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The mass of anti-de Sitter light-cones

Contents

1 Introduction

The Einstein equations, which provide the basis of general relativity, do not fall into a class of partial differential equations (PDEs) for which well-posedness results are known. However, they are tensorial equations which implies coordinate-independence and therefore leads to a great amount of gauge freedom. Because of this freedom one cannot expect solutions to be unique but rather uniqueness of them up to gauge-invariance.

Now, due to the fundamental work of Choquet-Bruhat in the 1950's [5] we know that imposing a *harmonic gauge condition* on the coordinates removes the disturbing degrees of freedom and splits the Einstein equations into a system of hyperbolic wave equations, so-called evolution equations, and a system of constraint equations, which are propagated by the evolution equations. For these evolution equations, which are hyperbolic equations, in turn well-posedness results are available.

As we are interested in characteristic hypersurfaces in this thesis we are going to restrict ourselves here to those aspects and note that for the characteristic Cauchy problem the constraint equations can be read as a system of ordinary differential equations (ODEs), which are quite easy to handle. For a more detailed discussion see e.g. [8] and references therein.

2 Preliminaries

2.1 Wave-map gauge

In order to split the Einstein equations into constraint and evolution equations as mentioned above we impose the *generalized wave-map gauge* [2, 6], which is characterized by the vanishing of the *generalized wave-gauge vector*

$$H^\lambda = 0\,, \tag{2.1}$$

which is defined as

$$H^\lambda := \Gamma^\lambda - V^\lambda\,, \quad \text{where} \quad V^\lambda := \hat{\Gamma}^\lambda + W^\lambda\,, \quad \Gamma^\lambda := g^{\alpha\beta}\Gamma^\lambda_{\alpha\beta}\,, \quad \hat{\Gamma}^\lambda := g^{\alpha\beta}\hat{\Gamma}^\lambda_{\alpha\beta}\,. \tag{2.2}$$

This gauge is a generalization of the classical harmonic gauge by Choquet-Bruhat. It was modified by Friedrich [6] who added the *gauge source functions* $W^\lambda = W^\lambda(x^\mu, g_{\mu\nu})$ which bring crucial flexibility in the characteristic case. These vector fields can be freely specified and are allowed to depend upon the coordinates chosen and the metric itself, but not upon derivatives thereof. In (2.2) and in what follows we decorate objects associated with some *target metric* $\hat{g}$ with the hat symbol "^".

2.2 Characteristic surfaces, adapted null coordinates and assumptions on the metric

As mentioned previously in this thesis we are interested in null hypersurfaces which are also called characteristic surfaces and problems restricted to them. It is natural to think of a (future) light-cone with a single point in space-time as vertex but we do not restrict ourselves to this case and will therefore generally talk about

characteristic surfaces which we denote by $\mathscr{N}$.
In order to treat our problem effectively we introduce coordinates adapted to the characteristic surface called *adapted null coordinates* $(x^0 = u,\, x^1 = r,\, x^A)$, $A \in \{2,3\}$ [1]. The coordinate $r > r_0$, where we accommodate a vertex at $r = r_0$ possibly different from zero, parametrizes the null geodesics issuing from r_0 and generating the null hypersurface which is characterized by $\{u = 0\}$. The x^A's on the other hand are local coordinates on the level sets $\{u = 0, r = \text{const.}\}$. The trace of the metric on the characteristic surface can then be written as (we will interchangeably use x^0 and u)

$$\overline{g} = \overline{g}_{\mu\nu} dx^\mu dx^\nu = \overline{g}_{00}(du)^2 + 2\nu_0 du dr + 2\nu_A du dx^A + \check{g}\,, \tag{2.3}$$

where we use the definitions

$$\nu_0 := \overline{g}_{0r}\,, \quad \nu_A := \overline{g}_{0A}\,, \quad \check{g} := \overline{g}_{AB} dx^A dx^B\,. \tag{2.4}$$

Here and throughout this thesis we use an overline to denote the restriction of a space-time object to $\mathscr{N}$.

In the spirit of the analysis in [4, leading towards Equation (26) there] we assume that we have given a metric $\check{g}$ and an affine parameter, which will be denoted by r, such that, for r large,

$$\overline{g}_{AB} = \mathring{h}_{AB} r^2 + (\overline{g}_{AB})_{-1}\, r + (\overline{g}_{AB})_0 + \mathcal{O}(r^{-1})\,, \tag{2.5}$$

where we use the symbol $\mathring{h}$ to denote the standard metric on the respective boundary manifold, e.g. a two-sphere or a torus, and $(\overline{g}_{AB})_{-i} = (\overline{g}_{AB})_{-i}\,(x^C)$, $i \in \mathbb{N}$, are some smooth tensors on that manifold. Further the $\mathcal{O}(r^{-1})$ terms remain $\mathcal{O}(r^{-1})$ under x^C-differentiation, and become $\mathcal{O}(r^{-2})$ under r-differentiation.

Note that the restriction of the inverse metric to $\mathscr{N}$ takes the form

$$\overline{g}^{\#} = 2\nu^0 \partial_u \partial_r + \overline{g}^{rr} \partial_r \partial_r + 2\overline{g}^{rA} \partial_r \partial_A + \overline{g}^{AB} \partial_A \partial_B\,, \tag{2.6}$$

where $\overline{g}^{AB}$ is the inverse of $\overline{g}_{AB}$ and

$$\nu^0 := \overline{g}^{0r} = \frac{1}{\nu_0}\,, \quad \overline{g}^{rA} = -\nu^0 \nu^A = -\nu^0 \overline{g}^{AB} \nu_B\,, \quad \overline{g}^{rr} = \left(\nu^0\right)^2 \left(\nu^A \nu_A - \overline{g}_{00}\right)\,. \tag{2.7}$$

Next, we note that the *null second fundamental form* of $\mathscr{N}$ is intrinsically defined and does not depend on transverse derivatives of the metric. In adapted null coordinates it reads

$$\chi_{AB} = \frac{1}{2} \partial_r \overline{g}_{AB}\,. \tag{2.8}$$

The expansion, also called divergence, of the characteristic surface will be denoted by

$$\tau := \chi_A{}^A = \overline{g}^{AB} \chi_{AB}\,, \tag{2.9}$$

while the trace-free part of the null second fundamental form

$$\sigma_A{}^B = \chi_A{}^B - \frac{1}{2} \tau \delta_A{}^B \tag{2.10}$$

[1] Here we have assumed space-time dimension $n + 1 = 4$ which will be the case of interest in the present work. Note that we have, for arbitrary space-time dimension $n + 1 \geq 3$, $A \in \{2, \ldots, n\}$

is called the *shear* of $\mathscr{N}$.

The constraint equations for the characteristic problem which are propagated by the evolution equations are called *Einstein wave-map gauge constraints*. They are derived, in arbitrary space-time dimension $n+1 \geq 3$, in [2] and we have added the contribution from the cosmological constant Λ:[2]

$$(\partial_r - \kappa)\,\tau + \frac{1}{n-1}\tau^2 = -|\sigma|^2 - 8\pi \overline{T}_{rr}\,, \tag{2.11}$$

$$\left(\partial_r + \frac{1}{2}\tau + \kappa\right)\nu^0 = -\frac{1}{2}\overline{V}^0\,, \tag{2.12}$$

$$(\partial_r + \tau)\,\xi_A = 2\check{\nabla}_B \sigma_A^B - 2\frac{n-2}{n-1}\partial_A \tau - 2\partial_A \kappa - 16\pi \overline{T}_{rA}\,, \tag{2.13}$$

$$\left(\partial_r + \frac{1}{2}\nu_0 \overline{V}^0\right)\nu^A = \frac{1}{2}\nu_0\left(\overline{V}^A - \xi^A - \overline{g}^{BC}\check{\Gamma}^A_{BC}\right)\,, \tag{2.14}$$

$$(\partial_r + \tau + \kappa)\,\zeta = \frac{1}{2}|\xi|^2 - \check{\nabla}_A \xi^A - \check{R} + \underbrace{8\pi\left(\overline{g}^{AB}\overline{T}_{AB} - \overline{T}\right)}_{=:S} + 2\Lambda\,, \tag{2.15}$$

$$\left(\partial_r + \frac{1}{2}\tau + \kappa\right)\overline{g}^{rr} = \frac{1}{2}\zeta - \overline{V}^r\,, \tag{2.16}$$

where

$$|\sigma|^2 := \sigma_A{}^B \sigma_B{}^A\,, \quad |\xi|^2 := \overline{g}^{AB}\xi_A \xi_B\,, \quad \xi^A := \overline{g}^{AB}\xi_B\,, \quad \overline{T} := \overline{g}^{\mu\nu}\overline{T}_{\mu\nu}\,, \tag{2.17}$$

and objects associated with the one-parameter family of Riemannian metrics $\check{g}$ are decorated with the check symbol "ˇ". In the case of $r_0 = 0$ one needs boundary conditions to integrate (2.11)-(2.16), which follow from regularity conditions at the tip of the light-cone, see [2, Section 4.5].

The function κ is defined through the equation

$$\nabla_{\partial_r}\partial_r = \kappa \partial_r \tag{2.18}$$

and reflects the freedom to choose the coordinate r which parametrizes the null geodesic generators of $\mathscr{N}$. The "auxiliary" fields ξ_A and ζ have been introduced to transform (2.11)-(2.16) into first-order equations. $\xi_A \equiv -2\overline{\Gamma}^r_{rA}$ represents connection coefficients, while the ζ is the divergence of the family of suitably normalized null generators normal to the spheres of constant radius r and transverse to the characteristic surface. In coordinates adapted to the light-cone as in [2] the space-time formula for ζ reads (compare [2, Equations (10.32) and (10.36)]; note, however, that there is a term $\tau\overline{g}^{11}/2$ missing at the right-hand side of the second equality in (10.36) there)

$$\zeta := (2\partial_r + 2\kappa + \tau)\overline{g}^{rr} + 2\overline{\Gamma}^r \equiv 2\overline{g}^{AB}\overline{\Gamma}^r_{AB} + \tau\overline{g}^{rr}\,. \tag{2.19}$$

2.3 Trautman-Bondi mass

In general relativity the concept of energy and finding a reasonable definition of it is an involved problem owing to the fact that the gravitational field, i.e. the space-time itself, contributes to the energy.

[2] Note that even though we will assume space-time dimension $n+1=4$ in the remainder of this work we list the wave-map gauge constraints in their form valid for arbitrary space-time dimensions $n+1 \geq 3$ here.

One possibility to define a notation of energy, due to Bondi et al. [1, 9, 10], is to define the mass of an (asymptotically) null or hyperboloidal hypersurface at a given moment of "retarded time" u at the cross-section $\mathring{M}$ where it intersects null infinity $\mathscr{I}^+$. For this Bondi et al. introduced — in the asymptotically flat case, with $\Lambda = 0$ — adapted coordinates, which we call Bondi-type coordinates throughout this thesis. In these coordinates the metric takes the form

$$\overline{g} = \overline{g}_{00} du^2 - 2e^{2\beta} dr\, du - 2r^2 U_A dx^A du + r^2 \underbrace{h_{AB} dx^A dx^B}_{=:h}, \qquad (2.20)$$

where the determinant of h_{AB} is r-independent. Bondi-type coordinates also demand the fields g_{00}, U_A, β and h_{AB} to fulfill certain asymptotic conditions. If this is the case we will decorate all fields and coordinates with a symbol "Bo" [3]. In the asymptotically flat case, with $\Lambda = 0$, and in space-time dimension $n + 1 = 4$ the *Trautman-Bondi mass* is then defined as

$$m_{\mathrm{TB}} = \frac{1}{4\pi} \int_{\mathring{M}} M \, d\mu_{\mathring{h}} \,, \qquad (2.21)$$

where[4] $d\mu_{\mathring{h}} = \sqrt{\det \mathring{h}_{AB}} dx^2 dx^3$, and where M denotes the *mass aspect function* $M : \mathring{M} \to \mathbb{R}$ which is defined as half the coefficient $\left(\overline{g}_{00}^{\mathrm{Bo}}\right)_1$ (compare, e.g., [1, 9]):

$$M := \frac{1}{2} \left(\overline{g}_{00}^{\mathrm{Bo}}\right)_1 \,. \qquad (2.22)$$

Here $(g_{00}^{\mathrm{Bo}})_k$ denotes the coefficient in front of $1/r^k$ in an asymptotic expansion of g_{00}^{Bo} for large r:

$$g_{00}^{\mathrm{Bo}} = -1 + \frac{\left(g_{00}^{\mathrm{Bo}}\right)_1}{r} + o(r^{-1}) \,.$$

Since we are interested in asymptotically Anti-de Sitter space-times, i.e. $\Lambda < 0$, we need an analogue of the Trautman-Bondi mass in this setup. We will assume that our data arise from a space-time with a smooth conformal completion at null infinity $\mathscr{I}^+$, and that the characteristic surface intersects $\mathscr{I}^+$ in a smooth cross-section $\mathring{M}$.

