DISSERTATION

Titel der Dissertation

Combinatorial Auctions with Bidding Constraints and Network Externalities

Verfasser

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Acknowledgments

Abstract

Combinatorial auctions allow to simultaneously sell multiple heterogeneous items with interdependencies. Examples for such auctions are spectrum auctions, where the government sells licenses for bands of the radio spectrum, and procurement auctions, where the auctioneer buys services or goods (see Blumrosen and Nisan [16] and Cramton et al. [27]). Another prominent example are auctions for sponsored search. Here a search engine company shows advertisements next to the search results on its website and the advertisers bid money for their ads being shown next to the search results for certain keywords (see Edelman et al. [38] and Lahaie et al. [78]).

One of the standard mechanisms for combinatorial auctions is the seminal VCG mechanism (Vickrey [104], Clarke [23], and Groves [57]). This mechanism has the property that every bidder is incentivized to bid his valuation; this property is called incentive compatibility. Moreover, it has other desirable properties if bidders indeed bid their valuation: (1) No bidder has a negative utility; this property is called individual rationality. (2) The allocation of the items selected by the mechanism maximizes social welfare; that is, the allocation maximizes the sum of the valuations. However, these properties can only be achieved under certain conditions. It is necessary that (1) the utility functions of the bidders are quasi-linear, this implies that they cannot have budget constraints, and that (2) the computation of the social welfare maximizing allocation is possible in the available time. Thus, in settings where this is not the case it is not recommended to use the VCG mechanism. These are exactly the settings for which we design new auction mechanisms in this dissertation.

(A) First, we study settings where bidders are *budget constrained*. We design incentive compatible, individually rational, and Pareto-optimal mechanisms for auctions with heterogeneous items and bidders with additive valuations and budget constraints. (1) Specifically, we first study settings with single-dimensional valuations and prove a positive result for randomized mechanisms and an impossibility result for deterministic mechanisms. While the positive result allows for private budgets, the negative result is for public budgets. (2) Next, we study settings with multi-dimensional valuations. Here, we prove an impossibility result that applies to deterministic and randomized mechanisms even if budgets are public. Together with the previous results this shows the power of randomization in certain settings

with heterogeneous items, but it also shows its limitations. (3) Furthermore, we study multiple keyword sponsored search auctions with budgets. Here, each keyword has multiple ad slots with so-called click-through rates. The bidders have single-dimensional additive valuations, which are linear in the click-through rates. Additionally, we assume that the number of slots per keyword assigned to a bidder is bounded. We give the first mechanism for multiple keywords and click-through rates differing among slots that is incentive compatible in expectation, individually rational in expectation, and that satisfies Pareto-optimality ex-post.

(B) Next, we study settings with *complex valuations* for which maximizing social welfare is NP-hard and is thus impracticable for large instances. We consider various classes of valuation functions and bidding functions and study the performance of the VCG mechanism when bidders are forced to choose their bids from a subclass of the class of valuation functions. Thus, the auctioneer restricts the class of bidding functions such that the VCG outcome can be computed efficiently. The performance of the mechanisms is measured in terms of the Price of Anarchy, which is the ratio of the maximum social welfare of all allocations and the minimum social welfare of outcomes when the bids form a strategic equilibrium. We show improved upper bounds on the Price of Anarchy for restrictions to additive bids and upper and lower bounds for restrictions to non-additive bids. Our bounds show that increased expressiveness can give rise to additional equilibria of poorer efficiency.

(C) Finally, we generalize the standard model for combinatorial auctions where the utility of a bidder depends solely on the item set assigned to him and on his payment. For instance, in online advertisement an advertiser might not want his ads to be displayed on the same page as the ads of his direct competitor. We propose and analyze several natural, simple graph-theoretic models for combinatorial auctions that incorporate *negative, conflict-based externalities*. Here bidders are embedded into a directed conflict graph, and Δ is the maximum out-degree of any node. We design $\mathcal{O}(\Delta \log \log \Delta / \log \Delta)$-approximate algorithms and $\mathcal{O}(\Delta)$-approximate incentive compatible mechanisms for social welfare maximization. These ratios are almost optimal given existing hardness results for the independent set problem. For the prominent application of sponsored search, we present several algorithms when the number of items is small—the most relevant scenario in practice. In particular, we show how to obtain an approximation ratio that is sublinear in Δ when the number of items is only logarithmic in the number of bidders. All our algorithms for sponsored search can be turned into incentive compatible mechanisms.

Keywords

Zusammenfassung

Kombinatorische Auktionen ermöglichen den zeitgleichen Verkauf mehrerer heterogener Güter mit Abhängigkeiten (bspw. Substitute). Beispiele für solche Auktionen sind Spektrumauktionen bei welchen Regierungen Lizenzen für Bänder des Frequenzspektrums verkaufen und Auftragsauktionen bei welchen der Auktionator Dienstleistungen oder Güter erwirbt (siehe Blumrosen and Nisan [16] und Cramton et al. [27]). Ein weiteres bedeutendes Beispiel sind Auktionen für *Sponsored-Search*. Hierbei bieten Werbung schaltende Unternehmen Geld an Suchmaschinen damit ihre Werbung neben den Suchergebnissen für bestimmte Suchbegriffe angezeigt wird (siehe Edelman et al. [38] and Lahaie et al. [78]).

Einer der Standardmechanismen für kombinatorische Auktionen ist der bekannte VCG-Mechanismus (Vickrey [104], Clarke [23], und Groves [57]). Dieser Mechanismus hat die Eigenschaft, dass Bieter motiviert werden genau ihre Bewertung für die Güter zu bieten. Diese Eigenschaft wird Anreizkompatibilität genannt. Darüber hinaus besitzt der Mechanismus weitere Eigenschaften wenn Bieter tatsächlich ihre Bewertung bieten: (1) Kein Bieter hat einen negativen Nutzen. Diese Eigenschaft wird individuelle Rationalität genannt. (2) Die Allokation der Güter die der Mechanismus bestimmt maximiert die Wohlfahrt. Das bedeutet, dass die Summe der Bewertungen maximiert wird. Jedoch können diese Eigenschaften nur unter bestimmten Voraussetzungen erreicht werden: (1) Es ist notwendig, dass die Nutzenfunktionen der Bieter quasilinear sind. Die Bieter dürfen daher keine Budgetbeschränkungen haben. (2) Außerdem ist es notwendig, dass die Berechnung der wohlfahrtsmaximierenden Allokation in der zur Verfügung stehenden Zeit möglich ist. In Fällen wo diese Voraussetzungen verletzt werden ist die Verwendung des VCG-Mechanismus nicht anzuraten. In dieser Dissertation werden für genau diese Fälle neue Auktionsmechanismen entwickelt.

(A) Zuerst befassen wir uns mit Szenarien in denen die Bieter Budgetbeschränkungen haben. Wir entwickeln anreizkompatible, individuell rationale und Pareto-optimale Mechanismen für Auktionen mit heterogenen Gütern und Bietern mit additiven Bewertungen und Budgetbeschränkungen. (1) Insbesondere berücksichtigen wir Szenarien mit eindimensionalen Bewertungen und zeigen positive Resultate für randomisierte Mechanismen. Zudem zeigen wir das Fehlen deterministischer Mechanismen. Die positiven Resultate gelten auch für private Budgetbe-

schränkungen und die negativen Resultate selbst für öffentliche Budgetbeschränkungen. (2) Des Weiteren berücksichtigen wir Szenarien mit mehrdimensionalen Bewertungen. Wir zeigen das Fehlen von deterministischen und randomisierter Mechanismen, selbst im Fall öffentlicher Budgetbeschränkungen. Zusammengenommen zeigt das die Wichtigkeit von Randomisierung in bestimmten Szenarien mit heterogenen Gütern, jedoch zeigt dies auch die Grenzen von Randomisierung auf. (3) Darüber hinaus berücksichtigen wir Sponsored-Search-Auktionen für mehrere Suchbegriffe und Bieter mit Budgetbeschränkungen. Hierbei hat die Ergebnisseite jedes Suchbegriffs mehrere Plätze für Werbung mit einer sogenannten Click-Through-Rate. Zusätzlich nehmen wir an, dass die Anzahl der Werbeplätze pro Suchbegriff die einem Bieter zugeteilt werden kann beschränkt ist. Wir präsentieren den ersten Mechanismus für mehrere Suchbegriffe und Click-Through-Raten die vom Werbeplatz abhängen, welcher in Erwartung anreizkompatibel und individuell rationell ist und der ex-post Pareto-Optimalität erfüllt.

(B) Als Nächstes befassen wir uns mit Szenarien mit komplexen Bewertungsfunktionen für die das Maximieren der Wohlfahrt NP-schwer und daher für große Instanzen nicht praktikabel ist. Wir berücksichtigen zahlreiche Klassen von Bewertungsfunktionen und Gebotsfunktionen und analysieren die Wohlfahrt des VCG-Mechanismus wenn Bieter gezwungen sind ihre Gebote aus einer Unterklasse der Klasse der Bewertungsfunktionen zu wählen. Hierbei beschränkt der Auktionator die Klasse der Gebotsfunktionen damit das VCG-Ergebnis effizient berechnet werden kann. Wir bewerten die durch die Mechanismen erreichte Wohlfahrt mit Hilfe des Price-of-Anarchy, dem Verhältnis der maximalen Wohlfahrt über alle Allokationen und der minimalen Wohlfahrt wenn die Gebote ein strategisches Gleichgewicht bilden. Wir zeigen verbesserte obere Schranken für den Price-of-Anarchy für Beschränkungen auf additive Gebote und obere und untere Schranken für Beschränkungen auf nicht-additive Gebote. Unsere Schranken zeigen, dass erhöhte Ausdrucksstärke zur Existenz zusätzlicher Gleichgewichte mit geringer Effizienz führen kann.

(C) Zuletzt verallgemeinern wir das Standardmodell für kombinatorische Auktionen in welchen der Nutzen eines Bieters einzig von den zugeordneten Gütern und den Kosten hierfür abhängt. Beispielsweise mögen manche Werbung schaltende Unternehmen im Bereich der Onlinewerbung nicht wenn ihre Werbung auf der selben Seite angezeigt wird, wie die Werbung eines direkten Konkurrenten. Wir definieren und analysieren mehrere natürliche und einfache graphentheoretische Modelle für kombinatorische Auktionen die *negative konfliktbasierte Externalitäten* aufweisen. Hierbei bilden Bieter einen gerichteten Konfliktgraphen und Δ ist der maximale Ausgangsgrad jedes Knotens. Wir entwickeln $\mathcal{O}(\Delta \log \log \Delta / \log \Delta)$-approximative Algorithmen und $\mathcal{O}(\Delta)$-approximative anreizkompatible Mechanismen bezüglich der maximalen Wohlfahrt. Diese Approximationsfaktoren sind aufgrund bestehender Komplexitätsresultate für das Stabilitätsproblem beinahe optimal.

Für Sponsored-Search-Auktionen zeigen wir mehrere Algorithmen für eine kleine Anzahl an Gütern, dem in der Praxis relevantesten Szenario. Insbesondere zeigen wir wie ein im maximalen Ausgangsgrad Δ sublinearer Approximationsfaktor erreicht werden kann wenn die Anzahl der Güter lediglich logarithmisch in der Anzahl der Bieter ist. Darüber hinaus können alle unsere Algorithmen für Sponsored-Search-Auktionen in anreizkompatible Mechanismen umgewandelt werden.

Schlagwörter

Algorithmische Spieltheorie, Kombinatorische Auktionen, Sponsored-Search, Approximative Algorithmen

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Introduction

When selling items to customers it is a challenging problem to determine a suitable price; the difficulty of this problem increases with the lack of information about the customers. We consider settings where the valuations of the customers are private and the seller has no information about their valuations at all. In this dissertation, we design new auction mechanisms that help the seller to elude their lack of information. Our focus lies on combinatorial auctions, which are auctions that simultaneously allocate a set of heterogeneous items to bidders who can bid a distinct value for each subset of items. Such auctions are frequently used between companies or between the government and companies. Notable examples are spectrum auctions where the government sells licenses for the right to use some specific bands of the radio spectrum to telecommunication companies and procurement auctions where the auctioneer buys services or goods from bidding companies, for instance, transportation services (see Blumrosen and Nisan [16] and Cramton et al. [27]).

Another prominent example are auctions for "sponsored search" (see Edelman et al. [38] and Lahaie et al. [78]). These auctions are a special case of combinatorial auctions where the valuations of the bidders are assumed to be strongly restricted. Here, the selling company operates a search engine for the World Wide Web; users can input keywords into a text field and the search engine outputs links to related websites together with short text samples. See Figure 1.1 for an example of a search engine's website. This service is free of charge for the users, and search engine companies earn money by displaying advertisements together with the search results on their web sites. The positions for the advertisements are slots shown above and beside the search results, and the advertisers (the customers) have to pay to the search engine company when a user clicks on their advertisement (pay-per-click). The probability of an advertisement being clicked on (or to be recognized by the user) and, thus, the valuation of the advertiser, depends beside the type and the look of the advertisement itself also on the location of its slot on the page. The decision which advertisement to display in which slot and the payments by the advertisers

Figure 1.1: Sponsored search result page for the keyword "car rental". The page displays eleven ads and ten search results; the bottom five search results are hidden in the figure. The area in which an ad is located is called "ad slot" (or simply "slot") and the probability of an ad being clicked on or being seen depends on the location of the slot. Source: Screenshot taken from *www.google.com*.
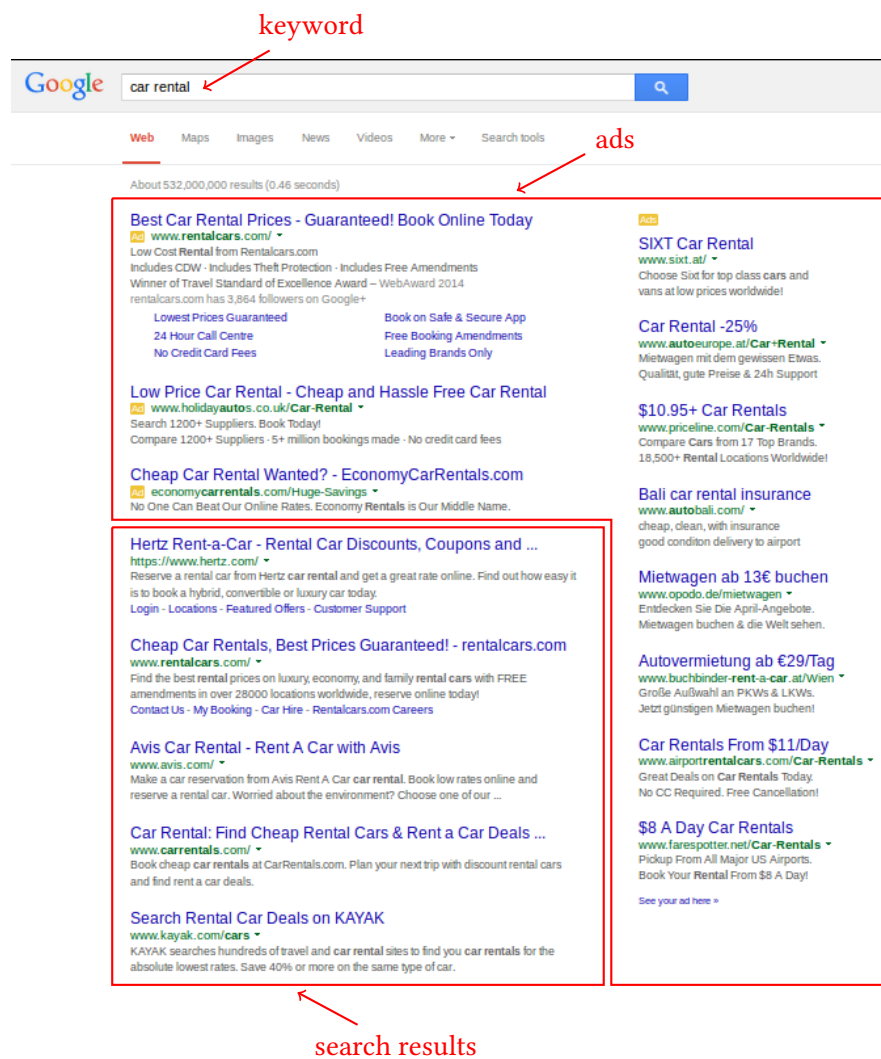
are computed by running an auction every time *before* showing search results to a user. Advertisers define a budget for a certain time period and the amount they are at most willing to pay (i.e., their bid) when their advertisement is clicked on. The influence of a slot's location on the valuations of the bidders and, thus, the influence on their willingness to pay, is supposed to be linear and it is determined by the search engine itself. Furthermore, it is assumed that the influence is independent of the bidders. The search engine company can now compute the optimal allocation of the slots to the advertisers and their payments. We will consider sponsored search auctions whenever those particular restrictions in the valuations and the bids help to strengthen our results.

1.1 The VCG Mechanism

One example of an auction mechanism used for combinatorial auctions is the seminal VCG mechanism by Vickrey [104], Clarke [23], and Groves [57] (see also Ausubel and Milgrom [7]), which generalizes second-price auctions to combinatorial settings. The mechanism selects the allocation that maximizes the social welfare and makes every bidder pay his externality; that is, bidders pay the loss in valuation they induce to the other bidders. This mechanism has many desirable properties as we discuss below but, unfortunately, they depend on the structure of the utility functions and the computation of the allocation maximizing the social welfare. In this dissertation, we develop and study new mechanisms for settings in which those limitations arise and, thus, the usage of the VCG mechanism is not recommended.

We will next present the VCG mechanism, discuss its properties, and use this as an opportunity to introduce the notation that we use throughout the dissertation. We refer the reader to Nisan [91] for additional details on the game-theoretic background. We are given a set of bidders $N = \{1, \ldots, n\}$ and a set of items $I = \{1, \ldots, m\}$. Each bidder $i \in N$ has a private valuation function $v_i : \mathcal{P}(I) \to \mathbb{R}_{\geq 0}$ that defines the value for each item set assigned to him,[1] and the goal of the VCG mechanism is to find an allocation $X = (X_1, \ldots, X_n)$ that maximizes the social welfare. To be specific, an allocation X partitions the item set such that no two bidders share an item (i.e., $\bigcup_{i \in N} X_i \subseteq I$ and $X_i \cap X_j = \emptyset$ for $i \neq j \in N$) and VCG is expected to return the allocation that maximizes the social welfare $SW(X) = \sum_{i \in N} v_i(X_i)$. Note that in some chapters an allocation X is represented by a matrix. Since the valuations are private the mechanism cannot directly access them and ask each bidder i to report his valuation; that is, bidder i is asked to place his bid $b_i : \mathcal{P}(I) \to \mathbb{R}_{\geq 0}$. Subsequently, an allocation rule $f : \mathcal{B} \to \mathcal{X}$, where $\mathcal{B}$ is the set of feasible bids $b = (b_1, \ldots, b_n)$ and $\mathcal{X}$ is the set of allocations, selects the allocation that maximizes $\sum_{i \in N} b_i(X_i)$. Thus, if we incentivize the bidders to bid their valuation VCG maximizes social welfare as intended.

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[1]In general, the valuation of a bidder can depend on the whole outcome and not only on his assigned item set. We will study such a setting in Chapter 5.

Before we introduce a payment rule p that motivates bidders to bid "truthfully" their valuation, we define their utilities. The utility of a bidder depends on his valuation $v_i(X_i)$ for his assigned item set X_i and on his payment p_i. As we will see later, it is crucial for the VCG mechanism that utilities have the form $u_i(X_i, p_i) = v_i(X_i) - p_i$; such utilities are called "quasi-linear".[2] The following payment rule $p : \mathcal{B} \to \mathbb{R}^n_{\geq 0}$, called Clarke pivot rule,[3] causes each bidder i to pay his *externality*, which is the loss of valuation that bidder i imposes on the other bidders (assuming the other bidders bid truthfully); that is,

$$p_i(b) = \sum_{i' \in N \setminus \{i\}} (b_{i'}(f_{i'}(b_{-i})) - b_{i'}(f_{i'}(b))), \tag{1.1}$$

where $f_{i'}(b_{-i})$ is bidder i''s item set when bidder i is excluded from the auction and $f_{i'}(b)$ is bidder i''s item set when bidder i takes part in the auction. In general, we denote by $f_i(b)$ a function that returns the assignment X_i given $f(b) = (X_1, \ldots, X_n)$; and we use the notations b_{-i} for $(b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$ and (b_i, b_{-i}) for $(b_1, \ldots, b_n)$. Theorem 1.1 shows that the Clarke pivot rule indeed incentivizes bidders to bid their valuation; that is, for all bids $(b_1, \ldots, b_n)$, the utility of each bidder i does not decrease when bidding his valuation:

$$u_i(f_i((v_i, b_{-i})), p_i((v_i, b_{-i}))) \geq u_i(f_i(b), p_i(b)); \tag{1.2}$$

in other words, bidding the valuation is a "dominant strategy". A mechanism that satisfies this condition is called *incentive compatible*. This property is not only desirable because it enables the auctioneer to compute the social welfare maximizing allocation but also because bidders know their preferred bid without strategic considerations.

Theorem 1.1 (Vickrey-Clarke-Groves). *The VCG mechanism described above is incentive compatible.*

Proof. Let us fix a bidder $i \in N$ and consider bids $b' = (b'_1, \ldots, b'_n)$ where every bidder has an arbitrary bid and, additionally, the bids $b = (v_i, b'_{-i})$ where bidder i bids his valuation and the other bidders do not change their bid. When the bidders

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[2] If we assume the utility of the auctioneer to be the sum of the payments and the utilities of the bidders to be quasi-linear, then social welfare also corresponds to the sum of the utilities including the auctioneer's utility.

[3] The VCG mechanism also allows the usage of other payment rules that add to each bidder's payment a term that does not depend on the bidder's bid. This is for instance necessary when valuations are negative like in procurement auctions. However, we will assume, in the following, that the Clarke pivot is used.

bid b, bidder i's utility is

$$u_i(f_i(b), p_i(b)) = v_i(f_i(b)) - p_i(b) \tag{1.3}$$

$$= b_i(f_i(b)) - \sum_{i' \in N \setminus \{i\}} b_{i'}(f_{i'}(b_{-i})) + \sum_{i' \in N \setminus \{i\}} b_{i'}(f_{i'}(b)) \tag{1.4}$$

$$= \sum_{i' \in N} b_{i'}(f_{i'}(b)) - \sum_{i' \in N \setminus \{i\}} b_{i'}(f_{i'}(b_{-i})). \tag{1.5}$$

While, when the bidders bid b', bidder i's utility is

$$u_i(f_i(b'), p_i(b')) = v_i(f_i(b')) - p_i(b') \tag{1.6}$$

$$= b_i(f_i(b')) - \sum_{i' \in N \setminus \{i\}} b'_{i'}(f_{i'}(b'_{-i})) + \sum_{i' \in N \setminus \{i\}} b'_{i'}(f_{i'}(b')) \tag{1.7}$$

$$= b_i(f_i(b')) - \sum_{i' \in N \setminus \{i\}} b_{i'}(f_{i'}(b_{-i})) + \sum_{i' \in N \setminus \{i\}} b_{i'}(f_{i'}(b')) \tag{1.8}$$

$$= \sum_{i' \in N} b_{i'}(f_{i'}(b')) - \sum_{i' \in N \setminus \{i\}} b_{i'}(f_{i'}(b_{-i})). \tag{1.9}$$

In the chains of equalities above, we used in (1.3) and (1.6) that the utilities are quasi-linear and we used in (1.4) and (1.7) the Clarke pivot rule (1.1).

Certainly,

$$u_i(f_i(b), p_i(b)) \geq u_i(f_i(b'), p_i(b')) \tag{1.10}$$

if and only if

$$\sum_{i' \in N} b_{i'}(f_{i'}(b)) \geq \sum_{i' \in N} b_{i'}(f_{i'}(b')), \tag{1.11}$$

and this inequality holds since $f(b)$ selects an allocation $X \in \mathcal{X}$ that maximizes $\sum_{i' \in N} b_{i'}(X_{i'})$. If the mechanism selects an allocation that is not optimal this inequality might not hold and the mechanism is not incentive compatible. $\square$

In the proof of Theorem 1.1 it is crucial that (1) utilities are quasi-linear and that (2) the allocation rule finds the allocation that maximizes the bids. Beside maximizing social welfare and being incentive compatible under the above conditions, VCG with Clarke pivot rule satisfies the following additional properties:

- The mechanism makes "no positive transfers" to the bidders; that is; the payments are non-negative.

- Taking part in the auction is "individually rational"; that is, the utility of each bidder who bids his valuation cannot be negative. This holds because the payments are bounded by the bids.

In this dissertation, we design new mechanisms for settings in which quasi-linearity does not apply to the utility functions or in which an optimal allocation

cannot be computed in feasible time and, thus, the usage of the VCG mechanism is not recommended (see Milgrom [84], Chapter 2.5 for other disadvantages of the VCG mechanism). We will describe these settings and our results in the next two sections.

1.2 Budget Constraints

We will first consider a setting where the quasi-linearity of the utilities is violated. To be specific, we study the case when bidders face budget constraints; that is, their utility drops when their payments exceed their budgets. Consequently, the VCG mechanism is not incentive compatible in this case and selecting a bid becomes cumbersome for the bidders as the following example shows. Moreover, the loss of incentive compatibility implies that the VCG mechanism is not maximizing social welfare anymore.

Example: Let us study the VCG mechanism with Clarke pivot rule for the following setting: We are given two bidders, 1 and 2, and three items, A, B and C. The bidders have the same valuation functions and budget constraints. Their budgets are $\beta_i = 1$ and their valuations are $v_i(X_i) = |X_i|$ for $i \in \{1, 2\}$. We assume that their utilities for a set of items X_i are $v_i(X_i) - p_i$ if $p_i \leq \beta_i$ and $-\infty$ otherwise, again for $i \in \{1, 2\}$. Here, $-\infty$ simply implies the infeasibility of payments that exceed the budget limits. These utilities are not quasi-linear.

Certainly, the bidders should not ignore their budget constraints and bid their valuation as in this case the bidders pay $|X_i|$ for their item sets X_i and the bidder who gets more than one item has a utility of $-\infty$. Thus, the VCG mechanism is not incentive compatible for these utility functions.

One could argue that a bidder should bid $\min\{v_i(X_i), \beta_i\}$ for all X_i; that is, the bidder bids his valuation when it does not exceed his budget and, otherwise, his budget. However, assume that bidder 1 bids $\min\{v_1(X_1), \beta_1\} = 1$ for each $X_1 \neq \emptyset$ and zero otherwise, and bidder 2 bids $|X_2| \cdot \beta_2 / |\{A, B, C\}| = |X_2|/3$ for each X_2. Note that both bids do not distinguish between the items. The VCG mechanism would assign one item to bidder 1 for a price of $1/3$ but two items to bidder 2 for a price of 0. If bidder 2 were also bidding $\min\{v_2(X_2), \beta_2\}$ then one of the bidders gets one item and the other bidder gets two items and both have a payment of zero; thus, in this case, the bidder who gets only one item could improve his utility by bidding $|X_i|/3$ as bidder 2 above. Hence, bidding $\min\{v_i(X_i), \beta_i\}$ is not a dominant strategy.

So should both bidders bid $|X_i| \cdot \beta_i / |\{A, B, C\}|$ if they do not care which specific items they receive? As one can easily see one of the bidders could improve his utility by fixing an interest set S_i being either $\{A, B\}$, $\{A, C\}$, or $\{B, C\}$, and bidding $|X_i \cap S_i|/2$.[4] Thus, also bidding $|X_i| \cdot \beta_i / |\{A, B, C\}|$ is not a good idea. In fact, the

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[4]Also the bid $\min\{|X_i|/2, \beta_i\}$, which does not distinguish between the items, can improve the utility of a bidder.

bids with interest sets above can be used to show that no dominant strategy exists because the best responses to such a strategy depend on the selected interest set.

This example shows the struggle a bidder faces in the VCG mechanism under budget restrictions and highlights the need for alternative mechanisms. We present such alternative mechanisms in Chapter 2 and 3. We aim for mechanism that are incentive compatible, individually rational, and make no positive transfers. Furthermore, we want the allocation selected by the mechanism to satisfy a suitable optimality criterion.[5] However, the budget constraints make the maximization of the social welfare infeasible for incentive compatible auctions without positive transfers.[6] We, thus, relax our optimality criterion to the requirement of Pareto-optimal allocations. Pareto-optimality guarantees that we cannot increase the utility of a bidder without decreasing the utility of another bidder or decreasing the sum of the payments, which is the utility of the auctioneer. Chapter 2 and 3 are inspired by the seminal work of Ausubel [5] and Dobzinski et al. [30], as our positive results are based on their (adapted) clinching auction. This type of mechanism increases a price per unit over time and sells an item to one of the bidders every time the demand of the other bidders falls below the supply. The demand of the bidders is computed by the mechanism considering their bids and budgets.

In Chapter 2, we study incentive compatible, individually rational, and Pareto optimal mechanisms for auctions with heterogeneous items and hard budget constraints as in the example above. That means that payments that exceed the budget constraints are infeasible or induce a utility of $-\infty$. We consider settings where bidders have valuation functions that are additive; that is, they are of the form $v_i(X_i) = \sum_{j \in X_i} v_{i,j}$.

We first study settings with single-dimensional valuations; that is, we assume that valuations are restricted to be $v_{i,j} = v_i \cdot \alpha_j$ for all bidders i and all items j where v_i only depends on the bidder i and α_j only depends on the item j. We prove a positive result for randomized mechanisms and an impossibility result for deterministic mechanisms. While the positive result allows for private budgets, the negative result is for public budgets.

Next, we study settings with multi-dimensional valuations. Thus, we consider additive valuations without these additional restrictions. Here, we prove an impossibility result that applies to deterministic and randomized mechanisms even if budgets are public. Taken together this shows the power of randomization in certain settings with heterogeneous items, but it also shows its limitations.

In Chapter 3, we study multiple keyword sponsored search auctions with budgets. Each keyword has multiple ad slots with a click-through rate. The bidders have additive valuations, which are linear in the click-through rate, and budgets, which

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[5]Note that without aiming for an optimal allocation such mechanisms are easy to design. The mechanism could simply ignore the bids and give the items to one of the bidders for free.

[6]This follows from Theorem 9.37 in Nisan [91] and from the infeasibility of VCG payments without positive transfers.

are restricting their overall payments. Thus, the valuations are similar to the setting with single-dimensional valuations studied in Chapter 2. However, we assume additionally that the number of slots per keyword assigned to a bidder is bounded. This additional constraint has to be treated separately from the budget constraints. We deal with those constraints by solving suitable linear programs during the execution of the auction and by the usage of a correlated randomization based on scheduling algorithms. We give the first mechanism for multiple keywords and click-through rates differing among slots that is incentive compatible in expectation, individually rational in expectation, and that satisfies Pareto-optimality ex-post.

Unfortunately, the impossibility results mentioned above indicate that our techniques cannot be extended to settings with more complex valuation functions such as those considered in the next section.

1.3   Complex Valuation Functions

In this section we assume that utility functions are quasi-linear and consider the case of more complex valuations. Recall that we also used a second condition in the proof of Theorem 1.1 for showing the incentive compatibility of the VCG mechanism; to show Equation (1.11), we assumed that the VCG mechanism can select the social welfare maximizing outcome. This is a rather optimistic assumption when valuation functions are complex and the number of items m is large. Note that even under strong restrictions on $\mathcal{B}$ (e.g., that the valuation functions are monotone and submodular set functions; see Lehmann et al. [80]) maximizing social welfare is NP-hard and, thus, no optimal algorithm that runs in polynomial time is known. Moreover, using an approximation algorithm might violate incentive compatibility.

We deal with such settings in Chapter 4 and Chapter 5 in different ways. In Chapter 4 we study the consequences of restricting the allowed bids such that the auctioneer can solve the optimization problem optimally, while the valuation functions remain unchanged. Furthermore, this solves another problem when valuations are complex; the bids can be expressed using fewer bits than needed for the valuations.[7] In Chapter 5, we consider another challenge. The valuation of a bidder for an item set depends on the assignment to the other bidders. Here we design mechanisms that deal with the computational hardness in a different way. The allocation rules of the mechanisms always select the social welfare maximizing allocations of a restricted set of allocations. Thus, Equation (1.11) in the proof of Theorem 1.1 is still valid. We will next describe the results in those chapters.

In Chapter 4, our results are inspired by the work of Christodoulou et al. [22] and Bhawalkar and Roughgarden [12] who study the replacement of the VCG mechanism for bidders with submodular or subadditive valuations with simultaneous

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[7]Note that valuation functions can need up to 2^m many numbers to be expressed, while additive bids can be expressed by using only m numbers. This implies that even if only small costs are caused for the bidders when they are computing the value of a single set of items, the VCG mechanism becomes impracticable for complex bids (see Milgrom [84], Chapter 2.5).

second price auctions for each item. Note that simultaneous second price auctions correspond to a VCG auction accepting only additive bids. Certainly, the considered auctions are not incentive compatible. Thus, they study the performance of those auctions in terms of the "Price of Anarchy" (Koutsoupias and Papadimitriou [76]); that is, they study the ratio between the optimal social welfare and the minimum social welfare when the bids form a (Bayesian) Nash equilibrium[8]. Thus, it is the task of the bidders to find a Nash equilibrium before bidding.

We generalize this idea and consider various classes of valuation functions and bidding functions. We thus study the performance of the VCG mechanism when bidders are forced to choose their bids from a subclass $\mathcal{B}$ of the class of valuation functions $\mathcal{V}$, such that the VCG outcome for the given bids can be computed efficiently. We show improved upper bounds on the welfare loss for restrictions to additive bids and upper and lower bounds for restrictions to non-additive bids. All our bounds apply to equilibrium concepts that can be computed in polynomial time as well as to learning outcomes, which are the result of repeated play of the same game induced by the auction rules and valuations. Our bounds show that increased expressiveness can give rise to additional equilibria of poorer efficiency.

In Chapter 5, we will generalize the standard model for combinatorial auctions where the utility of a bidder depends solely on the item set assigned to him and on his payment. For instance, in online advertisement an advertiser might not want his ads to be displayed on the same page as the ads of his direct competitor. We propose and analyze several natural, simple graph-theoretic models for combinatorial auctions that incorporate negative, conflict-based externalities. Here bidders are embedded into a directed conflict graph, and Δ is the maximum out-degree of any node. We design algorithms and incentive compatible mechanisms for social welfare maximization that obtain approximation ratios depending on Δ.

For combinatorial auctions, we apply two algorithmic techniques. The first is via "lottery"; that is, we eliminate conflicts by removing bidders/items independent of the bids received. It allows to conveniently apply incentive compatible mechanisms for conflict-free combinatorial auctions as a black-box, thereby increasing the approximation guarantee only by a factor of $\mathcal{O}(\Delta)$. The second is via a cone program relaxation. We design a polynomial-time approximation algorithm for combinatorial auctions where bidders have fractionally subadditive valuations; the approach again can be combined with algorithms for conflict-free combinatorial auctions and increases the approximation guarantee only by a factor of $\mathcal{O}(\Delta \log \log \Delta / \log \Delta)$. This is sublinear in Δ and improves upon our lottery algorithm under the same scenario. To the best of our knowledge, our results are the first to use cone programs in the context of mechanism design. These ratios are almost optimal given existing hardness results for the independent set problem.

For the prominent application of sponsored search, we present several algorithms for the case when the number of items is small, which is arguably the most

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[8]A Nash equilibrium is a combination of strategies (bids) where no agent (bidder) can increase his utility by changing his strategy if the others agents do not change their strategy either.

practically relevant scenario. Among other things, we show how to obtain an approximation ratio that is sublinear in Δ when the number of items is only logarithmic in the number of bidders. All our algorithms for sponsored search can be turned into incentive compatible mechanisms.

Auctions with Heterogeneous Items and Budget Limits

2.1 Introduction

A canonical problem in mechanism design is the design of economically efficient auctions that satisfy *individual rationality* and *incentive compatibility*. In settings with quasi-linear utilities these goals are achieved by the VCG mechanism by Vickrey [104], Clarke [23], and Groves [57]. In many practical situations, including settings in which the bidders have budget limits, quasi-linearity is violated and, thus, the VCG mechanism is not applicable.

Ausubel [5] describes an ascending-bid auction for homogeneous items that yields the same outcome as the sealed-bid Vickrey auction, but offers advantages in terms of simplicity, transparency, and privacy preservation. In his concluding remarks he points out that "when budgets impair the bidding of true valuations in a sealed-bid Vickrey auction, a dynamic auction may facilitate the expression of true valuations while staying within budget limits" (Ausubel [5], p. 1469). Dobzinski et al. [30] show that an adaptive version of Ausubel's "clinching auction" is indeed the unique mechanism that satisfies individual rationality, Pareto-optimality, and incentive compatibility in settings with *public* budgets. They use this fact to show that there can be no mechanism that achieves those properties for *private* budgets. An important restriction of Dobzinski et al.'s impossibility result for private budgets is that it only applies to *deterministic* mechanisms. In fact, as Bhattacharya et al. [11] show, there exists a *randomized* mechanism for homogeneous items that is individually rational, Pareto-optimal, and incentive compatible with private budgets.

As Ausubel [6] points out, "situations abound in diverse industries in which heterogeneous (but related) commodities are auctioned" (Ausubel [6], p. 602). He also describes an ascending-bid auction, the "crediting and debiting auction", that takes the place of the "clinching auction" when items are heterogeneous. Positive

and negative results for *deterministic* mechanisms and public budgets that apply to heterogeneous items are given in Colini-Baldeschi et al. [24], Fiat et al. [42], Goel et al. [53], and Lavi and May [79]. We focus on *randomized* mechanisms for heterogeneous items, and prove positive results for private budgets and negative results for public budgets. We thus explore the power and limitations of randomization in this setting.

2.1.1 Contribution

We analyze two settings with heterogeneous items and additive valuations. In the first setting the valuations are single-dimensional in that each bidder has a valuation, each item has a quality, and a bidder's valuation for an item is the product of the item's quality and the bidder's valuation.[1] In the second setting the valuations are multi-dimensional in that each bidder has an arbitrary, non-negative valuation for each item. In both cases we analyze whether a deterministic or randomized mechanism exists that satisfies individual rationality (IR), Pareto-optimality (PO), and incentive compatibility (IC). For both types of mechanisms we distinguish between settings with public budgets and settings with private budgets. For randomized mechanisms the corresponding properties can either be satisfied in expectation or they can be satisfied ex post. The former requires that the property is satisfied in expectation over the outcomes the randomized mechanism produces, while the latter requires that it is satisfied by every possible outcome of the mechanism.

(a) For single-dimensional valuations we present a *deterministic* mechanism for *divisible* items that is IR, PO, and IC with *public* budgets and a *randomized* mechanism for both *divisible* and *indivisible* items that is IR in expectation, PO ex post, and IC in expectation with *private* budgets. These mechanisms also satisfy another desirable property, namely "no positive transfers" (NPT), which requires that the individual payments are non-negative. We obtain these mechanisms through a general reduction from the setting with multiple, heterogeneous items to the setting of a single and, by definition, homogeneous item. This allows us to apply the mechanisms for this setting presented by Bhattacharya et al. [11]. The main difficulty in showing that the resulting deterministic and randomized mechanisms for multiple items have the desired properties is to show that they satisfy PO resp. PO ex post. For this we argue that the reduction preserves a certain structural property of the mechanisms for a single item. We connect this structural property to a novel "no trade" (NT) condition, and show that it is equivalent to PO resp. PO ex post.

(b) For single-dimensional valuations the impossibility result of Dobzinski et al. [30] implies that there can be no deterministic mechanism for indivisible items that

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[1]Such valuations arise whenever the bidders agree about the relative values of the items. One concrete example is an auction in which display ads are sold in bulks consisting of a certain number of impressions together with per-impression valuations. Another example are auctions in which display ads of different size are sold and the valuations are proportional to size. In both cases the respective per-item valuations are the product of the item's quality, either the number of impressions or the size, and the bidder's valuation, either per impression or per pixel.

is IR, PO, and IC for *private* budgets. We show that for heterogeneous items there can also be no deterministic mechanism for indivisible items that is IR, PO, and IC for *public* budgets. To this end we extend the "classic" result that IC mechanisms must satisfy "value monotonicity" (VM) and "payment identity" (PI) from settings without budgets to settings with public budgets. To establish the impossibility result, we then use NT and PI to derive a lower bound on the payments that conflicts with the upper bounds on the payments required by IR. Our impossibility result is tight in the sense that if any of the conditions is relaxed such a mechanism exists: (i) For *homogeneous*, indivisible items a deterministic mechanism is given by Dobzinski et al. [30]. (ii) For heterogeneous items we give a deterministic mechanism for *divisible* and a *randomized* mechanism for indivisible items as described above. We thus obtain a strong separation between deterministic mechanisms, that do *not* exist for *public* budgets, and randomized mechanisms, that exist for *private* budgets. This separation is stronger than in the homogeneous items setting, where a deterministic mechanism exists for public budgets.

(c) For multi-dimensional valuations the impossibility result of Fiat et al. [42] implies that there can be no deterministic mechanism for *indivisible* items that is IR, PO, and IC for public budgets. We show that there can also be no deterministic mechanism with these properties for *divisible* items. To prove this we observe that—just as in settings without budgets—every mechanism that satisfies IC with public budgets must satisfy "weak monotonicity" (WMON). Then we show that in certain settings this condition will be violated. For this we use that multi-dimensional valuations enable the bidders to manipulate the mechanism's outcome in a sophisticated manner. While all previous impossibility results in this area used bidders that either only overstate or only understate their valuations, we use a bidder that overstates his valuation for some item and understates his valuation for another item. We use our impossibility result for deterministic mechanisms to show that for both divisible and indivisible items there can be no randomized mechanism that is IR in expectation, PO in expectation, and IC in expectation with public budgets. This is the first impossibility result for randomized mechanisms in this domain. It also establishes an interesting separation between multi-dimensional valuations, where no such mechanism exists, and single-dimensional valuations, where such a mechanism exists.

2.1.2 Related Work

Homogeneous items were studied by Dobzinski et al. [30], Bhattacharya et al. [11], and Lavi and May [79]. Dobzinski et al. show that for both divisible and indivisible items there is a deterministic mechanism that is IR, PO, and IC with public budgets, and that no *deterministic* mechanisms can achieve this with *private* budgets. Bhattacharya et al. show that there is a randomized mechanism for both divisible and indivisible items that is IR in expectation, PO ex post, NPT ex post, and IC in expectation with private budgets. Lavi and May prove an impossibility result for non-additive valuations with decreasing marginals. The impossibility result of

Dobzinski et al. applies to both of our settings, but our impossibility results are stronger as they are for *public* budgets and, in the case of multi-dimensional valuations, also apply to *randomized* mechanisms. The positive results of Dobzinski et al. and Bhattacharya et al. do *not* apply to our settings as we study *heterogeneous* items, not homogeneous items. The impossibility result of Lavi and May does *not* apply to our settings as the valuations that we study are *additive*.

Heterogeneous items were first studied by Fiat et al. [42]. In their model each bidder has the same valuation for each item in a bidder-dependent interest set and zero for all other items. They give a deterministic mechanism for indivisible items that satisfies IR, NPT, PO, and IC when both, interest sets and budgets, are public. They also show that when the interest sets are private, then there can be no deterministic mechanism that satisfies IR, PO, and IC. The positive result of Fiat et al. does *not* apply to our settings as it is not always possible to express the valuations that we consider in terms of per-bidder valuations and interest sets. The impossibility result of Fiat et al. applies to our multi-dimensional setting and shows that there can be no *deterministic* mechanism that satisfies IR, PO, and IC with public budgets for *indivisible* items. Our impossibility result for this setting is stronger as it also applies to *randomized* mechanisms and *divisible* items.

Settings with heterogeneous items were subsequently, and in parallel to this paper, studied by Colini-Baldeschi et al. [24][2] and Goel et al. [53]. The former study problems in sponsored search in which the bidders are interested in a certain number of slots for each of a set of keywords. The slots are associated with click-through rates that are assumed to be identical across keywords. The latter study settings in which the bidders have identical valuations per item but the allocations must satisfy polyhedral or polymatroidal constraints. The settings studied in these papers are more general than the single-dimensional valuations setting studied here. On the one hand this implies that our impossibility result for this setting applies to their settings, showing that in their settings there can be no deterministic mechanism for *indivisible* items that is IC with public budgets. On the other hand this implies that their positive results apply to our setting. This shows the existence of deterministic mechanisms for divisible items and randomized mechanisms for both divisible and indivisible items that are IC with *public* budgets in our single-dimensional valuations setting. Our positive result for this setting is stronger as it shows the existence of a mechanism that is IC with *private* budgets. Finally, the impossibility results of Colini-Baldeschi et al. and Goel et al. either assume non-additive valuations or that the allocations satisfy arbitrary polyhedral constraints and therefore do *not* apply to the multi-dimensional valuations setting that we study here.

We summarize the results from this chapter and the related work along with open problems in Figure 2.1.

The clinching auction was also studied for more different settings after the results in this chapter were published. Goel et al. [51, 52] study the clinching auction in an online setting and under different liquidity constraints. Dobzinski and Paes

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[2]Note that the major result of this article is part of this thesis and can be found in Chapter 3.

Leme [32] study other efficiency measures than Pareto-optimality and Devanur et al. [28] study prior-free auctions for bidders with budget constraints that are based on the clinching auction.

Valuation functions (indivisible items)	randomized mechanisms		deterministic mechanisms	
	public budgets	private budgets	public budgets	private budgets
Homogeneous				
additive	+ [30]	+ [11]	+ [30]	− [30]
non-additive	?	?	−[24, 79]	− [30]
Heterogeneous & Additive				
public interest set	+[42]	?	+[42]	− [30]
private interest set	?	?	− [42]	− [30]
multi-keyword unit demand	+[24, 53]	?	⊖	− [30]
single-dimensional	⊕	⊕	⊖	− [30]
multi-dimensional	⊖	⊖	−[42]	− [30]

Valuation functions (divisible items)	randomized mechanisms		deterministic mechanisms	
	public budgets	private budgets	public budgets	private budgets
Homogeneous				
additive	+ [11, 30]	+ [11]	+ [11, 30]	− [30]
non-additive	?	?	−[53]	− [30]
Heterogeneous & Additive				
polymatroid constraints	+[53]	?	+[53]	− [30]
multi-keyword unit demand	+[24, 53]	?	+[24, 53]	− [30]
single-dimensional	⊕	⊕	⊕	− [30]
multi-dimensional	⊖	⊖	⊖	− [30]

Table 2.1: Summary of the results for indivisible items (upper table) and divisible items (lower table) when bidders have budget constraints. A plus (+ or ⊕) indicates a positive result, and a minus (− or ⊖) indicates a negative result. We use + and − for results from the related work and ⊕ and ⊖ for results from this chapter. A question mark (?) indicates that nothing is known for this setting.

2.2 Preliminaries

We are given a set $N = \{1, \ldots, n\}$ of n bidders and a set $I = \{1, \ldots, m\}$ of m items. We distinguish between settings with divisible items and settings with indivisible items. In both settings we use $\mathcal{X} = \prod_{i=1}^{n} \mathcal{X}_i$ for the allocation space. For divisible items $\mathcal{X}_i = [0, 1]^m$ for all bidders $i \in N$ and $x_{i,j} \in [0, 1]$ denotes the fraction of item $j \in I$ that is allocated to bidder $i \in N$ in allocation $X_i \in \mathcal{X}_i$. For indivisible items $\mathcal{X}_i = \{0, 1\}^m$ for all bidders $i \in N$ and $x_{i,j} \in \{0, 1\}$ indicates whether item $j \in I$ is allocated to bidder $i \in N$ in allocation $X_i \in \mathcal{X}_i$ or not. In both cases we require that $\sum_{i=1}^{n} x_{i,j} \leq 1$ for all items $j \in I$. We do *not* require that $\sum_{j=1}^{m} x_{i,j} \leq 1$ for all bidders $i \in N$; that is, we do *not* assume that the bidders have unit demand.

Each bidder $i \in N$ has a type $\theta_i = (v_i, \beta_i)$ consisting of a valuation function $v_i : \mathcal{X}_i \to \mathbb{R}_{\geq 0}$ and a budget $\beta_i \in \mathbb{R}_{\geq 0}$. We use $\Theta = \prod_{i=1}^{n} \Theta_i$ for the type space. We consider two settings with heterogeneous items, one with multi-dimensional valuations and one with single-dimensional valuations. In the first setting, each bidder $i \in N$ has a valuation $v_{i,j} \in \mathbb{R}_{\geq 0}$ for each item $j \in I$ and bidder i's valuation for allocation X_i is $v_i(X_i) = \sum_{j=1}^{m} x_{i,j} v_{i,j}$. In the second setting, each bidder $i \in N$ has a valuation $v_i \in \mathbb{R}_{\geq 0}$, each item $j \in I$ has a quality $\alpha_j \in \mathbb{R}_{\geq 0}$, and bidder i's valuation for allocation $X_i \in \mathcal{X}_i$ is $v_i(X_i) = \sum_{j=1}^{m} x_{i,j} \alpha_j v_i$. For simplicity we will assume in this setting that $\alpha_1 > \alpha_2 > \cdots > \alpha_m$ and that $v_1 > v_2 > \cdots > v_n > 0$.

A (direct) mechanism $M = (f, p)$ consisting of an allocation rule $f : \Theta \to \mathcal{X}$ and a payment rule $p : \Theta \to \mathbb{R}^n$ is used to compute an outcome (X, p) consisting of an allocation $X \in \mathcal{X}$ and payments $p \in \mathbb{R}^n$. Given $f(\theta) = X$ we define $f_i(\theta) = X_i$ and $f_{i,j}(\theta) = x_{i,j}$. A mechanism is deterministic if the computation of (X, p) is deterministic, and it is randomized if the computation of (X, p) is randomized. We allow the resulting allocation and payments to be arbitrarily correlated.

