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"Holonomy Reductions of Cartan Geometries"

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Abstract

This thesis is dedicated to reductions of holonomy on Cartan geometries. Given such a geometric structure the underlying manifold decomposes into initial submanifolds that in turn carry induced Cartan geometry structures.

We study the possible outcomes of this theory when applied to Riemannian manifolds by using tractor calculus and obtain an interpretation of a wide range of holonomy reductions in geometrical terms. They are characterized by a parallel distribution and a vector field with certain properties, together with a structure resulting of classical holonomy.

Abstract (Deutsch)

Diese Arbeit ist Holonomiereduktionen von Cartan Geometrien gewidmet. Ist eine solche geometrische Struktur gegeben, zerfällt die unterliegende Mannigfaltigkeit in initiale Teilmannigfaltigkeiten, die wiederum induzierte Cartan Geometrien tragen.

Wir untersuchen die Auswirkungen dieser Theorie unter Verwendung von Traktorbündeln und -konnexionen, wenn sie auf Riemann'sche Geometrie angewandt wird und erhalten eine geometrische Interpretation eines breiten Sprektrums von Holonomiereduktionen. Sie werden durch eine parallele Distribution und ein Vektorfeld, zusammen mit einer Struktur, die aus klassischer Holonomie resultiert, beschrieben.

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Contents

Introduction

Historically, there are two starting points for the ideas considered in this thesis.

One the one hand, Riemannian geometry is the very classical example in differential geometry that originates in the work of Gauss, who studied two-dimensional surfaces in three-dimensional Euclidean space. His student Bernhard Riemann extended Gauss' ideas to n-dimensional space in his habilitation thesis in 1854 and thereby laid the foundations of what we call today "Riemannian Geometry". After the introduction of the Levi-Civita-connection and the parallel transport in the beginning of the 20th century, Élie Cartan first considered the notion of "holonomy" in 1926, that extracts a Lie group out of parallel transport along closed curves. The relationship between the holonomy and the curvature of a Riemannian manifold is made explicit by the Ambrose-Singer-Theorem (1952). In 1955, the French mathematican Marcel Berger further developed this area by classifying all simply connected Riemannian manifolds, that are irreducible and non-symmetric, in terms of their holonomy [2].

On the other hand, Felix Klein laid the foundation for a new perspective towards geometry with his Erlangen program in 1872. Again, Élie Cartan contributed greatly to this theory by formulating his method of moving frames. Later, when this concept was described in terms of principal bundles, it was named after him: the Cartan geometry. Cartan geometries constitute an attempt to specify what is meant by the term "geometry" – they are certain principal bundles endowed with a differential form that has similar properties as a principal connection, but are more restrictive. In particular, two Lie groups are encoded in the definition of a Cartan geometry, namely the structure group of the principal bundle and a group containing the first one, that describes all isomorphisms of a model geometry. Together, these two Lie groups form the "type" of a Cartan geometry. These geometries include Euclidean geometry (of type $(\mathrm{Euc}(n), O(n))$), Projective geometry and Conformal geometry, both in their classical form on $\mathbb{R}^n$ and on the sphere, respectively, and on differentiable manifolds.

Only recently in 2011, these two ideas were merged by Čap, Gover and Hammerl [4]. They introduced a structure on a Cartan geometry called "holonomy reduction". This has proved to be particularly successful when applied to parabolic geometries that are special types of Cartan geometries. We obtain an immediate geometric geometric interpretation of holonomy reductions by applying tractor calculus, i.e. a vector bundle endowed with a linear connection naturally associated to the geometry.

We start by introducing the basic definitions and properties of principal bundles and vector bundles (without proofs). In Chapter 2, we shortly discuss the original idea of holonomy on Riemannian manifolds, and, more generally, on principal bundles, including a proof of the Ambrose-Singer-Theorem. Then we start a discussion of Cartan geometries by giving Klein geometries and G-structures as a motivation. Subsequently, we define Cartan geometries and observe their main properties. Then we turn to holonomy reductions of Cartan geometries and explain their implications on the structure and geometry of the underlying manifold.

The main part of the thesis is dedicated to the examination of holonomy reductions on Riemannian manifolds. As desired, we can recover the classical holonomy that is a special case of holonomy on principal bundles and that is discussed in Chapter 2.

However, we obtain more: Any holonomy reduction of a Riemannian Cartan geometry has a “type” that is a subgroup of $\mathrm{Euc}(n) = O(n) \ltimes \mathbb{R}^n$. We will give a geometric interpretation of all holonomy reductions of “type” $H \ltimes V$ where H is a Lie subgroup of $O(n)$ and $V \subset \mathbb{R}^n$ is a vector space that is invariant under the standard action of H. For $V = \mathbb{R}^n$ we obtain classical holonomy.

The case $V \subsetneq \mathbb{R}^n$ gives a parallel distribution on M whose rank is the same as the dimension of V and a vector field with certain properties (cf. 5.3.25). This is enough information to characterize the holonomy reduction $(O(V) \times O(V^\perp)) \ltimes V \subset \mathrm{Euc}(n)$. In order to obtain these structures we will give a characterization of holonomy reductions in terms of tractor bundles as an intermediate step.

Since H leaves V invariant, we have $H \subset O(V) \times O(V^\perp)$. The reduction $H \ltimes V \subset O(V) \times O(V^\perp) \ltimes V$ is similar to a classical holonomy reduction, hence induces another geometric structure on M that is compatible with the above distribution and the vector field.

Chapter 1

Background and fundamental concepts

In this chapter we will recall the most important properties of principal fiber bundles and connections in order to introduce the notation. We will omit most proofs.

1.1 Principal bundles

Notation Let $\pi : \mathcal{G} \to M$ be a fiber bundle. We will denote $\mathcal{G}|_U := \pi^{-1}(U)$ for $U \subset M$. In addition, let $\mathcal{G}_x := \mathcal{G}|_{\{x\}}$.

It is possible to describe certain geometries through a "symmetry group", for example, the Euclidean space that is $\mathbb{R}^n$ endowed with the standard inner product. We may alternatively characterize the Euclidean inner product by forming the group of all its isometries, that is denoted by $\mathrm{Euc}(n)$ (on the concrete form of $\mathrm{Euc}(n)$ cf. Example 3.1.1). Furthermore, it is well-known that we obtain an identification $\mathbb{R}^n = \mathrm{Euc}(n)/P$ via the canonical group action of $\mathrm{Euc}(n)$ on $\mathbb{R}^n$, where P is the stabilizer of some element in $\mathbb{R}^n$ under the action. If we choose $0 \in \mathbb{R}^n$, we obtain $P = O(n)$. Therefore, we may express Euclidean space as the pair of Lie groups $(\mathrm{Euc}(n), O(n))$, explicitely we have the canonical projection $\mathrm{Euc}(n) \to \mathrm{Euc}(n)/O(n)$ where $\mathrm{Euc}(n)$ acts by left multiplication on the total space and the action factors to the base space.

The first fundamental step in order to generalize the above idea to arbitrary manifolds endowed with certain geometries will be the concept of a principal bundle:

Definition Let $\pi : \mathcal{G} \to M$ be a fiber bundle over a smooth manifold M with a Lie group G as typical fiber.

Two bundle charts $\varphi_1 : \mathcal{G}|_{U_1} \to U_1 \times G$, $\varphi_2 : \mathcal{G}|_{U_2} \to U_2 \times G$, where U_1, U_2 are open in M, are called *G-compatible* if the composition $\varphi_2 \circ (\varphi_1)^{-1} : (U_1 \cap U_2) \times G \to (U_1 \cap U_2) \times G$ is of the form $(x, g) \mapsto (x, f(x)\, g)$ for a smooth function $f : M \to G$.

A bundle atlas is called *G-compatible* if all its elements are principal charts that are pairwise G-compatible.

Two G-compatible atlasses are *equivalent* (with respect to G), if their union is again G-compatible.

The fiber bundle is called *principal bundle* $\pi : \mathcal{G} \to M$ *with structure group* G (or shortly G-principal bundle) if it is endowed with an equivalence class of G-compatible bundle atlasses. The charts of one of the atlasses inside this equivalence class are called *principal charts*.

Some important properties of principal bundles are presented in the next proposition:

Proposition 1.1.1. *Let $\pi : \mathcal{G} \to M$ be a principal bundle with structure group G.*

(i) There is a canonical, smooth, right G-action ρ on $\mathcal{G}$ such that in the charts of a G-compatible atlas the action is given by right multiplication in the second component (by G-compatibility this fits together to form a global action on $\mathcal{G}$). The action ρ is free, leaves fibers invariant and acts fiberwise transitively.

(ii) The quotient of $\mathcal{G}$ by G can be identified with M such that the differentiable structure is preserved.

Proof. see [8, p.50]

Notation We denote $\rho\left(u,g\right) = \rho^u\left(g\right) = \rho_g\left(u\right)$ for $u \in \mathcal{G}$ and $g \in G$. In the following, we will often omit the ρ and denote $\rho\left(u,g\right)$ by $u \cdot g$ or ug.

Remark 1.1.2. *The fibers of a principal bundle $\mathcal{G}$ do not carry a group structure (in contrast to the fibers of a vector bundle). This is due to the fact that given two charts $\varphi, \psi : \mathcal{G}|_U \to U \times G$ of a principal bundle $\pi : \mathcal{G} \to M$, where $U \subset M$ open, the fiberwise chart change $\psi \circ \varphi^{-1}$ over $x \in U$ is of the form $(x,g) \mapsto (x, g_0 \cdot g)$ where $g_0, g \in G$. Left multiplication by a fixed element g_0 of G is in general not a group homomorphism of G, hence a chart does not endow $\mathcal{G}_x$ with a group structure that is compatible with all other charts.*

Definition Let $\pi : \mathcal{G} \to M$ be a G-principal bundle and $\pi' : \mathcal{G}' \to M'$ a G'-principal bundle. A *homomorphism of principal bundles* (ϕ, Φ) is given by a Lie group homomorphism $\phi : G \to G'$ and a smooth map $\Phi : \mathcal{G} \to \mathcal{G}'$ such that $\Phi\left(ug\right) = \Phi\left(u\right)\phi\left(g\right)$ for all $u \in \mathcal{G}$ and $g \in G$.

Such a homomorphism (ϕ, Φ) as in the definition above induces a smooth map $\bar{\Phi}$ between the underlying manifolds $M \to M'$, since $\pi' \circ \Phi$ factors through π:

Take $u_1, u_2 \in \mathcal{G}$ with $\pi\left(u_1\right) = \pi\left(u_2\right)$ then there exists a $g \in G$ with $u_2 = u_1 g$, therefore $\pi'\left(\Phi\left(u_2\right)\right) = \pi'\left(\Phi\left(u_1\right)\phi\left(g\right)\right) = \pi'\left(\Phi\left(u_1\right)\right)$. The factorized map $\bar{\Phi} : M \to M'$ is smooth and satisfies $\pi' \circ \Phi = \bar{\Phi} \circ \pi$.

If ϕ is injective, $M = M'$ and $\bar{\Phi} = \mathrm{id}_M$, Φ is called a *reduction of the structure group form G' to G* (or shortly we call $\mathcal{G}$ a G-reduction).

Notation We denote the Lie algebra of a Lie group G by $\mathfrak{g}$, similarly the Lie algebras of H and P will be denoted by $\mathfrak{h}$ and $\mathfrak{p}$, respectively.

We immediately see that we obtain distinguished objects on the total space of a principal bundle, firstly from the projection and secondly from the canonical G-action (see 1.1.1(i)).

Definition (i) Each G-principal bundle $\pi : \mathcal{G} \to M$ carries a canonical distribution $V\mathcal{G}$, that is defined by $V_u\mathcal{G} := (V\mathcal{G})_u := \ker(T_u\pi)$ for $u \in \mathcal{G}$. This is called the *vertical subbundle*.

(ii) The vector fields $\zeta_X(u) := \frac{d}{dt}|_0 \rho^{\exp(tX)}(u) = T_e\rho^u \cdot X$ on $\mathcal{G}$ where $u \in \mathcal{G}$, $X \in \mathfrak{g}$ and $t \in \mathbb{R}$, are called the *fundamental vector fields*.

These two objects are actually closely related:

Proposition 1.1.3. *Let $\pi : \mathcal{G} \to M$ be a principal G-bundle.*

(i) All fundamental vector fields have values in the vertical subbundle of $T\mathcal{G}$. Conversely, each element in the vertical subbundle $V_u\mathcal{G}$ is of the form $\zeta_X(u)$ for a unique $X \in \mathfrak{g}$, where $u \in \mathcal{G}$. In fact, $X \mapsto \zeta_X(u)$ for a fixed $u \in \mathcal{G}$ is a linear isomorphism $\mathfrak{g} \to V_u\mathcal{G}$.

(ii) All fundamental vector fields are G-equivariant, i.e.

$$(\rho_g)^* \zeta_X = \zeta_{\mathrm{Ad}(g^{-1})(X)}$$

for $g \in G$ and $X \in \mathfrak{g}$.

(iii) The flow of ζ_X for $X \in \mathfrak{g}$ is given by $\mathrm{Fl}_t^X = \rho_{\exp(tX)}$. It is defined for all $t \in \mathbb{R}$.

Proof. (i) and (ii) see [10, p.42]
(iii) For $u \in \mathcal{G}$, we have $u \cdot exp(0 \cdot X) = u$ and

$$\begin{aligned}\frac{d}{dt}|_{t=s}(u \cdot \exp(tX)) =& \frac{d}{dt}|_{t=0}(u \cdot \exp((t+s)X)) \\ =& \frac{d}{dt}|_{t=0}(u \cdot \exp(sX) \cdot \exp(tX)) = \zeta_X(u \cdot exp(sX)).\end{aligned}$$

□

In the following, we give two important examples. The first one was mentioned in a special case as a motiviation for the definition of a principal bundle.

The second example is the prototypical example of a principal bundle and may be formed for each smooth manifold. Reductions of it can be used to describe structures on the manifold. An example for such a structure is given in 1.2.6, and a more general treatment follows in 3.2.

Example 1.1.4. *1.* Homogenous spaces*: Let G be a Lie group and P a closed Lie subgroup of G. Then the canonical projection $\pi : G \to G/P$ is equipped with the structure of a P-principal bundle, such that the right action on G is given by right multiplication of P.*

The action is obviously free, leaves fibers invariant (for $g \in G$ and $p \in P$ we have $gpP = gP$) and acts transitively on each fiber (if $gP = g'P$ for $g, g' \in G$ there is a $p \in P$ such that $g' = gp$).

Since $\pi : G \to G/P$ is a surjective submersion, there are local sections $\sigma \in \Gamma(G|_U)$ of G over open subsets $U \subset G/P$ that can be used to define principal charts as $G|_U \to U \times P$, $g \mapsto (gP, \sigma(gP)g)$. Those are equivariant by definition and give rise to a G-compatible atlas.

2. Frame bundles*: Let M be a smooth manifold of dimension n. We define the frame bundle over M as $\mathcal{F}_{TM} := \bigsqcup_{x \in M} \mathrm{GL}(T_x M)$, the set of all ordered bases of tangent spaces of M. It is natural to define the projection $\mathcal{F}_{TM} \to M$ by $u \mapsto x$ for $u \in \mathrm{GL}(T_x M)$.*

   There is a canonical right action of $g \in \mathrm{GL}(n, \mathbb{R})$ on $u \in \mathcal{F}_{TM}$ that is given by $u \cdot g = \left\{ \sum_{j=1}^{n} g_j^1 u_j, \ldots, \sum_{j=1}^{n} g_j^n u_j \right\}$ where the basis u of a tangent space is given by $\{u_1, \ldots, u_n\}$ and g_j^i denotes the entry of the $n \times n$-matrix g in row i and column j.

   (i) The $\mathrm{GL}(n, \mathbb{R})$-action on $\mathcal{F}_{TM}$ is free, leaves fibers invariant and acts transitively on each fiber.

   (ii) There is a canonical identification of $\mathcal{F}_{TM}$ with

   $$\bigsqcup_{x \in M} \{ u : \mathbb{R}^n \to T_x M \mid u \text{ a linear isomorphism } \} .$$

   In this picture, the G-action is given by $(u \cdot g)(y) = u(g(y))$ for $g \in \mathrm{GL}(n, \mathbb{R})$ and $y \in \mathbb{R}^n$.

   (iii) $\mathcal{F}_{TM} \to M$ can be uniquely equipped with the structure of a fiber bundle, such that the projection is a surjective submersion with respect to the the differentiable structure on $\mathcal{F}_{TM}$.

   (iv) There is a bundle atlas with $\mathrm{GL}(n, \mathbb{R})$-equivariant charts, which implies that $\mathcal{F}_{TM}$ is a principal bundle.

   For proofs of the statements above see [8, p.55f].

1.1.1 Reduction of the structure group

We present a result that describes how a reduction of a given G-principal bundle $\pi : \mathcal{G} \to M$ looks like:

Lemma 1.1.5. *Suppose there is a subset $\mathcal{H} \subset \mathcal{G}$ and a Lie subgroup H of G, such that the four following conditions are satisfied:*

(i) $\pi\big|_{\mathcal{H}} : \mathcal{H} \to M$ is surjective.

(ii) $\mathcal{H}$ is invariant under the restriction of the principal action to H.

(iii) For each $x \in M$, H acts transitively on $\pi^{-1}(x) \cap \mathcal{H}$.

(iv) For each $x \in M$ there is a smooth local section of $\mathcal{G}$ defined on an open neighborhood of x that has values in $\mathcal{H}$.

Then the inclusion $\mathcal{H} \hookrightarrow \mathcal{G}$ is a reduction of the structure group from G to H.

Proof. see [6, p.19f]

1.1.2 Principal connections

Charles Ehresmann first introduced connections on principal bundles in 1950. They will be crucial for our viewpoint of geometry on principal bundles, and we will see that they are closely related to the concept of a Cartan connection on a principal bundle, however are much more general.

Definition A *principal connection* on a G-principal bundle $\pi : \mathcal{G} \to M$ with $\dim(M) = n$ is a smooth distribution $\mathfrak{H}$ of rank n on $\mathcal{G}$ that is invariant under the right action on $\mathcal{G}$, i.e. $\mathfrak{H}_{ug} = T_u\rho_g(\mathfrak{H}_u)$ for $u \in \mathcal{G}$ and $g \in G$, and complementary to the vertical bundle.

The distribution $\mathfrak{H}$ is often called *horizontal*, since it is complementary to the vertical bundle.

Firstly, a distinguished horizontal distribution $\mathfrak{H}$ provides a pointwise linear isomorphism $T_u\pi\big|_{\mathfrak{H}_u} : \mathfrak{H}_u \cong T_{\pi(u)}M$ for all $u \in \mathcal{G}$. This implies the following

Lemma 1.1.6. *(i) Each vector field $\xi \in \mathfrak{X}(M)$ can be uniquely lifted to a vector field $\xi_{\mathcal{G}}^{hor}$ on $\mathcal{G}$ that satisfies $T\pi \cdot \xi_{\mathcal{G}}^{hor} = \xi$ and $\xi_{\mathcal{G}}^{hor}(u) \in \mathfrak{H}_u$ for all $u \in \mathcal{G}$. The vector field $\xi_{\mathcal{G}}^{hor}$ is called the* horizontal lift *of ξ.*

(ii) Furthermore, since $\mathfrak{H}$ is G-invariant, the horizontal lifts are G-equivariant, i.e. $T_u\rho_g \cdot \xi_{\mathcal{G}}^{hor}(u) = \xi_{\mathcal{G}}^{hor}(ug)$ for $u \in \mathcal{G}$ and $g \in G$.

Proof. see [8, p.65]

Notation If it is clear to which bundle we lift ξ, we will just write ξ^{hor}.

In the following proposition we will present how a connection can be equivalently described:

Proposition 1.1.7. *Let $\pi : \mathcal{G} \to M$ be a G-principal bundle endowed with a principal connection $\mathfrak{H} \subset T\mathcal{G}$.*

A principal connection can be equivalently expressed by a differential form $\gamma \in \Omega^1(\mathcal{G}, \mathfrak{g})$ that satisfies both $\rho_g^*\gamma = \mathrm{Ad}\left(g^{-1}\right) \circ \gamma$ for all $g \in G$ and $\gamma(\zeta_X) = X$ for all fundamental vector fields corresponding to $X \in \mathfrak{g}$. It is called the principal connection form *associated to $\mathfrak{H}$.*

The horizontal distribution is given by $\mathfrak{H}_u := \ker(\gamma_u)$ where $u \in \mathcal{G}$. Conversely, the splitting $T\mathcal{G} = \mathcal{V}\mathcal{G} \oplus \mathfrak{H}$ induces the principal connection form γ as the projection onto the first component together with the trivialization of $V\mathcal{G}$ from 1.1.3.

Proof. see [8, p.63f]

Any such connection has a curvature, that is the vertical projection of the Lie bracket of the horizontal projections of vector fields. Its purpose is to measure the "involutivity" of the horizontal distribution, i.e. if the curvature vanishes, the horizontal distribution is involutive and hence there exists a foliation of the bundle coming from the connection. More precisely, we have the following

Definition Let $\mathfrak{H}$ be a principal connection on a principal bundle $\mathcal{G} \to M$ over a smooth manifold M. Then $T\mathcal{G} = \mathfrak{H} \oplus V\mathcal{G}$. Denote the projections onto the horizontal and the vertical subbundle by $\mathrm{pr}^{\mathfrak{H}}$ and pr^V, respectively. The curvature of $\mathfrak{H}$ is the 2-form on $\mathcal{G}$ with values in the vector-bundle $V\mathcal{G}$, given by $R(\xi, \eta) := -\,\mathrm{pr}^V\left(\left[\mathrm{pr}^{\mathfrak{H}}(\xi), \mathrm{pr}^V(\eta)\right]\right)$ where $\xi, \eta \in \mathfrak{X}(\mathcal{G})$.

Again it is possible the describe the curvature in terms of a differential form that has values in the Lie algebra $\mathfrak{g}$:

Proposition 1.1.8. *Let $\gamma \in \Omega^1(\mathcal{G}, \mathfrak{g})$ be a connection form on the G-principal bundle $\pi : \mathcal{G} \to M$ of a principal connection $\mathfrak{H}$.*

(i) There is a 2-form $\Omega \in \Omega^2(\mathcal{G}, \mathfrak{g})$ that contains all the information on the curvature $R \in \Omega^2(\mathcal{G}, V\mathcal{G})$ of $\mathfrak{H}$. It is related to R via the equation

$$R_u(\xi, \eta) = \zeta_{\Omega_u(\xi,\eta)}(u)$$

where $u \in \mathcal{G}$ and $\xi, \eta \in T_u\mathcal{G}$. The differential form Ω is called the curvature form *of γ.*

(ii) The following equation holds for all $\xi, \eta \in \mathfrak{X}(\mathcal{G})$:

$$\Omega(\xi, \eta) = d\gamma(\xi, \eta) + [\gamma(\xi), \gamma(\eta)],$$

where $[.,.]$ denotes the Lie bracket in $\mathfrak{g}$.

(iii) The curvature form Ω is G-equivariant, i.e. $\rho_g^*\Omega = \mathrm{Ad}\left(g^{-1}\right) \circ \Omega$ for $g \in G$.

(iv) Moreover, for $u \in \mathcal{G}$ and $\xi \in V_u\mathcal{G}$ we have $\Omega(\xi, .) = 0$.

Proof. see [5, p.39]

1.2 Associated bundles

A very important feature of principal bundles is that they induce other bundles with the help of actions of their structure group G. The additional ingredient we need is an action of the structure group on some other manifold (often a vector space). Then, many objects on a principal bundle, such as a connection, can be transferred to the induced bundles.

Let $\pi : \mathcal{G} \to M$ be a G-principal bundle and S a smooth manifold endowed with a smooth left G-action $\lambda : G \times S \to S$.

Notation We will often abbreviate the action by $\cdot$ or completely omit the symbol.

Consider the manifold $\mathcal{G} \times S$, that carries an induced G-action given by $(g, (u, s)) \mapsto \left(\rho(u, g), \lambda\left(g^{-1}, s\right)\right)$ where $g \in G$, $s \in S$ and $u \in \mathcal{G}$. By forming the quotient with respect to the G-action, we obtain a new space, that is denoted by $\mathcal{G} \times_G S$ (we will see below that it is again a manifold). Its elements are equivalence classes of pairs (u, s), that we denote by $[u, s]$. By definition, we have $\left[ug, g^{-1}s\right] = [u, s]$, hence in particular $[ug, s] = [u, gs]$.

Let $\pi^S : \mathcal{G} \times_G S \to M$ be given by $\pi^S([u, s]) = \pi(u)$. This is well-defined, since the right action on $\mathcal{G}$ leaves fibers invariant.

We want to show that $\pi^S : \mathcal{G} \times_G S \to M$ is again a smooth fiber bundle: Given a principal chart $\varphi : \mathcal{G}|_U \to U \times G$ over an open set $U \subset M$, we construct a chart for the associated bundle $\mathcal{G} \times_G S$. Let $\psi : (\mathcal{G} \times_G S)|_U \to U \times S$, $[u, s] \mapsto (\pi(u), \mathrm{pr}_2(\varphi(u)) \cdot s)$ where $\mathrm{pr}_2 : U \times G \to G$ is the projection on the second component. One can show that this is a well-defined diffeomorphism (for the details see [1, p.53f]). In summary, this shows

Proposition 1.2.1. *The space $\mathcal{G} \times_G S$ together with π^S carries the structure of a fiber bundle over M with typical fiber S.*

As mentioned before, many objects on the principal bundle carry over to associated bundles. In the following proposition, we give some examples of this correspondence.

Proposition 1.2.2. *Let $\pi : \mathcal{G} \to M$ be a G-principal bundle and S a manifold carrying a left G-action.*

(i) Each homomorphism (ϕ, Φ) of principal G-bundles between $\pi : \mathcal{G} \to M$ and another G-principal bundle $\pi' : \mathcal{G}' \to M'$, such that $\phi = \mathrm{id}_G$, induces a fiber bundle morphism $\mathcal{G} \times_G S \to \mathcal{G}' \times_G S$ that is characterized by $[u, s] \mapsto [\Phi(u), s]$ for $u \in \mathcal{G}$ and $s \in S$.

(ii) Conversely, for another manifold S' equally equipped with a left G-action and a smooth map $f : S \to S'$ that is equivariant, we obtain a smooth bundle map $\mathcal{G} \times_G S \to \mathcal{G} \times_G S'$ via $[u, s] \mapsto [u, f(s)]$ for $u \in \mathcal{G}$ and $s \in S$.

Proof. see [5, p.28,40]

The next correspondence will be a particularly important one, since it converts sections of associated bundles into equivariant functions, that are often easier to work with.

Proposition 1.2.3. *Let $\pi : \mathcal{G} \to M$ be a G-principal bundle and S a manifold carrying a G-action. The the set of smooth sections of the associated bundle $\mathcal{G} \times_G S$ and the equivariant, smooth functions $\mathcal{G} \to S$ are in bijective correspondence, namely via $\sigma(\pi(u)) = [u, f(u)]$ for $\sigma \in \Gamma(\mathcal{G} \times_G S)$ and $f \in C^\infty(\mathcal{G}, S)$ satisfying $f \circ \rho_g = \lambda_{g^{-1}} \circ f$ for $g \in G$.*

Proof. see [5, p.28]

Finally, the following proposition will connect the concept of associated bundles with the reduction of the structure group from subsection 1.1. This is one of the key points of the description of geometric structures as reductions of the frame bundle.

Proposition 1.2.4. *Given a reduction $\mathcal{H}$ of the structure group H of a G-principal $\pi : \mathcal{G} \to M$ and a G-action on the manifold S, then we obtain $\mathcal{H} \times_H S = \mathcal{G} \times_G S$ where in the first associated bundle we restricted the G-action to H.*

Proof. We denote the reduction by $\iota : \mathcal{H} \hookrightarrow \mathcal{G}$. Let $\Phi : \mathcal{H} \times_H S \to \mathcal{G} \times_G S$ be given by $\Phi([u, s]) := [\iota(u), s]$ for $u \in \mathcal{H}$ and $s \in S$. Similarly to the proof of 1.2.2(i) one shows that this is a well-defined fiber bundle morphism.

To prove its injectivity, suppose $[\iota(u_1), s_1] = [\iota(u_2), s_2]$ for $u_1, u_2 \in \mathcal{H}$ and $s_1, s_2 \in S$. Then there is an $h \in H$ such that $u_2 = u_1 h$ and thus $[\iota(u_1)h, s_2] = [\iota(u_1), hs_2] = [\iota(u_1), s_1]$. Therefore, $s_1 = hs_2$ and consequently $[u_2, s_2] = [u_1 h, h^{-1} s_1] = [u_1, s_1]$.

Now let $[u, s] \in \mathcal{G} \times_G S$. Then choose a $u_0 \in \mathcal{H}$ such that $\pi(\iota(u_0)) = \pi(u)$. There is a $g \in G$ such that $u = \iota(u_0)g$, thus $[u, s] = [\iota(u_0)g, s] = [\iota(u_0), gs]$ and consequently Φ is a bijective map.

We still have to show that the inverse of Φ is smooth. Take a local section σ of $\mathcal{H}$ and consider the map $\tau : \mathcal{G} \times_M \mathcal{G} \to G$ that gives for two elements $u, u' \in \mathcal{G}$, that lie in the same fiber, the unique element of G that satisfies $u = u'g$. By the implicit function theorem, this is a smooth map. Then the inverse of Φ is given by $\Phi^{-1}([u,s]) = [\sigma(\pi(u)), \tau(u, \iota(\sigma(\pi(u))))s]$. □

Note that in the case that S is a vectorspace, the map Φ from the above proof is a linear isomorphism.

On the other hand, we may describe reductions of the structure group as sections of an associated bundle:

Proposition 1.2.5. *Reductions of the structure group $j : \mathcal{G} \to \mathcal{F}$ from G to H are in bijective correspondence with smooth sections σ of the associated bundle $\hat{\pi}^{G/H} : \mathcal{F} \times_G (G/H) = \mathcal{F}/H \to M$.*

Sketch of Proof. We give a sketch of the proof. For the details, see [5, p.46]. First, note that there is a natural identification $\mathcal{F} \times_G (G/H) = \mathcal{F}/H$ via $[u, gH] \mapsto uH$ where $u \in \mathcal{F}$ and $g \in G$. It is straightforward to check that this is a well-defined bijection.

Since H acts freely on $\mathcal{G}$, the canonical projection $q : \mathcal{F} \to \mathcal{F}/H$ is a principal fiber bundle, such that the following diagram commutes by definition:

$$\begin{tikzcd}
\mathcal{F} \arrow[d, "q"'] \arrow[dd, bend left=60, "\hat{\pi}"] \\
\mathcal{F}/H \arrow[d, "\hat{\pi}^{G/H}"] \\
M
\end{tikzcd}$$

Let $\mathcal{G}$ be an H-subbundle of $\mathcal{F}$. Then we obtain the section of $\mathcal{F}/H$ at $x \in M$ by choosing a local section $\bar{\sigma}$ of $\mathcal{G}$ around x and taking $\sigma := q \circ \bar{\sigma}$ where $\bar{\sigma}$ is defined. This is well-defined, since two such sections piece together to a global section of $\mathcal{F}/H$: Consider two such sections $\bar{\sigma}_1, \bar{\sigma}_2 : U \to \mathcal{G}$ that are defined on the same open set $U \subset M$, then for each $x \in U$ there is an $h \in H$ with $\bar{\sigma}_1(x) \cdot h = \bar{\sigma}_2(x)$, hence $q(\bar{\sigma}_1(x)) = q(\bar{\sigma}_2(x))$.

On the other hand, for a local section $\sigma \in \Gamma(\mathcal{F}/H)$ let $\mathcal{G} := q^{-1}(\sigma(M))$. This defines an H-subbundle of $\mathcal{F}$ since the conditions from 1.1.5 are satisfied. This is straightforward to check. □

Finally, in order to illustrate how reductions of the structure group induce geometric structures, we present the following Example. This also shows how associated bundles are used to construct bijective correspondences.

Example 1.2.6. *Let $E \to M$ be an n-dimensional vector bundle over the manifold M and*

$$\mathcal{F}_E = \bigsqcup_{x \in M} \{u \text{ linear isomorphism } \mathbb{R}^n \to E_x\}$$

its linear frame bundle, that is a principal bundle with structure group $\mathrm{GL}(n, \mathbb{R})$.

<u>Claim:</u> An $O(n)$-reduction of $\mathcal{F}_E$ is equivalent to a positive definite bundle metric on E.

Proof: *A metric on E is a choice of an inner product on each fiber E_x for $x \in M$, that depends smoothly on x. More formally, it is a smooth section of the associated bundle $\mathcal{F}_E \times_{GL(n,\mathbb{R})} \mathcal{O}$, $\mathcal{O}$ is the space of inner products on $\mathbb{R}^n$ and a given metric g corresponds to the section $\sigma(\pi(u)) = \left[u, u^* g_{\pi(u)}\right]$.*

Note that by choosing an element b_0 in $\mathcal{O}$, we obtain an idenfication $\mathcal{O} = \mathrm{GL}(n,\mathbb{R})/H$ where $H = \mathrm{Stab}_{GL(n,\mathbb{R})}(b_0)$, since $\mathrm{GL}(n,\mathbb{R})$ acts transitively on $\mathcal{O}$, and hence

$$\begin{aligned}\mathcal{F}_E \times_{\mathrm{GL}(n,\mathbb{R})} \mathcal{O} &= \mathcal{F} \times_{\mathrm{GL}(n,\mathbb{R})} GL(n,\mathbb{R})/H \\ &= \left(\mathcal{F}_E \times_{GL(n,\mathbb{R})} GL(n,\mathbb{R})\right)/H = \mathcal{F}_E/H.\end{aligned}$$

Thus, a section of $\mathcal{F}_E \times_{GL(n,\mathbb{R})} \mathcal{O}$ corresponds to fiberwise H-Orbits in $\mathcal{F}_E$, that piece together to give a reduction of the structure group from $GL(n,\mathbb{R})$ to H.

Associated vector bundles play a particularly important role. These are formed as the associated bundle of a principal bundle together with a vector-space that carries a smooth G-representation.

Proposition 1.2.7. *Let $\pi : \mathcal{G} \to M$ be a G-principal bundle and V a finite-dimensional vector space endowed with a smooth G-representation.*

Then the associated bundle $\mathcal{G} \times_G V$ is a vector bundle of rank $\dim(V)$. *The vector space structure on its fiber is characterized by $[u,v] + r[u,w] := [u, v + rw]$ where $u \in \mathcal{G}$, $r \in \mathbb{R}$ and $v, w \in V$.*

The construction satisfies functorial properties such as

$$\mathcal{G} \times_G (V^*) = (\mathcal{G} \times_G V)^*,\ \mathcal{G} \times_G (V \otimes W) = (\mathcal{G} \times_G V) \otimes (\mathcal{G} \times_G W)$$
$$\text{and } \mathcal{G} \times_g (V \oplus W) = \mathcal{G} \times_G V \oplus \mathcal{G} \times_G W$$

where W is another finite-dimensional vector space.

Proof. see [5, p.28]

1.2.1 Induced connections

The interplay between principal and associated bundles continues: Let $\mathfrak{H}$ be a principal connection on the bundle $\pi : \mathcal{G} \to M$ with structure group G.

Definition A *fiber bundle connection* on a fiber bundle $\pi : F \to M$ is given by a distribution of rank $\dim(M)$ on the manifold F, that is complementary to the vertical distribution given by $VF = \ker(T\pi)$.

Analogously as before, connections on fiber bundles determine a distinguished lifting of vector fields.

Furthermore, connections on principal bundles $\mathcal{G}$ induce fiber bundle connections on associated bundles of $\mathcal{G}$:

Let S be a manifold endowed with a G-action. Firstly note that $q : \mathcal{G} \times S \to \mathcal{G} \times_G S$, $(u,s) \mapsto [u,s]$ is a surjective submersion, and in fact even a principal bundle. Hence for all (u,s) the map $T_{(u,s)}q : T_u\mathcal{G} \times T_sS \to T_{[u,s]}\mathcal{G} \times S$ is surjective.

Note that $(q \circ \pi^S)(u,s) = \pi^S([u,s]) = \pi(u) = (\pi \circ \mathrm{pr}_1)(u)$ where $u \in \mathcal{G}$, $s \in S$ and $\mathrm{pr}_1 : \mathcal{G} \times S \to \mathcal{G}$ denotes the projection on the first component. Now consider the subspace $\mathfrak{H}_u \times \{0\} \subset T_{(u,s)}(\mathcal{G} \times S)$. Restricted to this subspace, the projection $T_{(u,s)}q$ is injective, since $(\xi, 0) \in \mathfrak{H}_u \times \{0\} \cap \ker(Tq)$ implies that $T\pi \cdot Tq \cdot (\xi, 0) = T\pi \cdot \xi = 0$. But ξ is horizontal, hence $\xi = 0$.

Definition The induced connection on $\mathcal{G} \times_P S$ is given by $\mathfrak{H}_{[u,s]} := T_{(u,s)}q \cdot (\mathfrak{H}_u \times \{0_s\})$.

We still have to show that the induced connection is well-defined: Note that $q(ug, g^{-1}s) = q(u, s)$, thus $q \circ (\rho_g \times \lambda_{g^{-1}}) = q$. Hence we have

$$\begin{aligned}\mathfrak{H}_{[ug,g^{-1}s]} &= T_{(ug,g^{-1}s)}q \cdot (\mathfrak{H}_{ug} \times \{0\}) \\ &= T_{(ug,g^{-1}s)}q \cdot (T_u\rho_g \cdot \mathfrak{H}_u, T_s\lambda_{g^{-1}} \cdot \{0\}) = T_{(u,s)}q \cdot (\mathfrak{H}_u \times \{0\}).\end{aligned}$$

In the case of an associated vector bundle $\mathcal{V} := \mathcal{G} \times_G V$ induced connections admit a particularly nice form.

Definition and Proposition 1.2.8. *A fiber bundle connection on a vector bundle $\pi : \mathcal{V} \to M$ is called* linear connection*, if the horizontal lift for fixed $x \in M$ and $\xi \in T_xM$, depending on the point in $\mathcal{V}_x$ to where ξ is lifted, is linear for all x and ξ.*

A linear connection is equivalent to a differential operator

$$\nabla : \mathfrak{X}(M) \times \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V}),\ (\xi, \sigma) \mapsto \nabla_\xi \sigma.$$

that is $C^\infty(M)$-linear in $\mathfrak{X}(M)$ and $\mathbb{R}$-linear in $\Gamma(\mathcal{V})$, and additionally satisfies a Leibniz-rule for $f \in C^\infty(M)$, $\xi \in \mathfrak{X}$ and $\sigma \in \Gamma(\mathcal{V})$:

$$\nabla_\xi(f\sigma) = (\xi \cdot f)\sigma + f\nabla_\xi\sigma$$

Proof. see [5, p. 35f]

Recall from 1.2.3 that a section σ of $\mathcal{V} = \mathcal{G} \times_G V$ corresponds to an equivariant function $f : \mathcal{G} \to V$.

Proposition 1.2.9. *The derivative $\nabla_\xi\sigma$ of $\sigma \in \Gamma(\mathcal{V})$ with respect to a vector field $\xi \in \mathfrak{X}(M)$ corresponds to $\xi^{hor}_{\mathcal{G}} \cdot f \in C^\infty(\mathcal{G}, V)^G$, the derivative of f by the horizontal lift of ξ.*

Furthermore, the frame bundle $\mathcal{F}_E$ of a vector bundle E (for the definition see Example 1.2.6) has special properties. Note that its structure group is the general linear group of the same dimension as the fiber of E. The frame bundle $\mathcal{F}_M$ in Example 1.1.4 is in fact the frame bundle of TM.

Lemma 1.2.10. *Let M be a smooth dimensional manifold, E an n-dimensional vector bundle over M and $\mathcal{F}_E$ its frame bundle.*

(i) The canonical map $\mathcal{F}_E \times_{\mathrm{GL}(n,\mathbb{R})} \mathbb{R}^n \to E$ given by $[u, x] \mapsto u(x)$ is an isomorphism of vector bundles.

(ii) Any linear connection on E is induced by a unique principal connection on $\mathcal{F}_E$.

Proof. see [5, p.42]

Example 1.2.11. *Continuing 1.2.6, note that there is a unique principal connection on $\mathcal{F}_E$ that induces ∇. The above $O(n)$-reduction of $\mathcal{F}_E$ is compatible with this principal connection if and only if the metric on E is parallel with respect to ∇. This is easy to compute.*

Chapter 2

Holonomy of principal connections

In this chapter we want to give a short introduction to holonomy of principal connections. One should regard it as motivation for the concepts in Chapter 4. We follow Chapter 3 of [6].

Throughout this chapter we fix a G-principal bundle $\pi : \mathcal{G} \to M$ endowed with a principal connection form $\gamma \in \Omega^1 (\mathcal{G}, \mathfrak{g})$.

2.1 Horizontal lift of curves

Let $c : [a, b] \to M$ be a smooth curve, more precisely c should be smoothly extensible on an interval $(a - \epsilon, b + \epsilon)$ for an $\epsilon > 0$. Our aim is to construct a curve on $\mathcal{G}$, that covers c and has derivative in the horizontal distribution.

Form the pullback bundle

$$\mathrm{pr}_1 : c^*\mathcal{G} := \{(t, u) \in [a, b] \times \mathcal{G} \mid c\,(t) = \pi\,(u)\} \to [a, b],$$

that is again a G-principal bundle with the G-action in the second component of $c^*\mathcal{G}$. Here $[a, b]$ should be regarded as a manifold with boundary. It carries the differential form $(\mathrm{pr}_2)^*\,\gamma \in \Omega^1\,(c^*\mathcal{G}, \mathfrak{g})$, that is a principal connection because the map $\mathrm{pr}_2 : c^*\mathcal{G} \to \mathcal{G}$ is G-equivariant by definition of $c^*\mathcal{G}$, hence $c^*\gamma$ is G-equivariant since γ is, and the fibers, projection and G-action on $c^*\mathcal{G}$ remain the same as on $\mathcal{G}$, so fundamental vector fields are reproduced by $c^*\gamma$.

Consider the vector field $\bar{c}$ on $c^*\mathcal{G}$ given by

$$\bar{c}\,(t, u) = \left(1, (c'\,(t))^{hor}\,(u)\right) \in (Tc)^*\,(T\mathcal{G}) = T\,(c^*\mathcal{G})\,,$$

where $(c'\,(t))^{hor}$ is the horizontal lift of $c'\,(t)$. This is well-defined, since $T_t c \cdot 1 = c'\,(t) = T\pi\left((c'\,(t))^{hor}\,(u)\right)$. $\bar{c}$ obviously depends smoothly on t and one may take any local smooth section of $\mathcal{G}$ to show smooth dependence on u.

Definition Let $c : [a, b] \to M$ a smooth curve and $u \in \mathcal{G}_{c(t_0)}$ where $t_0 \in [a, b]$. Shrink the interval $[a, b]$ around t_0, such that $\mathrm{Fl}^{\bar{c}}_t(t_0, u)$ exists for all $t \in [a, b]$.

Then the horizontal lift of c at u is defined as $\tilde{c}_u(t) := \mathrm{pr}_2\left(\mathrm{Fl}_t^{\bar{c}}(t_0, u)\right)$ on $[a,b]$.

Let us state some basic properties of $\tilde{c}_u$.

Proposition 2.1.1. *(i) The curve $\tilde{c}_u$ is indeed a lift of c, i.e. $\pi \circ \tilde{c}_u = c$.*

(ii) The horizontal lift of $c : [a,b] \to \mathcal{G}$ is G-equivariant, i.e. for $g \in G$, $u \in \mathcal{G}_{c(t_0)}$ for $t_0, t \in [a,b]$ we have $\tilde{c}_{ug}(t) = c_u(t) \cdot g$.

(iii) The horizontal curve $\tilde{c}_u : [a,b] \to \mathcal{G}$ is horizontal, meaning that $\tilde{c}'_u(t) \in \mathfrak{H}_{\tilde{c}(t)}$ for all $t \in [a,b]$, where $\mathfrak{H}$ denotes the horizontal distribution of the given principal connection.

Proof. (i) Show that for $u \in \mathcal{G}_{c(0)}$ and $t \in (a,b)$ we have $\pi\left(\tilde{c}_u(t)\right) = c\left(t\right)$.

