n$ is a *topological manifold* of dimension n. The following result is a useful consequence of the Brouwer theorem on the invariance of domain: If $\varphi : T \to \tilde{T}$ is a one to one continuous mapping of n-dimensional topological manifolds, then φ is a homeomorphism onto an open set $\varphi(T)$ of $\tilde{T}$.

Definition 3.4.4. A subspace S of a smooth manifold M is called a *topological hypersurface* if for every $p \in S$ there exist a neighborhood U of p in M and a homeomorphism $\varphi : U \to V$, where $V \subseteq \mathbb{R}^n$ is open, such that $\varphi(U \cap S) = V \cap (\{0\} \times \mathbb{R}^{n-1})$.

Remark 3.4.5. Note that subsets of achronal sets are achronal. Also, the closure $\bar{S}$ of an achronal set S is achronal as well: assume there existed $p, q \in \bar{S}$ such that $p \ll q$. Choose $p_n, q_n \in S$ so that $p_n \to p$ and $q_n \to q$. By Corollary 2.3.17, $p_n \ll q_n$ for n big enough, which is a contradiction.

Lemma 3.4.6. *Let $S \subseteq M$ be achronal. Then:*

1. $\bar{S} \setminus S \subseteq \text{edge}(S)$*;*
2. $\text{edge}(S)$ *is closed.*

Proof. 1. Let $p \in \bar{S} \setminus S$ and let U be an open neighborhood of p. Consider a curve α through p going from $I^-(p, U)$ to $I^+(p, U)$. By the previous remark, $\bar{S}$ is achronal, hence α cannot meet $\bar{S}$ except in p. Since $p \notin S$, α has no intersection with S it follows that $p \in \text{edge}(S)$.

2. Let $p \in \overline{\text{edge}(S)}$. We want to show that $p \in \text{edge}(S)$. Consider a neighborhood U of p and choose an open neighborhood $V \subseteq U$ of p such that $V \subseteq I^+(I^-(p, U), U) \cap I^-(I^+(p, U), U)$. Then there exists some $q \in V \cap \text{edge}(S)$ thus a timelike curve $\alpha : [-1, 1] \to V$ with $p_1 := \alpha(-1) \in I^-(q, V)$ and $p_2 := \alpha(1) \in I^+(q, V)$, that does not meet S. Since $p_1 \in I^+(I^-(p, U), U)$, α can be extended to a timelike curve from $[-2, 1]$ to U with $\alpha(-2) \in I^-(p, U)$. Analogously, we can extend it to a timelike curve from $[-2, 2]$ to U such that $\alpha(2) \in I^+(p, U)$.

We show that $\alpha|_{[-2,-1]}$ does not meet S. Analogously, the same follows for $\alpha|_{[1,2]}$. By $p_1 \in I^-(q, V)$, we have that $q \in I^+(p_1, V)$. Since $q \in \text{edge}(S) \subseteq \bar{S}$, there exists $r \in S \cap I^+(p_1, V)$. Let β be a timelike curve connecting p_1 and r. Then $\alpha|_{[-2,-1]} \cup \beta$ is timelike and meets S twice, which is impossible since S is achronal.

Hence $\alpha : [-2, 2] \to U$ is a timelike curve from $I^-(p, U)$ to $I^+(p, U)$ that does not meet S thus $p \in \text{edge}(S)$. ∎

Proposition 3.4.7. *Let $S \subseteq M$ be achronal. Then the following are equivalent:*

(i) $S \cap \mathrm{edge}(S) = \emptyset$*;*

(ii) S *is a topological hypersurface.*

Proof. (i) $\Rightarrow$ (ii) Let $p \in S$. Then $p \notin \mathrm{edge}(S)$ hence there exists an open neighborhood U of p such that every timelike curve from $I^-(p,U)$ to $I^+(p,U)$ entirely contained in U, intersects S. Without loss of generality we can have on U a coordinate chart $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$, $\varphi = (x^0, ..., x^{n-1})$ with $\partial/\partial x^0$ future directed timelike. Using these coordinates we can obtain an open neighborhood $V \subseteq U$ of p such that:

1. $\varphi(V) = (a - \delta, b + \delta) \times N \subseteq \mathbb{R} \times \mathbb{R}^{n-1}$
2. $\{x \in V \mid x^0 = a\} \subseteq I^-(p,U)$ and $\{x \in V \mid x^0 = b\} \subseteq I^+(p,U)$.

Now let $y \in N \subseteq \mathbb{R}^{n-1}$. Then the curve $\alpha : [a,b] \to V$, $s \mapsto \varphi^{-1}(s,y)$, is timelike from $I^-(p,U)$ to $I^+(p,U)$ hence it meets S. Since S is achronal, there is a unique $h(y) \in (a,b)$ such that $\varphi^{-1}(h(y),y) \in S$.

Next we show that the function $h : N \to (a,b)$ is continuous. Let (y_m) be a sequence in N that converges to y and suppose that $h(y_m)$ does not converge to $h(y)$. Since $h(N) \subseteq [a,b]$, passing to a subsequence if necessary, $h(y_m)$ converges to $r \neq h(y)$. Now let $q := \varphi^{-1}(h(y),y) \in S$. The curve $s \mapsto \varphi^{-1}(s,y)$ is timelike and contains both, q and $\varphi^{-1}(r,y) \neq q$, thus $\varphi^{-1}(r,y) \in I^-(q,V) \cup I^+(q,V)$. This set is open by Theorem 2.3.10 and $\varphi^{-1}(h(y_m),y_m) \to \varphi^{-1}(r,y)$, hence there exists $m_0 \in N$ such that $\varphi^{-1}(h(y_{m_0}),y_{m_0}) \in I^-(q,V) \cup I^+(q,V)$, contradicting achronality of S.

Now write $\varphi = (\varphi_0, \varphi')$ and let $\psi : V \to \mathbb{R}^n$, $\psi(p) := (\varphi_0(p) - h(\varphi'(p)), \varphi'(p))$. Then ψ is continuous and $\psi^{-1}(x_0, x') = \varphi^{-1}(x_0 + h(x'), x')$. Since $\varphi(V)$ is open and $\psi \circ \varphi^{-1}(x) = (x_0 - h(x'), x')$ is continuous and injective on $\varphi(V)$, by the Brouwer theorem, $\psi : V \to \psi(V)$ is a homeomorphism and $\psi(V)$ is open as well, hence $\psi(V \cap S) = \psi \circ \varphi^{-1}(\{(h(y),y) \mid y \in N\}) = \{(0,y) \mid y \in N\} = \psi(V) \cap (\{0\} \times \mathbb{R}^{n-1})$. Thus S is a topological hypersurface.

(ii) $\Rightarrow$ (i) Let $p \in S$. By Theorem 2.3.10, we can assume that $M = \mathbb{R}^n_1$. Let (φ, U) be as in Definition 3.4.4, i.e., $\varphi : U \to V$ is a homeomorphism and $\varphi(U \cap S) = V \cap (\{0\} \times \mathbb{R}^{n-1}) =: V_1$. In particular, $\varphi_1 := \varphi|_{U \cap S} : U \cap S \to V_1$ is a homeomorphism. Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection $(x^0, x') \mapsto x'$. Since S is achronal and every vertical line $t \mapsto (t, x')$ is timelike, it can intersect it at most once. Hence $\pi|_S$ and therefore $\pi|_{U \cap S}$ is injective. Since $\pi \circ \varphi^{-1} : V_1 \to \pi(U \cap S)$ is bijective and continuous, by the Brower theorem on the invariance of domain, it is also a homeomorphism and thus $\pi(U \cap S)$ is open in $\mathbb{R}^{n-1}$. So $\pi : U \cap S \to \pi(U \cap S)$ is a homeomorphism.

Now let $f : \pi(U \cap S) \to \mathbb{R}$, $f(x') := pr_0 \circ \pi^{-1}(x')$. Then f is continuous and $U \cap S = \{(f(x'), x') \mid x' \in \pi(U \cap S)\}$. We can assume that U is connected and that $U \setminus S$ has just two components, $U^+ := \{(x^0, x') \in U \mid x^0 > f(x')\}$ and $U^- := \{(x^0, x') \in U \mid x^0 < f(x')\}$. The sets $I^-(p, U)$ and $I^+(p, U)$ are open and connected hence since S is achronal, they do not meet S. Thus they are contained in U^+ or U^-. The vertical straight line through p meets both $I^-(p, U)$ and $I^+(p, U)$, hence also both U^- and U^+. By $U^- \cap I^-(p, U) \neq \emptyset$, $I^-(p, U) \subseteq U^-$ and analogously, $I^+(p, U) \subseteq U^+$. Then every curve α in U that connects $I^-(p, U)$ to $I^+(p, U)$ must meet $S \cap U$, hence $p \notin \text{edge}(S)$. □

Corollary 3.4.8. *An achronal set S is a closed topological hypersurface if and only if* $\text{edge}(S)$ *is empty.*

Proof. Assume $\text{edge}(S) = \emptyset$. Since $S \cap \text{edge}(S) = \emptyset$, by Proposition 3.4.7, S is a topological hypersurface. From Lemma 3.4.6, we obtain $\bar{S} \setminus S \subseteq \text{edge}(S) = \emptyset$ hence $\bar{S} = S$.

Now let S be a closed topological hypersurface. By Proposition 3.4.7, $S \cap \text{edge} S = \emptyset$. Since $\text{edge}(S) \subseteq \bar{S}$ and $S = \bar{S}$, the edge is empty. □

A set $A \in M$ is called a *future set* if $I^+(A) \subseteq A$. The causal future $J^+(A)$ is a future set for any set A since $I^+(J^+(A)) \subseteq J^+(A)$.

Remark 3.4.9. Note that if A is a future set, its complement $M \setminus A$ is a *past set*: if there were $q \in M \setminus A$ such that $I^-(q) \nsubseteq M \setminus A$, there would exist some $p \in A$ such that $p \in I^-(q)$ hence $q \in I^+(p) \subseteq I^+(A) \subseteq A$, which is impossible.

Lemma 3.4.10. *Let $A \neq \emptyset$ be a future set in M. Then the boundary of ∂A is a closed achronal topological hypersurface.*

Proof. By Corollary 3.4.8, we need to show that ∂A is achronal and $\text{edge}(\partial A) = \emptyset$. Consider $p \in \partial A$ and let $q \in I^+(p)$. By Theorem 2.3.10, $I^-(q)$ is an open neighborhood of p thus $I^-(q) \cap A \neq \emptyset$. A is a future set thus $q \in I^+(A) \subseteq A$. By the openness of $I^+(A)$, $I^+(A) \subseteq A^\circ$ hence $I^+(p) \subseteq A^\circ$.

Let $p \in \partial A$ and let $q \in I^-(p)$. Then $I^+(q)$ is also an open neighborhood of $p \in \partial A$ thus $I^+(q) \cap (M \setminus A) \neq \emptyset$. By Remark 3.4.9, we have $q \in I^-(M \setminus A) \subseteq M \setminus A$ therefore $q \in (M \setminus A)^\circ$ and hence $I^-(p) \subseteq (M \setminus A)^\circ$. Since $I^+(p) \cap \partial A = \emptyset$ and $I^-(p) \cap \partial A = \emptyset$, $\forall p \in \partial A$, ∂A is achronal.

On the other hand, for $p \in \text{edge}(A)$ and α a timelike curve from $I^-(p)$ to $I^+(p)$, by the above, it has to go from $(M \setminus A)^\circ$ to B° and thus meet ∂A. Therefore $\text{edge}(\partial A) = \emptyset$. □

3.5 Cauchy hypersurfaces

Definition 3.5.1. A Cauchy hypersurface is a subset S of M which every inextendible timelike curve intersects exactly once.

In the smooth case, for spacelike hypersurfaces this definition of a Cauchy hypersurface is equivalent to the one in [HE73], and this remains true in the $C^{1,1}$-case, cf. Proposition 3.6.20.

Example 3.5.2. In $\mathbb{R}^n_1$, the hyperplanes t constant are Cauchy hypersurfaces but the future light cones $\Lambda^+(p)$ are not.

Lemma 3.5.3. *Let $S \in M$ be a closed set and let $\alpha : [0, b) \to M \setminus S$ be a past inextendible causal curve starting at p that does not meet S. Then:*

(i) For any $q \in I^+(p, M \setminus S)$ there exists a past inextendible timelike piecewise geodesic $\tilde{\alpha} : [0, b) \to M \setminus S$ starting at q that does not meet S;

(ii) If α is not a null geodesic, there exists a past inextendible timelike piecewise geodesic $\tilde{\alpha} : [0, b) \to M \setminus S$ starting at p that does not meet S.

Proof. Since α is past inextendible, we can assume without loss of generality that $b = \infty$ and that the sequence $(\alpha(n))_n$ does not converge. Let d be some metric on M inducing the topology of M.

