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Gino Angelo Velasco

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Abstract

The purpose of this research is to study time-frequency localized functions, the sampling, approximation and reconstruction of such functions in the Gabor setting, and to construct some adaptive transforms that can be applied to audio signal processing. We first investigate functions that satisfy some localization in a region in the time-frequency plane using the tools from time-frequency localization operators. We characterize a function's concentration in a region in the time-frequency plane and compare some measures of localization. We then consider the approximation of time-frequency localized functions using a local Gabor system. We obtain approximation estimates in terms of a time-frequency localization measure. We show that if a function lies on a subspace generated by eigenfunctions of a time-frequency localization operator on a bounded region, we can choose an enlargement of the region such that the local Gabor system over time-frequency points on the larger region satisfies a frame-like inequality on the subspace. This would allow for the construction of a time-frequency dictionary consisting of functions that are maximally concentrated in the region, and a family of these dictionaries forming a global frame. We also study the random sampling of functions that are localized in a region in the time-frequency plane. We determine the probability that a sampling inequality holds for time-frequency localized functions using sampling points in the region of concentration. Lastly, we present two adaptive time-frequency based transforms - via time-frequency localized subspaces and via nonstationary Gabor frames, and present their advantages in audio signal processing. We show that applying an approximate projection onto the time-frequency localized subspaces exhibits a reduction in the error in reconstructing a signal from the corresponding analysis coefficients. For non-stationary Gabor frames, we show that perfect reconstruction is easily realizable in the painless case, and we illustrate its applications in signal processing, e.g. obtaining an invertible constant-Q transform.

Zusammenfassung

Die vorliegende Arbeit befasst sich mit dem Studium Zeit-Frequenz-lokalisierter Funktionen, insbesondere deren Abtastung, Approximation und Rekonstruktion im Gabor Kontext, sowie der Konstruktion adaptiver Transformationen und deren Verwendung zur Verarbeitung von Audiosignalen. Zeit-Frequenz-Lokalisierungsoperatoren liefern uns Werkzeuge zur Beschreibung der Konzentration, beziehungsweise Lokalisation, von Funktionen in einer gewissen Region der Zeit-Frequenz-Ebene. Wir charakterisieren die Konzentration einer Funktion in solchen Regionen und vergleichen unterschiedliche Lokalisationsmaße, um schliesslich die Approximation Zeit-Frequenz-lokalisierter Funktionen durch lokale Gaborsysteme zu untersuchen. In einem geeigneten Maß der Zeit-Frequenz-Lokalisation knnen wir den Approximationsfehler nach oben abschtzen. Insbesondere zeigen wir, dass Funktionen, welche in einem von den Eigenfunktionen eines Lokalisierungsoperators generierten Unterraum liegen, in folgendem Sinne durch lokale Gaborsysteme beschrieben werden knnen: Vergrern wir die durch den Lokalisierungsoperator beschriebene Zeit-Frequenz-Region ausreichend, so erfllt das auf diese vergrerte Region eingeschrnkte lokale Gaborsystem eine Art Frame-Ungleichung fr Funktionen im von den Eigenfunktionen aufgespannten Unterraum. Dies erlaubt die Konstruktion von Zeit-Frequenz-Systemen aus, bezglich einer bestimmten Region, maximal konzentrierten Funktionen, so dass eine Familie solcher Funktionen ein globales Frame ist. Weiters untersuchen wir die zullige Abtastung Zeit-Frequenz-lokalisierter Funktionen und bestimmen die Wahrscheinlichkeit, mit welcher fr solche lokalisierten Funktionen eine Sampling-Ungleichung gilt, abhngig von der Region der Lokalisierung. Auerdem prsentieren wir zwei Methoden zur Konstruktion adaptiver, Zeit-Frequenz-basierter Transformationen durch (a) Zeit-Frequenz-lokalisierte Unterrume und (b) nichtstationre Gaborsysteme, sowie ihre Vorteile im Kontext der Audiosignalverarbeitung. Wir zeigen, dass nherungsweise Projektion auf Zeit-Frequenz-lokalisierte Unterrume zu einer Verminderung des Rekonstruktionsfehlers bezglich der zugehrigen Analysekoeffizienten fhrt. Auerdem diskutieren wir die Konstruktion nichtstationrer Gaborsysteme mit fehlerloser Rekonstruktion und ihre Anwendung in der Signalverarbeitung, unter anderem am Beispiel einer invertierbaren Constant-Q Transformation.

Introduction

Background and overview

Time-frequency localization is an ongoing active topic of research in harmonic analysis. While it is well known that no nontrivial function can be compactly supported simultaneously in time and frequency, functions that exhibit some localization in a compact region in the time-frequency plane have been studied using operators which localize a function's time-frequency content on bounded regions in the time-frequency plane. Landau, Slepian, and Pollak considered operators composed of consecutive time- and band-limiting steps, cf. [87, 73, 74], that yield the well-known prolate spheroidal functions as eigenfunctions. These functions satisfy some optimality in concentration in a rectangular region in the time-frequency domain.

In [25], Daubechies introduced time-frequency localization operators obtained by restricting the integral in the inversion formula to a subset of $\mathbb{R}^2$. The eigenfunctions and eigenvalues of these operators have been studied in [82, 47, 29]. The study of the properties of time-frequency operators and its connection with other mathematical topics have been a continued topic of research, e.g. [95, 23, 1, 57].

We make use of time-frequency localization operators to describe a function's local time-frequency content, and we compare and relate this measure of time-frequency concentration with measures that use a sharp cutoff in time and in frequency. As in the case of the prolate spheroidal wave functions, the eigenfunctions of time-frequency localization operators are somehow maximally concentrated in time-frequency in the region being considered. We investigate how a function that satisfies a localization criterion can be characterized by these eigenfunctions. Since the eigenfunctions are optimally concentrated in the region, we also show how a given time-frequency localized function can be approximated by its projection onto a subspace spanned by a finite number of these eigenfunctions. Such projection onto a time-frequency localized subspace is comparable to the method of time-varying filtering using Wigner distribution synthesis techniques, cf. [60].

Using Gabor frames, we also obtain local representations of time-frequency localized functions. That is, we approximate such functions using a local Gabor sampling set, namely, those which are inside some larger cover of the given region. This is influenced by the approximation result formulated by Daubechies in a seminal paper [26]. Similar estimates were also established in [80]. In contrast, the truncated Gabor expansions we use are over more general regions, i.e. not just a rectangular region.

Moreover, the error bounds to be used are in terms of time-frequency localization measure.

Sampling and time-frequency localization via time- and band-limiting operators were studied in [90, 63]. We likewise consider the approximation of time-frequency localized functions via eigenfunctions of a time-frequency localization operator from the local samples.

In [9], the problem of random sampling of band-limited functions was considered and the probability of obtaining a sampling inequality for band-limited functions was estimated. We also investigate the random sampling of time-frequency localized functions and estimate the probability that a sampling inequality holds involving only the relevant samples.

Families of time-frequency localization operators and coverings of the time-frequency plane can be used to construct global frames that satisfy some local properties, cf. [36, 39, 40]. Projection of the local Gabor system onto a subspace spanned by a finite number eigenfunctions of a time-frequency localization operator allows a construction of a time-frequency dictionary consisting of functions that are maximally concentrated in the region. By taking a family of such time-frequency dictionaries, we are also able to obtain an global adaptive frame for $L^2(\mathbb{R})$.

Adaptive time-frequency representations provide an alternative to the fixed resolution inherent in Gabor frames. Such works on adaptive representations include designing building blocks that adapt to specific signals, or having more flexible tilings of the time-frequency plane that yield different resolutions in different areas of the plane, cf. [94, 69, 34] among others.

Using the local Gabor systems and approximate projections onto the time-frequency localized subspaces, we obtain adaptive analysis-synthesis systems, cf. [41], that would provide the desired resolution in various frequency bands and at the same time provide arbitrarily good reconstruction quality. Numerical results will be compared to an adaptive method proposed in [77].

We also present the work on nonstationary Gabor frames [89, 7], that allow for adaptivity in time or in frequency. We show that for the painless case, the transform is perfectly invertible. We then show how the nonstationary Gabor frames can be used in signal processing applications, e.g. obtaining an invertible constant-Q transform.

Structure and contributions

Chapter 1 recalls some basic concepts on Fourier, time-frequency, and Gabor analysis. In Chapter 2, we review and summarize some aspects of time-frequency localization and prove some new observations. Chapter 3 deals with the approximation of time-frequency localized functions, wherein we obtain some theoretical estimates, and illustrate the results with numerical examples. Chapter 4 is concerned with the local random sampling of the STFT of time-frequency localized functions. In Chapter 5, we present two adaptive time-frequency representations, via approximate projections onto subspaces of eigenfunctions and via nonstationary Gabor frames, in a theoretical setting and with numerical experiments.

The following is a summary of the contributions from this research work:

- We use a time-frequency localization operator in measuring a function's local time-frequency content in a bounded region Ω in the time-frequency plane and prove a result that relates the concentration of f and $\hat{f}$ on intervals with the time-frequency concentration of f on a rectangle (Section 2.3.1).
- We prove a characterization of the function's time-frequency concentration Ω using eigenfunctions and eigenvalues of a time-frequency localization operator on Ω (Section 2.3.2).
- We obtain an approximation of a function localized in Ω by its projection onto the subspace generated by a finite number eigenfunctions of a time-frequency localization operator on Ω and obtain error bounds in terms of the time-frequency concentration of f in Ω (Section 2.3.2).
- We obtain an approximation of a function localized in Ω using a local Gabor system and obtain error bounds in terms of the function's time-frequency concentration in Ω (Section 3.1).
- We show that if a function lies on a subspace generated by eigenfunctions of a time-frequency localization operator, we can choose an enlargement of Ω such that the local Gabor system over time-frequency points on the larger region satisfies a frame-like inequality on the subspace (Section 3.2). We also present numerical results that exhibit the dependence of the approximation error on the enlargement of the region and the number of sampling points, and the performance of a reconstruction algorithm (Section 3.4).
- By projecting the local Gabor system onto a subspace spanned by a finite number eigenfunctions of a time-frequency localization operator, we obtain a time-frequency dictionary consisting of functions that are maximally concentrated in the region. And by considering a family of such time-frequency dictionaries such that the union of the regions cover the time-frequency plane, we are also able to obtain an adaptive frame for the whole $L^2(\mathbb{R})$ (Section 3.3). We also present numerical results that show the dependence of the condition numbers of the resulting frame operators from the time-frequency dictionary on the amount of overlap used between adjacent regions, and compare them with frame operators arising from quilted Gabor frames [34] (Section 3.4).

- We study the random sampling of time-frequency localized functions and estimate the probability that a sampling inequality holds using only the relevant samples from the region of concentration, analogous to the results in [9, 10] (Chapter 4).
- We make use of time-frequency localization operators corresponding to the desired partition into frequency bands. At the same time, the windows can also change over time as desired. Using approximate projections onto subspaces generated by the eigenvectors which are best-concentrated in each frequency band and time interval, we naturally obtain a smooth transition between adjacent frequency bands and time intervals. We provide some of numerical results in comparison to the method proposed in [77] (Section 5.1). This work has been published in [41]
- We obtain an adaptive time-frequency representation via nonstationary Gabor frames, and show invertibility in the painless case. We also present efficient implementations of the nonstationary Gabor transform in automatic adaptation to transients and in the construction of an invertible constant-Q transform (Section 5.2). This work has been published in [89, 7]

CHAPTER 1

Preliminaries

In this chapter, we recall some basic definitions and concepts that will be used in the succeeding chapters. We review the basics of Fourier analysis, the short-time Fourier transform, and Gabor theory.

For a function f in $\mathbb{R}^d$, $\int_{\mathbb{R}^d} f(t)\,dx$ is the usual Lebesgue integral on $\mathbb{R}^d$. For a measurable set $E \subseteq \mathbb{R}^d$, its measure is $|E| = \int_{\mathbb{R}^d} \chi_E(x)\,dx$, where χ_E denotes the characteristic function on E. If $p \in [1,\infty)$, the integral

$$\left(\int_{\mathbb{R}^d} |f(x)|^p\,dx\right)^{\frac{1}{p}}$$

is the L^p-norm, $\|f\|_p$, of f, and $L^p(\mathbb{R}^d)$, or simply L^p is the Banach space of all measurable functions such that $\|f\|_p < \infty$. If $p = \infty$, the space $L^\infty(\mathbb{R}^d)$ consists of essentially bounded measurable functions. Here, we take $\|f\|_\infty = \text{ess sup}_{x\in\mathbb{R}^d}|f(x)|$.

The space $L^1(\mathbb{R}^d)$ is also referred to as the space of integrable functions, while $L^2(\mathbb{R}^d)$ is referred to as the space of square integrable functions. Moreover, $L^2(\mathbb{R}^d)$ is a Hilbert space with inner product

$$\langle f, g\rangle = \int_{\mathbb{R}^d} f(x)\overline{g(x)}\,dx.$$

A Banach space is said to be *separable* if it contains a countable dense subset. The spaces $L^p(\mathbb{R}^d)$ are separable for any $p \in [1,\infty)$.

Given a collection $\{g_k\}_{k\in\mathbb{N}}$ in a Banach space and scalars $\{c_k\}_{k\in\mathbb{N}}$, we say the series

$$f = \sum_{k\in\mathbb{N}} c_k\, g_k = \sum_{k=1}^{\infty} c_k\, g_k$$

converges and is equal to f if

$$\left\| f - \sum_{k=1}^{N} c_k\, g_k \right\| \longrightarrow 0 \qquad \text{as } N \to \infty.$$

We note that the ordering of the terms is important; if the order of indices is changed, we are not guaranteed the convergence of the series. If the convergence of the series does not depend on the order of the terms, then it is called *unconditionally convergence*, otherwise it is called *conditional convergence*.

1.1. Fourier transform

We recall in this section some basic definitions and properties of the Fourier transform. For more details and proofs, one may refer to [54].

The *Fourier transform* of a function $f \in L^1(\mathbb{R}^d)$ is defined as

$$\mathcal{F}f(\omega) = \hat{f}(\omega) = \int_{\mathbb{R}^d} f(t)\, e^{-2\pi\omega\cdot t}\, dt, \;\; \omega \in \mathbb{R}^d. \tag{1.1}$$

Lemma 1.1 (Riemann-Lebesgue). *If $f \in L^1(\mathbb{R}^d)$, then $\hat{f}$ is uniformly continuous and $\lim_{|\omega|\to\infty} |\hat{f}(\omega)| = 0$.*

If we let $\mathcal{C}_0(\mathbb{R}^d)$ denote the Banach space of continuous functions vanishing at infinity, then from Lemma 1.1, we get the following mapping property of the Fourier transform:

$$\mathcal{F} \, : \, L^1(\mathbb{R}^d) \longrightarrow \mathcal{C}_0(\mathbb{R}^d).$$

Theorem 1.2 (Plancherel). *If $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, then*

$$\|f\|_2 = \|\hat{f}\|_2. \tag{1.2}$$

As a consequence of Theorem 1.2, $\mathcal{F}$ extends to a unitary operator on $L^2(\mathbb{R}^d)$ and satisfies *Parseval's formula*

$$\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle, \quad \text{for all } f,\, g \in L^2(\mathbb{R}^d). \tag{1.3}$$

An interpretation of Plancherel's theorem in signal analysis is that the Fourier transform is an energy-preserving transform.

Theorem 1.3 (Hausdorff-Young). *Let $1 \le p \le 2$ and let p' be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Then $\mathcal{F} \, : \, L^p(\mathbb{R}^d) \to L^{p'}(\mathbb{R}^d)$ and $\|\hat{f}\|_{p'} \le \|f\|_p$.*

The *convolution* of two functions $f,\, g \in L^1(\mathbb{R}^d)$ is the function $f * g$ defined by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(y)g(x-y)\, dy. \tag{1.4}$$

The norm and Fourier transform of a convolution of two functions satisfy the following properties:

$$\|f * g\|_1 \le \|f\|_1 \|g\|_1; \tag{1.5}$$

$$\widehat{(f * g)} = \hat{f} \cdot \hat{g} \qquad \text{and} \quad \widehat{(f \cdot g)} = \hat{f} * \hat{g} \tag{1.6}$$

Theorem 1.4 (Young). *If $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ and $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, then $f * g \in L^p(\mathbb{R}^d)$ and*

$$\|f * g\|_r \le (A_p A_q A_{r'})^d \|f\|_p \|g\|_q, \tag{1.7}$$

where $A_p = \left(\frac{p^{1/p}}{p'^{1/p'}} \right)^{1/2}$.

The *involution* f^* of f is defined by

$$f^*(x) = \overline{f(-x)} \tag{1.8}$$

and the *reflection operator* $\mathcal{I}$ by

$$\mathcal{I}f(x) = f(-x). \tag{1.9}$$

It follows that

$$\widehat{f^*} = \overline{\hat{f}} \qquad \text{and} \quad \widehat{\mathcal{I}f} = \mathcal{I}\hat{f}. \tag{1.10}$$

Using this notation, the convolution operator can be written as

$$(f * g)(x) = \langle f, \mathbf{T}_x g^* \rangle, \tag{1.11}$$

if both sides are defined.

Theorem 1.5 (Inversion formula)**.** *If* $f \in L^1(\mathbb{R}^d)$ *and* $\hat{f} \in L^1(\mathbb{R}^d)$*, then*

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\omega) e^{2\pi i x \cdot \omega} d\omega, \quad \textit{for all } x \in \mathbb{R}^d. \tag{1.12}$$

We consider two operators that occur frequently in the study of time- and frequency-representations of signals. Suppose T and F are subsets of $\mathbb{R}^d$ with finite measure. The *time-limiting* operator $\mathbf{Q}_T$ is given by

$$\mathbf{Q}_T f(t) = \chi_T(t) f(t), \tag{1.13}$$

while the *band-limiting* operator $\mathbf{Q}_F$ is given by

$$\mathbf{P}_F f(t) = \int_F \hat{f}(\omega) e^{2\pi i \omega \cdot t}\, d\omega = \mathcal{F}^{-1}(\chi_F \hat{f})(t). \tag{1.14}$$

We note that both operators are orthogonal projections on $L^2(\mathbb{R}^d)$. We say that f is *time-limited* to T if $f(t) = \mathbf{Q}_T f(t)$. Similarly, we say that f is *band-limited* to F if $f(t) = \mathbf{P}_F f(t)$.

1.2. The short-time Fourier transform

In this section, we discuss the short-time Fourier transform and mention some of its properties. The transform is designed to represent a function combining the time and frequency information at the same-time. The book [55] provides an excellent reference to the area of time-frequency (TF) analysis using the short-time Fourier transform as a main tool.

Before we define the short-time Fourier transform, we would need the following operators. The *translation* and *modulation* operators are given by

$$\mathbf{T}_x f(t) = f(t-x) \qquad \text{and} \qquad \mathbf{M}_\omega f(t) = f(t)\, e^{2\pi i \omega \cdot t}, \tag{1.15}$$

where $t, x, \omega \in \mathbb{R}^d$. Together, they form a *time-frequency shift operator* $\pi(z)$, $z = (x, w) \in \mathbb{R}^{2d}$, given by

$$\pi(z)f(t) = f(t-x)\, e^{2\pi i\omega\cdot t}. \tag{1.16}$$

Definition 1.6. Let φ be a non-zero function in $L^2(\mathbb{R}^d)$, called the *window function*. The *short-time Fourier transform* (STFT) of a function f with respect to φ is defined as

$$\mathcal{V}_\varphi f(x,\omega) = \int_{\mathbb{R}^d} f(t)\, \overline{\varphi(t-x)}\, e^{-2\pi i t\cdot\omega} dt, \tag{1.17}$$

where $x, \omega \in \mathbb{R}^d$.

The following lemma shows some equivalent forms of the STFT.

Lemma 1.7. *If f, $\varphi \in L^2(\mathbb{R}^d)$, then the STFT is uniformly continuous on $\mathbb{R}^{2d}$ and*

$$\begin{aligned}\mathcal{V}_\varphi f(x,\omega) &= \mathcal{F}(f\cdot \mathbf{T}_x\overline{\varphi})(\omega) \qquad (1.18)\\ &= \langle f, \mathbf{M}_\omega \mathbf{T}_x\varphi\rangle = \langle f, \pi(z)\varphi\rangle \\ &= \langle \hat{f}, \mathbf{T}_\omega \mathbf{M}_{-x}\hat{\varphi}\rangle\end{aligned}$$

Remark 1.8. Expressing the STFT in the form $\mathcal{V}_\varphi f(z) = \langle f, \pi(z)\varphi\rangle$ is useful in extending it for f lying in a Banach space $\boldsymbol{B}$, where $\langle\cdot,\cdot\rangle$ is defined by some form of duality. For instance, $\mathcal{V}_\varphi$ is well defined for all f in the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$, provided that φ is in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$.

Lemma 1.9 (Covariance Property)**.** *Whenever $\mathcal{V}_\varphi f$ is defined, we have*

$$\mathcal{V}_\varphi(\mathbf{T}_u\mathbf{M}_\eta f)(x,\omega) = e^{-2\pi i u\cdot\omega}\mathcal{V}_\varphi f(x-u, \omega-\eta) \tag{1.19}$$

for $x, u, \omega, \eta \in \mathbb{R}^{2d}$. In particular,

$$|\mathcal{V}_\varphi(\mathbf{T}_u\mathbf{M}_\eta f)(x,\omega)| = |\mathcal{V}_\varphi f(x-u, \omega-\eta)|. \tag{1.20}$$

Theorem 1.10 (Orthogonality relations for STFT)**.** *Let $f_1, f_2, \varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$. Then $\mathcal{V}_{\varphi_j} f_j \in L^2(\mathbb{R}^{2d})$ for $j = 1, 2$, and*

$$\langle \mathcal{V}_{\varphi_1} f_1, \mathcal{V}_{\varphi_2} f_2\rangle_{L^2(\mathbb{R}^{2d})} = \langle f_1, f_2\rangle \overline{\langle \varphi_1, \varphi_2\rangle}.$$

Corollary 1.11. *If $f, \varphi \in L^2(\mathbb{R}^d)$, then*

$$\|\mathcal{V}_\varphi f\|_2 = \|f\|_2\|\varphi\|_2.$$

In particular, if $\|\varphi\|_2 = 1$, then

$$\|f\|_2 = \|\mathcal{V}_\varphi f\|_2, \quad \textit{for all } f \in L^2(\mathbb{R}^d). \tag{1.21}$$

In this case, the STFT is an isometry from $L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^{2d})$.

Corollary 1.12 (Inversion formula for the STFT)**.** *Suppose that $\varphi, \gamma \in L^2(\mathbb{R}^d)$ and $\langle \varphi, \gamma\rangle \neq 0$. Then for all $f \in L^2(\mathbb{R}^d)$*

$$f = \frac{1}{\langle \gamma, \varphi\rangle} \iint_{\mathbb{R}^d} \mathcal{V}_\varphi f(z)\, \pi(z)\gamma dz. \tag{1.22}$$

The inversion formula for the STFT is well defined in the weak sense for all $f \in L^2(\mathbb{R}^d)$ for windows γ, φ with $\langle \gamma, \varphi \rangle \neq 0$. Weisz [91] proved the convergence of the Riemann sums of the inverse STFT of f under some conditions in the window functions.

Given a non-zero window φ and a function F on $\mathbb{R}^{2d}$. The formal adjoint $\mathcal{V}_\varphi^*$ of $\mathcal{V}_\varphi$ is given by

$$\mathcal{V}_\varphi^* F = \iint_{\mathbb{R}^{2d}} F(z)\, \pi(z)\varphi \, dz, \tag{1.23}$$

where the integral is defined weakly by

$$\begin{aligned}
\langle \mathcal{V}_\varphi^* F, f \rangle &= \iint_{\mathbb{R}^{2d}} F(z) \langle \pi(z)\varphi, f \rangle \, dz \\
&= \iint_{\mathbb{R}^{2d}} F(z) \overline{\mathcal{V}_\varphi f(z)} \, dz \\
&= \langle F, \mathcal{V}_\varphi f \rangle.
\end{aligned}$$

We have the following pointwise estimate on the STFT of $\mathcal{V}_\varphi^* F$:

Lemma 1.13. [55, Proposition 11.3.2] *Let φ be a non-zero window function and let F be a function on $\mathbb{R}^{2d}$. Then*

$$|\mathcal{V}_\varphi(\mathcal{V}_\varphi^* F)(x, \omega)| \leq (|\mathcal{V}_\varphi \varphi| * |F|)(x, \omega), \tag{1.24}$$

for all $(z) \in \mathbb{R}^{2d}$.

Proof:

$$\begin{aligned}
\mathcal{V}_\varphi \mathcal{V}_\varphi^* F(x, \omega) &= \langle \mathcal{V}_\varphi^* F, \mathbf{M}_\omega \mathbf{T}_x \varphi \rangle \\
&= \iint_{\mathbb{R}^{2d}} F(t, \xi) \overline{\mathcal{V}_\varphi(\mathbf{M}_\omega \mathbf{T}_x \varphi)(t, \xi)} \, dt d\xi \\
&= \iint_{\mathbb{R}^{2d}} F(t, \xi) \mathcal{V}_\varphi \varphi(x - t, \omega - \xi) e^{-2\pi i t \cdot (\omega - \xi)} \, dt d\xi.
\end{aligned}$$

By taking the absolute value of each side, the conclusion is obtained. ∎

Definition 1.14. Let $\varphi \in L^2(\mathbb{R}^d)$ be a window function with $\varphi\|_2 = 1$. The *spectrogram* of f with respect to φ is defined to be

$$\mathrm{SPEC}_\varphi f(z) = |\mathcal{V}_\varphi f(z)|^2.$$

From the definition, the spectrogram $\mathrm{SPEC}_\varphi f$ is non-negative. Moreover, by Lemma 1.9, it is covariant, and by Corollary 1.11, it is energy-preserving, i.e.

$$\begin{aligned}
&\mathrm{SPEC}_\varphi(\mathbf{T}_u \mathbf{M}_\eta f) = \mathrm{SPEC}_\varphi f(x - u, \omega - \eta), \\
&\iint_{\mathbb{R}^{2d}} \mathrm{SPEC}_\varphi f(z) \, dz = \|f\|_2^2.
\end{aligned}$$

Definition 1.15. Let $1 \leq p, q \leq \infty$. The *mixed-norm space* $L^{p,q}(\mathbb{R}^{2d})$ is the space of all measurable functions on $\mathbb{R}^{2d}$ such that the norm

$$\|F\|_{L^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |F(x,\omega)|^p \, dx \right)^{q/p} d\omega \right)^{1/q},$$

with the usual modification for $p = \infty$ or $q = \infty$, is finite.

Note that $L^{p,p}(\mathbb{R}^{2d}) = L^p(\mathbb{R}^{2d})$. We consider now the functions whose STFT lies in the mixed-norm space $L^{p,q}(\mathbb{R}^d)$.

Definition 1.16. Let φ be a nonzero function in $\mathcal{S}(\mathbb{R}^d)$. The *modulation space* $M^{p,q}(\mathbb{R}^d)$ is given by

$$M^{p,q}(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) \,:\, \mathcal{V}_\varphi f \in L^{p,q}(\mathbb{R}^d)\}.$$

If $p = q$, then we write $M^p(\mathbb{R}^d)$ for $M^{p,p}(\mathbb{R}^d)$.

The modulation space $M^{p,q}(\mathbb{R}^d)$ is a Banach space equipped with the norm $\|f\|_{M^{p,q}} := \|\mathcal{V}_\varphi f\|_{L^{p,q}}$, where a different choice for φ yields an equivalent norm. Since the STFT is an isometry from $L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^{2d})$, then $M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)$. The space $M^1(\mathbb{R}^d)$, consisting of functions whose STFT is integrable, coincides with Feichtinger's algebra $\boldsymbol{S}_0(\mathbb{R}^d)$, cf. [45]. It is the smallest Banach space that is invariant under modulations and translations. We recall some of its characterizations, cf. [55, Proposition 12.1.2].

Proposition 1.17. *The following conditions are equivalent:*

(1) $f \in \boldsymbol{S}_0(\mathbb{R}^d)$.
(2) $f \in L^2(\mathbb{R}^d)$ *and for one/all* $g \in \mathcal{S}(\mathbb{R}^d)$, *we have* $\mathcal{V}_g f \in L^1(\mathbb{R}^{2d})$.
(3) $f \in L^2(\mathbb{R}^d)$ *and for one/all* $g \in \boldsymbol{S}_0(\mathbb{R}^d)$, *we have* $\mathcal{V}_g f \in L^1(\mathbb{R}^{2d})$.

In the discussion of *time-frequency localization operators*, which are obtained by restricting the inversion formula for the STFT, in Chapter 2, properties of compact operators will come into play. We recall some characterizations and properties (cf. [84, 22]). Here, $\mathcal{H}$, $\mathcal{H}_1$, and $\mathcal{H}_2$ are complex separable Hilbert spaces.

Definition 1.18. The operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ is called *compact* if the image of the closed unit ball in $\mathcal{H}_1$ has compact closure in $\mathcal{H}_2$.

We will make use of the following properties of compact operators.

Theorem 1.19. (1) *If* $T : \mathcal{H} \to \mathcal{H}$ *is compact and* $L : \mathcal{H} \to \mathcal{H}$ *is bounded, then* TL *and* LT *are compact.*
(2) *Suppose* $T : \mathcal{H} \to \mathcal{H}$ *is a bounded operator. If* $\|Tx_n\|_{\mathcal{H}} \to 0$ *whenever* $x_n \to 0$ *weakly, i.e.* $\langle x_n, h \rangle \to 0$ *for any* $h \in \mathcal{H}$, *then* T *is compact.*

Theorem 1.20 (Spectral theorem for compact self-adjoint operators)**.** *Let $T : \mathcal{H} \to \mathcal{H}$ be a compact self-adjoint operator. Then there exists a sequence $\{\alpha_k\}_{k\in\mathbb{N}}$ of real numbers and a corresponding orthonormal sequence $\{\psi_k\}_{k\in\mathbb{N}}$ in $\mathcal{H}$ such that*

(1) $\lim_{k\to\infty} \alpha_k = 0$,
(2) $T\psi_k = \alpha_k\psi_k$ *for every* $k \in \mathbb{N}$, *i.e. each* ψ_k *is an eigenfunction of* T *with eigenvalue* α_k, *and*
(3) $Tf = \sum_{k=1}^{\infty} \alpha_k \langle f, \psi_k\rangle \psi_k$ *for every* $f \in \mathcal{H}$, *where the series converges in the norm of* $\mathcal{H}$.

Definition 1.21. Let $T : \mathcal{H} \to \mathcal{H}$ be a compact operator and let $\{s_k\}_{k\in\mathbb{N}}$ be the sequence of singular values of T (square roots of the eigenvalues of the non-negative self-adjoint operator T^*T). The operator T belongs to the *Schatten p-class* if $\{s_k\}_{k\in\mathbb{N}} \in \ell^p(\mathbb{N})$. The set of all Schatten p-class operators is denoted by $S^p(\mathcal{H})$. If $p = 1$, then T is called a *trace class* operator. If $p = 2$, then T is called a *Hilbert-Schmidt* operator.

A criterion for T to be trace class is given in the following theorem.

Theorem 1.22. *A compact operator $T : \mathcal{H} \to \mathcal{H}$ belongs to the trace class $S^1(\mathcal{H})$ if and only if*

$$\sum_{k\in\mathbb{N}} |\langle Te_k, e_k\rangle| < \infty$$

for every orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ of $\mathcal{H}$. In this case,

$$\|T\|_{S^1} = \sup_{\{e_k\}_{k\in\mathbb{N}}\ ONB} \sum_{k\in\mathbb{N}} |\langle Te_k, e_k\rangle|.$$

1.3. Gabor frames

Frames are a generalization of bases that offer added flexibility because of its redundancy. They were introduced by Duffin and Schaeffer in [43], and have since become an important tool in mathematics. Frames consisting of time-frequency shifts of a single function are called Gabor frames. For a more detailed discussion on frames, we recommend the books by Christensen [19, 20]. A good reference on Gabor frames is [49] and its sequel [50].

Definition 1.23. A sequence $\{f_i\}_{i\in I}$ in a separable Hilbert space $\mathcal{H}$ is called a *frame* if there exist constants A, $B > 0$ such that for all $f \in \mathcal{H}$

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2. \tag{1.25}$$

Any such constants A and B are called *frame bounds*. If $A = B$, then the sequence $\{f_i\}_{i\in I}$ is called a *tight frame*.

Remark 1.24. If there exists $B > 0$ such that at least just the right-hand inequality in (1.25) is satisfied, then $\{f_i\}_{i\in I}$ is called a *Bessel sequence*

Remark 1.25. An orthonormal basis is a normalized tight frame with $A = B = 1$.

Definition 1.26. Let $\{f_i\}_{i\in I} \subseteq \mathcal{H}$. The *coefficient operator* or *analysis operator* $\mathbf{C}$ is defined by

$$\mathbf{C}f = \{\langle f, f_i\rangle\}_{i\in I}.$$

The *reconstruction operator* or *synthesis operator* $\mathbf{D}$ is defined by

$$\mathbf{D}c = \sum_{i\in I} c_i f_i \in \mathcal{H},$$

for a finite sequence $c = \{c_i\}_{i\in I}$. The *frame operator* is defined on $\mathcal{H}$ by

$$\mathbf{S}f = \sum_{i\in I} \langle f, f_i\rangle f_i.$$

Proposition 1.27. *Let $\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$ with frame bounds $A, B > 0$.*

(1) $\mathbf{C}$ *is a bounded operator from $\mathcal{H}$ into $\ell^2(I)$ with closed range.*
(2) *The operators $\mathbf{C}$ and $\mathbf{D}$ are adjoint to each other, i.e. $\mathbf{D} = \mathbf{C}^*$. Consequently, $\mathbf{D}$ extends to a bounded operator from $\ell^2(I)$ into $\mathcal{H}$ and satisfies*

$$\Big\| \sum_{i\in I} c_i f_i \Big\| \le B^{\frac{1}{2}} \|c\|_2.$$

(3) *If $f = \sum_{i\in I} c_i f_i$ for some $c \in \ell^2(I)$, the for every $\varepsilon > 0$ there exists a finite subset $F_0 \subset I$ such that*

$$\Big\| f - \sum_{i\in F} c_i f_i \Big\| < \varepsilon \qquad \text{for all finite subsets } F \supseteq F_0,$$

i.e. $\sum_{i\in I} c_i f_i$ converges unconditionally to $f \in \mathcal{H}$.
(4) *The operator $\mathbf{S} = \mathbf{C}^*\mathbf{C} = \mathbf{D}\mathbf{D}^*$ maps $\mathcal{H}$ onto $\mathcal{H}$ and is a positive invertible operator.*
(5) *The sequence $\{\mathbf{S}^{-1} f_i\}_{i\in I}$ is a frame, called the dual frame, with bounds B^{-1}, $A^{-1} > 0$.*
(6) *Every $f \in \mathcal{H}$ has nonorthogonal expansions*

$$f = \sum_{i\in I} \langle f, \mathbf{S}^{-1} f_i\rangle f_i \qquad \text{and} \qquad f = \sum_{i\in I} \langle f, f_i\rangle \mathbf{S}^{-1} f_i, \tag{1.26}$$

where both sums converge unconditionally in $\mathcal{H}$.

For tight frames, the frame operator becomes $\mathbf{S} = A\mathbf{I}$. They have the advantage of having the same functions for analysis and synthesis.

The series expansion in Proposition 1.27(6) is useful if one can obtain the dual frames explicitly. Usually, it is more convenient to apply the following iterative method, called the *frame algorithm*, in reconstructing a function from its analysis coefficients.