The flow of the $\bar{c}$ is given by $\mathrm{Fl}_s^{\bar{c}}(t,u) = \left(t+s, \mathrm{Fl}_s^{\left(c'(t)\right)^{hor}}(u)\right)$, since, after evalulating at $s=0$ we have $\left(t, \mathrm{Fl}_0^{\left(c'(t)\right)^{hor}}(u)\right) = (t,u)$, and

$$\begin{aligned}\frac{d}{ds}\left(t+s, \mathrm{Fl}_s^{\left(c'(t)\right)^{hor}}(u)\right) &= \left(1, \left(c'\left(t\right)\right)^{hor}\left(\mathrm{Fl}_s^{\left(c'(t)\right)^{hor}}(u)\right)\right)\\ &= \bar{c}\left(\mathrm{Fl}_s^{\left(c'(t)\right)^{hor}}(u)\right).\end{aligned}$$

The fact that $\mathrm{Fl}_s^{\bar{c}}(t,u) \in c^*\mathcal{G}$ implies, by setting $s = t_0$, that

$$c\left(t\right) = c\left(\mathrm{pr}_1\left(t, \mathrm{Fl}_{t_0}^{\left(c'(t)\right)^{hor}}\right)\right) = \pi\left(\mathrm{Fl}_{t+t_0}^{\left(c'(t)\right)^{hor}}(u)\right) = \pi\left(\tilde{c}_u(t)\right).$$

(ii) <u>*Claim 1*</u>: Flows of invariant vector fields are equivariant.

proof of claim: Let $\xi \in \mathfrak{X}\left(\mathcal{G}\right)$ satisfying $\xi\left(ug\right) = T_u\rho_g \cdot \xi\left(u\right)$. Then we show $\mathrm{Fl}_t^{\xi}(ug) = \mathrm{Fl}_t^{\xi}(u)\, g$: Firstly, $\mathrm{Fl}_0^{\xi}(u)\, g = ug$, and secondly, $\frac{d}{dt}\left(\mathrm{Fl}_t^{\xi}(u)\, g\right) = \frac{d}{dt}\left(\rho_g\left(\mathrm{Fl}_t^{\xi}(u)\right)\right) = T\rho_g \cdot \xi\left(\mathrm{Fl}_t^{\xi}(u)\right) = \xi\left(\mathrm{Fl}_t^{\xi}(u)\, g\right)$. This implies Claim 1.

<u>*Claim 2*</u>: $\bar{c}$ is G-invariant.

proof of claim: Let $g \in G$, then the projection satisfies $\pi \circ \rho_g = \pi$, hence $T_{ug}\pi \circ T_u\rho_g = T_u\pi$. By invariance of the horizontal distribution we have $T_{ug}\pi|_{\mathfrak{H}_{ug}} \circ T_u\rho_g|_{\mathfrak{H}_u} = T_u\pi|_{\mathfrak{H}_u}$, and these three maps are all linear isomorphisms onto their image. Hence for $(t,u) \in c^*\mathcal{G}$ the vector field satisfies

$$\begin{aligned}\bar{c}\left(t, ug\right) &= \left(1, \left(c'\left(t\right)\right)^{hor}\left(ug\right)\right) = \left(1, \left(T_{ug}\pi|_{\mathfrak{H}_{ug}}\right)^{-1} \cdot c'\left(t\right)\right)\\ &= \left(1, T_u\rho_g \cdot \left(T_u\pi|_{\mathfrak{H}_u}\right)^{-1} \cdot c'\left(t\right)\right) = T_u\rho_g \cdot \bar{c}\left(t,u\right).\end{aligned}$$

This finishes the proof of (i).

(iii) can be seen from

$$\begin{aligned}\frac{d}{dt}\tilde{c}_u(t) &= \mathrm{pr}_2\left(\frac{d}{dt}\mathrm{Fl}_t^{\bar{c}}(t_0,u)\right) = \mathrm{pr}_2\left(\bar{c}\left(\mathrm{Fl}_t^{\bar{c}}(t_0,u)\right)\right)\\ &= \left(c'\left(t\right)\right)^{hor}\left(\mathrm{Fl}_{t_0+t}^{\left(c'(t)\right)^{hor}}(u)\right) \in \mathfrak{H}.\end{aligned}$$

$\square$

2.2 Holonomy groups

Consider a smooth loop $c : [a, b] \to M$. Then, since G acts freely and transitively on each fiber, there is a unique element g_c^u of G such that $\tilde{c}_u(b) = u g_c^u$ for all $u \in \mathcal{G}_a$.

Our aim is to collect all such elements of G in order to obtain a distinguished subgroup of G that is induced by the connection.

We move to a slightly different concept in order to simplify the subsequent definition, namely we consider a piecewise smooth curve $c : [a, b] \to M$, so c is continuous and there is a partition $\{a = t_0, t_1, \dots, t_{k-1}, t_k = b\}$ of the interval $[a, b]$ such that c is smooth on $[t_j, t_{j+1}]$ for $0 \leq j \leq k-1$.

For $u \in \mathcal{G}_{c(a)}$ define $\tilde{c}_u$ recursively by $\tilde{c}_u(t) := \tilde{c}_u(t)$ if $t \in [a, t_1]$ and $\tilde{c}_u(t) := \tilde{c}_{\tilde{c}(t_j)}(t)$ if $t \in [t_j, t_{j+1}]$. In order to see that $\tilde{c}_u$ is well-defined, note that if c is smooth on $[a, b]$ and $u \in \mathcal{G}_{t_0}$ and $t_1, t + t_1 \in [a, b]$ we have by definition $\tilde{c}_{\tilde{c}_u(t_1)}(t) = \tilde{c}_u(t + t_1)$.

Definition Let $u \in \mathcal{G}$ and $x := \pi(u)$. Then

$$\begin{aligned} \mathrm{Hol}_\gamma(u) := \{ g_c^u \in G \mid \exists \text{ piecewise smooth, closed curve } c \text{ such that} \\ \tilde{c}_u(b) = u \cdot g_c^u \} \end{aligned}$$

is called the *holonomy group* at u of γ.

Proposition 2.2.1. *The holonomy group* $\mathrm{Hol}_\gamma(u)$ *for* $u \in \mathcal{G}$ *is a subgroup of* G.

Proof. Consider two piecewise smooth loops $c : [a, b] \to M$, $\bar{c} : [b, d] \to M$ and $x = \pi(u) \in M$ where $u \in \mathcal{G}_a = \mathcal{G}_b$. Then their horizontal lifts $\tilde{c}$ and $\tilde{\bar{c}}$ yield elements $g_c^u, g_{\bar{c}}^u \in \mathrm{Hol}_\gamma(u)$ that are characterized by $\tilde{c}(b) = u g_c^u$ and $\tilde{\bar{c}}(d) = u g_{\bar{c}}^u$.

The concatenation $c * \bar{c} : [a, d] \to M$ is defined by

$$(c * \bar{c})(t) = \begin{cases} c(t) & \text{if } t \in [a, b] \\ \bar{c}(t) & \text{if } t \in [b, d] \end{cases}.$$

This is again a loop at $x \in M$ and its horizontal lift yields

$$\widetilde{(c * \bar{c})}_u(d) = \tilde{\bar{c}}_{\tilde{c}_u(b)}(d) = \tilde{\bar{c}}_{u g_c^u}(d) = \tilde{\bar{c}}_u(d) g_c^u = u g_{\bar{c}}^u g_c^u,$$

hence the product of two elements is again an element of the holonomy group.

Furthermore, let $c^{-1} : [a, b] \to M$ be given by $c^{-1}(t) := c(a + b - t)$. This is exactly c passed through backward. We slightly modify c^{-1} by shifting: $c^{-1} : [b, 2b - a] \to M$, $c^{-1}(t) := c(a + b - (t - (b - a))) = c(2b - t)$. Note that $(c^{-1})'(t) = -c'(2b - t)$. Then we have

$$\begin{aligned} \widetilde{(c * c^{-1})}_u(2b - a) = \widetilde{c^{-1}}_{\tilde{c}(b)}(2b - a) &= \mathrm{Fl}_{(2b-a)-b}^{-(c')^{hor}} \left(\mathrm{Fl}_{b-a}^{(c')^{hor}}(u) \right) \\ &= \mathrm{Fl}_{a-b}^{(c')^{hor}} \left(\mathrm{Fl}_{b-a}^{(c')^{hor}}(u) \right) = u. \end{aligned}$$

Thus $c^{-1}(2b - a) = u(g_u^c)^{-1}$. Hence for each element of $\mathrm{Hol}_\gamma(u)$ also its inverse is in $\mathrm{Hol}_\gamma(u)$.

□

Observe how holonomy groups at different points on the total space are related:

Proposition 2.2.2. *(i) Let $u \in \mathcal{G}$ and $g \in G$. Then we have* $\operatorname{Hol}_\gamma(ug) = g^{-1}\operatorname{Hol}_\gamma(u)\, g$.

(ii) Let $\bar{c} : [a,b] \to M$ be a piecewise smooth curve, $u \in \mathcal{G}_{c(a)}$ and $\bar{u} := \tilde{\bar{c}}_u(b)$. Then there is a canonical identification $\operatorname{Hol}_\gamma(\bar{u}) \cong \operatorname{Hol}_\gamma(u)$.

Proof. (i) If $c : [a,b] \to M$ is a piecewise smooth loop at $x := \pi(u)$, then $\tilde{c}_u(b) = ug^u_c$ and $\tilde{c}_{ug}(b) = ugg^{ug}_c$. We know from 2.1.1 (ii) that $ugg^{ug}_c = \tilde{c}_{ug}(b) = \tilde{c}_u(b)g = ug^u_c g$. We conclude $ug^u_c = ugg^{ug}_c g^{-1}$, thus $g^{ug}_c = g^{-1}g^u_c g$.

(ii) Let $c : [a,d] \to M$ be piecewise smooth loop at $x := c(a)$, then $g^u_c \in \operatorname{Hol}_\gamma(u)$. Form $\bar{c}^{-1}$, the curve in reverse direction, and shift it by $d-b$: $\bar{c}^{-1}(t) := \bar{c}(a+b-t+(b-d)) = \bar{c}(a+d-t)$, $[d-b+a,d] \to M$.

The concatenation $\bar{c}^{-1} * c * \bar{c} : [d-b+a,b]$ gives a loop based at $\bar{x} := c(b)$, since $\bar{c}^{-1}(d-b+a) = \bar{c}(a+d-d+b-a) = \bar{c}(b)$. Hence any loop at x can be made into a loop at $\bar{x}$. Note that since $\tilde{\bar{c}}_u(b) = \bar{u}$ we have $\widetilde{\bar{c}^{-1}}_{\bar{u}}(d) = u$. Thus $\tilde{c}_{\widetilde{\bar{c}^{-1}}_{\bar{u}}(d)}(a) = \tilde{c}_{\bar{u}}(a) = ug^u_c$ and therefore

$$(\widetilde{\bar{c}^{-1} * c * \bar{c}})_{\bar{u}}(b) = \tilde{\bar{c}}_{ug^u_c}(b) = \tilde{\bar{c}}_u(b)g^u_c = \bar{u}g^u_c.$$

Therefore, any element in $\operatorname{Hol}_\gamma(u)$ can be interpreted as an element in $\operatorname{Hol}_\gamma(\bar{u})$.

The converse direction is completely analogous. □

Theorem 2.2.3. *The holonomy group $\operatorname{Hol}_\gamma(u)$ for $u \in \mathcal{G}$ is a Lie subgroup of G.*

Proof. We give a short idea of the proof. For the full proof, see [6, p.49ff].

Firstly, we consider the restricted holonomy group

$$\begin{aligned}\operatorname{Hol}^0_\gamma(u) := \{g^c_u \in G \mid \exists c : [a,b] \to M \text{ a piecewise smooth,}\\ \text{nullhomotopic loop at } x = \pi(u) \text{ and } \tilde{c}_u(b) = ug^c_u\}.\end{aligned}$$

For each nullhomotopic curve there is a smooth $F : [a,b]^2 \to M$ such that c is the concatenation of the four boundary segments $t \mapsto F(t,a)$, $t \mapsto F(b,t)$, $t \mapsto F(a+b-t,b)$ and $t \mapsto F(a,a+b-t)$. Then retract this rectangle to the line $[a,b] \times \{0\}$, hence we obtain curves c_s for $s \in [0,1]$ that are the concatenation of $t \mapsto F(t,a)$, $t \mapsto F(b,(1-s)t)$, $t \mapsto F(a+b-t,(1-s)b+sa)$ and $t \mapsto F(a,(1-s)(a+b-t))$.

Mapping s to the evaluation of $(\widetilde{c_s})_u$ at its endpoint gives a piecewise smooth curve from ug^c_u to u, since for $s=1$ the curve c_s is just the concatenation of the line segment $t \mapsto F(t,a)$ and its inverse $t \mapsto F(a+b-t,a)$. This can be seen as a piecewise smooth curve in $\operatorname{Hol}^0_\gamma(u)$ from g^c_u to e.

Then apply the Theorem from [6, p.50] that claims that each subgroup H of G, where each element in H can be connected to e by a piecewise smooth path in H, is a Lie subgroup.

Furthermore, the map $[c] \mapsto g^c_u \cdot \operatorname{Hol}^0_\gamma(u)$ is a surjective homomorphism from the fundamental group $\pi_1(M,x)$ onto $\operatorname{Hol}_\gamma(u)/\operatorname{Hol}^0_\gamma(u)$. Since $\pi_1(M,x)$ is countable, this gives the necessary topological conditions in order to prove that $\operatorname{Hol}_\gamma(u)$ is a Lie subgroup of G. □

2.3 Curvature and Holonomy

In this section we will show that the curvature form has values only in the Lie-algebra $\mathfrak{hol}(u)$ of $\mathrm{Hol}_\gamma(u)$ for $u \in \mathcal{G}$. This result prepares the theorems following in the next sections.

Lemma 2.3.1. *Let $\xi, \eta \in \mathfrak{X}(\mathcal{G})$ be two vector fields on $\mathcal{G}$ with values in the horizontal distribution $\mathfrak{H}$. Then $\zeta_{\,\Omega(\xi,\eta)} = \mathrm{pr}^{\mathfrak{H}}([\xi,\eta]) - [\xi,\eta]$ where $\mathrm{pr}^{\mathfrak{H}} : T\mathcal{G} \to \mathfrak{H}$ denotes the projection onto the horizontal distribution according to the decomposition $T\mathcal{G} = \mathfrak{H} \oplus V\mathcal{G}$.*

Proof. We know from 1.1.8(ii) that the curvature form is given by

$$\begin{aligned}\Omega(\xi,\eta) &= d\gamma(\xi,\eta) + [\gamma(\xi), \gamma(\eta)] \\ &= \xi \cdot \gamma(\eta) - \eta \cdot \gamma(\xi) - \gamma([\xi,\eta]) + [\gamma(\xi), \gamma(\eta)],\end{aligned}$$

The first two and the last summand vanish, since ξ and η are horizontal.

Let $u \in \mathcal{G}$. Any tangent vector $\nu \in T_u\mathcal{G}$ is of the form $\mathrm{pr}^{\mathfrak{H}}(\nu) + \mathrm{pr}^V(\nu)$, where pr^V denotes the projection onto the vertical subspace of $T_u\mathcal{G}$. Moreover, there is a $X \in \mathfrak{g}$ such that $\mathrm{pr}^V(\nu) = \zeta_X(u)$. Since $\ker(\gamma_u) = \mathfrak{H}_u$ we obtain $\zeta_{\gamma_u(\nu)}(u) = \zeta_{\gamma_u(\zeta_X(u))}(u) = \zeta_X(u) = \mathrm{pr}^V(\nu)$.

Thus

$$\begin{aligned}\zeta_{\,\Omega(\xi,\eta)} &= \zeta_{-\gamma([\xi,\eta])} = -\,\mathrm{pr}^V([\xi,\eta]) \\ &= -\,[\xi,\eta] + \mathrm{pr}^{\mathfrak{H}}([\xi,\eta]).\end{aligned}$$

This finishes the proof. □

It is well-known (see [11, p.34]), that for $x \in M$ we have

$$[\xi,\eta](x) = \frac{d}{dt}|_0 \, \mathrm{Fl}^{\xi}_{-\sqrt{t}} \left(\mathrm{Fl}^{\eta}_{-\sqrt{t}} \left(\mathrm{Fl}^{\xi}_{\sqrt{t}} \left(\mathrm{Fl}^{\eta}_{\sqrt{t}}(x) \right) \right) \right),$$

where the right hand side of the expression is smooth up to $t = 0$.

This helps to prove the following

Proposition 2.3.2. *The curvature form Ω_u for $u \in \mathcal{G}$ has values in $\mathfrak{hol}_\gamma(u)$.*

Proof. Let $u \in \mathcal{G}$ and $\xi, \eta \in \mathfrak{X}(\mathcal{G})$. It is sufficient to consider vector fields with values in $\mathfrak{H}$, since Ω vanishes on the vertical subspace (see 1.1.8(iv)). Note that the horizontal lifts of coordinate vector fields on M form a pointwise basis for $\mathfrak{H}_u$, since $\mathfrak{H}_u \cong T_{\pi(u)}M$ via $T_u\pi$. Hence we may take $\xi := \partial_i^{hor}$ and $\eta := \partial_j^{hor}$, the coordinate vector fields associated to some chart of M and $1 \leq i, j \leq n$.

In particular, $[\partial_i, \partial_j] = 0$. This means that the flows of ∂_i and ∂_j commute, i.e. $\mathrm{Fl}^{\partial_i}_{-t} \circ \mathrm{Fl}^{\partial_j}_{-t} \circ \mathrm{Fl}^{\partial_i}_{t} \circ \mathrm{Fl}^{\partial_j}_{t} = \mathrm{id}_M$ for $t \in \mathbb{R}$ sufficiently small. Note that $[\xi,\eta]$ does not necessarily vanish, since the horizontal distribution may not be involutive.

However, from 1.1.8 and the fact, that ξ and η are horizontal, we obtain $\Omega_u(\xi(u), \eta(u)) = -\gamma([\xi,\eta])$.

Note that since ξ covers ∂_i, their flows satisfy $\pi \circ \mathrm{Fl}^{\xi}_t = \mathrm{Fl}^{\partial_i}_t \circ \pi$. The same holds for η and ∂_j. Let $f(t,u) := \left(\mathrm{Fl}^{\xi}_{-\sqrt{t}} \circ \mathrm{Fl}^{\eta}_{-\sqrt{t}} \circ \mathrm{Fl}^{\xi}_{\sqrt{t}} \circ \mathrm{Fl}^{\eta}_{\sqrt{t}} \right)(u')$ for $t \in \mathbb{R}$

and $u' \in \mathcal{G}$. Note that the curve $t \mapsto f(t,u)$ is piecewise horizontal. In addition,

$$\begin{aligned}\pi(f(t,u)) &= \pi\left(\mathrm{Fl}^{\xi}_{-\sqrt{t}}\left(\mathrm{Fl}^{\eta}_{-\sqrt{t}}\left(\mathrm{Fl}^{\xi}_{\sqrt{t}}\left(\mathrm{Fl}^{\eta}_{\sqrt{t}}(u)\right)\right)\right)\right)\\ &= \mathrm{Fl}^{\partial_i}_{-\sqrt{t}}\left(\pi\left(\mathrm{Fl}^{\eta}_{-\sqrt{t}}\left(\mathrm{Fl}^{\xi}_{\sqrt{t}}\left(\mathrm{Fl}^{\eta}_{\sqrt{t}}(u)\right)\right)\right)\right) = \dots\\ &= \mathrm{Fl}^{\partial_i}_{-\sqrt{t}}\left(\mathrm{Fl}^{\partial_j}_{-\sqrt{t}}\left(\mathrm{Fl}^{\partial_i}_{\sqrt{t}}\left(\mathrm{Fl}^{\partial_j}_{\sqrt{t}}(\pi(u))\right)\right)\right).\end{aligned}$$

The flows of ∂_i and ∂_j commute, hence this expression equals $\pi(u)$ for all t. In fact, this means that each $t \in \mathbb{R}$, that is sufficiently small, gives us a piecewise smooth, closed curve $t \mapsto \pi(f(t,u))$. Its horizontal lift is exactly $t \mapsto f(t,u)$.

In particular, $f(t,u) \in \pi^{-1}(\pi(u))$ and for each $t \in \mathbb{R}$ there exists a $g_t \in \mathrm{Hol}_\gamma(u)$ such that $f_t(u) = ug_t$. The group element g_t depends smoothly on t by the implicit function theorem. Thus

$$[\xi,\eta](u) = \frac{d}{dt}|_0 f_t = \frac{d}{dt}|_0 u g_t = T_e\rho^u \cdot \underbrace{\frac{d}{dt}|_0 g_t}_{\in \mathfrak{hol}_\gamma(u)} = \zeta_{\frac{d}{dt}|_0 g_t}(u),$$

hence $\Omega_u(\xi,\eta) = -\gamma_u([\xi,\eta]) = -\frac{d}{dt}|_0 g_t \in \mathfrak{hol}_\gamma(u)$. □

2.4 The reduction theorem

We will prove, that by fixing $u \in \mathcal{G}$ it is possible to consider a reduction $j : \mathcal{G}(u) \hookrightarrow \mathcal{G}$ that is induced by the connection. Furthermore, this subbundle is compatible with the principal connection, meaning that the horizontal distribution is tangential to the subbundle. In terms of the connection form γ this means that $\iota^*\gamma$ has values in $\mathfrak{h}$.

Theorem 2.4.1. *Let $\mathcal{G} \to M$ be a G-principal bundle where M is (path-) connected and γ a principal connection on $\mathcal{G}$. Fixing $u \in \mathcal{G}$, there is a reduction $\mathcal{G}(u) \hookrightarrow \mathcal{G}$ of the structure group to* $\mathrm{Hol}_\gamma(u)$*, such that the horizontal distribution of γ is tangent to $\mathcal{G}(u)$, thus defining a principal connection on $\mathcal{G}(u)$. This connection also has holonomy group* $\mathrm{Hol}_\gamma(u)$ *at u. The reduction $\mathcal{G}(u)$ is defined as the set of all points that can be connected to u by a horizontal curve.*

Proof. Let

$$\mathcal{G}(u) := \{u' \in \mathcal{G} \mid \exists c : [a,b] \to M \text{ horizontal curve}, c(a) = u, c(b) = u'\}.$$

In order to show that $\mathcal{G}(u)$ is a reduction of $\mathcal{G}$, we use 1.1.5.

(1) Let $x \in M$. Then there is a curve $c : [a,b] \to M$ such that $c(a) = \pi(u)$ and $c(b) = x$, since M is path-connected. The horizontal lift $\tilde{c}_u : [a,b] \to \mathcal{G}$ connects u with $\tilde{c}_u(b)$, i.e. $\tilde{c}_u(b) \in \mathcal{G}(u)$. Hence $\pi\big|_{\mathcal{G}(u)} : \mathcal{G}(u) \to M$ is surjective.

(2) Let $u' \in \mathcal{G}(u)$ and $g \in \mathrm{Hol}_\gamma(u)$. So, on the one hand, there is a horizontal curve $c : [a,b] \to \mathcal{G}$ that connects u and u'. Note that $c = (\widetilde{\pi \circ c})_u$.

On the other hand, there is a closed curve $c_g : [d,a] \to M$ such that $(\tilde{c_g})_u(a) = ug$. Consider the concatenation $\bar{c} := (\pi \circ c)^{-1} * c_g * (\pi \circ c)$,

$[d-b+a,a] \to M$, where $(\pi \circ c)^{-1}$ is shifted by $d-b$ (cf. proof of 2.2.2(ii)). This is a closed curve at $\pi(u')$, and its horizontal lift satisfies $\widetilde{\tilde{c}}_{u'}(b) = u'g$ (this is proved analogously as in 2.2.2(ii)).

This shows that acting by elements of $\mathrm{Hol}_\gamma(u)$ preserves $\mathcal{G}(u)$.

(3) In order to show fiberwise transitivity of the action, let $u'_1, u'_2 \in \mathcal{G}_x \cap \mathcal{G}(u)$ where $x \in M$. There are horizontal curves $c_1 : [a,b] \to \mathcal{G}$, $c_2 : [b,d] \to \mathcal{G}$ such that $c_1(a) = u = c_2(b)$ and $c_1(b) = u'_1$, $c_2(d) = u'_2$, respectively. Note that $(\widetilde{c_i})_u = c_i$.

The curve $c := (\pi \circ c_1)^{-1} * (\pi \circ c_2)$ on M is closed at x, and its horizontal lift yields

$$u'_1 g^c_{u'_1} = \tilde{c}_{u'_1}(d) = (\widetilde{c_2})_{\widetilde{(c_1^{-1})}_{u'_1}(b)}(d) = (\widetilde{c_2})_u(d) = u'_2,$$

where $g^c_{u'_1} \in \mathrm{Hol}_\gamma(u'_1) \cong \mathrm{Hol}_\gamma(u)$.

(4) We have to construct a local smooth section of $\mathcal{G}$ that has values in $\mathcal{G}(u)$. Choose a local frame for $\mathfrak{H}$, i.e. vector fields $\xi_1, \ldots, \xi_n \in \mathfrak{X}(U)$, where U is an open neighborhood of some $u_0 \in \mathcal{G}(u)$, such that for all $u' \in U$ the vector fields $\xi_1(u'), \ldots, \xi_n(u')$ are a basis of $\mathfrak{H}_{u'}$. This can be done by taking a local frame on M (e.g. coordinate vector fields) and lifting them horizontally. Then consider the map $\varphi(t^1, \ldots, t^n) := (\mathrm{Fl}^{\xi_1}_{t_1} \circ \cdots \circ \mathrm{Fl}^{\xi_n}_{t_n})(u_0)$, that is defined on an open neighborhood V of $0 \in \mathbb{R}^n$.

Note that

$$T_0\varphi \cdot e_i = \frac{d}{dt}|_0 \varphi(0, \ldots, 0, t, 0, \ldots, 0) = \frac{d}{dt}|_0 \,\mathrm{Fl}^{\xi_i}_t(u_0) = \xi_i(\varphi(0)) = \xi_i(u_0).$$

Thus $T_0\varphi \cdot (s_i) = \sum_{i=1}^n s_i \xi_i(u_0)$ for $s_1, \ldots, s_n \in \mathbb{R}$. This shows that $T_0\varphi : \mathbb{R}^n \to \mathfrak{H}_{u_0}$ is a linear isomorphism.

Since $T_{u_0}\pi\big|_{\mathfrak{H}_{u_0}} : \mathfrak{H}_{\mathrm{u}_\circ} \to T_{\pi(u)}M$ is a linear isomorphism, so is $T_0(\phi \circ \varphi)$. Thus $\psi := \pi \circ \varphi$ is a local diffeomorphism $V \to \psi(V)$ (possibly we have to shrink V).

Furthermore, for all $(t^1, \ldots, t^n) \in \mathbb{R}^n$ the curve $[0,1] \to \mathcal{G}$, $t \mapsto \varphi(t(t^1, \ldots, t^n))$ is horizontal, as can be seen by an analogous computation. Hence φ has values in $\mathcal{G}(u)$.

Therefore the map $\psi(V) \to \mathcal{G}(u)$, $y \mapsto \varphi(\psi^{-1}(y))$ gives a local section of $\mathcal{G}(u)$, since $\pi(\varphi(\psi^{-1}(y))) = \psi(\psi^{-1}(y)) = y$.

Next, consider the horizontal spaces. The claim is easy to see in terms of the local section from (4): For each u_0 in $\mathcal{G}(u)$ we have a smooth local section σ around $\pi(u_0)$ of $\mathcal{G}(u)$ with the property that $T_{\pi(u_0)}\sigma$ has values in $\mathfrak{H}_{u_0}$. Obviously this implies that $\mathfrak{H}_{u_0} \subset T_{u_0}\mathcal{G}(u)$ and the remaining claims follow easily. □

2.5 The Ambrose-Singer-theorem

The following theorem will conclude our short outline on holonomy groups of principal connections. It shows, that the curvature of a connection determines the Lie algebra of its holonomy group, hence the structure of the holonomy group itself.

Theorem 2.5.1. *Let $\mathcal{G} \to M$ be a G-principal bundle over a connected manifold M and $\gamma \in \Omega^1(\mathcal{G}, \mathfrak{g})$ a principal connection. Then the values $\Omega_{u'}(\xi, \eta)$ of the curvature form, where $u' \in \mathcal{G}$ and $\xi, \eta \in T_{u'}\mathcal{G}$ vary, span the Lie subalgebra $\mathfrak{hol}_\gamma(u)$ for $u \in \mathcal{G}$.*

Proof. Without loss of generality, we will consider $\mathcal{G}(u)$, the holonomy reduction reduction of $\mathcal{G}$, together with the connection γ restricted to $\mathcal{G}(u)$ (see 2.4.1). Let

$$\mathfrak{h} := \operatorname{span}\left\{\Omega_{u'}(\xi, \eta) \mid u' \in \mathcal{G}(u),\ \xi, \eta \in T_{u'}\mathcal{G}(u)\right\}.$$

By 2.3.2 we have $\mathfrak{h} \leq \mathfrak{hol}_\gamma(u) =: \mathfrak{hol}$. Hence we have to show $\mathfrak{hol} \leq \mathfrak{h}$.

Let $\mathfrak{N}$ be the distribution on $\mathcal{G}(u)$ that is defined by $\mathfrak{N}_{u'} := \mathfrak{H}_{u'} \oplus \{\zeta_X(u') \mid X \in \mathfrak{h}\}$ for $u' \in \mathcal{G}(u)$. This is smooth, since it is spanned by a smooth distribution and a set of vector fields.

<u>*Claim:*</u> $\mathfrak{N}$ is integrable.
proof of claim: We show that $\mathfrak{N}$ is involutive.

Let $\xi, \eta \in \mathfrak{X}(\mathcal{G}(u))$ vector fields on $\mathcal{G}(u)$ with values in the horizontal distribution. From 2.3.1 we know that their Lie bracket decomposes into vertical and horizontal component as $[\xi, \eta] = \operatorname{pr}^{\mathfrak{H}}([\xi, \eta]) - \zeta_{\Omega(\xi,\eta)}$.

Let $X, Y \in \mathfrak{h}$. Then $[\zeta_X, \zeta_Y] = \zeta_{[X,Y]} \in \mathfrak{N}$, since $\mathfrak{h}$ is a subalgebra of $\mathfrak{hol}$.

Finally, we have to compute $[\zeta_Y, \xi]$. Note that $\mathrm{Fl}^{\zeta_Y}_t = \rho_{\exp(tV)}$. We use the formula $[\zeta_Y, \xi](u') = (\mathcal{L}_{\zeta_Y})\xi(u') = \frac{d}{dt}|_0 \left(T_{\mathrm{Fl}^{\zeta_Y}_{-t}(u')} \mathrm{Fl}^{\zeta_Y}_t \cdot \xi\left(\mathrm{Fl}^{\zeta_Y}_{-t}(u')\right)\right)$ where $u' \in \mathcal{G}(u)$ (see [9, p.20]). Therefore,

$$\begin{aligned}
[\zeta_Y, \xi](u') &= \frac{d}{dt}|_0 \left(T\rho_{\exp(tY)} \cdot \xi\left(u' \exp(-tY)\right)\right) \\
&= \lim_{t \to 0} \frac{1}{t}\left(T\rho_{\exp(tY)} \cdot \xi\left(u' \exp(-tY)\right) - T\rho_{\exp(0)} \cdot \xi(u')\right) \\
&= \lim_{t \to 0} \frac{1}{t}\left(T\rho_{\exp(tY)} \cdot \xi\left(u' \exp(-tY)\right) - \xi(u')\right)
\end{aligned}$$

By right-invariance of the horizontal distribution, $T\rho_{\exp(tY)} \cdot \xi\left(u' \exp(-tY)\right)$ is again horizontal, hence the whole expression is horizontal. Therefore, $[\zeta_Y, \xi](u') \in \mathfrak{N}_{u'}$. *end of proof of claim*

Let N be the maximal integral manifold of $\mathfrak{N}$ in $\mathcal{G}(u)$ that contains u. However, each $u' \in \mathcal{G}(u)$ can be connected with u by a horizontal curve, and each horizontal curve starting at u must be contained in N. Consequently, $N = \mathcal{G}(u)$. In particular, $T\mathcal{G}(u) = \mathfrak{N}$ and hence $\mathfrak{h} = \mathfrak{hol}$, since the holonomy group is the structure group of $\mathcal{G}(u)$. This means, both $\mathfrak{h}$ and $\mathfrak{hol}$ parametrize each vertical subspace of $T\mathcal{G}$, hence $\mathfrak{h} = \mathfrak{hol}$ by dimensional reasons. $\square$

Chapter 3

Cartan geometries

This chapter deals with Cartan geometries. We follow 1.4.1, 1.3.6 and 1.5 of [5].

3.1 Klein geometries

As a motivation for the concept of Cartan geometries, we take look at homogenous spaces and recall their essential properties. In the following, let G be a Lie group and P a closed Lie subgroup of G.

Definition A *Klein-geometry* is a connected manifold M endowed with a smooth, transitive action λ of a Lie-group G.

Here, M should be viewed as a manifold endowed with a geometric structure whose isomorphisms are of the form $\lambda_g(x) = gx$ for $x \in M$ and $g \in G$.

The name "homogenous" should indicate that the space is uniform, i.e. that it looks the same around each point (even globally): Given two elements $x, y \in M$, then there is a $g \in G$ such that $g \cdot x = y$ where λ_g is a "structure-preserving" diffeomorphism.

Example 3.1.1. *Of course, any definition of geometry should include the Euclidean space, given by $\mathbb{R}^n$ that is endowed with the flat Riemannian metric δ. It can be realized as a Klein geometry, by considering the group* $\mathrm{Euc}(n)$ *of Euclidean motions, that is given by all maps $\mathbb{R}^n \to \mathbb{R}^n$ of the form $x \mapsto Ax + b$, where $A \in O(n)$ and $b \in \mathbb{R}^n$. The group* $\mathrm{Euc}(n)$ *may be viewed as the subgroup of* $\mathrm{GL}(n+1, \mathbb{R})$ *given by block-matrices of the form* $\begin{pmatrix} 1 & 0 \\ b & A \end{pmatrix}$*, hence it is a finite-dimensional Lie group.*

Claim: $\mathrm{Euc}(n)$ *is the isometry group of* $(\mathbb{R}^n, \delta)$*.*
proof of claim: On the one hand, if $\Phi \in \mathrm{Euc}(n)$ is of the form $\Phi(x) = Ax + b$, where $A \in O(n)$ and $b \in \mathbb{R}^n$, we have $D_x\Phi = A$ for all $x \in \mathbb{R}^n$. Thus $(\Phi^*\delta)_x(y_1, y_2) = \delta_{\Phi(x)}(Ay_1, Ay_2) = \delta(y_1, y_2)$. Thus each element of $\mathrm{Euc}(n)$ *is an isometry of $(\mathbb{R}^n, \delta)$.*

On the other hand, let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ satisfy $\Phi^*\delta = \delta$. Then for $t \in \mathbb{R}$ and

$x, y_1, y_2, z \in \mathbb{R}^n$ we obtain

$$\begin{aligned} 0 &= \frac{d}{dt}|_0 \delta_{x+tz}(y_1, y_2) = \frac{d}{dt}|_0 \delta_{\Phi(x+tz)}(D_{x+tz}\Phi \cdot y_1, D_{x+tz}\Phi \cdot y_2) \\ &= \delta_{\Phi(x)}(D_x^2\Phi \cdot (z, y_1), D_x\Phi \cdot y_2) + \delta_{\Phi(x)}(D_x^2\Phi \cdot (z, y_2), D_x\Phi \cdot y_1). \end{aligned}$$

This equation shows that the map $T : (y_1, y_2, z) \mapsto \delta_{\Phi(x)}(D_x^2\Phi \cdot (z, y_1), D_x\Phi \cdot y_2)$ is skew-symmetric in the first two arguments. Furthermore, it is symmetric in the last two arguments and trilinear. These properties imply that it vanishes:

$$\begin{aligned} T(y_1, y_2, z) &= -T(y_2, y_1, z) = -T(y_2, z, y_1) = T(z, y_2, y_1) \\ &= T(z, y_1, y_2) = -T(y_1, z, y_2) = -T(y_1, y_2, z). \end{aligned}$$

Since $D_x\Phi$ is a linear isomorphism, we obtain $D_x^2\Phi = 0$. This implies that $D\Phi = A$, a fixed linear map, that is by assumption an element of $O(n)$.

Now let $x \in \mathbb{R}^n$, and $c : \mathbb{R} \to \mathbb{R}^n$ the smooth curve given by $c(t) = tx$. Then

$$\Phi(x) = \Phi(c(1)) = \Phi(c(0)) + \int_0^1 (\Phi \circ c)'(0)dt = \Phi(0) + \int_0^1 Axdt. = \Phi(0) + Ax.$$

This shows the claim.

Furthermore, $\mathrm{Euc}(n)$ *obviously acts transitively on $\mathbb{R}^n$, thus the pair* $(\mathbb{R}^n, \mathrm{Euc}(n))$ *is a Klein geometry.*

After distinguishing a point $x \in M$, we obtain an identification $G/P = M$ via $gP \mapsto g \cdot x$, where P is the stabilizer of x in G. It is natural to consider the canonical projection $\pi : G \to G/P$, since in this picture the isomorphisms of the structure are exactly given by the factorized left-multiplications with fixed elements on G.

Example 3.1.2. *Fix* 0 *in the Euclidean space* $(\mathbb{R}^n, \delta)$. *The stabilizer of* 0 *in* $\mathrm{Euc}(n)$ *is obviously given by $O(n)$, thus we obtain the identification* $\mathbb{R}^n = \mathrm{Euc}(n)/O(n)$.

There is a canonical differential form on G which encodes the geometric structure of G/P:

Definition The *Maurer-Cartan-form* $\omega^{MC} = \omega \in \Omega^1(G, \mathfrak{g})$ on a Lie-group G is defined by $\omega(g)(\xi) := T_g\lambda_{g^{-1}} \cdot \xi$

First we summarize the characteristics of the space G/P and the Maurer-Cartan-form. The proofs are straightforward.

Proposition 3.1.3. *(i) The canonical projection $\pi : G \to G/P$ together with the right-multiplication of P on G is a principal bundle.*

(ii) The Maurer-Cartan-form ω is equivariant with respect to right multiplication on G, i.e. for all $g \in G$ we have $\rho_g^*\omega = Ad\left(g^{-1}\right) \circ \omega$.

(iii) For all $g \in G$ the linear map $\omega(g) : T_gG \to \mathfrak{g}$ is an isomorphism. In particular, the tangent bundle of G is trivial – ω is therefore called an absolute parallelism.

(iv) The Maurer-Cartan form reproduces the generators of left-invariant vector-fields, i.e. for all $X \in \mathfrak{g}$, $\omega(L_X) = X$.

(v) *The "Maurer-Cartan-equation" holds: For all* $\xi, \eta \in \mathfrak{X}(G)$ *we have*

$$d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)] = 0$$

Proof. [12, p.102,108,111,145].

Consider the left multiplication $\lambda_g : G \to G$ for $g \in G$. This map factors to the left action by g on $G/P \to G/P$, that is the "structure-preserving" isomorphism from before, hence this is compatible with our viewpoint.

The map $\lambda_g : G \to G$ is even an isomorphism of principal bundles: Firstly it commutes with the projection $(\pi \circ \lambda_g)(g') = gg'P = \lambda_g(g'P) = (\lambda_g \circ \pi)(g')$ for all $g' \in G$, and secondly, for all $p \in P$ and $g' \in G$ we have $\lambda_g(g'p) = gg'p = \lambda_g(g')p$. Clearly it is bijective and smooth.

In the next proposition we will see that the Maurer-Cartan-form can be used to distinguish these "structure-preserving" isomorphisms from the rest of the principal bundle isomorphisms.

Proposition 3.1.4. *Let* $\Phi : G \to G$ *be an isomorphism of principal bundles, i.e.* $\Phi(gp) = \Phi(g)p$ *for all* $g \in G$ *and* $p \in P$*. Then the following statements are equivalent:*

(i) *We have* $\Phi^*\omega = \omega$.

(ii) *There is a* $g \in G$ *such that* $\Phi = \lambda_g$.

Proof. $(ii) \Rightarrow (i)$ Let $g' \in G$ and $\xi \in T'_g G$ then $\left(\lambda_g^*\omega\right)_{g'}(\xi) = \omega_{gg'}(T_{gg'}\lambda_g \cdot \xi) = T_{gg'}\lambda_{(g')^{-1}g^{-1}} \cdot T_{g'}\lambda_g \cdot \xi = \omega_g(\xi)$.

$(i) \Rightarrow (ii)$ Recall that we assumed G/P connected. We make use of the fact that for a connected manifold N and two smooth functions $f_1, f_2 : N \to G$ which satisfy $f_1^*\omega = f_2^*\omega$ there is a $g \in G$ such that $f_2 = \lambda_g \circ f_1$ (see [12, p.115]). Let $N := G_0$, the connected component of G that contains $e \in G$. Then consider $\Phi\big|_{G_0} : G_0 \to G$ and $\mathrm{id}_G\big|_{G_0} : G_0 \to G$. By the fundamental theorem of calculus, we know that $\Phi\big|_{G_0}$ is of the form λ_g for a $g \in G$.

For $g'P \in G/P$ there exists a smooth curve $c : [0,1] \to G/P$ such that $c(0) = eP$ and $c(1) = g'P$. Next, we lift this curve to $\hat{c} : [0,1] \to G$ such that $\hat{c}(0) = e$, then $\hat{c}$ has values in G_0. Furthermore, $\hat{c}(1)P = g'P$, therefore there is a $p \in P$ such that $\hat{c}(1) = g'p$. Now $\Phi(g')p = \Phi(g'p) = \Phi(\hat{c}(1)) = g\hat{c}(1) = gg'p$, hence $\Phi(g') = gg'$. □

3.2 G-structures

As another motivation for the definition following in the next section, we consider G-structures with connections. We will see that both Klein geometries and G-structures fit in the definition of a Cartan geometry.

We use a smooth manifold M of dimension n as base space. Then consider the frame bundle $\pi : \mathcal{F} \to M$ (see 1.1.4 *2.*). There we offered two different interpretations of the frame bundle: Firstly, we may interpret the fiber as the set of all bases of the tangent space at the underlying point, and secondly, as the set of all linear isomorphisms between $\mathbb{R}^n$ and this tangent space. In the following, we will mostly use the second interpretation.

In the following, we will consider reductions of the frame bundle. The geometrical meaning of such a reduction is best illustrated by consider explicit examples:

A Riemannian metric g on M induces the *orthonormal frame bundle* defined by

$$\mathcal{F}_O := \left\{ u \in \mathcal{F} \mid g_{\pi(u)}(u(x), u(y)) = \langle x, y \rangle \; \forall x, y \in \mathbb{R}^n \right\} \subset \mathcal{F}.$$

An almost-complex structure on a $2n$-dimensional manifold M is given by a vector bundle homomorphism $J : TM \to TM$ with $J^2 = -\operatorname{id}_{TM}$. It induces the reduction

$$\mathcal{F}_{\mathbb{C}} = \bigsqcup_{x \in M} \left\{ u : \mathbb{C}^n \to T_x M \mid u \; \mathbb{R}\text{-linear and } u\,(iy) = J_x\,(u\,(y)) \text{ where } y \in \mathbb{C}^n \right\},$$

where we identified $\mathbb{R}^{2n} = \mathbb{C}^n$.

Definition Let M be a n-dimensional manifold. A G-structure with structure group H, where H is a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$, is a reduction of the structure group of $\mathcal{F} \to M$ from $\mathrm{GL}(n, \mathbb{R})$ to H.