Consider, now, characteristic data in Bondi-type coordinates which may only be defined for large values of r_{Bo}. The space-time metric on $\mathscr{N} = \{u^{\mathrm{Bo}} = 0\}$ can then be written as

$$\overline{g} = \overline{g}_{00}^{\mathrm{Bo}} du_{\mathrm{Bo}}^2 + 2\nu_0^{\mathrm{Bo}} du_{\mathrm{Bo}} dr_{\mathrm{Bo}} + 2\nu_A^{\mathrm{Bo}} du_{\mathrm{Bo}} dx_{\mathrm{Bo}}^A + \check{g}^{\mathrm{Bo}} \,, \qquad (2.23)$$

and assuming the same asymptotic conditions as Bondi et al. leads to [8, Equations (5.16)][5]

$$\lim_{r_{\mathrm{Bo}} \to \infty} \nu_{\mathrm{Bo}}^A = 0 \,, \qquad \lim_{r_{\mathrm{Bo}} \to \infty} \nu_{\mathrm{Bo}}^0 = 1 \,, \qquad \lim_{r_{\mathrm{Bo}} \to \infty} \left(r_{\mathrm{Bo}}^{-2} \overline{g}_{AB}^{\mathrm{Bo}}\right) = \mathring{h}_{AB} \,. \qquad (2.24)$$

[3] Note that the transformation from the affine parameter r, we described in Section 2.2, to r_{Bo} is given by (see [4, Equation (51) there]) the equation $r_{\mathrm{Bo}} = r - \tau_2/2 + \mathcal{O}(r^{-1})$.

[4] Bondi et al. introduced this formalism in the asymptotically flat case, where $\mathring{h}_{AB} \equiv s_{AB}$, the standard metric on S^2. In anticipation of other boundary topologies, e.g. a torus, we will use the symbol $\mathring{h}$ to denote the respective standard metric on that manifold.

[5] Note that the sign of ν^0 can be reversed by changing u to $-u$, which only changes the sign of ν_0 and ν_A in the metric, with no influence on the mass.

Note that the boundary conditions on $\nu^A_{\rm Bo}$ and $\nu^0_{\rm Bo}$ are necessary to determine the "global integration function" arising when solving the constraint equations (see Section 4). For the sake of generality we will allow general $\left(\nu^A_{\rm Bo}\right)_0(x^B)$ and $\left(\nu^0_{\rm Bo}\right)_0(x^B)$ when solving the respective constraint equations and calculating the first expression for the mass aspect function M in Section 5, cf. (5.6), but we will *not* do so when calculating our final formulae for the Trautman-Bondi mass $m_{\rm TB}$.

Further we will assume the definition of the Trautman-Bondi mass remaining to be given by (2.21) while the leading order of the asymptotic expansion of $\overline{g}^{\rm Bo}_{00}$ will change. This expansion can be calculated using the third equation in (2.7) in Bondi-type coordinates

$$\overline{g}^{\rm Bo}_{00} = \overline{g}^{\rm Bo}_{AB}\nu^A_{\rm Bo}\nu^B_{\rm Bo} - (\nu^{\rm Bo}_0)^2\overline{g}^{rr}_{\rm Bo}\,. \tag{2.25}$$

The expansions of the fields entering in this equation in turn can be obtained using the asymptotic conditions (2.24) and integrating the Einstein wave-map gauge constraints (2.11)-(2.16) in Bondi-type coordinates. We note that the Raychaudhuri equation (2.11) determines τ up to a global integration function $\tau_2(x^A)/r^2$ which, however, has a gauge character and vanishes in Bondi-type coordinates. We stress that it is not zero anymore, and indeed globally defined by the initial data, when an affine parametrization is used.

To integrate the wave-map gauge constraints (2.11)-(2.16) one also needs the components $\overline{V}^\mu$, which are determined by the wave-map gauge (2.1)-(2.2). We have, in adapted null coordinates [2, Appendix A, Equations (A.29)-(A.31)],

$$\overline{\Gamma}^0 \equiv \overline{g}^{\lambda\mu}\overline{\Gamma}^0_{\lambda\mu} \equiv \nu^0\left(\nu^0\overline{\partial_0 g_{11}} - \tau\right)\,, \tag{2.26}$$

$$\begin{aligned}\overline{\Gamma}^1 \equiv \overline{g}^{\lambda\mu}\overline{\Gamma}^1_{\lambda\mu} \equiv & -\partial_1\overline{g}^{11} + \overline{g}^{11}\nu^0\left(\frac{1}{2}\overline{\partial_0 g_{11}} - \partial_1\nu_0 - \tau\nu_0\right)\\ & + \nu^0\overline{g}^{AB}\check{\nabla}_B\nu_A - \frac{1}{2}\nu^0\overline{g}^{AB}\overline{\partial_0 g_{AB}}\,.\end{aligned} \tag{2.27}$$

$$\begin{aligned}\overline{\Gamma}^A \equiv \overline{g}^{\lambda\mu}\overline{\Gamma}^A_{\lambda\mu} \equiv & \,\nu^0\nu^A\left(\tau - \nu^0\overline{\partial_0 g_{11}}\right) + \nu^0\overline{g}^{AB}(\overline{\partial_0 g_{1B}} + \partial_1\nu_B - \partial_B\nu_0)\\ & - 2\nu^0\nu^B\chi_B{}^A + \check{\Gamma}^A,\end{aligned} \tag{2.28}$$

Now, from the restriction to $\mathscr{N}$, of (2.1) and the first equation of (2.2) one finds that the choice of the target metric only redefines the fields $\overline{\hat{\Gamma}}^\mu$ and $\overline{W}^\mu$ entering in the definition of $\overline{V}^\mu = \overline{\hat{\Gamma}}^\mu + \overline{W}^\mu$ without changing $\overline{V}^\mu$ itself. We stress that only $\overline{V}^\mu$ enters in the Einstein wave-map gauge constraint equations and therefore only the explicit form of those fields is relevant in the equations of interest to us.

We will derive the asymptotic expansions of all relevant fields in *Bondi-type coordinates on the characteristic surface*, as it is in those coordinates that the calculations appear to be simplest. Writing interchangeably r and $r_{\rm Bo}$, we then have [8, Equation (5.5)]

$$\varphi^{\rm Bo} = r_{\rm Bo}\,, \quad \overline{\partial_0 g^{\rm Bo}_{rr}} = 0\,, \quad \overline{\partial_0 g^{\rm Bo}_{rA}} = 0\,, \quad \overline{g}^{AB}_{\rm Bo}\overline{\partial_0 g^{\rm Bo}_{AB}} = 0\,, \tag{2.29}$$

where φ is defined by $\tau = 2\partial_r\log\varphi$, as well as

$$\overline{V}^0_{\rm Bo} = -\tau^{\rm Bo}\nu^0_{\rm Bo}\,, \tag{2.30}$$

$$\overline{V}^A_{\rm Bo} = \overline{g}^{CD}_{\rm Bo}\left(\check{\Gamma}^{\rm Bo}\right)^A_{CD} - \nu^0_{\rm Bo}\check{\nabla}^A\nu^{\rm Bo}_0 + \nu^0_{\rm Bo}\left(\partial_{r_{\rm Bo}} + \tau^{\rm Bo}\right)\nu^A_{\rm Bo}\,, \tag{2.31}$$

$$\overline{V}^r_{\rm Bo} = \nu^0_{\rm Bo}\check{\nabla}_A\nu^A_{\rm Bo} - \left(\partial_{r_{\rm Bo}} + \tau^{\rm Bo} + \nu^0_{\rm Bo}\partial_{r_{\rm Bo}}\nu^{\rm Bo}_0\right)\overline{g}^{rr}_{\rm Bo}\,. \tag{2.32}$$

As mentioned previously the Einstein wave-map gauge constraints form a hierarchical system of ODEs along the null generators of the characteristic surface and can therefore be solved step-by-step.

3 Matter terms

In this section we analyze the influence of the matter fields on the asymptotic expansion of the metric in Bondi-type coordinates. Our aim is to determine the decay rate of the energy-momentum tensor which is compatible with finite total mass. The decay rates for various components of the energy-momentum tensor have been chosen so that they do not affect the leading-order behavior, as arising in the vacuum case, of the solutions of the equations in which they appear.

For the convenience of the reader we repeat here the relevant equations in Bondi-type gauge (see [8, Equations (5.11)-(5.15)] with the contribution from the cosmological constant Λ added here), with $r \equiv r_{\text{Bo}}$:

$$\kappa^{\text{Bo}} - \frac{1}{2} r \left(|\sigma^{\text{Bo}}|^2 + 8\pi \overline{T}^{\text{Bo}}_{rr} \right) = 0\,, \tag{3.1}$$

$$(\partial_r + \frac{r}{2}(|\sigma^{\text{Bo}}|^2 + 8\pi \overline{T}^{\text{Bo}}_{rr}))\nu^0 = 0\,, \tag{3.2}$$

$$(\partial_r + \tau^{\text{Bo}})\xi^{\text{Bo}}_A - 2\check{\nabla}_B \sigma_A^{\text{Bo}\,B} + \partial_A \tau^{\text{Bo}} + r\partial_A(|\sigma^{\text{Bo}}|^2 + 8\pi \overline{T}^{\text{Bo}}_{rr}) = -16\pi \overline{T}^{\text{Bo}}_{rA}\,, \tag{3.3}$$

$$\partial_r \nu^A_{\text{Bo}} + (\check{\nabla}^A + \xi^A_{\text{Bo}})\nu^{\text{Bo}}_0 = 0\,, \tag{3.4}$$

$$(\partial_r + \tau^{\text{Bo}} + \frac{r}{2}(|\sigma^{\text{Bo}}|^2 + 8\pi \overline{T}^{\text{Bo}}_{rr}))\zeta^{\text{Bo}} + \check{R}^{\text{Bo}} - \frac{|\xi^{\text{Bo}}|^2}{2} + \check{\nabla}_A \xi^A_{\text{Bo}} = \underbrace{8\pi(\overline{g}^{AB}_{\text{Bo}}\overline{T}^{\text{Bo}}_{AB} - \overline{T}^{\text{Bo}})}_{S^{\text{Bo}}} + 2\Lambda\,, \tag{3.5}$$

$$\overline{g}^{rr}_{\text{Bo}} + (\tau^{\text{Bo}})^{-1}(\zeta^{\text{Bo}} - 2\nu^0_{\text{Bo}}\check{\nabla}_A \nu^A_{\text{Bo}}) = 0\,. \tag{3.6}$$

To simplify notation we will again drop the index "Bo" in the remainder of this section.

In Bondi-type coordinates, the relation

$$\tau = \frac{2}{r}$$

is independent of matter fields.

It follows from (3.1), which can be solved algebraically for κ in Bondi-type coordinates, that a term $O(r^{-\alpha_{rr}})$ in T_{rr} with $\alpha_{rr} > 2$ produces an $O(r^{-\alpha_{rr}+1})$ term in κ (see also the discussion in Section 4.1 and Equation (4.4)):

$$T_{rr} = O(r^{-\alpha_{rr}})\,,\ \alpha_{rr} > 2 \qquad \Longrightarrow \qquad \kappa = (\kappa)_{\text{vacuum}} + O(r^{-\alpha_{rr}+1})\,. \tag{3.7}$$

Next, from (3.2) we find

$$\nu^0 = (\nu^0)_{\text{vacuum}} + O(r^{-\alpha_{rr}+2})\,. \tag{3.8}$$

In the ξ_A-constraint equation (3.3), the assumption

$$T_{rA} = O(r^{-\alpha_{rA}+1})\,,\ 4 \neq \alpha_{rA}\,,\ 4 \neq \alpha_{rr}\,, \tag{3.9}$$

leads to

$$\xi_A = (\xi_A)_{\text{vacuum}} + O(r^{-\alpha_{rA}+2}) + O(r^{-\alpha_{rr}+2})\,, \tag{3.10}$$

where the values $\alpha_{rA} = 4$ and $\alpha_{rr} = 4$ have been excluded to avoid here a supplementary annoying discussion of logarithmic terms (we will discuss such terms in our more detailed analysis below):

$$\alpha_{rA} = 4 \text{ or } \alpha_{rr} = 4: \;\; \xi_A = (\xi_A)_{\text{vacuum}} + O(r^{-\alpha_{rA}+2}) + O(r^{-\alpha_{rr}+2}) + O(r^{-2}\log r)\,. \tag{3.11}$$

From now on we assume (3.9). To preserve the vacuum asymptotics $\xi_A = O(r^{-1})$ we will moreover require

$$\alpha_{rr} > 3\,, \quad \alpha_{rA} > 3\,. \tag{3.12}$$

(Anticipating, we have excluded the case $\alpha_{rr} = 3$, which introduces $1/r$ terms in ν^0, which lead subsequently to logarithmically divergent terms in ν^A. We further note that $\alpha_{rA} = 3$ will produce an additional $(\xi_A^{\text{Bo}})_1$-term that would not integrate away in m_{TB} and would remain as a $(\overline{T}^{\text{Bo}}_{rA})_2$-term in the final formula.)