We assume that the bidders are utility maximizers and as such need not report their types truthfully. We consider settings in which both the valuations and budgets are private and settings in which only the valuations are private and the budgets are public. When the budgets are public then they are known to the auctioneer and all bidders. Private valuations/budgets mean that only the bidder itself knows its valuation/budget, but not the other bidders or the auctioneer. In the private valuations and private budgets setting a report by bidder $i \in N$ with true type $\theta_i = (v_i, \beta_i)$ can be any type $\theta_i' = (v_i', \beta_i')$. In the private valuations but public budgets setting bidder $i \in N$ is restricted to reports of the form $\theta_i' = (v_i', \beta_i)$. In both settings, if mechanism $M = (f, p)$ is used to compute an outcome for reported types $\theta' = (\theta_1', \ldots, \theta_n')$ and the true types are $\theta = (\theta_1, \ldots, \theta_n)$ then the utility of bidder $i \in N$ is

$$u_i(f_i(\theta'), p_i(\theta'), \theta_i) = \begin{cases} v_i(f_i(\theta')) - p_i(\theta') & \text{if } p_i(\theta') \leq \beta_i, \text{ and} \\ -\infty & \text{otherwise.} \end{cases}$$

For deterministic mechanisms and their outcomes, we are interested in the following properties: (a) *Individual rationality (IR)*: A mechanism is IR if it always

produces an IR outcome. An outcome (X, p) for types $\theta = (v, \beta)$ is IR if it is (i) *bidder rational*: $u_i(X_i, p_i, \theta_i) \geq 0$ for all bidders $i \in N$ and (ii) *auctioneer rational*: $\sum_{i=1}^n p_i \geq 0$. (b) *Pareto-optimality (PO)*: A mechanism is PO if it always produces a PO outcome. An outcome (X, p) for types $\theta = (v, \beta)$ is PO if there is no other outcome (X', p') such that $u_i(X'_i, p'_i, \theta_i) \geq u_i(X_i, p_i, \theta_i)$ for all bidders $i \in N$ and $\sum_{i=1}^n p'_i \geq \sum_{i=1}^n p_i$, with at least one of the inequalities strict.[3] Note that we do not explicitly require that the alternative outcome is IR, but that only IR outcomes can dominate an IR outcome. That means that if we consider a PO *and* IR outcome then the two definitions are actually equivalent. (c) *No positive transfers (NPT)*: A mechanism satisfies NPT if it always produces an NPT outcome. An outcome (X, p) satisfies NPT if $p_i \geq 0$ for all bidders $i \in N$. (d) *Incentive compatibility (IC)*: A mechanism satisfies IC if for all bidders $i \in N$, all true types θ, and all reported types θ' we have $u_i(f_i(\theta_i, \theta'_{-i}), p_i(\theta_i, \theta'_{-i}), \theta_i) \geq u_i(f_i(\theta'_i, \theta'_{-i}), p_i(\theta'_i, \theta'_{-i}), \theta_i)$.

For randomized mechanisms we are naturally interested in randomized outcomes, which are distributions over deterministic ones. We then consider the expected utility a bidder gets and compare it to the expected utility that the bidder could get with other randomized outcomes. If a randomized outcome satisfies the above conditions in this way, we say it satisfies them *in expectation*. Alternatively, if each deterministic outcome in the support of a randomized outcome has this property, we say it satisfies the property *ex post*. For outcomes that are IR in expectation and PO in expectation only outcomes that are IR in expectation can be better. Hence our negative results for randomized mechanisms also apply under this alternative definition.

2.2.1 Comparison of PO in expectation and PO ex post

We show that neither PO in expectation implies PO ex post, nor PO ex post implies PO in expectation. Let us assume that we have two bidders, one or two (in)divisible items, and a uniformly distributed random variable $Y \sim \mathcal{U}(0, 1]$ that represents the random decisions taken by an auction.

Consider first the case with one item where bidder 1 has valuation $v_{1,1} = 1$ and budget $\beta_1 = 1$, bidder 2 has valuation $v_{2,1} = 2$ and budget $\beta_2 = 1$, and we have a value $\tilde{y} \in (0, 1]$. If we sell the item to bidder 2 for the payment $p_2 = 1$ (and $p_1 = 0$) for every realization y of Y with $y \neq \tilde{y}$ the outcome is PO in expectation. However, only if we sell the item to bidder 2 also for $y = \tilde{y}$ each deterministic outcome in the support of the randomized outcome is PO in expectation. Hence, PO in expectation does not imply PO ex post.

Next consider the case with two items where bidder 1 has valuations $v_{1,1} = 2$ and $v_{1,2} = 1$, and budget $\beta_1 = 0$, and bidder 2 has valuations $v_{2,1} = 1$ and $v_{2,2} = 2$, and budget $\beta_2 = 0$. Assume we sell both items to bidder 1 for a price of zero with probability $1/2$ and, otherwise, we sell both items to bidder 2 for a price of zero. Certainly, the randomized outcome is PO ex post. However, in expectation each

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[3]Both IR and PO are defined with respect to the reported types, and are satisfied with respect to the true types only if the mechanism also satisfies IC.

bidder gets half of each item and has a utility of $3/2$. Thus, the the randomized outcome is not PO in expectation as giving item 1 to bidder 1 and item 2 to bidder 2 and charging both bidders a payment of zero increases each bidder's utility to 2. Hence, PO ex post does not imply PO in expectation.

2.3 Single-Dimensional Valuations

In this section we present exact characterizations of PO resp. PO ex post outcomes and deterministic mechanisms that are IC with public budgets. We characterize PO and PO ex post by a simpler "no trade" condition, and extend the "classic" characterization results for deterministic mechanisms for single-dimensional valuations (see, e.g., Archer and Tardos [3] and Myerson [90]) that are IC without budgets to settings with public budgets. We then show our main positive result, that is, the existence of *randomized* mechanisms for divisible and indivisible items that are IR in expectation, PO *ex post*, and IC in expectation for *private* budgets. We complement this positive result with an impossibility result for *deterministic* mechanisms for indivisible items that applies even when budgets are *public*.

2.3.1 Exact Characterizations of PO and IC

We start by characterizing PO and PO ex post outcomes through a simpler "no trade" condition. In the deterministic setting we consider an outcome (X, p) and compare it to alternative allocations X'. In the randomized setting we consider a deterministic outcome (X, p) and compare it to possibly randomized allocations X'. In what follows, we use $x'_{i,j}$ to denote the expected fraction of item j that bidder i gets in X'. This allows us to treat the two settings in a unified manner.

We say that an outcome (X, p) for single-dimensional valuations satisfies *no trade (NT)* if (a) $\sum_{i \in N} x_{i,j} = 1$ for all $j \in I$, and (b) there is no allocation X' such that for $\delta_i = \sum_{j \in I}(x'_{i,j} - x_{i,j})\alpha_j$ for all $i \in N$, $W = \{i \in N \mid \delta_i > 0\}$, and $L = \{i \in N \mid \delta_i \leq 0\}$ we have $\sum_{i \in N} \delta_i v_i > 0$ and $\sum_{i \in W} \min(\beta_i - p_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i \geq 0$. The quantity $\delta_i v_i$ is how much valuation bidder i gains/loses when switching from allocation X to X'. The bidders in W are "winners", while the bidders in L are "losers". Winners are willing to increase their payment by at most $\min(\beta_i - p_i, \delta_i v_i)$, while losers would need to be paid $\delta_i v_i$. The definition says that there should be no alternative allocation that strictly increases the sum of the valuations and allows the winners to compensate the losers.

Here is an example: Consider a setting with two bidders and a single indivisible item. Suppose that the bidders have valuations 10 and 5 and budgets 6 and 4, respectively. Then the outcome (X, p) which gives the item to bidder 2 at a price of 4 does *not* satisfy NT. This is because the alternate allocation X' which gives the item to bidder 1 has $\delta_1 v_1 = 1 \cdot 10 = 10$ and $\delta_2 v_2 = (-1) \cdot 5 = -5$ and thus $\sum_{i \in N} \delta_i v_i > 0$. Moreover, $W = \{1\}$ and $L = \{2\}$ and $\sum_{i \in W} \min(\beta_i - p_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i = \min(6, 10) - 5 \geq 0$. Indeed we could re-assign the item from bidder 2 to bidder 1, increase bidder 1's payment by 5, and decrease bidder 2's payment by 5. In the

resulting outcome bidder 1 would have a strictly higher utility, bidder 2's utility would be unchanged, and the sum of payments would increase by one. Hence the original outcome was not PO.

Proposition 2.1. *An outcome (X, p) for single-dimensional valuations and either divisible or indivisible items that respects the budget limits is PO or PO ex post, respectively, if and only if it satisfies NT.*

Proof. We show the claim for the deterministic setting. The claim for the randomized setting follows by interpreting $x'_{i,j}$ as the the expected fraction of item j allocated to bidder i, p'_i as bidder i's expected payment, and u'_i as its expected utility.

First, we show that if (X, p) satisfies PO, then it satisfies NT. To this end, we show that if (X, p) does *not* satisfy NT, then it is *not* PO.

Case 1: $\neg$ NT because $\neg$ (a). We can assign the unassigned fraction of the item $j \in I$ for which $\sum_{i \in N} x_{i,j} < 1$ to any bidder $i \in N$ to get a contradiction to PO.

Case 2: $\neg$ NT because $\neg$ (b). There exists an allocation X' such that $\sum_{i \in N} \delta_i v_i > 0$ and $\sum_{i \in W} \min(\beta_i - p_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i \geq 0$. Consider the outcome (X', p') for which $p'_i = p_i + \min(\beta_i - p_i, \delta_i v_i)$ for all bidders $i \in W$ and $p'_i = p_i + \delta_i v_i$ for all bidders $i \in L$.

For all bidders $i \in N$ we have $u'_i \geq u_i$ because

$$u'_i = \sum_{j \in I} x_{i,j} \alpha_j v_i + \delta_i v_i - p_i - \min(\beta_i - p_i, \delta_i v_i) \geq u_i \quad \text{for } i \in W, \text{ and} \quad (2.1)$$

$$u'_i = \sum_{j \in I} x_{i,j} \alpha_j v_i + \delta_i v_i - p_i - \delta_i v_i = u_i \qquad \qquad \text{for } i \in L.$$

For the auctioneer we have $\sum_{i \in N} p'_i \geq \sum_{i \in N} p_i$ because

$$\sum_{i \in N} p'_i - \sum_{i \in N} p_i = \sum_{i \in W} p'_i + \sum_{i \in L} p'_i - \sum_{i \in N} p_i = \sum_{i \in W} (p_i + \min(\beta_i - p_i, \delta_i v_i))$$

$$+ \sum_{i \in L} (p_i + \delta_i v_i) - \sum_{i \in N} p_i = \sum_{i \in W} \min(\beta_i - p_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i \geq 0. \quad (2.2)$$

If $\sum_{i \in W} \min(\beta_i - p_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i > 0$, then inequality (2.2) is strict showing that $\sum_{i \in N} p'_i > \sum_{i \in N} p_i$. Otherwise, $\sum_{i \in W} \min(\beta_i - p_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i = 0$, and because $\sum_{i \in N} \delta_i v_i > 0$ we must have $\beta_i - p_i < \delta_i v_i$ for at least one bidder $i \in W$. For this bidder i inequality (2.1) is strict showing that $u'_i > u_i$. Hence in both cases (X, p) is *not* PO.

Next, we show that if (X, p) satisfies NT, then it is PO. To this end, we show that if (X, p) is *not* PO, then it does *not* satisfy NT. If (X, p) is *not* PO, then there exists an outcome (X', p') such that $u'_i \geq u_i$ for all bidders $i \in N$ and $\sum_{i \in N} p'_i \geq \sum_{i \in N} p_i$, with at least one of the inequalities strict.

If *not* all items are assigned completely in (X, p), then we have $\neg$ (a) and so (X, p) does *not* satisfy NT. Otherwise, if in (X, p) all items are assigned completely, then to show that (X, p) does *not* satisfy NT we have to show $\neg$ (b). To this end

consider the allocation X' and let $\delta_i = \sum_{j \in I}(x'_{i,j} - x_{i,j})\alpha_j$ for $i \in N$, let $W = \{i \in N \mid \delta_i > 0\}$, and let $L = \{i \in N \mid \delta_i \leq 0\}$.

We begin by showing that $\sum_{i \in W} \min(\beta_i - p_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i \geq 0$. For $i \in N$ we have $p'_i - p_i \leq \min(\beta_i - p_i, \delta_i v_i)$ because $p'_i \leq \beta_i$ implies $p'_i - p_i \leq \beta_i - p_i$, and $u'_i \geq u_i$ implies $p'_i - p_i \leq \delta_i v_i$. It follows that $\sum_{i \in W} \min(\beta_i - p_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i \geq \sum_{i \in W}(p'_i - p_i) + \sum_{i \in L}(p'_i - p_i) = \sum_{i \in N} p'_i - \sum_{i \in N} p_i \geq 0$.

Next we show that $\sum_{i \in N} \delta_i v_i > 0$. Since $u'_i \geq u_i$ for all $i \in N$ we have that $\sum_{i \in N} u'_i \geq \sum_{i \in N} u_i$. This implies $\sum_{i \in N}((\sum_{j \in I} x'_{i,j}\alpha_j v_i) - p'_i) \geq \sum_{i \in N}((\sum_{j \in I} x_{i,j}\alpha_j v_i) - p_i)$, and consequently, $\sum_{i \in N}(\sum_{j \in I}(x'_{i,j} - x_{i,j})\alpha_j v_i) \geq \sum_{i \in N} p'_i - \sum_{i \in N} p_i$. Thus, it follows by $\sum_{i \in N} p'_i \geq \sum_{i \in N} p_i$ that

$$\sum_{i \in N} \delta_i v_i \geq \sum_{i \in N} p'_i - \sum_{i \in N} p_i \geq 0. \tag{2.3}$$

If $u'_i > u_i$ for some $i \in N$, then $\sum_{i \in N} u'_i > \sum_{i \in N} u_i$ and, thus, the first inequality in (2.3) is strict. Otherwise, if $\sum_{i \in N} p'_i > \sum_{i \in N} p_i$, then the second inequality in (2.3) is strict. In both cases strictness of the inequality implies that $\sum_{i \in N} \delta_i v_i > 0$. $\square$

Next we characterize deterministic mechanisms for indivisible items that are IC with public budgets by "value monotonicity" and "payment identity". A deterministic mechanism $M = (f, p)$ for single-dimensional valuations and indivisible items that respects the publicly known budgets satisfies *value monotonicity (VM)* if for all $i \in N$, $\theta_i = (v_i, \beta_i)$, $\theta'_i = (v'_i, \beta_i)$, and $\theta_{-i} = (v_{-i}, \beta_{-i})$ we have that $v_i \leq v'_i$ implies $\sum_{j \in I} f_{i,j}(\theta_i, \theta_{-i})\alpha_j \leq \sum_{j \in I} f_{i,j}(\theta'_i, \theta_{-i})\alpha_j$. A deterministic mechanism $M = (f, p)$ for single-dimensional valuations and indivisible items that respects the publicly known budgets satisfies *payment identity (PI)* if for all $i \in N$ and $\theta = (v, \beta)$ with $c_{\gamma_t} \leq v_i < c_{\gamma_{t+1}}$ we have $p_i(\theta) = p_i((0, \beta_i), \theta_{-i}) + \sum_{s=1}^{t}(\gamma_s - \gamma_{s-1})c_{\gamma_s}(\beta_i, \theta_{-i})$, where $\gamma_0 < \gamma_1 < \ldots$ are the values $\sum_{j \in I} x_{i,j}\alpha_j$ can take and $c_{\gamma_s}(\beta_i, \theta_{-i})$ for $1 \leq s \leq t$ are the corresponding critical valuations. While VM ensures that stating a higher valuation cannot lead to a worse allocation, PI gives a formula for the payment in terms of the possible allocations and the critical valuations.

Proposition 2.2. *A deterministic mechanism $M = (f, p)$ for single-dimensional valuations and indivisible items that respects the publicly known budgets is IC if and only if it satisfies VM and PI.*

Proof. We begin by showing that if M satisfies VM and PI, then it satisfies IC. For a contradiction assume that M satisfies VM and PI, but that it does *not* satisfy IC. Then there exists $i \in N$, $\theta_i = (v_i, \beta_i)$, $\theta'_i = (v'_i, \beta_i)$, and $\theta_{-i} = (v_{-i}, \beta_{-i})$ with $v_i \neq v'_i$ such that $u_i(f_i(\theta'_i, \theta_{-i}), p(\theta'_i, \theta_{-i}), \theta_i) > u_i(f_i(\theta_i, \theta_{-i}), p(\theta_i, \theta_{-i}), \theta_i)$.

Let $c_{\gamma_t}(\beta_i, \theta_{-i}) \leq v_i < c_{\gamma_{t+1}}(\beta_i, \theta_{-i})$ and let $c_{\gamma_{t'}}(\beta_i, \theta_{-i}) \leq v'_i < c_{\gamma_{t'+1}}(\beta_i, \theta_{-i})$.

If $v_i > v'_i$ then since M satisfies VM and PI the utilities u_i and u'_i that bidder i gets from reports θ_i and θ'_i satisfy $u_i - u'_i = (\gamma_t - \gamma_{t'})v_i - \sum_{s=t'+1}^{t}(\gamma_s - \gamma_{s-1})c_{\gamma_s}(\beta_i, \theta_{-i}) \geq (\gamma_t - \gamma_{t'})v_i - \sum_{s=t'+1}^{t}(\gamma_s - \gamma_{s-1})v_i = 0$.

If $v_i < v_i'$ then since M satisfies VM and PI the utilities u_i' and u_i that bidder i gets from reports θ_i' and θ_i satisfy $u_i' - u_i = (\gamma_{t'} - \gamma_t)v_i - \sum_{s=t+1}^{t'}(\gamma_s - \gamma_{s-1})c_{\gamma_s}(\beta_i, \theta_{-i}) \leq (\gamma_{t'} - \gamma_t)v_i - \sum_{s=t+1}^{t'}(\gamma_s - \gamma_{s-1})v_i = 0$.

We conclude that in both cases bidder i is weakly better off when he reports truthfully. This contradicts our assumption that M does *not* satisfy IC.

Next we show that if M satisfies IC, then it satisfies VM. By contradiction assume that M satisfies IC, but that it does *not* satisfy VM. Then there exists $i \in N$, $\theta_i = (v_i, \beta_i)$, $\theta_i' = (v_i', \beta_i)$, and $\theta_{-i} = (v_{-i}, \beta_{-i})$ with $v_i < v_i'$ such that $\sum_{j \in I} f_{i,j}(\theta_i, \theta_{-i})\alpha_j > \sum_{j \in I} f_{i,j}(\theta_i', \theta_{-i})\alpha_j$. Since M satisfies IC bidder i with type θ_i does *not* benefit from reporting θ_i', and vice versa. Thus, it follows $\sum_{j \in I} f_{i,j}(\theta_i, \theta_{-i})\alpha_j v_i - p_i(\theta_i, \theta_{-i}) \geq \sum_{j \in I} f_{i,j}(\theta_i', \theta_{-i})\alpha_j v_i - p_i(\theta_i', \theta_{-i})$, and $\sum_{j \in I} f_{i,j}(\theta_i', \theta_{-i})\alpha_j v_i' - p_i(\theta_i', \theta_{-i}) \geq \sum_{j \in I} f_{i,j}(\theta_i, \theta_{-i})\alpha_j v_i' - p_i(\theta_i, \theta_{-i})$. By combining these inequalities we get $(\sum_{j \in I} f_{i,j}(\theta_i, \theta_{-i})\alpha_j - \sum_{j \in I} f_{i,j}(\theta_i', \theta_{-i})\alpha_j)(v_i - v_i') \geq 0$. Since $\sum_{j \in I} f_{i,j}(\theta_i, \theta_{-i})\alpha_j > \sum_{j \in I} f_{i,j}(\theta_i', \theta_{-i})\alpha_j$ this shows that $v_i \geq v_i'$ and gives a contradiction to our assumption that $v_i < v_i'$.

We conclude the proof by showing that if M satisfies IC, then it satisfies PI. For a contradiction assume that M satisfies IC, but that it does *not* satisfy PI. Then there exists $i \in N$, $\theta_i' = (v_i', \beta_i)$, and $\theta_{-i} = (v_{-i}, \beta_{-i})$ with $c_{\gamma_{t'}} \leq v_i' < c_{\gamma_{t'+1}}$ such that $p_i(\theta_i', \theta_{-i}) \neq p_i((0, \beta_i), \theta_{-i}) + \sum_{s=1}^{t'}(\gamma_s - \gamma_{s-1})c_{\gamma_s}(\beta_i, \theta_{-i})$, where the γ_s are the sum over the α's of all possible allocations in non-increasing order and the $c_{\gamma_s}(\beta_i, \theta_{-i})$ are the smallest valuations (or critical valuations) that make bidder i win γ_s.

Consider the smallest v_i' such that this is the case. For this v_i' we must have $v_i' = c_{\gamma_{t'}}(\beta_i, \theta_{-i}) > c_{\gamma_0}(\beta_i, \theta_{-i}) = 0$. We must have $v_i' = c_{\gamma_{t'}}(\beta_i, \theta_{-i})$ because by VM bidder i's allocation for all reports $\theta_i'' = (v_i'', \beta_i)$ with v_i'' such that $c_{\gamma_{t'}}(\beta_i, \theta_{-i}) \leq v_i'' < c_{\gamma_{t'+1}}(\beta_i, \theta_{-i})$ is the same and, thus, by IC he must face the same payment. We must have $c_{\gamma_{t'}}(\beta_i, \theta_{-i}) > c_{\gamma_0}(\beta_i, \theta_{-i}) = 0$ because for $v_i' = 0$ we have $p(\theta_i', \theta_{-i}) = p((0, \beta_i), \theta_{-i})$ by definition.

Case 1: $p_i(\theta_i', \theta_{-i}) > p_i((0, \beta_i), \theta_{-i}) + \sum_{s=1}^{t'}(\gamma_s - \gamma_{s-1})c_{\gamma_s}(\beta_i, \theta_{-i})$. Consider $\theta_i = (v_i, \beta_i)$ with $v_i < v_i'$ such that $c_{\gamma_{t'-1}}(\beta_i, \theta_{-i}) \leq v_i < c_{\gamma_{t'}}(\beta_i, \theta_{-i})$. Since $v_i < v_i'$ we have $p_i(\theta_i, \theta_{-i}) = p_i((0, \beta_i), \theta_{-i}) + \sum_{s=1}^{t'-1}(\gamma_s - \gamma_{s-1})c_{\gamma_s}(\beta_i, \theta_{-i})$. If bidder i's type is θ_i' then for the utilities u_i' and u_i that he gets for reports θ_i' and θ_i we have $u_i' - u_i < (\gamma_{t'} - \gamma_{t'-1})v_i' - (\gamma_{t'} - \gamma_{t'-1})c_{\gamma_{t'}}(\beta_i, \theta_{-i}) = 0$. This shows that bidder i with type θ_i' has an incentive to misreport his type as θ_i and contradicts our assumption that M satisfies IC.

Case 2: $p_i(\theta_i', \theta_{-i}) < p_i((0, \beta_i), \theta_{-i}) + \sum_{s=1}^{t'}(\gamma_s - \gamma_{s-1})c_{\gamma_s}(\beta_i, \theta_{-i})$. Let $\epsilon = p_i((0, \beta_i), \theta_{-i}) + \sum_{s=1}^{t'}(\gamma_s - \gamma_{s-1})c_{\gamma_s}(\beta_i, \theta_{-i}) - p_i(\theta_i', \theta_{-i})$ and consider $\theta_i = (v_i, \beta_i)$ with $v_i < v_i'$ such that $c_{\gamma_{t'-1}}(\beta_i, \theta_{-i}) \leq v_i < c_{\gamma_{t'}}(\beta_i, \theta_{-i})$. Since $v_i < v_i'$ we have $p_i(\theta_i, \theta_{-i}) = p_i((0, \beta_i), \theta_{-i}) + \sum_{s=1}^{t'-1}(\gamma_s - \gamma_{s-1})c_{\gamma_s}(\beta_i, \theta_{-i})$. If bidder i's type is θ_i then for the utilities u_i' and u_i that he gets from reports θ_i' and θ_i we have $u_i' - u_i = (\gamma_{t'} - \gamma_{t'-1})v_i - (\gamma_{t'} - \gamma_{t'-1})c_{\gamma_{t'}}(\beta_i, \theta_{-i}) + \epsilon$. Since this is true for all v_i with $c_{\gamma_{t'-1}}(\beta_i, \theta_{-i}) \leq v_i < c_{\gamma_{t'}}(\beta_i, \theta_{-i})$ we can choose v_i such that

$(\gamma_{t'} - \gamma_{t'-1})(v_i - c_{\gamma_{t'}}(\beta_i, \theta_{-i})) > -\epsilon$. We get $u'_i - u_i > 0$. This shows that bidder i with type θ_i has an incentive to misreport his type as θ'_i and contradicts our assumption that M satisfies IC. $\qquad\square$

2.3.2 Randomized Mechanisms for Indivisible and Divisible Items

We obtain our positive result through a reduction to the setting with a single (and thus homogeneous) item that allows us to apply the following proposition by Bhattacharya et al. [11]. The basic building block of the mechanisms mentioned in this proposition is the "adaptive clinching auction" for a single divisible item. It is described for two bidders by Dobzinski et al. [30] and as a "continuous time process" for arbitrarily many bidders in [11].

Proposition 2.3 (Bhattacharya et al. [11]). *For a single divisible item there exists a deterministic mechanism that satisfies IR, NPT, PO, and IC for public budgets. Additionally, for a single divisible or indivisible item there exists a randomized mechanism that satisfies IR in expectation, NPT ex post, PO ex post, and IC in expectation for private budgets.*

For indivisible items we reduce the multi-item setting to the single-item setting by applying the *randomized* mechanism for a single *indivisible* item in [11] to a single indivisible item for which bidder $i \in N$ has valuation $\tilde{v}_i = \sum_{j\in I} \alpha_j v_i$. We then map the single-item outcome $(\tilde{X}, \tilde{p})$ to an outcome (X, p) for the multi-item setting by setting $x_{i,j} = 1$ for all $j \in I$ if and only if $\tilde{x}_{i,1} = 1$ and setting $p_i = \tilde{p}_i$ for all $i \in N$.

A similar reduction works in the case of divisible items. The only difference is that in this case we use the *deterministic* or *randomized* mechanisms of [11] for a single *divisible* item, and then map the single-item outcome $(\tilde{X}, \tilde{p})$ into a multi-item outcome by setting $x_{i,j} = \tilde{x}_{i,1}$ for all $i \in N$ and all $j \in I$ and setting $p_i = \tilde{p}_i$ for all $i \in N$.

The main difficulty in proving that the resulting mechanisms have the claimed properties is to establish that they are PO/PO ex post. For this we argue that these particular ways of mapping the single-item outcome to a multi-item outcome preserve a specific structural property of the single-item outcome which remains to be sufficient for PO/PO ex post also in the multi-item setting.

Proposition 2.4. *For indivisible or divisible items, if $(\tilde{X}, \tilde{p})$ denotes the randomized outcome for a single item by the randomized mechanism given in [11], and (X, p) denotes the randomized outcome for the multi-item setting constructed as described above, then $\mathbf{E}\left[u_i(X_i, p_i, (v_i, \beta_i))\right] = \mathbf{E}\left[u_i(\tilde{X}_i, \tilde{p}_i, (\tilde{v}_i, \beta_i))\right]$ for all $i \in N$. Similarly, for divisible items, if $(\tilde{X}, \tilde{p})$ denotes the deterministic outcome for a single item of the deterministic mechanism in [11] and (X, p) denotes the deterministic outcome for the multi-item setting constructed as described above, then $u_i(X_i, p_i, (v_i, \beta_i)) = u_i(\tilde{X}_i, \tilde{p}_i, (\tilde{v}_i, \beta_i))$ for all $i \in N$.*

Proof. First suppose that the payments are deterministic. If $p_i > \beta_i$ then $\tilde{p}_i > \beta_i$ and $u_i(X_i, p_i, (v_i, \beta_i)) = u_i(\tilde{X}_i, \tilde{p}_i, (\tilde{v}_i, \beta_i)) = -\infty$. Otherwise, $u_i(X_i, p_i, (v_i, \beta_i)) = \sum_{j=1}^m (x_{i,j}\alpha_j v_i) - p_i = \tilde{x}_{i,1}\tilde{v}_i - \tilde{p}_i = u_i(\tilde{X}_i, \tilde{p}_i, (\tilde{v}_i, \beta_i))$.

Next suppose that the payments are randomized. If $\mathbf{Pr}\left[p_i > \beta_i\right] > 0$ then $\mathbf{Pr}\left[\tilde{p}_i > \beta_i\right] > 0$ and $\mathbf{E}\left[u_i(X_i, p_i, (v_i, \beta_i))\right] = \mathbf{E}\left[u_i(\tilde{X}_i, \tilde{p}_i, (\tilde{v}_i, \beta_i))\right] = -\infty$. Otherwise, $\mathbf{E}\left[u_i(X_i, p_i, (v_i, \beta_i))\right] = \mathbf{E}\left[\sum_{j=1}^m (x_{i,j}\alpha_j v_i) - p_i\right] = \mathbf{E}\left[\tilde{x}_{i,1}\tilde{v}_i - \tilde{p}_i\right] = \mathbf{E}\left[u_i(\tilde{X}_i, \tilde{p}_i, (\tilde{v}_i, \beta_i))\right]$. $\qquad\square$

Theorem 2.5. *For single-dimensional valuations, divisible or indivisible items, and private budgets there is a randomized mechanism that satisfies IR in expectation, NPT ex post, PO ex post, and IC in expectation. Additionally, for single-dimensional valuations and divisible items there is a deterministic mechanism that satisfies IR, NPT, PO, and IC for public budgets.*

Proof. IR or IR in expectation and IC or IC in expectation follow from Proposition 2.4 and the fact that the corresponding mechanisms in [11] are IR or IR in expectation and IC or IC in expectation, respectively. NPT or NPT ex post follows from the fact that the payments in our mechanisms and the mechanisms in [11] are the same, and the mechanisms in [11] satisfy NPT or NPT ex post, respectively. For PO (ex post) we argue that the structural property of the outcomes of the mechanisms in [11] that (a) $\sum_{i \in N} \tilde{x}_{i,1} = 1$ and (b) $\tilde{x}_{i,1} > 0$ and $\tilde{v}_{i'} > \tilde{v}_i$ imply $\tilde{p}_{i'} = \beta_{i'}$ (both ex post) is preserved by the mapping to the multi-item setting and remains to be sufficient for PO (ex post).

We first show that the property is preserved. For this observe that $\sum_{i \in N} \tilde{x}_{i,1} = 1$ implies that $\sum_{i \in N} x_{i,j} = 1$ for all $j \in I$ and that "$\tilde{x}_{i,1} > 0$ and $\tilde{v}_{i'} > \tilde{v}_i$ imply $\tilde{p}_{i'} = \beta_{i'}$" implies that "$\sum_{j \in I} x_{i,j} > 0$ and $v_{i'} > v_i$ imply $p_{i'} = \beta_{i'}$". Recall that $x'_{i,j}$ denotes the expected fraction of item j which bidder i gets.

Next we show that the property remains to be sufficient for PO (ex post). For this assume by contradiction that the outcome (X, p) is *not* PO (ex post). Then, by Proposition 2.1, there exists a (possibly randomized) X' such that $\sum_{i \in N} \delta_i v_i > 0$ and $\sum_{i \in W} \min(\beta_i - p_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i \geq 0$, where $\delta_i = \sum_{j \in I}(x'_{i,j} - x_{i,j})\alpha_j$, $W = \{i \in N \mid \delta_i > 0\}$, and $L = \{i \in N \mid \delta_i \leq 0\}$.

Since (X, p) satisfies condition (a), i.e., $\sum_{i \in N} x_{i,j} = 1$ for all $j \in I$, and X' is a valid allocation, i.e., $\sum_{i \in N} x'_{i,j} \leq 1$ for all $j \in I$, we have $\sum_{i \in N} \delta_i = \sum_{j \in I}\sum_{i \in N}(x'_{i,j} - x_{i,j})\alpha_j \leq 0$. Since $\sum_{i \in N} \delta_i v_i > 0$ we have $\sum_{i \in W} \delta_i v_i \geq \sum_{i \in N} \delta_i v_i > 0$ and, thus, $\sum_{i \in W} \delta_i > 0$. We conclude that $\sum_{i \in L} \delta_i = \sum_{i \in N} \delta_i - \sum_{i \in W} \delta_i < 0$ and, thus, $\sum_{i \in L} \delta_i v_i < 0$.

Since (X, p) satisfies condition (b), i.e., $\sum_{j \in I} x_{i,j} > 0$ and $v_{i'} > v_i$ imply $p_{i'} = \beta_{i'}$, there exists a t with $1 \leq t \leq n$ such that (1) $\sum_{j \in I} x_{i,j} \geq 0$ and $p_i = \beta_i$ for $1 \leq i \leq t$, (2) $\sum_{j \in I} x_{i,j} \geq 0$ and $p_i \leq \beta_i$ for $i = t + 1$, and (3) $\sum_{j \in I} x_{i,j} = 0$ and $p_i \leq \beta_i$ for $t + 2 \leq i \leq n$.

We complete the proof by distinguishing three cases, and showing that in each of the three cases we get a contradiction.

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Case 1: $t = n$. Then $\sum_{i\in W}\min(\beta_i - p_i, \delta_i v_i) = 0$ because $\beta_i = p_i$ for all $i \in N \supseteq W$ and, thus, $\sum_{i\in W}\min(\beta_i - p_i, \delta_i v_i) + \sum_{i\in L}\delta_i v_i < 0$.

Case 2: $t < n$ and $W \cap \{1,\ldots,t\} = \emptyset$. Then $\sum_{i\in W}\delta_i v_i \leq \sum_{i\in W}\delta_i v_{t+1}$ and $\sum_{i\in L}\delta_i v_i \leq \sum_{i\in L}\delta_i v_{t+1}$ because $\sum_{j\in I}x_{i,j} = 0$ for all $i \geq t+2$ and, thus, $\delta_i = 0$ for all $i \in L \setminus \{1,\ldots,t+2\}$. Thus, $\sum_{i\in N}\delta_i v_i = \sum_{i\in W}\delta_i v_i + \sum_{i\in L}\delta_i v_i \leq \sum_{i\in N}\delta_i v_{t+1} \leq 0$.

Case 3: $t < n$ and $W \cap \{1,\ldots,t\} \neq \emptyset$. Then $\sum_{i\in W}\min(p_i - \beta_i, \delta_i v_i) \leq \sum_{i\in W\setminus\{1,\ldots,t\}}\delta_i v_{t+1}$ and $\sum_{i\in L}\delta_i v_i \leq \sum_{i\in L}\delta_i v_{t+1}$ and, thus, $\sum_{i\in W}\min(p_i - \beta_i, \delta_i v_i) + \sum_{i\in L}\delta_i v_i \leq (\sum_{i\in N}\delta_i - \sum_{i\in W\cap\{1,\ldots,t\}}\delta_i)v_{t+1} < 0$. $\qquad\square$

2.3.3 Deterministic Mechanisms for Indivisible Items

The proof of our impossibility result uses the characterizations of PO outcomes and mechanisms that are IC with public budgets as follows: (a) PO is characterized by NT and NT induces a lower bound on the bidders' payments for a *specific* allocation, namely for the case that bidder 1 only gets item 2. (b) IC, in turn, is characterized by VM and PI. Now VM and PI can be used to extend the lower bound on the payments for the *specific* allocation to *all* possible allocations. (c) Finally, IR implies upper bounds on the payments that, with a suitable choice of valuations, conflict with the lower bounds on the payments induced by NT, VM, and PI.

Theorem 2.6. *For single-dimensional valuations, indivisible items, and public budgets there can be no deterministic mechanism $M = (f, p)$ that satisfies IR, PO, and IC.*

Proof. For a contradiction suppose that there is a mechanism $M = (f, p)$ that is IR, PO, and IC for all n and all m. Consider a setting with $n = 2$ bidders and $m = 2$ items in which $v_1 > v_2 > 0$ and $\beta_1 > \alpha_1 v_2$.

Observe that if bidder 1's valuation was $v_1' = 0$ and he reported his valuation truthfully, then since M satisfies IR his utility would be $u_1((0, \beta_1), \theta_{-1}, (0, \beta_1)) = -p_1((0, \beta_1), \theta_{-1}) \geq 0$. This shows that $p_1((0, \beta_1), \theta_{-1}) \leq 0$.

By PO, which by Proposition 2.1 is characterized by NT, bidder 1 with valuation $v_1 > v_2$ and budget $\beta_1 > \alpha_1 v_2$ must win at least one item because otherwise he could buy any item from bidder 2 and compensate him for his loss.

PO, respectively NT, also implies that bidder 1's payment for item 2 must be strictly larger than $\beta_1 - (\alpha_1 - \alpha_2)v_2$ because otherwise he could trade item 2 against item 1 and compensate bidder 2 for his loss.

By IC, which by Proposition 2.2 is characterized by VM and PI, bidder 1's payment for item 2 is given by $p_1(\{2\}) = p_1((0, \beta_1), \theta_{-1}) + \alpha_2 c_{\alpha_2}(\beta_1, \theta_{-1})$, where c_{α_2} is the critical valuation for winning item 2. Together with $p_1(\{2\}) > \beta_1 - (\alpha_1 - \alpha_2)v_2$ this shows that $c_{\alpha_2}(\beta_1, \theta_{-1}) > (1/\alpha_2)(\beta_1 - (\alpha_1 - \alpha_2)v_2 - p_1((0, \beta_1), \theta_{-1}))$.

IC, respectively VM and PI, also imply that bidder 1's payment for any non-empty set of items S in terms of the fractions $\gamma_t = \sum_{j\in S}\alpha_j > \cdots > \gamma_1 = \alpha_2 > \gamma_0 = 0$ and corresponding critical valuations $c_{\gamma_t}(\beta_1, \theta_{-1}) \geq \cdots \geq c_{\gamma_1}(\beta_1, \theta_{-1}) = c_{\alpha_2}(\beta_1, \theta_{-1})$ is $p_1(S) = p_1((0, \beta_1), \theta_{-1}) + \sum_{s=1}^{t}(\gamma_s - \gamma_{s-1})c_{\gamma_s}(\beta_1, \theta_{-1})$. Since

$c_{\gamma_s}(\beta_1, \theta_{-1}) \geq c_{\alpha_2}(\beta_1, \theta_{-1})$ for all s and $\sum_{s=1}^{t}(\gamma_s - \gamma_{s-1}) = \sum_{j \in S} \alpha_j$ we obtain $p_1(S) \geq p_1((0, \beta_1), \theta_{-1}) + (\sum_{j \in S} \alpha_j) c_{\alpha_2}(\beta_1, \theta_{-1})$.

Combining this lower bound on $p_1(S)$ with the lower bound on $c_{\alpha_2}(\beta_1, \theta_{-1})$ shows that $p_1(S) > (\sum_{j \in S} \alpha_j / \alpha_2)(\beta_1 - (\alpha_1 - \alpha_2)v_2)$.

For v_1 such that $(1/\alpha_2)(\beta_1 - (\alpha_1 - \alpha_2)v_2) > v_1 > v_2$ we know that bidder 1 must win some item, but for any non-empty set of items S the lower bound on bidder 1's payment for S contradicts IR. $\qquad \square$

2.4 Multi-Dimensional Valuations

In this section we obtain a partial characterization of deterministic mechanisms that are IC with public budgets by generalizing the "weak monotonicity" condition of Bikhchandani et al. [14] from settings without budgets to settings with budgets. We use this partial characterization together with a sophisticated misreport, in which a bidder understates his valuation for some item and overstates his valuation for another item, to prove that there can be no *deterministic* mechanism for *divisible* items that is IR, PO, and IC with *public* budgets. Afterwards, we use this result to show that there can be no *randomized* mechanism for either *divisible* or *indivisible* items that is IR in expectation, PO in expectation, and IC in expectation for *public* budgets.

2.4.1 Partial Characterization of IC

For settings *without budgets* every deterministic mechanism that is incentive compatible must satisfy what is known as *weak monotonicity (WMON)*, namely if X_i' and X_i are the allocations of bidder i for reports v_i' and v_i, then the difference in the valuations for the two allocations must be at least as large under v_i' as under v_i, i.e., $v_i'(f_i(\theta_i', \theta_{-i})) - v_i'(f_i(\theta_i, \theta_{-i})) \geq v_i(f_i(\theta_i', \theta_{-i})) - v_i(f_i(\theta_i, \theta_{-i}))$. We show that this is also true for deterministic mechanisms that respect the public budgets.

Proposition 2.7. *If a deterministic mechanism $M = (f, p)$ for multi-dimensional valuations and either divisible or indivisible items that respects the publicly known budget limits is IC, then it satisfies WMON.*

Proof. Fix $i \in N$ and $\theta_{-i} = (v_{-i}, \beta_{-i})$. By IC bidder i does not benefit from reporting $\theta_i' = (v_i', \beta_i)$ when his true type is $\theta_i = (v_i, \beta_i)$, nor does he benefit from reporting $\theta_i = (v_i, \beta_i)$ when his true type is $\theta_i' = (v_i', \beta_i)$. Thus, $v_i(f_i(\theta_i, \theta_{-i})) - p_i(\theta_i, \theta_{-i}) \geq v_i(f_i(\theta_i', \theta_{-i})) - p_i(\theta_i', \theta_{-i})$, and $v_i'(f_i(\theta_i', \theta_{-i})) - p_i(\theta_i', \theta_{-i}) \geq v_i'(f_i(\theta_i, \theta_{-i})) - p_i(\theta_i, \theta_{-i})$. By combining these inequalities we get $v_i'(f_i(\theta_i', \theta_{-i})) - v_i'(f_i(\theta_i, \theta_{-i})) \geq v_i(f_i(\theta_i', \theta_{-i})) - v_i(f_i(\theta_i, \theta_{-i}))$. $\qquad \square$

2.4.2 Deterministic Mechanisms for Divisible Items

We prove the impossibility result by analyzing a setting with two bidders and two items. This restriction is without loss of generality as the impossibility result for an

arbitrary number of bidders $n > 2$ and an arbitrary number of items $m > 2$ follows by setting $v_{i,j} = 0$ if $i > 2$ or $j > 2$. In our impossibility proof bidder 2 is not budget restricted (i.e., $\beta_2 > v_{2,1} + v_{2,2}$). Since bidders can misreport their valuations, it is not sufficient to study a single input to prove the impossibility. Hence, we study the outcome for three related cases, namely Case 1 where $v_{1,1} < v_{2,1}$ and $v_{1,2} < v_{2,2}$; Case 2 where $v_{1,1} > v_{2,1}$, $v_{1,2} < v_{2,2}$, and $\beta_1 > v_{1,1}$; and Case 3 where $v_{1,1} > v_{2,1}$, $v_{1,2} > v_{2,2}$, and additionally, $\beta_1 > v_{1,1}$, $v_{1,1}v_{2,2} > v_{1,2}v_{2,1}$, and $v_{2,1} + v_{2,2} > \beta_1$. We give a partial characterization of those cases, which allows us to analyze the rational behavior of the bidders.

Case 1 is easy: Bidder 2 is not budget restricted and has the highest valuation for both items; so he will get both items. Thus, in this case the utility for bidder 1 is zero.

Lemma 2.8 (Case 1). *Given $\beta_2 > v_{2,1} + v_{2,2}$, $v_{2,1} > v_{1,1}$ and $v_{2,2} > v_{1,2}$, then $x_{1,1} = 0$, $x_{1,2} = 0$, $x_{2,1} = 1$, $x_{2,2} = 1$, and $u_1 = 0$ in every IR and PO outcome selected by an IC mechanism.*

Proof. We divide the proof into the following parts: in (a) we show that $x_{1,1} = 0$, $x_{1,2} = 0$, $x_{2,1} = 1$, and $x_{2,2} = 1$, and in (b) we show that $u_1 = 0$.

To (a): Let us assume by contradiction that we have an IR and PO outcome where $x_{1,1} > 0$ or $x_{1,2} > 0$. IR requires that $p_2 \leq x_{2,1}v_{2,1} + x_{2,2}v_{2,2}$. Hence, bidder 2 can buy the fractions $x_{1,1}$ of item 1 and $x_{1,2}$ of item 2 for a payment p with $x_{1,1}v_{2,1} + x_{1,2}v_{2,2} > p \geq x_{1,1}v_{1,1} + x_{1,2}v_{1,2}$ from bidder 1. Because of $v_{2,1} > v_{1,1}$ and $v_{2,2} > v_{1,2}$ such a payment exists and bidder 2 has enough money, since $\beta_2 > v_{2,1} + v_{2,2}$ implies $\beta_2 > v_{2,1} + v_{2,2} = (x_{1,1} + x_{2,1})v_{2,1} + (x_{1,2} + x_{2,2})v_{2,2} > p_2 + p$. The utility of bidder 2 would increase and the utilities of bidder 1 and the auctioneer would not decrease. Contradiction to PO!

To (b): We have already shown before that bidder 1 gets no fraction of the items, and therefore, IR implies that his payments cannot be positive.

Let us consider the subcase where $v_{1,1} = v_{1,2} = 0$ and bidder 1 reports truthfully. The valuations of bidder 2 are positive. Because of IR the payment of bidder 2 cannot exceed his reported valuation, but (a) holds when his reported valuations are positive. Therefore, bidder 2 would have an incentive to understate his valuation when his payment was positive. Hence, IR of the auctioneer implies that the payment of both bidders is equal to zero. This means, that the utility of bidder 1 is zero in this case.

If there existed any other reported valuation of bidder 1, where he gets no items, but where his payments are negative, then he would have an incentive to lie, when his valuations are equal to zero. This would contradict IC! □

In Case 2, bidder 1 has the higher valuation for item 1, while bidder 2 has the higher valuation for item 2. Thus, bidder 1 gets item 1 and bidder 2 gets item 2. Since the only difference to Case 1 is that in Case 2 $v_{1,1} > v_{2,1}$ while in Case 1 $v_{1,1} < v_{2,1}$, the critical value whether bidder 2 gets item 1 or not is $v_{2,1}$, and thus in every IC mechanism, bidder 1 has to pay $v_{2,1}$ and his utility is $v_{1,1} - v_{2,1}$.

Lemma 2.9 (Case 2). *Given $\beta_2 > v_{2,1} + v_{2,2}$, $v_{1,1} > v_{2,1}$, $v_{2,2} > v_{1,2}$, and $\beta_1 > v_{1,1}$, then $x_{1,1} = 1$, $x_{1,2} = 0$, $x_{2,1} = 0$, $x_{2,2} = 1$, and $u_1 = v_{1,1} - v_{2,1}$ in every IR and PO outcome selected by an IC mechanism.*

Proof. We divide the proof into the following parts: in (a) we show that $x_{1,1} = 1$, $x_{1,2} = 0$, $x_{2,1} = 0$, and $x_{2,2} = 1$, and in (b) we show that $u_1 = v_{1,1} - v_{2,1}$.

To (a): Let us assume by contradiction that $x_{1,2} > 0$. Then, bidder 2 can buy these fractions of item 2 for a payment p with $x_{1,2}v_{2,2} > p \geq x_{1,2}v_{1,2}$, which exists because of $v_{2,2} > v_{1,2}$. IR and $\beta_2 > v_{2,1} + v_{2,2}$ ensure that bidder 2 has enough budget, because $\beta_2 > v_{2,1} + v_{2,2} = (x_{1,1} + x_{2,1})v_{2,1} + (x_{1,2} + x_{2,2})v_{2,2} \geq p_2 + x_{1,1}v_{2,1} + x_{1,2}v_{2,2} > p_2 + p$. The utility of the bidder 2 would increase, while the utilities of bidder 1 and the auctioneer would not decrease. Contradiction to PO!

Otherwise, let us assume that $x_{1,1} < 1$ and $x_{1,2} = 0$. Then, bidder 1 can buy the other fractions of item 1 for a payment p with $x_{2,1}v_{1,1} > p \geq x_{2,1}v_{2,1}$, which exists because of $v_{1,1} > v_{2,1}$. IR and $\beta_1 > v_{1,1}$ ensure that bidder 1 has enough budget, because $\beta_1 > v_{1,1} = (x_{1,1} + x_{2,1})v_{1,1} \geq p_1 + x_{2,1}v_{1,1} > p_1 + p$. The utility of bidder 1 would increase, while the utilities of bidder 2 and the auctioneer would not decrease. Contradiction to PO!

To (b): We show first that $p_1 \leq v_{2,1}$. Since $x_{1,1} = 1$ and $x_{1,2} = 0$, IR requires that $p_1 \leq v_{1,1}$. If $p_1 > v_{2,1}$, then bidder 1 has an incentive to lie. If he states that his valuation for item 1 is $v'_{1,1}$ with $p_1 > v'_{1,1} > v_{2,1}$, then the allocation of the items does not change, but he pays less because of IR. Contradiction to IC!

Now, we show that $p_1 \geq v_{2,1}$. Let us therefore assume by contradiction that $p_1 < v_{2,1}$. If we have $v'_{1,1}$ with $p_1 < v'_{1,1} < v_{2,1}$ instead of $v_{1,1}$, and all the other valuations are left unchanged, then Lemma 2.8 implies that $u'_1 = 0$. Hence, in this case bidder 1 can increase his utility when he lies and states that his valuation is $v_{1,1}$, because his utility would be $v'_{1,1} - p_1 > 0$. Contradiction to IC!

Since bidder 1 gets all fractions of item 1, no fraction of item 2, and has to pay $v_{2,1}$, his utility is $v_{1,1} - v_{2,1}$. $\qquad\square$

In Case 3, bidder 1 has a higher valuation than bidder 2 for both items, but he does not have enough budget to pay for both fully. In Lemma 2.10 we show that if bidder 1 does not spend his whole budget ($p_1 < \beta_1$) he must fully receive both items (specifically $x_{1,2} = 1$), since if not, he would buy more of them. Additionally, even if he spent his budget fully (i.e., $p_1 = \beta_1$) his utility u_i, which equals $x_{1,1}v_{1,1} + x_{1,2}v_{1,2} - \beta_1$, must be non-negative. Since $\beta_1 > v_{1,1}$ this implies that $x_{1,1}$ must be 1, i.e., he must receive item 1 fully, and $x_{1,2}$ must be non-zero.

