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Entropy in Infinite Measure Preserving Dynamical Systems

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Introduction

In 1959 Kolmogorov and Sinai established a substantial definition of entropy for dynamical systems of probability spaces. To extend this definition to infinite measure systems different attempts have been made. In this thesis we will treat three of them, namely Krengel's ([14], 1967), Parry's ([15], 1969) and the so-called Poisson entropy, introduced by Roy in 2005 (referring to [10]). These are the best established notions of entropy for dynamical systems with infinite measure. Of course having differing definition for the same quantity is something one tries to avoid, so investigations have been made to obtain some relations between them. In 2010 Janvresse, Meyerovitch, Roy and de la Rue published a paper, [10], which provides a proof of equality of Krengel's, Parry's and the Poisson entropy under quite weak assumptions. To be precise, they proved equality of Parry's and Poisson entropy. That Krengel's and Parry's entropy coincide under these assumptions had already been proven by Parry himself in [15]. Be aware that (at least some of) the assumptions are crucial: There is an example construed by Janvress and de la Rue in [9], 2012, for which Krengel's and Poisson entropy differ.
One of the main aims of the present master thesis is a detailed elaboration of the result of Janvresse, Meyerovitch, Roy and de la Rue, [10], mentioned above. In order to do so, we will in particular consider the theory of canonical systems of measure, which goes back to Rokhlin, [16], and also give proofs of some basic properties of them, which are quite hidden in the usual literature. The Poisson entropy is based on the theory of Poisson suspensions, to which we will devote a whole chapter, including a proof of ergodicity of the Poisson suspension, which is due to Zweimüller, [25] (a generalization of a proof for automorphisms by Grabinsky, [6]). To emphasise the need of a notion of entropy for infinite measure systems we will give some examples, for which we will also compute Krengel's entropy and finally apply some results of [10] to them.

1 Preliminaries

Let us briefly develop the set-out of this thesis and state some classical results for later use, where we omit well-known or purely topological proofs. We will assume basic knowledge of measure theory, integration theory and (higher) probability theory. For a treatment of those principles we refer the reader to [12] or [3].

Throughout the whole thesis we will only consider σ-finite measure spaces, that is there are countably many measurable sets of finite measure whose union already covers the whole space.

1.1 Names and notations

Let $(X, \mathfrak{B}, \mu)$ always denote a σ-finite measure space. For sets $A, B \in \mathfrak{B}$ we will write $A \subseteq B$ *modulo* μ *(or mod* μ*)* if only a subset of A of measure zero is not contained in B, i.e. $\mu(A \setminus B) = 0$. This gives also the definition of $A = B \bmod \mu$ by $A \subseteq B \bmod \mu$ and $B \subseteq A \bmod \mu$. Measurable sets on which the measure is zero are called *null-sets*. If something is valid everywhere beside on a null-set, we say it is true *almost everywhere (a.e.)*, respectively for *almost every* point. An *atom* of $\mathfrak{B}$ (or of μ) is a set $A \in \mathfrak{B}$ with $\mu(A) \neq 0$, such that $\forall B \in \mathfrak{B}$, $B \subset A$ either $A = B \bmod \mu$ or $B = \emptyset \bmod \mu$. By abuse of notation we will write $A \cap \mathfrak{F} := \{A \cap F \ : \ F \in \mathfrak{F}\}$, for a family $\mathfrak{F}$ of sets. For two

families $\mathfrak{F}$, $\mathfrak{E}$ of measurable sets we say $\mathfrak{F} \subseteq \mathfrak{E} \bmod \mu$, if for every element $F \in \mathfrak{F}$ there is an element $E \in \mathfrak{E}$ such that $E = F \bmod \mu$. Further we define $\mathfrak{F} = \mathfrak{E} \bmod \mu$ as $\mathfrak{F} \subseteq \mathfrak{E} \bmod \mu$ and $\mathfrak{F} \supseteq \mathfrak{E} \bmod \mu$.
If there is a topology on X we call the σ-algebra generated by all open sets *Borel-σ-algebra*, denoted by $\mathfrak{B}(X)$. If we are dealing with $\mathbb{R}$ or some subset of $\mathbb{R}$, the associated σ-algebra will always be the Borel-σ-algebra w.r.t. the usual topology and the measure the Lebesgue measure λ. Recall that a metric space is said to be *separable* if there is a countable dense set, and *complete* if every Cauchy-sequence converges.
Let $(Y, \mathfrak{A}, \eta)$ be another σ-finite measure space. A measurable function $f : X \longrightarrow Y$ is called *invertible* if it is injective and $f^{-1} : f(X) \longrightarrow X$ is measurable. The two measure spaces are called *isomorphic*, if there is an invertible surjective function $f : X \longrightarrow Y$, such that $\mu \circ f^{-1} = \eta$. Such an f is called an *isomorphism*. Functions which fulfill $\mu \circ f^{-1} = \eta$ are called *measure-preserving*. The spaces are called *essentially isomorphic*, denoted by $(X, \mathfrak{B}, \mu) \approx (Y, \mathfrak{A}, \eta)$, if there are measurable sets X_0 and Y_0 with $X_0 = X \bmod \mu$ and $Y_0 = Y \bmod \eta$, such that $(X_0, \mathfrak{B} \cap X_0, \mu|_{\mathfrak{B} \cap X_0})$ and $(Y_0, \mathfrak{A} \cap Y_0, \eta|_{\mathfrak{A} \cap Y_0})$ are isomorphic. Given an isomorphism $f : (X, \mathfrak{B}, \mu) \longrightarrow (Y, \mathfrak{A}, \eta)$, we have $\{A : A \text{ is an atom in } \mathfrak{B}\} = \{f^{-1}(C) : C \text{ is an atom in } \mathfrak{A}\}$.
A measure space is called *complete (w.r.t. the measure)* if every subset of a null-set is measurable, i.e. contained in the σ-algebra. For every measure space $(X, \mathfrak{B}, \mu)$ there exists a so-called *completion* $(X, \overline{\mathfrak{B}}, \overline{\mu})$, that is the smallest complete measure space with $\mathfrak{B} \subseteq \overline{\mathfrak{B}}$ and $\overline{\mu}|_{\mathfrak{B}} = \mu$.
By $\mathbb{N}$ we denote the natural numbers without zero, by $\mathbb{N}_0$ those with zero. Moreover, $\overline{\mathbb{N}}$ and $\overline{\mathbb{R}}$ stand for $\mathbb{N} \cup \{\infty\}$ and $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$, respectively.