Integration of (3.4) gives

$$\begin{aligned} \nu^A &= (\nu^A)_{\text{vacuum}} + O(r^{-\alpha_{rA}+1}) + O(r^{-\alpha_{rr}+1}) \\ &\Longleftrightarrow \nu_A = (\nu_A)_{\text{vacuum}} + O(r^{-\alpha_{rA}+3}) + O(r^{-\alpha_{rr}+3})\,. \end{aligned} \tag{3.13}$$

Finally, the asymptotic behavior $\xi_A = O(r^{-1})$ together with (3.5) and (3.6) show that: a term $O(r^{-\alpha_S})$ in S with $\alpha_S < 2$ would change the leading order behavior of ζ; $\alpha_S = 2$ would change the leading order term of ζ; $\alpha_S = 3$ would lead to a logarithmic term in ζ. This leads to

$$S = O(r^{-\alpha_S})\,,\; \alpha_S > 3\,,\; \alpha_{rr} \neq 5\,,$$

$$\Longrightarrow \quad \zeta = (\zeta)_{\text{vacuum}} + O(r^{-\alpha_S+1}) + O(r^{-\alpha_{rA}+1}) + O(r^{-\alpha_{rr}+3})\,, \tag{3.14}$$

$$g^{rr} = (g^{rr})_{\text{vacuum}} + O(r^{-\alpha_S+2}) + O(r^{-\alpha_{rA}+2}) + O(r^{-\alpha_{rr}+4})\,, \tag{3.15}$$

and note that a factor r^2 in the $O(r^{-\alpha_{rr}+3})$ terms in ζ arises from the $4\pi r \overline{T}^{\text{Bo}}_{rr}\zeta^{\text{Bo}}$ term in (3.5), taking into account the $2\Lambda r/3$ leading behavior of ζ.

We conclude that the leading order of all quantities of interest will be preserved if we assume that

$$\alpha_{rr} > 3\,,\; \alpha_{rA} > 3\,,\; \alpha_S > 3\,. \tag{3.16}$$

Keeping in mind our main assumptions, that all fields can be expanded in terms of inverse powers of r to the order needed to perform our expansions, possibly with some logarithmic coefficients, we will allow below matter fields for which (3.16) holds.

4 Solving the characteristic wave-map gauge constraints asymptotically

It was shown in [7] that it is possible to solve the Einstein wave-map gauge constraints (2.11)-(2.16) in terms of polyhomogeneous expansions of the solution at infinity, i.e., expansions in terms of inverse powers of r and of powers of $\log r$. Our goal is to obtain an expression for the Trautman-Bondi mass which is defined via an

expansion coefficient in Bondi-type coordinates, hence we will solve the equations in those coordinates[6].
For the convenience of the reader we stress that one would need boundary conditions on $\nu^0_{\rm Bo}$ and $\nu^A_{\rm Bo}$, cf. (2.24), to determine arising global integration functions when solving the respective constraint equations. To keep track of the influence of those integration functions we will not assume any boundary conditions for the moment and will apply them later.

We assume that for large r

$$\sigma_A^{{\rm Bo}B} = \left(\sigma_A^{{\rm Bo}B}\right)_2 r_{\rm Bo}^{-2} + \left(\sigma_A^{{\rm Bo}B}\right)_3 r_{\rm Bo}^{-3} + \mathcal{O}\left(r_{\rm Bo}^{-4}\right) . \tag{4.1}$$

Recall further the results of section 3 because of which we allow energy-momentum tensors satisfying

$$\overline{T}^{\rm Bo}_{rr} = \mathcal{O}\left(r_{\rm Bo}^{-4}\right) , \quad \overline{T}^{\rm Bo}_{rA} = \mathcal{O}\left(r_{\rm Bo}^{-3}\right) , \quad \overline{g}^{AB}_{\rm Bo}\overline{T}^{\rm Bo}_{AB} - \overline{T}^{\rm Bo} = \mathcal{O}\left(r_{\rm Bo}^{-3}\right) \tag{4.2}$$

in what follows. Note that the third equation in (4.2) includes the case we excluded in Section 3 previously and will lead to a logarithmic term in the asymptotic expansion of ζ. This term however will be of the order $\log r_{\rm Bo}/r^2_{\rm Bo}$ and will not influence our result for the Trautman-Bondi mass. It is accounted for in the correction term in (4.17). An analog statement holds for the fall-off behavior and the correction term in Equation (6.5) in affine coordinates, denoted by (u, r, x^A).

When solving the wave-map gauge constraints we keep in mind that we finally want to determine the expansion coefficient $\left(\overline{g}^{\rm Bo}_{00}\right)_1$. This determines how far the asymptotic expansions of intermediate results need to be calculated explicitly.

4.1 Solving equation (2.11)

The first equation of (2.29) implies $\tau^{\rm Bo} = 2r^{-1}_{\rm Bo}$ and using this one directly finds from (2.11) in Bondi-type coordinates, cf. (3.1),

$$\kappa^{\rm Bo} = \frac{1}{2} r_{\rm Bo} \left(|\sigma^{\rm Bo}|^2 + 8\pi \overline{T}^{\rm Bo}_{rr}\right) . \tag{4.3}$$

Note for further reference that this means

$$\left(\kappa^{\rm Bo}\right)_n = \frac{1}{2}\left[|\sigma^{\rm Bo}|^2_{n+1} + 8\pi \left(\overline{T}^{\rm Bo}_{rr}\right)_{n+1}\right] \tag{4.4}$$

for the expansion coefficients of $\kappa^{\rm Bo}$, where we have assumed that n is positive.

4.2 Expansion of $\nu^0_{\rm Bo}$

Inserting (2.30) and (4.3) into (2.12) in Bondi-type coordinates yields, cf. (3.2),

$$\left[\partial_{r_{\rm Bo}} + \frac{r_{\rm Bo}}{2}\left(|\sigma^{\rm Bo}|^2 + 8\pi\overline{T}^{\rm Bo}_{rr}\right)\right]\nu^0_{\rm Bo} = 0 , \tag{4.5}$$

[6]Note that even though we have given the wave-map gauge constraint equations in Bondi-type coordinates from [8] in the previous Section we will mention briefly how to obtain them in the following Subsections.

and from (4.1) we have

$$|\sigma^{\mathrm{Bo}}|^2 = \frac{|\sigma^{\mathrm{Bo}}|_4^2}{r_{\mathrm{Bo}}^4} + \frac{|\sigma^{\mathrm{Bo}}|_5^2}{r_{\mathrm{Bo}}^5} + \frac{|\sigma^{\mathrm{Bo}}|_6^2}{r_{\mathrm{Bo}}^6} + \mathcal{O}(r_{\mathrm{Bo}}^{-7}) \,. \tag{4.6}$$

Using this and (4.2) we find the solution of (4.5)

$$\begin{aligned} \nu^0_{\mathrm{Bo}} = (\nu^0_{\mathrm{Bo}})_0 \Bigg(1 + \frac{1}{4}\Big[|\sigma^{\mathrm{Bo}}|_4^2 + 8\pi \left(\overline{T}^{\mathrm{Bo}}_{rr}\right)_4 \Big] r_{\mathrm{Bo}}^{-2} \\ + \frac{1}{6}\Big[|\sigma^{\mathrm{Bo}}|_5^2 + 8\pi \left(\overline{T}^{\mathrm{Bo}}_{rr}\right)_5 \Big] r_{\mathrm{Bo}}^{-3} \Bigg) + \mathcal{O}(r_{\mathrm{Bo}}^{-4}) \,, \end{aligned} \tag{4.7}$$

where $(\nu^0_{\mathrm{Bo}})_0$ is a global integration function.

4.3 Expansion of ξ_A^{Bo}

We have $\partial_A \tau^{\mathrm{Bo}} = 0$ and obtain

$$\left(\partial_{r_{\mathrm{Bo}}} + \tau^{\mathrm{Bo}}\right) \xi_A^{\mathrm{Bo}} = 2\check{\nabla}_B \sigma_A^{\mathrm{Bo}B} - 2\partial_A \kappa^{\mathrm{Bo}} - 16\pi \overline{T}^{\mathrm{Bo}}_{rA} \tag{4.8}$$

for (2.13) in Bondi-type coordinates, cf. (3.3). Using again (4.2), (4.6) as well as [7, Equations (3.24)-(3.26)]

$$\check{\nabla}_B \sigma_A^{\mathrm{Bo}B} = (\Xi^{\mathrm{Bo}})_A^{(2)} r_{\mathrm{Bo}}^2 + (\Xi^{\mathrm{Bo}})_A^{(3)} r_{\mathrm{Bo}}^3 + \mathcal{O}(r_{\mathrm{Bo}}^{-4}) \,, \tag{4.9}$$

where

$$(\Xi^{\mathrm{Bo}})_A^{(2)} := \mathring{\nabla}_B \left(\sigma_A^{\mathrm{Bo}B}\right)_2 \,, \quad (\Xi^{\mathrm{Bo}})_A^{(3)} := \mathring{\nabla}_B \left(\sigma_A^{\mathrm{Bo}B}\right)_3 + \frac{1}{2}\mathring{\nabla}_A |\sigma^{\mathrm{Bo}}|_4^2 \,, \tag{4.10}$$

the solution of (4.8) reads

$$\begin{aligned} \xi_A^{\mathrm{Bo}} = 2(\Xi^{\mathrm{Bo}})_A^{(2)} r_{\mathrm{Bo}}^{-1} - 2\left[(\Xi^{\mathrm{Bo}})_A^{(3)} - \partial_A \left(\kappa^{\mathrm{Bo}}\right)_3 - 8\pi \left(\overline{T}^{\mathrm{Bo}}_{rA}\right)_3 \right] \frac{\log r_{\mathrm{Bo}}}{r_{\mathrm{Bo}}^2} \\ + C_A^{(\xi_B)} r_{\mathrm{Bo}}^{-2} + \mathcal{O}(r_{\mathrm{Bo}}^{-3}) \,, \end{aligned} \tag{4.11}$$

where the coefficients $C_A^{(\xi_B)} = C_A^{(\xi_B)}(x^C)$ are global integration functions.

4.4 Expansion of ν^A_{Bo}

Equation (2.14) in Bondi-type coordinates does not change due to Λ being non-zero and reads, cf. (3.4),

$$\partial_{r_{\mathrm{Bo}}} \nu^A_{\mathrm{Bo}} = -\left(\check{\nabla}^A + \xi^A_{\mathrm{Bo}}\right) \nu_0^{\mathrm{Bo}} \,. \tag{4.12}$$

Using the solutions we found for ν^0_{Bo}, (4.7) and recall that $\nu^0_{\mathrm{Bo}} = 1/\nu_0^{\mathrm{Bo}}$, ξ_A^{Bo}, (4.11), and the fact that $\overline{g}^{AB}_{\mathrm{Bo}}$ is of the form (recall (2.5) and footnote 3 in Section 2.3 where we stated that $r_{\mathrm{Bo}} = r - \tau_2/2 + \mathcal{O}(r^{-1})$)

$$\overline{g}^{AB}_{\mathrm{Bo}} = \mathring{h}^{AB} r_{\mathrm{Bo}}^{-2} + \left(\overline{g}^{AB}_{\mathrm{Bo}}\right)_3 r_{\mathrm{Bo}}^{-3} + \mathcal{O}(r_{\mathrm{Bo}}^{-4}) \,, \tag{4.13}$$

we find the solution of (4.12)

