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DISSERTATION

Titel der Dissertation

Foundations of Higher-Order Forcing

Verfasserin

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angestrebter akademischer Grad

Doktorin der Naturwissenschaften (Dr.rer.nat.)

Wien, im Juli 2015

Studienkennzahl lt. Studienblatt: A 791 405

Dissertationsgebiet lt. Studienblatt: Mathematik

Betreuer: O.Univ.-Prof. Sy-David Friedman

To Daniel and Dora

Acknowledgement

I am deeply grateful to my advisor Sy-David Friedman for his enthusiasm, advice, direction and kind support. I have vastly benefited from his encyclopedic knowledge of set theory and especially from guiding me through the deep waters of class forcing.

Furthermore I want to thank everybody who supported me with helpful discussions and insight, especially Andrew Brooke-Taylor, Victoria Gitman, Joel Hamkins, Peter Holy, Radek Honzik and all members of the KGRC for a wonderful working atmosphere.

As a beneficiary of the Doctoral Fellowship Program (DOC) I want to thank the Austrian Academy of Sciences (ÖAW) for their generous support.

Finally, I want to thank my parents and my husband for their never-ending support and love.

Abstract

Forcing notions can be classified via their size in a general way. Until now two different types were developed: set forcing and definable class forcing, where the forcing notion is a set or definable class respectively. Here, we want to introduce and study the next two steps in this classification by size, namely class forcing and definable hyperclass forcing (where the conditions of the forcing notion are themselves classes) in the context of (an extension of) Morse-Kelley class theory.

For class forcing, we adapt the existing account of class forcing over a ZFC model to a model (M, \mathcal{C}) of Morse-Kelley class theory. We give a rigorous definition of class forcing in such a model and show that the Definability Lemma (and the Truth Lemma) can be proven without restricting the notion of forcing. Furthermore we show under which conditions the axioms are preserved. We conclude by proving that Laver's Theorem does not hold for class forcings.

For definable hyperclass forcing, we use a symmetry between MK^{**} models and models of ZFC^- plus there exists a strongly inaccessible cardinal (called SetMK^{**}). This allows us to define hyperclass forcing in MK^{**} by going to the related SetMK^{**} model and use a definable class forcing there. We arrive at a definable class forcing extension from which we can go back to a model of MK^{**} . To use this construction we define a coding between MK^{**} and SetMK^{**} models and show how definable class forcing can be applied in the context of an ZFC^- model. We conclude by giving an application of this forcing in showing that every β -model of MK^{**} can be extended to a minimal β -model of MK^{**} with the same ordinals.

Zusammenfassung

Grundlegend kann die Erzwingungsmethode durch die Größe ihrer Grundstruktur klassifiziert werden. Die heute verwendeten Formen umfassen Größen im Bereich von Mengen und definierbaren Klassen. In dieser Arbeit werden die nächsten möglichen Schritte in dieser Hierarchie entwickelt: Klassen und Hyperklassen (Objekte deren Elemente Klassen sind) in der Theorie von Morse-Kelley.

Für Klassen werden die Definitionen und grundlegenden Ergebnisse der Erzwingungsmethode im Kontext von Morse-Kelley definiert und gezeigt.

Insbesondere wird das Definierbarkeitslemma und Wahrheitslemma bewiesen das hier im allgemeinen für jegliche Klassenerzwingung gilt. Als Anwendung beweisen wir, dass das Theorem von Laver nicht für Klassenerzwingung gilt.

Für Hyperklassen nutzen wir die Beziehung zwischen Modellen von MK^{**} und Modellen einer Variante von ZFC^- . Dies erlaubt uns Hyperklassenerzwingung in MK^{**} zu definieren, indem wir zu einem ZFC^- Modell gehen und dort definierbare Klassenerzwingung anwenden. Von der daraus gewonnenen Modellerweiterung können wir nun wieder zu einem Modell von MK^{**} zurückgehen. Für diesen Ansatz entwickeln wir eine Kodierung zwischen Modellen von MK^{**} und bestimmten Modellen von ZFC^- und zeigen, dass definierbare Klassenerzwingung in diesem Kontext durchführbar ist. Ein konkretes Beispiel einer Hyperklassenerzwingung wird gegeben, indem gezeigt wird, dass jedes β -Modell von MK^{**} zu einem minimalen β -Modell von MK^{**} erweitert werden kann (mit den selben Ordinalzahlen).

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Chapter 1

Introduction and Basic Definitions

When considering forcing notions with respect to their size, there are two different types: the original version of forcing, where the forcing notion is a set, called set forcing, and forcing in ZFC, where the forcing notion is a class, called definable class forcing. In this thesis we consider these two types as first steps in a hierarchy where forcing notions are classified via their size. We will define the next two steps in this hierarchy, class forcing and definable hyperclass forcing, and give applications for these forcings in the context of Morse-Kelley class theory.

The structure of this thesis will be as follows: In the first chapter we will present an overview over existing definitions and results regarding definable class forcing and give a short introduction to Morse-Kelley class theory (MK).

In the second chapter we develop class forcing in MK by defining all basic notions in the context where we now have two types of object, sets and classes, and proving the Forcing Theorem. In difference to definable class forcing, it is now possible to prove the Definability Lemma without having to restrict the forcing notion. Further, we will show which forcing notions preserve the axioms and we will show that Laver's Theorem¹ about the definability of the ground model in its forcing extensions does not hold for class forcing.

The third chapter is concerned with definable hyperclass forcing, i.e. a

¹see [Lav07]

forcing with class conditions. We will define this forcing indirectly by using a correspondence between certain models of MK and models of a version of ZFC^- (minus PowerSet) and showing that we can define definable hyperclass forcing by going to the related ZFC^- model and using definable class forcing there. To set this up we will show that this correspondence is indeed a coding between a variant of MK and certain models of the ZFC^- which allows us to go back and forth between them. Then we define definable hyperclass forcing and show how the problems of definable class forcing in the setting of ZFC^- can be handled. We conclude the chapter by giving an example of definable hyperclass forcing by showing that every β -model of a variant of MK can be extended to a minimal β -model of the same variant of MK with the same ordinals.

1.1 Definable Class Forcing

The idea of considering a forcing notion with a (proper) class of conditions instead of with a set of conditions was introduced by W. Easton in 1970. He needed the forcing notion to be a class to prove the theorem that the continuum function 2^κ , for κ regular, can behave in any reasonable way and as changes in the size of 2^κ are bounded by the size of a set-forcing notion, the forcing has to be a class. Two problems arise when considering a class sized forcing: the forcing relation might not be definable in the ground model and the extension might not preserve the axioms. As an example consider $Col(\omega, ORD)$ with conditions $p : n \rightarrow Ord$ for $n \in \omega$ which adds a cofinal sequence of length ω in the ordinals. Here Replacement fails². These problems were addressed in a general way in [Fri00] where class forcings are presented which are definable (with parameters) over a model $\langle M, A \rangle$.

In the following we give a short overview of the main definitions and results of [Fri00] and [Fri10] which we will use repeatedly in the following chapters of this thesis. For proofs and more details see there.

Definition 1.1. *The structure $\langle M, A \rangle$ is a ground model, if:*

- (a) *$\langle M, A \rangle$ is a transitive model of ZFC, i.e. M is a transitive model of ZFC, $A \subseteq M$ and Replacement holds in M for formulas mentioning A as a unary predicate.*

²A detailed analyses on how even the Definability Lemma for Class Forcings can fail can be found in [HKL⁺].

(b) (*Minimality*) $M \models V = L(A) = \bigcup\{L(A \cap V_\alpha) : \alpha \in ORD\}$, the smallest inner model containing each of the sets $A \cap V_\alpha$, $\alpha \in ORD$.

We need A as a class predicate to define the class P . Property (a) implies that $\langle M, A \rangle$ is *amenable*: for x in M , $A \cap x$ also belongs to M . Property (b) ensures that for every extension N of M with $N \models ZFC$, M is definable in $\langle N, A \rangle$, because M equals $L(A)^N$. From now on $\langle M, A \rangle$ will always denote a ground model. We say that $D \subseteq P$ is *dense below* p if $\forall q \leq p \exists r (r \leq q, r \in D)$ and $D \subseteq P$ is *predense* $\leq p \in P$ if every $q \leq p$ is compatible with an element of D .

Definition 1.2. $P \subseteq M$ is called a $\langle M, A \rangle$ -forcing, if it is a pre-ordering (a reflexive, transitive relation) with greatest element 1^P which is definable over $\langle M, A \rangle$. Let $G \subseteq P$. G is P -generic over $\langle M, A \rangle$ if G is compatible, upward-closed and $G \cap D \neq \emptyset$ whenever $D \subseteq P$ is dense and $\langle M, A \rangle$ -definable.

As in set forcing we define $M[G]$ for any $G \subseteq M$ as follows: A *name* is a set $\sigma \in M$, whose elements are of the form $\langle \tau, a \rangle$, τ a name and $a \in M$. Then $\sigma^G = \{\tau^G : \langle \tau, a \rangle \in \sigma \text{ for some } a \in G\}$ and $M[G] = \{\sigma^G : \sigma \text{ a name}\}$. Then a P -generic extension of $\langle M, A \rangle$ is a model $\langle M[G], A, G \rangle$ where G is P -generic over $\langle M, A \rangle$. P is an M -forcing if it is an $\langle M, A \rangle$ -forcing for some A . A *generic extension of M* is a model $\langle M[G], A, G \rangle$ for some choice of P , A and of G P -generic over $\langle M, A \rangle$. $X \subseteq M$ is *generic over M* if X is definable in a generic extension of M .

Definition 1.3. Suppose p belongs to P , $\varphi(x_1, \dots, x_m)$ is a formula and $\sigma_1, \dots, \sigma_m$ are names. We write $p \Vdash \varphi(\sigma_1, \dots, \sigma_m)$ iff for each G which is P -generic over $\langle M, A \rangle$ and $p \in G$, we have $\langle M[G], A, G \rangle \models \varphi(\sigma_1^G, \dots, \sigma_m^G)$.

As in set forcing the following holds:

Lemma 1.4.

- (i) $M \subseteq M[G]$, $M[G]$ is transitive and $ORD(M[G]) = ORD(M)$.
- (ii) $G \cap V_\alpha \in M[G]$ for each $\alpha \in ORD(M)$.

Next we state the two parts of the Forcing Theorem:

Lemma 1.5 (Definability for \Vdash). For any φ , the relation “ $p \Vdash \varphi(\sigma_1, \dots, \sigma_m)$ ” is definable in $\langle M, A \rangle$.

Lemma 1.6 (Truth Lemma). *If G is P -generic then*

$$\langle M[G], A, G \rangle \models \varphi(\sigma_1^G, \dots, \sigma_m^G) \Leftrightarrow \exists p \in G (p \Vdash \varphi(\sigma_1, \dots, \sigma_m)).$$

In set forcing we can show the Definability Lemma by defining a relation \Vdash^* for which the Definability and Truth Lemma holds and then showing that indeed \Vdash and \Vdash^* are the same. When doing this for class forcing, we have to restrict the forcing notions we consider:

Definition 1.7. *Let σ, τ, π be elements of \mathcal{N} .*

1. $p \Vdash^* \sigma \in \tau$ iff $\{q : \exists \langle \pi, r \rangle \in \tau \text{ such that } q \leq r, q \Vdash^* \sigma = \pi\}$ is dense below p .
2. $p \Vdash^* \sigma = \tau$ iff for all $\langle \pi, r \rangle \in \sigma \cup \tau$, $p \Vdash^* (\pi \in \sigma \leftrightarrow \pi \in \tau)$.
3. $p \Vdash^* \varphi \wedge \psi$ iff $p \Vdash^* \varphi$ and $p \Vdash^* \psi$.
4. $p \Vdash^* \neg \varphi$ iff $\forall q \leq p (\neg q \Vdash^* \varphi)$.
5. $p \Vdash^* \forall x \varphi$ iff for all σ , $p \Vdash^* \varphi(\sigma)$.

As P may now be a class the first two clauses now involve unbounded quantifiers and therefore lead to definitions of $p \Vdash^* \sigma \in \tau$, $p \Vdash^* \sigma = \tau$ whose quantifier complexity may increase with the ranks of σ, τ .

This problem can be resolved by using the property of pretameness, which helps to control the quantifier complexity of the relations $p \Vdash^* \sigma \in \tau$, $p \Vdash^* \sigma = \tau$:

Definition 1.8. *P is pretame iff whenever $\langle D_i | i \in a \rangle$, $a \in M$, is an $\langle M, A \rangle$ -definable sequence of dense classes and $p \in P$ then there is $q \leq p$ and $\langle d_i | i \in a \rangle \in M$ such that $d_i \subseteq D_i$ and d_i is predense $\leq q$ for each i .*

Now we can prove the definability of \Vdash^* and it follows that:

Lemma 1.9. *For a pretame P and G P -generic over M :*

$$\langle M[G], A, G \rangle \models \varphi(\sigma_1^G, \dots, \sigma_n^G) \text{ iff for some } p \in G, \quad p \Vdash^* \varphi(\sigma_1, \dots, \sigma_n).$$

Lemma 1.10. *The relations \Vdash^* and \Vdash are the same.*

Pretameness not only allows us to show that the Definability Lemma holds, it also serves to show that the axioms of ZFC^- are preserved. Indeed pretameness is equivalent to the preservation of ZFC^- :

Proposition 1.11. *Suppose that for each $p \in P$ there is $G \subseteq P$ such that $p \in G$, G is P -generic over $\langle M, A \rangle$ and $\langle M[G], A, G \rangle$ is a model of ZF-Power. Then P is pretame.*

Proposition 1.12. *Suppose that P is pretame. Then P is ZFC^- preserving.*

For the PowerSet Axiom we need a strengthening of pretameness, called tameness:

Definition 1.13. *A predense $\leq p$ partition is a pair (D_0, D_1) such that $D_0 \cup D_1$ is predense $\leq p$ and $p_0 \in D_0$, $p_1 \in D_1$ then p_0 and p_1 are incompatible. Suppose $\langle (D_0^i, D_1^i) \mid i \in a \rangle$, $\langle (E_0^i, E_1^i) \mid i \in a \rangle$ are sequences of predense $\leq p$ partitions, We say that they are equivalent $\leq p$ if for each $i \in a$, $\{q \mid q \text{ meets } D_0^i \text{ iff } q \text{ meets } E_0^i\}$ is dense $\leq p$. When $p = 1^P$ we omit “ $\leq p$ ”.*

To each sequence of predense $\leq p$ partitions $\vec{D} = \langle (D_0^i, D_1^i) \mid i \in a \rangle \in M$ and G P -generic over $\langle M, A \rangle$, $p \in G$ we can associate the function $f_{\vec{D}}^G : a \rightarrow 2$ defined by $f(i) = 0 \leftrightarrow G \cap D_0^i \neq \emptyset$. Then two such sequences are equivalent $\leq p$ exactly if their associated functions are equal, for each choice of G .

Definition 1.14. *P is tame iff P is pretame and for each $a \in M$ and $p \in P$ there is $q \leq p$ and $\alpha \in ORD(M)$ such that whenever $\vec{D} = \langle (D_0^i, D_1^i) \mid i \in a \rangle \in M$ is a sequence of predense $\leq q$ partitions, $\{r \mid \vec{D} \text{ is equivalent } \leq r \text{ to some } \vec{E} = \langle (E_0^i, E_1^i) \mid i \in a \rangle \text{ in } V_\alpha^M\}$ is dense below q .*

The idea behind this definition is simply that P is tame if P is pretame and in addition $1^P \Vdash$ “Power Set Axiom”. Indeed it follows by the Truth Lemma for P -forcing:

Theorem 1.15. *Suppose that M is countable. Then P is ZFC preserving iff P is tame.*

There are helpful connections between distributive forcings and tameness which are also used for cofinality preservation, so for example the following holds:

Definition 1.16. For regular $\kappa > \omega$, P is κ -distributive whenever $p \in P$ and $\langle D_i \mid i < \beta \rangle$ is an $\langle M, A \rangle$ -definable sequence of dense classes, $\beta < \kappa$ then there is $q \leq p$ meeting each D_i . P is tame below κ if the tameness conditions hold for P provided we impose the added restriction that $\text{Card}(a) < \kappa$.

Lemma 1.17. If P is κ -distributive then P is tame below κ and if P is κ -distributive and pretame then whenever $G \subseteq P$ is P -generic over $\langle M, A \rangle$ and $\text{cof}(\alpha) \geq \kappa$ in M we have $\text{cof}(\alpha) \geq \kappa$ in $M[G]$.

1.2 Morse-Kelley Class Theory

In ZFC we can only talk about classes as abbreviations for formulas as our only objects are sets. In class theories like Morse-Kelley (MK) or Gödel-Bernays (GB) the language is two-sorted, i.e. the objects are sets and classes and we have corresponding quantifiers for each type of object.³ We denote the classes by upper case letters and sets by lower case letters, the same will hold for class-names and set-names and so on. Hence atomic formulas for the \in -relation are of the form “ $x \in X$ ” where x is a set-variable and X is a set- or class-variable. The models \mathcal{M} of MK are of the form $\langle M, \in, \mathcal{C} \rangle$, where M is a transitive model of ZFC, \mathcal{C} the family of classes of \mathcal{M} (i.e. every element of \mathcal{C} is a subset of M) and \in is the standard \in relation (from now on we will omit mentioning this relation).

The axiomatizations of class theories which are often used and closely related to ZFC are MK and GBC. Their axioms which are purely about sets coincide with the corresponding ZFC axioms such as pairing and union and they share class axioms like the Global Choice Axiom. Their difference lies in the Comprehension Axiom in the sense that GB only allows quantification over sets whereas MK allows quantification over sets as well as classes. This results in major differences between the two theories which can be seen for example in their relation to ZFC: GB is a conservative extension of ZFC, meaning that every sentence about sets that can be proved in GB can already be proved in ZFC and so GB cannot prove “new” theorems about ZFC. MK on the other hand can do just that, in particular MK implies $\text{CON}(\text{ZFC})$ ⁴

³There is also an equivalent one-sorted formulation in which the only objects are classes and sets are defined as being classes which are elements of other classes. For reasons of clarity we will use the two-sorted version throughout the paper.

⁴This is because in MK we can form a Satisfaction Predicate for V and then by reflection we get an elementary submodel V_α of V . But any such V_α models ZFC.

and so MK is not conservative over ZFC. The consistency strength of MK is strictly stronger than that of ZFC but lies below that of ZFC + there is an inaccessible cardinal as $\langle V_\kappa, V_{\kappa+1} \rangle$ for κ inaccessible, is a model for MK in ZFC.

As said above we choose MK (and not GB) as underlying theory to define class-forcing. The reason lies mainly in the fact that within MK we can show the Definability Lemma for class-forcing without having to restrict the forcing notion whereas in GB this would not be possible. We use the following axiomatization of MK:

A) Set Axioms:

1. Extensionality for sets: $\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)$.
2. Pairing: For any sets x and y there is a set $\{x, y\}$.
3. Infinity: There is an infinite set.
4. Union: For every set x the set $\bigcup x$ exists.
5. Power set: For every set x the power set $P(x)$ of x exists.

B) Class Axioms:

1. Foundation: Every nonempty class has an \in -minimal element.
2. Extensionality for classes: $\forall z (z \in X \leftrightarrow z \in Y) \rightarrow X = Y$.
3. Replacement: If a class F is a function and x is a set, then $\{F(z) : z \in x\}$ is a set.
4. Class-Comprehension:

$$\forall X_1 \dots \forall X_n \exists Y Y = \{x : \varphi(x, X_1, \dots, X_n)\}$$

where φ is a formula containing class parameters in which quantification over both sets and classes are allowed.

5. Global Choice: There exists a global class well-ordering of the universe of sets.

There are different ways of axiomatizing MK, one of them is obtained by using the Limitation of Size Axiom instead of Global Choice and Replacement. Limitation of Size is an axiom that was introduced by von Neumann and says that for every $C \in \mathcal{M}$, C is a proper class if and only if there

is a one-to-one function from the universe of sets to C , i.e. all the proper classes have the same size. The two axiomatizations are equivalent: Global Choice and Replacement follow from Limitation of size and vice versa.⁵ A nontrivial argument shows that Limitation of Size does not follow from Replacement plus Local Choice.