1.2 Standard measure spaces

In some of the following sections we will have to restrict our consideration to measure spaces of a special kind. This restriction will still allow a wide class of measure spaces to apply. We adopt the definition from [1]:

Definition. A measurable space $(X, \mathfrak{B})$ is called *standard*, if X is a separable complete metric space and $\mathfrak{B}$ its Borel-σ-algebra. The same is done for measure spaces $(X, \mathfrak{B}, \mu)$.

Remark 1.1. Given a standard measurable space $(X, \mathfrak{B})$ and a set $A \in \mathfrak{B}$, then $(A, A \cap \mathfrak{B})$ is again a standard measurable space (see e.g. [1]).

The following result is well known in measure theory. A heuristical proof can be found in [1] and a rigorous one for probability spaces in [3] (Theorem 9.2.2).

Theorem 1.1. *Let $(X, \mathfrak{B}, \mu)$ be a σ-finite standard measure space without atoms. If $\mu(X) = \infty$, then the space is essentially isomorphic to $(\mathbb{R}, \mathfrak{B}(\mathbb{R}), \lambda)$. If $\mu(X) = 1$ then the space is essentially isomorphic to $([0,1], \mathfrak{B}([0,1]), \lambda|_{[0,1]})$, furthermore, if we release the condition on atoms, then the space is essentially isomorphic to $([0,1], \mathfrak{B}([0,1]), \nu)$, for some probability measure ν.*

For every standard measurable space $(X, \mathfrak{B})$ we can consider the set of all probability measures defined on this space, which we will denote by $\mathsf{P}(X, \mathfrak{B})$, and equip it with the σ-algebra $\mathfrak{G}$ generated by the functions N_B which evaluate the probability measures on a fixed set $B \in \mathfrak{B}$, that is $N_B(P) := P(B)$, for all $P \in \mathsf{P}(X, \mathfrak{B})$, i.e. $\mathfrak{G} := \sigma(N_B : B \in \mathfrak{B})$. Then $(\mathsf{P}(X, \mathfrak{B}), \mathfrak{G})$ is again a standard measurable space (referring to [1]).

The following very useful theorem is proven for instance in [1] (Theorem 1.0.8).

Theorem 1.2 (Disintegration)**.** *Let $\phi : (X, \mathfrak{B}, \mu) \longrightarrow (Y, \mathfrak{D}, \eta)$ be a measure-preserving map between two standard probability spaces. Then there is a $\mathfrak{D} \cap Y_0$-$\mathfrak{G}$-measurable function $y \mapsto \mu_y$, $Y_0 \longrightarrow \mathsf{P}(X, \mathfrak{B})$, on a set $Y_0 \in \mathfrak{D}$ with $Y_0 = Y$ mod η, such that $\forall B \in \mathfrak{B}$, $\forall A \in \mathfrak{D}$,*

$$\mu(B \cap \phi^{-1}(A)) = \int_A \mu_y(B)\, d\eta(y)$$

and $\mu_y(\phi^{-1}(\{y\})) = 1$ for all $y \in Y_0$.

Note that $\{y\} \in \mathfrak{D}$ for all $y \in Y$, since one-point-sets are closed sets (i.e. complements of open sets) and $\mathfrak{D}$ is the Borel-σ-algebra (hence contains the complement of every open set).