$$
\begin{aligned}
\nu^A_{\mathrm{Bo}} &= \left(\nu^A_{\mathrm{Bo}}\right)_0 + \mathring{h}^{AB}\check{\nabla}_B \left(\nu^0_{\mathrm{Bo}}\right)_0^{-1} r_{\mathrm{Bo}}^{-1} \\
&\quad + \frac{1}{2}\left[\left(\nu^0_{\mathrm{Bo}}\right)_0^{-1} \mathring{h}^{AB} \left(\xi_B\right)_1 + \left(\overline{g}^{AB}_{\mathrm{Bo}}\right)_3 \check{\nabla}_B \left(\nu^0_{\mathrm{Bo}}\right)_0^{-1}\right] r_{\mathrm{Bo}}^{-2} \\
&\quad - \frac{1}{3}\mathring{h}^{AB} \left(\xi_B\right)_{\log,2} \frac{\log r_{\mathrm{Bo}}}{r_{\mathrm{Bo}}^2} \\
&\quad - \frac{1}{3}\Bigg[\frac{1}{3}\mathring{h}^{AB} \left(\xi_B\right)_{\log,2} + \left(\overline{g}^{AB}_{\mathrm{Bo}}\right)_3 \left(\xi_B\right)_1 + \mathring{h}^{AB} \left(\xi_B\right)_2 - \left(\overline{g}^{AB}_{\mathrm{Bo}}\right)_4 \mathring{\nabla}_B \left(\nu^0_{\mathrm{Bo}}\right)_0^{-1} \\
&\qquad + \mathring{h}^{AB} \left(\nu^0_{\mathrm{Bo}}\right)_2 \mathring{\nabla}_B \left(\nu^0_{\mathrm{Bo}}\right)_0^{-1} + \left(\nu^0_{\mathrm{Bo}}\right)_0^{-1} \mathring{h}^{AB}\mathring{\nabla}_B \left(\nu^0_{\mathrm{Bo}}\right)_2 \Bigg] r_{\mathrm{Bo}}^{-3} \\
&\quad + o\left(r_{\mathrm{Bo}}^{-3}\right) \\
&= \left(\nu^A_{\mathrm{Bo}}\right)_0 + \mathring{h}^{AB}\check{\nabla}_B \left(\nu^0_{\mathrm{Bo}}\right)_0^{-1} r_{\mathrm{Bo}}^{-1} \\
&\quad + \left[\left(\nu^0_{\mathrm{Bo}}\right)_0^{-1} \mathring{h}^{AB}\mathring{\nabla}_C \left(\sigma_B^{\mathrm{Bo}C}\right)_2 + \frac{1}{2}\left(\overline{g}^{AB}_{\mathrm{Bo}}\right)_3 \check{\nabla}_B \left(\nu^0_{\mathrm{Bo}}\right)_0^{-1}\right] r_{\mathrm{Bo}}^{-2} \\
&\quad + \frac{2}{3}\mathring{h}^{AB} \left[\mathring{\nabla}_A \left(\sigma_B^{\mathrm{Bo}A}\right)_3 + \frac{1}{2}\mathring{\nabla}_B |\sigma^{\mathrm{Bo}}|^2_4 - \partial_B \left(\kappa^{\mathrm{Bo}}\right)_3 - 8\pi \left(\overline{T}^{\mathrm{Bo}}_{rB}\right)_3\right] \frac{\log r_{\mathrm{Bo}}}{r_{\mathrm{Bo}}^2} \\
&\quad - \frac{1}{3}\Bigg[\frac{2}{3}\mathring{h}^{AB} \left(-\mathring{\nabla}_A \left(\sigma_B^{\mathrm{Bo}A}\right)_3 - \frac{1}{2}\mathring{\nabla}_B |\sigma^{\mathrm{Bo}}|^2_4 + \partial_B \left(\kappa^{\mathrm{Bo}}\right)_3 + 8\pi \left(\overline{T}^{\mathrm{Bo}}_{rB}\right)_3\right) \\
&\qquad + 2\left(\overline{g}^{AB}_{\mathrm{Bo}}\right)_3 \mathring{\nabla}_A \left(\sigma_B^{\mathrm{Bo}A}\right)_2 + \mathring{h}^{AB} C_B^{(\xi_C)} - \left(\overline{g}^{AB}_{\mathrm{Bo}}\right)_4 \mathring{\nabla}_B \left(\nu^0_{\mathrm{Bo}}\right)_0^{-1} \\
&\qquad + \mathring{h}^{AB} \left(\nu^0_{\mathrm{Bo}}\right)_2 \mathring{\nabla}_B \left(\nu^0_{\mathrm{Bo}}\right)_0^{-1} + \left(\nu^0_{\mathrm{Bo}}\right)_0^{-1} \mathring{h}^{AB}\mathring{\nabla}_B \left(\nu^0_{\mathrm{Bo}}\right)_2 \Bigg] r_{\mathrm{Bo}}^{-3} \\
&\quad + o\left(r_{\mathrm{Bo}}^{-3}\right), \qquad (4.14)
\end{aligned}
$$

where $\left(\nu^A_{\mathrm{Bo}}\right)_0$ is a global integration function.

4.5 Expansion of ζ^{Bo}

Inserting $\tau^{\mathrm{Bo}} = 2r_{\mathrm{Bo}}^{-1}$ and (4.3) into (2.15) in Bondi-type coordinates yields, cf. (3.5),

$$
\begin{aligned}
&\left(\partial_{r_{\mathrm{Bo}}} + \frac{2}{r_{\mathrm{Bo}}} + \frac{r_{\mathrm{Bo}}}{2}\left(|\sigma^{\mathrm{Bo}}|^2 + 8\pi\overline{T}^{\mathrm{Bo}}_{rr}\right)\right)\zeta^{\mathrm{Bo}} = \\
&\qquad - \check{R}^{\mathrm{Bo}} + \frac{|\xi^{\mathrm{Bo}}|^2}{2} - \check{\nabla}_A \xi^A_{\mathrm{Bo}} + 8\pi\left(\overline{g}^{AB}_{\mathrm{Bo}}\overline{T}^{\mathrm{Bo}}_{AB} - \overline{T}^{\mathrm{Bo}}\right) + 2\Lambda\,. \qquad (4.15)
\end{aligned}
$$

In order to solve this equation we start by defining

$$
\zeta^{\mathrm{Bo}} := \zeta^{\mathrm{Bo}}_{\Lambda=0} + \delta\zeta^{\mathrm{Bo}}\,, \qquad (4.16)
$$

where $\zeta^{\mathrm{Bo}}_{\Lambda=0}$ is the solution of (4.15) in the case $\Lambda = 0$. Its asymptotic expansion is known ([7, eq. (3.40)] gives the formula in general coordinates for general $\mathring{R}$, while [8, eq. (5.23)] the one in Bondi-type coordinates with $\mathring{R} = 2$) and reads

$$
\zeta^{\mathrm{Bo}}_{\Lambda=0} = -\frac{\mathring{R}}{r_{\mathrm{Bo}}} + \left(\zeta^{\mathrm{Bo}}_{\Lambda=0}\right)_2 r_{\mathrm{Bo}}^{-2} + o\left(r_{\mathrm{Bo}}^{-2}\right), \qquad (4.17)
$$

where $\left(\zeta^{\mathrm{Bo}}_{\Lambda=0}\right)_2$ is a global integration function and $\mathring{R}$ denotes the leading order coefficient of the asymptotic expansion of $\check{R}$ in terms of r, which coincides with the Ricci scalar of the boundary metric $\lim_{r\to\infty} r^{-2}\overline{g}_{AB}dx^A dx^B$. The expansion of $\delta\zeta^{\mathrm{Bo}}$ on the other hand can be calculated by subtracting (4.15) from the corresponding equation in the case $\Lambda = 0$ (keep in mind that it is an equation for $\zeta^{\mathrm{Bo}}_{\Lambda=0}$ in this case), which gives

$$\left(\partial_{r_{\mathrm{Bo}}} + \tau^{\mathrm{Bo}} + \kappa^{\mathrm{Bo}}\right)\delta\zeta^{\mathrm{Bo}} = 2\Lambda\,. \tag{4.18}$$

This equation can be solved as usual by using (4.2), (4.3) as well as (4.6) and we end up with

$$\begin{aligned}
\delta\zeta^{\mathrm{Bo}} &= \frac{2\Lambda}{3} r_{\mathrm{Bo}} - \frac{2\Lambda(\kappa^{\mathrm{Bo}})_3}{3 r_{\mathrm{Bo}}} + \frac{2\Lambda(\kappa^{\mathrm{Bo}})_4}{3}\frac{\log\left(r_{\mathrm{Bo}}\right)}{r_{\mathrm{Bo}}^2} \\
&\quad + \frac{(\delta\zeta^{\mathrm{Bo}})_2}{r_{\mathrm{Bo}}^2} + o\big(r_{\mathrm{Bo}}^{-2}\big) \\
&= \frac{2\Lambda}{3} r_{\mathrm{Bo}} - \frac{\Lambda}{3}\left[|\sigma^{\mathrm{Bo}}|_4^2 + 8\pi\left(\overline{T}^{\mathrm{Bo}}_{rr}\right)_4\right] r_{\mathrm{Bo}}^{-1} \\
&\quad + \frac{\Lambda}{3}\left[|\sigma^{\mathrm{Bo}}|_5^2 + 8\pi\left(\overline{T}^{\mathrm{Bo}}_{rr}\right)_5\right]\frac{\log\left(r_{\mathrm{Bo}}\right)}{r_{\mathrm{Bo}}^2} \\
&\quad + \frac{(\delta\zeta^{\mathrm{Bo}})_2}{r_{\mathrm{Bo}}^2} + o\big(r_{\mathrm{Bo}}^{-2}\big)\,,
\end{aligned} \tag{4.19}$$

where $(\delta\zeta^{\mathrm{Bo}})_2$ is again a global integration function. Summing up the solution of (4.15) in Bondi-type coordinates reads

$$\begin{aligned}
\zeta^{\mathrm{Bo}} &= \frac{2\Lambda}{3} r_{\mathrm{Bo}} - \left(\mathring{R} + \frac{2\Lambda}{3}(\kappa^{\mathrm{Bo}})_3\right) r_{\mathrm{Bo}}^{-1} \\
&\quad + \frac{2\Lambda}{3}(\kappa^{\mathrm{Bo}})_4 \frac{\log r_{\mathrm{Bo}}}{r_{\mathrm{Bo}}^2} + \frac{(\zeta^{\mathrm{Bo}})_2}{r_{\mathrm{Bo}}^2} + o\big(r_{\mathrm{Bo}}^{-2}\big) \\
&= \frac{2\Lambda}{3} r_{\mathrm{Bo}} - \left(\mathring{R} + \frac{\Lambda}{3}\left[|\sigma^{\mathrm{Bo}}|_4^2 + 8\pi\left(\overline{T}^{\mathrm{Bo}}_{rr}\right)_4\right]\right) r_{\mathrm{Bo}}^{-1} \\
&\quad + \frac{\Lambda}{3}\left[|\sigma^{\mathrm{Bo}}|_5^2 + 8\pi\left(\overline{T}^{\mathrm{Bo}}_{rr}\right)_5\right]\frac{\log r_{\mathrm{Bo}}}{r_{\mathrm{Bo}}^2} + \frac{(\zeta^{\mathrm{Bo}})_2}{r_{\mathrm{Bo}}^2} + o\big(r_{\mathrm{Bo}}^{-2}\big)\,,
\end{aligned} \tag{4.20}$$

and we have combined the two integration functions $(\delta\zeta^{\mathrm{Bo}})_2$ and $\left(\zeta^{\mathrm{Bo}}_{\Lambda=0}\right)_2$ into $(\zeta^{\mathrm{Bo}})_2$.

4.6 Analyzing equation (2.16)

Inserting (2.32) into (2.16) in Bondi-type coordinates and keeping in mind that, by (4.5), $\partial_{r_{\mathrm{Bo}}}\nu^0_{\mathrm{Bo}} = -\kappa^{\mathrm{Bo}}$ one finds, cf. (3.6),

$$\overline{g}^{rr}_{\mathrm{Bo}} + (\tau^{\mathrm{Bo}})^{-1}\left(\zeta^{\mathrm{Bo}} - 2\nu^0_{\mathrm{Bo}}\check{\nabla}_A\nu^A_{\mathrm{Bo}}\right) = 0 \tag{4.21}$$

for (2.16) in Bondi-type coordinates and we note that this is an algebraic equation for $\overline{g}^{rr}_{\mathrm{Bo}}$. Inserting the asymptotic expansions (4.7), (4.14) and (4.20) we found for

$\nu^0_{\rm Bo}$, $\nu^A_{\rm Bo}$ and $\zeta^{\rm Bo}$ respectively we obtain the asymptotic expansion

$$\overline{g}^{rr}_{\rm Bo} = -\frac{\Lambda}{3} r^2_{\rm Bo} + \left(\nu^0_{\rm Bo}\right)_0 \check{\nabla}_A \left(\nu^A_{\rm Bo}\right)_0 r_{\rm Bo} + \left(\frac{\mathring{R}}{2} + \frac{\Lambda}{3}\left(\kappa^{\rm Bo}\right)_3\right) - \frac{\Lambda}{3}(\kappa^{\rm Bo})_4 \frac{\log r_{\rm Bo}}{r_{\rm Bo}} + \left(\left(\nu^0_{\rm Bo}\right)_0 \left[\mathring{\nabla}_A \left(\nu^A_{\rm Bo}\right)_2 + \left(\nu^0_{\rm Bo}\right)_2 \mathring{\nabla}_A \left(\nu^A_{\rm Bo}\right)_0\right] - \frac{1}{2}(\zeta^{\rm Bo})_2\right) r^{-1}_{\rm Bo} + o\left(r^{-1}_{\rm Bo}\right), \tag{4.22}$$

where, as before, $\mathring{R}$ is the Ricci scalar of the boundary metric $\lim_{r\to\infty} r^{-2}\overline{g}_{AB}dx^A dx^B$ and $\mathring{\nabla}_A$ is the respective covariant derivative. Here we kept the expression quite general to keep it as clear as possible.