In the definition of forcing we will use the following induction and recursion principles:

Proposition 1.18 (Induction). *Let (Ord, R) be well-founded and $\varphi(\alpha)$ a property of an ordinal α . Then it holds that*

$$\forall \alpha \in Ord ((\forall \beta \in Ord (\beta R \alpha \rightarrow \varphi(\beta))) \rightarrow \varphi(\alpha)) \rightarrow \forall \alpha \in Ord \varphi(\alpha)$$

Proof. Otherwise, as R is well-founded, there exists an R -minimal element α of Ord such that $\neg\varphi(\alpha)$. That is a contradiction. \square

Proposition 1.19 (Recursion). *For every well-founded binary relation R on Ord and every formula $\varphi(X, Y)$ satisfying $\forall X \exists! Y \varphi(X, Y)$, there is a unique binary relation S on $Ord \times V$ such that for every $\alpha \in Ord$ it holds that $\varphi(S_{<\alpha}, S_\alpha)$, where $S_\alpha = \{x \mid (\alpha, x) \in S\}$ and $S_{<\alpha} = \{(\beta, x) \in S \mid \beta R \alpha\}$.*

Proof. By induction on α it holds that for each γ there exists a unique binary relation S^γ on $Ord_{<\gamma} \times V$, where $Ord_{<\gamma} = \{\beta \in Ord \mid \beta R \gamma\}$, such that $\varphi(S_{<\alpha}^\gamma, S_\alpha^\gamma)$ holds for all $\alpha R \gamma$. Then it follows from Class-Comprehension that we can take $S = \bigcup_{\gamma \in Ord} S^\gamma$. \square

⁵This is because Global Choice is equivalent with the statement that every proper class is bijective with the ordinals.

Chapter 2

Class Forcing in MK

In this chapter we would like to broaden the approach for definable class forcing from the last chapter by changing the notion of ground model from a model M of ZFC with a class A to general models of class theory with an arbitrary collection of classes \mathcal{C} . We choose Morse-Kelley class theory as our underlying theory.¹

In the following we will define the relevant notions like names, interpretations and the extension for class forcing in Morse-Kelley. Then we will show that the forcing relation is definable in the ground model, that the Truth Lemma holds and we characterize P -generic extensions which satisfy the axioms of MK. We will show that Laver's Theorem fails for class forcings.

2.1 Generics, Names and the Extension

To lay out forcing in MK we have to redefine the basic notions like names, interpretation of names etc. to arrive at the definition of the forcing exten-

¹In [Chu80] R. Chuaqui follows a similar approach and defines forcing for Morse-Kelley class theory. However there is a significant difference between our two approaches. To show that the extension preserves the axioms Chuaqui restricts the generic G for an arbitrary forcing notion P in the following way: A subclass G of a notion of forcing P is *strongly P -generic over* a model (M, \mathcal{C}) of MK iff G is P -generic over (M, \mathcal{C}) and for all ordinals $\beta \in M$ there is a set $P' \in M$ such that $P' \subseteq P$ and for all sequences of dense sections $\langle D_\alpha : \alpha \in \beta \rangle$, there is a $q \in G$ satisfying

$$\forall \alpha (\alpha \in \beta \rightarrow \exists p (p \in P' \cap G \wedge \text{the greatest lower bound of } p \text{ and } q \text{ exists} \\ \text{and is an element of } D_\alpha)).$$

where a subclass D of a partial order P is a P -*section* if every extension of a condition in D is in D .

sion. As we work in a two-sorted theory we will define these notions for sets and classes respectively. Let us start with the definition of the forcing notions and its generics. We use the notation $(X_1, \dots, X_n) \in \mathcal{C}$ to mean $X_i \in \mathcal{C}$ for all i .

Definition 2.1. *Let $P \in \mathcal{C}$ and $\leq_P \in \mathcal{C}$ be a partial ordering with greatest element 1^P . We call $(P, \leq_P) \in \mathcal{C}$ an (M, \mathcal{C}) -forcing and often abbreviate it by writing P . With the above convention $(P, \leq_P) \in \mathcal{C}$ means that P and \leq_P are in \mathcal{C} .*

$G \subseteq P$ is P -generic over (M, \mathcal{C}) if

1. G is compatible: If $p, q \in G$ then for some r , $r \leq p$ and $r \leq q$.
2. G is upwards closed: $p \geq q \in G \rightarrow p \in G$.
3. $G \cap D \neq \emptyset$ whenever $D \subseteq P$ is dense, $D \in \mathcal{C}$.

Note that from now on we will assume M to be countable (and transitive) and \mathcal{C} to be countable to ensure that for each $p \in P$ there exists G such that $p \in G$ and G is P -generic.

We build the hierarchy of names for sets and classes in the following way (we will use capital greek letters for class-names and lower case greek letters for set-names):

Definition 2.2.

$$\mathcal{N}_0^s = \emptyset.$$

$$\mathcal{N}_{\alpha+1}^s = \{\sigma : \sigma \text{ is a subset of } \mathcal{N}_\alpha^s \times P \text{ in } M\}.$$

$$\mathcal{N}_\lambda^s = \bigcup \{\mathcal{N}_\alpha^s : \alpha < \lambda\}, \text{ if } \lambda \text{ is a limit ordinal.}$$

$$\mathcal{N}^s = \bigcup \{\mathcal{N}_\alpha^s : \alpha \in \text{ORD}(M)\} \text{ is the class of all set-names of } P.$$

$$\mathcal{N} = \{\Sigma : \Sigma \text{ is a subclass of } \mathcal{N}^s \times P \text{ in } \mathcal{C}\}.$$

Note that the \mathcal{N}_α^s (for $\alpha > 0$) are in fact proper classes (and indeed \mathcal{N} is a hyperclass) and therefore Definition 2.2 is an inductive definition of a sequence of proper classes of length the ordinals. The fact that with this definition we stay inside \mathcal{C} follows from Proposition 1.19.

Lemma 2.3.

- a) If $\alpha \leq \beta$ then $\mathcal{N}_\alpha^s \subseteq \mathcal{N}_\beta^s$.
- b) $\mathcal{N}^s \subseteq \mathcal{N}$.

Proof. a) By induction on β . For $\beta = 0$ there is nothing to prove.

Successor step $\beta \rightarrow \beta + 1$. Assume $\mathcal{N}_\alpha^s \subseteq \mathcal{N}_\beta^s$ for all $\alpha \leq \beta$. Let $\tau \in \mathcal{N}_\alpha^s$ for some $\alpha < \beta + 1$. Then we know by assumption that $\tau \in \mathcal{N}_\beta^s$. So by Definition 2.2 there is some $\gamma < \beta$ such that $\tau = \{\langle \pi_i, p_i \rangle \mid i \in I\}$ where for each $i \in I$, $\pi_i \in \mathcal{N}_\gamma^s$ and $p_i \in P$. By assumption $\pi_i \in \mathcal{N}_\beta^s$ for all $i \in I$ and so $\tau \in \mathcal{N}_{\beta+1}^s$.

Limit step λ . Assume $\mathcal{N}_\alpha^s \subseteq \mathcal{N}_\beta^s$ for all $\alpha \leq \beta < \lambda$. But by Definition 2.2, $\sigma \in \mathcal{N}_\lambda^s$ iff $\sigma \in \mathcal{N}_\beta^s$ for some $\beta < \lambda$ and so it follows that $\mathcal{N}_\alpha^s \subseteq \mathcal{N}_\lambda^s$ for all $\alpha \leq \lambda$.

- b) By Definition 2.2, $\Sigma \in \mathcal{N}$ iff Σ is a subclass of $\mathcal{N}^s \times P$ iff for every $\langle \tau, p \rangle \in \Sigma$, $\tau \in \mathcal{N}^s$ and $p \in P$ iff for every $\langle \tau, p \rangle \in \Sigma$ there is an ordinal α such that $\tau \in \mathcal{N}_\alpha^s$ and $p \in P$. Let $\sigma \in \mathcal{N}^s$, i.e. there is an ordinal β such that $\sigma \in \mathcal{N}_\beta^s$. Then it holds that for every $\langle \tau, p \rangle \in \sigma$ there is an ordinal $\alpha < \beta$ such that $\tau \in \mathcal{N}_\alpha^s$ and $p \in P$. So $\sigma \in \mathcal{N}$.

□

We define the interpretations of set- and class-names recursively.

Definition 2.4.

$$\sigma^G = \{\tau^G : \exists p \in G(\langle \tau, p \rangle \in \sigma)\} \text{ for } \sigma \in \mathcal{N}^s.$$

$$\Sigma^G = \{\sigma^G : \exists p \in G(\langle \sigma, p \rangle \in \Sigma)\} \text{ for } \Sigma \in \mathcal{N}.$$

According to the definitions above we define the extension of an MK model (M, \mathcal{C}) to be the extension of the set part and the extension of the class part:

Definition 2.5. $(M, \mathcal{C})[G] = (M[G], \mathcal{C}[G]) = (\{\sigma^G : \sigma \in \mathcal{N}^s\}, \{\Sigma^G : \Sigma \in \mathcal{N}\})$.

Definition 2.6. If P is a partial order with greatest element 1^P , we define the canonical P -names of $x \in M$ and $C \in \mathcal{C}$:

$$\check{x} = \{\langle \check{y}, 1^P \rangle \mid y \in x\}.$$

$$\check{C} = \{\langle \check{x}, 1^P \rangle \mid x \in C\}.$$

From these definitions the basic facts of forcing follow easily:

Lemma 2.7. *Let $\mathcal{M} = \langle M, \mathcal{C} \rangle$ be a model of MK, where M is a transitive model of ZFC and \mathcal{C} the family of classes of \mathcal{M} . Then it holds that:*

- a) $\forall x \in M (\check{x} \in \mathcal{N}^s \wedge \check{x}^G = x)$ and $\forall C \in \mathcal{C} (\check{C} \in \mathcal{N} \wedge \check{C}^G = C)$.
- b) $(M, \mathcal{C}) \subseteq (M, \mathcal{C})[G]$ in the sense that $M \subseteq M[G]$ and $\mathcal{C} \subseteq \mathcal{C}[G]$.
- c) $G \in (M, \mathcal{C})[G]$, i.e. $G \in \mathcal{C}[G]$
- d) $M[G]$ is transitive and $\text{Ord}(M[G]) = \text{Ord}(M)$.
- e) If (N, \mathcal{C}') is a model of MK, $M \subseteq N$, $\mathcal{C} \subseteq \mathcal{C}'$, $G \in \mathcal{C}'$ then $(M, \mathcal{C})[G] \subseteq (N, \mathcal{C}')$.

Proof. a): Using Definition 2.4 and Definition 2.6 we can easily show this by induction.

b): follows immediately from 1.

c): Let $\Gamma = \{\langle \check{p}, p \rangle : p \in P\}$. Then this is a name for G as $\Gamma^G = \{\check{p}^G \mid p \in G\} = \{p \mid p \in G\} = G$.

d) It follows from Definition 2.4 and Definition 2.5 that $M[G]$ is transitive. For every $\sigma \in \mathcal{N}^s$ the rank of σ^G is at most rank σ , so $\text{Ord}(M[G]) \subseteq \text{Ord}(M)$.

e) For each name $\Sigma \in \mathcal{N}$, $\Sigma \in (M, \mathcal{C})$ and therefore $\Sigma \in (N, \mathcal{C}')$. As $G \in \mathcal{C}'$ the interpretation of Σ in $(M, \mathcal{C})[G]$ is the same as in (N, \mathcal{C}') . □

2.2 Definability and Truth Lemmas

We will define the forcing relation and show that it is definable in the ground model and how it relates to truth in the extension. The main focus will be the Definability Lemma, since it now is possible to prove that it holds for all forcing notions in contrast to A -definable class-forcings in a ZFC setting (see [Fri00]). Note that when we talk about a formula $\varphi(x_1, \dots, x_m, X_1, \dots, X_n)$ we mean φ to be a second-order formula that allows second-order quantification and we always assume the model (M, \mathcal{C}) to be countable.

Definition 2.8. Suppose p belongs to P , $\varphi(x_1, \dots, x_m, X_1, \dots, X_n)$ is a formula, $\sigma_1, \dots, \sigma_m$ are set-names and $\Sigma_1, \dots, \Sigma_n$ are class-names. We write $p \Vdash \varphi(\sigma_1, \dots, \sigma_m, \Sigma_1, \dots, \Sigma_n)$ iff whenever $G \subseteq P$ is P -generic over (M, \mathcal{C}) and $p \in G$, we have $(M, \mathcal{C})[G] \models \varphi(\sigma_1^G, \dots, \sigma_m^G, \Sigma_1^G, \dots, \Sigma_n^G)$.

Lemma 2.9 (Definability Lemma). For any φ , the relation “ $p \Vdash \varphi(\sigma_1, \dots, \sigma_m, \Sigma_1, \dots, \Sigma_n)$ ” of $p, \vec{\sigma}, \vec{\Sigma}$ is definable in (M, \mathcal{C}) .

Lemma 2.10 (Truth Lemma). If G is P -generic over (M, \mathcal{C}) then

$$(M, \mathcal{C})[G] \models \varphi(\sigma_1^G, \dots, \sigma_m^G, \Sigma_1^G, \dots, \Sigma_n^G) \Leftrightarrow \exists p \in G (p \Vdash \varphi(\sigma_1, \dots, \sigma_m, \Sigma_1, \dots, \Sigma_n)).$$

Following the approach of set-forcing we introduce a new relation \Vdash^* and prove the Definability and Truth Lemma for this \Vdash^* . Then we will show that \Vdash^* equals the intended forcing relation \Vdash .

The definition of \Vdash^* consists of ten cases: six cases for atomic formulas, where the first two are for set-names, the second two for the “hybrid” of set- and class-names and the last two for class-names, one for \wedge and \neg respectively and two quantifier cases, one for first-order and one for second-order quantification. By splitting the cases in this way we can see very easily that it is enough to prove the Definability Lemma for set-names only (case one and two in the Definition) and then infer the general Definability Lemma by induction.

Definition 2.11. $D \subseteq P$ is dense below p if $\forall q \leq p \exists r (r \leq q, r \in D)$.

Definition 2.12. Let σ, γ, π be elements of \mathcal{N}^s and Σ, Γ elements of \mathcal{N} .

1. $p \Vdash^* \sigma \in \gamma$ iff $\{q : \exists \langle \pi, r \rangle \in \gamma \text{ such that } q \leq r, q \Vdash^* \pi = \sigma\}$ is dense below p .
2. $p \Vdash^* \sigma = \gamma$ iff for all $\langle \pi, r \rangle \in \sigma \cup \gamma$, $p \Vdash^* (\pi \in \sigma \leftrightarrow \pi \in \gamma)$.
3. $p \Vdash^* \sigma \in \Sigma$ iff $\{q : \exists \langle \pi, r \rangle \in \Sigma \text{ such that } q \leq r, q \Vdash^* \pi = \sigma\}$ is dense below p .
4. $p \Vdash^* \sigma = \Sigma$ iff for all $\langle \pi, r \rangle \in \sigma \cup \Sigma$, $p \Vdash^* (\pi \in \sigma \leftrightarrow \pi \in \Sigma)$.
5. $p \Vdash^* \Sigma \in \Gamma$ iff $\{q : \exists \langle \pi, r \rangle \in \Gamma \text{ such that } q \leq r, q \Vdash^* \pi = \Sigma\}$ is dense below p .

6. $p \Vdash^* \Sigma = \Gamma$ iff for all $\langle \pi, r \rangle \in \Sigma \cup \Gamma$, $p \Vdash^* (\pi \in \Sigma \leftrightarrow \pi \in \Gamma)$.
7. $p \Vdash^* \varphi \wedge \psi$ iff $p \Vdash^* \varphi$ and $p \Vdash^* \psi$.
8. $p \Vdash^* \neg \varphi$ iff $\forall q \leq p (\neg q \Vdash^* \varphi)$.
9. $p \Vdash^* \forall x \varphi$ iff for all σ , $p \Vdash^* \varphi(\sigma)$.
10. $p \Vdash^* \forall X \varphi$ iff for all Σ , $p \Vdash^* \varphi(\Sigma)$.

We have to show that \Vdash^* is definable within the ground model. For this it is enough to concentrate on the first two of the above cases, because we can reduce the definability of the \Vdash^* -relation for arbitrary second-order formulas to its definability for atomic formulas $\sigma \in \tau$, $\sigma = \tau$, where σ and τ are set-names. The rest of the cases then follow by induction. So let us restate Lemma 2.9 for the case of \Vdash^* and set-names:

Lemma 2.13 (Definability Lemma for the atomic cases of set-names). *The relation “ $p \Vdash^* \varphi(\sigma, \tau)$ ” is definable in (M, \mathcal{C}) for $\varphi = “\sigma \in \tau”$ and $\varphi = “\sigma = \tau”$.*

Proof. We will show by induction² on $\beta \in ORD$ that there are unique classes $X_\beta, Y_\beta \subseteq \beta \times M$ which define the \Vdash^* -relation for the first two cases of Definition 2.12 in the following way: for all $\alpha < \beta$, $R_\alpha = (X_\beta)_\alpha$, $S_\alpha = (Y_\beta)_\alpha$ where $(X_\beta)_\alpha = \{x \mid \langle \alpha, x \rangle \in X_\beta\}$ and

$$\begin{aligned}
 R_\alpha = \{ & (p, \sigma, \in, \tau) \mid p \in P, \sigma \text{ and } \tau \text{ are set } P\text{-names,} & (\star) \\
 & \text{rank}(\sigma) \text{ and } \text{rank}(\tau) < \alpha, \text{ for all } q \leq p \\
 & \text{there is } q' \leq q \text{ and } \langle \pi, r \rangle \in \tau \text{ such that} \\
 & q' \leq r \text{ and } (q', \pi, =, \sigma) \in S_\alpha \}
 \end{aligned}$$

and

$$\begin{aligned}
 S_\alpha = \{ & (p, \sigma, =, \tau) \mid p \in P, \sigma \text{ and } \tau \text{ are set } P\text{-names,} & (\star\star) \\
 & \text{rank}(\sigma) \text{ and } \text{rank}(\tau) < \alpha, \\
 & \text{for all } \langle \pi, r \rangle \in \sigma \cup \tau \text{ such that} \\
 & (p, \pi, \in, \sigma) \in R_\alpha \text{ iff } (p, \pi, \in, \tau) \in R_\alpha \}
 \end{aligned}$$

²To show how this induction works in the context of a class-theory we will not simply use Proposition 1.18 and 1.19, but rather give the complete construction.

To show that X_β and Y_β are definable we will define the classes R_α and S_α at each step by recursion on the tuple (p, σ, e, τ) according to the following well-founded partial order on $P \times \mathcal{N}^s \times \{“\in”, “=”\} \times \mathcal{N}^s$.

Definition 2.14. *Suppose $(p, \sigma, e, \tau), (q, \sigma', e', \tau') \in P \times \mathcal{N}^s \times \{“\in”, “=”\} \times \mathcal{N}^s$. Say that $(q, \sigma', e', \tau') < (p, \sigma, e, \tau)$ if*

- $\max(\text{rank}(\sigma'), \text{rank}(\tau')) < \max(\text{rank}(\sigma), \text{rank}(\tau))$, or
- $\max(\text{rank}(\sigma'), \text{rank}(\tau')) = \max(\text{rank}(\sigma), \text{rank}(\tau))$, and $\text{rank}(\sigma) \geq \text{rank}(\tau)$ but $\text{rank}(\sigma') < \text{rank}(\tau')$, or
- $\max(\text{rank}(\sigma'), \text{rank}(\tau')) = \max(\text{rank}(\sigma), \text{rank}(\tau))$, and $\text{rank}(\sigma) \geq \text{rank}(\tau) \leftrightarrow \text{rank}(\sigma') \geq \text{rank}(\tau')$, and e is “=” and e' is “ \in ”.

Note that clause 1 and 2 of Definition 2.12 always reduce the $<$ -rank of the members of $P \times \mathcal{N}^s \times \{“\in”, “=”\} \times \mathcal{N}^s$.