We can also disintegrate a space w.r.t. a sub-σ-algebra. This result is proved in a very general setting in [20], Theorem 44[1]:

Theorem 1.3. *Let $(X, \mathfrak{B}, \mu)$ be a σ-finite standard measure space without atoms, and let $\mathfrak{C}$ be a σ-finite sub-σ-algebra of $\mathfrak{B}$, then there exists a* disintegration w.r.t. $\mathfrak{C}$*, that is there is a family $\{\mu_x\}_{x\in X}$ of measures on $(X, \mathfrak{B})$ such that for every $B \in \mathfrak{B}$*

- *the map $x \mapsto \mu_x(B)$ is measurable w.r.t. $\mathfrak{C}$, and*
- *$\mu(B \cap C) = \int_C \mu_x(B)\, d\mu(x)$, for every $C \in \mathfrak{C}$.*

Moreover, this family of measures is unique w.r.t. those properties, that is, if there are two families $\{\mu_x\}_{x\in X}$ and $\{\mu'_x\}_{x\in X}$ of measures on $(X, \mathfrak{B})$, which fulfill the listed properties w.r.t. $\mathfrak{C}$, then $\mu_x = \mu'_x$ for μ-a.e. $x \in X$.

Remark 1.2. The measures obtained above are *conditional measures*[2] *w.r.t.* $\mathfrak{C}$, i.e. $\forall B \in \mathfrak{B}$, $\forall x \in X$:

$$\mu_x(B) = E_\mu[1_B|\mathfrak{C}](x) = \mu(B|\mathfrak{C})(x).$$

In particular, they are probability measures for a.e. $x \in X$, because $\mu_x(X) = E_\mu[1_X|\mathfrak{C}](x) = 1_X(x) = 1$ for a.e. $x \in X$, since 1_X is clearly $\mathfrak{C}$-measurable.

Furthermore, we will need a result on sub-σ-algebras, shown for probability spaces e.g. in (Theorem 6.5 and Corollary 5.22) and the last claim is given in [4] (Fact 1.22) for probability measures.

Theorem 1.4. *Let $\mathfrak{C}$ be a sub-σ-algebra of a σ-finite standard space $(X, \mathfrak{B}, \mu)$. Then there is a σ-finite standard space $(Y, \mathfrak{D}, \nu)$ and a measure-preserving map $\phi : X_0 \longrightarrow Y_0$, such that $\mathfrak{C} = \phi^{-1}(\mathfrak{D})$ mod μ, where $X_0 = X$ mod μ and $Y_0 = Y$ mod ν. If $\bigcap_{x\in C\in\mathfrak{C}} C = \{x\}$ for every $x \in X_0$, then ϕ is injective. In this case we even can deduce that $\phi : (X, \mathfrak{C}, \mu) \longrightarrow (Y, \mathfrak{D}, \nu)$ is an essential isomorphism.*

1.3 Some basic ergodic theory

We want to consider the evolution of a σ-finite measure space $(X, \mathfrak{B}, \mu)$ described by a measurable transformation $T : X \longrightarrow X$, which is *non-singular*, that is $\forall B \in \mathfrak{B}$: $\mu(B) = 0 \Rightarrow \mu(T^{-1}B) = 0$, i.e. it shall obey the natural rule, that the set of elements,

[1]Actually, Theorem 44 of [20] applies in the more general setting of Hausdorff spaces which can be written as a countable union (mod μ) of compact metrizable measurable sets on which the measure is finite. With our assumptions these conditions are fulfilled by Theorem 1.1.

[2]For a definition of conditional measures for not necessary finite measures spaces, we refer to [20].

which will be taken to a null set under T, must not have positive measure, descriptively spoken, T shall not annihilate information (e.g. mass). The measure $T\mu := \mu \circ T^{-1}$ is called *image-measure*. To represent this *dynamical system* we write $(X,\mathfrak{B},\mu,T)$. We say T is *measure-preserving*, respectively that μ is *T-invariant*, if $\mu(B) = \mu(T^{-1}B)$ for all $B \in \mathfrak{B}$, i.e. if $\mu = T\mu$.
If $T : X \longrightarrow X$ is an essential isomorphism, then we call T an *automorphism*. Note, that in this case, by definition, $\mathfrak{B} \subseteq T^{-1}(\mathfrak{B}) \subseteq T^{-1}(f(\mathfrak{B})) = \mathfrak{B}$ mod μ, hence $T^{-1}\mathfrak{B} = \mathfrak{B}$ mod μ and $T\mathfrak{B} = \mathfrak{B}$ mod μ.
A dynamical system $(X,\mathfrak{B},\mu,T)$ is called *conservative*, if beside null-sets there are no *wandering sets*, i.e. sets $W \in \mathfrak{B}$ with $W \cap \bigcup_{n=1}^{\infty} T^{-n}W = \emptyset$ mod μ. We say that $(X,\mathfrak{B},\mu,T)$ is *ergodic*, if for every *invariant set* $B \in \mathfrak{B}$, $T^{-1}B = B$ mod μ, we have $\mu(B) = 0$ or $\mu(B^c) = 0$. Let us denote the σ-algebra consisting of all invariant sets by $\mathfrak{I}$. Provided that T is measure-preserving, we call the system *weakly mixing* if $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} |\mu(A\cap T^{-k}B) - \mu(A)\mu(B)| = 0$, $\forall A,B \in \mathfrak{B}$. If μ is a probability measure, then weakly mixing implies ergodicity, since if we take e.g. $A = B$ invariant in the above equation, we get $0 = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} |\mu(A\cap T^{-k}B) - \mu(A)\mu(B)| = \mu(B) - \mu(B)\mu(B)$, thus $\mu(B) = 0$ or $\mu(B) = 1$.