Lemma 2.10 (Case 3, Part a). *Given $v_{1,1} > v_{2,1}$, $v_{1,2} > v_{2,2}$, $\beta_1 > v_{1,1}$, and $v_{1,1}v_{2,2} > v_{1,2}v_{2,1}$, if $p_1 < \beta_1$ then $x_{1,1} = 1$ and $x_{1,2} = 1$, else if $p_1 = \beta_1$ then $x_{1,1} = 1$ and $x_{1,2} > 0$, in every IR and PO outcome.*

Proof. We divide the proof into the following parts: in (a) we show that $x_{1,1} = 1$ and $x_{1,2} = 1$ if $p_1 < \beta_1$, in (b) we show that $x_{1,2} > (1 - x_{1,1})\frac{v_{2,1}}{v_{2,2}}$ if $p_1 = \beta_1$, and in (c) we show that $x_{1,1} = 1$ and $x_{1,2} > 0$ if $p_1 = \beta_1$.

To (a): Let us assume by contradiction that $p_1 < \beta_1$ and $x_{1,j} < 1$ for an item $j \in \{1,2\}$. Bidder 1 can increase his utility by buying $\min\{\frac{\beta_1-p_1}{p}, x_{2,j}\}$ fractions of item j for a unit price p with $v_{1,j} > p \geq v_{2,j}$ from bidder 2. Such a price exists, because of $v_{1,1} > v_{2,1}$ and $v_{1,2} > v_{2,2}$. Bidder 1 has enough money for the trade, because $p_1 + p\min\{\frac{\beta_1-p_1}{p}, x_{2,j}\} = \min\{\beta_1, p_1 + px_{2,j}\} \leq \beta_1$. The utility of bidder 1 would increase, and the utilities of bidder 2 and the auctioneer would not decrease. Contradiction to PO!

To (b): IR requires $\beta_1 = p_1 \leq v_{1,1}x_{1,1} + v_{1,2}x_{1,2}$, and therefore, $x_{1,2} \geq \frac{\beta_1-v_{1,1}x_{1,1}}{v_{1,2}}$. If $x_{1,1} = 1$, then $\beta_1 > v_{1,1}$ implies that $(1-x_{1,1})\frac{v_{2,1}}{v_{2,2}} = 0 < \frac{\beta_1-v_{1,1}}{v_{1,2}} = \frac{\beta_1-v_{1,1}x_{1,1}}{v_{1,2}}$. Otherwise, if $x_{1,1} = 0$, then $\beta_1 > v_{1,1}$ and $v_{1,1}v_{2,2} > v_{1,2}v_{2,1}$ imply that $(1-x_{1,1})\frac{v_{2,1}}{v_{2,2}} = \frac{v_{2,1}}{v_{2,2}} < \frac{\beta_1}{v_{1,2}} = \frac{\beta_1-v_{1,1}x_{1,1}}{v_{1,2}}$, and hence, $(1-x_{1,1})\frac{v_{2,1}}{v_{2,2}} < \frac{\beta_1-v_{1,1}x_{1,1}}{v_{1,2}}$ for all $x_{1,1} \in [0,1]$. Therefore, we have that $(1-x_{1,1})\frac{v_{2,1}}{v_{2,2}} < x_{1,2}$ for all possible values of $x_{1,1}$.

To (c): We split the proof into two parts. We assume by contradiction that either $p_1 = \beta_1$, $x_{1,1} \leq 1$ and $x_{1,2} = 0$, or that $p_1 = \beta_1$, $x_{1,1} < 1$ and $x_{1,2} > 0$.

Let us assume that $p_1 = \beta_1$, $x_{1,1} \leq 1$ and $x_{1,2} = 0$. According to $\beta_1 > v_{1,1}$, the utility of bidder 1 is negative. Contradiction to IR!

We will now investigate the other case and assume that $p_1 = \beta_1$, $x_{1,1} < 1$ and $x_{1,2} > 0$. Bidder 2 has the same valuation for $x_{1,2} = 1-x_{1,1}$ fractions of item 1 and $(1-x_{1,1})\frac{v_{2,1}}{v_{2,2}}$ fractions of item 2. The valuation of bidder 1 for $(1-x_{1,1})\frac{v_{2,1}}{v_{2,2}}$ fractions of item 2 is identical to the valuation for $(1-x_{1,1})\frac{v_{2,1}v_{1,2}}{v_{2,2}v_{1,1}}$ fractions of item 1. We know that $v_{2,1}v_{1,2} < v_{2,2}v_{1,1}$. That is, that the utility of bidder 1 is increased and the utilities of bidder 2 and the auctioneer are not decreased, when bidder 1 trades $(1-x_{1,1})\frac{v_{2,1}}{v_{2,2}}$ fractions of item 2 against $x_{2,1} = 1-x_{1,1}$ fractions of item 1. Fact (b) implies that bidder 1 actually has the required $(1-x_{1,1})\frac{v_{2,1}}{v_{2,2}}$ fractions of item 2. Contradiction to PO! $\qquad\square$

In Lemma 2.11, we show that actually $x_{1,2} < 1$, which, combined with Lemma 2.10, implies that $p_1 = \beta_1$. The fact that $x_{1,2} < 1$, i.e, that bidder 1 does not fully get item 1 *and* 2 is not surprising because he does not have enough budget to outbid bidder 2 on both items as $\beta_1 < v_{2,1} + v_{2,2}$. However, we are even able to determine the exact value of $x_{1,2}$, which is $(\beta_1 - v_{2,1})/v_{2,2}$.

Lemma 2.11 (Case 3, Part b). *Given* $\beta_2 > v_{2,1} + v_{2,2}$, $v_{1,1} > v_{2,1}$, $v_{1,2} > v_{2,2}$, $\beta_1 > v_{1,1}$, $v_{1,1}v_{2,2} > v_{1,2}v_{2,1}$, *and* $v_{2,1} + v_{2,2} > \beta_1$, *then* $p_1 = \beta_1$ *and* $x_{1,2} = (\beta_1 - v_{2,1})/v_{2,2} < 1$ *in every IR and PO outcome selected by an IC mechanism.*

Proof. We divide the proof into the following parts: in (a) we show that $p_1 = \beta_1$ and $x_{1,2} < 1$, in (b) we show that $\frac{\beta_1-v_{2,1}}{v_{2,2}} \geq x_{1,2} \geq \frac{\beta_1-v_{2,1}}{v_{1,2}}$, and in (c) we show that $x_{1,2} = \frac{\beta_1-v_{2,1}}{v_{2,2}}$.

To (a): Lemma 2.10 implies that the utility of bidder 1 is $v_{1,1} + x_{1,2}v_{1,2} - p_1$. We know that $v_{2,1} + v_{2,2} > \beta_1$. Hence, we can select a sufficiently small $\epsilon > 0$

such that $v_{2,1} + v_{2,2} - \epsilon > \beta_1$. Because of $v_{1,1} > v_{2,1}$ and $\beta_1 > v_{1,1}$, we know that $v_{2,2} - \epsilon > 0$. Let us consider the case where we have $v'_{1,2}$ with $v_{2,2} > v'_{1,2} > v_{2,2} - \epsilon$ instead of $v_{1,2}$ and all other valuation are left unchanged. In this case, the utility of bidder 1 is $v_{1,1} - v_{2,1}$, because of Lemma 2.9 and since $v_{2,2} > v'_{1,2}$ holds. Therefore, IC implies that

$$v_{1,1} - v_{2,1} \geq v_{1,1} + x_{1,2}v'_{1,2} - p_1. \tag{2.4}$$

Let us assume by contradiction that $x_{1,2} = 1$, then inequality (2.4) implies $p_1 \geq v_{2,1} + v'_{1,2} > v_{2,1} + v_{2,2} - \epsilon > \beta_1$, which contradicts the budget constraint. Therefore, $x_{1,2} < 1$, and hence, Lemma 2.10 implies that $p_1 = \beta_1$.

To (b): Lemma 2.10 and (a) show that the utility of bidder 1 is $v_{1,1} + x_{1,2}v_{1,2} - \beta_1$. We select a sufficiently small $\epsilon > 0$, such that $v_{2,1} + v_{2,2} - \epsilon > \beta_1$ and consider the case where $v'_{1,2} = v_{2,2} - \epsilon$ and all other valuations are unchanged. Lemma 2.9 implies that the utility of bidder 1 is $v_{1,1} - v_{2,1}$ in this case. Hence, IC implies that

$$v_{1,1} - v_{2,1} \geq v_{1,1} + x_{1,2}v'_{1,2} - \beta_1, \qquad \text{and} \tag{2.5}$$
$$v_{1,1} + x_{1,2}v_{1,2} - \beta_1 \geq v_{1,1} - v_{2,1}. \tag{2.6}$$

Inequality (2.5) implies that $\frac{\beta_1 - v_{2,1}}{v_{2,2} - \epsilon} = \frac{\beta_1 - v_{2,1}}{v'_{1,2}} \geq x_{1,2}$. Since this inequality has to hold for all sufficiently small $\epsilon > 0$, we know that $\frac{\beta_1 - v_{2,1}}{v_{2,2}} \geq x_{1,2}$. Inequality (2.6) implies that $\frac{\beta_1 - v_{2,1}}{v_{1,2}} \leq x_{1,2}$.

To (c): Let us assume by contradiction that the inequality $\frac{\beta_1 - v_{2,1}}{v_{2,2}} \geq x_{1,2}$ implied by (b) is strict, and $\gamma > 0$ is defined such that $\frac{\beta_1 - v_{2,1}}{v_{2,2}} = x_{1,2} + \gamma$. We select arbitrary $\epsilon > 0$ and δ with $v_{2,2}\left(\frac{\beta_1 - v_{2,1}}{\beta_1 - v_{2,1} - \gamma v_{2,2}} - 1\right) > \delta > 0$ which fulfill $v_{1,2} - \epsilon - \delta = v_{2,2}$. Such variables ϵ and δ exist because of $v_{1,2} > v_{2,2}$, and since it holds by Lemma 2.10 that $x_{1,2} > 0$; this implies together with the definition of γ that $\frac{\beta_1 - v_{2,1}}{\beta_1 - v_{2,1} - \gamma v_{2,2}} > 1$. We consider the alternative case where $v'_{1,2} = v_{1,2} - \epsilon$ and all other valuations are unchanged. We use $x'_{1,2}$ for the fraction of item 2 assigned to bidder 1 in this case. By (b) it follows that $\frac{\beta_1 - v_{2,1}}{v'_{1,2}} \leq x'_{1,2}$, and hence, $\frac{\beta_1 - v_{2,1}}{v_{2,2} + \delta} \leq x'_{1,2}$. Furthermore, Lemma 2.10 and (a) imply that $p_1 = \beta_1$ and $x_{1,1} = 1$ in both cases. Now, IC requires that $v_{1,1} + x_{1,2}v_{1,2} - \beta_1 \geq v_{1,1} + x'_{1,2}v_{1,2} - \beta_1$, and consequently, $x_{1,2} \geq x'_{1,2}$; therefore, $\frac{\beta_1 - v_{2,1}}{v_{2,2}} - \gamma \geq \frac{\beta_1 - v_{2,1}}{v_{2,2} + \delta}$. But this inequality can be transformed to $\delta \geq v_{2,2}\left(\frac{\beta_1 - v_{2,1}}{\beta_1 - v_{2,1} - \gamma v_{2,2}} - 1\right)$. Contradiction! $\qquad \square$

We combine these characterizations of Case 3 with (a) the WMON property shown in Proposition 2.7 and (b) a sophisticated way of bidder 2 to misreport: He *overstates* his valuation for item 1 by a value ϵ and *understates* his valuation for item 2 by a value $0 < \xi < \epsilon$, but by such small values that Case 3 continues to hold. Thus, by Lemma 2.10, $x_{2,1}$ remains 0 (whether bidder 2 misreports or not), and thus, the WMON condition implies that $x_{2,2}$ does *not* increase. However, by the dependence of $x_{1,2}$ on $v_{2,1}$ and $v_{2,2}$ shown in Lemma 2.11, $x_{1,2}$, and thus also

$x_{2,2}$ changes when bidder 2 misreports. This gives a contradiction to the assumption that such a mechanism exists.

Theorem 2.12. *There is no deterministic IC mechanism for divisible items which selects for any given input with public budgets an IR and PO outcome.*

Proof. Let us assume by contradiction that such a mechanism exists and consider an input for which $\beta_2 > v_{2,1} + v_{2,2}$, $v_{1,1} > v_{2,1}$, $v_{1,2} > v_{2,2}$, $\beta_1 > v_{1,1}$, $v_{1,1}v_{2,2} > v_{1,2}v_{2,1}$, and $v_{2,1} + v_{2,2} > \beta_1$ holds. Such an input exists, for example $v_{1,1} = 4$, $v_{1,2} = 5$, $v_{2,1} = 3$, and $v_{2,2} = 4$ with budgets $\beta_1 = 5$ and $\beta_2 = 8$ is such an input. Lemma 2.10 and 2.11 imply that $x_{1,1} = 1$, $x_{2,1} = 0$, $x_{1,2} = \frac{\beta_1 - v_{2,1}}{v_{2,2}}$, $x_{2,2} = 1 - x_{1,2}$, and $p_1 = \beta_1$. Let us consider an alternative valuation by bidder 2. We define $v'_{2,1} = v_{2,1} + \epsilon$ and $v'_{2,2} = v_{2,2} - \xi$ for arbitrary $\epsilon, \xi > 0$ that satisfy $\epsilon > \xi$, and which are sufficiently small such that the inequalities in Case 3 still hold. In particular, $v_{1,1}v'_{2,2} > v_{1,2}v'_{2,1}$ holds, and we denote the fraction of item 2 assigned to bidder 2 for the alternated valuations by $x'_{2,2}$. By Proposition 2.7, IC implies WMON, and therefore, $x'_{2,2}v'_{2,2} - x_{2,2}v'_{2,2} \geq x'_{2,2}v_{2,2} - x_{2,2}v_{2,2}$. It follows that $x_{2,2} \geq x'_{2,2}$, and by Lemma 2.11, $\frac{\beta_1 - v_{2,1}}{v_{2,2}} \leq \frac{\beta_1 - v'_{2,1}}{v'_{2,2}}$. Hence, the budget of bidder 1 has to be large enough, such that $\beta_1 \geq \frac{v_{2,2}v'_{2,1} - v_{2,1}v'_{2,2}}{v_{2,2} - v'_{2,2}} = \frac{v_{2,1}\xi + v_{2,2}\epsilon}{\xi} > v_{2,1} + v_{2,2}$, but $\beta_1 < v_{2,1} + v_{2,2}$ holds by assumption. Contradiction! $\qquad\square$

2.4.3 Randomized Mechanisms for Divisible and Indivisible Items

We exploit the fact that randomized mechanisms for both divisible and indivisible items are essentially equivalent to deterministic mechanisms for divisible items.

We show that for bidders with budget constraints every randomized mechanism $\bar{M} = (\bar{f}, \bar{p})$ for divisible or indivisible items can be mapped bidirectionally to a deterministic mechanism $M = (f, p)$ for divisible items with identical expected utility for all the bidders and the auctioneer when the same reported types are used as input. To turn a randomized mechanism for *divisible* or *indivisible* items into a deterministic mechanism for *divisible* items simply compute the expected values of p_i and $x_{i,j}$ for all i and j and return them. To turn a deterministic mechanism for *divisible* items into a randomized mechanism for *divisible* or *indivisible* items simply assign the items with probability $x_{i,j}$ and keep the same payment as the deterministic mechanism.

Proposition 2.13. *Every randomized mechanism $\bar{M} = (\bar{f}, \bar{p})$ for bidders with finite budgets, a rational auctioneer, and a limited amount of divisible or indivisible items can be mapped bidirectionally to a deterministic mechanism $M = (f, p)$ for divisible items such that $u_i(f_i(\theta'), p_i(\theta'), \theta_i) = \mathbf{E}\left[u_i(\bar{f}_i(\theta'), \bar{p}_i(\theta'), \theta_i)\right]$ and $\sum_{i \in N} p_i(\theta') = \mathbf{E}\left[\sum_{i \in N} \bar{p}_i(\theta')\right]$ for all bidders i, all true types $\theta = (v, \beta)$, and reported types $\theta' = (v', \beta')$.*

Proof. Let us map $\bar{M} = (\bar{f}, \bar{p})$ to $M = (f, p)$ that assigns for each bidder $i \in N$ and item $j \in I$ a fraction of $\mathbf{E}\left[\bar{f}_{i,j}(\theta')\right]$ of item j to bidder i, and makes each bidder $i \in N$ pay $\mathbf{E}\left[\bar{p}_i(\theta')\right]$. The expectations exist because the feasible fractions of items and the feasible payments have an upper bound and a lower bound. For the other direction, we map $M = (f, p)$ to $\bar{M} = (\bar{f}, \bar{p})$ that randomly picks for each item $j \in I$ a bidder $i \in N$ to which it assigns item j in a way such that bidder i is picked with probability $f_{i,j}(\theta')$, and makes each bidder $i \in N$ pay $p_i(\theta')$. Since $f(\theta') = \mathbf{E}\left[\bar{f}(\theta')\right]$ and $p(\theta') = \mathbf{E}\left[\bar{p}(\theta')\right]$, $\sum_{j \in I}(f_{i,j}(\theta')v_{i,j}) - p_i(\theta') = \mathbf{E}\left[\sum_{j \in I}(\bar{f}_{i,j}(\theta')v_{i,j}) - \bar{p}_i(\theta')\right]$ for all $i \in N$ and $\sum_{i \in N} p_i(\theta') = \mathbf{E}\left[\sum_{i \in N} \bar{p}_i(\theta')\right]$. $\square$

This proposition implies the non-existence of randomized mechanisms stated in Theorem 2.14.

Theorem 2.14. *There can be no randomized mechanism for divisible or indivisible items that is IR in expectation, PO in expectation, and IC in expectation, and that satisfies the public budget constraint ex post.*

Proof. For a contradiction suppose that there is such a randomized mechanism. Then, by Proposition 2.13, there must be a deterministic mechanism for divisible items and public budgets that satisfies IR, PO, and IC. This gives a contradiction to Theorem 2.12. $\square$

2.5 Conclusion

In this chapter, we analyzed individually rational, Pareto-optimal, and incentive compatible mechanisms for settings with heterogeneous items. Our main accomplishments as follows. We design *randomized* mechanisms that achieve these properties for *private* budgets and a restricted class of additive valuations. Furthermore, we give an impossibility result for *randomized* mechanisms and *public* budgets when valuations are additive. We are able to circumvent the impossibility result in the restricted setting because our argument for the impossibility result is based on the ability of a bidder to overstate his valuation for one and understate his valuation for another item, which is not possible in the restricted setting. Note that we study another restricted setting that is more general in Chapter 3; this setting is motivated by multiple keyword sponsored search auctions. A promising direction for future work is to identify further restrictions for which the known impossibility results do not apply.

On Multiple Keyword Sponsored Search Auctions with Budgets

3.1   Introduction

Sponsored search auctions (ad-words auctions) are used by firms such as Google, Yahoo, and Microsoft for selling ad slots on search result pages (see Edelman et al. [38] and Lahaie et al. [78]). The ad slots for a query are auctioned between those advertisers that bid on the *keywords* of the query. The ad slots are ordered from top to bottom of the page. The value of an ad slot increases with the probability that the slot is clicked, also called click-through rate (CTR). The true valuation of the bidders is private knowledge and it is assumed to depend linearly on the CTR. Moreover, valuations are assumed to be additive; that is, the total valuation of a bidder is equal to the sum of his valuations for all the slots that are assigned to him.

A further key ingredient of a sponsored search auction is that bidders specify a budget which bounds the maximum payment chargeable for the ads in a given time frame (e.g., a day), effectively linking the different keywords. The introduction of budgets dramatically changes the nature of the auction. For instance, the VCG mechanism by Vickrey [104], Clarke [23], and Groves [57], which was designed to maximize social welfare, might not be feasible since the required payment by a bidder can exceed his budget. Moreover, it was observed in the seminal paper by Dobzinski et al. [30] that maximizing social welfare cannot be achieved for budgeted auctions. Thus, they suggest to consider the weaker optimality criterion of Pareto-optimality: If an outcome is Pareto-optimal then it is impossible to make a bidder better off without making another bidder or the auctioneer worse off.

Dobzinski et al. [30] study budgeted multi-unit auctions with additive valuations; thus, their setting corresponds to sponsored search auctions where each keyword has only one slot and all slots have identical CTR. They give an incentive compatible (IC) auction based on Ausubel's ascending clinching auction (Ausubel

[5]) that produces a Pareto-optimal (PO) and individually rational (IR) outcome if budgets are *public*. They also show that this assumption is strictly needed; that is, that no deterministic mechanism for *private* budgets exists if we insist on incentive compatibility, individual rationality, and Pareto-optimality. This impossibility result for *deterministic* mechanisms was strengthened for our setting to *public* budgets in Dütting et al. [36] (see also Chapter 2). However, the question was open what optimality result can be achieved for *randomized* mechanisms. Thus, our question to study is whether IC, IR, and PO auctions for selling ad slots can be achieved with randomized mechanisms.

3.1.1 Contribution

We give a positive answer to the above question. The results consider auctions for keywords with many slots. The participants are selfish agents that report valuations and have budgets that are public knowledge.

We show that the multi-unit auction of Dobzinski et al. [30] can be generalized to a sponsored search auction for multiple keywords having multiple slots, and budget limits for each bidder. We specifically model the case of several slots with different CTR, available for each keyword, and a bound on the number of slots per keyword (usually one) that can be allocated to a bidder. We first provide an IC, IR, and PO deterministic auction that provides a fractional allocation for the case of one keyword with divisible slots. Note that the impossibility result by Dütting et al. [36] does not hold for divisible slots. In contrast, the impossibility result by Dobzinski et al. [30] for multi-unit auctions applies also to this setting, and achieving IC, IR, and PO deterministic auctions is only possible if budgets are *public*. Thus, we restrict ourselves to the public budget case in this chapter. In our auction each bidder submits his valuation and budget at the beginning of the auction. The outcome of the auction defines a fractional allocation of the slots to the bidders and satisfies Pareto-optimality. We then show how to probabilistically round this fractional allocation for the divisible case to an integer allocation for the indivisible case with multiple keywords (i.e., the ad-words setting) and get an auction that is IC in expectation, IR in expectation, and PO ex post/in expectation.

3.1.2 Related Work

Ascending clinching auctions are used in the FCC spectrum auctions (see Ausubel [5], Ausubel and Milgrom [8], and Milgrom [85]). For a motivation of sponsored search auctions see Edelman et al. [38] and Lahaie et al. [78].

We first compare our results with those of a recent work by Goel et al. [53] that was developed independently at the same time. They studied IC auctions with feasible outcomes that must obey public polymatroid constraints and bidders with identical or separable valuations (see their Lemma 3.10) and public budgets. The problem of auctions with polymatroid constraints was first studied by Bikhchandani et al. [13] for unbudgeted bidders and concave utilities. The auction in Goel et al.

[53] is an adaption of the ascending auction in Bikhchandani et al. [13] to the case of budgeted bidders. The polymatroid constraints generalize on one hand the multi-unit case in Dobzinski et al. [30] and the multiple slots with different CTR model presented in this chapter. On the other hand, the PO ascending auction in Goel et al. [53] only returns outcomes for divisible items whereas in Section 3.5 of this chapter we demonstrate that these outcomes can be rounded to *outcomes for indivisible items* if we allow the auction to yield incentive compatibility in expectation.

There are three extensions of Dobzinski et al. [30]: (1) Fiat et al. [42] studied an extension to a combinatorial setting, where items are distinct and different bidders may be interested in different items. The auction presented in Fiat et al. [42] is IC, IR, and PO for additive valuations and single-valued bidders (i.e., every bidder does not distinguish between the keywords in his public interest set). This result was generalized in Colini-Baldeschi et al. [24] to a sponsored search setting with multiple slots for each keyword but identical CTR for all slots.[1] (2) Bhattacharya et al. [11] dealt with private budgets, and gave an auction for one infinitely divisible item, where bidders cannot improve their utility by underreporting their budget. This leads to a randomized IC in expectation auction for one infinitely divisible item with both private valuations and budgets. (3) Several papers (Aggarwal et al. [2], Ashlagi et al. [4], Dütting et al. [37], and Fujishige and Tamura [45]) studied *envy-free* outcomes that are bidder optimal, respectively PO, in a one-keyword sponsored search auction. In this setting they give (under certain conditions on the input) an IC auction with both private valuations and budgets.

There are three impossibility results for non-additive valuations: Lavi and May [79] show that there is no IC, IR, and PO deterministic mechanism for indivisible items and bidders with *monotone* valuations. This result was strengthened in Colini-Baldeschi et al. [24] to an impossibility result that applies to bidders with non-negative and diminishing marginal valuations. In Goel et al. [53] the same impossibility result for *divisible* items and bidders with monotone and concave utility functions was given.

3.2 Preliminaries

We have n bidders and m slots. We call the set of bidders $N := \{1, \ldots, n\}$ and the set of slots $I := \{1, \ldots, m\}$. Each bidder $i \in N$ has a private *valuation* $v_i \geq 0$, a public *budget* $\beta_i \geq 0$, and a public *slot constraint* $\kappa_i \in \mathbb{N}_{>0}$. Each slot $j \in I$ has a public *quality* $\alpha_j \in \mathbb{Q}_{\geq 0}$. The slots are ordered such that $\alpha_j \geq \alpha_{j'}$ if $j > j'$, where ties are broken in some arbitrary but fixed order. We assume that the number of slots m fulfills $m = \sum_{i \in N} \kappa_i$ as we could add dummy-bidders with valuation $v_i = 0$, if $m > \sum_{i \in N} \kappa_i$, or we could add dummy-items with quality $\alpha_j = 0$, if $m < \sum_{i \in N} \kappa_i$.

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[1] This result was published together with the results presented in this chapter.

Divisible case. In the divisible case we assume that there is only one keyword with infinitely divisible slots. Thus the goal is to assign each bidder i a fraction $x_{i,j} \geq 0$ of each slot j and charge him a payment p_i. An allocation matrix $X = (x_{i,j})_{(i,j) \in N \times I}$ and a payment vector p are called an *outcome* (X, p). We call $c_i = \sum_{j \in I} \alpha_j x_{i,j}$ the *weighted capacity* allocated to bidder i. An outcome is *feasible* if it fulfills the following conditions: (1) the sum of the fractions assigned to a bidder does not exceed his *slot constraint* ($\sum_{j \in I} x_{i,j} \leq \kappa_i \ \forall i \in N$); (2) each of the slots is fully assigned to the bidders ($\sum_{i \in N} x_{i,j} = 1 \ \forall j \in I$); and (3) the payment of a bidder does not exceed his budget limit ($\beta_i \geq p_i \ \forall i \in N$).

Indivisible case. We additionally have a set R of *keywords*, where $|R|$ is public, and each keyword has the set of slots I. The goal is to assign each slot $j \in I$ of keyword $r \in R$ to one bidder $i \in N$ while obeying various constraints. An allocation $X = (x_{i,j,r})_{(i,j,r) \in N \times I \times R}$ where $x_{i,j,r} = 1$ if slot j is assigned to bidder i in keyword r, and $x_{i,j,r} = 0$ otherwise, and a payment vector p form an *outcome* (X, p). We call $c_i = \sum_{j \in I} \frac{\alpha_j}{|R|}(\sum_{r \in R} x_{i,j,r})$ the *weighted capacity* allocated to bidder i. An outcome is *feasible* if it fulfills the following conditions: (1) the number of slots of a keyword that are assigned to a bidder does not exceed his *slot constraint* ($\sum_{j \in I} x_{i,j,r} \leq \kappa_i \ \forall i \in N, \forall r \in R$); (2) each slot is assigned to exactly one bidder ($\sum_{i \in N} x_{i,j,r} = 1 \ \forall j \in I, \forall r \in R$); and (3) the payment of a bidder does not exceed his budget limit ($\beta_i \geq p_i \ \forall i \in N$).

Properties of the auctions. The *utility* u_i of bidder i for a feasible outcome (X, p) is $c_i v_i - p_i$; the *utility* of the auctioneer (or mechanism) is $\sum_{i \in N} p_i$. A feasible outcome (X', p') is *Pareto-superior* to the feasible outcome (X, p) if (1) the utility of no bidder in (X', p') is less than his utility in (X, p), (2) the utility of the auctioneer in (X', p') is no less than his utility in (X, p), and (3) at least one bidder or the auctioneer is better off in (X', p') compared with (X, p).

We study auctions that select feasible outcomes obeying the following conditions: (*Bidder rationality*) $u_i \geq 0$ for all bidders $i \in N$, (*Auctioneer rationality*) the utility of the auctioneer fulfills $\sum_{i \in N} p_i \geq 0$, and (*No-positive-transfer*) $p_i \geq 0$ for all bidders $i \in N$. An auction that on all inputs outputs an outcome that is both bidder rational and auctioneer rational is called *individually rational (IR)*.

A feasible outcome is *Pareto-optimal* (PO) if there is no other feasible outcome (X', p') that is Pareto-superior to (X, p). An auction is said to be *Pareto-optimal* (PO) if the outcome it produces is PO. An auction is *incentive compatible* (IC) if it is a dominant strategy for all bidders to reveal their true valuation.

A randomized auction is an auction that returns a random outcome. A randomized auction is *IC in expectation*, *IR in expectation*, respectively *PO in expectation* if the above conditions hold in expectation when comparing the random outcome of the auction with all other randomized outcomes. A randomized outcome satisfies PO *ex post*, if every deterministic outcome that is possibly drawn from the ran-

domized outcome satisfies the conditions for PO in expectation when comparing it against all randomized outcomes.[2]

3.3 Characterization of Pareto-optimality

In this section we present a novel characterization of PO outcomes that allows to address the case of multiple divisible slots with different CTR. Like previous characterizations of PO for other settings [11, 30, 42] our characterization ensures that no bidder can resell items (i.e., weighted capacity) to another bidder to increase his utility. However, in our setting, we have to consider that transferring weighted capacity between two bidders might result in the fractional exchange of slots between many bidders. We use the characterization to prove the PO of the auction given in Section 3.4.

Given a feasible outcome (X, p), a *swap* between two bidders i and i' is a fractional exchange of slots, i.e., if there are slots j and j' and a constant $\tau > 0$ with $x_{i,j} \geq \tau$ and $x_{i',j'} \geq \tau$ then a swap between i and i' can give a new feasible outcome (X', p) with $x'_{i,j} = x_{i,j} - \tau$, $x'_{i',j'} = x_{i',j'} - \tau$, $x'_{i,j'} = x_{i,j'} + \tau$, and $x'_{i',j} = x_{i',j} + \tau$. If $\alpha_j < \alpha_{j'}$ then the swap increases i's weighted capacity and reduces i''s weighted capacity. To characterize PO outcomes we first define for each bidder i the set N_i of bidders such that for every bidder a in N_i there exists a swap between i and a that increases i's weighted capacity. Given a feasible outcome (X, p) we use $h'(i) := \max\{j \in I | x_{i,j} > 0\}$ for the slot with the highest quality that is assigned to bidder i and $l'(i) := \min\{j \in I | x_{i,j} > 0\}$ for the slot with the lowest quality that is assigned to bidder i. To consider the case of slots with equal α-value we define $h(i) := \min\{j \in I | \alpha_j = \alpha_{h'(i)}\}$ and $l(i) := \min\{j \in I | \alpha_j = \alpha_{l'(i)}\}$. Thus, N_i is the set of all the bidders $a \in N$ with $h(a) > l(i)$. To model sequences of swaps we define furthermore $N_i^1 := N_i$ and $N_i^k := \bigcup_{a \in N_i^{k-1}} N_a$ for $k > 1$.

Since we have only n bidders, $\bigcup_{k=1}^{n} N_i^k = \bigcup_{k=1}^{n'} N_i^k$ for all $n' \geq n$. We define $\tilde{N}_i := \bigcup_{k=1}^{n} N_i^k \setminus \{i\}$ as the set of *desired (recursive) trading partners* of i. See Figure 3.1 for an example with five bidders. The bidders a in $\tilde{N}_i$ are all the bidders such that through a sequence of trades that "starts" with i and "ends" with a, bidder i could increase his weighted capacity, bidder a could decrease his weighted capacity, and the capacity of the remaining bidders involved in the sequence would be unchanged.

The following lemma shows the non-reflexive transitivity of $\tilde{N}_i$.

Lemma 3.1. *Given an arbitrary allocation, if $b \in \tilde{N}_a$, $c \in \tilde{N}_b$, and $a \neq c$ then $c \in \tilde{N}_a$.*

Proof. Let us assume that $b \in \tilde{N}_a$, $c \in \tilde{N}_b$, and $a \neq c$. It follows that there exists an integer k_b with $b \in N_a^{k_b}$ and an integer k_c with $c \in N_b^{k_c}$. We first show by induction that $N_b^l \subseteq N_a^{k_b+l}$ for all $l \geq 1$. For $l = 1$ we have $N_b^1 = N_b \subseteq \bigcup_{i \in N_a^{k_b}} N_i = N_a^{k_b+1}$.

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slot
number
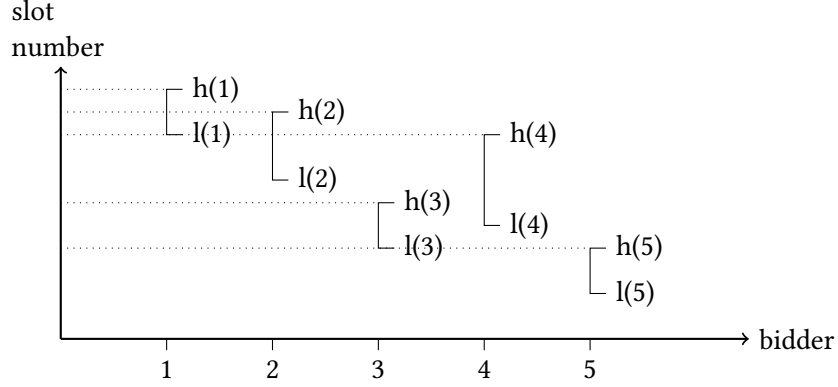
Bidder i	N_i	N_i^2	N_i^3	$\tilde{N}_i$
1	$\{1,2\}$	$\{1,2,4\}$	$\{1,2,3,4\}$	$\{2,3,4\}$
2	$\{1,2,4\}$	$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{1,3,4\}$
3	$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{1,2,4\}$
4	$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{1,2,3\}$
5	$\{1,2,3,4,5\}$	$\{1,2,3,4,5\}$	$\{1,2,3,4,5\}$	$\{1,2,3,4\}$

Figure 3.1: Example of desired trading partners.

For $l > 1$ we assume inductively that $N_b^{l-1} \subseteq N_a^{k_b+l-1}$. Then $N_b^l = \bigcup_{i \in N_b^{l-1}} N_i \subseteq \bigcup_{i \in N_a^{k_b+l-1}} N_i = N_a^{k_b+l}$. Thus, $N_b^{k_c} \subseteq N_a^{k_b+k_c}$. Since $c \in N_b^{k_c}$ it follows that $c \in N_a^{k_b+k_c}$, and moreover, since $a \neq c$ it follows that $c \in \tilde{N}_a$. $\qquad\square$

Given a feasible outcome (X, p) we use $B := \{i \in N | \beta_i > p_i\}$ to denote the set of bidders who have a positive remaining budget (i.e., each bidder i with budget β_i larger than payment p_i). Let $\tilde{v}_i = \min_{a \in \tilde{N}_i}(v_a)$, if $\tilde{N}_i \neq \emptyset$, and $\tilde{v}_i = \infty$ else. Note that $\tilde{v}_i$ depends on $\tilde{N}_i$ and, thus, on the current allocation. As we show below the $\tilde{v}_i$-value and the remaining budget $\beta_i - p_i$ for each bidder i suffice to decide whether a given allocation is PO or not.

We say that a feasible outcome (X, p) contains a *trading swap sequence* (δ, a) (for short *tss*), where $\delta > 0$ is the swapped amount of weighted capacity and a is a sequence $a_0, a_1, \ldots, a_k$ of bidders in N, if the following conditions hold:

(S1) the sequence has no cycles, i.e., $a_\ell \neq a_{\ell'}$ if $\ell \neq \ell'$,

(S2) bidder a_0 has a higher valuation than bidder a_k, i.e., $v_{a_0} > v_{a_k}$,

(S3) we can swap weighted capacity δ from $a_{\ell+1}$ to a_ℓ for all $\ell \in \{0, 1, \ldots, k-1\}$ in a certain way, in particular, it holds for all $\ell \in \{0, 1, \ldots, k-1\}$ that $(\alpha_{h'(a_{\ell+1})} - \alpha_{l'(a_\ell)}) \cdot \min\{x_{a_\ell, l'(a_\ell)}, x_{a_{\ell+1}, h'(a_{\ell+1})}\} \geq \delta$, and
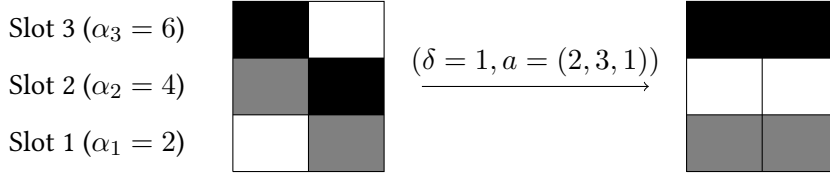
Bidder i	color	valuation v_i	budget β_i	payment p_i
1	gray	1	2	0
2	black	2	1	0
3	white	3	0	0

Figure 3.2: Example of a trading swap sequence that transfers weighted capacity from bidder 1 to bidder 2. Before the tss slot 1 is assigned to bidder 1 and 3, slot 2 is assigned to bidder 1 and 2, and slot 3 is assigned to bidder 2 and 3. The tss swaps the half of slot 3 assigned to bidder 3 with the half of slot 2 assigned to bidder 2, and the half of slot 2 assigned to bidder 1 with the half of slot 1 assigned to bidder 3.

(S4) bidder a_0 has a remaining budget that could compensate bidder a_k's loss of weighted capacity δ, i.e., $\beta_{a_0} - p_{a_0} \geq \delta v_{a_k}$.

Furthermore, we say that the outcome (X', p') results from the outcome (X, p) through the tss (δ, a) where the length of the sequence a is $k + 1$ if

(A1) $p'_i = p_i$ for all $N \setminus \{a_0, a_k\}$, $p'_{a_0} = p_{a_0} + \delta v_{a_k}$, and $p'_{a_k} = p_{a_k} - \delta v_{a_k}$,

(A2) $x'_{a_\ell, l'(a_\ell)} = x_{a_\ell, l'(a_\ell)} - \tau_\ell$, $x'_{a_\ell, h'(a_{\ell+1})} = x_{a_\ell, h'(a_{\ell+1})} + \tau_\ell$, $x'_{a_{\ell+1}, l'(a_\ell)} = x_{a_{\ell+1}, l'(a_\ell)} + \tau_\ell$, and $x'_{a_{\ell+1}, h'(a_{\ell+1})} = x_{a_{\ell+1}, h'(a_{\ell+1})} - \tau_\ell$, where $\tau_\ell = \delta/(\alpha_{h'(a_{\ell+1})} - \alpha_{l'(a_\ell)})$ for all $\ell \in \{0, 1, \ldots, k-1\}$, and all other entries of X' are identical to the entries of X.

In the remainder of this section we prove the following characterization of Pareto-optimal outcomes.

Theorem 3.2. *A feasible outcome is Pareto-optimal, if and only if it contains no trading swap sequence.*

We show first that if an outcome (X, p) is PO, then it contains no trading swap sequence.

Proposition 3.3. *An outcome (X', p') that results from the outcome (X, p) through a trading swap sequence (δ, a) is Pareto-superior to (X, p).*

Proof. Let $k + 1$ be the length of the sequence a. By (A1) the sum of the payments, which is equal to the utility of the auctioneer, does not change. By (A1) and (A2), for all bidders except a_0 and a_k neither the payment, nor the weighted capacity changes, and thus, their utility does not change. By (A2) the weighted capacity assigned to a_k decreases by δ and by (A1) the payment of a_k decreases by δv_{a_k}. Thus, the utility of a_k does not change. Finally, the utility of a_0 increases because $p'_{a_0} - p_{a_0} = v_{a_k}\delta$ by (A1) and $v_{a_k}\delta < v_{a_0}\delta$ by (S2) implies $p'_{a_0} - p_{a_0} < v_{a_0}\delta$, and by (A2) a_0's weighted capacity increases by δ. $\qquad\square$

Next we show that if a feasible outcome does not contain a trading swap sequence then it is PO. We show this in two steps, namely, Proposition 3.4 proves that the non-existence of a trading swap sequence depends on a certain condition for the $\tilde{v}_i$'s of the bidders, and Proposition 3.7 shows that if this condition is fulfilled, then the outcome is PO.

Proposition 3.4. *A feasible outcome contains no trading swap sequences, if and only if (a) $\tilde{v}_i \geq v_i$ for each bidder $i \in B$ and (b) $\tilde{v}_i > 0$ for each bidder $i \in N$ with $v_i > 0$.*

Proof. (A) We show first that given a feasible outcome (X, p) that contains a tss (δ, a) where a has length $k + 1$, it either holds that $a_0 \in B$ with $\tilde{v}_{a_0} < v_{a_0}$ or that $a_0 \in N$ with $\tilde{v}_{a_0} = 0$ and $v_{a_0} > 0$. This is a consequence of the definition of a tss as follows. Since $\delta > 0$ it follows by (S3) that $\alpha_{h'(a_{\ell+1})} > \alpha_{l'(a_\ell)}$ for all $\ell \in \{0, 1, \ldots, k - 1\}$, and thus, $a_{\ell+1} \in N_{a_\ell}$ for all $\ell \in \{0, \ldots, k - 1\}$. It follows that $a_k \in \tilde{N}_{a_0}$ by Lemma 3.1. Hence, by (S2) holds that $v_{a_0} > v_{a_k} \geq \tilde{v}_{a_0}$. If $a_0 \in B$ this directly proves (A). If $a_0 \in N \setminus B$ then $\beta_{a_0} = p_{a_0}$ and thus by (S4) holds $v_{a_k} = 0$. It follows $v_{a_0} > \tilde{v}_{a_0} = 0$, which proves (A) for $a_0 \in N \setminus B$.

(B) We will show now the other direction, i.e., that given a feasible outcome (X, p) such that there exists a bidder $a_0 \in B$ with $\tilde{v}_{a_0} < v_{a_0}$ or a bidder $a_0 \in N$ with $\tilde{v}_{a_0} = 0$ and $v_{a_0} > 0$, there exists a tss in (X, p). Thus, there is a bidder $a_0 \in N$ with $\tilde{v}_{a_0} < v_{a_0}$. We select the smallest $k \in \{1, 2, \ldots, n\}$ for which there is a bidder $a_k \in N_{a_0}^k$ who has $v_{a_k} = \tilde{v}_{a_0}$. We define for all $\ell \in \{1, 2, \ldots, k - 1\}$ the bidder a_ℓ such that $a_\ell \in N_{a_0}^\ell$ and $a_{\ell+1} \in N_{a_\ell}$. By using this construction there cannot be a cycle in the sequence a which proves (S1) for any tss with sequence a. Furthermore, it holds that $v_{a_k} = \tilde{v}_{a_0}$ which proves together with $\tilde{v}_{a_0} < v_{a_0}$ (S2) for any tss with sequence a. We next define the value δ_S as the largest possible value for δ in (S3). We define $\tau_\ell = \min\{x_{a_\ell, l'(a_\ell)}, x_{a_{\ell+1}, h'(a_{\ell+1})}\}$ for all $\ell \in \{0, 1, \ldots, k - 1\}$ and $\delta_S = \min_{\ell \in \{0,1,\ldots,k-1\}} \tau_\ell \cdot (\alpha_{h'(a_{\ell+1})} - \alpha_{l'(a_\ell)})$. Observe that $\delta_S > 0$ because $a_{\ell+1} \in N_{a_\ell}$ for all $\ell \in \{0, 1, \ldots, k - 1\}$ implies $\alpha_{h'(a_{\ell+1})} > \alpha_{l'(a_\ell)}$. We next define $\delta_B = (\beta_{a_0} - p_{a_0})/v_{a_k}$ if $a_0 \in B$ and $v_{a_k} > 0$ and we define $\delta_B = \delta_S$ otherwise. Thus, $[0, \delta_B]$ are feasible values for δ in (S4). Hence, it follows that $(\min\{\delta_S, \delta_B\}, a)$ satisfies all conditions for a tss. $\qquad\square$

To show Proposition 3.7 we first need to extend the sets $\tilde{N}_i$ to deal with bidders that occupy slots with identical quality. For this purpose we introduce the sets T_i. The containment relation on the sets T_i gives a total order on these sets (Lemma

3.5). Additionally these sets are "tight" in the sense that no other feasible allocation can assign more weighted capacity to them (Lemma 3.6). This fact is crucial when showing Proposition 3.7.

Definition 3.1. *If $h(i) = l(i)$ let $L_i = \{u \in N | (v_u \geq v_i) \wedge (h(u) = l(u) = h(i))\}$, and let $L_i = \emptyset$ otherwise. Let $T_i = \tilde{N}_i \cup \{i\} \cup L_i$ for all $i \in N$.*

The next lemma shows that for a given allocation X the relation "$\subseteq$" defines a total order on the sets T_i with $i \in N$.

Lemma 3.5. *Given bidders $i, u \in N$, then $T_i \subseteq T_u$ or $T_u \subseteq T_i$.*

Proof. We can restrict our analysis to the case $i \neq u$ as otherwise $T_i = T_u$. Let us first assume that $h(u) = l(u) = h(i) = l(i)$. It follows that $N_i = N_u$, and thus, $\tilde{N}_i \cup \{i, u\} = \tilde{N}_u \cup \{i, u\}$. Let us assume additionally that $v_u \geq v_i$. Then $u \in L_i$ and $L_u \subseteq L_i$, and thus, $T_u = \tilde{N}_u \cup \{u\} \cup L_u \subseteq \tilde{N}_u \cup \{i, u\} \cup L_u \subseteq \tilde{N}_i \cup \{i\} \cup L_i = T_i$. For the same arguments $v_i \geq v_u$ implies $T_i \subseteq T_u$.

Next assume that $h(u) = l(u) = h(i) = l(i)$ does not apply. If both $h(u) \leq l(i)$ and $h(i) \leq l(u)$ then $h(u) \geq l(u) \geq h(i) \geq l(i) \geq h(u)$ implies that all "$\geq$" are "$=$" which gives us a contradiction. Thus we can assume w.l.o.g. that $h(u) > l(i)$. It follows that $u \in N_i \subseteq \tilde{N}_i \cup \{i\}$ and $i \neq u$ implies $u \in \tilde{N}_i$. By Lemma 3.1 it follows that all $c \in \tilde{N}_u$ with $c \neq i$ satisfy $c \in \tilde{N}_i$, and thus, $\tilde{N}_u \subseteq \tilde{N}_i \cup \{i\}$. Furthermore, all $x \in L_u$ satisfy $h(x) = h(u) > l(i)$, and thus, $x \in N_i \subseteq \tilde{N}_i \cup \{i\}$. Hence, we obtain $T_u = \tilde{N}_u \cup \{u\} \cup L_u \subseteq \tilde{N}_i \cup \{i\} \subseteq T_i$. $\qquad\square$

Moreover, we can show in the next lemma for any $i \in N$ that given the set T_i determined by a feasible allocation X, no other allocation allocates more weighted capacity to the set of bidders T_i.

Lemma 3.6. *Given a feasible allocation X and the set T_i for a bidder $i \in N$ determined by X, then for any other feasible allocation X' it holds that $\sum_{u \in T_i} \sum_{j \in I} \alpha_j x_{u,j} \geq \sum_{u \in T_i} \sum_{j \in I} \alpha_j x'_{u,j}$.*

Proof. We first fix an $i \in N$. Let $\kappa = \sum_{u \in T_i} \kappa_u$ and let $a = \min_{u \in T_i} l(u)$. Since $l(i) = l(u)$ for all $u \in L_i$ it follows that **(Fact a)** $a = \min_{u \in \tilde{N}_i \cup \{i\}} l(u)$. Recall that since X is a feasible allocation it holds that $\sum_{u \in N} x_{u,j} = 1$ for all $j \in I$ and $\sum_{j \in I} x_{u,j} \leq \kappa_u$ for all $u \in N$ by definition. Furthermore, recall that we assume that $\sum_{u \in N} \kappa_u = m$. Thus,

$$\sum_{u \in T_i} \sum_{j \in I} x_{u,j} \leq \sum_{u \in T_i} \kappa_u = \kappa \quad \text{and}$$

$$\sum_{u \in N} \kappa_u = m = \sum_{j \in I} 1 = \sum_{j \in I} \sum_{u \in N} x_{u,j} = \sum_{u \in T_i} \sum_{j \in I} x_{u,j} + \sum_{u \in N \setminus T_i} \sum_{j \in I} x_{u,j}$$

$$\leq \sum_{u \in T_i} \sum_{j \in I} x_{u,j} + \sum_{u \in N \setminus T_i} \kappa_u.$$

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It follows that **(Fact b)** $\kappa = \sum_{j \in I} \sum_{u \in T_i} x_{u,j}$.

Notice that **(Fact c)** for all $j \in I$ with $\alpha_j > \alpha_a$ it holds that $\sum_{u \in T_i} x_{u,j} = 1$: Assume by contradiction that $\sum_{u \in T_i} x_{u,j} < 1$. Then there exists a $w \in N \setminus T_i$ with $x_{w,j} > 0$ because feasibility of X implies $\sum_{u \in N} x_{u,j} = 1$. Thus, $h(w) > a = l(u)$ for some $u \in \tilde{N}_i \cup \{i\}$ by Fact a, which implies $w \in \tilde{N}_i \cup \{i\} \subseteq T_i$ by Lemma 3.1. This leads to a contradiction to the assumption $w \in N \setminus T_i$.

Next we argue that **(Fact d)** for all $j \in I$ with $\alpha_j < \alpha_a$ it holds that $\sum_{u \in T_i} x_{u,j} = 0$. This is the case because otherwise there would be a bidder $u \in T_i$ with $x_{u,j} > 0$ which implies $l(u) < a$. This is a contradiction to the definition of a.