“Successor step $\beta \rightarrow \beta + 1$.” We know that there are unique classes X_β, Y_β such that for all $\alpha < \beta$, $R_\alpha = (X_\beta)_\alpha, S_\alpha = (Y_\beta)_\alpha$ and (\star) and $(\star\star)$ hold. We want to show that there are unique classes $X_{\beta+1}, Y_{\beta+1}$ such that for all $\alpha < \beta + 1$, $R_\alpha = (X_{\beta+1})_\alpha, S_\alpha = (Y_{\beta+1})_\alpha$ and (\star) and $(\star\star)$ hold. So let for all $\alpha < \beta$ $(X_{\beta+1})_\alpha = (X_\beta)_\alpha = R_\alpha$ and $(Y_{\beta+1})_\alpha = (Y_\beta)_\alpha = S_\alpha$ and define $(X_{\beta+1})_\beta = R_\beta$ and $(Y_{\beta+1})_\beta = S_\beta$ uniquely as follows:

- A) $(p, \sigma, “\in”, \tau) \in R_\beta$ if and only if for all $q \leq p$ there is $q' \leq q$ and $\langle \pi, r \rangle \in \tau$ such that $q' \leq r$ and $(q', \pi, “=”, \sigma) \in S_\beta$.
- B) $(p, \sigma, “=”, \tau) \in S_\beta$ if and only if for all $\langle \pi, r \rangle \in \sigma \cup \tau$: $(p, \pi, “\in”, \sigma) \in R_\beta$ iff $(p, \pi, “\in”, \tau) \in R_\beta$.

These definitions clearly satisfy (\star) and $(\star\star)$ and to see that they are indeed inductive definitions over the well-order defined in Definition 2.14, we consider the following three cases for each of the definitions A) and B):

1. $\text{rank}(\sigma) < \text{rank}(\tau)$
2. $\text{rank}(\tau) < \text{rank}(\sigma)$
3. $\text{rank}(\sigma) = \text{rank}(\tau)$

Ad A.1: $(q', \pi, "=", \sigma) < (p, \sigma, "\in", \tau)$ because $\text{rank}(\sigma), \text{rank}(\pi) < \text{rank}(\tau)$ (first clause of Definition 2.14).

Ad A.2: $(q', \pi, "=", \sigma) < (p, \sigma, "\in", \tau)$ because $\max(\text{rank}(\pi), \text{rank}(\sigma)) = \max(\text{rank}(\sigma), \text{rank}(\tau))$ and $\text{rank}(\sigma) \geq \text{rank}(\tau)$ and $\text{rank}(\pi) < \text{rank}(\sigma)$ (second clause of Definition 2.14).

Ad A.3: $(q', \pi, "=", \sigma) < (p, \sigma, "\in", \tau)$ because $\max(\text{rank}(\pi), \text{rank}(\sigma)) = \max(\text{rank}(\sigma), \text{rank}(\tau))$ and $\text{rank}(\sigma) \geq \text{rank}(\tau)$ and $\text{rank}(\pi) < \text{rank}(\sigma) = \text{rank}(\tau)$ (second clause of Definition 2.14).

Ad B.1: $(p, \pi, "\in", \sigma) < (p, \sigma, "=", \tau)$ because $\text{rank}(\sigma), \text{rank}(\pi) < \text{rank}(\tau)$ and $(p, \pi, "\in", \tau) < (p, \sigma, "=", \tau)$ because $\max(\text{rank}(\pi), \text{rank}(\tau)) = \max(\text{rank}(\sigma), \text{rank}(\tau))$ and $\text{rank}(\sigma) < \text{rank}(\tau)$ and $\text{rank}(\pi) < \text{rank}(\tau)$ (third clause of Definition 2.14).

Ad B.2: $(p, \pi, "\in", \sigma) < (p, \sigma, "=", \tau)$ because of the second clause of Definition 2.14 and $(p, \pi, "\in", \tau) < (p, \sigma, "=", \tau)$ because $\text{rank}(\pi), \text{rank}(\tau) < \text{rank}(\sigma)$.

Ad B.3: $(p, \pi, "\in", \sigma) < (p, \sigma, "=", \tau)$ and $(p, \pi, "\in", \tau) < (p, \sigma, "=", \tau)$ because $\max(\text{rank}(\pi), \text{rank}(\tau)) = \max(\text{rank}(\sigma), \text{rank}(\tau))$ and $\text{rank}(\sigma) \geq \text{rank}(\tau)$ and $\text{rank}(\pi) < \text{rank}(\sigma), \text{rank}(\tau)$ (both second clause of Definition 2.14).

“Limit step λ .” We know that for every $\beta < \lambda$ there are unique classes X_β, Y_β such that for all $\alpha < \beta$, $R_\alpha = (X_\beta)_\alpha, S_\alpha = (Y_\beta)_\alpha$ and (\star) and $(\star\star)$ hold. We have to show that there are unique classes $X_\lambda, Y_\lambda \subseteq \lambda \times M$, λ limit, such that for all $\beta < \lambda$, $R_\beta = (X_\lambda)_\beta, S_\beta = (Y_\lambda)_\beta$ and (\star) and $(\star\star)$ hold respectively. We define the required classes as follows:

$$\begin{aligned} \langle \alpha, x \rangle \in X_\lambda &\leftrightarrow \exists \langle \langle R_\gamma, S_\gamma \rangle \mid \gamma \leq \alpha \rangle \exists X, Y ((\forall \gamma \leq \alpha ((X)_\gamma = R_\gamma \text{ and} \\ & (Y)_\gamma = S_\gamma \text{ and they satisfy } (\star) \text{ and } (\star\star) \text{ resp.}) \wedge \\ & (x \in (X)_\gamma \text{ for some } \gamma \leq \alpha)) \end{aligned}$$

$$\begin{aligned} \langle \alpha, x \rangle \in Y_\lambda &\leftrightarrow \exists \langle \langle R_\gamma, S_\gamma \rangle \mid \gamma \leq \alpha \rangle \exists X, Y ((\forall \gamma \leq \alpha ((X)_\gamma = R_\gamma \text{ and} \\ & (Y)_\gamma = S_\gamma \text{ and they satisfy } (\star) \text{ and } (\star\star) \text{ resp.}) \wedge \\ & (x \in (Y)_\gamma \text{ for some } \gamma \leq \alpha)) \end{aligned}$$

From the proof of the successor step we see that the sequence $\langle\langle R_\gamma, S_\gamma \rangle \mid \gamma \leq \alpha \rangle$ is unique for every $\alpha < \lambda$ and therefore X_λ, Y_λ are also unique. This definition is possible only in Morse-Kelly with its version of Class-Comprehension and not in Gödel-Bernays, because we are quantifying over class variables (in fact we only need Δ_1^1 Class-Comprehension). \square

The general Definability Lemma now follows immediately from this Lemma and Definition 2.12. We now turn to the Truth Lemma.

In the following a capital greek letter denotes a name from \mathcal{N} (and therefore can be a set- or a class-name), whereas a lower case greek letter is a name from \mathcal{N}^s (and therefore can only be a set-name).

Lemma 2.15.

- a) If $p \Vdash^* \varphi$ and $q \leq p$ then $q \Vdash^* \varphi$
- b) If $\{p \mid q \Vdash^* \varphi\}$ is dense below p then $p \Vdash^* \varphi$.
- c) If $\neg p \Vdash^* \varphi$ then $\exists q \leq p (q \Vdash^* \neg \varphi)$.

Proof. a) By induction on φ : Let φ be $\Sigma \in \Gamma$, then by Definition 2.2 $D = \{q' : \exists \langle \pi, r \rangle \in \Gamma \text{ such that } q' \leq r, q' \Vdash^* \pi = \Sigma\}$ is dense below p . Then for all $q \leq p$, D is also dense below q and therefore $q \Vdash^* \varphi$. The other cases follow easily.

b) By induction on φ . Let φ be $\Sigma \in \Gamma$ and $\{q \mid q \Vdash^* \Sigma \in \Gamma\}$ is dense below p . From Definition 2.12 it follows that $\{q \mid \{s : \exists \langle \pi, r \rangle \in \Gamma \text{ such that } s \leq r, s \Vdash^* \pi = \Sigma\} \text{ is dense below } q\}$ is dense below p and from a well-known fact it follows that $D = \{s : \exists \langle \pi, r \rangle \in \Gamma \text{ such that } s \leq r, s \Vdash^* \pi = \Sigma\}$ is dense below p . Again by Definition 2.12 we get as desired $p \Vdash^* \Sigma \in \Gamma$.

The other cases follow easily; for the case of negation we will use the fact that if $\{p \mid q \Vdash^* \neg \varphi\}$ is dense below p then $\forall q \leq p (\neg q \Vdash^* \varphi)$, using a).

c) follows directly from b). \square

Now, the proofs for the Truth Lemma and $\Vdash^* = \Vdash$ follow similarly to the proofs in set-forcing (note that a name $\Sigma \in \mathcal{N}$ can also be a set-name and therefore we don't need to mention the cases for set-names explicitly):

Lemma 2.16 (Truth Lemma). *If G is P -generic then*

$$(M, \mathcal{C})[G] \models \varphi(\Sigma_1^G, \dots, \Sigma_m^G) \Leftrightarrow \exists p \in G (p \Vdash^* \varphi(\Sigma_1, \dots, \Sigma_m)).$$

Proof. By induction on φ .

$\Sigma \in \Gamma$. “ \rightarrow ” Assume $\Sigma^G \in \Gamma^G$ then choose a $\langle \pi, r \rangle \in \Gamma$ such that $\Sigma^G = \pi^G$ and $r \in G$. By induction there is a $p \in G$ with $p \leq r$ and $p \Vdash^* \pi = \Sigma$. Then for all $q \leq p$, $q \Vdash^* \pi = \Sigma$ and by Definition 2.2 $p \Vdash^* \Sigma \in \Gamma$.

“ \leftarrow ”: Assume $\exists p \in G (p \Vdash^* \Sigma \in \Gamma)$. Then $\{q : \exists \langle \pi, r \rangle \in \Gamma \text{ such that } q \leq r, q \Vdash^* \sigma = \pi\} = D$ is dense below p and so by genericity $G \cap D \neq \emptyset$. So there is a $q \in G$, $q \leq p$ such that $\exists \langle \pi, r \rangle \in \Gamma$ with $q \leq r$, $q \Vdash^* \pi = \Sigma$. By induction $\pi^G = \Sigma^G$ and as $r \geq q$, $r \in G$ and therefore $\pi^G \in \Gamma^G$. So $\Sigma^G \in \Gamma^G$.

$\Sigma = \Gamma$. “ \rightarrow ” Assume $\sigma^G = \Gamma^G$. Then for all $\langle \pi, r \rangle \in \Sigma \cup \Gamma$ with $r \in G$ it holds that $\pi^G \in \Sigma^G \leftrightarrow \pi^G \in \Gamma^G$. Let $D = \{p \mid \text{either } p \Vdash^* \Sigma = \Gamma \text{ or for some } \langle \pi, r \rangle \in \Sigma \cup \Gamma, p \Vdash^* \neg(\pi \in \Sigma \leftrightarrow \pi \in \Gamma)\}$. Then D is dense: By contradiction, let $q \in P$ and assume that there is no $p \leq q$ such that $p \in D$. But if there is no $p \leq q$ such that for some $\langle \pi, r \rangle \in \Sigma \cup \Gamma, p \Vdash^* \neg(\pi \in \Sigma \leftrightarrow \pi \in \Gamma)$ then by Lemma 2.15 $q \Vdash^* (\pi \in \Sigma \leftrightarrow \pi \in \Gamma)$ for all $\langle \pi, r \rangle \in \Sigma \cup \Gamma$ and therefore $q \Vdash^* \Sigma = \Gamma$. So there is a $p \leq q$ such that $p \in D$. Since the filter G is generic, there is a $p \in G \cap D$. If $p \Vdash^* \neg(\pi \in \Sigma \leftrightarrow \pi \in \Gamma)$ for some $\langle \pi, r \rangle \in \Sigma \cup \Gamma$ then by induction $\neg(\pi^G \in \Sigma^G \leftrightarrow \pi^G \in \Gamma^G)$ for some $\langle \pi, r \rangle \in \Sigma \cup \Gamma$. But this is a contradiction to $\Sigma^G = \Gamma^G$ and so $P \Vdash^* \Sigma = \Gamma$.

“ \leftarrow ” Assume that there is $p \in G (p \Vdash^* \Sigma = \Gamma)$. By Definition 2.2 it follows that for all $\langle \pi, r \rangle \in \Sigma \cup \Gamma$ $P \Vdash^* (\pi \in \Sigma \leftrightarrow \pi \in \Gamma)$. Then by induction $\pi^G \in \Sigma^G \leftrightarrow \pi^G \in \Gamma^G$ for all $\langle \pi, r \rangle \in \Sigma \cup \Gamma$. So $\Sigma^G = \Gamma^G$.

$\varphi \wedge \psi$ “ \rightarrow ” Assume that $(M, \mathcal{C})[G] \models \varphi \wedge \psi$ iff $(M, \mathcal{C})[G] \models \varphi$ and $(M, \mathcal{C})[G] \models \psi$. Then by induction $\exists p \in G P \Vdash^* \varphi$ and $\exists q \in G, q \Vdash^* \psi$ and we know that $\exists r \in G (r \leq p \text{ and } r \leq q)$ such that $r \Vdash^* \varphi$ and $r \Vdash^* \psi$ and so by Definition 2.2 $r \Vdash^* \varphi \wedge \psi$.

“ \leftarrow ” Assume $\exists p \in G, p \Vdash^* \varphi \wedge \psi$, then $p \Vdash^* \varphi$ and $p \Vdash^* \psi$. So $(M, \mathcal{C})[G] \models \varphi$ and $(M, \mathcal{C})[G] \models \psi$ and therefore $(M, \mathcal{C})[G] \models \varphi \wedge \psi$.

$\neg\varphi$ “ \rightarrow ” Assume that $(M, \mathcal{C})[G] \models \neg\varphi$. $D = \{p \mid p \Vdash^* \varphi \text{ or } p \Vdash^* \neg\varphi\}$ is dense (using Lemma 2.15 and Definition 2.2). Therefore there is a $p \in G \cap D$ and by induction $p \Vdash^* \neg\varphi$.

“ \leftarrow ” Assume that there is $p \in G$ such that $p \Vdash^* \neg\varphi$. If $(M, \mathcal{C}) \models \varphi$ then by induction hypothesis there is a $q \in G$ such that $q \Vdash^* \varphi$. But

then also $r \Vdash^* \varphi$ for some $r \leq p, q$ and this is a contradiction because of Definition 2.2. So $(M, \mathcal{C}) \models \neg\varphi$.

$\forall X\varphi$ “ \rightarrow ” Assume that $(M, \mathcal{C})[G] \models \forall X\varphi$. Following the lines of the “ \rightarrow ”-part of the proof for $\Sigma = \Gamma$, there is a dense $D = \{p \mid \text{either } p \Vdash^* \forall X\varphi \text{ or for some } \sigma, p \Vdash^* \neg\varphi(\sigma)\}$. By induction we show that the second case is not possible and so it follows that $p \Vdash^* \forall X\varphi$.

“ \leftarrow ” By induction.

□

Lemma 2.17. $\Vdash^* = \Vdash$

Proof. $p \Vdash^* \varphi(\sigma_1, \dots, \sigma_n) \rightarrow p \Vdash \varphi(\sigma_1, \dots, \sigma_n)$ follows directly from the Truth Lemma. For the converse we use Lemma 2.15 c) and note that we assumed the existence of generics. Then from $\neg p \Vdash^* \varphi(\sigma_1, \dots, \sigma_n)$ it follows that for some $q \leq p$, $q \Vdash^* \neg\varphi(\sigma_1, \dots, \sigma_n)$ and so $\neg p \Vdash \varphi(\sigma_1, \dots, \sigma_n)$. □

2.3 The Extension fulfills the Axioms

We have shown that in MK we can prove the Definability Lemma without restricting the forcing notion as we have to do when working with A -definable class-forcing in ZFC (see [Fri00]). Unfortunately we do not have the same advantage when proving the preservation of the axioms. For example, when proving the Replacement Axiom we have to show that the range of a set under a class function is still a set and this does not hold in general for class-forcings. In [Fri00] two properties of forcing notions are introduced, namely pretameness and tameness. Pretameness is needed to prove the Definability Lemma and show that all axioms except Power Set are preserved. For the Power Set Axiom this restriction needs to be strengthened to tameness. Let us give the definitions in the MK context:

Definition 2.18 (Pretameness). $D \subseteq P$ is *predense* $\leq p \in P$ if every $q \leq p$ is compatible with an element of D .

P is *pretame* if and only if whenever $\langle D_i \mid i \in a \rangle$ is a sequence of dense classes in \mathcal{M} , $a \in M$ and $p \in P$ then there exists a $q \leq p$ and $\langle d_i \mid i \in a \rangle \in M$ such that $d_i \subseteq D_i$ and d_i is predense $\leq q$ for each i .

Definition 2.19. $q \in P$ meets $D \subseteq P$ if q extends an element in D .

A predense $\leq p$ partition is a pair (D_0, D_1) such that $D_0 \cup D_1$ is predense $\leq p$ and $p_0 \in D_0, p_1 \in D_1 \rightarrow p_0, p_1$ are incompatible. Suppose $\langle (D_0^i, D_1^i) \mid i \in a \rangle, \langle (E_0^i, E_1^i) \mid i \in a \rangle$ are sequences of predense $\leq p$ partitions. We say that they are equivalent $\leq p$ if for each $i \in a$, $\{q \mid q \text{ meets } D_0^i \leftrightarrow q \text{ meets } E_0^i\}$ is dense $\leq p$. When $p = 1^P$ we omit $\leq p$.

To each sequence of predense $\leq p$ partitions $\vec{D} = \langle (D_0^i, D_1^i) \mid i \in a \rangle \in M$ and G is P -generic over $\langle M, \mathcal{C} \rangle$, $p \in G$ we can associate the function

$$f_D^G : a \rightarrow 2$$

defined by $f(i) = 0 \leftrightarrow G \cap D_0^i \neq \emptyset$. Then two such sequences are equivalent $\leq p$ exactly if their associated functions are equal, for each choice of G .

Definition 2.20 (Tameness). P is tame iff P is pretame and for each $a \in M$ and $p \in P$ there is $q \leq p$ and $\alpha \in \text{ORD}(M)$ such that whenever $\vec{D} = \langle (D_0^i, D_1^i) \mid i \in a \rangle \in M$ is a sequence of predense $\leq q$ partitions, $\{r \mid \vec{D} \text{ is equivalent } \leq r \text{ to some } \vec{E} = \langle (E_0^i, E_1^i) \mid i \in a \rangle \text{ in } V_\alpha^M\}$ is dense below q .

Theorem 2.21. Let (M, \mathcal{C}) be a model of MK. Then, if G is P -generic over (M, \mathcal{C}) and P is tame then $(M, \mathcal{C})[G]$ is a model of MK.

Proof. Extensionality and Foundation follow because $M[G]$ is transitive (see Lemma 2.7 d) ; axiom 2 and 3 from Definitions 2.2 and 2.4. For Pairing, let σ_1^G, σ_2^G be such that $\sigma_1, \sigma_2 \in \mathcal{N}^s$. Then the interpretation of the name $\sigma = \{\langle \sigma_1, 1^P \rangle, \langle \sigma_2, 1^P \rangle\}$ in the extension gives the desired $\sigma^G = \{\sigma_1^G, \sigma_2^G\}$. Infinity follows because ω exists in (M, \mathcal{C}) and the notion of ω is absolute to any model, $\omega \in (M, \mathcal{C})[G]$. Union follows as in the set-forcing case.

Replacement: This follows as in [Fri00] from the property of pretameness and we give the proof to make clear where the property of pretameness is needed: Suppose that $F : \sigma^G \rightarrow M[G]$. Then for each σ_0 of rank $< \text{rank } \sigma$ the class $D(\sigma_0) = \{p \mid \text{for some } \tau, q \Vdash \sigma_0 \in \sigma \rightarrow F(\sigma_0) = \tau\}$ is dense below p , for some $p \in G$ which forces that F is a total function on σ . We now use pretameness to “shrink” this class to a set: so for each $q \leq p$ there is an $r \leq q$ and $\alpha \in \text{Ord}(M)$ such that $D_\alpha(\sigma_0) = \{s \mid s \in V_\alpha^M \text{ and for some } \tau \text{ of rank } < \alpha, s \Vdash \sigma_0 \in \sigma \rightarrow F(\sigma_0) = \tau\}$ is predense $\leq r$ for each σ_0 of rank $< \text{rank } \sigma$. Then it follows by genericity that there is a $q \in G$ and $\alpha \in \text{Ord}(M)$ such that $q \leq p$ and $D_\alpha(\sigma_0)$ is predense $\leq q$ for each σ_0 of rank $< \text{rank } \sigma$. So let

$\pi = \{\langle \tau, r \rangle \mid \text{rank } \tau < \alpha, r \in V_\alpha^M, r \Vdash \tau \in \text{ran}(F)\}$ and then it follows that $\text{ran}(F) = \pi^G \in M[G]$.