Remark 1.3. *Let $(X,\mathfrak{B},\mu,T)$ be a σ-finite, conservative and ergodic dynamical system. Then every set $A \in \mathfrak{B}$ with $\mu(A) \neq 0$ is a sweep-out set, that is $\bigcup_{n=0}^{\infty} T^{-n}A = X$ mod μ.*
If additionally T is measure-preserving and $\mu(X) = \infty$, then every σ-finite sub-σ-algebra $\mathfrak{C}$ of $\mathfrak{B}$, which is sub-invariant, i.e. $T^{-1}\mathfrak{C} \subseteq \mathfrak{C}$, has no atoms. (In particular, for every $x \in X$ with $\{x\} \in \mathfrak{B}$ we get $\mu(\{x\}) = 0$.)

Proof. Let $B \in \mathfrak{B}$ be an arbitrary set with $\mu(B) \neq 0$. Since $(X,\mathfrak{B},\mu)$ is conservative, we have $B \subseteq \bigcup_{n=1}^{\infty} T^{-n}B$ mod μ and therefore $\bigcup_{n=0}^{\infty} T^{-n}B$ is T-invariant. Thus, due to ergodicity, $\bigcup_{n=0}^{\infty} T^{-n}B = X$ mod μ , since the case $\bigcup_{n=0}^{\infty} T^{-n}B = \emptyset$ mod μ is forbidden by $\mu(B) \neq 0$.
Let $\mathfrak{C}$ be a σ-finite sub-invariant σ-algebra and C be an atom of $\mathfrak{C}$, $\mu(C) \neq 0$ by definition. By assumption $\mathfrak{C}$ is σ-finite, so let $C_n \in \mathfrak{C}$ with $\mu(C_n) < \infty$, $\forall n \in \mathbb{N}$, and $\bigcup_{n\in\mathbb{N}} C_n = X$, then there is an $m \in \mathbb{N}$ such that $\mu(C \cap C_m) \neq 0$, but since C is an atom of $\mathfrak{C}$ and $C_m \in \mathfrak{C}$, we get $C \cap C_m = C$ mod μ, i.e. $C \subseteq C_m$ mod μ. Therefore we have $\mu(C) < \infty$. The same argument leads to $C \subseteq T^{-m}C$ mod μ, for some $m \in \mathbb{N}$, $m \geq 1$, since by conservativity $C \cap \bigcup_{n=1}^{\infty} T^{-n}C \neq \emptyset$ mod μ and $T^{-n}C \in \mathfrak{C}$, $\forall n \in \mathbb{N}$, by sub-invariance. But then we get $C = T^{-m}C$ mod μ, because $\mu(T^{-m}C) = \mu(C)$ and $\mu(C) < \infty$. Now, by the first part of this proof, $X = \bigcup_{n=0}^{\infty} T^{-n}C = \bigcup_{n=0}^{m-1} T^{-n}C$ mod μ, hence $\mu(X) = m \cdot \mu(C) < \infty$, a contradiction. ☐

1.3.1 Ergodic Theorems

We now state a main theorem of ergodic theory. For a proof we refer to [27] or [12] (Satz 20.14).

Theorem 1.5 (Birkhoff's Ergodic Theorem)**.** *Let $(X,\mathfrak{B},\mu,T)$ be a measure-preserving dynamical system with $\mu(X) = 1$. Then for every $f \in L_1(\mu) := \{f : X \longrightarrow \mathbb{R}$ measurable$|$ $\int |f|\, d\mu < \infty\}/\{f = 0$ a.e.$\}$ there is an $f^\star \in L_1(\mu)$ such that*

$$\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k \xrightarrow{n\to\infty} f^\star \qquad a.e,$$

$f^\star = f^\star \circ T$ a.e. (in particular $f^\star$ is $\mathfrak{I}$-measurable) and $\int_A f^\star \, d\mu = \int_A f \, d\mu$, for every invariant set $A \in \mathfrak{I}$.
Especially, if in addition the system is ergodic, then

$$\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k \xrightarrow{n\to\infty} \int f \, d\mu = E[f] \quad \text{a.e.}$$

For σ-finite systems we get the following. (A nice proof can be found in [24].)

Theorem 1.6 (Hopf-Stepanov's Ratio Ergodic Theorem). *Given a σ-finite measure-preserving dynamical system and $L_1(\mu)$-functions f and g on X such that g is non-negative and $\int g \, d\mu > 0$, then one can find a real-valued measurable function $Q(f,g)$ on X, depending on f and g, such that*

$$\frac{\sum_{k=0}^{n-1} f \circ T^k}{\sum_{k=0}^{n-1} g \circ T^k} \xrightarrow{n\to\infty} Q(f,g) \quad \text{a.e. on } \left\{ x \in X \ : \ \sup_{n\in\mathbb{N}} \frac{1}{n}\sum_{k=0}^{n-1} g(T^k(x)) > 0 \right\}.$$

If $(X,\mathfrak{B},\mu,T)$ is conservative, then $Q(f,g) = Q(f,g)\circ T$ a.e. and $\int_A Q(f,g)\cdot g \, d\mu = \int_A f \, d\mu$, for all $A \in \mathfrak{I}$.
Especially, if in addition the system is ergodic, then $Q(f,g) = \frac{\int f \, d\mu}{\int g \, d\mu}$ a.e.