Frame algorithm: Given a relaxation parameter $0 < \lambda < \frac{2}{B}$, set $\delta = \max\{|1-\lambda A|, |1-\lambda B|\} < 1$. Let $f_0 = 0$ and define recursively

$$f_{n+1} = f_n + \lambda \mathbf{S}(f - f_n). \tag{1.27}$$

Then $\lim_{n\to\infty} f_n = f$ with a geometric rate of convergence, i.e.

$$\|f - f_n\|_2 \leq \delta^n \|f\|_2.$$

Definition 1.28. A sequence $\{f_i\}_{i\in I}$ in a Hilbert space $\mathcal{H}$ is called a *Riesz sequence*, if and only if there exist constants A', $B' > 0$ such that the inequalities

$$A'\|c\|_2 \leq \Big\| \sum_{i\in I} c_i f_i \Big\| \leq B'\|c\|_2 \tag{1.28}$$

hold for all finite sequences $c = \{c_i\}_{i\in I}$. For a Riesz sequence, the coefficients in the frame expansions (1.26) are unique. A Riesz sequence is called a *Riesz basis* for $\mathcal{H}$ if $\overline{\text{span}}\{f_i\}_{i\in I} = \mathcal{H}$.

We now consider a special type of frames called *Gabor frames*.

Definition 1.29. Given a non-zero window function $g \in L^2(\mathbb{R}^d)$ and lattice $\Lambda \in \mathbb{R}^{2d}$, the set of time-frequency shifts

$$\mathcal{G}(g, \Lambda) = \{g_\lambda := \pi(\lambda)g \, : \, \lambda \in \Lambda\} \tag{1.29}$$

is called a *Gabor system*. If such a Gabor system is a frame for $L^2(\mathbb{R}^d)$, then it is called a *Gabor frame*.

The Gabor frame operator has the form

$$\mathbf{S}f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda)g. \tag{1.30}$$

If we want to emphasize the dependence on the window φ in (1.30), we write $\mathbf{S}_{g,g}$ instead of $\mathbf{S}$.

Proposition 1.30. *If $\mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$, then there exists a dual window $\gamma \in L^2(\mathbb{R}^d)$ such that the dual frame of $\mathcal{G}(g, \Lambda)$ is $\mathcal{G}(\gamma, \Lambda)$. Every $f \in L^2(\mathbb{R}^d)$ can be represented as*

$$f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda)\gamma \tag{1.31}$$

$$= \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)\gamma\rangle \pi(\lambda)g. \tag{1.32}$$

The representation (1.31)*, or equivalently* (1.32)*, is called the Gabor expansion of $f \in L^2(\mathbb{R}^d)$.*

Corollary 1.31. *If $\mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$ with dual window $\gamma = \mathbf{S}^{-1}g \in L^2(\mathbb{R}^d)$, then the inverse frame operator is given by*

$$\mathbf{S}^{-1}_{g,g} f = \mathbf{S}_{\gamma,\gamma} f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)\gamma\rangle \pi(\lambda)\gamma.$$

For a separable lattice $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$, with *lattice parameters* a and b, we also write $\mathcal{G}(g,\Lambda)$ as $\mathcal{G}(g,a,b)$. We next mention some results concerning the density of Gabor frames over a separable lattice.

Corollary 1.32. (1) *If $\mathcal{G}(g,a,b)$ is a frame for $L^2(\mathbb{R}^d)$, then $ab \le 1$.*
(2) *The Gabor system $\mathcal{G}(g,a,b)$ is a frame for $L^2(\mathbb{R}^d)$ and $ab = 1$ if and only if $\mathcal{G}(g,a,b)$ is a Riesz basis for $L^2(\mathbb{R}^d)$.*
(3) *The Gabor system $\mathcal{G}(g,a,b)$ is an orthonormal basis for $L^2(\mathbb{R}^d)$ if and only if $\mathcal{G}(g,a,b)$ is a tight frame, $\|g\|_2 = 1$ and $ab = 1$.*

In some special cases, the Gabor frame operator can be simplified obtaining explicit simple examples of Gabor frames. An example is the following "painless non-orthogonal expansion," cf. [28].

Theorem 1.33. *Suppose that $g \in L^\infty(\mathbb{R}^d)$ is supported on the cube $Q_L = [0,L]^d$. If $a \le L$ and $b \le \frac{1}{L}$, then the frame operator is the multiplication operator*

$$\mathbf{S}f(x) = \left(\frac{1}{b} \sum_{k\in\mathbb{Z}^d} |g(x-ak)|^2 \right) f(x).$$

Consequently, $\mathcal{G}(g,a,b)$ is a frame with frame bounds $\frac{A}{b^d}$ and $\frac{B}{b^d}$ if and only if

$$A \le \sum_{k\in\mathbb{Z}^d} |g(x-ak)|^2 \le B \qquad a.e. \tag{1.33}$$

Furthermore, $\mathcal{G}(g,a,b)$ is a tight frame if and only if $\sum_{k\in\mathbb{Z}^d} |g(x-ak)|^2 =$constant almost everywhere.

If Λ is not necessarily a lattice, but a general countable subset of $\mathbb{R}^{2d}$, we say that the set $\mathcal{G}(g,\Lambda) = \{\pi(\lambda)g : \lambda \in \Lambda\}$ is an *irregular Gabor system*. If it is a frame, then we say that $\mathcal{G}(g,\Lambda)$ is an *irregular Gabor frame*. Christensen, Deng, and Heil [21] provide some necessary density conditions on the set of points Λ for the Gabor system $\mathcal{G}(g,\Lambda)$ to be a frame.

For $z = (t,\omega) \in \mathbb{R}^{2d}$, we denote by $Q_h(z)$ the unit cube in $\mathbb{R}^{2d}$ centered at z with side lengths h, i.e.

$$Q_h(z) = \prod_{k=1}^{d} [t_1 - h/2, t_1 + h/2) \times \prod_{k=1}^{d} [\omega_k - h/2, \omega_k + h/2).$$

We also write $Q(z) = Q_1(z)$.

Definition 1.34. Let $\Lambda = \{\lambda_k\}_{k\in I}$ be a countable set of points in $\mathbb{R}^{2d}$.

(1) Λ is said to be *separated* if $\inf_{j\neq k} |\lambda_j - \lambda_k| > 0$. Any constant $\delta > 0$ such $|\lambda_j - \lambda_k| > 0$, $j \neq k$, is called a *separation constant.*
(2) Λ is said to be *relatively separated* if it is a finite union of separated sets of points.

We denote by $\nu^+(h)$ and $\nu^-(h)$ the largest and smallest number of points in Λ that lie in any cube $Q_h(z)$, i.e.

$$\nu^+(h) = \sup_{z\in\mathbb{R}^{2d}} \#(\Lambda \cap Q_h(z)), \qquad \nu^-(h) = \inf_{z\in\mathbb{R}^{2d}} \#(\Lambda \cap Q_h(z)),$$

and we define the *upper and lower Beurling densities* of Λ as

$$D^+(\Lambda) = \limsup_{h\to\infty} \frac{\nu^+(h)}{h^{2d}} \quad \text{and} \quad D^-(\Lambda) = \liminf_{h\to\infty} \frac{\nu^-(h)}{h^{2d}},$$

respectively. If $D^+(\Lambda) = D^-(\Lambda)$, then Λ is said to have *uniform Beurling density*

$$D(\Lambda) = D^+(\Lambda) = D^-(\Lambda).$$

A characterization of the density of Λ and the separation of its points can be seen in [19] stated as follows.

Lemma 1.35. [19, Lemma 7.1.3] *Let Λ be a countable subset of $\mathbb{R}^{2d}$. The following are equivalent:*

(1) $D^+(\Lambda) < \infty$
(2) Λ *is relatively separated.*
(3) *For some (and therefore every) $h > 0$, there is an $N_h(\Lambda) \in \mathbb{N}$ such that*

$$\sup_{m\in\mathbb{Z}^{2d}} \#(\Lambda \cap Q_h(hm)) < N_h.$$

Definition 1.36. The *Wiener amalgam space* $W(L^p, \ell^q)$, $1 \le p, q \le \infty$ is the Banach space of all measurable functions f with norm

$$\|f\|_{W(L^p,\ell^q)} = \Big\| \|f \cdot \chi_{Q(\cdot)}\|_{L^p} \Big\|_{\ell^q} < \infty.$$

A sampling estimate for functions in the Wiener amalgam space was shown e.g. in [52, Lemma 3.2.11]. A similar proof can be used to show that it holds in the irregular case as well, cf. [6].

Lemma 1.37. [6, Proposition 2.2.3] *Let Λ be a relatively separated set in $\mathbb{R}^{2d}$. Then there is a constant $C_\Lambda = N_1(\Lambda)$ such that for all $p \in [1,\infty)$,*

$$\sum_{\lambda\in\Lambda} |f(\lambda)|^p \le C_\Lambda \|f\|_{W(L^\infty,\ell^p)}$$

for all continuous functions $f \in W(L^\infty, \ell^p)$.

Lemma 1.37 above can be used to show that a Bessel condition for the Gabor system $\mathcal{G}(g,\Lambda)$, cf. [6].

Theorem 1.38. [6, Theorem 2.2.6] *Let $g \in \boldsymbol{S}_0$ and let Λ be a relatively separated set in $\mathbb{R}^{2d}$. Then the Gabor system $\mathcal{G}(g,\Lambda)$ forms a Bessel sequence in $L^2(\mathbb{R}^d)$, i.e. there exists $B > 0$ such that for all $f \in L^2(\mathbb{R}^d)$,*

$$\sum_{\lambda\in\Lambda} |\mathcal{V}_g f(\lambda)|^2 \le B\, \|f\|_2^2.$$

REMARK 1.39. A suitable choice for B is

$$B = N_1(\Lambda)\|\mathcal{V}_{\varphi_0}\varphi_0\|_{W(L^\infty,\ell^1)}\|g\|^2_{\boldsymbol{S}_0}C_{g,\varphi_0}, \tag{1.34}$$

where φ_0 is the Gaussian window $\varphi_0(t) = e^{-\pi t^2}$ and $C_{g,\varphi_0} = \sup\Big\{C > 0 \,:\, \sum_{n\in\mathbb{N}} |b_n| \leq C\|g\|_{\boldsymbol{S}_0},\, g = \sum_{n\in\mathbb{N}} b_n \mathbf{M}_{\eta_m}\mathbf{T}_{y_m}\varphi_0\Big\}$.

We mention the result of Feichtinger and Gröchenig in [46] that provides conditions for an irregular Gabor frame.

Theorem 1.40. [46, Theorem 6.1] *Let $g \in \boldsymbol{S}_0$. Then there is an open set $U \in \mathbb{R}^{2d}$ such that the Gabor system $\mathcal{G}(g,\Lambda)$ is a frame for $L^2(\mathbb{R}^d)$ provided Λ is a relatively separated and $\cup_{\lambda\in\Lambda}(\lambda + U) = \mathbb{R}^{2d}$.*

1.4. Wavelet theory

Let $\psi \in L^2(\mathbb{R})$ and $(\alpha,\beta) \in \mathbb{R}^+ \times \mathbb{R}$. We define the *wavelet system* by

$$\psi_{\alpha,\beta}(t) = \frac{1}{\sqrt{\alpha}}\psi\left(\frac{t-\beta}{\alpha}\right) = \mathbf{T}_\beta\mathbf{D}_\alpha\psi, \tag{1.35}$$

where $\mathbf{D}_\alpha$ denotes the *dilation operator* given by $\mathbf{D}_\alpha f(t) = \frac{1}{\sqrt{\alpha}} f(\frac{t}{\alpha})$.

The *wavelet transform* is then defined as

$$\mathbf{W}_\psi f(\alpha,\beta) = \langle f, \mathbf{T}_\beta\mathbf{D}_\alpha\psi\rangle = \left(f * \mathbf{D}_\alpha\overline{\mathcal{I}\psi}\right)(\beta). \tag{1.36}$$

If ψ is localized around τ_0, then $\psi_{\alpha,\beta}(t)$ is centered at $\alpha\cdot\tau_0 + \beta$. The frequency center is at η/α, where η is the center of $\hat{\psi}$.

CHAPTER 2

Time-frequency localized functions

The uncertainty principle, in its several forms, sets a restriction on the time-frequency behavior of a function. While a signal cannot have all its energy lying in a compact region in the time-frequency plane, signals that have highly concentrated time-frequency content are used in many applications.

Landau, Pollak, and Slepian, in [87, 73, 74, 86], developed the study of band-limited functions that are concentrated on a finite time interval. They made use of compositions of time- and band-limiting operators and considered the eigenvalue problem associated with these operators. They investigated concentrations of band-limited functions on finite length intervals, and the "dimension" of the set of band-limited signals that are approximately time-limited on an interval. The optimal orthogonal system that represented band-limited and essentially time-limited functions consists of the prolate spheroidal wave functions (PSWF). Among the results obtained is that if f is essentially time-limited to $[-T/2, T/2]$ and bandlimited to $[-\Omega, \Omega]$, then it is well approximated by its projection on the span of the first $[2T\Omega]$ PSWF eigenfunctions of the operator $\mathbf{P}_{[-\Omega,\Omega]}\mathbf{Q}_{[-T/2,T/2]}$.

The time-frequency localization operators that will be considered here would allow for localization on more general regions of the time-frequency plane. They were introduced and studied by Daubechies in [25], and Ramanathan and Topiwala in [82]. These operators can be used to extract and localize components of a signal from its representation in the time-frequency plane. They go by the names STFT multipliers [48] or Toeplitz operators [30]. They have appeared in physics as tools in quantization procedures [13] called anti-Wick operators, and in the approximation of pseudodifferential operators [24].

These operators are built by restricting the integral in the inversion formula from the STFT coefficients to a subset of $\mathbb{R}^{2d}$. Its properties, connections with other mathematical topics, and applications have been covered in various works, e.g. [82, 47, 29, 95, 23, 1, 57, 39, 40].

We recall time-frequency localization operators and their properties, such as boundedness and compactness, and review eigenvalues and eigenfunctions of these operators. Then we use these operators to measure the time-frequency content of functions on the compact region. We likewise show that if a function is highly concentrated on a compact region in the time-frequency plane, then it is well approximated on a subspace of eigenfunctions of a time-frequency localization operator. We also compare measures of localization from time-frequency localization operators and time- and band-limiting operators. Most of the new results in this chapter and in Chapter 3 are presented in the joint work with M. Dörfler [42].

2.1. Time-frequency localization operators and their properties

Definition 2.1. Let φ be a given window function and σ a bounded nonnegative function on $\mathbb{R}^{2d}$. The *time-frequency localization operator* $\mathbf{H}_{\sigma,\varphi}$ with window φ and symbol σ is formally defined as

$$\mathbf{H}_{\sigma,\varphi} f = \iint_{\mathbb{R}^{2d}} \sigma(z)\mathcal{V}_\varphi f(z)\pi(z)\varphi \, dz = \mathcal{V}^*\sigma\mathcal{V}f.$$

The integral is defined strongly e.g. in $L^2(\mathbb{R}^d)$ if $\sigma \in L^1(\mathbb{R}^d)$ and $\varphi \in L^2(\mathbb{R}^d)$. Indeed, if $K_n \subset \mathbb{R}^{2d}$, $n \geq 1$ is a nested exhausting sequence of compact sets and if we define f_n^σ to be

$$f_n^\sigma = \iint_{K_n} \sigma(z)\mathcal{V}_\varphi f(z)\pi(z)\varphi \, dz,$$

then by the Cauchy-Schwarz inequality, we estimate for $h \in L^2(\mathbb{R}^d)$ that

$$\begin{aligned}|\langle f_n^\sigma, h\rangle| &= \left|\iint_{K_n} \sigma(z)\mathcal{V}_\varphi f(z)\overline{\mathcal{V}_\varphi h(z)}\, dz\right| \\ &\leq \|\mathcal{V}_\varphi f\|_\infty \|\mathcal{V}_\varphi h\|_\infty \|\sigma\|_1 \\ &\leq \|f\|_2\|\varphi\|_2^2\|h\|_2\|\sigma\|_1.\end{aligned}$$

So for each n, f_n^σ is a well-defined element of $L^2(\mathbb{R}^d)$ with $\|f_n^\sigma\|_2 \leq \|f\|_2\|\varphi\|_2^2\|\sigma\|_1$. We estimate similarly that

$$\begin{aligned}|\langle \mathbf{H}_{\sigma,\varphi}f - f_n^\sigma, h\rangle| &= \left|\iint_{\mathbb{R}^{2d}} \sigma(z)\mathcal{V}_\varphi f(z)\overline{\mathcal{V}_\varphi h(z)}\, dz - \iint_{K_n} \sigma(z)\mathcal{V}_\varphi f(z)\overline{\mathcal{V}_\varphi h(z)}\, dz\right| \\ &= \left|\iint_{K_n^c} \sigma(z)\mathcal{V}_\varphi f(z)\overline{\mathcal{V}_\varphi h(z)}\, dz\right| \\ &\leq \|f\|_2\|\varphi\|_2^2\|h\|_2 \iint_{K_n^c} |\sigma(z)|\, dz\end{aligned}$$

Since this is true for all $h \in L^2(\mathbb{R}^d)$, we have

$$\begin{aligned}\|\mathbf{H}_{\sigma,\varphi}f - f_n^\sigma\|_2 &= \sup_{\|h\|_2=1} |\langle \mathbf{H}_{\sigma,\varphi}f - f_n^\sigma, h\rangle| \\ &\leq \|f\|_2\|\varphi\|^2 \iint_{K_n^c} |\sigma(z)|\, dz.\end{aligned}$$

Since $\sigma \in L^1(\mathbb{R}^d)$ the right-hand side approaches 0 as n increases.

We note that if $\sigma \equiv 1$ and $\|\varphi\|_2 = 1$, then by the inversion formula of the STFT, $\mathbf{H}_{\sigma,\varphi}f = f$. If σ is compactly supported on $\Omega \subseteq \mathbb{R}^{2d}$, then $\mathbf{H}_{\sigma,\varphi}f$ is interpreted as the part of f that lies essentially in Ω.

It is usually more convenient to use the alternative weak definition of $\mathbf{H}_{\sigma,\varphi}$ given by

$$\langle \mathbf{H}_{\sigma,\varphi} f, g\rangle = \langle \sigma \mathcal{V}_\varphi f, \mathcal{V}_\varphi g\rangle. \tag{2.1}$$

This definition extends to symbols σ in $\mathcal{S}'(\mathbb{R}^d)$. Boundedness and Schatten class properties of time-frequency localization operators between various spaces in terms of properties of the symbol σ and the window φ have been studied in various works such as [95, 48, 23, 15, 92, 11]. Some of their results are summarized in Table 1.

Symbol	Window	Localization Operator
$L^\infty(\mathbb{R}^{2d})$	$L^2(\mathbb{R}^d)$	$B(L^2(\mathbb{R}^d))$
$L^p(\mathbb{R}^{2d})$, $1 \le p < \infty$	$L^2(\mathbb{R}^d)$	$S^p(L^2(\mathbb{R}^d))$
$M^{\infty,\infty}(\mathbb{R}^{2d})$	$M^1(\mathbb{R}^d) = \boldsymbol{S}_0(\mathbb{R}^d)$	$B(M^{p,q}(\mathbb{R}^d))$, $1 \le p, q \le \infty$
$M^{p,\infty}(\mathbb{R}^{2d})$, $1 \le p < \infty$	$M^1(\mathbb{R}^d) = \boldsymbol{S}_0(\mathbb{R}^d)$	$S^p(L^2(\mathbb{R}^d))$

TABLE 1. Time-frequency localization operators and symbols and window functions

For the purpose of this research, we shall keep our focus on time-frequency localization operators $\mathbf{H}_{\sigma,\varphi}$ with symbol $\sigma = \chi_\Omega$ where χ_Ω is the characteristic function on Ω, a compact set in $\mathbb{R}^{2d}$ or at least a bounded set in $\mathbb{R}^{2d}$ with $|\Omega| < \infty$, and window function $\varphi \in L^2(\mathbb{R}^d)$ with $\|\varphi\|_2 = 1$. In this case, we also write the localization operator as $\mathbf{H}_{\Omega,\varphi}$. Let us show the well-known boundedness, compactness, and trace class properties of $\mathbf{H}_{\Omega,\varphi}$, see e.g. [14, 95, 48].

Theorem 2.2. *Let Ω be a compact region of $\mathbb{R}^{2d}$ and $\varphi \in L^2(\mathbb{R}^d)$, with $\|\varphi\|_2 = 1$. Then $\mathbf{H}_{\Omega,\varphi}$ is a bounded operator on $L^2(\mathbb{R}^{2d})$ with norm $\|\mathbf{H}_{\Omega,\varphi}\|_{B(L^2(\mathbb{R}^d))} \le 1$. Moreover, $\mathbf{H}_{\Omega,\varphi}$ is a compact operator and even trace class.*

Proof: We first note that $\mathcal{V}_\varphi$ and $\mathcal{V}_\varphi^*$ are bounded operators from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^{2d})$ and $L^2(\mathbb{R}^{2d})$ to $L^2(\mathbb{R}^d)$, respectively, with operator norm 1. For $f \in L^2(\mathbb{R}^d)$,

$$\begin{aligned} \|\mathbf{H}_{\Omega,\varphi} f\|_2 &= \|\mathcal{V}_\varphi^*(\chi_\Omega \cdot \mathcal{V}_\varphi f)\|_2 \\ &\le \|\mathcal{V}_\varphi^*\|_{L^2\to L^2} \|\chi_\Omega\|_\infty \|\mathcal{V}_\varphi\|_{L^2\to L^2} \|f\|_2 \\ &= \|f\|_2, \end{aligned}$$

so $\mathbf{H}_{\Omega,\varphi} \in B(L^2(\mathbb{R}^d))$ and $\|\mathbf{H}_{\Omega,\varphi}\|_{B(L^2(\mathbb{R}^d))} \le 1$.

We now show that $\mathbf{H}_{\Omega,\varphi}$ is a compact operator. We first denote by $\mathcal{M}_\Omega : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, $F(z) \mapsto (\mathcal{M}_\Omega F)(z) := \chi_\Omega(z) \cdot \mathcal{V}_\varphi(z)$ the multiplication operator with the function χ_Ω. Since $\mathbf{H}_{\Omega,\varphi} = \mathcal{V}_\varphi^* \mathcal{M}_\Omega \mathcal{V}_\varphi$, and $\mathcal{V}_\varphi^*$ is bounded, it suffices to show that $\mathcal{M}_\Omega \mathcal{V}_\varphi$ is compact.

Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $L^2(\mathbb{R}^d)$ that is weakly convergent to 0. We show that $\|\mathcal{M}_\Omega \mathcal{V}_\varphi f_n\|_2 \to 0$ as $n \to \infty$. We calculate

$$\|\mathcal{M}_\Omega \mathcal{V}_\varphi f_n\|_2^2 = \|\chi_\Omega \mathcal{V}_\varphi f_n\|_2^2 = \iint_{\mathbb{R}^{2d}} |\chi_\Omega(z)|^2 |\mathcal{V}_\varphi f_n(z)|^2 \, dz = \iint_\Omega |\mathcal{V}_\varphi f_n(z)|^2 \, dz.$$

Since $\{f_n\}_{n\in\mathbb{N}}$ converges weakly to 0, i.e. $\langle f_n, g\rangle \to 0$ for every $g \in L^2(\mathbb{R}^d)$, we have for every $z \in \Omega$, $|\mathcal{V}_\varphi f_n(z)|^2 = |\langle f_n, \pi(z)\varphi\rangle|^2 \to 0$ as $n \to \infty$. This means that the integrand converges to 0 pointwise in Ω.

Recall that every weakly convergent sequence is norm bounded, i.e. there exists a $C > 0$ such that $\|f_n\|_2 \le C$ for all $n \in \mathbb{N}$. So

$$|\mathcal{V}_\varphi f_n(z)|^2 \le \|\varphi\|_2^2 \|f_n\|_2^2 \le C^2.$$

By the Dominated Convergence Theorem, $\|\mathcal{M}_\Omega \mathcal{V}_\varphi f_n\|_2^2 \to 0$ as $n \to 0$. Hence, $\mathcal{M}_\Omega \mathcal{V}_\varphi$ is compact, which implies that $\mathbf{H}_{\Omega,\varphi}$ is compact.

To show that $\mathbf{H}_{\Omega,\varphi}$ is trace class, we let $\{e_k\}_{k=1}^\infty$ be an arbitrary orthonormal basis of $L^2(\mathbb{R}^d)$, and we calculate

$$\begin{aligned}
\sum_{k=1}^\infty |\langle \mathbf{H}_{\Omega,\varphi} e_k, e_k\rangle| &= \sum_{k=1}^\infty |\langle \mathcal{V}_\varphi^*(\chi_\Omega \cdot \mathcal{V}_\varphi e_k), e_k\rangle| \\
&= \sum_{k=1}^\infty |\langle \chi_\Omega \cdot \mathcal{V}_\varphi e_k, \mathcal{V}_\varphi e_k\rangle| \\
&= \sum_{k=1}^\infty \Big| \iint_{\mathbb{R}^{2d}} \chi_\Omega(z) \mathcal{V}_\varphi e_k(z) \overline{\mathcal{V}_\varphi e_k(z)}\, dz \Big| \\
&= \sum_{k=1}^\infty \iint_\Omega |\mathcal{V}_\varphi e_k(z)|^2\, dz \\
&\overset{\text{(Fubini)}}{=} \iint_\Omega \sum_{k=1}^\infty |\mathcal{V}_\varphi e_k(z)|^2\, dz \\
&= \iint_\Omega \sum_{k=1}^\infty |\langle e_k, \pi(z)\varphi\rangle|^2\, dz \\
&= \iint_\Omega \|\pi(z)\varphi\|_2^2\, dz = |\Omega|\, \|\varphi\|_2^2 = |\Omega|,
\end{aligned}$$

where the last line follows from the fact that $\{e_k\}_{k=1}^\infty$ is an orthonormal basis of $L^2(\mathbb{R}^d)$. Therefore, $\mathbf{H}_{\Omega,\varphi}$ is trace class with $\|\mathbf{H}_{\Omega,\varphi}\|_{S^1} = |\Omega|$. ■

The STFT of $\mathbf{H}_{\Omega,\varphi}$, using (1.24), satisfies the following pointwise estimate:

$$|\mathcal{V}_\varphi(\mathbf{H}_{\Omega,\varphi} f)(z)| = |\mathcal{V}_\varphi(\mathcal{V}_\varphi^*(\chi_\Omega \mathcal{V}_\varphi f))(z)| \le (|\mathcal{V}_\varphi \varphi| * (\chi_\Omega |\mathcal{V}_\varphi f|))(z). \tag{2.2}$$

The estimate above is useful in establishing the norm estimates involving $\mathbf{H}_{\Omega,\varphi} f$. For instance, for $\sigma = \chi_\Omega$ and $\varphi \in \boldsymbol{S}_0(\mathbb{R}^d)$, then $\mathbf{H}_{\Omega,\varphi}$ is a bounded operator say from $M^p(\mathbb{R}^d)$ into $M^p(\mathbb{R}^d)$ (see Table 1) since (2.2) gives

$$\begin{aligned}
\|\mathbf{H}_{\Omega,\varphi} f\|_{M^p(\mathbb{R}^d)} &= \|\mathcal{V}_\varphi(\mathbf{H}_{\Omega,\varphi} f)\|_{L^p(\mathbb{R}^{2d})} \\
&\le \||\mathcal{V}_\varphi \varphi| * |\chi_\Omega \mathcal{V}_\varphi f|\|_{L^p(\mathbb{R}^{2d})}
\end{aligned}$$

$$\leq \|\mathcal{V}_\varphi \varphi\|_{L^1(\mathbb{R}^{2d})} \|\chi_\Omega \mathcal{V}_\varphi f\|_{L^p(\mathbb{R}^{2d})} \qquad \text{(Young's inequality (1.7))}$$
$$\leq \|\varphi\|_{\boldsymbol{S}_0(\mathbb{R}^d)} \|\mathcal{V}_\varphi f\|_{L^p(\mathbb{R}^{2d})} = C\|f\|_{M^p(\mathbb{R}^{2d})}.$$

2.2. Eigenvalues and eigenfunctions

Since the time-frequency localization operator $\mathbf{H}_{\Omega,\varphi} = \mathcal{V}_\varphi^* \chi_\Omega \mathcal{V}_\varphi$ that we consider is a compact and self-adjoint operator, the spectral theorem gives the following spectral representation:

$$\mathbf{H}_{\Omega,\varphi} f = \sum_{k=1}^{\infty} \alpha_k \langle f, \psi_k \rangle \psi_k, \tag{2.3}$$

where $\{\alpha_k\}_{k=1}^\infty$ are the positive eigenvalues arranged in a non-increasing manner and $\{\psi_k\}_{k=1}^\infty$ is the corresponding orthonormal set of eigenfunctions.

The operator $\mathbf{H}_{\Omega,\varphi}$ is useful in studying the optimization problem

$$\text{Maximize} \iint_\Omega |\mathcal{V}_\varphi f(z)|^2 \, dz, \; \|f\|_2 = 1, \tag{2.4}$$

which aims to look for the function that has a spectrogram that is well concentrated in Ω. Since

$$\langle \mathbf{H}_{\Omega,\varphi} f, f \rangle = \iint_\Omega \mathcal{V}_\varphi f(z) \langle \pi(z)\varphi, f \rangle \, dz = \iint_\Omega |\mathcal{V}_\varphi f(z)|^2 \, dz,$$

it follows that the first eigenfunction ψ_1 satisfies

$$\alpha_1 = \langle \mathbf{H}_{\Omega,\varphi} \psi_1, \psi_1 \rangle = \iint_\Omega |\mathcal{V}_\varphi \psi_1(z)|^2 dz = \max_{\|f\|_2=1} \iint_\Omega |\mathcal{V}_\varphi f(z)|^2 dz, \tag{2.5}$$

which solves (2.4). Moreover, the min-max lemma for self-adjoint operators states that

$$\alpha_k = \iint_\Omega |\mathcal{V}_\varphi \psi_k(z)|^2 \, dz = \max_{f \perp \psi_1, \ldots, \psi_{k-1}, \, \|f\|_2=1} \iint_\Omega |\mathcal{V}_\varphi f(z)|^2 dz. \tag{2.6}$$

So the eigenvalues of $\mathbf{H}_{\Omega,\varphi}$ determines the number of orthogonal functions that have a well-concentrated spectrogram in Ω.

For the case where φ is a normalized Gaussian and Ω is a disk centered at the origin, Daubechies [25] showed that the eigenfunctions of the corresponding time-frequency localization operator are the Hermite functions. The behavior of the eigenvalues α_k was also described including its exponential decay in the index k and the width of the plunge region. It was shown that there are $\approx |\Omega|$ eigenvalues greater than or equal to $\frac{1}{2}$.

Figure 1 shows a disk in the time-frequency plane and the eigenvalues of the resulting time-frequency localization operator with a normalized Gaussian as the window function. We see in Figure 2 the spectrograms of four eigenfunctions.

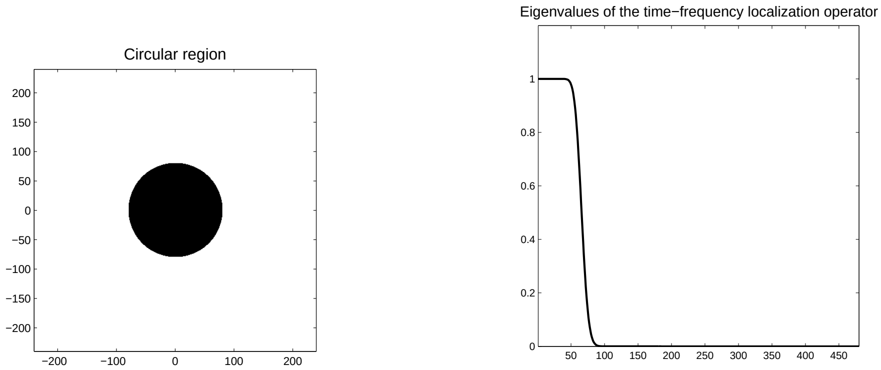

FIGURE 1. A circular region and the eigenvalues of a time-frequency localization operator

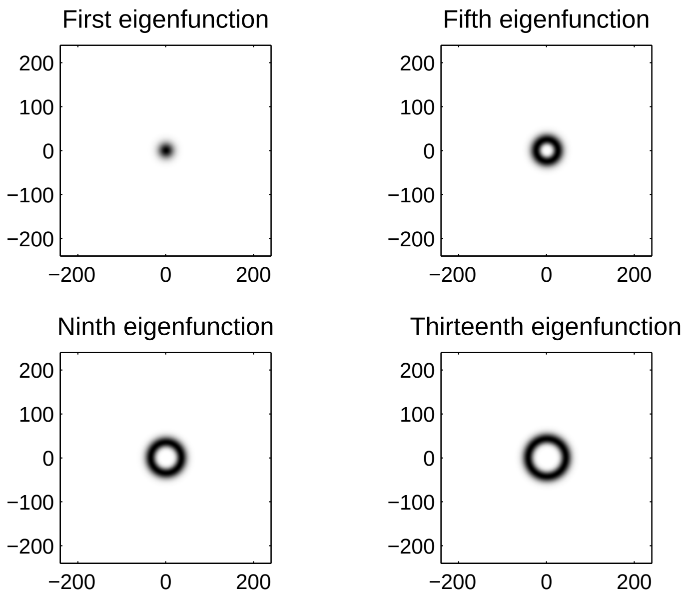

FIGURE 2. Spectrograms of eigenfunctions of a time-frequency operator with Gaussian window over a circular region

We mention the standard estimate for the distribution of the eigenvalues of $\mathbf{H}_{\Omega,\varphi}$, which appears e.g. in [72]. The version presented in [2, Lemma 3.3] is the following:

$$\left|\#\{k\ :\ \alpha_k^\Omega > 1-\delta\} - |\Omega|\right| \leq \max\left\{\frac{1}{\delta}, \frac{1}{1-\delta}\right\} \left| \int_\Omega \int_\Omega |\mathcal{V}_\varphi \varphi(z-z')|^2 \, dz \, dz' - |\Omega| \right|. \quad (2.7)$$

Upon a dilation of the region Ω, it turns out that the number of eigenvalues of close to 1 is asymptotically equal to the area of the region. Given a dilation $r\Omega$ of the region, denote by $\alpha_k^{r\Omega}$ the kth eigenvalue of the time-frequency localization operator

$\mathbf{H}_{r\Omega,\varphi}$. The distribution of the eigenvalues satisfies

$$\lim_{r\to\infty} \frac{\#\{k\,:\,\alpha_k^{r\Omega} > 1-\delta\}}{|r\Omega|} = 1. \tag{2.8}$$

The asymptotic distribution (2.8) was proved by Ramanathan and Topiwala in [82] with the assumption that the region Ω has C^1 boundary. Generalities and refinements of the result appear in [47, 29, 58, 59, 2, 3].

We note that these properties on eigenvalues and eigenfunctions are analogous to that of the localization operators of Landau, Pollak, and Slepian. For those localization operators consisting of time- and band-limiting operators, the eigenfunctions are the prolate spheroidal wave functions. Along with new results in time- and band-limiting, the works of Landau, Pollak, and Slepian have been compiled in recent the book [62].

2.3. Time-frequency concentration on a region

2.3.1. Localization measures. We consider various measures of a function's concentration in compact sets in $\mathbb{R}^d$ or in the time-frequency plane.

Definition 2.3. Let φ be a window function in $L^2(\mathbb{R}^d)$ with $\|\varphi\|_2 = 1$, let T and F be compact intervals in $\mathbb{R}^d$, and let Ω be a compact subset of $\mathbb{R}^{2d}$.

(1) A function $f \in L^2(\mathbb{R}^d)$ is *ε-concentrated* in T if

$$\int_T |f(t)|^2\,dt \geq (1-\varepsilon)\|f\|_2^2, \tag{2.9}$$

or equivalently,

$$\int_{T^c} |f(t)|^2\,dt \leq \varepsilon\|f\|_2^2. \tag{2.10}$$

If $\hat{f}$ is ε-concentrated in F, then we also say that f is *ε-band-limited* in F.