Note that $m - a + 1 = \sum_{j=a}^{m} 1 \geq \sum_{j=a}^{m} \sum_{u \in T_i} x_{u,j} = \sum_{j=1}^{m} \sum_{u \in T_i} x_{u,j} = \kappa$, where the first inequality follows from feasibility of X, the second equality follows from Fact d, and the third equality follows from Fact b. Thus, $m - \kappa + 1 \geq a$. Together with the ordering of the slots by α, this implies **(Fact e)** $\alpha_j \geq \alpha_a$ for all $j \geq m - \kappa + 1$.

We now define $j^* = \min\{j \in I | \alpha_j > \alpha_a\}$. By the following arguments we obtain the next sequence of equalities: The first equality follows from Fact d; the second equality follows from Fact b and Fact c; and the fourth equality follows from Fact e and $\alpha_j \leq \alpha_a$ for all $j \leq j^* - 1$ implied by the definition of j^*.

$$\sum_{j \in I: \alpha_j = \alpha_a} \sum_{u \in T_i} x_{u,j} = \sum_{j \in I} \sum_{u \in T_i} x_{u,j} - \sum_{j \in I: \alpha_j > \alpha_a} \sum_{u \in T_i} x_{u,j}$$
$$= \kappa - (m - j^* + 1) = \sum_{j=m-\kappa+1}^{j^*-1} 1 = (1/\alpha_a) \sum_{j=m-\kappa+1}^{j^*-1} \alpha_j$$

In the next sequence of equalities the second equality follows from Fact c and Fact d, and the third equality from the above sequence.

$$\sum_{u \in T_i} \sum_{j \in I} x_{u,j} \alpha_j = \sum_{j \in I} \alpha_j \sum_{u \in T_i} x_{u,j}$$
$$= \sum_{j=j^*}^{m} \alpha_j + \alpha_a \Big(\sum_{j \in I: \alpha_j = \alpha_a} \sum_{u \in T_i} x_{u,j} \Big) = \sum_{j=m-\kappa+1}^{m} \alpha_j$$

Thus, the bidders in T_i have an aggregated weighted capacity equal to the weighted capacity of the allocation where the most valuable slots from m down to $m - \kappa + 1$ and no fraction of a slot below are occupied by T_i. This is the "optimal allocation" for T_i, i.e., for any other feasible allocation X' it holds that $\sum_{u \in T_i} \sum_{j \in I} \alpha_j x_{u,j} \geq \sum_{u \in T_i} \sum_{j \in I} \alpha_j x'_{u,j}$. $\square$

Now we use the previous lemmata in the next proposition that gives a sufficient condition for the Pareto-optimality of a feasible outcome (X, p).

Proposition 3.7. [3] *Given a feasible outcome (X, p), if (a) $\tilde{v}_i \geq v_i$ for each $i \in B$ and (b) $\tilde{v}_i > 0$ for each $i \in N$ with $v_i > 0$ then the respective feasible outcome is Pareto-optimal.*

Proof. Let us assume by contradiction that we have a feasible outcome (X', p') that is Pareto-superior to (X, p) and (a) and (b) hold. The utility of the auctioneer does not decrease. Thus, the sum of the payments of the bidders fulfills $\sum_{i \in N} p'_i \geq \sum_{i \in N} p_i$. If $\sum_{i \in N} p'_i > \sum_{i \in N} p_i$ then an outcome (X', p'') where $\sum_{i \in N} p''_i = \sum_{i \in N} p_i$ exists, which is Pareto-superior compared to (X, p) as well: simply give the additional payments back to some of the bidders. We can therefore restrict our analysis to the cases with **(Fact a)** $\sum_{i \in N} p'_i = \sum_{i \in N} p_i$.

In the following parts of the proof we study a set N^* that contains all bidders with positive valuations, and we define a sequence of subsets of N^* ordered by "$\subseteq$" that starts with $\emptyset$ and ends with N^*. First we show that all bidders in N^* who have not spent all their budget in (X, p) and appear the first time in a subset S of the sequence have the lowest valuation among all bidders in S. Then we show that all bidders in N^* who spent all their budget in (X, p) cannot get more weighted capacity in X' than in X. Furthermore, we use Lemma 3.6 to show that no subset of bidders in the sequence can get more weighted capacity in X' than in X. After this, we can show by induction over the sequence that the social welfare of X' cannot be higher than the social welfare of X. This leads immediately to a contradiction to the assumed Pareto-superiority of (X', p') over (X, p).

For all $i \in N$ let the set T_i from Definition 3.1 be determined by X. We define $N^* = \bigcup_{i \in N : v_i > 0} T_i$ and show first some facts about N^* and $N \setminus N^*$. By Lemma 3.5, "$\subseteq$" induces a total order on the sets T_i and thus there is a "largest" T_{i^*} in this order. For this set T_{i^*} it holds that $T_{i^*} = N^*$. Thus, **(Fact b)** there exists an $i \in N$ with $v_i > 0$ for which $N^* = T_i$. Let $i \in N$ be a bidder with $v_i > 0$. By (b) $\tilde{v}_i > 0$ holds, implying that no u with $v_u = 0$ is in $\tilde{N}_i$. Furthermore, no u with $v_u = 0$ is in L_i by the definition of L_i. Thus, no u with $v_u = 0$ is in T_i. It follows that **(Fact c)** $N^* = \{i \in N | v_i > 0\}$. The bidders $i \in N \setminus N^*$ have $v_i = 0$ and by Pareto-superiority of (X', p') over (X, p) it follows that $x'_i v_i - p'_i \geq x_i v_i - p_i$, and thus, **(Fact d)** $p_i \geq p'_i$ for all $i \in N \setminus N^*$.

Now we introduce an ordered sequence of subsets of N^* that we use later in an induction. By Lemma 3.5 the relation "$\subseteq$" forms a total order on the sets T_i with $i \in B^* := B \cap N^*$. Reorder the bidders, such that $T_1, \ldots, T_{|B^*|}$ are the sets T_i with $i \in B^*$ ordered by "$\subseteq$" and that T_1 is the smallest set. We define $T_0 = \emptyset$ and $T_{|B^*|+1} = N^*$. Furthermore, we let $v_{T_i} = \min_{u \in T_i} v_u$ for $i = 1, \ldots, |B^*| + 1$, $v_{T_0} = v_{T_1}$, and $v_{T_{|B^*|+2}} = 0$. We will use ΔT_i for $T_i \setminus T_{i-1}$. It is easy to see that **(Fact e)** for a bidder $u \in \Delta T_i \cap B^*$ it holds that $v_u = v_{T_i}$: Since $u \in \Delta T_i \cap B^*$ it follows that $u \notin T_0, \ldots, T_{i-1}$. Thus, $T_u = T_{i'}$ for some $i' \geq i$ because $u \in B^*$ and $u \in T_u$, and moreover, $T_i \subseteq T_{i'} = T_u = \tilde{N}_u \cup \{u\} \cup L_u$ since $i' > i$ implies

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that $T_i \subseteq T_{i'}$. The definition of L_u implies that all $w \in L_u$ satisfy $v_w \geq v_u$ and (a) implies that all $w \in \tilde{N}_u$ satisfy $v_w \geq v_u$. Thus all $w \in T_i$ satisfy $v_w \geq v_u$ which implies $v_u = \min_{x \in T_i} v_x = v_{T_i}$.

We partition N^* into three sets, namely $N^* \cap B$, C^+, and C^-. Recall that $c_i = \sum_{j \in I} \alpha_j x_{i,j}$ and $c'_i = \sum_{j \in I} \alpha_j x'_{i,j}$ for all $i \in N$. Formally we define $C = N^* \setminus B$, $C^+ := \{i \in C | c'_i > c_i\}$, and $C^- := C \setminus C^+$ and show **(Fact f)**: $C^+ = \emptyset$. This implies that the bidders with positive valuations who spent their full budget under X cannot get more weighted capacity under X'. The sequence of inequalities below follows by the following arguments: The first inequality holds since $p_u = \beta_u \geq p'_u$ for all $u \in C \supseteq C^+$; the second inequality follows from Pareto-superiority of (X', p') compared to (X, p); and the third inequality follows because $\Delta T_i \setminus C^+ = (\Delta T_i \cap B^*) \cup (\Delta T_i \cap C^-)$, all $u \in \Delta T_i \cap B^*$ have $v_u = v_{T_i}$ by Fact e, all $u \in C^-$ have $c_u \geq c'_u$, and all $u \in T_i$ have $v_{T_i} \leq v_u$.

$$\sum_{u \in \Delta T_i} (p_u - p'_u) \geq \sum_{u \in \Delta T_i \setminus C^+} (p_u - p'_u) \geq \sum_{u \in \Delta T_i \setminus C^+} v_u(c_u - c'_u)$$

$$\geq \sum_{u \in \Delta T_i \setminus C^+} v_{T_i}(c_u - c'_u) = \sum_{u \in \Delta T_i} v_{T_i}(c_u - c'_u) + \sum_{u \in \Delta T_i \cap C^+} v_{T_i}(c'_u - c_u). \quad (3.1)$$

The next sequence of inequalities holds for the following reason: The first inequality follows from Fact a and Fact d; the second inequality follows by summing (3.1) for $i = 1, \ldots, |B^*| + 1$; and the third inequality holds since $\sum_{i=1}^{|B^*|+1} \sum_{u \in \Delta T_i} v_{T_i}(c_u - c'_u) = \sum_{i=1}^{|B^*|+1} (v_{T_i} - v_{T_{i+1}}) \sum_{u \in T_i}(c_u - c'_u) \geq 0$ as $\sum_{u \in T_i}(c_u - c'_u) \geq 0$ for all T_i by Lemma 3.6, which applies by Fact b also for $T_{|B^*|+1}$.

$$0 \geq \sum_{i=1}^{|B^*|+1} \sum_{u \in \Delta T_i} (p_u - p'_u) \geq \sum_{i=1}^{|B^*|+1} \sum_{u \in \Delta T_i} v_{T_i}(c_u - c'_u)$$

$$+ \sum_{i=1}^{|B^*|+1} \sum_{u \in \Delta T_i \cap C^+} v_{T_i}(c'_u - c_u) \geq \sum_{i=1}^{|B^*|+1} \sum_{u \in \Delta T_i \cap C^+} v_{T_i}(c'_u - c_u). \quad (3.2)$$

It follows that $\sum_{i=1}^{|B^*|+1} \sum_{u \in \Delta T_i \cap C^+} v_{T_i}(c'_u - c_u) \leq 0$. Since $v_u > 0$ for all $u \in N^*$ and $T_i \subseteq N^*$ it holds that $v_{T_i} > 0$. Thus, since $c'_u > c_u$ for all $u \in C^+$ it follows that C^+ has to be empty and $C = C^-$.

We can prove now by induction that **(Fact g)** $\sum_{u \in T_i} v_u(c_u - c'_u) \geq v_{T_i} \sum_{u \in T_i}(c_u - c'_u)$ for all T_i. For $i = 0$ we have that $T_0 = \emptyset$ and thus the claim holds. For $i > 0$ we have that $\sum_{u \in \Delta T_i \setminus B^*} v_u(c_u - c'_u) \geq v_{T_i} \sum_{u \in \Delta T_i \setminus B^*}(c_u - c'_u)$ because for all $u \in C^- = C \supseteq \Delta T_i \setminus B^*$ (by Fact f) holds $c_u - c'_u \geq 0$ and for all $u \in T_i$ holds $v_u \geq v_{T_i}$. Furthermore, we have that $\sum_{u \in \Delta T_i \cap B^*} v_u(c_u - c'_u) \geq v_{T_i} \sum_{u \in \Delta T_i \cap B^*}(c_u - c'_u)$ because of Fact e and by induction it holds that $\sum_{u \in T_{i-1}} v_u(c_u - c'_u) \geq v_{T_{i-1}} \sum_{u \in T_{i-1}}(c_u - c'_u)$. It follows that $\sum_{u \in T_i} v_u(c_u - c'_u) \geq v_{T_i} \sum_{u \in T_i}(c_u - c'_u)$ for all T_i.

Now we finish the proof by generating a contradiction. By Lemma 3.6, which applies by Fact b also for $T_{|B^*|+1}$, it holds that $\sum_{u \in T_i}(c_u - c'_u) \geq 0$ for all $i = 1, \ldots, |B^*| + 1$, and thus, $v_{T_i} \sum_{u \in T_i}(c_u - c'_u) \geq 0$. Consequently, it holds that $\sum_{u \in N} v_u c_u \geq \sum_{u \in N} v_u c'_u$ because $\sum_{u \in T_i} v_u(c_u - c'_u) \geq v_{T_i} \sum_{u \in T_i}(c_u - c'_u) \geq 0$ by Fact g, $T_{|B^*|+1} = N^*$, and $v_u = 0$ for all $u \in N \setminus N^*$ by Fact c. Thus, the social welfare under (X, p) is at least as large as under (X', p'). This implies that $\sum_{u \in N}(v_u c_u - p_u) \geq \sum_{u \in N}(v_u c'_u - p'_u)$ by Fact a. Pareto-superiority of (X', p') compared to (X, p) implies that for one bidder $w \in N$ it holds that $v_w c_w - p_w < v_w c'_w - p'_w$. Hence, $\sum_{u \in N \setminus \{w\}}(v_u c_u - p_u) > \sum_{u \in N \setminus \{w\}}(v_u c'_u - p'_u)$, which implies that for a $u \in N \setminus \{w\}$ it holds that $v_u c_u - p_u > v_u c'_u - p'_u$. This, contradicts our assumption that (X', p') is Pareto-superior. $\qquad\square$

By Proposition 3.4 and Proposition 3.7 we know that a feasible outcome that contains no trading swap sequence is Pareto-optimal. Moreover, by Proposition 3.3 an outcome resulting from a trading swap sequence is Pareto-superior, and thus, a feasible outcome that contains a trading swap is not Pareto-optimal. This concludes the proof of Theorem 3.2.

3.4 Multiple Keyword Auction for the Divisible Case

We describe next our deterministic clinching auction for divisible slots and show that it is IC, IR, and PO. We assume throughout this section that $v_i \in \mathbb{N}_{\geq 0}$ and $\beta_i \in \mathbb{Q}_{\geq 0}$ for all $i \in N$.[4] Furthermore, we assume that the input N is ordered such that all $i \in N$ with bid $v_i = 0$ are in the order before all $i \in N$ with bid $v_i > 0$, and all $i \in N$ with $v_i > 0$ are ordered independently of their bids. This order is used by the for-loop in line 3 of Algorithm 3.1 and is needed to show PO in Theorem 3.10; it is necessary to avoid the existence of trading swaps that do not require monetary compensation. Finally, recall that we assume that $m = \sum_{i \in N} \kappa_i$. If $m > \sum_{i \in N} \kappa_i$ we could add dummy-bidders with valuation $v_i = 0$ and budget $\beta_i = 0$; that is, they have to pay no money and they are not competing with the other bidders. If $m < \sum_{i \in N} \kappa_i$ we could add dummy-items with quality $\alpha_j = 0$, that is, they have no value for any bidder. Thus, the slot constraints imply $\sum_{j \in I} x_{i,j} = \kappa_i$ for all $i \in N$.

The auction repeatedly increases a price "per weighted capacity" and gives different weights to different slots depending on their CTR. To perform the check whether all remaining unsold weighted capacity can still be sold we solve suitable linear programs. We will show that if the outcome of the auction did not fulfill the characterization of Pareto-optimality given in Section 3.3, i.e., if it contained a trading swap sequence, then one of the linear programs solved by the auction would not have computed an optimal solution. Since this is not possible, it will follow that the outcome is PO. A formal description of the auction is given in Algorithm 3.1.

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[4]All the arguments go through if we simply assume that $v_i \in \mathbb{Q}_{\geq 0}$ for all $i \in N$ and there exists a publicly known value $z \in \mathbb{R}_{>0}$ such that for all bidders i and i' either $v_i = v_{i'}$ or $|v_i - v_{i'}| \geq z$.

Algorithm 3.1: Clinching auction for divisible slots

Input: $N, I, \alpha, \kappa, v, \beta$

1   $\pi \leftarrow 0; \; c_i \leftarrow 0, \; p_i \leftarrow 0, \; d_i \leftarrow \infty \; \forall i \in N$

2   **while** $\sum_{i \in N} c_i < \sum_{j \in I} \alpha_j$ **do**

     `/* unsold weighted capacity exists                */`

3   **forall the** i' *in* N *with* $d_{i'} > D_{i'}(\beta_{i'} - p_{i'}, \pi + 1)$ **do**

       `/* demand of bidder `i'` has to be updated     */`

       `/* solve linear program for bidder `i'`         */`

4       $(X, \gamma) \leftarrow$ Compute solution of LP 3.1 for (c, d, i')

       `/* update weighted capacity, payment, and demand`

         `variable of bidder `i'`                        */`

5       $(c_{i'}, p_{i'}) \leftarrow (c_{i'} + \gamma_{i'}, p_{i'} + \gamma_{i'} \pi)$

6       $d_{i'} \leftarrow D_{i'}(\beta_{i'} - p_{i'}, \pi + 1)$

     `/* increase price                                 */`

7     $\pi \leftarrow \pi + 1$

8   **return** (X, p)

The input values of Algorithm 3.1 are the bids v, budget limits β, and slot constraints κ that the bidders communicate to the auctioneer at the beginning of the auction, and information about the qualities of the slots α. We assume that bidders bid their valuation because Proposition 3.9 shows that bidding the valuation is a dominant strategy; thus, we use v_i for bidder i's bid and valuation. Note that the auction is a so called "one-shot auction"; the bidders are asked once for their bids at the beginning of the auction and then they cannot input any further data. The state of the auction is defined by the current price π, the weighted capacity c_i that bidder $i \in N$ has clinched so far, and the payment p_i that has been charged so far to bidder i. Moreover, the demand of bidder i for weighted capacity is computed by the mechanism based on i's remaining budget $\beta_i - p_i$, the current price π, and the bid v_i: The demand $D_i(\beta_i - p_i, \pi)$ is $(\beta_i - p_i)/\pi$ if $v_i \geq \pi > 0$, it is infinite if $\pi = 0$, and it is zero otherwise.

At the beginning of the auction the price π is zero and the demand variable for each bidder i is set to $d_i = \infty$. Furthermore, in Linear Program 3.1, bidder i' and the coefficients c and d change during the auction, while the coefficients α and κ are fixed. For each iteration of the while-loop the auction first solves Linear Program 3.1 for one of the bidders i' who has $d_{i'} > D_{i'}(\beta_{i'} - p_{i'}, \pi + 1)$. It sells bidder i' the respective $\gamma_{i'}$ for price π. Next it updates bidder i''s demand variable $d_{i'}$. If bidder i' reported a valuation $v_{i'}$ less than $\pi + 1$ the auction sets $d_{i'} = 0$ and bidder i' cannot get further weighted capacity. Otherwise bidder i' might get further weighted capacity in the next iteration of the while-loop but has to pay a price for it of at least $\pi + 1$. The auction continues the previous step until d_i of each bidder i corresponds to his demand for price $\pi + 1$. Then it sets π to $\pi + 1$.

Linear Program 3.1.

minimize $\gamma_{i'}$

$$s.t.: (a) \qquad \sum_{i \in N} x_{i,j} = 1 \quad \forall j \in I \qquad\qquad \triangleright \textit{assign all slots}$$

$$(b) \qquad \sum_{j \in I} x_{i,j} = \kappa_i \quad \forall i \in N \qquad\qquad \triangleright \textit{slot constraint}$$

$$(c) \qquad \sum_{j \in I} x_{i,j}\,\alpha_j - \gamma_i = c_i \quad \forall i \in N \qquad \triangleright \textit{assign value to } \gamma_i$$

$$(d) \qquad\qquad \gamma_i \le d_i \quad \forall i \in N \qquad\qquad \triangleright \textit{demand constraint}$$

$$(e) \qquad\qquad x_{i,j} \ge 0 \quad \forall i \in N, \forall j \in I$$

$$(f) \qquad\qquad \gamma_i \ge 0 \quad \forall i \in N$$

The crucial point of the auction is that it sells only weighted capacity $\gamma_{i'}$ to bidder i' at a certain price π, if it can sell $\sum_{j \in I} \alpha_j - \sum_{i \in N} c_i - \gamma_{i'}$ to the other bidders but not more. The auction computes $\gamma_{i'}$ by solving an LP. We use an LP as there are two types of constraints to consider: the slot constraint in line (b) of the LP, which constrains "unweighted" capacity, and the demand constraint in line (d) of the LP, which is implied by the budget limit, and which constrains weighted capacity. In the homogeneous item setting studied by Dobzinski et al. [30] and Bhattacharya et al. [11] there are no slot constraints and the demand constraints are unweighted (i.e., $\alpha_j = 1$ for all $j \in I$). Thus, no LP is needed to decide what amount to sell to whom.

To illustrate the mechanism we give the following example.

Example: There are two slots with qualities $\alpha_1 = 1$ and $\alpha_2 = 2$. Bidder 1 has valuation $v_1 = 1$, budget $\beta_1 = 1$, and slot constraint $\kappa_1 = 1$. Bidder 2 has valuation $v_2 = 2$, budget $\beta_2 = 0.5$, and slot constraint $\kappa_2 = 1$. The auction starts for both bidders with a price of zero and thus their demand is infinite. First we solve an LP for bidder 1. He is assigned a weighted capacity of one for price zero, since the most weighted capacity that we can assign to bidder 2 is the quality of slot 2. Then by updating his demand variable we implicitly set the price of bidder 1 to one. Next, we solve an LP for bidder 2. After this we sell a weighted capacity of one to bidder 2, since the most weighted capacity that we can assign to bidder 1 is the quality of slot 2 and he can also afford just an additional weighted capacity of one. Then we set the price of bidder 2 implicitly to one and continue with the next iteration. We solve an LP for bidder 1; bidder 2 can only afford an additional weighted capacity of one half. Hence, we have to sell the other half that is left to bidder 1. Next we sell the remaining half to bidder 2. Each bidder gets a weighted capacity of 1.5 and pays 0.5. The only possible allocation is that each bidder gets half of the first slot and half of the second slot.

It is crucial for the progress and the correctness of the mechanism that there is a feasible solution for each LP we try to solve.

Lemma 3.8. *There exists a feasible solution for all the linear programs that Algorithm 3.1 has to solve.*

Proof. We show the claim by induction on the linear programs that Algorithm 3.1 solves. Let LP_t be the t-th such LP. There is a feasible solution for LP_1 as the demand d_i of every bidder is unlimited. Hence, we can set X such that $\sum_{j \in I} x_{i,j} = \kappa_i \; \forall i \in N$ and can make γ_i as large as necessary for every bidder i. Next let us inductively assume that there was a feasible solution for LP_t. As there exists a feasible solution for LP_t, we obtain an optimal solution (X, γ) by solving LP_t. After the call, $c_{i'}$ is increased by $\gamma_{i'}$, and thus, (X, γ^f) with $\gamma_i^f = \gamma_i$ for $i \neq i'$ and $\gamma_{i'}^f = 0$ for $i = i'$ is a feasible solution of LP_{t+1}, which uses the new c-values. Since $\gamma_{i'}^f = 0$, we know that (X, γ^f) is a feasible solution for LP_{t+1} even if the demand variable $d_{i'}$ was decreased. Thus the inductive claim holds. $\square$

The previous lemma implies that the final allocation X gives a feasible solution for the final LP. Thus, X fulfills conditions (1) and (2) for a feasible outcome. Condition (3) is also fulfilled as by the definition of the demand of a bidder, the auction guarantees that $\beta_i \geq p_i$ for all $i \in N$. Thus, the outcome (X, p) computed by the auction is a feasible outcome. As no bidder is assigned weighted capacity if the price is above his valuation and the mechanism never pays the bidders, the auction is IR. As it is an increasing price auction, it is also IC. We show this formally in the next proposition.

Proposition 3.9. *The auction in Algorithm 3.1 is individually rational and incentive compatible.*

Proof. Since no bidder will ever pay a higher price than his reported valuation and the demand is set so that $\beta_i \geq p_i$, individual rationality follows.

We next show incentive compatibility and use b for the bids and v for the real valuations. First observe that a bidder i with $v_i = 0$ cannot increase his utility by bidding $b_i > 0$. Bidder i's utility is zero when bidding $b_i = 0$ and cannot become positive by bidding $b_i > 0$.

Let us now consider a bidder i with $v_i > 0$. We first show that our ordering assumption causes no problems. Observe that if i bids $b_i = 0$ then i is selected in an earlier or the same iteration of the for-loop during the first iteration of the while-loop when the price for the bidder is zero. That is, the set of bidders processed before i when $b_i = 0$ is a subset of the set of bidders processed before i when $b_i = v_i$. Thus, all the bidders with positive bid still have infinite demand and the optimal solution of the LP for i cannot increase. More formally, assume first that i bids his valuation $b_i = v_i$, let the solution of the first LP for bidder i be (X, γ), and let the parameters of the LP be d and c. Next consider the case where i bids $b_i = 0$, let the first LP for bidder i be LP', and let the parameters of LP' be d' and c'. It holds that $c' \leq c$

and $d' \geq d$. Moreover, for all u with $c_u > c'_u$ holds $d'_u = \infty$. It follows that (1) $\gamma' := \gamma + c - c' \leq d'$ by $\gamma \leq d$ and (2) $0 \leq \gamma \leq \gamma'$. Thus, (X, γ') is a feasible solution for LP′ with objective value $\gamma'_i = \gamma_i$. It follows that the optimal value is at most γ_i. Since bidder i cannot obtain weighted capacity in the next iterations of the while-loop if he bids $b_i = 0$ and a bidder never pays a higher price than his reported valuation it follows that bidding $b_i = 0$ does not increase i's utility.

Again consider a bidder i with $v_i > 0$ and recall that by the construction of the auction, each bidder i with $v_i > 0$ never pays a higher price than his reported valuation. If bidder i's reported valuation is b_i and $0 < b_i < v_i$, his demand variable d_i is zero for all prices larger than b_i. Thus, his utility cannot increase by reporting b_i as the weighted capacity he gets for each price $\pi \leq b_i$ cannot increase and he will lose all weighted capacity that he clinched at a price larger than b_i. Moreover, if his reported valuation is $b_i > v_i$, he gets the same weighted capacity for each price $\pi \leq v_i$. He might receive additional weighted capacity at a price larger v_i, but this cannot increase his utility. Thus, the auction is IC. □

We show finally that the outcome (X, p) our auction computes does not contain any trading swap sequence, and thus, by Theorem 3.2 it is PO. The proof shows that every trading swap sequence in (X, p) would lead to a superior solution to one of the linear programs solved by the mechanism. Since the mechanism found an optimal solution this leads to a contradiction.

Theorem 3.10. *The outcome returned by the auction in Algorithm 3.1 is Pareto-optimal.*

Proof. We will show that the outcome does not contain any tss. Let $(X^{\mathrm{f}}, p^{\mathrm{f}})$ be the final outcome computed by the auction and assume by contradiction that there exists a tss (δ, a) where the length of a is $k + 1$. We define $u = a_0$ and $w = a_k$. Consider the outcome (X', p') that results from $(X^{\mathrm{f}}, p^{\mathrm{f}})$ through the tss (δ', a) where $\delta' = \delta/2$ and that is Pareto-superior to $(X^{\mathrm{f}}, p^{\mathrm{f}})$ by Proposition 3.3. Define $c_i^{\mathrm{f}} := \sum_{j \in I} \alpha_j x_{i,j}^{\mathrm{f}}$ and $c'_i := \sum_{j \in I} \alpha_j x'_{i,j}$ for all bidders i. Note that $c'_w = c_w^{\mathrm{f}} - \delta'$, $c'_u = c_u^{\mathrm{f}} + \delta'$, and $c'_i = c_i^{\mathrm{f}} \; \forall i \in N \setminus \{u, w\}$ by (A2).

We will show that (X', p') can be used to construct a smaller feasible solution to one of the linear programs solved by the algorithm. Since the linear program has found the minimal solution this leads to a contradiction with the assumption that there exists a tss in $(X^{\mathrm{f}}, p^{\mathrm{f}})$. The value c_w of bidder w increases only when an LP was trying to minimize γ_w and returns a non-zero value for γ_w. Since $c_w^{\mathrm{f}} > c'_w$, there exists a unique LP solved for bidder w by the auction that has parameters c^* and d^*, for which $c_w^* \leq c'_w$, and where the solution (X^*, γ^*) satisfies $c_w^* + \gamma_w^* > c'_w$. We name the linear program LP* and show the contradiction for it. Let π^* be the price and p^* be the payment vector at the time when we solve LP*.

We first show that the outcome of the auction $(X^{\mathrm{f}}, p^{\mathrm{f}})$ corresponds to a feasible solution $(X^{\mathrm{f}}, \gamma^{\mathrm{f}})$ for LP* where $\gamma_i^{\mathrm{f}} = c_i^{\mathrm{f}} - c_i^*$ for all $i \in N$ and that $(X^{\mathrm{f}}, \gamma^{\mathrm{f}})$ actually fulfills a stronger version of Constraint (d).

Claim 3.11. *The solution $(X^{\mathrm{f}}, \gamma^{\mathrm{f}})$ is feasible for LP*. It holds that (1) if $\pi^* > 0$ then $\gamma_i^{\mathrm{f}} \leq d_i^* - \frac{\beta_i - p_i^{\mathrm{f}}}{\pi^*}$ for all $i \in N$ with $d_i^* > D_i(\beta_i - p_i^*, \pi^* + 1)$ and (2) for all $i \in N$ with $d_i^* \leq D_i(\beta_i - p_i^*, \pi^* + 1)$ and $v_i \geq \pi^* + 1$ it holds that $\gamma_i^{\mathrm{f}} \leq d_i^* - \frac{\beta_i - p_i^{\mathrm{f}}}{\pi^* + 1}$.*

Proof. Let i be a bidder in N. First we show that $(X^{\mathrm{f}}, \gamma^{\mathrm{f}})$ fulfills the constraints of LP*. Since the outcome $(X^{\mathrm{f}}, \mathrm{p}^{\mathrm{f}})$ is derived from the final linear program executed by the algorithm, X^{f} fulfills the Constraints (a), (b), and (e). Constraint (c) holds by definition of γ^{f} and Constraint (f) holds because $\gamma^{\mathrm{f}} = c^{\mathrm{f}} - c^* \geq 0$. It remains to show that (d) is fulfilled as well. Recall that if $\pi^* = 0$ the case $d_i^* > D_i(\beta_i - p_i^*, \pi^* + 1)$ takes place before the demand of bidder i has been updated in the first iteration of the while-loop. Thus, if $\pi^* = 0$ and $d_i^* > D_i(\beta_i - p_i^*, \pi^* + 1)$ then $d_i^* = \infty$ and (d) holds. Moreover, $d_i^* \leq D_i(\beta_i - p_i^*, \pi^* + 1)$ and $v_i < \pi^* + 1$ imply $d_i^* = D_i(\beta_i - p_i^*, \pi^* + 1) = 0$. It follows that γ_i^{f} has to be zero by the condition of the for-loop, and thus, (d) holds. Otherwise Constraint (d) will follow from (1) and (2).

For (1) and (2) notice that $(\beta_i - p_i^{\mathrm{f}})$ is the remaining budget of bidder i at the end of the auction, which is, the money *not* spent by i, and that bidder i clinched $\gamma_i^{\mathrm{f}} = c_i^{\mathrm{f}} - c_i^*$ "weighted capacity" after LP* was solved.

To (1): Consider first the case where $\pi^* > 0$ and $d_i^* > D_i(\beta_i - p_i^*, \pi^* + 1)$. Note that bidder i has a remaining budget of $d_i^* \pi^*$ when LP* is solved. Thus, bidder i pays $d_i^* \pi^* - (\beta_i - p_i^{\mathrm{f}})$ for all the "weighted capacity" γ_i^{f} that was not clinched before LP* was solved. Moreover, the price that he pays per "weighted capacity" in this and the following iterations is at least π^*. It follows that $\gamma_i^{\mathrm{f}} \pi^* \leq d_i^* \pi^* - (\beta_i - p_i^{\mathrm{f}})$.

To (2): Consider next a bidder $i \in N$ with $d_i^* \leq D_i(\beta_i - p_i^*, \pi^* + 1)$ and $v_i \geq \pi^* + 1$. By line 6 it holds that d_i^* is equal to $D_i(\beta_i - p_i^*, \pi^*)$ or $D_i(\beta_i - p_i^*, \pi^* + 1)$. Since $D_i(\beta_i - p_i^*, \pi^* + 1) \leq D_i(\beta_i - p_i^*, \pi^*)$ it holds that $d_i^* = D_i(\beta_i - p_i^*, \pi^* + 1)$. Note that every such bidder has a remaining budget of $d_i^*(\pi^* + 1)$ when LP* is solved. Thus, bidder i pays $d_i^*(\pi^* + 1) - (\beta_i - p_i^{\mathrm{f}})$ for all the "weighted capacity" γ_i^{f} that was not clinched before LP* was solved. Moreover, the price that he pays per "weighted capacity" in this and the following iterations is at least $\pi^* + 1$. It follows that $\gamma_i^{\mathrm{f}}(\pi^* + 1) \leq d_i^*(\pi^* + 1) - (\beta_i - p_i^{\mathrm{f}})$. $\qquad \square$

Next we define $\gamma_i' = c_i' - c_i^*$ for all $i \in N$ and show that (X', γ') is a feasible solution of LP* and that $\gamma_w' < \gamma_w^*$, thus leading to a contradiction. By (A2) and Claim 3.11 it holds that X' satisfies Constraints (a) and (b) for LP*. By the definition of γ' Constraint (c) also holds. Constraint (e) is satisfied for X' by Claim 3.11 and since (S3) holds for (δ, a), and thus, also for (δ', a). Constraint (f) is satisfied since $c_i' \geq c_i^*$ for all $i \in N$: LP* was selected such that $c_w' \geq c_w^*$, and we have $c_i' \geq c_i^{\mathrm{f}} \geq c_i^*$ for all $i \in N \setminus \{w\}$. We next show that also Constraint (d) is satisfied. First note that for all $i \in N \setminus \{u, w\}$ we know that $\gamma_i' = \gamma_i^{\mathrm{f}}$, and thus, Constraint (d) holds for such i by Claim 3.11. For $i = w$, it holds that $c_i' < c_i^{\mathrm{f}}$, and thus, $\gamma_w' < \gamma_w^{\mathrm{f}} \leq d_w^*$ by Claim 3.11. Hence Constraint (d) also holds for $i = w$. For $i = u$, we know that $\gamma_u' = \gamma_u^{\mathrm{f}} + \delta'$ as $c_u' = c_u^{\mathrm{f}} + \delta'$ and we have to show that $d_u^* \geq \gamma_u'$.

We first consider the case $\pi^* = 0$. Since $v_u > v_w$ and $v_w \geq 0$ we know that $v_u \geq \pi^* + 1$. Assume first that $d_u^* > D_u(\beta_u - p_u^*, \pi^* + 1)$. Thus, the demand of

bidder u was not updated when LP* was called and is still infinite. Hence, $d_u^* \geq \gamma_u'$. Next assume that $d_u^* \leq D_u(\beta_u - p_u^*, \pi^* + 1)$. Thus, the demand of bidder u was already updated when LP* was called for w; by the ordering of the input this implies that $v_w > 0$, i.e., $v_w \geq \pi^* + 1$. Hence, $\beta_u - p_u^\mathrm{f} \geq p_u' - p_u^\mathrm{f} = v_w \delta' \geq (\pi^* + 1)\delta'$. By Claim 3.11 and $v_u \geq \pi^* + 1$ it follows that $d_u^* \geq \gamma_u^\mathrm{f} + \frac{\beta_u - p_u^\mathrm{f}}{\pi^* + 1} \geq \gamma_u^\mathrm{f} + \delta' = \gamma_u'$.

Next, we consider the case $\pi^* > 0$. Since we solve LP* for w we know that $d_w^* > 0$ when LP* is solved, and thus, $v_w \geq \pi^*$. Hence, $\beta_u - p_u^\mathrm{f} \geq (p_u^\mathrm{f} + v_w \delta) - p_u^\mathrm{f} = v_w \delta \geq \pi^* \delta$. Assume first that $d_u^* > D_u(\beta_u - p_u^*, \pi^* + 1)$. By Claim 3.11 it follows that $d_u^* \geq \gamma_u^\mathrm{f} + \frac{\beta_u - p_u^\mathrm{f}}{\pi^*} \geq \gamma_u^\mathrm{f} + \delta = \gamma_u' + \delta - \delta' > \gamma_u'$. Next assume that $d_u^* \leq D_u(\beta_u - p_u^*, \pi^* + 1)$. By Claim 3.11 and $v_u \geq \pi^* + 1$ it follows that $d_u^* \geq \gamma_u^\mathrm{f} + \frac{\beta_u - p_u^\mathrm{f}}{\pi^* + 1} \geq \gamma_u^\mathrm{f} + \delta \frac{\pi^*}{\pi^* + 1} \geq \gamma_u^\mathrm{f} + \delta \frac{1}{2} = \gamma_u^\mathrm{f} + \delta' = \gamma_u'$.

It remains to show that $\gamma_w' < \gamma_w^*$. Recall that by the definition of LP* it holds that $c_w^* + \gamma_w^* > c_w'$, while, by definition of γ_w', $c_w' = c_w^* + \gamma_w'$. Thus $\gamma_w' < \gamma_w^*$, which leads to the desired contradiction. $\qquad\square$

3.5 Randomized Clinching Auction for the Indivisible Case

We will now use the outcome computed by the deterministic auction for *divisible* slots to give a randomized auction for multiple keywords with *indivisible* slots that ensures that bidder i receives at most κ_i slots for each keyword. The randomized auction has to assign to every slot $j \in I$ exactly one bidder $i \in N$ for each keyword $r \in R$. We call a distribution over outcomes for the indivisible case *Pareto-superior* to another such distribution, if the expected utility of a bidder or the auctioneer is higher, while the expected utilities of the others are at least as large. If a distribution has no Pareto-superior distribution, we call it *Pareto-optimal*. The basic idea is as follows: given the PO outcome for the *divisible* case, we construct a *distribution over outcomes* of the *indivisible* case such that the expected utility of every bidder and of the auctioneer is the same as the utility of the bidder and the auctioneer in the *divisible* case. The mechanism for the *indivisible* case would, thus, first call the mechanism for the *divisible* case (with the same input) and then convert the resulting outcome (X^d, p^d) into a representation of a distribution over PO outcomes for the *indivisible* case. It then samples from this representation to receive the outcome that it outputs. As during all these steps the expected utility of the bidders and the auctioneer remains unchanged and the mechanism for the *divisible* case is IR and IC, the mechanism for the *indivisible* case is IR in expectation and IC in expectation. To show that the final outcome is PO in expectation and also PO ex post we use the following lemma.

Lemma 3.12. *For every probability distribution over feasible outcomes in the indivisible case there exists a feasible outcome in the divisible case such that the utilities of the bidders and the auctioneer in the divisible case equal their corresponding expected utilities in the indivisible case.*

Proof. We show first that for every feasible outcome (X, p) in the indivisible case there exists feasible outcome (X^d, p) in the divisible case where all the bidders and the auctioneer have the same utility. The utility of the auctioneer stays unchanged because we leave the payments unchanged. We set $x_{i,j}^d = \frac{1}{|R|} \sum_{r \in R} x_{i,j,r}$ for all $i \in N$ and $j \in I$. The utility of bidder i is the same for (X, p) and (X^d, p) because the utility of bidder i is $\sum_{j \in I} (\frac{\alpha_j}{|R|} \sum_{r \in R} x_{i,j,r}) v_i - p_i = \sum_{j \in I} \alpha_j x_{i,j}^d v_i - p_i$ for (X, p). The slot constraint for (X, p) implies $\kappa_i \geq \max_{r \in R} \sum_{j \in I} x_{i,j,r} \geq \sum_{j \in I} (\frac{1}{|R|} \sum_{r \in R} x_{i,j,r}) = \sum_{j \in I} x_{i,j}^d$ for all $i \in N$, and therefore it implies the slot constraint in (X^d, p). Since all the slots are fully assigned to the bidders in (X, p), and consequently for (X^d, p), it follows that (X^d, p) is feasible.

Given a probability distribution over feasible outcomes for the indivisible case, transform each feasible outcome that has a non-zero probability into a feasible outcome for the divisible case. Then create a new outcome for the divisible case by adding up all of these feasible outcomes for the divisible case weighted by the probability distribution. Since the weights are created by a probability distribution, they add up to one, and thus, the resulting combined outcome fulfills Conditions (1) and (2) of a feasible outcome. As the payment is identical to the payment for the indivisible case, Condition (3) is also fulfilled. □

Lemma 3.12 implies that any probability distribution over feasible outcomes in the *indivisible* case that is Pareto-superior to the distribution generated by our auction would lead to a feasible outcome for the *divisible* case that is Pareto-superior to (X^d, p^d). This is not possible as (X^d, p^d) is PO. Thus, the mechanism for the indivisible case described above is PO in expectation. Additionally, since our auction selects only outcomes having a positive probability, each realized outcome is ex-post Pareto-optimal: if in the *indivisible* case there existed an in expectation Pareto-superior randomized outcome $(\bar{X}^*, \bar{p}^*)$ to one of the outcomes that gets chosen with a positive probability in our auction $(\bar{X}, \bar{p})$, then a randomized outcome (X^*, p^*) would exist that is in expectation Pareto-superior to the randomized outcome of the auction (X, p). The randomized outcome (X^*, p^*) equals (X, p) if an allocation other than $(\bar{X}, \bar{p})$ gets drawn from (X, p), and it samples from $(\bar{X}^*, \bar{p}^*)$ otherwise. Thus, by Lemma 3.12 a Pareto-superior outcome would exist in the *divisible* case. By the same argument as above this would lead to a contradiction.

We still need to explain how to use the PO outcome (X^d, p^d) for the *divisible* case to give a probability distribution for the *indivisible* case such that the expected utility of every bidder for the probability distribution is equal to their utility in the divisible case. We will use the following steps: (a) We will reduce the computation of the probability distribution to a scheduling problem with preemption on uniform processors with the objective to minimize the finishing time. (b) We use Birkhoff's theorem [101] to show that an optimal schedule exists and has finishing time one. (c) Then we argue that an algorithm by Gonzalez and Sahni [55] can be used to compute a schedule with finishing time one. This schedule represents a probability distribution on feasible outcomes in the indivisible case and we show how to use it

to sample from the probability distribution. Computing the probability distribution and sampling from it can be done in time linear in the number of slots m.

We first define the input and the output clearly. For the computation of the probability distribution the input is the set of slots I, the set of bidders N, the slot constraints κ_i for all $i \in N$, the qualities α_j for all $j \in I$, and a feasible divisible outcome (X^d, p^d) that also defines the weighted capacities c_i^d for all $i \in N$. The output is a function that gives us for each number $t \in (0, 1]$ an allocation $X(t)$ of slots to bidders, where each bidder $i \in N$ gets κ_i slots. The allocation $X(t)$ is a binary matrix where $(X(t))_{i,j} = 1$ if and only if slot j is assigned to bidder i. For a random number T that is uniformly distributed on $(0, 1]$, the expected weighted capacity $\mathbf{E}\left[\sum_{j\in I}(X(T))_{i,j}\alpha_j\right]$ for each bidder $i \in N$ has to be equal to c_i^d. Given the allocation function $X(t)$ it suffices to draw $|R|$ numbers $t_1, \ldots, t_{|R|}$ uniformly from $(0, 1]$, use the allocation $X(t_r)$ for keyword r, and set $p = p^d$. The expected utility of all bidders $i \in N$ is equal to their utility in (X^d, p^d) because $\mathbf{E}\left[(\sum_{j\in I}\frac{\alpha_j}{|R|}\sum_{r\in R}(X(T))_{i,j})v_i - p_i\right] = \left(\frac{1}{|R|}\sum_{r\in R}\mathbf{E}\left[\sum_{j\in I}(X(T))_{i,j}\alpha_j\right]\right)v_i - p_i = (\frac{1}{|R|}\sum_{r\in R}c_i^d)v_i - p_i = c_i^d v_i - p_i$.

(a) In the scheduling problem we consider, we have m jobs with length $l_1, \ldots, l_m$ and m processors with speed $s_1, \ldots, s_m$ as input. Jobs can be processed on multiple processors, but not at the same time, and preemption is allowed. In a feasible schedule every job has to be finished. That is, if $t_{\lambda,j}$ is the time length that job λ is processed on processor j in the schedule then $\sum_{j=1}^m t_{\lambda,j}s_j = l_\lambda$ for all $\lambda \in \{1, \ldots, m\}$. The finishing time of a schedule is the time it takes until every job is finished. The goal in the scheduling problem is to find a schedule with minimal finishing time.

We first show how to convert the input for the computation of the probability distribution to the input of the scheduling problem. Recall that we suppose $m = \sum_{i=1}^n \kappa_i$. We set $\lambda_0 := 0$, replace each bidder $i \in N$ with κ_i jobs $\lambda_{i-1}+1, \ldots, \lambda_{i-1}+\kappa_i =: \lambda_i$ having length $\frac{c_i^d}{\kappa_i}$, set $\Lambda(i) := \{\lambda_{i-1}+1, \ldots, \lambda_i\}$, and set $\Lambda := \bigcup_{i\in N}\Lambda(i)$. Furthermore, each slot $j \in I$ is a processor with speed α_j.

Next we show that a schedule with finishing time one gives us the desired allocation function. Suppose that we have an allocation of jobs to processors on the interval $(0, 1]$ where no job is processed on multiple processors at the same time. As each processor represents one of the slots in I, we get an allocation function $X(t)$ if we replace for each time $t \in (0, 1]$ and for each bidder $i \in N$ the jobs in $\Lambda(i)$ with bidder i. As $|\Lambda(i)| = \kappa_i$ each bidder i gets κ_i slots assigned and each slot is assigned to one bidder. For $T \sim \mathcal{U}(0, 1]$ we have $\mathbf{E}\left[(X(T))_{i,j}\right] = \sum_{\lambda\in\Lambda(i)}t_{\lambda,j}$ for all $i \in N$ and $j \in I$. Thus, $\mathbf{E}\left[\sum_{j\in I}(X(T))_{i,j}\alpha_j\right] = \sum_{j\in I}\mathbf{E}\left[(X(T))_{i,j}\right]\alpha_j = \sum_{j\in I}\sum_{\lambda\in\Lambda(i)}t_{\lambda,j}\alpha_j = \sum_{\lambda\in\Lambda(i)}\sum_{j\in I}t_{\lambda,j}\alpha_j = \sum_{\lambda\in\Lambda(i)}\frac{c_i^d}{\kappa_i} = c_i^d$.

(b) We now argue that the minimal finishing time t is one for scheduling problems when the inputs are generated by the above reduction. First we restate Birkhoff's theorem, which we use for the argument.

Theorem 3.13 (Birkhoff's theorem [101]). *Each doubly stochastic matrix is a convex combination of permutation matrices.*

Recall that $|\Lambda| = |I|$ because $|\Lambda| = \sum_{i \in N} \kappa_i = m = |I|$. We build an $m \times m$-dimensional square matrix T as follows. We assign for each bidder $i \in N$ each job $\lambda \in \Lambda(i)$ to processor $j \in I$ for time $t_{\lambda,j} = \frac{x_{i,j}^d}{\kappa_i}$. The matrix T has the entries $t_{\lambda,j}$ where $\lambda \in \Lambda$ and $j \in I$. We show next that T is doubly stochastic, that is, the entries of the matrix are non-negative and for each column and for each row the sum of the entries is one. The sums are non-negative because $x_{i,j}^d \geq 0$ and $\kappa_i > 0$ for all $i \in N$ and $j \in I$. As the allocation X^d is feasible and $\sum_{j \in I} x_{i,j}^d = \kappa_i$ for all $i \in N$ because $m = \sum_{i \in N} \kappa_i$, it follows that for each column $j \in I$ of T it holds $\sum_{\lambda \in \Lambda} t_{\lambda,j} = \sum_{i \in N} \sum_{\lambda \in \Lambda(i)} \frac{x_{i,j}^d}{\kappa_i} = \sum_{i \in N} x_{i,j}^d \sum_{\lambda \in \Lambda(i)} \frac{1}{\kappa_i} = \sum_{i \in N} x_{i,j}^d = 1$, and for each row $\lambda \in \Lambda(i)$ with $i \in N$ of T it holds $\sum_{j \in I} t_{\lambda,j} = \sum_{j \in I} \frac{x_{i,j}^d}{\kappa_i} = \frac{1}{\kappa_i} \sum_{j \in I} x_{i,j}^d = 1$. Since T is doubly stochastic, we can decompose T by Birkhoff's theorem into a convex combination of permutation matrices. In a permutation matrix there is one entry in each column and each row that is one and all other entries are zero. Let k be the number of permutation matrices in the convex combination and ζ_l be the coefficient of the l'th permutation matrix P_l for all $l \in \{1, \ldots, k\}$. We construct our schedule in the following way: for the time interval $(\sum_{s=1}^{l-1} \zeta_s, \sum_{s=1}^{l} \zeta_s]$ we assign job λ to processor j if $(P_l)_{\lambda,j} = 1$. Every job $\lambda \in \Lambda(i)$ for all $i \in N$ is finished because $\sum_{j \in I} (\sum_{l=1}^{k} (P_l)_{\lambda,j} \zeta_l) \alpha_j = \sum_{j \in I} t_{\lambda,j} \alpha_j = \sum_{j \in I} \frac{x_{i,j}^d}{\kappa_i} \alpha_j = \frac{1}{\kappa_i} \sum_{j \in I} x_{i,j}^d \alpha_j = \frac{c_i^d}{\kappa_i}$. By the definition of a permutation matrix, it follows that each job is computed on exactly one processor at the same time, and each processor computes exactly one job at the same time. The finishing time is one because the time intervals of the schedule are the coefficients of a convex combination, and thus, $\sum_{l=1}^{k} \zeta_l = 1$. As the schedule has no idle time, the finishing time cannot be less than one. Thus, every optimal schedule has finishing time exactly one.