Example 3.2.1. *Let us examine Riemannian structures in detail:*

Claim: *A Riemannian metric g on a n-dimensional manifold M is equivalent to a G-structure on M with structure group $O(n)$.*
proof of claim: Let us start with a metric g. We saw above how to construct the orthonormal frame bundle $\mathcal{F}_O$ out of it, but we still have to show that this a reduction of the structure group from $\mathrm{GL}(n, \mathbb{R})$ *to $O(n)$. We use 1.1.5: (i) The projection π restricted to $\mathcal{F}_O$ is obviously surjective. (ii) Let $u \in \mathcal{F}_O$ and $h \in O(n)$. Then for $x, y \in \mathbb{R}^n$ we have*

$$g_{\pi(u \cdot h)}((u \cdot h)(x), (u \cdot h)(y)) = g_{\pi(u)}(u(hx), u(hy)) = \langle hx, hy \rangle = \langle x, y \rangle$$

, and thus $u \cdot h \in \mathcal{F}_O$. (iii) Now, suppose $u, u' \in \mathcal{F}_O$ such that $\pi(u) = \pi(u')$. Then $u^{-1} \circ u' : \mathbb{R}^n \to \mathbb{R}^n$ that is compatible with the inner product, since for $x, y \in \mathbb{R}^n$ we have $\langle (u^{-1} \circ u')(x), (u^{-1} \circ u')(y) \rangle = g_{\pi(u)}(u'(x), u'(y)) = \langle x, y \rangle$. Thus $u^{-1} \circ u' = h \in O(n)$, and hence $u' = u \cdot h$. Thus, $O(n)$ acts transitively on each fiber of $\mathcal{F}_O$. (iv) Take a smooth local section σ of $\mathcal{F}$. Each value of σ can be interpreted as a basis of the tangent space to M. Now note that Gram-Schmidt orthonormalization is a smooth procedure, and apply it to each value of σ with respect to g. Then we obtain a smooth local section of $\mathcal{F}_O$.

On the other hand, if we have a given reduction $\mathcal{F}_O$ of $\mathcal{F} \to M$, we can define a Riemannian metric on M by setting $g_x(\xi, \eta) := \langle u^{-1}(\xi), u^{-1}(\eta) \rangle$ for an arbitrary $u \in \mathcal{F}_O|_x$ and $\xi, \eta \in T_x M$. This definition is independent of the choice of $u \in \mathcal{F}_O|_x$, since each other element in $\mathcal{F}_O|_x$ is of the form $u \cdot h$ for $h \in O(n)$ and $\langle h^{-1}(u^{-1}(\xi)), h^{-1}(u^{-1}(\eta)) \rangle = \langle u^{-1}(\xi), u^{-1}(\eta) \rangle$. Furthermore, one can show smoothness of g by taking a local smooth section of $\mathcal{F}_O$.

If we start with a Riemannian metric, then construct the corresponding orthonormal frame bundle, and then again build a metric from it, by definition we regain the initial metric. The other direction is also true per construction.

Note that there is a canonical 1-form on the frame bundle:

Definition Let $\theta \in \Omega^1(\mathcal{F}, \mathbb{R}^n)$ be given by $\theta_u(\xi) = u^{-1}(T_u \pi \cdot \xi)$, where $u \in \mathcal{F}$ and $\xi \in T_u \mathcal{F}$. This is called the *soldering form*.

The soldering form interacts nicely with the structure of the principal bundle:

Proposition 3.2.2. *1. The soldering form θ is equivariant with respect to the right action on $\mathcal{F}$ and the standard action on $\mathbb{R}^n$: Let $g \in \mathrm{GL}(n, \mathbb{R})$, then $\rho_g^* \theta = g^{-1} \circ \theta$.*

2. It is strictly horizontal, meaning that for $\xi \in T\mathcal{F}$ we have $\theta(\xi) = 0$ if and only if $\xi \in V\mathcal{F}$.

Proof. (i) Let $u \in \mathcal{F}$ and $\xi \in T_u\mathcal{F}$. Note that $\pi \circ \rho_g = \pi$, hence

$$\begin{aligned} \left(\rho_g^* \theta\right)_u (\xi) &= \theta_{u \circ g} \left(T_u \rho_g \cdot \xi\right) = g^{-1} \left(u^{-1} \left(T_{ug}\pi \cdot T_u \rho_g \cdot \xi\right)\right) \\ &= g^{-1} \left(u^{-1} \left(T_u \pi \cdot \xi\right)\right) = g^{-1} \left(\theta_u (\xi)\right). \end{aligned}$$

(ii) Suppose $\xi \in T_u\mathcal{F}$. If $\theta_u(\xi)$ vanishes, we have $u^{-1}(T_u\pi \cdot \xi) = 0$. Since u is an isomorphism, $T_u\pi \cdot \xi = 0$. Hence by definition, ξ lies in the vertical subbundle. The other implication is clear. □

Proposition 3.2.3. *Let H be a subgroup of $\mathrm{GL}(n, \mathbb{R})$ and M a n-dimensional manifold. G-Structures with structure group H are in bijective correspondence with H-principal bundles $\mathcal{G} \to M$ endowed with a 1-form $\Theta \in \Omega^1(\mathcal{G}, \mathbb{R})$ that is equivariant and strictly horizontal.*

Proof. Consider a given reduction $\iota : \mathcal{G} \hookrightarrow \mathcal{F}$, then $\pi\big|_{\mathcal{G}} : \mathcal{G} \to M$ is an H-principal bundle. In addition, $\Theta := \iota^*\theta$ is a 1-form. It inherits equivariancy from θ, since ι is equivariant. Furthermore, ι is an injective immersion, hence $T\iota$ is injective. Thus for $\xi \in T\mathcal{G}$ we have

$$\Theta(\xi) = 0 \iff \theta(T\iota \cdot \xi) = 0 \iff \xi \in \ker(T\pi) \iff \xi \in \ker\left(T\pi\big|_{\mathcal{G}}\right).$$

On the other hand, let $\mathcal{G} \to M$ be an H-principal bundle and $\Theta \in \Omega^1(\mathcal{G}, \mathbb{R}^n)$ an equivariant, strictly horizontal 1-form. In particular, this implies that $V_u\mathcal{G} = \ker(\Theta_u)$ for all $u \in \mathcal{G}$. Hence Θ_u factors to an injective linear map $\tilde{\Theta}_u : T_u\mathcal{G}/V_u\mathcal{G} \to \mathbb{R}^n$. For dimensional reasons, it is bijective. The same argument shows that $\widetilde{T_u\pi} : T_u\mathcal{G}/V_u\mathcal{G} \to T_{\pi(u)}M$ is a linear isomorphism.

Now, let $\iota : \mathcal{G} \to \mathcal{F}$, $\iota(u) := \widetilde{T_u\pi} \circ \left(\tilde{\Theta}_u\right)^{-1}$ that is a linear isomorphism between $\mathbb{R}^n$ and $T_{\pi(u)}M$, hence an element of $\mathcal{F}$. The map ι is smooth and respects the fibers. Furthermore, we have $\tilde{\Theta}_{uh} \circ T\rho_h = h^{-1} \circ \tilde{\Theta}_u$ for $h \in H$ by equivariancy. Hence

$$\begin{aligned} \iota(uh) &= \widetilde{T_{uh}\pi} \circ \left(\tilde{\Theta}_{uh}\right)^{-1} = \widetilde{T_{uh}\pi} \circ \left(h^{-1} \circ \tilde{\Theta}_u \circ T\rho_{h^{-1}}\right)^{-1} \\ &= \widetilde{T_{uh}\pi} \circ T_u\rho_h \circ \tilde{\Theta}_u^{-1} \circ h = \widetilde{T_u\pi} \circ \tilde{\Theta}_u^{-1} \circ h = \iota(u) \cdot h. \end{aligned}$$

This shows equivariance of ι, and in particular it implies that ι is injective.

We still have to show that the correspondence is bijective. Suppose $\iota : \mathcal{G} \hookrightarrow \mathcal{F}$ is a given H-reduction. Then we obtain the differential form $\Theta := \iota^*\theta$ on $\mathcal{G}$. This in turn induces the reduction $\iota' : \mathcal{G} \to \mathcal{F}$ that is characterized by $\iota'(u) \circ T_u\pi = \Theta_u$. However,

$$\Theta_u = \theta_{\iota(u)} \circ T_u\iota = \iota(u) \circ T_u\pi \circ T_u\iota = \iota(u) \circ T_u\pi.$$

Hence we have $\iota' = \iota$.

On the other hand, a given Θ on $\mathcal{G} \to M$ induces the map ι characterized by $\iota(u) \circ T_u\pi = \Theta_u$. Let $\Theta' := \iota^*\Theta$. But then

$$\Theta'_u = (\iota^*\theta)_u = \theta_{\iota(u)} \circ T_u\iota = \iota(u) \circ T_u\pi \circ T\iota = \iota(u) \circ T_u\pi = \Theta_u.$$

□

This shows that G-structures are - just as Klein geometries - principal bundles over M.

Definition A connection on a G-structure $\mathcal{G} \hookrightarrow \mathcal{F}$ is a principal connection on $\mathcal{G}$.

Let $\mathcal{G}$ be a G-structure equipped with a connection $\gamma \in \Omega^1(\mathcal{G}, \mathfrak{h})$. Then, together with the soldering form on $\mathcal{G}$, we obtain a differential form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{h} \oplus \mathbb{R}^n)$, by simply setting $\omega := \gamma + \Theta$.

Example 3.2.4. *G-structures with structure group $O(n)$ carry a canonical connection, namely the Levi-Civita connection.*

Proposition 3.2.5. *Let $\iota : \mathcal{G} \hookrightarrow \mathcal{F}$ be a G-structure with structure group H. We know from 1.2.5 that it determines a unique section of $\mathcal{F}/H$. The section σ is parallel with respect to the induced connection coming from γ if and only if the horizontal distribution of γ is tangent to $\mathcal{G}$ in each point.*

Proof. Let $j : \mathcal{G} \to \mathcal{F}$ be a reduction such that $j^*\gamma$ is a principal connection. Let $\sigma : M \to \mathcal{F}/H$ be the corresponding section. By the identification $\mathcal{F} \times_G (G/H) = \mathcal{F}/H$, σ has values in an associated bundle and thus corresponds to an equivariant function $f : \mathcal{G} \to G/H$. Therefore one can view elements $\hat{u} \in q^{-1}(\sigma(M)) = \mathcal{G}$ as $[\hat{u}, eH] = \hat{u}H = \sigma(p(\hat{u})) = [\hat{u}, f(\hat{u})]$. Hence we see that f must map $\hat{u}$ to eH. Let $\xi \in \mathfrak{X}(M)$, $u \in \mathcal{G}$ and $\hat{u} := j(u)$, then the horizontal lifts are related via $\xi^{hor}_{\mathcal{F}}(\hat{u}) = T_u j \cdot \xi^{hor}_{\mathcal{G}}(u)$. This implies

$$\left(\xi^{hor}_{\mathcal{F}} \cdot f\right)(\hat{u}) = T_{j(u)}f \cdot T_u j \cdot \xi^{hor}_{\mathcal{G}}(u) = T_u(f \circ j) \cdot \xi^{hor}_{\mathcal{G}}(u) = 0,$$

since $f \circ j$ is constant.

But $\xi^{hor}_{\mathcal{F}} \cdot f$ corresponds to $\nabla_\xi\sigma$, hence σ is parallel.

Conversely, let $\sigma : M \to \mathcal{F}/H = \mathcal{F} \times_G G/H$ be a parallel section (represented again by the equivariant function f). This means that for all $\xi \in \mathfrak{X}(M)$ we have $\xi^{hor}_{\mathcal{F}} \cdot f = 0$.

Then the reduction $\mathcal{G}$ is defined as

$$\begin{aligned}\{u \in \mathcal{F} \mid \sigma(\pi(u)) = uH\} &= \{u \in \mathcal{F} \mid \sigma(\pi(u)) = [u, eH]\} \\ &= \{u \in \mathcal{F} \mid f(u) = eH\} = f^{-1}(eH).\end{aligned}$$

Thus if we can show that for each $u \in \mathcal{G}$ the equivariant map f is regular, we have $T_u\mathcal{G} = \ker(T_u f)$ that contains ξ^{hor}_u by definition.

Indeed for $X \in \mathfrak{g}$ and $u \in \mathcal{G}$ we have

$$\begin{aligned}T_u f \cdot \zeta_X(u) &= \frac{d}{dt}|_0 f(u \cdot \exp(tX)) = \frac{d}{dt}|_0 \exp(-tX)f(u) \\ &= \frac{d}{dt}|_0 \exp(-tX)H = -X + \mathfrak{g},\end{aligned}$$

and thus $T_u f$ is surjective. □

3.3 Cartan geometries

Think of the following construction as "curved" versions of Klein geometries (such as Riemannian structures are curved versions of Euclidean space):

Definition Let G be a Lie group and P a Lie subgroup of G. A *Cartan-geometry of type* (G, P) is a principal fiber bundle $\mathcal{G} \to M$ over a C^∞-manifold M with structure group P, that is equipped with a one-form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, which satisfies the following conditions:

1. For $p \in P$ we have $\rho_p^* \omega = Ad\left(p^{-1}\right) \circ \omega$.

2. For $u \in \mathcal{G}$, $\omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism.

3. By inserting a fundamental vector-field $\zeta_X \in \mathfrak{X}(\mathcal{G})$ for $X \in \mathfrak{p}$ into ω we reproduce its generator, i.e. $\omega(\zeta_X) = X$.

The differential form ω is called the *Cartan connection*.

Example 3.3.1. *In the case of a homogenous space $G \to G/P$ the fundamental vector fields corresponding to elements from $\mathfrak{p}$ are exactly the left-invariant ones. Thus, the Maurer-Cartan form of a Klein geometry satisfies the conditions 1.-3. above. Klein geometries are called homogenous models of Cartan geometries.*

Example 3.3.2. *G-structures with connections are Cartan geometries:*

Let $\mathcal{G} \hookrightarrow \mathcal{F}$ be a G-structure on the manifold M with structure group P, and γ a connection on $\mathcal{G}$. Let G be the Lie group $P \ltimes \mathbb{R}^n$. Its Lie algebra is isomorphic to $\mathfrak{p} \oplus \mathbb{R}^n$ as a vector space and as a representation of P. We consider, equivalently, the bundle $\mathcal{G} \to M$ equipped with the 1-form $\omega = \gamma + \Theta \in \Omega^1(\mathcal{G}, \mathfrak{p} \oplus \mathbb{R}^n)$.

1. *Both γ and Θ are P-equivariant, hence so is ω.*

2. *Let $u \in \mathcal{G}$. Since $\mathcal{G}$ carries a connection γ, we can decompose the tangent space $T_u\mathcal{G} = \mathfrak{H}_u \oplus V_u\mathcal{G}$ into a horizontal and a vertical part. The soldering form Θ_u vanishes on $V_u\mathcal{G}$ and is bijective on $\mathfrak{H}_u$, whereas the connection γ_u is bijective on $V_u\mathcal{G}$. By definition, $\mathfrak{H}_u = \ker(\gamma_u)$, thus ω_u is a linear isomorphism on $T_u\mathcal{G}$.*

3. *Let $X \in \mathfrak{p}$. We obtain $\omega(\zeta_X) = \gamma(\zeta_X) + \Theta(\zeta_X) = X + 0$.*

Thus $\mathcal{G}$ is a Cartan geometry of type $(P \ltimes \mathbb{R}^n, P)$.

As mentioned above, the following concept will be the main tool to capture the difference between the homogenous model and a general Cartan geometry:

Definition The *curvature* K of a Cartan-geometry $\mathcal{G} \to M, \omega$ of type (G, P) is defined by $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$, $K(\xi, \eta) := d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$, where $[.,.]$ denotes the Lie bracket on $\mathfrak{g}$.

Remark 3.3.3. *Since the Cartan connection provides a trivialization of the tangent bundle, we may fix one $X \in \mathfrak{g}$ and look at the constant vector field $\omega^{-1}(X) \in \mathfrak{X}(\mathcal{G})$, that is simply defined by $\left(\omega^{-1}(X)\right)(u) := (\omega(u))^{-1}(X)$ for $u \in \mathcal{G}$. This is obviously smooth, hence indeed a vector field.*

We may rewrite the curvature entirely in terms of the Lie algebra :

Definition The *curvature function* of a Cartan geometry is defined by $\kappa : \mathcal{G} \to \wedge^2 \mathfrak{g}^* \otimes \mathfrak{g}$, $\kappa(u)(X,Y) := K(u)\left(\omega^{-1}(X)(u), \omega^{-1}(Y)(u)\right)$.

Conversely, we can express the curvature as $K_u(\xi,\eta) = \kappa(u)(\omega_u(\xi), \omega_u(\eta))$, hence we do not lose any information about K when passing to the curvature function.
Also note that there is an induced P-action on $\wedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ that comes from the adjoint action of P on $\mathfrak{g}$.

Now observe the following properties of the curvature and the curvature function:

Lemma 3.3.4. *(i) The curvature is horizontal, i.e. for a vertical vector field corresponding to $X \in \mathfrak{p}$ and for $\eta \in \mathfrak{X}(\mathcal{G})$ the curvature vanishes: $K(\zeta_X, \eta) = 0$. In particular, κ factorizes to a map $\mathcal{G} \to \wedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$*

(ii) For $p \in P$ we have $\rho_p^* K = Ad\left(p^{-1}\right) \circ K$.

(iii) For $p \in P$ and $u \in \mathcal{G}$ we have $\kappa(up) = p^{-1} \cdot \kappa(u)$..

Proof. 1. Choose X, η as above. Since by definition of ω, $i_{\zeta_X}\omega$ is constant and thus has vanishing exterior derivative, we may start by applying the Cartan formula to

$$d\omega(\zeta_X, \eta) = (i_{\zeta_X} d\omega)(\eta) = (i_{\zeta_X} d\omega)(\eta) + d(i_{\zeta_X}\omega)(\eta) = (\mathcal{L}_{\zeta_X}\omega)(\eta).$$

By the defining properties of the Cartan connection and by 1.1.3(iii):

$$\begin{aligned}
\mathcal{L}_{\zeta_X}\omega &= \frac{d}{dt}|_0 \left(\left(Fl_t^{\zeta_X}\right)^* \omega\right) = \frac{d}{dt}|_0 \left(\rho_{\exp(tX)}\right)^* \omega \\
&= \frac{d}{dt}|_0 \left(Ad(\exp(-tX)) \circ \omega\right) = \frac{d}{dt}|_0 \exp(-t \cdot ad(X)) \circ \omega \\
&= ad(-X) \circ \omega.
\end{aligned}$$

From these observations we conclude

$$d\omega(\zeta_X, \eta) = -ad(X)(\omega(\eta)) = -[X, \eta],$$

hence $K(\zeta_X, \eta) = 0$.
Furthermore, for $X \in \mathfrak{g}$ and $H \in \mathfrak{p}$ we calculate

$$\kappa(X, H) = K\left(\omega^{-1}(X), \omega^{-1}(H)\right) = K\left(\omega^{-1}(X), \zeta_H\right) = 0,$$

hence $\kappa : \mathcal{G} \to \wedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ is well-defined.

2. Let $u \in \mathcal{G}, \xi, \eta \in \mathfrak{X}(\mathcal{G}), p \in P$:

$$\begin{aligned}
\left(\rho_p^* K\right)(u)(\xi, \eta) =& d\omega(up)(T_u\rho_p \cdot \xi, T_u\rho_p \cdot \eta) \\
&+ [\omega(up)(T_u\rho_p \cdot \xi), \omega(up)(T_u\rho_p \cdot \eta)] \\
=& \left(\rho_p^*(d\omega)\right)(u)(\xi, \eta) + [\left(\rho_p^*\omega\right)(u)(\xi), \left(\rho_p^*\omega\right)(u)(\eta)]
\end{aligned}$$

By naturality of the exterior derivative and since $Ad\left(p^{-1}\right)$ is a Lie algebra homomorphism we obtain

$$
\begin{aligned}
\left(\rho_p^* K\right)(u)(\xi,\eta) = d\left(\rho_p^*\omega\right)(u)(\xi,\eta) + \left[\left(\rho_p^*\omega\right)(u)(\xi), \left(\rho_p^*\omega\right)(u)(\eta)\right] \\
=& d\left(Ad\left(p^{-1}\right)\circ\omega\right)(u)(\xi,\eta) \\
&+ \left[Ad\left(p^{-1}\right)(\omega(u)(\xi)), Ad\left(p^{-1}\right)(\omega(u)(\eta))\right] \\
=& Ad\left(p^{-1}\right)(d\omega(u)(\xi,\eta)) + Ad\left(p^{-1}\right)([\omega(u)(\xi), \omega(u)(\eta)]) \\
=& Ad\left(p^{-1}\right)(K(u)(\xi,\eta))
\end{aligned}
$$

□

Definition Let $(\mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G, P). If the curvature function $\kappa : \mathcal{G} \to \wedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ only takes values in the subspace $\wedge^2(\mathfrak{g}\mathfrak{p})^* \otimes \mathfrak{p}$, the Cartan geometry is called *torsion-free*.

Remark 3.3.5. *In the case of a G-structure $\mathcal{G}$ with structure group P on a manifold M, note that $\mathcal{G} \times_P \mathbb{R}^n = \mathcal{F} \times_{\mathrm{GL}(n,\mathbb{R})} \mathbb{R}^n = TM$, where $\mathcal{F}$ is the frame bundle and n is the dimension of M. If $\mathcal{G}$ is endowed with a principal connection γ, this induces a linear connection ∇ on TM.*

However, the G-structure is equivalent to the Cartan geometry $\mathcal{G} \to M$, $\omega := \theta + \gamma$. One can show that $(\mathcal{G} \to M, \omega)$ is torsion-free in sense of the above definition, if and only if ∇ is torsion-free, i.e. $\nabla_\xi \eta - \nabla_\eta \xi = [\xi, \eta]$ where $\xi, \eta \in \mathfrak{X}(M)$. The details of this computation are given in [5, p.44].

3.4 Tractor Bundles

We saw in 1.2 that P-representations induce vector bundles. In the case that $\mathcal{G}$ carries a principal connection, the vector bundle inherits a linear connection. However, the Cartan connection ω is not a principal connection and in general does not yield a linear connection.

Thus we restrict to the special case of a G-representation. We will conclude this section on Cartan geometries by showing that this suffices to construct a canonical linear connection on the associated vector bundle.

Definition Let $\mathcal{G}$ be a Cartan geometry of type (G, P), and $\bar{\rho} : G \to GL(V)$ a representation of G on a finite-dimensional vector space V. Then we call $\mathcal{G} \times_P V = \hat{\mathcal{G}} \times_G V$ (see 1.2.4) the corresponding tractor bundle.

3.4.1 The Adjoint Tractor Bundle

Definition Let $(\mathcal{G} \to M, \omega)$ be a Cartan geometry. The tractor bundle $\mathcal{A}M := \mathcal{G} \times_P \mathfrak{g}$ corresponding to the adjoint action on $\mathfrak{g}$ is called the adjoint tractor bundle.

We start our survey on tractor bundles with focus on the adjoint tractor bundle, which interacts particularly nicely with other tractor bundles. Recall that $\mathcal{G} \times_P \mathfrak{g}/\mathfrak{p} = TM$. Note that the canonical projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{p}$ is P-equivariant and therefore induces a bundle map $\Pi : \mathcal{G} \times_P \mathfrak{g} = \mathcal{A}M \to \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p} = TM$

Proposition 3.4.1. *Let $\mathcal{G}$ be a Cartan geometry and $\mathcal{V}M := \mathcal{G} \times_P V$ a tractor bundle of $\mathcal{G}$.*

(i) There is a bundle map $\{.,.\} : \mathcal{A}M \times \mathcal{A}M \to \mathcal{A}M$ that turns each fiber $\mathcal{A}_xM$, where $x \in M$, into a Lie algebra isomorphic to $\mathfrak{g}$.

(ii) There is a bijective correspondence between the set of sections of $\mathcal{A}M$ and the P-invariant vector fields $\mathfrak{X}(\mathcal{G})^P$ on $\mathcal{G}$, i.e. $\xi \in \mathfrak{X}(\mathcal{G})^P \iff \forall u \in \mathcal{G}, h \in H : \xi(uh) = T_u\rho_h \cdot \xi(u)$. This induces a Lie bracket $[.,.]$ on $\Gamma(\mathcal{A}M)$ such that $\Pi([\sigma_1, \sigma_2]) = [\Pi(\sigma_1), \Pi(\sigma_2)]$, where $\sigma_1, \sigma_2 \in \Gamma(\mathcal{A}M)$ and the bracket on the right hand side is the usual Lie bracket of vector fields.

(iii) There is a map $\bullet : \mathcal{A}M \times \mathcal{V}M \to \mathcal{V}M$ that turns each fiber of $\mathcal{V}_xM$, where $x \in M$, into an $\mathcal{A}_xM$-module, i.e. it satisfies $\{\sigma_1, \sigma_2\} \bullet t = \sigma_1 \bullet \sigma_2 \bullet t - \sigma_2 \bullet \sigma_1 \bullet t$ for $\sigma_1, \sigma_2 \in \Gamma(\mathcal{A}M)$ and $t \in \Gamma(\mathcal{V}M)$.

Proof. As for (i), let $p \in P$ and $X, Y \in \mathfrak{g}$ and recall that $\mathrm{Ad}(p) \cdot [X, Y] = [\mathrm{Ad}(p) \cdot X, \mathrm{Ad}(p) \cdot Y]$, so the Lie-bracket is H-equivariant and thus induces a bundle map $\{\ ,\ \}$. It is easy to see that the induced charts of the associated bundle $\mathcal{G} \times_P \mathfrak{g}$ provide Lie-algebra-isomorphisms for each fiber: Let $x \in M$ and $u \in \pi^{-1}(x)$. The fiber $\mathcal{A}_xM$ can be written as $\{[u, X] \mid X \in \mathfrak{g}\}$, and a chart around x restricted to $\pi^{-1}(x)$ is given by $[u, X] \mapsto (\pi(u), \bar{\rho}(\mathrm{pr}_2(\varphi(u)), X))$, where φ is a principal chart for $\mathcal{G}$ around x. But here $\bar{\rho}$ is the adjoint action and we know from above that this is a Lie-algebra-isomorphism. However, note there is no canonical identification of $\mathcal{A}_x$ with $\mathfrak{g}$.

(ii) First, let us establish the bijective correspondence. We know $\Gamma(\mathcal{A}M) = \Gamma(\mathcal{G} \times_P \mathfrak{g}) \cong C^\infty(\mathcal{G}, \mathfrak{g})^P$. Furthermore, the Cartan connection trivializes the tangent bundle of $\mathcal{G}$, more precisely $\omega : T\mathcal{G} \xrightarrow{\cong} \mathcal{G} \times \mathfrak{g}$, therefore each vector field $\xi \in \mathfrak{X}(\mathcal{G})$ can be identified with the smooth function $\omega(\xi) := \omega \circ \xi \in C^\infty(\mathcal{G}, \mathfrak{g})$, and $\omega(\xi)$ is P-equivariant if and only if ξ is P-invariant: Let $u \in \mathcal{G}$ and $p \in P$, then

$$\omega(\xi)(up) = \omega(\xi(up)) = \omega(T\rho_p \cdot \xi(u)) = \mathrm{Ad}(p^{-1}) \cdot \omega(\xi)(u),$$

which proves the one implication; the other follows analogously.
Therefore $\mathfrak{X}(\mathcal{G})^P \cong C^\infty(\mathcal{G}, \mathfrak{g})^P \cong \Gamma(\mathcal{A}M)$.

Because of the naturality of the Lie bracket, the P-invariant vector fields form a Lie-subalgebra of $\mathfrak{X}(\mathcal{G})$ - this induces our Lie bracket $[\ ,\ ]$ on $\Gamma(\mathcal{A}M)$.

The projection $\Pi : \mathcal{A}M \to TM$ induces a map $\Gamma(\mathcal{A}M) \to \Gamma(TM) = \mathfrak{X}(M)$, that we again denote by Π. It will be useful to know, how the projection Π looks in the correspondence with $\mathfrak{X}(\mathcal{G})^P$:

<u>*claim:*</u> Let $\xi \in \mathfrak{X}(\mathcal{G})^P$. Since it is right-invariant, $\bar{\xi}(x) := T_u\pi \cdot \xi(u)$ for $x \in M$ and an arbitrary $u \in \pi^{-1}(x)$ is well defined. The projection via Π of the section of $\mathcal{A}M$ corresponding to ξ is exactly $\bar{\xi}$.
proof of claim: ξ corresponds to the smooth, equivariant function $\mathcal{G} \to \mathfrak{g}$, $u \mapsto \omega(\xi(u))$, therefore to the section
$\sigma(\pi(u)) = [u, \omega(\xi(u))]$ where $u \in \mathcal{G}$. Now let $x \in M$ and $u \in \pi^{-1}(x)$, then

$$(\Pi(\sigma))(x) = \Pi(\sigma(x)) = \Pi([u, \omega(\xi(u))]) = [u, \omega(\xi(u)) + \mathfrak{p}].$$

Recall that in the identification $\mathcal{G} \times_P \mathfrak{g}/\mathfrak{p} = TM$ we treat $[u, X + \mathfrak{p}]$ as $T_u\pi\left(\omega_u^{-1}(X)\right)$, so here $[u, \omega(\xi(u))]$ is exactly $T_u\pi \cdot \xi(u)$. *end of proof of claim*

Now we calculate the desired identity: Let $\xi_1, \xi_2 \in \mathfrak{X}(\mathcal{G})^P$. Then naturality of the Lie bracket implies

$$\Pi([\xi_1, \xi_2]) = T\pi \cdot [\xi_1, \xi_2] = [T\pi \cdot \xi_1, T\pi \cdot \xi_2] = [\Pi(\xi_1), \Pi(\xi_2)],$$

so the same identity holds with ξ_j replaced by $\sigma_j \in \Gamma(\mathcal{A}M)$.

(iii) We consider the derivative of the representation $\bar{\rho}' : \mathfrak{g} \to \mathfrak{gl}(V)$. Let $t \in \mathbb{R}$, $g \in G$, $v \in V$ and $X \in \mathfrak{g}$. Note that $\exp(t \cdot \mathrm{Ad}(g)(X)) = g \cdot \exp(tX) \cdot g^{-1}$, since conj_g is a Lie-group-homomorphism. Thus

$$\begin{aligned}
&\bar{\rho}'(\mathrm{Ad}(g)(X))(\bar{\rho}(g)(v)) \\
&= \frac{d}{dt}|_0 \left(\bar{\rho}(\exp(t \cdot \mathrm{Ad}(g)(X)))(\bar{\rho}(g)(v))\right) \\
&= \frac{d}{dt}|_0 \left(\rho\left(g \exp(tX) g^{-1}\right)(\bar{\rho}(g)(v))\right) \\
&= \frac{d}{dt}|_0 \left(\bar{\rho}(g)\bar{\rho}(\exp(tX))\bar{\rho}\left(g^{-1}\right)\bar{\rho}(g) \cdot v\right) \\
&= \frac{d}{dt}|_0 \left(\bar{\rho}(g)\bar{\rho}(\exp(tX))(v)\right) = \bar{\rho}(g)\left(\bar{\rho}'(X)(v)\right).
\end{aligned}$$

Therefore, the map $\mathfrak{g} \times V \to V$, $(X, v) \mapsto \bar{\rho}'(X)(v)$ is a G-equivariant with respect to the G-actions $\mathrm{Ad} \times \bar{\rho}$ on $\mathfrak{g} \times V$ and $\bar{\rho}$ on V, respectively, hence in particular P-equivariant. Therefore it induces a bundle map $\bullet : \mathcal{A}M \times \mathcal{V}M \to \mathcal{V}M$. □

3.4.2 The fundamental derivative

The next construction works for P-representations, thus on general associated vector bundles:

Definition and Lemma 3.4.2. *Let $(\mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G, P), $\bar{\rho} : P \to GL(V)$ a representation of P and $\mathcal{V}M := \mathcal{G} \times_P V$ the corresponding associated bundle. Let $\mathcal{D} : \Gamma(\mathcal{A}M) \times \Gamma(\mathcal{V}M) \to \Gamma(\mathcal{V}M)$, $\mathcal{D}(\sigma, \tau) := \mathcal{D}_\sigma\tau$ the section of $\mathcal{V}M$ that corresponds to $\xi \cdot \phi \in C^\infty(\mathcal{G}, V)^P$, where $\xi \in \mathfrak{X}^P(\mathcal{G})$ is the vector field that corresponds to $\sigma \in \Gamma(\mathcal{A}M)$; and $\phi \in C^\infty(\mathcal{G}, V)^P$ is the smooth equivariant function, that corresponds to $\tau \in \Gamma(\mathcal{V}M)$.*

Proof. We have to prove $\xi \cdot \phi$ is P-equivariant: Let $u \in \mathcal{G}$ and $p \in P$. Then

$$(\xi \cdot \phi)(up) = \xi(up) \cdot \phi = T\phi \cdot T\rho_p \cdot \xi(u) = \bar{\rho}\left(p^{-1}\right)(\xi(u) \cdot \phi),$$

where the last equality holds, since ϕ is P-equivariant, i.e. $\phi(\rho_p(u)) = \phi(up) = \bar{\rho}\left(p^{-1}\right)(\phi(u))$, and $\bar{\rho}\left(p^{-1}\right) : V \to V$ is linear.

□

Remark 3.4.3. *By exactly the same procedure as in Proposition 3.4.1 (iii) we obtain an operator $\bullet : (\mathcal{G} \times_P \mathfrak{p}) \times \mathcal{V}M \to \mathcal{V}M$, because $\bar{\rho}'$ is an equivariant map $\mathfrak{p} \times V \to V$. In the case that $\bar{\rho}$ is in fact a G-representation, this $\bullet$ is just the restriction of the $\bullet$ from Proposition 3.4.1 to $(\mathcal{G} \times_P \mathfrak{p}) \times \mathcal{V}M \subset \mathcal{A}M \times \mathcal{V}M$.*

Proposition 3.4.4. *Let $(\mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G, P), and $\bar{\rho} : P \to GL(V)$ a representation of P.*

(i) Let $\sigma \in \Gamma(\mathcal{A}M)$ and $f : M \to \mathbb{R}$ a smooth function, i.e. a section of the vector bundle $M \times \mathbb{R} = \mathcal{G} \times_P \mathbb{R}$ with the trivial representation $\bar{\rho} : P \to GL(\mathbb{R})$, $\bar{\rho} \equiv \mathrm{id}_{\mathbb{R}}$, then $\mathcal{D}_\sigma f = \Pi(\sigma) \cdot f$.

(ii) Let $\sigma \in \Gamma(\mathcal{G} \times_P \mathfrak{p}) \subset \Gamma(\mathcal{A}M)$, then for $\tau \in \Gamma(\mathcal{V}M)$ we have $\mathcal{D}_\sigma \tau = -s \bullet \tau$.

Proof. (i) Let $x \in M$. The function f can be viewed as the section $x \mapsto (x, f(x))$ which is identified with $[u, f(x)]$, where u is an arbitrary element of $\pi^{-1}(x)$. Therefore, the equivariant function that corresponds to the section is given by $f \circ \pi$. Let $\xi \in \mathfrak{X}(\mathcal{G})^P$ be the vector field corresponding to σ. The derivative $\xi \cdot (f \circ \pi) = (T\pi \cdot \xi) \cdot f$, whereas $T\pi \cdot \xi = \Pi(\sigma)$, which gives the claim.

(ii) The section σ corresponds to the equivariant function $\phi : \mathcal{G} \to \mathfrak{p}$, such that $\sigma(\pi(u)) = [u, \phi(u)]$ for $u \in \mathcal{G}$; and furthermore to an vector field $\xi \in \mathfrak{X}(\mathcal{G})^P$. By assumption $\omega(\xi)$ has values in $\mathfrak{p}$, and is therefore of the form $\xi(u) = \zeta_{\omega_u(\zeta(u))}(u)$ for $u \in \mathcal{G}$.
Let $f : \mathcal{G} \to V$ be the equivariant function corresponding to τ, and $p := \exp(tX)$ for $X \in \mathfrak{p}$ and $t \in \mathbb{R}$. By equivariance of f we have $f(u \cdot \exp(tX)) = \bar{\rho}(\exp(-tX))(f(u))$ for $u \in \mathcal{G}$. Therefore

$$\begin{aligned}(\zeta_X \cdot f)(u) = T_u f \cdot T_e \rho^u \cdot X &= \frac{d}{dt}|_0 f(u \cdot \exp(tX)) \\ &= \frac{d}{dt}|_0 \bar{\rho}(\exp(-tX))(f(u)) = -\bar{\rho}'(X)(f(u)).\end{aligned}$$

Altogether,

$$\begin{aligned}(\mathcal{D}_\sigma \tau)(\pi(u)) &= \left(\zeta_{\omega_u(\xi)\cdot f}(u)\right) = -\bar{\rho}'(\omega_u(\xi))(f(u)) = -\bar{\rho}'(\phi(u))(f(u)) \\ &= -[u, \phi(u)] \bullet [u, f(u)] = -(\sigma \bullet \tau)(\pi(u))\end{aligned}$$

□

Now we turn back to tractor bundles. In the following theorem we construct the desired linear connection.

Theorem 3.4.5. *Let $\mathcal{V}M = \mathcal{G} \times_p V$ a tractor bundle of the Cartan geometry $(\mathcal{G} \to M, \omega)$ together with a G-representation $\bar{\rho} : G \to \mathrm{GL}(V)$. Then for $\sigma \in \Gamma(\mathcal{A}M)$ and $\tau \in \Gamma(\mathcal{V}M)$,*

$$\nabla_{\Pi(\sigma)} \tau := \mathcal{D}_\sigma \tau + \sigma \bullet \tau$$

defines a linear connection on the tractor bundle.

Proof. According to 1.2.8 we have to prove $\mathbb{R}$-bilinearity, $C^\infty(M)$-linearity in the first argument and the Leibniz rule for ∇.

Firstly, take a look on the identifications that are needed to define the operators: The correspondence between $\Gamma(\mathcal{V}M)$ and $C^\infty(\mathcal{G}, V)^P$ is $C^\infty(M)$-linear. The projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{p}$ is $\mathbb{R}$-linear, thus Π is $C^\infty(M)$-linear. Similarly the derivative $\bar{\rho}' : \mathfrak{g} \times V \to V$ is $\mathbb{R}$-bilinear, thus the operator $\bullet$ is $C^\infty(M)$-bilinear. The fundamental derivative $\mathcal{D}$ is a usual derivative after identifying $\Gamma(\mathcal{A}M)$

with $\mathfrak{X}^P(\mathcal{G})$ and $\Gamma(\mathcal{V}M)$ with $C^\infty(\mathcal{G},\mathfrak{g})^P$. As above these identifications are $C^\infty(M)$-linear. The derivative itself is $C^\infty(M)$-linear (even $C^\infty(\mathcal{G})$-linear) in the $\mathfrak{X}^P(\mathcal{G})$-argument and satisfies the usual Leibniz rule in the $C^\infty(\mathcal{G},\mathfrak{g})^P$.

This immediately implies $C^\infty(M)$-linearity in $\Pi(\sigma)$ and $\mathbb{R}$-linearity in τ.

As for the Leibniz rule, we have for $\sigma \in \Gamma(\mathcal{A}M)$ (corresponding to $\xi \in \mathfrak{X}^P(\mathcal{G})$), $\tau \in \Gamma(\mathcal{V}M)$ (corresponding to $\phi \in C^\infty(\mathcal{G},V)^P$)) and $f \in C^\infty(M)$

$$\begin{aligned}
\nabla_{\Pi(\sigma)}(f\tau) &= \mathcal{D}_\sigma(f\tau) + \sigma \bullet (f\tau) \\
&= \xi \cdot ((f\circ\pi)\phi) + ((f\circ\pi))(\sigma\bullet\tau) \\
&= (\xi\cdot((f\circ\pi)))\phi + ((f\circ\pi))(\xi\cdot\phi) + ((f\circ\pi))(\sigma\bullet\tau) \\
&= (\Pi(\sigma)\cdot f)\tau + f(\nabla_{\Pi(\sigma)}\tau).
\end{aligned}$$

Finally, we have to observe that $\mathcal{D}_\sigma\tau + \sigma\bullet\tau$ depends only on $\Pi(\sigma)$ instead of σ: Let $\sigma_1, \sigma_2 \in \Gamma(\mathcal{A}M)$ such that $\Pi(\sigma_1) = \Pi(\sigma_2)$. Then σ_1 and σ_2 differ by a section σ' that has values in $\mathcal{G}\times_P \mathfrak{p} \subset \mathcal{A}M$. By 3.4.4(ii) we have $\mathcal{D}_{\sigma'}\tau + \sigma'\bullet\tau = 0$ hence by $\mathbb{R}$-linearity of the expression (see above) we obtain $\mathcal{D}_{\sigma_1}\tau + \sigma_1\bullet\tau = \mathcal{D}_{\sigma_1}\tau + \sigma_1\bullet\tau$.

□

Chapter 4

Holonomy of Cartan Geometries

The following chapter will constitute the theoretical key section of this thesis, that is a detailed examination of the theoretical part of [4]. We will generalize the concept of holonomy to Cartan geometries and consider implications of holonomy reductions for the geometric structure of the underlying manifold.

Hereafter let $(\pi : \mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G, P).

4.1 Holonomy reductions

We have a principal bundle $\mathcal{G}$ to which we might try to apply the concept of holonomy. However, ω is not a principal connection, since it is pointwise a linear isomorphism, hence its kernel is trivial. It is invariant under the G-action though and reproduces the generators of fundamental vector fields coming from elements in $\mathfrak{p}$, and indeed it induces a principal connection on a canonical associated bundle of $\mathcal{G}$. The details are established in the following proposition:

Proposition 4.1.1. *(i) The associated bundle $\hat{\pi} : \hat{\mathcal{G}} := \mathcal{G} \times_P G \to M$ is a G-principal bundle and there is a canonical, P-equivariant embedding $\iota : \mathcal{G} \hookrightarrow \hat{\mathcal{G}}$.*

(ii) There is a unique principal connection $\hat{\omega} \in \Omega^1\left(\hat{\mathcal{G}}, \mathfrak{g}\right)$ on $\hat{\mathcal{G}}$ such that $\iota^*\hat{\omega} = \omega$.

(iii) Conversely, if $\gamma \in \Omega^1\left(\hat{\mathcal{G}}, \mathfrak{g}\right)$ is a principal connection and $\iota^*\gamma$ is pointwise injective, $\iota^*\gamma$ is a Cartan connection on $\mathcal{G}$.

Proof. (i) G acts canonically on $\mathcal{G} \times_P G$ by right multiplication in the second component. This is well-defined, since right and left multiplication commute. Furthermore, the action is free, leaves fibers invariant and is transitive on each fiber. For a given principal chart $\varphi\big|_U : \mathcal{G}|_U \to U \times P$ over an open set $U \in M$, one can use the induced charts on the associated bundle as principal charts $\psi\big|_U : \hat{\mathcal{G}}|_U \to U \times G$ for $\hat{\mathcal{G}}$, since they are equivariant: Let $u \in \mathcal{G}|_U$ and $g, g' \in G$, then

$$\begin{aligned}\psi\left([u,g]\cdot g'\right)=\psi\left([u,gg']\right)&=\left(\pi\left(u\right),\varphi\left(u\right)\cdot gg'\right)\\&=\left(\pi\left(u\right),\varphi\left(u\right)g\right)g'=\psi\left([u,g]\right)\cdot g'.\end{aligned}$$

The inclusion is given by $\iota\left(u\right):=[u,e]$, where $u\in\mathcal{G}$. This is injective. In the charts $\varphi\big|_U$ and $\psi\big|_U$ one can see that ι is also infinitesimally injective, namely for $x\in U$ and $p\in P$ we have

$$\begin{aligned}\psi\left(\iota\left(\varphi^{-1}\left(x,p\right)\right)\right)&=\psi\left(\left[\varphi^{-1}\left(x,p\right),e\right]\right)=\left(\pi\left(\varphi^{-1}\left(x,p\right)\right),\varphi\left(\varphi^{-1}\left(x,p\right)\right)\cdot e\right)\\&=\left(x,p\right).\end{aligned}$$

Therefore ι is given by the inclusion $P\hookrightarrow G$ in the appropriate charts φ, ψ for $\mathcal{G}|_U$ and $\hat{\mathcal{G}}|_U$, respectively.

Equivariancy follows from $\iota\left(up\right)=[up,e]=[u,p]=[u,e]\,p=\iota\left(u\right)p$ for $u\in\mathcal{G}$ and $p\in P$.