Power Set: This follows from tameness as shown in [Fri00].

Class-Comprehension: Let $\Gamma = \{\langle \sigma, p \rangle \in \mathcal{N}^s \times P \mid p \Vdash \varphi(\sigma, \Sigma_1, \dots, \Sigma_n)\}$. Because of the Definability Lemma, we know that $\Gamma \in \mathcal{N}$. By Definition 2.2 and 2.4, $\Gamma^G = \{\sigma^G \mid \exists p \in G(\langle \sigma, p \rangle \in \Gamma)\}$ and we need to check that this equals the desired $Y = \{x \mid (\varphi(x, \Sigma_1^G, \dots, \Sigma_n^G))^{(M, \mathcal{C})[G]}\}$. So let $\sigma^G \in \Gamma^G$. Then by the definition of Γ^G we know that $p \Vdash \varphi(\sigma, \Sigma_1, \dots, \Sigma_n)$ and because of the Truth Lemma it follows that $(M, \mathcal{C})[G] \models \varphi(\sigma^G, \Sigma_1^G, \dots, \Sigma_n^G)$. For the converse, let $x \in Y$. By the Truth Lemma, $\exists p \in G(p \Vdash \varphi(\pi, \Sigma_1, \dots, \Sigma_n))$, where π is a name for x . By definition of Γ , $\langle \pi, p \rangle \in \Gamma$.

Global Choice: Let $<_M$ denote the well-order of M and let σ_x, σ_y be the least names for some $x, y \in M[G]$. As the names are elements of M , we may assume that $\sigma_x <_M \sigma_y$. So we define the relation $<_G$ in $M[G]$ using M and $<_M$ as parameters, so that $x <_G y$ iff $\sigma_x <_M \sigma_y$ for the corresponding least names of x and y . Let $R = \{(x, y) \mid x, y \in M[G] \text{ and } x <_G y\}$. Then by Class-Comprehension the class R exists. \square

As mentioned in Chapter 1, [Fri00] gives us a simple sufficient condition for tameness that translates directly into the context of MK:

Definition 2.22. *For regular, uncountable $\kappa > \omega$, P is κ -distributive if whenever $p \in P$ and $\langle D_i \mid i < \beta \rangle$ are dense classes, $\beta < \kappa$ then there is a $q \leq p$ meeting each D_i (p meets D if $p \leq q \in D$ for some q).*

P is tame below κ if the tameness conditions hold for P with the added restriction that $\text{Card}(a) < \kappa$.

Lemma 2.23. *If P is κ -distributive then P is tame below κ .*

Proof. Analogous to set-forcing³. \square

2.4 Laver's Theorem

In the following we will give an example which shows that a fundamental theorem that holds for set-forcing can be violated by tame class-forcings.

Laver's Theorem (see [Lav07]) shows that for a set-generic extension $V \subseteq V[G]$, $V \models ZFC$ with the forcing notion $P \in V$ and G P -generic over

³See [Fri00], page 37.

V , V is definable in $V[G]$ from parameter $V_{\delta+1}$ (of V) and $\delta = |P|^+$ in $V[G]$. This result makes use of the fact that every such forcing extension has the approximation and cover properties as defined in [Ham03] and relies on certain results for such extensions.

In general, the same does not hold for class-forcing. In fact there are class-forcings such that the ground model is not even second-order definable from set-parameters:

Theorem 2.24. *There is an MK-model (M, \mathcal{C}) and a first-order definable, tame class-forcing \mathbb{P} with G \mathbb{P} -generic over (M, \mathcal{C}) such that the ground model M is not definable with set-parameters in the generic extension $(M, \mathcal{C})[G]$.*

Proof. We are starting from L . For every successor cardinal α , let P_α be the forcing that adds one Cohen set to α : P_α is the set of all functions p such that

$$\text{dom}(p) \subset \alpha, \quad |\text{dom}(p)| < \alpha, \quad \text{ran}(p) \subset \{0, 1\}.$$

Let P be the Easton product of the P_α for every successor α : A condition $p \in P$ is a function $p \in L$ of the form $p = \langle p_\alpha : \alpha \text{ successor cardinal} \rangle \in \prod_{\alpha \text{ succ.}} P_\alpha$ (p is stronger than q if and only if $p \supset q$) and p has Easton support: for every inaccessible cardinal κ , $|\{\alpha < \kappa \mid p(\alpha) \neq \emptyset\}| < \kappa$. Then P is the forcing which adds one Cohen set to every successor cardinal.

Let $\mathbb{P} = P \times P = \prod_{\alpha \text{ succ.}} P_\alpha \times \prod_{\alpha \text{ succ.}} P_\alpha$ be the forcing that adds simultaneously two Cohen sets to every successor cardinal.⁴ Note that $\prod_{\alpha \text{ succ.}} P_\alpha \times \prod_{\alpha \text{ succ.}} P_\alpha$ is isomorphic to $\prod_{\alpha \text{ succ.}} P_\alpha \times P_\alpha$. Let G be \mathbb{P} -generic. Then $G = \prod_{\alpha \text{ succ.}} G_0(\alpha) \times G_1(\alpha)$ and we let $G_0 = \prod_{\alpha \text{ succ.}} G_0(\alpha)$ and $G_1 = \prod_{\alpha \text{ succ.}} G_1(\alpha)$ with G_0, G_1 P -generic over L . We consider the extension $L[G_0] \subseteq L[G_0][G_1]$ and we will show, that $L[G_0]$ is not definable in $L[G_0][G_1]$ from parameters in $L[G_0]$.

The reason that we cannot apply Laver's and Hamkins' results of [Lav07] to this extension is that it does not fulfill the δ approximation property⁵: As the forcing adds a new set to every successor, the δ approximation property cannot hold at successor cardinals δ : the added Cohen set is an element of

⁴It follows by a standard argument that \mathbb{P} is pretame (and indeed tame) over (M, \mathcal{C}) , see [Fri00].

⁵A pair of transitive classes $M \subseteq N$ satisfies the δ approximation property (with $\delta \in \text{Card}^N$) if whenever $A \subseteq M$ is a set in N and $A \cap a \in M$ for any $a \in M$ of size less than δ in M , then $A \in M$. For models of set theory equipped with classes, the pair $M \subseteq N$ satisfies the δ approximation property for classes if whenever $A \subseteq M$ is a class of N and $A \cap a \in M$ for any a of size less than δ in M , then A is a class of M .

the extension and a subset of the ground model and all of its $< \delta$ approximations are elements of the ground model but the whole set is not.

Note that the forcing is weakly homogeneous, i.e. for every $p, q \in \mathbb{P}$ there is an automorphism π on \mathbb{P} such that $\pi(p)$ is compatible with q . This is because every P_α is weakly homogeneous (let $\pi(p) \in P_\alpha$ such that $\text{dom}(\pi(p)) = \text{dom}(p)$ and $\pi(p)(\lambda) = q(\lambda)$ if $\lambda \in \text{dom}(p) \cap \text{dom}(q)$ and $\pi(p)(\lambda) = p(\lambda)$ otherwise, then π is order preserving and a bijection) and therefore also P is weakly homogeneous (define π componentwise using the projection of p to p_α). Similar for $P \times P$.

To show that $L[G_0]$ is not definable in $L[G_0][G_1]$ with parameters, assume to the contrary that there is a set-parameter a_0 such that $L[G_0]$ is definable by the second-order formula $\varphi(x, a_0)$ in $L[G_0][G_1]$ from a_0 . Let α be such that $a_0 \in L[G_0 \upharpoonright \alpha, G_1 \upharpoonright \alpha]$. Now consider $a = G_0(\alpha^+)$, the Cohen set which is added to α^+ in the first component of \mathbb{P} . a is P_{α^+} -generic over $L[G_0 \upharpoonright \alpha, G_1 \upharpoonright \alpha]$ and as a is an element of $L[G_0]$ the formula φ holds for a . So we also know that there is a condition $q \in G$ such that $q \Vdash \varphi(\dot{a}, a_0)$.

Now we construct another generic $G^* = G_0^* \times G_1^*$ which produces the same extension but also an element for which φ holds and which is not an element of $L[G_0]$. This new generic adds the same sets as G , but we switch G_0 and G_1 at α^+ so that the set added by $G_1(\alpha^+)$ is now added in the new first component G_0^* . However we have to make sure that the new generic respects q so that φ is again forced in the extension. We achieve this by fixing the generic G on the length of $q(\alpha^+)$ (we can assume that the length is the same on G_0 and G_1).

It follows that $q \in G_0^* \times G_1^*$ and because of weakly homogeneity $G_0^* \times G_1^*$ is generic and $L[G_0][G_1] = L[G_0^*][G_1^*]$. Because of the construction of G^* , the formula $\varphi(x, a_0)$ holds for the set $b = G_0^*(\alpha^+)$ but b is not an element of $L[G_0]$. That is a contradiction!

□

We have seen that there are different ways of approaching class-forcing, namely on the one hand as definable from a class parameter A in a ZFC model (M, A) and on the other hand in the context of an MK model (M, \mathcal{C}) . That presents us with three notions of genericity: set-genericity, A -definable class genericity and class-genericity. In the next chapter we define the next step in this hierarchy of genericity which is definable hyperclass-genericity.

Chapter 3

Definable Hyperclass Forcing in MK^{**}

In the context of ZFC we can talk about definable class forcings as done in [Fri00], where we deal directly with the class forcing notion as it is definable from a class predicate. Here we want to develop a way of defining definable hyperclass forcings in MK, i.e. forcings with class conditions, but we will choose an indirect approach, which will allow us to reduce the technical problems as much as possible to the context of definable class forcing. So instead of talking directly about hyperclasses, we will use a correspondence between models of a variant of MK (called MK^*) and models of a variant of ZFC^- (called SetMK^*). We get an idea of how such a model of SetMK^* looks by considering the following model of MK: $\langle V_\kappa, V_{\kappa+1} \rangle$ where κ is strongly inaccessible. Similar to this model we will show how to define a model of SetMK^* with a strongly inaccessible cardinal κ which is the largest cardinal such that the sets of the MK^* model are elements of V_κ and the classes are elements of V_{κ^*} , where κ^* is the height of the SetMK^* model. We will then force over such a model with a definable class forcing which will give us an extension of the SetMK^* model. From this extension we can then go back to a model of MK^* and this is the definable hyperclass-generic extensions of the original MK^* model.

$$\begin{array}{ccc}
\text{In SetMK}^*: & M^+ & \xrightarrow[\text{forcing}]{\text{def. class}} & M^+[G] \\
& \uparrow & & \downarrow \\
\text{In MK}^*: & \mathcal{M} & \xrightarrow[\text{forcing}]{\text{def. hyperclass}} & \mathcal{M}[G]
\end{array}$$

In the following we will describe how we can go from MK^* to SetMK^* and vice versa and show that the basic properties of class forcing over a model of SetMK^* hold. Then we give an application of definable hyperclass forcing regarding minimal models of MK^{**} .

But before we develop the relation between these models further we will impose a restriction on the models we are considering.

Definition 3.1. *A model \mathcal{M} of Morse-Kelley class theory is a β -model of MK if a class is well-founded in \mathcal{M} if and only if it is true that the class is well-founded.*

We introduce this restriction for two reasons: First, we will define a coding which allows us to go from a β -model of MK^* to a transitive model of SetMK^* and this coding only works in the intended way if we know that every well-founded class in the model is really well-founded (see section 3.1). Secondly we will prove a theorem about minimal models and such a notion only makes sense if we work with minimal β -models. So from now on, we will always talk about β -models of (variants of) MK.

3.1 Coding between MK^* and SetMK^*

The associated model of set theory will be a model of ZFC^- (i.e. minus the Power Set Axiom) where we understand such a model to include the Collection (or Bounding) Principle¹. To ensure this we have to add the Class-Bounding Principle, a “class version” of the Bounding Principle, and we call the resulting axiomatic system MK^* :

Definition 3.2. *The axioms of MK^* consist of the axioms of MK plus the*

¹Note that in ZFC minus Power Set the Bounding Principle does not follow from Replacement. This is used in [Zar82], where he showed that in ZF^- the different formulations of the Axiom of Choice are not equivalent. As for MK, work done in [GH] shows that for example ultrapower constructions don’t work without first adding a version of Class-Bounding.

Class-Bounding Axiom

$$\forall x \exists A \varphi(x, A) \rightarrow \exists B \forall x \exists y \varphi(x, (B)_y)$$

where $(B)_y = \{z \mid (y, z) \in B\}$.

Note that as we have Global Choice, this is equivalent to AC_∞ :

$$\forall x \exists A \varphi(x, A) \rightarrow \exists B \forall x \varphi(x, (B)_x).$$

Equivalently, SetMK* will include the set version of Bounding (here called Set-Bounding):

$$\forall x \in a \exists y \varphi(x, y) \rightarrow \exists b \forall x \in a \exists y \in b \varphi(x, y)$$

As we will show in the proof of Theorem 3.7 and the proof of Theorem 3.12, Set-Bounding in SetMK* follows from Class-Bounding in MK* and vice versa.

We are now going to show how to translate the theory of MK* to a first-order set theory SetMK*. The axioms of SetMK* are:

1. ZFC⁻ (including Set-Bounding).
2. There is a strongly inaccessible cardinal κ .
3. Every set can be mapped injectively into κ .

We can construct a transitive model M^+ of SetMK* out of any β -model (M, \mathcal{C}) of MK* by taking all sets which are coded by a pair (M_0, R) , where M_0 belongs to \mathcal{C} and R is a binary relation within \mathcal{C} . We will show that M^+ is the unique model of SetMK* with largest cardinal κ such that $M = V_\kappa^{M^+}$ and the elements of \mathcal{C} are the subsets of M in M^+ .

To describe the coding between SetMK* and MK* we will define what a coding pair (M_0, R) is and what it means for a coding pair (M_0, R) to code a set x in a model of SetMK*.

Definition 3.3. *A pair (M_0, R) is a coding pair in the β -model $\mathcal{M} = (M, \mathcal{C})$ if M_0 is an element of \mathcal{C} with a distinguished element a , $R \in \mathcal{C}$ and R is a binary relation on M_0 with the following properties:*

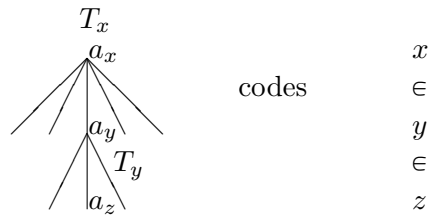
- a) $\forall z \in M_0 \exists! n$ such that z has R -distance n from a , i.e. there is an R -chain $(zRz_{n-1}R \dots Rz_1Ra)$,
- b) if $x, y, z \in M_0$ with $y \neq z$, yRx , zRx then $(M_0, R) \upharpoonright y$ is not isomorphic to $(M_0, R) \upharpoonright z$, where $(M_0, R) \upharpoonright y$ denotes the R -transitive closure below y (i.e. y together with all elements which are connected to y via an R -chain), respectively for z ,
- c) if $y, z \in M_0$ are on level n (i.e. have the same R -distance n from a) and $y \neq z$ then $vRy \rightarrow \neg(vRz)$,
- d) R is well-founded.

Note that in the definition of the codes in (M, \mathcal{C}) we need the assumption that (M, \mathcal{C}) is a β -model as for a class to code a set in M^+ it has to be well-founded not only in the MK model but “in the real world”.

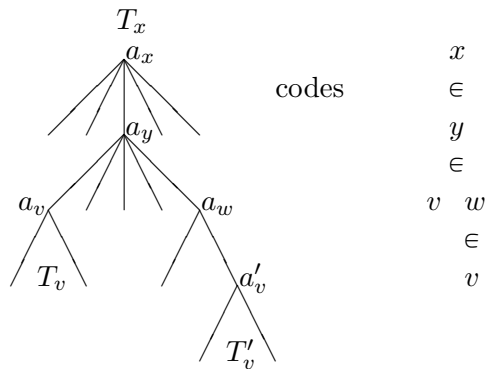
The meaning of the definition becomes clearer when we view the coding pair as a tree T whose nodes are exactly the distinct elements of M_0 , the top node is a and R is the extension relation of the tree. A tree T' with top node a' is a subtree of T if a' is a node of T and T' contains all T -nodes (not only immediately) below a' . If T' is a subtree of T such that a' lies directly below a then T' is called a direct subtree of T . Then property b) states that for every node x distinct direct subtrees are not isomorphic and property c) implies that the trees below two distinct points on the same level are disjoint (and not only on the next level).

The idea behind the coding pairs is, that every coding pair will define a unique set x in the SetMK* model. Note that at the same time every x in M^+ can correspond to different coding pairs in \mathcal{M} .

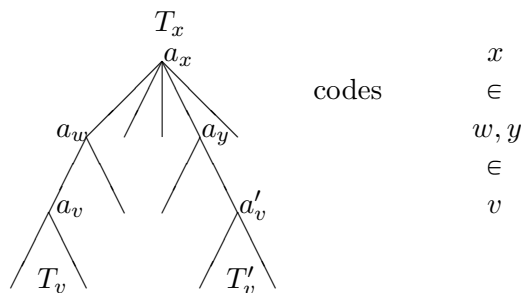
In the following we will give some intuition on what such a correspondence between coding pairs in \mathcal{M} and sets in M^+ should look like: Every $x \in M^+$ is coded by a tree T_x where x is associated to the top node a_x of T_x , the elements $y \in x$ are associated to the nodes on the first level below a_x so that every node on this level gives rise to a subtree T_y which codes y so that the elements of y are associated to the nodes on the second level below a_x and so on:



Note that there are only countably many levels but a level can have class many elements. If two elements a_y, a_z have the same R_x predecessor (i.e. are connected to the same node on the previous level) their subtrees T_y, T_z will never be isomorphic and therefore don't code the same element of M^+ (by property b) of Definition 3.3). But it can happen that there are isomorphic subtrees on different levels or on the same level but not connected to the same node on the level above. This can be made clear in the following two examples: First let $y \in x, v \in y$ and $w \in y$ and $v \in w$. Then there are two isomorphic trees T_v and T'_v both coding v but on different levels:



Secondly let $v \in w, v \in y$ and $w, y \in x$. Again there are two isomorphic trees T_v and T'_v coding v but this time on the same level:

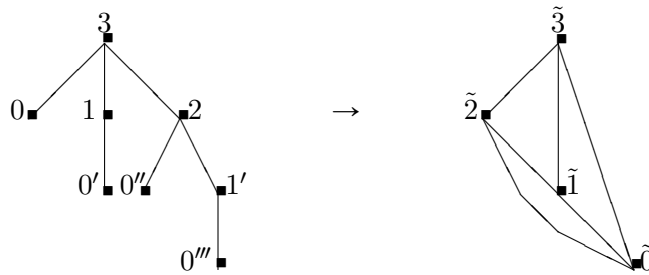


To show that Definition 3.3 indeed defines a coding, we have to show that there is a correspondence between x and its coding pair. As we want to include non-transitive sets we will work with $(TC(\{x\}), \in)$ (note that we used the transitive closure of $\{x\}$ rather than the transitive closure of x as the transitive closure of two different sets could be the same). As we have seen, the coding tree will have a lot of isomorphic subtrees, for example many different pairs $(a_i, \{\})$ coding the empty set. So the tree T_x itself will not be isomorphic to $(TC(\{x\}), \in)$ and we will have to collapse (M_x, R_x) to a structure $(M_x, R_x)/\approx$ in which we have identified all these isomorphic subtrees. We define this quotient of the coding pair in the following way:

Definition 3.4. For a coding pair (M_0, R) , let $[a] = \{b \in M_0 \mid (M_0, R) \upharpoonright b \text{ isomorphic to } (M_0, R) \upharpoonright a\}$ be the equivalence class of all the top nodes of subtrees of the coding tree T which are isomorphic to the subtree T_a (here $(M_0, R) \upharpoonright b$ denotes the “sub-coding pair” which is the subtree T_b as detailed in Definition 3.3). By Global Choice let \tilde{a} be a fixed representative of this class. Then let $\tilde{M}_0 = \{\tilde{a} \mid [a] \text{ for all } a \in M_0\}$ and define the relation \tilde{R} as follows: $\tilde{a}\tilde{R}\tilde{b}$ iff $\exists a_0, b_0$ such that $a_0 \in [a]$ and $b_0 \in [b]$ and $a_0 R b_0$.