(c) We can use the scheduling algorithm by Gonzalez and Sahni [55] that minimizes the finishing time to compute an optimal schedule with finishing time one. The schedules computed by the algorithm have at most $2(m-1)$ preemptions, and the computation has a time complexity that is linear in the number of jobs $|\Lambda| = m$. The algorithm outputs a schedule for each job, which is represented by a list of the processors on which the job gets processed; the lists contain the start time and the end time of the allocations of the jobs to the processors. Thus, we can represent the allocation function $X(t)$ by merging the lists of all the jobs in $\Lambda(i)$ to a list for each bidder $i \in N$. We can evaluate the i'th row of $X(t)$ for a certain t by traversing the list of bidder i and setting $(X(t))_{i,j} = 1$ if and only if processor j is in the list and $t \in (a, b]$ where a is the start time and b is the end time in the list entry. To sample from the probability distribution for the indivisible case we pick $|R|$ random numbers t_r, $1 \leq r \leq |R|$, uniformly at random from $(0, 1]$ and set for each bidder i and each slot j the value $x_{i,j,r} = X(t_r)_{i,j}$.

The following theorem summarizes the results in this section.

Theorem 3.14. *A PO and IR outcome for the divisible case can be converted in polynomial time without a change of the (expected) utilities into a randomized outcome for the indivisible case that is PO in expectation, PO ex post, and IR in expectation. This results in a mechanism that is PO in expectation, PO ex post, IR in expectation, and IC in expectation.*

3.6 Conclusion

We design an auction for sponsored search where the slots of multiple keywords are auctioned at the same time. We assume that the number of slots is the same for all keywords and that the click-through rates only depend on the position of the slot and not on the keyword. Furthermore, we assume that bidders are interested in all keywords but can only obtain a limited number of slots of each keyword. Each bidder defines a budget limit and the auction cannot charge him a payment above this limit. The auction that we design for this setting is incentive compatible in expectation, individually rational in expectation, and finds outcomes that are Pareto-optimal ex post/in expectation.

The possibility to generalize our results to more complex sponsored search settings is restricted by the following impossibility results. We have shown in Theorem 2.6 in Chapter 2 that no deterministic auction can achieve incentive compatibility, individual rationality, and Pareto-optimality for the setting studied in this chapter. This follows by considering a single keyword and bidders who are interested in all slots of this keyword. Furthermore, no deterministic auction can achieve incentive compatibility, individual rationality, and Pareto-optimality if the bidders can bid different amounts for the different keywords; this follows from the impossibility result for multidimensional valuations in Fiat et al. [42]. Moreover, this result has been extended to randomized mechanisms that achieve incentive compatible in expectation, individual rationality in expectation, and Pareto-optimality in expectation in Theorem 2.14 in Chapter 2. However, as shown in Colini-Baldeschi et al. [24][5] there exists an auction for bidders that have interest sets for the keywords; this result holds under the constraint that all slots have the same click-through rate. The auction generalizes an auction mechanism by Fiat et al. [42]. It is an interesting open question if the auction defined in this chapter can be extended to settings where slots have different click-through rates and bidders have interest sets for keywords.

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[5]The results presented in this chapter are published in the same publication.

4

Valuation Compressions in VCG-Based Combinatorial Auctions

4.1 Introduction

For combinatorial auctions the incentive compatible mechanism that maximizes social welfare is the VCG mechanism (Vickrey [104], Clarke [23], and Groves [57]). Unfortunately, for many valuation spaces computing the VCG allocation and payments is a computationally hard problem. This is, for example, the case for subadditive, fractionally subadditive, and submodular valuations (see Lehmann et al. [80]). We thus study the performance of the VCG mechanism in settings in which the bidders are forced to use bids from a subspace of the valuation space for which the allocation and payments can be computed efficiently. This is obviously the case for additive bids, where the VCG-based mechanism can be interpreted as a separate second-price auction for each item. But it is also the case for the syntactically defined bidding space OXS, which stands for ORs of XORs of singletons, and the semantically defined bidding space GS, which stands for gross substitutes. For OXS bids polynomial-time algorithms for finding a maximum weight matching in a bipartite graph such as the algorithms of Tarjan [103] and Fredman and Tarjan [44] can be used. For GS bids there is a fully polynomial-time approximation scheme due to Kelso and Crawford [74] and polynomial-time algorithms based on linear programming by Vries et al. [105] and convolutions of $M^\natural$-concave functions (see Murota [87, 88] and Murota and Tamura [89]).

One consequence of restrictions of this kind, that we refer to as *valuation compressions,* is that there is typically no longer a truthful dominant-strategy equilibrium that maximizes social welfare. We therefore analyze the *Price of Anarchy*; that is, the ratio between the optimal welfare and the worst possible social welfare at

equilibrium. We focus on equilibrium concepts such as correlated equilibria and coarse correlated equilibria, which can be computed in polynomial time (see Jiang and Leyton-Brown [71] and Papadimitriou and Roughgarden [95]), and naturally emerge from learning processes in which the bidders minimize external or internal regret (see Cesa-Bianchi et al. [20], Foster and Vohra [43], Hart and Mas-Colell [62], and Littlestone and Warmuth [81]).

4.1.1 Contribution

We start our analysis by showing that for restrictions from subadditive valuations to additive bids deciding whether a pure Nash equilibrium exists is NP-hard. This shows the necessity to study other bidding functions or other equilibrium concepts.

We then define a smoothness notion for mechanisms that we refer to as *relaxed smoothness*. This smoothness notion is weaker in some aspects and stronger in another aspect than the weak smoothness notion of Syrgkanis and Tardos [102]. It is weaker in that it allows a bidder's deviating bid to depend on the distribution of the bids of the other bidders. It is stronger in that it disallows the bidder's deviating bid to depend on his own bid. The former gives us more power to choose the deviating bid, and thus has the potential to lead to better bounds. The latter is needed to ensure that the bounds on the welfare loss extend to coarse correlated equilibria and minimization of external regret.

We use relaxed smoothness to prove an upper bound of 4 on the Price of Anarchy with respect to (coarse) correlated equilibria, additive bids, and subadditive valuations. Similarly, we show that the average welfare obtained by minimization of internal and external regret converges to $1/4$-th of the optimal welfare. The proofs of these bounds are based on the arguments used by Feldman et al. [41]. The bounds improve the previously known bounds for these solution concepts by a logarithmic factor. We also use relaxed smoothness to prove bounds for restrictions to non-additive bids. For subadditive valuations the bounds are $\mathcal{O}(\log(m))$ resp. $\Omega(1/\log(m))$, where m denotes the number of items. For fractionally subadditive valuations the bounds are 2 resp. $1/2$. The proofs require novel techniques as non-additive bids lead to non-additive prices for which most of the techniques developed in prior work fail. The bounds extend the corresponding bounds of Bhawalkar and Roughgarden [12] and Christodoulou et al. [22] from additive to non-additive bids.

Finally, we prove lower bounds on the Price of Anarchy. By showing that VCG-based mechanisms satisfy the *outcome closure property* of Milgrom [86] we show that the Price of Anarchy with respect to pure Nash equilibria weakly increases with expressiveness. We thus extend the lower bound of 2 from Christodoulou et al. [22] from additive to non-additive bids. This shows that our upper bounds for fractionally subadditive valuations are tight. We prove a lower bound of 2.4 on the Price of Anarchy with respect to pure Nash equilibria that applies to restrictions from subadditive valuations to OXS bids. Together with the upper bound of 2 of Bhawalkar and Roughgarden [12] for restrictions from subadditive valuations to additive bids

this shows that more expressiveness can give rise to additional equilibria of poorer efficiency. Note that non-efficient equilibria can also exist when the valuation and bidding space coincide.

Bids	Valuations	
	less general	subadditive
additive	$[2, 2]$	$[2, \mathbf{4}]$
more general	$[\mathbf{2}, \mathbf{2}]$	$[\mathbf{2.4}, \mathcal{O}(\mathbf{\log(m)})]$

Table 4.1: Summary of our results (bold) and the related work (regular) for coarse correlated equilibria and minimization of external regret through repeated play. The range indicates upper and lower bounds on the Price of Anarchy.

4.1.2 Related Work

The Price of Anarchy of restrictions to additive bids is analyzed in Bhawalkar and Roughgarden [12], Christodoulou et al. [22], and Feldman et al. [41] for second-price auctions and in Feldman et al. [41] and Hassidim et al. [63] for first price auctions. The case where all items are identical, but additional items contribute less to the valuation and bidders are forced to place additive bids is analyzed in Keijzer et al. [73] and Markakis and Telelis [83]. Smooth games are defined and analyzed in Roughgarden [98, 99]. The smoothness concept is extended to mechanisms in Syrgkanis and Tardos [102]. An upper bound on the efficiency of the best Nash equilibrium of a mechanism is given by Benisch et al. [10]; this bound increases with some measure for expressiveness. Babaioff et al. [9] study, independently of our work, the Price of Anarchy when bidding languages are restricted. A technique for obtaining lower bounds on the Price of Anarchy that is based on the complexity of computation and communication is introduced in Roughgarden [97]. Other results on the hardness of computing equilibria of combinatorial auctions are given by Cai and Papadimitriou [19].

4.2 Preliminaries

Combinatorial Auctions. In a *combinatorial auction* there is a set of n *bidders* $N = \{1, \ldots, n\}$ and a set of m *items* $I = \{1, \ldots, m\}$. Each bidder $i \in N$ employs preferences over bundles of items, represented by a *valuation function* $v_i : \mathcal{P}(I) \to \mathbb{R}_{\geq 0}$. We use V_i for the *class of valuation functions* of bidder i, and $V = \prod_{i \in N} V_i$ for the class of joint valuations. We write $v = (v_i, v_{-i}) \in V$, where v_i denotes bidder i's valuation and v_{-i} denotes the valuations of all bidders other than i. We assume that the valuation functions are *normalized* and *monotone*, i.e., $v_i(\emptyset) = 0$ and $v_i(S) \leq v_i(T)$ for all $S \subseteq T$.

We use the same notation for the bidding functions $b_i : \mathcal{P}(I) \to \mathbb{R}_{\geq 0}$. We use B_i for the *class of bidding functions* of bidder i, and $B = \prod_{i \in N} B_i$ for the class of joint bids. We write $b = (b_i, b_{-i}) \in B$, where b_i denotes bidder i's bid and b_{-i} denotes the bids of all bidders other than i. Furthermore, we take the same assumptions for the bidding functions; we assume that they are *normalized* and *monotone*.

A mechanism $M = (f, p)$ is defined by an *allocation rule* $f : B \to \mathcal{X}(I)$ and a *payment rule* $p : B \to \mathbb{R}_{\geq 0}^n$, where B is the *class of bidding functions* and $\mathcal{X}(I)$ denotes the *set of allocations* consisting of all possible partitions X of the set of items I into n sets $X_1, \ldots, X_n$. Given $f(b) = X$ we set $f_i(b) = X_i$ for all $i \in N$ and $b \in B$. We define the *social welfare* of an *allocation* $X \in \mathcal{X}(I)$ as the sum $SW(X) = \sum_{i \in N} v_i(X_i)$ of the bidders' valuations and use $\mathrm{OPT}(v)$ to denote the maximal achievable social welfare. We say that an allocation rule f is *efficient* if for all bids b it chooses the allocation $f(b)$ that maximizes the sum of the bidder's bids, i.e., $\sum_{i \in N} b_i(f_i(b)) = \max_{X \in \mathcal{X}(I)} \sum_{i \in N} b_i(X_i)$. We assume *quasi-linear preferences*, i.e., bidder i's *utility* under mechanism M given valuations v and bids b is $u_i(b, v_i) = v_i(f_i(b)) - p_i(b)$.

We focus on the *Vickrey-Clarke-Groves (VCG)* mechanism [23, 57, 104]. Define $\hat{b}_{-i}(S) = \max_{X \in \mathcal{X}(S)} \sum_{j \neq i} b_j(X_j)$ for all $S \subseteq I$. The VCG mechanism starts from an efficient allocation rule f and computes the payment of each bidder i as $p_i(b) = \hat{b}_{-i}(I) - \hat{b}_{-i}(I \setminus f_i(b))$. As the payment $p_i(b)$ only depends on the bundle $f_i(b)$ allocated to bidder i and the bids b_{-i} of the bidders other than i, we also use $p_i(f_i(b), b_{-i})$ to denote bidder i's payment.

If the bids are additive then the VCG prices are additive, i.e., for every bidder i and every bundle $S \subseteq I$ we have $p_i(S, b_{-i}) = \sum_{j \in S} \max_{k \neq i} b_k(\{j\})$. Furthermore, the set of items that a bidder wins in the VCG mechanism are the items for which he has the highest bid, i.e., bidder i wins item j against bids b_{-i} if $b_i(\{j\}) \geq \max_{k \neq i} b_k(\{j\}) = p_i(\{j\}, b_{-i})$ (ignoring ties). Many of the complications in this chapter come from the fact that these two observations do *not* apply to non-additive bids.

Valuation Compressions. Our main object of study in this chapter are *valuation compressions*, i.e., restrictions of the class of bidding functions B to a strict subclass of the class of valuation functions V.[1] Specifically, we consider valuations and bids from the following hierarchy due to Lehmann et al. [80],

$$\mathrm{OS} \subset \mathrm{OXS} \subset \mathrm{GS} \subset \mathrm{SM} \subset \mathrm{XOS} \subset \mathrm{CF},$$

where OS stands for additive, GS for gross substitutes, SM for submodular, and CF for subadditive.

The classes OXS and XOS are syntactically defined. Define OR ($\vee$) as $(u \vee w)(S) = \max_{T \subseteq S} (u(T) + w(S \setminus T))$ and XOR ($\otimes$) as $(u \otimes w)(S) = \max(u(S), w(S))$. Define XS as the class of valuations that assign the same value

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[1]This definition is consistent with the notion of *simplification* used in Dütting et al. [35] and Milgrom [86].

to all bundles that contain a specific item and zero otherwise. Then OXS is the class of valuations that can be described by ORs of XORs of XS valuations and XOS is the class of valuations that can be described by XORs of ORs of XS valuations.

Another important class is the class β-XOS, where $\beta \geq 1$, of β-fractionally subadditive valuations. A valuation v_i is β-fractionally subadditive if for every subset of items T there exists an additive valuation a_i such that (a) $\sum_{j \in T} a_i(\{j\}) \geq v_i(T)/\beta$ and (b) $\sum_{j \in S} a_i(\{j\}) \leq v_i(S)$ for all $S \subseteq T$. It can be shown that the special case $\beta = 1$ corresponds to the class XOS, and that the class CF is contained in $\mathcal{O}(\log(m))$-XOS (see, e.g., Theorem 5.2 in Bhawalkar and Roughgarden [12]). Functions in XOS are called *fractionally subadditive.*

Solution Concepts.   We use game-theoretic reasoning to analyze how bidders interact with the mechanism; a desirable criterion is stability according to some solution concept. In the *complete information* model that we consider the bidders are assumed to know each others' valuations.

The static solution concepts that we consider in this complete information setting are:
$$\text{DSE} \subset \text{PNE} \subset \text{MNE} \subset \text{CE} \subset \text{CCE} \,,$$

where DSE stands for dominant strategy equilibrium, PNE for pure Nash equilibrium, MNE for mixed Nash equilibrium, CE for correlated equilibrium, and CCE for coarse correlated equilibrium.

In our analysis we only need the definitions of pure Nash and coarse correlated equilibria. Bids $b \in B$ constitute a *pure Nash equilibrium (PNE)* for valuations $v \in V$ if for every bidder $i \in N$ and every bid $b'_i \in B_i$, $u_i((b_i, b_{-i}), v_i) \geq u_i((b'_i, b_{-i}), v_i)$. A distribution $\mathcal{B}$ over bids $b \in B$ is a *coarse correlated equilibrium (CCE)* for valuations $v \in V$ if for every bidder $i \in N$ and every pure deviation $b'_i \in B_i$, $\mathbf{E}_{b \sim \mathcal{B}}\left[u_i((b_i, b_{-i}), v_i)\right] \geq \mathbf{E}_{b \sim \mathcal{B}}\left[u_i((b'_i, b_{-i}), v_i)\right]$.

The dynamic solution concept that we consider in this setting is regret minimization. A sequence of bids $b^1, \ldots, b^T$ incurs *vanishing average external regret* if for all bidders i, $\sum_{t=1}^{T} u_i((b_i^t, b_{-i}^t), v_i) \geq \max_{b'_i} \sum_{t=1}^{T} u_i((b'_i, b_{-i}^t), v_i) - o(T)$ holds, where $o(\cdot)$ denotes the little-oh notation. The empirical distribution of bids in a sequence of bids that incurs vanishing external regret converges to a coarse correlated equilibrium (see, e.g., Blum and Mansour [15]).

Price of Anarchy.   We quantify the welfare loss from valuation compressions by means of the *Price of Anarchy (PoA)*.

The PoA with respect to PNE for valuations $v \in V$ is defined as the worst ratio between the optimal social welfare $\text{OPT}(v)$ and the welfare $SW(f(b))$ of a PNE $b \in B$,
$$\text{PoA}(v) = \sup_{b: \text{ PNE}} \frac{\text{OPT}(v)}{SW(f(b))} \,.$$

Similarly, the PoA with respect to MNE, CE, and CCE for valuations $v \in V$ is the worst ratio between the optimal social welfare $SW(f(b))$ and the expected welfare

$\mathbf{E}_{b\sim\mathcal{B}}\left[SW(f(b))\right]$ of a MNE, CE, or CCE $\mathcal{B}$,

$$\text{PoA}(v) = \sup_{\mathcal{B}:\ \text{MNE, CE or CCE}} \frac{\text{OPT}(v)}{\mathbf{E}_{b\sim\mathcal{B}}\left[SW(f(b))\right]}\ .$$

We require that the bids b_i for a given valuation v_i are *conservative* (this requirement is also called *no-overbidding*), i.e., $b_i(S) \leq v_i(S)$ for all bundles $S \subseteq I$. Similar assumptions are made in the related work (Bhawalkar and Roughgarden [12], Christodoulou et al. [22], and Feldman et al. [41]); they are motivated by the risk of overpaying, for instance, due to uncertainty or strategic retaliation.

4.3 Hardness Result for PNE with Additive Bids

Our first result is that deciding whether there exists a pure Nash equilibrium of the VCG mechanism for restrictions from subadditive valuations to additive bids is NP-hard. The proof of this result is by reduction from 3-Partition (see Garey and Johnson [47]) and uses an example with no pure Nash equilibrium by Bhawalkar and Roughgarden [12]. The same decision problem is simple for $V \subseteq$ XOS because pure Nash equilibria are guaranteed to exist (see Christodoulou et al. [22]).

Theorem 4.1. *Suppose that $V = CF$, $B = OS$, that the VCG mechanism is used, and that bidders bid conservatively. Then it is NP-hard to decide whether there exists a PNE.*

Proof. Given an instance of 3-Partition consisting of a multiset of $3m$ positive integers $w_1,\ldots,w_{3m} \in (W/4, W/2)$ that sum up to mW, we construct in polynomial time an instance of a combinatorial auction in which the bidders have subadditive valuations as follows:

The set of bidders is $C_1,\ldots,C_m$ and $D_1,\ldots,D_m$. The set of items is $\mathcal{I} \cup \mathcal{J}$, where $\mathcal{I} = \{1,\ldots,3m\}$ and $\mathcal{J} = \{J_1,\ldots,J_{3m}\}$. Let $\mathcal{J}_i = \{J_i, J_{m+i}, J_{2m+i}\}$. Every bidder C_i has valuations

$$v_{C_i}(S) = \max\{v_{\mathcal{I},C_i}(S), v_{\mathcal{J},C_i}(S)\}, \qquad \text{where}$$
$$v_{\mathcal{I},C_i}(S) = \sum_{e\in\mathcal{I}\cap S} w_e, \qquad \text{and}$$
$$v_{\mathcal{J},C_i}(S) = \begin{cases} 10W & \text{if } |\mathcal{J}_i \cap S| = 3, \\ 5W & \text{if } |\mathcal{J}_i \cap S| \in \{1,2\}, \\ 0 & \text{otherwise.} \end{cases}$$

Every bidder D_i has valuations

$$v_{D_i}(S) = \begin{cases} 16W & \text{if } |\mathcal{J}_i \cap S| = 3, \\ 8W & \text{if } |\mathcal{J}_i \cap S| \in \{1,2\}, \\ 0 & \text{otherwise.} \end{cases}$$

The valuations for the items in $\mathcal{J}$ are motivated by an example for valuations without a PNE in [12]. Note that our valuations are subadditive.

We show first that if there is a solution of our 3-Partition instance then the corresponding auction has a PNE. Let us assume that $P_1, \ldots, P_m$ is a solution of 3-Partition. We obtain a PNE when every bidder C_i bids w_j for each $j \in P_i$ and zero for the other items; and every bidder D_i bids $4W$ for each item in $\mathcal{J}_i$. The first step is to show that no bidder C_i would change his strategy. The utility of C_i is W, because C_i's payment is zero. As the valuation function of C_i is the maximum of his valuation for the items in $\mathcal{I}$ and the items in $\mathcal{J}$ we can study the strategies for $\mathcal{I}$ and $\mathcal{J}$ separately. If C_i changed his bid and won another item in $\mathcal{I}$, C_i would have to pay his valuation for this item because there is a bidder C_j bidding on it, and, thus, his utility would not increase. As C_i bids conservatively, C_i could win at most one item of the items in $\mathcal{J}_i$. His value for the item would be $5W$, but the payment would be D_i's bid of $4W$. Thus, his utility would not be larger than W if C_i won an item of $\mathcal{J}$. Hence, C_i does not want to change his bid. The second step is to show that no bidder D_i would change his strategy. This follows since the utility of every bidder D_i is $16W$, and this is the highest utility that D_i can obtain.

We will now show two facts that follow if the auction is in a PNE: (1) We first show that in every PNE every bidder C_i must have a utility of at least W. To see this denote the bids of bidder D_i for the items J_i, J_{m+i}, J_{2m+i} in $\mathcal{J}_i$ by $d_1, d_2,$ and d_3 and assume w.l.o.g. that $d_1 \leq d_2 \leq d_3$. As bidder D_i bids conservatively, $d_2 + d_3 \leq 8W$, and, thus, $d_1 \leq 4W$. If bidder C_i bade $c_1 = 5W$ for J_1, C_i would win J_1 and his utility would be at least W, because C_i has to pay D_i's bid for J_1. As C_i's utility in a PNE cannot be worse, his utility in a PNE has to be at least W. (2) Next we show that in a PNE bidder C_i cannot win any of the items in $\mathcal{J}_i$. For a contradiction suppose that bidder C_i wins at least one of the items in $\mathcal{J}_i$ by bidding $c_1, c_2,$ and c_3 for the items in $\mathcal{J}_i$. Then bidder D_i does not win the whole set $\mathcal{J}_i$ and his utility is at most $8W$. As bidder C_i bids conservatively, $c_i + c_j \leq 5W$ for $i \neq j \in \{1, 2, 3\}$. Then, $c_1 + c_2 + c_3 \leq 7.5W$. Bidder D_i can however bid $c_1 + \epsilon$, $c_2 + \epsilon$, $c_3 + \epsilon$ for some $\epsilon > 0$ without violating conservativeness, to win all items in $\mathcal{J}_i$ for a utility of at least $16W - 7.5W > 8W$. Thus, D_i's utility increases when D_i changes his bid, i.e., the auction is not in a PNE.

Now we use fact (1) and (2) to show that our instance of 3-Partition has a solution if the auction has a PNE. Let us assume that the auction is in a PNE. By (1) we know that every bidder C_i gets at least utility W. Furthermore, by (2) we know that every bidder C_i wins only items in $\mathcal{I}$, pays zero and has exactly utility W. Recall that all w_e with $e \in \mathcal{I}$ satisfy $W/4 < w_c < W/2$. Thus, the valuation of a bidder C_i is *larger* than $4 \cdot W/4 = W$ for a subset of $\mathcal{I}$ with *more* than 3 items and is smaller than $2 \cdot W/2 = W$ for a subset of $\mathcal{I}$ with *less* than 3 items. Hence, every bidder C_i gets exactly 3 items in $\mathcal{I}$ and the assignment of the items in $\mathcal{I}$ corresponds to a solution of 3-Partition. $\qquad\square$

4.4 Smoothness Notion and Extension Results

Next we define a smoothness notion for mechanisms. It is weaker in some aspects and stronger in another aspect than the weak smoothness notion in Syrgkanis and Tardos [102]. It is weaker because it allows bidder i's deviating bid a_i to depend on the marginal distribution $\mathcal{B}_{-i}$ of the bids b_{-i} of the bidders other than i. This gives us more power in choosing the deviating bid, which might lead to better bounds. It is stronger because it does *not* allow bidder i's deviating bid a_i to depend on his own bid b_i. This allows us to prove bounds that extend to coarse correlated equilibria and not just correlated equilibria.

Definition 4.1. *A mechanism is relaxed (λ, μ_1, μ_2)-smooth for $\lambda, \mu_1, \mu_2 \geq 0$ if for every valuation profile $v \in V$, every distribution over bids $\mathcal{B}$, and every bidder i there exists a bid $a_i(v, \mathcal{B}_{-i})$ such that*

$$\sum_{i \in N} \mathop{\mathbf{E}}_{b_{-i} \sim \mathcal{B}_{-i}} \left[u_i((a_i, b_{-i}), v_i) \right]$$
$$\geq \lambda \cdot \mathrm{OPT}(v) - \mu_1 \cdot \sum_{i \in N} \mathop{\mathbf{E}}_{b \sim \mathcal{B}} \left[p_i(f_i(b), b_{-i}) \right] - \mu_2 \cdot \sum_{i \in N} \mathop{\mathbf{E}}_{b \sim \mathcal{B}} \left[b_i(f_i(b)) \right] .$$

Theorem 4.2. *If a mechanism is relaxed (λ, μ_1, μ_2)-smooth, then the Price of Anarchy under conservative bidding with respect to coarse correlated equilibria is at most*

$$\frac{\max\{\mu_1, 1\} + \mu_2}{\lambda} .$$

Proof. Fix valuations v. Consider a coarse correlated equilibrium $\mathcal{B}$. For each b from the support of $\mathcal{B}$ denote the allocation for b by $f(b) = (f_1(b), \ldots, f_n(b))$. Let $a = (a_1, \ldots, a_n)$ be defined as in Definition 4.1. Then,

$$\mathop{\mathbf{E}}_{b \sim \mathcal{B}} \left[SW(f(b)) \right] = \sum_{i \in N} \mathop{\mathbf{E}}_{b \sim \mathcal{B}} \left[u_i(b, v_i) \right] + \sum_{i \in N} \mathop{\mathbf{E}}_{b \sim \mathcal{B}} \left[p_i(f_i(b), b_{-i}) \right]$$
$$\geq \sum_{i \in N} \mathop{\mathbf{E}}_{b_{-i} \sim \mathcal{B}_{-i}} \left[u_i((a_i, b_{-i}), v_i) \right] + \sum_{i \in N} \mathop{\mathbf{E}}_{b \sim \mathcal{B}} \left[p_i(f_i(b), b_{-i}) \right]$$
$$\geq \lambda \, \mathrm{OPT}(v) - (\mu_1 - 1) \sum_{i \in N} \mathop{\mathbf{E}}_{b \sim \mathcal{B}} \left[p_i(f_i(b), b_{-i}) \right] - \mu_2 \sum_{i \in N} \mathop{\mathbf{E}}_{b \sim \mathcal{B}} \left[b_i(f_i(b)) \right] ,$$

where the first equality uses the definition of $u_i(b, v_i)$ as the difference between $v_i(f_i(b))$ and $p_i(f_i(b), b_{-i})$, the first inequality uses the fact that $\mathcal{B}$ is a coarse correlated equilibrium, and the second inequality holds because $a = (a_1, \ldots, a_n)$ is defined as in Definition 4.1.

Since the bids are conservative this can be rearranged to give

$$(1 + \mu_2) \mathop{\mathbf{E}}_{b \sim \mathcal{B}} \left[SW(f(b)) \right] \geq \lambda \, \mathrm{OPT}(v) - (\mu_1 - 1) \sum_{i \in N} \mathop{\mathbf{E}}_{b \sim \mathcal{B}} \left[p_i(f_i(b), b_{-i}) \right] .$$

For $\mu_1 \leq 1$ the second term on the right hand side is lower bounded by zero and the result follows by rearranging terms. For $\mu_1 > 1$ we use that $\mathbf{E}_{b \sim \mathcal{B}} \left[p_i(f_i(b), b_{-i}) \right]$ $\leq \mathbf{E}_{b \sim \mathcal{B}} \left[v_i(f_i(b)) \right]$ to lower bound the second term on the right hand side and the result follows by rearranging terms. $\qquad \square$

Theorem 4.3. *If a mechanism is relaxed (λ, μ_1, μ_2)-smooth and $(b^1, \dots, b^T)$ is a sequence of conservative bids with vanishing external regret, then*

$$\frac{1}{T} \sum_{t=1}^{T} SW(b^t) \geq \frac{\lambda}{\max\{\mu_1, 1\} + \mu_2} \cdot \mathrm{OPT}(v) - o(1) \,.$$

Proof. Fix valuations v. Consider a sequence of bids $b^1, \dots, b^T$ with vanishing average external regret. For each b^t in the sequence of bids denote the corresponding allocation by $f(b^t) = (f_1(b^t), \dots, f_n(b^t))$. Let $\delta_i^t(a_i) = u_i((a_i, b_{-i}^t), v_i) - u_i(b^t, v_i)$ and let $\Delta(a) = \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{n} \delta_i^t(a_i)$. Let $a = (a_1, \dots, a_n)$ be defined as in Definition 4.1, where $\mathcal{B}$ is the empirical distribution of bids. Then,

$$\frac{1}{T} \sum_{t=1}^{T} SW(b^t) = \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{n} u_i((b_i^t, b_{-i}^t), v_i) + \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{n} p_i(f_i(b^t), b_{-i}^t)$$

$$= \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{n} u_i((a_i, b_{-i}^t), v_i) + \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{n} p_i(f_i(b^t), b_{-i}^t) - \Delta(a)$$

$$\geq \lambda \, \mathrm{OPT}(v) - (\mu_1 - 1) \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{n} p_i(f_i(b^t), b_{-i}^t)$$

$$- \mu_2 \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{n} b_i(f_i(b^t)) - \Delta(a) \,,$$

where the first equality uses the definition of $u_i((b_i^t, b_{-i}^t), v_i)$ as the difference between $v_i(f_i(b^t))$ and $p_i(f_i(b^t), b_{-i}^t)$, the second equality uses the definition of $\Delta(a)$, and the third inequality holds because $a = (a_1, \dots, a_n)$ is defined as in Definition 4.1.

Since the bids are conservative this can be rearranged to give

$$(1 + \mu_2) \frac{1}{T} \sum_{t=1}^{T} SW(b^t) \geq \lambda \, \mathrm{OPT}(v) - (\mu_1 - 1) \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{n} p_i(f_i(b^t), b_{-i}^t) - \Delta(a) \,.$$

For $\mu_1 \leq 1$ the second term on the right hand side is lower bounded by zero and the result follows by rearranging terms provided that $\Delta(a) = o(1)$. For $\mu_1 > 1$ we use that $\frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{n} p_i(f_i(b^t), b_{-i}^t) \leq \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{n} v_i(f_i(b^t))$ to lower bound the second term on the right hand side and the result follows by rearranging terms provided that $\Delta(a) = o(1)$.

The term $\Delta(a)$ is bounded by $o(1)$ because the sequence of bids $b^1, \ldots, b^T$ incurs vanishing average external regret and, thus,

$$\Delta(a) \leq \frac{1}{T} \sum_{i=1}^{n} \left[\max_{b'_i} \sum_{t=1}^{T} u_i((b'_i, b^t_{-i}), v_i) - \sum_{t=1}^{T} u_i(b^t, v_i) \right] \leq \frac{1}{T} \sum_{i=1}^{n} o(T) \,.$$

$\square$

4.5 Upper Bounds for CCE and Minimization of External Regret for Additive Bids

We first show how the argument of Feldman et al. [41] can be adopted to prove that for restrictions from $V = CF$ to $B = OS$ the VCG mechanism is relaxed $(1/2, 0, 1)$-smooth. Using Theorem 4.2 we obtain an upper bound of 4 on the Price of Anarchy with respect to coarse correlated equilibria. Using Theorem 4.3 we conclude that the average social welfare for sequences of bids with vanishing external regret converges to at least $1/4$ of the optimal social welfare. We thus improve the best known bounds by a logarithmic factor.

Proposition 4.4. *Suppose that $V = CF$ and that $B = OS$. Then the VCG mechanism is relaxed $(1/2, 0, 1)$-smooth under conservative bidding.*

To prove this result we need two auxiliary lemmata.

Lemma 4.5. *Suppose that $V = CF$, that $B = OS$, and that the VCG mechanism is used. Then for every bidder i, every bundle Q_i, and every distribution $\mathcal{B}_{-i}$ on the bids b_{-i} of the bidders other than i there exists a conservative bid a_i such that*

$$\mathop{\mathbf{E}}_{b_{-i} \sim \mathcal{B}_{-i}} [u_i((a_i, b_{-i}), v_i)] \geq \frac{1}{2} \cdot v_i(Q_i) - \mathop{\mathbf{E}}_{b_{-i} \sim \mathcal{B}_{-i}} [p_i(Q_i, b_{-i})] \,.$$

Proof. Consider bids b_{-i} of the bidders $-i$. The bids b_{-i} induce a price $p_i(\{j\}) = \max_{k \neq i} b_k(\{j\})$ for each item j and $p_i(S) = \sum_{j \in S} p(\{j\})$ for each subset $S \subseteq I$. Let T be a maximal subset of items from Q_i such that $v_i(T) \leq p_i(T)$. Define the *truncated* prices q_i as follows:

$$q_i(\{j\}) = \begin{cases} p_i(\{j\}) & \text{for } j \in Q_i \setminus T, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The distribution $\mathcal{B}_{-i}$ on the bids b_{-i} induces a distribution $\mathcal{C}_i$ on the prices p_i as well as a distribution $\mathcal{D}_i$ on the truncated prices q_i.

We would like to allow bidder i to draw his bid b_i from the distribution $\mathcal{D}_i$ on the truncated prices q_i. For this we need that (1) the truncated prices are additive and that (2) the truncated prices are conservative. The first condition is satisfied because additive bids lead to additive prices. To see that the second condition is

satisfied assume by contradiction that for some set $S \subseteq Q_i \setminus T$, $q_i(S) > v_i(S)$. As $p_i(S) = q_i(S)$ it follows that

$$v_i(S \cup T) \leq v_i(S) + v_i(T) \leq p_i(S) + p_i(T) = p_i(S \cup T),$$

which contradicts our definition of T as a maximal subset of Q_i for which $v_i(T) \leq p_i(T)$.

Consider an arbitrary bid b_i from the support of $\mathcal{D}_i$. Let $f_i(b_i, p_i)$ be the set of items won with bid b_i against prices p_i. Additionally, let $g_i(b_i, q_i)$ be the subset of items from Q_i won with bid b_i against the truncated prices q_i. As $p_i(\{j\}) = q_i(\{j\})$ for $j \in Q_i \setminus T$ and $p_i(\{j\}) \geq q_i(\{j\})$ for $j \in T$ we have $g_i(b_i, q_i) \subseteq f_i(b_i, p_i) \cup T$. Thus, using the fact that v_i is subadditive, $v_i(g_i(b_i, q_i)) \leq v_i(f_i(b_i, p_i)) + v_i(T)$. By the definition of the prices p_i and the truncated prices q_i we have $p_i(Q_i) - q_i(Q_i) = p_i(T) \geq v_i(T)$. By combining these inequalities we obtain

$$v_i(f_i(b_i, p_i)) + p_i(Q_i) \geq v_i(g_i(b_i, q_i)) + q_i(Q_i).$$

Taking expectations over the prices $p_i \sim \mathcal{C}_i$ and the truncated prices $q_i \sim \mathcal{D}_i$ gives

$$\mathop{\mathbf{E}}_{p_i \sim \mathcal{C}_i} [v_i(f_i(b_i, p_i)) + p_i(Q_i)] \geq \mathop{\mathbf{E}}_{q_i \sim \mathcal{D}_i} [v_i(g_i(b_i, q_i)) + q_i(Q_i)].$$

Next we take expectations over $b_i \sim \mathcal{D}_i$ on both sides of the inequality. Then we bring the $p_i(Q_i)$ term to the right and the $q_i(Q_i)$ term to the left. Finally, we exploit that the expectation over $q_i \sim \mathcal{D}_i$ of $q_i(Q_i)$ is the same as the expectation over $b_i \sim \mathcal{D}_i$ of $b_i(Q_i)$ to obtain

$$\mathop{\mathbf{E}}_{b_i \sim \mathcal{D}_i} \left[\mathop{\mathbf{E}}_{p_i \sim \mathcal{C}_i} [v_i(f_i(b_i, p_i))] \right] - \mathop{\mathbf{E}}_{b_i \sim \mathcal{D}_i} [b_i(Q_i)]$$
$$\geq \mathop{\mathbf{E}}_{b_i \sim \mathcal{D}_i} \left[\mathop{\mathbf{E}}_{q_i \sim \mathcal{D}_i} [v_i(g_i(b_i, q_i))] \right] - \mathop{\mathbf{E}}_{p_i \sim \mathcal{C}_i} [p_i(Q_i)]. \quad (4.1)$$

Now, using the fact that b_i and q_i are drawn from the same distribution $\mathcal{D}_i$, we can lower bound the first term on the right-hand side of the preceding inequality by

$$\mathop{\mathbf{E}}_{b_i \sim \mathcal{D}_i} \left[\mathop{\mathbf{E}}_{q_i \sim \mathcal{D}_i} [v_i(g_i(b_i, q_i)] \right] =$$
$$\frac{1}{2} \cdot \mathop{\mathbf{E}}_{b_i \sim \mathcal{D}_i} \left[\mathop{\mathbf{E}}_{q_i \sim \mathcal{D}_i} [v_i(g_i(b_i, q_i)) + v_i(g_i(q_i, b_i))] \right] \geq \frac{1}{2} \cdot v_i(Q_i), \quad (4.2)$$

where the inequality in the last step comes from the fact that the subset $g_i(b_i, q_i)$ of Q_i won with bid b_i against prices q_i and the subset $g_i(q_i, b_i)$ of Q_i won with bid q_i against prices b_i form a partition of Q_i,[2] and, thus, because v_i is subadditive, it must be that $v_i(g_i(b_i, q_i)) + v_i(g_i(q_i, b_i)) \geq v_i(Q_i)$.

---

[2]We assume tie-breaking in bidder i's favor, which can be achieved by an infinitesimal increase of bidder i's bid.

Note that bidder i's utility for bid b_i against bids b_{-i} is given by his valuation for the set of items $f_i(b_i, p_i)$ minus the price $p_i(f_i(b_i, p_i))$. Note further that the price $p_i(f_i(b_i, p_i))$ that he faces is at most his bid $b_i(f_i(b_i, p_i))$. Finally note that his bid $b_i(f_i(b_i, p_i))$ is at most $b_i(Q_i)$ because b_i is drawn from $\mathcal{D}_i$. Together with inequality (4.1) and inequality (4.2) this shows that

$$
\mathop{\mathbf{E}}_{b_i \sim \mathcal{D}_i} \left[\mathop{\mathbf{E}}_{b_{-i} \sim \mathcal{B}_{-i}} \left[u_i((b_i, b_{-i}), v_i) \right] \right] \geq \mathop{\mathbf{E}}_{b_i \sim \mathcal{D}_i} \left[\mathop{\mathbf{E}}_{p_i \sim \mathcal{C}_i} \left[v_i(f_i(b_i, p_i)) - b_i(Q_i) \right] \right]
$$
$$
\geq \frac{1}{2} \cdot v_i(Q_i) - \mathop{\mathbf{E}}_{p_i \sim \mathcal{C}_i} \left[p_i(Q_i) \right] .
$$

Since this inequality is satisfied in expectation if bid b_i is drawn from distribution $\mathcal{D}_i$ there must be a bid a_i from the support of $\mathcal{D}_i$ that satisfies it. $\qquad\square$

Lemma 4.6. *Suppose that $V = CF$, that $B = OS$, and that the VCG mechanism is used. Then for every partition $Q_1, \ldots, Q_n$ of the items and all bids b,*

$$
\sum_{i \in N} p_i(Q_i, b_{-i}) \leq \sum_{i \in N} b_i(f_i(b)) .
$$

Proof. For every bidder $i \in N$ and item $j \in Q_i$ we have $p_i(\{j\}, b_{-i}) = \max_{k \neq i} b_k(\{j\})$ $\leq \max_k b_k(\{j\})$. Hence an upper bound on the sum $\sum_{i \in N} p_i(Q_i, b_{-i})$ is given by $\sum_{i \in N} \max_k b_k(\{j\})$. The VCG mechanism selects allocation $f_1(b), \ldots, f_n(b)$ such that $\sum_{i \in N} b_i(f_i(b))$ is maximized. The claim follows. $\qquad\square$

We can now prove Proposition 4.4.

Proof of Proposition 4.4. The claim follows by applying Lemma 4.5 to every bidder i and the corresponding optimal bundle O_i, summing over all bidders i, and using Lemma 4.6 to bound $\mathbf{E}_{b_{-i} \sim \mathcal{B}_{-i}} \left[\sum_{i \in N} p_i(O_i, b_{-i}) \right]$ by $\mathbf{E}_{b \sim \mathcal{B}} \left[\sum_{i \in N} b_i(f_i(b)) \right]$. $\qquad\square$

An important observation is that the proof of the previous proposition requires that the class of price functions, which is induced by the class of bidding functions via the formula for the VCG payments, is contained in B. While this is the case for additive bids that lead to additive (or "per item") prices this is *not* the case for more expressive bids. In fact, as we will see in the next section, even if the bids are from OXS, the least general class from the hierarchy of Lehmann et al. [80] that strictly contains the class of additive bids, then the class of price functions that is induced by B is no longer contained in B. This shows that the techniques that led to the results in this section *cannot* be applied to the more expressive bids that we study next.

4.6 A Lower Bound for PNE with Non-Additive Bids

For non-additive bids we start our analysis with the following separation result: While for restrictions from subadditive valuations to additive bids the bound is 2 for pure Nash equilibria (see Bhawalkar and Roughgarden [12]), we show that for restrictions from subadditive valuations to OXS bids the corresponding bound is at least 2.4. This shows that more expressiveness can lead to strictly worse bounds. The proof uses a setting with 2 bidders and 6 items.

Theorem 4.7. *Suppose that $V = CF$, that $OXS \subseteq B \subseteq XOS$, and that the VCG mechanism is used. Then there exist valuations v such that the PoA with respect to PNE under conservative bidding is at least 2.4.*

The proof of this theorem makes use of the following auxiliary lemma.

Lemma 4.8. *If $b_i \in XOS$, then for any $J \subseteq I$,*

$$\max_{S \subseteq J, |S| = |J| - 1} b_i(S) \geq \frac{|J| - 1}{|J|} \cdot b_i(J).$$

Proof. As $b_i \in XOS$ there exists an additive bid a_i such that $\sum_{j \in J} a_i(\{j\}) = b_i(J)$ and for every $S \subseteq J$ we have $b_i(S) \geq \sum_{j \in S} a_i(\{j\})$. There are $|J|$ many ways to choose $S \subseteq J$ such that $|S| = |J| - 1$ and these $|J|$ many sets will contain each of the items $j \in J$ exactly $|J| - 1$ times. Thus, $\sum_{S \subseteq J, |S| = |J| - 1} b_i(S) \geq (|J| - 1) \cdot b_i(J)$. For any set $T \in \arg\max_{S \subseteq J, |S| = |J| - 1} b_i(S)$, using the fact that the maximum is at least as large as the average, we therefore have $b_i(T) \geq (|J| - 1)/|J| \cdot b_i(J)$. $\square$

Proof of Theorem 4.7. There are 2 bidders and 6 items. The items are divided into two sets I_1 and I_2, each with 3 items. The valuations of bidder $i \in \{1, 2\}$ are given by (all indices are modulo two)

$$v_i(S) = \begin{cases} 12 & \text{for } S \subseteq I_i, |S| = 3 \\ 6 & \text{for } S \subseteq I_i, 1 \leq |S| \leq 2 \\ 5 + 1\epsilon & \text{for } S \subseteq I_{i+1}, |S| = 3 \\ 4 + 2\epsilon & \text{for } S \subseteq I_{i+1}, |S| = 2 \\ 3 + 3\epsilon & \text{for } S \subseteq I_{i+1}, |S| = 1 \\ \max_{j \in \{1,2\}} \{v_i(S \cap I_j)\} & \text{otherwise.} \end{cases}$$

The variable ϵ is a sufficiently small positive number. The valuation v_i of bidder i is subadditive, but not fractionally subadditive. (The problem for bidder i is that the valuation for I_i is too high given the valuations for $S \subset I_i$.)

The welfare maximizing allocation awards set I_1 to bidder 1 and set I_2 to bidder 2. The resulting welfare is $v_1(I_1) + v_2(I_2) = 12 + 12 = 24$.

We claim that the following bids $b = (b_1, b_2)$ are contained in OXS and constitute a pure Nash equilibrium:

$$b_i(S) = \begin{cases} 0 & \text{for } S \subseteq I_i \\ 5 + 1\epsilon & \text{for } S \subseteq I_{i+1}, |S| = 3 \\ 4 + 2\epsilon & \text{for } S \subseteq I_{i+1}, |S| = 2 \\ 3 + 3\epsilon & \text{for } S \subseteq I_{i+1}, |S| = 1 \\ \max_{j \in \{1,2\}} \{b_i(S \cap I_j)\} & \text{otherwise.} \end{cases}$$

Given b VCG awards set I_2 to bidder 1 and set I_1 to bidder 2 for a welfare of $v_1(I_2) + v_2(I_1) = 2 \cdot (5 + \epsilon) = 10 + 2\epsilon$, which is by a factor $2.4 - 12\epsilon/(25 + 5\epsilon)$ smaller than the optimum welfare. For showing the PoA of 2.4 we consider the limit of the PoA as ϵ approaches 0.

We can express b_i as ORs of XORs of XS bids as follows: Let $I_i = \{a, b, c\}$ and $I_{i+1} = \{d, e, f\}$. Let h_d, h_e, h_f and ℓ_d, ℓ_e, ℓ_f be XS bids that value d, e, f at $3 + 3\epsilon$ and $1 - \epsilon$, respectively. Then $b_i(T) = (h_d(T) \otimes h_e(T) \otimes h_f(T)) \vee \ell_d(T) \vee \ell_e(T) \vee \ell_f(T)$.

To show that b is a Nash equilibrium we can focus on bidder i (by symmetry) and on deviating bids a_i that win bidder i a subset S of I_i (because bidder i currently wins I_{i+1} and $v_i(S) = \max\{v_i(S \cap I_1), v_i(S \cap I_2)\}$ for sets S that intersect both I_1 and I_2).

Note that the price that bidder i faces on the subsets S of I_i are superadditive: For $|S| = 1$ the price is $(5 + \epsilon) - (4 + 2\epsilon) = 1 - \epsilon$, for $|S| = 2$ the price is $(5 + \epsilon) - (3 + 3\epsilon) = 2 - 2\epsilon$, and for $|S| = 3$ the price is $5 + \epsilon$.

Case 1: $S = I_i$. We claim that this case cannot occur. To see this observe that because $a_i \in$ XOS, Lemma 4.8 shows that there must be a 2-element subset T of S for which $a_i(T) \geq 2/3 \cdot a_i(S)$. On the one hand this shows that $a_i(S) \leq 9$ because otherwise $a_i(T) \geq 2/3 \cdot a_i(S) > 6$ in contradiction to our assumption that a_i is conservative. On the other hand to ensure that VCG assigns S to bidder i we must have $a_i(S) \geq a_i(T) + (3 + 3\epsilon)$ due to the subadditivity of the prices. Thus $a_i(S) \geq 2/3 \cdot a_i(S) + (3 + 3\epsilon)$ and, hence, $a_i(S) \geq 9(1 + \epsilon)$. We conclude that $9 \geq a_i(S) \geq 9(1 + \epsilon)$, which gives a contradiction.

Case 2: $S \subset I_i$. In this case bidder i's valuation for S is 6 and his payment is at least $1 - \epsilon$ as we have shown above. Thus, $u_i(a_i, b_{-i}) \leq 5 + \epsilon = u_i(b_i, b_{-i})$, i.e., the utility does not increase with the deviation. $\square$

4.7 Upper Bounds for CCE and Minimization of External Regret for Non-Additive Bids

Our next group of results concerns upper bounds for the Price of Anarchy for restrictions to non-additive bids. For β-fractionally subadditive valuations we show that the VCG mechanism is relaxed $(1/\beta, 1, 1)$-smooth. By Theorem 4.2 this implies that the PoA with respect to coarse correlated equilibria is at most 2β. By Theorem 4.3

this implies that the average social welfare obtained in sequences of repeated play with vanishing external regret converges to $1/(2\beta)$ of the optimal social welfare. For subadditive valuations, which are $\mathcal{O}(\log(m))$-fractionally subadditive, we thus obtain bounds of $\mathcal{O}(\log(m))$ resp. $\Omega(1/\log(m))$. For fractionally subadditive valuations, which are 1-fractionally subadditive, we thus obtain bounds of 2 resp. $1/2$. We thus extend the results of Bhawalkar and Roughgarden [12] and Christodoulou et al. [22] from additive to non-additive bids.

Proposition 4.9. *Suppose that $V \subseteq \beta\text{-}XOS$ and that $OS \subseteq B \subseteq XOS$, then the VCG mechanism is relaxed $(1/\beta, 1, 1)$-smooth under conservative bidding.*

We will prove that the VCG mechanism satisfies the definition of relaxed smoothness point-wise. For this we need two auxiliary lemmata.