5 The Trautman-Bondi mass in Bondi-type coordinates

We return to (2.25), which we repeat here for convenience of the reader,

$$\overline{g}^{\rm Bo}_{00} = \overline{g}^{\rm Bo}_{AB}\nu^A_{\rm Bo}\nu^B_{\rm Bo} - (\nu^{\rm Bo}_0)^2 \overline{g}^{rr}_{\rm Bo}, \tag{5.1}$$

and we note again (cf. (4.13)) that $\overline{g}^{AB}_{\rm Bo}$ is of the form

$$\overline{g}^{AB}_{\rm Bo} = \mathring{h}^{AB} r^{-2}_{\rm Bo} + \left(\overline{g}^{AB}_{\rm Bo}\right)_3 r^{-3}_{\rm Bo} + \mathcal{O}\left(r^{-4}_{\rm Bo}\right). \tag{5.2}$$

Using this and (4.22) leads us to

$$\begin{aligned}
\overline{g}^{\rm Bo}_{00} = & \left(\mathring{h}_{AB}\left(\nu^A_{\rm Bo}\right)_0 \left(\nu^B_{\rm Bo}\right)_0 - \left(\nu^0_{\rm Bo}\right)_0^{-2} (\overline{g}^{rr}_{\rm Bo})_{-2}\right) r^2_{\rm Bo} + \left(\overline{g}^{\rm Bo}_{AB}\right)_{-1} \left(\nu^A_{\rm Bo}\right)_0 \left(\nu^B_{\rm Bo}\right)_0 r_{\rm Bo} \\
& + \left(\nu^A_{\rm Bo}\right)_0 \left[2\mathring{h}_{AB}\left(\nu^B_{\rm Bo}\right)_2 + \left(\overline{g}^{\rm Bo}_{AB}\right)_0 \left(\nu^B_{\rm Bo}\right)_0\right] \\
& - \left(\nu^0_{\rm Bo}\right)_0^{-1} \left[\left(\nu^0_{\rm Bo}\right)_0^{-1} (\overline{g}^{rr}_{\rm Bo})_0 + 2\,(\overline{g}^{rr}_{\rm Bo})_{-2} \left(\nu^{\rm Bo}_0\right)_2\right] \\
& - \left(\nu^0_{\rm Bo}\right)_0^{-2} (\overline{g}^{rr}_{\rm Bo})_{\log,1} \frac{\log r_{\rm Bo}}{r_{\rm Bo}} \\
& + \Bigg(\left(\nu^A_{\rm Bo}\right)_0 \left[2\mathring{h}_{AB}\left(\nu^B_{\rm Bo}\right)_3 + 2\left(\overline{g}^{\rm Bo}_{AB}\right)_{-1}\left(\nu^B_{\rm Bo}\right)_2 + \left(\overline{g}^{\rm Bo}_{AB}\right)_1\left(\nu^B_{\rm Bo}\right)_0\right] \\
& \qquad - \left(\nu^0_{\rm Bo}\right)_0^{-1} \left[\left(\nu^0_{\rm Bo}\right)_0^{-1} (\overline{g}^{rr}_{\rm Bo})_1 + 2\,(\overline{g}^{rr}_{\rm Bo})_{-1}\left(\nu^{\rm Bo}_0\right)_2 + 2\,(\overline{g}^{rr}_{\rm Bo})_{-2}\left(\nu^{\rm Bo}_0\right)_3\right]\Bigg) r^{-1}_{\rm Bo} \\
& + o\left(r^{-1}_{\rm Bo}\right),
\end{aligned} \tag{5.3}$$

where we can directly read off an expression for M:

$$\begin{aligned}
M := & \frac{1}{2}\left(\overline{g}^{\rm Bo}_{00}\right)_1 \\
= & \frac{1}{2}\Bigg(\left(\nu^A_{\rm Bo}\right)_0 \left[2\mathring{h}_{AB}\left(\nu^B_{\rm Bo}\right)_3 + 2\left(\overline{g}^{\rm Bo}_{AB}\right)_{-1}\left(\nu^B_{\rm Bo}\right)_2 + \left(\overline{g}^{\rm Bo}_{AB}\right)_1\left(\nu^B_{\rm Bo}\right)_0\right] \\
& \qquad - \left(\nu^0_{\rm Bo}\right)_0^{-1} \left[\left(\nu^0_{\rm Bo}\right)_0^{-1} (\overline{g}^{rr}_{\rm Bo})_1 + 2\,(\overline{g}^{rr}_{\rm Bo})_{-1}\left(\nu^{\rm Bo}_0\right)_2 + 2\,(\overline{g}^{rr}_{\rm Bo})_{-2}\left(\nu^{\rm Bo}_0\right)_3\right]\Bigg)
\end{aligned} \tag{5.4}$$

Now, using that $\nu_0^{\mathrm{Bo}} = 1/\nu^0_{\mathrm{Bo}}$ (cf. (2.7)) and (4.7), we have

$$\begin{aligned}\nu_0^{\mathrm{Bo}} &= \left(\nu^0_{\mathrm{Bo}}\right)_0^{-1}\left(1-\left(\nu^0_{\mathrm{Bo}}\right)_2 r_{\mathrm{Bo}}^{-2} - \left(\nu^0_{\mathrm{Bo}}\right)_3 r_{\mathrm{Bo}}^{-3}\right) + \mathcal{O}\left(r_{\mathrm{Bo}}^{-4}\right), \\ &= \left(\nu^0_{\mathrm{Bo}}\right)_0^{-1}\left(1-\frac{1}{4}\left[|\sigma^{\mathrm{Bo}}|_4^2 + 8\pi\left(\overline{T}^{\mathrm{Bo}}_{rr}\right)_4\right]r_{\mathrm{Bo}}^{-2}\right. \\ &\qquad\left. -\frac{1}{6}\left[|\sigma^{\mathrm{Bo}}|_5^2 + 8\pi\left(\overline{T}^{\mathrm{Bo}}_{rr}\right)_5\right]r_{\mathrm{Bo}}^{-3}\right) + \mathcal{O}\left(r_{\mathrm{Bo}}^{-4}\right). \end{aligned} \tag{5.5}$$

Inserting this and the expansion coefficients of $\overline{g}^{rr}_{\mathrm{Bo}}$ and ν^A_{Bo} we calculated before M reads

$$\begin{aligned} M = &\left(\nu^A_{\mathrm{Bo}}\right)_0\left[\mathring{h}_{AB}\left(\nu^B_{\mathrm{Bo}}\right)_3 + \left(\overline{g}^{\mathrm{Bo}}_{AB}\right)_{-1}\left(\nu^B_{\mathrm{Bo}}\right)_2 + \frac{1}{2}\left(\overline{g}^{\mathrm{Bo}}_{AB}\right)_1\left(\nu^B_{\mathrm{Bo}}\right)_0\right] \\ &- \left(\nu^0_{\mathrm{Bo}}\right)_0^{-1}\frac{1}{2}\left(\mathring{\nabla}_A\left(\nu^A_{\mathrm{Bo}}\right)_2 - \frac{1}{2}\left(\left(\zeta^{\mathrm{Bo}}\right)_2 + \check{\nabla}_A\left(\nu^A_{\mathrm{Bo}}\right)_0\left[|\sigma^{\mathrm{Bo}}|_4^2 + 8\pi\left(\overline{T}^{\mathrm{Bo}}_{rr}\right)_4\right]\right)\right. \\ &\left.+\left(\nu^0_{\mathrm{Bo}}\right)_0^{-1}\frac{\Lambda}{4}\left[|\sigma^{\mathrm{Bo}}|_5^2 + 8\pi\left(\overline{T}^{\mathrm{Bo}}_{rr}\right)_5\right]\right) - \frac{1}{8}\left[|\sigma^{\mathrm{Bo}}|_4^2 + 8\pi\left(\overline{T}^{\mathrm{Bo}}_{rr}\right)_4\right]\mathring{\nabla}_A\left(\nu^A_{\mathrm{Bo}}\right)_0 \end{aligned} \tag{5.6}$$

We return, now, to the definition of the Trautman-Bondi mass, (2.21), and assume in the remainder of the present work that the boundary conditions on ν^0_{Bo} and ν^A_{Bo}, introduced in (2.24), hold. For the convenience of the reader we repeat them here:

$$\lim_{r_{\mathrm{Bo}}\to\infty}\nu^A_{\mathrm{Bo}} = 0\,, \quad \lim_{r_{\mathrm{Bo}}\to\infty}\nu^0_{\mathrm{Bo}} = 1\,.$$

With these boundary conditions and using the fact that the remaining divergence term in (5.6) will integrate out to zero we find

$$\begin{aligned} m_{\mathrm{TB}} &= \frac{1}{16\pi}\int_{\mathring{M}}\left(\zeta^{\mathrm{Bo}}\right)_2 d\mu_{\mathring{h}} + \frac{\Lambda}{12\pi}\int_{\mathring{M}}\left(\nu_0^{\mathrm{Bo}}\right)_3 d\mu_{\mathring{h}} \\ &= \frac{1}{16\pi}\int_{\mathring{M}}\left(\zeta^{\mathrm{Bo}}\right)_2 d\mu_{\mathring{h}} - \frac{\Lambda}{72\pi}\int_{\mathring{M}}\left[|\sigma^{\mathrm{Bo}}|_5^2 + 8\pi\left(\overline{T}^{\mathrm{Bo}}_{rr}\right)_5\right] d\mu_{\mathring{h}}\,. \end{aligned} \tag{5.7}$$

6 The Trautman-Bondi mass expressed through characteristic data

To continue, we want to relate the fields occurring in Bondi-type coordinates to their representation in coordinates where r is an affine parameter along the radial null outgoing geodesics of $\overline{g}$. We start with $(\zeta^{\mathrm{Bo}})_2$ and follow the argumentation in [4, leading to Equation (51) there], which we repeat here for the convenience of the reader,

$$r_{\mathrm{Bo}} = r - \frac{\tau_2}{2} + \mathcal{O}(r^{-1})\,. \tag{6.1}$$

For ζ, as given by the space-time formula (2.19), we then find the following transformation formulae:

$$\begin{aligned}
\tau^{\mathrm{Bo}}(r^{\mathrm{Bo}}) &= \frac{\partial r}{\partial r^{\mathrm{Bo}}}\tau(r(r^{\mathrm{Bo}})) = \frac{2}{r_{\mathrm{Bo}}}\,, \qquad (6.2)\\
\zeta^{\mathrm{Bo}} &= 2(\overline{g}^{\mathrm{Bo}})^{AB}(\overline{\Gamma}^{\mathrm{Bo}})^{r_{\mathrm{Bo}}}_{AB} + \tau^{\mathrm{Bo}}(\overline{g}^{\mathrm{Bo}})^{r_{\mathrm{Bo}}r_{\mathrm{Bo}}}\\
&= 2(\overline{g}^{\mathrm{Bo}})^{AB}\Big(\frac{\partial r_{\mathrm{Bo}}}{\partial x^k}\frac{\partial x^i}{\partial x^A_{\mathrm{Bo}}}\frac{\partial x^j}{\partial x^B_{\mathrm{Bo}}}\overline{\Gamma}^k_{ij} + \frac{\partial r_{\mathrm{Bo}}}{\partial r}\frac{\partial^2 r}{\partial x^A_{\mathrm{Bo}}\partial x^B_{\mathrm{Bo}}}\Big)\\
&\quad +\tau\frac{\partial r}{\partial r_{\mathrm{Bo}}}\frac{\partial r_{\mathrm{Bo}}}{\partial x^i}\frac{\partial r_{\mathrm{Bo}}}{\partial x^j}\overline{g}^{ij}\\
&= 2\overline{g}^{AB}\frac{\partial r_{\mathrm{Bo}}}{\partial r}\frac{\partial r}{\partial x^A_{\mathrm{Bo}}}\frac{\partial r}{\partial x^B_{\mathrm{Bo}}}\kappa + 2\overline{g}^{AB}\frac{\partial r_{\mathrm{Bo}}}{\partial x^C}\frac{\partial r}{\partial x^A_{\mathrm{Bo}}}\frac{\partial r}{\partial x^B_{\mathrm{Bo}}}\underbrace{\overline{\Gamma}^C_{11}}_{=0}\\
&\quad +\frac{\partial r_{\mathrm{Bo}}}{\partial r}\zeta + 2\overline{g}^{AB}\frac{\partial r_{\mathrm{Bo}}}{\partial x^C}\tilde{\Gamma}^C_{AB} + 2\underbrace{\overline{g}^{AB}\chi_{AB}}_{=\tau}\frac{\partial r_{\mathrm{Bo}}}{\partial x^C}\nu^0\nu^C\\
&\quad -2\overline{g}^{AB}\frac{\partial r_{\mathrm{Bo}}}{\partial r}\frac{\partial r}{\partial x^B_{\mathrm{Bo}}}\xi_A + 2\tau\overline{g}^{AB}\frac{\partial r_{\mathrm{Bo}}}{\partial x^A}\frac{\partial r}{\partial x^B_{\mathrm{Bo}}} + 4\overline{g}^{AB}\frac{\partial r_{\mathrm{Bo}}}{\partial x^C}\frac{\partial r}{\partial x^B_{\mathrm{Bo}}}\sigma_A{}^C\\
&\quad +2\overline{g}^{AB}\frac{\partial r_{\mathrm{Bo}}}{\partial r}\frac{\partial^2 r}{\partial x^A_{\mathrm{Bo}}\partial x^B_{\mathrm{Bo}}} + \tau\frac{\partial r}{\partial r_{\mathrm{Bo}}}\frac{\partial r_{\mathrm{Bo}}}{\partial x^A}\frac{\partial r_{\mathrm{Bo}}}{\partial x^B}\overline{g}^{AB}\\
&= \frac{\partial r_{\mathrm{Bo}}}{\partial r}\zeta + 2\frac{\partial r_{\mathrm{Bo}}}{\partial r}\Delta_{\breve{g}}r + O(r_{\mathrm{Bo}}^{-3})\,, \qquad (6.3)
\end{aligned}$$

where $\Delta_{\breve{g}}$ is the Laplace operator of the two-dimensional metric $\breve{g}_{AB}dx^A_{\mathrm{Bo}}dx^B_{\mathrm{Bo}}$.