(2) A function $f \in L^2(\mathbb{R}^d)$ is *(ε,φ)-concentrated* in Ω if the *time-frequency concentration* $\mathcal{E}_{\Omega,\varphi}(f)$ given by

$$\mathcal{E}_{\Omega,\varphi}(f) = \iint_\Omega |\mathcal{V}_\varphi f(z)|^2 dz = \langle \mathbf{H}_{\Omega,\varphi} f, f\rangle,$$

satisfies

$$\mathcal{E}_{\Omega,\varphi}(f) \geq (1-\varepsilon)\|f\|_2^2,$$

or equivalently, the *time-frequency (concentration) remainder* $\mathcal{E}^{\mathrm{rem}}_{\Omega,\varphi}(f)$ satisfies

$$\mathcal{E}^{\mathrm{rem}}_{\Omega,\varphi}(f) := \langle (I - \mathbf{H}_{\Omega,\varphi})f, f\rangle \leq \varepsilon\|f\|_2^2.$$

(3) A function $f \in L^2(\mathbb{R}^d)$ is *ε-localized* with respect to an operator L if

$$\|Lf - f\|_2^2 \leq \varepsilon\|f\|_2^2.$$

REMARK 2.4.

(1) If f is ε-concentrated in T, with $0 \leq \varepsilon \leq 1/4$, then most of the energy of f is concentrated in T. In this case, we also say that f is (ε-)essentially concentrated/time-limited in T. Similarly, respective to the other definitions, we also say that f is (ε-)essentially band-limited in F, (ε-)essentially concentrated in Ω, (ε-)essentially localized with respect to L.
(2) In terms of the time-limiting operator $\mathbf{P}_T$, f is ε-concentrated in T if and only if f is ε-localized with respect to $\mathbf{P}_T$, i.e. $\|\mathbf{P}_T f - f\|_2^2 \leq \varepsilon \|f\|_2^2$. Similarly, f is ε-band-limited in F if and only if f is ε-localized with respect to the band-limiting operator $\mathbf{Q}_F$, i.e. $\|\mathbf{Q}_F f - f\|_2^2 \leq \varepsilon \|f\|_2^2$
(3) In [72], Landau introduced the notion of ε-approximated eigenvalues and eigenfunctions. α is said to be an ε-approximated eigenvalue of L if there exists f with $\|f\|_2 = 1$, such that $\|Lf - \alpha f\|_2 \leq \varepsilon$; f is called an ε-approximated eigenfunction corresponding to α. So a function $f \in L^2(\mathbb{R}^d)$ that is ε-localized with respect to L is a $\sqrt{\varepsilon}$-approximated eigenfunction of L corresponding to 1.

Unlike $\mathbf{P}_T$ and $\mathbf{Q}_F$, $\mathbf{H}_{\Omega,\varphi}$ is not a projection so we do not get an immediate result as in Remark 2.4(2) for $\mathbf{H}_{\Omega,\varphi}$. Instead, we have the following comparison.

Lemma 2.5. *Let φ be a window function in $L^2(\mathbb{R}^d)$ with $\|\varphi\|_2 = 1$, and Ω be a compact set in $\mathbb{R}^{2d}$. If $f \in L^2(\mathbb{R}^d)$ is (ε, φ)-concentrated in Ω, then f is also ε-localized with respect to $\mathbf{H}_{\Omega,\varphi}$. On the other hand, if f is ε-localized with respect to $\mathbf{H}_{\Omega,\varphi}$, then f is $(\varepsilon + \sqrt{\varepsilon}, \varphi)$-concentrated in Ω.*

Proof: Since $\|\mathbf{H}_{\Omega,\varphi} f\|_2 \leq \|f\|_2$, i.e. $\|\mathbf{H}_{\Omega,\varphi}\|_{B(L^2(\mathbb{R}^d))} \leq 1$, we have

$$\langle \mathbf{H}_{\Omega,\varphi}^2 f, f \rangle \leq \langle \mathbf{H}_{\Omega,\varphi} f, f \rangle,$$

or equivalently,

$$\langle (I - \mathbf{H}_{\Omega,\varphi})^2 f, f \rangle \leq \langle (I - \mathbf{H}_{\Omega,\varphi}) f, f \rangle.$$

Since $\mathbf{H}_{\Omega,\varphi}$ is self-adjoint, the left-hand side is equal to $\|\mathbf{H}_{\Omega,\varphi} f - f\|_2^2$, so the first statement is obtained.

For the second statement, we observe that

$$\begin{aligned} 2\langle (I - \mathbf{H}_{\Omega,\varphi}) f, f \rangle &= \|\mathbf{H}_{\Omega,\varphi} f - f\|_2^2 + \|f\|_2^2 - \|\mathbf{H}_{\Omega,\varphi} f\|_2^2 \\ &\leq \|\mathbf{H}_{\Omega,\varphi} f - f\|_2^2 + (\|\mathbf{H}_{\Omega,\varphi} f - f\|_2 + \|\mathbf{H}_{\Omega,\varphi} f\|_2)^2 - \|\mathbf{H}_{\Omega,\varphi} f\|_2^2 \\ &= 2\|\mathbf{H}_{\Omega,\varphi} f - f\|_2^2 + 2\|\mathbf{H}_{\Omega,\varphi} f - f\|_2 \|\mathbf{H}_{\Omega,\varphi} f\|_2. \end{aligned}$$

So we have

$$\langle (I - \mathbf{H}_{\Omega,\varphi}) f, f \rangle \leq \|\mathbf{H}_{\Omega,\varphi} f - f\|_2^2 + \|\mathbf{H}_{\Omega,\varphi} f - f\|_2 \|f\|_2,$$

and the result follows. ∎

We note that measuring the localization of a function in Ω via a time-frequency localization operator would naturally depend on the window function φ. The next result shows how the time-frequency concentration changes given a change in the window function.

Lemma 2.6. *Given window functions φ and φ'. Then for every $f \in L^2(\mathbb{R}^d)$,*

$$|\mathcal{E}_{\Omega,\varphi}(f) - \mathcal{E}_{\Omega,\varphi'}(f)| \leq (\|\varphi - \varphi'\|_2^2 + 2\|\varphi\|_2\|\varphi - \varphi'\|_2)\|f\|_2^2.$$

Proof: From the boundedness of the time-frequency localization operators in $L^2(\mathbb{R}^d)$, we get the following estimate:

$$\begin{aligned}
|\mathcal{E}_{\Omega,\varphi}(f) - \mathcal{E}_{\Omega,\varphi'}(f)| &= |\langle \mathbf{H}_{\Omega,\varphi} f, f\rangle - \langle \mathbf{H}_{\Omega,\varphi'} f, f\rangle| \\
&= |\langle \mathcal{V}_\varphi^* \mathcal{M}_\Omega \mathcal{V}_\varphi f, f\rangle - \langle \mathcal{V}_{\varphi'}^* \mathcal{M}_\Omega \mathcal{V}_{\varphi'} f, f\rangle| \\
&= |\langle (\mathcal{V}_\varphi^* \mathcal{M}_\Omega \mathcal{V}_\varphi - \mathcal{V}_{\varphi'}^* \mathcal{M}_\Omega \mathcal{V}_{\varphi'}) f, f\rangle| \\
&= |\langle (\mathcal{V}_{\varphi-\varphi'}^* \mathcal{M}_\Omega \mathcal{V}_{\varphi-\varphi'} + \mathcal{V}_{\varphi-\varphi'}^* \mathcal{M}_\Omega \mathcal{V}_{\varphi'} + \mathcal{V}_{\varphi'}^* \mathcal{M}_\Omega \mathcal{V}_{\varphi-\varphi'}) f, f\rangle| \\
&\leq |\langle \mathcal{V}_{\varphi-\varphi'}^* \mathcal{M}_\Omega \mathcal{V}_{\varphi-\varphi'} f, f\rangle| + |\langle \mathcal{V}_{\varphi-\varphi'}^* \mathcal{M}_\Omega \mathcal{V}_{\varphi'} f, f\rangle| \\
&\quad + |\langle \mathcal{V}_{\varphi'}^* \mathcal{M}_\Omega \mathcal{V}_{\varphi-\varphi'} f, f\rangle| \\
&= |\langle \mathbf{H}_{\Omega,\varphi-\varphi'} f, f\rangle| + |\langle \mathcal{V}_{\varphi-\varphi'}^* \mathcal{M}_\Omega \mathcal{V}_{\varphi'} f, f\rangle| + |\langle \mathcal{V}_{\varphi'}^* \mathcal{M}_\Omega \mathcal{V}_{\varphi-\varphi'} f, f\rangle| \\
&\leq \|\mathbf{H}_{\Omega,\varphi-\varphi'} f\|_2 \|f\|_2 + \|\mathcal{V}_{\varphi-\varphi'}^* \mathcal{M}_\Omega \mathcal{V}_{\varphi'} f\|_2 \|f\|_2 \\
&\quad + \|\mathcal{V}_{\varphi'}^* \mathcal{M}_\Omega \mathcal{V}_{\varphi-\varphi'} f\|_2 \|f\|_2 \\
&\leq (\|\chi_\Omega\|_\infty \|\varphi - \varphi'\|_2^2 + 2\|\varphi'\|_2 \|\varphi - \varphi'\|_2) \|f\|_2^2
\end{aligned}$$

■

We now show how the concentration of f and $\hat{f}$ on intervals is related to the concentration of f on a rectangular region in the time-frequency plane. We shall make use of the following lemma.

Lemma 2.7. *For any $0 < a < A$, the following inequalities hold:*

$$(1) \int_{Q_A^c} \int_{\mathbb{R}^d} |\mathcal{V}_\varphi f(x,\omega)|^2 d\omega dx \leq \int_{Q_{A-a}^c} |f(t)|^2 dt \int_{Q_a} |\varphi(t)|^2 dt + \|f\|_2^2 \int_{Q_a^c} |\varphi(t)|^2 dt$$

$$(2) \int_{Q_A} \int_{\mathbb{R}^d} |\mathcal{V}_\varphi f(x,\omega)|^2 d\omega dx \leq \int_{Q_{A+a}} |f(t)|^2 dt \int_{Q_a} |\varphi(t)|^2 dt + \|f\|_2^2 \int_{Q_a^c} |\varphi(t)|^2 dt$$

Proof: To prove (1), we write the STFT of f with respect to φ as a Fourier transform and apply Plancherel's theorem:

$$\begin{aligned}
\int_{Q_A^c} \int_{\mathbb{R}^d} |\mathcal{V}_\varphi f(x,\omega)|^2 d\omega dx &= \int_{Q_A^c} \int_{\mathbb{R}^d} |f \cdot \mathbf{T}_x \overline{\varphi}(t)|^2 dt dx \\
&= \int_{Q_A^c} \int_{\mathbb{R}^d} |f(t+x)|^2 |\varphi(t)|^2 dt dx
\end{aligned}$$

$$= \int_{Q_A^c}\int_{\mathbb{R}^d} |f(t+x)|^2 |\varphi(t)\chi_{Q_a}(t)|^2 dt dx + \int_{Q_A^c}\int_{\mathbb{R}^d} |f(t+x)|^2 |\varphi(t)\chi_{Q_a^c}(t)|^2 dt dx.$$

By Fubini's theorem, we can interchange the order of integration in the first term of the last equality and estimate it as follows:

$$\begin{aligned}\int_{Q_A^c}\int_{\mathbb{R}^d} |f(t+x)|^2 |\varphi(t)\chi_{Q_a}(t)|^2 dt dx &= \int_{Q_A^c}\int_{Q_a} |f(t+x)|^2 |\varphi(t)|^2 dt dx \\ &\leq \int_{\mathbb{R}^d}\int_{Q_{A-a}^c} |f(x)|^2 |\varphi(t)|^2 dx dt.\end{aligned}$$

For the second term, we set $\mathcal{I}\varphi(t) = \varphi(-t)$ and we obtain the following:

$$\begin{aligned}\int_{Q_A^c}\int_{\mathbb{R}^d} |f(t+x)|^2 |\varphi(t)\chi_{Q_a^c}(t)|^2 dt dx &\leq \int_{Q_A^c} (|f|^2 * |\mathcal{I}(\varphi \cdot \chi_{Q_a^c})|^2)(x) dx \\ &\leq \||f|^2 * |\mathcal{I}(\varphi \cdot \chi_{Q_a^c})|^2\|_1 \\ &\leq \|f\|_2^2 \, \|\varphi \cdot \chi_{Q_a^c}\|_2^2.\end{aligned}$$

Let us now prove (2). Again, we make use of Plancherel's theorem and Fubini's theorem.

$$\begin{aligned}\int_{Q_A}\int_{\mathbb{R}^d} |\mathcal{V}_\varphi f(x,\omega)|^2 d\omega dx &= \int_{Q_A}\int_{\mathbb{R}^d} |f(x+t)|^2 |\varphi(t)|^2 dt dx \\ &= \int_{Q_A}\int_{Q_a} |f(x+t)|^2 |\varphi(t)|^2 dt dx \\ &\quad + \int_{Q_A}\int_{Q_a^c} |f(x+t)|^2 |\varphi(t)|^2 dt dx \\ &= \int_{Q_a} |\varphi(t)|^2 \int_{Q_A} |f(x+t)|^2 dx dt \\ &\quad + \int_{Q_a^c} |\varphi(t)|^2 \int_{Q_A} |f(x+t)|^2 dx dt \\ &\leq \int_{Q_a} |\varphi(t)|^2 \int_{Q_{A+a}} |f(x)|^2 dx dt + \|f\|_2^2 \int_{Q_a^c} |\varphi(t)|^2 dt\end{aligned}$$

■

Proposition 2.8. *Let φ be a window function in $L^2(\mathbb{R}^d)$ with $\|\varphi\|_2 = 1$. Suppose that φ is ε_1-concentrated in $Q_a = [-a,a]^d$ with Fourier transform $\hat{\varphi}$ that is ε_2-concentrated in $Q_b = [-b,b]^d$.*

(1) *If f is $\frac{\varepsilon}{2}$-concentrated in Q_{A-a} and $\hat{f}$ is $\frac{\varepsilon}{2}$-concentrated in Q_{B-b}, then f is $(\varepsilon + \varepsilon_1 + \varepsilon_2, \varphi)$-concentrated in $Q_A \times Q_B$.*

(2) *If f is (ε, φ)-concentrated in $Q_A \times Q_B$, then f and $\hat{f}$ are $(\varepsilon+\varepsilon_1)$- and $(\varepsilon+\varepsilon_2)$-concentrated in Q_{A+a} and Q_{B+b}, respectively.*

Proof:

(1) Using Lemma 2.7(1), also for $\hat{f}$ and $\hat{\varphi}$, we obtain the following inequality which gives the desired result:

$$\begin{aligned}\int_{Q_A^c}\int_{Q_B^c} |\mathcal{V}_\varphi f(x,\omega)|^2 d\omega dx \le &\int_{Q_{A-a}^c} |f(t)|^2 dt \int_{Q_a} |\varphi(t)|^2 dt \\ &+ \int_{Q_{B-b}^c} |\hat{f}(\omega)|^2 d\omega \int_{Q_b} |\hat{\varphi}(\omega)|^2 d\omega \\ &+ \|f\|_2^2 \Big(\int_{Q_a^c} |\varphi(t)|^2 dt + \int_{Q_b^c} |\hat{\varphi}(\omega)|^2 d\omega \Big).\end{aligned}$$

(2) By assumption, we have

$$\begin{aligned}(1-\varepsilon)\|f\|_2^2 &\le \int_{Q_A}\int_{Q_B} |\mathcal{V}_\varphi f(x,\omega)|^2 d\omega dx \\ &\le \int_{Q_A}\int_{\mathbb{R}^d} |\mathcal{V}_\varphi f(x,\omega)|^2 d\omega dx,\end{aligned}$$

and using Lemma 2.7(2), we obtain

$$\begin{aligned}(1-\varepsilon)\|f\|_2^2 &\le \int_{Q_{A+a}} |f(t)|^2 dt \int_{Q_a} |\varphi(t)|^2 dt + \|f\|_2^2 \int_{Q_a^c} |\varphi(t)|^2 dt \\ &\le \int_{Q_{A+a}} |f(t)|^2 dt + \|f\|_2^2 \int_{Q_a^c} |\varphi(t)|^2 dt.\end{aligned}$$

Since the second term in the inequality above is less than $\varepsilon_1 \|f\|_2^2$ by assumption, transposing the term yields

$$(1-\varepsilon-\varepsilon_1)\|f\|_2^2 \le \int_{Q_{A+a}} |f(t)|^2 dt.$$

The case for $\hat{f}$ being $(\varepsilon+\varepsilon_2)$-concentrated on Q_{B+b} is proved similarly, applying Lemma 2.7(2) to $\hat{f}$ and $\hat{\varphi}$. ■

The various notions of concentration lead to different versions of the uncertainty principle. In terms of our definition of a function's concentration on an interval T and the concentration of its Fourier transform on F, the uncertainty principle by Donoho and Stark, cf. [32], is as follows.

Theorem 2.9 (Donoho-Stark)**.** *Suppose that $f \in L^2(\mathbb{R}^d)$, $f \neq 0$, is ε_T-concentrated in $T \subseteq \mathbb{R}^d$ and ε_F-band-limited in $F \subseteq \mathbb{R}^d$. Then*

$$|T||F| \ge (1 - \sqrt{\varepsilon_T} - \sqrt{\varepsilon_F})^2.$$

For an uncertainty principle in terms of the STFT, we have the following weak uncertainty principle, cf. [55], whose proof follows immediately from the function's concentration in Ω.

Proposition 2.10. [55, Proposition 3.3.1] *Suppose that* $\|f\|_2 = \|\varphi\|_2 = 1$ *and that* $\Omega \subseteq \mathbb{R}^{2d}$. *If* $\varepsilon \geq 0$ *and* f *is* (ε, φ)*-concentrated in* Ω, *then* $|\Omega| \geq 1 - \varepsilon$.

Proof: Since $|\mathcal{V}_\varphi f(z)| = |\langle f, \pi(z)\varphi\rangle| \leq \|f\|_2\|\varphi\|_2 = 1$ for all $z \in \mathbb{R}^{2d}$, and f is (ε, φ)-concentrated in Ω, it follows that

$$1 - \varepsilon \leq \langle \mathbf{H}_{\Omega,\varphi} f, f\rangle = \iint_\Omega |\mathcal{V}_\varphi f(z)|^2\, dz \leq \|\mathcal{V}_\varphi\|_\infty^2 |\Omega| \leq |\Omega|.$$

■

Remark 2.11. A sharper estimate on the size of Ω was also proved in [55, Theorem 3.3.3], yielding $|\Omega| \geq 2^d(1 - \varepsilon)^2$.

2.3.2. Functions concentrated in Ω.

We denote by $\mathscr{C}(\Omega, \varepsilon, \varphi)$ the set of functions in $L^2(\mathbb{R}^d)$ that are (ε, φ)-concentrated in a compact subset Ω of $\mathbb{R}^{2d}$:

$$\mathscr{C}(\Omega, \varepsilon, \varphi) = \{f \in L^2(\mathbb{R}^d) \,:\, \mathcal{E}_{\Omega,\varphi}(f) \geq (1 - \varepsilon)\|f\|_2^2\}.$$

Each eigenfunction ψ_k of $\mathbf{H}_{\Omega,\varphi}$ with eigenvalue $\alpha_k \geq (1 - \varepsilon)$ from the spectral representation (2.3) is in $\mathscr{C}(\Omega, \varepsilon, \varphi)$. Indeed, $\langle \mathbf{H}_{\Omega,\varphi}\psi_k, \psi_k\rangle = \alpha_k \geq (1-\varepsilon)$. Moreover, if we let

$$V_N = \operatorname{span}\{\psi_k \,:\, k = 1, \ldots, N\}$$

be the span of the first N eigenfunctions of the time-frequency localization operator $\mathbf{H}_{\Omega,\varphi}$, then for $f = \sum_{k=1}^N \langle f, \psi_k\rangle\psi_k \in V_N$, we have

$$\langle \mathbf{H}_{\Omega,\varphi} f, f\rangle = \sum_{k=1}^N \alpha_k |\langle f, \psi_k\rangle|^2 \geq \alpha_N \sum_{k=1}^N |\langle f, \psi_k\rangle|^2 = \alpha_N\|f\|_2^2, \tag{2.11}$$

i.e. f is $(1 - \alpha_N, \varphi)$-concentrated in Ω. So for a properly chosen N, functions in V_N are in $\mathscr{C}(\Omega, \varepsilon, \varphi)$.

In contrast, functions which are $(1 - \alpha_N, \varphi)$-concentrated in Ω need not lie in V_N. The following proposition characterizes a function that is (ε, φ)-concentrated on Ω.

Proposition 2.12. *Let* φ, Ω *and* ε *be given and let* N_0 *be the integer such that* $\alpha_{N_0} \geq 1-\varepsilon$ *and* $\alpha_{N_0+1} < 1-\varepsilon$. *Furthermore, let* f_{ker} *denote the orthogonal projection of* f *onto the kernel* $\ker(\mathbf{H}_{\Omega,\varphi})$ *of* $\mathbf{H}_{\Omega,\varphi}$. *A function* f *in* $L^2(\mathbb{R}^d)$ *is* (ε, φ)*-concentrated on* Ω *if and only if*

$$\sum_{k=1}^{N_0} (\alpha_k + \varepsilon - 1)|\langle f, \psi_k\rangle|^2 \geq \sum_{k=N_0+1}^{\infty} (1 - \varepsilon - \alpha_k)|\langle f, \psi_k\rangle|^2 + (1 - \varepsilon)\|f_{ker}\|_2^2$$

Proof: The eigenfunctions $\{\psi_k\}_k$ form an orthonormal subset in $L^2(\mathbb{R}^d)$, possibly incomplete if $\ker(\mathbf{H}_{\Omega,\varphi}) \neq \{0\}$; hence, we can write $f = \sum_{j=1}^\infty \langle f, \psi_j\rangle\psi_j + f_{ker}$, where

$f_{ker} \in \ker(\mathbf{H}_{\Omega,\varphi})$ and, as in (2.11), $\langle \mathbf{H}_{\Omega,\varphi} f, f\rangle = \sum_{k=1}^{\infty} \alpha_k |\langle f, \psi_k\rangle|^2$. So the function f is (ε,φ)-concentrated on Ω if and only if

$$\sum_{k=1}^{\infty} \alpha_k |\langle f, \psi_k\rangle|^2 \geq (1-\varepsilon)\left(\sum_{k=1}^{\infty} |\langle f, \psi_k\rangle|^2 + \|f_{ker}\|_2^2\right),$$

and the conclusion follows. ∎

Remark 2.13. Despite the interpretation of $\mathbf{H}_{\Omega,\varphi} f$ as the part of f that essentially lies in Ω, it is possible that the resulting function $\mathbf{H}_{\Omega,\varphi} f$ is not (ε,φ)-concentrated in Ω. In fact, for every eigenfunction ψ_k with corresponding eigenvalue $\alpha_k < 1-\varepsilon$,

$$\langle \mathbf{H}_{\Omega,\varphi}(\mathbf{H}_{\Omega,\varphi}\psi_k), \mathbf{H}_{\Omega,\varphi}\psi_k\rangle = \alpha_k^3 = \alpha_k \|\mathbf{H}_{\Omega,\varphi}\psi_k\|_2^2,$$

i.e. $\mathbf{H}_{\Omega,\varphi}\psi_k$ is not (ε,φ)-concentrated in Ω.

Remark 2.14. We emphasize that $\mathscr{C}(\Omega,\varepsilon,\varphi)$ is not a linear space. Indeed, consider the eigenfunction ψ_M corresponding to the eigenvalue $\alpha_M > 1-\varepsilon$. Let $h = \sum_{k\in\mathbb{Z}} c_k \psi_k$ such that the sequence $\{c_k\}_{k\in\mathbb{Z}}$ satisfies the following conditions:

$$0 < c_M < \frac{1-\varepsilon}{\alpha_M}, \ \sum_{k\in\mathbb{Z}} c_k^2 = 1, \ \text{ and } \ \sum_{k\in\mathbb{Z}} \alpha_k c_k^2 = 1-\eta\varepsilon, \ 1 < \eta < \frac{1}{\varepsilon}.$$

It follows that $\|h\|_2 = 1$ and $\langle \mathbf{H}_{\Omega,\varphi} h, h\rangle = 1-\eta\varepsilon < 1-\varepsilon$ so that $h \notin \mathscr{C}(\Omega,\varepsilon,\varphi)$. Choose δ such that $0 < \delta \leq \dfrac{2c_M(\alpha_M - (1-\varepsilon))}{\varepsilon(\eta-1)}$, and let $f = \psi_M + \delta h$. We calculate

$$\begin{aligned}\langle \mathbf{H}_{\Omega,\varphi} f, f\rangle &= \langle \mathbf{H}_{\Omega,\varphi}\psi_M, \psi_M\rangle + 2\delta \operatorname{Re}\langle \mathbf{H}_{\Omega_\varphi}\psi_M, h\rangle + \delta^2 \langle \mathbf{H}_{\Omega,\varphi} h, h\rangle \\ &= \alpha_M + 2\delta\alpha_M c_M + \delta^2(1-\eta\varepsilon).\end{aligned}$$

It follows from the conditions above that the right-hand side of the equation is greater than or equal to $(1+2\delta c_M+\delta^2)(1-\varepsilon)$, which is equal to $\|f\|_2^2(1-\varepsilon)$. So $f \in \mathscr{C}(\Omega,\varepsilon,\varphi)$, but $f - \psi_M = \delta h \notin \mathscr{C}(\Omega,\varepsilon,\varphi)$.

While a function f that is (ε,φ)-concentrated in Ω does not necessarily lie in some subspace V_N of eigenfunctions of $\mathbf{H}_{\Omega,\varphi}$, it can be approximated using a finite number of such eigenfunctions. Let $\mathcal{P}_{V_N}$ denote the orthogonal projection onto the subspace V_N. We note that approximations of band-limited functions via projections onto eigenspaces of time- and band-limited functions were presented in [90, 63, 10].

Proposition 2.15. *Let f be (ε,φ)-concentrated on $\Omega \subset \mathbb{R}^{2d}$. For fixed $c > 1$, let ψ_k, $k = 1,\ldots,N$, be all eigenfunctions of $\mathbf{H}_{\Omega,\varphi}$ corresponding to eigenvalues $\alpha_k > \frac{c-1}{c}$. Then*

(1) $\|\mathcal{P}_{V_N} f\|_2^2 \geq (1-c\varepsilon)\|f\|_2^2$,
(2) $\left\| f - \mathcal{P}_{V_N} f\right\|_2^2 < c\varepsilon\|f\|_2^2$, *and*
(3) $\mathcal{E}_{\Omega,\varphi}(\mathcal{P}_{V_N} f) \geq \alpha_N(1-c\varepsilon)\|f\|_2^2$.

Proof: Without loss of generality, we assume that $\|f\|_2 = 1$. We have, by assumption:

$$\langle \mathbf{H}_{\Omega,\varphi} f, f\rangle = \sum_{k=1}^{\infty} \alpha_k |\langle f, \psi_k\rangle|^2 = \iint_\Omega |\mathcal{V}_\varphi f(z)|^2 dz \geq (1-\varepsilon)\|f\|_2^2$$

We argue by contradiction; to this end, assume that $\sum_{k=1}^{N} |\langle f, \psi_k\rangle|^2 = K < 1 - c\varepsilon$. Furthermore

$$\|f\|_2^2 = 1 = \sum_{k=1}^{\infty} |\langle f, \psi_k\rangle|^2 + \|f_{ker}\|_2^2\,,$$

hence

$$\sum_{k=N+1}^{\infty} |\langle f, \psi_k\rangle|^2 = 1 - K - \|f_{ker}\|_2^2.$$

We then have

$$\sum_{k=N+1}^{\infty} \alpha_k |\langle f, \psi_k\rangle|^2 < \frac{c-1}{c} \cdot (1 - K - \|f_{ker}\|_2^2)$$

such that

$$\begin{aligned}\sum_{k=1}^{\infty} \alpha_k |\langle f, \psi_k\rangle|^2 <& K + \frac{c-1}{c} \cdot (1 - K - \|f_{ker}\|_2^2)\\ =& \frac{c-1+K}{c} - \frac{c-1}{c}\|f_{ker}\|_2^2\\ <& 1 + \frac{1-c\varepsilon-1}{c} - \frac{c-1}{c}\|f_{ker}\|_2^2 < 1-\varepsilon,\end{aligned}$$

which is a contradiction. Hence, $\sum_{k=1}^{N} |\langle f, \psi_k\rangle|^2$ must be greater than or equal to $1 - c\varepsilon$.

The second inequality follows from the decomposition of f into $f = \mathcal{P}_{V_N} + (f - \mathcal{P}_{V_N} f)$, which gives

$$\|f - \mathcal{P}_{V_N} f\|_2^2 = 1 - \|\mathcal{P}_{V_N} f\|_2^2 \leq 1 - (1 - c\varepsilon) = c\varepsilon.$$

And for the third inequality, we have

$$\mathcal{E}_{\Omega,\varphi}(\mathcal{P}_{V_N} f) = \langle \mathbf{H}_{\Omega,\varphi} \mathcal{P}_{V_N} f, \mathcal{P}_{V_N} f\rangle = \sum_{k=1}^{N} \alpha_k |\langle f, \psi_k\rangle|^2 \geq \alpha_N \|\mathcal{P}_{V_N} f\|_2^2 \geq \alpha_N (1 - c\varepsilon).$$

■

Remark 2.16. Projections onto subspaces generated by eigenfunctions of compact self-adjoint operators have been used as time-frequency filters. In [61], Hlawatsch, et. al. used eigenfunctions of a linear operator via the Wigner distribution. Dörfler [33], on the other hand, used Gabor multipliers to obtain the projection operators onto time-frequency localized subspaces.

Remark 2.17. In [70, 71], Jaming, et. al. investigated the approximation of essentially time- and band-limited functions via expansions in the Hermite, Legendre, and Chebyshev bases.

2.3.3. Spectrogram of a subspace and accumulated spectrograms. Given an N-dimensional subspace V of $L^2(\mathbb{R})$, $\mathcal{P}_V$ the orthogonal projection onto V with projection kernel κ_V, i.e. $\mathcal{P}_V f(t) = \int_{\mathbb{R}} \kappa_V(t,y)f(y)\,dy$, recall that if $\{e_k\}_{k=1}^N$ is an orthonormal basis of V, then $\kappa_V(t,y) = \sum_{k=1}^N e_k(t)\overline{e_k(y)}$. The kernel κ_V is independent of the choice of orthonormal basis for V.

In [60], different quadratic signal representations, e.g. the Wigner distribution, spectral energy density, ambiguity function, were extended to a linear signal space. We consider here the *spectrogram* $\mathrm{SPEC}_\varphi V$ *of the subspace* V with window function φ defined as

$$\mathrm{SPEC}_\varphi V(x,\omega) = \iint_{\mathbb{R}^{2d}} \kappa_V(t,y)\overline{\varphi(t-x)}\,\varphi(y-x)\,e^{-2\pi i\omega\cdot(t-y)}\,dt\,dy.$$

If the subspace V is the subspace V_N consisting of the first N eigenfunctions $\psi_1,\dots,\psi_N$ of $\mathbf{H}_{\Omega,\varphi}$ corresponding to the N largest eigenvalues $\{\alpha\}_{k=1}^N$, then $\kappa_{V_N}(t,y) = \sum_{k=1}^N \psi_k(t)\overline{\psi_k(y)}$ and

$$\begin{aligned}\mathrm{SPEC}_\varphi V(x,\omega) &= \iint_{\mathbb{R}^{2d}} \sum_{k=1}^N \psi_k(t)\overline{\psi_k(y)}\;\overline{\varphi(t-x)}\,\varphi(y-x)\,e^{-2\pi i\omega(t-y)}\,dt\,dy\\ &= \sum_{k=1}^N \int_{\mathbb{R}^d} \psi_k(t)\,\overline{\varphi(t-x)}\,e^{-2\pi i\omega\cdot t}\,dt \int_{\mathbb{R}^d} \overline{\psi_k(y)}\,\varphi(y-x)\,e^{2\pi i\omega\cdot y}\,dy\\ &= \sum_{k=1}^N \mathcal{V}_\varphi\psi_k(x,\omega)\,\overline{\mathcal{V}_\varphi\psi_k(x,\omega)} = \sum_{k=1}^N |\mathcal{V}_\varphi\psi_k(x,\omega)|^2.\end{aligned}$$

Similar to the definition of a function f's concentration $\mathcal{E}_{\Omega,\varphi}(f)$, we define the *time-frequency concentration of a subspace* V_N in Ω as

$$\mathcal{E}_{\Omega,\varphi}(V_N) := \frac{1}{N}\iint_{\Omega} \mathrm{SPEC}_g V_N(x,\omega)\,dx\,d\omega.$$

If the ψ_ks are eigenfunctions of the localization operator $\mathbf{H}_{\Omega,\varphi}$, then $\mathcal{E}_{\Omega,\varphi}(V_N) = \frac{1}{N}\sum_{k=1}^N \alpha_k$. We can see that $\alpha_N \le \mathcal{E}_{\Omega,\varphi}(V_N) \le \alpha_1$. The min-max characterization of the eigenvalues of compact operators implies that any N-dimensional subset cannot be better concentrated in Ω, i.e. if V_N' is any N-dimensional subspace of $L^2(\mathbb{R})$, then $\mathcal{E}_{\Omega,\varphi}(V_N') \le \mathcal{E}_{\Omega,\varphi}(V_N)$.

In [2], Abreu, Gröchenig, and Romero showed that the corresponding spectrograms of the first $\lceil\Omega\rceil$ eigenfunctions of $\mathbf{H}_{\Omega,\varphi}$ approximately form a partition of unity on Ω. Define the *accumulated spectrogram* of Ω with respect to φ as the spectrogram of the subspace $V_{\lceil\Omega\rceil}$ consisting of the eigenfunctions ψ_k, $k = 1,\dots,\lceil\Omega\rceil$ of $\mathbf{H}_{\Omega,\varphi}$, i.e. $\mathrm{SPEC}_\varphi V_{\lceil\Omega\rceil}(z)$. They derived the following asymptotic, non-asymptotic, and weak L^2 estimates for the accumulated spectrogram.

Theorem 2.18. [2] *Let $\varphi \in L^2(\mathbb{R}^d)$, $\|\varphi\|_2 = 1$, and let $\Omega \subset \mathbb{R}^{2d}$ be compact.*

(1) *The accumulated spectrogram $\operatorname{SPEC}_\varphi V_{\lceil R\Omega \rceil}(R\cdot)$ converges to the characteristic function χ_Ω in $L^1(\mathbb{R}^{2d})$ as $R \to \infty$.*
(2) *If φ satisfies $\|\varphi\|^2_{M^*} := \int_{\mathbb{R}^{2d}} |z|\, |\mathcal{V}_\varphi \varphi(z)|^2\, dz < \infty$ and Ω has finite perimeter given by $|\partial\Omega|$, then*

$$\frac{1}{|\Omega|} \| \operatorname{SPEC}_\varphi V_{\lceil \Omega \rceil} - \chi_\Omega * |\mathcal{V}_\varphi \varphi|^2 \|_1 \leq \left(\frac{1}{|\Omega|} + 4\|g\|_{M^*} \sqrt{\frac{|\partial\Omega|}{|\Omega|}} \right).$$

(3) *If g and Ω satisfy $1 \leq \|g\|^2_{M^*} |\partial\Omega| < \infty$, then*

$$|\{ z \in \mathbb{R}^{2d} \,:\, | \operatorname{SPEC}_\varphi V_{\lceil \Omega \rceil}(z) - \chi_\Omega(z)| > \delta \}| \lesssim \frac{1}{\delta^2} \|g\|^2_{M^*} |\partial\Omega|, \; \delta > 0.$$

CHAPTER 3

Sampling and approximation of time-frequency localized functions

The STFT of a function provides a continuous joint time-frequency representation for a function f, wherein by the inversion formula, we are able to recover f from the information encoded in $\mathcal{V}_\varphi f(z)$, $z \in \mathbb{R}^{2d}$. This representation is highly redundant. By the use of Gabor frames, we are able to obtain a discrete representation of f from the samples of the STFT without information loss.