Corollary 1.1. *Let $(X,\mathfrak{B},\mu,T)$ be measure-preserving, conservative, ergodic and $\mu(X) = \infty$. Then*

$$\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k \to 0 \qquad \text{a.e. for } n \to \infty$$

for every $f \in L_1(\mu)$.
In particular, for $C \in \mathfrak{B}$ with $\mu(C) < \infty$, we get

$$\frac{1}{n}\sum_{k=0}^{n-1} \mu(C \cap T^{-k}C) \to 0 \qquad \text{for } n \to \infty.$$

Proof. By Hopf-Stepanov's Ratio Ergodic Theorem 1.6,

$$\frac{\sum_{k=0}^{n-1} f \circ T^k}{\sum_{k=0}^{n-1} g \circ T^k} \to \frac{\int f \, d\mu}{\int g \, d\mu} \qquad \text{a.e. for } n \to \infty$$

for all $f, g \in L_1(\mu)$ with $\int g \, d\mu \neq 0$.
In particular,

$$\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k \le \frac{\sum_{k=0}^{n-1} f \circ T^k}{\sum_{k=0}^{n-1} 1_B \circ T^k} \to \frac{\int f \, d\mu}{\mu(B)} \qquad \text{a.e. for } n \to \infty$$

for every $B \in \mathfrak{B}$ with $\mu(B) < \infty$, $f \in L(\mu)$, but since X is assumed to be infinite and σ-finite, we can choose B arbitrary large, which makes $\frac{\int f \, d\mu}{\mu(B)}$ arbitrary small, i.e.

$$\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k \to 0 \qquad \text{a.e. for } n \to \infty.$$

Let $C \in \mathfrak{B}$ be such that $\mu(C) < \infty$, then $1_C \in L_1(\mu)$, thus

$$\frac{1}{n}\sum_{k=0}^{n-1} 1_{C\cap T^{-k}C} = 1_C \cdot \frac{1}{n}\sum_{k=0}^{n-1} 1_C \circ T^k \to 0 \qquad \text{a.e. for } n \to \infty.$$

Hence by dominated convergence

$$\frac{1}{n}\sum_{k=0}^{n-1}\mu(C\cap T^{-k}C)\to 0 \qquad \text{for } n\to\infty.$$

$\square$

1.3.2 Induced transformations

The theory of induced transformations, and in particular first return maps, can be found in most standard books on ergodic theory (see e.g. [1], [5]), therefore we do not reproduce this well-known theory here but only state some results which will be useful later on.

Let $(X,\mathfrak{B},\mu,T)$ be a conservative non-singular dynamical system, then every set $A\in\mathfrak{B}$ of positive measure is recurrent, i.e. $A\subseteq\bigcup_{n=1}^{\infty}T^{-n}A$, therefore the *first return time map* $\varphi_A : X\longrightarrow\overline{\mathbb{N}}$, $\varphi_{T,A}(x):=\inf\{n\in\mathbb{N}: T^n x\in A\}$ is finite for a.e. $x\in X$. The *first return map* of A is defined as

$$T_A : A\longrightarrow A,\ T_A x := T^{\varphi_{T,A}(x)}x \text{ for a. e. } x\in A.$$

If T is an automorphism, we also define $\varphi_{T^{-1},A}(x):=\inf\{n\in\mathbb{N}: T^{-n}x\in A\}$.
The first return time map gives rise to a new dynamical system $(A,\mathfrak{B}\cap A,\mu|_A,T_A)$ which shares many properties with the original one.

Proposition 1.1. *Let $(X,\mathfrak{B},\mu,T)$ be a non-singular dynamical system and $A\in\mathfrak{B}$ a sweep-out set. If $(X,\mathfrak{B},\mu,T)$ is conservative, then so is $(A,\mathfrak{B}\cap A,\mu|_A,T_A)$. If $(X,\mathfrak{B},\mu,T)$ is ergodic, then so is $(A,\mathfrak{B}\cap A,\mu|_A,T_A)$. If $\mu(A)<\infty$ and $(X,\mathfrak{B},\mu,T)$ is measure-preserving, then $(A,\mathfrak{B}\cap A,\mu|_A,T_A)$ is measure-preserving.*

Under some assumption, the reverse implications are true, too, which is actually the more interesting statement in some situations in ergodic theory (referring to [27] for details).

The following identity will be useful for some calculations. (For a proof we refer to [26].)

Lemma 1.1. *Let $(X,\mathfrak{B},\mu,T)$ be a σ-finite measure-preserving dynamical system and $A\in\mathfrak{B}$ a sweep-out set with $\mu(A)<\infty$. Then the first return time map $\varphi_{T,A}$ on A fulfills*

$$\mu(A\cap\{\varphi_{T,A}>n\})=\mu(A^c\cap\{\varphi_{T,A}=n\})$$

for every $n\in\mathbb{N}$.

1.3.3 Natural extensions

Sometimes it is useful to look at a wider system than the given one, especially if the original system lacks some properties.