To continue we need the asymptotic expansion of ζ and therefore solve the respective constraint equation (2.15). Note that we have already done this in Bondi-type coordinates, but we also need the result in affine coordinates.

We begin with the same procedure as in Section 4.5 and define

$$\zeta := \zeta_{\Lambda=0} + \delta\zeta\,, \qquad (6.4)$$

where $\zeta_{\Lambda=0}$ is the solution of (2.15) in the case $\Lambda = 0$. Its asymptotic expansion is known and reads [7, Equation (3.40)]

$$\zeta_{\Lambda=0} = -\frac{\mathring{R}}{r} + (\zeta_{\Lambda=0})_2\, r^{-2} + \mathrm{o}(r^{-2})\,, \qquad (6.5)$$

with $(\zeta_{\Lambda=0})_2$ being a global integration function. We assume that the relevant fields satisfy analog fall-off behavior in affine coordinate r as we assumed in Bondi-type coordinate r_{Bo} (cf. Equations (4.1) and (4.2)). The equation for $\delta\zeta$ reads

$$(\partial_r + \tau + \kappa)\,\delta\zeta = 2\Lambda\,. \qquad (6.6)$$

From now on we choose the coordinate r so that $\kappa = 0$. We start by solving the Raychaudhuri equation (2.11) in this gauge and obtain the expansion of τ

$$\begin{aligned}
\tau &= \frac{2}{r} + \frac{\tau_2}{r^2} + \frac{\tau_3}{r^3} + \frac{\tau_4}{r^4} + \mathcal{O}(r^{-5})\\
&= \frac{2}{r} + \frac{\tau_2}{r^2} + \frac{2\left[|\sigma|_4^2 + 8\pi\left(\overline{T}_{rr}\right)_4\right] + \tau_2^2}{2r^3}\\
&\quad + \frac{2\left[|\sigma|_5^2 + 8\pi\left(\overline{T}_{rr}\right)_5\right] + 2\tau_2\left[|\sigma|_4^2 + 8\pi\left(\overline{T}_{rr}\right)_4\right] + \tau_2^3}{4r^4} + \mathcal{O}(r^{-5})\,, \qquad (6.7)
\end{aligned}$$

where τ_2 is a global integration function and $|\sigma|^2_n$ are the expansion coefficients of $|\sigma|^2$

$$|\sigma|^2 = \frac{|\sigma|^2_4}{r^4} + \frac{|\sigma|^2_5}{r^5} + \mathcal{O}\big(r^{-6}\big)\,. \tag{6.8}$$

Using (6.7) we find from (6.6)

$$\begin{aligned}\delta\zeta = {}& \Lambda\left(\frac{2r}{3} - \frac{\tau_2}{3} + \frac{\tau_2^2 - 2\tau_3}{3r} + \frac{3\tau_2\tau_3 - \tau_2^3 - 2\tau_4}{3}\frac{\log r}{r^2}\right)\\ &+ \frac{\delta\zeta_2}{r^2} + \mathrm{o}\big(r^{-2}\big)\,,\end{aligned} \tag{6.9}$$

where $\delta\zeta_2$ is again a global integration function. Summing up and combining the two integration functions $(\zeta_{\Lambda=0})_2$ and $\delta\zeta_2$ into ζ_2 the solution of (2.15) reads

$$\begin{aligned}\zeta = {}& \frac{2\Lambda}{3}r - \frac{\Lambda\tau_2}{3} - \left(\mathring{R} + \frac{\Lambda\big(2\tau_3 - \tau_2^2\big)}{3}\right) r^{-1}\\ &+ \frac{\Lambda\left(3\tau_2\tau_3 - \tau_2^3 - 2\tau_4\right)}{3}\frac{\log r}{r^2} + \frac{\zeta_2}{r^2} + \mathrm{o}\big(r^{-2}\big)\,.\end{aligned} \tag{6.10}$$

Using this and the asymptotic expansion of $\Delta_{\breve{g}} r$ (compare [4, Equation (51)])

$$\Delta_{\breve{g}} r = \frac{\Delta_{\mathring{h}}\tau_2}{2r^2} + \mathcal{O}\big(r^{-3}\big) \tag{6.11}$$

and expressing (6.3) in terms of r_{Bo} one obtains

$$\begin{aligned}(\zeta^{\mathrm{Bo}})_2 = {}& \zeta_2 + \frac{\mathring{R}}{2}\tau_2 + \Delta_{\mathring{h}}\tau_2\\ &+ \frac{\Lambda}{3}\left(-\frac{|\sigma^{\mathrm{Bo}}|^2_5 + 8\pi\left(\overline{T}^{\mathrm{Bo}}_{rr}\right)_5}{3} + \tau_2\Big[|\sigma|^2_4 + 8\pi\big(\overline{T}_{rr}\big)_4\Big]\right)\,,\end{aligned} \tag{6.12}$$

$$\zeta^{\mathrm{Bo}}_{\log,2} = \frac{\Lambda}{3}\Big(2\tau_2\Big[|\sigma|^2_4 + 8\pi\big(\overline{T}_{rr}\big)_4\Big] - |\sigma|^2_5\Big)\,. \tag{6.13}$$

Inserting (6.12) into (5.6) and using the boundary conditions on ν^0_{Bo} and ν^A_{Bo}, introduced in (2.24), we find

$$\begin{aligned}M = {}& \frac{1}{4}\left(\zeta_2 + \frac{\mathring{R}}{2}\tau_2 + \Delta_{\mathring{h}}\tau_2\right) - \frac{1}{2}\mathring{\nabla}^A\left(\nu^{\mathrm{Bo}}_A\right)_0\\ &- \frac{\Lambda}{12}\left(|\sigma^{\mathrm{Bo}}|^2_5 + 8\pi\left(\overline{T}^{\mathrm{Bo}}_{rr}\right)_5 - \tau_2\Big[|\sigma|^2_4 + 8\pi\big(\overline{T}_{rr}\big)_4\Big]\right)\,.\end{aligned} \tag{6.14}$$

We now calculate the expansions of $|\sigma|^2$ and $|\sigma^{\mathrm{Bo}}|^2$ to insert explicit expressions for the coefficients occurring in M. For $|\sigma|^2$ one obviously has

$$\begin{aligned}|\sigma|^2 &= \frac{|\sigma|^2_4}{r^4} + \frac{|\sigma|^2_5}{r^5} + \mathcal{O}\big(r^{-6}\big)\\ &= \frac{\big(\sigma_A{}^B\big)_2\big(\sigma_B{}^A\big)_2}{r^4} + 2\frac{\big(\sigma_A{}^B\big)_2\big(\sigma_B{}^A\big)_3}{r^5} + \mathcal{O}\big(r^{-6}\big)\,,\end{aligned} \tag{6.15}$$

and performing a coordinate transformation and replacing the dependence on r with r_{Bo} we obtain

$$
\begin{aligned}
|\sigma^{\mathrm{Bo}}|^2 = & \frac{\left(\sigma_A{}^B\right)_2\left(\sigma_B{}^A\right)_2}{r_{\mathrm{Bo}}^4} + 2\frac{\left(\sigma_A{}^B\right)_2\left(\sigma_B{}^A\right)_3 - \left(\sigma_A{}^B\right)_2\left(\sigma_B{}^A\right)_2 \tau_2}{r_{\mathrm{Bo}}^5} \\
& + \mathcal{O}\left(r_{\mathrm{Bo}}^{-6}\right) \\
= & \frac{|\sigma|_4^2}{r_{\mathrm{Bo}}^4} + \frac{|\sigma|_5^2 - 2|\sigma|_4^2\tau_2}{r_{\mathrm{Bo}}^5} + \mathcal{O}\left(r_{\mathrm{Bo}}^{-6}\right). \qquad (6.16)
\end{aligned}
$$

Therefore we end up with the following formula for the mass aspect expressed through characteristic data

$$
\begin{aligned}
M = & \frac{1}{4}\left(\zeta_2 + \frac{\mathring{R}}{2}\tau_2 + \Delta_{\mathring{h}}\tau_2\right) - \frac{1}{2}\mathring{\nabla}^A\left(\nu_A^{\mathrm{Bo}}\right)_0 \\
& + \frac{\Lambda}{12}\left(\tau_2\left[3|\sigma|_4^2 + 8\pi\left(\overline{T}_{rr}\right)_4\right] - |\sigma|_5^2 - 8\pi\left(\overline{T}_{rr}^{\mathrm{Bo}}\right)_5\right). \qquad (6.17)
\end{aligned}
$$

Using again the definition of the Trautman-Bondi mass and bearing in mind that the divergence terms will vanish when integrating over $\mathring{M}$ we find

$$
\begin{aligned}
m_{\mathrm{TB}} = & \frac{1}{16\pi}\int_{\mathring{M}}\left(\zeta_2 + \frac{\mathring{R}}{2}\tau_2\right) d\mu_{\mathring{h}} \\
& + \frac{\Lambda}{48\pi}\int_{\mathring{M}}\left(\tau_2\left[3|\sigma|_4^2 + 8\pi\left(\overline{T}_{rr}\right)_4\right] - |\sigma|_5^2 - 8\pi\left(\overline{T}_{rr}^{\mathrm{Bo}}\right)_5\right) d\mu_{\mathring{h}}. \qquad (6.18)
\end{aligned}
$$

7 The Trautman-Bondi mass expressed in explicit geometric terms

We are ready to prove our final formula for the Trautman-Bondi mass, which will be in terms of geometric fields defined on a characteristic surface parametrized by an affine parameter r ranging from r_0 to infinity. In the case of a light-cone we take $r_0 = 0$, but we allow non-zero r_0 to cover other situations of interest.

We first note the asymptotic expansion of $\sqrt{\det \overline{g}_{AB}}$ for large r, which is obtained by using the considerations in [7, leading to equation (3.13) there] and our result for the expansion of τ, (6.7):

$$
\begin{aligned}
\sqrt{\det \overline{g}_{AB}} = r^2\sqrt{\det \mathring{h}_{AB}}\Bigg(1 - \frac{\tau_2}{r} + & \frac{\tau_2^2 - 2\left[|\sigma|_4^2 + 8\pi\left(\overline{T}_{rr}\right)_4\right]}{4r^2} \\
& + \frac{2\tau_2\left[|\sigma|_4^2 + 8\pi\left(\overline{T}_{rr}\right)_4\right] - \left[|\sigma|_5^2 + 8\pi\left(\overline{T}_{rr}\right)_5\right]}{6r^3} \\
& + \mathcal{O}(r^{-4})\Bigg). \qquad (7.1)
\end{aligned}
$$

Using this, $d\mu_{\breve g} = \sqrt{\det \overline{g}_{AB}}dx^2dx^3$ and the expansion (6.10) of ζ we find

$$\begin{aligned}
\int_{\mathring{M}} \zeta d\mu_{\breve g} = {}& \frac{2\Lambda}{3}r^3 \underbrace{\int_{\mathring{M}} d\mu_{\mathring h}}_{=:\mu_{\mathring h}(\mathring{M})} - \Lambda r^2 \int_{\mathring{M}} \tau_2 d\mu_{\mathring h} - r\int_{\mathring{M}} \mathring{R} d\mu_{\mathring h} \\
& - \Lambda r \int_{\mathring{M}} \left(\Big[|\sigma|_4^2 + 8\pi \big(\overline{T}_{rr}\big)_4 \Big] - \frac{1}{2}\tau_2^2 \right) d\mu_{\mathring h} \\
& - \frac{\Lambda}{3}\log r \int_{\mathring{M}} \left(\Big[|\sigma|_5^2 + 8\pi \big(\overline{T}_{rr}\big)_5 \Big] - 2\Big[|\sigma|_4^2 + 8\pi \big(\overline{T}_{rr}\big)_4 \Big]\tau_2 \right) d\mu_{\mathring h} \\
& + \int_{\mathring{M}} \Big[\zeta_2 + \mathring{R}\tau_2 \Big] d\mu_{\mathring h} - \frac{\Lambda}{12}\int_{\mathring{M}} \tau_2^3 d\mu_{\mathring h} \\
& + \frac{\Lambda}{18}\int_{\mathring{M}} \left(19\tau_2 \Big[|\sigma|_4^2 + 8\pi \big(\overline{T}_{rr}\big)_4 \Big] - 2\Big[|\sigma|_5^2 + 8\pi \big(\overline{T}_{rr}\big)_5 \Big] \right) d\mu_{\mathring h} + \mathrm{o}(1)\,.
\end{aligned} \tag{7.2}$$

From (2.15) with $\kappa = 0$ and the Gauss–Bonnet theorem we have,

$$\int_{\mathring{M}} (\partial_r + \tau)\zeta d\mu_{\breve g} = -4\pi\chi(\mathring{M}) + \int_{\mathring{M}} \left(\frac{1}{2}|\xi|^2 + S \right) d\mu_{\breve g} + 2\Lambda \int_{\mathring{M}} d\mu_{\breve g}\,, \tag{7.3}$$

where $\chi(\mathring{M})$ is the Euler characteristic of $\mathring{M}$. The integral in the last term of that equation is the area of the constant-r sections of $\mathscr{N}$, and we define the volume function $V(r)$ to be its integral

$$V(r) := \int_{\tilde r = r_0}^{r} \frac{dV(\tilde r)}{d\tilde r} d\tilde r := \int_{\tilde r = r_0}^{r} \int_{\mathring{M}} d\mu_{\breve g} d\tilde r\,. \tag{7.4}$$

With that and using $\partial_r \sqrt{\det \overline{g}_{AB}} = \tau \sqrt{\det \overline{g}_{AB}}$ we find

$$\partial_r \int_{\mathring{M}} \zeta d\mu_{\breve g} = -4\pi\chi(\mathring{M}) + 2\Lambda \frac{dV(r)}{dr} + \int_{\mathring{M}} \left(\frac{1}{2}|\xi|^2 + S \right) d\mu_{\breve g}\,, \tag{7.5}$$

which we can integrate in r starting from $r = r_0$

$$\begin{aligned}
&\lim_{r\to\infty} \left(\int_{\mathring{M}} \zeta d\mu_{\breve g} + 4\pi\chi(\mathring{M})r - 2\Lambda V(r) \right) \\
&\quad = \lim_{r\to r_0} \int_{\mathring{M}} \zeta d\mu_{\breve g} + 4\pi\chi(\mathring{M})r_0 + \int_{r=r_0}^{\infty} \int_{\mathring{M}} \left(\frac{1}{2}|\xi|^2 + S \right) d\mu_{\breve g} dr\,.
\end{aligned} \tag{7.6}$$

We leave the symbol $\lim_{r\to r_0}$ in the last equation to accommodate a vertex at $r = r_0$, where ζ is singular, but note that light-surfaces emanating from smooth space co-dimension-two submanifolds will also be of interest to us. One needs to make sure to use appropriate boundary conditions for the lower bound of the integration depending on what kind of characteristic surface is studied. In the case of a light-cone, i.e. $r_0 = 0$, for example the necessary boundary conditions follow from regularity at the tip of the cone as has been discussed in [2, Section 4.5].