Lemma 4.10. *Suppose that $V \subseteq \beta\text{-}XOS$, that $OS \subseteq B \subseteq XOS$, and that the VCG mechanism is used. Then for all valuations $v \in V$, every bidder i, and every bundle of items $Q_i \subseteq I$ there exists a conservative bid $a_i \in B_i$ such that for all conservative bids $b_{-i} \in B_{-i}$,*

$$u_i((a_i, b_{-i}), v_i) \geq \frac{v_i(Q_i)}{\beta} - p_i(Q_i, b_{-i}).$$

Proof. Fix valuations v, bidder i, and bundle Q_i. As $v_i \in \beta\text{-}XOS$ there exists a conservative, additive bid $a_i \in OS$ such that $\sum_{j \in X_i} a_i(\{j\}) \leq v_i(X_i)$ for all $X_i \subseteq Q_i$, and $\sum_{j \in Q_i} a_i(\{j\}) \geq \frac{v_i(Q_i)}{\beta}$. Consider conservative bids b_{-i}. Suppose that for bids (a_i, b_{-i}) bidder i wins items X_i and bidders $-i$ win items $I \setminus X_i$. As VCG selects an allocation that maximizes the sum of the bids,

$$a_i(X_i) + \hat{b}_{-i}(I \setminus X_i) \geq a_i(Q_i) + \hat{b}_{-i}(I \setminus Q_i).$$

We have chosen a_i such that $a_i(X_i) \leq v_i(X_i)$ and $a_i(Q_i) \geq v_i(Q_i)/\beta$. Thus,

$$v_i(X_i) + \hat{b}_{-i}(I \setminus X_i) \geq a_i(X_i) + \hat{b}_{-i}(I \setminus X_i)$$
$$\geq a_i(Q_i) + \hat{b}_{-i}(I \setminus Q_i) \geq \frac{v_i(Q_i)}{\beta} + \hat{b}_{-i}(I \setminus Q_i).$$

Subtracting $\hat{b}_{-i}(I)$ from both sides gives

$$v_i(X_i) - p_i(X_i, b_{-i}) \geq \frac{v_i(Q_i)}{\beta} - p_i(Q_i, b_{-i}).$$

As $u_i((a_i, b_{-i}), v_i) = v_i(X_i) - p_i(X_i, b_{-i})$ this shows that $u_i((a_i, b_{-i}), v_i) \geq v_i(Q_i)/\beta - p_i(Q_i, b_{-i})$ as claimed. $\square$

Lemma 4.11. *Suppose that $OS \subseteq B \subseteq XOS$ and that the VCG mechanism is used. For every allocation $Q_1, \ldots, Q_n$ and all conservative bids $b \in B$ and allocation $X_1, \ldots, X_n$ selected by the VCG mechanism for bids b,*

$$\sum_{i=1}^{n} \left(p_i(Q_i, b_{-i}) - p_i(X_i, b_{-i}) \right) \leq \sum_{i=1}^{n} b_i(X_i).$$

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Proof. We have $p_i(Q_i, b_{-i}) = \hat{b}_{-i}(I) - \hat{b}_{-i}(I \setminus Q_i)$ and $p_i(X_i, b_{-i}) = \hat{b}_{-i}(I) - \hat{b}_{-i}(I \setminus X_i)$ because the VCG mechanism is used. Thus,

$$\sum_{i=1}^{n} \left(p_i(Q_i, b_{-i}) - p_i(X_i, b_{-i}) \right) = \sum_{i=1}^{n} \left(\hat{b}_{-i}(I \setminus X_i) - \hat{b}_{-i}(I \setminus Q_i) \right). \qquad (4.3)$$

We have $\hat{b}_{-i}(I \setminus X_i) = \sum_{k \neq i} b_k(X_k)$ and $\hat{b}_{-i}(I \setminus Q_i) \geq \sum_{k \neq i} b_k(X_k \cap (I \setminus Q_i))$ because $(X_k \cap (I \setminus Q_i))_{i \neq k}$ is a feasible allocation of the items $I \setminus Q_i$ among the bidders $-i$. Thus,

$$\begin{aligned}
\sum_{i=1}^{n} \left(\hat{b}_{-i}(I \setminus X_i) - \hat{b}_{-i}(I \setminus Q_i) \right) &\leq \sum_{i=1}^{n} \left(\sum_{k \neq i} b_k(X_k) - \sum_{k \neq i} b_k(X_k \cap (I \setminus Q_i)) \right) \\
&\leq \sum_{i=1}^{n} \left(\sum_{k=1}^{n} b_k(X_k) - \sum_{k=1}^{n} b_k(X_k \cap (I \setminus Q_i)) \right) \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} b_k(X_k) - \sum_{i=1}^{n} \sum_{k=1}^{n} b_k(X_k \cap (I \setminus Q_i)).
\end{aligned}$$
$$(4.4)$$

The second inequality holds due to the monotonicity of the bids. Since XOS $=$ 1-XOS for every bidder k and bid $b_k \in$ XOS there exists a bid $a_{k,X_k} \in$ OS such that $b_k(X_k) = a_{k,X_k}(X_k) = \sum_{j \in X_k} a_{k,X_k}(\{j\})$ and $b_k(X_k \cap (I \setminus Q_i)) \geq a_{k,X_k}(X_k \cap (I \setminus Q_i)) = \sum_{j \in X_k \cap (I \setminus Q_i)} a_{k,X_k}(\{j\})$ for all k. As $Q_1, \ldots, Q_n$ is a partition of I every item is contained in exactly one of the sets $Q_1, \ldots, Q_n$ and hence in $n-1$ of the sets $I \setminus Q_1, \ldots, I \setminus Q_n$. By the same argument for every bidder k and set X_k every item $j \in X_k$ is contained in exactly $n-1$ of the sets $X_k \cap (I \setminus Q_1), \ldots, X_k \cap (I \setminus Q_n)$. Thus, for every fixed k we have that $\sum_{i=1}^{n} b_k(X_k \cap (I \setminus Q_i)) \geq (n-1) \cdot \sum_{j \in X_k} a_{k,X_k}(\{j\}) = (n-1) \cdot a_{k,X_k}(X_k) = (n-1) \cdot b_k(X_k)$. It follows that

$$\begin{aligned}
\sum_{i=1}^{n} \sum_{k=1}^{n} b_k(X_k) &- \sum_{i=1}^{n} \sum_{k=1}^{n} b_k(X_k \cap (I \setminus Q_i)) \\
&\leq n \cdot \sum_{k=1}^{n} b_k(X_k) - (n-1) \cdot \sum_{k=1}^{n} b_k(X_k) = \sum_{i=1}^{n} b_i(X_i). \qquad (4.5)
\end{aligned}$$

The claim follows by combining inequalities (4.3), (4.4), and (4.5). $\qquad \square$

We can now give the proof of Proposition 4.9.

Proof of Proposition 4.9. Applying Lemma 4.10 to the optimal bundles $O_1, \ldots, O_n$ and summing over all bidders i, we obtain

$$\sum_{i \in N} u_i((a_i, b_{-i}), v) \geq \frac{1}{\beta} \text{OPT}(v) - \sum_{i \in N} p_i(O_i, b_{-i}).$$

Applying Lemma 4.11 we obtain

$$\sum_{i \in N} u_i((a_i, b_{-i}), v) \geq \frac{1}{\beta} \mathrm{OPT}(v) - \sum_{i \in N} p_i(f_i(b), b_{-i}) - \sum_{i \in N} b_i(f_i(b)). \qquad \square$$

4.8 More Lower Bounds for PNE with Non-Additive Bids

We conclude by proving matching lower bounds for the VCG mechanism and restrictions from fractionally subadditive valuations to non-additive bids. We prove this result by showing that the VCG mechanism satisfies the *outcome closure property* of Milgrom [86], which implies that the set of PNE for a less general bidding space is contained in the set of equilibria for a more general bidding space.

We say that a mechanism satisfies *outcome closure* for a given class V of valuation functions and a restriction of the class B of bidding functions to a subclass B' of bidding functions if for every $v \in V$, every i, every conservative $b'_{-i} \in B'_{-i}$, and every conservative $b_i \in B_i$ there exists a conservative $b'_i \in B'_i$ such that $u_i((b'_i, b'_{-i}), v_i) \geq u_i((b_i, b'_{-i}), v_i)$.

Proposition 4.12. *If a mechanism satisfies outcome closure for a given class V of valuation functions and a restriction of the class B of bidding functions to a subclass B', then the Price of Anarchy with respect to pure Nash equilibria under conservative bidding for B is at least as large as for B'.*

Proof. It suffices to show that the set of PNE for B' is contained in the set of PNE for B. To see this assume by contradiction that, for some $v \in V$, $b' \in B'$ is a PNE for B' but not for B. As b' is not a PNE for B there exists a bidder i and a bid $b_i \in B_i$ such that $u_i((b_i, b'_{-i}), v_i) > u_i((b'_i, b'_{-i}), v_i)$. By outcome closure, however, there must be a bid $b''_i \in B'_i$ such that $u_i((b''_i, b'_{-i}), v_i) \geq u_i((b_i, b'_{-i}), v_i)$. It follows that $u_i((b''_i, b'_{-i}), v_i) > u_i((b'_i, b'_{-i}), v_i)$, which contradicts our assumption that b' is a PNE for B'. $\qquad \square$

Next we use outcome closure to show that the Price of Anarchy in the VCG mechanism with respect to pure Nash equilibria weakly increases with expressiveness for classes of bidding functions below XOS.

Proposition 4.13. *Suppose that $V \subseteq CF$, that $OS \subseteq B' \subseteq B \subseteq XOS$, and that the VCG mechanism is used. Then the Price of Anarchy with respect to pure Nash equilibria under conservative bidding for B is at least as large as for B'.*

Proof. By Proposition 4.12 it suffices to show that the VCG mechanism satisfies outcome closure for V and the restriction of B to B'. For this fix valuations $v \in V$, bids $b'_{-i} \in B'_{-i}$, and consider an arbitrary bid $b_i \in B_i$ by bidder i. Denote the bundle that bidder i gets under (b_i, b'_{-i}) by X_i and denote his payment by $p_i = p_i(X_i, b'_{-i})$.

Since $b_i \in B_i \subseteq$ XOS there exists a bid $b_i' \in$ OS $\subseteq B'$ such that

$$\sum_{j \in X_i} b_i'(\{j\}) = b_i(X_i) \qquad \text{and,}$$

$$\sum_{j \in S} b_i'(\{j\}) \leq b_i(S) \qquad \text{for all } S \subseteq X_i.$$

By setting $b_i'(\{j\}) = 0$ for $j \notin X_i$ we ensure that b_i' is conservative. Recall that the VCG mechanism assigns bidder i the bundle of items that maximizes his reported utility. We have that $b_i'(X_i) = b_i(X_i)$ and that $b_i'(T) \leq b_i(T)$ for all $T \subseteq I$. We also know that the prices $p_i(T, b_{-i})$ for all $T \subseteq I$ do not depend on bidder i's bid. Hence bidder i's reported utility for X_i under b' is as high as under b and his reported utility for every other bundle T under b' is no higher than under b. This shows that bidder i wins bundle X_i and pays p_i under bids (b_i', b_{-i}'). $\qquad \square$

Hence the lower bound of 2 for pure Nash equilibria and additive bids of Christodoulou et al. [22] translates into a lower bound of 2 for pure Nash equilibria and non-additive bids.

Theorem 4.14. *Suppose that OXS $\subseteq V \subseteq$ CF, that OS $\subseteq B \subseteq$ XOS, and that the VCG mechanism is used. Then the PoA with respect to PNE under conservative bidding is at least* 2.

Note that the previous result applies even if valuation and bidding space coincide, and the VCG mechanism has an efficient, dominant-strategy equilibrium. This is because the VCG mechanism admits other, non-efficient equilibria and the Price of Anarchy metric does not restrict to dominant-strategy equilibria if they exist.

4.9 Conclusion

For many valuation spaces computing the VCG allocation and payments is a computationally hard problem. We thus study the VCG mechanism in settings in which the allocation and payments can be computed efficiently. This is the case for additive bids, where the VCG-based mechanism can be interpreted as a separate second-price auction for each item, and for the bidding spaces OXS and GS. We measure the performance of the VCG mechanism by the Price of Anarchy; that is, by the ratio between the optimal social welfare and the worst possible social welfare at equilibrium. For subadditive bids we prove an upper bound of 4 on the Price of Anarchy with respect to (coarse) correlated equilibria and additive bids. Furthermore, we prove upper bounds and lower bounds for bids that are more general and (fractionally) subadditive valuations. These bounds show that increased expressiveness can give rise to additional equilibria of poorer efficiency.

Our analysis leaves a number of interesting open questions, both regarding the computation of equilibria and regarding improved upper and lower bounds. Interesting questions regarding the computation of equilibria include whether or not

mixed Nash equilibria can be computed efficiently for restrictions from subadditive to additive bids or whether pure Nash equilibria can be computed efficiently for restrictions from fractionally subadditive valuations to additive bids. A particularly interesting open problem regarding improved bounds is whether the welfare loss for computable equilibrium concepts and learning outcomes can be shown to be strictly larger for restrictions to non-additive, say OXS, bids than for restrictions to additive bids. This would show that additive bids are not only sufficient for the best possible bound but also necessary.

5

Combinatorial Auctions with Conflict-Based Externalities

5.1 Introduction

Combinatorial auctions are an important area in algorithmic mechanism design due to their wide-spread applications in resource allocation and e-commerce, such as spectrum or ad-word auctions (see Cramton et al. [27] and Lahaie et al. [78]). In the standard model of combinatorial auctions, a set of items is assigned to a set of bidders in order to maximize social welfare, which is given by the total valuation of the bidders for their assigned items. The standard model assumes that each bidder values exclusively the set of items assigned to him; his valuation is *independent of the assignment of the other items to the other bidders.* In many applications, however, such an assumption is not justified since the bidders' preferences have a significant dependence on how items are assigned to other bidders.

One of the most popular special cases of combinatorial auctions are sponsored search auctions, where a search engine company assigns ad slots on a search result page to advertisers. Obviously, for a car-rental company such an ad slot is of much smaller value if an ad of another popular rental company is shown right next to it; this implies a *negative externality.* For an advertiser there might be a number of such competitors, and an assignment yields value to the bidder only if the ads of competitors are not displayed simultaneously. The existence of negative externalities in sponsored search has been confirmed empirically (see Gomes et al. [54]). Moreover, similar negative externalities also arise in other prominent applications of combinatorial auctions, for example, in secondary spectrum auctions where interferences induce negative externalities, or when selling luxury goods, where the value of a buyer for items from an exclusive brand drops when other bidders also obtain items from the same brand. These examples motivate a natural and simple graph-based model of externalities: (1) Each bidder is a node in a directed graph, and

a directed edge indicates that a bidder sees another bidder as a competitor. (2) Assigning an item to a bidder yields value only if none of the competitors receive any item (or just any "similar" or "better" item).

Negative externalities in auctions have recently received some attention, but—perhaps surprisingly—the natural and simple idea sketched above has not been analyzed in a rigorous and general fashion. In this chapter, we study approximation algorithms and incentive compatible mechanisms for this model and a number of natural variants. More formally, we assume that there is a directed conflict graph on the set of bidders with maximum out-degree Δ. Each edge (i, j) indicates a conflict: bidder i has no value for any assignment in which bidder j receives an item. Additionally, we also consider cases where conflicts arise only among certain pairs of items, or cases with different values for assignments that include or avoid certain conflicts. Our algorithms cope with externalities via new extensions of algorithmic techniques for independent set problems *in combination with* algorithms for conflict-free combinatorial auctions. We also provide additional results for the prominent special case of sponsored search.

5.1.1 Contribution

We study approximation algorithms and incentive compatible mechanisms for combinatorial auctions with conflict-based externalities. For combinatorial auctions with bidder conflicts, we prove two results.

First, we give a reduction to combinatorial auctions without conflicts in Section 5.3. For any α-approximation algorithm for the unconflicted problem, we obtain an $\mathcal{O}(\alpha\Delta)$-approximation algorithm for the model with bidder conflicts. If the original algorithm is an incentive compatible mechanism, we can also turn our algorithm into an incentive compatible mechanism for the model with conflicts. Our reduction also preserves the use of randomization (deterministic, incentive compatible in the universal sense, incentive compatible in expectation). Moreover, it extends to auctions with bidder and item conflicts, which will be introduced in Section 5.2.

Second, if we do not insist on incentive compatibility, we can extend algorithms for the unconflicted problem to the case with bidder conflicts and increase the approximation ratio only by $o(\Delta)$. A lower bound of the independent set problem implies that the increase has to be at least $\Omega(\Delta/\log^4 \Delta)$ (see Chan [21]), even for single-parameter unit-demand valuations, because social welfare optimization in this case generalizes the weighted independent set (WIS) problem. Here we combine an approach for independent set based on semidefinite programming with the standard approach for combinatorial auctions based on linear programming, to design a cone programming relaxation and a rounding scheme. For fractionally subadditive (FSA) valuations (see Chapter 4 or Dobzinski and Schapira [33] for a definition), we can turn an LP-based α-approximation algorithm for the unconflicted problem into an $\mathcal{O}(\alpha \cdot \Delta \log \log \Delta / \log \Delta)$-approximation algorithm for the problem with bidder conflicts (Theorem 5.9). The dependence on Δ mirrors the best-known approxi-

mation ratio for the weighted independent set problem. This implies, for example, approximation ratios of $\mathcal{O}(\Delta \log \log \Delta / \log \Delta)$ for sponsored search, unit-demand, or more general, gross-substitute valuations and bidder conflicts. It is an interesting open problem if this approach can be turned into an incentive compatible mechanism, or if it can be generalized to auctions with bidder and item conflicts as the results above.

We then focus on sponsored search with bidder conflicts. Even in this special case, the hardness bound of $\Omega(\Delta / \log^4 \Delta)$ applies. We consider a restriction to a small number of slots that is natural in the context of sponsored search. For the case of $m \in \mathcal{O}(\log n)$ slots, where n is the number of bidders, we present an algorithm based on semidefinite programming that obtains an $\mathcal{O}(\Delta \sqrt{\log \log \Delta / \log \Delta})$-approximation (Theorem 5.18). Furthermore, we get an $\mathcal{O}(\log m)$-approximation based on partial enumeration that runs in time $\mathcal{O}((m(\Delta + 1))^m)$ (Theorem 5.19). The advantage of these algorithms is that both can be turned into in expectation incentive compatible mechanisms with the same approximation guarantee. The algorithm based on partial enumeration also extends to the model with bidder and item conflicts.

5.1.2 Related Work

The study of auctions with externalities was initiated in the seminal work by Jehiel et al. [70], who investigated the single-item setting. Ghosh and Mahdian [48] investigated externalities in online advertising using a probabilistic model of externalities. Combinatorial auctions with externalities were presented by Krysta et al. [77] and Conitzer and Sandholm [25] and it was shown that it is NP-hard to determine an allocation that maximizes social welfare. For sponsored search, Ghosh and Sayedi [49] studied the setting where each advertiser has two valuations, one if his ad is shown exclusively and one if it is shown together with other ads. Their model is a special case of our model for sponsored search with bidder conflicts where each advertiser is modeled by two bidders. Further work on analyzing equilibria in single-item auctions with externalities can be found in Constantin et al. [26], Funk [46], Giotis and Karlin [50], Gomes et al. [54], Jehiel and Modovanu [69], and Paes Leme et al. [94]. Gomes et al. [54] also give empirical evidence that externalities exist in real-life sponsored search auctions. A different line of work considered bidder-independent externalities in the click-through rates of sponsored search auctions (Aggarwal et al. [1], Kempe and Mahdian [75], and Roughgarden and Tardos [100]). All this work considered only the unit-demand setting.

Our model of sponsored search with bidder conflicts has been proposed and studied before by Papadimitriou and Garcia-Molina [96]. They consider an approach based on exact optimization algorithms using ILP, implement incentive compatibility using VCG, and experimentally evaluate their approach with respect to running time and revenue on a dataset from Yahoo! Webscope. However, they do not consider polynomial-time algorithms, provable approximation ratios, or extensions to combinatorial auctions with more general valuation functions.

Our work is related to approximation algorithms for weighted independent set, a central problem in the study of approximation algorithms and computational hardness over the past four decades. The literature on the problem is too vast to survey here, we just mention a number of directly related results. The problem is known to be NP-hard to approximate within a ratio of $n^{1-\epsilon}$ [64], and even in undirected Δ-regular graphs it remains hard for a ratio of $\mathcal{O}(\Delta/\log^4 \Delta)$ (Chan [21]). In the unweighted version, the trivial greedy algorithm obtains a ratio of Δ in graphs with maximum degree Δ, and its idea has been extended and adjusted to bounds based on average degree (Halldórsson and Radhakrishnan [60]), or to weighted independent set using notions of weighted degree (Kako et al. [72]). It is, however, not obvious how to prove non-trivial ratios for such algorithms in directed graphs with bounded out-degree, which are arising in our application. The best-known approximation algorithms in graphs with maximum degree Δ obtain ratios of $\mathcal{O}(\Delta \log\log \Delta/\log \Delta)$ and are given by Halldórsson [59] and Halperin [61]. They are based on rounding of suitable semidefinite programming relaxations, and below we build on these techniques and their analysis to provide algorithms for our case. For a survey on some of the work on approximation algorithms, see, for instance, Halldórsson [58]. In addition, independent set has been studied from the perspective of fixed-parameter tractability and has been shown to be W[1]-hard (Downey and Fellows [34]). However, there exist special classes of graphs such that cardinality-constrained versions of the problem can be solved in polynomial time (Cai et al. [18]).

More recently, the study of asymmetric and edge-weighted versions of independent set has found interest, especially in the context of combinatorial auctions. Closely related to our study are secondary spectrum auctions, where bidders are wireless devices that strive to obtain channel access under interference constraints. In these scenarios, bidders become vertices in a conflict graph. Each channel is an item that can be given to any subset of bidders representing an independent set in the graph. This model has been initially studied by Zhou et al. [107] for deterministic algorithms, single-parameter valuations and undirected conflict graphs. More recently, near-optimal ratios were obtained even for multiple items, general valuations, and edge-weighted and directed conflict graphs stemming from realistic interference models (Hoefer et al. [67]). Instead of maximum degree, the ratios depend linearly on a graph parameter termed inductive independence number (Ye and Borodin [106]), and the algorithms round suitable linear and convex programs to maintain incentive compatibility (Hoefer and Kesselheim [66]). In addition, mechanisms with incentive compatibility concepts based on more restrictive forms of randomization have been studied by Hoefer and Kesselheim [65] and Hoefer et al. [68].

5.2 Preliminaries

We consider several conflict-based models of externalities in sponsored search and combinatorial auctions with increasing level of generality. In all models, we have

a bidder set $N = \{1, \ldots, n\}$ and an item set $I = \{1, \ldots, m\}$. Each item can be given to at most one bidder. For each bidder $i \in N$, there is a valuation function $v_i(X_i) \geq 0$ that captures the value for receiving item set $X_i \subseteq I$. This valuation is extended (due to externalities and conflicts) to $v_i^c(X)$, a valuation that depends on the complete allocation $X = (X_1, \ldots, X_n)$ of items. Our goal in all models is to find an allocation X that maximizes social welfare $SW(X) = \sum_{i \in N} v_i^c(X)$. Let us proceed with details on our assumptions and definitions for $v_i(X_i)$ and $v_i^c(X)$ in the different models.

Sponsored Search with Bidder Conflicts. We have a set N of n bidders, which is the vertex set V of a *(bidder) conflict graph* $G = (V, E)$. The graph is directed, and each bidder $i \in N$ has out-degree at most $\Delta \geq 0$. We assign a set I of slots to the bidders. Each slot $k \in I$ has a click-through rate $\alpha_k \geq 0$. Bidder i has a valuation per click of $v_i \geq 0$ and is interested in one slot. The unconflicted value of allocating slot set X_i to bidder i is thus $v_i(X_i) = \max_{k \in X_i} v_i \cdot \alpha_k$, a unit-demand valuation with free disposal.

Given an allocation $X = (X_1, \ldots, X_n)$ of slots to bidders, slot $k \in X_i$ is *useless* if there is any slot $\ell \in X_j$ with $(i, j) \in E$, i.e., i has no use for any slot if a competitor j receives a slot. Intuitively, this models the situation that advertiser i is not interested in showing its ad together with an ad from a competitor. Formally, for an allocation X bidder i has a set D_i of *useless slots* where $D_i = \emptyset$ if $\bigcup_{j:(i,j)\in E} X_j = \emptyset$ and $D_i = X_i$ otherwise. Then,

$$v_i^c(X) = v_i(X_i \setminus D_i) = \begin{cases} \max_{k \in X_i} v_i \cdot \alpha_k & \text{if } \bigcup_{j:(i,j)\in E} X_j = \emptyset \\ 0 & \text{otherwise.} \end{cases} \quad (5.1)$$

Observe that the introduction of conflicts turns social welfare maximization NP-hard. Even in the special case when all $v_i = 1$ and all $\alpha_k = 1$, it generalizes the maximum independent set problem.

Sponsored Search with Bidder and Slot Conflicts. In this extension, we change the way bidders are in conflict. We again have a (bidder) conflict graph G with maximum degree Δ, slots, click-through rates and unit-demand valuations. In addition, we have a second conflict structure among slots. There is a directed *item conflict graph* $G_I = (V_I, E_I)$ on the slots, where an edge $(k, \ell) \in E_I$ implies that slot ℓ can make slot k *useless*. We denote the maximum out-degree of any vertex in G_I by Δ_I.

An intuitive example for this extension are *ordered conflicts*, where ad slots are ordered on a page top-down, and a bidder has a conflict only if a competitor receives a slot *above* him. This can be modeled by numbering slots in the top-down order and $E_I = \{(k, \ell) \mid k, l \in I, k > \ell\}$.[1] We also extend the previous model with bidder conflicts only, where G_I was a complete directed graph.

Given an allocation $X = (X_1, \ldots, X_n)$ of slots to bidders, slot $k \in X_i$ is useless if there is any slot $\ell \in X_j$ with $(i, j) \in E$ and $(k, \ell) \in E_I$, i.e., i has no use for

---

[1]Note that we use a different order for the slots in Chapter 3.

slot k if j receives slot ℓ. The valuation $v_i^c(X)$ is then determined by the best slot in X_i that is not useless. Formally, the set D_i of useless slots is now defined as $D_i = \{k \in X_i \mid \exists \ell \in X_j : (i,j) \in E \text{ and } (k,\ell) \in E_I\}$. Then,

$$v_i^c(X) = v_i(X_i \setminus D_i) = \max_{k \in X_i \setminus D_i} v_i \cdot \alpha_k. \tag{5.2}$$

Combinatorial Auctions with Bidder Conflicts. We extend the sponsored search model with unit-demand valuations to general valuation functions. We again have a *(bidder) conflict graph* G with maximum out-degree Δ, but here we allocate a set I of m items to the bidders. Each bidder $i \in N$ has a valuation function $v_i : \mathcal{P}(I) \to \mathbb{R}_{\geq 0}$ on the set of items. Given an allocation $X = (X_1, \ldots, X_n)$ of items to bidders, set X_i is useless if there is $X_j \neq \emptyset$ for some $(i,j) \in E$, i.e., i has no use for any set of items if j receives an item. Formally, $v_i^c(X) = v_i(X_i \setminus D_i)$ using the same definition of D_i for useless items as above. We here extend to general v_i instead of the unit-demand case in (5.1).

Combinatorial Auctions with Bidder and Item Conflicts. We also consider the extension to conflicts among items in combinatorial auctions, where we have a conflict graph G among bidders with maximum out-degree Δ and an *item conflict graph* G_I on the items with maximum degree Δ_I. Each bidder $i \in N$ has a valuation function $v_i : \mathcal{P}(I) \to \mathbb{R}_{\geq 0}$ on the set of items. Given an allocation $X = (X_1, \ldots, X_n)$ of items to bidders, item $k \in X_i$ is useless if there is an item $\ell \in X_j$ with $(i,j) \in E$ and $(k,\ell) \in E_I$. Formally, $v_i^c(X) = v_i(X_i \setminus D_i)$ using the same definition of D_i for useless items as for sponsored search above. We here extend to general v_i instead of the unit-demand case in (5.2).

Combinatorial Auctions with Bidder Conflicts and Conflict Value. Recall that we assume in *Combinatorial Auctions with Bidder Conflicts* that the valuation of a bidder $i \in N$ drops to $v_i^c(X) = 0$ as soon as a competitor receives any item and $\bigcup_{j:(i,j)\in E} X_j \neq \emptyset$. We can generalize this assumption to a second valuation function $w_i(X_i)$:

$$v_i^c(X) = \begin{cases} v_i(X_i) & \text{if } \bigcup_{j:(i,j)\in E} X_j = \emptyset \\ w_i(X_i) & \text{otherwise.} \end{cases}$$

This extension can be reduced to the model with bidder-conflicts only. Given an instance of a combinatorial auction with bidder conflicts and conflict value we build an instance without conflict value as follows: For each bidder $i \in N$, we add an auxiliary bidder i_c, where $v_{i_c}(X_i) = w_i(X_i)$. In the conflict graph, we add the edges (i, i_c) and (i_c, i). This increases Δ by exactly 1. Now if bidder i is conflicted, we can take all items assigned to it and assign them to bidder i_c instead. In this way, we can transform any allocation into an instance without conflict value and obtain the same social welfare. It is straightforward to observe that social welfare

maximization in both instances is equivalent. This, however, does not directly apply to incentive compatibility.

Our aim is to provide approximation ratios depending on the maximum degrees Δ and Δ_I. Note that one might be tempted to apply a similar transformation also to reduce the model with bidder and item conflicts to the one with bidder conflicts only. Here we could introduce a separate "agent" for each bidder-item pair or pairs of bidders and item sets. Apart from incentive compatibility, this also creates problems in the resulting conflict graph for the agents. For general valuations, we would have to introduce an exponential number of agents (since we represent all subsets of items), and the maximum degree in this graph would also increase exponentially. Even for unit-demand valuations, where we only need $n \cdot m$ agents, the degree of the graph can become as large as $\Delta \cdot \Delta_I$. Thus, if we apply the algorithms for bidder conflicts to this case, the resulting guarantees are much worse than the ones that we obtain by working directly on the model with bidder and item conflicts.

There are numerous further ways to extend our models, for instance, to combinations of item conflicts and conflict values, weighted conflicts, etc., and studying their properties are interesting directions for future work.

5.3 Combinatorial Auction with Bidder and Item Conflicts via Lottery

In this section, we present results for combinatorial auctions with bidder and item conflicts. We assume that Δ is bounded and G_I is arbitrary. At the end of this section, we will discuss how the techniques can be extended to the scenario in which Δ_I is bounded, G is arbitrary, and bidders have fractionally subadditive valuations.

First, we show that a (randomized) α-approximation algorithm f for combinatorial auctions without conflicts can be turned into a randomized $(4\Delta\alpha)$-approximation algorithm f^c for combinatorial auctions with conflicts. The results apply to arbitrary restrictions on the valuations (e.g., submodular valuations).

Given an allocation $X = (X_1, \ldots, X_n)$ of items to bidders, we show how to compute a random set $N^c \subseteq N$ such that (1) if $(i, j) \in E$ then $i \notin N^c$ or $j \notin N^c$, (2) $\sum_{i \in N} v_i(X_i) \leq (4\Delta) \mathbf{E}_{N^c} \left[\sum_{i \in N^c} v_i(X_i) \right]$, and (3) the selection of N^c does not depend on the valuations.

We will use pairwise independent distributions which we define below. Note that such a distribution always exists as one can pick the elements in N independently with probability q.

Definition 5.1. *We call a distribution $\mathcal{D}$ over subsets of a set N "pairwise independent with probability q" if for $N^R \sim \mathcal{D}$ and $i \neq j \in N$ holds that $\mathbf{Pr}\left[i \in N^R\right] = q$ and $\mathbf{Pr}\left[\{i, j\} \subseteq N^R\right] = \mathbf{Pr}\left[i \in N^R\right] \cdot \mathbf{Pr}\left[j \in N^R\right]$.*

The random set N^c computed by Algorithm 5.1 is constructed in the following way. First, in line 1 the algorithm picks a random subset from a pairwise independent distribution with probability $1/(2\Delta)$. Next, in line 5 the algorithm resolves all

---

Algorithm 5.1: Conflict-free random set

---

1 Pick a random subset N^R from a distribution over subsets of N that is pairwise independent with probability $1/(2\Delta)$
2 $N^c \leftarrow N^R$
3 **foreach** *bidder i in N^R* **do**
4   **if** *a bidder $j \in N^R$ with $(i,j) \in E$ exists* **then**
5   delete i from N^c

6 **return** N^c

---

the remaining conflicts between the bidders. We show that every bidder is in N^c with probability at least $1/(4\Delta)$.

Lemma 5.1. *In Algorithm 5.1, every bidder is in the returned set N^c with a probability of at least $1/(4\Delta)$.*

Proof. For all $i \in N$ let Q_i be the event that $i \in N^R$ in line 1. Thus, the probability for this event is $\mathbf{Pr}\left[Q_i\right] = 1/(2\Delta)$. The probability that a bidder $i \in N^R$ gets deleted in line 5 given that it was selected in N^R is $\mathbf{Pr}\left[\bigcup_{j\in N(i)} Q_j \mid Q_i\right]$, where $N(i)$ denotes the set of out-neighbors of bidder $i \in N$.

$$\mathbf{Pr}\left[\bigcup_{j\in N(i)} Q_j \;\middle|\; Q_i\right] = \frac{\mathbf{Pr}\left[(\bigcup_{j\in N(i)} Q_j) \cap Q_i\right]}{\mathbf{Pr}\left[Q_i\right]} = \frac{\mathbf{Pr}\left[\bigcup_{j\in N(i)}(Q_j \cap Q_i)\right]}{\mathbf{Pr}\left[Q_i\right]}$$

$$\overset{(*)}{\leq} \frac{\sum_{j\in N(i)} \mathbf{Pr}\left[Q_j \cap Q_i\right]}{\mathbf{Pr}\left[Q_i\right]} \overset{(**)}{=} \frac{\sum_{j\in N(i)} \mathbf{Pr}\left[Q_j\right]\mathbf{Pr}\left[Q_i\right]}{\mathbf{Pr}\left[Q_i\right]} = \sum_{j\in N(i)} \mathbf{Pr}\left[Q_j\right] \leq \frac{1}{2}.$$

In the above (in)equalities $(*)$ follows from Boole's inequality and $(**)$ follows from pairwise independence.

Thus, the probability that $i \in N^c$ at the end of the loop is

$$\mathbf{Pr}\left[Q_i\right] \mathbf{Pr}\left[\left(\bigcup_{j\in N(i)} Q_j\right)^C \;\middle|\; Q_i\right] \tag{5.3}$$

$$= \mathbf{Pr}\left[Q_i\right]\left(1 - \mathbf{Pr}\left[\bigcup_{j\in N(i)} Q_j \;\middle|\; Q_i\right]\right) \geq \frac{1}{4\Delta}.$$

$\square$

The next theorem shows that an α-approximation algorithm for combinatorial auctions without conflicts can be used to give a randomized $(4\Delta\alpha)$-approximation algorithm for combinatorial auctions with conflicts.

Theorem 5.2. *A (randomized) α-approximation algorithm f for combinatorial auctions without conflicts can be turned into a randomized $(4\Delta\alpha)$-approximation algorithm f^c for combinatorial auctions with Δ bounded and G_I arbitrary.*

Proof. We define our $(4\Delta\alpha)$-approximation algorithm f^c as follows: f^c first calls Algorithm 5.1 to compute a random subset N^c and then calls f for the bidders in N^c.

Assume that if we use the α-approximation algorithm f on the set of bidders $N' \subseteq N$ and if we set $X_i(N') = \emptyset$ for all $i \notin N'$ then the algorithm returns the allocation $(X_1(N'), \ldots, X_n(N'))$. Furthermore, assume that the optimal allocation is $\text{OPT}(N') = (\text{OPT}_1(N'), \ldots, \text{OPT}_n(N'))$ given the constraints $\text{OPT}_i(N') = \emptyset$ if $i \notin N'$. Then

$$
\underset{N^c}{\mathbf{E}}\left[\sum_{i \in N^c} v_i(X_i(N^c))\right] \geq \underset{N^c}{\mathbf{E}}\left[\frac{1}{\alpha}\sum_{i \in N^c} v_i(\text{OPT}_i(N^c))\right]
$$

$$
\overset{(*)}{\geq} \frac{1}{\alpha}\underset{N^c}{\mathbf{E}}\left[\sum_{i \in N^c} v_i(\text{OPT}_i(N))\right] = \frac{1}{\alpha}\sum_{i \in N} v_i(\text{OPT}_i(N)) \cdot \mathbf{Pr}\left[i \in N^c\right]
$$

$$
\geq \frac{1}{4\Delta\alpha}\sum_{i \in N} v_i(\text{OPT}_i(N)).
$$

Inequality $(*)$ holds because $\text{OPT}(N^c)$ gives the bidders in N^c the maximal social welfare. $\qquad\square$

Since no bidder can alter N^c by changing his valuation, it also holds that we can turn an in the universal sense incentive compatible (resp. incentive compatible in expectation) α-approximation mechanism (f, p) for combinatorial auctions without conflicts into an in the universal sense incentive compatible (resp. incentive compatible in expectation) $(4\Delta\alpha)$-approximation mechanism (f^c, p^c) for combinatorial auctions with conflicts by using the same approach as above. That is, we can first call Algorithm 5.1 to compute a random subset N^c and can then use (f, p) only for the bidders in N^c. Note that bidders not in N^c cannot change their utility by changing their bid; it is always zero. Furthermore, bidders in N^c behave like in (f, p), i.e., they bid truthfully their valuation so as to maximize their own utilities for each realization of N^c. The approximation guarantee follows directly from Theorem 5.2.

Corollary 5.3. *An in the universal sense incentive compatible (resp. incentive compatible in expectation) α-approximation mechanism (f, p) for combinatorial auctions without conflicts can be turned into an in the universal sense incentive compatible (resp. incentive compatible in expectation) $(4\Delta\alpha)$-approximation mechanism (f^c, p^c) for combinatorial auctions with Δ bounded and G_I arbitrary.*

5.3.1 Derandomization of the Algorithms

We also show how the results above can be generalized to deterministic approximation algorithms and deterministic maximal-in-range mechanisms. The cornerstone

for the derandomization is to show that there exists a pairwise independent distribution of subsets of N with probability $1/2^{\lceil \log_2 \Delta \rceil + 1}$ that has a domain with a cardinality that is polynomial in n; this follows from [82, Section 1.2]. The idea is to represent the distribution by a randomization over a family of 2-universal hash functions that assigns each bidder u.a.r. values from the set $\{t_1, \ldots, t_{2\Delta}\}$ and to consider the subset of bidders with value t_1. Furthermore, this implies that also a distribution over the random sets N^c computed by Algorithm 5.1 exists that has a domain with a cardinality that is polynomial in n.

Lemma 5.4 (Luby and Wigderson [82]). *Given a set N with $|N| = n$ and an integer Δ with $1 \leq \Delta \leq n - 1$ then there exists a pairwise independent distribution over subsets of N with probability $1/2^{\lceil \log_2 \Delta \rceil + 1}$ and a domain with a cardinality in $\mathcal{O}(n^2)$.*

Note that we can use the pairwise independent distribution of Lemma 5.4 in Algorithm 5.1. Moreover, instead of picking a random subset from this distribution in Algorithm 5.1, we can iterate over the domain of the distribution in polynomial time; this gives us at least the same social welfare and proves Corollary 5.5.

Corollary 5.5. *A deterministic α-approximation algorithm f for combinatorial auctions without conflicts can be turned into a deterministic $(16\Delta\alpha/3)$-approximation algorithm f^c for combinatorial auctions with Δ bounded and G_I arbitrary.*

Proof. Note that $1/(2\Delta) \geq 1/2^{\lceil \log_2 \Delta \rceil + 1} > 1/(4\Delta)$. By slightly modifying the proof of Lemma 5.1, we can show that every bidder is in N^c with a probability of at least $3/(16\Delta)$. In particular, let us assume that Q_i is the event that $i \in N^R$ in line 1 of Algorithm 5.1 like in the proof of Lemma 5.1. Then, the probability that $i \in N^c$ at the end of the algorithm is at least $\mathbf{Pr}[Q_i] \cdot (1 - \sum_{j \in N(i)} \mathbf{Pr}[Q_j]) \geq y \cdot (1 - \Delta y)$ where $y = \mathbf{Pr}[Q_i]$ for all $i \in N$. Since the slope of $y \cdot (1 - \Delta y)$ is positive for $y \leq 1/(2\Delta)$, it follows that $i \in N^c$ at the end of the algorithm with a probability of at least $1/(4\Delta) \cdot (1 - \Delta(1/(4\Delta))) = 3/(16\Delta)$. □

Furthermore, we can extend the results to *maximal-in-range* algorithms (see Dobzinski and Dughmi [29]) which are important for the design of incentive compatible approximation mechanisms.

Definition 5.2. *An algorithm f is called "maximal-in-range" if it exists a subset $\mathcal{X}'$ of the set of allocations $\mathcal{X}$ for which $f(v_1, \ldots, v_n) \in \arg\max_{X \in \mathcal{X}'}(\sum_{i \in N} v_i(X_i))$ for all possible valuations.*

Given a maximal-in-range algorithm f for combinatorial auctions without conflicts that is a deterministic α-approximation algorithm we show in Algorithm 5.2 how to construct a maximal-in-range algorithm f^c for combinatorial auctions with conflicts that is a deterministic $(16\Delta\alpha/3)$-approximation algorithm. Hence, Theorem 5.6 follows. Note that Algorithm 5.2 calls f always for the same set of bidders and set of items, and only the valuations of the bidders change; thus, the target set $\mathcal{X}'$ of f is the same in each call.

Algorithm 5.2: Maximal-in-range algorithm f^c over range $\mathcal{X}'_D$.

---

1   Let D be the domain of a distribution over the random set N^c computed by Algorithm 5.1 using the distribution from Lemma 5.4 such that $|D| \in \mathcal{O}(n^2)$

2   Let $\mathcal{X}' \subseteq \mathcal{X}$ be the target set of f

3   Set $\mathrm{OPT} \leftarrow (\emptyset, \dots, \emptyset)$

4   **foreach** $N' \in D$ **do**

5   **forall the** $i \in N$ **do** set $v_i^{N'} \leftarrow v_i$ if $i \in N'$ and $v_i^{N'} \leftarrow 0$ else

6   Set $\mathrm{OPT}(N') \leftarrow f(v^{N'})$

7   **if** $\sum_{i \in N} v_i^{N'}(\mathrm{OPT}_i(N')) \geq \sum_{i \in N} v_i(\mathrm{OPT}_i)$ **then**

8   Set OPT to the following assignment:
         (1) all $i \in N'$ get the same items as in $\mathrm{OPT}(N')$
         (2) others get no items

9   **return** OPT

Theorem 5.6. *Given a maximal-in-range deterministic α-approximation algorithm f for combinatorial auctions without conflicts, Algorithm 5.2 defines a maximal-in-range, deterministic and incentive compatible $(16\Delta\alpha/3)$-approximation mechanism (f^c, p^c) for combinatorial auctions with conflicts.*

Proof. Let us assume that OPT was set to its final value when $N' = N^*$. Furthermore, assume that N^c is a random subset computed by Algorithm 5.1 with a probability of at least $3/(16\Delta)$ by Corollary 5.5. It follows that

$$
\begin{aligned}
\sum_{i \in N} v_i(\mathrm{OPT}_i) &= \sum_{i \in N} v_i^{N^*}(\mathrm{OPT}_i) = \sum_{i \in N} v_i^{N^*}(\mathrm{OPT}_i(N^*)) \\
&\geq \mathop{\mathbf{E}}_{N^c} \left[\sum_{i \in N} v_i^{N^c}(\mathrm{OPT}_i(N^c)) \right] \overset{(*)}{\geq} \mathop{\mathbf{E}}_{N^c} \left[\sum_{i \in N} v_i^{N^c}(\mathrm{OPT}_i(N)) \right] \\
&= \sum_{i \in N} v_i(\mathrm{OPT}_i(N)) \cdot \mathbf{Pr}\left[i \in N^c \right] \geq \frac{3}{16\Delta} \cdot \sum_{i \in N} v_i(\mathrm{OPT}_i(N)).
\end{aligned}
$$

Inequality $(*)$ holds because $\mathrm{OPT}(N^c) \in \arg\max_{X \in \mathcal{X}'} \sum_{i \in N} v_i^{N^c}(X_i)$ and $\mathrm{OPT}(N) \in \mathcal{X}'$. Since the maximum social welfare is at most $\alpha \cdot \sum_{i \in N} v_i(\mathrm{OPT}_i(N))$ the claimed approximation factor follows.

We still have to show that the algorithm is maximal-in-range. For each $X \in \mathcal{X}'$ and $N' \in D$ let $X^{N'}$ be the allocation when all bidders in $i \in N'$ get the same set of items as in X and all other bidders get no items. Define $\mathcal{X}'_D := \{X^{N'} | (X, N') \in$

$\mathcal{X}' \times D$}. We show next that f^c is maximal on the subset $\mathcal{X}'_D \subseteq \mathcal{X}$. It holds that

$$
\sum_{i \in N} v_i(\text{OPT}_i) = \max_{N' \in D} \sum_{i \in N} v_i^{N'}(\text{OPT}_i(N'))
$$
$$
= \max_{N' \in D} \max_{X \in \mathcal{X}'} \sum_{i \in N} v_i^{N'}(X_i) = \max_{X^{N'} \in \mathcal{X}'_D} \sum_{i \in N} v_i((X^{N'})_i) .
$$

The pricing scheme for an incentive compatible mechanism is given in [91, Proposition 9.31]. □

5.3.2 Bounded Out-degree in the Item Conflict Graph

We extend our results to the case where Δ_I is bounded, G is arbitrary and bidders have fractionally subadditive valuations. The algorithms for this case are similar to what we give above, but instead of eliminating bidder conflicts by removing bidders, they eliminates item conflicts by removing items.

The requirement of fractionally subadditive valuations is necessary as the following example illuminates: Suppose there are two items which are *total complements* and have item conflicts. If we remove either of the two, the social welfare drops to zero, while the optimal allocation attains a positive social welfare by allocating both items to the same bidder.

Also, note that an $\mathcal{O}(\Delta)$-approximation algorithm for the bounded-Δ case and an $\mathcal{O}(\Delta_I)$-approximation algorithm for this case can be combined to yield an $\mathcal{O}\left(\min\{\Delta, \Delta_I\}\right)$-approximation algorithm, resulting in the following theorem.

Theorem 5.7. *Suppose that in Theorem 5.2, Corollary 5.3, Corollary 5.5 and Theorem 5.6, f works for the class of fractionally subadditive valuations or a subclass. Moreover, note that the original approximation guarantees of f^c in the four theorems/corollaries are in the form of $C\Delta\alpha$, where C is either 4 or $16/3$. Then f^c can be modified so that the approximation guarantees improve to $C \cdot \min\{\Delta, \Delta_I\} \cdot \alpha$.*

Next, we present the proofs for bounded Δ_I that are missing for Theorem 5.7; that is, we present the proofs for the case when Δ_I is bounded, G is arbitrary, and valuations are fractionally subadditive. Again, the results apply to arbitrary restrictions on the valuations (e.g., submodular valuations).

The missing proofs for Theorem 5.7 will use the following proposition by Feige [39] for fractionally subadditive valuations. In the proposition it is not required that items in I are selected independently for I'.

Proposition 5.8 (Prop. 2.3 in Feige [39]). *Let $k \geq 1$ be an integer and let v be an arbitrary fractionally subadditive utility function. For a set I, consider a distribution over subsets $I' \subset I$ such that each item in I is included in I' with probability at least $1/k$. Then $\mathbf{E}\left[v(I')\right] \geq v(I)/k$.*

Proof of Theorem 5.2 for bounded Δ_I. The idea is to restrict the item set I to a random set I^c that is independent of the valuations of the bidders and then to call

the α-approximation algorithm for the restricted item set I^c. Note that we can use Algorithm 5.1 also for items. Thus, by the same arguments as for bidders we obtain a random set of items I^c where items have no conflicts and where every item is in I^c with probability at least $1/(4\Delta_I)$. Let $(X_1(I'), \ldots, X_n(I'))$ be the allocation that f returns when we restrict the set of items to $I' \subseteq I$. Furthermore, let $(\mathrm{OPT}_1(I'), \ldots, \mathrm{OPT}_n(I'))$ be the optimal allocation of the item set I'. It holds that

$$
\mathop{\mathbf{E}}_{I^c}\left[\sum_{i \in N} v_i(X_i(I^c))\right] \geq \mathop{\mathbf{E}}_{I^c}\left[\frac{1}{\alpha}\sum_{i \in N} v_i(\mathrm{OPT}_i(I^c))\right] \overset{(*)}{\geq} \mathop{\mathbf{E}}_{I^c}\left[\frac{1}{\alpha}\sum_{i \in N} v_i(\mathrm{OPT}_i(I) \cap I^c)\right]
$$

$$
\geq \frac{1}{\alpha}\sum_{i \in N}\mathop{\mathbf{E}}_{I^c}\left[v_i(\mathrm{OPT}_i(I) \cap I^c)\right] \overset{(**)}{\geq} \frac{1}{4\Delta_I \alpha}\sum_{i \in N} v_i(\mathrm{OPT}_i(I)).
$$

Above, inequality $(*)$ follows because $\mathrm{OPT}_i(I^c)$ is optimal for item set I^c, and inequality $(**)$ follows by Proposition 5.8. $\qquad\square$

Proof of Corollary 5.3 for bounded Δ_I. Again, we restrict the item set I to a random set I^c as in the proof of Theorem 5.2 for bounded Δ_I and then we call the mechanism (f, p) for the restricted item set. The approximation guarantee follows from Theorem 5.2 for bounded Δ_I and incentive compatibility in the universal sense (resp. incentive compatibility in expectation) follows since truthful bidding is a dominant strategy for each realization of I^c. $\qquad\square$

Proof of Corollary 5.5 for bounded Δ_I. Note that in Theorem 5.2 we use the same randomization technique for N^c and for I^c. Thus, we can apply our derandomization technique for N^c also to I^c. $\qquad\square$

Proof of Theorem 5.6 for bounded Δ_I. Given a maximal-in-range algorithm f for combinatorial auctions without conflicts that is a deterministic α-approximation algorithm we show in Algorithm 5.3 how to construct a maximal-in-range algorithm f^c for combinatorial auctions with conflicts that is a deterministic $(16\Delta_I\alpha/3)$-approximation algorithm. As Algorithm 5.2, Algorithm 5.3 calls f always for the same set of bidders and set of items; thus, the target set $\mathcal{X}'$ of f is the same in each call.