We define the following relations between dynamical systems:

Definition. A non-singular dynamical system $(X',\mathfrak{B}',\mu',T')$ is called an *extension* of $(X,\mathfrak{B},\mu,T)$ (resp. the latter is called a *factor* of the first), if there are measurable sets

$Y = X \bmod \mu$ and $Y' = X' \bmod \mu'$ with $TY \subseteq Y$, $T'Y' \subseteq Y'$ and there exists a measure-preserving map $\pi : Y' \longrightarrow Y$ such that $\pi \circ T' = T \circ \pi$ on Y'. The map π is called *factor map*. If π is an isomorphism, then the systems are called *isomorphic*, denoted by $(X, \mathfrak{B}, \mu, T) \approx (X', \mathfrak{B}', \mu', T')$.

We will see that for every measure-preserving standard measure space there exists a unique minimal extension which is invertible, that is, the transformation on the system is an automorphism.

Definition. An invertible extension $(\widetilde{X}, \widetilde{\mathfrak{B}}, \widetilde{\mu}, \widetilde{T})$ of $(X, \mathfrak{B}, \mu, T)$ is called *natural extension*, if it is minimal in the following sense:

$$\sigma(\{\widetilde{T}^n \pi^{-1} B | B \in \mathfrak{B}, n \in \mathbb{N}_0\}) = \widetilde{\mathfrak{B}} \mod \widetilde{\mu},$$

where π denotes the factor map $\pi : \widetilde{X} \longrightarrow X$.

To construct such an extension, one needs the famous Kolmogorov's Extension Theorem in the following general version, proven in [23].

Theorem 1.7 (Kolmogorov's Extension)**.** *Let $(X^{(k)}, \mathfrak{B}^{(k)})$ be standard measurable spaces, for each $k \in \mathbb{N}$, and denote by $(X_n, \mathfrak{B}_n)$ the product space up to the n-th space, i.e. $X_n := X^{(1)} \times \ldots \times X^{(n)}$, $\mathfrak{B}_n := \sigma\big(\{(p_n^{(k)})^{-1} B^{(k)} | B^{(k)} \in \mathfrak{B}^{(k)}, k = 1, \ldots, n\}\big)$, where $p_n^{(k)} : X_n \longrightarrow X^{(k)}$ is the canonical projection. Now, let us consider measures μ_n on $(X_n, \mathfrak{B}_n)$ for every $n \in \mathbb{N}$, which are* self-consistent*, that is*

$$\mu_n(E_n) = \mu_m(E_n \times X^{(n+1)} \times \ldots \times X^{(m)}), \quad \forall m \in \mathbb{N} \text{ with } m > n$$

for every $E_n \in \mathfrak{B}_n$. If one of the measures μ_n is σ-finite, then there exists a unique measure μ on the infinite product space $(X, \mathfrak{B})$ such that $\mu \circ {p^{(k)}}^{-1} = \mu_k$ for all $k \in \mathbb{N}$, where $X := \prod_{k=1}^{\infty} X^{(k)}$, $\mathfrak{B} := \sigma\big(\{(p^{(k)})^{-1} B^{(k)} | B^{(k)} \in \mathfrak{B}^{(k)}, k \in \mathbb{N}\}\big)$, and $p^{(k)}$ denotes for the canonical projection $X \longrightarrow X^{(k)}$.

Theorem 1.8 (Natural Extension)**.** *For every σ-finite standard measure space $(X, \mathfrak{B}, \mu, T)$ with a measure-preserving transformation T there exists a natural extension, and this extension is unique modulo essential isomorphism.*
The natural extension can be construed as follows:

$$\widetilde{X} := \{(x_0, x_1, x_2, \ldots) | x_i \in X, Tx_{i+1} = x_i, i \in \mathbb{N}_0\},$$

$$\widetilde{\mathfrak{B}} := \sigma(B_i' : B \in \mathfrak{B}) \text{ for } B_i' := \{(x_0, x_1, x_2, \ldots) \in \widetilde{X} | x_i \in B\},$$

$$\widetilde{\mu}(B_i') := \mu(B)$$

and

$$\widetilde{T}(x_0, x_1, x_2 \ldots) := (Tx_0, x_0, x_1, \ldots) \text{ for } (x_0, x_1, x_2, \ldots) \in \widetilde{X}.$$

Clearly, $(\widetilde{X}, \widetilde{\mathfrak{B}}, \widetilde{\mu})$ is a σ-finite standard measure space and $\widetilde{T}$ a measure-preserving automorphism.

A proof of this Theorem can be found e.g. in [1], Theorem 3.1.6 (Uniqueness), Theorem 3.1.5 (Existence), or a more constructive one in [13], Chapter 10, §.4.

One reason why this kind of extension is useful, is that crucial properties like ergodicity and conservativity carry over. More precisely, we have the following result (proven e.g. in [1] Theorem 3.1.7 or for probability measures in [13] Chapter 10, §.4, Theorem 1).