In this chapter, we will investigate how well the frame expansion of a function f captures its time-frequency localization. In particular, we will consider in Section 3.1 truncations of the Gabor frame expansion of f: $\sum_{\lambda \in \Lambda \cap \Omega^*} \langle f, g_\lambda \rangle \tilde{g_\lambda}$. We shall recall, among others, the result of Daubechies in [26] showing under certain conditions that a function can be reasonably approximated by a truncated version of the frame expansion assuming that a function is essentially localized in time and in frequency. For a compact region Ω in the time-frequency plane, we obtain a similar approximation for f by a truncated Gabor frame expansion where the error will be expressed in terms of the concentration of f in Ω.

In Section 3.2, we consider the case where the functions come from the subspace of eigenfunctions of a time-frequency localization operator over Ω. Projecting the local time-frequency dictionary from the truncated Gabor expansions yields a frame for the subspace. If we take a family of such dictionaries corresponding to compact regions that would collectively cover the time-frequency plane, we would then obtain a global frame for $L^2(\mathbb{R})$. These will be illustrated by numerical experiments in Section 3.4.

3.1. Local Gabor approximation

In [26] (see also [20, Theorem 9.8.1]), Daubechies proved the following theorem which shows that if a function is essentially limited to a finite time interval and to a finite range in frequency, then it can essentially be represented by a finite number of expansion coefficients.

Theorem 3.1. [26, Theorem 3.1] *Suppose that the Gabor systems $\{\mathbf{M}_{mb}\mathbf{T}_{na}g\}_{m,n\in\mathbb{Z}}$ and $\{\mathbf{M}_{mb}\mathbf{T}_{na}h\}_{m,n\in\mathbb{Z}}$ form a pair of dual frames for $L^2(\mathbb{R})$ with upper frame bounds B and D, respectively, and that for some constants $C > 0, \alpha > 1/2$, the decay conditions*

$$|h(t)| \leq C(1+t^2)^{-\alpha},\ t \in \mathbb{R}, \qquad |\hat{h}(\omega)| \leq C(1+\omega^2)^{-\alpha},\ \omega \in \mathbb{R},$$

hold. Then for any $\varepsilon > 0$*, there exist numbers* y_ε, $\xi_\varepsilon > 0$ *such that for all* y, $\xi > 0$,

$$\Big\| f - \sum_{(m,n)\in B(y+y_\varepsilon,\xi+\xi_\varepsilon)} \langle f, \mathbf{M}_{mb}\mathbf{T}_{na}h\rangle \mathbf{M}_{mb}\mathbf{T}_{na}g \Big\|_2 \leq \sqrt{BD}(\|(I-\mathbf{Q}_{[-y,y]})f\|_2 + \|(I-\mathbf{P}_{[-\xi,\xi]})f\|_2 + \varepsilon\|f\|_2)$$

for all $f \in L^2(\mathbb{R})$.

Eldar and Matusiak in [80] also provided an approximation for a function f using a truncated Gabor expansion where the error is estimated via the function's respective concentration on a finite time and a finite frequency interval.

Theorem 3.2. [80, Theorem III.1] *Let* f *be a function supported on the interval* $[-\beta/2, \beta/2]$ *and* ε_ξ*-bandlimited to* $[-\xi/2, \xi/2]$*. Suppose* $\mathcal{G}(g, a, b)$ *is a Gabor frame with* g *compactly supported on* $[-\alpha/2, \alpha/2]$, $a = \mu\alpha$*, and* $b = 1/\alpha$ *for some* $\mu \in (0, 1)$*, and suppose that* $\gamma \in \boldsymbol{S}_0$ *is the dual atom. Then for every* $\varepsilon_B > 0$*, there exists an* $L_0 < \infty$*, depending on* γ *and the essential bandwidths of* g *and* f*, such that*

$$\Big\| f - \sum_{k=-K_0}^{K_0} \sum_{l=-L_0}^{L_0} \langle f, \mathbf{M}_{bl}\mathbf{T}_{ak}g\rangle \mathbf{M}_{bl}\mathbf{T}_{ak}\gamma \Big\|_2 \leq \tilde{C}_0(\varepsilon_\xi + \varepsilon_B)\|f\|_2, \tag{3.1}$$

where $\tilde{C}_0 = C_{a,b}^2 \|\gamma\|_{\boldsymbol{S}_0} \|g\|_{\boldsymbol{S}_0}$ *with* $C_{a,b} = (1 + 1/a)^{1/2}(1 + 1/b)^{1/2}$.

Remark 3.3. In Theorem 3.2, the number of frequency coefficients L_0 is determined by the essential bandwidth of g, i.e. if g_c is bandlimited to the interval $[-B/2, B/2]$ and $\|g - g_c\|_{\boldsymbol{S}_0} \leq \|g\|_{\boldsymbol{S}_0}$, then $L_0 = \left\lceil \frac{\xi + B}{2b} \right\rceil - 1$.

We shall show a result analogous to Theorem 3.1, this time involving the localization of f with respect to $\mathbf{H}_{\Omega,\varphi}$ or the time-frequency concentration $\mathcal{E}_{\Omega,\varphi}(f)$ of f. We need the following lemma which gives an upper bound on the inner product of a time-frequency localized function with a time-frequency shifted copy of the window function φ. Note that while $\mathbf{H}_{\Omega,\varphi}f$ is interpreted as the part of f in Ω, the uncertainty principle prohibits its STFT to have nonzero values only in Ω, and there will always be points $z \in \mathbb{R}^2 \setminus \Omega$ at which $|\mathcal{V}_\varphi \mathbf{H}_{\Omega,\varphi}f(z)| \neq 0$. It can be shown, however, that $|\mathcal{V}_\varphi \mathbf{H}_{\Omega,\varphi}f(z)|$ decays fast with respect to the distance of z from Ω. Daubechies proved this result in [25] for the case where the window function is the Gaussian $\varphi_0(t) = e^{-\pi t^2}$, showing that the pointwise magnitude of the STFT decays exponentially (see Lemma 3.4 below), using the property involving the STFT of time-frequency shifts of the Gaussian [55, Lemma 1.5.2]:

$$\langle \mathbf{T}_x\mathbf{M}_\omega\varphi_0, \mathbf{T}_u\mathbf{M}_\eta\varphi_0\rangle = \tfrac{1}{\sqrt{2}} \exp[\pi i(u-x)(\eta+\omega) - \tfrac{\pi}{2}(u-x)^2 - \tfrac{\pi}{2}(\eta-\omega)^2]$$

Lemma 3.4. [25, Section III] *For any* δ *between* 0 *and* 1*, one has*

$$|\langle \mathbf{H}_{\Omega,\varphi_0}f, \pi(z)\varphi_0\rangle| \leq \tfrac{1}{\sqrt{2}}\delta^{-\frac{1}{2}}\|f\|_2 \exp[-\tfrac{\pi}{2}(1-\delta)\operatorname{dist}(z,\Omega)^2].$$

A similar result involving windows with milder decay conditions is the following.

Lemma 3.5. *Let φ, $g \in L^2(\mathbb{R})$ such that $\|\varphi\|_2 = 1$ and $|\mathcal{V}_\varphi g(z)| \le C(1+|z|^{2s})^{-1}$, for some $C > 0$ and $s > 1$, for all $z \in \mathbb{R}^2$. For any δ between 0 and 1, one has*

$$|\mathcal{V}_\varphi \mathbf{H}_{\Omega,\varphi} f(z)| = |\langle \mathbf{H}_{\Omega,\varphi} f, \pi(z)g\rangle| \le C_s \delta^{-\frac{1}{2s}} \|f\|_2 (1 + (1-\delta)\mathrm{dist}(z,\Omega)^s)^{-1},$$

where $C_s = \frac{C\sqrt{2}\,\pi}{\sqrt{s\sin(\pi/s)}}$.

Remark 3.6. An example of the inequality $|\mathcal{V}_\varphi g(z)| \le C(1+|z|^{2s})^{-1}$ being satisfied for all $z \in \mathbb{R}^2$ is when φ and g are in the Schwartz space $\mathcal{S}(\mathbb{R})$. Moreover, in that case, for every $s > 0$, there is a C for which the inequality is satisfied. We also note that another (equivalent) form for a polynomial decay of the STFT that appear in the literature is $|\mathcal{V}_\varphi g(z)| \le C'(1+|z|^2)^{-s}$.

Proof: If z, $z' \in \mathbb{R}^2$, then $|\langle \pi(z')\varphi, \pi(z)g\rangle| = |\langle \varphi, \pi(z-z')g\rangle| \le C_s'(1+|z-z'|^{2s})^{-1}$.

For $0 < \delta < 1$,

$$\begin{aligned}
|\langle \mathbf{H}_{\Omega,\varphi} f, \pi(z)g\rangle| &\le \iint_\Omega |\langle f, \pi(z')\varphi\rangle|\,|\langle \pi(z')\varphi, \pi(z)g\rangle|\,dz' \\
&\le C \iint_\Omega |\langle f, \pi(z')\varphi\rangle| \frac{1}{1+|z-z'|^{2s}}\,dz' \\
&\le C \iint_\Omega |\langle f, \pi(z')\varphi\rangle| \frac{1}{\sqrt{1+\delta|z-z'|^{2s}}} \frac{1}{\sqrt{1+(1-\delta)|z-z'|^{2s}}}\,dz' \\
&\le C\sqrt{2} \frac{1}{1+(1-\delta)\inf\limits_{z'\in\Omega}|z-z'|^s} \left(\iint_{\mathbb{R}^2} \frac{1}{1+\delta|z-z'|^{2s}}\,dz' \right)^{\frac{1}{2}} \cdot \\
&\qquad \left(\iint_{\mathbb{R}^2} |\langle f, \pi(z')\varphi\rangle|^2\,dz' \right)^{\frac{1}{2}} \\
&= \frac{C\sqrt{2}\,\pi}{\sqrt{s\sin(\pi/s)}} \delta^{-\frac{1}{2s}} (1+(1-\delta)\inf_{z'\in\Omega}|z-z'|^s)^{-1} \|\varphi\|_2 \|f\|_2,
\end{aligned}$$

and the conclusion follows. ■

Theorem 3.7. *Let φ, $g \in L^2(\mathbb{R})$ such that $\|\varphi\|_2 = 1$ and $|\mathcal{V}_\varphi g(z)| \le C(1+|z|^{2s})^{-1}$, for some $C > 0$ and $s > 1$, for all $z \in \mathbb{R}^2$. Suppose that the Gabor system $\mathcal{G}(g,\Lambda)$ forms a frame with the system $\{\tilde{g_\lambda}\}_{\lambda\in\Lambda}$ as a dual frame, with respective upper frame bounds B and D. Let Ω be a compact subset of $\mathbb{R}^2$. Then, for any $\varepsilon > 0$, there exists $\Omega_\varepsilon \subset \mathbb{R}^2$ such that for all $\Omega^* \supset \Omega_\varepsilon$,*

$$\left\| f - \sum_{\lambda\in\Lambda\cap\Omega^*} \langle f, g_\lambda\rangle \tilde{g_\lambda} \right\|_2 \le C' \Big(\|f - \mathbf{H}_{\Omega,\varphi} f\|_2 + \varepsilon \|f\|_2 \Big), \tag{3.2}$$

for all $f \in L^2(\mathbb{R})$.

Proof: Let $f \in L^2(\mathbb{R})$, and consider compact set $\Omega \subset \mathbb{R}^2$. For any $\Omega^* \supset \Omega$, since $\mathcal{G}(g,\Lambda)$ is a frame with dual frame $\{\tilde{g_\lambda}\}_{\lambda\in\Lambda}$, we have

$$\begin{aligned}\left\| f - \sum_{\lambda\in\Lambda\cap\Omega^*} \langle f, g_\lambda\rangle \tilde{g_\lambda}\right\|_2 &= \left\| \sum_{\lambda\notin\Lambda\cap\Omega^*} \langle f, g_\lambda\rangle \tilde{g_\lambda}\right\|_2 \\ &= \sup_{\|h\|_2=1} \left| \left\langle \sum_{\lambda\notin\Lambda\cap\Omega^*} \langle f, g_\lambda\rangle \tilde{g_\lambda}, h \right\rangle \right| \\ &\leq \sup_{\|h\|_2=1} \sum_{\lambda\notin\Lambda\cap\Omega^*} |\langle f, g_\lambda\rangle|\, |\langle \tilde{g_\lambda}, h\rangle|.\end{aligned}$$

Since

$$\sum_{\lambda\notin\Lambda\cap\Omega^*} |\langle f, g_\lambda\rangle|\, |\langle \tilde{g_\lambda}, h\rangle| = \sum_{\lambda\notin\Lambda\cap\Omega^*} |\langle (\mathbf{H}_{\Omega,\varphi} + (I - \mathbf{H}_{\Omega,\varphi}))f, g_\lambda\rangle|\, |\langle \tilde{g_\lambda}, h\rangle|,$$

we obtain

$$\begin{aligned}\sum_{\lambda\notin\Lambda\cap\Omega^*} |\langle f, g_\lambda\rangle|\, |\langle \tilde{g_\lambda}, h\rangle| &\leq \sum_{\lambda\notin\Lambda\cap\Omega^*} |\langle \mathbf{H}_{\Omega,\varphi} f, g_\lambda\rangle|\, |\langle \tilde{g_\lambda}, h\rangle| \\ &\quad + \sum_{\lambda\notin\Lambda\cap\Omega^*} |\langle (I - \mathbf{H}_{\Omega,\varphi}) f, g_\lambda\rangle|\, |\langle \tilde{g_\lambda}, h\rangle| \\ &\leq \left(\sum_{\lambda\notin\Lambda\cap\Omega^*} |\langle \mathbf{H}_{\Omega,\varphi} f, g_\lambda\rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{\lambda\notin\Lambda\cap\Omega^*} |\langle \tilde{g_\lambda}, h\rangle|^2 \right)^{\frac{1}{2}} \\ &\quad + \left(\sum_{\lambda\notin\Lambda\cap\Omega^*} |\langle (I - \mathbf{H}_{\Omega,\varphi}) f, g_\lambda\rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{\lambda\notin\Lambda\cap\Omega^*} |\langle \tilde{g_\lambda}, h\rangle|^2 \right)^{\frac{1}{2}}.\end{aligned}$$

So that

$$\begin{aligned}\left\| f - \sum_{\lambda\in\Lambda\cap\Omega^*} \langle f, g_\lambda\rangle \tilde{g_\lambda}\right\|_2 &\leq \sup_{\|h\|_2=1} \left(\sum_{\lambda\notin\Lambda\cap\Omega^*} |\langle \mathbf{H}_{\Omega,\varphi} f, g_\lambda\rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{\lambda\notin\Lambda\cap\Omega^*} |\langle \tilde{g_\lambda}, h\rangle|^2 \right)^{\frac{1}{2}} \\ &\quad + \sup_{\|h\|_2=1} \left(\sum_{\lambda\notin\Lambda\cap\Omega^*} |\langle (I - \mathbf{H}_{\Omega,\varphi}) f, g_\lambda\rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{\lambda\notin\Lambda\cap\Omega^*} |\langle \tilde{g_\lambda}, h\rangle|^2 \right)^{\frac{1}{2}}\end{aligned}$$

Using Lemma 3.5 and the assumption that $\mathcal{G}(g,\Lambda)$, $\{\tilde{g_\lambda}\}_{\lambda\in\Lambda}$ have upper frame bounds B and D, respectively, we get

$$\begin{aligned}\left\| f - \sum_{\lambda\in\Lambda\cap\Omega^*} \langle f, g_\lambda\rangle \tilde{g_\lambda}\right\|_2 &\leq C 2^{\frac{1}{s}} \sqrt{D}\, \|f\|_2 \left(\sum_{\lambda\notin\Lambda\cap\Omega^*} (1 + \tfrac{1}{2}\mathrm{dist}(\lambda,\Omega)^s)^{-2} \right)^{\frac{1}{2}} \\ &\quad + \sqrt{BD}\, \|(I - \mathbf{H}_{\Omega,\varphi}) f\|_2.\end{aligned}$$

This estimate holds true for all $\Omega^* \supset \Omega$. By the convergence of the above series, if Ω is fixed, then for a given $\varepsilon > 0$, one can find $\Omega_\varepsilon \supset \Omega$ such that for all $\Omega^* \supset \Omega_\varepsilon$, the series is less than ε^2. Finally, taking $C' = \max\{C2^{\frac{1}{s}}\sqrt{D}, \sqrt{BD}\}$, we get the conclusion of the theorem. ■

REMARK 3.8. We emphasize that by Theorem 3.7, we are able to obtain an approximation estimate that holds *uniformly* for for all $f \in \mathscr{C}(\Omega, \varepsilon_0, \varphi)$, i.e. for any $\varepsilon > 0$, there exists a Ω_ε such that for all $\Omega^* \supset \Omega_\varepsilon$ and all $\tilde{\varepsilon} > \varepsilon_0 + \varepsilon$,

$$\left\| f - \sum_{\lambda \in \Lambda \cap \Omega^*} \langle f, g_\lambda \rangle \tilde{g_\lambda} \right\|_2 \leq C' \tilde{\varepsilon} \|f\|_2, \tag{3.3}$$

for all $f \in \mathscr{C}(\Omega, \varepsilon_0, \varphi)$. In contrast, for a fixed f, such an inequality can be obtained more simply from the strong operator convergence of the Gabor frame operator.

We show an example for the case where the window function is the Gaussian φ_0 and the region Ω is the disk $B(O, R)$ with center at the origin O and with radius R. We will make use of the decay of the STFT of $\mathbf{H}_{\Omega,\varphi}$ in Lemma 3.4. First, we prove the following lemma that gives an estimate on the decay of the tail of the sum of samples of the two-dimensional Gaussian outside the disk $B(O, R^*)$. Let $Q(j) = [j_1 - \frac{1}{2}, j_1 + \frac{1}{2}) \times [j_2 - \frac{1}{2}, j_2 + \frac{1}{2})$, $j = (j_1, j_2) \in \mathbb{Z}^2$.

Lemma 3.9. *Let Λ be a relatively separated set of points in $\mathbb{R}^2$ with $\sup_{z \in \mathbb{R}^2} \#(\Lambda \cap Q(z)) =: N_\Lambda < \infty$. Fix $R > 0$. If $R^* > R$, then*

$$\sum_{\lambda \in \Lambda,\, |\lambda| > R^*} \exp(-\tfrac{\pi}{2}(|\lambda| - R)^2) \leq C_\Lambda \exp\big(-\tfrac{\pi}{4}(\tfrac{(R^*)^2}{4} - R^2)\big), \tag{3.4}$$

where $C_\Lambda = 8\exp(\frac{5\pi}{4}) N_\Lambda$.

Proof: Let $R^* > R$ and define the sets

$$\begin{aligned} \mathscr{J}_{R^*} &= \{j \in \mathbb{Z}^2 : Q(j) \cap (\mathbb{R}^2 \setminus B(O, R^*)) \neq \varnothing\} \quad \text{and} \\ \Lambda_{R^*, j} &= \{\lambda \in \Lambda : |\lambda| > R^*,\ \lambda \in Q(j)\}. \end{aligned}$$

We are then able to rewrite the left-hand side of (3.4) as

$$\sum_{\lambda \in \Lambda,\, |\lambda| > R^*} \exp(-\tfrac{\pi}{2}(|\lambda| - R)^2) = \sum_{j \in \mathscr{J}_{R^*}} \sum_{\lambda \in \Lambda_{R^*, j}} \exp(-\tfrac{\pi}{2}(|\lambda| - R)^2). \tag{3.5}$$

If $\lambda, z \in Q(j)$, then $|\lambda| \geq |z| - \sqrt{2}$. And since $-(|z| - R - \sqrt{2})^2 \leq -(\frac{(|z|-R)^2}{2} - 2)$, we have

$$e^{-\frac{\pi}{2}(|\lambda| - R)^2} \leq e^{-\frac{\pi}{2}(|z| - R - \sqrt{2})^2} \leq e^{\pi} \exp\big(-\tfrac{\pi}{4}(|z| - R)^2\big).$$

Using the inequalities $-(|z| - R)^2 \leq -\frac{|z|^2}{2} + R^2$ and $-(R^* - \sqrt{2})^2 \leq -\frac{(R^*)^2}{2} + 2$, we estimate (3.5) as follows:

$$\begin{aligned} \sum_{j \in \mathscr{J}_{R^*}} \sum_{\lambda \in \Lambda_{R^*, j}} \exp(-\tfrac{\pi}{2}(|\lambda| - R)^2) &\leq \sum_{j \in \mathscr{J}_{R^*}} \sum_{\lambda \in \Lambda_{R^*, j}} \iint_{Q(j)} \exp\big(-\tfrac{\pi}{4}(|z| - R)^2\big)\, dz \\ &\leq N_\Lambda \iint_{\mathbb{R}^2 \setminus B(O, R^* - \sqrt{2})} \exp\big(-\tfrac{\pi}{4}(|z| - R)^2\big)\, dz \end{aligned}$$

$$\begin{aligned}
&\leq N_\Lambda\, e^{\pi} e^{\frac{\pi R^2}{4}} \iint_{|z|>R^*-\sqrt{2}} \exp\big(-\tfrac{\pi|z|^2}{8}\big)\, dz \\
&= 8N_\Lambda\, e^{\pi} e^{\frac{\pi R^2}{4}} \exp\big(-\tfrac{\pi(R^*-\sqrt{2})^2}{8}\big) \\
&\leq 8N_\Lambda\, e^{\frac{5\pi}{4}} \exp\big(-\tfrac{\pi}{4}(\tfrac{(R^*)^2}{4} - R^2)\big).
\end{aligned}$$

By taking $C_\Lambda = 8N_\Lambda\, e^{\frac{5\pi}{4}}$, we get the conclusion of the lemma. ∎

Example 3.10. Suppose Ω is the disk centered at the origin with radius R and suppose that the Gabor system $\{\pi(\lambda)\varphi_0 \,:\, \lambda \in \Lambda\}$ forms a frame with the system $\{\widetilde{\varphi_{0,\lambda}} \,:\, \lambda \in \Lambda\}$ as a dual frame, having respective upper frame bounds B and D. Then, for any ε between 0 and 1, there exists R_ε such that

$$\Big\| f - \sum_{\lambda\in\Lambda,\, |\lambda|\leq R+R_\varepsilon} \langle f, \pi(\lambda)\varphi_0\rangle \widetilde{\varphi_{0,\lambda}} \Big\|_2 \leq C_{\Lambda,B,D}\Big(\|f - \mathbf{H}_{\Omega,\varphi_0} f\|_2 + \varepsilon\|f\|_2 \Big), \tag{3.6}$$

for all $f \in L^2(\mathbb{R})$. Here, we can take $R_\varepsilon \geq -R + \sqrt{4R^2 - \frac{32}{\pi}\ln\varepsilon}$.

Proof: Following the proof of Theorem 3.7, we have for $\Omega^* \supset \Omega$,

$$\begin{aligned}
\Big\| f - \sum_{\lambda\in\Lambda\cap\Omega^*} \langle f, \pi(\lambda)\varphi_0\rangle \widetilde{\varphi_{0,\lambda}} \Big\|_2 &\leq \sqrt{D}\Big(\sum_{\lambda\notin\Lambda\cap\Omega^*} |\langle \mathbf{H}_{\Omega,\varphi_0} f, \pi(\lambda)\varphi_0\rangle|^2 \Big)^{\frac{1}{2}} \\
&\quad + \sqrt{BD}\|f - \mathbf{H}_{\Omega,\varphi_0} f\|_2.
\end{aligned}$$

We can take Ω^* to be a disk centered at the origin with radius $R^* := R + R_\varepsilon > R$. We use Lemma 3.4 (with $\delta = \frac{1}{2}$) and Lemma 3.9 to estimate the first term on the right side as follows:

$$\begin{aligned}
\sqrt{D}\Big(\sum_{\lambda\notin\Lambda\cap\Omega^*} |\langle \mathbf{H}_{\Omega,\varphi_0} f, \pi(\lambda)\varphi_0\rangle|^2 \Big)^{\frac{1}{2}} &\leq \sqrt{D}\|f\|_2 \Big(\sum_{\lambda\notin\Lambda\cap\Omega^*} \exp(-\tfrac{\pi}{2}\operatorname{dist}(\lambda,\Omega)^2) \Big)^{\frac{1}{2}} \\
&= \sqrt{D}\|f\|_2 \sqrt{C_\Lambda} \exp\big(-\tfrac{\pi}{8}(\tfrac{(R^*)^2}{4} - R^2)\big)
\end{aligned}$$

Now, $\exp\big(-\frac{\pi}{8}(\frac{(R^*)^2}{4} - R^2)\big) \leq \varepsilon$ whenever $R^* \geq \sqrt{4R^2 - \frac{32}{\pi}\ln\varepsilon}$. So we can take $R_\varepsilon \geq -R + \sqrt{4R^2 - \frac{32}{\pi}\ln\varepsilon}$, and $C_{\Lambda,B,D} := \max\{\sqrt{DC_\Lambda}, \sqrt{BD}\}$. ∎

Remark 3.11. In Theorem 3.1, the enlargement of the rectangular region, i.e. adding y_ε and ξ_ε, depends only on ε, the desired precision of the approximation. In Theorem 3.7, however, the generality of the region of concentration Ω and the possible nonuniformity of the samples in Λ make the enlargement dependent on the region.

3.2. Local Gabor approximation of a function in a TF-localized subspace

Consider the subspace V_N spanned by the first N eigenfunctions of the localization operator $\mathbf{H}_{\Omega,\varphi}$, Ω a compact subset of $\mathbb{R}^2$ and $\varphi \in L^2(\mathbb{R})$ with $\|\varphi\|_2 = 1$, corresponding to the eigenvalues arranged in descending order. Let g be a window function in $L^2(\mathbb{R})$ such that $\|g\|_2 = 1$ and $|\mathcal{V}_\varphi g(z)| \leq C(1+|z|^{2s})^{-1}$ for some $C > 0$ and $s > 1$. Let the Gabor system $\mathcal{G}(g,\Lambda)$ form a frame with lower and upper frame bounds A and B, respectively, and let $\{\tilde{g_\lambda}\}_{\lambda\in\Lambda}$ be its dual frame.

Since every $f \in V_N$ is $(1-\alpha_N)$-concentrated in Ω, it follows that (3.3) holds for all $f \in V_N$, where $\varepsilon_0 = 1-\alpha_N$. We note however that $\tilde{\varepsilon}$ is bounded below by ε_0, which is fixed. We can improve the estimate since we are considering only the elements of V_N, wherein the error bound approaches 0 as the set Ω^* gets larger.

Proposition 3.12. *For any $\varepsilon > 0$, there exists an $\Omega^* \supset \Omega$ such that*

$$\left\| f - \sum_{\lambda\in\Lambda\cap\Omega^*} \langle f, g_\lambda\rangle \tilde{g_\lambda} \right\|_2 \leq \varepsilon \|f\|_2, \quad \textit{for all } f \in V_N. \tag{3.7}$$

Proof: Let $\varepsilon > 0$ and $f \in V_N$. Then

$$\begin{aligned} f - \sum_{\lambda\in\Lambda\cap\Omega^*} \langle f, g_\lambda\rangle \tilde{g_\lambda} &= \sum_{k=1}^N \langle f, \psi_k\rangle \psi_k - \sum_{\lambda\in\Lambda\cap\Omega^*} \left(\sum_{k=1}^N \langle f, \psi_k\rangle \langle \psi_k, g_\lambda\rangle \right) \tilde{g_\lambda} \\ &= \sum_{k=1}^N \langle f, \psi_k\rangle \left(\psi_k - \sum_{\lambda\in\Lambda\cap\Omega^*} \langle \psi_k, g_\lambda\rangle \tilde{g_\lambda} \right) \\ &= \sum_{k=1}^N \langle f, \psi_k\rangle \left(\sum_{\lambda\notin\Lambda\cap\Omega^*} \langle \psi_k, g_\lambda\rangle \tilde{g_\lambda} \right). \end{aligned}$$

So we have

$$\begin{aligned} \left\| f - \sum_{\lambda\in\Lambda\cap\Omega^*} \langle f, g_\lambda\rangle \tilde{g_\lambda} \right\|_2^2 &= \left\| \sum_{k=1}^N \langle f, \psi_k\rangle \left(\sum_{\lambda\notin\Lambda\cap\Omega^*} \langle \psi_k, g_\lambda\rangle \tilde{g_\lambda} \right) \right\|_2^2 \\ &\leq \left(\sum_{k=1}^N |\langle f, \psi_k\rangle| \left\| \sum_{\lambda\notin\Lambda\cap\Omega^*} \langle \psi_k, g_\lambda\rangle \tilde{g_\lambda} \right\|_2 \right)^2 \\ &\leq \sum_{k=1}^N |\langle f, \psi_k\rangle|^2 \sum_{k=1}^N \left\| \sum_{\lambda\notin\Lambda\cap\Omega^*} \langle \psi_k, g_\lambda\rangle \tilde{g_\lambda} \right\|_2^2 \\ &= \|f\|_2^2 \sum_{k=1}^N \sup_{\|h\|_2=1} \left| \sum_{\lambda\notin\Lambda\cap\Omega^*} \langle \psi_k, g_\lambda\rangle \langle \tilde{g_\lambda}, h\rangle \right|^2 \\ &\leq \|f\|_2^2 \sum_{k=1}^N \sup_{\|h\|_2=1} \left(\sum_{\lambda\notin\Lambda\cap\Omega^*} |\langle \psi_k, g_\lambda\rangle|^2 \right) \left(\sum_{\lambda\notin\Lambda\cap\Omega^*} |\langle \tilde{g_\lambda}, h\rangle|^2 \right) \end{aligned}$$

$$\leq A^{-1}\|f\|_2^2 \sum_{k=1}^{N} \sum_{\lambda \notin \Lambda \cap \Omega^*} |\langle \psi_k, g_\lambda \rangle|^2.$$

We consider $|\langle \psi_k, g_\lambda \rangle|$ and note that $|\langle \psi_k, g_\lambda \rangle| = \frac{1}{\alpha_k} |\langle \mathbf{H}_{\Omega,\varphi} \psi_k, g_\lambda \rangle|$. Since g satisfies $|\mathcal{V}_\varphi g(z)| \leq C(1 + |z|^{2s})^{-1}$, it follows from Lemma 3.5 that

$$|\langle \psi_k, g_\lambda \rangle| \leq \frac{1}{\alpha_k} C_s 2^{\frac{1}{2s}} (1 + \tfrac{1}{2}\operatorname{dist}(\lambda, \Omega)^s)^{-1},$$

where δ is taken to be $\frac{1}{2}$, which gives us

$$\left\| f - \sum_{\lambda \in \Lambda \cap \Omega^*} \langle f, g_\lambda \rangle \tilde{g_\lambda} \right\|_2^2 \leq A^{-1}\|f\|_2^2 C_s^2 2^{\frac{1}{s}} \left(\sum_{k=1}^{N} \frac{1}{\alpha_k^2} \right) \sum_{\lambda \notin \Lambda \cap \Omega^*} (1 + \tfrac{1}{2}\operatorname{dist}(\lambda, \Omega)^s)^{-2}.$$

The right-hand side of the above equation approaches 0 as Ω^* gets larger. In particular, given $\varepsilon > 0$, one can choose Ω^* so that the sum

$$\sum_{\lambda \notin \Lambda \cap \Omega^*} (1 + \tfrac{1}{2}\operatorname{dist}(\lambda, \Omega)^s)^{-2} < \varepsilon^2 / \left(A^{-1} C_s^2 2^{\frac{1}{s}} \sum_{k=1}^{N} \frac{1}{\alpha_k^2} \right)$$

which gives the conclusion of the proposition. ■

Remark 3.13. In [33], Dörfler considered the truncated frame expansion

$$\mathbf{S}_R f = \sum_{\lambda \in M_R \cap \Lambda} \langle f, \pi(\lambda) g_t \rangle \pi(\lambda) g_t,$$

where $\mathcal{G}(g_t, \Lambda)$ is a tight Gabor frame and M_R is a region such that $M_R \subseteq B_R(0)$. It was shown that for a fixed $R_0 > 0$ and for any $\varepsilon > 0$, there exists R_1 such that for all $f \in \operatorname{ran}(\mathbf{S}_{R_0})$,

$$\|f - \mathbf{S}_R f\|_2^2 < \varepsilon \|f\|_2^2$$

for all $R > R_1$. In contrast, Proposition 3.12 holds for f in the subspace V_N and we obtain more explicit relations between ε and the enlargement of the region, especially if the region has known shape.

In the next proposition, we obtain yet another error estimate for the approximation of $f \in L^2(\mathbb{R})$ wherein this time, the error bound is expressed in terms of the error between f and its projection onto the subspace V_N.

Proposition 3.14. *Let $\mathcal{G}(g, \Lambda)$ be a Gabor frame with $|\mathcal{V}_\varphi g(z)| \leq C(1 + |z|^{2s})^{-1}$ and frame bounds A and B. Then, for all $N > 0$ and all $\varepsilon > 0$, there exists a set $\Omega^* \supset \Omega$ in $\mathbb{R}^2$, such that for all $f \in L^2(\mathbb{R})$ with corresponding projection $f_N := \mathcal{P}_{V_N} f$ onto the TF-localization subspace V_N,the following estimate holds:*

$$\left\| f - \sum_{\lambda \in \Lambda \cap \Omega^*} \langle f, g_\lambda \rangle \tilde{g_\lambda} \right\|_2 \leq \left(1 + \sqrt{\tfrac{B}{A}}\right) \|f - f_N\|_2 + \varepsilon \|f\|_2. \tag{3.8}$$

Proof: Since

$$\left\| f - \sum_{\lambda \in \Lambda \cap \Omega^*} \langle f, g_\lambda \rangle \tilde{g_\lambda} \right\|_2 \leq \|f - f_N\|_2 + \left\| f_N - \sum_{\lambda \in \Lambda \cap \Omega^*} \langle f_N, g_\lambda \rangle \tilde{g_\lambda} \right\|_2 + \left\| \sum_{\lambda \in \Lambda \cap \Omega^*} \langle f_N - f, g_\lambda \rangle \tilde{g_\lambda} \right\|_2$$

the result follows from Proposition 3.12 and the boundedness of the associated analysis and synthesis operators. ■

As a corollary, we obtain the following result for local approximation by Gabor frame elements for functions with known time-frequency concentration in a given set Ω.

Corollary 3.15. *Let f be (ε, φ)-concentrated on $\Omega \subset \mathbb{R}^2$. For fixed $c > 1$, let ψ_k, $k = 1, \dots, N$, be all eigenfunctions of $\mathbf{H}_{\Omega,\varphi}$ corresponding to eigenvalues $\alpha_k > \frac{c-1}{c}$. Then, for all $\tilde{\varepsilon} > \left(1 + \sqrt{\frac{B}{A}}\right) \cdot \sqrt{c\varepsilon}$, there exists a set $\Omega^* \supset \Omega$ in $\mathbb{R}^2$, such that*

$$\left\| f - \sum_{\lambda \in \Lambda \cap \Omega^*} \langle f, g_\lambda \rangle \tilde{g_\lambda} \right\|_2 \leq \tilde{\varepsilon} \|f\|_2. \tag{3.9}$$

Proof: The result follows immediately from Proposition 3.14 and Proposition 2.15(2). ■

3.2.1. Local TF-dictionaries and reconstruction from samples. We now look at some properties of the local time-frequency dictionary corresponding to the enlarged region Ω^* covering Ω, and show a reconstruction procedure for a function in the subspace V_N from the local samples. We recall the following theorem by Feichtinger and Zimmermann, c.f. [52]:

Theorem 3.16. [52, Theorem 3.6.16] *Let (g, γ) be a Λ-dual pair in $L^2(\mathbb{R}^d)$. Consider a closed subspace $V \subseteq L^2(\mathbb{R}^d)$, and assume that $J \subseteq \Lambda$ is an index set such that for some $\varepsilon < 1$,*

$$\left\| f - \sum_{\lambda \in J} \langle f, \pi(\lambda) g \rangle \pi(\lambda) \gamma \right\|_2 \leq \varepsilon \|f\|_2 \text{ for all } f \in V. \tag{3.10}$$

Then $f \in V$ can be completely reconstructed from $\{\langle f, \pi(\lambda) g \rangle\}_{\lambda \in J}$.