(i) We are now only working in an open submanifold $M \setminus S$ of M and the relation $\ll$ is that of $M \setminus S$. Let $p_0 := q \gg p$. Since $\alpha(1) \leq \alpha(0) = p \ll q$, by Proposition 2.3.14, $\alpha(1) \ll p_0$. Choose a point $p_1 \in M \setminus S$ on a timelike curve from $\alpha(1)$ to p_0 such that $0 < d(p_1, \alpha(1)) < 1$. Then $\alpha(2) \leq \alpha(1) \ll p_1$ so we may now choose some $p_2 \in M \setminus S$ on a timelike curve from $\alpha(2)$ to p_1 such that $d(p_2, \alpha(2)) < 1/2$. Continuing inductively in this way, we get a sequence $p_k \in M \setminus S$ such that $\alpha(k) \ll p_k \ll p_{k-1}$ and $d(p_k, \alpha(k)) < 1/k$. Joining the p_k's by past timelike segments, we obtain a timelike curve $\tilde{\alpha}$ in $M \setminus S$ such that $\tilde{\alpha}(k) = p_k$. Assume now that $\tilde{\alpha}$ is past extendible in M. Then there exists $p_\infty \in M$

such that $\tilde{\alpha}(k) \to p_\infty$ as $k \to \infty$. Then it follows:

$$d(\alpha(k), p_\infty) \leq d(\alpha(k), p_k) + d(p_k, p_\infty) \to 0$$

hence $\alpha(k) \to p_\infty$, contradiction to α being past inextendible.

(ii) To avoid the variational calculus-based proof of [ON83, Lemma 14.30] we need the following argument:

Lemma 3.5.4. *Let S be a closed set and let $\alpha : [0, \infty) \to M \setminus S$ be a past directed causal curve which is not a null geodesic. Then there exists $a > 0$ such that $\alpha(a) \ll \alpha(0)$ (with $\ll$ the relation on $M \setminus S$).*

Proof. Suppose to the contrary that there is no point on the curve α which can be timelike related to $\alpha(0)$ within $M \setminus S$. Using Theorem 1.7.1 we can cover α by totally normal neighborhoods U_i with $U_i \subseteq M \setminus S$ since $M \setminus S$ is open. Let $t_0 = 0 < t_1 < t_2$ ... such that $\alpha|_{[t_i,t_{i+1}]} \subseteq U_{i+1}$. By our assumption, it follows that $\alpha|_{[t_0,t_1]}$ lies in $\partial J^-(\alpha(0), U_1)$. Hence, by Corollary 2.3.13, $\alpha|_{[t_0,t_1]}$ is a null geodesic. Iterating this procedure we obtain that α is a null geodesic, a contradiction. □

In particular, $p \in I^+(\alpha(a), M \setminus S)$. Applying (i) to $\alpha|_{[a,\infty]}$, we obtain a past directed past inextendible timelike curve $\tilde{\alpha}$ starting at p. □

Proposition 3.5.5. *Let $S \subseteq M$ be a Cauchy hypersurface. Then:*

(i) S is a closed achronal topological hypersurface.

(ii) Every inextendible causal curve meets S.

Proof. (i) Let α be a timelike curve that intersects S twice. Then so does every inextendible timelike extension $\tilde{\alpha}$ of α, contradicting the assumption that S is a Cauchy hypersurface.

Now we show that M is a disjoint union of $S, I^-(S)$ and $I^+(S)$, i.e., $M = I^-(S) \dot{\cup} S \dot{\cup} I^+(S)$: let $p \in M$ and let α be a inextendible timelike curve through p. Let q be the intersection point of α with S. Then $p \in I^-(S) \cup S \cup I^+(S)$. Now assume $q \in I^\pm(S) \cap S$ or $q \in I^-(S) \cap I^+(S)$. Then there exists a timelike curve that intersects S twice, contradicting (i). Hence the union is disjoint. In particular, $S = M \setminus (I^-(S) \cup I^+(S))$ is closed.

Next, $S = \partial I^+(S) = \partial I^-(S)$: Since $S \cup I^+(S) = M \setminus I^-(S)$, as well as $S \cup I^-(S) = M \setminus I^+(S)$, it follows that $\partial I^+(S) = \overline{I^+(S)} \cap \overline{M \setminus I^+(S)} \subseteq (I^+(S) \dot\cup S) \cap (I^-(S) \dot\cup S) = S$. On the other hand, $S \subseteq \partial I^+(S)$ always holds.

Finally, $\mathrm{edge}(S) = \emptyset$: we show that every timelike curve α from $I^-(S)$ to $I^+(S)$ meets S. If α didn't meet S, we would get that $\alpha([a,b]) = (\alpha([a,b]) \cap I^+(S)) \dot\cup (\alpha([a,b]) \cap I^-(S))$, a contradiction to the connectedness of $\alpha([a,b])$. Then the result follows by Corollary 3.4.8.

(ii) Assume there is an inextendible causal curve α that does not meet S. Since $M = I^-(S) \dot\cup S \dot\cup I^+(S)$, without loss of generality α runs in $I^+(S)$. Choose p on α and let $q \in I^+(p, M \setminus S)$. By Lemma 3.5.3 (i), there is a past inextendible timelike curve $\tilde\alpha : [0,b) \to M \setminus S$ that does not meet S, hence it has to remain in $I^+(S)$. Extending $\tilde\alpha$ to the the future, we obtain a curve that lies entirely in $I^+(S)$ hence it is inextendible timelike and does not meet S, contradicting the fact that S is a Cauchy hypersurface. □

3.6 Cauchy developments and Cauchy horizons

Definition 3.6.1. Let S be an achronal set. The future Cauchy development of S is the set $D^+(S)$ of all points $p \in M$ with the property that every past inextendible causal curve through p meets S. The past Cauchy development is defined analogously. The Cauchy development is then given by $D(S) := D^+(S) \cup D^-(S)$.

Definition 3.6.2. Let S be an achronal set. The future Cauchy horizon $H^+(S)$ of S is defined as

$$H^+(S) := \overline{D^+(S)} \setminus I^-(D^+(S)),$$

with a coresponding definition for the past Cauchy horizon $H^-(S)$. One defines the Cauchy horizon of S as $H(S) = H^-(S) \cup H^+(S)$.

Note that both, Cauchy development and Cauchy horizon, are defined with locally Lipschitz causal curves (contrary to [HE73, ON83]). That this does not affect our considerations is shown in Lemma 3.6.7.

Example 3.6.3. 1. Let $M = \mathbb{R}^n_1$ and $A = \{x^0 = c\}$. Then $D^\pm = \{x^0 \geq (\leq) c\} = J^\pm(A)$ so $D(A) = \mathbb{R}^n_1$.

2. Let $M = \mathbb{R}^1_1 \times S^1 \setminus \{p\}$. For a spacelike circle S, the future Cauchy development is the union of S and the open region between S and the null geodesics α and β while

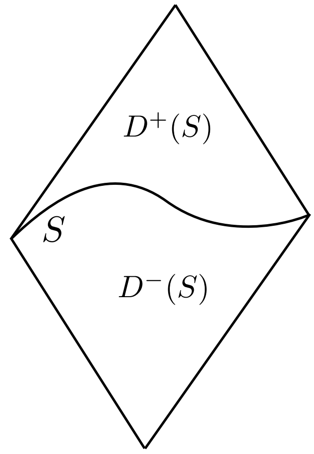

Figure 3.1: Cauchy development

the past Cauchy development is just $J^-(S)$.

Remark 3.6.4. Some authors define Cauchy developments with timelike curves instead of causal ones, see for example [Chr11]. When defined with causal curves, the Cauchy development is open for closed acausal topological hypersurfaces, cf. Proposition 3.6.12. It is also easier to work with in the case of continuous metrics, cf. [CG12]. On the other hand, it prevents piecewise null hypersurfaces from being Cauchy hypersurfaces which is one of the reasons some authors prefer the definition with timelike curves. The analogous definition of future Cauchy horizon then leads in general to essentially different sets for continuous metrics. Note that, physically, the Cauchy horizon $H^+(S)$ represents the limit of the region in spacetime that can be predicted from S as can be seen in Figure 3.6.

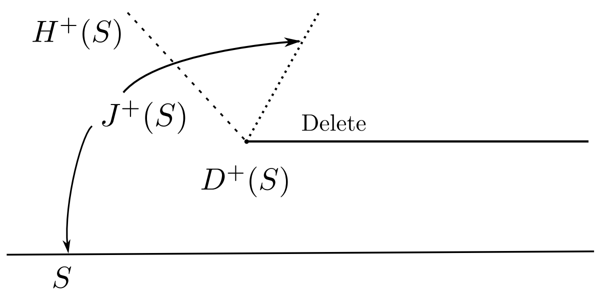

Figure 3.2: Cauchy horizon

Lemma 3.6.5. *Let $S \subseteq M$ be achronal. Then:*

(i) $S \subseteq D^\pm(S) \subseteq S \cup I^\pm(S)$

(ii) $D^+(S) \cap I^-(S) = \emptyset$

(iii) $D^+(S) \cap D^-(S) = S$

(iv) $D(S) \cap I^{\pm}(S) = D^{\pm}(S) \setminus S$.

Proof. (i) Let $p \in S$. Since every causal curve through p meets S at p then $p \in D^{\pm}(S)$. Now let α be an inextendible timelike curve through p. Then α meets S hence $p \in S \cup I^{\pm}(S)$.

(ii) Assume to the contrary, that there exists $q \in D^+(S) \cap I^-(S)$. Since $q \in I^-(S)$, there exist $p \in S$ and a past directed timelike curve α from p to q. Extending α from q on, one obtains an intersection with S, a contradiction to S being achronal.

(iii) Using (i) and (ii), we have:

$$S \subseteq D^+(S) \cap D^-(S) \subseteq D^+(S) \cap (S \cup I^-(S)) = D^+(S) \cap S = S.$$

(iv) Again, from (i) and (ii) and the fact that, since S is achronal, $I^+(S) \subseteq M \setminus S$, it follows:

$$D(S) \cap I^+(S) = D^+(S) \cap I^+(S) \subseteq D^+(S) \setminus S \subseteq D(S) \cap ((S \cup I^+(S)) \setminus S) = D(S) \cap I^+(S).$$

□

Lemma 3.6.6. *Let S be achronal. Then:*

(i) S is a Cauchy hypersurface if and only if $D(S) = M$.

(ii) Every past directed causal curve that starts in $D^+(S)$ and leaves it, has to intersect S.

(iii) Every inextendible causal curve through any $p \in S$ intersects both $I^-(S)$ and $I^+(S)$.

Proof. (i) ($\subseteq$) Let S be a Cauchy hypersurface in M. Then M is the disjoint union of S, $I^+(S)$ and $I^-(S)$. In particular, $D^{\pm}(S) = S \cup I^{\pm}(S)$: by Lemma 3.6.5 (i), $D^{\pm}(S) \subseteq S \cup I^{\pm}(S)$. Also, $S \subseteq D^+(S)$. It remains to show $I^+(S) \subseteq D^+(S)$. Let $q \in I^+(S)$ and let α be a past extendible causal curve through q. Let $\tilde{\alpha} = \alpha \cup \beta$ be a future inextendible extension of α. By Lemma 3.5.5, $\tilde{\alpha}$ intersects S. If β intersected S, say in q_1, then $q_1 \in J^+(q) \subseteq J^+(I^+(S)) = I^+(S)$. Therefore there exists a timelike curve $\tilde{\beta}$ from S to

$q_1 \in S$, a contradiction to S being a Cauchy hypersurface. Hence α itself intersects S and $q \in D^+(S)$. Thus $D(S) = D^+(S) \cup D^-(S) = M$.

($\supseteq$) By the definition of $D(S)$, every inextendible timelike curve meets S. Since S is achronal, it can meet S at most once, hence exactly once.

(ii) Let $\alpha : [0, b] \to M$ be a past directed causal curve with $\alpha(0) \in D^+(S)$ and $\alpha(b) \notin D^+(S)$. Hence there exists a past inextendible causal curve γ that starts at $\alpha(b)$ and does not intersect S. Then the concatenation $\alpha \cup \gamma$ is a past inextendible causal curve that starts at $\alpha(0) \in D^+(S)$ hence it must intersect S. Thus α intersects S.