(ii) Let $u\in\mathcal{G}$ and $\hat{u}:=\iota\left(u\right)$. We have $\left(T_{\hat{u}}\iota\right)^{-1}\left(V_{\hat{u}}\hat{\mathcal{G}}\right)=V_u\mathcal{G}$ since $\hat{\pi}\circ\iota=\pi$, hence for dimensional reasons $T_ui\left(T_u\mathcal{G}\right)+V_{\hat{u}}\hat{\mathcal{G}}=T_{\hat{u}}\hat{\mathcal{G}}$ and $T_u\iota\left(T_u\mathcal{G}\right)\cap V_{\hat{u}}\hat{\mathcal{G}}=T_u\iota\left(V_u\mathcal{G}\right)$. Now define for $\xi\in T_u\mathcal{G}$ the principal connection $\hat{\omega}_{\hat{u}}\left(T_u\iota\cdot\xi\right):=\omega_u(\xi)$ and for $X\in\mathfrak{g}$ let $\hat{\omega}_{\hat{u}}\left(\zeta_X\left(\hat{u}\right)\right):=X$.

In order to show that $\hat{\omega}$ is well-defined, let $\zeta_X\left(u\right)\in V_u\mathcal{G}$ for $X\in\mathfrak{g}$. Then note that for $p\in P$ we have

$$\left(\iota\circ\rho^u\right)\left(p\right)=\iota\left(up\right)=[up,e]=[u,p]=[u,e]\cdot p=\rho^{\hat{u}}\left(p\right).\tag{4.1}$$

Therefore, $T_{\hat{u}}\iota\cdot\zeta_X\left(u\right)=T_u\iota\cdot T_e\rho^u\cdot X=T_{\hat{u}}\rho^{\hat{u}}\cdot X=\zeta_X\left(\hat{u}\right)$, so the definitions of $\hat{\omega}$ coincide on $T_u\iota\left(V_u\mathcal{G}\right)$.

Furthermore, $\hat{\omega}$ is equivariant, what is obvious on the vertical subspace, whereas for $\xi\in T_u\iota\left(T_u\mathcal{G}\right)$ we have

$$\begin{aligned}\left(\left(\rho_p\right)^*\hat{\omega}_{\hat{u}}\right)\left(T_u\iota\cdot\xi\right)=\hat{\omega}_{\hat{u}p}\left(T_u\rho_p\cdot T_u\iota\cdot\xi\right)&\overset{(4.1)}{=}\hat{\omega}_{\hat{u}p}\left(T_{\hat{u}p}\iota\cdot T_u\rho_p\cdot\xi\right)\\&=\omega_u\left(T_u\rho_p\cdot\xi\right)=\mathrm{Ad}\left(p^{-1}\right)\circ\omega_u\left(\xi\right)\\&=\mathrm{Ad}\left(p^{-1}\right)\circ\hat{\omega}_{\hat{u}}\left(T_u\iota\cdot\xi\right)\end{aligned}$$

Therefore we may extend $\hat{\omega}$ equivariantly to $\hat{\mathcal{G}}$ in order to obtain a principal connection: From above we know that $\hat{\omega}_{up}\circ T_u\rho_p=\mathrm{Ad}(p^{-1})\circ\hat{\omega}_u$ for all $u\in\iota(\mathcal{G})$ and $p\in P$. Let $u\in\hat{\mathcal{G}}$. Then there is a $g\in G$ such that $gu\in\iota(\mathcal{G})$. Then define $\hat{\omega}_u:=\mathrm{Ad}(g)\circ\hat{\omega}_{ug}\circ T_u\rho_g$. The differential form $\hat{\omega}$ is well-defined because of the above equivariance-property.

We still have to see that $\hat{\omega}$ is a principal connection. Equivariance is easy to check: Let $u\in\hat{\mathcal{G}}$, $g\in G$ such that $ug\in\iota(\mathcal{G})$ and $h\in G$. Note that $(uh)(h^{-1}g)\in\iota(\mathcal{G})$. Now by definition

$$\begin{aligned}\hat{\omega}_{uh}\circ T\rho_h&=\mathrm{Ad}(h^{-1}g)\circ\hat{\omega}_{ug}\circ T\rho_{h^{-1}g}\circ T\rho_h\\&=\mathrm{Ad}(h^{-1})\circ\mathrm{Ad}(g)\circ\hat{\omega}_{ug}\circ T\rho_g=\mathrm{Ad}(h^{-1})\circ\hat{\omega}_u.\end{aligned}$$

Also, it reproduces the generators of fundamental vector fields: Firstly, note that for $u \in \hat{\mathcal{G}}$, $X \in \mathfrak{g}$ and $g \in G$ we have

$$\begin{aligned}T_u\rho_g \cdot \zeta_X(u) &= \frac{d}{dt}|_0 \rho_g(u \cdot \exp(tX)) = \frac{d}{dt}|_0 u\exp(tX)g = \frac{d}{dt}|_0 ugg^{-1}\exp(tX)g \\ &= \frac{d}{dt}|_0 ug\exp(\mathrm{Ad}(g^{-1})(tX)) = \frac{d}{dt}|_0 \rho_{\exp(tAd(g^{-1}))(X)}(ug) \\ &= \zeta_{\mathrm{Ad}(g^{-1})(X)}(ug).\end{aligned}$$

Therefore for $u \in \hat{\mathcal{G}}$ and $g \in G$ such that $ug \in \iota(\mathcal{G})$ we have

$$\begin{aligned}\hat{\omega}_u(\zeta_X(u)) &= \mathrm{Ad}(g)(\hat{\omega}_{ug}(T_u\rho_g \cdot \zeta_x(u))) = \mathrm{Ad}(g)(\hat{\omega}_{ug}(\zeta_{\mathrm{Ad}(g^{-1}(X)}(ug)))) \\ &= \mathrm{Ad}(g)(\mathrm{Ad}(g^{-1})(X)) = X.\end{aligned}$$

Uniqueness follows from $T_u\iota\,(T_u\mathcal{G}) + V_{\hat{u}}\hat{\mathcal{G}} = T_{\hat{u}}\hat{\mathcal{G}}$.

(iii) Let $p \in P$, $u \in \mathcal{G}$ and $X \in \mathfrak{g}$. Firstly, $\iota^*\gamma$ is equivariant:

$$\begin{aligned}(\rho_p)^*(\iota^*\gamma) &= (\iota^*\gamma) \circ T\rho_p = \gamma \circ T\iota \circ T\rho_p \overset{(i)}{=} \gamma \circ T\rho_p \circ T\iota \\ &= \mathrm{Ad}\left(p^{-1}\right) \circ \gamma \circ T\iota = \mathrm{Ad}\left(p^{-1}\right) \circ (\iota^*\gamma)\end{aligned}$$

Secondly, it reproduces the generators of fundamental vector fields:

$$(\iota^*\gamma)_u(\zeta_X(u)) = \gamma_{\iota(u)}(T_u\iota \cdot \zeta_X(u)) = \gamma_u(\zeta_X(\iota(u))) = X$$

And finally, $(\iota^*\gamma)_u : T_u\mathcal{G} \to \mathfrak{g}$ is injective, therefore bijective due to dimensional reasons. □

Remark 4.1.2. *Note that one can view the space of principal connections on a given principal bundle as an affine space. In this sense, one can interpret the above proposition as saying that the set of Cartan connections on $\mathcal{G}$ is an open subset of the space of principal connections on $\hat{\mathcal{G}}$, since the condition to be pointwise injective is an open condition.*

This can be explained as follows: Injectivity is an open condition on linear maps $T_xM \to V$, where $x \in M$, a k-dimensional manifold, and V an n-dimensional vector-space, since after the choice of a basis of T_xM and V, we obtain a map $\mathbb{R}^k \to \mathbb{R}^n$. Injectivity of this map means that it has a $k\times k$-submatrix whose determinant does not vanish. Since the determinant is continuous and the set $\mathbb{R} \setminus \{0\}$ is open in $\mathbb{R}$, also its preimage under the determinant, the set of injective maps, is open. Thus the set of one-forms, consisting of pointwise linear maps from tangent spaces into a fixed vector space, that are pointwise injective, is open.

We want to introduce a notion of holonomy for Cartan geometries.

Recall Example 1.2.6: We showed that a parallel metric on a vector bundle is equivalent to a parallel section of the bundle associated to its frame bnundles via the space of inner products on $\mathbb{R}^n$. By choosing a distinguished inner product, we obtained a reduction of the structure group of the frame bundle from the general linear group to the stablizer of the distinguished inner product.

Now we try to realize a similar concept on Cartan Geometries:

We will consider H-reductions of the principal bundle $\hat{\mathcal{G}}$ where H is a subgroup of G. These reductions can be described by sections of an associated bundle with fiber G/H. We will face problems with H being determined only up to conjugation. Thus we replace the fiber G/H by an "abstract homogenous space" $\mathcal{O}$, i.e. a manifold $\mathcal{O}$, on that G acts smoothly and transitively from the left. The choice of an $\alpha \in \mathcal{O}$ yields $H_\alpha := \mathrm{Stab}_G(\alpha)$ and an identification $G/H = \mathcal{O}$.

Definition A holonomy reduction of $(\mathcal{G} \to M, \omega)$ is a parallel, smooth section σ of $\hat{\mathcal{G}} \times_G \mathcal{O} = \mathcal{G} \times_P \mathcal{O}$ with respect to the connection induced by $\hat{\omega}$, where $\mathcal{O}$ is a G-homogenous space. The reduction σ is said to be of *G-type* $\mathcal{O}$.

In the following we will explore the consequences of a holonomy reduction for the original Cartan geometry $(\pi : \mathcal{G} \to M, \omega)$.

On the group level, we see the other side of the problem: We observe how different elements of the orbit interact. Given two $\alpha, \alpha' \in \mathcal{O}$, they are linked by an element $g \in G$ as $\alpha' = g \cdot \alpha$. Hence their stabilizers in G, that we denote by H_α and $H_{\alpha'}$, respectively, are conjugate:

$$\begin{aligned} H_{\alpha'} &= \{g' \in G \mid g' \cdot \alpha' = \alpha'\} = \{g' \in G \mid g'g \cdot \alpha = g \cdot \alpha\} \\ &= \left\{g' \in G \mid g^{-1}g'g \cdot \alpha = \alpha\right\} = \left\{gg'g^{-1} \mid g' \in G,\, g' \cdot \alpha = \alpha\right\} \\ &= gH_\alpha g^{-1}. \end{aligned}$$

This issue is well-known for principal bundles (see 2.2.2(ii)).

We would like to obtain a reduction of $\mathcal{G}$ from P to $P \cap H_\alpha$. However, if we choose a different element in $\mathcal{O}$, this has a stabilizer of the form $gH_\alpha g^{-1}$ for $g \in G$. The intersections $H_\alpha \cap P$ and $H_{g\alpha} \cap P = gH_\alpha g^{-1} \cap P$ need not be of the same dimension though.

So we conclude that we cannot hope to obtain a global reduction of the structure group $\mathcal{H} \subset \mathcal{G}$ in general.

The issue of "relative position" is best illustrated by considering an explicit example:

Example 4.1.3. *Let $G := \mathrm{SL}(n, \mathbb{R})$, $P := \mathrm{Stab}_G(l)$, the stabilizer of a line $l \subset \mathbb{R}^n$ containing 0 and $\mathcal{O}$ the space of non-degenerate, symmetric bilinear forms on $\mathbb{R}^n$ with signature (p, q), where $p + q = n$.*

Real, symmetric bilinear forms are characterized (up to base change) by their rank and signature, hence G acts transitively on $\mathcal{O}$.

However, we must not forget the additional structure given by the distinguished line l in $\mathbb{R}^n$. Now different choices of inner products have different relative positions with respect to l: The restriction of the chosen inner product to $l \times l$ has image either $\mathbb{R}_0^+$, $\mathbb{R}_0^-$ or $\{0\}$. As long as $0 < p, q < n$, all three possiblities occur.

Therefore, the choice of an element in $\mathcal{O}$ is not canonical.

4.2 Tractor bundles

In this section we see how parallel sections of certain vector bundles induce holonomy reductions. To this end, we need vector bundles that are associated to $\hat{\mathcal{G}}$. These are exactly the tractor bundles we introduced in Section 3.4.

Note that since $\iota : \mathcal{G} \hookrightarrow \hat{\mathcal{G}}$ is a reduction of the structure group, we have $\mathcal{G} \times_P V = \hat{\mathcal{G}} \times_G V$ for all G-representations V. Furthermore, from 4.1.1 we know that $\hat{\mathcal{G}}$ carries a canonical principal connection $\hat{\omega}$. This induces a linear connection $\nabla^{\hat{\omega}}$ on $\hat{\mathcal{G}} \times_G V = \mathcal{G} \times_P V$.

However, in 3.4.5 we already constructed a canonical linear connection ∇ on $\mathcal{G} \times_P V$. Firstly, we show that these two concepts are the same:

Proposition 4.2.1. *The two linear connections $\nabla^{\hat{\omega}}$ and ∇ on $\mathcal{G} \times_P V$ coincide. In the following we will denote it by ∇.*

Proof. Let $\xi \in \mathfrak{X}(M)$ and $\tau \in \Gamma(\mathcal{G} \times_P V) = \Gamma(\hat{\mathcal{G}} \times_G V)$. The section τ corresponds to equivariant functions $f : \mathcal{G} \to V$ and $\hat{f} : \hat{\mathcal{G}} \to V$. With the identification from 1.2.4 we have for $[\iota(u), \hat{f}(\iota(u))] = \tau(\pi(u)) = [u, f(u)] = [\iota(u), f(u)]$ for $u \in \mathcal{G}$ and thus $\hat{f} \circ \iota = f$.

Now choose a P-equivariant lift $\tilde{\xi} \in \mathfrak{X}(\mathcal{G})^P$. Note that $\Pi(\tilde{\xi}) = \xi$ (as we observed in the proof of 3.4.1(i)). The horizontal lift of ξ to $\hat{\mathcal{G}}$ at $\iota(u) \in \hat{\mathcal{G}}$ is given by the expression

$$\hat{\xi}^{hor}(\iota(u)) = T_u\iota \cdot \tilde{\xi}(u) - \zeta_{\omega(\tilde{\xi}(u))}(\iota(u)),$$

since $T_u\pi \cdot \hat{\xi}^{hor}(\iota(u)) = T_u\pi\tilde{\xi}(u) = \xi(\pi(u))$ and $\hat{\omega}(T_u\iota \cdot \tilde{\xi}(u) - \zeta_{\omega(\tilde{\xi}(u))}(\iota(u))) = \omega(\tilde{\xi}(u)) - \omega(\tilde{\xi}(u)) = 0$.

The expression $(\nabla^{\hat{\omega}}_{\xi}\tau)(\pi(u))$ is equivalent to the equivariant function $(\hat{\xi}^{hor} \cdot \hat{f})(\iota(u))$. We compute

$$\begin{aligned}
(\hat{\xi}^{hor} \cdot \hat{f})(\iota(u)) &= T_{\iota(u)}\hat{f} \cdot (T_u\iota \cdot \tilde{\xi}(u) - \zeta_{\omega(\tilde{\xi}(u))}(\iota(u))) \\
&= (\tilde{\xi} \cdot f)(u) - \frac{d}{dt}|_0 \hat{f}(\iota(u) \cdot \exp(t\omega(\tilde{\xi}(u)))) \\
&= (\tilde{\xi} \cdot f)(u) - \frac{d}{dt}|_0 \bar{\rho}(\exp(-t\omega(\tilde{\xi}(u))))(f(u)) \\
&= (\tilde{\xi} \cdot f)(u) + \bar{\rho}'(\omega(\tilde{\xi}(u)))(f(u)),
\end{aligned}$$

where $\bar{\rho} : G \to \mathrm{GL}(V)$ is the G-representation and $\bar{\rho}'$ its derivative $\mathfrak{g} \to \mathfrak{gl}(V)$.

The first summand of the result of the computation above is the fundamental derivative of τ with respect to the section σ of $\mathcal{A}M$ that corresponds to $\tilde{\xi}$. This is the desired term.

This section σ is explicitely given by $\sigma(\pi(u)) = [u, \omega(\tilde{\xi}(u))]$. Thus the second summand corresponds to $\sigma \bullet \tau$. Hence we have proved that $\nabla^{\hat{\omega}}$ is the same linear connection as the one from 3.4.5. $\square$

Next, we will link the above concept with holonomy reductions of Cartan geometries.

Proposition 4.2.2. *Assume M is connected and let $\bar{\rho} : G \to \mathrm{GL}\,(V)$ be a finite-dimensional G-representation and σ a section of the corresponding tractor bundle $\mathcal{V}$ that is parallel with respect to the tractor connection ∇. Then σ corresponds to an G-equivariant function $s : \hat{\mathcal{G}} \to V$ whose image $s(\hat{\mathcal{G}})$ is a G-orbit in V.*

Proof. We know from 1.2.3 that sections of associated bundles correspond to equivariant functions. That σ is parallel means that for all $\xi \in \mathfrak{X}\,(M)$ we have

$\nabla_\xi \sigma = 0$, i.e. s satisfies $\xi^{hor} \cdot s = 0$ for the horizontal lift $\xi^{hor} \in \mathfrak{X}(\hat{\mathcal{G}})$ of ξ (cf. 1.2.9).

By fixing an element $\hat{u} \in \hat{\mathcal{G}}$ we see that the G-orbit of $s(\hat{u})$ in V is contained in the image of s: Let $g \in G$ then $g \cdot s(\hat{u}) = s\left(\hat{u} g^{-1}\right)$.

On the other hand, let $\hat{u}' \in \hat{\mathcal{G}}$ be an arbitrary element. Connect the base points of $\hat{u}$ and $\hat{u}'$ by a smooth curve that we lift to a horizontal curve $c : [0,1] \to \hat{\mathcal{G}}$ such that $c(0) = \hat{u}$. Now $c(1)$ is in the same fiber of $\hat{\mathcal{G}}$ hence there is a $g \in G$ such that $\hat{u}' = c(1)\, g$. But as we conclude from above that $c' \cdot s = 0$ the function s is constant along the curve c, i.e.

$$s(\hat{u}') = s(c(1)\, g) = g^{-1} s(c(1)) = g^{-1} s(c(0)) = g^{-1} s(\hat{u}).$$

□

Now given a parallel section σ of a tractor bundle as in 4.2.2, let $\mathcal{O}$ denote the G-orbit from 4.2.2. Observe that by definition $\mathcal{O}$ is a G-invariant subset of V, therefore $\hat{\mathcal{G}} \times_G \mathcal{O} \subset \hat{\mathcal{G}} \times_G V$ is a well-defined smooth subbundle such that the induced connections are compatible with the inclusion (this is obvious from the definition of the induced horizontal subspace). We know σ has values in $\mathcal{G} \times_P \mathcal{O}$ hence defines a holonomy reduction of type $\mathcal{O}$.

Example 4.2.3. *We carry on with our previous Example 4.1.3 and consider the vector space $\left(S^2\mathbb{R}^n\right)^*$. This is the vector space of symmetric bilinear forms on $\mathbb{R}^n$. We already mentioned before, that the bilinear forms are determined, up to base change, by their rank and signature, hence the decomposition into G-orbits is given by*

$$\bigsqcup_{0 \le p \le r,\ 0 \le r \le n} \left\{ b \in \left(S^2\mathbb{R}^n\right)^* \mid b \text{ has rank } r \text{ and signature } p \right\}.$$

The homogenous space $\mathcal{O}$ from Example 4.1.3 is exactly such a G-orbit, and a holonomy reduction of this type can be interpreted as a parallel section of the vector bundle $\hat{\mathcal{G}} \times_G S^2(\mathbb{R}^n)^*$.

4.3 Structure of the underlying manifold

Recall that our problem was related to the fact that the choice of different elements of $\mathcal{O}$ have different stablizers, that are all conjugated by elements of G, but might have different intersections with P.

Now consider the case of $\alpha, \alpha' \in \mathcal{O}$ with $\alpha' = g \cdot \alpha$ for a $g \in P$. Let H be the stabilizer of α in G, then the stabilizer H' of α' is given by $H' = gHg^{-1}$.

But in addition, we clearly have $gPg^{-1} = P$ and thus $H' \cap P = gHg^{-1} \cap gPg^{-1} = g(H \cap P)\, g^{-1}$. Hence the pairs $(H', H' \cap P)$ and $(H, H \cap P)$ (consisting of two nested subgroups of G) are simultaneously isomorphic by conjugation.

Let us consider the G-equivariant function $s : \hat{\mathcal{G}} \to \mathcal{O}$ corresponding to a holonomy reduction σ of type $\mathcal{O}$ (cf. 1.2.3) and recall that there is the canonical P-reduction $\iota(\mathcal{G})$ of $\hat{\mathcal{G}}$. Combining these two objects, we observe that for $x \in M$ the image of the fiber $\iota(\mathcal{G}_x)$ under s is a P-orbit in $\mathcal{O}$ – by fixing any $u \in \mathcal{G}_x$ we conclude $s(\iota(\mathcal{G}_x)) = s(\iota(u \cdot P)) = s(P \cdot \iota(u)) = P \cdot s(\iota(u))$.

Definition (i) Let $x \in M$. The P-orbit in $\mathcal{O}$ given by $s(\iota(\mathcal{G}_x))$ is called the P-type of x with respect to the holonomy reduction σ.

(ii) Let $i \in P\backslash\mathcal{O}$ be a P-orbit in $\mathcal{O}$. Then the set

$$M_i := \{x \in M \mid s(\iota(\mathcal{G}_x)) = i\} \subset M$$

is called the curved orbit of type i.

(iii) The decomposition $M = \bigsqcup_{i \in P\backslash\mathcal{O}} M_i$ is called the curved-orbit-decomposition of M.

4.4 The homogenous model

Following up the last definition, let us explicitly compute these objects for the homogenous model. We will need these information particularly for Lemma 4.5.1.

To begin with, we collect some useful facts about the homogenous model in the following lemma.

Lemma 4.4.1. *(i) The extension $\hat{G} := G \times_P G$ can be canonically trivialized as $\hat{G} = G/P \times G$, where the projection corresponding to $\hat{\pi}$ is the projection onto the first component. The G-action in the trivialization becomes right multiplication in the second component, thus the trivialization provides a global principal chart.*

(ii) The fundamental vector field corresponding to $X \in \mathfrak{g}$ in the trivialization is given by $\zeta_X(gP, g') = (0_{gP}, T_e\lambda_{g'} \cdot X) = (0, L_X)$, where L_X denotes the left-invariant vector field with respect to $X \in \mathfrak{g}$.

(iii) G is embedded into $G/P \times G$ via $\iota(g) = (gP, g)$.

(iv) The principal connection $\hat{\omega}$ is the flat connection with respect to the trivialization from (i).

Proof. In the following, let $g, g', g'' \in G$ and $p \in P$. (i) Let $\Phi : G/P \times G \to G \times_P G$ be the map defined by $(gP, g') \mapsto [g, g^{-1}g']$. Its inverse map is evidently given by $\Phi^{-1}([g, g']) = (gP, gg')$, which is well-defined, since $\Phi^{-1}([gp, p^{-1}g']) = (gpP, gpp^{-1}g') = (gP, gg')$.

As for the projection, we have $\hat{\pi}(\Phi(gP, g')) = \hat{\pi}([g, g^{-1}g']) = gP$; whereas the right action is given by

$$\begin{aligned}(gP, g') \cdot g'' :&= \Phi^{-1}(\Phi(gP, g') \cdot g'') = \Phi^{-1}([g, g^{-1}g']\, g'') \\ &= \Phi^{-1}([g, g^{-1}g'g'']) = (gP, g'g'').\end{aligned}$$

(ii) Since the action of g'' on (gP, g') can be written as $\rho^{(gP,g')}(g'') = (gP, g') \cdot g'' = (gP, g'g'') = (\mathrm{const}_{gP} \times \lambda_{g'})(g'')$, the derivative is

$$\zeta_X(gP, g') = T_e\rho^{(gP,g')} \cdot X = (T_e \,\mathrm{const}_{gP} \times T_e\lambda_{g'}) \cdot X = (0_{gP}, T_e\lambda_{g'} \cdot X).$$

(iii) is obvious from 4.1.1 and the proof of (i).

(vi) The principal connection $\hat{\omega}$ is determined by $\iota^*\hat{\omega} = \omega$. We want to calculate $\ker(\hat{\omega}_{\iota(g)})$ for $g \in G$, so assume $(\xi', \eta') \in T_{\iota(g)}(G/P \times G)$. But

$T_{\iota(g)}(G/P\times G) = T_g\iota(T_gG) + V_{\iota(g)}(G/P\times G)$ and $T_g\iota = T_g\pi \times \mathrm{id}_{T_gG}$, so we can write $\xi' = T_g\pi\cdot\xi$ for $\xi\in T_gG$. Then, let $\eta := \eta' - \xi \in T_gG$. We get

$$\hat\omega_{\iota(g)}(\xi',\eta') = \hat\omega(T_g\pi\cdot\xi, \xi+\eta) = \hat\omega\underbrace{(T_g\pi\cdot\xi,\xi)}_{=T_g\iota\cdot\xi} + \hat\omega(0,\eta) = \omega_g(\xi) + T_g\lambda_{g^{-1}}\cdot\eta.$$

The last equation follows since we know from (ii) that $(0,\eta)$ is horizontal and the form of the fundamental vector fields, therefore

$$\hat\omega(0,\eta) = \hat\omega\left(0, T_e\lambda_g\cdot T_g\lambda_{g^{-1}}\cdot\eta\right) = \hat\omega\left(\zeta_{T_g\lambda_{g^{-1}}\cdot\eta}\right)(g) = T_g\lambda_{g^{-1}}\cdot\eta.$$

Recall that the Maurer Cartan form is given by $\omega_g(\xi) = T_g\lambda_{g^{-1}}\cdot\xi$. Consequently, we can determine the horizontal space by

$$\hat\omega_{\iota(g)}(\xi',\eta') = 0 \iff T_g\lambda_{g^{-1}}\cdot\xi = -T_g\lambda_{g^{-1}}\cdot\eta \iff \xi = -\eta \iff \eta' = 0.$$

Hence by equivariancy (and since G acts only on the second component), $T(G/P)\times\{0\}\subset T(G/P\times G)$ is the horizontal distribution. In particular, the principal connection on $\hat G$ is given by $\hat\omega_{(gP,g')} = \omega_{g'}\circ \mathrm{pr}_2$. □

Next, we describe the possible H-reductions of the homogenous model.

Lemma 4.4.2. *Let σ be a holonomy reduction of type G/H of the homogenous model $G\to G/P$, where $H\subset G$ is a closed subgroup. Then there is a $g\in G$ such that the corresponding reduction of $\hat{\mathcal{G}}$ has the form $G/P\times gH\subset G/P\times G$. It corresponds to the equivariant function $[g',g'']\mapsto g''(g')^{-1}gH$, $\hat G\to G/H$.*

Proof. Let $\mathcal{H}$ be the reduction of $\hat G\cong G/P\times G$ from G to H, that corresponds to σ. The connection is flat on $\hat G$ and must stay a principal connection under the reduction, hence the reduction must be of the form $G/P\times F$ where $F\subset G$ is a set diffeomorphic to H. Indeed, if we know one element $(g'P,g)\in\mathcal{H}$, we can immediately conclude that $G/P\times\{g\}\subset\mathcal{H}$ and, by H-equivariancy, we obtain $\mathcal{H} = G/P\times gH$.

Next, we view the reduction of $\hat G$ as a section of $\hat G/H$. Note that $\hat G/H = G\times_P(G/H)\cong G/P\times G/H$.

The image of $\mathcal{H}$ under the projection $\hat G\to\hat G/H$ is gives the section $\sigma : G/P\to\hat G/H$, $\sigma(g'P) = (g'P, gH) = \left[g', (g')^{-1}gH\right]$.

Therefore, the G-equivariant function corresponding to σ is given by $s:\hat G\to G/H$, $[g',g'']\mapsto (g'')(g')^{-1}gH$.

□

Let us start with the most obvious case and fix a subgroup $H\subset G$. Consider the reduction of the form $G/P\times H\subset G/P\times G$, equivalent to the holonomy reduction $\sigma\in\Gamma\left(\hat G/H\right)$, $\sigma(gP) = (gP,H)$ for $g\in G$.

We want a more explicit description of the P-type decomposition of the underlying manifold G/P. The fiber of $gP\in G/P$ in G is $gP\subset G$, therefore the embedded fiber in $\hat G$ is given by

$$\iota(gP) = \iota(\{gp\mid p\in P\}) = \{(gpP, gp)\in G/P\times G\mid p\in P\} = \{gP\}\times gP.$$

Hence the P-orbit of gP in $\mathcal{O} = G/H$ is $s(\{gP\} \times gP) = \{p^{-1}g^{-1} \mid p \in P\} \cdot H = P \cdot g^{-1}H$. Thus the set of curved orbits in G/H is parametrized by the set of double cosets $P\backslash (G/H)$.

In order to determine the curved orbit of $gP \in G/P$, we consider the map $G/P \to P\backslash (G/H)$, $g'P \mapsto P(g')^{-1} H$ that assignes to each element of the underlying manifold G/P its P-orbit in $\mathcal{O}$. Obviously, this is well-defined and factors over the canonical projection $G/P \to H\backslash (G/P)$. The inverse of the factorized map is given by $P(g')^{-1} H \mapsto Hg'P$, so it is a bijection. From that we can read off all other elements in G/P that have the same P-type as gP, that form its curved orbit $H \cdot gP \subset G/P$.

Now observe that by fixing an element gP in $H \cdot gP$, the curved orbit itself inherits a geometry: H acts freely and transitively on $H \cdot gP \subset G/P$, and the stabilizer of gP in H is given by $H \cap gPg^{-1}$, since $h \in H$ stabilizes gP if and only if $h \cdot gP = gP$, which is equivalent to $h \in gPg^{-1}$. So, we can identify $H \cdot gP$ with $H/\left(H \cap gPg^{-1}\right)$.

If we choose another point in the same H-orbit, namely one of the form $hgP \in H \cdot gP$ for $h \in H$, we obtain an isomorphic structure: The stabilizer of hgP is given by $\mathrm{Stab}_H(hgP) = h\,\mathrm{Stab}_H(gP)\,h^{-1} = h\left(H \cap gPg^{-1}\right)h^{-1} = H \cap hgP(hg)^{-1}$. Hence we obtain the structure $H \cdot gP = H/\left(H \cap hgP(hg)^{-1}\right)$. This is isomorphic to $H/\left(H \cap gPg^{-1}\right)$ via $h'\left(H \cap hgP(hg)^{-1}\right) \mapsto h^{-1}h'hH \cap gPg^{-1}$, where $h' \in H$.

Note that given an abstract homogenous space $\mathcal{O}$. Choosing a distinguished element $\alpha \in \mathcal{O}$ gives the identification $\mathcal{O} = G/H$ where H is the stabilizer of α in G.

The following proposition summarizes the information on holonomy reductions of the homogenous model that was collected in the above paragraphs.

Proposition 4.4.3. *(i) The P-type of $gP \in G/P$ is given by $P \cdot g^{-1}H \subset P\backslash(G/H)$, so the curved orbits are parametrized by $P\backslash(G/H)$. In the abstract homogenous space, this means that the P-type of gP is $P \cdot g^{-1}\alpha$.*

(ii) The curved orbit corresponding to the P-orbit $P \cdot gH$ is given by

$$(G/P)_{Pg^{-1}H} = H \cdot gP \subset G/P,$$

where H is the stabilizer of α in G. Furthermore,

$$H \cdot gP = H/\left(H \cap gPg^{-1}\right) = g^{-1}Hg/g^{-1}Hg \cap P.$$

Example 4.4.4. *Let us look again at our example (see 4.1.3 and 4.2.3), i.e. let $G := \mathrm{SL}(n, \mathbb{R})$, $P := \mathrm{Stab}_G(l)$, the stabilizer of a line $l \subset \mathbb{R}^n$ through 0.*

Now we fix a subgroup $H := SO(p,q)$ of G (that gives us $G/H = \mathcal{O}$, where $\mathcal{O}$ is the space of symmetric, non-degenerate inner products of signature (p,q)). We assume $0 < p, q$ in order to avoid trivialities. The group H is the stabilizer of the standard inner product of signature (p,q) on $\mathbb{R}^n$, that we denote by b_0.

We consider the homogenous model $G \to G/P$ of the Cartan geometry of type (G,P), where $G/P = \mathbb{R}P^{n-1}$ is the space of lines in $\mathbb{R}^n$. An element

$gP \in G/P$ corresponds to the line $g\,(l) \in G/P$ for $g \in G$. Proposition 4.4.3 shows that the curved orbit, that contains gP, is given by $H \cdot gP \mathrel{\widehat{=}} H \cdot g\,(l)$.

We observed in 4.4.3 that the curved orbit decomposition is given by $H\backslash(G/P) = H\backslash\mathbb{R}P^{n-1}$.

Claim: *There are three H-orbits, that we denote by $(G/P)_+$, $(G/P)_-$ and $(G/P)_0$. An element $l \in G/P$ is classified by $b_0\big|_{l\times l}$ that is positive definite, negative definite or vanishes.*

proof of claim: *First, note that $A \in SO(p,q)$ implies $b_0(Al, Al) = b_0(l,l)$ for all $l \in \mathbb{R}P^{n-1}$, hence the H-action leaves the sets $(G/P)_+$, $(G/P)_-$ and $(G/P)_0$ invariant.*

Let $l \in (G/P)_+$ and choose $v \in l$ such that $b_0(v,v) = 1$. Complete v to a orthonormal basis of $\mathbb{R}^n$ such that the base change to the standard basis has determinant 1. Then $v \mapsto e_1$ under this base change. This shows that H acts transitively on $(G/P)_+$.

The analogous argument works for $(G/P)_-$.

Let $\mathbb{R}v = l \in (G/P)_0$. Then, since b_0 is non-degenerate there is a $w \in \mathbb{R}^n$ such that $b_0(v,w) = 1$. Consider $w' := w - \frac{1}{2}b_0(w,w)v$, that lies in the span of v and w. Then $b_0(w',w') = b_0(w,w) - 2\frac{1}{2}b_0(w,w)b_0(v,w) + \frac{1}{4}b_0(w,w)^2 b_0(v,v) = 0$.

On the span of v and w' the inner product b_0 is of signature $(1,1)$. Thus, by choosing an appropriate basis of $\{v,w\}^\perp$, we can complete v, w to a basis of $\mathbb{R}^n$. Starting with another null vector $\tilde{v}$, we can obtain a basis in the same way. In addition, we require that the basis has the same orientation as above. The base change is in $SO(p,q)$, since b_0 looks the same in both bases. Thus H acts transitively on $(G/P)_0$.

Now let us look at the orbits in more detail:

(1) *Firstly, look at $(G/P)_+ := \left\{ l \in G/P \mid b_0\,(l,l) = \mathbb{R}_0^+ \right\}$.*

In order to describe the structure on the curved orbit, we choose a line in $(G/P)_+$, for example the line $l := \langle\, e_1 \,\rangle$ spanned by the first unit vector.

We show that its stabilizer in H is given by $O\,(p-1,q)$: Let $g := \begin{pmatrix} a & b \\ c & D \end{pmatrix} \in \mathrm{SL}\,(n,\mathbb{R})$ be a $(1, n-1)$-block matrix with $g\,(l) = l$, i.e. $\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \end{pmatrix}$ where $\lambda \neq 0$. Hence $c = 0$ and $a > 0$. On the other hand, such a matrix $\begin{pmatrix} a & b \\ 0 & D \end{pmatrix}$ clearly fixes the line l. Now g should additionally satisfy $g^T g = I_{p,q}$, where $I_{p,q}$ is the diagonal matrix with 1 in the first p diagonal entries, and -1 in the remaining q entries.

Hence we have the equation

$$\begin{pmatrix} a & b \\ 0 & D \end{pmatrix}^T \begin{pmatrix} a & b \\ 0 & D \end{pmatrix} = \begin{pmatrix} a & 0 \\ b^T & D \end{pmatrix} \begin{pmatrix} a & b \\ 0 & D \end{pmatrix} = \begin{pmatrix} a^2 & ab \\ ab^T & D^T D \end{pmatrix} = I_{p,q}.$$

We conclude $D \in O\,(p-1,q)$ and $a = \pm 1$, and thus $b = 0$. Since $H = SO(p,q)$, we obtain $a = \det(D)$.

Therefore, we have $(G/P)_+ = SO(p,q)/O(p-1,q)$. This gives a Riemannian metric of signature $(p-1,q)$ on $(G/P)_+$ (see [5, p.7f]).

(2) *The case of the curved orbit* $(G/P)_{-} = \left\{l \in G/P \mid b_0(l,l) = \mathbb{R}_0^-\right\}$ *is completely analogous to Case (1). We obtain the structure* $SO(p,q)/O(p,q-1) = (G/P)_{-}$*, that is endowed with a metric of signature* $(p, q-1)$.

(3) *The third case is more involved, hence we will only sketch the construction of the structure on the orbit (for the details see [5, p.13f]). The curved orbit* $(G/P)_0$ *is given by* $\{l \in G/P \mid b_0(l,l) = \{0\}\}$.

Consider the projection $\pi : C := \{x \in \mathbb{R}^n \setminus \{0\} \mid b_0(x,x) = 0\} \to (G/P)_0$*, that maps* x *to the line that contains* x*. The tangent space of* x *to* C *is given by* $x^{\perp}$*, and since* x *is null,* $x \in x^{\perp}$*. The tangent map of the projection,* $T_x\pi : T_xC \to T_{\pi(x)}(G/P)_0$*, satisfies* $T_x\pi \cdot x = 0$*, since* $\pi(t \cdot x) = \pi(x)$ *for all* $t \in \mathbb{R}$*. Therefore,* $T_x\pi$ *factors to a map* $\widetilde{T_x\pi} : x^{\perp}/\mathbb{R}x \to T_{\pi(x)}(G/P)_0$*. One can show that* $T_x\pi$ *is a linear isomorphism.*

Moreover, $x^{\perp}$ *inherits an inner product of signature* $(p-1, q-1)$ *from the surrounding space* $\mathbb{R}^n$*, that factors to* $x^{\perp}/\mathbb{R}x$*, since for* $y_1, y_2 \in x^{\perp}$ *we have* $b_0(y_1+x, y_2+x) = b_0(y_1,y_2) + b_0(y_1,x) + b_0(x,y_2) + b_0(x,x) = b_0(y_1,y_2)$*. The isomorphism* $\widetilde{T_x\pi}$ *carries the inner product over to* $T_{\pi(x)}(G/P)_0$.

Let us check whether this inner product is well-defined: Choose another element of C*, that is mapped to* $\pi(x)$*, then this has to be of the form* λx *where* $\lambda \in \mathbb{R}^+$*. Consider the curves* $c(t) := x + t\xi$ *and* $\tilde{c}(t) := \lambda c(t)$ *for* $\xi \in x^{\perp} = (\lambda x)^{\perp}$*, then these curves satisfy* $c(0) = x$ *and* $\tilde{c}(0) = \lambda x$*, respectively, and* $c'(0) = \xi$ *whereas* $\tilde{c}'(0) = \lambda\xi$*. Thus the tangent maps* $T_x\pi$ *and* $T_{\lambda x}\pi$ *differ only by the positive factor* λ*, hence the inner product on* $T_{\pi(x)}(G/P)_0$ *is uniquely defined up to a positive scalar. This gives a conformal structure on* $(G/P)_0$.

The group action of $SO(p,q)$ *leaves the inner product* b_0 *invariant, and hence acts by conformal isometries on* $(G/P)_0$*. One can show that it is exactly the isomorphim group of this structure.*

Finally, to complete our discussion on the homogenous model, we consider other H-reductions of G/P. We will see that the whole structure of the decomposition only changes by the action of an element of G. This shows that there is only one H-reduction of the homogenous model up to the G-action.

Let g_0 be a fixed element of G and consider the H-reduction $G/P \times g_0H \subset G/P \times G$ (cf. 4.4.2) In order to establish the P-type of a given $gP \in G/P$, we compute

$$s(\iota(G_{gP})) = s(\iota(gP)) = s([gP, e]) = P^{-1} \cdot (g)^{-1} g_0 H \subset G/H.$$

Therefore, the map $T : G/P \to P\backslash(G/H)$, that assignes to each element in G/P its P-type, is given by $gP \mapsto Pg^{-1}g_0H$.

Again we compute the curved orbit, that corresponds to the P-orbit $Pg^{-1}H$, thus we have to compute $T^{-1}\left(Pg^{-1}H\right)$:

An element $g'P \in G/P$ is mapped to $Pg^{-1}H$ if and only if $P(g')^{-1}g_0H = Pg^{-1}H$. This is equivalent to the existence of an $h \in H$ and a $p \in P$ such that $(g')^{-1}g_0 = pg^{-1}h \Leftrightarrow g' = g_0h^{-1}gp^{-1}$. Hence the curved orbit is given by $(G/P)_{Pg^{-1}H} = \{g'P \in G/P \mid \exists h \in H : g'P = g_0hgP\} = g_0HgP$. This is exactly the curved orbit (corresponding to the P-orbit $Pg^{-1}H$) from the reduction above, translated by the action of g_0.

As for the structures on the curved orbits, we rewrite the curved orbit as $(G/P)_{Pg^{-1}H} = g_0HgP = \operatorname{conj}_{g_0}(H)g_0gP$. It is easy to see that $\operatorname{conj}_{g_0}(H)$ is again a subgroup of G that stabilizes an element of $\mathcal{O}$. We abbreviate $H_0 := \operatorname{conj}_{g_0}(H)$.

We see immediately that we can write down the curved orbit as a homogenous space: $H_0 \cdot g_0gP = H_0/\operatorname{Stab}_{H_0}(g_0gP)$ for $gP \in G/P$. An element $g_0hg_0^{-1}$ of H_0, where $h \in H$, stabilizes g_0gP if and only if $g_0hg_0^{-1}g_0gP = g_0gP$, hence if h stabilizes gP. From above, we know that $\operatorname{Stab}_H(gP) = H \cap gPg^{-1}$. Therefore, $\operatorname{Stab}_{H_0}(g_0gP) = g_0 \operatorname{Stab}_H(gP)g_0^{-1} = g_0(H \cap gPg^{-1})g_0^{-1} = H_0 \cap (g_0g)P(g_0g)^{-1}$.

In summary, we have proved the following

Proposition 4.4.5. *The curved orbit of P-type $Pg^{-1}H$ induced by the holonomy reduction $G/P \times g_0H$ is of the form $g_0HgP \subset G/P$. It can be written as*

$$\begin{aligned}\operatorname{conj}_{g_0}(H) \cdot g_0gP &= \operatorname{conj}_{g_0}(H)/\operatorname{conj}_{g_0}(H \cap gPg^{-1})\\ &= \operatorname{conj}_{g_0}(H)/(\operatorname{conj}_{g_0}(H) \cap (g_0g)P(g_0g)^{-1}).\end{aligned}$$

Therefore it suffices to consider one holonomy reduction of a certain type of the homogenous model. The curved orbits of all other such holonomy reductions are obtained by the left action of an element of G. Their geometric structure is isomorphic to that of the original curved orbit by conjugation with the respective G-element.

4.5 The Curved-Orbit-Decomposition

The next section, that contains our main theorem, will explain the name "curved-orbit-decomposition". In order to to compare different Cartan geometries of the same type, we start with the "Comparison-Lemma":

Lemma 4.5.1 (Comparison)**.** *Let $(\pi : \mathcal{G} \to M, \omega)$ and $(\pi' : \mathcal{G}' \to M', \omega')$ be Cartan geometries of type (G, P), and $s : \hat{\mathcal{G}} \to \mathcal{O}$ and $s' : \hat{\mathcal{G}}' \to \mathcal{O}$ the equivariant functions corresponding to holonomy reductions of $\mathcal{G}$ and $\mathcal{G}'$ of the same type $\mathcal{O}$. For a P-orbit $i \in P\backslash\mathcal{O}$, such that the curved orbits M_i, M_i' are not empty, and for $x \in M_i$ and $x' \in M_i'$ there exist*

(i) a local diffeomorphism ϕ between open neighborhoods U, U' of x and x', respectively, such that $\phi(x) = x'$; and

(ii) a diffeomorphism $\Phi : \pi^{-1}(U) \to \pi'^{-1}(U')$ that is equivariant with respect to the P-actions, such that the following diagram is commutative:

$$\begin{tikzcd} U \arrow[d, "\phi"'] & \pi^{-1}(U) \arrow[l, "\pi"'] \arrow[r, "s"] \arrow[d, "\Phi"] & \mathcal{O} \\ U' & (\pi')^{-1}(U') \arrow[l, "\pi'"] \arrow[ur, "s'"'] & \end{tikzcd}$$

(iii) Moreover, $\phi(M_j \cap U) = M_j' \cap U'$ for all $j \in P\backslash\mathcal{O}$.