Note that if $a_0 \approx a_1$ and $b_0 R a_0$ then there is b_1 with $b_1 R a_1$ such that $b_0 \approx b_1$ as the isomorphism between T_{a_0} and T_{a_1} will restrict to the trees T_{b_0} and T_{b_1} .

The following example shows how this quotient structure looks for a possible coding tree of the set 3:



As one can see, the resulting structure $(\tilde{M}_3, \tilde{R}_3)$ is then isomorphic to $(TC(\{3\}), \in)$. In the following we will show that this construction works in general:

Lemma 3.5. Let (M_0, R) be a coding pair. Then the quotient structure (\tilde{M}_0, \tilde{R}) as defined in Definition 3.4 is extensional and well-founded.

Proof. By Class-Bounding $\tilde{R} \in \mathcal{C}$ and \tilde{R} is well-founded as we can always find an R -minimal element a , build the equivalence class $[a]$ and find its representative \tilde{a} . Then \tilde{a} is \tilde{R} minimal as otherwise there exists \tilde{a}' such that $\tilde{a}'\tilde{R}\tilde{a}$ and therefore there is $a'_0 \in [a']$ such that a'_0Ra .

To show that \tilde{R} is extensional, let $\tilde{y}, \tilde{z} \in \tilde{M}_0$ with $\tilde{y} \neq \tilde{z}$ and assume that they have the same extension $\{\tilde{x} \mid \tilde{x}\tilde{R}\tilde{y}\} = \{\tilde{x} \mid \tilde{x}\tilde{R}\tilde{z}\}$. Going back to (M_0, R) this means that the elements of the related equivalence classes $[y], [z]$ have the same isomorphism types of children, i.e. for every $x_0, y_0, z_0 \in M_0$ with $x_0Ry_0, y_0 \in [y]$ and $z_0 \in [z]$ we can find x_1 with x_1Rz_0 such that $x_0, x_1 \in [x]$. By using property *b*) of Definition 3.3 it follows that the $[y] = [z]$, because we do not have multiplicities in (M_0, R) , i.e. isomorphic subtrees that are connected to the same R -predecessor. It follows that $\tilde{y} = \tilde{z}$. \square

Note that the quotient structure always has a fixed top node which is the representative of the equivalence class of the distinguished node of (M_0, R) , which has the distinguished node as its only element.

It follows from Mostowski's Theorem that there is a unique transitive structure with the \in -relation that is isomorphic to (\tilde{M}_0, \tilde{R}) . This structure then has the form $(TC(\{x\}), \in)$ for a unique set x .

Definition 3.6. A coding pair (M_x, R_x) is called a coding pair for x , if x is the unique set such that $(\tilde{M}_x, \tilde{R}_x)$ is isomorphic to $(TC(\{x\}), \in)$.

In the following we will use this coding to associate a transitive model of $SetMK^*$ to each β -model of MK^* and vice versa.

Theorem 3.7. Let $\mathcal{M} = (M, \mathcal{C})$ be a β -model of MK^* and

$$M^+ = \{x \mid \text{there is a coding pair } (M_x, R_x) \text{ for } x\}$$

Then M^+ is the unique, transitive set that obeys the following properties:

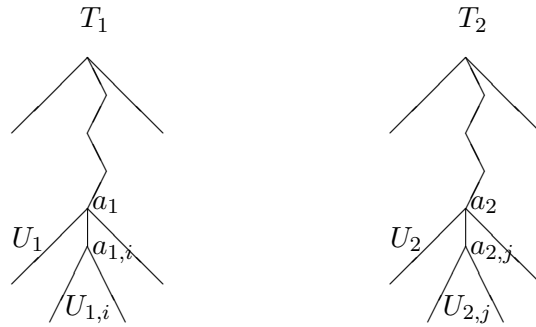
- a) $M^+ \models SetMK^*$,
- b) $\mathcal{C} = P(M) \cap M^+$,
- c) $M = V_\kappa^{M^+}$, κ is the largest cardinal in M^+ and strongly inaccessible in M^+ .

The coding between \mathcal{M} and M^+ is the key to prove the theorem. So before proving this theorem we will prove two useful fact about the coding.

As we have seen there can be more than one coding pair for an $x \in M^+$. Of course these coding pairs are isomorphic because they are all built according to Definition 3.3 but we also would like to know that they are isomorphic in \mathcal{M} . For elements of M^+ that can be coded by sets in \mathcal{M} this is trivial but for elements that are coded by proper classes we have to show the following:

Lemma 3.8 (Coding Lemma 1). *Let $\mathcal{M} = (M, \mathcal{C})$ be a transitive β -model of MK^* . Let $N_1, N_2 \in \mathcal{C}$ and R_1, R_2 be well-founded binary relations in \mathcal{C} such that (N_1, R_1) and (N_2, R_2) are coding pairs as described in Definition 3.3. Then if there is an isomorphism between (N_1, R_1) and (N_2, R_2) there is such an isomorphism in \mathcal{C} .*

Proof. Let T_1, T_2 be the coding trees associated to the coding pairs $(N_1, R_1), (N_2, R_2)$. Assume to the contrary that there is an isomorphism between T_1 and T_2 but not one in \mathcal{C} . It follows that the tree below the top node of T_1 is isomorphic to the tree below the top node of T_2 , but there is no such isomorphism in \mathcal{C} . Then, as T_1 and T_2 are well-founded we can choose a T_1 -minimal node a_1 of T_1 such that for some node a_2 of T_2 the tree U_1 (the tree T_1 below and including a_1) is isomorphic to U_2 (the tree T_2 below and including a_2) but there is no isomorphism in \mathcal{C} . Because of the minimality of a_1 we know that for every node $a_{1,i}$ of U_1 just below a_1 and every node $a_{2,j}$ of U_2 just below a_2 , if $U_{1,i}$ is isomorphic to $U_{2,j}$ then there is an isomorphism in \mathcal{C} . Moreover the property “ $U_{1,i}, U_{2,j}$ are isomorphic” is expressible in (M, \mathcal{C}) .



Now we can apply the Class Bounding Principle of MK^* to get a class B so that for each $a_{1,i}, a_{2,j}$ for which $U_{1,i}, U_{2,j}$ are isomorphic, $(B)_c$ is such an isomorphism for some set c . Using the global well-order of M we can

choose a unique $c(a_{1,i}, a_{2,j})$ for each relevant pair $\langle a_{1,i}, a_{2,j} \rangle$ and combine the isomorphisms $(B)_{c(a_{1,i}, a_{2,j})}$ to get an isomorphism between U_1 and U_2 in \mathcal{C} , which is a contradiction. \square

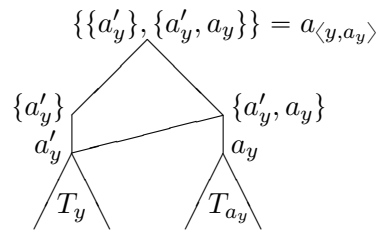
So all coding trees of the same element of M^+ are isomorphic in \mathcal{C} . For the converse it is obvious that two isomorphic coding trees code the same element in M^+ as they give rise to the same $(\tilde{M}_x, \tilde{R}_x)$.

The next lemma shows that we are able to see something of the coding in M^+ :

Lemma 3.9 (Coding Lemma 2). *For all $x \in M^+$ there is a one-to-one function $f \in M^+$ such that $f : x \rightarrow M_x$, where (M_x, R_x) is a coding pair for x .*

Proof. Let T_x be a coding tree for x and for each $y \in x$ let T_y is the subtree of T_x with top node a_y lying just below the top node of T_x such that T_y codes y . Note that the choice of a_y is unique after having fixed the tree T_x .

To show that $f = \{\langle y, a_y \rangle \mid y \in x\}$ belongs to M^+ , we have to find a coding tree for f . Firstly we construct a coding tree $T_{\langle y, a_y \rangle}$ for every $\langle y, a_y \rangle$ with $y \in x$. As a_y is a set in M , it is a set in M^+ and therefore coded by some T_{a_y} . So we can build $T_{\langle y, a_y \rangle}$ by connecting the trees T_y and T_{a_y} . To make sure that the relation $R_{\langle y, a_y \rangle}$ on the new tree is well-defined we can relabel the nodes of the tree T_{a_y} and so we get the following picture:



In this way we code every pair $\langle y, a_y \rangle$ with $y \in x$ and we can now join all the codes to code f .

Let (M_f, R_f) be the following pair: $M_f = \bigcup_{z \in x} M_{\langle z, a_z \rangle} \cup \{a_f\}$ where $a_f \in M$ and $a_f \notin M_{\langle z, a_z \rangle}$ for every $z \in x$. Then R_f is the binary relation

which is defined using R_x as parameter:

$$R_f = \{ \langle v, w \rangle \mid \text{for some } y \in x \text{ either } \langle v, w \rangle \in R_{\langle y, a_y \rangle} \text{ or } v = a_{\langle y, a_y \rangle} \text{ and } w = a_f \}$$

M_f and R_f are well-defined because of Class Comprehension in MK^* and so f is coded by the tree T_f which is ordered by R_z below every a_z and by putting $a_{\langle y, a_y \rangle}$ below a_f otherwise.

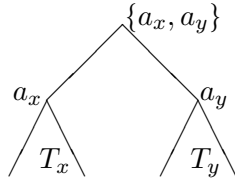
□

Now we give the proof of Theorem 3.7.

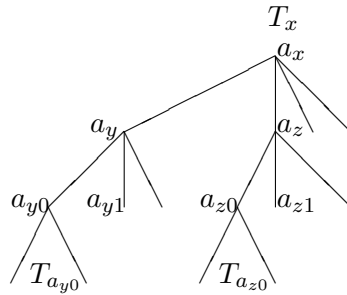
Proof. a) We show that if \mathcal{M} is a β -model of MK^* then $M^+ \models \text{Set}MK^*$. The first step is proving that M^+ satisfies ZFC^- with Set-Bounding.

Observe that M^+ is transitive: Let $x \in M^+$. Then for every $y \in x$ there is a coding tree for y (namely the corresponding subtree of T_x). Therefore $y \in M^+$ and so $x \subseteq M^+$. From transitivity it follows that Extensionality and Foundation hold in M^+ ; Infinity follows as $\omega \in M^+$.

Pairing: Let x, y be coded by T_x, T_y respectively. Then $\{x, y\}$ is coded by the tree:



Union: Let x be coded by T_x :



The obvious way to code $\bigcup x$ would be to join the $a_{y0}, a_{y1}, \dots, a_{z0}, a_{z1}, \dots$

together by one top node $a_{\cup x}$. But in general this is not a coding tree by reasons of isomorphism: Our coding trees have the property that subtrees which are connected to the same node on the next level above are all pairwise non-isomorphic. In this case that means that the trees T_{a_y}, T_{a_z}, \dots are pairwise non-isomorphic, as are the trees $T_{a_{y0}}, T_{a_{y1}}, \dots$ and the trees $T_{a_{z0}}, T_{a_{z1}}, \dots$ and so on. But, as we explained before, it can happen that some of the $T_{a_{y_i}}$ are isomorphic to, for example, some of the $T_{a_{z_j}}$. So if we connect these trees by a top node the resulting tree would have isomorphic subtrees connected by the same node on the next level and therefore would not be a coding tree. This problem can easily be resolved by taking equivalence classes of the subtrees of T_x from the second level below a_x (where two trees are equivalent if they are isomorphic). Then we take a representative from each equivalence class and connect them to the top node $a_{\cup x}$ (as before, this is possible by Class Comprehension in MK^* and Coding Lemma 1).

To prove Comprehension and Bounding we need to take a closer look at how formulas in M^+ translate to formulas in \mathcal{M} :

Lemma 3.10. *For each first-order formula φ there is a formula ψ of second-order class theory such that for all $x_1, \dots, x_n \in M^+$, $M^+ \models \varphi(x_1, \dots, x_n)$ if and only if $\mathcal{M} \models \psi(c_1, \dots, c_n)$ for any choice of codes c_1, \dots, c_n for x_1, \dots, x_n .*

Proof. The proof is by induction over the complexity of the formula φ . For the first atomic case assume that $M^+ \models y \in x$. Let c_x and c_y be codes for x and y respectively and let T_x, T_y be the associated coding trees. As we know that $y \in x$ it follows that there is a direct subtree $T_{y'}$ of T_x such that $T_{y'}$ is a coding tree for y (“direct subtree” means a subtree whose top node lies just below the top node of the original tree). As $T_{y'}$ and T_y are both codes for y they are isomorphic and by Coding Lemma 1 we know that they are isomorphic in \mathcal{M} . So $\mathcal{M} \models$ “ c_y is isomorphic to a direct subtree of c_x ” and this therefore is the desired ψ .

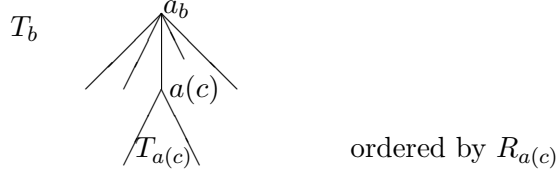
For the second atomic case assume that $M^+ \models y = x$. Let c_x and c_y be codes for x and y respectively. As $y = x$, c_y is also a code for x and again by Coding Lemma 1 we know that the codes are isomorphic in \mathcal{M} thus giving us the desired ψ .

The cases of $\neg\varphi$, $\varphi_1 \wedge \varphi_2$ follow easily by using the induction hypothesis.

For the quantor case assume that $M^+ \models \forall x \varphi$. Let c_x be a code for x . By induction hypothesis let ψ be the second-order formula associated to φ such that $\mathcal{M} \models \psi$. Then $M^+ \models \forall x \varphi$ translates to $\mathcal{M} \models \forall c_x \psi$. \square

Comprehension: Let $a, x_1, \dots, x_n \in M^+$ and let $\varphi(x, x_1, \dots, x_n, a)$ be any first-order formula. We will show that $b = \{x \in a : M^+ \models \varphi(x, x_1, \dots, x_n, a)\}$ is an element of M^+ by using Class Comprehension in \mathcal{M} to find the corresponding $B \in \mathcal{C}$ and build from it a coding tree for b .

Let $T_{x_1}, \dots, T_{x_n}, T_a$ be codes for the corresponding elements of M^+ and let ψ be the formula corresponding to φ provided by Lemma 3.10. Assume that b is non-empty, i.e. that there is x_0 in a such that φ holds. Therefore there is a c_0 such that $\psi(c_0, T_{x_1}, \dots, T_{x_n}, T_a)$ holds. Let c be a variable that varies over the level directly below the top level of T_a so that each $T_{a(c)}$ denotes a direct subtree of T_a . Then by Class Comprehension there is a class B such that if $\psi(T_{a(c)}, T_{x_1}, \dots, T_{x_n}, T_a)$ holds then $(B)_c$ is the direct subtree $T_{a(c)}$ of T_a and if not then $(B)_c$ is T_{c_0} . So let T_b be the coding tree with top node a_b and whose direct subtrees are all of the $(B)_c$:



Then T_b codes $b \in M^+$ with $b = \{x \in a : \varphi(x, x_1, \dots, x_n, a)\}$.

Bounding: We have to show that for $a \in M^+$ and φ a first-order formula

$$M^+ \models \forall x \in a \exists y \varphi(x, y) \rightarrow \exists b \forall x \in a \exists y \in b \varphi(x, y).$$

So assume that $\forall x \in a \exists b \varphi(x, y)$. Let T_y, T_a be coding trees for y and a respectively and let ψ be the second-order formula corresponding to φ provided by Lemma 3.10. By Class-Bounding in MK* we know that

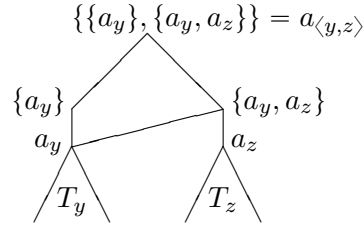
$$\exists B \forall T_x \text{ direct subtree of } T_a \exists y' \psi(T_x, (B)_{y'}),$$

where $(B)_{y'} = \{z \mid (y', z) \in B\}$. By Class Comprehension we can join to-

gether all the section $(B)_{y'}$ which are coding trees $T_{(B)_{y'}}$ to obtain a tree T_b with top node a_b such that the $T_{(B)_{y'}}$ are the direct subtrees of T_b . It follows that in \mathcal{M} there is a tree T_b such that for every tree T_x subtree of T_a there is a $T_{(B)_{y'}}$ direct subtree of T_b such that $\psi(T_x, T_{(B)_{y'}})$ and the tree T_b gives us the desired b in M^+ .

Replacement: Follows from Comprehension and Bounding.

Choice: We have to show that every element of M^+ can be well-ordered (we aim for the strongest version of the axiom of Choice in a set-theory without Power Set (see [Zar82]). So let $x \in M^+$ and let T_x be a coding tree for x with top node a_x . We know that the direct subtrees T_y of T_x code the elements y of x and their top nodes a_y are elements of M . As we have a well-order of M we can well-order the class $B = \{a_y \mid a_y \text{ is the top node of a direct subtree } T_y \text{ of } T_x\}$. We call this well-order W . Now we can build a tree for every pair $\langle a_y, a_z \rangle \in W$ by using the trees T_y, T_z analogous as we did in the proof of Coding Lemma 2:



So for every $\langle a_y, a_z \rangle \in W$ we get a coding tree for the pair $\langle y, z \rangle$ with $y, z \in x$. As we have shown in the proof of Coding Lemma 2 we can now join together the trees by a single top node a_w using Class Comprehension. We now get a tree T_w which is a coding tree for an element w of M^+ and w is a well-order of x .

Remark 3.11. *The next two results below (b and c) will show, that there even is a global choice function for the sets in $V_\kappa^{M^+}$ for κ an inaccessible cardinal, as there is a class which well-orders M and we will show that every class in \mathcal{C} is an element of M^+ .*

b) We have to show that $\mathcal{C} = P(M) \cap M^+$. So assume that $X \in \mathcal{C}$ and $y \in X$. Then $y \in M$ and so can be coded by the following tree: y is the top node of the tree T_y . On the first level below the top node there are nodes for every element of y which are named by pairwise different elements z_i of $M \setminus \{y\}$. On the first level below such an z_i there are nodes for every element in z_i named by pairwise different elements v_j of $M \setminus \{y, z_i\}$ and so on. So T_y is a coding tree for y and therefore $y \in M^+$. This can be done for all $y \in X$ and by Class Comprehension the trees T_y can be connected to a tree T_X with top node a_X . Then the pair (M_X, R_X) gives a code for X with $M_X = \bigcup_{y \in X} M_y \cup \{a_X\}$ and

$$R_X = \{\langle v, w \rangle \mid \text{for some } y \in X \text{ either } \langle v, w \rangle \in R_y \text{ or } v = a_y \text{ and } w = a_X\}$$

Therefore $X \in M^+$.

For the converse, let $x \in M^+$ and $x \subseteq M$. Then there exists a coding pair (M_x, R_x) of x such that $(\tilde{M}_x, \tilde{R}_x) \cong (TC\{x\}, \in)$ (see Lemma 3.5). As $(\tilde{M}_x, \tilde{R}_x)$ is in \mathcal{C} , has rank $\text{Ord}(M)$ and we can build $TC(\{x\})$ by transfinite induction from $(\tilde{M}_x, \tilde{R}_x)$, we can decode x in \mathcal{C} and so $x \in \mathcal{C}$.

c) Now we will show that there is a strongly inaccessible cardinal κ in M^+ which is the largest cardinal in M^+ and the elements of M (the sets in \mathcal{M}) are exactly the elements of $V_\kappa^{M^+}$.