Let us assume that OPT was set to its final value when $I' = I^*$. Furthermore, assume that I^c is a random subset computed by Algorithm 5.1. It follows that

$$
\sum_{i \in N} v_i(\mathrm{OPT}_i) = \sum_{i \in N} v_i^{I^*}(\mathrm{OPT}_i(I^*))
$$

$$
\geq \mathop{\mathbf{E}}_{I^c}\left[\sum_{i \in N} v_i^{I^c}(\mathrm{OPT}_i(I^c))\right] \overset{(*)}{\geq} \mathop{\mathbf{E}}_{I^c}\left[\sum_{i \in N} v_i^{I^c}(\mathrm{OPT}_i(I))\right]
$$

$$
= \sum_{i \in N}\mathop{\mathbf{E}}_{I^c}\left[v_i(\mathrm{OPT}_i(I) \cap I^c)\right] \overset{(**)}{\geq} \frac{3}{16\Delta_I}\sum_{i \in N} v_i(\mathrm{OPT}_i(I)).
$$

Algorithm 5.3: Maximal-in-range algorithm f^c over range $\mathcal{X}'_D$ when Δ_I is bounded.

---

1 Let D be the domain of a distribution over the random set I^c computed by Algorithm 5.1 using the distribution from Lemma 5.4 such that $|D| \in \mathcal{O}(m^2)$
2 Let $\mathcal{X}' \subseteq \mathcal{X}$ be the target set of f
3 Set $\mathrm{OPT} \leftarrow (\emptyset, \dots, \emptyset)$
4 **foreach** $I' \in D$ **do**
5 $\quad$ Define $v_i^{I'}(S) := v_i(S \cap I')$ for all $S \subseteq I$ and $i \in N$
6 $\quad$ Set $\mathrm{OPT}(I') \leftarrow f(v^{I'})$
7 $\quad$ **if** $\sum_{i \in N} v_i^{I'}(\mathrm{OPT}_i(I')) \geq \sum_{i \in N} v_i(\mathrm{OPT}_i)$ **then**
8 $\quad\quad$ $\mathrm{OPT}_i \leftarrow \mathrm{OPT}_i(I') \cap I'$ for all $i \in N$

9 **return** OPT

Inequality $(*)$ holds because $\mathrm{OPT}(I^c) \in \arg\max_{X \in \mathcal{X}'} \sum_{i \in N} v_i^{I^c}(X_i)$ and $\mathrm{OPT}(I) \in \mathcal{X}'$, and inequality $(**)$ follows from Proposition 5.8. Since the maximum social welfare it at most $\alpha \cdot \sum_{i \in N} v_i(\mathrm{OPT}_i(I))$ the claimed approximation factor follows.

We still have to show that the algorithm is maximal-in-range. For each $X \in \mathcal{X}'$ and $I' \in D$ let $X^{I'} := (X_1 \cap I', \dots, X_n \cap I')$. Define $\mathcal{X}'_D := \{X^{I'} | (X, I') \in \mathcal{X}' \times D\}$. The next sequence of equalities shows that f^c is maximal on the subset $\mathcal{X}'_D \subseteq \mathcal{X}$.

$$\sum_{i \in N} v_i(\mathrm{OPT}_i) = \max_{I' \in D} \sum_{i \in N} v_i^{I'}(\mathrm{OPT}_i(I'))$$
$$= \max_{I' \in D} \max_{X \in \mathcal{X}'} \sum_{i \in N} v_i^{I'}(X_i) = \max_{X^{I'} \in \mathcal{X}'_D} \sum_{i \in N} v_i((X^{I'})_i)$$

Again, the pricing scheme for an incentive compatible mechanism is given in [91, Proposition 9.31]. $\qquad\square$

5.4 Combinatorial Auctions with Bidder Conflicts via Cone Program Relaxation

We design an approximation algorithm via cone programming relaxation for maximizing social welfare in combinatorial auctions with bidder conflicts. Due to the definition of v_i^c, this problem is equivalent to maximizing $\sum_{i \in N} v_i(X_i)$ subject to the constraint that for any $(i, j) \in E$, either $X_i = \emptyset$ or $X_j = \emptyset$. We will show the following result.

Theorem 5.9. *For combinatorial auctions with bidder conflicts, suppose that the conflict graph G has out-degree at most Δ, and the bidders have fractionally subadditive (FSA) valuations. If there is a demand oracle for each bidder (we will define this later),*

then there exists an $\mathcal{O}\left(\Delta \cdot (\log\log\Delta)/(\log\Delta)\right)$-approximation algorithm of social welfare that runs in $\mathrm{poly}(m, n)$-time.

We include some standard facts of combinatorial auctions without conflicts in Section 5.4.1. As this is one of the first applications of cone programming relaxations in the context of algorithmic mechanism design, we include some standard facts of cone programs in Section 5.4.2. The algorithm and its analysis will be given in Sections 5.4.3 and 5.4.5 respectively. In the remainder of this section, we use S to denote a subset of I.

5.4.1 Combinatorial Auction without Conflicts

The optimal social welfare in a combinatorial auction without conflicts can be represented by the following integral linear program ILP-NC:

$$\max \sum_{i \in N} \sum_{S \neq \emptyset} v_i(S) \cdot x_{i,S} \qquad\qquad \text{(ILP-NC)}$$

$$\text{subject to} \qquad \sum_{S \neq \emptyset} x_{i,S} \leq 1 \qquad\qquad \forall i \in N,$$

$$\sum_{S:k \in S} \sum_{i \in N} x_{i,S} \leq 1 \qquad\qquad \forall k \in I,$$

$$x_{i,S} \in \{0, 1\} \qquad\qquad \forall i \in N \text{ and } S \subseteq I.$$

In general, solving ILP-NC is NP-hard. The usual remedy is to solve its linear programming relaxation LPR-NC, i.e., relaxing the constraint $x_{i,S} \in \{0, 1\}$ to $x_{i,S} \in [0, 1]$, to obtain a *fractional* solution, and *round* it to an integral solution that attains a good approximation guarantee.

There are $\Omega\left(2^m n\right)$-many variables in LPR-NC, but it can be solved in $\mathrm{poly}(m, n)$-time if there is a *demand oracle* for each bidder: Given the prices $p_1, p_2, \ldots, p_m$ of the items, the demand oracle of bidder i returns a set $S \subseteq I$ that maximizes $v_i(S) - \sum_{k \in S} p_k$. The demand oracles serve as *separation oracles* for the dual of LPR-NC; thus, they allow solving LPR-NC efficiently by using the ellipsoid method (see Nisan and Segal [92]).

For the general class of valuations, the approximation guarantee is worse than $\Omega\left(m^{1/2-\epsilon}\right)$ for any positive constant ϵ, unless P=NP. For some restricted classes of valuations, the approximation guarantee can be much better. For FSA valuations, the approximation guarantee is $\frac{e}{e-1} \approx 1.58$ (see [33]). Indeed, the fair contention resolution algorithm for FSA valuations (see Feige and Vondrák [40, Section 1.2]) takes any feasible point of LPR-NC as input, and gives an allocation such that each bidder obtains expected welfare of at least $\left(1 - \frac{1}{e}\right)$ times his part in the objective function value in LPR-NC at the feasible point.

5.4.2 Cone Programs in a Nutshell

Cone programs (CP) are a generalization of the more familiar linear programs (LP) and semi-definite programs (SDP). We list the relevant definitions and properties of CP here. They are extracted from Chapters 2 and 5 in Boyd and Vandenberghe [17].

A closed set $K \subseteq \mathbb{R}^q$ is a *proper cone* if (a) for any real numbers $a_1, a_2 \geq 0$ and for any $k_1, k_2 \in K$, $a_1 k_1 + a_2 k_2 \in K$; (b) it has nonempty interior; and (c) it is pointed, i.e., if $x, -x \in K$, then $x = 0$. Two examples of proper cones are the non-negative orthant (the set of points with non-negative coordinates), and the set of symmetric positive semi-definite (SPSD) matrices.

There is a natural partial ordering on $\mathbb{R}^q$ associated with any proper cone K, which is denoted by $\succeq_K$: For any $x_1, x_2 \in \mathbb{R}^q$, $x_1 \succeq_K x_2$ if and only if $(x_1 - x_2) \in K$. The corresponding *strict* partial ordering, $\succ_K$, is defined as follows: For any $x_1, x_2 \in \mathbb{R}^q$, $x_1 \succ_K x_2$ if and only if $(x_1 - x_2)$ is an interior point of K.

The *dual cone* of a proper cone K, is the set $K^* := \{z \mid k \cdot z \geq 0 \; \forall k \in K\}$, where $k \cdot z$ is the inner product of k and z. The dual cone of the non-negative orthant is the non-negative orthant itself, and the dual cone of the set of SPSD matrices is the set of SPSD matrices itself [17, Examples 2.23 and 2.24]. The two proper cones are said to be *self-dual*.

There are various forms of CP, but they can be shown to be equivalent. In this chapter, we will use two of the forms of CP. The *standard form* of CP is as follows; note that $c, x \in \mathbb{R}^q$, A is an $\ell \times q$ real matrix, $b \in \mathbb{R}^\ell$ and K is a proper cone in $\mathbb{R}^q$:

$$\min c \cdot x \qquad \qquad \text{(CP-STD)}$$
$$\text{subject to} \qquad Ax = b$$
$$x \succeq_K 0$$

The above CP has two constraints. The first one, $Ax = b$, is called *non-conic constraint*. The second one, $x \succeq_K 0$ or equivalently $x \in K$, is called *conic constraint*.

LP is a special case of CP, in which K is the non-negative orthant. SDP is a special case of CP, in which K is the set of SPSD matrices.

As in the cases of LP and SDP, there is a dual for CP too, which is also a CP. Before describing the dual, we note that the non-conic constraint $Ax = b$ can be broken into ℓ equality constraints $A_h x = b_h$, where A_h is the h-th row of the matrix A and b_h is the h-th entry of the vector b. Each such equality constraint in the primal will associate to one distinct real variable in the dual, but the conic constraint will not associate to any dual variable. This is important since for our problem, we will introduce a CP with exponentially many variables but only $\text{poly}(m, n)$ equality constraints. Then its dual will have only $\text{poly}(m, n)$ dual variables, which is a necessary feature for using ellipsoid method to solve it in $\text{poly}(m, n)$-time.

The dual of CP-STD, in an *inequality form* of CP, is (see [17, Example 5.12])

$$\max b \cdot y \qquad\qquad \text{(CP-DUAL-INEQ)}$$
$$\text{subject to} \qquad c \succeq_{K^*} A^\mathsf{T} y$$

We may solve CP-STD by solving its dual CP-DUAL-INEQ if strong duality holds between them. While strong duality always holds for LP, it may not hold for SDP and CP. The standard method to determine strong duality of CP is to check that *Slater's condition* holds, i.e., there exists an x such that $Ax = b$ and $x \succ_K 0$.

5.4.3 Algorithm

Halperin [61] designed an SDP and a rounding scheme for WIS with approximation guarantee $\mathcal{O}\left(\Delta \cdot \frac{\log \log \Delta}{\log \Delta}\right)$. We conglomerate his SDP with LPR-NC for our problem.

(ICP-C)

$$\max \sum_{i \in N} \sum_{S \neq \emptyset} v_i(S) \cdot x_{i,S}$$

subject to

$$\sum_{S \neq \emptyset} x_{i,S} \leq 1, \qquad \forall i \in N$$

$$\sum_{S:k \in S} \sum_{i \in N} x_{i,S} \leq 1, \qquad \forall k \in I$$

$$\frac{1 + w_i}{2} = \sum_{S \neq \emptyset} x_{i,S}, \qquad \forall i \in N$$

$$(1 + w_i)(1 + w_j) = 0, \quad \forall (i,j) \in E \quad (5.4)$$

$$w_i \in \pm 1, \qquad \forall i \in N$$

$$x_{i,S} \in \{0,1\}, \qquad \forall i \in N, S \subseteq I.$$

(CPR-C)

$$\max \; Z := \sum_{i \in N} \sum_{S \neq \emptyset} v_i(S) \cdot x_{i,S}$$

subject to

$$\sum_{S \neq \emptyset} x_{i,S} \leq 1$$

$$\sum_{S:k \in S} \sum_{i \in N} x_{i,S} \leq 1$$

$$\frac{1 + w_0 \cdot w_i}{2} = \sum_{S \neq \emptyset} x_{i,S} \qquad (5.5)$$

$$(w_0 + w_i) \cdot (w_0 + w_j) = 0 \qquad (5.6)$$

$$\|w_0\| = \|w_i\| = 1. \qquad (5.7)$$

$$x_{i,S} \geq 0.$$

Note that our problem is equivalent to solving the integer program ICP-C shown above. As the constraint (5.4) involves a product of variables, LP relaxation is not admissible. We relax ICP-C to CPR-C, a "mixture" of an LP and an SDP; note that in CPR-C, $w_0, w_i \in \mathbb{R}^{n+1}$. CPR-C is a CP. In Section 5.4.4, we will show that strong duality holds between CPR-C and its dual, and we can solve the dual of CPR-C in poly(m, n)-time by using the ellipsoid method, assuming that we have a demand oracle for each bidder.

Let $(Z^*, \{x^*\}, \{w^*\})$ be the optimal solution to CPR-C. Our algorithm, Algorithm 5.4, partitions the bidders according to the value of $1 + w_0^* \cdot w_i^*$ into three sets

---

Algorithm 5.4: Approximation Algorithm via Cone Program Relaxation.

1 Solve CPR-C to obtain the solution $(Z^*, \{x^*\}, \{w^*\})$.

2 Set $\tau \leftarrow \frac{3 \log \log \Delta}{4 \log \Delta}$, which is less than $1/2$.

3 **Partition the bidders into three sets** N_0, N_1, N_2:
$N_0 = \{i \mid 0 \le 1 + w_0^* \cdot w_i^* \le 2\tau\}$,
$N_1 = \{i \mid 2\tau < 1 + w_0^* \cdot w_i^* \le 1\}$,
$N_2 = \{i \mid 1 < 1 + w_0^* \cdot w_i^* \le 2\}$

4 **Consider bidders in** N_2:.

5 Let $J_2 = N_2$. Solve LPR-NC with the set of bidders restricted to J_2. Round the solution of LPR-NC, i.e., allocate items I to bidders in J_2, using the fair contention resolution algorithm in [40, Section 1.2]. Let $\mathcal{A}_2$ denote this allocation.

6 **Consider bidders in** N_1:

7 Project all vectors w_i^* corresponding to bidders in N_1 to the space orthogonal to w_0^*, and normalize them. Let w_i' denote the projected normal vectors. Note that the space orthogonal to w_0^* has dimension n, so we can treat each w_i' as an n-dimensional vector.

8 Choose a random n dimensional vector $r = (r_1, r_2, \ldots, r_n)$, where each r_i follows the standard normal distribution with density function
$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \forall x \in \mathbb{R}$.

9 Let $\gamma := (1 - 2\tau)/(2 - 2\tau)$ and $N_1' := \left\{ i \in N_1 \mid w_i' \cdot r \ge \sqrt{\frac{2\gamma}{1-\gamma} \log \Delta} \right\}$.

10 Let $J_1 := N_1' \setminus \{i \in N_1' \mid \exists j \in N_1' \text{ such that } (i, j) \in E\}$. Solve LPR-NC with the set of bidders restricted to J_1. Round the solution of LPR-NC, i.e., allocate items I to bidders in J_1, using the fair contention resolution algorithm. Let $\mathcal{A}_1$ denote this allocation.

11 **Consider bidders in** N_0:

12 Each bidder $i \in N_0$ chooses a *target set* X_i with the following distribution: each non-empty set S is chosen with probability $q_{i,S} = \frac{x_{i,S}^*}{2\tau\Delta}$, and the empty set is chosen with probability $q_{i,\emptyset} = 1 - \sum_{S \ne \emptyset} q_{i,S}$. (Since $2\tau\Delta > 1$, q is a valid probability distribution.)

13 With the target sets generated, use the fair contention resolution algorithm to allocate each bidder i a set of goods $T_i \subseteq X_i$, where $\{T_i\}_{i \in N_0}$ are disjoint.

14 Conflict handling: For each bidder $i \in N_0$, if there exists another bidder j such that $(i, j) \in E$ and $T_j \ne \emptyset$, set the allocation of bidder i to the empty set

15 Let $\mathcal{A}_0$ denote this allocation.

16 **Return the best allocation among** $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$.

N_0, N_1 and N_2. Items are allocated to either of the three sets; the best one is chosen. The methods of allocating items to N_1 and N_2 are well motivated by Halperin's algorithm.

The bidders in N_2 have high values of $1 + w_0^* \cdot w_i^*$, so (5.6) will guarantee that N_2 is an independent set. Thus we can use established approximation algorithms for combinatorial auction without conflicts.

The bidders in N_1 have suitably high values of $1 + w_0^* \cdot w_i^*$. The aim is to select a sufficiently good independent subset of N_1, and then proceed as in N_2. The selection is done via randomized vector rounding w.r.t. the space orthogonal to w_0^*.

The bidders in N_0 will have low values of $\frac{1+w_0^* \cdot w_i^*}{2} = \sum_{S \neq \emptyset} x_{i,S}^*$. Note that $x_{i,S}^*$ are typically viewed as probability densities. Low values of $\frac{1+w_0^* \cdot w_i^*}{2}$ allow room to "expand" these densities by a factor of $1/\tau$, where $\tau < \frac{1}{2}$. However, to handle conflicts, we ought to "dwell" these densities by a factor of $1/(2\Delta)$ afterwards. Then we apply the fair contention resolution algorithm with the "expanded then dwelled" probability densities to obtain a sufficiently good allocation to N_0 in expectation.

Remark.   We conglomerate an LP and an SDP to design our CP relaxation. As LP is a subclass of SDP, using an SDP suffices to express all the constraints. However, using this approach introduces exponentially many dual variables.

5.4.4   Solving CPR-C in Polynomial Time

First, we refer the readers to Grötschel et al. [56] (Chapters 2–4) for details of the ellipsoid method. We will use the following result from fundamental linear algebra:

Lemma 5.10 (Grötschel et al. [56], Section 0.1). *Let M be a symmetric $q \times q$ real matrix. M is positive semi-definite if and only if there exists $w_1, w_2, \ldots, w_q \in \mathbb{R}^q$ such that for $1 \leq k \leq q$, $1 \leq \ell \leq q$, $M_{k\ell} = w_k \cdot w_\ell$. Furthermore, M is positive definite if and only if the vectors $w_1, w_2, \ldots, w_q$ are linearly independent.*

For our problem, we define the following proper cone K: K consists of all points $(\{x_{i,S}\}, \{\alpha_i\}, \{\beta_k\}, M)$, where $\{x_{i,S}\}, \{\alpha_i\}, \{\beta_k\}$ are vectors of dimension $(2^m - 1)n$, n, m respectively, and all of them have non-negative entries; M is a symmetric positive semi-definite $(n+1) \times (n+1)$ real matrix. This matrix can be represented by a vector that lists the coefficients of the matrix in some order and the standard inner product of two such matrices is the standard inner product of the associated vectors[17, Appendix A.1.1]. Since the non-negative orthant is a proper cone [17, Examples 2.14] and the set of symmetric positive semi-definite matrices is a proper cone [17, Examples 2.15] it follows that K is a proper cone. Furthermore, since the non-negative orthant is self-dual [17, Examples 2.23] and the set of symmetric positive semi-definite matrices is self-dual [17, Examples 2.24] it follows that K is self-dual.

Using Lemma 5.10 and introducing slack variables $\{\alpha_i\}$ and $\{\beta_k\}$, we can rewrite CPR-C in the standard CP form CP-STD:

$$\min - \sum_{i \in N} \sum_{S \neq \emptyset} v_i(S) \cdot x_{i,S} \qquad \text{(CPR-C')}$$

subject to
$$- \sum_{S \neq \emptyset} x_{i,S} - \alpha_i = -1 \qquad \forall i \in N, \qquad \textbf{(u_i)}$$

$$- \sum_{S : k \in S} \sum_{i \in N} x_{i,S} - \beta_k = -1 \qquad \forall k \in I, \qquad \textbf{(p_k)}$$

$$2 \sum_{S \neq \emptyset} x_{i,S} - M_{0,i} = 1 \qquad \forall i \in N, \qquad \textbf{(z_i)}$$

$$M_{0,i} + M_{0,j} + M_{\min\{i,j\},\max\{i,j\}} = -1 \qquad \forall (i,j) \in E, \qquad \textbf{(y_{i,j})}$$

$$M_{0,0} = 1, \qquad \textbf{(q_0)}$$

$$M_{i,i} = 1 \qquad \forall i \in N, \qquad \textbf{(q_i)}$$

$$(\{x_{i,S}\}, \{\alpha_i\}, \{\beta_k\}, M) \succeq_K 0.$$

Instead of solving CPR-C' directly, we will solve its dual. Each equality constraint in CPR-C' will associate to a variable in the dual. We have written the variables down on the right of their corresponding constraints.

To ensure that solving the dual of CPR-C' is equivalent to solving CPR-C', we need to check that strong duality holds between them. We do this at the end of this subsection.

The dual of CPR-C' is

$$\max - \sum_{i \in N} u_i - \sum_{k \in I} p_k + \sum_{i \in N} z_i - \sum_{(i,j) \in E} y_{i,j} + q_0 + \sum_{i \in N} q_i \quad \text{(CPR-C-DUAL)}$$

subject to
$$v_i(S) - u_i - \sum_{k \in S} p_k + 2z_i \leq 0 \qquad \forall i \in N, S \subseteq I,$$

$$u_i \geq 0 \qquad \forall i \in N,$$

$$p_k \geq 0 \qquad \forall k \in I,$$

$$-Q \text{ is SPSD.}$$

where Q is the symmetric $(n+1) \times (n+1)$-matrix determined as follows:

$$Q_{i,i} = q_i, \qquad \forall i \in N \cup \{0\}$$

$$Q_{0,i} = Q_{i,0} = -z_i + \sum_{j:(i,j) \in E} y_{i,j} + \sum_{j:(j,i) \in E} y_{j,i}, \qquad \forall i \in N$$

$$Q_{i,j} = Q_{j,i} = \begin{cases} 0 & \text{if } (i,j) \notin E \text{ and } (j,i) \notin E \\ y_{i,j} & \text{if } (i,j) \in E \text{ and } (j,i) \notin E \\ y_{j,i} & \text{if } (i,j) \notin E \text{ and } (j,i) \in E \\ y_{i,j} + y_{j,i} & \text{if } (i,j) \in E \text{ and } (j,i) \in E \end{cases}, \quad \forall \text{ distinct } i, j \in N.$$

The final step is to design a poly-time separation oracle.

- If $u_i < 0$ for some $i \in N$ or $p_k < 0$ for some $k \in I$, we have an obvious separation hyperplane.

- Since the dimension of $-Q$ is poly(n), we can use a standard algorithm to check whether it is SPSD in poly(n) time, and obtain a separation hyperplane if $-Q$ is not SPSD. See Oliveira Filho [93, Example 2] for details.

- If $v_i(S) - u_i - \sum_{k \in S} p_k + 2z_i > 0$ for some $i \in N$ and $S \subseteq I$, then we can use the demand oracle of bidder i to find $S = S^*$ that maximizes $v_i(S) - \sum_{k \in S} p_k$. Then $v_i(S^*) - u_i - \sum_{k \in S^*} p_k + 2z_i > 0$, which provides us a separation hyperplane. This is almost identical to the separation oracle used in the ellipsoid method for solving LPR-NC in [92].

Strong duality. While strong duality always holds for LP, it does not always hold for CP. To check strong duality, we verify that the primal program CPR-C' satisfies *Slater's condition* (see Boyd and Vandenberghe [17, Section 5.2.3]), i.e., find a feasible point which satisfies all equality constraints, and *strictly satisfies* the conic constraint. In other words, we need to find a feasible point $(\{x_{i,S}\}, \{\alpha_i\}, \{\beta_k\}, M)$ which satisfies all equality constraints, and such that $x_{i,S} > 0$ for all $i \in N, S \subseteq I$, $\alpha_i > 0$ for all $i \in N$, $\beta_k > 0$ for all $k \in I$, and M is positive *definite*.

Here, we only consider the cases $n \geq 2$; the auction with $n = 1$ bidder is trivial.

Consider the point with $x_{i,S} = \frac{1}{4(2^m-1)n^2}$ for all $i \in N$ and $S \subseteq I$. Then $\alpha_i = 1 - (2^m - 1) \cdot \frac{1}{4(2^m-1)n^2} > 0$ and $\beta_k = 1 - 2^{m-1}n \cdot \frac{1}{4(2^m-1)n^2} > 0$. Also, $\forall i \in N, M_{0,i} = 2(2^m-1) \cdot \frac{1}{4(2^m-1)n^2} - 1 = \frac{1}{2n^2} - 1$, and $\forall i, j \in N$ where $i \neq j$, we choose $M_{i,j} = -1 - M_{0,i} - M_{0,j} = 1 - \frac{1}{n^2}$. Recall that $\forall i \in N, M_{0,0} = M_{i,i} = 1$.

For notational convenience, let $\epsilon = \frac{1}{2n^2}$, i.e., $\forall i \in N, M_{0,i} = \epsilon - 1$; $\forall i, j \in N$ where $i \neq j$, $M_{i,j} = 1 - 2\epsilon$. By Lemma 5.10, to check that M is positive definite, equivalently, we find linearly independent $w_0, w_1, \ldots, w_n \in \mathbb{R}^{n+1}$ such that $\forall i, j \in N \cup \{0\}, w_i \cdot w_j = M_{i,j}$.

Let

$$w_0 = (1, 0, 0, \ldots, 0)$$
$$w_1 = (\epsilon - 1, a_1, 0, 0, \ldots, 0)$$
$$w_2 = (\epsilon - 1, b_1, a_2, 0, 0, \ldots, 0)$$
$$w_3 = (\epsilon - 1, b_1, b_2, a_3, 0, 0, \ldots, 0)$$
$$\vdots$$
$$w_n = (\epsilon - 1, b_1, b_2, b_3, \ldots, b_{n-1}, a_n).$$

Note that $\forall i \in N \cup \{0\}$, w_i has $n - i$ trailing zeroes. Also, $\forall i \in N$, the first entry of w_i is $\epsilon - 1$, followed by $b_1, b_2, \ldots, b_{i-1}$, and then followed by a_i and the trailing

zeroes. This ensures that $w_0 \cdot w_0 = 1 = M_{0,0}$ and $w_0 \cdot w_i = \epsilon - 1 = M_{0,i}$ for all $i \in N$.

We will determine $a_1, b_1, a_2, b_2, a_3, b_3, \ldots, a_{n-1}, b_{n-1}, a_n$ in this order. We will show that for every $i \in N$, $a_i \geq \frac{\sqrt{133}}{12n}$, and b_i's are negative with $|b_i| \leq \frac{1}{3n^3}$.

Since $w_1 \cdot w_1 = 1$, $(\epsilon - 1)^2 + (a_1)^2 = 1$ and thus

$$a_1 = \sqrt{2\epsilon - \epsilon^2} = \sqrt{\frac{1}{n^2} - \frac{1}{4n^4}} = \sqrt{\frac{1}{n^2}\left(1 - \frac{1}{4n^2}\right)} \geq \sqrt{\frac{15}{16n^2}} > \frac{\sqrt{133}}{12n}.$$

Since for $i > 1$, $w_1 \cdot w_i = 1 - 2\epsilon$, $(\epsilon - 1)^2 + a_1 b_1 = 1 - 2\epsilon$. Hence $a_1 b_1 = -\epsilon^2$ and thus b_1 is negative with

$$|b_1| = \epsilon^2/a_1 = \frac{1}{4n^4 a_1} < \frac{3}{\sqrt{133}n^3} < \frac{1}{3n^3}.$$

Next, we proceed by induction. Suppose that for some $q \geq 1$, $a_1, a_2, \ldots, a_q \geq \frac{\sqrt{133}}{12n}$ and $b_1, b_2 \ldots, b_q$ are negative with $|b_1|, |b_2|, \ldots, |b_q| \leq \frac{1}{3n^3}$.

Since $w_{q+1} \cdot w_{q+1} = 1$, $(\epsilon - 1)^2 + \sum_{\ell=1}^{q}(b_\ell)^2 + (a_{q+1})^2 = 1$ and thus

$$a_{q+1} = \sqrt{2\epsilon - \epsilon^2 - \sum_{\ell=1}^{q}(b_\ell)^2} \geq \sqrt{\frac{1}{n^2} - \frac{1}{4n^4} - n \cdot \frac{1}{9n^6}}$$

$$\geq \sqrt{\frac{1}{n^2} - \frac{1}{4n^4} - \frac{1}{18n^4}} = \sqrt{\frac{1}{n^2}\left(1 - \frac{11}{36n^2}\right)} \geq \frac{\sqrt{133}}{12n}.$$

Since for $i > q+1$ holds that $w_{q+1} \cdot w_i = 1 - 2\epsilon$ it follows that $(\epsilon-1)^2 + \sum_{\ell=1}^{q}(b_\ell)^2 + a_{q+1}b_{q+1} = 1 - 2\epsilon$. Hence

$$0 > a_{q+1}b_{q+1} = -\epsilon^2 - \sum_{\ell=1}^{q}(b_\ell)^2 \geq -\frac{1}{4n^4} - n \cdot \frac{1}{9n^6} \geq -\frac{1}{4n^4} - \frac{1}{18n^4} = -\frac{11}{36n^4},$$

and thus b_{q+1} is negative with

$$|b_{q+1}| \leq \frac{11}{36n^4 a_{q+1}} \leq \frac{11}{36n^4} \cdot \frac{12n}{\sqrt{133}} = \frac{11}{3\sqrt{133}n^3} < \frac{1}{3n^3}.$$

This completes the induction.

Since all a_i's are strictly positive, $w_0, w_1, w_2, \ldots, w_n$ are linearly independent.

5.4.5   Analysis

For the analysis we need some notation. For any set N' of bidders, let $Z^*(N') := \sum_{i \in N'} \sum_{S \neq \emptyset} v_i(S) \cdot x_{i,S}^*$. Note that $Z^*(N_0) + Z^*(N_1) + Z^*(N_2) = Z^*$. Given any independent set $J \subseteq N$, let $\mathsf{LPR}(J)$ denote the program LPR-NC with the set of items I and the set of bidders restricted to J. Let $L^*(J)$ denote the optimal value of

LPR(J). Let $x^*(J)$ denote the vector $\{x^*_{i,S}\}_{i \in J, S \subseteq I}$, i.e., it is the vector x^* restricted to bidders in J.

We will show that the expected social welfare attained by $\mathcal{A}_q$ is at least $\Omega((\log \Delta / (\Delta \log \log \Delta)) \cdot Z^*(N_q)$ for $q = 0, 1, 2$. Our analysis uses some results in Halperin [61, Lemma 5.2].

Analysis of Line 4. Constraints (5.6) guarantee that $J_2 = N_2$ is an independent set. Since $x^*(J_2)$ is a feasible point of LPR(J_2), $L^*(J_2) \geq Z^*(J_2) = Z^*(N_2)$. The fair contention resolution algorithm gives an allocation which is at least $\left(1 - \frac{1}{e}\right) L^*(J_2)$ $\geq \left(1 - \frac{1}{e}\right) Z^*(N_2)$ in expectation.

Analysis of Line 6. We note that the selection of an independent set J_1 is identical to the corresponding part in Halperin's algorithm, modulo that since we are dealing with a directed graph, we can remove fewer bidders from N'_1.[2] Thus, we can follow closely Halperin's analysis to show that

$$\mathbf{E}\left[Z^*(J_1)\right] = \Omega\left(\frac{\Delta^{2\tau}}{\Delta \sqrt{\log \Delta}}\right) \cdot Z^*(N_1) = \Omega\left(\frac{\log \Delta}{\Delta \log \log \Delta}\right) \cdot Z^*(N_1).$$

Since $x^*(J_1)$ is a feasible point of LPR(J_1), $L^*(J_1) \geq Z^*(J_1)$. The fair contention resolution algorithm gives an allocation which is at least $\left(1 - \frac{1}{e}\right) L^*(J_1)$ in expectation. Thus, the expected social welfare attained by $\mathcal{A}_1$ is at least $\left(1 - \frac{1}{e}\right) \mathbf{E}\left[L^*(J_1)\right] = \Omega\left(\frac{\log \Delta}{\Delta \log \log \Delta}\right) \cdot Z^*(N_1)$.

Analysis of Line 11. Observe that $2\tau\Delta > 1$ for sufficiently large Δ, so the vector q, which contains $\{q_{i,S}\}_{i \in N_0, S \subseteq I}$, is a feasible point of LPR(N_0). The sets $\{T_i\}_{i \in N_0}$ are generated by the fair contention resolution algorithm w.r.t. the vector q, so $\mathbf{E}\left[v_i(T_i)\right]$, the expected welfare of bidder i (modulo conflicts), is at least $\left(1 - \frac{1}{e}\right) \frac{Z^*(\{i\})}{2\tau\Delta}$.

To handle conflicts, the algorithm resets the allocation of some bidders to the empty set. We will show that for each bidder $i \in N_0$, at least half of his expected welfare (modulo conflicts) is retained after conflict handling.

For every $i \in N_0$, let F_i be the event that $\forall j$ with $(i, j) \in E$, $X_j = \emptyset$. Then $\bar{F}_i$ is the event: $\exists j$ such that $(i, j) \in E$ and $X_j \neq \emptyset$. We note that before conflict handling, for all $i \in N_0$, $\mathbf{E}\left[v_i(T_i) \mid F_i\right] \geq \mathbf{E}\left[v_i(T_i) \mid \bar{F}_i\right]$. We will prove the above inequality formally in Lemma 5.11, but it is indeed intuitive. The target set of each bidder is the set of items the bidder is competing for. Thus, the above inequality depicts that bidder i gets more when facing less competition from bidders he conflicts with.

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[2]Halperin's analysis is for undirected graphs, but his proof can be reused for directed graphs with little modification. In [61, Lemma 5.2], if there is an edge between vertices in N'_1, both vertices of the edge are removed from N'_1. For directed graphs it suffices to remove the outgoing vertex only, so the bound provided in the lemma is also applicable.

Lemma 5.11. *Let X_i, X_j, T_i be defined as in line 11 of Algorithm 5.4. For every $i \in N_0$, let F_i be the event that $\forall j$ with $(i, j) \in E$, $X_j = \emptyset$. Then $\mathbf{E}\left[v_i(T_i) \mid F_i\right] \geq \mathbf{E}\left[v_i(T_i) \mid \bar{F}_i\right]$.*

Proof. Let C_i be the set of bidders j with $(i, j) \in E$. Fix the target sets of bidders who are not in C_i. For an item $k \in X_i$ and every $i' \in N_0$, let $p_{i'}(k) := \sum_{S:k \in S} x^*_{i',S}$. Let $A(k) := \{i' \mid k \in X_{i'}\}$.

In the fair contention resolution algorithm (see [40, Section 1.2]), if $|A(k)| = 1$, i.e., $A(k) = \{i\}$, then $\mathbf{Pr}\left[k \in T_i\right] = 1$; if $|A(k)| > 1$, then

$$\mathbf{Pr}\left[k \in T_i\right] = \frac{1}{\sum_{i' \in N_0} p_{i'}(k)} \left(\sum_{i' \in A(k) \setminus \{i\}} \frac{p_{i'}(k)}{|A(k)| - 1} + \sum_{i' \notin A(k)} \frac{p_{i'}(k)}{|A(k)|} \right).$$

Recall that we are fixing the target sets of bidders who are not in C_i. If F_i holds, then $\forall j \in C_i$, $j \notin A(k)$. However, if $\bar{F}_i$ holds, some bidders in C_i may get into $A(k)$, i.e.,

$$(|A(k)| \text{ when } F_i \text{ holds}) \leq (|A(k)| \text{ when } \bar{F}_i \text{ holds})$$

no matter what target sets the bidders in C_i choose. Then it follows that

$$\mathbf{Pr}\left[k \in T_i \mid F_i\right] \geq \mathbf{Pr}\left[k \in T_i \mid \bar{F}_i\right].$$

As each item is allocated independently, and each item in X_i is allocated to bidder i with higher probability when F_i holds, the lemma follows. □

Note that $\mathbf{E}\left[v_i(T_i)\right] = \mathbf{E}\left[v_i(T_i) \mid F_i\right] \cdot \mathbf{Pr}\left[F_i\right] + \mathbf{E}\left[v_i(T_i) \mid \bar{F}_i\right] \cdot \mathbf{Pr}\left[\bar{F}_i\right]$, i.e., $\mathbf{E}\left[v_i(T_i)\right]$ is a weighted average of the two conditional expectations. Since the first conditional expectation is larger than the second one,

$$\mathbf{E}\left[v_i(T_i) \mid F_i\right] \geq \mathbf{E}\left[v_i(T_i)\right] \geq \left(1 - \frac{1}{e}\right) \frac{Z^*(\{i\})}{2\tau\Delta}.$$

Next, note that $\mathbf{Pr}\left[F_i\right] = 1 - \mathbf{Pr}\left[\bar{F}_i\right] = 1 - \mathbf{Pr}\left[\bigcup_{j:(i,j) \in E} (X_j \neq \emptyset)\right]$ is at least

$$1 - \sum_{j:(i,j) \in E} \mathbf{Pr}\left[X_j \neq \emptyset\right] = 1 - \sum_{j:(i,j) \in E} \frac{1}{2\tau\Delta} \sum_{S \neq \emptyset} x^*_{j,S} \geq 1 - \sum_{j:(i,j) \in E} \frac{1}{2\tau\Delta} \cdot \tau \geq 1/2,$$

since i conflicts with at most Δ bidders.

Bidder i's allocation is set to $\emptyset$ during conflict handling only if $\bar{F}_i$ holds. By the last two paragraphs, the expected welfare of bidder i after conflict handling is at least

$$\mathbf{E}\left[v_i(T_i) \mid F_i\right] \cdot \mathbf{Pr}\left[F_i\right] + 0 \cdot \mathbf{Pr}\left[\bar{F}_i\right] \geq \left(1 - \frac{1}{e}\right) \frac{Z^*(\{i\})}{2\tau\Delta} \cdot \frac{1}{2}$$

$$= \Omega\left(\frac{\log\Delta}{\Delta \log\log\Delta}\right) \cdot Z^*(\{i\}). \quad (5.8)$$

Then the expected social welfare is at least $\sum_{i \in N_0} \Omega\left(\frac{\log\Delta}{\Delta \log\log\Delta}\right) \cdot Z^*(\{i\}) = \Omega\left(\frac{\log\Delta}{\Delta \log\log\Delta}\right) \cdot Z^*(N_0)$.

Analysis of Line 16. The final step is to choose the best allocation among $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_2$, which, by the analysis of the previous three steps, is at least

$$\frac{1}{3}\left[\left(1 - \frac{1}{e}\right) Z^*(N_2) + \Omega\left(\frac{\log \Delta}{\Delta \log\log \Delta}\right) \cdot (Z^*(N_1) + Z^*(N_0))\right]$$

$$= \Omega\left(\frac{\log \Delta}{\Delta \log\log \Delta}\right) \cdot Z^*. \quad (5.9)$$

This concludes the analysis of the algorithm. Observe that we obtain the following proposition as a notable special case.

Proposition 5.12. *There is a poly-time $\mathcal{O}\left(\Delta \cdot \frac{\log\log \Delta}{\log \Delta}\right)$-approximation algorithm for the WIS problem in a directed graph G with out-degree at most Δ.*

Remark. By Theorem 5.2, we have an $\mathcal{O}\left(\min\{\Delta, \Delta_I\}\right)$-approximation algorithm for combinatorial auctions with bidder and item conflicts, in which bidders have FSA valuations. An interesting open problem is whether we can improve the approximation guarantee to be sublinear in $\min\{\Delta, \Delta_I\}$.

5.5 Sponsored Search with Limited Number of Slots

In this section we consider sponsored search with bidder conflicts in more detail. Some of our results extend to ordered conflicts and more general graph-based slot conflicts. In light of the application, we focus on the case with a small number m of slots. Note that a trivial enumeration solves the problem in time $\mathcal{O}(n^m)$. Moreover, it is unlikely that significantly faster algorithms exist that solve the problem exactly, even for $m \leq \log n$; it is W[1]-hard to decide Log-Independent-Set, i.e., given $k \leq \log n$, deciding if G has an independent set of size at least k cannot be done in time $f(k) \cdot n^c$ for constant c unless FPT = W[1] (see Downey and Fellows [34]). Thus, we present two approximation algorithms. The first one uses semidefinite programming and has polynomial running time for $m \in \mathcal{O}(\log n)$. The second one is a partial enumeration approach and runs in polynomial time if, in addition, $m \in \mathcal{O}((\log n)/(\log\min\{\Delta + 1, \log n\}))$.

5.5.1 Sponsored Search via Semidefinite Programming

We study sponsored search with bidder conflicts and $m \in \mathcal{O}(\log n)$. We assume for simplicity that $n \geq m \geq 6$. Note that, if $m > n$, we could add $(m - n)$ dummy bidders with valuation zero. Additionally, we assume consistent tie-breaking among the bidders with the same valuation. Recall that in this setting bidders have unit demand, and thus we can represent an allocation X of slots to bidders by a matching M_X in a bipartite bidder-slot-graph. We define $v_i(M_X) = v_i(X_i)$ for all $i \in N$. We call a matching M_X *conflict-free* if $D_i \cap X_i = \emptyset$ for all $i \in N$. Note that for every matching there exists a conflict-free matching with the same social welfare; we

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Algorithm 5.5: Sponsored search auction with conflicts

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1. Assign all bidders in N independently with probability $1/2$ to set N_1 and set $N_2 \leftarrow N \setminus N_1$. With $v_1 \geq v_2 \geq \cdots \geq v_n$ and $h = |N_1|$ define the functions $\phi : \{1, \ldots, h\} \to \{1, \ldots, n\}$ and $\chi : \{1, \ldots, n - h\} \to \{1, \ldots, n\}$ such that $N_1 = \{\phi(1), \ldots, \phi(h)\}$ and $\phi(j) < \phi(j + 1)$ for $j \in \{1, \ldots, h - 1\}$ and $N_2 = \{\chi(1), \ldots, \chi(n - h)\}$ and $\chi(j) < \chi(j + 1)$ for $j \in \{1, \ldots, n - h - 1\}$
2. Set $q \leftarrow 1$ with probability $\frac{1}{2}$ and set $q \leftarrow 2$ otherwise
3. **if** $n - h \geq \lceil \frac{m}{4} \rceil + 1$ **then** $t \leftarrow \chi(\lceil \frac{m}{4} \rceil + 1)$ **else** $t \leftarrow \infty$ and $v_t \leftarrow 0$
4. **if** $q = 1$ **then**
5.   Set $r_1 \leftarrow v_t$
6.   Set $N_1^1 \leftarrow \{\phi(j) | j \in \{1, \ldots, h\}$ and $\phi(j) < t\}$
7.   **if** $t \leq m + 1$ **then** set $\mathcal{A}$ to the set of all subsets of N_1^1 **else** $\mathcal{A} \leftarrow \emptyset$
8. **else**
9.   Set $r_2 \leftarrow v_t \cdot \frac{1}{8} R(\Delta)$
10.   Set $N_1^2 \leftarrow \{\phi(j) | j \in \{1, \ldots, h\}$ and $v_{\phi(j)} \geq r_2\}$
11.   Set $S \leftarrow$ independent set in N_1^2 computed by using the WIS algorithm described in Proposition 5.12 giving bidders in N_1^2 in random order and with equal weights
12.   Set $\mathcal{A} \leftarrow \{S\}$
13. Add m dummy-bidders without conflicts and with valuation r_q to each set in $\mathcal{A}$ and to N
14. For each set $A \in \mathcal{A}$ let $\mathcal{M}(A)$ define all the conflict-free matchings of bidders in A to slots; define $\mathcal{M} = \bigcup_{A \in \mathcal{A}} \mathcal{M}(A)$
15. Select allocation $M' \in \arg\max_{M \in \mathcal{M}} \sum_{i \in N} v_i(M)$
16. Every real-bidder a in N pays $p_a \leftarrow \max_{M \in \mathcal{M}} \sum_{i \in N \setminus \{a\}} (v_i(M) - v_i(M'))$

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simply unassign all the slots in $\bigcup_{i \in N} D_i \cap X_i$. Furthermore, we define the expected social welfare $SW(M) := \mathbf{E}\left[\sum_{i \in N} v_i(M)\right]$ for a (randomized) matching M. In the following, we also use the notation $R(\Delta) = \sqrt{\log\log \Delta / \log \Delta}$.

The auction mechanism is presented in Algorithm 5.5 and its approximation guarantee is analyzed in Theorem 5.13.

Theorem 5.13. *The matching M' computed in Algorithm 5.5 is in expectation an $\mathcal{O}(\Delta \cdot R(\Delta))$-approximation of the optimal social-welfare.*

We show for the random index t defined in Algorithm 5.5, that if the optimal conflict-free assignment of bidders to slots OPT was restricted to a random subset OPT$''$ of the $t - 1$ most valuable edges, where each of those edges is picked with probability $1/2$ if $t \leq m + 1$ and is discarded otherwise, then $SW(\text{OPT}'') \geq SW(\text{OPT})/16$. Thus, it suffices to compare the performance of a mechanism with OPT$''$. We run two different algorithms, Algorithm 1 and Algorithm 2, each with probability $1/2$, and receive at least $1/2$ of the maximum of their social welfares SW_1 and SW_2, respectively.

If Algorithm 1 performs very well, i.e., if $SW_1 > SW(\text{OPT}'')/(\Delta R(\Delta))$, we achieve the result promised in Theorem 5.13. Algorithm 1 tries out all possibilities to find the best non-conflicting matching for bidders in N_1^1. If Algorithm 1 does not perform very well, we can show that OPT'' must get at least a quarter of its social welfare from bidders in $N_1^2 \setminus N_1^1$. In this case, we build an (unweighted) independent set S of all bidders in N_1^2 using the WIS algorithm described in Proposition 5.12, which guarantees that the number of bidders in S is at least an $\mathcal{O}(1/(\Delta R(\Delta)^2))$-fraction of the optimal number for bidders in $N_1^2 \setminus N_1^1$. As in OPT'' every bidder in $N_1^2 \setminus N_1^1$ contributes at most with valuation r_1 to $SW(\text{OPT}'')$ and in Algorithm 2 every bidder in S contributes at least with valuation r_2 to SW_2, the overall approximation ratio of Algorithm 2 is $\mathcal{O}(\Delta R(\Delta)^2 \cdot r_1/r_2) = \mathcal{O}(\Delta R(\Delta))$. We show first, in Lemma 5.14, that the condition in line 7 holds with probability at least $3/4$.

Lemma 5.14. *It holds that* $\mathbf{Pr}\left[t \leq m + 1\right] \geq 3/4$.

Proof. Note that $t \leq m + 1$ if and only if $|N_2 \cap \{1, \ldots, m+1\}| \geq \lceil \frac{m}{4} \rceil + 1$. This happens with probability

$$1 - \frac{1}{2^{m+1}} \sum_{\ell=0}^{\lceil \frac{m}{4} \rceil} \binom{m+1}{\ell},$$

which is at least $3/4$ when $m \geq 6$. $\qquad\square$

Proof of Theorem 5.13. Assume that the social-welfare-maximizing conflict-free matching of bidders in N to slots is given by OPT. The valuation of the dummy bidders will not be considered in the social welfare as they were only included to guarantee incentive compatibility.

We first analyze the random partition of N into N_1 and N_2 by the mechanism. Let m^* be the number of edges in OPT and let us denote those edges by $(i(1), j(1)), \ldots, (i(m^*), j(m^*))$ such that they are ordered by their value, i.e., $v_{i(1)} \cdot \alpha_{j(1)} \geq \cdots \geq v_{i(m^*)} \cdot \alpha_{j(m^*)}$. Let OPT' be the random subset of OPT where all the edges but the $t-1$ most valuable ones are discarded, i.e., $\text{OPT}' = \{(i(1), j(1)), \ldots, (i(t-1), j(t-1))\}$. Furthermore, let OPT'' be the random subset of OPT' where (1) all the edges that contain bidders in N_2 are discarded and (2) if $t > m+1$ all edges are discarded. We will show that $SW(\text{OPT}) \leq 16 \cdot SW(\text{OPT}'')$. Since, $N_2 \subseteq N$ it holds that $t \geq \lceil m/4 \rceil + 1 \geq m/4 + 1$, and thus, it follows by $m^* \leq m$ that

$$\frac{SW(\text{OPT})}{SW(\text{OPT}')} = \frac{\sum_{s=1}^{t-1} v_{i(s)} \cdot \alpha_{j(s)} + \sum_{s=t}^{m^*} v_{i(s)} \cdot \alpha_{j(s)}}{\sum_{s=1}^{t-1} v_{i(s)} \cdot \alpha_{j(s)}}$$

$$\leq 1 + \frac{\sum_{s=t}^{m^*} v_{i(t)} \cdot \alpha_{j(t)}}{\sum_{s=1}^{t-1} v_{i(t)} \cdot \alpha_{j(t)}} = \frac{m^*}{t-1} \leq 4. \quad (5.10)$$

Now, for all $i \in N$ let E_i be the event that bidder i is not in N_1 and let T be the event that $t > m+1$. By Lemma 5.14, it holds for each bidder $i \in N$ that

$\mathbf{Pr}\left[E_i \cup T\right] \leq \mathbf{Pr}\left[E_i\right] + \mathbf{Pr}\left[T\right] \leq 1/2 + 1/4 = 3/4$. Thus, $SW(\mathrm{OPT}'') = \sum_{s=1}^{t-1} v_{i(s)} \cdot \alpha_{j(s)} \cdot (1 - \mathbf{Pr}\left[E_{i(s)} \cup T\right]) \geq (1/4) \cdot SW(\mathrm{OPT}')$. It follows that $SW(\mathrm{OPT}'') \geq (1/16) \cdot SW(\mathrm{OPT})$.

We will now compare the outcome M' of the mechanism with OPT''. Let M_1 or M_2 be the matching computed by the mechanism under the condition $q = 1$ or $q = 2$, respectively. It holds that $2 \cdot SW(M') \geq \max\{SW(M_1), SW(M_2)\}$. Then the following claim completes the proof.

Claim 1. *For some constant $c > 1$ it holds that*

$$c \cdot \Delta R(\Delta) \cdot \max\{SW(M_1), SW(M_2)\} \geq SW(\mathrm{OPT}'').　\qquad (5.11)$$

Proof. Notice that if $SW(\mathrm{OPT}'') < \max\{4, \Delta R(\Delta)\} \cdot SW(M_1)$ then (5.11) is satisfied. Thus, we assume that $SW(\mathrm{OPT}'') \geq \max\{4, \Delta R(\Delta)\} \cdot SW(M_1)$. Moreover, we assume that $t \leq m + 1$, as otherwise, $SW(\mathrm{OPT}'') = 0$.