Proof. We will construct charts for M that are nicely compatible with the curved orbits. These will allow to compare the two curved orbit decompositions.

First of all, choose a linear subspace $\mathfrak{g}_- \subset \mathfrak{g}$, such that $\mathfrak{g}_- \oplus \mathfrak{p} = \mathfrak{g}$ as a vector space. Recall that each $X \in \mathfrak{g}_-$ induces a vector field $\omega^{-1}(X) \in \mathfrak{X}(\mathcal{G})$.

Let $\alpha \in i$ and choose $u_0 \in \mathcal{G}$, such that $\pi(u_0) = x$ and $s(u_0) = \alpha$. Then consider the map

$$\Psi : X \mapsto \mathrm{Fl}_1^{\omega^{-1}(X)}(u_0).$$

<u>*Claim 1:*</u> There is an open neighborhood W of 0 in $\mathfrak{g}_-$, such that $\Psi : X \mapsto \mathrm{Fl}_1^{\omega^{-1}(X)}(u_0)$ defines a smooth map $W \to \mathcal{G}$ and $\psi := \pi \circ \Psi$ is a diffeomorphism onto an open subset of M.

proof of claim: Consider the manifold $\mathcal{G} \times \mathfrak{g}$ and the vector field $\Xi \in \mathfrak{X}(\mathcal{G} \times \mathfrak{g})$ given by $\Xi(u, X) := \left(\omega(X)^{-1}(u), 0\right)$ where $u \in \mathcal{G}$ and $X \in \mathfrak{g}$. The flow of Ξ at $(u, X) \in \mathcal{G} \times \mathfrak{g}$ is given by $c(t) := (c_1(t), c_2(t)) := \left(\mathrm{Fl}_t^{\omega^{-1}(X)}(u), X\right)$, since, $c(0) = \left(\mathrm{Fl}_0^{\omega^{-1}(X)}(u), X\right) = (u, X)$, and the derivative of c is given by

$$c'(t) = \left(\omega^{-1}(X)\left(\mathrm{Fl}_t^{\omega^{-1}(X)}(u)\right), 0\right) = \left(\omega^{-1}(c_2(t))(c_1(t)), 0\right).$$

Now consider $(u_0, 0) \in \mathcal{G} \times \mathfrak{g}$. There is an open neighborhood $\tilde{W}$ of $(u_0, 0)$ in $\mathcal{G} \times \mathfrak{g}$ and an $\varepsilon > 0$ such that the flow of Ξ exists for all elements of $\tilde{W}$ at time $t < \varepsilon$. Let $\bar{W} := \tilde{W} \cap (\{u_0\} \times \mathfrak{g}_-)$. This shows existence of the above claimed neighborhood of 0 in $\mathfrak{g}_-$. Also, since flows are smooth, the map $\Psi : \bar{W} \to \mathcal{G}$ given by $\Psi(X) := \mathrm{pr}_2\left(\mathrm{Fl}_1^{\Xi}(u_0, X)\right) = \mathrm{Fl}_1^{\omega^{-1}(X)}(u_0)$ is smooth.

In order to show that ψ is a diffeomorphism, consider $T_0\psi : \mathfrak{g}_- \to T_xM$. Suppose for $Y \in \mathfrak{g}_-$ that $T_0\psi \cdot Y = 0$, then $0 = T_{u_0}\pi \cdot T_0\Psi \cdot Y$. Hence $T_0\Psi \cdot Y$ is vertical, but

$$\begin{aligned} 0 = T_0\Psi \cdot Y = \frac{d}{dt}|_0 \left(\mathrm{Fl}_1^{\omega^{-1}(t \cdot Y)}(u_0)\right) &= \frac{d}{dt}|_0 \left(\mathrm{Fl}_1^{t \cdot \omega^{-1}(Y)}(u_0)\right) \\ &= \frac{d}{dt}|_0 \left(\mathrm{Fl}_t^{\omega^{-1}(Y)}(u_0)\right) = \omega^{-1}(Y)(u_0), \end{aligned}$$

so $Y \in \mathfrak{p}$, therefore $Y = 0$. Thus $T_0\psi$ is injective, and by dimensional reasons it is also bijective. This shows that ψ is a local diffeomorphism around 0, hence we can shrink $\bar{W}$ to W, a neighborhood around 0, on which ψ is a diffeomorphism. *end of proof*

Let $U := \psi(W) \subset M$, and $\tau : U \to \mathcal{G}$ and $\hat{\tau} : U \to \hat{\mathcal{G}}$ such that $\tau(\psi(X)) = \Psi(X)$ and $\hat{\tau}(\psi(X)) = \iota(\Psi(X)) \cdot \exp(-X)$ for $X \in \mathfrak{g}_-$. The map τ is not only smooth but also a local section of $\mathcal{G}$, since by definition $\pi(\tau(\psi(X))) = \pi(\Psi(X)) = \psi(X)$, and the same argument shows that also $\hat{\tau}$ is a section.

Then fix $X \in W \subset \mathfrak{g}_-$ and consider the curve $c : [0, 1] \to \hat{\mathcal{G}}$, $c(t) = \hat{\tau}(\psi(tX))$, that is well-defined and smooth.

<u>*Claim 2:*</u> The curve $\tilde{c}$ is horizontal.
proof of claim: By definition,

$$\tilde{c}(t) = \iota\left(\mathrm{Fl}_1^{\omega^{-1}(tX)}(u_0)\right) \cdot \exp(-tX) = \iota\left(\mathrm{Fl}_t^{\omega^{-1}(X)}(u_0)\right) \cdot \exp(-tX),$$

thus

$$\begin{aligned}\hat{\omega}_{\tilde{c}(t)}\left(\tilde{c}'(t)\right) &= \hat{\omega}_{\tilde{c}(t)}\left(T\rho\left(T\iota\cdot\omega^{-1}(X)\left(\mathrm{Fl}_t^{\omega^{-1}(X)}(u_0)\right), T_{-tX}\exp\cdot(-X)\right)\right)\\ &= \hat{\omega}_{\tilde{c}(t)}\left(T\rho_{\mathrm{Fl}_t^{\omega^{-1}(X)}(u_0)}\cdot\frac{d}{dt}\left(\exp(-tX)\right)\right)\\ &\quad+\ \hat{\omega}_{\tilde{c}(t)}\left(T\rho^{\exp(-tX)}\cdot\omega^{-1}(X)\left(\mathrm{Fl}_t^{\omega^{-1}(X)}(u_0)\right)\right)\end{aligned}$$

Using equivariancy of $\hat{\omega}$, the first summand gives

$$\begin{aligned}&\hat{\omega}_{\iota\left(\mathrm{Fl}_t^{\omega^{-1}(X)}(u_0)\right)\cdot\exp(-tX)}\left(T\rho_{\iota\left(\mathrm{Fl}_t^{\omega^{-1}(X)}(u_0)\right)}\cdot\frac{d}{ds}\left(\exp(-sX)\right)\right)\\ &= \hat{\omega}_{\iota\left(\mathrm{Fl}_t^{\omega^{-1}(X)}(u_0)\right)}\left(T\rho_{\iota\left(\mathrm{Fl}_t^{\omega^{-1}(X)}(u_0)\right)}\cdot T\rho^{\exp(tX)}\cdot\frac{d}{ds}\left(\exp(-sX)\right)\right)\\ &= \hat{\omega}_{\iota\left(\mathrm{Fl}_t^{\omega^{-1}(X)}(u_0)\right)}\left(T\rho_{\iota\left(\mathrm{Fl}_t^{\omega^{-1}(X)}(u_0)\right)}\cdot\frac{d}{ds}|_{s=t}\exp\left((t-s)X\right)\right)\\ &= \hat{\omega}_{\iota\left(\mathrm{Fl}_t^{\omega^{-1}(X)}(u_0)\right)}\left(T\rho_{\iota\left(\mathrm{Fl}_t^{\omega^{-1}(X)}(u_0)\right)}\cdot(-X)\right)\\ &= \hat{\omega}_{\iota\left(\mathrm{Fl}_t^{\omega^{-1}(X)}(u_0)\right)}\left(\zeta_{-X}\left(\iota\left(\mathrm{Fl}_t^{\omega^{-1}(X)}(u_0)\right)\right)\right) = -X,\end{aligned}$$

whereas the second summand results in

$$\begin{aligned}&\mathrm{Ad}\left(\exp(-tX)\right)\hat{\omega}_{\tilde{c}(t)}\left(\omega^{-1}(X)\left(\mathrm{Fl}_t^{\omega^{-1}(X)}(u_0)\right)\right) = \mathrm{Ad}\left(\exp(-tX)\right)\cdot X\\ &\quad= T_e\,\mathrm{conj}_{\exp(tX)}(X) = \frac{d}{ds}|_{s=0}\,\mathrm{conj}_{\exp(tX)}\left(\exp(sX)\right)\\ &\quad= \frac{d}{ds}|_{s=0}\exp\left((t+s-t)X\right) = X.\end{aligned}$$

Therefore $\tilde{c}$ is horizontal. *end of proof*

Since s is constant along horizontal curves, we see that

$$\alpha = s\left(u_{u_0}\right) = s\left(\tilde{c}(0)\right) = s\left(\tilde{c}(1)\right) = s\left(\Psi(X)\cdot\exp(-X)\right) = s\left(\Psi(X)\right)\exp(-X)$$

and therefore $s\left(\Psi(X)\right) = \alpha\cdot\exp(X)$.

Now, we can apply the same procedure to $\pi' : \mathcal{G}' \to M'$ for $u_0' \in \mathcal{G}'$ with $s'\left(u_0'\right) = \exp(-X)\cdot\alpha$ and $\pi'\left(u_0'\right) = x'$, and we obtain again $s'\left(\Psi'(X)\right) = \alpha$. By shrinking W, such that both ψ and ψ' are diffeomorphisms on W, define $\phi := \psi'\circ\psi^{-1}$. Since $\psi(0) = \pi\left(\Psi(0)\right) = \pi(u) = x$ we get $\phi(x) = \psi'\left(\psi^{-1}(x)\right) = \psi'(0) = x'$. That proves (i).

There is a uniquely determined, P-equivariant diffeomorphism $\Phi : \pi^{-1}(U) \to \pi'^{-1}(U')$ that satisfies $\Phi\circ\tau = \tau'\circ\phi$. Indeed, both τ and τ' are injective, so we can just map $\tau(\tilde{x}) \mapsto \tau'\left(\phi(\tilde{x})\right)$ and extend this equivariantly.

Because $s'\left(\Psi'(X)\right) = \exp(-X)\cdot\alpha = s\left(\Psi(X)\right)$, we have $s\circ\Psi = s'\circ\Phi\circ\Psi$ and therefore, using the injectivity of Ψ and Ψ', $s = s'\circ\Phi$. This proves (ii).

Finally, since Φ covers ϕ, we have $\Phi\left(\mathcal{G}_{\tilde{x}}\right) = \mathcal{G}'_{\phi(\tilde{x})}$ for any $\tilde{x} \in U$, so the P-orbits $s\left(\mathcal{G}_{\tilde{x}}\right)$ and $s'\left(\mathcal{G}'_{\phi(\tilde{x})}\right)$ coincide. This completes the proof of (ii).

□

So far, we see that the decomposition of the underlying manifolds into orbits locally looks alike up to diffeomorphism. Using this, we will compare an arbitrary Cartan geometry with its homogenous model to verify that curved orbits are initial submanifolds and to carry over geometric structures to them.

The pair consisting of H_α and its distinguished subgroup $P_\alpha := G_\alpha \cap P \subset G_\alpha$ is isomorphic to $(G_{\alpha'}, P_{\alpha'})$ by g-conjugation and hence may be identified. We denote elements of this isomorphism class by (G_i, P_i).

Before we come to our main result, we recall the notion of an initial submanifold:

Remark 4.5.2. *Recall that a k-dimensional initial submanifold is a subset N of a smooth, n-dimensional manifold M, such that for all $x \in N$ there is a chart $\varphi : U \to U'$ around x in M, with $\varphi\left((U \cap N)_x\right) = U' \cap \left(\mathbb{R}^k \times \{0\}\right)$. For an arbitrary subset $A \subset M$, A_x denotes the set of all $y \in A$ that can be joined to x with a smooth curve in M lying in A.*

Lemma 4.5.3. *Let M be a smooth manifold endowed with a G-action λ, where G is a Lie group. Then, the G-orbits are initial submanifolds of M.*

Proof. see [9, p.47]

Theorem 4.5.4. *Let $(\mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G, H) endowed with a holonomy reduction of type $\mathcal{O}$ such that $s : \hat{\mathcal{G}} \to \mathcal{O}$ is its corresponding equivariant function. Let $M_i \subset M$ be a non-empty curved orbit where $i \in P\backslash\mathcal{O}$.*

(i) Locally, the curved orbit M_i looks like an orbit of a group action: For $\alpha \in i$ and $x \in M$, there exist open neighborhoods U of x and U' of $eP \in G/P$ and a diffeomorphism $\varphi : U \to U'$ such that $\varphi(x) = eP$ and $\varphi(M_i \cap U) = (G_\alpha \cdot eP) \cap U'$. Therefore, M_i is an initial submanifold.

(ii) The curved orbit M_i itself carries a canonical Cartan geometry structure $(\mathcal{G}_i \to M_i, \omega_i)$ of type (G_i, P_i). If we choose a representative $\alpha \in i$, we obtain an embedding $\iota_\alpha : \mathcal{G}_i \to \mathcal{G}\mid_{M_i}$ such that $\iota_\alpha^* \omega = \omega_i$.

(iii) By choosing $\alpha \in i$, we can relate the curvatures K of $\mathcal{G}$ and K_i of $\mathcal{G}_i$ via $K_i = \iota_\alpha^* K$, whereas the curvature functions satisfy for $u \in \mathcal{G}_\alpha$

$$\kappa\left(\iota_\alpha(u)\right)\Big|_{\bigwedge^2 \mathfrak{g}_\alpha/\mathfrak{g}_\alpha \cap \mathfrak{p}} = \kappa_i(u).$$

Proof. (i) By using Lemma 4.5.1 we can compare our given Cartan geometry with its homogenous model $G \to G/P$ carrying the holonomy reduction $s' : \hat{G} \to \mathcal{O}$ that is determined by $s'(eP, e) = \alpha$. Now since eP lies inside the curved orbit $(G/P)_i$, the Lemma provides a diffeomorphism ϕ between open neighborhoods U and U' of x and eP, respectively, that maps x to eP and satisfies $\varphi(M_i \cap U) = (G/P)_i \cap U'$. But as we saw in 4.4.3 (ii) the curved orbit $(G/P)_i$ is exactly $G_\alpha \cdot eP \subset G/P$.

Lemma 4.5.3 shows that therefore, the curved orbits are initial submanifolds.

(ii) First note that we can pullback $\mathcal{G}$ and $\hat{\mathcal{G}}$ by the inclusion $j_i : M_i \hookrightarrow M$. Generally, the pullbacks of principal fiber bundles are again principal fiber bundles of the same structure group, since the fibers remain the same. Now we start with a fixed $\alpha \in i$. The holonomy reduction can be equivalently described by the reduction $s^{-1}(\alpha) \subset \hat{\mathcal{G}}$. Now form the pullback $\mathcal{G}_\alpha := {j_i}^* \left(\iota^{-1}\left(s^{-1}(\alpha)\right)\right) \subset {j_i}^* \mathcal{G}$.

Claim 1: $\mathcal{G}_\alpha$ is a P_α-principal bundle over M_i.
proof of claim: As in (i) we compare $\mathcal{G}$ with its homogenous model. Lemma 4.5.1 provides a local, P-equivariant diffeomorphism $\Phi : \hat{\pi}^{-1}(U) \to \hat{\pi}'^{-1}(U')$, where U and U' are open neighborhoods of $x \in M_i$ and $eP \in G/P$ that satisfy $s^{-1}(\alpha) = \Phi\left(s'^{-1}(\alpha)\right)$. Moreover, $\mathcal{G}$ is embedded into $\hat{\mathcal{G}}$ hence we may w.l.o.g. assume $\mathcal{G}$ to be the homogenous model G. The holonomy reduction has the form $s(gP, g') = (g')^{-1} \cdot \alpha$, so $s^{-1}(\alpha) = G/P \times G_\alpha$. Since $\iota(g) = (gP, g)$, we have $\iota^{-1}\left(s^{-1}(\alpha)\right) = G_\alpha$. This is a P_α-reduction of G, thus also its pullback bundle $j_i^* G_\alpha$. *end of proof*

Claim 2: The $\mathfrak{g}_\alpha$-valued 1-form $\omega_\alpha := {j_i}^* \omega \in \Omega^1({j_i}^* \mathcal{G}_\alpha, \mathfrak{g}_\alpha)$ is a Cartan connection of type (G_α, P_α).
proof of claim: First consider the values of $\hat{\omega}$ on $\mathcal{H} := s^{-1}(\alpha) \subset \hat{\mathcal{G}}$. By parallelity of s we have the horizontal distribution $\mathfrak{H}_u \subset T_u s^{-1}(\alpha)$ for $u \in \mathcal{H}$. Hence by dimensional reasons $T_u \mathcal{H} = \mathfrak{H}_u \oplus V_u s^{-1}(\alpha)$ and since $s^{-1}(\alpha)$ is a G_α-reduction of $\hat{\mathcal{G}}$, the principal connection $\hat{\omega}$ takes values only in $\mathfrak{g}_\alpha$. Through $\omega = \iota^* \hat{\omega}$ we see that also $j_i^* \omega$ has values in $\mathfrak{g}_\alpha$.

Equivariance of ω_α can be computed straightforwardly. The inclusion j_i is by definition equivariant, so we have for $u \in \mathcal{G}_\alpha$, $p \in P_\alpha$ and $\xi \in T_u \mathcal{G}_\alpha$

$$\begin{aligned}\left((\rho_p)^* (\omega_\alpha)\right)_u (\xi) &= (\omega_\alpha)_{up} (T_u \rho_p \cdot \xi) = \omega_{j_i(up)} (T_{up} j_i \cdot T_u \rho_p \cdot \xi) \\ &= \omega_{j_i(u)p} (T_u \rho_p \cdot T_u j_i \cdot \xi) = \mathrm{Ad}\left(p^{-1}\right) \cdot (\omega_\alpha)_u (\xi).\end{aligned}$$

The proof of the reproduction property again uses the equivariance of j_i as well as the properties of ω: Let $u \in \mathcal{G}_\alpha$ and $X \in \mathfrak{g}_\alpha$ then

$$(\omega_\alpha)_u (\zeta_X(u)) = \omega_{j_i(u)} (T_u j_i \cdot T_e \rho^u \cdot X) = \omega_{j_i(u)} \left(T_e \rho^{j_i(u)} \cdot X\right) = X.$$

Finally, we have to show that ω_α is an absolute parallelism. $(\omega_\alpha)_u : T_u \mathcal{G}_\alpha \to \mathfrak{g}_\alpha$ is injective, since $T_u j_i$ and $\omega_{j_i(u)}$ are, but the dimension of $\mathcal{G}_\alpha$ must be the same as G_α, its homogenous model, hence $(\omega_\alpha)_u$ is also surjective. *end of proof*

So far we chose a fixed α in the orbit i. If we take another $\alpha' := p \cdot \alpha$, where $p \in P$, then we obtain another P_α-reduction $\mathcal{G}_{\alpha'}$ of $\mathcal{G}$. Note that there is a canonical Lie group isomorphism $\mathrm{conj}_p : G_\alpha \to G_{\alpha'}$, hence also a Lie algebra isomorphism $\mathrm{Ad}(p) : \mathfrak{g}_\alpha \to \mathfrak{g}_{\alpha'}$, such that

$$\begin{tikzcd} G_\alpha \arrow[r, "\mathrm{conj}_g"] \arrow[d, "\mathrm{Ad}"'] & G_{\alpha'} \arrow[d, "\mathrm{Ad}'"] \\ \mathrm{GL}(\mathfrak{g}_\alpha) \arrow[r, "\mathrm{conj}_{(T_e \mathrm{conj}_g)}"'] & \mathrm{GL}(\mathfrak{g}_{\alpha'}) \end{tikzcd}$$

commutes.

Hence the quadruples $(G_\alpha, P_\alpha, \mathfrak{g}_\alpha, \mathrm{Ad})$ and $\left(G_{\alpha'}, P_{\alpha'}, \mathfrak{g}_{\alpha'}, \mathrm{Ad}'\right)$ are canonically isomorphic. We will show that also the corresponding Cartan geometries $(\mathcal{G}_\alpha, \omega_\alpha)$ and $(\mathcal{G}_{\alpha'}, \omega_{\alpha'})$ are canonically isomorphic with respect to the isomorphism above, hence we may identify those Cartan geometries to obtain a geometry $\mathcal{G}_i \to M_i$.

Claim 3: The restriction of the action $\rho_{p^{-1}}\big|_{\mathcal{G}_\alpha} : \mathcal{G}_\alpha \to \mathcal{G}_{\alpha'}$ together with $\mathrm{conj}_p : P_\alpha \to P_{\alpha'}$ is an isomorphism of Cartan geometries.

proof of claim: We have to check for $u \in \mathcal{G}_\alpha = {j_i}^* \left(\iota^{-1}\left(s^{-1}(\alpha)\right)\right)$ that $up^{-1} \in \mathcal{G}_{\alpha'}$. The right action leaves fibers invariant, therefore it is compatible with the pullback. We compute for $u \in \mathcal{G}_\alpha$ and $p' \in P_\alpha$

$$s\left(\iota\left(up^{-1}\right)\right) = s\left(\iota(u)\, p^{-1}\right) = p \cdot s\left(\iota(u)\right) = p \cdot \alpha = \alpha',$$

hence $\rho_{p^{-1}}(\mathcal{G}_\alpha) = \mathcal{G}_{\alpha'}$, and $\rho_{p^{-1}}(up') = up'p^{-1} = up^{-1}pp'p^{-1} = \rho_{p^{-1}}(u) \cdot \mathrm{conj}_p(p')$.

Also, for $u \in \mathcal{G}_\alpha$ and $\xi \in T_u\mathcal{G}_\alpha$ we have

$$\begin{aligned}(\omega_{\alpha'})_{up^{-1}}\left(T\rho_{p^{-1}} \cdot \xi\right) &= \omega_{j_i(up^{-1})}\left(Tj_i \cdot T\rho_{p^{-1}} \cdot \xi\right) = \omega_{j_i(u)p^{-1}}\left(T\rho_{p^{-1}} \cdot Tj_i \cdot \xi\right)\\ &= \mathrm{Ad}(p) \cdot \omega_{j_i(u)}\left(Tj_i \cdot \xi\right) = \mathrm{Ad}(p) \cdot (\omega_\alpha)_u(\xi).\end{aligned}$$

end of proof

However, the different $\mathcal{G}_\alpha$'s in the isomorphism class $\mathcal{G}_i$ are embedded differently into $\mathcal{G}$. Each choice of $\alpha \in i$ yields an inclusion, that from now we call $\iota_\alpha : \mathcal{G}_\alpha \hookrightarrow \mathcal{G}$.

As for (iii), we use compatiblity of ω and ω_i and naturality of the exterior derivative to compute for $\xi, \eta \in T\mathcal{G}_\alpha$

$$\begin{aligned}\left(\iota_\alpha^* K\right)(\xi, \eta) =& d\omega\left(T\iota_\alpha \cdot \xi, T\iota_\alpha \cdot \eta\right) + \left[\omega\left(T\iota_\alpha \cdot \xi\right), \omega\left(T\iota_\alpha \cdot \eta\right)\right]\\ =& d\omega_i(\xi, \eta) + \left[\omega_i(\xi), \omega_i(\eta)\right] = K_i(\xi, \eta).\end{aligned}$$

We use $\iota_\alpha^* K = K_i$ and $\iota_\alpha^* \omega = \omega_i$ to calculate for $u \in \mathcal{G}_i$ and $X, Y \in \mathfrak{g}_\alpha$:

$$\begin{aligned}\kappa_i(u)(X, Y) =& (K_i)_u\left((\omega_i)_u^{-1}(X), (\omega_i)_u^{-1}(Y)\right)\\ =& K_{\iota_\alpha(u)}\left(T_u j_i \cdot \left(\omega_u \circ T_u j_i\right)^{-1}(X), T_u j_i \cdot \left(\omega_u \circ T_u j_i\right)^{-1}(Y)\right)\\ =& K_{\iota_\alpha(u)}\left(\omega_{\iota_\alpha(u)}^{-1}(X), \omega_{\iota_\alpha(u)}^{-1}(Y)\right) = \kappa\left(\iota_\alpha(u)\right)(X, Y)\end{aligned}$$

This completes the proof.

□

Chapter 5

Holonomy reductions of Riemannian Cartan geometries

First of all, we show that a Riemannian metric on a manifold M induces a Cartan geometry of type $(\mathrm{Euc}\,(n)\,, O\,(n))$ where $n \in \mathbb{N}$ is the dimension of M. Conversely, one can reconstruct the metric from the Cartan geometry. Thoughout this chapter, we will denote $G := \mathrm{Euc}(n)$ and $P := O(n)$.

We are interested in the question, what candidates for subgroups of $\mathrm{Euc}(n)$ for holonomy reductions there are, and what kind of structures they imply.

Before we start investigating different types of holonomy reductions, we collect some information about the Cartan description of Riemannian metrics.

5.1 Cartan Geometries of Riemannian type

We first show that a Cartan geometry of type (G, P) is equivalent to a Riemannian metric g on M:

Proposition 5.1.1. *Let M be an n-dimensional manifold. Then a Riemannian metric g on M is equivalent to a torsion-free Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, P).*

Proof. We saw in 3.2.1 that Riemannian metrics on M are equivalent to reductions of the structure group of the frame bundle $\mathcal{F}$ from $\mathrm{GL}(n, \mathbb{R})$ to $O(n)$. This is done by forming the orthonormal frame bundle with respect to g. The orthonormal frame bundle carries a canonical principal connection – the Levi-Civita-connection with respect to the underlying Riemannian metric. The corresponding connection on TM is torsion-free. Therefore, we have to show:

Claim: A Cartan geometry $(\mathcal{G} \to M, \omega)$ of type $(\mathrm{Euc}(n), O(n))$ is equivalent to a G-structure $j : \mathcal{F}_O \hookrightarrow \mathcal{F}_M$ with structure group $O(n)$ endowed with a connection $\gamma \in \Omega^1(\mathcal{F}_O, \mathfrak{o}(n))$.

Given a G-structure $\mathcal{F}_O$ with connection γ, define the Cartan geometry by $\mathcal{G} := \mathcal{F}_O$ and $\omega := (j^*\theta) + \gamma$, where $\theta \in \Omega^1(\mathcal{G}, \mathbb{R}^n)$ is the soldering form on $\mathcal{F}_M$.

Note that the adjoint representation of $\mathrm{Euc}(n)$ restricted to $O(n)$ is the standard representation on $\mathbb{R}^n$ and the adjoint representation of $O(n)$ on $\mathfrak{o}(n)$.

Thus ω is equivariant with respect to the action on $\mathcal{G}$, since θ and γ are. Furthermore, γ reproduces the generators of fundamental vector fields, whereas θ vanishes on the vertical subspace. And finally, suppose that for $u \in \mathcal{G}$ and $\xi \in T_u\mathcal{G}$ we have $\omega_u(\xi) = 0$. Therefore $\gamma_u(\xi) = 0$ and $\theta_u(\xi) = 0$. Since θ is strictly horizontal, we conclude $\xi \in V_u\mathcal{G}$. But γ reproduces the generators of fundamental vector fields, and thus $\xi = 0$. This shows that ω_u is injective, hence by dimensional reasons bijective. Altogether we have shown that ω is a Cartan connection on $\mathcal{G} \to M$.

On the other hand, suppose we have a given Cartan geometry $\mathcal{G}$. For $u \in \mathcal{G}$ consider the map $\mathrm{pr}^{\mathbb{R}^n} \circ \omega_u$ where $\mathrm{pr}^{\mathbb{R}^n} : \mathbb{R}^n \oplus \mathfrak{o}(n) \to \mathbb{R}^n$ denotes the projection on the $\mathbb{R}^n$-component. This map factorizes over $T_u\pi$ to a map $\widetilde{\mathrm{pr}^{\mathbb{R}^n} \circ \omega_u} : T_{\pi(u)}M \to \mathbb{R}^n$. Taking the inverse of this map, we obtain an element of the frame bundle of M. Then denote the map $u \mapsto (\widetilde{\mathrm{pr}^{\mathbb{R}^n} \circ \omega_u})^{-1}$ by $j : \mathcal{G} \to \mathcal{F}$.

In order to show that j is injective, let $u \in \mathcal{G}$ and $g \in O(n)$. Then

$$\begin{aligned} j(ug) &= (\widetilde{\mathrm{pr}^{\mathbb{R}^n} \circ \omega_{ug}})^{-1} = (\mathrm{pr}^{\mathbb{R}^n} \circ \widetilde{\mathrm{Ad}(g^{-1})} \circ \omega_u)^{-1} \\ &= (g^{-1} \circ \widetilde{\mathrm{pr}^{\mathbb{R}^n}} \circ \omega_u)^{-1} = (g \circ (\widetilde{\mathrm{pr}^{\mathbb{R}^n} \circ \omega_u}))^{-1} = j(u) \circ g \end{aligned}$$

The above equation shows that j is a homomorphism, thus is injective. Furthermore, it is easy to see that $\gamma := \mathrm{pr}^{\mathfrak{o}(n)} \circ \omega$ is a principal connection by using the characterizing properties of the Cartan connection.

Finally, we have to show that above constructions are inverse: Start with a given Cartan geometry $\mathcal{G}$. We embed it into the frame bundle of M via j and define $\gamma := \mathrm{pr}^{\mathfrak{o}(n)} \circ \omega$. We have to show that $(j^*\theta) + \gamma = \omega$. The $\mathfrak{o}(n)$-part is clear by construction. As for the $\mathbb{R}^n$-part, we have for $u \in \mathcal{G}$

$$(j^*\theta)_u = \theta_{j(u)} \circ T_u j = (j(u))^{-1} \circ T_{j(u)}\pi \circ T_u j = (\widetilde{\mathrm{pr}^{\mathbb{R}^n} \circ \omega_u}) \circ T_u\pi = \mathrm{pr}^{\mathbb{R}^n} \circ \omega_u.$$

Conversely, starting with a G-structure $i : \mathcal{F}_O \hookrightarrow \mathcal{F}_M$, we have to show that the embedding j obtained from $\omega := i^*\theta + \gamma$ is the same as i. Let $u \in \mathcal{F}_O$, then

$$\begin{aligned} j(u) &= (\widetilde{\mathrm{pr}^{\mathbb{R}^n} \circ \omega_u})^{-1} = (\widetilde{i^*\theta})^{-1} = (i(u)^{-1} \circ \widetilde{T_{i(u)}\pi} \circ T_u i)^{-1} \\ &= (i(u)\widetilde{^{-1} \circ T_u}\pi)^{-1} = i(u). \end{aligned}$$

Finally, by recalling 3.3.5, we conclude that the Cartan geometry has to be torsion-free. $\square$

5.1.1 The groups $\mathrm{Euc}(n)$ and $O(n)$

We first describe the groups G and P in a coordinate-independent way. Note that G (and in particular P) acts canonically on $\mathbb{R}^{n+1}$ in the following manner: There is an injective group homomorphism

$$O(n) \ltimes \mathbb{R}^n \hookrightarrow \mathrm{GL}\,(n+1, \mathbb{R})\,,\ (A, v) \mapsto \begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix},$$

a matrix with blocks of sizes 1 and n. This admits the standard representation on $\mathbb{R}^{n+1}$.
Whenever we compute with elements of G, we will assume them to be given in the above block-matrix-form.

Notation We will denote the vector space $\mathbb{R}^{n+1}$ by V and its standard basis by $\{e_0, \ldots, e_n\}$.

Proposition 5.1.2. *(i) There is a $\varphi_0 \in V^*$ that is G-invariant. In particular, the G-action leaves* $\ker(\varphi_0)$ *invariant.*

(ii) There is an inner product on $\ker(\varphi_0)$ *that is invariant under G.*

(iii) There is a $v_0 \in V$ such that $P = \mathrm{Stab}_G(v_0)$ and $\varphi_0(v_0) = 1$.

Proof. Let $g := \begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix} \in G$ where $v \in \mathbb{R}^n$ and $A \in O(n)$.

(i) For $\varphi_0 := \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix} \in \mathbb{R}^{n+1*}$ we have

$$g \cdot \varphi_0 = \varphi_0 \circ g^{-1} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -v & A^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

(ii) In coordinates $e_0, \ldots, e_n$ we have $\ker(\varphi_0) = \{0\} \times \mathbb{R}^n$, i.e. for $w \in \ker(\varphi_0)$ we have

$$\begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix} \cdot \begin{pmatrix} 0 \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ Aw \end{pmatrix}$$

Therefore the restricted action is exactly the standard $O(n)$-action on $\mathbb{R}^n$, and this leaves the standard inner product invariant.

(iii) Let $v_0 := e_0 \in \mathbb{R}^{n+1}$, then clearly $\varphi_0(v) = 1$. An element $g := \begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix} \in G$ acts trivially on v, if and only if $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ v \end{pmatrix}$. This is equivalent to $v = 0$, hence $g \in P$.

$\square$

The above proposition implies the following

Corollary 5.1.3. *The vector space V decomposes into $\langle v_0 \rangle \oplus \ker(\varphi_0)$ as a P-module.*

Proof. From 5.1.2(iii) we know that $v_0 \notin \ker(\varphi_0)$, hence by dimensional reasons $\langle v_0 \rangle \oplus \ker(\varphi_0) = V$. Furthermore, v_0 is P-invariant, and $\ker(\varphi_0)$ is even G-invariant, therefore in particular P-invariant. Conseqently, the P-action preserves the decomposition. $\square$

5.1.2 The canonical tractor bundle

In the following we will construct canonical objects from $(\pi : \mathcal{G} \to M, \omega)$.

Notation If we consider two or more Cartan geometries at once, we expand the notation by a ^{M} where M is the underlying manifold of the Cartan geometry, in order to avoid ambiguities.

We have given the principal bundles $\mathcal{G}$ and $\hat{\mathcal{G}}$. This allows us to form the canonical associated bundle $\mathcal{T} := \mathcal{G} \times_P V = \hat{\mathcal{G}} \times_G V$ and to transfer the principal connection $\hat{\omega}$ from $\hat{\mathcal{G}}$ to a linear connection ∇ on $\mathcal{T}$.

Recall that the subspace $\ker(\varphi_0)$ carries a natural inner product, coming from the G-action (see 5.1.2). This yields the following

Proposition 5.1.4. *Let $\mathcal{G} \to M$ be a Cartan geometry of type (G, P). Then the standard tractor bundle $\mathcal{T}$ admits the decomposition*

$$\mathcal{G} \times_P V = (\mathcal{G} \times_P \langle v_0 \rangle) \oplus (\mathcal{G} \times_P \ker(\varphi_0)) .$$

Furthermore, $\mathcal{G} \times_P \ker(\varphi_0)$ inherits a canonical metric h, and we have

(i) $(\mathcal{G} \times_P \ker(\varphi_0), h) = (TM, g)$, where g denotes the Riemannian metric on M, and

(ii) $\mathcal{G} \times_P \langle v_0 \rangle = M \times \mathbb{R}$. We will denote this line bundle by $\mathcal{L}$ and its canonical section $\pi(u) \mapsto [u, v_0]$, where $u \in \mathcal{G}$, by $\mathbb{1}$.

Proof. We know from 5.1.3 that the decomposition is P-invariant, hence it carries over to the associated bundle.

(i) Note that $\mathcal{G}$ is the orthonormal frame bundle of (M, g) (see 5.1.1). There is a canonical isomorphism $\mathcal{G} \times_P \ker(\varphi_0) \to TM$ given by $[u, y] \mapsto u(y)$ where $u \in \mathcal{G}$ and $y \in \ker(\varphi_0) = \mathbb{R}^n$. This is well-defined, smooth and commutes with the projections. It is fiberwise linear, since $u \in \mathcal{G}$ is a linear isometry $\mathbb{R}^n \to T_{\pi(u)}M$. Bijectivity also follows from that as well.

Furthermore, the isomorphism is an isometry with respect to the induced metric h on $\mathcal{G} \times_P \ker(\varphi_0)$ and the Riemannian metric g on TM: The metric h is defined by $h([u, x], [u, y]) := \langle x, y \rangle$, where $u \in \mathcal{G}$, $x, y \in \ker(\varphi_0) = \mathbb{R}^n$ and $\langle\, , \rangle$ denotes the standard inner product on $\mathbb{R}^n$. Thus, $h([u, x], [u, y]) = \langle x, y \rangle = g_{\pi(u)}(u(x), u(y))$, since u is an isometry.

(ii) The isomorphism $\mathcal{G} \times_P \langle v_0 \rangle \to M \times \mathbb{R}$ is given by $[u, v] \mapsto (\pi(u), \varphi_0(v))$ where $v \in \langle v_0 \rangle$. It is well-defined, commutes with the the projections and is smooth. Furthermore, it is fiberwise a linear isomorphism. □

5.1.3 The homogenous model

In this section, we will consider the tractor bundle of the homogenous model of the Cartan geometry of type (G, P), that is the space $\mathbb{R}^n = G/P$ endowed with the Euclidean metric.

In the next lemma we note that associated bundles of the homogenous model are trivial:

Lemma 5.1.5. *Let (G, P) be a pair of Lie-groups such that P is closed in G and $\bar{\rho} : G \to \mathrm{GL}(V)$ is a representation of G on an n-dimensional vector-space V.*

(i) The associated bundle satisfies $G \times_P V = (G/P) \times V$ via $[g, v] \mapsto (gP, \bar{\rho}(g)v)$ where $g \in G$ and $v \in V$.

(ii) The flat connection on $G \times_P G$ with respect to the canonical trivialization $G/P \times G$, induces the flat connection on $G \times_P V$ with respect to the trivialization from (i).

Proof. (i) From 1.2.4 we know that $G \times_P V = (G \times_P G) \times_G V$. Also, $G \times_P G$ can be canonically trivialized (see 4.4.1(i)) as $G \times_P G = (G/P) \times G$, hence we have $G \times_P V = (G/P \times G) \times_G V$.

Let $g, g' \in G$ and consider the map $(G/P \times G) \times_G V \to G/P \times V$, given by $[(gP, g'), v] \mapsto (gP, (g')\, v)$. This is well defined, since $[(gP, g'g''), (g'')^{-1} v]$ is mapped to $(gP, g'g''(g'')^{-1} v) = (gP, g'v)$ for $g'' \in G$. The map is fiberwise linear. Furthermore, if $g'v = 0$ we have $v = 0$, thus the map is fiberwise injective, hence bijective. Thus we obtain a vector bundle isomorphism between $G \times_P V$ and $G/P \times V$.

In order to write down the isomorphism explicitely, let $[g, v] \in G \times_P V$. This is identified with $[[g, e], v] \in (G \times_P G) \times_G V$. The element $[g, e]$ corresponds to(gP, g) in the trivialization. Therefore, we have $(G/P \times G) \times_G V \ni [(gP, g), v] = [(gP, e), gv]$ and this is mapped to $(gP, gv) \in G/P \times V$.

(ii) Let $\sigma \in \Gamma(G \times_P V)$. On the one hand, σ corresponds to an equivariant function $f : (G/P \times G) \to V$ that is characterized by $\sigma(gP) = [(gP, g'), f(gP, g')]$ for $g, g' \in G$. On the other hand, since $G \times_P V$ is trivial, the section σ can be interpreted as a smooth function $\bar{f} : G/P \to V$.

Then $\sigma(gP) = [(gP, g'), f(gP, g)]$ is under the trivialization from (i) given by $(gP, g' \cdot f(gP, g')) = (gP, f(gP, e))$, where the last equality follows from equivariance of f. Hence we obtain $\bar{f}(gP) = f(gP, e)$.

Let $\xi \in \mathfrak{X}(G/P)$, then its flat horizontal lift to $G \times_P G = G/P \times G$ is given by $(\xi, 0) \in T(G/P) \times TG$. Thus, the induced connection ∇ is computed as follows: The section $\nabla_\xi \sigma$ corresponds to the function $Tf \cdot (\xi, 0) = T\bar{f} \cdot \xi$. The last term gives the derivative of σ with respect to the flat connection on $G \times_P V = G/P \times V$.

Therefore, these two connections coincide. $\square$

In particular, the standard tractor bundle of the homogenous model is given by $\mathcal{T} = G \times_P V = G/P \times V = \mathbb{R}^n \times \mathbb{R}^{n+1}$. Therefore, we will interpret sections of $\mathcal{T}$ as smooth functions $G/P \to V$.

Next, we will compute the line bundle $\mathcal{L}$ in the tractor bundle:

Proposition 5.1.6. *Under the canonical trivialization Φ of the canonical tractor bundle $\mathcal{T}$ (see 5.1.5), the line bundle $\mathcal{L}$ is given by*

$$\left\{ \left(v, \begin{pmatrix} \lambda \\ \lambda v \end{pmatrix} \right) \in \mathbb{R}^n \times \mathbb{R}^{n+1} \mid v \in \mathbb{R}^n, \lambda \in \mathbb{R} \right\}.$$

Proof. Let $g \in G$ be of the form $\mathrm{g} = \begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix}$ where $v \in \mathbb{R}^n$ and $A \in O(n)$. Then $gP \in G/P$ corresponds to $v \in \mathbb{R}^n$.

The line bundle is given by $\mathcal{L} := G \times_P \langle v_0 \rangle$. The element $[g, \lambda v_0] \in \mathcal{L}$, where $\lambda \in \mathbb{R}$, is mapped by Φ to $(gP, g\lambda v_0) = (gP, \lambda g v_0)$.
Here we have $\lambda g v_0 = \lambda \begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda v \end{pmatrix}$. Hence, $(gP, \lambda g v_0) = \left(v, \begin{pmatrix} \lambda \\ \lambda v \end{pmatrix} \right)$.
This shows the claim. $\square$

5.1.4 The tractor connection

Next, we compute the tractor connection in the decomposition of Proposition 5.1.4:

Proposition 5.1.7. *For a Cartan geometry $(\pi : \mathcal{G} \to M, \omega)$ of type $(\mathrm{Euc}(n), O(n))$, and its canonical tractor bundle $\mathcal{T} = \mathcal{G} \times_P \mathbb{R}^{n+1} = \mathcal{L} \oplus TM$ (cf. 5.1.4), the corresponding tractor connection is given by*

$$\nabla_\xi \begin{pmatrix} \phi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi \cdot \phi \\ \phi\xi + \nabla^{LC}_\xi \eta \end{pmatrix},$$

where $\phi \in C^\infty(M, \mathbb{R})$, $\xi, \eta \in \mathfrak{X}(M)$ and ∇^{LC} denotes the Levi-Civita-connection on TM.

Proof. We use the theorem 3.4.5 to prove the statement.

We consider the adjoint tractor bundle $\mathcal{A}M := \mathcal{G} \times_P \mathfrak{g} \to M$. The adjoint action of $O(n)$ on $\mathfrak{euc}(n)$ is given by

$$\mathrm{Ad}\,(A, (B, X)) = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 0 & 0 \\ X & B \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ AX & ABA^{-1} \end{pmatrix},$$

where $A \in O(n)$ and $(B, X) \in \mathfrak{euc}(n) = \mathfrak{o}(n) \oplus \mathbb{R}^n$ (as a vector space).

Let $\sigma \in \Gamma(\mathcal{T})$. Firstly, $\mathcal{T} = \mathcal{L} \oplus TM$, thus $\sigma = (\phi, \eta)$ for $\phi \in C^\infty(M, \mathbb{R})$ and $\eta \in \mathfrak{X}(M)$. We denote the equivariant function corresponding to σ by $f : \mathcal{G} \to \mathbb{R}^{n+1}$. Furthermore, we denote the function f composed with the projection on the first component by f_0, and f composed with the projection onto the last n components by f_n. Then we have $\eta(\pi(u)) = [u, f_n(u)]$ and $\phi(\pi(u)) = [u, f_0(u)]$ for $u \in \mathcal{G}$.

Recall that there is a surjective bundle map $\Pi : \mathcal{G} \times_P \mathfrak{g} \to \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p} = \mathcal{G} \times_P \mathbb{R}^n = TM$ that is given by $\Pi(\tau(\pi(u))) = \Pi([u, t(u)]) = [u, t(u) + \mathfrak{p}] = [u, X(u)] = u(X(u))$.