When the first term in the last line vanishes, we can infer non-negativity of the left-hand side by assuming the *dominant energy condition* for non-vanishing matter fields. This condition implies then [8]

$$S := 8\pi \overline{g}^{AB}\overline{T}_{AB} - \overline{T} \geq 0 \tag{7.7}$$

which means that the right-hand side of (7.6) is manifestly non-negative. Furthermore it is reasonable to assume that it does not diverge from which we can deduce, keeping (7.6) in mind, that the divergent terms in $2\Lambda V(r)$ and $4\pi\chi(\mathring{M})r$ need to cancel those in the expression on the right-hand side of (7.2) exactly. To make this precise we continue by calculating an explicit expression for the volume function $V(r)$. We start by using again (7.1) and find

$$\begin{aligned}\frac{dV(r)}{dr} = \int_{\mathring{M}} d\mu_{\breve{g}} &= r^2\mu_{\mathring{h}}(\mathring{M}) - r\int_{\mathring{M}} \tau_2 d\mu_{\mathring{h}} \\ &+ \frac{1}{2}\int_{\mathring{M}}\left(\frac{1}{2}\tau_2^2 - \left[|\sigma|_4^2 + 8\pi\big(\overline{T}_{rr}\big)_4\right]\right) d\mu_{\mathring{h}} \\ &+ \frac{1}{6r}\int_{\mathring{M}}\left(2\left[|\sigma|_4^2 + 8\pi\big(\overline{T}_{rr}\big)_4\right]\tau_2 - \left[|\sigma|_5^2 + 8\pi\big(\overline{T}_{rr}\big)_5\right]\right) d\mu_{\mathring{h}} \\ &+ \mathcal{O}\big(r^{-2}\big)\,. \end{aligned} \tag{7.8}$$

It follows that there exist constants so that the function $V(r)$ has an asymptotic expansion of the form

$$V(r) = \frac{1}{3}r^3\mu_{\mathring{h}}(\mathring{M}) + V_{-2}r^2 + V_{-1}r + V_{\log}\log r + V_0 + V_1 r^{-1} + \mathcal{O}\big(r^{-1}\big)\,.$$

We define the *renormalized volume* V_{ren} as "the finite left-over in the expansion":

$$V_{\text{ren}} := V_0\,.$$

One can think of V_{ren} as the global integration function arising from integrating the equation for dV/dr.

Integrating (7.8) we obtain in fact

$$\begin{aligned}-2\Lambda V(r) = \Lambda\Bigg[&-\frac{2}{3}r^3\mu_{\mathring{h}}(\mathring{M}) + r^2\int_{\mathring{M}}\tau_2 d\mu_{\mathring{h}} \\ &+ r\int_{\mathring{M}}\left(\left[|\sigma|_4^2 + 8\pi\big(\overline{T}_{rr}\big)_4\right] - \frac{1}{2}\tau_2^2\right) d\mu_{\mathring{h}} \\ &+ \frac{1}{3}\log r\int_{\mathring{M}}\left(\left[|\sigma|_5^2 + 8\pi\big(\overline{T}_{rr}\big)_4\right] - 2\left[|\sigma|_4^2 + 8\pi\big(\overline{T}_{rr}\big)_4\right]\tau_2\right) d\mu_{\mathring{h}}\Bigg] \\ &- 2\Lambda V_{\text{ren}} + \mathcal{O}\big(r^{-1}\big) \\ = \Lambda\Bigg[&-\frac{2}{3}\left(\left(r - \frac{\tau_2}{2}\right)^3 + \left(\frac{\tau_2}{2}\right)^3\right)\mu_{\mathring{h}}(\mathring{M}) - 2V_{\text{ren}} \\ &+ r\int_{\mathring{M}}\left(\left[|\sigma|_4^2 + 8\pi\big(\overline{T}_{rr}\big)_4\right]\right) d\mu_{\mathring{h}} \\ &+ \frac{1}{3}\log r\int_{\mathring{M}}\left(\left[|\sigma|_5^2 + 8\pi\big(\overline{T}_{rr}\big)_4\right] - 2\left[|\sigma|_4^2 + 8\pi\big(\overline{T}_{rr}\big)_4\right]\tau_2\right) d\mu_{\mathring{h}}\Bigg] \\ &+ \mathcal{O}\big(r^{-1}\big)\,, \end{aligned} \tag{7.9}$$

thus

$$V_{\rm ren} := \lim_{r\to\infty}\Bigg[V(r) - \frac{r^3}{3}\mu_{\mathring{h}}(\mathring{M}) + \frac{r^2}{2}\int_{\mathring{M}}\tau_2 d\mu_{\mathring{h}} + \frac{r}{2}\int_{\mathring{M}}\left(\Big[|\sigma|_4^2 + 8\pi\big(\overline{T}_{rr}\big)_4\Big] - \frac{1}{2}\tau_2^2\right)d\mu_{\mathring{h}} + \frac{1}{6}\log r\int_{\mathring{M}}\left(\Big[|\sigma|_5^2 + 8\pi\big(\overline{T}_{rr}\big)_5\Big] - 2\Big[|\sigma|_4^2 + 8\pi\big(\overline{T}_{rr}\big)_4\Big]\tau_2\right)d\mu_{\mathring{h}}\Bigg]. \tag{7.10}$$

Now, by (7.6) and using (7.2) and (7.9),

$$\begin{aligned}\lim_{r\to\infty}\left(\int_{\mathring{M}}\zeta d\mu_{\breve{g}} + 4\pi\chi(\mathring{M})r - 2\Lambda V(r)\right) = \\ = \lim_{r\to\infty}\left(-r\int_{\mathring{M}}\mathring{R}d\mu_{\mathring{h}} + 4\pi\chi(\mathring{M})r\right) - \frac{\Lambda}{12}\int_{\mathring{M}}\tau_2^3 d\mu_{\mathring{h}} \\ +\frac{\Lambda}{18}\int_{\mathring{M}}\left(19\tau_2\Big[|\sigma|_4^2 + 8\pi\big(\overline{T}_{rr}\big)_4\Big] - 2\Big[|\sigma|_5^2 + 8\pi\big(\overline{T}_{rr}\big)_5\Big]\right)d\mu_{\mathring{h}} \\ +\int_{\mathring{M}}\left(\zeta_2 + \mathring{R}\tau_2\right)d\mu_{\mathring{h}} - 2\Lambda V_{\rm ren} \\ = \lim_{r\to r_0}\int_{\mathring{M}}\zeta d\mu_{\breve{g}} + 4\pi\chi(\mathring{M})r_0 + \int_{r=r_0}^{\infty}\int_{\mathring{M}}\left(\frac{1}{2}|\xi|^2 + S\right)d\mu_{\breve{g}}dr\,.\end{aligned} \tag{7.11}$$

Next we rewrite (6.18) as

$$\begin{aligned}16\pi m_{\rm TB} = \int_{\mathring{M}}\left(\zeta_2 + \mathring{R}\tau_2\right)d\mu_{\mathring{h}} - \int_{\mathring{M}}\frac{\mathring{R}}{2}\tau_2 d\mu_{\mathring{h}} \\ +\frac{\Lambda}{3}\int_{\mathring{M}}\left(\tau_2\Big[3|\sigma|_4^2 + 8\pi\big(\overline{T}_{rr}\big)_4\Big] - |\sigma|_5^2 - 8\pi\left(\overline{T}^{\rm Bo}_{rr}\right)_5\right)d\mu_{\mathring{h}}\,,\end{aligned} \tag{7.12}$$

and find, by (7.11) and (7.12), using $\int_{\mathring{M}}\mathring{R}d\mu_{\mathring{h}} = 4\pi\chi(\mathring{M})$,

$$\begin{aligned}16\pi m_{\rm TB} = \lim_{r\to r_0}\int_{\mathring{M}}\zeta d\mu_{\breve{g}} + 4\pi\chi(\mathring{M})r_0 + \int_{r=r_0}^{\infty}\int_{\mathring{M}}\left(\frac{1}{2}|\xi|^2 + S\right)d\mu_{\breve{g}}dr \\ -\int_{\mathring{M}}\frac{\mathring{R}}{2}\tau_2 d\mu_{\mathring{h}} + 2\Lambda V_{\rm ren} + \frac{\Lambda}{12}\int_{\mathring{M}}\tau_2^3 d\mu_{\mathring{h}} \\ -\frac{\Lambda}{18}\int_{\mathring{M}}\left(19\tau_2\Big[|\sigma|_4^2 + 8\pi\big(\overline{T}_{rr}\big)_4\Big] - 2\Big[|\sigma|_5^2 + 8\pi\big(\overline{T}_{rr}\big)_5\Big]\right)d\mu_{\mathring{h}} \\ +\frac{\Lambda}{3}\int_{\mathring{M}}\left(\tau_2\Big[3|\sigma|_4^2 + 8\pi\big(\overline{T}_{rr}\big)_4\Big] - |\sigma|_5^2 - 8\pi\left(\overline{T}^{\rm Bo}_{rr}\right)_5\right)d\mu_{\mathring{h}}\,.\end{aligned} \tag{7.13}$$

$$\begin{aligned}= \lim_{r\to r_0}\int_{\mathring{M}}\zeta d\mu_{\breve{g}} + 4\pi\chi(\mathring{M})r_0 + \int_{r=r_0}^{\infty}\int_{\mathring{M}}\left(\frac{1}{2}|\xi|^2 + S\right)d\mu_{\breve{g}}dr \\ -\int_{\mathring{M}}\frac{\mathring{R}}{2}\tau_2 d\mu_{\mathring{h}} + 2\Lambda V_{\rm ren} + \frac{\Lambda}{12}\int_{\mathring{M}}\tau_2^3 d\mu_{\mathring{h}} \\ -\frac{\Lambda}{18}\int_{\mathring{M}}\Bigg(\tau_2\Big[|\sigma|_4^2 + 104\pi\big(\overline{T}_{rr}\big)_4\Big] + 2\Big[2|\sigma|_5^2 - 8\pi\big(\overline{T}_{rr}\big)_5\Big] \\ +48\pi\left(\overline{T}^{\rm Bo}_{rr}\right)_5\Bigg)d\mu_{\mathring{h}}\,.\end{aligned} \tag{7.14}$$

We continue with a generalisation of the arguments leading to equation (43) in [4]. Indeed, we allow the case $r_0 \neq 0$. Next, for further reference, we allow an asymptotic behaviour for small r for light-cones emanating from a submanifold of general space co-dimension m, and not only a light-cone. Finally the following calculations, up to the resulting expansion of τ, (7.23), are performed for arbitrary space-time dimensions $n+1 \geq 3$.