(iii) Let $\alpha : [0, \infty) \to M$ be a past inextendible causal curve that starts at $\alpha(0) = p \in D(S)^\circ$. By Lemma 3.6.5 (i), we get $D(S) \subseteq I^-(S) \cup I^+(S) \cup S$. If $p \in I^-(S)$, there is nothing to prove. Now let $p \in S \cup I^+(S)$. By Theorem 2.3.10, every neighborhood of p intersects $I^+(p)$. Since $D(S)$ is a neighborhood of p, there exists $q \in I^+(p) \cap D(S) = I^+(p) \cap D^+(S)$. By Lemma 3.5.3 (i), there is a past inextendible timelike curve $\tilde{\alpha} : [0, \infty) \to M$ starting at q. In particular, by the proof of 3.5.3 (i), it follows that for every $s \in [0, \infty)$ there exists $k \in \mathbb{N}$ such that $\alpha(k) \in I^-(\tilde{\alpha})(s)$: indeed, for $k > s$ one obtains $\alpha(k) \ll p_k = \tilde{\alpha}(k) \ll \tilde{\alpha}(s)$. Now, $q \in D^+(S)$, hence $\tilde{\alpha}$ meets S in some point $\tilde{\alpha}(s)$. The corresponding $\alpha(k)$ is thus in $I^-(S)$. $\square$

Lemma 3.6.7. *Let S be a closed achronal hypersurface. Then the Cauchy development defined with Lipschitz curves, $D^+(S)$, coincides with the one defined with piecewise C^1-curves, $D^+_{C^1}(S)$.*

Proof. Obviously, $D^+(S) \subseteq D^+_{C^1}(S)$. Now suppose there existed some $p \in D^+_{C^1}(S) \setminus D^+(S)$. Then there would exist a past inextendible Lipschitz causal curve γ from p such that $\gamma \cap S = \emptyset$. By Theorem 1.7.1, we may cover γ by totally normal neighborhoods $U_1, ..., U_N, ...$ such that $\gamma([s_i, s_{i+1}]) \subseteq U_{i+1}, \forall i$. Then we distinguish two cases: If $\gamma([s_i, s_{i+1}]) \subseteq \partial J^+(\gamma(s_i), U_i)$ for all i, then by Corollary 2.3.13 γ is a piecewise null geodesic and therefore piecewise C^1, a contradiction. The second possibility is that $\exists i$, $\exists t \in (s_i, s_{i+1})$ such that $\gamma(s_i) \ll \gamma(t)$. But then Lemma 3.5.3 (ii) gives a contradiction. $\square$

Lemma 3.6.8. *Let S be a closed achronal set. Then $\overline{D^+(S)}$ is the set of all points p such that every past inextendible timelike curve through p meets S.*

Proof. Let $X := \{p \in M \mid \text{every past inextendible timelike curve through } p \text{ meets } S\}$.

$\overline{D^+(S)} \subseteq X$: Assume $p \in \overline{D^+(S)} \setminus X$. Then there exists a past inextendible timelike curve $\alpha : [0, b) \to M$, $\alpha(0) = p$, that does not meet S. In particular, $p \notin S$. Since S is closed, there exists a totally normal neighborhood U of p such that $U \cap S = \emptyset$. Choose $s \in [0, b)$ such that for $q := \alpha(s)$ it follows that $p \in I^+(q, U)$. $I^+(q, U)$ is an open neighborhood of $p \in \overline{D^+(S)}$, thus there exists $r \in I^+(q, U) \cap D^+(S)$. By Theorem 2.3.10, there exists a past directed timelike geodesic β from r to q in U. As $U \cap S = \emptyset$, β cannot intersect S. But $\beta \cup \alpha|_{[s,b]}$ is a past inextendible timelike curve starting at $r \in D^+(S)$ hence it must meet S. Thus $\alpha|_{[s,b]}$ meets S, contradicting our assumption.

$X \subseteq \overline{D^+(S)}$: Let $p \notin \overline{D^+(S)}$ and choose $q \in I^-(p, M \setminus \overline{D^+(S)})$. In fact, $q \in M \setminus \overline{D^+(S)}$ so $q \notin D^+(S)$. Hence there exists a past inextendible causal curve $\alpha \in M$ starting at q that does not intersect S. Since $q \in I^-(p, M \setminus \overline{D^+(S)})$, using Lemma 3.6.5 (i), we get $p \in I^+(q, M \setminus \overline{D^+(S)}) \subseteq I^+(q, M \setminus S)$. By Lemma 3.5.3 (i), there exists a past inextendible timelike curve through p that does not intersect S. Thus $p \notin X$. □

Lemma 3.6.9. *Let S be a closed achronal set. Then $\partial D^\pm(S) = S \cup H^\pm(S)$.*

Proof. First we show $S \subseteq \partial D^+(S)$. Assume $p \in S \cap D^+(S)^\circ$. By Theorem 2.3.10, there exists $q \in D^+(S)^\circ \cap I^-(p)$. Let α be a past inextendible timelike curve starting at q. Then α meets S at some point r. Thus $r \in I^-(q)$ and $q \in I^-(p)$ so $r \le q \ll p$. By Proposition 2.3.14, $r \ll p$, $r, p \in S$, contradicting achronality of S.

$H^+(S) \subseteq \partial D^+(S)$: by the definition of a Cauchy horizon, we have $H^+(S) \subseteq \overline{\partial D^+(S)}$. Assume $p \in H^+(S) \cap D^+(S)^\circ$. As above, $I^+(p) \cap D^+(S) \neq \emptyset$, which is a contradiction to $p \in H^+(S)$.

$\partial D^+(S) \subseteq S \cup H^+(S)$: Assume $p \in \partial D^+(S) \setminus (S \cup H^+(S))$. Then $p \in \overline{D^+(S)} \setminus S$. By Lemma 3.6.8, $p \in I^+(S)$. In addition, $p \in \overline{D^+(S)} \setminus H^+(S)$, so by Definition 3.6.2 there exists $q \in I^+(p) \cap D^+(S)$. In particular, $p \in I^-(q)$, hence $I^+(S) \cap I^-(q)$ is an open neighborhood of p. It suffices to show that $I^+(S) \cap I^-(q) \subseteq D^+(S)$, for then $p \in D^+(S)^\circ$ in contradiction to $p \in \partial D^+(S)$. Let $r \in I^+(S) \cap I^-(q)$ and let α be a past inextendible causal curve starting at r. Since $r \in I^-(q)$, there exists a past directed timelike curve β from q to r. From $r \in I^+(S)$, we see that β is entirely contained in $I^+(S)$. Since S is achronal, $S \cap I^+(S) = \emptyset$, so β does not meet S. But $\beta \cup \alpha$ meets S as $q \in D^+(S)$ thus α meets S and $r \in D^+(S)$. □

The next result shows how in the case of achronal sets, Cauchy developments and global hyperbolicity are closely related (cf. [Chr11, Thm. 2.9.9])

Theorem 3.6.10. *Let S be achronal. Then $D(S)^\circ$ is globally hyperbolic.*

Proof. We first show that the causality condition holds on $D(S)^\circ$: Suppose there exists a closed causal curve α through $p \in D(S)^\circ$. Circling along α repeatedly leads to an inextendible causal curve $\hat{\alpha}$ which meets S repeatedly, contradicting achronality of S.

Next we show that the strong causality condition holds on $D(S)^\circ$: Suppose that strong causality is violated at $p \in D(S)^\circ$. Then there exists a neighborhood U of p such that for every n there exists an inextendible causal curve α_n parameterized by h-arc length such that $\alpha_n(0) \in B(p, \frac{1}{n})$, a sequence s_n such that $\alpha_n(s_n) \to p$ and $0 < t_n < s_n$ so that $\alpha_n(t_n) \notin U$. Note that we may without loss of generality assume, changing time orientation if necessary, that $p \in I^-(S) \cup S$.

Then there exists a causal curve α through p that is an accumulation curve of the α_n's such that, passing to a subsequence if necessary, the α_n's converge uniformly to α on compact subsets of $\mathbb{R}$. By the assumption, $p \in D(S)$ thus by Lemma 3.6.6 (iii) there exist $s_\pm \in \mathbb{R}$ such that $\alpha(s_-) \in I^-(S)$ and $\alpha(s_+) \in I^+(S)$. By achronality of S and the fact that α is future directed, it follows that $s_- < s_+$. The sets $I^\pm(S)$ are open and $\alpha_n(s_\pm) \to \alpha(s_\pm)$ thus $\alpha_n(s_\pm) \in I^\pm(S)$ for n large enough.

The simplest case to exclude is the one where the sequence $\{s_n\}$ is bounded. There exists $t > 0$ such that $t_n > t$, $\forall n$: Choose $r > 0$ such that $B_h(p, r) \subseteq U$. If $d_h(\alpha_n(0), p) < \frac{r}{2}$ then $t_n > \frac{r}{2}$ hence take $t := \frac{r}{2}$.

Then $s_n \geq t_n \geq t > 0$, $\forall n$. So there exists $s_* \in \mathbb{R}$ such that, passing again to a subsequence if necessary, it follows that $s_n \to s_*$. Thus

$$\begin{aligned} d(\alpha_n(s_*), p) &\leq d(\alpha_n(s_*), \alpha_n(s_n)) + d(\alpha_n(s_n), p) \\ &\leq L|s_n - s_*| + d(\alpha_n(s_n), p) \to 0 \end{aligned}$$

hence $\alpha_n(s_*) \to p$. Since $\alpha_n|_{[0,s_*]}$ converges uniformly to $\alpha|_{[0,s_*]}$, there exists an inextendible periodic causal curve α' through p obtained by repeatedly circling from p to p along $\alpha|_{[0,s_*]}$. By Lemma 3.6.6, α' meets all of $S, I^+(S)$ and $I^-(S)$, which is a contradiciton to the achronality of S.

Note that $p \in I^-(S)$ cannot occur. If p were in $I^-(S)$, it would follow that $s_n \leq s_+$ for n large enough. Otherwise, for n large, we could follow α_n in the future from $\alpha_n(s_+) \in I^+(S)$ to $\alpha_n(s_n) \in I^-(S)$ (since $\alpha_n(s_n) \to p$ and $p \in I^-(S)$), which cannot happen since S is achronal. But then the sequence s_n would be bounded which we have excluded.

Thus the only possibility remaining is $p \in S$. We need the following result (see [Had10, Prop. 4.1.1], [HP69, Lemma 2.10]):

Lemma 3.6.11. *Let S be chronological and suppose that strong causality fails at $p \in S$. Then there exists an inextendible null geodesic α through p such that strong causality fails at $\alpha(t)$ for all t.*

Proof. Assume strong causality fails at p. Then there exists an open neighborhood O of p such that $\overline{O}$ is compact and there exist neighborhoods $V_k \subseteq O$ and causal curves β_k from $p_k \in V_k$ to $r_k \in V_k$ which leave V_k. Without loss of generality we may assume that there exists a totally normal neighborhood U of p, such that $\overline{O} \Subset U$. Let q_k be the first intersection of β_k with ∂U. Since ∂U is compact, we may assume that $q_k \to q \in \partial O$, as $k \to \infty$, so $q \in U$. Therefore, there exists a unique geodesic α_{pq} from p to q.

We next show that α_{pq} is a null geodesic. The curves β_k are causal, thus $q_k \in J^+(p_k, U)$, for all k. Since $p_k \to p, q_k \to q$, by Proposition 2.3.22 we get $p \leq q$, so the vector $\overrightarrow{pq}$ is causal. It remains to show that $\overrightarrow{pq}$ is null. Suppose $\overrightarrow{pq}$ is timelike. Then by Corollary 2.3.17 there exists $k \in \mathbb{N}$ and there is a neighborhood W of q such that $r_k \ll q_k$ for all $r_k \in V_k$ and for all $q_k \in W$. Since $r_k \ll q_k \ll r_k$, by Proposition 2.3.14 we conclude that $r_k \ll r_k$, contradicting the chronology condition on S.

We next show that the strong causality fails at each point of α_{pq}: we will first show this in q. Choose neighborhoods $\tilde{V}_k$ of q such that $\tilde{V}_k \subseteq U'$ for a fixed totally normal neighborhood U' of q, and such that $r_k \notin U'$, $\forall k \in \mathbb{N}$. Now let $\tilde{q}_k \in I^+(q) \cap \tilde{V}_k$. Then $q \in I^-(\tilde{q}_k)$ and we may assume without loss of generality (by choosing a subsequence if necessary) that $q_k \in I^-(\tilde{q}_k) \setminus V_k$ for all $k \in \mathbb{N}$. In fact, $I^-(\tilde{q}_k)$ is an open neighborhood of p since $p \leq q \ll \tilde{q}_k$ and $r_k \to p$ so it follows that without loss of generality we may assume that $r_k \in I^-(\tilde{q}_k)$ for all $k \in \mathbb{N}$.

Now let $\tilde{\beta}_k$ be a future directed timelike curve from r_k to $\tilde{q}_k$ and let $\check{\beta}_k := \beta_k \cup \tilde{\beta}_k$. Then $\tilde{\beta}_k$ is a future directed causal curve from q_k to q_k which leaves $\tilde{V}_k$ hence contradicting the strong causality assumption at q.