Now choose $\tau \in \Gamma(\mathcal{A}M)$ such that $\Pi \circ \tau = \xi$. We obtain an equivariant function $\mathcal{G} \to \mathfrak{euc}(n)$ corresponding to τ. Note that $\mathfrak{euc}(n) \cong \mathfrak{o}(n) \oplus \mathbb{R}^n$ as a vectorspace, thus the function has two components that we denote by $B : \mathcal{G} \to \mathfrak{o}(n)$ and $X : \mathcal{G} \to \mathbb{R}^n$.

Let us start by computing $\tau \bullet \sigma$. Therefore we have to take the derivative of the standard action ρ of $\mathrm{Euc}(n)$ on $\mathbb{R}^{n+1}$. Let $\begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix} \in \mathrm{Euc}(n)$ and $(\lambda, y) \in \mathbb{R}^{n+1}$, where $A \in O(n)$, $v, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.

We have

$$\left(\rho\left(\begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix}\right)\right) \begin{pmatrix} \lambda \\ y \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda v + Ay \end{pmatrix}.$$

Let $(B, X) \in \mathfrak{euc}(n)$ and $t \mapsto (A(t), v(t)) \in \mathrm{Euc}(n)$ a smooth curve such that $0 \mapsto e$ and the derivative equals (B, X) at 0. Then

$$\frac{d}{dt}|_0 \rho\left(\begin{pmatrix} 1 & 0 \\ v(t) & A(t) \end{pmatrix}\right) \begin{pmatrix} \lambda \\ y \end{pmatrix} = \frac{d}{dt}|_0 \begin{pmatrix} \lambda \\ \lambda v(t) + A(t)y \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda X + By \end{pmatrix}.$$

This map $\rho' : \mathfrak{g} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ carries over to the bundles $\mathcal{A}M$ and $\mathcal{T}$, a map $\bullet : \mathcal{A}M \times \mathcal{T} \to \mathcal{T}$. We obtain for $u \in \mathcal{G}$

$$(\tau \bullet \sigma)(\pi(u)) = [u, \rho'(t(u)), f(u)] = \left[u, \rho'\left(\begin{pmatrix} 0 & 0 \\ X(u) & B(u) \end{pmatrix}\right)\begin{pmatrix} f_0(u) \\ f_n(u) \end{pmatrix}\right]$$
$$= \left[u, \begin{pmatrix} 0 \\ f_0(u) X(u) + B(u) f_n(u) \end{pmatrix}\right] = \left[u, \begin{pmatrix} 0 \\ \phi(\pi(u)) X(u) + B(u) f_n(u) \end{pmatrix}\right],$$

since $f_0 = \phi \circ \pi$.

Secondly, we want to compute the fundamental derivative $D_\tau \sigma$. Recall that $\omega : T\mathcal{G} \to \mathcal{G} \times \mathfrak{g}$ gives a trivialization of the tangent bundle of $\mathcal{G}$, thus we have a bijective correspondence between vector fields on $\mathcal{G}$ and smooth functions $\mathcal{G} \to \mathfrak{g}$. If we restrict this correspondence to equivariant functions on $C^\infty(\mathcal{G}, \mathfrak{g})$ we obtain exactly the P-invariant vector fields on $\mathcal{G}$.

In our case, $\omega = \theta + \gamma$, where $\theta \in \Omega^1(\mathcal{G}, \mathbb{R}^n)$ is the soldering form on $\mathcal{G}$ and $\gamma \in \Omega^1(\mathcal{G})$ denotes the Levi-Civita-connection on $\mathcal{G}$. Note that γ extracts the vertical part of a tangent vector, whereas θ projects to the underlying manifold M, hence is completely determined by the horizontal component of the tangent vector.

Let $u \in \mathcal{G}$ and $\xi \in T\mathcal{G}$, then ξ is of the form $\zeta_B + (T_u\pi \cdot \xi)^{hor}$ where $B \in \mathfrak{o}(n)$. Then $\omega_u(\xi) = \left(B, u^{-1}\left(T_u\pi \cdot \xi^{hor}\right)\right)$.

Thus the inverse is given by $\omega^{-1}(B(u), X(u)) = \zeta_{B(u)} + (u(X(u)))^{hor}$.

We have to compute the derivative of f in direction $\zeta_{B(u)} + (u(X(u)))^{hor}$. We start with the component f_0 of f: By definition, $f_0 = \phi \circ \pi$, hence

$$T_u f_0 \cdot \left(\zeta_{B(u)} + (u(X(u)))^{hor}\right) = T_{\phi(u)}\phi \cdot T_u\pi \cdot \left(\zeta_{B(u)} + (u(X(u)))^{hor}\right)$$
$$= T_{\pi(u)}\phi \cdot (u(X(u))) = T_{\pi(u)}\phi \cdot (\Pi(\tau))(\pi(u)) = (\xi \cdot \phi)(\pi(u)).$$

The f_n-component of f yields the following:

$$T_u f_n \cdot (u(X(u)))^{hor} = T_u f_n \cdot \xi^{hor}(u).$$

Recall that this expression is by definition the equivariant function corresponding to $\nabla^{LC}_\xi \eta$.

Furthermore,

$$T_u f_n \cdot \zeta_{B(u)}(u) = \frac{d}{dt}|_0 f_n(u \cdot \exp(tB(u))) = \frac{d}{dt}|_0 \exp(-tB(u)) \cdot f_n(u)$$
$$\left(\frac{d}{dt}|_0 \exp(-tB(u))\right) \cdot f_n(u) = -B(u) f_n(u).$$

Recall from above that the adjoint action of $O(n)$ on $\mathfrak{g}/\mathfrak{p} = \mathbb{R}^n$. In addition, this action is linear.

Summarizing the computations from above, we obtain by using 3.4.5

$$
\begin{aligned}
&(\nabla_\xi \sigma)(\pi(u)) \\
&= \begin{pmatrix} (\xi \cdot \phi)(\pi(u)) \\ [u, \phi(\pi(u)) X(u) + B(u) f_n(u) - B(u) f_n(u)] + \left(\nabla_\xi^{LC} \eta\right)(\pi(u)) \end{pmatrix} \\
&= \begin{pmatrix} (\xi \cdot \phi)(\pi(u)) \\ \phi(\pi(u)) \xi(u) + \left(\nabla_\xi^{LC} \eta\right)(\pi(u)) \end{pmatrix}.
\end{aligned}
$$

□

5.2 Reductions of Riemannian Holonomy

Now we start to consider holonomy reductions of a Riemannian Cartan geometry $(\mathcal{G} \to M, \omega)$.

The first obvious idea is to take a closed Lie subgroup $H \subset O(n)$ and consider a holonomy reduction according to $H \ltimes \mathbb{R}^n$, meaning that is has type $\mathrm{Euc}(n)/(H \ltimes \mathbb{R}^n)$. In particular, if H leaves some subspace $V \subset \mathbb{R}^n$ invariant, we have $H \subset O(V) \times O(V^\perp)$. In the end of this section we take a closer look at the case $H = O(V) \times O(V^\perp)$.

5.2.1 General Observations

Note that the curved-orbit-decomposition will consist of only one orbit. This is due to the fact that the group $O(n)$ acts transitively on $\mathrm{Euc}(n)/(H \ltimes \mathbb{R}^n)$ for every subgroup $H \subset O(n)$:

Lemma 5.2.1. *Let $H \subset O(n)$ be a subgroup. The left multiplication of $O(n)$ on $\mathrm{Euc}(n)/(H \ltimes \mathbb{R}^n)$ is transitive. Consequently, each holonomy reduction of a Riemannnian Cartan geometry with holonomy group $H \ltimes \mathbb{R}^n$ has exactly one curved orbit.*

Proof. Consider

$$\begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix} (H \ltimes \mathbb{R}^n) \in \mathrm{Euc}(n)/(H \ltimes \mathbb{R}^n)$$

for $A \in O(n)$ and $v \in \mathbb{R}^n$. Note that $\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in O(n)$ and $\begin{pmatrix} 1 & 0 \\ A^{-1}v & I \end{pmatrix} \in H \ltimes \mathbb{R}^n$, and in addition we have

$$\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 1 & 0 \\ A^{-1}v & I \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix}.$$

Therefore we obtain

$$\begin{pmatrix} 1 & 0 \\ 0 & A^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix} (H \ltimes \mathbb{R}^n) = \begin{pmatrix} 1 & 0 \\ A^{-1}v & I \end{pmatrix} (H \ltimes \mathbb{R}^n) = H \ltimes \mathbb{R}^n.$$

By definition, we have at most as many curved orbits in M as P-orbits in $G/(H \ltimes \mathbb{R}^n)$. □

Nevertheless, the holonomy reduction implies some kind of structure, that we will investigate. We start by describing the extended bundle $\hat{\mathcal{G}}$ more explicitly. It is built from the orthonormal frame bundle by forming the associated bundle $\hat{\mathcal{G}} = \mathcal{G} \times_{O(n)} (O(n) \ltimes \mathbb{R}^n)$.

Thus in order to understand this bundle, we must view $O(n) \ltimes \mathbb{R}^n$ as an $O(n)$-module. First note the following

Remark 5.2.2. *For two fiber bundles $\pi_1 : \mathcal{G}_1 \to M$ and $\pi_2 : \mathcal{G}_2 \to M$ over the same manifold M it is possible to define the "fibered product". This should be another fiber bundle over M, such that the fiber over $x \in M$ is given by $(\mathcal{G}_1)_x \times (\mathcal{G}_2)_x$. Technically, this can be realized as the pullback bundle $\pi_1^* \mathcal{G}_2$ (or, equivalently, $\pi_2^* \mathcal{G}_1$). Its projection π is defined as $\pi_1 \circ \mathrm{pr}_1$, as illustrated in the following commutative diagram:*

$$\begin{tikzcd} \pi_1^* \mathcal{G}_2 \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, "\pi"] & \mathcal{G}_2 \arrow[d, "\pi_2"] \\ \mathcal{G}_1 \arrow[r, "\pi_1"'] & M \end{tikzcd}$$

We denote this bundle by $\mathcal{G}_1 \times_M \mathcal{G}_2$. The fiber over $x \in M$ is indeed given by

$$(\mathcal{G}_1 \times_M \mathcal{G}_2)_x = \mathrm{pr}_1^{-1}\left(\pi_1^{-1}(\{x\})\right) = \mathrm{pr}_1^{-1}\left((\mathcal{G}_1)_x\right) = (\mathcal{G}_1)_x \times (\mathcal{G}_2)_x .$$

Let us investigate what happens if an associated bundle is formed with an action that decomposes into a product:

Lemma 5.2.3. *Let S_1, S_2 be manifolds, each endowed with a left P-action, where P is the structure group of the principal bundle $\pi : \mathcal{G} \to M$. Then $\mathcal{G} \times_P (S_1 \times S_2) = (\mathcal{G} \times_P S_1) \times_M (\mathcal{G} \times_P S_2)$.*

Proof. Let

$$\Phi : \mathcal{G} \times_P (S_1 \times S_2) \to (\mathcal{G} \times_P S_1) \times_M (\mathcal{G} \times_P S_2)\ , [u, (s_1, s_2)] \mapsto ([u, s_1], [u, s_2]) .$$

This is indeed a well-defined fiber-bundle-isomorphism. □

Lemma 5.2.4. *(i) As an $O(n)$-space, $O(n) \ltimes \mathbb{R}^n$ is isomorphic to $O(n) \times \mathbb{R}^n$ with left multiplication in the first component and standard action in the second.*

(ii) The extended bundle $\hat{\mathcal{G}}$ can be identified with the fibered product $\mathcal{G} \times_M TM$.

(iii) Under the identifiation from (ii), the embedding $\iota : \mathcal{G} \hookrightarrow \hat{\mathcal{G}}$ is of the form $\mathrm{id}_{\mathcal{G}} \times 0_{TM}$ where 0_{TM} denotes the zero-section of TM.

Proof. (i) Let $\begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \in O(n)$. It acts on $\begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix} \in \mathrm{Euc}(n)$ by

$$\begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ Bv & BA \end{pmatrix} .$$

(ii) By (i) and 5.2.3, we have

$$\begin{aligned} \hat{\mathcal{G}} &= \mathcal{G} \times_{O(n)} \mathrm{Euc}(n) = \mathcal{G} \times_{O(n)} (O(n) \times \mathbb{R}^n) \\ &= \left(\mathcal{G} \times_{O(n)} O(n)\right) \times_M \left(\mathcal{G} \times_{O(n)} \mathbb{R}^n\right) = \mathcal{G} \times_M TM. \end{aligned}$$

(iii) Let $u \in \mathcal{G}$. Then $\iota(u) = [u, e] \in \mathcal{G} \times_{O(n)} \mathrm{Euc}(n)$. This element corresponds to $[u, (e, 0)] \in \mathcal{G} \times_{O(n)} (O(n) \times \mathbb{R}^n)$, and is further identified with $([u, e], [u, 0])$. We obtain the element $(u, 0) \in \mathcal{G} \times TM$. □

5.2.2 Equivalence to structures on the Riemannian manifold

Now we assume that we have a given holonomy reduction $\hat{\mathcal{H}} \subset \hat{\mathcal{G}}$ of the Cartan geometry $(\pi : \mathcal{G} \to M, \omega)$ of type $\mathrm{Euc}(n) / (H \ltimes \mathbb{R}^n)$. We form the intersection $\mathcal{H} := \hat{\mathcal{H}} \cap \iota(\mathcal{G}) \subset \iota(\mathcal{G}) = \mathcal{G} \times \{0_{TM}\} \cong \mathcal{G}$, and obtain the following diagram:

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\iota"] & \hat{\mathcal{G}} \\ \mathcal{H} \arrow[u, "j"] \arrow[r, "\bar{\iota}"'] & \hat{\mathcal{H}} \arrow[u, "\hat{j}"'] \end{tikzcd}$$

We will denote the respective inclusions as in the diagram.

Proposition 5.2.5. *The inclusion $j : \mathcal{H} \hookrightarrow \iota(\mathcal{G}) = \mathcal{G} \times \{0_{TM}\} \cong \mathcal{G}$ is a reduction of the structure group from $O(n)$ to H. It is compatible with the connection γ, meaning $j^*\gamma$ has values in the Lie algebra $\mathfrak{h}$ of H.*

Proof. We use 1.1.5 to show the claim.

1. Let $x \in M$, $u \in \iota(\mathcal{G}_x) \subset \mathcal{G}$ and $u' \in \hat{\mathcal{H}}_x \subset \mathcal{G}$. There is an element $\begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix} \in \mathrm{Euc}(n)$ such that

$$u' = u \cdot \begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix} = u \cdot \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 1 & 0 \\ A^{-1}v & I \end{pmatrix}$$

   Thus, by invariance of $\mathcal{G}$ and $\hat{\mathcal{H}}$, we have found an element

$$u \cdot \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} = u' \begin{pmatrix} 1 & 0 \\ A^{-1}v & I \end{pmatrix}^{-1}$$

   in the intersection $\mathcal{H} = \iota(\mathcal{G}) \cap \hat{\mathcal{H}}$. Therefore $\pi\big|_{\mathcal{H}}$ is surjective.

2. The subset $\mathcal{H}$ is H-invariant, since for $u \in \mathcal{H}$ and $h \in H$ we have $h \in H \ltimes \mathbb{R}^n$, hence $uh \in \hat{\mathcal{H}}$, and on the other hand $h \in O(n)$, thus $uh \in \mathcal{G}$. Consequently, $uh \in \mathcal{H}$.

3. We have to show that H acts fiberwise transitively on $\mathcal{H}$. Let $u, u' \in \mathcal{H}$ with $\pi(u) = \pi(u')$. Since both u and u' are elements of $\mathcal{G}$ and $\hat{\mathcal{H}}$, there are elements $A \in O(n)$, $v \in \mathbb{R}^n$ and $B \in H$ such that

$$u' = u \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \text{ and } u = u' \begin{pmatrix} 1 & 0 \\ v & B \end{pmatrix}.$$

   Consequently,

$$u = u' \begin{pmatrix} 1 & 0 \\ v & B \end{pmatrix} = u \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v & B \end{pmatrix} = u \begin{pmatrix} 1 & 0 \\ Av & AB \end{pmatrix}.$$

The $\mathrm{Euc}(n)$-action on $\hat{\mathcal{G}}$ is free, therefore $\begin{pmatrix} 1 & 0 \\ Av & AB \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & I \end{pmatrix}$. Therefore, $v = 0$ and $A = B^{-1} \in H$. This gives an element in H that maps u to u'.

4. Finally, we have to show that around each $x \in M$ there is an open neighborhood U and a local smooth section of $\mathcal{G}|_U$ that has values in $\mathcal{H}$. Choose U such that there is a smooth local section σ of $\mathcal{G}|_U$ and a local section $\hat{\sigma}$ of $\hat{\mathcal{H}}|_U$.

   By the implicit function theorem, there is a smooth function $f : U \to \mathrm{Euc}(n)$ such that $\hat{\sigma}(y) = (\iota \circ \sigma)(y) \cdot f(y)$ for all $y \in U$. The function f is of the form $f(y) = \begin{pmatrix} 1 & 0 \\ v(y) & A(y) \end{pmatrix}$, where $v : U \to \mathbb{R}^n$ and $A : U \to O(n)$ are smooth.

   We rewrite $\begin{pmatrix} 1 & 0 \\ v(y) & A(y) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & A(y) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ A(y)^{-1} v(y) & I \end{pmatrix}$, where the first factor is an element of $O(n)$ and the second one is an element of $H \ltimes \mathbb{R}^n$.

   Then $(\iota \circ \sigma) \cdot \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} = \hat{\sigma} \cdot \begin{pmatrix} 1 & 0 \\ -A^{-1}v & I \end{pmatrix} =: \tau$.

   Since $\mathcal{G}$ is $O(n)$-invariant, τ is a section of $\mathcal{G}$, and since $\hat{\mathcal{H}}$ is $H \ltimes \mathbb{R}^n$-invariant, it is a section of $\hat{\mathcal{H}}$. Hence it has values in $\mathcal{H}$.

We still have to show that $j^*\gamma$ has values in $\mathfrak{h}$.

The Cartan connection ω decomposes into the soldering form $\theta \in \Omega^1(\mathcal{G}, \mathbb{R}^n)$ and the connection $\gamma \in \Omega^1(\mathcal{G}, \mathfrak{o}(n))$. However,

$$j^*\theta + j^*\gamma = j^*\omega = j^*(\iota^*\hat{\omega}) = \bar{\iota}^*\left(\hat{j}^*\hat{\omega}\right),$$

where $\bar{\iota} : \mathcal{H} \hookrightarrow \hat{\mathcal{H}}$ and $\hat{j} : \hat{\mathcal{H}} \hookrightarrow \hat{\mathcal{G}}$ are the inclusions.

Since $\hat{\mathcal{H}}$ is a holonomy reduction, it is compatible with the connection $\hat{\omega}$ on $\hat{\mathcal{G}}$, meaning $\hat{j}^*\hat{\omega}$ has values in $\mathfrak{h} \oplus \mathbb{R}^n$. Consequently, $j^*\gamma$ has values in $\mathfrak{h}$. □

It is straightforward to show that, conversely, a reduction of the structure group of $\mathcal{G}$ that is compatible with the connection, induces a holonomy reduction.

Proposition 5.2.6. *Let $j : \mathcal{H} \hookrightarrow \mathcal{G}$ be a reduction of the structure group from $O(n)$ to H, such that $j^*\gamma$ has values in $\mathfrak{h}$. Then $\mathcal{H}$ induces a holonomy reduction $\hat{j} : \hat{\mathcal{H}} \hookrightarrow \hat{\mathcal{G}}$ of type* $\mathrm{Euc}(n)/(H \ltimes \mathbb{R}^n)$.

Proof. Let $\hat{\mathcal{H}} := \mathcal{H} \times_H (H \ltimes \mathbb{R}^n)$ and $\bar{\iota} : \mathcal{H} \hookrightarrow \hat{\mathcal{H}}$ be given by $\bar{\iota}(u) := [u, e]$. We define the embedding $\hat{j}$ of $\hat{\mathcal{H}}$ into $\hat{\mathcal{G}}$ by

$$\hat{j}\left(\left[u, \begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix}\right]\right) := \left[j(u), \begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix}\right],$$

where $u \in \mathcal{H}$, $v \in \mathbb{R}^n$ and $A \in H$.

Let $u, u' = u \cdot B \in \mathcal{H}$, where $B \in H$, and furthermore $A, A' \in H$ and $v, v' \in \mathbb{R}^n$. The map $\hat{j}$ is well-defined and injective, since, by using injectivity and equivariance of j, we obtain

$$\begin{aligned}
&\hat{j}\left(\left[u,\begin{pmatrix}1 & 0\\ v & A\end{pmatrix}\right]\right)=\hat{j}\left(\left[u',\begin{pmatrix}1 & 0\\ v' & A'\end{pmatrix}\right]\right)\\
\iff &\left[j\left(u\right),\begin{pmatrix}1 & 0\\ v & A\end{pmatrix}\right]=\left[j\left(u\right)B,\begin{pmatrix}1 & 0\\ v' & A'\end{pmatrix}\right]\\
\iff &\begin{pmatrix}1 & 0\\ v & A\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & B\end{pmatrix}\begin{pmatrix}1 & 0\\ v' & A'\end{pmatrix}\\
\iff &\left[u,\begin{pmatrix}1 & 0\\ v & A\end{pmatrix}\right]=\left[u',\begin{pmatrix}1 & 0\\ v' & A'\end{pmatrix}\right].
\end{aligned}$$

The inclusion j is infinitesimally injective, thus so is $\hat{j}$.

We still have to show that $j^*\hat{\omega}$ has values in $\mathfrak{h}\oplus\mathbb{R}^n$:
First note that $T\bar{\iota}\cdot\mathfrak{H}^\gamma\cap V\hat{\mathcal{H}}=\{0\}$: Suppose $\xi\in\mathfrak{H}^\gamma$ such that $T\bar{\iota}\cdot\xi\in V\hat{\mathcal{H}}$. By definition, $T\bar{\iota}\cdot\xi$ is mapped to 0 by the derivative of the projection, hence, since $\bar{\iota}$ is a bundle map, ξ itself lies in the vertical subspace of $T\mathcal{H}$. However, $T\mathcal{H}=\mathfrak{H}^\gamma\oplus V\mathcal{H}$, hence $\xi=0$.

Therefore, by dimensional reasons, we obtain

$$T\hat{\mathcal{H}}|_{\bar{\iota}(\mathcal{H})}=\left(T\bar{\iota}\cdot\mathfrak{H}^\gamma\right)\oplus\left(V\hat{\mathcal{H}}|_{\bar{\iota}(\mathcal{H})}\right).$$

On the one hand, an element $\xi\in V\hat{\mathcal{H}}$ inserted into $\hat{j}^*\hat{\omega}$ will give an element of $\mathfrak{h}\oplus\mathbb{R}^n$. On the other hand, we have

$$\bar{\iota}^*\hat{j}^*\omega=j^*\iota^*\hat{\omega}=j^*\omega=j^*\gamma+j^*\theta$$

where $j^*\gamma$ has values in $\mathfrak{h}$ and $j^*\theta$ has values in $\mathbb{R}^n$. Thus for $\xi\in\mathfrak{H}^\gamma$ we have $\left(\hat{j}^*\hat{\omega}\right)(T\bar{\iota}\cdot\xi)=\left(\bar{\iota}^*\hat{j}^*\hat{\omega}\right)(\xi)\in\mathfrak{h}\oplus\mathbb{R}^n$, hence altogehter

$$\left(\hat{j}^*\hat{\omega}\right)\left(T\hat{\mathcal{H}}|_{\bar{\iota}(\mathcal{H})}\right)\in\mathfrak{h}\oplus\mathbb{R}^n.$$

Finally, by equivariant extension, the claim follows for all elements in $T\hat{\mathcal{H}}$. □

We still have to check that the two constructions from 5.2.5 and 5.2.6 are inverse to each other:

Proposition 5.2.7. *Let $(\pi:\mathcal{G}\to M,\omega)$ be a Cartan geometry of type $(\mathrm{Euc}(n),O(n))$ and $H\subset O(n)$ a closed Lie subgroup. Holonomy reductions of type $\mathrm{Euc}(n)/(H\ltimes\mathbb{R}^n)$ of the Cartan geometry are equivalent to reductions of the structure group of $\mathbb{G}$ from $O(n)$ to H.*

Proof. Start with a reduction $j:\mathcal{H}\hookrightarrow\mathcal{G}$, that is compatible with the connection γ. The holonomy reduction in 5.2.6 was defined as $\hat{\mathcal{H}}:=\mathcal{H}\times_H(H\ltimes\mathbb{R}^n)$ together with the embedding $\hat{j}:\hat{\mathcal{H}}\hookrightarrow\hat{\mathcal{G}}$, $\hat{j}\left([u,(h,x)]\right):=\left[j\left(u\right),(h,x)\right]$ where $u\in\mathcal{H}$, $h\in H$ and $x\in\mathbb{R}^n$.

In 5.2.4 we saw that $\hat{\mathcal{G}}=\mathcal{G}\times_M TM$. In this picture, the above embedding is of the form $\hat{j}\left([u,(h,x)]\right)=(uh,u\left(x\right))$. In 5.2.5, we reduced the bundle $\hat{\mathcal{H}}$ by intersecting it with $\iota\left(\mathcal{G}\right)=\mathcal{G}\times\{0_{TM}\}$, where 0_{TM} denotes the zero-section of TM. Therefore, we obtain the subset $\{uh\mid u\in\mathcal{H},\ h\in H\}=\mathcal{H}$ of $\mathcal{G}$. This is the set we had started with.

On the other hand, let $\hat{j} : \hat{\mathcal{H}} \hookrightarrow \hat{\mathcal{G}}$ be a reduction of the structure group from $\mathrm{Euc}\,(n)$ to $(H \ltimes \mathbb{R}^n)$, that is compatible with the connection $\hat{\omega}$ on $\hat{\mathcal{G}}$. In 5.2.5 we formed the induced reduction $\mathcal{H} := \hat{\mathcal{H}} \cap \iota\,(\mathcal{G})$ together with the inclusion $j : \mathcal{H} \hookrightarrow \iota\,(\mathcal{G}) \cong \mathcal{G}$. Additionally, we have the inclusion $\bar{\iota} : \mathcal{H} \hookrightarrow \hat{\mathcal{H}}$. In turn, by 5.2.6, $\mathcal{H}$ induces the reduction $\mathcal{H} \times_H (H \ltimes \mathbb{R}^n) \hookrightarrow \hat{\mathcal{G}}$, $[u, (h, x)] \mapsto [j\,(u)\,, (h, x)]$.

Let $u \in \mathcal{H}$, $h \in H$ and $x \in \mathbb{R}^n$. We have $\bar{\iota}\,(u) \in \hat{\mathcal{H}}$, hence the embedded elements are of the form $[j\,(u)\,, (h, x)] = [j(u), e] \cdot (h, x) = \iota\,(j\,(u)) \cdot (h, x) = \hat{j}\,(\bar{\iota}(u)) \cdot (h, x) \in \hat{j}(\hat{\mathcal{H}})$. Hence $\mathcal{H} \times_H (H \ltimes \mathbb{R}^n)$ is embedded into $\hat{\mathcal{G}}$ exactly in the same way as $\hat{\mathcal{H}}$, thus the two reductions are isomorphic. □

Thus we have proved that the original concept of Riemannian holonomy fits into our definition of a holonomy reduction. All these reductions are well-known as a certain type of G-structures endowed with a connection.

Hence holonomy reductions of Riemannian Cartan geometries generalize this concept.

Remark 5.2.8. *Note that we did not make use of the fact that $\mathcal{G}$ is the orthonormal frame bundle over M. Let $\mathcal{G}$ be a reduction of the frame bundle of M, that has structure group $P \subset \mathrm{GL}\,(n, \mathbb{R})$, together with ω of the form $\theta + \gamma$, where θ is the soldering form and γ a principal connection on $\mathcal{G}$ (i.e. $\mathcal{G}$ is a G-structure endowed with a connection γ) and H a closed subgroup of P. Then a holonomy reduction of type $(P \ltimes \mathbb{R}^n)/(H \ltimes \mathbb{R}^n)$, where $H \subset P$, is equivalent to a H-reduction of $\mathcal{G}$. The proof is completely analogous to the proof of 5.2.7.*

5.2.3 Reductions of type $O(V) \times O(V^\perp)$

At last, we consider an explicit type of reductions. Let V be a k-dimensional linear subspace of $\mathbb{R}^n$. We showed in Proposition 5.2.7 that holonomy reductions of type $O(V) \times O(V^\perp) \ltimes \mathbb{R}^n$ are equivalent to reductions of the structure group of the orthonormal frame bundle from $O(n)$ to $O(V) \times O(V^\perp)$ that are compatible with the connection. Without loss of generality we assume $V := \mathbb{R}^k \times \{0\}$.

Note that the group $H := O(k) \times O(n-k)$ is the stabilizer of the subspace $V \subset \mathbb{R}^n$ in $O(n)$. In particular, it stabilizes the complement $V^\perp$.

Definition Let $\mathcal{V} \to M$ be a vector bundle endowed with a linear connection ∇. A subbundle $\mathcal{V}_0$ of $\mathcal{V}$ is called *parallel* if for all $\xi \in \mathfrak{X}(M)$ and $\sigma \in \Gamma(\mathcal{V}_0)$ the covariant derivative $\nabla_\xi \sigma$ is again a section of $\mathcal{V}_0$.

We can immediately see, what kind of structure is induced by a H-reduction that is compatible with the connection:

Proposition 5.2.9. *Let (M, g) be a Riemannian manifold. A G-structure of type $O(k) \times O(n-k)$ on the orthonormal frame bundle $\mathcal{G}$, that is compatible with the Levi-Civita-connection, is equivalent to a parallel distribution E of rank k on M.*

Proof. Let $j : \mathcal{H} \hookrightarrow \mathcal{G}$ be a reduction of the structure group from $O(n)$ to $O(k) \times O(n-k)$. We have $\mathcal{H} \times_H \mathbb{R}^n = \mathcal{G} \times_{O(n)} \mathbb{R}^n = TM$. Let $E := \mathcal{H} \times_H \mathbb{R}^k \subset TM$.

We know $j^*\gamma \in \Omega^1(\mathcal{H}, \mathfrak{h})$, where γ is the Levi-Civita-connection on $\mathcal{G}$ and $\mathfrak{h} = \mathfrak{o}(k) \oplus \mathfrak{o}(n-k)$ the Lie algebra of H. Both connections induce the Levi-Civita-connection ∇ on TM. Therefore the covariant derivative $\nabla_\xi \eta$ for $\xi \in \mathfrak{X}(M)$ and $\eta \in \Gamma(E)$ corresponds to taking the derivative of the equivariant function $\mathcal{H} \to \mathbb{R}^k$ that is induced by η with respect to the horizontal lift of ξ. This is again an equivariant function $\mathcal{H} \to \mathbb{R}^k$. Thus $\nabla_\xi \sigma$ has values in E and hence E is parallel.

Conversely, suppose we have a given parallel distribution E. Then let $\mathcal{H} := \{u \in \mathcal{G} \mid [u, x] \in E \,\forall x \in \mathbb{R}^k\}$. We have to show that this is an H-reduction. We use 1.1.5:

(1) Let $x \in M$ and $u \in \mathcal{G}_x$. Choose an orthonormal basis of E_x, that is of the form $[u, v_1], \dots [u, v_k]$, where $v_i \in \mathbb{R}^n$. Complete it to an orthonormal basis $B = \{v_1, \dots, v_n\}$. Take the base change $A \in O(n)$ from the standard basis to B, i.e. $Ae_i = v_i$. Then $[uA, e_i] = [u, Ae_i] = [u, v_i] \in E \,\forall i \leq k$. Hence $uA \in \mathcal{H}_x$ and therefore $\pi\big|_{\mathcal{H}} : \mathcal{H} \to M$ is surjective.

(2) Let $u \in \mathcal{H}$ and $A \in H$, i.e. A stabilizes the subspace $\mathbb{R}^k \subset \mathbb{R}^n$. Thus $[uA, x] = [u, Ax] \in \mathcal{T}_0$ for all $x \in \mathbb{R}^k$.

(3) Let $u, u' \in \mathcal{H}$ with $u' = uA$ for a $A \in O(n)$. We have to show that $A \in H$. We know that $[u, x] \in E$ and $[u', x] = [uA, x] = [u, Ax] \in E$ for all $x \in \mathbb{R}^k$. If A did not stabilize $\mathbb{R}^k$, the dimension of $E_{\pi(u)}$ would be larger than k. Hence $A \in H$.

(4) The procedure from (1) can be conducted in a smooth way. Take a local orthonormal frame of E that is of the form $[\sigma(x), v_1(x)], \dots, [\sigma(x), v_k(x)]$ where $\sigma : U \to \mathcal{G}$ is a local section of $\mathcal{G}$, $x \in U$ and $v_1, \dots, v_k : U \to \mathbb{R}^n$ are smooth functions. Complete it to an orthonormal frame

$$[\sigma(x), v_1(x)], \dots, [\sigma(x), v_k(x)], [\sigma(x), v_{k+1}(x)], \dots [\sigma(x), v_n(x)]$$

of TM. Then let $A(x) \in O(n)$ be the base change satisfying $A(x)e_i = v_i(x)$. By the implicit function theorem, A depends smoothly on x. Finally, $\sigma \cdot A$ is a local section of $\mathcal{H}$, since $[(\sigma \cdot A)(x), e_i] = [\sigma(x), A(x)e_i] = [\sigma(x), v_i(x)] \forall x \in U \,\forall i \leq k$.

Next, we show that these two constructions are inverse to each other:
Firstly, let $\mathcal{H}$ be an H-reduction and form $E := \mathcal{H} \times_H \mathbb{R}^k$. Then let

$$\bar{\mathcal{H}} := \left\{ u \in \hat{\mathcal{G}} \mid [u, x] \in E \,\forall x \in \mathbb{R}^k \right\}.$$

We have to show that $\mathcal{H} = \bar{\mathcal{H}}$.

On the one hand, let $u \in \mathcal{H}$. By definition, $[u, x] \in E$ for all $x \in \mathbb{R}^k$, thus $x \in \bar{\mathcal{H}}$.

On the other hand, let $u \in \bar{\mathcal{H}}$. We take an element $u' \in \mathcal{H}$ such that $\pi(u) = \pi(u')$, hence there is a $A \in O(n)$ such that $u = u'A$. We conclude that for all $x \in \mathbb{R}^k$ we have $[u', Ax] = [u'A, x] = [u, x] \in E$. Therefore, A has to stabilize the subspace $\mathbb{R}^k \subset \mathbb{R}^n$, since otherwise E would have dimension higher than k. Thus $A \in H$ and $u = u'A \in \mathcal{H}$.

Now, let us start with a given subbundle E that has rank k. Let $\mathcal{H} := \left\{u \in \mathcal{G} \mid [u,x] \in E \,\forall x \in \mathbb{R}^k\right\}$, and form $\bar{E} := \mathcal{H} \times_H \mathbb{R}^k$. We have to show that $\bar{E} = E$.

Let $[u,x] \in \bar{E}$ where $u \in \mathcal{H}$ and $x \in \mathbb{R}^k$. By definition, $[u,y] \in E$ for all $y \in \mathbb{R}^k$, hence in particular $[u,x] \in E$.

On the other hand, consider an element $[u,x] \in E$ where $u \in \mathcal{G}$ and $x \in \mathbb{R}^n$. Choose an element $u' = uA \in \mathcal{H}$ where $A \in O(n)$. Then $\left[u', A^{-1}x\right] = \left[u'A^{-1}, x\right] = [u,x] \in E$, hence $A^{-1}x \in \mathbb{R}^k$ again by dimensional reasons. Therefore, $A \in H$ and $[u,x] = \left[u', A^{-1}x\right] \in \bar{E}$.

We still have to prove that the reduction $\mathcal{H}$ is compatible with the connection. Note that $\mathfrak{h} = \mathfrak{o}(k) \oplus \mathfrak{o}(n-k) = \mathrm{Stab}_{\mathfrak{o}(n)}(\mathbb{R}^k)$. From above, we know that $E = \mathcal{H} \times_H \mathbb{R}^k$. We want to show that the horizontal distribution satisfies $\mathfrak{H}_u \subset T_u\mathcal{H}$ for all $u \in \mathcal{H}$.

Suppose indirectly that there is a $u \in \mathcal{H}$ such that $\mathfrak{H}_u \not\subset T_u\mathcal{H}$. Take a horizontal curve $c : I \to \mathcal{G}$ such that $c(0) = u$ and $c(t) \notin \mathcal{H}$ for all $t > 0$, such that I is an open interval containing 0.

Furthermore, choose a local section τ of $\mathcal{H}$ around $\pi\,(u)$ satisfying $\tau\,(\pi\,(u)) = u$. By the inverse function theorem, we can find a smooth map $A : I \to G$ such that $c(t) = \tau\,(\pi\,(c\,(t)))\,A\,(t)$. We see that $A(0) = e$ and $A^{-1}\,(t) \notin H$ for all $t > 0$, hence $X := \frac{d}{dt}|_0 A^{-1}\,(t) \notin \mathfrak{h}$.

Now choose $x \in \mathbb{R}^{k+1}$ such that $X \cdot x \notin \mathbb{R}^{k+1}$ and $f : \mathcal{G} \to \mathbb{R}^{n+1}$ an equivariant function such that $f\,(u) = x$. The function f corresponds to a vector field on M with values in E. By definition of E, the covariant derivative of this vector field along any vector field ξ on M again has values in E. This covariant derivative corresponds to $\xi^{hor} \cdot f$, hence $\xi^{hor} \cdot f$ again has values in $\mathbb{R}^k$.

We obtain

$$
\begin{aligned}
\mathbb{R}^k \ni (c'\,(0) \cdot f) &= \frac{d}{dt}|_0 f\,(c\,(t)) = \frac{d}{dt}|_0 A^{-1}\,(t)\, f\,(\tau\,(\pi\,(c\,(t)))) \\
&= \left(\frac{d}{dt}|_0 A^{-1}\,(t)\right) f\,(\tau\,(\pi\,(c\,(0)))) + A^{-1}\,(0) \left(\frac{d}{dt}|_0 f\,(\tau\,(\pi\,(c\,(t))))\right) \\
&= X f\,(u) + \frac{d}{dt}|_0 f\,(\tau\,(\pi\,(c\,(t)))) .
\end{aligned}
$$

However, $f\,(u) \in \mathbb{R}^k$ and τ is a section of $\mathcal{H}$, hence also the second summand is in $\mathbb{R}^k$.

Therefore we have a contradiction, since $X \cdot x \notin \mathbb{R}^k$ but the other two terms are elements of $\mathbb{R}^k$. $\square$

Finally, we observe two immediate properties of parallel distributions.

Proposition 5.2.10. *Let M be a Riemannian manifold and E a parallel on M. Then*

(i) E is involutive, hence integrable, and

(ii) its orthogonal complement $E^{\perp}$ is parallel.

Proof. (i) Let $\xi, \eta \in \Gamma(E)$. Then, since the Levi-Civita-connection is torsion-free, we have

$$[\xi, \eta] = \nabla_\xi \eta - \nabla_\eta \xi,$$

and since E is parallel, both summands on the right hand side are sections of E.

(ii) Let $\eta \in \Gamma(E^\perp)$ and $\zeta \in \mathfrak{X}(M)$. Then for each $\xi \in \Gamma(E)$ we have

$$0 = \zeta \cdot 0 = \zeta \cdot g(\xi, \eta) = g(\nabla_\zeta \xi, \eta) + g(\xi, \nabla_\xi \eta) = g(\xi, \nabla_\zeta \eta),$$

since E is parallel. Hence $\nabla_\zeta \eta$ is again a section of $E^\perp$. $\square$

This shows that a $O(V) \times O(V^\perp)$-reduction implies a product structure on the tangent bundle. As a consequence, the manifold M itself is locally isometric to a product of Riemannian manifolds (see the de Rham decomposition theorem, e.g. [8, p.187]).

5.3 Reductions of type $O(V) \times O(V^\perp) \ltimes V$

In this section we consider an explicit subgroup, namely $O(V) \times O(V^\perp) \ltimes V$, where $V \subset \mathbb{R}^n$ is a k-dimensional vector space. We will give a geometrical characterization of these holonomy reductions in terms of a certain vector field on the underlying manifold together with the parallel distribution that we studied in 5.2.9.

Having examined holonomy reductions of this type in detail, we can make conclusions for holonomy reductions of type $H \ltimes V$, where H is a subgroup of $O(n)$ that leaves V invariant, since

$$H \ltimes V \subset O(V) \times O(V^\perp) \ltimes V \subset \mathrm{Euc}(n).$$

In 5.2.7 we showed that the holonomy reduction $H \ltimes \mathbb{R}^n \subset \mathrm{Euc}(n)$ is a usual reduction of the structure group from $O(n)$ to H. Here we have $H \subset O(V) \times O(V^\perp)$, hence H is reducible in the sense that its action leaves the decomposition $V \oplus V^\perp$ invariant. By the *de Rham decomposition theorem* (see [8, p.187]) the manifold is locally isometric to a product of Riemannian manifolds. The reduction $H \ltimes V \subset O(V) \times O(V^\perp) \ltimes V$ endows this product with a structure coming from Riemannian holonomy.

For notational reasons, we take $V = \mathbb{R}^k$ and thus consider

$$H := (O(k) \times O(n-k)) \ltimes \mathbb{R}^k$$

where $k < n$.

5.3.1 Groups and Orbits

We start by describing the group H as the stabilizer of an object on the canonical representation $\mathbb{R}^{n+1}$ of G:

Proposition 5.3.1. *There is a $k+1$-dimensional linear subspace $W_0 \subset V$ such that $H = \mathrm{Stab}_G(W_0)$ and $\dim(\ker(\varphi_0) \cap W) = k$.*

Proof. We conduct the proof in coordinates $e_0, ..., e_n$ of $V = \mathbb{R}^{n+1}$, such that φ_0 extracts the coefficient of e_0 (see 5.1.2).

Let $W_0 := \langle\, e_0, \ldots, e_k \,\rangle$. Since $\ker(\varphi_0) = \langle\, e_1, \ldots, e_n \,\rangle$, the intersection with W_0 has dimension k. An element of H is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ v' & C & 0 \\ 0 & 0 & D \end{pmatrix}$$

in $1{+}k{+}(n{-}k)$-block matrix form , where $v' \in \mathbb{R}^k$, $C \in O(k)$ and $D \in O(n{-}k)$. It stabilizes W_0 since for $1 \le i \le k$ we have

$$\begin{pmatrix} 1 & 0 & 0 \\ v' & C & 0 \\ 0 & 0 & D \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ v' \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ v' & C & 0 \\ 0 & 0 & D \end{pmatrix} \begin{pmatrix} 0 \\ e_i \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ Ce_i \\ 0 \end{pmatrix}.$$

.

Conversely, let $g := \begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix} \in G$, where $A \in O(n)$ and $v \in \mathbb{R}^n$, such that g stabilizes W_0. Then $\begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ v \end{pmatrix} \in W_0$, thus the last $n-k$ components of v must vanish.

Furthermore, for $1 \le i \le k$ we have $\begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix} \begin{pmatrix} 0 \\ e_i \end{pmatrix} = \begin{pmatrix} 0 \\ Ae_i \end{pmatrix} \in W_0$, hence A maps the linear subspace $L := \langle\, e_1, \ldots, e_k \,\rangle \subset \mathbb{R}^n$ to itself. Thus also $A^{-1}(L) = L$. Let $v \in L^{\perp} = \langle\, e_{k+1}, \ldots, e_n \,\rangle \subset \mathbb{R}^n$, then we have $\langle\, Av, w \,\rangle = \langle\, v, A^{-1}w \,\rangle = 0$ for all $w \in L$. Therefore $A\left(L^{\perp}\right) \subset L^{\perp}$. Hence A preserves the decomposition $\mathbb{R}^n = L \oplus L^{\perp}$. Consequently, it is of the form $\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}$ for $C \in O(k)$ and $D \in O(n-k)$. $\square$

The G-orbit

We consider the G-orbit of W_0 in the space of linear, $k{+}1$-dimensional subspaces of V, the Grassmannian $\mathrm{Gr}(k+1, V)$. First we recall from linear algebra:

Lemma 5.3.2. *Let V be a finite-dimensional vector space and $W \subset V$ a linear subspace with annihilator $W^{\circ} \subset V^*$.*

(i) $V^*/W^{\circ} = W^*$

(ii) $(V/W)^* = W^{\circ}$

(iii) The linear subspaces of V are in bijective correspondence to the quotients of V^* via $W \mapsto V^*/W^{\circ}$.