Theorem 1.9. *Let $(X,\mathfrak{B},\mu)$ be a σ-finite standard measure space and T a measure-preserving, conservative transformation on it. Then T is ergodic if and only if $\widetilde{T}$ is ergodic. Moreover, if T is ergodic, then $\widetilde{T}$ is also conservative.*

1.4 μ-partitions

Let $(X,\mathfrak{B},\mu)$ be a σ-finite measure space. By a *μ-partition* of X we understand a family of pairwise disjoint sets of $\mathfrak{B}$, such that the union of all these sets is modulo μ equal to X.

Definition. Let α, β be μ-partitions of X. We call α *finer* than β, denoted by $\alpha \succeq \beta$, if every element of α is contained in an element of β. The relation $\alpha \succeq \beta \bmod \mu$, means that there is a measurable set X_0 with $X = X_0 \bmod \mu$ such that $\alpha \cap X_0 \succeq \beta \cap X_0$.
The *common refinement* of α and β is defined as $\alpha \vee \beta := \{A \cap B :\ A \in \alpha,\ B \in \beta\}$. For finitely or countably many μ-partitions $\alpha_1, \alpha_2, \ldots$ of X we write $\bigvee_{k=1}^{n} \alpha_n = \alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n$ for $n \in \mathbb{N}$ and $\bigvee_{k=1}^{\infty} \alpha_n = \alpha_1 \vee \alpha_2 \vee \ldots$, respectively.
We say, that $\alpha = \beta \bmod \mu$, if there is an $X_0 \in \mathfrak{B}$ with $X_0 = X \bmod \mu$ such that $\alpha \cap X_0 = \beta \cap X_0$.

Remark 1.4. Note, that $\alpha \succeq \beta$ means that every element $B \in \beta$ is, up to a fixed null-set, a union of elements $A_l \in \alpha$ for l in some (not necessarily countable) index set L. Indeed, if $\alpha \succeq \beta$, then by disjointness of the elements of β, for every $B \in \beta$ and $A \in \alpha$ either $A \subseteq B$ or $A \cap B = \emptyset$ (since there is an $B' \in \beta$ such that $A \subseteq B'$, by assumption), therefore $B = B \cap \bigcup_{A \in \alpha} A = \bigcup_{A \in \alpha,\ A \subseteq B} A \cap B = \bigcup_{A \in \alpha,\ A \subseteq B} A \bmod \mu$. (This equality holds only mod μ, since $\bigcup_{A \in \alpha} A$ is only modulo μ equal to X.)
Note, that the common refinement $\alpha \vee \beta$ is again a μ-partition of X.
Moreover, $\alpha \succeq \beta \pmod{\mu} \Leftrightarrow \alpha \vee \beta = \alpha \pmod{\mu}$. And $\alpha = \beta \pmod{\mu}$ if and only if $\alpha \succeq \beta \pmod{\mu}$ and $\beta \succeq \alpha \pmod{\mu}$.

We will look at a wider class of μ-partitions than countable ones (i.e. μ-partitions which have finite or countably infinite many elements), which will bear some advantages.

Definition. A μ-partition α is called *measurable* if there exists an at most countable set $\Delta = \{D_1, D_2, \ldots\}$ with $D_n \in \mathfrak{B}$ for $n = 1, 2, \ldots$, which generates α, that is, α consists of all (nonempty) sets of the form $\bigcap_{n=1}^{\infty} S_n$, where either $S_n = D_n$ or $S_n = D_n^c$ for $D_n \in \Delta$.

Example 1.1.

1. Clearly, every countable μ-partition α is measurable, by just taking α itself as generating set Δ.

2. Let $(X,\mathfrak{B},\mu)$ be a standard space, then $\varepsilon := \{\{x\} : x \in X\}$ is a measurable partition. Indeed, a metric space is separable if and only if its topology contains a countable basis (2^{nd} countability axiom), i.e. there exists a countable family of open sets, such that every open set is a union of elements of that family. Therefore, since $\{x\}$ is closed, it has to be an intersection of complements of elements of that countable family.
Moreover, we get a trivial ordering $\varepsilon \succeq \alpha \succeq \{X, \emptyset\}$ for every μ-partition α of X. If we restrict ε to a set $B \in \mathfrak{B}$, then we will denote this by $\varepsilon_B := \varepsilon \cap B = \{\{x\} : x \in B\}$. (Confer example 4 below.)

3. The common refinement of at most countably many measurable μ-partitions is obviously measurable: Let Δ_n denote a countable, generating set of a μ-partition α_n then $\bigcup_{n=1}^{\infty} \Delta_n$ is countable and generates $\bigvee_{n=1}^{\infty} \alpha_n$.