Finally we show that strong causality fails at each point $\hat{q}$ on α_{pq}. Note that there exists a neighborhood $\hat{U} \subset O$ with the same properties as O above so that any such $\hat{q}$ lies on the boundary of $\hat{U}$. We may assume that $\{\alpha_{pq}\} \cap \partial \hat{U} = \{\hat{q}\}$ and $V_k \subseteq \hat{U}$ for all $k \in \mathbb{N}$. Also we may replace the initial part of β_k from p_k to q_k by the geodesic $\alpha_{p_k q_k}$ connecting p_k and

q_k in U. Let $\hat{q}_k$ be the first intersection of $\alpha_{p_k q_k}$ with $\partial\hat{U}$. Then the causal vector $\overrightarrow{p_k q_k}$ is the initial velocity of $\alpha_{p_k q_k}$ and by construction, there exists some $c_k \in [0,1]$ such that $\overrightarrow{p_k \hat{q}_k} = c_k \overrightarrow{p_k q_k}$. $\partial\hat{U}$ is compact thus without loss of generality we have that $c_k \to c \in [0,1]$ and $\hat{q}_k \to \breve{q} \in \partial\hat{U}$. Hence $\overrightarrow{p\breve{q}} = c\overrightarrow{pq}$ and therefore $\breve{q} \in \{\alpha_{pq}\}$. Also, $\breve{q} \in \partial\hat{U} \cap \{\alpha_{pq}\} = \{\hat{q}\}$, i.e., $\breve{q} = \hat{q}$ thus the strong causality fails at $\breve{q} = \hat{q}$.

Hence by the above, there is a totally normal neighborhood $\tilde{U}$ of q such that $\tilde{\beta}_k$ starting from q_k leaves $\tilde{U}$ and re-enters $\tilde{V}_k$. Therefore, we may repeat the construction from the beginning (when we proved that strong causality fails at p) with q instead of p and $\tilde{U}$ instead of O. Since $\hat{\beta}_k$ coincides with β_k within $\tilde{U}$ (before it leaves it), the new sequence of boundary points $\{s_k\}_{k\in\mathbb{N}}$ was obtained by the α_k themselves. Hence we obtain

$$s = \lim_{k\to\infty} s_k$$

and a null geodesic α_{qs} along which the strong causality fails. We want to show that α_{qs} extends α_{pq} as an unbroken null geodesic. Suppose this is not the case. Then by Proposition 2.3.16, it follows that $p \ll r$ and thus by Corollary 2.3.17 there exists a neighborhood $\tilde{W}$ and some $k \in \mathbb{N}$ such that for all $y_k \in V_k$ and for all $s_k \in \tilde{W}$, $y_k \ll s_k$. Then for some $s_k \in \tilde{W} \cap \{\beta_k\}$, using Proposition 2.3.14 we have:

$$s_k \leq y_k \ll s_k \ \Rightarrow s_k \ll s_k.$$

By iterating this procedure both into the future and into the past to obtain an inextendible null geodesic α along which strong causality fails. □

Now by Lemma 3.6.6, since α is null (hence causal) it must intersect both $I^-(S)$ and $I^+(S)$ which means that there is a point in $I^-(S) \cap D(S)^\circ$ at which strong causality fails and we have already excluded this case.

Finally we prove compactness of the causal diamonds $J^+(p) \cap J^-(q)$, $p, q \in D(S)^\circ$. Consider a sequence $p_n \in J^+(p) \cap J^-(q)$. Then either $p_n \in I^-(S) \cup S$ for all $n \geq n_0$, or there exists a subsequence, still denoted by p_n, such that $p_n \in I^+(S)$. In the second case we can reduce the analysis to the first one by changing time-orientation, passing to a subsequence and renaming p and q hence leading to $p \in I^-(S) \cup S$. By definition, there exists a future directed causal curve $\hat{\alpha}_n$ from p to q passing through p_n:

$$\hat{\alpha}_n(t_n) = p_n. \tag{3.6}$$

Let α_n be any h-arc length parameterized future directed inextendible causal curves which are extensions of $\hat{\alpha}_n$ with $\alpha_n(0) = p$. Denote by α an inextendible accumulation curve of the curves α_n. Then α is a future inextendible causal curve through $p \in (D^-(S) \cup S) \cap D(S)^\circ$ ($p \notin I^+(S)$ because otherwise we would have $p_n \in I^+(S)$ and we have already excluded that). By Lemma 3.6.6 there exists t' such that $\alpha(t') \in I^+(S)$. Without loss of generality we have that the α_n's converge uniformly to α on $[0, t']$ which implies that the α_n's enter $I^+(S)$ for n large enough. Hence the sequence t_n defined by (3.6) is bounded, i.e., $0 \leq t_n \leq t'$. Otherwise, we would have:

$$t_n > t' \;\Rightarrow I^-(S) \ni p_n = \alpha(t_n) \geq \alpha(t') \in I^+(S).$$

Passing to another subsequence, we have $t_n \to \tilde{t}$, for some $\tilde{t} \in \mathbb{R}$. Then

$$\begin{aligned} d_h(\alpha(\tilde{t}), p_n) &\leq d_h(\alpha(\tilde{t}), \alpha(t_n)) + d_h(\alpha(t_n), \alpha_n(t_n)) \\ &\leq d_h(\alpha(\tilde{t}), \alpha(t_n)) + \sup_{s \in [0,t']} d_h(\alpha(s), \alpha_n(s)) \to 0, \end{aligned}$$

thus $p_n \to \alpha(\tilde{t}) \in J^+(p) \cap J^-(q)$, which we wanted to prove. □

The following result has recently been established in [CG12] even for continuous metrics in the case of acausal spacelike C^1-hypersurfaces (cf. [CG12, Prop. 2.4.]):

Proposition 3.6.12. *Let S be a closed acausal topological hypersurface. Then $D(S)$ is open.*

Proof. We first show that $S \subseteq D(S)^\circ$. Assume there is $p \in S \setminus D(S)^\circ$. Then there exist an open set U and a totally normal open neighborhood V such that $p \in U \subseteq \overline{U} \Subset V \subseteq I(S)$. Since $p \notin D(S)^\circ$, there is a sequence $r_n \in M \setminus D(S)$ such that $r_n \to p$. Without loss of generality we may assume that $r_n \in U$ for all n. As $r_n \in M \setminus D(S) \subseteq M \setminus S$, r_n must be in $I^+(S)$ or $I^-(S)$. Again without loss of generality, we have that $r_n \in I^+(S) \cap U$. Since $r_n \notin D^+(S)$, there exists a past directed past inextendible causal curve α_n starting at r_n that does not intersect S. By Proposition 2.3.22 (iv), as $\overline{U}$ is contained in V and α_n is a past inextendible causal curve starting at $r_n \in U$, it must leave $\overline{U}$. Let q_n be the first intersection of α_n with ∂U. Since ∂U is compact, a subsequence of q_n, without loss of generality, q_n itself, converges to $q \in \partial U$. Also, $q_n \leq r_n$ hence by Proposition 2.3.22 (iii) $q \leq p$. As $p \in U$ and $q \in \partial U$, we have that $q < p$. Then there are three possibilities:

1. $q \in I^+(S)$. Then there exists $q' \in S$ such that $q' \ll q$. Since $q' \ll q < p$ by Proposition 2.3.14, $q' \ll p$, contradiction to S being achronal.

2. $q \in S$. Then $q < p \in S$ hence a contradiction to achronality of S.

3. $q \in I^-(S)$. By Corollary 2.3.17 $I^-(S)$ is open hence there exists n such that $q_n = \alpha(t_n) \in I^-(S)$. Thus α_n is a causal curve from $r_n \in I^+(S)$ to $q_n \in I^-(S)$. Then there are timelike curves β and γ from $s_1 \in S$ to r_n and from q_n to $s_2 \in S$ respectively. The curve $\beta \cup \alpha_n \cup \gamma$ is causal and intersects S twice, again contradiction to S being achronal.

Now assume that $D(S)$ is not open. By changing the time orientation if necessary, we may without loss of generality assume that there is a point $p \in D^+(S)$, a sequence of points $p_n \to p$ and a sequence of inextendible causal curves α_n through p_n that do not intersect S. An accumulation curve α of the α_n's is an inextendible causal curve through $p \in D^+(S)$ hence by definition, it must meet S. Then for n large enough, α_n's need to intersect $D(S)^\circ$ hence also meet S. □

Lemma 3.6.13. *Let $S \in M$ be achronal. Then $H^\pm(S)$ is closed and achronal.*

Proof. Since $I^\mp(D^\pm(S))$ is open, $H^\pm = \overline{D^\pm(S)} \setminus I^\mp(D^\pm(S))$ is closed. Since $I^+(H^+(S))$ is open, by Definition 3.6.2, $I^+(H^+(S)) \cap D^+(S) = \emptyset$ hence $I^+(H^+(S)) \cap \overline{D^+(S)} = \emptyset$ and therefore $I^+(H^+(S)) \cap H^+(S) = \emptyset$. □

Proposition 3.6.14. *Let S be a closed acausal topological hypersurface. Then*

1. $H^+(S) = I^+(S) \cap \partial D^+(S) = \overline{D^+(S)} \setminus D^+(S)$;

2. $H^+(S) \cap S = \emptyset$;

3. *$H^+(S)$ is either empty or a closed achronal topological hypersurface;*

4. *$H^+(S)$ is generated by past inextendible null geodesics that are entirely contained in $H^+(S)$.*

Proof. 1. By the definition of a Cauchy horizon and Lemma 3.6.8, it follows that

$$H^+(S) \subseteq \overline{D^+(S)} \subseteq I^+(S) \cup S. \tag{3.7}$$

Also, $H^+(S) \cap D^+(S) = \emptyset$: assume $p \in H^+(S) \cap D^+(S)$. By Proposition 3.6.12, $D(S)$ is open hence by Theorem 2.3.10, $I^+(p) \cap D(S) \neq \emptyset$. S is achronal, thus $I^+(p) \cap D^-(S) = \emptyset$: if there was $q \in I^+(p) \cap D^-(S)$ there would exist a timelike curve α connecting p and q. An inextendible extension $\tilde{\alpha}$ of α intersects S either before p since $p \in D^+(S)$, or after q since $q \in D^-(S)$, which is impossible. Thus $I^+(p) \cap D^+(S) \neq \emptyset$, contradicting $p \in H^+(S)$. So $H^+(S) \cap D^+(S) = \emptyset$ hence $H^+(S) \cap S = \emptyset$ and by (3.7), $H^+(S) \subseteq I^+(S)$. Using Lemma 3.6.9 and the fact that $I^+(S) \cap S = \emptyset$ since S is achronal, we get:

$$I^+(S) \cap \partial D^+(S) = I^+(S) \cap (S \cup H^+(S)) = I^+(S) \cap H^+(S) = H^+(S),$$

hence the first equality.

Now, by $H^+(S) \cap D^+(S) = \emptyset$ we have that $H^+(S) \subseteq \overline{D^+(S)} \setminus D^+(S)$. Conversely, let $p \in \overline{D^+(S)} \setminus D^+(S)$ and $q \in I^+(p)$. Then there is a past directed timelike curve α from q to p. Since $p \in \overline{D^+(S)}$, by the openness of $I^-(S)$ and Lemma 3.6.5, $p \notin I^-(S)$ and $p \notin D^+(S)$ so $p \in S \cup I^-(S)$. Hence α does not intersect S. By $p \notin D^+(S)$, there exists an inextendible causal curve β starting at p and does not intersect S. Then the concatenation $\beta \cup \alpha$ starting at q does not meet S and it is inextendible causal hence $q \notin D^+(S)$.

2. Using $I^+(S) \cap S = \emptyset$, we have $H^+(S) \cap S = S \cap I^+(S) \cap \partial D^+(S) = \emptyset$.

3. Let $A := D^+(S) \cup I^-(S)$. We show that A is a past set, i.e., $I^-(A) \subseteq A$. It suffices to show $I^-(D^+(S)) \subseteq A$ since $I^-(A) = I^-(D^+(S)) \cup I^-(I^-(S))$. Assume $q \in I^-(D^+(S))$ and let γ be a past directed timelike curve from $p \in D^+(S)$ to q. If $q \in D^+(S)$, there is nothing to prove. Otherwise, there exists a past directed timelike curve α starting at q that does not meet S. As $p \in D^+(S)$, $\gamma \cup \alpha$ meets S hence γ meets S in some point r. Thus $q \in S \subseteq D^+(S)$ if $q = r$ or $q \in I^-(r)$ if $r \ll q$.

4. Suppose $p \in H^+(S) = \overline{D^+(S)} \setminus D^+(S)$. Since $p \notin D^+(S)$ there exists an inextendible causal curve α starting at p that does not meet S. By Lemma 3.6.8, no such curve can be timelike, or deformed to a timelike curve, that starts at p and does not meet S. By Lemma 3.5.3 (ii), α is a null geodesic. It remains to show that α runs in $H^+(S)$.