(iv) Each linear subspace $L \subset V^*$ is of the form W° for a unique $W \subset V$.

Proof. (i) Let $p : V^* \to W^*$, $p(f) = f\big|_W$ for $f \in V^*$. Then p is surjective and has kernel $\ker(p) = \left\{f \in V^* : f\big|_W = 0\right\} = W^{\circ}$, therefore $V^*/W^{\circ} \cong W^*$.

(ii) Let $q : W^{\circ} \to (V/W)^*$ given by $f \mapsto \tilde{f}$, where for $f \in W^{\circ}$, $\tilde{f}$ is the factorized map over the canonical projection $V \to V/W$. The map q is linear and if $f \in \ker(q)$ then for all $v \in V$ we have $0 = \tilde{f}(v + W) = f(v)$, therefore

$f = 0$. Hence q is injective and surjectivity follows from dimensional reasons.

(iii) Denote $\Phi : W \mapsto V^*/W$ and $\Psi : \{\text{quotients of } V^*\} \to \{\text{subspaces of } V\}$, $V^*/L \mapsto L^\circ$. The maps are inverse to each other since

$$\Psi\left(\Phi\left(W\right)\right) = \Psi\left(V^*/W^\circ\right) = \left(W^\circ\right)^\circ = W$$

and $\Phi\left(\Psi\left(V^*/L\right)\right) = \Phi\left(L^\circ\right) = V^*/\left(\left(L^\circ\right)^\circ\right) = V^*/L$.

(iv) Construct a basis $\{v_1, \dots, v_n\}$ of V such that its dual basis $\{f_1, \dots, f_n\}$ gives a basis of L by $\{f_1, \dots, f_k\}$. Now $L = \langle\, v_{k+1}, \dots, v_n \,\rangle^\circ$ since: $f \in L \Rightarrow f = \sum_{i=1}^k \alpha_i f_i$ where $\alpha_i \in \mathbb{R} \Rightarrow \forall j > k : f\left(v_j\right) = 0$. Conversely, it is clear that $f_j \in \langle\, v_{k+1}, \dots, v_n \,\rangle^\circ$ if and only if $j \le k$, therefore in this case $f_j \in L$. $\square$

Proposition 5.3.3. *The G-orbit in* $\mathrm{Gr}(k+1, V)$ *that contains W_0 is given by*

$$\mathcal{O} := \{W \subset V \mid \varphi_0|_W \neq 0\} = \{W \subset V \mid \dim\left(\ker\left(\varphi_0\right) \cap W\right) = k\}.$$

Proof. We first prove the second equality: For $W \in \mathrm{Gr}(k+1, V)$ we claim $\varphi_0\big|_W \neq 0 \iff \dim\left(W \cap \ker\left(\varphi_0\right)\right) = k$.

($\Rightarrow$) Note that $W \cap \ker(\varphi_0) = \{w \in W \mid \varphi_0(w) = 0\} = \ker(\varphi_0\big|_W)$. Thus $\varphi_0\big|_W \neq 0 \iff \ker(\varphi_0\big|_W) \subsetneq W \iff \ker(\varphi_0) \cap W \subsetneq W$. Hence $W \cap \ker\left(\varphi_0\right)$ can have at most dimension k, but since $\dim(W) + \dim(\ker\left(\varphi_0\right)) = \dim(V) + \dim\left(W \cap \ker\left(\varphi_0\right)\right)$ we have $\dim(W \cap \ker\left(\varphi_0\right)) = (k+1) + n - (n+1) = k$.

The other implication ($\Leftarrow$) is clear from $\ker(\varphi_0) \subsetneq W_0$

Now we prove the remaining equality:
($\subset$) Let $g = \begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix} \in G$, where $A \in O(n)$ and $v \in \mathbb{R}^n$, then

$$g\left(W_0\right) = \langle\, ge_0, \dots, ge_k \,\rangle = \left\langle \begin{pmatrix} 1 \\ v \end{pmatrix}, \begin{pmatrix} 0 \\ Ae_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ Ae_k \end{pmatrix} \right\rangle$$

and $\varphi_0 \begin{pmatrix} 1 \\ v \end{pmatrix} = 1$, so $\varphi_0\big|_{g(W_0)} \neq 0$.

($\supset$) Let $W \in \mathrm{Gr}(k+1, V)$ such that $\varphi_0\big|_W \neq 0$. We construct a basis of W as follows: Take an orthonormal basis $\{v_1, \dots, v_k\}$ of $W \cap \ker\left(\varphi_0\right)$ and complete it to a basis of W by some $w \in W$. Necessarily $w \notin \ker\left(\varphi_0\right)$ therefore its 0-component $w_0 \neq 0$. Let $v_0 := \frac{1}{w_0} w$, that must be of the form $v_0 = \begin{pmatrix} 1 \\ v \end{pmatrix}$ where $v \in \mathbb{R}^n$.

On the other hand, complete $\{v_1, \dots, v_k\}$ to an orthonormal basis $\{v_1, \dots, v_n\}$ of $\ker\left(\varphi_0\right)$ and denote the the base change in $\ker\left(\varphi_0\right)$ from $\{e_1, \dots, e_n\}$ to $\{v_1, \dots, v_n\}$ by $A \in O(n)$ (both bases are orthonormal with respect to the inner product on $\ker(\varphi_0)$). Hence

$$W = \langle w, v_1, \dots, v_k \rangle = \left\langle \begin{pmatrix} 1 \\ v \end{pmatrix}, \begin{pmatrix} 0 \\ Ae_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ Ae_k \end{pmatrix} \right\rangle = \begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix} \left(W_0\right).$$

$\square$

Proposition 5.3.4. *The orbit $\mathcal{O}$ is an open subset of the manifold* $\mathrm{Gr}\left(k+1, V\right)$.

Proof. Note that $\mathrm{Gr}(k+1, \ker(\varphi_0)) \subset \mathrm{Gr}(k+1, V)$ is a closed submanifold. Furthermore, an element $W \in \mathrm{Gr}(k+1, V)$ satisfies $W \notin \mathrm{Gr}(k+1, \ker(\varphi_0))$ if and only if $W \subsetneq \ker(\varphi_0)$. This is equivalent to $\varphi_0\big|_W \neq 0$. Thus $\mathcal{O}$ is the complement of $\mathrm{Gr}(k+1, \ker(\varphi_0))$ and hence an open subset of $Gr(k+1, V)$. $\square$

The above proposition shows that $\mathcal{O}$ is indeed a smooth manifold, namely an open subset of the manifold $\mathrm{Gr}(k+1, V)$.

In the following, we will describe a Cartan geometry endowed with a holonomy reduction of type $\mathcal{O}$ in detail. We start by investigating the $O(n)$-orbits in $\mathcal{O}$, in order to determine the different P-types. Then we compute the curved orbits in the homogenous model. By taking a detour to the canonical tractor bundle, we will be able to transfer certain results from the homogenous model to any Riemannian Cartan geometry that is equipped with such a holonomy reduction. Finally, we find an equivalent geometrical interpretation for holonomy reductions of this type.

The P-orbits

Next, we want to describe the P-orbits in $\mathcal{O}$ in an invariant way.

Let $W \in \mathcal{O}$. We know that $\ker(\varphi_0) \cap W$ has dimension k. Take $(\ker(\varphi_0) \cap W)^{\perp}$ in $\ker(\varphi_0)$ that is endowed with an inner product (cf. 5.1.2). Then

$$V = W \oplus (\ker(\varphi_0) \cap W)^{\perp},$$

since $w \in W \cap ((W \cap \ker(\varphi_0))^{\perp})$ implies that $w \in W \cap \ker(\varphi)$ and $w \in (W \cap \ker(\varphi))^{\perp}$. Thus $w = 0$. Furthermore, $\dim(W + (\ker(\varphi_0) \cap W)^{\perp}) = k+1+n-k = n+1$, thus the above equation holds.

Denote the projection on the second component of the direct sum by $\mathrm{pr}_2 : V \to (\ker(\varphi_0) \cap W)^{\perp}$. Then we can compute the norm of the projection of v_0 (cf. 5.1.2), since $(\ker(\varphi_0) \cap W)^{\perp} \subset \ker(\varphi_0)$ carries an inner product.

We define

Definition

$$\mathcal{O}_r := \{W \in \mathcal{O} \mid ||\,\mathrm{pr}_2(v_0)|| = r\}$$

for $r \in \mathbb{R}_0^+$.

Proposition 5.3.5. *The $\mathcal{O}_r$'s for $r \in \mathbb{R}_0^+$ are exactly the P-orbits in $\mathcal{O}$.*

Proof. We conduct the proof in the standard coordinates $\{e_0, \dots, e_n\}$ of $V = \mathbb{R}^{n+1}$ (see 5.1.2). Recall that $\ker(\varphi_0) = \langle\, e_1, \dots, e_n \,\rangle$ in these coordinates.

We prove the following three claims in order to establish the proposition:

(i) $W_r := \left\langle \begin{pmatrix} 1 \\ re_n \end{pmatrix}, \begin{pmatrix} 0 \\ e_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ e_k \end{pmatrix} \right\rangle \in \mathcal{O}_r.$

(ii) $P \cdot W \subset \mathcal{O}_r$ for each $W \in \mathcal{O}_r$. In particular $P \cdot W_r \subset \mathcal{O}_r$.

(iii) $P \cdot W_r = \mathcal{O}_r$.

proof of (i): We have $\ker(\varphi_0) \cap W_r = \langle\, e_1, \dots, e_k \,\rangle$, thus $(\ker(\varphi_0) \cap W_r)^{\perp} = \langle\, e_{k+1}, \dots, e_n \,\rangle$.

In order to compute the projection of v_0 we have to solve the following equation

$$\begin{pmatrix}1\\0\end{pmatrix} = \alpha\begin{pmatrix}1\\re_n\end{pmatrix} + \begin{pmatrix}0\\x\end{pmatrix} + \begin{pmatrix}0\\y\end{pmatrix},$$

where $\alpha \in \mathbb{R}$, $x \in W \cap \ker(\varphi_0)$ and $y \in (W \cap \ker(\varphi_0))^\perp = \langle\, e_{k+1}, \ldots, e_n \,\rangle$. We immediately conclude $\alpha = 1$ and $x = 0$, therefore $y = -re_n$. This gives $\mathrm{pr}_2(v_0)$, thus $||\,\mathrm{pr}_2(v_0)|| = ||-re_n|| = r$.

proof of (ii): Let $p \in P$ and $W \in \mathcal{O}_r$. We consider the transformation $p : V \to V$. Note that p preserves the vector v_0, i.e. $p(v_0) = v_0$, and the inner product on $\ker(\varphi_0)$. Therefore, for any linear subspace U of $\ker(\varphi_0)$ we have $p(U^\perp) = (p(U))^\perp$, in particular this holds for $U = W \cap \ker(\varphi_0)$. Thus it maps the decomposition $W \oplus (W \cap \ker(\varphi_0))^\perp$ to $p(W) \oplus (p(W) \cap \ker(\varphi_0))^\perp$ of V. Denote the projections on the second component by pr_2^W and $\mathrm{pr}_2^{p(W)}$, respectively. Otherwise said, p satisfies $p \circ \mathrm{pr}_2^W = \mathrm{pr}_2^{p(W)} \circ p$.

Altogether, we have $||\,\mathrm{pr}_2^{p(W)}(v_0)|| = ||\,\mathrm{pr}_2^{p(W)}(p(v_0))|| = ||p(\mathrm{pr}_2^W(v_0)|| = ||\,\mathrm{pr}_2^W(v_0)|| = r$, since p is orthogonal on $\ker(\varphi_0)$. Thus $p(W) \in \mathcal{O}_r$.

proof of (iii): We will show that $\mathcal{O} \subset \bigsqcup_{r\geq 0} P \cdot W_r$. Combined with $\mathcal{O} = \bigsqcup_{r\geq 0} \mathcal{O}_r$ and (ii) this proves equality of $P \cdot W_r$ and $\mathcal{O}_r$.

In the proof of 5.3.3 we saw that elements W of $\mathcal{O}$ are of the form $\left\langle \begin{pmatrix}1\\v\end{pmatrix}, \begin{pmatrix}0\\Ae_1\end{pmatrix}, \ldots, \begin{pmatrix}0\\Ae_k\end{pmatrix} \right\rangle$ for $v \in \mathbb{R}^n$ and $A \in O(n)$. Acting with $\begin{pmatrix}1 & 0\\0 & A^{-1}\end{pmatrix}$ on W gives the subspace $\left\langle \begin{pmatrix}1\\A^{-1}v\end{pmatrix}, \begin{pmatrix}0\\e_1\end{pmatrix}, \ldots, \begin{pmatrix}0\\e_k\end{pmatrix} \right\rangle$ of $\mathbb{R}^{n+1}$. Denote the first k components of $A^{-1}v$ by $v_1 \in \mathbb{R}^k$ and the remaining $n-k$ components by $v_2 \in \mathbb{R}^{n-k}$, then there is $r \geq 0$ and $D \in O(n-k)$ such that $Dv_2 = re_{n-k}$. Let additionally the $1 \times k \times n-k$-block-matrix $\begin{pmatrix}1 & 0 & 0\\0 & I_k & 0\\0 & 0 & D\end{pmatrix}$ act on the subspace to obtain $\left\langle \begin{pmatrix}1\\v_1\\re_{n-k}\end{pmatrix}, \begin{pmatrix}0\\e_1\\0\end{pmatrix}, \ldots, \begin{pmatrix}0\\e_k\\0\end{pmatrix} \right\rangle = W_r$. Hence $\mathcal{O} \subset \bigcup_{r\geq 0} P \cdot W_r$.

It is clear from (ii) and the fact that $\mathcal{O}_r \cap \mathcal{O}_{r'} = \emptyset$ for $r \neq r'$ that the union is disjoint. This completes the proof. $\square$

Thus, our P-types are characterized by the non-negative, real numbers. We will denote the curved orbits in M by M_r, where M_r is the collection of all points that have P-type $\mathcal{O}_r$.

Example 5.3.6. *Consider the special case $k = 0$. Then we have $H = O(n) \times \{0\}$ and $\mathcal{O} = \{W \in \mathrm{Gr}(1, k+1) \mid \dim(\ker(\varphi_0)) \cap W = 0\}$. Note that each element $W \in \mathcal{O}$ is generated by an element of the form $e_0 + w$ for a unique $w \in \{0\} \times \mathbb{R}^n$. Each $w \in \{0\} \times \mathbb{R}^n$ induces a line $\langle e_0 + w\rangle \in \mathcal{O}$, hence $\mathcal{O} = \mathbb{R}^n$. The Lie group $G = \mathrm{Euc}(n)$ acts on $\mathcal{O}$ via $(A, v) \cdot \langle e_0 + w\rangle = \langle e_0 + Aw + v\rangle$, thus the induced action on $\mathbb{R}^n$ is the standard action of $O(n) \ltimes \mathbb{R}^n$ on $\mathbb{R}^n$.*

The P-orbits in $\mathbb{R}^n$ are exactly the spheres around 0.

This indeed fits together with the $\mathcal{O}_r$'s defined above: $\mathbb{R}^{n+1}$ decomposes into $\langle e_0 + w \rangle \oplus (\ker(\varphi_0) \cap \langle e_0 + w \rangle)^{\perp} = \langle e_0 + w \rangle \oplus \ker(\varphi_0)$. Then v_0 is projected onto $-w \in \ker(\varphi_0)$ since $v_0 = (e_0 + w) - w$. Hence the orbits in $\mathcal{O} = \mathbb{R}^n$ are formed according to $||w||$ where $w \in \mathbb{R}^n$. These are the spheres around 0, as seen above.

5.3.2 The induced subbundle of the tractor bundle

There is a $k+1$-dimensional subbundle $\mathcal{T}_0$ of $\mathcal{T}$, that is induced by a holonomy reduction of type $\mathcal{O}$:

The holonomy reduction can be interpreted as a reduction $j : \mathcal{H} \to \hat{\mathcal{G}}$ of the structure group from $\mathrm{Euc}(n)$ to H. We know from 5.3.1 that H is the stabilizer of a $k+1$-dimensional subspace W_0 in V. Hence we obtain $\mathcal{T}_0 := \mathcal{H} \times_H W_0 \subset \mathcal{H} \times_H V = \mathcal{T}$.

The pullback connection of $\hat{\omega}$ on $\mathcal{H}$ is principal and we can show, that the induced connection on $\mathcal{T}$ coincides with the tractor connection:

Lemma 5.3.7. *The connection $\hat{\omega}$ on $\hat{\mathcal{G}}$ and its pullback on $\mathcal{H}$ induce the same connection on $\mathcal{T}$.*

Proof. We have $\mathcal{T} = \mathcal{H} \times_H V = \hat{\mathcal{G}} \times_G V$. Let $\sigma \in \Gamma(\mathcal{T})$, then σ corrsponds to the equivariant functions $f^H : \mathcal{H} \to V$ and $f^G : \hat{\mathcal{G}} \to V$, respectively. The functions satisfy $f^G \circ j = f^H$.

For $\xi \in \mathfrak{X}(M)$, denote the horizontal lifts of ξ to $\mathcal{H}$ and $\hat{\mathcal{G}}$ by ξ_H^{hor} and ξ_G^{hor}, respectively. Then $Tj \cdot \xi_H^{hor} = \xi_G^{hor}$.

Consequently, $\xi_G^{hor} \cdot f^G = Tf^G \cdot Tj \cdot \xi_H^{hor} = Tf^H \cdot \xi_H^{hor}$. Therefore, $j^*\hat{\omega}$ and $\hat{\omega}$ induce the same connection on $\mathcal{T}$. $\square$

From the above Lemma we can immediately conclude that $\mathcal{T}_0$ is parallel: Since the pullback of the connection to the reduction induces the same connection as $\hat{\omega}$ on $\mathcal{T}$, the covariant derivative corresponds to taking the derivative of the equivariant function $\mathcal{H} \to \mathbb{R}^{k+1}$ with respect to the horizontal lift of ξ. This is again an equivariant function $\mathcal{H} \to \mathbb{R}^{k+1}$, hence $\nabla_\xi \sigma$ has values in $\mathcal{T}_0$.

Question: How does $\mathcal{T}_0$ behave with respect to the decomposition $\mathcal{T} = \mathcal{L} \oplus TM$ (cf. 5.1.4)?

Recall from 5.3.1 that $\dim(\ker(\varphi_0) \cap W_0) = k$, and since $\ker(\varphi_0)$ is G-invariant and hence H-invariant, we have

$$\dim((\mathcal{H} \times_H W_0) \cap (\mathcal{H} \times_H \ker(\varphi_0))) = \dim(\mathcal{T}_0 \cap TM) = k.$$

Furthermore, this implies that there can be no $x \in M$ such that $(\mathcal{T}_0)_x \subset \mathcal{T}_x$, since in this case the dimension of the above intersection would be $k+1$. Equivalently expressed, this means that $\mathrm{pr}^{\mathcal{L}} : \mathcal{T} \to \mathcal{L}$, the projection onto $\mathcal{L}$ with respect to the decomposition $\mathcal{T} = \mathcal{L} \oplus TM$, cannot vanish at any point.

Now we will show that these properties of $\mathcal{T}_0$ already suffice to characterize the holonomy reduction:

Proposition 5.3.8. *Let (M, g) be a Riemannian manifold of dimension n, $k < n$ and $\mathcal{T}_0$ be a $k+1$-dimensional parallel subbundle of the canonical tractor bundle $\mathcal{T}$ on M, such that $\mathrm{pr}^{\mathcal{L}}\left((\mathcal{T}_0)_x\right) \neq \{0\}$ for all $x \in M$. Then*

$$\mathcal{H} = \{u \in \hat{\mathcal{G}} \mid [u, x] \in \mathcal{T}_0 \; \forall x \in \mathbb{R}^{k+1}\}$$

is a reduction of the structure group of $\hat{\mathcal{G}}$ from $\mathrm{Euc}(n)$ to $(O(k) \times O(n-k)) \ltimes \mathbb{R}^k$.

Proof. We make use of 1.1.5 so we have to verify conditions (i) - (iv) from there.

(i) Let $x \in M$ and $u \in \mathcal{G}_x \subset \hat{\mathcal{G}}_x$. There is a $[u, x_0] \in (\mathcal{T}_0)_x$ such that $\mathrm{pr}_0(x) \neq 0$ and $g_x(x_0, x_0) = 1$, where pr_0 denotes the projection $\mathbb{R}^{n+1} \to \mathbb{R}$ on the first component. Complete $[u, x_0]$ to a basis of $(\mathcal{T}_0)$, that is of the form $[u, x_0], [u, x_1], \ldots, [u, x_k]$, such that $\mathrm{pr}_0(x_i) = 0$ for all $1 \leq i \leq k$ and the set $\{[u, x_1], \ldots, [u, x_k]\}$ is orthonormal with respect to g_x. By 5.1.4, $x_1, \ldots, x_k$ are orthonormal in $\{0\} \times \mathbb{R}^n$ with respect to the standard inner product, thus we can complete them to an orthonormal basis of $\{0\} \times \mathbb{R}^n$. Denote the base change from the standard basis of $\mathbb{R}^{n+1}$ to the above constructed basis by g. This is an element of $\mathrm{Euc}(n)$, since the first standard basis element is mapped to x_0 and the rest of the elements are in $\mathbb{R}^n$ and form an orthonormal basis.

   Then, $ug \in \mathcal{H}$ since $[ug, e_i] = [u, ge_i] = [u, x_i] \in (\mathcal{T}_0)_x$ for all $i \in \{0, \ldots, k\}$.

(ii) Let $u \in \mathcal{H}$ and $h \in O(k) \times O(n-k) \ltimes \mathbb{R}^k$. Then h stabilizes the subspace $\mathbb{R}^{k+1} \subset \mathbb{R}^{n+1}$, thus $[uh, x] = [u, hx] \in \mathcal{T}_0$ for all $x \in \mathbb{R}^{k+1}$.

(iii) Let $u, u' \in \mathcal{H}$ with $u' = ug$ for a $g \in G$. We have to show that $g \in H$. We know that $[u, x] \in \mathcal{T}_0$ and $[u', x] = [ug, x] = [u, gx] \in \mathcal{T}_0$ for all $x \in \mathbb{R}^{k+1}$. If g did not stabilize $\mathbb{R}^{k+1}$, the dimension of $(\mathcal{T}_0)_{\pi(u)}$ would be larger than $k+1$. Hence $g \in H$.

(iv) Let $\sigma : U \to \mathcal{G}$ be a local smooth section of $\mathcal{G} \subset \hat{\mathcal{G}}$ where $U \subset M$ is an open subset.

   The projection $\mathrm{pr}^{\mathcal{L}} : \mathcal{T} \to \mathcal{L}$ is a smooth surjective homomorphism of vector bundles. Thus there is a smooth section of $\mathcal{T}_0$ which gets mapped to the constant function $\mathbb{1}$. This corresponds to a smooth map $x_0 : U \to \mathbb{R}^{n+1}$ such that $||x_0(x)|| = 1$ and $\mathrm{pr}^{\mathcal{L}}([\sigma(x), x_0(x)]) \neq 0$ for all $x \in U$.

   Analogously as in (i), but in a smooth way, we complete $x \mapsto [\sigma(x), x_0(x)]$ to a local frame of $\mathcal{T}|_U$. Denote the respective base change by $g : U \to G$. Then, exactly as in (i), we obtain the local section σg of $\hat{\mathcal{G}}$ that has values in $\mathcal{H}$.

□

This allows us to construct reductions of the structure group from parallel subbundles of the canonical tractor bundle. Conversely, given a reduction of the structure group $\mathcal{H}$ from G to $O(k) \times O(n-k) \ltimes \mathbb{R}^{k+1}$ we build parallel subbundles of the tractor bundle by forming the associated bundle $\mathcal{H} \times_H \mathbb{R}^{k+1}$.

In order to prove that holonomy reductions of type $O(k) \times O(n-k) \ltimes \mathbb{R}^k$ are equivalent to certain parallel tractor bundles, we have to show the following

Lemma 5.3.9. *Consider the Lie algebra representation $\mathfrak{euc}(n) \to \mathfrak{gl}\left(\mathbb{R}^{n+1}\right)$. Then $\mathfrak{h} := \mathfrak{o}(k) \oplus \mathfrak{o}(n-k) \oplus \mathbb{R}^k$ is exactly the stabilizer of $\mathbb{R}^{k+1}$ in $\mathfrak{euc}(n)$.*

Proof. On the one hand, let $\begin{pmatrix} 0 & 0 & 0 \\ X & A & 0 \\ 0 & 0 & B \end{pmatrix} \in \mathfrak{h}$. Then $\begin{pmatrix} 0 & 0 & 0 \\ X & A & 0 \\ 0 & 0 & B \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ X \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 \\ X & A & 0 \\ 0 & 0 & B \end{pmatrix} \begin{pmatrix} 0 \\ e_i \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ Ae_i \\ 0 \end{pmatrix} \in \mathbb{R}^{k+1}$.

On the other hand, suppose indirectly that there is an element of $\mathfrak{euc}(n)$ that stabilizes $\mathbb{R}^{k+1}$ but is not in $\mathfrak{h}$. It must be of the form $\begin{pmatrix} 0 & 0 & 0 \\ X & A & D \\ Y & -D^T & B \end{pmatrix}$ for $Y \neq 0$ or $D \neq 0$.

However, $\begin{pmatrix} 0 & 0 & 0 \\ X & A & D \\ Y & -D^T & B \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ X \\ Y \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 \\ X & A & D \\ Y & -D^T & B \end{pmatrix} \begin{pmatrix} 0 \\ e_i \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ Ae_i \\ -D^T e_i \end{pmatrix}$. Therefore, it cannot stabilize $\mathbb{R}^{k+1}$. □

This allows us to show

Theorem 5.3.10. *Let (M, g) be a Riemannian manifold of dimension n and $(\mathcal{G} \to M, \omega)$ the corresponding Cartan geometry (cf. 5.1.1) and $k < n$. A holonomy reduction of type $\mathcal{O}$ (where $\mathcal{O}$ is as in 5.3.3) is equivalent to a $k+1$-dimensional, parallel subbundle T_0 of $\mathcal{T}$ such that $\mathrm{pr}^{\mathcal{L}}((\mathcal{T}_0)_x) \neq \{0\}$ for all $x \in M$.*

Proof. Denote $H := O(k) \times O(n-k) \ltimes \mathbb{R}^k$.

At first, we show that on the one hand, given a tractor bundle as in the above claim, the reduction of the structure group from 5.3.8 and on the other hand, forming the tractor bundle out of a reduction of the structure group, are inverse to each other.

Firstly, let $\mathcal{H}$ be an H-reduction. Then form $\mathcal{T}_0 := \mathcal{H} \times_H \mathbb{R}^{k+1}$. Finally, as in 5.3.8, let $\bar{\mathcal{H}} := \left\{u \in \hat{\mathcal{G}} \mid [u, x] \in \mathcal{T}_0 \; \forall x \in \mathbb{R}^{k+1}\right\}$. We have to show that $\mathcal{H} = \bar{\mathcal{H}}$.

On the one hand, let $u \in \mathcal{H}$. By definition, $[u, x] \in \mathcal{T}_0$ for all $x \in \mathbb{R}^{k+1}$, thus $x \in \bar{\mathcal{H}}$.

On the other hand, let $u \in \bar{\mathcal{H}}$. We take an element $u' \in \mathcal{H}$ such that $\pi(u) = \pi(u')$, hence there is a $g \in G$ such that $u = u'g$. We conclude, that for all $x \in \mathbb{R}^{k+1}$ we have $[u', gx] = [u'g, x] = [u, x] \in \mathcal{T}_0$. Therefore, g has to stabilize the subspace $\mathbb{R}^{k+1} \subset \mathbb{R}^{n+1}$, since otherwise $\mathcal{T}_0$ would have dimension higher than $k+1$. Thus $g \in H$ and $u = u'g \in \mathcal{H}$.

Now, let us start with a given sub-vector bundle $\mathcal{T}_0$ that has the same properties as described in 5.3.8. Let $\mathcal{H} := \left\{u \in \hat{\mathcal{G}} \mid [u, x] \in \mathcal{T}_0 \; \forall x \in \mathbb{R}^{k+1}\right\}$, and form $\bar{\mathcal{T}}_0 := \mathcal{H} \times_H \mathbb{R}^{k+1}$. We have to show that $\bar{\mathcal{T}}_0 = \mathcal{T}_0$.

Let $[u, x] \in \bar{\mathcal{T}}_0$ where $u \in \mathcal{H}$ and $x \in \mathbb{R}^{k+1}$. By definition, $[u, y] \in \mathcal{T}_0$ for all $y \in \mathbb{R}^{k+1}$, hence in particular $[u, x] \in \mathcal{T}_0$.

On the other hand, consider an element $[u,x] \in \mathcal{T}_0$ where $u \in \hat{\mathcal{G}}$ and $x \in \mathbb{R}^{n+1}$. Choose an element $u' = ug \in \mathcal{H}$ where $g \in G$. Then $\left[u', g^{-1}x\right] = \left[u'g^{-1}, x\right] = [u,x] \in \mathcal{T}_0$, hence $g^{-1}x \in \mathbb{R}^{k+1}$ again by dimensional reasons. Therefore, $g \in H$ and $[u,x] = \left[u', g^{-1}x\right] \in \bar{\mathcal{T}}_0$.

There is one thing left to prove: we saw before that given a reduction of the structure group that is compatible with the prinicipal connection on $\hat{\mathcal{G}}$ induces a parallel sub-tractor bundle. However, we have to show that in our situation also the converse is true, i.e. that $\mathcal{H}$ from 5.3.8 is compatible with the connection $\hat{\omega}$ on $\hat{\mathcal{G}}$.

From above, we know that $\mathcal{T}_0 = \mathcal{H} \times_H \mathbb{R}^{k+1}$. We want to show that the horizontal distribution satisfies $\mathfrak{H}_u \subset T_u\mathcal{H}$ for all $u \in \mathcal{H}$.

Suppose indirectly that there is a $u \in \mathcal{H}$ such that $\mathfrak{H}_u \not\subset T_u\mathcal{H}$. Take a horizontal curve $c : I \to \hat{\mathcal{G}}$ such that $c(0) = u$ and $c(t) \notin \mathcal{H}$ for all $t > 0$, such that I is an open interval containing 0.

Furthermore, choose a local section τ of $\mathcal{H}$ around $\pi(u)$ satisfying $\tau(\pi(u)) = u$. By the inverse function theorem, we can find a smooth map $g : I \to G$ such that $c(t) = \tau(\pi(c(t)))\, g(t)$. We see that $g(0) = e$ and $g^{-1}(t) \notin H$ for all $t > 0$, hence $X := \frac{d}{dt}|_0 g^{-1}(t) \notin \mathfrak{h}$.

Now choose $x \in \mathbb{R}^{k+1}$ such that $X \cdot x \notin \mathbb{R}^{k+1}$ and $\hat{f} : \hat{\mathcal{G}} \to \mathbb{R}^{n+1}$ an equivariant function such that $\hat{f}(u) = x$. The function $\hat{f}$ corresponds to a section of $\mathcal{T}$ with values in $\mathcal{T}_0$. By definition of $\mathcal{T}_0$, the covariant derivative of this section along any vector field ξ on the underlying manifold again has values in $\mathcal{T}_0$. This covariant derivative corresponds to $\xi^{hor} \cdot \hat{f}$, hence $\xi^{hor} \cdot \hat{f}$ again has values in $\mathbb{R}^{k+1}$.

We obtain

$$\begin{aligned}
\mathbb{R}^{k+1} \ni \left(c'(0) \cdot \hat{f}\right) &= \frac{d}{dt}|_0 \hat{f}(c(t)) = \frac{d}{dt}|_0 g^{-1}(t)\, \hat{f}(\tau(\pi(c(t)))) \\
&= \left(\frac{d}{dt}|_0 g^{-1}(t)\right) \hat{f}(\tau(\pi(c(0)))) + g^{-1}(0) \left(\frac{d}{dt}|_0 \hat{f}(\tau(\pi(c(t))))\right) \\
&= X\hat{f}(u) + \frac{d}{dt}|_0 \hat{f}(\tau(\pi(c(t)))).
\end{aligned}$$

However, $\hat{f}(u) \in \mathbb{R}^{k+1}$ and τ is a section of $\mathcal{H}$, hence also the second summand is in $\mathbb{R}^{k+1}$.

Therefore we have a contradiction, since $X \cdot x \notin \mathbb{R}^{k+1}$ but the other two terms are elements of $\mathbb{R}^{k+1}$. □

Altogether this shows that holonomy reductions with holonomy group $O(k) \times O(n-k) \ltimes \mathbb{R}^k$ on a Riemannian geometry are equivalent to certain parallel subbundles of the tractor bundle.

Now we want to understand the structures that are induced on the tangent bundle.

Denote $E := \mathcal{T}_0 \cap TM$. We already noted before that this bundle has constant rank k, thus we obtain a distribution on M.

Since v_0 is not H-invariant, the dimension of the intersection of $\mathcal{L}$ and $\mathcal{T}_0$ will in general not be constant. By dimensional reasons there are only two possibilities for each $x \in M$: either $\mathcal{L}_x \subset (\mathcal{T}_0)_x$ or $\mathcal{L}_x \cap (\mathcal{T}_0)_x = \{0\}$. In the following we will investigate how the intersection $\mathcal{L}_x \cap (\mathcal{T}_0)_x$ behaves for $x \in M$ with respect to the curved orbits.

Moreover, we can consider the projection $q : \mathcal{L} \oplus TM \to TM$. If $(\mathcal{L}_x \cap (\mathcal{T}_0)_x) = \{0\}$ for $x \in M$, we have $\dim(q((\mathcal{T}_0)_x)) = k+1$ and $E_x \subset q((\mathcal{T}_0)_x)$. Since T_xM carries a metric g_x, we obtain the decomposition $q((\mathcal{T}_0)_x) = E_x \oplus E_x^{\perp}$, where $E_x^{\perp}$ denotes the orthogonal complement of E_x in $q((\mathcal{T}_0)_x)$. The vector space $E_x^{\perp}$ has dimension 1. We denote it by L_x. Furthermore, we denote the orthogonal complement of $q((\mathcal{T}_0)_x)$ in T_xM by F_x, hence we have $T_xM = E_x \oplus L_x \oplus F_x$.

Note that this decomposition depends smoothly on x, thus we have outside of

$$S := \{x \in M \mid \mathcal{L}_x \subset (\mathcal{T}_0)_x\}$$

the following decomposition:

$$TM|_{M\setminus S} = E|_{M\setminus S} \oplus L \oplus F.$$

Later we will show that $S = M_0$ (cf. 5.3.22).

Also the other curved orbits can be characterized in terms of the relative positions of $\mathcal{L}$ and $\mathcal{T}_0$: Note that $\mathcal{T}$ decomposes into $\mathcal{T}_0 \oplus E^{\perp}$ where $E^{\perp}$ is the orthogonal complement of E in TM. The intersection is trivial, since $\mathcal{T}_0 \cap TM = E$ and $E^{\perp} \subset TM$. Furthermore, we have $\dim(\mathcal{T}_0 + E^{\perp}) = \dim(\mathcal{T}_0) + \dim(E^{\perp}) = k+1+n-k = n+1$, and thus $\mathcal{T} = \mathcal{T}_0 \oplus E^{\perp}$. Denote the projection $\mathcal{T} \to E^{\perp}$ with respect to this decomposition by pr. Later we will see that $x \in M_r$ if and only if $||\operatorname{pr}(\mathbb{1}_x)|| = r$, where $\mathbb{1}$ is the canonical section of $\mathcal{L}$ and the norm is computed with respect to the Riemannian metric on M.

5.3.3 The homogenous model

Firstly, we will thoroughly examine the case of the homogenous model, so that we are later able to generalize results via Comparison (cf. 4.5.1). Here we follow the theoretical observations on holonomy reductions of the homogenous model in 4.4.3.

The curved orbits

Thus we consider the H-action on G/P to determine the curved-orbit-decomposition in order to study the reduction $G/P \times H \subset G/P \times G$.

Notation Let M be a Riemannian manifold endowed with a holonomy reduction of type $\mathcal{O}$ (where $\mathcal{O}$ is as in Proposition 5.3.3). We denote the curved orbit corresponding to $\mathcal{O}_r$ (see Proposition 5.3.5) by M_r for $r \in \mathbb{R}_0^+$.

Proposition 5.3.11. *The H-orbits in $G/P = \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ are given by*

$$(G/P)_0 = \mathbb{R}^k \times \{0\} \text{ and } (G/P)_r = \mathbb{R}^k \times S_r^{n-k-1},$$

where $r > 0$ and S_r^{n-k-1} denotes the $n-k-1$-dimensional sphere of radius r in $\mathbb{R}^{n-k}$ with respect to the standard inner product restricted to R^{n-k}.

Proof.
$$\begin{pmatrix}1 & 0 & 0\\ v' & C & 0\\ 0 & 0 & D\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ x & I\end{pmatrix}P=\begin{pmatrix}1 & 0 & 0\\ v'+Cx_1 & C & 0\\ Dx_2 & 0 & D\end{pmatrix}P,$$

where $x=\begin{pmatrix}x_1\\ x_2\end{pmatrix}$ for $x_1, v' \in \mathbb{R}^k$, $x_2 \in \mathbb{R}^{n-k}$, $C \in O(k)$ and $D \in O(n-k)$. Thus $(C, D, v')\cdot(x_1, x_2) = (Cx_1 + v', Dx_2)$.

Therefore, the H-orbit that contains $0 \in \mathbb{R}^n$ is given by

$$\left\{(C\cdot 0+v', D\cdot 0) \mid C\in O(k), D\in O(n-k), v'\in\mathbb{R}^k\right\} = \mathbb{R}^k\times\{0\}\subset\mathbb{R}^n.$$

Now let $r \in \mathbb{R}^+$ and consider the H-orbit that contains re_n. On the one hand, let $(C, D, v') \in H$. Then $(C, D, v')\cdot re_n = (v', rDe_{n-k}) \in \mathbb{R}^k \times S_r^{n-k-1}$. Assume on the other hand $(x_1, x_2) \in \mathbb{R}^k \times S_r^{n-k-1}$. The orthogonal group acts transitively on the spheres, hence we can choose $D \in O(n-k-1)$ such that $D\,(re_{n-k}) = x_2$. Then $(I_k, D, x_1)\cdot re_n = (x_1, x_2)$.

We still have to prove that $x \in (G/P)_r$ has P-type $\mathcal{O}_r$. Take $re_n \in (G/P)_r$ for $r \in \mathbb{R}_0^+$, then $re_n \widehat{=} \begin{pmatrix}1 & 0\\ re_n & I\end{pmatrix}P$. Note that its P-type is given by $P\cdot g^{-1}H$ (cf. 4.4.3), and

$$\begin{aligned}g^{-1}(W_0) &= \begin{pmatrix}1 & 0\\ -re_n & I_n\end{pmatrix}\left\langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\e_1\end{pmatrix}, \dots, \begin{pmatrix}0\\e_n\end{pmatrix}\right\rangle\\ &= \left\langle \begin{pmatrix}1\\-re_n\end{pmatrix}, \begin{pmatrix}0\\e_1\end{pmatrix}, \dots, \begin{pmatrix}0\\e_k\end{pmatrix}\right\rangle = W_r \text{ (see 5.3.5).}\end{aligned}$$

This proves that re_n has P-type $\mathcal{O}_r$, thus the whole curved orbit $(G/P)_r$ has this P-type. □

Note that G/P decomposes into infinitely many orbits, where one is of dimension k and the others have dimension $n-1$.

Next we consider the induced structures on the curved orbits (cf 4.4.3(ii)). We compute the stabilizers H_α for representatives α in each $\mathcal{O}_r$ for $r \in \mathbb{R}_0^+$ and then consider the geometry $H_\alpha/(H_\alpha \cap P)$.

Proposition 5.3.12. *The singular orbit $(G/P)_0$ inherits a Riemannian structure, whereas the other orbits $(G/P)_r$ admit the structure of a global Riemannian product of dimension $k\times(n-k-1)$ for $r>0$.*

Proof. We already know from Proposition 5.3.11 how the curved orbits look like. We choose the elements $re_n \in (G/P)_r = \mathbb{R}^k \times S_r^{n-k-1}$, i.e. for $r = 0$ we have $0 \in (G/P)_0 = \mathbb{R}^k \times \{0\}$, and compute their stabilizers in H in order to write the orbits as homogenous spaces.

The element $re_n \in (G/P)_r$ corresponds to $g_rP \in G/P$ where $g_r = \begin{pmatrix}1 & 0\\ re_n & I_n\end{pmatrix}$. Note that

$$\begin{pmatrix}1 & 0 & 0\\ v & C & 0\\ 0 & 0 & D\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & I_k & 0\\ re_{n-k} & 0 & I_{n-k}\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ v & C & 0\\ rDe_n & 0 & D\end{pmatrix},$$

where $v \in \mathbb{R}^k$, $C \in O(k)$ and $D \in O(n-k)$ and $e_{n-k} \in \mathbb{R}^{n-k}$ denotes the unit vector with 1 in the last entry.

Thus if $r = 0$, we look for all elements in H that stabilize g_0P. These are precisely the elements of H with $v = 0$, i.e.

$$\begin{aligned}(G/P)_0 = H \cdot g_0P &= ((O(k) \times O(n-k)) \ltimes \mathbb{R}^k)/(O(k) \times O(n-k)) \\ &= \mathrm{Euc}(k)/O(k) \times \{0\}.\end{aligned}$$

This is the Euclidean space of dimension k embedded into the Euclidean space of dimension n.

In the case $r > 0$, the elements of H that stabilize g_rP are those satisfying $v = 0$ and $rDe_{n-k} = re_{n-k}$. Thus $D \in O(n-k-1)$ and we obtain

$$\begin{aligned}(G/P)_r = H \cdot g_rP &= ((O(k) \times O(n-k)) \ltimes \mathbb{R}^k)/(O(k) \times O(n-k-1)) \\ &= \mathrm{Euc}(k)/O(k) \times O(n-k)/O(n-k-1).\end{aligned}$$

This is $\mathbb{R}^k \times S^{n-k-1}$ endowed with the product of the two respective standard metrics. □

Hence we have one singular curved orbit, $(G/P)_0$, that has dimension k, while all other curved orbits have dimension $n-1$ and satisfy $(G/P)_r = \{x \in \mathbb{R}^n \mid d\left((G/P)_0, x\right) = r\}$ for $r \in \mathbb{R}_0^+$, where d measures the distance between a point and a closed set in $\mathbb{R}^n$. More precisely, $d(A, y) := \inf_{x \in A} d(x, y)$ where A is a closed set in $\mathbb{R}^n$ and $y \in \mathbb{R}^n$.

Furthermore, any curved orbit $(G/P)_r$ is a global product of $(G/P)_0$ and a sphere, and its Riemannian metric decomposes into these two parts. They should be interpreted as the "parallel" part to $(G/P)_0$ and the "spherical" part that represents the distance to a point on $(G/P)_0$.

Also, note that the Riemannian structures that are induced by the holonomy reduction are the respective restrictions of the Riemannian metric on the surrounding space G/P.

This is clear in view of the construction of the curved orbits (cf. 4.4.3): We obtained the structures of the curved orbits as subsets of the original homogenous space for different $g \in G$:

$$\begin{tikzcd} g^{-1}Hg \arrow[r, hook] \arrow[d] & G \arrow[d] \\ g^{-1}Hg/g^{-1}Hg \cap P \arrow[r, hook] & G/P \end{tikzcd}$$

The Maurer-Cartan-form on G that is pull-backed along the inclusion gives the Maurer-Cartan-form of the subgroup $g^{-1}Hg \subset G$. Therefore the structures on the curved orbits must clearly be compatible with the surrounding structure of G/P.

The other holonomy reductions of the homogenous model are easily computed by applying the action of an element of $\mathrm{Euc}(n)$ (cf. 4.4.5): Let $(A, v) \in O(n) \ltimes \mathbb{R}^n$. Then $(A, v) \cdot (G/P)_0 = A(\mathbb{R}^k \times \{0\}) + v$. Note that this gives all possible k-dimensional affine subspaces of the Euclidean space, equipped with the Euclidean metric of $\mathbb{R}^n$ restricted to the subspace.