First we note (recall (6.7)),

$$\tau = \begin{cases} \dfrac{n-1}{r} + \dfrac{\tau_2}{r^2} + \mathcal{O}\big(r^{-3}\big)\,, & \text{for large } r\ ; \quad (7.15a) \\ \dfrac{m-1}{r} + \mathcal{O}(1)\,, & \text{for small } r\ . \quad (7.15b) \end{cases}$$

Here the behaviour for small r is the one which occurs when the set $\{r=0\}$ has space co-dimension m (e.g., $m=n$ for a light-cone emanating from a point). If $r_0 > 0$ we assume that τ is smooth up-to-boundary when $r = r_0$ is approached. Next, let

$$\tau_1 := \frac{n-1}{r}\,. \tag{7.16}$$

(This is the value of τ for a light-cone in Minkowski space-time.) Let $\delta\tau := \tau - \tau_1$ denote the deviation of τ from the Minkowskian value, then

$$\delta\tau = \begin{cases} \dfrac{\tau_2}{r^2} + \mathcal{O}\big(r^{-3}\big)\,, & \text{for large } r\ ; \quad (7.17a) \\ \dfrac{m-n}{r} + \mathcal{O}(1)\,, & \text{for small } r\ , \quad (7.17b) \end{cases}$$

From the Raychaudhuri equation (2.11) with $\kappa = 0$ one finds that $\delta\tau$ satisfies the equation

$$\frac{d\delta\tau}{dr} + \delta\tau\left(\frac{\delta\tau}{n-1} + \frac{2}{r}\right) = -|\sigma|^2 - 8\pi\overline{T}_{rr}\,. \tag{7.18}$$

Defining now $\Phi := r^2\Psi$, (7.18) is equivalent to

$$\frac{1}{\Phi}\frac{d\,(\Phi\,\delta\tau)}{dr} = -|\sigma|^2 - 8\pi\overline{T}_{rr}\,, \tag{7.19}$$

if and only if

$$\frac{1}{\Phi}\frac{d\Phi}{dr} = \frac{\delta\tau}{n-1} + \frac{2}{r}\,. \tag{7.20}$$

This gives, by (7.17), the following three equivalent expressions for the function Ψ:

$$\Psi(r,x^A) = \begin{cases} \exp\left(-\displaystyle\int_r^\infty \frac{\delta\tau}{n-1}(s,x^A)ds + C_1(x^A)\right)\ ; & (7.21a) \\ \exp\left(\displaystyle\int_{r_0}^r \frac{\delta\tau}{n-1}(s,x^A)ds + C_2(x^A)\right)\ ; & (7.21b) \\ r^{\frac{m-n}{n-1}}\exp\left(\dfrac{1}{n-1}\displaystyle\int_0^r \left[\delta\tau(s,x^A) - \frac{m-n}{s}\right]ds + C_3(x^A)\right)\,, & (7.21c) \end{cases}$$

for some functions $C_i(x^A)$, where in the middle term we have assumed that $r_0 > 0$, and with the last equality holding when $\delta\tau(r,x^A) \sim (m-n)r^{-1}$ for small r, as assumed in (7.17b). (Both Φ and Ψ are auxiliary functions which are only needed to derive (7.23) below, and there is some freedom in their definition. In particular either of the functions $C_i(x^A)$, $i = 1, 2, 3$, can be chosen to be zero if convenient for a specific problem at hand, and we note that the $C_i(x^A)$'s cancel out in the final expression for τ in any case.) We stress that in the special case of a light-cone — as it has been the case in [4] — we have $m = n$ and treating the case for small r separately is not necessary. In this case (7.21b) coincides with (7.21c).

Thus, using $\delta\tau = \tau - \tau_1$ and (7.16),

$$\frac{\Psi(r_0, x^A)}{\Psi(r, x^A)} = \begin{cases} \exp\left(-\frac{1}{n-1}\int_{r_0}^{r}\left(\tau(s, x^A) - \frac{n-1}{s}\right)ds\right) \;; & \text{(7.22a)} \\ \left(\frac{r_0}{r}\right)^{\frac{m-n}{n-1}} \exp\left(\frac{1}{n-1}\int_{r}^{r_0}\left(\tau(s, x^A) - \frac{m-n}{s}\right)ds\right) \;, & \text{(7.22b)} \end{cases}$$

with the last equality holding when $\delta\tau(r, x^A) \sim (m - n)r^{-1}$ for small r. Note that both expressions for Ψ, (7.21a) and (7.21b), lead to (7.22a).

Integrating (7.19) and using (7.22a), without denoting the dependence on coordinates x^A explicitly in what follows,

$$\begin{aligned} \tau = \frac{n-1}{r} - r^{-2}\Bigg[&\Psi(r)^{-1}\int_{\tilde r = r_0}^{r}\left(|\sigma(\tilde r)|^2 + 8\pi \overline{T}_{rr}(\tilde r)\right)\Psi(\tilde r)\tilde r^2 d\tilde r \\ &- \frac{\Psi(r_0)}{\Psi(r)}\left(\tau(r_0) - \frac{n-1}{r_0}\right)r_0^2\Bigg] \,, \end{aligned} \tag{7.23}$$

and we can directly read off the expression for τ_2 in space-time dimension $n+1 = 4$, which is the case of interest in the present work and which we assume from now on:

$$\begin{aligned} \tau_2 = &- \lim_{r\to\infty}\left[\Psi(r)^{-1}\int_{\tilde r = r_0}^{r}\left(|\sigma(\tilde r)|^2 + 8\pi \overline{T}_{rr}(\tilde r)\right)\Psi(\tilde r)\tilde r^2 d\tilde r\right] \\ &- \left[\lim_{r\to\infty}\Psi(r)^{-1}\right] \times \lim_{r\to r_0}\left[\Psi(r)\left(\frac{2}{r} - \tau\right)r^2\right] . \end{aligned} \tag{7.24}$$

Returning now to (7.14), inserting the result for τ_2 we just found, where we use (7.21b) with $C_2(x^A) = 0$ for all occurrences of $\Psi(r)$, and using further

$$d\mu_{\check g} = \mathrm{e}^{-\int_r^\infty \frac{\tilde r \tau - 2}{\tilde r} d\tilde r} r^2 d\mu_{\mathring h} \tag{7.25}$$

we obtain the following formula for the Trautman-Bondi mass m_{TB} of a null hypersurface $\mathscr{N} = [r_0, \infty) \times \mathring{M}$:

$$
\begin{aligned}
m_{\rm TB} \;=\; & \frac{1}{16\pi}\int_{r=r_0}^{\infty}\int_{\mathring{M}}\Bigg(\frac{1}{2}|\xi|^2 + S \\
& \qquad + \left[\frac{\mathring{R}}{2} + \frac{\Lambda}{18}\Big(|\sigma|^2_4 + 104\pi\big(\overline{T}_{rr}\big)_4\Big)\right]\Big(|\sigma|^2 + 8\pi\overline{T}_{rr}\Big)e^{\int_r^\infty \frac{\tilde r\tau - 2}{2\tilde r}d\tilde r}\Bigg)d\mu_{\breve g}dr \\
& + \frac{1}{16\pi}\Bigg[4\pi\chi(\mathring{M})r_0 \\
& + \lim_{r\to r_0}\Bigg(\int_{\mathring{M}}\Bigg[\zeta + \Bigg(\frac{\mathring{R}}{2} + \frac{\Lambda}{18}\Big(|\sigma|^2_4 + 104\pi\big(\overline{T}_{rr}\big)_4\Big)\Bigg)\left(\frac{2}{r} - \tau\right)e^{\int_{r_0}^\infty \frac{r\tau-2}{2r}dr}\Bigg]d\mu_{\breve g}\Bigg)\Bigg] \\
& + \frac{\Lambda}{192\pi}\int_{\mathring{M}}\tau_2^3 d\mu_{\mathring{h}} \\
& + \frac{\Lambda V_{\rm ren}}{8\pi} - \frac{\Lambda}{48\pi}\int_{\mathring{M}}\left(\frac{1}{3}\Big[2|\sigma|^2_5 - 8\pi\left(\overline{T}_{rr}\right)_5\Big] + 8\pi\left(\overline{T}^{\rm Bo}_{rr}\right)_5\right)d\mu_{\mathring{h}}\,.
\end{aligned}
\tag{7.26}
$$

To obtain this equation, it is irrelevant which form of Ψ in (7.21) we take, provided that the same formula is consistently used throughout. For example, if $\Psi(r)$ is given by (7.21a) with $C_1(x^A) = 0$, then $\lim_{r\to\infty}\Psi(r)$ equals one, independently of whether $r_0 = 0$ (so that the null hypersurface is singular at r_0) or $r_0 \neq 0$ (in which case the set $\{r = r_0\}$ has space co-dimension one).

Now, in the special case of a light-cone, where $m = n = 3$, $\mathring{R} = 2$ and $r_0 = 0$,[7] (7.26) simplifies to

$$
\begin{aligned}
m_{\rm TB} \;=\; & \frac{1}{16\pi}\int_0^{\infty}\int_{\mathring{M}}\Bigg(\frac{1}{2}|\xi|^2 + S \\
& \qquad + \left[1 + \frac{\Lambda}{18}\Big(|\sigma|^2_4 + 104\pi\big(\overline{T}_{rr}\big)_4\Big)\right]\Big(|\sigma|^2 + 8\pi\overline{T}_{rr}\Big)e^{\int_r^\infty \frac{\tilde r\tau - 2}{2\tilde r}d\tilde r}\Bigg)d\mu_{\breve g}dr \\
& + \frac{\Lambda}{192\pi}\int_{\mathring{M}}\tau_2^3 d\mu_{\mathring{h}} + \frac{\Lambda V_{\rm ren}}{8\pi} \\
& - \frac{\Lambda}{48\pi}\int_{\mathring{M}}\left(\frac{1}{3}\Big[2|\sigma|^2_5 - 8\pi\left(\overline{T}_{rr}\right)_5\Big] + 8\pi\left(\overline{T}^{\rm Bo}_{rr}\right)_5\right)d\mu_{\mathring{h}}\,.
\end{aligned}
\tag{7.27}
$$

Assuming further a conformally smooth compactification and vacuum, we obtain the striking identity when $\Lambda \le 0$:

$$
\begin{aligned}
m_{\rm TB} + \frac{|\Lambda| V_{\rm ren}}{8\pi} \;=\; & \frac{1}{16\pi}\int_0^{\infty}\int_{\mathring{M}}\left(\frac{1}{2}|\xi|^2 + |\sigma|^2 e^{\int_r^\infty \frac{\tilde r\tau - 2}{2\tilde r}d\tilde r}\right)d\mu_{\breve g}dr \\
& - \frac{|\Lambda|}{192\pi}\int_{\mathring{M}}\tau_2^3 d\mu_{\mathring{h}}\,.
\end{aligned}
\tag{7.28}
$$

Note that all the terms on the right-hand side are positive for $\Lambda < 0$ when the dominant energy condition holds, since τ_2 is then negative.

[7] Recall that $\mathring{R} = 2$ when $\mathring{M}$ is a two-sphere, $\mathring{R} = 0$ for a torus, and $\mathring{R} < 0$ for higher genus topologies of $\mathscr{I} \approx \mathbb{R} \times \mathring{M}$. In the case of a smooth-light cone the cross-sections are spherical for small r, whence everywhere, so $\mathring{R} = 2$.

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Abstract

We derive a formula for the Trautman-Bondi mass of smooth, complete null hypersurfaces in *asymptotically anti-de Sitter space-times*. To do so we extend the analysis in [4], where P. T. Chruściel and T.-T. Paetz obtained a manifestly positive formula for the Trautman-Bondi mass of globally smooth, null-geodesically complete light-cones in asymptotically Minkowskian space-times.
After introducing definitions and known results and techniques we analyze the influence of matter fields on the asymptotic expansions of the metric in Bondi-type coordinates. The aim of this discussion is to determine the decay rate of the energy-momentum tensor which is compatible with finite total mass. Following the wave-map gauge constraint equations on a null hypersurface are solved in the spirit of [7]. Here one solves a system of ordinary differential equations in terms of polyhomogeneous expansions of the solution at infinity. After some technical discussions these expansions finally lead to our resulting formula.

Zusammenfassung

In dieser Arbeit leiten wir eine Formel für die Trautman-Bondi Masse von glatten, vollständigen, nullartigen Hyperflächen in Raumzeiten ab, welche dasselbe asymptotische Verhalten aufweisen wie anti-de Sitter Raumzeiten. Um dies zu bewerkstelligen, erweitern wir die Analyse von P. T. Chruściel und T.-T. Paetz, die in [4] eine Formel für die Trautman-Bondi Masse von global glatten, null-geodätisch vollständigen Licht-Kegeln in Raumzeiten erhalten haben, die dasselbe asymptotische Verhalten aufweisen wie die Minkowski-Raumzeit.
Wir beginnen damit, Definitionen, bekannte Resultate und Techniken einzuführen und diskutieren anschließend den Einfluss von Materiefeldern auf die asymptotischen Entwicklungen der Komponenten des Metriktensors in sogennaten Bondi-artigen Koordinaten. Das Ziel dieser Analyse ist zu bestimmen wie schnell die Komponenten des Energie-Momentum-Tensors, mit zunehmender Entfernung, mindestens abfallen müssen, um eine endliche Gesamtmasse zu erhalten. Im Anschluss daran lösen wir, ähnlich wie T.-T. Paetz in [7], die Zwangsgleichungen im Wellenbild auf einer nullartigen Hyperfläche. Dazu löst man ein System gewöhnlicher Differentialgleichungen in Termen von polyhomogenen Entwicklungen der Lösung im Unendlichen. Diese Entwicklungen führen, nach einigen technischen Betrachtungen, schließlich zu unserem Ergebnis.

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