Since α does not meet S, it cannot intersect $D^+(S)$. By (i), it suffices to show that α is contained in $\overline{D^+(S)}$. Assume there exists some t such that $\alpha(t) \notin \overline{D^+(S)}$. Then there exists a normal neighborhood U of $\alpha(t)$ such that $U \cap \overline{D^+(S)} = \emptyset$. By Theorem 2.3.10, we can connect $\alpha(t)$ with some $q \in U$ by a future directed timelike curve β. As $q \notin D^+(S)$, there exists an inextendible causal curve γ starting at q that does not meet S. Hence

$\alpha \cup \beta \cup \gamma$ is an inextendible causal curve which is not a null geodesic thus by Lemma 3.5.3 (ii), it cannot meet S. Therefore there is an inextendible timelike curve $\tilde{\alpha}$ starting at p that does not intersect S: contradiction to our assumption. □

Lemma 3.6.15. *Let S be a spacelike hypersurface and let $p \in S$. Then there exists a neighborhood V of p such that $V \cap S$ is a Cauchy hypersurface in V.*

Proof. Let $\hat{g}_\varepsilon$ be smooth metrics approximating g from the outside as in Prop. 2.2.5. Then given any compact neighborhood W of p in M there exists some $\varepsilon > 0$ such that $W \cap S$ is spacelike for $\hat{g}_\varepsilon$. From the smooth theory (e.g., [BGPf07, Lemma A.5.6]) we obtain that there exists a neighborhood $V \subseteq W$ such that $V \cap S$ is a Cauchy hypersurface in V for $\hat{g}_\varepsilon$, and consequently also for g. □

Lemma 3.6.16. *Let S be an achronal set in M and let $p \in D(S)^\circ \setminus I^-(S)$. Then $J^-(p) \cap D^+(S)$ is compact.*

Proof. Consider a sequence of points $r_n \in J^-(p) \cap D^+(S)$. For every n, there is a past directed causal curve $\tilde{\alpha}_n$ from p to r_n such that $\tilde{\alpha}_n(t_n) = r_n$. If there is a convergent subsequence of r_n to p, there is nothing to prove. Otherwise, let α_n be an h-arc length parametrized past directed past inextendible causal curve which extends $\tilde{\alpha}_n$. An accumulation curve α of the curves α_n is a past directed past inextendible causal curve through $p \in D(S)^\circ$ hence by Lemma 3.6.6, there exists t_- such that $\alpha(t_-) \in I^-(S)$. Since the α_n's converge to α uniformly on $[0, t_-]$, the α_n's enter $I^-(S)$ for n large enough. Similarly to the proof of Theorem 3.6.10, the sequence t_n has to be bounded, otherwise we would obtain a contradiciton to the achronality of S. Hence $0 \leq t_n \leq \tilde{t}$. Passing to a subsequence, we obtain $t_n \to \hat{t}$, for some $\hat{t} \in \mathbb{R}$. Therefore $r_n \to \alpha(\hat{t})$, as we wanted to show. □

The following two results follow from [BGPf07, Lemma A.5.3] and [BGPf07, Cor. A.5.4]:

Lemma 3.6.17. *Let K be a compact subset of M and let $A \subseteq M$ be such that, $\forall p \in M$, $A \cap J^+(p)$, respectively $A \cap J^-(p)$, is relatively compact in M. Then $A \cap J^+(K)$, respectively $A \cap J^-(K)$, is relatively compact in M.*

Proof. We only consider the case of $A \cap J^+(K)$, the result analogously follows for $A \cap J^-(K)$. Consider an open covering $I^-(p)$, $p \in M$, of M and cover K by finitely many such sets, $K \subseteq I^-(p_1) \cup ... \cup I^-(p_k)$. By Corollary 2.3.18,

$$J^-(K) \subseteq J^-(I^-(p_1) \cup ... \cup I^-(p_k)) \subseteq J^-(p_1) \cup ... \cup J^-(p_k).$$

By the assumption, each $A \cap J^+(p_i)$, $i = 1, ..., k$, is relatively compact hence $A \cap J^-(K) \subset \bigcup_{i=1}^k (A \cap J^+(p_i))$ is contained in a compact set. □

Corollary 3.6.18. *Let S be a Cauchy hypersurface in a globally hyperbolic manifold M and let K be compact in M. Then $S \cap J^\pm(K)$ and $J^\mp(S) \cap J^\pm(K)$ are compact.*

Proof. Since S is a Cauchy hypersurface and therefore $D(S)^\circ = M$, by Lemma 3.6.16 we have that $J^+(S) \cap J^-(p)$ is compact for all $p \in M$. Then we may apply Lemma 3.6.17 to $A := J^+(S)$ to obtain that $J^-(K) \cap J^+(S)$ is relatively compact in M. By Proposition 3.3.5 and Corollary 3.3.6, we know that $J^+(S)$ and the relation $\leq$ are closed, hence $J^-(K) \cap J^+(S)$ is closed and thus compact.

S is a closed subset of $J^+(S)$ hence $J^-(K) \cap S$ is also compact. The statements for $J^+(K) \cap S$ and $J^+(K) \cap J^-(S)$ follow analogously. □

We give a proof of the following result, again to avoid the variational calculus-based argument in [ON83, Lemma 14.42].

Lemma 3.6.19. *Any achronal spacelike hypersurface S is acausal.*

Proof. Let $\alpha : [0, 1] \to M$ be a future directed causal curve with endpoints $\alpha(0)$ and $\alpha(1)$ in S. If α is not a null-geodesic, by Proposition 2.3.14, we can connect $\alpha(0)$ with $\alpha(1)$ also by a timelike curve, which is a contradiction to the achronality of S. Now let α be a null geodesic. By Lemma 3.6.15, there exists a neighborhood U around $\alpha(0)$ in which $S \cap U$ is a Cauchy hypersurface. Since α is C^2 and causal, it must be transversal to S, so it contains points in $J^+(S, U) \setminus S$. Then we can connect any such point with some point in $S \cap U$ by a timelike curve within U. Concatenating this curve with the remainder of α, we obtain a curve that is not entirely null and meets S twice. As above, this gives a contradiction to achronality. □

Proposition 3.6.20. *Let S be a spacelike hypersurface in M. Then S is a Cauchy hypersurface if and only if every inextendible causal curve intersects S precisely once.*

Proof. Let S be a Cauchy hypersurface and let α be an inextendible causal curve. By Lemmas 3.5.5(i), 3.6.19, α intersects S at most once. Also, by Lemma 3.5.5 (ii), it has to intersect S at least once, hence the result. □

A very important consequence of the Proposition 3.6.12 is the following:

Lemma 3.6.21. *If M contains a Cauchy hypersurface, then M is globally hyperbolic.*

Proof. By Theorem 3.6.10, $D(S)^\circ$ is globally hyperbolic. By Lemma 3.6.6, $D(S) = M$, hence also $D(S)^\circ = M$. □

Now let S be a spacelike hypersurface in M with a Lorentzian metric g. By $N(S)$ we denote the set of vectors perpendicular to S with respect to the metric g and by $(N(S), \pi)$ the normal bundle of S in M, where $\pi : N(S) \to S$ is the map carrying each vector $v \in T_p(S)^\perp$ to $p \in S$. The exponential map with respect to the metric g generalizes in the following way: the normal exponential map

$$\exp^\perp : N(S) \to M$$

assigns to a vector $v \in N(S)$ the point $c_v(1)$ in M, where c_v is the geodesic with initial velocity v. Thus $\exp^\perp$ carries radial lines in T_pS to geodesics of M that are normal to S at p.

The remaining statements serve to justify that in the proof of the main result in Section 4.5 we may without loss of generality assume S to be achronal. This is done using a covering argument, as in [HE73, ON83]. A key ingredient in adapting this construction to the $C^{1,1}$-setting is the following consequence of [Min13, Th. 1.39]:

Theorem 3.6.22. *Let M be a smooth manifold with a $C^{1,1}$ Lorentzian metric and let S be a semi-Riemannian submanifold of M. Then the normal bundle $N(S)$ is Lipschitz. Moreover, there exist neighborhoods U of the zero section in $N(S)$ and V of S in M such that*

$$\exp^\perp : U \to V$$

is a bi-Lipschitz homeomorphism.

Concerning curve-lengths in normal neighborhoods, [Min13, Th. 1.23] gives:

Proposition 3.6.23. *Let U be a normal neighborhood of $p \in M$. If $p \ll q$ for a point $q \in U$, then the radial geodesic segment σ is the unique longest timelike curve in U connecting p and q.*

Lemma 3.6.24. *Let S be a connected closed spacelike hypersurface in M.*

(i) If the homomorphism of fundamental groups $i_\sharp : \pi_1(S) \to \pi_1(M)$ *induced by the inclusion map* $i : S \hookrightarrow M$ *is onto, then* S *separates* M *(i.e.,* $M \setminus S$ *is not connected).*

(ii) If S *separates* M*, then* S *is achronal.*

The proof carries over from [ON83, Lemma 14.45] using Theorem 3.6.22, Theorem 2.3.10 and a result from intersection theory, namely, that a closed curve which intersects a closed hypersurface S precisely once and there transversally, is not freely homotopic to a closed curve which does not intersect S, cf. [GP74, p. 78]. The only change to [ON83, Lemma 14.45] is that for the curve σ we take a geodesic, which automatically is a C^1-curve (in fact, even C^2), so that the intersection theory argument applicable.

Theorem 3.6.25. *Let* S *be a closed, connected, spacelike hypersurface in* M*. Then there exists a Lorentzian covering* $\rho : \tilde{M} \to M$ *and an achronal closed spacelike hypersurface* $\tilde{S}$ *in* $\tilde{M}$ *which is isometric under* ρ *to* S*.*

The proof carries over from [ON83, Prop. 14.48] using Lemma 3.6.24.

The results 3.1.3-3.2.4, 3.3.5-3.6.9, 3.6.13, 3.6.14, 3.6.16, 3.6.20 and 3.6.21 follow from [ON83, Section 14].

Chapter 4

Singularity Theorems

The purpose of this chapter is to discuss the singularities of spacetimes and prove the Hawking singularity theorem in regularity $C^{1,1}$. The proof is based on the results from $C^{1,1}$ causality theory that were obtained in the previous chapters and regularization techniques from [CG12]. We will base our approach on [HE73], [KSSV15] and [ON83].

4.1 What Is a Singularity?

Unfortunately, it is not an easy task to give a precise meaning to what singularities actually are. Intuitively, one can imagine a singularity as a location in spacetime where the curvature explodes, where the metric tensor is not defined or is not regular enough. However such points can be cut out from the spacetime and the remaining manifold is then considered as a non-singular spacetime. This is a natural thing to do since spacetimes are solutions of Einstein's equations, which are not defined where curvature is infinite. Thus the problem of seeing whether the given spacetime has a singularity is detecting if some points have been cut out. This is done by the concept of geodesic incompleteness, namely by proving the existence of at least one incomplete causal geodesic, i.e., a geodesic that can only be extended for a finite time as measured by an observer travelling along it. Hence one can distinguish three types of geodesic incompleteness: timelike, null and spacelike. Again intuitively speaking, a spacelike singularity would mean that the matter is compressed to a point and a timelike singularity is the one where certain light rays come from a region with infinite curvature.

The significance of spacelike incompleteness is not very clear since nothing moves along spacelike curves. However, timelike geodesic incompleteness is very important from the physical point of view since it represents the possible existence of an observer or a particle whose histories cease to exist after a finite interval of proper time. Similarly, null geodesics represent the histories of zero restmass particles. Hence, if we want to talk about singularity-free spacetimes, we should at least exclude timelike and null geodesic incompleteness. However, not all spacetimes can be classified as the ones having an incomplete timelike geodesic or the ones that have an incomplete null geodesic, as seen in the example by Geroch in [Ge68], where he constructed a geodesically complete spacetime which contains an inextendible timelike curve of bounded acceleration and finite length. Therefore, we need a stronger condition so that we could say that a spacetime has no singularities. In particular, a generalization of an affine parameter to all curves is needed, not only geodesics. This is taken care of in the following way:

Let $\alpha : I \to M$ be any locally Lipschitz curve through $p \in M$ and choose $t_0 \in I$ and any orthonormal basis $\{e_i\}$ for $T_{\alpha(t_0)}M$ that is parallelly propagated along α in order to to obtain a basis for $T_{\alpha(t)}M$ for almost all $t \in I$. Then the tangent vector $v \in T_{\alpha(t)}M$ expressed in terms of the basis is $v = \sum_i v^i(t)E_i$, where E_i is the parallel field along α such that $E_i(t_0) = e_i$. Then the *generalized affine parameter* is defined by:

$$\lambda = \int_{t_0}^{t} \sqrt{\sum_i v^i v^i} dt.$$

Clearly, λ depends on p and the chosen orthonormal basis. The important property is that α has finite arc length with respect to the generalized affine parameter λ if and only if it has finite arc length with respect to any other generalized affine parameter obtained by choosing another orthonormal basis, see [HE73, Section 8.1]. This justifies the following definition:

Definition 4.1.1. The spacetime (M, g) is called b-complete if every locally Lipschitz curve that has finite arc length as measured with respect to the generalized affine parameter has an endpoint in M.

Hence a spacetime is said to be singularity-free if it is b-complete.