We note that reconstruction from restricted samples of the Gabor coefficients for functions on a closed subspace in [52] was motivated by the problem of reconstructing a bandlimited function from samples of its STFT on a strip in the time-frequency plane covering the frequency band of the function.

By Proposition 3.12, the above theorem is satisfied on the subspace V_N, where $J = \Lambda \cap \Omega^*$. If $\mathbf{S}^{\text{loc}}$ is the operator

$$\mathbf{S}^{\text{loc}} \,:\, f \longmapsto \sum_{\lambda \in \Lambda \cap \Omega^*} \langle f, g_\lambda \rangle \tilde{g_\lambda},$$

then it follows from Proposition 3.12 that $\|f - \mathbf{S}^{\text{loc}} f\|_2 < \varepsilon \|f\|_2$ for all $f \in V_N$. With $\mathcal{P}_{V_N}$ as the orthogonal projection onto V_N, we have $\|\operatorname{Id} - \mathcal{P}_{V_N}\mathbf{S}^{\text{loc}}\|_{\text{Op}} < \varepsilon$. Using the Neumann series to obtain the operator $L = \sum_{k=0}^{\infty} (\operatorname{Id} - \mathcal{P}_{V_N}\mathbf{S}^{\text{loc}})^k$, we have $L\mathcal{P}_{V_N}\mathbf{S}^{\text{loc}} = \operatorname{Id}$ on V_N. Moreover, as a consequence of Proposition 3.12 is the following frame-like inequality for functions in V_N.

Proposition 3.17. *If $\varepsilon < 1$ and inequality (3.7) is satisfied, then for all $f \in V_N$,*

$$A(1-\varepsilon)^2 \|f\|_2^2 \le \sum_{\lambda \in \Lambda \cap \Omega^*} |\langle f, g_\lambda \rangle|^2 \le B \|f\|_2^2, \tag{3.11}$$

where A and B are lower and upper frame bounds, respectively, for $\mathcal{G}(g, \Lambda)$. This implies that the system $\{\mathcal{P}_{V_N} g_\lambda\}_{\lambda \in \Lambda \cap \Omega^*}$ forms a frame for V_N. More generally, the system $\{\pi(\mu)\mathcal{P}_{V_N}\pi(\lambda)g\}_{\lambda \in \Lambda \cap \Omega^*}$, where $\mu \in \mathbb{R}^2$, forms a frame for the subspace $V_{N,\mu} := \{\pi(\mu) f \,:\, f \in V_N\}$.

Proof: From Proposition 3.12, we get

$$\|f\|_2 - \left\| \sum_{\lambda \in \Lambda \cap \Omega^*} \langle f, g_\lambda \rangle \tilde{g_\lambda} \right\|_2 \le \left\| f - \sum_{\lambda \in \Lambda \cap \Omega^*} \langle f, g_\lambda \rangle \tilde{g_\lambda} \right\|_2 \le \varepsilon \|f\|_2.$$

And we obtain

$$\begin{aligned} (1-\varepsilon)^2 \|f\|_2^2 &\le \left\| \sum_{\lambda \in \Lambda \cap \Omega^*} \langle f, g_\lambda \rangle \tilde{g_\lambda} \right\|_2^2 \\ &\le \frac{1}{A} \sum_{\lambda \in \Lambda \cap \Omega^*} |\langle f, g_\lambda \rangle|^2 \\ &\le \frac{B}{A} \|f\|_2^2. \end{aligned}$$

For the subspace $V_{N,\mu}$, we first note that

$$\|f\|_2 = \|\pi(\mu) f\|_2 \quad \text{and} \quad \langle f, \mathcal{P}_{V_N} g_\lambda \rangle = \langle \pi(\mu) f, \pi(\mu)\mathcal{P}_{V_N} g_\lambda \rangle.$$

The inequality in (3.11) can then be reformulated as

$$A(1-\varepsilon)^2 \|\pi(\mu) f\|_2^2 \le \sum_{\lambda \in \Lambda \cap \Omega^*} |\langle \pi(\mu) f, \pi(\mu)\mathcal{P}_{V_N} g_\lambda \rangle|^2 \le B \|\pi(\mu) f\|_2^2,$$

for all $f \in V_N$, or $\pi(\mu) f \in V_{N,\mu}$. ∎

3.2.1.1. *Local TF-dictionaries and pseudoframes for subspaces.* Pseudoframes for subspaces were introduced in by Li and Ogawa in [75] that aims to provide a more flexible representation for functions in a subspace since it does not require the analysis and synthesis sequences to lie in the subspace. It can be used e.g. for optimal noise suppression, cf. [76].

A Bessel sequence $\{x_n\}$ with respect to a subspace V of a separable Hilbert space $\mathcal{H}$ is said to be a *pseudoframe for the subspace* V (PFFS) with respect to a Bessel sequence $\{x_n^*\}$ in $\mathcal{H}$ (called a *dual pseudoframe* to $\{x_n\}$ for V) if

$$\forall f \in V, \quad f = \sum_n \langle f, x_n \rangle x_n^*. \tag{3.12}$$

By Proposition 3.17, since $\{\mathcal{P}_{V_N} g_\lambda\}_{\lambda \in \Lambda \cap \Omega^*}$ is a frame for V_N, there is a dual frame $\{\widetilde{g_{\lambda,V_N}}\}_{\lambda \in \Lambda \cap \Omega^*}$ such that for all $f \in V_N$

$$f = \sum_{\lambda \in \Lambda \cap \Omega^*} \langle f, \mathcal{P}_{V_N} g_\lambda \rangle \widetilde{g_{\lambda,V_N}}. \tag{3.13}$$

Moreover, since $f \in V_N$, $\langle f, \mathcal{P}_{V_N} g_\lambda \rangle = \langle f, g_\lambda \rangle$, so $\{g_\lambda\}_{\lambda \in \Lambda \cap \Omega^*}$ is a pseudoframe for V_N with respect to $\{\widetilde{g_{\lambda,V_N}}\}_{\lambda \in \Lambda \cap \Omega^*}$. We also note that by [75, Theorems 2 and 3], a dual pseudoframe may be obtained via $\widetilde{g_{\lambda,V_N}} = \mathcal{P}_{V_N}(\mathbf{C}_{g,\Lambda,\Omega^*}\mathcal{P}_{V_N})^\dagger g_\lambda$, where $\mathbf{C}_{g_\lambda} f = \{\langle f, g_\lambda \rangle\}_{\lambda \in \Lambda \cap \Omega^*}$ and $L^\dagger$ denotes the pseudoinverse of L. In the language of [51], $\{g_\lambda\}_{\lambda \in \Lambda \cap \Omega^*}$ is a family of local atoms that provide an atomic decomposition for V_N.

3.2.1.2. *Local TF-dictionaries and generalized sampling.* The reconstruction of a function $f \in V_N$ from the samples $\langle f, g_\lambda \rangle$ translates to obtaining samples from inner products with respect to one set of functions and reconstructing with another given set of functions. This is the problem dealt with in consistent sampling involving bases, which was later extended to frames via generalized sampling, cf. [44, 4, 5].

In this case, if $\mathbf{C}_{g_\lambda}$ denotes the analysis operator of $\{g_\lambda\}_{\lambda \in \Lambda \cap \Omega^*}$ with $\mathbf{C}_{g_\lambda}^*$ as the synthesis operator, and $\mathbf{C}_{\psi_k}$ denotes the analysis operator of $\{\psi_k\}_{k=1}^N$ with $\mathbf{C}_{\psi_k}^*$ as the synthesis operator, then the reconstruction of f is given by $\mathbf{C}_{\psi_k}(\mathbf{C}_{g_\lambda}^*\mathbf{C}_{\psi_k})^\dagger \mathbf{C}_{g_\lambda}^* f$.

3.3. Global frames from TF-localization

In [36], Dörfler and Gröchenig showed that finitely many eigenfunctions of $\mathbf{H}_{\Omega,\varphi}$ generate a multi-window Gabor frame for $L^2(\mathbb{R})$ (see also the works of Dörfler and Romero in [39, 40]). We restate the result in [36] for the case where the symbol is χ_Ω, Ω a compact subset of $\mathbb{R}^2$, $\varphi \in \boldsymbol{S}_0(\mathbb{R})$ and $g \in L^2(\mathbb{R})$ such that $\|\varphi\|_2 = \|g\|_2 = 1$ and $|\mathcal{V}_\varphi g(z)| \leq C(1 + |z|^{2s})^{-1}$ for some $C > 0$ and $s > 1$. Note that by Proposition 1.17, g is also in $\boldsymbol{S}_0(\mathbb{R}^d)$. Given two non-negative functions h_1 and h_2, we write $h_1 \asymp h_2$ if there exist constants K_1, $K_2 \geq 0$ such that $K_1 h_1 \leq h_2 \leq K_2 h_1$.

Lemma 3.18. [36, Lemma 9] *Suppose* $\sum_{\mu\in\tilde{\Lambda}} \mathbf{T}_\mu \chi_\Omega \asymp 1$. *Let* $\{\psi_k\}_{k\in\mathbb{N}}$ *be the orthonormal system of eigenfunctions of* $\mathbf{H}_{\Omega,\varphi}$. *Then there exists* $N \in \mathbb{N}$ *such that* $\cup_{k=1}^N \mathcal{G}(\psi_k, \tilde{\Lambda})$ *is a multi-window Gabor frame for* $L^2(\mathbb{R})$.

We use this result to obtain another family of time-frequency dictionaries that form a frame for $L^2(\mathbb{R})$.

Proposition 3.19. *Suppose* $\mathcal{G}(g,\Lambda)$ *is a Gabor frame,* Ω *is a compact subset of* $\mathbb{R}^2$ *and* $\tilde{\Lambda}$ *a lattice such that* $\sum_{\mu\in\tilde{\Lambda}} \mathbf{T}_\mu \chi_\Omega \asymp 1$. *Let* $\{\psi_k\}_{k\in\mathbb{N}}$ *be the orthonormal system of eigenfunctions of* $\mathbf{H}_{\Omega,\varphi}$. *Then there exists* $N \in \mathbb{N}$ *and a region* $\Omega^* \supset \Omega$ *such that* $\cup_{\lambda\in\Lambda\cap\Omega^*} \mathcal{G}(\mathcal{P}_{V_N} g_\lambda, \tilde{\Lambda})$, *where* $V_N = \operatorname{span}\{\psi_k\}_{k=1}^N$, *is a multi-window Gabor frame for* $L^2(\mathbb{R})$.

Proof: By Lemma 3.18, there exists $N \in \mathbb{N}$, such that $\cup_{k=1}^N \mathcal{G}(\psi_k, \tilde{\Lambda})$ is a multi-window Gabor frame for $L^2(\mathbb{R})$. Proposition 3.12 and Proposition 3.17 tell us that there exists $\Omega^* \supset \Omega$ such that for any $\mu \in \mathbb{R}^2$, $\{\pi(\mu)\mathcal{P}_{V_N} g_\lambda\}_{\lambda\in\Lambda\cap\Omega^*}$ is a frame for $V_{N,\mu} = \{\pi(\mu)f \,:\, f \in V_N\}$. Let $\mu \in \tilde{\Lambda}$. For any $f \in L^2(\mathbb{R})$, we have

$$\begin{aligned}
\sum_{\lambda\in\Lambda\cap\Omega^*} |\langle f, \pi(\mu)\mathcal{P}_{V_N} g_\lambda\rangle|^2 &= \sum_{\lambda\in\Lambda\cap\Omega^*} |\langle \pi(\mu)^* f, \mathcal{P}_{V_N} g_\lambda\rangle|^2 \\
&= \sum_{\lambda\in\Lambda\cap\Omega^*} |\langle \mathcal{P}_{V_N}\pi(\mu)^* f, \mathcal{P}_{V_N} g_\lambda\rangle|^2 \\
&\asymp \|\mathcal{P}_{V_N}\pi(\mu)^* f\|_2^2 \qquad \text{(by Proposition 3.17)} \\
&= \sum_{k=1}^N |\langle \mathcal{P}_{V_N}\pi(\mu)^* f, \psi_k\rangle|^2 \\
&= \sum_{k=1}^N |\langle f, \pi(\mu)\psi_k\rangle|^2.
\end{aligned}$$

And we obtain the following equivalent expressions:

$$\sum_{\mu\in\tilde{\Lambda}} \sum_{\lambda\in\Lambda\cap\Omega^*} |\langle f, \pi(\mu)\mathcal{P}_{V_N} g_\lambda\rangle|^2 \asymp \sum_{\mu\in\tilde{\Lambda}} \sum_{k=1}^N |\langle f, \pi(\mu)\psi_k\rangle|^2 \asymp \|f\|_2^2, \tag{3.14}$$

where the second equivalence follows from Lemma 3.18. Hence, the conclusion follows. ■

We note that the equivalence $\sum_{\lambda\in\Lambda\cap\Omega^*} |\langle f, \pi(\mu)\mathcal{P}_{V_N} g_\lambda\rangle|^2 \asymp \sum_{k=1}^N |\langle f, \pi(\mu)\psi_k\rangle|^2$ can be written explicitly, using (3.11), as

$$A(1-\varepsilon)^2 \sum_{k=1}^N |\langle f, \pi(\mu)\psi_k\rangle|^2 \leq \sum_{\lambda\in\Lambda\cap\Omega^*} |\langle f, \pi(\mu)\mathcal{P}_{V_N} g_\lambda\rangle|^2 \leq B \sum_{k=1}^N |\langle f, \pi(\mu)\psi_k\rangle|^2, \tag{3.15}$$

where $0 < \varepsilon < 1$. Consequently, the first equivalence in (3.14) is as follows:

$$A_\varepsilon \sum_{\mu\in\tilde{\Lambda}} \sum_{k=1}^{N} |\langle f, \pi(\mu)\psi_k\rangle|^2 \leq \sum_{\mu\in\tilde{\Lambda}} \sum_{\lambda\in\Lambda\cap\Omega^*} |\langle f, \pi(\mu)\mathcal{P}_{V_N} g_\lambda\rangle|^2 \leq B \sum_{\mu\in\tilde{\Lambda}} \sum_{k=1}^{N} |\langle f, \pi(\mu)\psi_k\rangle|^2,$$

where $A_\varepsilon = A(1-\varepsilon)^2$.

We can generalize the above proposition, wherein instead of translating a single region Ω to cover $\mathbb{R}^2$, we consider a family of regions $\Omega_\mu \subset \mathbb{R}^2$ such that $\sum_{\mu\in\tilde{\Lambda}} \chi_{\Omega_\mu} \asymp 1$. It follows from [40, Theorem 5.10] that we can choose N_μ such that

$$\|f\|_2^2 \asymp \sum_{\mu\in\tilde{\Lambda}} \sum_{k=1}^{N_\mu} |\langle f, \psi_k^\mu\rangle|^2, \;\; f \in L^2(\mathbb{R}) \tag{3.16}$$

and obtain the following theorem.

Theorem 3.20. *Let $\{\Omega_\mu\}_{\mu\in\tilde{\Lambda}}$ be a family of compact regions in $\mathbb{R}^2$ such that $\sum_{\mu\in\tilde{\Lambda}} \chi_{\Omega_\mu} \asymp 1$, and let $\varphi \in \boldsymbol{S}_0(\mathbb{R})$ such that $\|\varphi\|_2 = 1$. Corresponding to each $\mu \in \tilde{\Lambda}$, we let $g^\mu \in L^2(\mathbb{R})$ such that $\|g^\mu\|_2 = 1$ and $|\mathcal{V}_\varphi g^\mu(z)| \leq C_\mu(1+|z|^{2s_\mu})^{-1}$ for some $C_\mu > 0$ and $s_\mu > 1$, and let $\mathcal{G}(g^\mu, \Lambda_\mu)$ be a frame for $L^2(\mathbb{R})$ with frame bounds A_μ and B_μ. Denote by V_{N_μ} the span of the first N_μ eigenfunctions $\{\psi_k^\mu\}_{k=1}^{N_\mu}$ of $\mathbf{H}_{\Omega_\mu,\varphi}$ corresponding to the N_μ largest eigenvalues, where each N_μ is chosen so that (3.16) holds. If $0 < \varepsilon_\mu < 1$ such that $0 < \inf_{\mu\in\tilde{\Lambda}} A_\mu(1-\varepsilon_\mu)^2 \leq \sup_{\mu\in\tilde{\Lambda}} B_\mu < \infty$, then there exist $\Omega_\mu^* \supset \Omega_\mu$ such that $\bigcup_{\mu\in\tilde{\Lambda}}\{\mathcal{P}_{V_{N_\mu}}\pi(\lambda)g^\mu\}_{\lambda\in\Lambda_\mu\cap\Omega_\mu^*}$ is a frame for $L^2(\mathbb{R})$.*

Remark 3.21. This global system forming a frame obtained from local systems is comparable to quilted Gabor frames introduced by Dörfler in [34], the difference being the the projection of the time-frequency dictionary elements onto the time-frequency localized subspaces. In [83], Romero proved results concerning frames for general spline-type spaces from portions of given frames which provided existence conditions for quilted Gabor frames.

Proof: For each $\mu \in \tilde{\Lambda}$, by Proposition 3.17, there exists $\Omega_\mu^* \supset \Omega_\mu$ such that $\{\mathcal{P}_{V_{N_\mu}}\pi(\lambda)g^\mu\}_{\lambda\in\Lambda_\mu\cap\Omega_\mu^*}$ is a frame for V_{N_μ} and as in (3.15), the following inequality holds for all $f \in L^2(\mathbb{R})$:

$$A_\mu(1-\varepsilon_\mu)^2 \sum_{k=1}^{N_\mu} |\langle f, \psi_k^\mu\rangle|^2 \leq \sum_{\lambda\in\Lambda_\mu\cap\Omega_\mu^*} |\langle f, \mathcal{P}_{V_{N_\mu}}\pi(\lambda)g^\mu\rangle|^2 \leq B_\mu \sum_{k=1}^{N_\mu} |\langle f, \psi_k^\mu\rangle|^2.$$

By the assumption that $0 < \tilde{A} := \inf_{\mu\in\tilde{\Lambda}} A_\mu(1-\varepsilon_\mu)^2 \leq \tilde{B} := \sup_{\mu\in\tilde{\Lambda}} B_\mu < \infty$ and the equivalence in (3.16), we get

$$\tilde{A} \sum_{\mu\in\tilde{\Lambda}} \sum_{k=1}^{N_\mu} |\langle f, \psi_k^\mu\rangle|^2 \leq \sum_{\mu\in\tilde{\Lambda}} \sum_{\lambda\in\Lambda_\mu\cap\Omega_\mu^*} |\langle f, \mathcal{P}_{V_{N_\mu}}\pi(\lambda)g^\mu\rangle|^2 \leq \tilde{B} \sum_{\mu\in\tilde{\Lambda}} \sum_{k=1}^{N_\mu} |\langle f, \psi_k^\mu\rangle|^2,$$

and finally $\sum_{\mu\in\tilde{\Lambda}} \sum_{\lambda\in\Lambda_\mu\cap\Omega_\mu^*} |\langle f, \mathcal{P}_{V_{N_\mu}}\pi(\lambda)g^\mu\rangle|^2 \asymp \|f\|_2^2$. ■

3.4. Numerical examples

In this section, we consider examples in the finite discrete case ($\mathbb{C}^L$, $L = 480$) that illustrate the results in the previous sections. The experiments were done in MATLAB using the NuHAG Matlab toolbox available in the following website: `http://www.univie.ac.at/nuhag-php/mmodule/`.

3.4.1. Experiment 1. We first examine the approximation of time-frequency localized signals by a local Gabor system, in particular, functions lying in the N-dimensional subspace V_N of eigenfunctions of $\mathbf{H}_{\Omega,\varphi}$, as shown in Proposition 3.12. In this example, we take Ω to be a disk centered at the origin with radius 80 and φ to be a normalized Gaussian.

Figure 1 shows the STFT of a signal in V_N and the sample points taken over circular regions with varying radii, each containing Ω. In each case, the sampling points are obtained by restricting a lattice with parameters $a = b = 20$ over the circular region. The error of the approximation $\|\mathcal{P}_{V_N} - \mathbf{S}^{\text{loc}}\mathcal{P}_{V_N}\|_{\text{Op}}$, where $\mathbf{S}^{\text{loc}}$ is a truncated tight frame operator, is shown in Table 1 below.

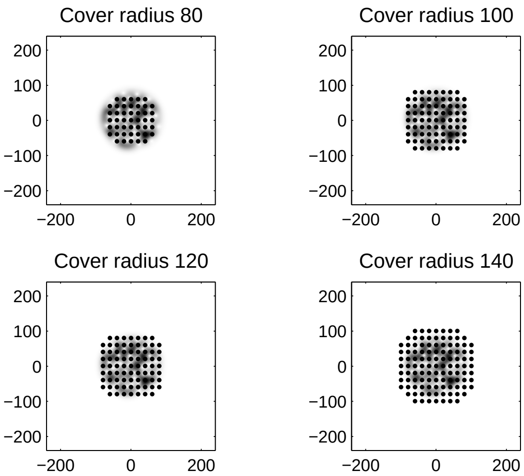

FIGURE 1. Sampling points over various enlargements of the covering region.

Cover radius	No. of samp. pts.	Op. norm error
80	45	0.9650
100	77	0.1105
120	109	0.0194
140	145	0.0031

TABLE 1. Error $\|\mathcal{P}_{V_N} - \mathbf{S}^{\text{loc}}\mathcal{P}_{V_N}\|_{\text{Op}}$ over varying radii for the disk Ω

We saw in Proposition 3.17 that if $\varepsilon < 1$, corresponding to the operator norm $\|\mathcal{P}_{V_N} - \mathbf{S}^{\mathrm{loc}}\mathcal{P}_{V_N}\|_{\mathrm{Op}}$ being less than 1, then the local Gabor system projected into V_N forms a frame for V_N so perfect reconstruction is possible by the frame algorithm (1.27). The performance of the reconstruction algorithm is shown in Figure 2. As expected, the larger the covering region, the faster the convergence.

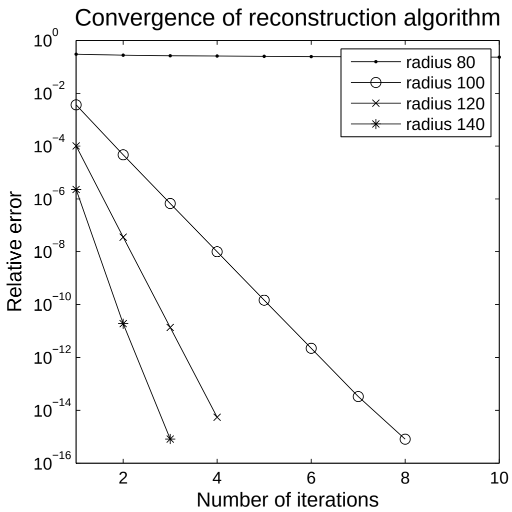

FIGURE 2. Convergence of the reconstruction algorithm from the local samples with the same lattice parameters but with varying radii of the covering regions.

3.4.2. Experiment 2. In this next experiment, we look at an example of how the collection of local Gabor systems can form a frame given that the sum of the characteristic functions over the regions is bounded above and below by a positive number. Figure 3 shows ten regions in the TF-plane and Figure 4 shows its sum.

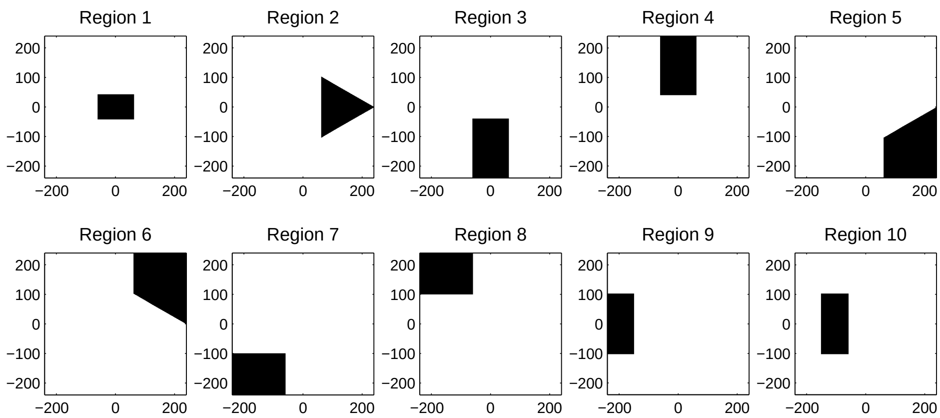

FIGURE 3. Ten regions that partition the time-frequency plane.

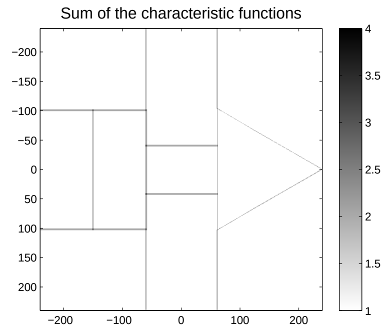

FIGURE 4. Sum of the characteristic functions over the ten regions.

Sample points are then taken over sets that contain each region, where different lattices are used for each set. The lattice parameters assigned to each set are summarized in Table 2, and the sample points are depicted in Figure 5. The left image shows sample points obtained by restricting each lattice over the regions themselves, while the samples in the right image are obtained from the restriction over larger sets containing each region, thus producing more overlap. Tight windows are used corresponding to each set of restricted lattice points.

Region	(a, b)	Region	(a, b)
1	$(20, 20)$	6	$(15, 15)$
2	$(16, 20)$	7	$(12, 15)$
3	$(20, 16)$	8	$(12, 12)$
4	$(16, 16)$	9	$(10, 12)$
5	$(15, 16)$	10	$(10, 10)$

TABLE 2. Lattice parameters over the different regions.

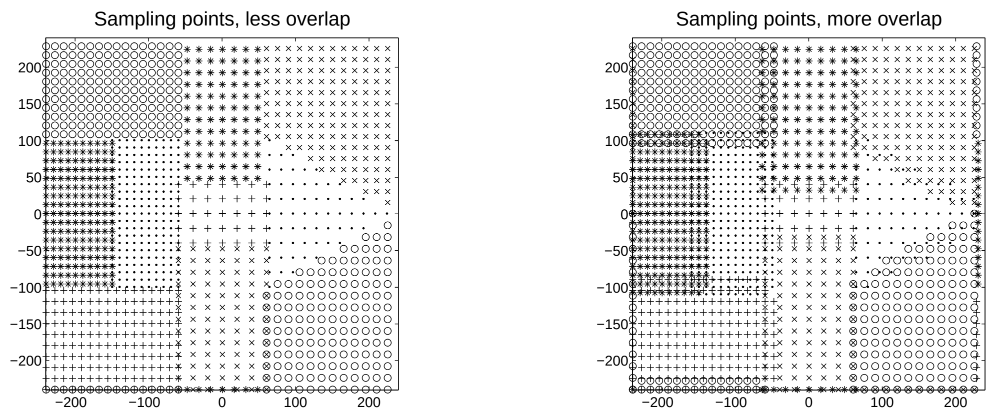

FIGURE 5. Sampling points on the different local patches.

We form a quilted Gabor frame from the collection of local Gabor systems. And by projecting each local Gabor system onto the local subspace corresponding to each region, we likewise obtain a global frame as in Theorem 3.20. The average of the relative error $\frac{\|f - \mathbf{S}_i f\|_2}{\|f\|_2}$ when the frame operators $\mathbf{S}_1$ and $\mathbf{S}_2$, corresponding to the quilted Gabor frame (i.e. without projection) and the global frame (i.e. with projection), respectively, are applied to a random signal f are shown in Table 3.

	without projection	with projection
Less overlap	0.2610	0.1687
More overlap	0.5840	0.1709

Table 3. Average of the error in applying the frame operator to a random signal (average of 1000 attempts).

In both cases of less and more overlap, projecting onto the TF-localized subspaces decreases the relative error between the signal and the approximation by the frame operator. Note that in both quilted Gabor frame and the global frame with projection, having more overlap increases the relative error since we are just comparing f with $\mathbf{S}_i f$. Since we are dealing with frames, perfect reconstruction (up to numerical error) is possible via the frame algorithm (1.27).

We first compare the respective condition numbers of the frame operators for the cases of less and more overlap. The values are shown in Table 4. Once again, in both quilted Gabor frame and the global frame with projection, having more overlap improves the condition number. Note that the large condition number for the frame operator corresponding to the global frame with less overlap can be attributed to the lower frame bound in Theorem 3.20, which is related to the set Ω_μ^* that covers the region Ω_μ - a smaller region Ω_μ^* implies a smaller lower frame bound.

	without projection	with projection
Less overlap	5.1429	16.0406
More overlap	3.5472	1.9845

Table 4. Condition numbers of the resulting frame operators.

Figure 6 compares the convergence of the frame algorithm for the four cases considered.

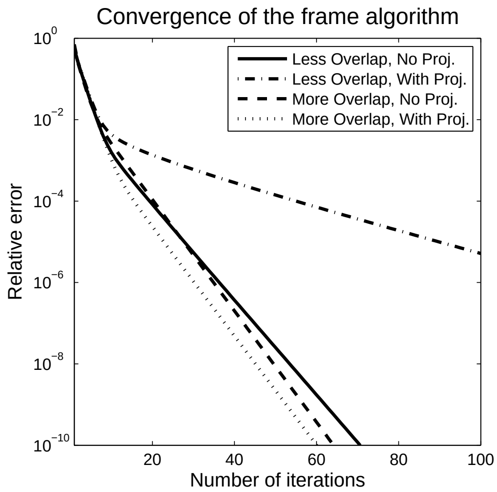

FIGURE 6. Convergence of the frame algorithm.

CHAPTER 4

Random sampling of time-frequency localized functions

In [9, 10], R. Bass and K. Gröchenig studied the random sampling of band-limited functions. They investigated the probability that a sampling inequality holds from random local sampling points. Given the space $\mathcal{B}$ of bandlimited functions,

$$\mathcal{B} = \{f \in L^2(\mathbb{R}) \,:\, \operatorname{supp} \hat{f} \subseteq [-1/2, 1/2]\},$$

we let $C_R = [-R/2, R/2]$, and we define the subset

$$\mathcal{B}(R, \varepsilon) = \{f \in \mathcal{B} \,:\, \int_{C_R} |f(x)|^2 \, dx \geq (1 - \varepsilon)\|f\|_2^2\}.$$

Theorem 4.1. [10, Theorem 1] *Let $\{x_j \,:\, j \in \mathbb{N}\}$ be a sequence of independent and identically distributed random variables that are uniformly distributed in C_R. Suppose that $R \geq 2$, that $\varepsilon \in (0, 1)$ and $\nu \in (0, 1/2)$ are small enough, and that $0 < \delta < 1$. There exists a constant κ so that if the number of samples r satisfies*

$$r \geq R\frac{1 + \nu/3}{\nu^2} \ln \frac{2R}{\delta},$$

then the sampling inequality

$$\frac{r}{R}\left(\frac{1}{2} - \varepsilon - \nu - 12\varepsilon\kappa\right)\|f\|_2^2 \leq \sum_{j=1}^{r} |f(x_j)| \leq r\|f\|_2^2 \qquad \text{for all } f \in \mathcal{B}(R, \varepsilon)$$

holds with probability at least $1 - \delta$. The constant κ can be taken to be $\kappa = e^{d\pi}$.

Remark 4.2. Führ and Xian [53] extended the results to the setting of finitely generated shift-invariant spaces.

We follow the approach in [10] for functions that are (ε, φ)-concentrated on a compact region Ω in the time-frequency plane, where $\varphi \in L^2(\mathbb{R})$ with $\|\varphi\|_2 = 1$. We first consider random sampling for functions in the finite-dimensional space V_N of eigenfunctions of $\mathbf{H}_{\Omega,\varphi}$.

Proposition 4.3. *Let $\Lambda_\Omega = \{\lambda_j\}_{j\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables that are uniformly distributed in Ω. Then*

$$\mathbb{P}\left(\inf_{f\in V_N, \|f\|_2=1} \frac{1}{r}\sum_{j=1}^{r}(|\mathcal{V}_\varphi f(\lambda_j)|^2 - \frac{1}{|\Omega|}\langle \mathbf{H}_{\Omega,\varphi} f, f\rangle) \leq -\frac{\nu}{|\Omega|}\right) \leq N \exp\left(-\frac{\nu^2 r}{|\Omega|(1 + \nu/3)}\right) \tag{4.1}$$

for $\nu \geq 0$.

First part of the proof. Let $f = \langle c, \psi \rangle = \sum_{k=1}^{N} c_k \psi_k \in V_N$, so that

$$|\mathcal{V}_\varphi f(\lambda_j)|^2 = \sum_{k,l=1}^{N} c_k \overline{c_l} \langle \psi_k, \pi(\lambda_j)\varphi \rangle \overline{\langle \psi_l, \pi(\lambda_j)\varphi \rangle}.$$

We define the $N \times N$ rank-one matrix T_j as follows:

$$(T_j)_{kl} := \langle \psi_k, \pi(\lambda_j)\varphi \rangle \overline{\langle \psi_l, \pi(\lambda_j)\varphi \rangle}. \tag{4.2}$$

Note that $|\mathcal{V}_\varphi f(\lambda_j)|^2 = \langle c, T_j c \rangle$. Since each random variable λ_j is uniformly distributed over Ω, and ψ_k is the kth eigenfunction of the time-frequency localization operator $\mathbf{H}_{\Omega,\varphi}$, the expectation of the kl-th entry is

$$\mathbb{E}\left((T_j)_{kl}\right) = \frac{1}{|\Omega|} \iint_\Omega \langle \psi_k, \pi(z)\varphi \rangle \overline{\langle \psi_l, \pi(z)\varphi \rangle}\, dz = \frac{1}{|\Omega|} \langle \mathbf{H}_{\Omega,\varphi} \psi_k, \psi_l \rangle \tag{4.3}$$

$$= \frac{1}{|\Omega|} \alpha_k \delta_{kl} \quad k, l = 1, \ldots, N, \tag{4.4}$$

where δ_{kl} is Kronecker's delta. The expectation of T_j is the diagonal matrix

$$\mathbb{E}(T_j) = \frac{1}{|\Omega|} \operatorname{diag}(\alpha_k) =: \frac{1}{|\Omega|} \Delta. \tag{4.5}$$

Now, the expression inside the left-hand side of (4.1) can be rewritten as

$$\inf_{f \in V_N, \|f\|_2 = 1} \frac{1}{r} \sum_{j=1}^{r} \left(|\mathcal{V}_\varphi f(\lambda_j)|^2 - \frac{1}{|\Omega|} \langle \mathbf{H}_{\Omega,\varphi} f, f \rangle \right) \tag{4.6}$$

$$= \inf_{\|c\|_2 = 1} \frac{1}{r} \sum_{j=1}^{r} \left(\langle c, T_j c \rangle - \langle c, \mathbb{E}(T_j) c \rangle \right) \tag{4.7}$$

$$= \alpha_{\min} \left(\frac{1}{r} \sum_{j=1}^{r} (T_j - \mathbb{E}(T_j)) \right), \tag{4.8}$$

where $\alpha_{\min}(U)$ denotes the smallest eigenvalue of a self-adjoint matrix U.

We now apply a matrix Bernstein inequality due to Tropp [88]. Let $\alpha_{\max}(A)$ be the largest singular value of a matrix A so that $\|A\| = \alpha_{\max}(A^* A)^{1/2}$ is the operator norm with respect to the ℓ^2-norm.