In particular, they are all totally geodesic:

Definition Let (M, g) be a Riemannian manifold. A submanifold $N \subset M$ is called *totally geodesic* if for all $x \in N$ and $\xi \in T_x N$ the geodesic (in M) with initial data (x, ξ) stays at least for a short time in N.

Also, since the action of an element of G preserves the Euclidean metric, the other curved orbits remain the sets of fixed distance to $(G/P)_0$.

The tractor bundle

Next, we compute the subbundle $\mathcal{T}_0$ of the tractor bundle explicitly.

Consider the H-reduction $\mathcal{H} := G/P \times H$ of $\hat{G} = G/P \times G$. We have $\mathcal{T}_0 = \mathcal{H} \times_H W_0 = (G/P \times H) \times_H W_0$.

In 5.1.5(i) we saw that associated bundles of trivial principal bundles admit a trivialization as well, i.e. $(G/P \times H) \times_H V \cong G/P \times V$ via $[(g'P, h), v] \mapsto (g'P, hv)$, where V is an arbitrary vector space endowed with a G-action and $g' \in G$, $h \in H$ and $v \in V$.

In particular, we can compute $\mathcal{T}_0$ in the trivialization and answer the question for which $v \in G/P = \mathbb{R}^n$ we have $\mathcal{L}_v \subset (T_0)_v$. The form of the line bundle $\mathcal{L}$ in the trivialization was computed in 5.1.6.

For $v \in \mathbb{R}^n$ we have $\mathcal{L}_v \subset (\mathcal{T}_0)_v$ if and only if $\begin{pmatrix} 1 \\ v \end{pmatrix} \in W_0$. This is true if there are real numbers $(\alpha_0, \dots, \alpha_k)$ such that

$$\begin{pmatrix} 1 \\ v \end{pmatrix} = \alpha_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_1 \begin{pmatrix} 0 \\ e_1 \end{pmatrix} + \cdots + \alpha_k \begin{pmatrix} 0 \\ e_k \end{pmatrix}.$$

One sees immediately, that $\alpha_0 = 1$, hence this is the case if and only if $v \in \langle\, e_1, \dots, e_k \,\rangle$, that is a k-dimensional affine subspace in $\mathbb{R}^n$.

Summarizing this information, we obtain the following

Proposition 5.3.13. *Consider the holonomy reduction $G/P \times H \subset G/P \times G$ of the homogenous model $G \to G/P$. Then we have*

$$S := \{v \in \mathbb{R}^n \mid \mathcal{L}_v \subset (\mathcal{T}_0)_v\} = \mathbb{R}^k \times \{0\} = (G/P)_0 \,.$$

Recall the situation $V = \langle v_0 \rangle \oplus \ker(\varphi_0)$ together with a $W \in \mathcal{O}$, that is a $k+1$-dimensional subspace $W \subset V$ such that $\varphi_0\big|_W \neq 0$. There we observed that the P-orbits in $\mathcal{O}$ are exactly those W, for that $||\operatorname{pr}(v_0)||$ is constant, where $\operatorname{pr} : V \to E^\perp$ with respect to the decomposition $V = W \oplus E^\perp$, where $E = W \cap \ker(\varphi_0)$.

Proposition 5.3.14. *An element $x \in G/P$ lies in the curved orbit $(G/P)_r$ if and only if $||\operatorname{pr}(\mathbb{1}_x)|| = r$, where $\mathbb{1}$ is the section $(\pi(u)) = [u, v_0]$ (for $u \in \mathcal{G}$) and $||.||$ denotes the norm on $T_x M$ that is induced by g_x.*

Proof. We compute the projection in the canonical trivialization: The tractor bundle $\mathcal{T}_0$ is given by $G/P \times (\mathbb{R}^{k+1} \times \{0\})$ and $E = G/P \times (\{0\} \times \mathbb{R}^k \times \{0\})$, thus

$E^\perp = G/P\times(\{0\}\times\mathbb{R}^{n-k})$. Therefore the projection pr simply is the projection of the second component to the last $n-k$ components.

Recall from 5.1.6 that the line bundle $\mathcal{L}_x \cong \left\{\left(x, \begin{pmatrix}\lambda\\ \lambda x\end{pmatrix}\right) \mid \lambda\in\mathbb{R}\right\}$ in the trivialization $\mathcal{T}\cong G/P\times\mathbb{R}^{n+1}$. In particular, the distinguished element $v_0(x)$ in $\mathcal{L}$ is $\left(x, \begin{pmatrix}1\\ x\end{pmatrix}\right)$.

Thus $||\,\mathrm{pr}(v_0(x))||$ is just the norm of the last $n-k$ components of x. Proposition 5.3.11 immediately implies that this characterizes the curved orbits. $\square$

The decomposition of the tangent bundle

Proposition 5.3.13 shows that for x outside of $S=(G/P)_0$ we have $\mathcal{L}_x \not\subset (\mathcal{T}_0)_x$. Thus projecting $q:\mathcal{L}\oplus TG/P\to TG/P$ yields the $k+1$-dimensional subbundle $q((\mathcal{T}_0)_x)$ of T_xG/P. This contains E_x, thus we can distinguish a line L_x that is the orthogonal complement of E_x in $q((\mathcal{T}_0)_x)$. The whole construction is smooth, thus we get a smooth line bundle L on $G/P\setminus S$.

Furthermore, denote the orthogonal complement of $q((\mathcal{T}_0)_x)$ in T_xG/P by F_x. This yields a distribution of rank $n-k-1$ on $G/P\setminus S$.

The next proposition will show how these distributions interact with the curved orbits:

Proposition 5.3.15. *(i) We have $TS = E|_S$.*

(ii) For all $v\in(G/P)_r$ we have $T_v(G/P)_r\perp L_v$.

(iii) E_v and F_v are tangential to the curved orbit $(G/P)_r$ for $v\in(G/P)_r$. In particular, $T_v(G/P)_r = E_v\oplus F_v$.

Proof. First we have to compute the projection in coordinates. In this proof, restrict all bundles over G/P to bundles over $(G/P)\setminus S$.

Let $v\in(G/P)\setminus(G/P)_0$. We have $\mathcal{L}=\left\{\left(v,\begin{pmatrix}\lambda\\ \lambda v\end{pmatrix}\right)\in V \mid v\in G/P,\ \lambda\in\mathbb{R}\right\}$ and $T\,(G/P)=\{0\}\times\mathbb{R}^n$. An element $x=\begin{pmatrix}x_1\\ x_2\end{pmatrix}\in\mathcal{T}_v=V$ is of the form $\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\alpha_1\begin{pmatrix}1\\ v\end{pmatrix}+\alpha_2\begin{pmatrix}0\\ y\end{pmatrix}$ for $\alpha_1,\alpha_2\in\mathbb{R}$ and $y\in\mathbb{R}^n$. Then $\alpha_1=x_1$ and $q\,(v,x)=\alpha_2 y = x_2-\alpha_1 v = x_2 - x_1 v$.

(i) We have $\mathcal{T}_0 = G/P\times W_0$ and $TM = G/P\times(\{0\}\times\mathbb{R}^n)$. Therefore, the intersection $E=\mathcal{T}_0\cap TM$ is the subbundle $G/P\times(\mathbb{R}^k\times\{0\})$ in $G/P\times\mathbb{R}^n$. Restricted to $(G/P)_0=\mathbb{R}^k\times\{0\}$, this gives $S\times(\mathbb{R}^k\times\{0\})$, that is exactly the tangent bundle of S in $T(G/P)$.

(ii) We have

$$q(v,(\mathcal{T}_0)_v) = \langle\, q(v,e_0),\dots q(v,e_k)\,\rangle = \langle\, -v, e_1,\dots,e_k\,\rangle\,.$$

Thus $n(v) := v - \sum_{1\le i\le k}\langle\, v,e_i\,\rangle\, e_i$ is orthogonal to $\mathbb{R}^k\times\{0\}$ and contained in $q((\mathcal{T}_0)_x)$. This is just the projection of v to the last $n-k$ components, that we denote by $\mathrm{pr}^{n-k}(v)$. This is a nonzero element of L. We know $(G/P)_r = \mathbb{R}^k\times S_r^{n-k-1}$ (see 5.3.11), hence if $v\in(G/P)_r$, we have

$n(v) \in \{0\} \times S_r^{n-k-1}$ and thus $n(v) \in (T_v(G/P)_r)^\perp$.

(iii) This follows from (ii), since $T(G/P)_r = L^\perp = E \oplus F$. □

5.3.4 Comparison

Now we move on to holonomy reductions of type $\mathcal{O}$ (where $\mathcal{O}$ is as in 5.3.3) of Riemannian Cartan geometries over smooth manifolds.

We can conclude from the homogenous model that each holonomy reduction induces a distinguished curved orbit M_0 that is (if non-empty) of dimension k, while all others have dimension $n-1$: First, note that all curved orbits are submanifolds of M. This is easy to see by applying the Comparison-Lemma 4.5.1: Comparison with the holonomy reduction of the homogenous model immediately provides a submanifold chart for M_0. For the other curved orbits M_r, where $r \in \mathbb{R}^+$, similarly use the local diffeomorphism obtained from comparison with the homogenous model and then compose with a submanifold chart for the respective orbit $(G/P)_r = \mathbb{R}^k \times S_r^{n-k-1}$.

We know from Theorem 4.5.4 that the structures on the curved orbits are of the same type as those of the homogenous model. Thus we apply 5.3.12 and see that the distinguished orbit M_0 is a Riemannian manifold and the other curved orbits M_r are Riemannian manifolds as well, that in addition decompose locally into a product of dimension $k + (n-k)$. Furthermore, by construction of the Cartan geometries $(\mathcal{G}_i, \omega_i)$ in the proof of 4.5.4, their structures are compatible with the surrounding Riemannian metric g, meaning that for each $r \in \mathbb{R}_0^+$ the Riemannian metric g_r coming from the reduction satisfies $j_r^* g = g_r$, where j_r is the inclusion of M_r into M.

We want to show that some properties of the curved orbits of the homogenous model remain true in the curved case. To that effect we make use of the explicit form of $\mathfrak{g}$. We take a closer look at the local diffeomorphism ϕ from the Comparison-Lemma 4.5.1. It depends on the choice of a complement $\mathfrak{g}_-$ of $\mathfrak{p}$ in $\mathfrak{g}$.

In our case, $\mathfrak{g} = \mathfrak{o}(n) \oplus \mathbb{R}^n$ as a P-module and $\mathfrak{p} = \mathfrak{o}(n)$. Indeed, it is the unique P-invariant complement of $\mathfrak{p}$ in $\mathfrak{g}$. Furthermore, the Cartan connection ω is of the form $\theta + \gamma$, where $\theta \in \Omega^1(\mathcal{G}, \mathbb{R}^n)$ is the canonical soldering form and $\gamma \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is a principal connection on $\mathcal{G}$, that is the equivalent to the Levi-Civita-connection on the orthonormal frame bundle (cf. 5.1.1). This emphasizes the naturality of the decomposition $\mathfrak{g} = \mathfrak{o}(n) \oplus \mathbb{R}^n$. Hence in the following we will always choose $\mathfrak{g}_- = \mathbb{R}^n$ as a natural complement to $\mathfrak{p}$.

Notation For two holonomy reductions on Riemannian manifolds M and M', one has to choose $x \in M_r$ and $y \in M'_r$ in order to obtain a diffeomorphism from 4.5.1 between open neighborhoods of x and y, respectively. We will denote it by ϕ_y^x. Note that ϕ_y^x still depends on the choice of orthonormal bases of T_xM and T_yM'.

Let $(\pi : \mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G, P) that is endowed with a holonomy reduction of type $\mathcal{O}$. For Comparison, consider the homogenous model endowed with the holonomy reduction $G/P \times H \subset G/P \times G$.

Proposition 5.3.16. *If $x \in M_r$ and $y \in (G/P)_r$ for $r \in \mathbb{R}_0^+$ there are open neighborhoods U of x and V of y such that each $\phi_x^y : V \to U$ is of the form $y' \mapsto \exp_x(f_0(y' - y))$ where f_0 is an isometry between $\mathbb{R}^n$ and $T_x M$. In particular, the map ϕ_x^y as in the Proposition above is defined exactly where the exponential map based at x is a diffeomorphism.*

Proof. First of all, we have to choose $u_0 \in \mathcal{G}_x$ and $g_0 \in G_y$ such that $s(u_0)$, $s^{G/P}(g_0) \in \mathcal{O}_r$ (where s and $s^{G/P}$ are the equivariant function corresponding to the holonomy reductions on $\mathcal{G}$ and G, respectively). Recall from 4.5.1 that the local diffeomorphism ϕ_y^x is defined such that the following diagram commutes:

$$\begin{tikzcd}
G \arrow[d] & & \mathbb{R}^n = \mathfrak{g}_- \arrow[ll, "\mathrm{Fl}_1^{\omega_{MC}^{-1}(\cdot)}(g_0)"'] \arrow[rr, "\mathrm{Fl}_1^{\omega^{-1}(\cdot)}(u_0)"] \arrow[dll, "\psi'"'] \arrow[dl, "g_0"] \arrow[dr, "u_0"'] \arrow[drr, "\psi"] & & \mathcal{G} \arrow[d, "\pi"] \\
\mathbb{R}^n \arrow[rrrr, bend right, "\phi_x^y"'] & T_y\mathbb{R}^n & & T_x M & M
\end{tikzcd}$$

where ψ and ψ' are the local diffeomorphisms as in the proof of 4.5.1 (the are defined such that the diagram commutes).

Note that the construction of the local diffeomorphism in the proof of 4.5.1 is a generalization of the construction of the exponential function (see [3, p.56]). One can define the exponential map as $\exp_x(\xi) := \pi\left(\mathrm{Fl}_1^{\omega^{-1}\left((u_0)^{-1}(\xi)\right)}(u_0)\right)$ where $u_0 \in \pi^{-1}(x)$ and ξ is sufficiently close to $0 \in T_x M$ such that the flow is defined.

Hence we have $\psi(X) = \pi(\mathrm{Fl}_1^{\omega^{-1}(X)}(u_0)) = \exp_x(u_0(X))$ for $X \in \mathfrak{g}_-$ that is sufficiently close to 0, and analogously $\psi' = \exp_y \circ g_0$. Therefore,

$$\phi_x^y = \psi \circ \psi'^{-1} = \exp_x \circ u_0 \circ g_0^{-1} \circ \exp_y^{-1},$$

and $f_0 := u_0 \circ g_0^{-1}$ is an isometry $\mathbb{R}^n \to T_x M$.

For G/P the exponential map is given by $\exp_y(v) = y + v$ where $y \in G/P$ and $v \in \mathbb{R}^n$, hence $\exp_y^{-1}(y') = y' - y$. Thus $\phi_x^y(y') = \exp_x(f_0(y' - y))$. $\square$

Now we are able to show that the distinguished curved orbit M_0 is totally geodesic:

Proposition 5.3.17. *Let (M, g) be a Riemannian manifold endowed with a holonomy reduction of type $\mathcal{O}$ as in 5.1.2. Then the k-dimensional curved orbit M_0 (if non-empty) is totally geodesic.*

Proof. This follows immediately from the form of the local diffeomorphism in the Comparison-Lemma 4.5.1: Let $x \in M_0$ and $\xi \in T_x M_0$, and denote the geodesic through x with initial velocity ξ by c. We want to show that c stays inside of M_0 for some interval $(-\epsilon, \epsilon)$ where $\epsilon > 0$.

Compare with the homogenous model: By 5.3.16, we can locally write $M_0 = \exp_x(f_0(U \times \{0\}))$ where $f_0 : \mathbb{R}^n \to T_x M$ is an isometry and U is a neighborhood of 0 in $\mathbb{R}^k$. Since $T_x M = T_0 \exp_x \cdot T f_0 \cdot (\mathbb{R}^k \times \{0\}) = f_0(\mathbb{R}^k \times \{0\})$, the tangent vector ξ is of the form $f_0(\zeta)$ for some $\zeta \in \mathbb{R}^k \times \{0\}$.

Let $\epsilon > 0$ such that the geodesic c is defined on $(-\epsilon, \epsilon)$ and of the form $c(t) = \exp_x(t\xi) = \exp_x(f_0(t\zeta))$ where $-\epsilon \leq t \leq \epsilon$. This is obviously inside of M_0. □

Now we turn to the other curved orbits. We will see that locally they can be interpreted as sets of fixed distance from M_0 – or from each other – as we already observed in the case of the homogenous model.

Proposition 5.3.18. *Let $x_0 \in M_r$ for $r \in \mathbb{R}_0^+$. Then there is a neighborhood U of x_0 in M such that $M_{r'} \cap U = \{x \in U \mid d(x, M_{r'}) = |r - r'|\}$ for all $r' \in \mathbb{R}_0^+$.*

Proof. Let U' be a totally normal neighborhood of x (a neighborhood that is normal for each of its elements). By a normal neighborhood we mean an the image of an open neighborhood of 0 in the tangent space of the point, on that the exponential map is a diffeomorphism, under exp.

In the first step, we shrink U': It is well-known, that there is a $\rho > 0$ such that the ball $B_\rho(x)$ is geodesically convex, i.e. for each pair of points (x', x'') in $B_\rho(x)$ there is a unique minimizing geodesic between x' and x'' that stays inside of $B_\rho(x)$ (see [7, p.85]). Let $\rho \in \mathbb{R}^+$ such that $B_\rho(x_0)$ is geodesically convex and contained in U. Then we define $U := B_{\frac{\rho}{2}}(x_0) \cap \{x \in M_{r'} \mid |r - r'| < \frac{\rho}{2}\}$. Note that M is locally foliated into the M_r's by comparison, thus U is an open neighborhood of x_0.

<u>*Claim 1:*</u> For all $x \in M_{r'} \cap B_\rho(x_0)$ we have $d(x, M_r) \geq |r - r'|$.

proof of claim: Let $y := \begin{pmatrix} 0 \\ r'e_{n-k} \end{pmatrix} \in \mathbb{R}^k \times S_{r'}^{n-k-1} = (G/P)_{r'}$ and consider $V := \phi^{-1}(U)$ where $\phi := \phi_x^y$ (ϕ^{-1} is defined on U since the inverse of the exponential map based at x is always defined on a totally normal neighborhood around x).

(a) Let $x_1 \in M_r \cap U$, then x_1 is of the form $\phi(y_1)$ where $y_1 \in \mathbb{R}^k \times S_r^{n-k-1} = (G/P)_r$. Therefore, and by 5.3.16, we obtain

$$d(x, x_1) = d(x, \exp_x(f(y_1 - y))) = ||f(y - 1 - y)|| = ||y_1 - y|| \geq |r - r'|,$$

where $f : \mathbb{R}^n \to T_xM$ is the isometry such that $\phi(y') = \exp_x(f(y' - y))$.

(b) Let $x_1 \in M_r \setminus B_\rho(x_0)$, then we have

$$\rho \leq d(x_0, x_1) \leq d(x_0, x) + d(x, x_1) < \frac{\rho}{2} + d(x, x_1),$$

hence $d(x, x_1) > \frac{\rho}{2}$ and $x \in U$, thus by definition $|r - r'| < \frac{\rho}{2}$. Consequently, $d(x, x_1) > |r - r'|$.

<u>*Claim 2:*</u> For all $x \in M_{r'} \cap U$ there is a $x_1 \in M_r \cap U$ such that $d(x, x_1) = |r - r'|$.

proof of claim: Again consider y as above and $\phi = \phi_x^y$. Let $y_1 := \begin{pmatrix} 0 \\ re_{n-k} \end{pmatrix} \in \mathbb{R}^k \times S_r^{n-k-1} = (G/P)_r$. Then $x_1 := \phi(y_1)$ is in $M_r \cap U$ and

$$d(x, x_1) = d(x, \exp_x(f(y_1 - y))) = ||f(y_1 - y)|| = ||y_1 - y|| = |r' - r|.$$

□

This shows how to interpret the curved orbit decomposition geometrically. Consequently in the case $M_0 \neq \emptyset$, given the submanifold M_0, we can locally reconstruct the decomposition into curved orbits. Our aim will be to to reconstruct the whole holonomy reduction locally. For that, we use tractor bundles as an auxiliary construction.

Above we obtained a clear picture of the behaviour of the tractor bundle in the homogenous model. The crucial point will now be to carry this over to any Cartan geometry of type $(\mathrm{Euc}\,(n), O\,(n))$. To this end we will note that the Comparison-map from 4.5.1 induces a local vector bundle isomorphism between the canonical tractor bundles and that this isomorphism preserves the examined subbundles.

Proposition 5.3.19. *Let $\mathcal{G} \to M$ be a Cartan geometry, endowed with a holonomy reduction $\mathcal{H}$ of type $\mathcal{O}$ and $\mathcal{H}^{G/P}$ a holonomy reduction of type $\mathcal{O}$ of the homogenous model. The Comparison-Lemma 4.5.1 gives a local diffeomorphism ϕ between neighborhoods U of $y \in (G/P)_r$ and U' of $x \in M_r$ for $r \in \mathbb{R}_0^+$. There is a vector-bundle-isomorphism $\tilde{\Phi} : \mathcal{T}^{G/P}|_U = (G \times_P V)|_U \to \mathcal{T}^M|_{U'} = (\mathcal{G} \times_P V)|_{U'}$, that covers the local diffeomorphism ϕ, such that*

- *(i) $\tilde{\Phi}$ maps the canonical line bundle $\mathcal{L}^{G/P}|_U := (G \times_P \langle v_0 \rangle)|_U$ onto $\mathcal{L}^M|_{U'} = (\mathcal{G} \times_P \langle v_0 \rangle)|_{U'}$.*
- *(ii) $\tilde{\Phi}$ maps the tangent bundle $T\,(G/P)\,|_U$ onto $TM|_{U'}$, hence preserves the decompositions of $\mathcal{T}^{G/P}|_U$ and $\mathcal{T}^M|_{U'}$. Furthermore, it is compatible with the Riemannian metrics g on M and delta on G/P, i.e. $\tilde{\Phi}^* g = \delta^{G/P}$.*
- *(iii) $\tilde{\Phi}$ maps the induced subbundle $\mathcal{T}_0^{G/P}|_U := (\mathcal{H}^{G/P} \times_H W_0)|_U$ onto the subbundle $\mathcal{T}_0^M|_{U'} = (\mathcal{H} \times_H W_0)|_{U'}$. In particular, it preserves the decomposition of the tangent bundle.*

Remark 5.3.20. *Note that $\tilde{\Phi}|_{TU}$ is not necessarily the derivative of ϕ. If this is the case, the Cartan geometry $\mathcal{G}$ is locally flat and thus locally isomorphic to $\mathbb{R}^n$. However, $\tilde{\Phi}|_{TU}$ yields a local orthonormal frame on U by taking $\tilde{\Phi}|_{TU}(e_i)$ for $1 \leq i \leq n$.*

Proof. $\phi : U \to U'$ is a diffeomorphism between open sets $U \subset G/P$ and $U' \subset M$, that is covered by a P-equivariant $\Phi : G|_U \to \mathcal{G}|_{U'}$, an isomorphism of principal bundles.

(i) Let $\tilde{\Phi} : (G \times_P V)\,|_U \to (\mathcal{G} \times_P V)\,|_{U'}$ be given by $\tilde{\Phi}\,([g, x]) := [\Phi\,(g)\,, x]$. From now on, in the whole proof we will speak about bundles with restriced base set U or U', respectively, but for the sake of readability, we will omit the restrictions $|_{U(')}$.

The map $\tilde{\Phi}$ is well-defined, since $\left[gp, p^{-1}x\right] \in G \times_P V$ is mapped to

$$\left[\Phi\,(gp)\,, p^{-1}x\right] = \left[\Phi\,(g)\,p, p^{-1}x\right] = \left[\Phi\,(g)\,, x\right].$$

Furthermore, it is obviously fiberwise linear. Since Φ is an isomorphism of principal bundles, $\tilde{\Phi}$ is a vector-bundle-isomorphism. It covers ϕ, since Φ does.

(ii) The line bundles are given by $\mathcal{L}^{G/P} = G \times_P \langle v_0 \rangle$ and $\mathcal{L}^M = \mathcal{G} \times_P \langle v_0 \rangle$, thus an element $[g, \lambda v_0] \in \mathcal{L}^{G/P}$ is mapped to $\tilde{\Phi}\,([g, \lambda v_0]) = [\Phi\,(g)\,, \lambda v_0] \in \mathcal{L}^M$.

By the same argument as above, $\tilde{\Phi}$ maps $T(G/P) \cong G \times_P \ker(\varphi_0)$ onto $TM \cong \mathcal{G} \times_P \ker(\varphi_0)$.

By 5.1.4, the Riemannian metrics are equivalent to the canonical metrics $h^{G/P}$ and h^M on $\mathcal{G} \times_P \ker(\varphi_0)$ and $\mathcal{G} \times_P \ker(\varphi_0)$, respectively, that come from the standard inner product on $\ker(\varphi_0)$, hence for $[g,x], [g,y] \in G \times_P \ker(\varphi_0)$ we have

$$\begin{aligned} h^{G/P}([g,x],[g,y]) &= \langle x, y \rangle = h^M([\Phi(g), x], [\Phi(g), y]) \\ &= h^M\left(\tilde{\Phi}([g,x]), \tilde{\Phi}([g,y])\right). \end{aligned}$$

(iii) The proof of the last statement is rather involved compared to (i) and (ii). This results from the fact, that we constructed $\tilde{\Phi}$ from the P-equivariant map Φ, and that $v_0 \in V$ and $\varphi_0 \in \mathbb{R}^{n+1*}$ are P-invariant. However, W_0 is not invariant under P, hence we have to take a detour to show H-equivariancy of $\tilde{\Phi}$.

At first, consider the map $\Phi' : \hat{G} = G \times_P G \to \hat{\mathcal{G}} = \mathcal{G} \times_P G$ given by $\Phi'([g,g']) := [\Phi(g), g']$. It is well-defined, since Φ is P-equivariant. Also, it preserves fibers, is bijective and equivariant with respect to the G-actions on $\hat{G}$ and $\hat{\mathcal{G}}$.

Now, analogously as in (i), Φ' induces an isomorphism of vector bundles $\tilde{\Phi}' : \hat{G} \times_G V \to \hat{\mathcal{G}} \times_G V$, defined by $\tilde{\Phi}'([\hat{g}, x]) := [\Phi'(\hat{g}), x]$.

However, $\hat{G} \times_G V = G \times_P V$ and $\hat{\mathcal{G}} \times_G V = \mathcal{G} \times_P V$. We show $\tilde{\Phi} = \tilde{\Phi}'$:
Take an element $[g,x] \widehat{=} [[g,e], x]$ in $G \times_P V = \hat{G} \times_G V$. Under $\tilde{\Phi}$ it is mapped to $\tilde{\Phi}([g,x]) = [\Phi(g), x] \widehat{=} [[\Phi(g), e], x] = [\Phi'([g,e]), x] = \tilde{\Phi}'([[g,e],x])$.

<u>*Claim:*</u> The map Φ' maps $\mathcal{H}^{G/P} \subset \hat{G}$ onto $\mathcal{H} \subset \hat{\mathcal{G}}$.
proof of claim: The holonomy reductions $\mathcal{H}^{G/P}$ and $\mathcal{H}$ may equivalently be expressed by equivariant functions $s^{G/P} : \hat{G} \to \mathcal{O}$ and $s^M : \hat{\mathcal{G}} \to \mathcal{O}$ (they are related via $\left(s^{G/P}\right)^{-1}(W_0) = \mathcal{H}^{G/P}$ and $\left(s^M\right)^{-1}(W_0) = \mathcal{H}$).

From the Comparison-Lemma 4.5.1 we know that $s^{G/P} \circ \Phi' = s^M$. Thus

$$\begin{aligned} \Phi'\left(\mathcal{H}^{G/P}\right) &= \Phi'\left(\left(s^{G/P}\right)^{-1}(W_0)\right) = \left((\Phi')^{-1}\right)^{-1}\left(\left(s^{G/P}\right)^{-1}(W_0)\right) \\ &= \left(\left(s^{G/P} \circ (\Phi')^{-1}\right)\right)^{-1}(W_0) = \left(s^M\right)^{-1}(W_0) = \mathcal{H}. \end{aligned}$$

This proves the claim.

Now let $g \in \mathcal{H}^{G/P}$ and $x \in W_0$. Then we have

$$\tilde{\Phi}([g,x]) = \tilde{\Phi}'([g,x]) = [\Phi'(g), x] \in \mathcal{T}_0,$$

since $\Phi'(g) \in \mathcal{H}$ and $x \in W_0$. □

In the origin of Comparison, the canonical tractor bundle isomorphism from above admits a particularly nice form:

Corollary 5.3.21. *Let $y \in (G/P)_r$ and $x \in M_r$ for $r \in \mathbb{R}_0^+$. By 5.3.16, comparison mapping y to x is of the form $\phi(y) = \exp_x(f_0(y' - y))$ where f_0 :*

$\mathbb{R}^n \to T_xM$ is a linear isometry. By 5.3.19, ϕ induces a canonical $\tilde{\Phi} : \mathcal{T}^{G/P} \to \mathcal{T}^M$ locally around y and x.

Then $\tilde{\Phi}|_{T_y(G/P)} : T_y(G/P) \to T_xM$, since $\tilde{\Phi}$ preserves the decomposition $\mathcal{T} = \mathcal{L} \oplus TM$, has the form $\tilde{\Phi}|_{T_yG/P} = f_0$. In particular, $f_0(E_y^{G/P}) = E_x$.

Proof. Recall the construction of $\Phi : G|_U \to \mathcal{G}|_{U'}$ from the proof of 4.5.1, where U' is a normal neighborhood around x and U is the preimage of it under ϕ. For the construction of the comparison map we have to choose $g_0 \in G_y$ and $u_0 \in \mathcal{G}_x$, such that $s^{G/P}(g_0) \in \mathcal{O}_r$ and $s(u_0) \in \mathcal{O}_r$ where $s^{G/P}$ and s denote the equivariant functions on $\hat{G}$ and $\hat{\mathcal{G}}$ that correspond to the holonomy reductions, respectively. Now $f_0 = u_0 \circ g_0^{-1}$ (cf. proof of 5.3.16). The map Φ_y is constructed such that it maps g_0 to u_0.

Hence $\tilde{\Phi}([g_0, e_i]) = [\Phi(g_0), e_i] = [u_0, e_i]$ for $1 \le i \le n$. Using the canonical identification $\mathcal{G} \times_P \mathbb{R}^n = TM$ given by $[u, x] \mapsto u(x)$ we see that $\tilde{\Phi}(g_0(e_i)) = u_0(e_i)$, hence $\tilde{\Phi}(x) = u_0(g_0^{-1}(x)) = f_0(x)$ for all $x \in \mathbb{R}^n$.

In particular, since $\tilde{\Phi}$ preserves both $\mathcal{T}_0$, and $T(G/P)$ and TM, we have $\tilde{\Phi}(E^{G/P}) = E$. Hence $f_0(E_y^{G/P}) = \tilde{\Phi}(E_y^{G/P}) = E_x$. □

This provides an efficient tool to generalize properties of holonomy reductions on the homogenous model.

Corollary 5.3.22. *The curved orbit M_0 can be characterized as*

$$M_0 = \{x \in M \mid \mathcal{L}_x \subset (\mathcal{T}_0)_x\}.$$

Proof. The claim follows straightforwardly from 5.3.19.

Endow the homogenous model with the holonomy reduction $G/P \times H \subset G/P \times G$. Let $x \in M_0$, then by 4.5.1 there is a diffeomorphism $\phi : U \to U'$, where U is a neighborhood of $0 \in G/P$ and U' is a neighborhood of x, such that ϕ maps the $(G/P)_r \cap U$ to $M_r \cap U'$ for $r \in \mathbb{R}_0^+$. By 5.3.19 there is an isomorphism of tractor bundles $\tilde{\Phi} : \mathcal{T}^{G/P}|_U \to \mathcal{T}^M|_{U'}$ that covers ϕ, such that $\tilde{\Phi}(\mathcal{L}_0^{G/P}) = \mathcal{L}_x^M$ and $\tilde{\Phi}((\mathcal{T}_0^{G/P})_0) = (\mathcal{T}_0^M)_x$. Furthermore, by 5.3.13, we have $\mathcal{L}_0^{G/P} \subset (\mathcal{T}_0^{G/P})_0$, thus $\mathcal{L}_0^M \subset (\mathcal{T}_0^M)_0$.

On the other hand, let $x \in M$ such that $\mathcal{L}_x \subset (\mathcal{T}_0)_x$. Suppose indirectly that $x \in M_r$ for $r > 0$. Again there is, locally around x and any $y \in (G/P)_r$, an isomorphism of tractor bundles that is compatible with the subbundles. However, $(\mathcal{L}_y^{G/P} \cap (\mathcal{T}_0^{G/P})_y = \{0\}$ and hence $\mathcal{L}_x^M \cap (\mathcal{T}_0^M)_x = \{0\}$. This is a contradiction. □

By reformulating the last Corollary 5.3.22, we obtain

$$M \setminus M_0 = \{x \in M \mid \mathcal{L}_x \cap (\mathcal{T}_0)_x = \{0\}\}.$$

Recall the canonical decomposition from the end of section 5.3.2 of the tangent bundle $TM|_{M\setminus S} = E|_{M\setminus S} \oplus L \oplus F$ where $S = \{x \in M \mid \mathcal{L}_x \subset (\mathcal{T}_0)_x\}$.

In 5.3.15 we observed how this decomposition behaves with respect to the curved orbits in the homogenous model. Now we are able to generalize the result to any Riemannian Cartan geometry.

Proposition 5.3.23. *Let $(\mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G, P) that carries a holonomy reduction $\mathcal{H}$ of type $\mathcal{O}$. Then* *(i)* *we have* $TM_0 = E|_{M_0}$, *and*

(ii) *for* $r > 0$, *we have* $L|_{M_r} \perp TM_r$. *Thus we obtain* $TM_r = E|_{M_r} \oplus F|_{M_r}$.

(iii) *We have* $r(x) = ||\operatorname{pr}(\mathbb{1}_x)||$ *for all* $x \in M_r$, *where* $\operatorname{pr} : \mathcal{T}_0 \oplus E^\perp \to E^\perp$ *is the projection discussed in the end of section 5.3.2.*

Proof. Let $\mathcal{H}^{G/P} := G/P \times H \subset G/P \times G = \hat{G}$ be a holonomy reduction of the homogenous model. Then the tractor-bundle-isomorphism $\tilde{\Phi}$ from 5.3.19 preserves the decompositions of $T(G/P)$ and TM:

Firstly, the subbundle E is given by the intersection of $\mathcal{T}_0^{G/P}$ and $T(G/P)$, and $\mathcal{T}_0^{G/P}$ and TM, respectively. In 5.3.19 (iii) and (iv) we proved that $\tilde{\Phi}$ maps $T(G/P)$ onto TM and $\mathcal{T}_0^{G/P}$ onto $\mathcal{T}_0^M$. Thus the first component of the decomposition is preserved.

Secondly, 5.3.19(ii) and (iii) show that $\tilde{\Phi}$ preserves the decompositions, hence it commutes with the projections $q^{G/P}$ and q^M: $\tilde{\Phi} \circ q^{G/P} = q^M \circ \tilde{\Phi}$. Therefore, we have $\tilde{\Phi}\left(q^{G/P}\left(\mathcal{T}_0^{G/P}\right)\right) = q^M\left(\tilde{\Phi}\left(\mathcal{T}_0^{G/P}\right)\right) = q^M\left(\mathcal{T}_0^M\right)$ by 5.3.19(iii).

Finally, by 5.3.19 (iii) the isomorphim $\tilde{\Phi}$ is compatible with the Riemannian metrics δ on G/P and g on M. Thus if $\tilde{\Phi}(A) \subset B$, also $\tilde{\Phi}\left(A^\perp\right) \subset B^\perp$ for $y \in G/P$ and subspaces A of $T_y(G/P)$ and B of $T_{\phi(y)}M$.

Since L and $q(\mathcal{T}_0)^\perp$ are defined by forming orthogonal complements, the decomposition is invariant under $\tilde{\Phi}$.

(i) Compare with the homogenous model at $x \in M_0$ and $y = 0 \in G/P$. We know that the Comparison map is of the form $\phi(y') = \exp_x(f_0(y'))$ and $\phi(\mathbb{R}^k \times \{0\}) = M_0$ in a neighborhood of 0 (see 5.3.16). Now from 5.3.21 we know that $f_0(\mathbb{R}^k \times \{0\}) = f_0(E_y) = E_x$. Furthermore, we have $T_y\phi = f_0$. Thus

$$T_xM_0 = T_x(\phi(\mathbb{R}^k \times \{0\})) = T_x\phi \cdot E_y = f_0(\mathbb{R}^k \times \{0\}) = E_x.$$

(ii) Again we use Comparison. Let $x \in M_r$ and $y \in (G/P)_r$. We know that $L_y \perp T_y(G/P)_r$. For the Comparison map $\phi(y') = \exp_x(f_0(y - y'))$ we have $T_y\phi = f_0 = \tilde{\Phi}_y$ (cf. 5.3.16 and 5.3.21). Above we saw that $\tilde{\Phi}_y(L_y) = L_x$ and furthermore we know that $T_y\phi(T_y(G/P)_r) = T_yM_r$. Thus

$$L = f_0(L_y) \perp f_0(T_y(G/P)_r) = T_xM_r.$$

(iii) Let $x \in M_r$ for some $r \in \mathbb{R}_0^+$ and $y \in (G/P)_r$. Comparison of x and y gives a map ϕ of the usual form and induces a local vector bundle isomorphism $\tilde{\Phi}$ that preserves the subbundles $\mathcal{T}_0$ and E. Thus in particular it commutes with the projection pr.

Furthermore, $\tilde{\Phi}(\mathbb{1}_{(\pi(u))}) = \tilde{\Phi}([u, v_0]) = [\Phi(u), v_0] = \mathbb{1}_{\phi(\pi(u))}$. In the origin y of Comparison we have $\tilde{\Phi}|_{T_y(G/P)} = T_y\phi$.

The claim is true for the homogenous model (see 5.3.14), hence we have

$$r = ||\operatorname{pr}^{G/P}(\mathbb{1}_y^{G/P})|| = ||\tilde{\Phi}(\operatorname{pr}^{G/P}(\mathbb{1}_y))|| = ||\operatorname{pr}(\tilde{\Phi}(\mathbb{1}_y))|| = ||\operatorname{pr}(\mathbb{1}_x)||.$$

$\square$

Corollary 5.3.24. *Let $\phi(y') = \exp_x(f_0(y-y'))$ be a Comparison map of $x \in M_r$ and $y \in (G/P)_r$ for any $r \in \mathbb{R}_0^+$ and some $f_0 : \mathbb{R}^n \to T_xM$ a linear isometry as shown in 5.3.16. Then $f_0(E_y^{G/P}) = E_x$ and if $r > 0$ moreover $f_0(L_y^{G/P}) = L_x$ holds.*

Proof. This was shown in the proof of 5.3.23. □

5.3.5 A geometrical characterization

We finally give a geometrical characterization of the holonomy reduction.

We can show that a vector field on M, that satisfies the right properties, is equivalent to a holonomy reduction. Recall that in 5.3.10 we showed that a holonomy reduction is equivalent to a certain subbundle of the canonical tractor bundle.

Theorem 5.3.25. *Let M be a Riemannian manifold that is endowed with a parallel, rank-k distribution E. Then the following two objects are equivalent:*

(a) A parallel subbundle $\mathcal{T}_0 \subset \mathcal{T}$ of rank $k+1$ such that $\mathcal{T}_0 \cap TM = E$; and

(b) a smooth vector field $n \in \Gamma(E^\perp)$, that satisfies

(i) $S := \{x \in M \mid n(x) = 0\}$ is empty or a smooth integral manifold of E, and

(ii) $\nabla_\xi n = \mathrm{pr}^{E^\perp}(\xi)\ \forall \xi \in \mathfrak{X}(M)$, where $\mathrm{pr}^{E^\perp}$ denotes the projection $E \oplus E^\perp \to E^\perp$.

Proof. $(a) \to (b)$ The given $\mathcal{T}_0$ is equivalent to a holonomy reduction of the Cartan geometry (see 5.3.10). Hence we can use everything we know about holonomy reductions.

Let $n(x) := \mathrm{pr}(\mathbb{1}(x))$, where $\mathrm{pr} : \mathcal{T}_0 \oplus E^\perp \to E^\perp$. This is per definition a smooth section of $E^\perp$. We know from 5.3.22 that

$$\begin{aligned} S &= \{x \in M \mid n(x) = 0\} = \{x \in M \mid \mathbb{1}_x \in (\mathcal{T}_0)_x\} \\ &= \{x \in M \mid \mathcal{L}_x \subset (\mathcal{T}_0)_x\} = M_0. \end{aligned}$$

This is either empty or a smooth integral manifold of E.

Furthermore, the decomposition of $\mathbb{1}$ in terms of $\mathcal{T} = \mathcal{T}_0 \oplus E^\perp$ is per definition of n given by

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -n \end{pmatrix} + \begin{pmatrix} 0 \\ n \end{pmatrix},$$

where the first component denotes the $\mathcal{L}$-part and the second component denotes the TM-part of an element of $\mathcal{T}$. Therefore $\begin{pmatrix} 1 \\ -n \end{pmatrix} \in \mathcal{T}_0$.

The bundle $\mathcal{T}_0$ is parallel, hence for any $\xi \in \mathfrak{X}(M)$

$$\nabla_\xi \begin{pmatrix} 1 \\ -n \end{pmatrix} = \begin{pmatrix} 0 \\ \xi - \nabla_\xi^{LC} n \end{pmatrix}$$

is a section of $\mathcal{T}_0$. Thus $\xi - \nabla_\xi^{LC} n$ is a section of $\mathcal{T}_0 \cap TM = E$.

Furthermore, $E^\perp$ is parallel since E is and hence $\nabla_\xi n \in \Gamma(E^\perp)$. Therefore $\nabla_\xi n = \mathrm{pr}^{E^\perp}(\xi)$.

$(b) \to (a)$ For the given vector field n, let $\mathcal{T}_0 = \left\langle \begin{pmatrix} 1 \\ -n \end{pmatrix} \right\rangle \oplus \begin{pmatrix} 0 \\ E \end{pmatrix}$ in the decomposition $\mathcal{T} = \mathcal{L} \oplus TM$. Since n is smooth, the bundle $\mathcal{T}_0$ is smooth.

Obviously we have $\mathcal{T}_0 \cap TM = E$. It is left to show that $\mathcal{T}_0$ is parallel: The distribution E is parallel, hence the derivative of a section of this part of $\mathcal{T}_0$ is again a section of $\mathcal{T}_0$. Let $\xi \in \mathfrak{X}(M)$, then

$$\nabla_\xi \begin{pmatrix} 1 \\ -n \end{pmatrix} = \begin{pmatrix} 0 \\ \xi - \nabla^{LC}_\xi n \end{pmatrix} = \begin{pmatrix} 0 \\ \xi - \mathrm{pr}^{E^\perp}(\xi) \end{pmatrix}.$$

Obviously this is again a section of $\mathcal{T}_0$.

We still have to see that this correspondence is bijective. Start with a given $\mathcal{T}_0$. Then let $n(x) = \mathrm{pr}(\mathbb{1}_x)$. We construct the tractor by adding $\begin{pmatrix} 1 \\ -n \end{pmatrix}$ to E, but $\begin{pmatrix} 1 \\ -n \end{pmatrix} \in \mathcal{T}_0$ by definition of n. Hence the constructed tractor bundle is the same as the original one.

Now start with a given n. Construct $\mathcal{T}_0 = \left\langle \begin{pmatrix} 1 \\ -n \end{pmatrix} \right\rangle \oplus \begin{pmatrix} 0 \\ E \end{pmatrix}$. Then since $\begin{pmatrix} 1 \\ -n \end{pmatrix} \in \mathcal{T}_0$, the decomposition of $\mathbb{1}$ in $\mathcal{T}_0 \oplus E^\perp$ is given by $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -n \end{pmatrix} + \begin{pmatrix} 0 \\ n \end{pmatrix}$. Hence $n(x) = \mathrm{pr}(\mathbb{1}_x)$. $\square$

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