Let κ be $\text{Ord}(M)$. Then as $\kappa \subseteq M$ and $\kappa \in M^+$ it follows from b) that κ is a class in \mathcal{C} . Let $f : \beta \rightarrow \kappa$ with β is a ordinal less than κ be a function in \mathcal{C} . From the Class Bounding Principle it follows that f is bounded in κ . So κ is regular in \mathcal{M} and therefore regular in M^+ . Moreover, again by b), any subset of an ordinal β of M which belongs to M^+ is a class in \mathcal{C} and indeed a set in M , so the power set of β in M^+ equals the power set of β in M and so κ is strongly inaccessible. It follows that if $x \in M$ then $x \in V_\kappa^{M^+}$. For the converse let $x \in V_\kappa^{M^+}$ and let (M_x, R_x) be a coding pair and T_x the associate coding tree for x . By Coding Lemma 2 any coding tree of a set is a set, so T_x is an element of M . Clause 3 of the axioms of SetMk^* follows directly from Coding Lemma 2 and so κ is the largest cardinal in M^+ .

That M^+ is unique follows from its construction: Let M^{++} be another such model of SetMk^* (i.e. it is transitive, $\mathcal{C} = P(M) \cap M^{++}$ and $M =$

$V_\kappa^{M^{++}}$ with κ largest cardinal in M^{++} and strongly inaccessible cardinal in M^{++}). Then M^+ and M^{++} have the same largest cardinal κ , they have the same subsets of κ and as every set in both models can be coded by a subset of κ they are the same.

This concludes the proof of Theorem 3.7. \square

The converse of Theorem 3.7 follows by the corresponding axioms in the $SetMK^*$ model:

Theorem 3.12. *Let N be a transitive model of $SetMK^*$ that has a strongly inaccessible cardinal κ that is the largest cardinal, let $\mathcal{C} = P(M) \cap N$ and M is defined to be V_κ^N . Then $\mathcal{M} = (M, \mathcal{C})$ is a β -model of MK^* and the model M^+ derived from \mathcal{M} by Theorem 3.7 equals N .*

Proof. We have to show that (M, \mathcal{C}) fulfills the axioms of MK^* : Extensionality, Pairing, Infinity, Union, Power Set, and Foundation follow directly by the corresponding axioms of $SetMK^*$. By the definition of M and \mathcal{C} it follows that every set is a class and elements of classes are sets.

For the remaining axioms, note that there is an easy converse for Lemma 3.10: For each formula φ of second-order class theory there is a first-order formula ψ such that for all $x_1, \dots, x_n \in \mathcal{M}$, $\mathcal{M} \models \varphi(x_1, \dots, x_n)$ if and only if $N \models \psi(x_1, \dots, x_n)$. This holds because by assumption all elements of \mathcal{M} are elements of \mathcal{C} or M and therefore elements of N and so φ and ψ are the same where the statement that x is a set in \mathcal{M} translates to $x \in V_\kappa^N$ and the statement that X is a class in \mathcal{M} translates to $X \in P(M) \cap N$. So for Class Comprehension we have to show that the following holds:

$$\forall X_1 \dots \forall X_n \exists Y Y = \{x : \varphi(x, X_1, \dots, X_n)\}$$

where φ is a formula containing class parameters in which quantification over both sets and classes is allowed. By the definition of M and \mathcal{C} this statement is exactly the Comprehension Axiom of N where ψ is the first-order formula corresponding to φ : $y = \{x \in V_\kappa^N : N \models \psi(x, x_1, \dots, x_n, V_\kappa^N)\}$.

For Class Bounding we have to show:

$$\forall x \exists A \varphi(x, A) \rightarrow \exists B \forall x \exists y \varphi(x, (B)_y)$$

where $(B)_y = \{z \mid (y, z) \in B\}$. So assume that $\forall x \exists A \varphi(x, A)$ holds in \mathcal{M} .

Then translating this to N we know by Set-Bounding that

$$\forall x \in V_\kappa^{M^+} \exists A \in P(M) \cap M^+ \psi(x, A) \rightarrow \exists b \forall x \in V_\kappa^{M^+} \exists y \in b \psi(x, y)$$

where ψ is the first-order formula corresponding to φ . By Set-Comprehension we can form a set b_0 from b such that $b_0 = \{y \mid y \in b \wedge y \subseteq V_\kappa^N\}$. Then there is a function $f \in N$ from V_κ^N onto b_0 (as b_0 has size less or equal κ) and so f is also an element of \mathcal{M} . Then we can define the class $(B)_z = \{w \mid w \in f(z)\}$ and therefore also $B = \{(z, w) \mid z \in V_\kappa^N \wedge w \in f(z)\}$. So Class-Bounding holds.

For Global Choice we have to show that there is a well-ordering of M . We know that every element of N can be well-ordered and so V_κ^N can be well-ordered. The well-order is therefore an element of \mathcal{C} .

(M, \mathcal{C}) has to be a β -model: Any well-founded relation in (M, \mathcal{C}) corresponds to a well-founded relation in N and because N is a transitive model of ZF^- , well-foundedness is absolute (we can define a rank function into the “real” ordinals which witnesses the well-foundedness in V).

Finally when we build the M^+ of \mathcal{M} according to Theorem 3.7, M^+ and N are both transitive, have the same largest cardinal κ and the same subsets of κ and are therefore equal. \square

Remark 3.13. *We can also use this switching between models of MK^* and $SetMK^*$ for class-forcing: Instead of doing class-forcing over MK^* we go to $SetMK^*$ and do a set-forcing there. Note that by doing this indirect version of class-forcing we don't lose the tameness requirement for the forcing: Assume the class-forcing is not tame (as for example a forcing which collapses the universe to ω). Then we go to $M^+ \models SetMK^*$ and force with the associated set-forcing. But such a forcing destroys the inaccessibility of κ and therefore the preservation of PowerSet in the MK^* extension $\mathcal{M}[G]$.*

Corollary 3.14.

$$M^+ = \bigcup_{C \in \mathcal{C}} L_{\kappa^*}(C).$$

where κ^* is the height of M^+ and

$$\begin{aligned} L_0(C) &= TC(\{C\}) \\ L_{\beta+1}(C) &= Def(L_\beta(C)) \\ L_\lambda(C) &= \bigcup_{\beta < \lambda} L_\beta(C), \lambda \text{ limit.} \end{aligned}$$

Proof. Let $x \in M^+$. Then there is a coding pair (M_x, R_x) for x such that $(\tilde{M}_x, \tilde{R}_x)$ is isomorphic to $(TC(\{x\}), \epsilon)$. As \tilde{M}_x and \tilde{R}_x are elements of \mathcal{C} we can code the pair $(\tilde{M}_x, \tilde{R}_x)$ by a class $C_x \in \mathcal{C}$. As C_x is an element of M^+ , $L_{\kappa^*}(C_x)$ is an inner model in M^+ . But now we can decode x in $L_{\kappa^*}(C_x)$ as we can build $(TC(\{x\}))$ by transfinite induction from $(\tilde{M}_x, \tilde{R}_x)$. So $x \in L_{\kappa^*}(C_x)$.

For the converse, let $x \in \bigcup_{C \in \mathcal{C}} L_{\kappa^*}(C)$, i.e. there is an $C_x \in \mathcal{C}$ such that $x \in L_{\kappa^*}(C_x)$. As $L_{\kappa^*}(C_x)$ is an inner model of \mathcal{M} , x is an element of \mathcal{C} and by Theorem 3.7 b) it is an element of M^+ . \square

3.2 Hyperclass Forcing and Forcing in SetMK**

In the last section we have seen how to move back and forth between a model of MK* and its associated SetMK* model. Now we will use this relation between a model of class theory and a model of set theory to define hyperclass forcing. A hyperclass is a collection whose elements are classes. The key idea is that instead of trying to formalize forcing for a definable hyperclass forcing notion, we can go to the associated model of SetMK* where the forcing notion is now a class and so we force with a definable class forcing there and then go back to a new MK* model. First let us define the relevant notions:

Definition 3.15. Let $\mathcal{M} = (M, \mathcal{C})$ be a model of MK* and for $\mathbb{P} \subseteq \mathcal{C}$ let $(\mathbb{P}, \leq) = \mathbb{P}$ be an \mathcal{M} -definable partial ordering with a greatest element $1^{\mathbb{P}}$. $P, Q \in \mathbb{P}$ are compatible if for some R , $R \leq P$ and $R \leq Q$. A definable hyperclass $D \subseteq \mathbb{P}$ is dense if $\forall P \exists Q (Q \leq P \text{ and } Q \in D)$. Then a $\mathbb{G} \subseteq \mathcal{C}$ is called a \mathbb{P} -generic hyperclass over \mathcal{M} iff \mathbb{G} is a pairwise compatible, upward-closed subcollection of \mathbb{P} which meets every dense subcollection of \mathbb{P} which is definable over \mathcal{M} .

We will assume that for each $P \in \mathbb{P}$ there exists \mathbb{G} such that $P \in \mathbb{G}$ and

\mathbb{G} is \mathbb{P} -generic over \mathcal{M} (this is always possible if the model \mathcal{M} is countable).

To define the structure $(M, \mathcal{C})[\mathbb{G}]$ where \mathbb{G} is a \mathbb{P} -generic hyperclass over (M, \mathcal{C}) we will use Theorem 3.7 and Proposition 3.12. By Theorem 3.7 we go to the model $M^+ \models \text{SetMK}^*$. As \mathbb{P} is a subcollection of \mathcal{C} in \mathcal{M} it becomes a subclass of $P(M) \cap M^+$ and is an M^+ -definable class, \mathbb{G} remains a pairwise compatible, upward-closed subclass of \mathbb{P} which meets every dense subclass of \mathbb{P} which is definable over M^+ and therefore is definable class-generic over M^+ . Then we define names, their interpretation and the extension of M^+ as usual: A \mathbb{P} -name in M^+ is a set in M^+ consisting of pairs (τ, p) where τ is a \mathbb{P} -name in M^+ and p belongs to \mathbb{P} (as we are in the set model we now denote the elements of \mathbb{P} with lower-case letters). Then $\mathcal{N} = \cup\{\mathcal{N}_\alpha \mid \alpha \in \text{Ord}(M^+)\}$ is the collection of all names where $\mathcal{N}_0 = \emptyset$, $\mathcal{N}_{\alpha+1} = \{\sigma \mid \sigma \text{ is a subset of } \mathcal{N} \times P \text{ in } M^+\}$ and $\mathcal{N}_\lambda = \cup\{\mathcal{N}_\alpha \mid \alpha < \lambda\}$ for a limit ordinal λ . For a \mathbb{P} -name σ its interpretation is $\sigma^{\mathbb{G}} = \{\tau^{\mathbb{G}} \mid p \in \mathbb{G} \text{ for some } (\tau, p) \in \sigma\}$. Then $M^+[\mathbb{G}]$ is the set of all such $\tau^{\mathbb{G}}$. Finally we can define the extension of \mathcal{M} :

Definition 3.16. *Let $\mathcal{M} = (M, \mathcal{C})$ be a β -model of MK^* , \mathbb{P} be a definable hyperclass forcing and $\mathbb{G} \subseteq \mathbb{P}$ be a \mathbb{P} -generic hyperclass over \mathcal{M} . Let $M^+[\mathbb{G}] \models \text{SetMK}^*$ with largest cardinal κ with $M^+[\mathbb{G}]$ transitive. Then $\mathcal{M}[\mathbb{G}] = (M, \mathcal{C})[\mathbb{G}]$ is the β -model of MK^* derived from $M^+[\mathbb{G}]$ by Theorem 3.12, whose sets are the elements of $V_\kappa^{M^+[\mathbb{G}]}$ and whose classes are the subsets of $V_\kappa^{M^+[\mathbb{G}]}$ in $M^+[\mathbb{G}]$, where κ is the largest cardinal of $M^+[\mathbb{G}]$ and is strongly inaccessible. Such a model is called a definable hyperclass-generic outer model of \mathcal{M} .*

This definition assumes that the definable class-forcing \mathbb{P} again produces a model of SetMK^* with the same largest cardinal κ where κ is strongly inaccessible (we say in short that \mathbb{P} does not change κ). Unfortunately the assumption that SetMK^* is preserved is not as straightforward as it might seem. Definable class-forcing was developed by [Fri00]. There the concept of pretameness and tameness of a forcing notion is introduced and it is shown that such a forcing has a definable forcing relation and preserves the axioms. In the case of SetMK^* we now have the added problem that we are not forcing over a model of full ZFC but rather over ZFC^- , i.e. without the Power Set Axiom. This can cause problems when we use concepts like the hierarchy of the V_α , for example to prove that pretame class-forcings

preserve the Replacement (or in our case the Set-Bounding) Axiom. So we cannot simply transfer the results of [Fri00] but have to prove the Definability Lemma and the preservation of the axioms again without making use of the Power Set Axiom.

To define definable class-forcing in SetMK* first note that the following still holds: Let M^+ be a transitive model of SetMK*, P be a M^+ -definable forcing notion and G P -generic over M^+ . Then $M^+[G]$ is transitive and $Ord(M^+[G]) = Ord(M^+)$. It follows from the definition of the interpretation of names and the definition of $M^+[G]$ that if $y \in \sigma^G$ then $y = \tau^G$ for some $\tau \in TC(\sigma)$ and therefore $M^+[G]$ is transitive. Furthermore for every $x \in Ord(M^+)$ there exists a name σ for x (i.e. $x = \sigma^G$ as defined above) with name-rank of $\sigma =$ the least $\alpha \in Ord(M)$ such that $\sigma \in \mathcal{N}_{\alpha+1}$ and by induction the von Neumann rank of σ^G is at most the name rank of σ . So we know that if “new” sets are added by the forcing they have size at most the “old” sets from M^+ and so $Ord(M^+[G]) \subseteq Ord(M^+)$.

We will first treat the case where we already assume that the forcing relation is definable and P is a pretame class-forcing and then show how we can ensure that in general pretame class-forcings preserve the axioms and the Definability Lemma holds.

Proposition 3.17. *Let M^+ be a model of SetMK* and let P be a pretame definable class-forcing over M^+ that does not change κ and whose forcing relation is definable. Let $G \subseteq P$ be definable class-generic over M^+ . Then $M^+[G]$ is a model of SetMK*.*

Proof. Extensionality, Pairing, Comprehension, Infinity, Foundation and Choice still hold by the proof for definable class-forcing over full ZFC. We have to show that Set-Bounding holds in $M^+[G]$, i.e.

$$M^+[G] \models \forall x \in a \exists y \varphi(x, y) \rightarrow \exists b \forall x \in a \exists y \in b \varphi(x, y)$$

Let σ be a name for a . We can extend any p for which $p \Vdash \forall x \in \sigma \exists y \varphi(x, y)$ to force that there is an isomorphism between σ and an ordinal α (by using AC) and so we can assume without loss of generality that σ is $\check{\alpha}$ where $\alpha \in Ord$ and therefore $p \Vdash \forall x < \alpha \exists y \varphi(x, y)$. Then for such a fixed p and for each $x < \alpha$ we can define by the Definability of the forcing relation $D_x = \{q \leq p \mid \exists \tau q \Vdash \varphi(x, \tau)\}$ where D_x is dense below p . By pretameness there is a $q \leq p$ and $\langle d_x \mid x < \alpha \rangle \in M^+$ such that for all $x < \alpha$, d_x is pretense

$\leq q$ and by genericity there is such a q in G . Then we know that for all pairs $\langle x, r \rangle$ where $x < \alpha$ and $r \in d_x$ there is τ such that $r \Vdash \varphi(x, \tau)$. By the Set-Bounding principle in M^+ we get a set $T \in M^+$ such that $\forall(x, r)$ with $r \in d_x \exists \tau \in T$ such that $r \Vdash \varphi(x, \tau)$. Finally let π be a name for $\{\tau^G \mid \tau \in T\}$, i.e. $\pi = \{\langle \tau, 1^{\mathbb{P}} \rangle \mid \tau \in T\}$. Then, because the generic below q hits every d_x , $\varphi(x, \tau)$ will hold for some $\tau \in T$. It follows that $q \Vdash \forall x < \alpha \exists y \in \pi \varphi(x, y)$. Then Union follows with the use of Set-Bounding. \square

With this proposition we have shown that in a model of MK^* we can force with a definable hyperclass-forcing \mathbb{P} and preserve MK^* , provided \mathbb{P} translates to a pretame class-forcing in $\text{Set}MK^*$ which preserves the inaccessibility of κ and whose forcing relation is definable. But in practice we don't usually know if the forcing relation is definable, even if we know that P is pretame due to the absence of a suitable hierarchy (like the V -hierarchy which suffices when forcing over ZF-models). So we will introduce a preparatory forcing which does not add any new sets but converts the $\text{Set}MK^*$ model M^+ into a model of the form $L_\alpha[A]$ for some generic class predicate $A \subseteq ORD$ preserving $\text{Set}MK^*$ (relative to A). This will allow us to use the relativized L hierarchy and therefore adapt the proof of the Definability Lemma for a pretame class-forcing and the fact that it preserves the axioms. Such a preparatory forcing presents us with two difficulties: first we have to show that its forcing relation is definable and the forcing is pretame, so that we can infer from Proposition 3.17 that it preserves the axioms. Secondly we have to show that the predicate A , that was added by the forcing, can be coded into a subset of κ so as to avoid problems when going back to the MK^* model.

To prove the pretameness of such a forcing we have to add a new axiom to $\text{Set}MK^*$, namely a variant of Dependent Choice. To ensure that this axiom holds in M^+ , we will add its class version to MK^* and show that it is transformed to the appropriate set version using the coding introduced in the last section.

Definition 3.18. *Let MK^{**} consist of the axioms of MK^* plus Dependent Choice for Classes (we denote this with DC_∞):*

$$\forall \vec{X} \exists Y \varphi(\vec{X}, Y) \rightarrow \forall X \exists \vec{Z} (Z_0 = X \wedge \forall i \in ORD \varphi(\vec{Z} \upharpoonright i, Z_i))$$

where \vec{X} is an α -length sequence of classes for some $\alpha \in ORD$, \vec{Z} is an

ORD-length sequence of classes and $Z \upharpoonright i$ is the sequence of the “previously chosen” Z_j , $j < i$.

In the resulting SetMK** model M^+ , DC_∞ becomes a form of κ -Dependent Choice:

$$\forall \vec{x} \exists y \varphi(\vec{x}, y) \rightarrow \forall x \exists \vec{z} (z_0 = x \wedge \forall i < \kappa \varphi(\vec{z} \upharpoonright i, z_i))$$

where \vec{x} is a $< \kappa$ -length sequence of sets, \vec{z} is a κ -length sequences of sets and $z \upharpoonright i$ is the sequence of the “previously chosen” z_j , $j < i$.

The coding between MK** and SetMK** works exactly as in the MK* case, we only have to prove that it transforms DC_∞ into DC_κ and vice versa.

Proposition 3.19. 1. Let $\mathcal{M} = (M, \mathcal{C})$ be a β -model of MK**. Then we can define a model

$$M^+ = \{x \mid \text{there is a coding pair } (M_x, R_x) \text{ that codes } x\}$$

Then M^+ is the unique, transitive set that obeys the following properties:

- a) $M^+ \models \text{SetMK}^{**}$,
- b) $\mathcal{C} = P(M) \cap M^+$,
- c) $M = V_\kappa^{M^+}$, κ is the largest cardinal in M^+ and strongly inaccessible in M^+ .

- 2. Let M^+ be a model of SetMK** that has a strongly inaccessible cardinal κ , let $\mathcal{C} = P(M) \cap M^+$ and $M = V_\kappa^{M^+}$. Then $\mathcal{M} = (M, \mathcal{C})$ is a model of MK**.