Theorem 4.4. [88, Theorem 1.4] *Let X_j be a finite sequence of independent, random, self-adjoint $N \times N$-matrices. Suppose that $\mathbb{E}(X_j) = 0$ and $\|X_j\| \le B$ a.s. and let $\sigma^2 = \left\| \sum_{j=1}^{r} \mathbb{E}(X_j^2) \right\|$. Then for all $t \ge 0$,*

$$\mathbb{P}\left(\alpha_{\max}\left(\sum_{j=1}^{r} X_j \right) \ge t \right) \le N \exp\left(- \frac{t^2/2}{\sigma^2 + Bt/3} \right). \tag{4.9}$$

We take $X_j = T_j - \mathbb{E}(T_j)$ and compute $\|X_j\|$ and $\| \sum_{j=1}^{r} \mathbb{E}(X_j^2) \|$.

Lemma 4.5. *If* $X_j = T_j - \mathbb{E}(T_j)$, *then*

(1) $\|X_j\| \leq 1$,
(2) $\mathbb{E}(X_j^2) \leq \frac{1}{|\Omega|}\Delta$, *and*
(3) $\sigma^2 = \Big\| \sum_{j=1}^{r} \mathbb{E}(X_j^2) \Big\| \leq \frac{r}{|\Omega|}$.

Proof:

(1) The matrix norm of X_j is estimated as follows:

$$\begin{aligned}\|X_j\| = \|T_j - \mathbb{E}(T_j)\| &= \sup_{\|c\|_2=1} |\langle c, T_j c\rangle - \langle c, \mathbb{E}(T_j)c\rangle| \\ &= \sup_{\|f\|_2=1} \left| |\mathcal{V}_\varphi f(\lambda_j)|^2 - \frac{1}{|\Omega|}\langle \mathbf{H}_{\Omega,\varphi} f, f\rangle \right| \\ &\leq \|f\|_2^2 \|\varphi\|_2^2 = 1\end{aligned}$$

(2) To find $\mathbb{E}(X_j^2)$, we use (4.5) and obtain

$$\begin{aligned}\mathbb{E}(X_j^2) &= \mathbb{E}(T_j^2) - \frac{1}{|\Omega|}\mathbb{E}(T_j\Delta) - \frac{1}{|\Omega|}\mathbb{E}(\Delta T_j) + \frac{1}{|\Omega|^2}\Delta^2 \\ &= \mathbb{E}(T_j^2) - \frac{1}{|\Omega|}\mathbb{E}(T_j)\Delta - \frac{1}{|\Omega|}\Delta\mathbb{E}(T_j) + \frac{1}{|\Omega|^2}\Delta^2 \\ &= \mathbb{E}(T_j^2) - \frac{1}{|\Omega|^2}\Delta^2.\end{aligned}$$

Now we compare T_j^2 and T_j.

$$\begin{aligned}(T_j^2)_{km} &= \sum_{l=1}^{N} (T_j)_{kl}(T_j)_{lm} \\ &= \sum_{k=1}^{N} \overline{\langle \psi_k, \pi(\lambda_j)\varphi\rangle}\langle \psi_l, \pi(\lambda_j)\varphi\rangle\overline{\langle \psi_l, \pi(\lambda_j)\varphi\rangle}\langle \psi_m, \pi(\lambda_j)\varphi\rangle \\ &= \left(\sum_{l=1}^{N} |\langle \psi_l, \pi(\lambda_j)\varphi\rangle|^2\right)(T_j)_{km} \\ &= \|\mathcal{P}_{V_N}\varphi\|_2^2 (T_j)_{km} \leq \|\varphi\|_2^2 (T_j)_{km} = (T_j)_{km}\end{aligned}$$

We thus have $T_j^2 \leq T_j$ and $\mathbb{E}(T_j^2) \leq \mathbb{E}(T_j) = \frac{1}{|\Omega|}\Delta$, so the expectation of X_j^2 gives

$$\mathbb{E}(X_j^2) = \mathbb{E}(T_j^2) - \frac{1}{|\Omega|^2}\Delta^2 \leq \frac{1}{|\Omega|}\Delta.$$

(3) $\sigma^2 = \Big\| \sum_{j=1}^{r} \mathbb{E}(X_j^2) \Big\| \leq \frac{r}{|\Omega|}\|\Delta\| \leq \frac{r}{|\Omega|}$.

∎

End of proof of Proposition 4.3. It follows from Theorem 4.4, taking $t = r\nu/|\Omega|$, that

$$\mathbb{P}\left(\alpha_{\min}\left(\sum_{j=1}^{r}(T_j - \mathbb{E}(T_j)\right) \leq -\frac{\nu r}{|\Omega|}\right) \leq N \exp\left(-\frac{\nu^2 r^2 |\Omega|^{-2}}{|\Omega|^{-1} r + |\Omega|^{-1}\nu r/3}\right).$$

Together with (4.8), we obtain the conclusion of the proposition. ■

We now observe a relation between the lower sampling inequality for the space V_N to that for functions that are (ε, φ)-concentrated in Ω.

Lemma 4.6. *Let $\varphi \in \boldsymbol{S}_0$, with $\|\varphi\|_2 = 1$ and $\Lambda_\Omega = \{\lambda_r\}_{j=1}^r$ a finite relatively separated set of points in Ω. If the inequality*

$$\frac{1}{r}\sum_{j=1}^{r} |\mathcal{V}_\varphi p(\lambda_j)|^2 \geq \frac{\langle \mathbf{H}_{\Omega,\varphi} p, p\rangle - \nu \|p\|_2^2}{|\Omega|} \tag{4.10}$$

holds for all $p \in V_N$, then the inequality

$$\sum_{j=1}^{r} |\mathcal{V}_\varphi(\lambda_j)|^2 \geq A\|f\|_2^2 \tag{4.11}$$

holds for all f that are (ε, φ)-concentrated in Ω with constant

$$A = \frac{r}{|\Omega|}\left(\alpha_N - \frac{\alpha_N \varepsilon}{1-\alpha_N} - \nu\right) - 2B\sqrt{\frac{\varepsilon}{1-\alpha_N}},$$

where B is a constant dependent on the covering index

$$N_0 = \sup_{m \in \mathbb{Z}^2} \#(\Lambda \cap Q_1(m))$$

and the window function φ.

Remark 4.7. For $A > 0$, we need $r \geq |\Omega|\left(\dfrac{2B\sqrt{\frac{\varepsilon}{1-\alpha_N}}}{\alpha_N - \frac{\alpha_N \varepsilon}{1-\alpha_N}} - \nu\right)$.

Proof: Since $f = \mathcal{P}_{V_N} f + (I - \mathcal{P}_{V_N})f$, we have

$$\left(\sum_{j=1}^{r} |\mathcal{V}_\varphi f(\lambda_j)|^2\right)^{1/2} \geq \left(\sum_{j=1}^{r} |\mathcal{V}_\varphi \mathcal{P}_{V_N} f(\lambda_j)|^2\right)^{1/2} - \left(\sum_{j=1}^{r} |\mathcal{V}_\varphi (I - \mathcal{P}_{V_N}) f(\lambda_j)|^2\right)^{1/2}.$$

Squaring both sides of the inequality and applying Theorem 1.38, where B is the Bessel bound given in (1.34),we get

$$\begin{aligned}\sum_{j=1}^{r} |\mathcal{V}_\varphi f(\lambda_j)|^2 \geq &\sum_{j=1}^{r} |\mathcal{V}_\varphi \mathcal{P}_{V_N} f(\lambda_j)|^2 \\ &- 2\left(\sum_{j=1}^{r} |\mathcal{V}_\varphi \mathcal{P}_{V_N} f(\lambda_j)|^2\right)^{1/2}\left(\sum_{j=1}^{r} |\mathcal{V}_\varphi (I - \mathcal{P}_{V_N}) f(\lambda_j)|^2\right)^{1/2}\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{r}|\mathcal{V}_\varphi(I-\mathcal{P}_{V_N})f(\lambda_j)|^2\\
\geq &\sum_{j=1}^{r}|\mathcal{V}_\varphi\mathcal{P}_{V_N}f(\lambda_j)|^2-2B\|\mathcal{P}_{V_N}f\|_2\|(I-\mathcal{P}_{V_N})f\|_2\\
\geq &\sum_{j=1}^{r}|\mathcal{V}_\varphi\mathcal{P}_{V_N}f(\lambda_j)|^2-2B\sqrt{\frac{\varepsilon}{1-\alpha_N}}\,\|f\|_2^2,
\end{aligned}
$$

where the last inequality follows from $\|\mathcal{P}_{V_N}f\|_2\leq\|f\|_2$ and Proposition 2.15(2). By hypothesis (4.10) and Proposition 2.15, we obtain

$$
\begin{aligned}
\sum_{j=1}^{r}|\mathcal{V}_\varphi f(\lambda_j)|^2 &\geq \sum_{j=1}^{r}|\mathcal{V}_\varphi\mathcal{P}_{V_N}f(\lambda_j)|^2-2B\sqrt{\frac{\varepsilon}{1-\alpha_N}}\,\|f\|_2^2\\
&\geq \frac{r}{|\Omega|}\langle\mathbf{H}_{\Omega,\varphi}\mathcal{P}_{V_N}f,\mathcal{P}_{V_N}f\rangle-\frac{r\,\nu}{|\Omega|}\|\mathcal{P}_{V_N}f\|_2^2-2B\sqrt{\frac{\varepsilon}{1-\alpha_N}}\,\|f\|_2^2\\
&\geq \frac{r}{|\Omega|}\alpha_N\left(1-\frac{\varepsilon}{1-\alpha_N}\right)\|f\|_2^2-\frac{r\,\nu}{|\Omega|}-2B\sqrt{\frac{\varepsilon}{1-\alpha_N}}\,\|f\|_2^2.
\end{aligned}
$$

So we can take A as

$$
A=\frac{r}{|\Omega|}\left(\alpha_N-\frac{\alpha_N\varepsilon}{1-\alpha_N}-\nu\right)-2B\sqrt{\frac{\varepsilon}{1-\alpha_N}}.
$$

■

For the succeeding results, let Ω be a compact set that would need at most $|\Omega|+\epsilon_1$ cubes $Q(m)=[m_1-1/2,m_1+1/2]\times[m_2-1/2,m_2+1/2]$, where $m=(m_1,m_2)\in\mathbb{Z}^2$ and $\epsilon_1\geq 0$, to cover it.

Lemma 4.8. *Let $\Lambda_\Omega=\{\lambda_j\}_{j=1}^r$ be a finite sequence of independent and identically distributed random variables that are uniformly distributed in Ω. Let $a>|\Omega|^{-1}$. Then*

$$
\mathbb{P}(N_0>ar)\leq(|\Omega|+\epsilon_1)\exp\Big(-r(a\ln(a|\Omega|)-(a-|\Omega|^{-1}))\Big).
$$

Proof: If $N_0>ar$, then for at least one m, $Q(m)$ must contain at least ar points from Λ_Ω. So we have

$$
\mathbb{P}(N_0>ar)\leq(|\Omega|+\epsilon_1)\sup_{m\in\mathbb{Z}^2}\mathbb{P}(\#(\Lambda_\Omega\cap Q(m))>ar). \tag{4.12}
$$

We fix $m\in\mathbb{Z}^2$. For any $b>0$, it follows from Chebyshev's inequality that

$$
\begin{aligned}
\mathbb{P}(\#(\Lambda_\Omega\cap Q(m))>ar)&=\mathbb{P}\Big(\sum_{j=1}^{r}\chi_{Q(m)}(\lambda_j)>ar\Big)\\
&\leq e^{-bar}\,\mathbb{E}\exp\Big(b\sum_{j=1}^{r}\chi_{Q(m)}(\lambda_j)\Big).
\end{aligned}
$$

Since the λ's are uniformly distributed over Λ_Ω, it follows that $\chi_{Q(m)}(\lambda_j) = 1$ with probability at most $|\Omega|^{-1}$ and otherwise is 0. And by the independence,

$$\begin{aligned}\mathbb{P}(\#(\Lambda_\Omega \cap Q(m)) > ar) &\le e^{-bar} \prod_{j=1}^{r} \mathbb{E}\exp(b\chi_{Q(m)}(\lambda_j)) \\ &\le e^{-bar}\left((1-|\Omega|^{-1}) + e^b|\Omega|^{-1}\right)^r = e^{-bar}\left(1+(e^b-1)|\Omega|^{-1}\right)^r \\ &\le e^{-bar}\left(\exp\left((e^b-1)|\Omega|^{-1}\right)\right)^r.\end{aligned}$$

We choose $b = \ln(a|\Omega|)$ that optimizes the last term, which becomes

$$\exp\Big(-r\big(a\ln(a|\Omega|) - (a - |\Omega|^{-1})\big)\Big).$$

Substituting this expression in (4.12) gives the desired result. ■

We now combine the result in Proposition 4.3 with the estimates obtained in Lemma 4.6 and Lemma 4.8, and choose appropriate values of the parameters ε and ν to obtain the next theorem. We take $\alpha_N = 1/2$ so that N is around $|\Omega|$, say $N = |\Omega| + \epsilon_2$. From the Bessel bound B in (1.34), we have $N_1(\Lambda) = N_0$ and we let $C_\varphi = B/N_0$.

Theorem 4.9. *Let $\Lambda_\Omega = \{\lambda_j\}_{j\in\mathbb{N}}$ be a sequence of identically distributed random variables that are uniformly distributed in Ω, and let φ be a window function in $\boldsymbol{S}_0$ with $\|\varphi\|_2 = 1$. Suppose*

$$\varepsilon < \frac{1}{4(1+6\sqrt{2}C_\varphi)^2} \qquad \textit{and} \qquad \nu < \frac{1}{2} - (1+6\sqrt{2}C_\varphi)\sqrt{\varepsilon}.$$

If we let

$$A = \frac{r}{|\Omega|}\Big(\frac{1}{2} - \varepsilon - \nu - 6\sqrt{2}C_\varphi\sqrt{\varepsilon}\Big), \tag{4.13}$$

then the sampling inequality

$$A\|f\|_2^2 \le \sum_{j=1}^{r} |\mathcal{V}_\varphi(\lambda_j)|^2 \le r\|f\|_2^2, \tag{4.14}$$

for all (ε,φ)-concentrated functions, holds with probability at least

$$1 - (|\Omega|+\epsilon_2)\exp\Big(-\frac{\nu^2 r}{|\Omega|(1+\nu/3)}\Big) - (|\Omega|+\epsilon_1)\exp\Big(-\frac{r}{|\Omega|}(3\ln 3 - 2)\Big). \tag{4.15}$$

Proof: Since $|\mathcal{V}_\varphi f(\lambda_j)| = |\langle f, \pi(\lambda_j)\varphi\rangle| \le \|f\|_2$, the right-hand side of (4.14) follows immediately. We take $a = 3|\Omega|^{-1}$. Let

$$V_1 = \Big\{\inf_{f\in V_N,\, \|f\|_2=1} \frac{1}{r}\sum_{j=1}^{r}\Big(|\mathcal{V}_\varphi f(\lambda_j)|^2 - \frac{1}{|\Omega|}\langle \mathbf{H}_{\Omega,\varphi} f, f\rangle\Big) \le -\frac{\nu}{|\Omega|}\Big\}$$

and let

$$V_2 = \{N_0 > ar\}.$$

It follows from Proposition 4.3 and Lemma 4.8 that the probability of $(V_1 \cup V_2)^c$ is bounded below by (4.15). And by Lemma 4.6, we have that

$$\sum_{j=1}^{r} |\mathcal{V}_\varphi f(\lambda_j)|^2 \geq A\|f\|_2^2$$

for all (ε, φ)-concentrated functions f such that $(V_1 \cup V_2)^c$ holds. With $N_0 = 3|\Omega|^{-1}$, the lower bound in (4.11) becomes $A = \frac{r}{|\Omega|}\Big(\frac{1}{2} - \varepsilon - \nu - 6\sqrt{2}C_\varphi\sqrt{\varepsilon}\Big)$. The assumptions on ε and ν would guarantee that $A > 0$. ■

With $N = |\Omega| + \epsilon_2$ and $0 < \nu < 1/2 - (1 + 6\sqrt{2}C_\varphi)\sqrt{\varepsilon}$, if δ is given and

$$r \geq \max\left\{|\Omega|\frac{1+\nu/3}{\nu^2}\ln\frac{2(|\Omega|+\epsilon_2)}{\delta},\ \frac{|\Omega|}{3\ln 3 - 2}\ln\frac{2(|\Omega|+\epsilon_1)}{\delta}\right\} = |\Omega|\frac{1+\nu/3}{\nu^2}\ln\frac{2(|\Omega|+\epsilon_2)}{\delta},$$

then the probability in (4.15) will be larger than $1 - \delta$.

CHAPTER 5

Adaptive time-frequency representations and applications

This chapter consists of the results presented in [41, 89, 7]. These are joint works with P. Balazs, M. Dörfler, T. Grill, N. Holighaus, and F. Jaillet.

Adaptivity in the time-frequency representation of functions is often desired in applications, cf. [8]. While Gabor frames already provide more flexibility over Gabor Riesz bases, the rigid structure of Gabor frames may still be to be too restrictive in some applications, as it exhibits a fixed time-frequency resolution in its representation. Real life signals may have various components with with distinct time-frequency localization properties. Gabor frames may be adapted to certain properties of the signal. Such adaptation may be achieved for instance in opting to have diverse windows with certain desirable properties, or varying the sampling process instead of having a regular structure of the sampling set.

We present two adaptive time-frequency representations and illustrate their advantages in audio signal processing with numerical examples. The first method is obtained by an approximate projection of the relevant atoms onto a system of weighted vectors which are optimally concentrated inside the desired regions of adaptation. The second method is via nonstationary Gabor frames where the a set of more general windows are used instead of just regular translates of a single window. The transform thus obtained would allow for adaptivity in either time or frequency.

5.1. Approximate projections onto time-frequency subspaces

Assume that we are given a partition of $\mathbb{R}^2$, i.e. a family of sets $\Omega_\mu \subset \mathbb{R}^2$ such that $\sum_\mu \chi_{\Omega_\mu} \equiv 1$, and a window function $\varphi \in L^2(\mathbb{R})$ such that $\|\varphi\|_2 = 1$. Then, using the spectral decomposition of each time-frequency localization operator $\mathbf{H}_{\Omega_\mu,\varphi}$, we obtain

$$f = \sum_\mu \mathbf{H}_{\Omega_\mu,\varphi} f = \sum_\mu \sum_{k=1}^{\infty} \alpha_k^\mu \langle f, \psi_k^\mu \rangle \, \psi_k^\mu .$$

Now assume further that a tight Gabor frame $\mathcal{G}(g^\mu, \Lambda^\mu)$ is assigned to each set Ω_μ. Expanding f with respect to each of these frames, we obtain:

$$f = \sum_\mu \sum_{\lambda \in \Lambda^\mu} \langle f, \pi(\lambda) g^\mu \rangle \sum_{k=1}^{\infty} \alpha_k^\mu \langle \pi(\lambda) g^\mu, \psi_k^\mu \rangle \, \psi_k^\mu \tag{5.1}$$

We make the following observations from the properties of time-frequency localization operators as reviewed in Chapter 2:

- The largest eigenvalues α_k^μ of a localization operator typically are close to 1 and then drop to 0 very fast (in fact, the sequence $\{\alpha_k^\mu\}_k$ has exponential decay), cf. [29]. Around $|\Omega_\mu|$ eigenvalues lie above 0.5. Consequently, one can safely discard elements with index $k > N_\mu$ in (5.1).
- On the other hand, from Lemma 3.5 the inner product $\langle \pi(\lambda) g^\mu, \psi_k^\mu \rangle$ is shown to decay fast with respect to the distance of λ from Ω_μ, e.g. for a Gaussian window g^μ, the decay is exponential, while milder decay conditions lead to a polynomial decay. Therefore, all $\pi(\lambda) g^\mu$ with $\mathrm{dist}(\lambda, \Omega_\mu) \geq b$ for some b can be omitted from the equation (5.1).

We thus choose an appropriate N_μ, an extension size or overlap b and set $\Omega^* = \Omega_\mu \cup \{z \in \mathbb{R}^2 \setminus \Omega_\mu \,:\, \mathrm{dist}(z, \Omega_\mu) < b\}$. We then propose to use the following approximate reconstruction formula:

$$\tilde{f} = \sum_\mu \sum_{\lambda \in \Lambda^\mu \cap \Omega_\mu^*} \langle f, \pi(\lambda) g^\mu \rangle \sum_{k=1}^{N_\mu} \alpha_k^\mu \langle \pi(\lambda) g^\mu, \psi_k^\mu \rangle \psi_k^\mu \tag{5.2}$$

Observe that the sum $\sum_{k=1}^{N_\mu} \alpha_k^\mu \langle \cdot, \psi_k^\mu \rangle \psi_k^\mu$, which we denote by $\mathbf{H}_{\Omega_\mu,\varphi}^{N_\mu}$ since it is a truncation of the spectral decomposition of $\mathbf{H}_{\Omega_\mu,\varphi}$, is a weighted sum of the first N_μ eigenfunctions of the time-frequency localization operator $\mathbf{H}_{\Omega_\mu,\varphi}$. We can then treat $\mathbf{H}_{\Omega_\mu,\varphi}^{N_\mu}$ as an approximate projection onto the subspace V_{N_μ} spanned by $\{\psi_k^\mu\}_{k=1}^{N_\mu}$.

We now obtain the following error estimate:

Proposition 5.1. *Let a partition of* $\mathbb{R}^2$ *be given,* $\Omega_\mu \subset \mathbb{R}^2$ *such that* $\sum_\mu \chi_{\Omega_\mu} \equiv 1$, *and let the windows* g^μ *satisfy a joint polynomial decay condition of the form* $|\mathcal{V}_\varphi g^\mu(z)| \leq C(1 + |z|^{2s})^{-1}$, $s > 1$, *for all* $z \in \mathbb{R}^2$. *Let* $\tilde{f}$ *be the approximate reconstruction of* f *in* (5.2). *Then, the reconstruction error is bounded by* $\|\tilde{f} - f\|_2 \leq \sum_\mu \mathrm{err}_\mu \|f\|_2$, *where for all* μ *and some* $0 < \delta < 1$, *the following estimate holds:*

$$\mathrm{err}_\mu \leq \left(\sum_{k=N_\mu+1}^{\infty} (\alpha_k^\mu)^2 \right)^{\frac{1}{2}} + \left(C_\mu \sum_{\lambda \notin \Lambda^\mu \cap \Omega_\mu^*} \left(1 + (1-\delta)(\inf_{z \in \Omega} |z - \lambda|^s) \right)^{-2} \right)^{\frac{1}{2}} \tag{5.3}$$

Remark 5.2. It should be noted that the sum of err_μ over all μ can be shown to be finite for appropriate choices of N_μ and Ω_μ^*. Here, we prefer to state the explicit *local* errors, since their expression is more informative in showing directly the influence of the parameters N_μ and b.

Proof: Let $\tilde{f}_\mu = \sum_{\lambda\in\Omega^\mu\cap\Omega_\mu{}^*}\langle f,\pi(\lambda)g^\mu\rangle \mathbf{H}^{N_\mu}_{\Omega_\mu,\varphi}\pi(\lambda)g^\mu$. We can estimate the left-hand side of (5.2) as follows:

$$\begin{aligned}\|f-\tilde{f}\|_2 &= \Big\|\sum_\mu \mathbf{H}_{\Omega_\mu,\varphi}f - \sum_\mu \tilde{f}_\mu\Big\|_2 \le \sum_\mu \|\mathbf{H}_{\Omega_\mu,\varphi}f-\tilde{f}_\mu\|_2\\ &\le \sum_\mu \Big(\|\mathbf{H}_{\Omega_\mu,\varphi}f - \mathbf{H}^{N_\mu}_{\Omega_\mu,\varphi}f\|_2 + \|\mathbf{H}^{N_\mu}_{\Omega_\mu,\varphi}f - \tilde{f}_\mu\|_2\Big).\end{aligned}$$

We obtain error bounds for $\|\mathbf{H}_{\Omega_\mu,\varphi}f - \mathbf{H}^{N_\mu}_{\Omega_\mu,\varphi}f\|_2$ and $\|\mathbf{H}^{N_\mu}_{\Omega_\mu,\varphi}f - \tilde{f}_\mu\|_2$ separately. For the first expression, we have

$$\begin{aligned}\|\mathbf{H}_{\Omega_\mu,\varphi}f - \mathbf{H}^{N_\mu}_{\Omega_\mu,\varphi}f\|_2 &= \Big\|\sum_{k>N_\mu}\alpha_k^\mu\langle f,\psi_k^\mu\rangle\psi_k^\mu\Big\|_2\\ &\le \Big(\sum_{k>N_\mu}(\alpha_k^\mu)^2\Big)^{\frac12}\Big(\sum_{k>N_\mu}|\langle f,\psi_k^\mu\rangle|^2\Big)^{\frac12}\\ &\le \Big(\sum_{k>N_\mu}(\alpha_k^\mu)^2\Big)^{\frac12}\|f\|_2.\end{aligned}$$

On the other hand, for $\|\mathbf{H}^{N_\mu}_{\Omega_\mu,\varphi}f - \tilde{f}_\mu\|_2$ we calculate

$$\begin{aligned}\|\mathbf{H}^{N_\mu}_{\Omega_\mu,\varphi}f - \tilde{f}_\mu\|_2 &= \Big\|\sum_{\lambda\notin\Lambda^\mu\cap\Omega_\mu^*}\langle f,\pi(\lambda)g^\mu\rangle\sum_{k=1}^{N_\mu}\alpha_k^\mu\langle f,\psi_k^\mu\rangle\psi_k^\mu\Big\|_2\\ &\le \sum_{\lambda\notin\Lambda^\mu\cap\Omega_\mu^*}|\langle f,\pi(\lambda)g^\mu\rangle|\sum_{k=1}^{N_\mu}\alpha_k^\mu|\langle\pi(\lambda)g^\mu,\psi_k^\mu\rangle|\\ &= \sum_{\lambda\notin\Lambda^\mu\cap\Omega_\mu^*}|\langle f,\pi(\lambda)g^\mu\rangle|\sum_{k=1}^{N_\mu}|\langle\mathbf{H}_{\Omega_\mu,\varphi}\psi_k^\mu,\pi(\lambda)g^\mu\rangle|\\ &\le \sum_{\lambda\notin\Lambda^\mu\cap\Omega_\mu^*}|\langle f,\pi(\lambda)g^\mu\rangle|C\delta^{-\frac{1}{2s}}N_\mu(1+(1-\delta)\operatorname{dist}(\lambda,\Omega_\mu)^s)^{-1}\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(Lemma 3.5)}\\ &\le C\delta^{-\frac{1}{2s}}N_\mu\Big(\sum_{\lambda\notin\Lambda^\mu\cap\Omega_\mu^*}|\langle f,\pi(\lambda)g^\mu\rangle|^2\Big)^{\frac12}\cdot\\ &\qquad\Big(\sum_{\lambda\notin\Lambda^\mu\cap\Omega_\mu^*}(1+(1-\delta)\operatorname{dist}(\lambda,\Omega_\mu)^s)^{-2}\Big)^{\frac12}.\end{aligned}$$

If B_μ is the frame bound of the tight Gabor frame $\mathcal{G}(g^\mu,\Lambda_\mu)$, then by taking $C_\mu = \sqrt{B_\mu}C_s\delta^{-\frac{1}{2s}}N_\mu$, we obtain the conclusion of the proposition. ■

5.1.1. Derived Algorithm.

5.1.1.1. *Computation of* $\mathbf{H}^{N_\mu}_{\Omega_\mu,\varphi}$. To obtain the eigenvectors and eigenvalues needed for the approximation in (5.2), we work with discrete versions of the localization operators $\mathbf{H}_{\Omega_\mu,\varphi}$. To this end, consider the tight Gabor frame $\mathcal{G}(g_t,\Lambda)$. We define the Gabor multiplier $\mathsf{H}_{m_\mu,\Lambda}$ as follows:

$$\mathsf{H}_{m_\mu,\Lambda} f = \sum_{\lambda\in\Lambda} m_\mu(\lambda)\langle f,\pi(\lambda)g_t\rangle \pi(\lambda)g_t, \tag{5.4}$$

where the masks m_μ are obtained by letting $m_\mu(\lambda) := 1$, if $\lambda\in\Omega_\mu$ and 0 otherwise. Then $\mathsf{H}_{m_\mu,\Lambda}$ is a discretization of the operator $\mathbf{H}_{\Omega_\mu,\varphi}$ and it can be shown that its spectral decomposition accurately approximates $\mathbf{H}_{\Omega_\mu,\varphi}$, for sufficiently dense lattice Λ, cf. [48, 35].

In applications $\mathsf{H}_{m_\mu,\Lambda}$ is a matrix whose size depends on the signal length L and it may be cumbersome to find the eigenfunctions and eigenvalues directly. However, as observed e.g. in [39], the size of the corresponding *Gramian matrix* Γ_{m_μ}, given by

$$\Gamma_{m_\mu} := G_{\sqrt{m_\mu}}\cdot G^*_{\sqrt{m_\mu}}, \tag{5.5}$$

where $G_{\sqrt{m_\mu}}$ is the operator $f\mapsto [\sqrt{m_\mu(\lambda)}\langle f,\pi(\lambda)g\rangle]_{\lambda\in\Lambda\cap \mathrm{supp}(m)}$, mapping $\mathbb{C}^L$ to $\mathbb{C}^K$, is $K\times K$ with K being the number of lattice points λ_μ inside the support of the mask m_μ, which is usually small enough for the computation of the spectral decomposition to be a feasible task.

Writing $\mathsf{H}_{m_\mu,\Lambda}$ as a composition of $G_{\sqrt{m_\mu}}$ and its adjoint $G^*_{\sqrt{m_\mu}}$, the eigenfunctions of $\mathsf{H}_{m_\mu,\Lambda} = G^*_{\sqrt{m_\mu}}\cdot G_{\sqrt{m_\mu}}$ may be obtained from the eigenfunctions of the Gramian Γ_{m_μ} by

$$\psi^\mu_k = \frac{1}{s_j}\, G^*_{\sqrt{m_\mu}}\cdot u_j,\ \ j=1,\dots,K, \tag{5.6}$$

where $G_{\sqrt{m_\mu}} f = \sum_{k=1}^{K} s_j\langle f,\psi^\mu_k\rangle_{\mathbb{C}^L} u_j$ is the singular value decomposition of $G_{\sqrt{m_\mu}}$. Furthermore, in (5.5) only the largest N_μ eigenfunctions u_j need to be computed.

5.1.1.2. *Choosing* N_μ *and* Ω^*. For each μ, N_μ eigenfunctions $\{\psi^\mu_k\}_j$ of $\mathsf{H}_{m_\mu,\Lambda}$, associated to the eigenvalues α^μ_k greater than a threshold t_μ must be chosen. If the sets Ω_μ are of the same area, then we just take the same value of N_μ for each μ. Choosing N_μ such that $\alpha^\mu_{N_\mu} < 10^{-m}$, the first expression in the error estimate (5.3) is bounded by 10^{-m} due to the exponential decay of the eigenvalues.

Second, we choose a rectangular extension Ω^*_μ of Ω_μ by increasing its sides by a margin also of size b, such that in the second expression of (5.3), the value $\inf_{z\in\Omega}|z-\lambda|$ is sufficiently big for all $\lambda\notin\Lambda^\mu\cap\Omega^*_\mu$.

5.1.2. Numerical Experiments. We look at examples in the finite discrete case $\mathbb{C}^L$, $L = 144$. The experiments were done in MATLAB using the NuHAG

Matlab toolbox available in `http://www.univie.ac.at/nuhag-php/mmodule/`. The time-frequency plane will be partitioned into four parts, dividing the time axis at $t_{\text{cut}} = L/2$, and the frequency axis into bands corresponding to the frequencies above and below $\omega_{\text{cut}} = L/4$. We note that these frequency bands extend to the negative frequencies in a symmetric manner about the frequency 0.

The following tight Gabor frames will then be associated to the four regions:

(1) $\mathcal{G}(g_t^1, 12, 4)$ at the region Ω_1 (lower frequency region and time $t \leq L/2$);
(2) $\mathcal{G}(g_t^2, 16, 6)$ at the region Ω_2 (lower frequency region and time $t > L/2$);
(3) $\mathcal{G}(g_t^3, 8, 16)$ at the region Ω_3 (higher frequency region and time $t \leq L/2$); and
(4) $\mathcal{G}(g_t^4, 9, 12)$ at the region Ω_4 (higher frequency region and time $t > L/2$).

The signal will be analyzed using the these tight Gabor frames and applied with weighted functions over regions that cover our partitions. We shall reconstruct using the method introduced in [77] and our proposed method, and compare the approximation quality from the two methods.

For the approximate reconstruction [77], weight functions W_T^1 and W_T^2, depending only on time, and W_F^1 and W_F^2, depending only on frequency, shall be applied to the analysis coefficients. These weight functions are defined as follows:

$$W_T^1(t) := \begin{cases} 1 & \text{if } 1 \leq t \leq t_1 \\ \frac{t-t_2}{t_1-t_2} & \text{if } t_1 \leq t \leq t_2 \\ 0 & \text{elsewhere} \end{cases},$$

where $t_1 \leq t_{\text{cut}} \leq t_2$, $W_T^2 := 1 - W_T^1$, i.e. $W_T^1(t) + W_T^2(t) = 1$ for each t,

$$W_F^1(\omega) := \begin{cases} 1 & \text{if } -\omega_1 \leq \omega \leq \omega_1 \\ \frac{\omega-\omega_2}{\omega_1-\omega_2} & \text{if } \omega_1 \leq t \leq \omega_2 \\ \frac{\omega+\omega_2}{\omega_2-\omega_1} & \text{if } -\omega_2 \leq t \leq -\omega_1 \\ 0 & \text{elsewhere} \end{cases}$$

, where $\omega_1 \leq \omega_{\text{cut}} \leq \omega_2$, $W_F^2 := 1 - W_F^1$, i.e. $W_F^1(\omega) + W_F^2(\omega) = 1$ for each ω. Figure 1 shows the four weight functions. We note that varying the t_i and ω_i amounts to varying the overlap of the weight functions. In the experiment, the overlap value $b := t_2 - t_{\text{cut}} = t_{\text{cut}} - t_1$ for the weight function in time shall also be used for the weight function in frequency so that $b = \omega_2 - \omega_{\text{cut}} = \omega_{\text{cut}} - \omega_1$.

Recall that from [77], the reconstruction formula is given by

$$\tilde{f}_W = \sum_{k=1}^{4} \sum_{\lambda \in \Lambda^k} W_{TF}^k(t, \omega) \langle f, \pi(\lambda) g_t^k \rangle \pi(\lambda) g_t^k, \tag{5.7}$$

where W_{TF}^k corresponds to $W_T^1 \cdot W_F^1$ for $k = 1$, $W_T^2 \cdot W_F^1$ for $k = 2$, $W_T^1 \cdot W_F^2$ for $k = 3$, and $W_T^2 \cdot W_F^2$ for $k = 4$.

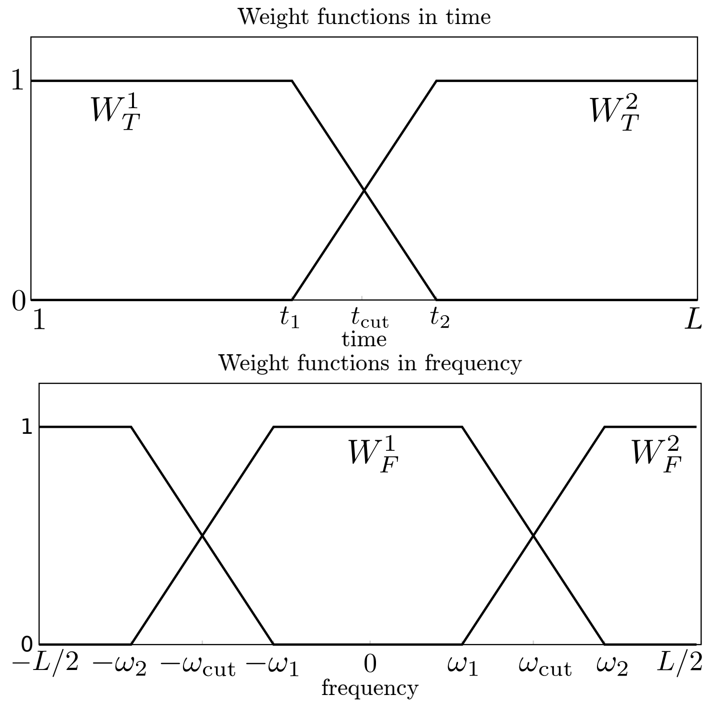

FIGURE 1. Weight functions W_T^1, W_T^2, W_F^1, and W_F^2.