Next, we define by SW_u the optimal social welfare for bidders in N_1^1 when bidder conflicts are ignored. Furthermore, note that Theorem 5.2 implies that $4\Delta \cdot SW(M_1) \geq SW_u$. Thus,

$$\begin{aligned} SW(\mathrm{OPT}'') &\geq \Delta R(\Delta) \cdot SW(M_1) \\ &\geq \Delta R(\Delta) \cdot SW_u / (4\Delta) \geq (R(\Delta)/4) \cdot SW_u \end{aligned} \quad (5.12)$$

Let us now partition the matching OPT'' into OPT_1 that contains the edges to bidders in N_1^1, OPT_2 that contains the edges to bidders in $N_1^2 \setminus N_1^1$, and $\mathrm{OPT}_3 := \mathrm{OPT}'' \setminus (\mathrm{OPT}_1 \cup \mathrm{OPT}_2)$. Thus, $SW(\mathrm{OPT}'') = SW(\mathrm{OPT}_1) + SW(\mathrm{OPT}_2) + SW(\mathrm{OPT}_3)$.

As the matching OPT_1 is considered when computing M_1, $SW(M_1) \geq SW(\mathrm{OPT}_1)$, and thus, by the assumption taken above holds that $SW(\mathrm{OPT}_1) \leq SW(M_1) \leq SW(\mathrm{OPT}'') / \max\{4, \Delta R(\Delta)\} \leq SW(\mathrm{OPT}'')/4$.

Furthermore, $SW(\mathrm{OPT}_3) \leq SW(\mathrm{OPT}'')/2$, as otherwise,

$$SW(\mathrm{OPT}'') < 2 \cdot SW(\mathrm{OPT}_3) < 2 \cdot r_2 \cdot \sum_{j=1}^{t-1} \alpha_j = 2 \cdot \frac{1}{8} \cdot R(\Delta) \cdot v_t \cdot \sum_{j=1}^{t-1} \alpha_j \leq \frac{R(\Delta)}{4} SW_u,$$

which contradicts (5.12). Hence, $SW(\mathrm{OPT}_2) \geq SW(\mathrm{OPT}'')/4$. It follows that

$$\begin{aligned} \frac{SW(\mathrm{OPT}_2)}{SW(M_2)} &= \frac{\sum_{(i,j) \in \mathrm{OPT}_2} v_i \cdot \alpha_j}{\sum_{(i,j) \in M_2} v_i \cdot \alpha_j} \leq \frac{r_1 \cdot \sum_{j=1}^{|\mathrm{OPT}_2|} \alpha_j}{r_2 \cdot \sum_{j=1}^{|M_2|} \alpha_j} \\ &\leq \frac{8}{R(\Delta)} \cdot \left(1 + \frac{\sum_{j=|M_2|+1}^{|\mathrm{OPT}_2|} \alpha_{|M_2|}}{\sum_{j=1}^{|M_2|} \alpha_{|M_2|}}\right) = \frac{8}{R(\Delta)} \cdot \frac{|\mathrm{OPT}_2|}{|M_2|} \\ &\leq \frac{8}{R(\Delta)} \cdot c' \cdot (\Delta \cdot R(\Delta)^2) = 8c' \cdot \Delta \cdot R(\Delta), \quad (5.13) \end{aligned}$$

where $c' \Delta R(\Delta)^2$ is the approximation factor of Proposition 5.12. $\qquad \square$

□

We show that the runtime of the mechanism is polynomial in n and Δ for certain restrictions on the number of slots m, and that the mechanism is in the universal sense incentive compatible.

Proposition 5.15. *If $m \in \mathcal{O}(\log n)$ the mechanism takes time $\mathrm{poly}(n, \Delta)$.*

Proof. We have to show that line 15 and 16 can be computed in polynomial time in n and Δ.

We first argue that $|\mathcal{A}|$ is polynomial in n. Consider the case where $q = 1$. If $t > m + 1$ then $\mathcal{A} = \emptyset$; otherwise, $\mathcal{A} = \mathcal{P}(N_1^1)$ and $|N_1^1| < t \le m + 1$. Moreover, if $q = 2$ then $|\mathcal{A}| = 1$. Thus, $|\mathcal{A}|$ is bounded by 2^m which is polynomial in n.

Next, assume that we are given some $A \in \mathcal{A}$. Given $q = 1$, we can ignore A if bidders in A have conflicts, because we know that there exists a conflict-free set of bidders in $\mathcal{A}$ that is optimal. Moreover, given $q = 2$ we know that the bidders in all sets in $\mathcal{A}$ have no conflicts. Thus, we can assume that the bidders in A are conflict-free. It follows that computing $\arg\max_{M \in \mathcal{M}(A)} \sum_{i \in C} v_i(M)$ can be done in polynomial time in n for all $C \subseteq N$; for all $i \in \{1, \dots, m\}$ the bidder in A with the i-th largest index has to be matched to the i-th slot. □

The crucial idea for showing incentive compatibility is to prove that no bidder has an incentive to alter the set of matchings $\mathcal{M}$.

Lemma 5.16. *No bidder has an incentive to report a non-truthful bid that alters $\mathcal{M}$.*

Proof. We can restrict the proof to bidders in N_1 as the other bidders always have utility zero. Assume that all bidders bid truthful. In both cases, $q = 1$ and $q = 2$, bidders not in N_1^q are in no matching in $\mathcal{M}$, and thus, they are not in M' and their utility is zero. However, they have no incentive to increase their bid because there are m competing dummy-bidders that have a valuation that is at least the same as theirs. Thus, if they increase their valuation, their utility cannot increase because they have to pay their externality. Furthermore, in both cases, $q = 1$ and $q = 2$, bidders in N_1^q have two possibilities: (i) Bidding high enough to stay in N_1^q and (ii) bidding below the value that is necessary for staying in N_1^q. In (i), if a bidder bids high enough to stay in N_1^q, he cannot affect N_1^q. Moreover, he cannot affect the outcome of the WIS algorithm by his bid because we randomized the order of the bidders. Thus, he cannot influence whether he belongs to a subset in $\mathcal{A}$ and, in turn, he cannot influence $\mathcal{M}$. In (ii), if a bidder bids below the value that is necessary for staying in N_1^q, he will receive nothing and has utility zero. Thus, no bidder has an incentive to change his bid if this alters $\mathcal{M}$. □

Thus, even though the range of allocations $\mathcal{M}$ depends on the valuations of the bidders (i.e., this is no maximal-in-range algorithm) no bidder has an incentive to change $\mathcal{M}$.

Lemma 5.17. *The mechanism is in the universal sense incentive compatible.*

Proof. We can assume that all random decision are taken before the bidders report their bids. We first fix a bidder a. Since by Lemma 5.16 bidder a has no incentive to report a non-truthful bid that changes $\mathcal{M}$ we can restrict the analysis to bidder a's non-truthful bids that do not change $\mathcal{M}$. Thus, we can consider $\mathcal{M}$ as fixed. The utility of bidder a for a matching M' is given by $u_a(M') = v_a(M') - (\max_{M \in \mathcal{M}} \sum_{i \in N \setminus \{a\}} (v_i(M) - v_i(M'))) = \sum_{i \in N} v_i(M') - \max_{M \in \mathcal{M}} \sum_{i \in N \setminus \{a\}} v_i(M)$ which is maximized when a bids truthful. $\square$

The following theorem summarizes the results in this section.

Theorem 5.18. *For sponsored search with bidder conflicts and $m \in \mathcal{O}(\log n)$, Algorithm 5.5 is an in the universal sense incentive compatible mechanism that yields an $\mathcal{O}\left(\Delta \sqrt{\frac{\log \log \Delta}{\log \Delta}}\right)$-approximation of social welfare and runs in time $\mathrm{poly}(n, \Delta)$.*

Proof. The theorem follows from Theorem 5.13, Proposition 5.15, and Lemma 5.17. $\square$

5.5.2 Sponsored Search via Partial Enumeration

We treat a slightly more general *small-supply* case with $m \leq n/(\Delta + 1)$. For this case we observe that the problem can be solved optimally in linear time when all bidders i have uniform values $v_i = v$. For non-uniform values v_i, we will strive for an incentive compatible mechanism that solves the problem approximately but much faster than the trivial enumeration that solves the problem exactly in $\mathcal{O}(n^m)$ time. Note that there is an m-approximation algorithm that assigns slot 1 to the highest bidder, obtains value $\max_{k,i} \alpha_k \cdot v_i$, and runs in time $\mathcal{O}(n)$. Thus, we obtain the following trade-off.

Theorem 5.19. *In sponsored search with bidder and slot conflicts, there is an in the universal sense incentive compatible mechanism that yields an $\mathcal{O}(\log m)$-approximation of social welfare and runs in time $\mathcal{O}(n + (m(\Delta + 1))^m)$.*

We first prove the existence of an approximation algorithm achieving the claimed approximation ratio.

Proposition 5.20. *In sponsored search with bidder conflicts, there is an $\mathcal{O}(\log m)$-approximation algorithm that runs in time $\mathcal{O}(n + (m(\Delta + 1))^m)$.*

Proof. The algorithm is extremely simple for uniform values $v_i = v$ for all $i \in N$ if $m \leq n/(\Delta + 1)$. Initially, every bidder is active. We assign slot 1 to the bidder i with smallest out-degree, label i and its all out-neighbors to be inactive. We repeat this procedure with slots $2, 3, \ldots, m$. Since $m \leq n/(\Delta + 1)$, we will be able to assign all slots in this way. This yields an optimum solution and takes time $\mathcal{O}(n)$. If the v_i are different, we apply logarithmic scaling. Let $v_{\max} = \max_{i \in N} v_i$. We consider $\lceil \log_2(2m) \rceil$ classes, where class k contains bidders i with value $v_i \in (v_{\max}/2^k, v_{\max}/2^{k-1}]$. The unclassified bidders have a value which is at most $v_i \leq$

$v_{\max}/(2m)$. Thus, by discarding this set of bidders, we discard at most $1/2$ of the optimum value.

For the remaining bidders, we pick $k \in \{1, 2, \ldots, \lceil \log_2(2m) \rceil\}$ uniformly at random and consider $V_k = \{i \in N \mid v_i > v_{\max}/2^k\}$, the union of all bidders in classes $1, \ldots, k$. Let $n_k = |V_k|$. If $n_k/(\Delta + 1) \geq m$, then we can apply the above algorithm for identical values to V_k. Otherwise, if $n_k/(\Delta + 1) \leq m$, then $n_k \leq (\Delta + 1)m$, and a complete enumeration takes time at most $\mathcal{O}((m(\Delta + 1))^m)$. In either case, we obtain the optimum for V_k under the assumption that every bidder has value $v_{\max}/2^k$, and hence at least half of the value that the optimum gets from bidders in class k. In expectation over the random choice of k, this shows that we recover an $\mathcal{O}(\log m)$-fraction of the optimum.

The highest valuation can be found in time $\mathcal{O}(n)$. Computing the threshold and reducing the set of considered bidders can be done in time $\mathcal{O}(n)$. Applying the previous algorithm can be done in time $\mathcal{O}(n)$, and the enumeration takes time $\mathcal{O}((m(\Delta + 1))^m)$. $\square$

Note that for a particular choice of k, the algorithm described in the proof of Proposition 5.20 is applied in the induced subgraph of V_k and produces an optimum solution under the assumption that all nodes have the same valuation. If this results from the greedy algorithm for the independent set of bidders, it also remains an optimum solution with arbitrary additional slot conflicts. If this results from enumeration, we can apply the enumeration also for additional slot conflicts in the same asymptotic running time. Thus, we obtain the same running time and approximation ratio also for sponsored search with bidder and slot conflicts.

By the sampling arguments in Dobzinski et al. [31] and Hoefer and Kesselheim [65] we can turn the algorithm into an in the universal sense incentive compatible mechanism with the same asymptotic running time and approximation ratio. The idea is as follows. First, choose a random bit q. If $q = 0$, partition N into N_1 and N_2 randomly and set $v_{\max}$ be the highest valuation in N_1. However, we run the algorithm in Proposition 5.20 on N_2 only; if a bidder $i \in N_2$ gets assigned slot ℓ he has to pay $\alpha_\ell \cdot v_{\max}/2^k$. If $q = 1$, we keep the best slot and remove all others, and run a second price auction among all bidders in N. This ensures that the claimed approximation ratio even if there is a dominant bidder, i.e., a bidder who contributes at least a constant fraction of the optimal social welfare. This completes the proof of Theorem 5.19.

5.6 Conclusion

We introduced models for goods with negative graph-based externalities and designed combinatorial auctions for selling the goods. Our work lead to an algorithm for approximating the optimal social welfare that has an approximation ratio that is sublinear in the out-degree Δ of the conflict graph. Furthermore, we designed incentive compatible mechanisms having an approximation ratio that is linear in Δ. For sponsored search, we showed incentive compatible mechanisms under the

restriction that the number of slots is small compared to the number of bidders. In particular, we showed an incentive compatible mechanism with approximation ratio sublinear in Δ when $m \in \mathcal{O}(\log n)$. This result depends strongly on the single-dimensionality of the problem, which enabled us to select independent sets without considering the valuations of the bidders.

One obvious question remains open for future research: Does an incentive compatible combinatorial auction with an approximation ratio sublinear in Δ that takes time $\mathcal{O}(n)$ exist? Note that the answer for this question is unknown even for the special case of sponsored search that we studied in Section 5.5.1 if we assume that the number of items is not in $\mathcal{O}(\log n)$. Moreover, the answer is unknown for combinatorial auctions even if we assume that the number of items m is in $\mathcal{O}(\log n)$ as we do in Section 5.5.1. Beside this, we consider the study of auctions for goods with positive and negative externalities as an interesting future research direction.

Bibliography

[1] Gagan Aggarwal, Jon Feldman, S. Muthukrishnan, and Martin Pál. "Sponsored Search Auctions with Markovian Users". In: *Proceedings of the 4th Workshop on Internet & Network Economics (WINE)*. Vol. 5385. LNCS. Springer, 2008, pp. 621–628. DOI: 10.1007/978-3-540-92185-1_68.

[2] Gagan Aggarwal, S. Muthukrishnan, Dávid Pál, and Martin Pál. "General Auction Mechanism for Search Advertising". In: *Proceedings of the 18th International World Wide Web Conference (WWW)*. ACM, 2009, pp. 241–250. DOI: 10.1145/1526709.1526742.

[3] Aaron Archer and Éva Tardos. "Truthful Mechanisms for One-Parameter Agents". In: *Proceedings of the 42nd Annual IEEE Symposium on Foundations of Computer Science (FOCS)*. IEEE Computer Society, 2001, pp. 482–491. DOI: 10.1109/SFCS.2001.959924.

[4] Itai Ashlagi, Mark Braverman, Avinatan Hassidim, Ron Lavi, and Moshe Tennenholtz. "Position Auctions with Budgets: Existence and Uniqueness". In: *The B.E. Journal of Theoretical Economics* 10.1 (2010). DOI: 10.2202/1935-1704.1648.

[5] Lawrence M. Ausubel. "An Efficient Ascending-Bid Auction for Multiple Objects". In: *The American Economic Review* 94.5 (2004), pp. 1452–1475. DOI: 10.1257/0002828043052330.

[6] Lawrence M. Ausubel. "An Efficient Dynamic Auction for Heterogeneous Commodities". In: *The American Economic Review* 96.3 (2006), pp. 602–629. DOI: 10.1257/aer.96.3.602.

[7] Lawrence M. Ausubel and Paul Milgrom. "The Lovely but Lonely Vickrey Auction". In: *Combinatorial Auctions*. Ed. by Peter Cramton, Yoav Shoham, and Richard Steinberg. Cambridge, Massachusetts: MIT Press, 2006. Chap. 1, pp. 17–40. ISBN: 0-262-03342-9.

[8] Lawrence M. Ausubel and Paul R. Milgrom. "Ascending Auctions with Package Bidding". In: *Advances in Theoretical Economics* 1.1 (2002). DOI: 10.2202/1534-5963.1019.

[9]   Moshe Babaioff, Brendan Lucier, Noam Nisan, and Renato Paes Leme. "On the efficiency of the walrasian mechanism". In: *Proceedings of the 15th ACM Conference on Economics and Computation (EC)*. ACM, 2014, pp. 783–800. DOI: 10.1145/2600057.2602850.

[10]   Michael Benisch, Norman Sadeh, and Tuomas Sandholm. "A theory of expressiveness in mechanisms". In: *Proceedings of the 23th Conference Artificial Intelligence (AAAI)*. AAAI Press, 2008.

[11]   Sayan Bhattacharya, Vincent Conitzer, Kamesh Munagala, and Lirong Xia. "Incentive Compatible Budget Elicitation in Multi-unit Auctions". In: *Proceedings of the 21th ACM-SIAM Symposium on Discrete Algorithms (SODA)*. SIAM, 2010, pp. 554–572. DOI: 10.1137/1.9781611973075.47.

[12]   Kshipra Bhawalkar and Tim Roughgarden. "Welfare Guarantees for Combinatorial Auctions with Item Bidding". In: *Proceedings of the 22nd ACM-SIAM Symposium on Discrete Algorithms (SODA)*. SIAM, 2011, pp. 700–709. DOI: 10.1137/1.9781611973082.55.

[13]   Sushil Bikhchandani, Sven de Vries, James Schummer, and Rakesh V. Vohra. "Ascending Auctions for Integral (Poly)Matroids with Concave Nondecreasing Separable Values". In: *Proceedings of the 19th ACM-SIAM Symposium on Discrete Algorithms (SODA)*. SIAM, 2008, pp. 864–873.

[14]   Sushil Bikhchandani, Ron Lavi, Ahuva Mu, Noam Nisan, and Arunava Sen. "Weak Monotonicity Characterizes Deterministic Dominant-Strategy Implementation". In: *Econometrica* 74.4 (2006), pp. 1109–1132. DOI: 10.1111/j.1468-0262.2006.00695.x.

[15]   Avrim Blum and Yishay Mansour. "Learning, Regret Minimization, and Equilibria". In: *Algorithmic Game Theory*. Ed. by Noam Nisan, Tim Roughgarden, Éva Tardos, and Vijay V. Vazirani. New York: Cambridge University Press, 2007. Chap. 4, pp. 79–101. ISBN: 978-0-521-87282-9.

[16]   Liad Blumrosen and Noam Nisan. "Combinatorial Auctions". In: *Algorithmic game theory*. Ed. by Noam Nisan, Tim Roughgarden, Éva Tardos, and Vijay V. Vazirani. New York: Cambridge University Press, 2007. Chap. 11, pp. 267–299. ISBN: 978-0-521-87282-9.

[17]   Stephen P. Boyd and Lieven Vandenberghe. *Convex Optimization*. 7th ed. New York: Cambridge University Press, 2009. ISBN: 978-0-521-83378-3.

[18]   Leizhen Cai, Siu Man Chan, and Siu On Chan. "Random Separation: A New Method for Solving Fixed-Cardinality Optimization Problems". In: *Proceedings of the 2nd International Workshop on Parameterized and Exact Computation (IWPEC)*. Vol. 4169. LNCS. Springer, 2006, pp. 239–250. DOI: 10.1007/11847250_22.

[19] Yang Cai and Christos Papadimitriou. "Simultaneous Bayesian Auctions and Computational Complexity". In: *Proceedings of the 15th ACM Conference on Economics and Computation (EC)*. ACM, 2014, pp. 895–910. DOI: 10.1145/2600057.2602877.

[20] Nicolò Cesa-Bianchi, Yishay Mansour, and Gilles Stoltz. "Improved Second-order Bounds for Prediction with Expert Advice". In: *Machine Learning* 66.2-3 (2007), pp. 321–352. DOI: 10.1007/s10994-006-5001-7.

[21] Siu On Chan. "Approximation Resistance from Pairwise Independent Subgroups". In: *Proceedings of the 45th ACM Symposium on Theory of Computing (STOC)*. ACM, 2013, pp. 447–456. DOI: 10.1145/2488608.2488665.

[22] George Christodoulou, Annamária Kovács, and Michael Schapira. "Bayesian Combinatorial Auctions". In: *Proceedings of the 35th International Colloquium on Automata, Languages and Programming (ICALP)*. Vol. 5126. LNCS. Springer, 2008, pp. 820–832. DOI: 10.1007/978-3-540-70575-8_67.

[23] Edward H. Clarke. "Multipart Pricing of Public Goods". In: *Public Choice* 11 (1971), pp. 17–33. DOI: 10.1007/BF01726210.

[24] Riccardo Colini-Baldeschi, Monika Henzinger, Stefano Leonardi, and Martin Starnberger. "On Multiple Keyword Sponsored Search Auctions with Budgets". In: *Proceedings of the 39th International Colloquium on Automata, Languages and Programming (ICALP)*. Vol. 7392. LNCS. Springer, 2012, pp. 1–12. DOI: 10.1007/978-3-642-31585-5_1.

[25] Vincent Conitzer and Tuomas Sandholm. "Computing Optimal Outcomes under an Expressive Representation of Settings with Externalities". In: *Journal of Computer and System Sciences* 78.1 (2012), pp. 2–14. DOI: 10.1016/j.jcss.2011.02.009.

[26] Florin Constantin, Malvika Rao, Chien-Chung Huang, and David C. Parkes. "On Expressing Value Externalities in Position Auctions". In: *Proceedings of the 25th Conference Artificial Intelligence (AAAI)*. AAAI Press, 2011.

[27] Peter Cramton, Yoav Shoham, and Richard Steinberg, eds. Cambridge, Massachusetts: MIT Press, 2006. ISBN: 0-262-03342-9.

[28] Nikhil R. Devanur, Bach Q. Ha, and Jason D. Hartline. "Prior-free auctions for budgeted agents". In: *Proceedings of the 14th ACM Conference on Electronic Commerce (EC)*. ACM, 2013, pp. 287–304. DOI: 10.1145/2492002.2482554.

[29] Shahar Dobzinski and Shaddin Dughmi. "On the Power of Randomization in Algorithmic Mechanism Design". In: *SIAM Journal on Computing* 42.6 (2013), pp. 2287–2304. DOI: 10.1137/090780146.

[30] Shahar Dobzinski, Ron Lavi, and Noam Nisan. "Multi-unit Auctions with Budget Limits". In: *Games and Economic Behavior* 74.2 (2012), pp. 486–503. DOI: 10.1016/j.geb.2011.08.003.

[31] Shahar Dobzinski, Noam Nisan, and Michael Schapira. "Truthful randomized mechanisms for combinatorial auctions". In: *Journal of Computer and System Sciences* 78.1 (2012), pp. 15–25. DOI: 10.1016/j.jcss.2011.02.010.

[32] Shahar Dobzinski and Renato Paes Leme. "Efficiency Guarantees in Auctions with Budgets". In: *Proceedings of the 41st International Colloquium on Automata, Languages and Programming (ICALP)*. Vol. 8572. LNCS. Springer, 2014, pp. 392–404. DOI: 10.1007/978-3-662-43948-7_33.

[33] Shahar Dobzinski and Michael Schapira. "An Improved Approximation Algorithm for Combinatorial Auctions with Submodular Bidders". In: *Proceedings of the 17th ACM-SIAM Symposium on Discrete Algorithms (SODA)*. ACM, 2006, pp. 1064–1073. DOI: 10.1145/1109557.1109675.

[34] Rodney G. Downey and M. R. Fellows. *Parametrized Complexity*. New York: Springer, 1999. ISBN: 978-0-387-94883-6. DOI: 10.1007/978-1-4612-0515-9.

[35] Paul Dütting, Felix Fischer, and David C. Parkes. "Simplicity-Expressiveness Tradeoffs in Mechanism Design". In: *Proceedings of the 12th ACM Conference on Electronic Commerce (EC)*. ACM, 2011, pp. 341–350. DOI: 10.1145/1993574.1993632.

[36] Paul Dütting, Monika Henzinger, and Martin Starnberger. "Auctions with Heterogeneous Items and Budget Limits". In: *Proceedings of the 8th Workshop on Internet & Network Economics (WINE)*. Vol. 7695. LNCS. Springer, 2012, pp. 44–57. DOI: 10.1007/978-3-642-35311-6_4.

[37] Paul Dütting, Monika Henzinger, and Ingmar Weber. "An Expressive Mechanism for Auctions on the Web". In: *Proceedings of the 20th International World Wide Web Conference (WWW)*. ACM, 2011, pp. 127–136. DOI: 10.1145/1963405.1963427.

[38] Benjamin Edelman, Michael Ostrovsky, and Michael Schwarz. "Internet Advertising and the Generalized Second Price Auction: Selling Billions of Dollars Worth of Keywords". In: *American Economic Review* 97.1 (2005), pp. 242–259. DOI: 10.1257/aer.97.1.242.

[39] Uriel Feige. "On Maximizing Welfare When Utility Functions Are Subadditive". In: *SIAM Journal on Computing* 39.1 (2009), pp. 122–142. DOI: 10.1137/070680977.

[40] Uriel Feige and Jan Vondrák. "The Submodular Welfare Problem with Demand Queries". In: *Theory of Computing* 6.1 (2010), pp. 247–290. DOI: 10.4086/toc.2010.v006a011.

[41] Michal Feldman, Hu Fu, Nick Gravin, and Brendan Lucier. "Simultaneous Auctions are (almost) Efficient". In: *Proceedings of the 45th ACM Symposium on Theory of Computing (STOC)*. ACM, 2013, pp. 201–210. DOI: 10.1145/2488608.2488634.

[42] Amos Fiat, Stefano Leonardi, Jared Saia, and Piotr Sankowski. "Single Valued Combinatorial Auctions with Budgets". In: *Proceedings of the 12th ACM Conference on Electronic Commerce (EC)*. ACM, 2011, pp. 223–232. DOI: 10.1145/1993574.1993609.

[43] Dean P. Foster and Rakesh V. Vohra. "Calibrated Learning and Correlated Equilibrium". In: *Games and Economic Behavior* 21 (1997), pp. 40–55. DOI: 10.1006/game.1997.0595.

[44] Michael L. Fredman and Robert Endre Tarjan. "Fibonacci Heaps and their Uses in Improved Network Optimization Algorithms". In: *Journal of the ACM* 34.3 (1987), pp. 596–615. DOI: 10.1145/28869.28874.

[45] Satoru Fujishige and Akihisa Tamura. "A Two-Sided Discrete-Concave Market with Possibly Bounded Side Payments: An Approach by Discrete Convex Analysis". In: *Mathematics of Operations Research* 32.1 (2007), pp. 136–155. DOI: 10.1287/moor.1070.0227.

[46] Peter Funk. "Auctions with Interdependent Valuations". In: *International Journal of Game Theory* 25.1 (1996), pp. 51–64. DOI: 10.1007/BF01254384.

[47] Michael R. Garey and David S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. New York: W. H. Freeman and Company, 1979. ISBN: 0-7167-1044-7.

[48] Arpita Ghosh and Mohammad Mahdian. "Externalities in Online Advertising". In: *Proceedings of the 17th International World Wide Web Conference (WWW)*. ACM, 2008, pp. 161–168. DOI: 10.1145/1367497.1367520.

[49] Arpita Ghosh and Amin Sayedi. "Expressive Auctions for Externalities in Online Advertising". In: *Proceedings of the 19th International World Wide Web Conference (WWW)*. ACM, 2010, pp. 371–380. DOI: 10.1145/1772690.1772729.

[50] Ioannis Giotis and Anna Karlin. "On the Equilibria and Efficiency of the GSP Mechanism in Keyword Auctions with Externalities". In: *Proceedings of the 4th Workshop on Internet & Network Economics (WINE)*. Vol. 5385. LNCS. Springer, 2008, pp. 629–638. DOI: 10.1007/978-3-540-92185-1_69.

[51] Gagan Goel, Vahab Mirrokni, and Renato Paes Leme. "Clinching auctions beyond hard budget constraints". In: *Proceedings of the 15th ACM Conference on Economics and Computation (EC)*. ACM, 2014, pp. 167–184. DOI: 10.1145/2600057.2602851.

[52] Gagan Goel, Vahab Mirrokni, and Renato Paes Leme. "Clinching Auctions with Online Supply". In: *Proceedings of the 24th ACM-SIAM Symposium on Discrete Algorithms (SODA)*. SIAM, 2013, pp. 605–619. DOI: 10.1137/1.9781611973105.44.

[53]   Gagan Goel, Vahab Mirrokni, and Renato Paes Leme. "Polyhedral Clinching Auctions and the Adwords Polytope". In: *Proceedings of the 44th ACM Symposium on Theory of Computing (STOC)*. ACM, 2012, pp. 107–122. DOI: 10.1145/2757277.

[54]   Renato Gomes, Nicole Immorlica, and Evangelos Markakis. "Externalities in Keyword Auctions: An Empirical and Theoretical Assessment". In: *Proceedings of the 5th Workshop on Internet & Network Economics (WINE)*. Vol. 5929. LNCS. Springer, 2009, pp. 172–183. DOI: 10.1007/978-3-642-10841-9_17.

[55]   Teofilo Gonzalez and Sartaj Sahni. "Preemptive Scheduling of Uniform Processor Systems". In: *Journal of the ACM* 25.1 (1978), pp. 92–101. DOI: 10.1145/322047.322055.

[56]   Martin Grötschel, László Lovász, and Alexander Schrijver. "Geometric Algorithms and Combinatorial Optimization". In: Algorithms and Combinatorics 2. Berlin Heidelberg: Springer, 1993. ISBN: 3-540-56740-2. DOI: 10.1007/978-3-642-78240-4.

[57]   Theodore Groves. "Incentives in Teams". In: *Econometrica* 41 (1973), pp. 617–631. DOI: 10.2307/1914085.

[58]   Magnús Halldórsson. "A survey on independent set approximations". In: *Proceedings of the 1st International Workshop on Approximation Algorithms for Combinatorial Optimization Problems (APPROX)*. Vol. 1444. LNCS. Springer, 1998, pp. 1–14.

[59]   Magnús M. Halldórsson. "Approximations of Weighted Independent Set and Hereditary Subset Problems". In: *Journal of Graph Algorithms and Applications* 4.1 (2000), pp. 1–16. DOI: 10.7155/jgaa.00020.

[60]   Magnús Halldórsson and Jaikumar Radhakrishnan. "Greed is Good: Approximating Independent Sets in Sparse and Bounded-Degree Graphs". In: *Algorithmica* 18 (1997), pp. 145–163. DOI: 10.1007/BF02523693.

[61]   Eran Halperin. "Improved Approximation Algorithms for the Vertex Cover Problem in Graphs and Hypergraphs". In: *SIAM Journal on Computing* 31.5 (2002), pp. 1608–1623. DOI: 10.1137/S0097539700381097.

[62]   Sergiu Hart and Andreu Mas-Colell. "A Simple Adaptive Procedure Leading to Correlated Equilibrium". In: *Econometrica* 68 (2000), pp. 1127–1150. DOI: 10.1111/1468-0262.00153.

[63]   Avinatan Hassidim, Haim Kaplan, Yishay Mansour, and Noam Nisan. "Non-price Equilibria in Markets of Discrete Goods". In: *Proceedings of the 12th ACM Conference on Electronic Commerce (EC)*. ACM, 2011, pp. 295–296. DOI: 10.1145/1993574.1993619.

[64]   Johan Håstad. "Clique is Hard to Approximate within $n^{1-\varepsilon}$". In: *Acta Mathematica* 182.1 (1999), pp. 105–142. DOI: 10.1007/BF02392825.

[65]   Martin Hoefer and Thomas Kesselheim. "Brief Announcement: Universally Truthful Secondary Spectrum Auctions". In: *Proceedings of the 25th ACM Symposium on Parallelism in Algorithms and Architectures (SPAA)*. ACM, 2013, pp. 99–101. DOI: 10.1145/2486159.2486163.

[66]   Martin Hoefer and Thomas Kesselheim. "Secondary Spectrum Auctions for Symmetric and Submodular Bidders". In: *Proceedings of the 13th ACM Conference on Electronic Commerce (EC)*. ACM, 2012, pp. 657–671. DOI: 10.1145/2739041.

[67]   Martin Hoefer, Thomas Kesselheim, and Berthold Vöcking. "Approximation Algorithms for Secondary Spectrum Auctions". In: *ACM Transactions on Internet Technology* 14.2–3 (2014), p. 16. DOI: 10.1145/2663496.

[68]   Martin Hoefer, Thomas Kesselheim, and Berthold Vöcking. "Truthfulness and Stochastic Dominance with Monetary Transfers". In: *Proceedings of the 14th ACM Conference on Electronic Commerce (EC)*. ACM, 2013, pp. 567–582. DOI: 10.1145/2492002.2482548.

[69]   Philippe Jehiel and Benny Modovanu. "Auctions with Downstream Interaction Among Buyers". In: *The RAND Journal of Economics* 31.4 (2000), pp. 768–791. URL: http://www.jstor.org/stable/2696358.

[70]   Philippe Jehiel, Benny Moldovanu, and Ennio Stacchetti. "Multidimensional Mechanism Design for Auctions with Externalities". In: *Journal of Economic Theory* 85 (1999), pp. 258–293. DOI: 10.1006/jeth.1998.2501.

[71]   Albert Xin Jiang and Kevin Leyton-Brown. "Polynomial-time Computation of Exact Correlated Equilibrium in Compact Games". In: *Proceedings of the 12th ACM Conference on Electronic Commerce (EC)*. ACM, 2011, pp. 119–126. DOI: 10.1145/1993574.1993593.

[72]   Akihisa Kako, Takao Ono, Tomio Hirata, and Magnús Halldórsson. "Approximation Algorithms for the Weighted Independent Set Problem in Sparse Graphs". In: *Discrete Applied Mathematics* 157 (2009), pp. 617–626. DOI: 10.1016/j.dam.2008.08.027.

[73]   Bart de Keijzer, Evangelos Markakis, Guido Schäfer, and Orestis Telelis. "Inefficiency of Standard Multi-unit Auctions". In: *Proceedings of the 21st*. Vol. 8125. LNCS. Springer, 2013, pp. 385–396. DOI: 10.1007/978-3-642-40450-4_33.

[74]   Alexander S. Kelso and Vincent P. Crawford. "Job Matching, Coalition Formation, and Gross Substitutes". In: *Econometrica* 50.6 (1982), pp. 1483–1504. URL: http://www.jstor.org/stable/1913392.

[75]   David Kempe and Mohammad Mahdian. "A Cascade Model for Externalities in Sponsored Search". In: *Proceedings of the 4th Workshop on Internet & Network Economics (WINE)*. Vol. 5385. LNCS. Springer, 2008, pp. 585–596. DOI: 10.1007/978-3-540-92185-1_65.

[76] Elias Koutsoupias and Christos Papadimitriou. "Worst-Case Equilibria". In: *Computer Science Review* 3.2 (2009), pp. 65–69. DOI: 10.1016/j.cosrev.2009.04.003.

[77] Piotr Krysta, Tomasz P. Michalak, Tuomas Sandholm, and Michael Wooldridge. "Combinatorial auctions with externalities". In: *Proceedings of the 9th International Conference on Autonomous Agents & Multi-Agent Systems (AAMAS)*. IFAAMAS, 2010, pp. 1471–1472.

[78] Sébastien Lahaie, David M. Pennock, Amin Saberi, and Rakesh V. Vohra. "Sponsored Search Auctions". In: *Algorithmic Game Theory*. Ed. by Noam Nisan, Tim Roughgarden, Éva Tardos, and Vijay V. Vazirani. New York: Cambridge University Press, 2007. Chap. 28, pp. 699–716. ISBN: 978-0-521-87282-9.

[79] Ron Lavi and Marina May. "A Note on the Incompatibility of Strategy-Proofness and Pareto-Optimality in Quasi-Linear Settings with Public Budgets". In: *Economics Letters* 115.1 (2012), pp. 100–103. DOI: 10.1016/j.econlet.2011.11.043.

[80] Benny Lehmann, Daniel Lehmann, and Noam Nisan. "Combinatorial Auctions with Decreasing Marginal Utilities". In: *Games and Economic Behavior* 55 (2005), pp. 270–296. DOI: 10.1016/j.geb.2005.02.006.

[81] Nick Littlestone and Manfred K. Warmuth. "The Weighted Majority Algorithm". In: *Information and Computation* 108.2 (1994), pp. 212–261. DOI: 10.1006/inco.1994.1009.

[82] Michael Luby and Avi Wigderson. "Pairwise Independence and Derandomization". In: *Foundations and Trends in Theoretical Computer Science* 1.4 (2005), pp. 237–301. DOI: 10.1561/0400000009.

[83] Evangelos Markakis and Orestis Telelis. "Uniform Price Auctions: Equilibria and Efficiency". In: *Proceedings of the 5th International Symposium on Algorithmic Game Theory (SAGT)*. Vol. 7615. LNCS. Springer, 2012, pp. 227–238. DOI: 10.1007/978-3-642-33996-7_20.

[84] Paul Milgrom. *Putting Auction Theory to Work*. New York: Cambridge University Press, 2004. ISBN: 9780521551847.

[85] Paul Milgrom. "Putting Auction Theory to Work: The Simultaneous Ascending Auction". In: *Journal of Political Economy* 108.2 (2000), pp. 245–272. DOI: 10.1086/262118.

[86] Paul Milgrom. "Simplified Mechanisms with an Application to Sponsored-Search Auctions". In: *Games and Economic Behavior* 70 (2010), pp. 62–70. DOI: 10.1016/j.geb.2008.12.003.

[87] Kazuo Murota. *Matrices and Matroids for Systems Analysis*. Algorithms and Combinatorics. Berlin Heidelberg: Springer, 2000. ISBN: 978-3-540-66024-8. DOI: 10.1007/978-3-642-03994-2.

[88]   Kazuo Murota. "Valuated Matroid Intersection II: Algorithms". In: *SIAM Journal of Discrete Mathematics* 9 (1996), pp. 562–576. DOI: `10.1137/S0895480195280009`.

[89]   Kazuo Murota and Akihisa Tamura. "Application of M-Convex Submodular Flow Problem to Mathematical Economics". In: *Japan Journal of Industrial and Applied Mathematics* 20.3 (2003), pp. 257–277. DOI: `10.1007/BF03167422`.

[90]   Roger B. Myerson. "Optimal Auction Design". In: *Mathematics of Operations Research* 6.1 (1981), pp. 58–73. DOI: `10.1287/moor.6.1.58`.

[91]   Noam Nisan. "Introduction to Mechanism Design (for Computer Scientists)". In: *Algorithmic Game Theory*. Ed. by Noam Nisan, Tim Roughgarden, Éva Tardos, and Vijay V. Vazirani. New York: Cambridge University Press, 2007. Chap. 9, pp. 209–241. ISBN: 978-0-521-87282-9.

[92]   Noam Nisan and Ilya Segal. "The Communication Requirements of Efficient Allocations and Supporting Prices". In: *Journal of Economic Theory* 129.1 (2006), pp. 192–224. DOI: `10.1016/j.jet.2004.10.007`.

[93]   Fernando Mário de Oliveira Filho. *The Ellipsoid Method*. 2014. URL: `http://www.ime.usp.br/~fmario/sdp/ell.pdf`.

[94]   Renato Paes Leme, Vasilis Syrgkanis, and Éva Tardos. "Sequential Auctions and Externalities". In: *Proceedings of the 23rd ACM-SIAM Symposium on Discrete Algorithms (SODA)*. SIAM, 2012, pp. 869–886. DOI: `10.1137/1.9781611973099.70`.

[95]   Christos H. Papadimitriou and Tim Roughgarden. "Computing Correlated Equilibria in Multi-Player Games". In: *Journal of the ACM* 55 (3 2008), p. 14. DOI: `10.1145/1379759.1379762`.

[96]   Panagiotis Papadimitriou and Hector Garcia-Molina. "Sponsored search auctions with conflict constraints". In: *Proceedings of the 5th ACM International Conference on Web Search and Data Mining (WSDM)*. ACM, 2012, pp. 283–292. DOI: `10.1145/2124295.2124332`.

[97]   Tim Roughgarden. "Barriers to Near-Optimal Equilibria". In: *Proceedings of the 55th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*. IEEE Computer Society, 2014, pp. 71–80. DOI: `10.1109/FOCS.2014.16`.

[98]   Tim Roughgarden. "Intrinsic robustness of the price of anarchy". In: *Proceedings of the 41st ACM Symposium on Theory of Computing (STOC)*. ACM, 2009, pp. 513–522. DOI: `10.1145/2209249.2209274`.

[99]   Tim Roughgarden. "The Price of Anarchy in Games of Incomplete Information". In: *Proceedings of the 13th ACM Conference on Electronic Commerce (EC)*. ACM, 2012, pp. 862–879. DOI: `10.1145/2229012.2229078`.

[100] Tim Roughgarden and Éva Tardos. "Do externalities degrade GSP's efficiency". In: *Proceedings of the 8th Workshop on Ad Auctions*. 2012.

[101] Alexander Schrijver. "Combinatorial Optimization: Polyhedra and Efficiency". In: Algorithms and Combinatorics 24. Berlin Heidelberg: Springer, 2003. Chap. 18. ISBN: 978-3-540-44389-6.

[102] Vasilis Syrgkanis and Éva Tardos. "Composable and Efficient Mechanisms". In: *Proceedings of the 45th ACM Symposium on Theory of Computing (STOC)*. ACM, 2013, pp. 211–220. DOI: 10.1145/2488608.2488635.

[103] Robert Endre Tarjan. *Data Structures and Network Algorithms*. Philadelphia: Society for Industrial and Applied Mathematics, 1983. ISBN: 0-89871-187-8.

[104] William Vickrey. "Counterspeculation, Auctions, and Competitive Sealed Tenders". In: *Journal of Finance* 16.1 (1961), pp. 8–37. DOI: 10.1111/j.1540-6261.1961.tb02789.x.

[105] Sven de Vries, Sushil Bikhchandani, James Schummer, and Rakesh V. Vohra. "Linear programming and Vickrey auctions". In: *Mathematics of the Internet: E-Auctions and Markets*. Ed. by Brenda Dietrich and Rakesh V. Vohra. The IMA Volumes in Mathematics and its Applications 127. New York: Springer, 2002, pp. 75–116. ISBN: 978-0-387-95359-5.

[106] Yuli Ye and Allan Borodin. "Elimination Graphs". In: *ACM Transactions on Algorithms* 8.2 (2012), 14:1–14:23. DOI: 10.1145/2151171.2151177.

[107] Xia Zhou, Sorabh Gandhi, Subhash Suri, and Haitao Zheng. "eBay in the Sky: Strategy-proof wireless spectrum auctions". In: *Proceedings of the 14th Annual International Conference on Mobile Computing and Networking (MOBICOM)*. ACM, 2008, pp. 2–13. DOI: 10.1145/1409944.1409947.

Bibliographic Note

This dissertation is based on the following conference and journal publications. The publications are the product of joint research with the co-authors listed below.

1. Chapter 2 is based on:
   Paul Dütting, Monika Henzinger, and Martin Starnberger. "Auctions with Heterogeneous Items and Budget Limits". In: *Proceedings of the 8th Workshop on Internet & Network Economics (WINE)*. vol. 7695. LNCS. Springer, 2012, pp. 44–57. DOI: 10.1007/978-3-642-35311-6_4

   - This work is accepted to the *ACM Transactions on Economics and Computation.*

2. Chapter 3 is based on selected results in:
   Riccardo Colini-Baldeschi, Monika Henzinger, Stefano Leonardi, and Martin Starnberger. "On Multiple Keyword Sponsored Search Auctions with Budgets". In: *Proceedings of the 39th International Colloquium on Automata, Languages and Programming (ICALP)*. vol. 7392. LNCS. Springer, 2012, pp. 1–12. DOI: 10.1007/978-3-642-31585-5_1

   - This work was also presented at the *8th Ad Auction Workshop (AAW 2012)* in Valencia, Spain and it is accepted to the *ACM Transactions on Economics and Computation.*

3. Chapter 4 is based on selected results in:
   Paul Dütting, Monika Henzinger, and Martin Starnberger. "Valuation Compressions in VCG-Based Combinatorial Auctions". In: *Proceedings of the 9th Conference on Web and Internet Economics (WINE)*. vol. 8289. LNCS. Springer, 2013, pp. 146–159. DOI: 10.1007/978-3-642-45046-4_13

4. Chapter 5 is based on:
   Yun Kuen Cheung, Monika Henzinger, Martin Hoefer, and Martin Starnberger. *Combinatorial Auctions with Conflict-Based Externalities.* Accepted to the 11th Conference on Web and Internet Economics (WINE). 2015

   - This work was presented at the *Annual International Conference of the German Operations Research Society (OR 2015)* in Vienna, Austria.

Curriculum Vitae

Education

Since July 2011	**Doctoral Program in Computer Science** University of Vienna, Austria Graduate School in Computational Optimization Thesis Supervisor: Monika Henzinger Thesis: Combinatorial Auctions with Bidding Constraints and Network Externalities
May 2011	**Master in Business Informatics** Vienna University of Technology, Austria Specialization in Econometrics (passed with distinction)
October 2010	**Diploma in Technical Mathematics** (5-year program) Vienna University of Technology, Austria Specialization in Mathematical Economics (passed with distinction)
July 2007	**Bachelor in Economics and Computer Science** Vienna University of Technology, Austria (passed with distinction)
Winter 2007	**Erasmus Exchange Term** KTH Royal Institute of Technology, Stockholm, Sweden

Employment

Since Feb. 2011	**University of Vienna** Research Assistant at the Research Group *Theory and Applications of Algorithms* (Head: Monika Henzinger)

- Graduate School in Computational Optimization
  (funded by the University of Vienna)

- Project *Challenges in Sponsored Search Auctions* (funded by the Vienna Science and Technology Fund (WWTF))
- Project *Multiplex: Foundational Research on MULTI-level comPLEX networks and systems* (funded by the European Union's Seventh Framework Programme (FP7/2007–2013))

Publications

1. Yun Kuen Cheung, Monika Henzinger, Martin Hoefer, and Martin Starnberger. *Combinatorial Auctions with Conflict-Based Externalities.* Accepted to the 11th Conference on Web and Internet Economics (WINE). 2015

2. Sayan Bhattacharya, Wolfgang Dvořák, Monika Henzinger, and Martin Starnberger. "Welfare Maximization with Friends-of-Friends Network Externalities". In: *Proceedings of the 32nd Symposium on Theoretical Aspects of Computer Science (STACS).* vol. 30. Leibniz International Proceedings in Informatics (LIPIcs). Schloss Dagstuhl (Leibniz-Zentrum für Informatik), 2015, pp. 90–102. DOI: `10.4230/LIPIcs.STACS.2015.90`

3. Luděk Cigler, Wolfgang Dvořák, Monika Henzinger, and Martin Starnberger. "Limiting Price Discrimination when Selling Products with Positive Network Externalities". In: *Proceedings of the 10th Conference on Web and Internet Economics (WINE).* vol. 8877. LNCS. Springer, 2014, pp. 44–57. DOI: `10.1007/978-3-319-13129-0_4`

4. Paul Dütting, Monika Henzinger, and Martin Starnberger. "Valuation Compressions in VCG-Based Combinatorial Auctions". In: *Proceedings of the 9th Conference on Web and Internet Economics (WINE).* vol. 8289. LNCS. Springer, 2013, pp. 146–159. DOI: `10.1007/978-3-642-45046-4_13`

5. Paul Dütting, Monika Henzinger, and Martin Starnberger. "Auctions with Heterogeneous Items and Budget Limits". In: *Proceedings of the 8th Workshop on Internet & Network Economics (WINE).* vol. 7695. LNCS. Springer, 2012, pp. 44–57. DOI: `10.1007/978-3-642-35311-6_4`

   - Accepted to the *ACM Transactions on Economics and Computation.*

6. Riccardo Colini-Baldeschi, Monika Henzinger, Stefano Leonardi, and Martin Starnberger. "On Multiple Keyword Sponsored Search Auctions with Budgets". In: *Proceedings of the 39th International Colloquium on Automata, Languages and Programming (ICALP).* vol. 7392. LNCS. Springer, 2012, pp. 1–12. DOI: `10.1007/978-3-642-31585-5_1`

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Scientific Talks and Poster Presentations

Sept. 2015   Combinatorial Auctions with Conflict-Based Externalities. *The Annual International Conference of the German Operations Research Society (OR 2015)*, University of Vienna, Sept. 2, 2015. (Talk)

Dec. 2014   Limiting Price Discrimination when Selling Products with Positive Network Externalities. *The 10th Conference on Web and Internet Economics (WINE 2014)*, Academy of Mathematics and Systems Science (Beijing), Dec. 17, 2014. (Talk)

Nov. 2014   Uniform Price Strategies to Exploit Positive Network Externalities. *Multiplex project meeting*, University of Vienna, Nov. 24, 2014. (Talk)

June 2014   Multiple Keyword Sponsored Search Auctions with Budgets. *Austrian Computer Science Day 2014*, TU Graz, June 6, 2014. (Poster)

Dec. 2013   Uniform Price Strategies to Exploit Positive Network Externalities. *The 9th Conference on Web and Internet Economics (WINE 2013)*, Harvard University, Dec. 12, 2013. (Poster)

Dec. 2012   Auctions with Heterogeneous Items and Budget Limits. *The 8th Workshop on Internet & Network Economics (WINE 2012)*, University of Liverpool, Dec. 10, 2012. (Talk)

June 2012   On Multiple Keyword Sponsored Search Auctions with Budgets. *Ad Auction Workshop 2012*, Universitat Politècnica de València, June 8, 2012 (Talk)

Academic Services

Reviewing
- The 8th International Symposium on Algorithmic Game Theory (SAGT 2015)
- The 5th Innovations in Theoretical Computer Science Conference (ITCS 2014)
- The 19th Annual International Computing and Combinatorics Conference (COCOON 2013)

Teaching Activities

Summer 2014   *Algorithms and Data Structures*, University of Vienna.

Winter 2013   *Introduction to Programming - Practical Training in C++*, University of Vienna.

Summer 2013   *Algorithmic Game Theory (TA)*, University of Vienna.

Winter 2012   *Introduction to Programming - Practical Training in C++*, University of Vienna.

Summer 2006   *Web Engineering (TA)*, Vienna University of Technology.