4.2 Singularity theorems

The phenomenon of some kind of irregularities of spacetimes or a singular behavior of solutions of the Einstein field equations was noticed in the very early years of General Relativity. However, it was believed that these singularities were the result of the high degree of symmetry or were unphysical in some way. This changed considerably with the first modern singularity theorem due to Penrose, who showed in his 1965 paper [Pen65] that deviations from spherical symmetry could not prevent gravitational collapse. In this paper, the concept of closed trapped surface was introduced and the notion of geodesic incompleteness was used to characterize a singular spacetime. Shortly afterwards Hawking realized that by considering a closed trapped surface to the past one could show that an approximately homogeneous and isotropic cosmological solution must have an initial singularity. This initiated the development of modern singularity theorems, one of the greatest achievements within general relativity. (See the recent review paper [SeGa14] for details.) All the resulting theorems have the same general structure described by Senovilla in [Se98] as a "pattern singularity theorem".

Pattern Singularity Theorem. *If a spacetime with a C^2-metric satisfies*

(i) a condition on the curvature

(ii) a causality condition

(iii) an appropriate initial and/or boundary condition

then it contains endless but incomplete causal geodesics.

However, the conclusion of the singularity theorems is their weak point. In fact, they only show that the spacetime is timelike or null geodesically incomplete but say little about the nature of the singularity. In particular, it is not claimed that the curvature blows up (see, however [Cl82, Cl93] as well as [SeGa14, Sec. 5.1.5] and the references therein) and it could be that the singularity is simply a result of the differentiability dropping below C^2.

However, as already indicated by Senovilla in [Se98] and in Chapter 2, $C^{1,1}$ regularity of the metric is the natural differentiability class from the point of view of the singularity theorems. In [HE73, Sec. 8.4] Hawking and Ellis presented a scheme of a proof of Hawking's singularity theorem based on an approximation of the $C^{1,1}$-metric by a 1-parameter family of smooth metrics. However the C^2-differentiability assumption plays a key role in many places in the singularity theorems and it is not obvious that these can all be dealt with without having further information about the nature of the approximation. In addition, the existence of normal coordinates and normal neighborhoods, as well as the existence of maximal curves is of high importance when proving the singularity theorems. In [Se98, Sec. 6.1], Senovilla lists where explicitly the C^2-assumption enters the proofs of the singularity theorems, indicating the number of technical difficulties a proof in the $C^{1,1}$-case would have to overcome.

As we have developed the key elements of causality theory in the previous chapters, we now approach the singularity theorems for $C^{1,1}$-metrics. Indeed, we will show that the tools now available allow one to prove singularity theorems with $C^{1,1}$-regularity and we illustrate this by providing a rigorous proof of Hawking's theorem in the $C^{1,1}$-regularity class. To be precise we establish the following result:

Theorem 4.2.1. *Let (M, g) be a $C^{1,1}$-spacetime. Assume*

(i) For any smooth timelike local vector field X, $\mathrm{Ric}(X, X) \geq 0$.

(ii) There exists a compact spacelike hypersurface S in M.

(iii) The future convergence $\mathbf{k}$ of S is everywhere strictly positive.

Then M is future timelike geodesically incomplete.

Remark 4.2.2.

(i) Since g is $C^{1,1}$, its Ricci-tensor is of regularity L^∞. In particular, it is in general only defined almost everywhere. For this reason, we have cast the curvature condition (i) in the above form. For any smooth vector field X defined on an open set $U \subseteq M$, $\mathrm{Ric}(X, X) \in L^\infty(U)$, so $\mathrm{Ric}(X, X) \geq 0$ means that $\mathrm{Ric}_p(X(p), X(p)) \geq 0$ for almost all $p \in U$. Since any timelike $X \in T_pM$ can be extended to a smooth timelike vector field in a neighborhood of p, (i) is equivalent to the usual pointwise condition ($\mathrm{Ric}(X, X) \geq 0$ for any timelike $X \in TM$) if the metric is C^2.

(ii) Concerning (iii) in the Theorem, our conventions (in accordance with [ON83]) are that $\mathbf{k} = tr\, S_U/(n-1)$ and $S_U(V) = -\nabla_V U$ is the shape operator of S, where U is the future pointing unit normal, ∇ denotes the connection on M and V is any vector field on the embedding $S \hookrightarrow M$.

(iii) In the physics literature, the negative of what we call the future convergence is often denoted as the expansion of S.

(iv) Finally, we note that an analogous result for past timelike incompleteness holds if the convergence in (iii) of the Theorem is supposed to be everywhere strictly negative.

In proving this theorem we will follow the basic strategy outlined in [HE73, Sec. 8.4]. However, in our proof we will make extensive use of the results of $C^{1,1}$-causality theory.

4.3 Regularization techniques

In the proof of Theorem 4.2.1 for $C^{1,1}$-metrics, regularization techniques play an important role, as was already clearly pointed out in [HE73, Sec. 8.4]. However, we shall see at several places below that a straightforward regularization via convolution in charts (as in [HE73, Sec. 8.4]) is insufficient to actually reach the desired conclusions. Rather, techniques adapted to the causal structure as introduced in [CG12] will be needed. This remark, in particular, applies to the results on the existence of maximizing curves (Lemma 4.4.2 and Proposition 4.4.3) below as well as to the proof of the main result in Section 4.5. The key result that we are going to use is Proposition 2.2.5.

One essential assumption in the singularity Theorem 4.2.1 is the curvature condition (i) for the $C^{1,1}$-metric g. We now derive from it a (weaker) curvature condition for any approximating sequence $\check{g}_\varepsilon$ as in Proposition 2.2.5, which is vital in our proof of the main theorem. This should be compared to condition (4) on p. 285 of [HE73].

Lemma 4.3.1. *Let M be a smooth manifold with a $C^{1,1}$-Lorentzian metric g and smooth background Riemannian metric h. Let K be a compact subset of M ($K \Subset M$) and suppose that* $\mathrm{Ric}(X,X) \geq 0$ *for every g-timelike smooth local vector field X. Then*

$$\begin{aligned} &\forall C>0\ \forall\delta>0\ \forall\kappa<0\ \exists\varepsilon_0>0\ \forall\varepsilon<\varepsilon_0\ \forall X\in TM|_K \\ &\textit{with}\ \ g(X,X)\leq\kappa\ \ \textit{and}\ \|X\|_h\leq C\ \textit{we have}\ \ \mathrm{Ric}_\varepsilon(X,X)>-\delta. \end{aligned} \tag{4.1}$$

Here Ric_ε *is the Ricci-tensor corresponding to a metric $\check{g}_\varepsilon$ as in Proposition 2.2.5.*

Proof. Let (U_i,ψ_i) ($i\in\mathbb{N}$) be a countable and locally finite collection of relatively compact charts of M and denote by $(\zeta_i)_i$ a subordinate partition of unity with $\mathrm{supp}(\zeta_i)\Subset U_i$ (i.e.,

$\mathrm{supp}(\zeta_i)$ is a compact subset of U_i) for all i. Moreover, choose a family of cut-off functions $\chi_i \in \mathcal{D}(U_i)$ with $\chi_i \equiv 1$ on a neighborhood of $\mathrm{supp}(\zeta_i)$. Finally, let $\rho \in \mathcal{D}(\mathbb{R}^n)$ be a non-negative test function with unit integral and define the standard mollifier $\rho_\varepsilon(x) := \varepsilon^{-n}\rho\left(\frac{x}{\varepsilon}\right)$ $(\varepsilon > 0)$. By f_* (resp. f^*) we denote push-forward (resp. pullback) under a smooth map f. It then follows from (2.2) in the proof of Proposition 2.2.5 that

$$\check{g}_\varepsilon - \sum_i \chi_i\, \psi_i^*\Big(\big(\psi_{i\,*}(\zeta_i\, g)\big) * \rho_{\eta(\lambda_i(\varepsilon),i)}\Big) \to 0 \text{ in } C^2(M). \tag{4.2}$$

Since $\eta(\lambda_i(\varepsilon), i) \to 0$ as $\varepsilon \to 0$ and $\{X \in TM|_K \mid \|X\|_h \le C\}$ is compact, we conclude that in order to establish the result it will suffice to assume that $M = \mathbb{R}^n$, $\|\,.\,\|_h = \|\,.\,\|$ is the Euclidean norm, to replace $\check{g}_\varepsilon$ by $g_\varepsilon := g * \rho_\varepsilon$ (component-wise convolution), and prove (4.1) for Ric_ε calculated from g_ε.

We first claim that

$$R_{\varepsilon jk} - R_{jk} * \rho_\varepsilon \to 0 \;\text{ uniformly on compact sets.} \tag{4.3}$$

We have $R_{jk} = \partial_{x^i}\Gamma^i_{kj} - \partial_{x^k}\Gamma^i_{ij} + \Gamma^i_{ij}\Gamma^m_{kj} - \Gamma^i_{km}\Gamma^m_{ij}$. In this expression, all terms involving at most first derivatives of g are uniform limits of the corresponding terms in $R_{\varepsilon jk}$, while the remaining terms are of the form $g^{im}a_{ijkm}$, where a_{ijkm} consists of second derivatives of g. These observations imply that (4.3) will follow from the following mild variant of the Friedrichs lemma:

Claim: Let $f \in \mathscr{C}^0(\mathbb{R}^n)$, $a \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$. Then $(f \cdot a) * \rho_\varepsilon - (f * \rho_\varepsilon)\cdot(a * \rho_\varepsilon) \to 0$ locally uniformly.

In fact,

$$\begin{aligned} ((f\cdot a) * \rho_\varepsilon - (f * \rho_\varepsilon)\cdot(a*\rho_\varepsilon))(x) &= \int (f(y) - (f*\rho_\varepsilon(x)))a(y)\rho_\varepsilon(x-y)\,dy \\ = \int (f(y) - f(x))a(y)\rho_\varepsilon(x-y)\,dy &+ \int (f(x) - (f*\rho_\varepsilon(x)))a(y)\rho_\varepsilon(x-y)\,dy, \end{aligned} \tag{4.4}$$

so for any $L \Subset \mathbb{R}^n$ we obtain

$$\begin{aligned} \sup_{x\in L} |(f\cdot a)*\rho_\varepsilon - (f*\rho_\varepsilon)\cdot(a*\rho_\varepsilon)|(x) \le \Big(\max_{\substack{x\in L\\ |x-y|\le\varepsilon}} |f(y)-f(x)|\Big)\cdot \sup_{d(y,L)\le\varepsilon}|a(y)| \\ + \Big(\sup_{x\in L}|f(x) - f_\varepsilon(x)|\Big)\cdot \sup_{d(y,L)\le\varepsilon}|a(y)| \to 0 \end{aligned} \tag{4.5}$$

as $\varepsilon \to 0$, so (4.3) follows.

Since g is uniformly continuous on K there exists some $r > 0$ such that for any p, $x \in K$ with $\|p - x\| < r$ and any $X \in \mathbb{R}^n$ with $\|X\| \leq C$ we have $|g_p(X,X) - g_x(X,X)| < -\kappa$. Now let $p \in K$ and let $X \in \mathbb{R}^n$ be any vector such that $g_p(X,X) \leq \kappa$ and $\|X\| \leq C$. Then on the open ball $B_r(p)$ the constant vector field $x \mapsto X$ (i.e., the map that assigns to each $x \in B_r(p)$ this same vector $X \in \mathbb{R}^n$), which we again denote by X, is g-timelike.

Let

$$\tilde{R}_{jk}(x) := \begin{cases} R_{jk}(x) & \text{for } x \in B_r(p) \\ 0 & \text{otherwise} \end{cases} \tag{4.6}$$

By our assumption and the fact that $\rho \geq 0$ we then have $(\tilde{R}_{jk}X^jX^k) * \rho_\varepsilon \geq 0$ on $\mathbb{R}^n$. Moreover, for $\varepsilon < r$ it follows that $(R_{jk} * \rho_\varepsilon)(p) = (\tilde{R}_{jk} * \rho_\varepsilon)(p)$. Thus for such ε we have

$$\begin{aligned} |R_{\varepsilon jk}(p)X^jX^k - ((\tilde{R}_{jk}X^jX^k) * \rho_\varepsilon)(p)| &= |(R_{\varepsilon jk}(p) - (R_{jk} * \rho_\varepsilon)(p))X^jX^k| \\ &\leq C^2 \sup_{x \in K} |R_{\varepsilon jk}(x) - R_{jk} * \rho_\varepsilon(x)|. \end{aligned} \tag{4.7}$$

Using (4.3) we conclude from this estimate that, given any $\delta > 0$ we may choose ε_0 such that for all $\varepsilon < \varepsilon_0$, all $p \in K$ and all vectors X with $g_p(X,X) \leq \kappa$ and $\|X\| \leq C$ we have $R_{\varepsilon jk}(p)X^jX^k > -\delta$, which is (4.1). □

4.4 Existence of maximal curves

The next key step in proving the main result is to establish the existence of geodesics maximizing the distance to a spacelike hypersurface. To prove this statement we will employ a net $\check{g}_\varepsilon$ ($\varepsilon > 0$) of smooth Lorentzian metrics whose lightcones approximate those of g from the inside as in Proposition 2.2.5. We first need some auxiliary results.