We now compare the errors in approximating f using the methods described above. Figure 2 shows the average of the root mean square (RMS) of the error given by

$$\text{err}(f_{\text{rec}}) = \frac{\|f - f_{\text{rec}}\|_2}{\|f\|_2} = \sqrt{\frac{\sum_{n=1}^{L}(f[n] - f_{\text{rec}}[n])^2}{\sum_{n=1}^{L}(f[n])^2}},$$

of 50 random signals against the amount of overlap b. The solid line is from the weight function method in [77] while the non-solid lines result from the proposed approximate projection method. Each of the non-solid lines uses a different number of eigenfunctions: 45, 50, and 55 eigenfunctions, corresponding to the eigenvalue thresholds 0.1016, 0.0243, and 0.0040. In both methods, we see the dependence of the approximate reconstruction on the overlap amount. In the case of our proposed method, the second term on error bound of (5.3) approaches 0 as the overlap increases. Moreover, the approximate projection method has the added possibility of improving the approximation error by increasing the number of eigenfunctions in the reconstruction. The dependence of the proposed method on the number of eigenfunctions in the subspace is depicted in Figure 3.

Finally, we point out that the separation between the distinct regions chosen for the different desired resolutions, that is Ω_μ, $\mu = 1, \ldots, 4$, is much sharper using the approximate projection method. This fact is illustrated in Figure 4, where we show the results of applying one of the local systems to random white noise. Depicted are

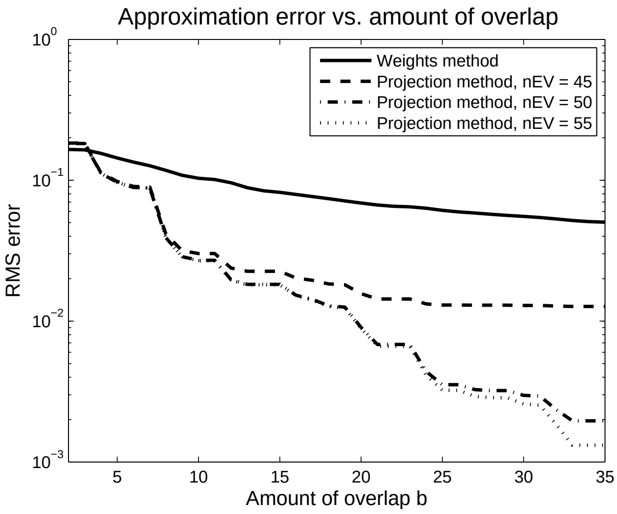

FIGURE 2. Approximation error vs. amount of overlap.

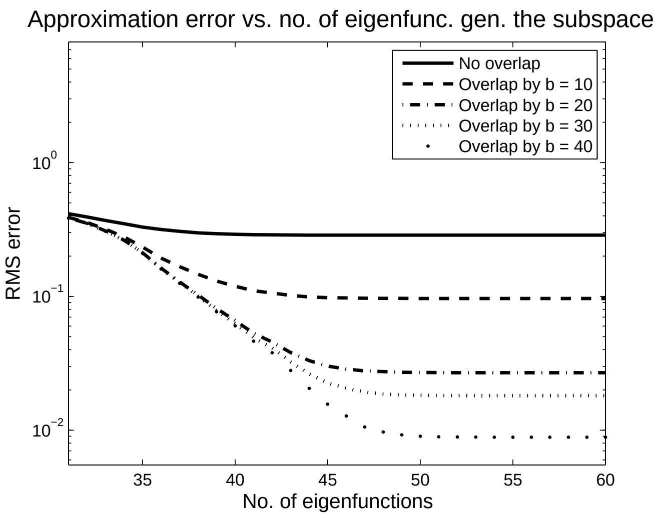

FIGURE 3. Approximation error vs. number of eigenfunctions in the subspace.

the spectrograms of the results for the systems corresponding to low frequencies, first signal part and high frequencies, second signal part, respectively. For both methods, the set of parameters providing the best approximation quality is used. It can clearly be seen, that the approximate projection method significantly reduces the spill outside the region of interest which is quite considerable in the weight function method.

5.2. Nonstationary Gabor frames

In this section, we present an approach to fast adaptive time-frequency transforms, that is based on a generalization of painless nonorthogonal expansions [28]. It allows

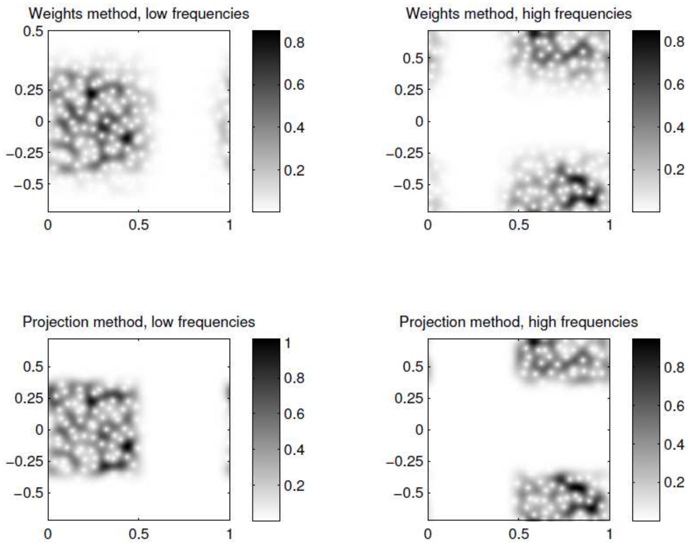

FIGURE 4. Concentration of local systems within Ω_μ. The spectrograms of local systems applied to random noise are shown.

for adaptivity of the analysis windows *and* the sampling points. Since the resulting frames locally resemble classical Gabor frames and share some of their structure, they are called *nonstationary Gabor frames*. The corresponding transform is likewise referred to as *nonstationary Gabor transform* (NSGT). This concept relies on ideas introduced in [67], and presented in [68].

The central feature of painless expansions is the diagonality of the frame operator associated with the proposed analysis system. This idea is used here to yield painless nonstationary Gabor frames and will allow for both mathematical accuracy in the sense of perfect reconstruction (the frame operator is invertible) and numerical feasibility by means of an FFT-based implementation. The construction of painless nonstationary Gabor frames relies on three intuitively accessible properties of the windows and time-frequency shift parameters used.

(1) The signal f of interest is localized at time- (or frequency-)positions n by means of multiplication with a *compactly supported* (or limited bandwidth, respectively) window function g_n.
(2) The Fourier transform is applied on the localized pieces $f \cdot g_n$. The resulting spectra are sampled densely enough in order to perfectly reconstruct $f \cdot g_n$ from these samples.
(3) Adjacent windows overlap to avoid loss of information. At the same time, unnecessary overlap is undesirable. In other words, we assume that $0 < A \leq \sum_{n\in\mathbb{Z}} |g_n(t)|^2 \leq B < \infty$, a.e., for some positive A and B.

We will show that these requirements lead to invertibility of the frame operator and therefore to perfect reconstruction. Moreover, the frame operator is diagonal and its inversion is straight-forward. Further, the dual frame has the same structure as the original one. Because of these pleasant consequences following from the three above-mentioned requirements, the frames satisfying all of them will be called *painless nonstationary Gabor frames* and we refer to this situation as the *painless case.* Since Gabor transforms, as opposed to wavelet transforms, are in a certain sense symmetric with respect to Fourier transform, our approach leads to adaptivity in either time or frequency.

5.2.1. Resolution changing over time. As opposed to standard Gabor analysis, where time translation is used to generate atoms, the setting of nonstationary Gabor frames allows for changing, hence adaptive, windows in different time positions. Then, for each time position, we build atoms by *regular* frequency modulation. Using a set of functions $\{g_n\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$ and frequency sampling step b_n, for $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$, we define atoms of the form:

$$g_{m,n}(t) = g_n(t)e^{2\pi i m b_n t} = \mathbf{M}_{mb_n} g_n(t),$$

implicitly assuming that the functions g_n are well-localized and centered around time-points a_n. This is similar to the standard Gabor scheme, however, with the possibility to vary the window g_n for each position a_n. Thus, sampling of the time-frequency plane is done on a grid which is irregular over time, but regular over frequency at each temporal position.

Figure 5 shows an example of such a sampling grid. Note that some results exist in Gabor theory for semi-regular sampling grids, as for example in [18]. Our study uses a more general setting, as the sampling grid is in general not separable and, more importantly, the window can evolve over time. To get a first idea of the effect of nonstationary Gabor frames, the reader may take a look at Figure 6 and Figure 7, which show regular Gabor transforms and a nonstationary Gabor transform of the same signal. Note that the NSGT in Figure 7 was adapted to transients and the components are well resolved.

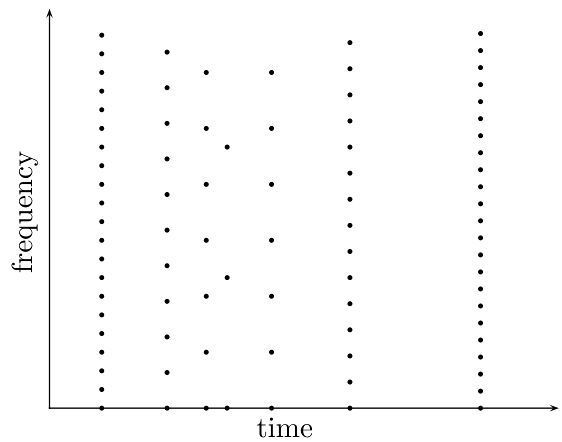

FIGURE 5. Example of a sampling grid of the time-frequency plane when building a decomposition with time-frequency resolution evolving over time

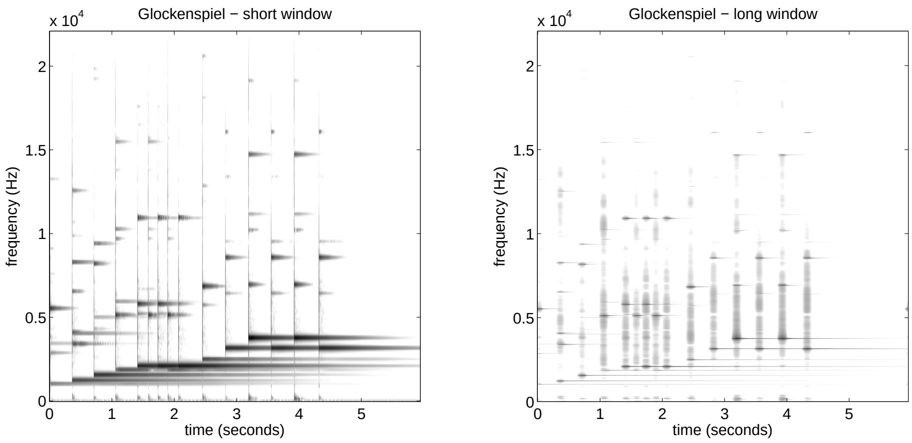

FIGURE 6. Glockenspiel (Example 1). Gabor representations with short window (11.6 ms), resp. long window (185.8 ms).

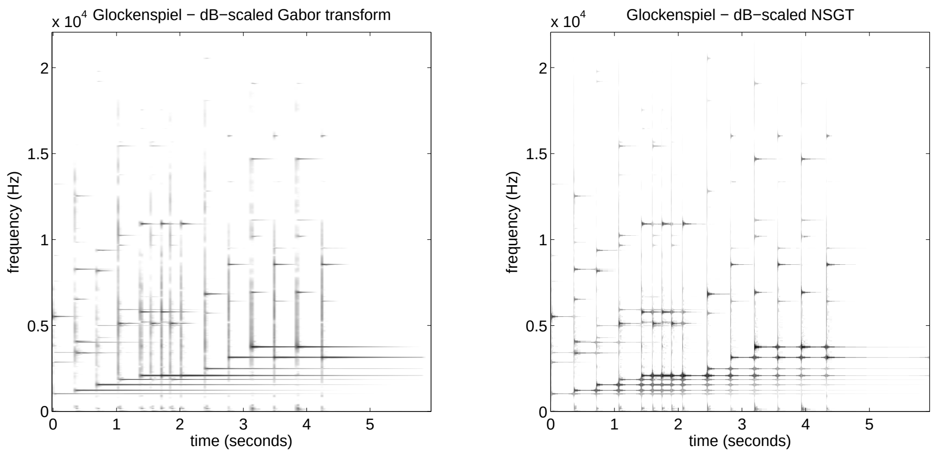

FIGURE 7. Glockenspiel (Example 1). Regular Gabor representation with a Hann window of 58 ms length and a nonstationary Gabor representation using Hann windows of varying length.

In the current situation, the analysis coefficients may be written as

$$c_{m,n} = \langle f, \mathbf{M}_{mb_n} g_n \rangle = (\widehat{f \cdot \overline{g_n}})(mb_n), \; m, n \in \mathbb{Z}.$$

REMARK 5.3. If we set $g_n(t) = g(t - na)$ for a fixed time-constant a and $b_n = b$ for all n, we obtain the case of classical painless nonorthogonal expansions for regular Gabor systems.

5.2.2. Resolution changing over frequency. An analog construction in the frequency domain leads to irregular sampling over frequency, together with windows featuring adaptive bandwidth. Then, sampling is regular over time. An example of the sampling grid in such a case is given in Figure 8.

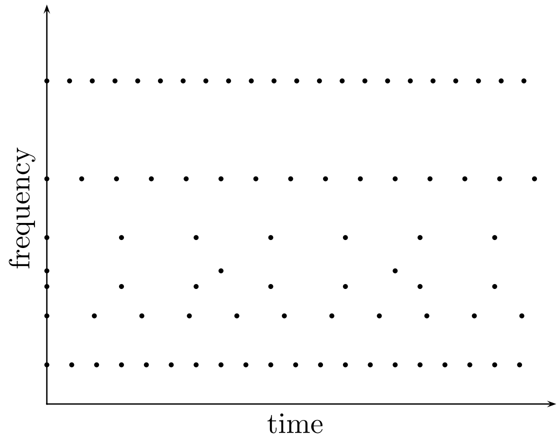

FIGURE 8. Example of a sampling grid of the time-frequency plane when building a decomposition with time-frequency resolution changing over frequency

In this case, we introduce a family of functions $\{h_m\}_{m\in\mathbb{Z}}$ of $L^2(\mathbb{R})$, and for $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$, we define atoms of the form:

$$h_{m,n}(t) = h_m(t - na_m). \tag{5.8}$$

Therefore $\widehat{h_{m,n}}(\nu) = \widehat{h_m}(\nu) \cdot e^{-2\pi i n a_m \nu}$ and the analysis coefficients may be written as

$$c_{m,n} = \langle f, h_{m,n}\rangle = \langle \hat{f}, \mathcal{F}(\mathbf{T}_{na_m} h_m)\rangle = \mathcal{F}^{-1}(\hat{f} \cdot \overline{\widehat{h_m}})(na_m).$$

Hence, the situation is completely analog to the one described in the previous section, up to a Fourier transform.

In practice we will choose each function h_m as a well localized band-pass function with center frequency b_n.

5.2.3. Link between nonstationary Gabor frames, wavelet frames and filterbanks: To obtain wavelet frames, the wavelet transform in (1.35) is sampled at sampling points (β_n, α_m). A typical discretization scheme is $(n\beta_0, \alpha_0^m)$, cf. [79]. Then, the frame elements are $\psi_{m,n}(t) = \mathbf{T}_{n\beta_0}\mathbf{D}_{\alpha_0^m}\psi(t)$. Comparing this expression to (5.8) and setting $h_m = \mathbf{D}_{\alpha_0}^m \psi$ and $a_m = \beta_0$, we see that a wavelet frame with this discretization scheme corresponds to a nonstationary Gabor transform.

Another possibility for sampling the continuous wavelet transform uses $\alpha = \alpha_0^m$ and $\beta = n\beta_0\alpha_0^m$, cf. [27]. Again, we obtain a correspondence to nonstationary Gabor frames by setting $h_m = \mathbf{D}_{\alpha_0}^m \psi$ and $a_m = \beta_0 \cdot \alpha_0^m$.

Beyond the setting of wavelets, any *filter bank* [79], even with non-constant downsampling factors D_m, can be written as a nonstationary Gabor frame. A filter bank is a set of time-invariant, linear filters $\mathfrak{h}_m$, i.e. Fourier multipliers. The response of a filter bank for the signal f and sampling period T_0 is given (in the continuous case) by

$$c_{m,n} = (f * \mathfrak{h}_m)(nD_mT_0) = \int\limits_{\mathbb{R}} f(t)\mathfrak{h}_m(nD_mT_0 - t)\, dt = \langle f, h_{m,n}\rangle ,$$

where $h_{m,n}(t) = \overline{\mathfrak{h}(nD_mT_0 - t)}$. Setting $h_m = \overline{\mathcal{I}\,\mathfrak{h}_m}$ and choosing $a_m = D_mT_0$ this construction is realized with nonstationary Gabor frames using (5.8). If the filters are band-limited and the down-sampling factors are small enough, then the conditions for the painless case are met and the corresponding reconstruction procedure can be applied.

5.2.4. Invertibility of the frame operator and reconstruction. In this subsection we give the precise conditions under which painless nonstationary Gabor frames are constructed. The first two basic conditions, namely compactly supported windows and sufficiently dense frequency sampling points, lead to diagonality of the associated frame operator $\mathbf{S}$. The third condition, the controlled overlap of adjacent windows, then leads to boundedness and invertibility of $\mathbf{S}$. The following theorem generalizes the results given for the classical case of painless nonorthogonal expansions [28, 55].

Theorem 5.4. *For every $n \in \mathbb{Z}$, let the function $g_n \in L^2(\mathbb{R})$ be compactly supported with* $\operatorname{supp}(g_n) \subseteq [c_n, d_n]$ *and let b_n be chosen such that $d_n - c_n \le \frac{1}{b_n}$. Then the frame operator*

$$\mathbf{S} : f \mapsto \sum_{m,n} \langle f, g_{m,n}\rangle g_{m,n}$$

of the system

$$g_{m,n}(t) = g_n(t)\, e^{2\pi i m b_n t},\ m \in \mathbb{Z} \text{ and } n \in \mathbb{Z},$$

is given by a multiplication operator of the form

$$\mathbf{S}f(t) = \left(\sum_n \frac{1}{b_n}|g_n(t)|^2\right) f(t).$$

Proof: Note that,

$$\begin{aligned}\langle \mathbf{S}f, f\rangle &= \sum_n \sum_m \left| \int_{\mathbb{R}} f(t)\,\overline{g_n(t)}\, e^{-2\pi i m b_n t} dt\right|^2 \\ &= \sum_n \sum_m \left| \int_{c_n}^{d_n} f(t)\,\overline{g_n(t)}\, e^{-2\pi i m b_n t} dt\right|^2,\end{aligned}$$

due to the compact support property of the g_n. Let $I_n = [c_n, c_n + b_n^{-1}]$ for all n and χ_I denote the characteristic function of the interval I. Taking into account the compact support of g_n again, it is obvious that

$$f\,\overline{g_n} = \chi_{I_n} \sum_l \mathbf{T}_{lb_n^{-1}}(f\overline{g_n}),$$

with the b_n^{-1}-periodic function $\sum_l \mathbf{T}_{lb_n^{-1}}(f\,\overline{g_n})$. Hence, with $W_{m,n}(t) = e^{-2\pi i m b_n t}$,

$$\left|\int_{c_n}^{d_n} f(t)\,\overline{g_n(t)}\,W_{m,n}(t)\,dt\right|^2 = \left|\int_{I_n} f(t)\,\overline{g_n(t)}\,W_{m,n}(t)\,dt\right|^2, \\ = |\langle f\,\overline{g_n}, W_{m,n}\rangle_{L^2(I_n)}|^2$$

and applying Parseval's identity to the sum over m yields

$$\langle \mathbf{S}f, f\rangle = \sum_n \sum_m |\langle f\,\overline{g_n}, W_{m,n}\rangle_{L^2(I_n)}|^2 \\ = \sum_n \frac{1}{b_n}\|f\,\overline{g_n}\|^2 = \left\langle \sum_n \frac{1}{b_n}|g_n|^2 f, f\right\rangle.$$

■

While in general, the inversion of $\mathbf{S}$ can be numerically cumbersome, in the special case described in Theorem 5.4, the invertibility of the frame operator is easy to check and inversion is a simple multiplication.

Corollary 5.5. *Under the conditions given in Theorem 5.4, the system of functions $g_{m,n}$ forms a frame for $L^2(\mathbb{R})$ if and only if $\sum_n \frac{1}{b_n}|g_n(t)|^2 \asymp 1$. In this case, the canonical dual frame elements are given by:*

$$\tilde{g}_{m,n}(t) = \frac{g_n(t)}{\sum_l \frac{1}{b_l}|g_l(t)|^2} e^{2\pi i m b_n t}, \tag{5.9}$$

and the associated canonical tight frame elements can be calculated as:

$$\mathring{g}_{m,n}(t) = \frac{g_n(t)}{\sqrt{\sum_l \frac{1}{b_l}|g_l(t)|^2}} e^{2\pi i m b_n t}.$$

Remark 5.6. The optimal lower and upper frame bounds are explicitly given by $A_{opt} = \operatorname{essinf} \sum_n \frac{1}{b_n}|g_n(t)|^2$ and $B_{opt} = \operatorname{esssup} \sum_n \frac{1}{b_n}|g_n(t)|^2$.

We next state the results of Theorem 5.4 and Corollary 5.5 in the Fourier domain. This is the basis for adaptation over frequency.

Corollary 5.7. *For every $m \in \mathbb{Z}$, let the function h_m be band-limited to* $\operatorname{supp}(\widehat{h_m}) = [c_m, d_m]$ *and let a_m be chosen such that $d_n - c_n \le \frac{1}{a_m}$. Then the frame operator of the system*

$$h_{m,n}(t) = h_m(t - na_m)\,, m \in \mathbb{Z}, n \in \mathbb{Z}$$

is given by a convolution operator of the form

$$\langle \mathbf{S}f, f\rangle = \left\langle \mathcal{F}^{-1}\left(\sum_m \frac{1}{a_m}|\widehat{h_m}|^2\right) * f, f\right\rangle \tag{5.10}$$

for $f \in L^2(\mathbb{R})$. Hence, the system of functions $h_{m,n}$ forms a frame of $L^2(\mathbb{R})$ if and only if $\forall \nu \in \mathbb{R}$, $\sum_m \frac{1}{a_m}|\widehat{h_m}(\nu)|^2 \asymp 1$. The elements of the canonical dual frame are *given by*

$$\tilde{h}_{m,n}(t) = \mathbf{T}_{na_m}\mathcal{F}^{-1}\left(\frac{\widehat{h_m}}{\sum_l \dfrac{1}{a_l}|\widehat{h_l}|^2}\right)(t) \tag{5.11}$$

and the canonical tight frame is given by

$$\mathring{h}_{m,n}(t) = \mathbf{T}_{na_m}\mathcal{F}^{-1}\left(\frac{\widehat{h_m}}{\sqrt{\sum_l \frac{1}{a_l}|\widehat{h_l}|^2}}\right)(t). \tag{5.12}$$

Proof: We deduce the form of the frame operator in the current setting from the proof of Theorem 5.4 by setting

$$\langle \mathbf{S}f, f\rangle = \langle \widehat{\mathbf{S}f}, \hat{f}\rangle = \sum_{m,n} |\langle \hat{f}, \widehat{h_{m,n}}\rangle|^2$$

and the rest of the corollary is equivalent to Corollary 1. ■

Remark 5.8. Classical Gabor frames are intimately related to modulation spaces, see [55] for an extensive discussion and relevant references. The characterization of modulation spaces depends on the joint time-frequency localization of the analysis window. Painless nonstationary Gabor frames characterize modulation spaces, if, in a addition to compactness in one domain (time or frequency), the windows g_k exhibit a uniform decay in the sense *time-frequency molecules*, see [56, Theorem 22], i.e., letting $\xi = (a_k, l/b_k)$, $k, l \in \mathbb{Z}$, we require $|\mathcal{V}_\varphi g_k(z)| \leq C(1+|z-\xi|)^{-r}$ for some $r > 2$. Then, the corresponding frame operator is invertible on all modulation spaces M^p, $1 \leq p \leq \infty$, and the ℓ^p-norm of the corresponding coefficient sequence is equivalent to the modulation space norm.

5.2.5. Discrete, time-adaptive Gabor transform. For the practical implementation, the equivalent theory may be developed in a finite discrete setting using the Hilbert space $\mathbb{C}^L$. Since this is largely straight-forward from simple matrix multiplication, we only state the main result. Given a set of functions $\{g_n\}_{n\in\{0,\ldots,N-1\}}$, a set of integers (number of frequency samples for each time position) $\{M_n\}_{n\in\{0,\ldots,N-1\}}$ associated with the set of real values $\{b_n = \frac{L}{M_n}\}_{n\in\{0,\ldots,N-1\}}$, the discrete, nonstationary Gabor system is given by

$$g_{m,n}[k] = g_n[k] \cdot e^{\frac{2\pi i m b_n k}{L}} = g_n[k] \cdot W_L^{mb_nk}.$$

for $n = 0, \ldots, N-1$, $m = 0, \ldots, M_n - 1$ and all $k = 0, \ldots L-1$. Note that in practice, $g_{m,n}[k]$ will have zero-values for most k, allowing for efficient FFT-implementation: since $M_n = \frac{L}{b_n}$, we have $g_{m,n}[k] = g_n[k] \cdot e^{\frac{2\pi i m k}{M_n}}$ and the nonstationary Gabor coefficients are given by an FFT of length M_n for each g_n.

The number of elements of $\{g_{m,n}\}$ is $P = \sum_{n=0}^{N-1} M_n$. Let $\mathbf{G}$ be the $L \times P$ matrix such that its p-th column is $g_{m,n}$, for $p = m + \sum_{k=0}^{n-1} M_k$.

Corollary 5.9. *The frame operator* $\mathbf{S} = \mathbf{G} \cdot \mathbf{G}^*$ *is an* $L \times L$ *matrix with entries:*

$$\mathbf{S}_{k,j} = \sum_{n \in \mathcal{N}_{(k-j)}} M_n \, g_n[k] \, \overline{g_n[j]}$$

where $\mathcal{N}_p = \{n \in [0, N-1] \,|\, p = 0 \mod M_n\}$ *for* $p \in [-L, L]$. *Therefore, if appropriate support conditions are met,* $\mathbf{S}$ *is a diagonal matrix.*

5.2.5.1. *Numerical complexity.* Assuming that the windows g_n have support of length L_n, let $M = \max_n \{M_n\}$ be the maximum FFT-length. We consider the painless case where $L_n \leq M_n \leq M$. The number of operations is

(1) Windowing: L_n operations for the n-th window.
(2) FFT: $\mathcal{O}(M_n \cdot \log(M_n))$ for the n-th window.

Then the number of operations for the discrete NSGT is

$$\mathcal{O}\left(\sum_{n=0}^{N-1} M_n \cdot \log(M_n) + L_n\right) = \mathcal{O}(N \cdot (M \log(M) + M))$$
$$= \mathcal{O}(N \cdot (M \log(M)))$$

Similar to the regular Gabor case, the number of windows N will usually depend linearly on the signal length L while the maximum FFT-length M is assumed to be independent of L. In that case, the discrete NSGT is a linear cost algorithm.

For the construction of the dual windows in the painless case, the computation involves multiplication of the window functions by the inverse of the diagonal matrix $\mathbf{S}$ and results in $\mathcal{O}(2 \sum_{n=0}^{N-1} L_n) = \mathcal{O}(N \cdot M)$ operations. Lastly, the inverse NSGT has numerical complexity $\mathcal{O}(N \cdot (M \log(M)))$, as in the NSGT, since it entails computing the IFFT of each coefficient vector, multiplying with the corresponding dual windows and evaluating the sum.

Technical framework: All subsequently presented simulations were done in MATLAB R2009b on a 2 Gigahertz Intel Core 2 Duo machine with 2 Gigabytes of RAM running Kubuntu 9.04. The CQTs were computed using the code published with [85], available for free download at `http://www.elec.qmul.ac.uk/people/anssik/cqt/`. The constant-Q nonstationary Gabor transform (CQ-NSGT) algorithms are available at `http://univie.ac.at/nonstatgab/`.

5.2.5.2. *Application: automatic adaptation to transients.* In real-life applications, NSGT has the potential to represent local signal characteristics, e.g. transient sound events, in a more appropriate way than pre-determined, regular transform schemes. Since the appropriateness of a representation depends on the specific application, any adaptation procedure must be designed specifically. For the implementation itself, however, two observations generally remain true: First, the general nonstationary framework needs to be restricted to a well defined set of choices. Second, some measure is needed to determine the most suitable of the possible choices. For example, in the case of a sparsity measure, the most sparse representation will be chosen. To show that good results are achieved even when using quite simple adaptation methods, we describe a procedure suitable for signals consisting mainly of transient and sinusoidal components. The adaptation measure proposed is based on onset detection, i.e. estimating where transients occur in the signal. The transform setting is what we call *scale frames*: the analysis procedure uses a single window prototype and a countable set of dilations thereof.

For evaluation, the representation quality is measured by comparison of the number of representation coefficients leading to certain root mean square (RMS) reconstruction errors, for both NSGT and regular Gabor transforms. The results are especially convincing for sparse music signals with high energy transient components. Other possible adaptation methods might be based on time-frequency concentration, sparsity or entropy measures [94],[69],[78].

Scale frames: In the following paragraphs, we propose a family of nonstationary Gabor frames that allows for exponential changes in time-frequency resolution along time positions. To avoid heavy notation and since the formalism necessary for the discrete, finite case could obscure the principal idea, we describe the continuous case construction. Suitable standard sampling then yields discrete, finite frames with equivalent characteristics.

The basic idea is to build a sequence of windows g_n from a single, continuous window prototype g with support on an interval of length 1 in such a way that the resulting g_n satisfy Corollary 5.5. The window sequence will be unambiguously determined by a sequence of scales. Once this *scale sequence* is known, it is a simple task to choose modulation parameters b_n satisfying the necessary conditions.

As a scale sequence, we allow any integer-valued sequence $\{s_n\}_{n\in\mathbb{Z}}$ such that $|s_n - s_{n-1}| \in \{0, 1\}$, where the latter restriction is set in order to avoid sudden changes of window length. Then, g_n is, up to translation, given by a dilation of the prototype g:

$$\mathbf{D}_{2^{s_n}}(g)(t) = \sqrt{2^{-s_n}}g(2^{-s_n}t)$$

This implies that a change of scale from one time step to the next corresponds to the use of a window either half or twice as long. More precisely, for every time step n, set $s = \min\{s_{n-1}, s_n\}$ and fix an overlap of $2/3 \cdot 2^s$, if $s_n \neq s_{n-1}$ and $1/3 \cdot 2^s$, if $s_n = s_{n-1}$. Explicitly,

$$g_n = \mathcal{T}_n \mathbf{D}_{2^{s_n}}(g),$$

with recursively defined time shift operators $\mathcal{T}_n$ given by

$$\mathcal{T}_0 = \mathbf{T}_0, \mathcal{T}_n = \begin{cases} \mathbf{T}_{2^s 5/6}\mathcal{T}_{n-1}, & \text{if } s_n \neq s_{n-1} \\ \mathbf{T}_{2^{s+1}/3}\mathcal{T}_{n-1}, & \text{else.} \end{cases}$$

Defining the time shifts in this manner, we achieve exactly the desired overlap as illustrated in Figure 9.

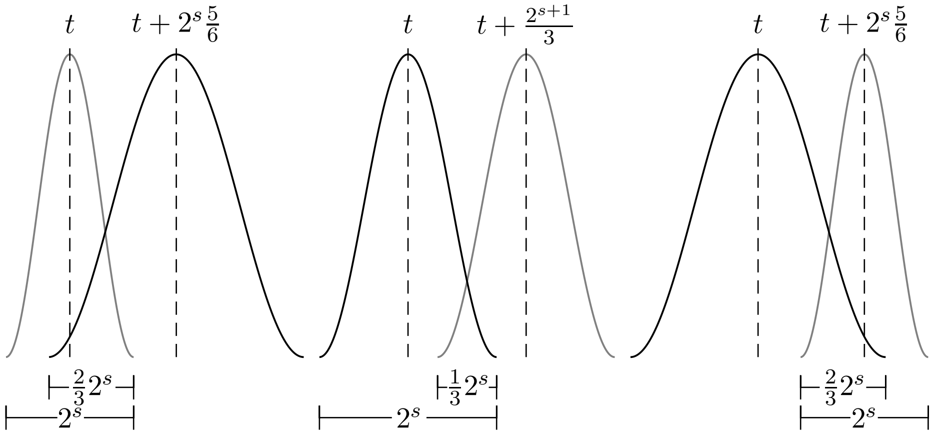

FIGURE 9. Illustration of scale frame overlaps and time shifts.

By construction, each g_n has non-zero overlap with its neighbors g_{n-1} and g_{n+1} and at any point on the real line, at most two windows are non-zero. After performing a preliminary transient detection step, the construction of the adapted frame reduces to the determination of a scale sequence.

In the subsequent figures and experiments we used the Hann window as prototype, but other window choices are possible. The described concept can easily be generalized by admitting other overlap factors and scaling ratio than the ones specified above. The parameters have to be chosen with some care, though. Otherwise the resulting frames might be badly conditioned, with a big or even infinite condition number $\frac{B}{A}$, caused by accumulation points for the time shifts or gaps between windows.

Frame construction from a sequence of onsets: In this paragraph, we assume that the signals of interest are mainly comprised of transient and sinusoidal components, an assumption met, e.g. by piano music. The instant a piano key is hit corresponds to a percussive, transient sound event, directly followed by harmonic components, concentrated in frequency. An intuitive adaptation to signals of this type would use high time resolution at the positions of transients. This corresponds to applying minimal scale at the transients and steadily increasing the scale with the distance from the closest transient. The transients' positions can be determined, e.g. by so-called onset detection procedures [31] which, if used carefully, work to a high degree of accuracy. Once the transient positions are known, the construction of a corresponding scale frame yields good nonstationary representations for sufficiently sparse signals.

Application of onset-based scale frames: We applied the procedure proposed above to various signals, mainly piano music. For this presentation, we selected three examples, all of them sampled at 44.1 kHz and consisting of a single channel. Some more examples and corresponding results as well as the source sound files can be found on the associated web-page `http://univie.ac.at/nonstatgab/`.

- Example 1: The widely used Glockenspiel signal shown in Figure 7.
- Example 2: An excerpt from a solo jazz piano piece performed by Herbie Hancock, characterized by its calmness and varied rhythmical pattern, resulting in irregularly spaced low-energy transients. See Figure 10.
- Example 3: A short excerpt of György Ligeti's piano concert. With highly percussive onsets in the piano and Glockenspiel voices and some orchestral background, this is the most polyphonic of our examples. See Figure 11.

For comparison, the plots in Figures 7, 10 and 11 also show standard Gabor coefficients with comparable (average) window overlap. A Hann window of 2560 samples length was chosen for the computation of regular Gabor transforms. The comparison shows that for the three signals, the NSGT features a better concentration of transient energy than a regular Gabor transform, while keeping, or even improving, frequency resolution.