Proof. For 1.: Using the proof of Theorem 3.7 it only remains to show that M^+ is a model of κ -Dependent Choice, where κ is strongly inaccessible in M^+ : $M^+ \models \forall \vec{x} \exists y \varphi(\vec{x}, y) \rightarrow \forall x \exists \vec{z} (z_0 = x \wedge \forall i < \kappa \varphi(\vec{z} \upharpoonright i, z_i))$ where \vec{x}, \vec{z} are κ -length sequences. So assume that $M^+ \models \forall \vec{x} \exists y \varphi(\vec{x}, y)$. From what we have show above, we know that \vec{x} is an ordinal length sequence of elements in \mathcal{M} and also y is an element of \mathcal{M} (as these can be classes we will write them with upper case letters in \mathcal{M}). Let ψ be the second-order formula associated to φ , i.e. ψ is the formula that says exactly the same as φ only that its variables can be classes. Then by DC_∞ we have that $\forall \vec{X} \exists Y \psi(\vec{X}, Y) \rightarrow \forall X \exists \vec{Z} (Z_0 = X \wedge \forall i \in \text{ORD} \psi(\vec{Z} \upharpoonright i, Z_i))$ where \vec{X}, \vec{Z}

are sequences of classes with ordinal length and $Z \upharpoonright i$ is the sequence of the previously “chosen” Z_j , $j < i$. As before all the classes mentioned here are elements of M^+ where \vec{Z} is a κ -length sequence and so we have proven the κ -Dependent Choice.

For 2.: Again we only have to proof the case of DC_∞ and this is an direct analog to the proof of the Comprehension Axiom in the proof of Proposition 3.12.

□

Lemma 3.20. *Let M^+ be a model of $SetMK^{**}$ with largest cardinal κ and P be an M^+ -definable class forcing notion. Then if P is $\leq \kappa$ -closed it is $\leq \kappa$ -distributive.*

Proof. Let $p \in P$ and $\langle D_i \mid i < \beta \rangle$ is an M^+ definable sequence of dense classes, $\beta \leq \kappa$, and we want to show that there is a $q \leq p$ meeting each D_i (q meets D_i if $q \leq q_i \in D_i$ for some q_i). As we have shown that P is $\leq \kappa$ -closed we want to construct a descending sequence $p_0 \geq p_1 \geq \dots \geq p_i \geq \dots$ ($i < \beta$) with $p_i \in D_i$ for all $i < \beta$. Here we need the $SetMK^{**}$ version of the Dependent Choice Axiom we added to MK^* : Recall that κ -Dependent Choice says that $\forall \vec{x} \exists y \varphi(\vec{x}, y) \rightarrow \forall x \exists \vec{z} (z_0 = x \wedge \forall i < \kappa \varphi(\vec{z} \upharpoonright i, z_i))$ where \vec{x} is a $< \kappa$ -length sequence of sets, \vec{z} is a κ -length sequences of sets and $z \upharpoonright i$ is the sequence of the previously “chosen” z_j , $j < i$. If we take $\varphi(\vec{x}, y)$ to mean that “ \vec{x} is a descending sequence of conditions, $x_i \in D_i$ for $i < \text{length } \vec{x}$, y is a lower bound for \vec{x} and $y \in D_{\text{length } \vec{x}}$ ” then we know that we can find a descending sequence $p_0 \geq p_1 \geq \dots \geq p_i \geq \dots$ ($i < \beta$) with $p_i \in D_i$ for all $i < \beta$ such that there is an $q \in P$ with $q \leq p$ and $q \leq p_i$ for all $i < \beta$ and so q meets all D_i . □

Theorem 3.21. *Let M^+ be a model of $SetMK^{**}$ with largest cardinal κ and let κ^* denote the height of M^+ . Then there is an M^+ -definable forcing P such that the Definability Lemma holds and P is pretame, which adds a class predicate $A \subseteq \kappa^*$ such that $M^+ = L_{\kappa^*}[A]$ and $(M^+, A) \models SetMK^{**}$ relativized to A .*

Proof. Let $P = \{p : \beta \rightarrow 2 \mid \beta < \kappa^*, p \in M^+\}$ and let G be P -generic over M^+ . Let $\bigcup G = g : \kappa^* \rightarrow 2$ and $A = \{\gamma < \kappa^* \mid g(\gamma) = 1\}$. Note that G is an amenable predicate, i.e. $G \cap a$ belongs to M^+ for every $a \in M^+$ and P is $\leq \kappa$ -closed, as for every $\lambda \leq \kappa$ and every descending sequence

$p_0 \geq p_1 \geq \dots \geq p_i \geq \dots$ ($i < \lambda$) there is $q = \bigcup_{i < \lambda} p_i \in P$ such that $\forall i < \lambda q \leq p_i$.

To show that the forcing relation is definable in the ground model, we will concentrate on the atomic cases “ $p \Vdash \sigma \in \tau$ ” and “ $p \Vdash \sigma = \tau$ ”. Then the other cases follow by induction. For $p \Vdash \sigma \in \tau$ first consider the case where the length of p is larger than the ranks of σ and τ (i.e. there is an γ such that $\text{rank } \sigma, \text{rank } \tau < \gamma$ and $\text{Dom}(p) > \gamma$). Then the question if $\sigma^G \in \tau^G$ is already decided by p , meaning that $\sigma^G \in \tau^G$ exactly when $\sigma^p \in \tau^p$ with $\tau^p = \{\pi^p \mid \langle \pi, q \rangle \in \tau, p \leq q\}$ as p “has no holes” and therefore a condition that extends p will never change the decisions made below the length of p . This now defines the forcing relation because P doesn’t add any new sets and therefore σ^p and τ^p are already elements of the ground model. If p is not large enough to decide if σ^G is an element of τ^G , then we have to check that every q that extends p decides that this is the case so we get the definition “ $p \Vdash \sigma \in \tau \leftrightarrow \forall q \leq p (|q| > \text{rank } \sigma, \text{rank } \tau \rightarrow \sigma^q \in \tau^q)$ ”. The definitions for the “=” case can be given the same way and so the forcing is definable. The Truth Lemma then follows from Definability by the usual arguments.

Next we want to show that P is pretame: As P is $\leq \kappa$ -closed, we know by Lemma 3.20 that P is $\leq \kappa$ -distributive. Then P is also pretame for sequences of dense classes of length $\leq \kappa$ and therefore P is pretame.

We have shown that P doesn’t add any new sets to the extension but a subclass $A \subset \kappa^*$. So the forcing just reorganizes M^+ and adds A as a predicate. Then every set of ordinals from M^+ is copied into an interval of the generic and so every set of ordinals and therefore also every set is coded by A . Also as A adds no new sets it holds that $L_{\kappa^*}[A] \subseteq M^+$. It follows that $M^+[G] = L_{\kappa^*}[A]$ and therefore already $M^+ = L_{\kappa^*}[A]$.

It remains to show that $(M^+, A) \models (\text{SetMK}^{**})^A$, i.e. SetMK^{**} holds for formulas which can mention A as a predicate. As P preserves the strongly inaccessibility of κ it follows by Proposition 3.17 that $M^+[G] \models \text{SetMK}^*$ and that means that $(M^+, A) \models \text{SetMK}^*$. But as the Comprehension and Bounding can mention the generic this implies that $(M^+, A) \models (\text{SetMK}^*)^A$. For the DC_κ note that by adding A we now have a global well-order of the extension. That means that if we have a $< -\kappa$ sequence \vec{x} in $M^+[G]$ such that $\forall \vec{x} \exists y \varphi(\vec{x}, y)$ and we want to find a κ -length sequence \vec{z} such that $\forall x \exists \vec{z} (z_0 = x \wedge \forall i < \kappa \varphi(\vec{z} \upharpoonright i, z_i))$ we can just take z_i to be least so that $\varphi(\vec{z} \upharpoonright i, z_i)$ for each i . \square

As our ultimate goal is to go back to an MK^{**} model, we want to show that the predicate A can be coded into a subset of κ :

Theorem 3.22. *Let (M^+, A) be a model of $SetMK^{**}$ relativized to a predicate A , with largest cardinal κ and let κ^* denote the height of (M^+, A) , where A is the generic predicate added by the forcing P in Theorem 3.21 and $M^+ = L_{\kappa^*}[A]$. Then we can force that there is a $X \subseteq \kappa$ such that $L_{\kappa^*}[A] \subseteq L_{\kappa^*}[X]$, $SetMK^{**}$ is preserved and κ remains strongly inaccessible.*

Proof. To get A definable in $M^+[X]$, for some $X \subseteq \kappa$, we want to use an almost disjoint forcing which codes the predicate A into such an X . The forcing will be along the following lines: we will need to define a family S of almost disjoint sets (i.e. for $x, y \subseteq \kappa$, x and y are almost disjoint if $x \cap y$ is bounded in κ) A_β which we will use to code the predicate $A \subseteq \kappa^*$ into an X . We will define A_β to be the least subset of κ (i.e. least in the canonical well-order of $L_{\kappa^*}[A \cap \beta]$) in $L_{\kappa^*}[A \cap \beta]$ which is distinct from the $A_{\bar{\beta}}$ for $\bar{\beta} < \beta$. The idea is that we can decode A in $L_{\kappa^*}[X]$ if we know the A_β 's. But as A is a proper class we don't know that we can always find such distinct A_β 's. So we will have to assume that the cardinality of β is at most κ not only in $L_{\kappa^*}[A]$ but also in $L_{\kappa^*}[A \cap \beta]$ because now to find an A_β distinct from each $A_{\bar{\beta}}$, $\bar{\beta} < \beta$, we can list these $A_{\bar{\beta}}$'s as $\langle A_i \mid i < \kappa \rangle$ and obtain A_β by diagonalization. To fulfill that assumption however we have to "reshape" A into a predicate A' that has the property that if $\beta < \kappa^*$ then the cardinality of β is $\leq \kappa$ in $L_{\kappa^*}[A' \cap \beta]$. Then we can code A as the even part of A' to get $(M^+, A') \models (SetMK^{**})^{A'}$ and finally code A' by a subset of κ .

So the proof consists of two steps: First we have to show that we can reshape A and then we have to force with an almost disjoint forcing to show that the reshaped predicate A' can be coded into a subset of κ , preserving $SetMK^{**}$ in each step.

Step 1: We add a reshaped predicate A' over $(L_{\kappa^*}[A], A)$ by the following forcing:

$$P = \{p : \beta \rightarrow 2 \mid \kappa \leq \beta < \kappa^*, \forall \gamma \leq \beta (L_{\kappa^*}[A \cap \gamma, p \upharpoonright \gamma] \models |\gamma| \leq \kappa)\}$$

The main obstacle is to show that P is definably-distributive, i.e. we have to show that for a $p \in P$ and (M^+, A) -definable sequences of dense classes of set-length $\langle D_i \mid i < \alpha \rangle$ for all $\alpha \leq \kappa$, there is a $q \leq p$ meeting each D_i

with $q \in P$.

Claim 3.23. *P is definably-distributive.*

Proof. Note that it suffices to show definable-distributivity for κ ; so we consider an (M^+, A) -definable sequence of dense classes $\langle D_i \mid i < \kappa \rangle$. We want to define a descending sequence of conditions $p \geq p_0 \geq p_1 \geq \dots$ where $p_i \geq q$, $q \in P$ and $p_{i+1} \in D_i$ for each $i < \kappa$. To show that the p_i are indeed conditions we have to show that $L_{\kappa^*}[A \cap \gamma, p_i \upharpoonright \gamma] \models |\gamma| \leq \kappa$ for every $\gamma \leq |p_i|$. In the following we will use the fact that a condition is always extendible to any length $< \kappa^*$: $\forall p \forall \beta < \kappa^* \exists q \leq p, |q| \geq \beta, q \in P$. This holds because there is an $x \subseteq \kappa$ such that β is coded by x and $p * x \in P$ and has length $|p| + \kappa$. If this is still below β we can lengthen p further by a sequence of 0's: $q = p * x * \vec{0}$. This will again be an element of P as we know from the information in the code x of β that the ordinals will collapse.

First, we assume that the sequence of dense classes is Σ_1 -definable, i.e. $\{(q, i) \mid q \in D_i\}$ is Σ_1 -definable with parameter.

As we have seen that every condition is extendible, we can extend p to catch a parameter $x \in L_{|p|}[A]$ such that the sequence of the D_i is Σ_1 -definable with parameter x . Let p_0 be this extension of p . Then, as we have Global Choice, we can consider the $<_{(M^+, A)}$ -least pair (q_0, w_0) such that $q_0 \leq p_0$ and w_0 witnesses “ $q_0 \in D_0$ ”. Then we choose p_1 such that p_1 is a condition which extends q_0 such that $w_0 \in L_{|p_1|}[A \cap |p_1|]$. Now we define p_2 in the same way: Choose (q_1, w_1) such that $q_1 \leq p_1$ and w_1 witnesses “ $q_1 \in D_1$ ”. Then let $p_2 \leq q_1$ such that $w_1 \in L_{|p_2|}[A \cap |p_2|]$. Define the rest of the successor cases (p_{n+1}, w_{n+1}) similarly.

For the first of the limit cases, let $p_\omega = \bigcup_{n < \omega} p_n$ and we claim that $p_\omega \in P$. So we have to show that $\forall \gamma \leq |p_\omega|$, γ collapses to κ using only $A \cap \gamma$ and $p_\omega \upharpoonright \gamma$. We know that if $\gamma < |p_\omega|$ then $\gamma < |p_n|$ for some n . So we only have to consider the case where $\gamma = |p_\omega|$. It follows from the construction of the p_n 's that the sequence $\langle p_n \mid n < \omega \rangle$ is definable over $L_{|p_\omega|}[A \cap |p_\omega|, p_\omega]$ and is a cofinal sequence in p_ω , i.e. it converges to p_ω . Then also the sequence of the lengths of the p_n 's, $\langle |p_n| \mid n < \omega \rangle$ is definable over $L_{|p_\omega|}[A \cap |p_\omega|, p_\omega]$ and converges to $|p_\omega|$. As we know that $|p_n|$ collapses to κ for every $n < \omega$, we know that in $L_{|p_\omega|}[A \cap |p_\omega|, p_\omega]$ $|p_\omega|$ definably collapses to κ . So $L_{|p_\omega|+1}[A \cap |p_\omega|, p_\omega] \models |p_\omega|$ is collapsed to κ . The other limit cases can be handled in the same way.

Now we go to the Σ_2 -definable case. Note that we cannot simply copy the construction of the p_n -sequence because the witness q_{n+1} we need for the definition of the next p_{n+1} will now be a solution to a Π_1 -statement and will therefore not be absolute in the other models. But we know that for $V = L_{\kappa^*}[A]$ it holds that $\forall \alpha < \kappa^* \exists \beta \leq \kappa^*, \alpha < \beta$ such that $L_\beta[A]$ is Σ_n -elementary in $L_{\kappa^*}[A]$. This holds because for a pair α, n we can take the Σ_n -Skolem Hull N of α in $L_{\kappa^*}[A]$. Then in M we have a solution for every Σ_n -property with parameters $< \alpha$, M is transitive and bounded by Class-Bounding. Then there is a $\beta \leq \kappa^*$ such that M is equal to $L_\beta[A]$.

So we can always find models that are Σ_1 -elementary submodels of (M^+, A) in which we can carry out the definition of the sequence of conditions: As before we choose for every $n < \omega$ a pair (q_n, w_n) such that $q_n \leq p_n$ such that w_n witnesses “ $q_n \in D_n$ ” and then let $p_{n+1} \leq q_n$ such that $w_n \in L_{|p_{n+1}|}[A \cap |p_{n+1}|, p_{n+1}]$ and $L_{|p_{n+1}|}[A \cap |p_{n+1}|, p_{n+1}]$ is an Σ_1 -elementary submodel of $L_{\kappa^*}[A]$. This also holds in the limit case by using the same construction we did for the Σ_1 case where again the model $L_{|p_\omega|+1}[A \cap |p_\omega|, p_\omega]$ is an Σ_1 -elementary submodel of $L_{\kappa^*}[A]$. The same can be done for all the Σ_m -definable cases. \square

Now that we know that P is $\leq \kappa$ -distributive, we know that P is $\leq \kappa$ -pretame and therefore $(M^+, A, A') \models (SetMK^{**})^{A, A'}$ (similar to proof of Theorem 3.21 by using Proposition 3.17 and the fact that there is a global well-order of the extension). Then we can code A to be the even part of A' and we get a model $(M^+, A') \models (SetMK^{**})^{A'}$. It remains to show that A' can be coded into a subset of κ .

Step 2: Code A' into $X \subseteq \kappa$. As we know that A' is reshaped we can define a collection of sets $\mathcal{S} = \langle A_\beta \mid \beta < \kappa^* \rangle$ in the following way: let A_β be the least $B \subseteq \kappa$ in $L_{\kappa^*}[A' \cap \beta]$ such that $B \notin \{A_{\bar{\beta}} \mid \bar{\beta} < \beta\}$. \mathcal{S} can be turned into a collection $\mathcal{S}' = \langle A'_\beta \mid \beta < \kappa^* \rangle$ of almost disjoint sets A'_β by mapping every set to the set of codes of its proper initial segments: $B \subseteq \kappa$ is mapped to $B' = \{\text{Code}(B \cap \alpha) \mid \alpha < \kappa\} \subseteq \kappa$. Then for two distinct subsets B and C of κ , $|B' \cap C'| < \kappa$ and therefore they are almost disjoint. We want to show that we can code A' by a subset X of κ by showing that $X \cap A'_\beta$ is bounded if and only if $\beta \in A'$. This can be done by a forcing Q with the conditions (g, S) where $S \subseteq A'$, $|S| < \kappa$ and g is an element of ${}^{<\kappa}2$. Extension is defined by: $(g, S) \geq (h, T)$ iff h extends g , $S \subseteq T$ and if $\beta \in S$ and $h(\gamma) = 1$

for a $\gamma \in A'_\beta$ then $g(\gamma) = 1$. Note that two conditions with the same first component $\langle g, S \rangle$ and $\langle g, T \rangle$ are compatible because we can always find a common extension $\langle g, S \cup T \rangle$. Thus a function which maps every element of a definable antichain into its first component is injective (as otherwise the conditions would be compatible). So we have injectively mapped a definable class to a set as there are only κ many first components. By Bounding such a function exists as a set and so Q is set-c.c., i.e. every definable antichain is only set-sized. Then Q is pretame, as every definable dense class can be seen as an antichain. Now let G be a Q -generic, $G_0 = \bigcup \{g \mid (g, S) \in G\}$ and $X = \{\gamma \mid G_0(\gamma) = 1\}$. We argue that we can find the almost disjoint sets in $L_{\kappa^*}[X]$ because A' is reshaped and therefore it holds for any β that $|\beta| \leq \kappa$ in $L_{\kappa^*}[A' \cap \beta]$. So after X has decoded $A' \cap \beta$ it can find A'_β and then continue the decoding in the following way: $\beta \in A'$ if there is an $(g, S) \in G$ with $\beta \in S$ and by the definition of extension if $G_0(\gamma) = 1$ for a $\gamma \in A'_\beta$ then $g(\gamma) = 1$. So $X \cap A'_\beta = \{\gamma \mid g(\gamma) = 1\} \cap A'_\beta$ and that is bounded and therefore we have a code of A' by X via

$$X \cap A'_\beta \text{ is bounded if and only if } \beta \in A'.$$

As this forcing is κ -closed (i.e. closed for $< \kappa$ sequences), κ stays regular and therefore strongly inaccessible and by Proposition 3.17 SetMK* is preserved and by Proposition 3.17 SetMK* is preserved. \square

We have seen how definable hyperclass-forcing can be carried out over a model \mathcal{M} of MK**: First we go to the related SetMK** model M^+ (Theorem 3.7). Then in order to be able to force over this model, we change M^+ to a model $L_{\kappa^*}[A]$ for a generic predicate A (Theorem 3.21). Finally we showed how to code A into a subset $X \subseteq \kappa$ to avoid having an undefinable predicate once we go back to the extension of the original MK** model (Theorem 3.22). At this point we can force with any desirable pretame definable class-forcing over $L_{\kappa^*}[X]$, go back to MK** and get the desired definable hyperclass-forcing over MK**.

So we have given a template which allows us to do definable hyperclass-forcing over MK**. In the following we will show how to use this template to produce minimal β -models of MK**.