Lemma 4.4.1. *Let (M,g) be a $C^{1,1}$-spacetime that is globally hyperbolic. Let h be a Riemannian metric on M and let $K \Subset M$. Then there exists some $C > 0$ such that the h-length of any causal curve taking values in K is bounded by C.*

Proof. It follows, e.g., from the proof of [ON83, Lemma 14.13] that (M,g) is non-totally imprisoning, i.e., there can be no inextendible causal curve that is entirely contained in K. Now suppose that, contrary to the claim, there exists a sequence σ_k of causal curves valued in K whose h-lengths tend to infinity. Parametrizing σ_k by h-arclength we may assume that $\sigma_k : [0, a_k] \to K$, where $a_k \to \infty$. Also, without loss of generality we may assume that $\sigma_k(0)$ converges to some $q \in K$. Then by point of [M08, Th. 3.1(1)][1] one may extract a subsequence σ_{k_j} that converges locally uniformly to an inextendible causal curve σ in K, thereby obtaining a contradiction to non-total imprisonment. $\square$

Lemma 4.4.2. *Let (M, g) be a globally hyperbolic $C^{1,1}$-spacetime and let g_ε $(\varepsilon > 0)$ be a net of smooth Lorentzian metrics such that g_ε converges locally uniformly to g as $\varepsilon \to 0$, and let $K \Subset M$. Then for each $\delta > 0$ there exists some $\varepsilon_0 > 0$ such that for each $\varepsilon < \varepsilon_0$ and each g-causal curve σ taking values in K, the lengths of σ with respect to g and g_ε, respectively, satisfy:*

$$L_g(\sigma) - \delta < L_{g_\varepsilon}(\sigma) < L_g(\sigma) + \delta. \tag{4.8}$$

Proof. Since $g_\varepsilon \to g$ uniformly on K, given any $\eta > 0$ there exists some $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ and all $X \in TM|_K$ with $\|X\|_h = 1$ we have

$$\|X\|_g - \eta \leq \|X\|_{g_\varepsilon} \leq \|X\|_g + \eta. \tag{4.9}$$

Consequently, for any $X \in TM|_K$ we have

$$\|X\|_g - \eta\|X\|_h \leq \|X\|_{g_\varepsilon} \leq \|X\|_g + \eta\|X\|_h. \tag{4.10}$$

Now if $\sigma : [a, b] \to K$ is any g-causal curve it follows that, for $\varepsilon < \varepsilon_0$,

$$\begin{aligned} L_g(\sigma) - \eta L_h(\sigma) &= \int_a^b \|\sigma'(t)\|_g \, dt - \eta \int_a^b \|\sigma'(t)\|_h \, dt \leq \int_a^b \|\sigma'(t)\|_{g_\varepsilon} \, dt = L_{g_\varepsilon}(\sigma) \\ &\leq L_g(\sigma) + \eta L_h(\sigma). \end{aligned} \tag{4.11}$$

Finally, by Lemma 4.4.1 there exists some $C > 0$ such that $L_h(\sigma) \leq C$ for any σ as above. Hence, picking $\eta < \delta/C$ establishes the claim. $\square$

Proposition 4.4.3. *Let (M, g) be a future timelike-geodesically complete $C^{1,1}$-spacetime. Let S be a compact spacelike acausal hypersurface in M, and let $p \in D^+(S) \setminus S$. Then*

(i) $d_{\check{g}_\varepsilon}(S, p) \to d(S, p) \quad (\varepsilon \to 0)$.

(ii) There exists a timelike geodesic γ perpendicular to S from S to p with $L(\gamma) = d(S, p)$.

[1] Note that the required result remains valid for $C^{1,1}$-metrics (in fact, even for continuous metrics): this follows exactly as in [CG12, Th. 1.6]

Here we have dropped the subscript from the time separation function $d_g(S,p)$ and the length $L_g(\gamma)$ of the $C^{1,1}$-metric g to simplify notations. Also we remark that the proof of (i) below neither uses geodesic completeness of M nor compactness of S and hence the $\check{g}_\varepsilon$-distance converges even on general M for any closed spacelike acausal hypersurface S.

Proof. (i) Since $p \notin S$ we have $c := d(S,p) > 0$. Let $0 < \delta < c$. Then there exists a g-causal curve $\alpha : [0,b] \to M$ from S to p with $L_g(\alpha) > d(S,p) - \delta$. In particular, α is not a null curve, hence there exist $t_1 < t_2$ such that $\alpha|_{[t_1,t_2]}$ is nowhere null. In what follows we adapt the argument from [Chr11, Lemma 2.4.14] to the present situation. Without loss of generality we may assume that $t_2 = b$. By Theorem 1.7.1 we may find $0 = s_0 < s_1 < \cdots < s_N = b$ and totally normal neighborhoods U_i $(1 \le i \le N)$ such that $\alpha([s_i, s_{i+1}]) \subseteq U_i$ for $0 \le i < N$. By Proposition 2.3.14 we obtain that $\alpha(s_{N-1}) \ll \alpha(b)$, hence by Proposition 3.6.23, the radial geodesic σ_N from $\alpha(s_{N-1})$ to p is longer than $\alpha|_{[s_{N-1},b]}$, and it is timelike. Next, we connect $\alpha(s_{N-2})$ via a timelike radial geodesic σ_{N-1} to some point on σ_N that lies in U_{N-1}. Concatenating σ_{N-1} with σ_N gives a timelike curve longer than $\alpha|_{[s_{N-2},b]}$. Iterating this procedure we finally arrive at a timelike piecewise geodesic σ from $\alpha(0) = \sigma(0) \in S$ to p of length $L_g(\sigma) \ge L_g(\alpha) > d(S,p) - \delta$.

Since $L_{\check{g}_\varepsilon}(\sigma) \to L_g(\sigma)$, we conclude that $L_{\check{g}_\varepsilon}(\sigma) > d(S,p) - \delta$ for ε sufficiently small. Moreover, σ is g-timelike and piecewise C^2, hence is $\check{g}_\varepsilon$-timelike for small ε. Therefore, $d_{\check{g}_\varepsilon}(S,p) \ge L_{\check{g}_\varepsilon}(\sigma) > d(S,p) - \delta$ for ε small.

Conversely, if σ is any $\check{g}_\varepsilon$-causal curve from S to p then σ is also g-causal, hence lies entirely in the set $K := J^-(p) \cap J^+(S, D(S))$. Since $D(S)$ is globally hyperbolic by Theorem 3.6.10 and Proposition 3.6.12, K is compact by Corollary 3.6.18. Then by Lemma 4.4.2 (applied to the globally hyperbolic spacetime $(D(S), g)$), for ε sufficiently small we have

$$L_{\check{g}_\varepsilon}(\sigma) < L_g(\sigma) + \delta \le d(S,p) + \delta. \tag{4.12}$$

Consequently, $d_{\check{g}_\varepsilon}(S,p) \le d(S,p) + \delta$ for ε sufficiently small. Together with the above this shows (i).

(ii) Since $\check{g}_\varepsilon$ has narrower lightcones than g, for each ε the point p lies in $D^+_{\check{g}_\varepsilon}(S) \setminus S$. Also, we may assume ε to be so small that S is $\check{g}_\varepsilon$-spacelike as well as $\check{g}_\varepsilon$-acausal. Then by smooth causality theory (e.g., [ON83, Th. 14.44]) there exists a $\check{g}_\varepsilon$-geodesic γ_ε that is $\check{g}_\varepsilon$-perpendicular to S and satisfies $L_{\check{g}_\varepsilon}(\gamma_\varepsilon) = d_{\check{g}_\varepsilon}(S,p)$. Let h be some background Riemannian metric on M and let $\gamma_\varepsilon(0) =: q_\varepsilon \in S$, $\gamma_\varepsilon'(0) =: v_\varepsilon$. Without loss of generality we may suppose $\|v_\varepsilon\|_h = 1$. Since $\{v \in TM \mid \pi(v) \in S,\ \|v\|_h = 1\}$ is compact, there exists a sequence $\varepsilon_j \searrow 0$ such that $q_{\varepsilon_j} \to q \in S$ and $v_{\varepsilon_j} \to v \in T_qM$. Denote by γ_v the g-geodesic with $\gamma(0) = q$, $\gamma'(0) = v$. To see that γ is g-orthogonal to S, let $w \in T_qS$ and pick any sequence $w_j \in T_{q_{\varepsilon_j}}S$ converging to w. Then $g(v, w) = \lim \check{g}_{\varepsilon_k}(v_{\varepsilon_j}, w_j) = 0$. Consequently, γ is g-timelike.

Since g is timelike geodesically complete, γ_v is defined on all of $\mathbb{R}$, so by standard ODE-results (cf., e.g., Section 1.4) for any $a > 0$ there exists some j_0 such that for all $j \geq j_0$ the curve γ_{ε_j} is defined on $[0, a]$ and $\gamma_{\varepsilon_j} \to \gamma$ in $C^1([0, a])$ (in fact, it follows directly from this and the geodesic equation that this convergence even holds in $C^2([0, a])$).

For each j, let $t_j > 0$ be such that $\gamma_{\varepsilon_k}(t_j) = p$. Then by (i) we obtain

$$d(S,p) = \lim d_{\check{g}_{\varepsilon_k}}(S,p) = \lim \int_0^{t_j} \|\gamma'_{\varepsilon_j}(t)\|_{\check{g}_{\varepsilon_k}}\, dt = \lim t_j \|v_{\varepsilon_j}\|_{\check{g}_{\varepsilon_k}} = \|v\|_g \lim t_j, \tag{4.13}$$

so $t_j \to \frac{d(S,p)}{\|v\|_g} =: a$. Finally, for j sufficiently large, all γ_{ε_k} are defined on $[0, 2a]$ and we have $p = \gamma_{\varepsilon_k}(t_j) \to \gamma(a)$, so $p = \gamma(a)$, as well as

$$d(S,p) = \lim \int_0^{t_j} \|\gamma'_{\varepsilon_j}(t)\|_{\check{g}_{\varepsilon_k}}\, dt = \int_0^a \|\gamma'(t)\|_g\, dt = L_0^a(\gamma). \tag{4.14}$$

□

4.5 Proof of the main result

To prove Theorem 4.2.1, we first note that without loss of generality we may assume S to be connected. Moreover, by Theorem 3.6.25 we may also assume S to be achronal, and thereby acausal by Lemma 3.6.19 (replacing, if necessary, M by a suitable Lorentzian covering space $\tilde{M}$ and S by its isometric image $\tilde{S}$ in $\tilde{M}$). Note that since the light cones of $\check{g}_\varepsilon$ approximate those of g from the inside it follows that for ε small S is a spacelike acausal hypersurface with respect to $\check{g}_\varepsilon$ as well.

We prove the theorem by contradiction and assume that (M, g) is future timelike geodesically complete. Hence we may apply Proposition 4.4.3 to obtain (using the notation from the proof of that result) for any $p \in D^+(S) \setminus S$:

(A) $\exists$ g-geodesic $\gamma \perp_g S$ realizing the time separation to p, i.e., $L(\gamma) = d(S, p)$.

(B) $\exists\ \check{g}_\varepsilon$-geodesics $\gamma_\varepsilon \perp_{\check{g}_\varepsilon} S$ realizing the time separation to p, i.e., $L_{\check{g}_\varepsilon}(\gamma_\varepsilon) = d_{\check{g}_\varepsilon}(S,p)$.

(C) $\exists\ \varepsilon_j \searrow 0$ such that $\gamma_{\varepsilon_j} \to \gamma$ in $C^1([0,a])$ for all $a > 0$ (in fact, even in $C^2([0,a])$).

We proceed in several steps.

Step 1. $D^+(S)$ is relatively compact.

The future convergence of S is given by $\mathbf{k} = 1/(n-1)\mathrm{tr}S_U$, with $S_U(V) = -\nabla_V U$ and U the future pointing g-unit normal on S. Analogously, for each ε_j as in (C) we obtain the future convergence $\mathbf{k}_j$ of S with respect $\check{g}_{\varepsilon_k}$, and we denote the future-pointing $\check{g}_{\varepsilon_k}$-unit normal to S and the corresponding shape operator by U_j and S_{U_j}, respectively. By Proposition 2.2.5 (i), $\mathbf{k}_j \to \mathbf{k}$ uniformly on S. Let $m := \min_S \mathrm{tr}S_U = (n-1)\min_S \mathbf{k}$, and $m_j := \min_S \mathrm{tr}S_{U_j} = (n-1)\min_S \mathbf{k}_j$. By assumption, $m > 0$, and by the above we obtain $m_j \to m$ as $j \to \infty$.

Let
$$b := \frac{n-1}{m} \tag{4.15}$$
and assume that there exists some $p \in D^+(S) \setminus S$ with $d(S,p) > b$. We will show that this leads to a contradiction.