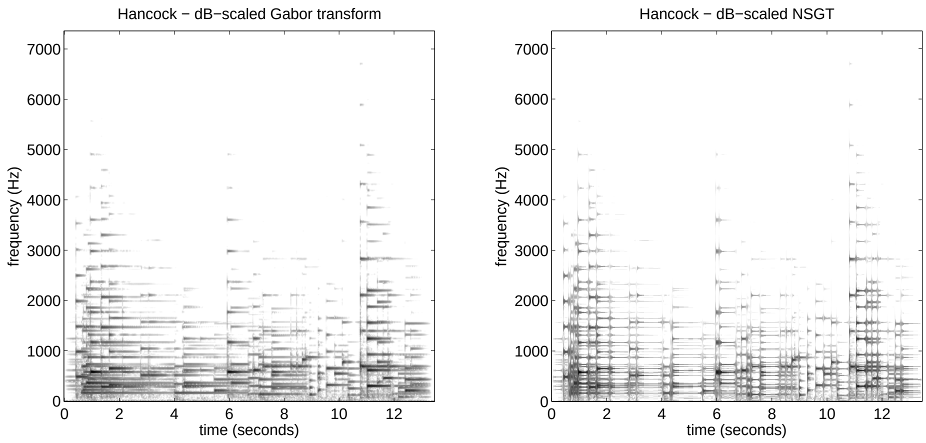

FIGURE 10. Hancock (Example 2). Regular and nonstationary Gabor representations.

Efficiency in sparse reconstruction: The onset detection procedure and a subsequent scale frame analysis were applied, along with a regular Gabor decomposition, to the Glockenspiel and Ligeti signals. As a test of the representations' sparsity, the signals were synthesized from their corresponding coefficients, modified by hard thresholding followed by reconstruction using the dual frame. Then the numbers of largest magnitude coefficients needed for a certain relative root mean square (RMS) reconstruction error for each representation were compared. The RMS error of a vector f and its reconstruction f_{rec} is given by

$$RMS(f, f_{rec}) = \sqrt{\frac{\sum_{k=0}^{L-1} |f[k] - f_{rec}[k]|^2}{\sum_{k=0}^{L-1} |f[k]|^2}}.$$

All transforms are of redundancy about $\frac{5}{3}$. The results for NSGT and different regular Gabor transform schemes are listed in Figure 12. On the Glockenspiel signal the

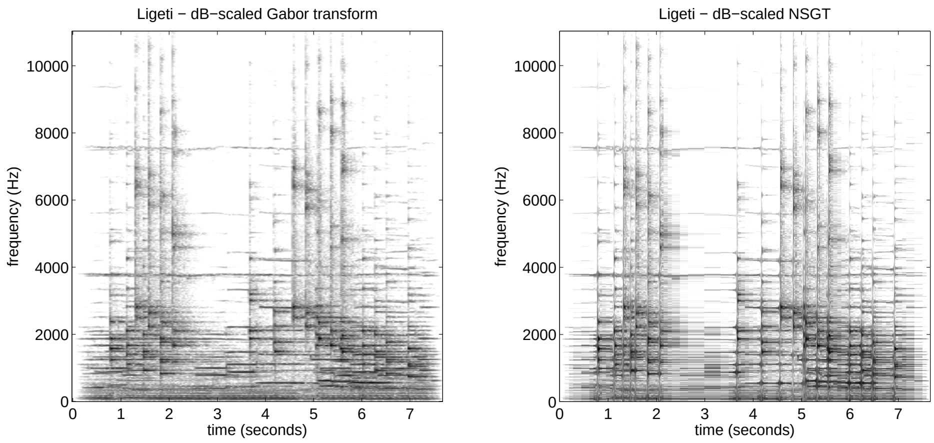

FIGURE 11. Ligeti (Example 3). Regular and nonstationary Gabor representations.

NSGT method performs vastly better than the ordinary Gabor transform. For Ligeti, the differences are not as significant, but still the NSGT-based procedure shows better overall results.

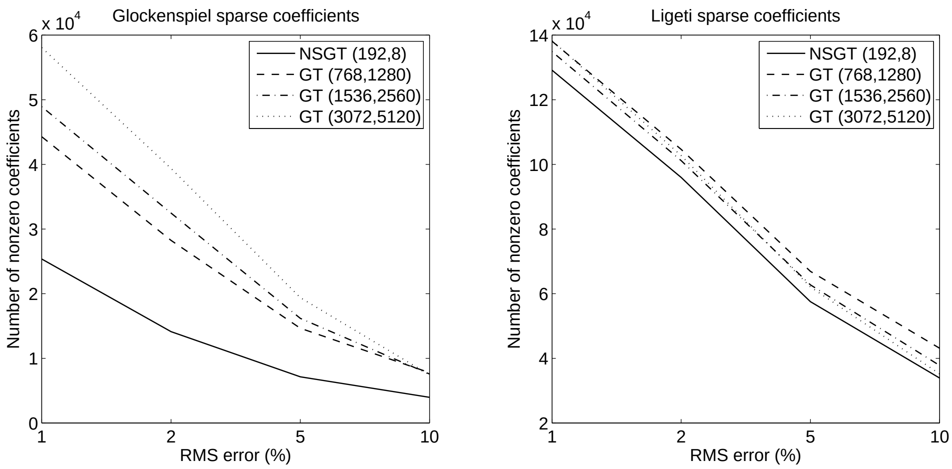

FIGURE 12. RMS error in sparse representations of Example 1 and Example 3. Parameters (in parentheses) are hop size and window length in the regular case (GT) or shortest window length and number of scales for the nonstationary case (NSGT). The values are estimated to be the optimal numbers of coefficients necessary to achieve reconstruction with less than the respective error.

5.2.6. Implementation of a discrete, frequency-adaptive Gabor Transform. Since our construction of Gabor frames with adaptivity in the frequency domain relies on the fact that analysis windows h_m possess compact bandwidth, an FFT-based implementation is highly efficient. We take the input signal's Fourier transform and treat the procedure in complete analogy to the situation developed for time-adaptive transforms, i.e. $h_{m,n}[k] = \mathbf{T}_{na_m} h_m[k]$ and $\widehat{h_{m,n}}[j] = \mathbf{M}_{-na_m}\widehat{h_m}[j]$.

As observed in Section 5.2.3, we are able to obtain wavelet frames using Gabor frames that exhibit nonstationarity in the frequency domain. Moreover, we may design general transforms with flexible frequency resolution, such as a constant-Q transform. While various other adjustments (e.g. Mel- or Bark-scaled transforms) are feasible, we will focus our discussion on the constant-Q case.

REMARK 5.10. Note that for real-valued signals the symmetry of their FFT can be exploited to further reduce the computational effort. We particularly refer to the LTFAT routines `filterbankrealdual.m` and `filterbankrealtight.m`.

5.2.6.1. *A constant-Q transform via nonstationary Gabor frames.* The constant-Q transform (CQT), introduced by Brown [16], transforms a time signal into the time-frequency domain, where the center frequencies of the frequency bins are geometrically spaced. Since the *Q-factor* (the ratio of the center frequencies to the window's bandwidth) is constant, the representation allows for a better frequency resolution at lower frequencies and a better time resolution at the higher frequencies. This is sometimes preferable to the fixed resolution of the standard Gabor transform, for which the frequency bins are linearly spaced. In particular, this kind of resolution is often desired in the analysis of musical signals, since the transform can be set to coincide the temperament, e.g. semitone or quartertone, used in Western music.

The originally introduced constant-Q transform, however, is not invertible and is computationally more intensive than the DFT. A more computationally efficient approach was presented in the sequel [17]. In the paper, for the nth time slice of the signal f, the coefficient vector $c_{m,n}$, equal to inner product of the signal f with the *time*-limited window $h_{m,n}$ is computed in the Fourier side via $\langle \hat{f}, \widehat{h_{m,n}} \rangle$, taking advantage of the sparsity of the frequency domain kernel or *spectral kernel.* Note that in contrast, we compute the coefficient vector for each frequency bin, making use of *band*-limited window functions.

Perfect reconstruction wavelet transforms with rational dilation factors were proposed in [12]. Since they are based on iterated filter banks, these methods are computationally too expensive for long, real-life signals, when high Q-factors, such as 12-96 bins per octave, are required.

In [85], Klapuri and Schörkhuber presented a computation of the CQT that shows improved efficiency and flexibility to the method proposed in [17], among others. However, the approximate inversion introduced in [85] still gives an RMS error of around 10^{-3}. The lack of perfect invertibility prevents the convenient modification of CQT-coefficients with subsequent resynthesis required in complex music processing tasks such as masking or transposition. By allowing adaptive resolution in frequency, we can construct an invertible nonstationary Gabor transform with a constant Q-factor on the relevant frequency bins.

Setting: For the frame elements in the transform, we consider functions $h_m \in \mathbb{C}^L$, $m = 1, \ldots, M$ having center frequencies (in Hz) at $\xi_m = \xi_{\min} 2^{\frac{m-1}{B}}$, as in the CQT. Here, B is the number of frequency bins per octave, and $\xi_{\min}$ and $\xi_{\max}$ are the desired minimum and maximum frequencies, respectively. In the experiments, we restrict $\xi_{\max}$ to be less than the Nyquist frequency and there should exist an $M \in \mathbb{N}$ satisfying $\xi_{\max} \leq \xi_{\min} 2^{\frac{M-1}{B}} < \xi_s/2$, where ξ_s denotes the sampling frequency. In this case, we take $M = \lceil B \log_2(\xi_{\max}/\xi_{\min}) + 1 \rceil$, where $\lceil z \rceil$ is the smallest integer greater than or equal to z. While in the CQT no 0-frequency is present, the NSGT provides all necessary freedom to use additional center frequencies. Since the signals of interest are real-valued, we put filters at center frequencies beyond the Nyquist frequency in a symmetric manner. This results in the following values for the center frequencies:

$$\xi_m = \begin{cases} 0, & m = 0 \\ \xi_{\min} 2^{\frac{m-1}{B}}, & m = 1, \ldots, M \\ \xi_s/2, & m = M+1 \\ \xi_s - \xi_{2M+2-m}, & m = M+2, \ldots, 2M+1. \end{cases}$$

For the corresponding bandwidth Ω_m of h_m, we set $\Omega_m = \xi_{m+1} - \xi_{m-1}$, for $m = 1, \ldots, M$, and $\Omega_0 = 2\xi_1 = 2\xi_{\min}$. By construction, these result in a constant Q-factor $Q = (2^{\frac{1}{B}} - 2^{-\frac{1}{B}})^{-1}$ for $m = 2, \ldots, M-1$. And we can write each Ω_m as follows:

$$\Omega_m = \begin{cases} 2\xi_{\min}, & m = 0 \\ \xi_2, & m = 1, 2M+1 \\ \xi_m/Q, & m = 2, \ldots, M-1 \\ (\xi_s - 2\xi_{M-1})/2, & m = M, M+2 \\ \xi_s - 2\xi_M, & m = M+1 \\ \xi_{2M+2-m}/Q, & m = M+3, \ldots, 2M. \end{cases}$$

If we use a Hann window $\hat{h}$, supported on $[-1/2, 1/2]$, then we can obtain each h_m via $\widehat{h_m}[j] = \hat{h}((j\frac{\xi_s}{L} - \xi_m)/\Omega_m)$, where $j = 0, \ldots, L-1$. Letting $a_m \leq \frac{\xi_s}{\Omega_m}$, we define $h_{m,n}$ by their Fourier transform $\widehat{h_{m,n}} = \mathbf{M}_{-na_m}\widehat{h_m}$, $n = 0, \ldots, \lfloor \frac{L}{a_m} \rfloor - 1$. Figure 13 illustrates the time-frequency sampling grid of the set-up, where the center frequencies are geometrically spaced and sampling points regularly spaced.

The support conditions on $\widehat{h_m}$ imply that the sum $\sigma = \sum_{m=0}^{2M+1} \frac{L}{a_m} |\widehat{h_m}|^2$ is finite and bounded away from 0. The frame operator is therefore invertible and we can apply inversion from painless nonorthogonal expansions.

Note that we consider the bandwidth to be the support of the window in frequency. This makes sense in the considered painless case. Very often, see e.g. [85], the bandwidth is taken as the width between the points, where the filter response drops to half of the maximum, i.e. the -3dB-bandwidth. This definition would also make

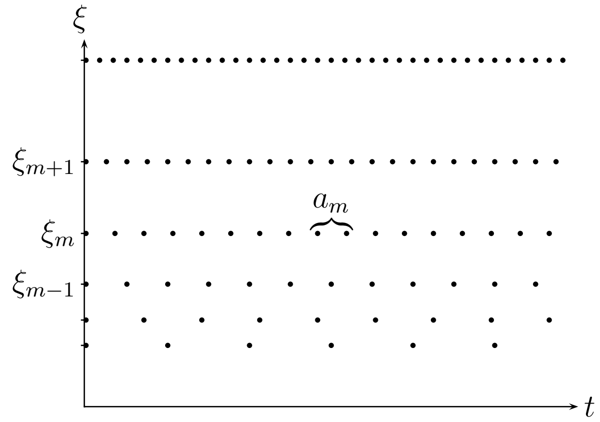

FIGURE 13. Exemplary sampling grid of the time-frequency plane for a constant-Q nonstationary Gabor system.

sense in a non-compactly supported case. For the chosen filters, Hann windows, the Q-factor considering the −3dB-bandwidth is just double of the one considered above.

We see in Figure 14 the standard Gabor transform spectrogram and the constant-Q NSGT spectrogram of the Glockenspiel signal, the latter being very similar to the CQT spectrogram obtained from the original algorithm [16] but with the additional property that the signal can be perfectly reconstructed from the coefficients. Figures 15 and 16 compare the standard Gabor transform spectrogram and the constant-Q NSGT spectrogram of two additional test signals, both sampled at 44.1 kHz:

- Example 4: A recording of Bach's Little Fugue in G Minor, BWV578 performed by Christopher Herrick on a pipe organ. Low frequency noise and the characteristic structure of pipe organ notes are resolved very well by a CQT. See Figure 15.
- Example 5: An excerpt from a duet between violin and piano. Written by John Zorn and performed by Sylvie Courvoisier and Mark Feldman, the sample is made up of three short segments: A frantic sequence of violin and piano notes, a slow violin melody with piano backing and an inharmonic part with chirp component. See Figure 16.

Efficiency: The computation time of the nonstationary Gabor transform was found to be better than a recent fast CQT implementation [85], as seen in Figure 17. The two plots show mean values for computation time in seconds and the corresponding variance over 50 iterations, with varying window lengths and number of frequency bins, respectively. The outlier, drawn in gray, in Figure 17 (left) at the prime number 600569 illustrates dependence of the current CQ-NSGT implementation on the signal length's prime factor structure, analogous to FFT.

It is again reasonable to assume that the number of filters is bounded, independently of L, while the number of temporal points depend on L. As the role of M and N is switched in the assumption in subsection 5.2.5.1 for the complexity, we arrive at a

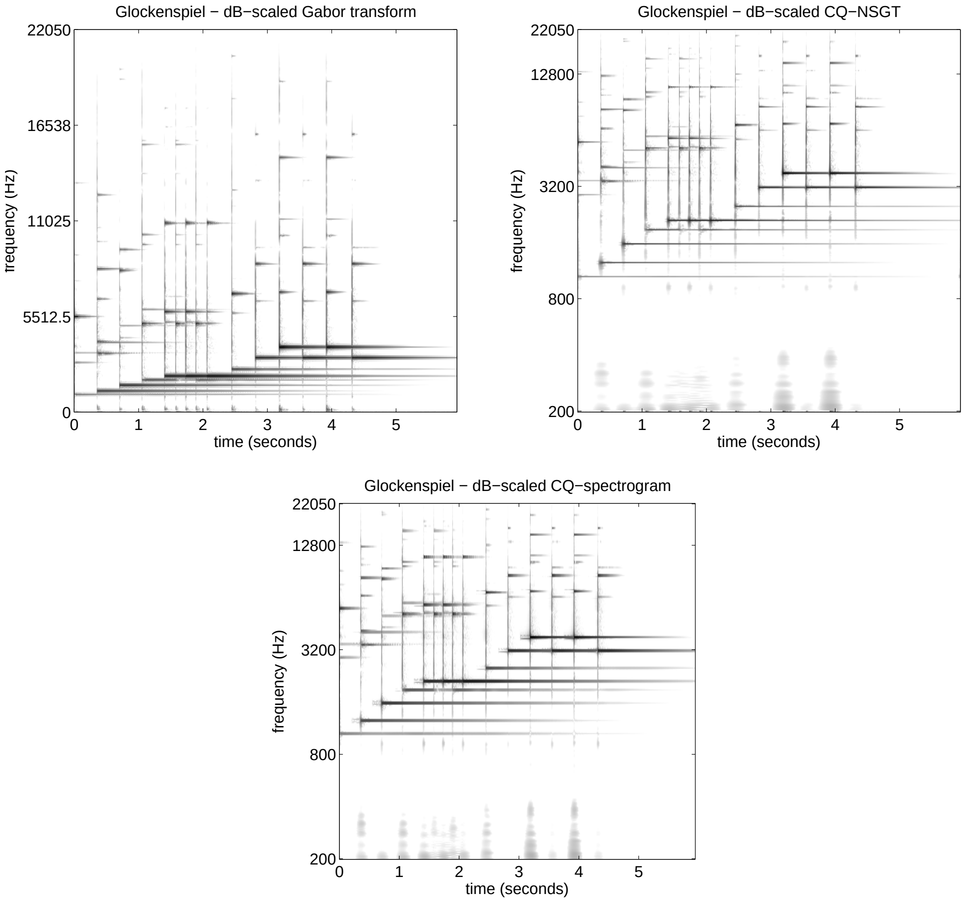

FIGURE 14. Glockenspiel (Example 1). Regular Gabor, constant-Q nonstationary Gabor and constant-Q representations of the signal. The transform parameters were $B = 48$ and $\xi_{\min} = 200$ Hz.

complexity of $\mathcal{O}(L \log L)$. This is also the complexity of the FFT of the whole signal. So the overall complexity of the frequency-dependent nonstationary Gabor transform is $\mathcal{O}(L \log L)$. The advantage of the method in terms of computation efficiency thus decreases as longer signals are considered.

Experiments on Applications: Our experiments show applications of the CQ-NSGT in musical contexts, where the property of a logarithmic frequency scale renders the method often superior to the traditional STFT. Corresponding sound examples can be found at `http://univie.ac.at/nonstatgab/cqt/`.

Transposition: A useful property of continuous constant-Q decompositions is the fact that the transposition of a harmonic structure, like a note including overtones, corresponds to a simple translation of the logarithmically scaled spectrum. Approximately, this is also the case for the finite, discrete CQ-NSGT. In this experiment, we transposed a piano chord simply by shifting the inner frequency bins accordingly. By inner frequency bins, we refer to all bins with constant Q-factor. This excludes the

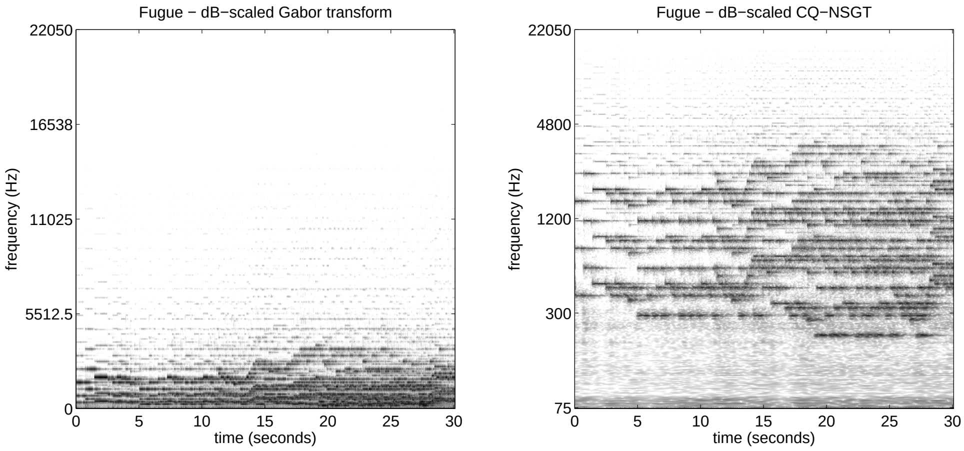

FIGURE 15. Bach's Little Fugue (Example 4). Regular and constant-Q nonstationary Gabor representations of the signal. The transform parameters were $B = 48$ and $\xi_{\min} = 75$ Hz.

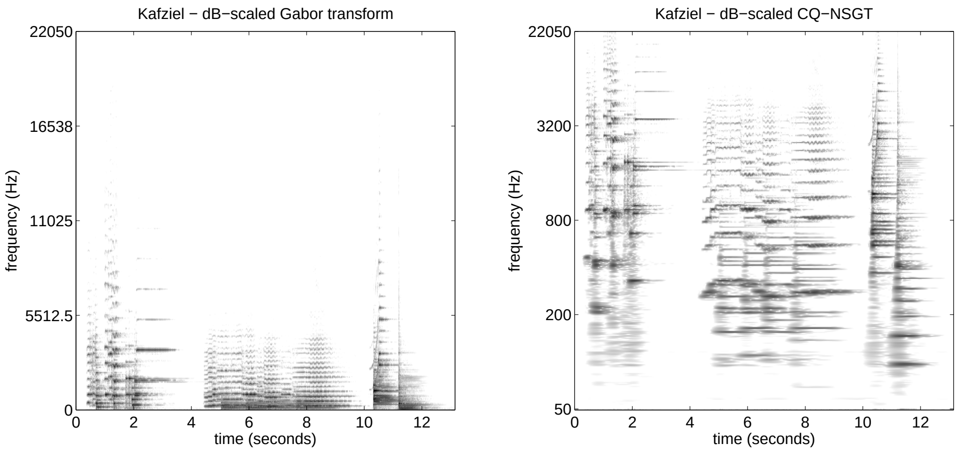

FIGURE 16. Violin and piano duet (Example 5). Regular and constant-Q nonstationary Gabor representations of the signal. The transform parameters were $B = 48$ and $\xi_{\min} = 50$ Hz.

0-frequency and Nyquist frequency bins. The onset portion of the signal has been damped, since inharmonic components, such as transients, produce audible artifacts when handled in this way. In Figure 18, we show spectrograms of the original and modified chords, shifted by 20 bins. This corresponds to an upwards transposition by 5 semitones.

Masking: In the masking experiment, we show that the perfect reconstruction property of CQ-NSGT can be used to cut out components from a signal by directly

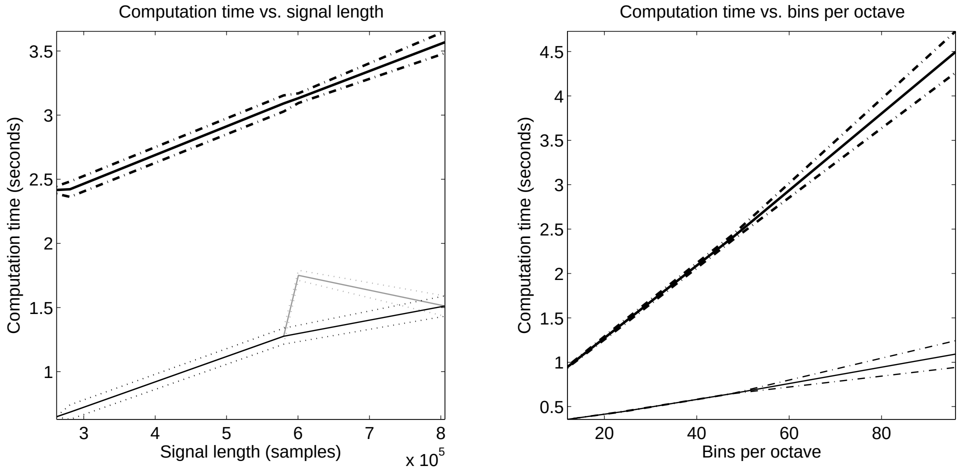

FIGURE 17. Comparison of computation time of CQT (top curves) and NSGT (bottom curves). The figure on the left shows the computation times for signals of various lengths with the number of bins per octave fixed at $B = 48$, while the figure on the right shows the computation times for the Glockenspiel signal, varying the number of bins per octave. In both figures, the solid lines represent the mean time (in seconds) and the dashed/dotted lines signify the mean time with corresponding variance. The minimum frequency for all cases $\xi_{\min}$ was chosen at 50 Hz.

modifying the time-frequency coefficients. The advantage of considerably higher spectral resolution at low frequencies (with a chosen application-specific temporal resolution at higher frequencies) compared to the STFT, makes the CQ-NSGT a very powerful, novel tool for masking or isolating time-frequency components of musical signals. Our example shows in Figure 19 a mask for extracting – or inversely, suppressing – a note from the Glockenspiel signal depicted in Figure 7 The mask was created as a gray-scale bitmap using an ordinary image manipulation program and then resampled in order to conform to the irregular time-frequency grid of the CQ-NSGT. Figure 19 shows the mask spectrogram, along with the spectrograms of the synthesized, processed signal and remainder.

5.2.7. Further work involving nonstationary Gabor frames. There has been a considerable amount of work on nonstationary Gabor frames since its introduction in [67, 68] and its initial development in [89, 7] that range from additional applications to generalizations and structural properties.

General existence and perturbation results of such frames were proved by Dörfler and Matusiak in [38]. Moreover, they constructed nonstationary Gabor frames having non-compactly supported windows or "almost painless" nonstationary Gabor frames. Due to the more complicated structure of such frames compared to the regular Gabor frame case, computing for the canonical dual frames, which would entail the inversion

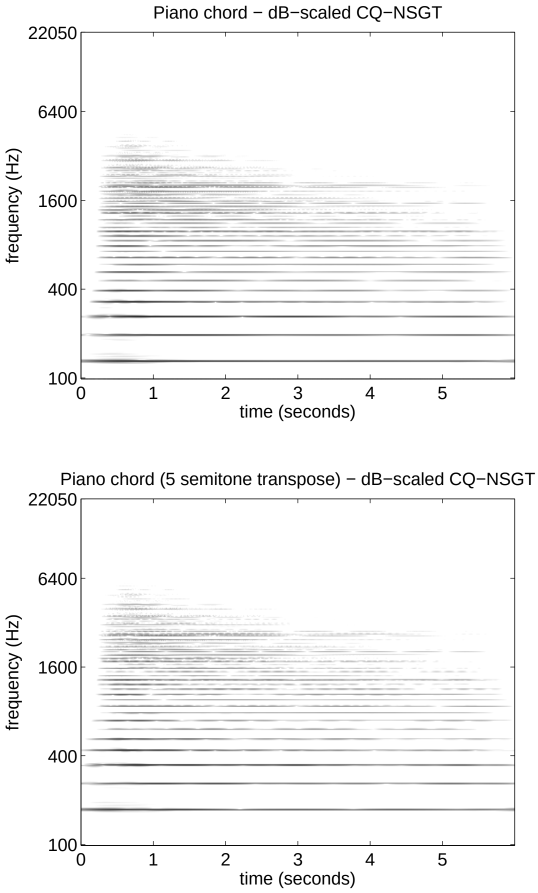

FIGURE 18. Piano chord signal and upwards transposition by 5 semitones, corresponding to a circular shift of the inner bins by 20. The transform parameters were $B = 48$ and $\xi_{\min} = 100$ Hz.

of the frame operator, is computationally more cumbersome. In their succeeding paper [37], Dörfler and Matusiak proposed the use of approximate dual frames obtaining good approximate reconstruction.

In [64], Holighaus studied further the structure of nonstationary Gabor systems and their dual systems. Following the Walnut representation (cf. [55, Theorem 6.3.2] for the regular Gabor case) for the frame operator of nonstationary Gabor systems in [38], he proved a Walnut-like representation for some inverse nonstationary Gabor frame operators, that leads to a dual nonstationary Gabor frame having the same support conditions. He also obtained characterizations for a pair of nonstationary Gabor frames forming dual frames.

The joint work with Holighaus, et. al. [65] extends the results in [89] concerning the constant-Q nonstationary Gabor transform. By introducing a preprocessing step of slicing the signal into pieces of finite length, thus the name "sliced constant-Q transform" or sliCQ, the resulting algorithm allows for real-time processing.

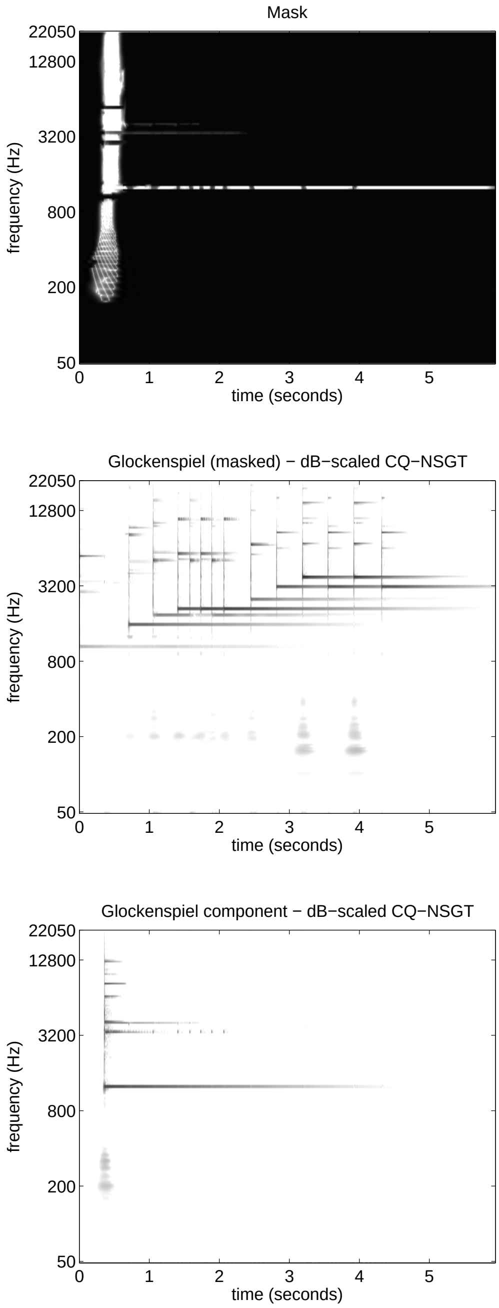

FIGURE 19. Note extraction from the Glockenspiel signal by masking. The CQ-NSGT coefficients of the Glockenspiel signal were weighted with the mask shown on top. The remaining signal and extracted component are depicted in the middle and bottom respectively. The transform parameters were $B = 24$ and $\xi_{\min} = 50$ Hz.

Additional work in applications include that of the joint work with Holzapfel, et. al. [66] wherein a nonstationary Gabor transform was used in beat tracking in music signals that produced statistically significant improvements on a large dataset.

Necciari, et. al. [81] applied the frequency side nonstationary Gabor transform with windows equidistantly spaced on the psychoacoustic "ERB" frequency scale, obtaining perfect reconstruction using fast iterative methods.

Following the results on nonstationary Gabor frames, Wiesmeyr [93] applied a warping of the frequency axis to obtain a transform given a desired frequency scaling. The resulting continuous warped transform is a generalization of the continuous STFT and wavelet transform.

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Curriculum Vitæ

Name:	Gino Angelo Velasco
Nationality:	Filipino
Email:	gino.velasco@univie.ac.at, gamvelasco@math.upd.edu.ph

Education

Universität Wien:
Doktor der Naturwissenschaften (Dr. rer. nat.) Mathematik (*in progress*)

University of the Philippines Diliman:
Master of Science (M.S.) in Mathematics, 2006
Bachelor of Science (B.S.) in Mathematics, 2001

Work experience

2001-present	University of the Philippines Diliman, Instructor
2010 & 2011	Audio-Miner. Mathematical Signal Analysis and Modeling for Manipulation of Sound Objects, Vienna Science and Technology Fund (WWTF), Project Member

Publications

1. Oscillation of Fourier transforms and Markov-Bernstein inequalities, with Noli N. Reyes and Szilard Gy. Révész, *Journal of Approximation Theory*, Vol. 145, No. 1, pp.100-110, 2007.
2. Sparse regression in time-frequency representations of complex audio, with Monika Dörfler, Arthur Flexer, and Volkmar Klien, Proceedings of the 7th Sound and Music Computing Conference (SMC'10), Barcelona, Spain, 2010.
3. Constructing an invertible constant-Q transform with nonstationary Gabor frames, with Monika Dörfler, Thomas Grill and Nicki Holighaus, Proceedings of the 14th International Conference on Digital Audio Effects (DAFx-11), Paris, France, pp.93-99, 2011. (Best Student Paper Award - Gold)

4. Theory, implementation and applications of nonstationary Gabor frames, with Peter Balazs, Monika Dörfler, Nicki Holighaus and Florent Jaillet, *Journal of Computational and Applied Mathematics*, Vol. 236, No. 6, pp. 1481-1496, 2011.
5. Advantages of nonstationary Gabor transforms in beat tracking, with Andre Holzapfel, Nicki Holighaus, Arthur Flexer and Monika Dörfler, Proceedings of the 1st International ACM Workshop on Music Information Retrieval with User-Centered and Multimodal Strategies (ACM MIRUM), Scottsdale, Arizona, USA, 2011.
6. Approximate reconstruction of bandlimited functions for the integrate and fire sampler, with Hans G. Feichtinger, José C. Príncipe, Alexander Singh Alvarado, and José Luis Romero, *Advances in Computational Mathematics*, Vol. 36, No. 1, pp. 67-78, 2012.
7. A framework for invertible, real-time constant-Q transforms, with Monika Dörfler, Thomas Grill and Nicki Holighaus, *IEEE Transactions on Audio, Speech and Language Processing*, Vol. 21, No. 4, pp. 775-785, 2013.
8. Adaptive Gabor frames by projection onto time-frequency subspaces, with Monika Dörfler, Proceedings of the 2014 IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP 2014), Florence, Italy, 4-9 May 2014.

Conference/ Workshop Presentations

1. Oscillation of Fourier transforms (joint work with Noli N. Reyes and Szilard Gy. Révész), oral presentation at the Annual Convention of the Mathematical Society of the Philippines, Davao City, Philippines, 27-28 May 2006.
2. Sparse regression in time-frequency representations of complex audio (joint work with Monika Dörfler, Arthur Flexer, and Volkmar Klien), oral presentation at the 7th Sound and Music Computing Conference (SMC'10), Barcelona, Spain, 21-24 July 2010.
3. Multistep nonstationary Gabor frames for adaptive signal analysis (joint work with Nicki Holighaus), poster presentation at the Strobl11 Conference - From Abstract to Computational Harmonic Analysis, Strobl, Austria, 13-19 June 2011.
4. Constructing an invertible constant-Q transform with nonstationary Gabor frames (joint work with Monika Dörfler, Thomas Grill and Nicki Holighaus), oral presentation at the 14th International Conference on Digital Audio Effects (DAFx-11), Paris, France, 19-23 September 2011.
5. The constant-Q transform via Gabor frames (joint work with Monika Dörfler, Thomas Grill and Nicki Holighaus), oral presentation at ESI12 Modern Methods of Time-Frequency Analysis II Workshop 6: Time-frequency methods for the applied sciences, Erwin Schrödinger Institute, Vienna, Austria, 03-08 December 2012.
6. Some approximation estimates for time-frequency localization functions (joint work with Monika Dörfler and Hans G. Feichtinger), poster presentation at the Asian Mathematical Conference 2013 (AMC 2013), Busan, South Korea, 30 June - 4 July 2013.

7. A continuous time-frequency representation via warping (joint work with Christoph Wiesmeyr and Nicki Holighaus), poster presentation at the Asian Mathematical Conference 2013 (AMC 2013), Busan, South Korea, 30 June - 4 July 2013.
8. Adaptive Gabor frames by projection onto time-frequency subspaces (joint work with Monika Dörfler), oral presentation at the 2014 IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP 2014), Florence, Italy, 4-9 May 2014.

Scholarships

- ÖAD (Austrian Exchange Service) scholarship via ASEA-Uninet, September 2006
- ÖAD (Austrian Exchange Service) Technology Doctorate Scholarship for Southeast Asia, November 2007-December 2009, July 2010-May 2011