3.3 Minimal β -Models of MK^{**}

As an application of definable hyperclass forcing we will show that every β -model of MK^{**} can be extended to a minimal β -model of MK^{**} via the use of SetMK^{**} models. Here a minimal model $M(S)$ of SetMK^{**} is the least transitive model of SetMK^{**} containing a real S and equivalently a minimal β -model $\mathcal{M}(S)$ of MK^{**} is the least β -model of MK^{**} containing a real S .² For that we will use and modify the template developed in the last section: We start with an arbitrary β -model $\mathcal{M} = (M, \mathcal{C})$ of MK^{**} and from that we get the corresponding model M^+ of SetMK^{**} (by Theorem 3.7) with $M = V_\kappa^{M^+}$ and $\mathcal{C} = P(M) \cap M^+$ where κ is strongly inaccessible in M^+ . Let κ^* denote the height of M^+ and apply Theorem 3.21 to arrive at $M^+ = L_{\kappa^*}[A]$ where $A \subseteq \kappa^*$ and (M^+, A) satisfies SetMK^{**} relative to A . We now show that we can extend M^+ to a minimal model of SetMK^{**} and then go back to an MK^{**} model, which will be a minimal β -model of MK^{**} .

Theorem 3.24. *Every β -model of MK^{**} can be extended to a minimal β -model of MK^{**} with the same ordinals.*

Proof. First we will code the predicate A into a subset of κ by using Theorem 3.22 with a small modification in the “reshaping” forcing. Instead of forcing that each $\gamma < \kappa^*$ collapses in $L_{\kappa^*}[A \cap \gamma, p \upharpoonright \gamma]$, we will force it to already collapse instantly in the next level, i.e. in $L_{\gamma+1}[A \cap \gamma, p \upharpoonright \gamma]$. So the forcing will be:

$$P = \{p : \beta \rightarrow 2 \mid \kappa \leq \beta < \kappa^*, \forall \gamma \leq \beta (L_{\gamma+1}[A \cap \gamma, p \upharpoonright \gamma] \models |\gamma| \leq \kappa)\}$$

The proof that P is definably-distributive then works in exactly the same way. As in Theorem 3.22 we can code A to be the even part of the predicate A' added by the reshaping forcing which in turn can be coded into an $X \subseteq \kappa$ by an almost disjoint forcing. This gives us that there are no SetMK^{**} models containing X of height between κ and κ^* : In the reshaping forcing we destroyed the Replacement axiom level by level relative to A and in the

²We can see here that it is vital to restrict ourselves to β -models in order to talk about minimal models of MK by comparing this to the situation in ZFC: There it also only makes sense to talk about minimal models containing a real for well-founded models (and not for ill-founded models). So by making the transformation from MK to SetMK we have to restrict ourselves to β -models.

almost disjoint coding we can now choose the codes instantly level-by-level (i.e. every code for γ appears in $L_{\gamma+1}[X]$). So A' can be recovered level-by-level from X and therefore Replacement is also destroyed level-by-level relative to X . We arrive at a SetMK** model $L_{\kappa^*}[X]$, with $X \subseteq \kappa$, which is the least transitive ZFC^- model containing X (again κ remains regular and indeed strongly inaccessible, because the almost disjoint coding is κ -closed).

We will extend this to a minimal model of SetMK** in two steps: First we extend $L_{\kappa^*}[X]$ to a model $L_{\kappa^*}[Y]$ such that no cardinal $\bar{\kappa} < \kappa^*$ can serve as a “source” for a SetMK** model (i.e. is the largest cardinal of a SetMK** model containing $Y \cap \bar{\kappa}$) and second we show that we can add a real S such that in $L_{\kappa^*}[S]$ there are no SetMK** models containing S below κ^* . Then it only remains to show that from $L_{\kappa^*}[S]$ we can go back to a minimal β -model of MK**.

Step 1: With the modification of Theorem 3.22, we have shown that there are no SetMK** models containing X between κ and κ^* . But it could still be that there exist cardinals below κ which are sources for SetMK** models. We will destroy these cardinals by shooting a club through a “fat-stationary” set which has no such cardinals and then force all limit cardinals to belong to this club.

So let $S = \{\bar{\kappa} < \kappa \mid \bar{\kappa} \text{ is a limit cardinal and for all } \bar{\beta} > \bar{\kappa}, \text{ if } L_{\bar{\beta}}[X \cap \bar{\kappa}] \models ZFC^- \text{ then } L_{\bar{\beta}}[X \cap \bar{\kappa}] \not\models \bar{\kappa} \text{ is strongly inaccessible}\}$.

Definition 3.25. S is fat-stationary if for every club C in $L_{\kappa^*}[X]$, $S \cap C$ contains closed subsets of any order type less than κ .

We prove the following:

Lemma 3.26. S is fat-stationary and there is a κ -distributive (i.e. $< \kappa$ distributive) forcing of size κ that adds a club $C \subseteq S$.

Proof. First we will show that S is stationary with respect to clubs in $L_{\kappa^*}[X]$. So suppose C is a club in $L_{\alpha}[X]$ for an $\alpha < \kappa^*$. We build an increasing sequence $\langle M_n \mid n < \omega \rangle$ of sufficiently elementary submodels of $L_{\alpha}[X]$ in the following way: Let M_0 be the Σ_1 -Skolem Hull of $\omega \cup \{X, C\}$ in $L_{\alpha}[X]$. Then $C \in M_0$ and $\kappa_0 = \sup(M_0 \cap \kappa)$ is a cardinal. Next, let M_1 be the Σ_1 -Skolem Hull of $\kappa_0 + 1 \cup \{X, C\}$ in $L_{\alpha}[X]$ and $\kappa_1 = \sup(M_1 \cap \kappa)$. Repeat this construction for all $n < \omega$. Then this sequence of elementary

submodels is definable over $M_\omega = \bigcup_{n < \omega} M_n$ and $\kappa_\omega = \sup_{n < \omega} \kappa_n < \kappa$ is a cardinal in C as C is closed, unbounded in κ . Also κ_ω is an element of S because if $L_{\bar{\alpha}}[X \cap \kappa_\omega]$ is the transitive collapse of M_ω then there are no ZFC^- models containing $X \cap \kappa_\omega$ of height $< \bar{\alpha}$ (by elementarity), of height $= \bar{\alpha}$ because $\langle \kappa_n \mid n < \omega \rangle$ is definable over it (and so κ_ω becomes definably singular) and any ZFC^- model containing $X \cap \kappa_\omega$ of height $> \bar{\alpha}$ sees that κ_ω has cofinality ω (as the κ_n -sequence is an element of it).

To show that S is fat-stationary we can use the same proof as for stationarity except one uses a longer δ -sequence of elementary submodels, for δ a limit cardinal less than κ .

Now for the second part of the Lemma we can force with a set-forcing to add a club. Here we will closely follow the proof of the ZFC version of this claim, as proven in [AS83] (see there for more details). Let $Q = \{p \mid p \text{ is a closed, bounded subset of } S\}$ be a forcing notion ordered by end-extensions: $q \leq p$ iff $p = q \cap (\sup(p) + 1)$. For G Q -generic over $L_{\kappa^*}[X]$ let $C = \bigcup G$. Then C is closed and unbounded and a subset of S . To show that Q is κ -distributive we have to show that for every $\tau < \kappa$ and sequence $\mathcal{D} = \langle D_i \mid i < \tau \rangle$ of open, dense subsets of Q , $\bigcap_{i < \tau} D_i$ is dense in Q . Now we can define a sequence of elementary substructures $\langle M_\alpha \mid \alpha < \kappa \rangle$ of $L_{\kappa^*}[X]$ such that $c_\alpha = M_\alpha \cap \kappa$ is an ordinal and $\langle c_\alpha \mid \alpha < \kappa \rangle$ is an increasing and continuous sequence cofinal in κ . Let E be the collection of the c_α , $\alpha < \kappa$. Because S is fat-stationary, $S \cap E$ contains a closed subset A of order-type $\tau + 1$. Then in the model M_α , with $\alpha = \sup(A)$, we can define an increasing sequence $\langle p_i \mid i < \tau \rangle$, such that $p_i \in Q$ and $p_{i+1} \in D_i \cap M_\alpha$. We can define $p_\tau = \bigcup_{i < \tau} p_i \cup \{\alpha\}$ and this will be in $\bigcap_{i < \tau} D_i$. Note that this (set-) forcing is an element of $L_{\kappa^*}[X]$ and therefore preserves ZFC^- . Furthermore, as this forcing doesn't add sets of size $< \kappa$, κ stays strongly inaccessible and SetMK^* is preserved because of Proposition 3.17. \square

Let X' be the join of X with the club we added. Then $X' \subseteq \kappa$ and the resulting model is $L_{\kappa^*}[X']$.

Lemma 3.27. *We can force all limit cardinals to belong to C with a forcing of size κ such that κ remains strongly inaccessible.*

Proof. Enumerate C as follows: $C = \langle \bar{\kappa}_i \mid i < \kappa \rangle$. We may assume that each $\bar{\kappa}_i$ is a strong limit cardinal (as κ is strongly inaccessible we can thin out C). Then we can build an Easton product of collapses, where we collapse

every $\bar{\kappa}_{i+1}$ to the successor of $\bar{\kappa}_i$ and therefore ensure that all limit cardinals below κ are limits of cardinals in C and therefore are themselves in C .

So for $i < \kappa$ consider $Col_i(\bar{\kappa}_i^+, \bar{\kappa}_{i+1})$, where the conditions are functions p with $dom(p) \subset \bar{\kappa}_i^+$, $|dom(p)| < \bar{\kappa}_i^+$ and $range(p) \subset \bar{\kappa}_{i+1}$. Cardinals below $\bar{\kappa}_i^+$ and above $\bar{\kappa}_{i+1}^{\bar{\kappa}_i}$ are preserved (the size of the forcing is $\bar{\kappa}_{i+1}^{\bar{\kappa}_i}$) and in the extension we have a function which maps $\bar{\kappa}_i^+$ onto $\bar{\kappa}_{i+1}$.

Now we can build the Easton product (product with Easton support) of these collapses for every $i < \kappa$: A condition p in this forcing is a function such that $p = \langle p_i \mid i < \kappa \rangle \in \prod_{i < \kappa} Col_i(\bar{\kappa}_i^+, \bar{\kappa}_{i+1})$ and the forcing is ordered by end-extension. p has Easton support, i.e. for every inaccessible cardinal λ , $|\{\alpha < \lambda \mid p(\alpha) \neq \emptyset\}| < \lambda$. As usual with Easton Products the forcing notion P can be split into two parts $P(\leq \lambda) = \prod_{i \leq \lambda} Col_i(\bar{\kappa}_i^+, \bar{\kappa}_{i+1})$ and $P(> \lambda) = \prod_{\lambda < i < \kappa} Col_i(\bar{\kappa}_i^+, \bar{\kappa}_{i+1})$ for every regular cardinal λ . For this reason and as each $\bar{\kappa}_i$ is a strong limit, each collapse from $\bar{\kappa}_{i+1}$ to $\bar{\kappa}_i^+$ will not be affected by the other collapses and κ remains regular and strong limit. Furthermore, as this forcing is in $L_{\kappa^*}[X']$ (it is of size κ) it preserves SetMK**.

Because of the unboundedness of C , every limit cardinal is also a limit of cardinals in C and therefore, as C is closed, it is an element of C . \square

We conclude Step 1 by choosing X'' to be the join of X' and the above Easton product. Then we arrive at a model $L_{\kappa^*}[X'']$ with $X'' \subseteq \kappa$ such that for every cardinal $\bar{\kappa} < \kappa^*$ there is no model of ZFC⁻ containing $X'' \cap \bar{\kappa}$ in which $\bar{\kappa}$ is inaccessible and therefore $\bar{\kappa}$ is not a source for a SetMK** model.

Step 2: We want to extend the results from the last step to hold for all ordinals, i.e. for all ordinals $\alpha < \kappa^*$ there is no SetMK** model of height $< \kappa^*$ containing a real S in which α is strongly inaccessible. This makes use of Jensen coding and a result about admissibility spectra which is connected to it. We will use these results as black boxes and will only state the main definitions and theorems here:

Theorem 3.28 (Jensen Coding). *Suppose that $\langle M, A \rangle$ is a transitive model of ZFC, i.e. M is a transitive model of ZFC, $A \subseteq M$ and Replacement holds in M for formulas mentioning A as a unary predicate. Then there is an $\langle M, A \rangle$ -definable class forcing P such that if $G \subseteq P$ is P -generic over $\langle M, A \rangle$, then:*

- a) $\langle M[G], A, G \rangle \models ZFC$.

b) For some $R \subseteq \omega$, $M[G] \models V = L[R]$ and $\langle M[G], A, G \rangle \models A, G$ are definable from the parameter R .

The very elaborate proof of this result uses Jensen's fine structure theory and, very roughly, the forcing involved consists of three components: an almost disjoint coding at successor cardinals, a variation thereof at limit cardinals and a reshaping forcing.³

Definition 3.29. Let T be the theory of ZF without Power Set and with Replacement restricted to Σ_1 formulas. Then $\Lambda(R)$ for a real R denotes the admissibility spectrum of R and is defined as the class of all ordinals α such that $L_\alpha[R] \models T$, i.e. the class of all R -admissible ordinals.

Theorem 3.30 (S.-D. Friedman).⁴ Suppose φ is Σ_1 and $L \models \varphi(\kappa)$ whenever κ is an L -cardinal. Then there exists a real $R <_L 0^\sharp$ such that $\Lambda(R) \subseteq \{\alpha \mid L \models \varphi(\alpha)\}$ and R is cardinal preserving over L .

We will use these theorems to prove the following lemma:

Lemma 3.31. We can extend the model $L_{\kappa^*}[X'']$ to be of the form $L_{\kappa^*}[S]$ for a real S such that $L_{\kappa^*}[S] \models \text{SetMK}^{**}$ and whenever $\bar{\alpha} < \kappa^*$ is an ordinal there is no model of SetMK^{**} of height $< \kappa^*$ containing S in which $\bar{\alpha}$ is strongly inaccessible.

Proof. First we add a real R to the resulting model of Step 1 and get a model $L_{\kappa^*}[R] \models \text{SetMK}^{**}$. This can be done by using Jensen coding over the model $L_{\kappa^*}[X'']$. Although we start from a model of ZFC^- rather than ZFC our model is of the form $L_{\kappa^*}[X'']$ and therefore we can use the standard pretameness argument for Jensen coding to show that ZFC^- is preserved⁵. Also, κ will still be inaccessible in the extension because Jensen coding preserves inaccessibles.⁶ Note that the result from Step 1 still holds: In $L_{\kappa^*}[R]$ we have that if $\bar{\kappa} < \kappa^*$ is a cardinal then there is no transitive model of SetMK^{**} containing R in which $\bar{\kappa}$ is inaccessible as otherwise there would have been such a model containing $X'' \cap \bar{\kappa}$ as the latter is coded by R in $L_{\bar{\kappa}}[R]$.

³An detailed account of this can be found in [BJW82], a simplified version of the proof can be found in [Fri00].

⁴See [Fri00], Theorem 7.5, p. 142.

⁵See [Fri00], Chapter 4.

⁶This follows from an property called diagonal distributivity (see [Fri00], p. 37).

Now we use Theorem 3.30 relativized to the real R to produce a new real S such that this holds for ordinals $\bar{\kappa}$. Theorem 3.30 works in the context of ZFC^- for the same reasons as for Jensen coding. Note that $L_{\kappa^*}[R] \models \varphi(\bar{\kappa})$ for every $L_{\kappa^*}[R]$ -cardinal $\bar{\kappa}$ where $\varphi(\alpha)$ is the following Σ_1 property with parameter R : “Either $L_\alpha[R] \models$ there is a largest cardinal or there is $\beta > \alpha$ such that $L_\beta[R] \models \alpha$ is singular and for all γ with $\alpha < \gamma < \beta$, $L_\gamma[R] \not\models ZFC^-$ ”. This property says that either α is a successor or we can “see” the singularity of α before we see a ZFC^- model for which it could be a source. Then by Theorem 3.30 there exists a real S generic over $L_{\kappa^*}[R]$ such that $L_{\kappa^*}[S] \models SetMK^{**}$ and $\Lambda(S) \subseteq \{\alpha \mid L[R] \models \varphi(\alpha)\}$. As α which is inaccessible in a model of ZFC^- containing S is S -admissible, we get the desired property for all ordinals. \square

We now have a minimal model $L_{\kappa^*}[S]$ of $SetMK^{**}$, i.e. the least transitive model of $SetMK^{**}$ containing S . It only remains to show that by going back to MK^{**} we arrive at a minimal β -model of MK^{**} . To see that consider the model $(L_\kappa[S], \mathcal{C})$ where \mathcal{C} consists of the subsets of $L_\kappa[S]$ in $L_{\kappa^*}[S]$. This is a β -model of MK^{**} by Proposition 3.12 and it is the least such model containing S because otherwise there exists a β -model $(N, \mathcal{C}') \subset (L_\kappa[S], \mathcal{C})$, $(N, \mathcal{C}') \models MK^{**}$ containing S that would give rise to a model N^+ of $SetMK^{**}$. If we then go to the $L[S]$ of N^+ we arrive at a model $L_\alpha[S]$ for some $\alpha < \kappa^*$ which is a model of $SetMK^{**}$. This is a contradiction to the minimality of $L_{\kappa^*}[S]$. \square

Chapter 4

Open Questions

This thesis opens up a wider area of further research and related open questions.

For example it would be interesting to transfer some more basic facts from set forcing in ZFC to the context of MK. One example would be to not only define forcing in MK via the approach of countable transitive models as done in this thesis, but also for example via the Boolean-valued approach.

Question: How can a Boolean valued approach to forcing be defined for class forcing in MK?

In the definition of definable hyperclass forcing we used the restriction to β -models of MK^* to make the coding of a transitive $\text{Set}MK^*$ model work. It would be interesting to investigate what happens if we drop this restriction:

Question: How can definable hyperclass forcing be defined for an arbitrary model of MK^{**} ?

Dropping the β -model assumption for the coding would mean to work only internally of the MK^{**} model and restricting ourselves to just work with coding pairs. We are confident that this can be done, but there are many details to be worked out.

In this thesis we consider three variants of the axioms of Morse-Kelley; the standard form MK, the extension via Class-Bounding, here called MK^* and the additional extension with Dependent Choice, called MK^{**} . The

obvious question presents itself, how they are related:

Question: Assuming just the consistency of MK, are there models of MK that don't satisfy MK* and models of MK* that don't satisfy MK**?

Apart from these specific questions which arise from the framework of this thesis, there is a wide field of possible future research. It will be interesting to see how results from forcing (for example cardinal preservation) translate to hyperclass forcing and what other application can be found:

Question: What further applications can be found for definable hyperclass forcing in MK** using the template we provided in this work? In general, what results from set and class forcing will hold in the context of hyperclass forcing?

Another fruitful topic is the connection between Morse-Kelley and second order arithmetic.

Question: What results and questions can be transferred from the context of Morse-Kelley class theory to second order arithmetic and vice versa?

For example, the question about minimal β -models of MK** can be translated to minimal β -models of second order arithmetic (plus Dependent Choice). Starting with a β -model of second order arithmetic we can go to a related model of ZFC^- where the inaccessible cardinal κ is now simply \aleph_0 . Then the question about models below the largest cardinal becomes trivial and the result of a minimal β -model follows much easier than in the context of MK**.

Of course, definable hyperclass forcing is not the last step in considering a hierarchy of forcing notions via their size. One could ask further:

Question: What would a general hyperclass forcing look like and in which context can it be developed (a hyperclass theory)? What would a hyperhyperclass forcing look like, i.e. a forcing where conditions are hyperclasses?

In this thesis we developed two further steps in this hierarchy after set

forcing and definable class forcing. We hope that it will serve as a basis for further fruitful research!

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Conference and Invited Talks

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- Winter Semester 2014 Vorlesung Grundkurs Logik, collaboration and lecture under supervision of Prof. Ramharter.
- Summer Semester 2014 Übung Logik, 2 Semesterwochenstunden.
- Summer Semester 2014 Vorlesung Grundkurs Logik, collaboration and lecture under supervision Prof. Ramharter.
- Winter Semester 2013 Übung Logik, 2 Semesterwochenstunden.
- Summer Semester 2013 Übung Logik, 2 Semesterwochenstunden.
- Winter Semester 2012 Übung Logik, 2 Semesterwochenstunden.
- Winter Semester 2011 Übung Logik, 2 Semesterwochenstunden.

Summer Semester 2011 Logik-Übung für LehramtskandidatInnen, 1 Semesterwochenstunde, Gruppe 1, Gruppe 4 und Gruppe 5.

Winter Semester 2010 Übung Logik, 2 Semesterwochenstunden.