Since each γ_{ε_j} as in (C) is maximizing until $p = \gamma_{\varepsilon_j}(t_j)$, it contains no $\check{g}_{\varepsilon_k}$-focal point to S before t_j. Setting $\tilde{t}_j := (1-\frac{1}{j})t_j$ it follows that $\exp^\perp_{\check{g}_\varepsilon}$ is non-singular on $[0,\tilde{t}_j]\gamma'_{\varepsilon_j}(0) = [0,\tilde{t}_j]v_{\varepsilon_j}$. As this set is compact there exist open neighborhoods W_j of $[0,\tilde{t}_j]v_{\varepsilon_j}$ in the normal bundle $N_{\check{g}_{\varepsilon_k}}(S)$ and V_j of $\gamma_{\varepsilon_j}([0,\tilde{t}_j])$ in M such that $\exp^\perp_{\check{g}_{\varepsilon_k}} : W_j \to V_j$ is a diffeomorphism. Due to $D_{\check{g}_{\varepsilon_k}}(S)$ being open, we may also assume that $V_j \subseteq D_{\check{g}_{\varepsilon_k}}(S)$.

On V_j we introduce the Lorentzian distance function $r_j := d_{\check{g}_{\varepsilon_k}}(S,.)$ and set $X_j := -\mathrm{grad}(r_j)$. Denote by $\tilde{\gamma}_j$ the re-parametrization of γ_{ε_j} by $\check{g}_{\varepsilon_k}$-arc length:

$$\tilde{\gamma}_j : [0, \tilde{t}_j \|v_{\varepsilon_j}\|_{\check{g}_{\varepsilon_k}}] \to M \quad \tilde{\gamma}_j(t) := \gamma_{\varepsilon_j}(t/\|v_{\varepsilon_j}\|_{\check{g}_{\varepsilon_k}}). \tag{4.16}$$

Then since $\tilde{\gamma}_j$ is maximizing from S to p in $D^+_{\check{g}_{\varepsilon_k}}(S)$, hence in particular in $V_j \cap J^+_{\check{g}_{\varepsilon_k}}(S)$, it follows that $X_j(\tilde{\gamma}_j(t)) = \tilde{\gamma}'_j(t)$ for all $t \in [0, \tilde{t}_j\|v_{\varepsilon_j}\|_{\check{g}_{\varepsilon_k}}]$. Next we define the shape operator corresponding to the distance function r_j by $S_{r_j}(Y) := \nabla^{\check{g}_{\varepsilon_k}}_Y(\mathrm{grad}(r_j))$ for $Y \in \mathfrak{X}(V_j)$. Then $S_{r_j}|_{S\cap V_j} = S_{U_j}|_{S\cap V_j}$ and the expansion $\tilde{\theta}_j := -\mathrm{tr}S_{r_j}$ satisfies the Raychaudhuri equation (cf., e.g., [Nat05])
$$X_j(\tilde{\theta}_j) + \mathrm{tr}(S^2_{r_j}) + \mathrm{Ric}_{\check{g}_{\varepsilon_k}}(X_j, X_j) = 0 \tag{4.17}$$

on V_j. Consequently, we obtain for $\theta_j(t) := \tilde\theta_j \circ \tilde\gamma_j(t)$:

$$\frac{d(\theta_j^{-1})}{dt} \geq \frac{1}{n-1} + \frac{1}{\theta_j^2}\mathrm{Ric}_{\check g_{\varepsilon_k}}(\tilde\gamma_j', \tilde\gamma_j'). \tag{4.18}$$

Now since by (C) the $\tilde\gamma_j$ converge in C^1 to the g-timelike geodesic γ, it follows that there exist $\kappa < 0$ and $C > 0$ such that for all j sufficiently large we have $g(\tilde\gamma_j'(t), \tilde\gamma_j'(t)) \leq \kappa$ as well as $\|\tilde\gamma_j'(t)\|_h \leq C$ for all $t \in [0, \tilde t_j \|v_{\varepsilon_j}\|_{\check g_{\varepsilon_k}}]$.

We are therefore in the position to apply Lemma 4.3.1 to obtain that, for any $\delta > 0$,

$$\frac{d(\theta_j^{-1})}{dt} > \frac{1}{n-1} - \frac{\delta}{\theta_j^2} \tag{4.19}$$

for j large enough. Pick any c with $b < c < d(S,p)$ and fix $\delta > 0$ so small that

$$b < \frac{n-1}{\alpha m} < c, \tag{4.20}$$

where m is as in (4.15) and $\alpha := 1 - (n-1)m^{-2}\delta$. Analogously, let $\alpha_j := 1 - (n-1)m_j^{-2}\delta$, so that $\alpha_j \to \alpha$ as $j \to \infty$. Setting $d_j := \tilde t_j \|v_{\varepsilon_j}\|_{\check g_{\varepsilon_k}}$, θ_j is defined on $[0, d_j]$. Note that, for j large, (4.20) implies the right hand side of (4.19) to be strictly positive at $t = 0$. Thus θ_j^{-1} is initially strictly increasing and $\theta_j(0) < 0$, so (4.19) entails that $\theta_j^{-1}(t) \in [-m_j^{-1}, 0)$ on its entire domain. From this we conclude that θ_j has no zero on $[0, d_j]$, i.e., that θ_j^{-1} exists on all of $[0, d_j]$. It then readily follows, again using (4.19), that $\theta_j^{-1}(t) \geq f_j(t) := -m_j^{-1} + t\frac{\alpha_j}{n-1}$ on $[0, d_j]$. Hence θ_j^{-1} must go to zero before f_j does, i.e., $\theta_j^{-1}(t) \to 0$ as $t \nearrow T$ for some positive $T \leq \frac{n-1}{\alpha_j m_j}$.

Here we note that due to $\lim d_j = \lim t_j \|v_{\varepsilon_j}\|_{\check g_{\varepsilon_k}} = d(S,p)$, for j sufficiently large we have by (4.20)

$$\frac{n-1}{\alpha_j m_j} < c < d_j. \tag{4.21}$$

This, however, means that $\theta_j^{-1} \to 0$ within $[0, d_j]$, contradicting the fact that θ_j is smooth, hence bounded, on this entire interval.

Together with (A) this implies that $D^+(S)$ is contained in the compact set $\beta(S \times [0,b])$ where

$$\beta : \ S \times [0,b] \to M, \quad (q,t) \mapsto \exp^g(t\,U(q)), \tag{4.22}$$

Hence also the future Cauchy horizon $H^+(S) = \overline{D^+(S)} \setminus I^-(D^+(S))$ is compact.

From here, employing the causality results developed in Chapter 2 and Chapter 3 we may conclude the proof exactly as in [ON83, Th. 14.55B]. For completeness, we give the full argument.

Step 2. The future Cauchy horizon of S is nonempty.

Assume to the contrary that $H^+(S) = \emptyset$. Then $I^+(S) \subseteq D^+(S)$: for $p \in S$, a future-directed timelike curve γ starting at p lies initially in $D^+(S)$ (using Proposition 3.6.12, or Lemma 3.6.15). Hence if γ leaves $D^+(S)$, it must meet $\partial D^+(S)$ and by Lemma 3.6.9 it also meets $H^+(S)$ (since S is achronal it can't intersect S again). But then $H^+(S)$ wouldn't be empty, contrary to our assumption. Hence $I^+(S) \subseteq D^+(S)$. By Step 1, then, $I^+(S) \subseteq \{p \in M \mid d(S,p) \leq b\}$ and hence $L(\gamma) \leq b$ for any timelike future-directed curve emanating from S, which is a contradiction to timelike geodesic completeness of M.

Step 3. The following extension of (A) holds:

(A') $\forall\, q \in H^+(S)\ \exists\ g$-geodesic $\gamma \perp_g S$ realizing the time separation and $L(\gamma) = d(S,q) \leq b$.

Consider the set $B \subseteq N(S)$ consisting of the zero section and all future pointing causal vectors v with $\|v\| \leq b$. B is compact by the compactness of S.

By definition there is a sequence q_k in $D^+(S)$ that converges to q. For any q_k there is a geodesic as in (A) and hence a vector $v_k \in B$ with $\exp_p(v_k) = q_k$. By the compactness of B we may assume that $v_k \to v$ for some $v \in B$ and hence by continuity $q_k \to \exp_p(v)$. Moreover, we have by construction that $\|v_k\| = d(S, q_k)$. Since d is lower semicontinuous (Lemma 3.2.4), $\|v\| \geq d(S,q)$.

As γ_v is perpendicular to S, hence timelike, our completeness assumption implies that it is defined on $[0,1]$. Thus it runs from S to q and has length $\|v\|$, which implies $d(S,q) = \|v\| \leq b$.

Step 4. The map $p \mapsto d(S,p)$ is strictly decreasing along past pointing generators of $H^+(S)$.

By Proposition 3.6.14 (iii), $H^+(S)$ is generated by past-pointing inextendible null geodesics. Suppose that $\alpha : I \to M$ is such a generator, and let $s, t \in I$, $s < t$. Using (A') we obtain a past pointing timelike geodesic γ from $\alpha(t)$ to $\gamma(0) \in S$ of length $d(S, \alpha(t))$. Then arguing as in the proof of Proposition 4.4.3 (i) we may construct a timelike curve σ from $\alpha(s)$ to $\gamma(0)$ that is strictly longer than the concatenation of $\alpha|_{[s,t]}$ and γ. Therefore,

$$d(S, \alpha(s)) \geq L(\sigma) > L(\alpha|_{[s,t]} + \gamma) = L(\gamma) = d(S, \alpha(t)). \tag{4.23}$$

Step 5. (M, g) is *not* future timelike geodesically complete.

By step 1, $H^+(S)$ is compact and by Lemma 3.2.4 $p \mapsto d(S, p)$ is lower semicontinuous, hence attains a finite minimum at some point q in $H^+(S)$. But then taking a past pointing generator of $H^+(S)$ emanating from q according to Proposition 3.6.14 (iii) gives a contradiction to step 4. □

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Abstract

This thesis studies the causality theory with low differentiability metrics, in particular, with metrics of $C^{1,1}$ regularity class. One of the key tools for studying local causality and therefore, singularity theory is the exponential map. In smooth pseudo-Riemannian geometry, the fact that the exponential map is a local diffeomorphism is of central importance for many fundamental constructions such as normal coordinates, normal neighborhoods, injectivity radius and comparison methods. There has for some time been considerable interest in determining the lowest degree of differentiability where one could expect the standard results of causality theory to remain valid. A reasonable candidate is given by the $C^{1,1}$ regularity class as it represents the threshold of the unique solvability of the geodesic equation.

Hence our aim is to show that the exponential map of a $C^{1,1}$ pseudo-Riemannian metric retains its maximal possible regularity, namely, that is a local bi-Lipschitz homeomorphism. This will allow us to prove the existence of totally normal neighborhoods and establish the key results of local causality theory.

The next goal is to further develop causality theory for $C^{1,1}$ metrics. We also study the global structure of spacetimes, reviewing the causality conditions that can be imposed on a spacetime and the main properties of Cauchy developments and Cauchy horizons.

The last part is devoted to the study of singularity theorems. Having the key elements of causality theory for $C^{1,1}$ metrics developed, we prove the Hawking singularity theorem in this regularity.

Kurzfassung

Diese Dissertation befasst sich mit der Kausalitätstheorie von Metriken niedriger Differenzierbarkeitsklasse, insbesondere solchen der Klasse $C^{1,1}$.

Eines der Hauptwerkzeuge um lokale Kausalität und damit Singularitätentheorie zu studieren ist die Exponentialabbildung. In der pseudo-Riemannschen Geometrie mit glatter Metrik ist die Exponentialabbildung ein lokaler Diffeomorphismus, was von zentraler Bedeutung für viele grundlegende Konstruktionen wie Normalkoordinaten, normale Umgebungen, Injektivitätsradius und Vergleichssätze ist. In jüngster Zeit gab es verstärkte Bemühungen, die niedrigste Differenzierbarkeitsklasse zu finden, für welche die üblichen Ergebnisse der Kausalitätstheorie noch gültig sind. Ein plausibler Kandidat dafür ist die Klasse $C^{1,1}$, welche die Grenze darstellt für die die Geodätengleichung noch eindeutig lösbar ist.

Das erste Ziel dieser Arbeit ist deshalb zu zeigen, dass die Exponentialabbildung einer $C^{1,1}$ pseudo-Riemannschen Metrik so regulär wie möglich ist, d.h. dass sie ein lokaler bi-Lipschitz-Homöomorphismus ist. Damit zeigen wir die Existenz von total normalen Umgebungen und erhalten zentrale Aussagen der lokalen Kausalitätstheorie in diesem Kontext.

Das nächste Ziel ist die Weiterentwicklung der Kausalitätstheorie für $C^{1,1}$-Metriken. Dabei untersuchen wir die globale Struktur von Raumzeiten und betrachten mögliche Kausalitätsbedingungen und die wichtigsten Eigenschaften von Cauchy-Entwicklungen und Cauchy-Horizonten.

Der letzte Teil ist dem Studium von Singularitätentheoremen gewidmet. Mit den zuvor entwickelten Grundbausteinen der Kausalitätstheorie für $C^{1,1}$-Metriken beweisen wir das Singularitätentheorem von Hawking für diese Regularitätsklasse.

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