4. Let α be a measurable μ-partition of some measure space $(X, \mathfrak{B}, \mu)$ and $C \in \mathfrak{B}$ with $\mu(C) > 0$. Then $\alpha \cap C$ is a measurable μ-partition of $(C, \mathfrak{B} \cap C, \mu|_C)$. Indeed, let $\Delta = \{D_1, D_2, D_3 \ldots\}$ be a generating set of α, then
$$\Delta_C := \{D_1 \cap C, D_1^c \cap C, D_2, D_3, \ldots\}$$
is a countable generating set of $\alpha \cap C$: Given $A \in \alpha$, $A = \bigcap_{n=1}^{\infty} S_n$ with either $S_n = D_n$ or $S_n = D_n^c$. If $S_1 = D_1$, we have $(D_1 \cap C) \cap (D_1^c \cap C)^c \cap S_2 \cap \ldots = (D_1 \cap C \cap D_1 \cap S_2 \cap \ldots) \cup (D_1 \cap C \cap C^c \cap S_2 \ldots) = D_1 \cap C \cap D_1 \cap S_2 \cap \ldots = A \cap C$, and if $S_1 = D_1^c$, then $(D_1 \cap C)^c \cap (D_1^c \cap C) \cap S_2 \cap \ldots = A \cap C$

5. Let $f : X \longrightarrow [0,1]$ be a measurable map. Then $\alpha := \{f^{-1}(\{r\}) : r \in [0,1]\}$ is a measurable μ-partition of X, since, e.g., the set $\Delta := \{f^{-1}(J) : J$ interval in $[0,1]$ with rational endpoints$\}$ is at most countable and generates α. (Obviously α is a μ-partition of X: $\bigcup_{A \in \alpha} A = \bigcup_{r \in [0,1]} f^{-1}(\{r\}) = f^{-1}([0,1]) = X$ and disjointness follows since f is a function.)
In fact, the above form of a measurable μ-partition is characterizing ([3], Lemma 10.8.2): *A μ-partition α of X is measurable if and only if there is a measurable map $f : X \longrightarrow [0,1]$ such that $\alpha = \{f^{-1}(\{r\}) : r \in [0,1]\}$.*
Indeed, let $\Delta = \{D_1, D_2, \ldots\}$ be an at most countable collection of measurable sets, which generates the μ-partition α. We define a function $g : X \longrightarrow \{0,1\}^{\mathbb{N}}$ by $g(x) := (1_{D_n}(x))_{n \in \mathbb{N}}$, taking the usual (Borel-)σ-algebra on $\{0,1\}^{\mathbb{N}}$, generated by the cylinder-sets $[a_1, \ldots, a_m] := \{(\omega_n)_{n \in \mathbb{N}} \in \{0,1\}^{\mathbb{N}} : \omega_k = a_k,\ k = 1, \ldots, m\}$. In this setting the function g is measurable, because $\forall m \in \mathbb{N}$: $\forall a_1, \ldots, a_m \in \{0,1\}^{\mathbb{N}}$ we have $g^{-1}([a_1, \ldots, a_m]) = \bigcap_{k=1}^{m} R_k \in \mathfrak{B}$, with $R_k := D_k$ if $a_k = 1$ and $R_k := D_k^c$, if $a_k = 0$. Furthermore, $\{g^{-1}((\omega_n)_{n \in \mathbb{N}}) : (\omega_n)_{n \in \mathbb{N}} \in \{0,1\}^{\mathbb{N}}\} = \left\{\bigcap_{n \in \mathbb{N}} S_n : S_n = D_n \text{ or } S_n = D_n^c, \text{ for } D_n \in \Delta\right\} = \alpha$. Now let $h : \{0,1\}^{\mathbb{N}} \longrightarrow [0,1]$ be a measurable injective map and set $f := h \circ g$. Then f is a measurable map from X to $[0,1]$ and $\{f^{-1}(\{r\}) : r \in [0,1]\} = \{g^{-1}(h^{-1}(\{r\})) : r \in [0,1]\} = \{g^{-1}((\omega_n)_{n \in \mathbb{N}}) : (\omega_n)_{n \in \mathbb{N}} \in \{0,1\}^{\mathbb{N}}\} = \alpha$.

Notation. For a measurable μ-partition γ, the following σ-algebra will be of interest
$$\widetilde{\sigma}(\gamma) := \left\{ \bigcup_{l \in L} C_l \,\middle|\, \bigcup_{l \in L} C_l \in \mathfrak{B},\ C_l \in \gamma,\ L \text{ an arbitrary not necessarily countable index set} \right\}.$$
Sometimes we will call $\widetilde{\sigma}(\gamma)$ the σ-algebra generated by γ, if there is no confusion with $\sigma(\gamma)$ due to the context.

Note that $\widetilde{\sigma}(\gamma)$ is bigger than $\sigma(\gamma)$ in general. Consider for example a standard measure space $(X, \mathfrak{B}, \mu)$ with $\gamma = \varepsilon_X$, then $\sigma(\gamma) \neq \mathfrak{B}$, but $\widetilde{\sigma}(\gamma) = \mathfrak{B}$.

One of the main advantages of measurable μ-partition (compaired with countable μ-partition) is that every σ-finite sub-σ-algebra is generated by a measurable μ-partition in the above sense. More precisely, we obtain

Lemma 1.2. *Let $(X, \mathfrak{B}, \mu)$ be a σ-finite standard measure space, then every σ-finite sub-σ-algebra $\mathfrak{C}$ of $\mathfrak{B}$ is modulo μ generated by a measurable μ-partition γ of X, i.e. $\widetilde{\sigma}(\gamma) = \mathfrak{C}$ mod μ. Moreover, this μ-partition is unique modulo μ, in that if there is another measurable μ-partition α of X with $\widetilde{\sigma}(\alpha) = \mathfrak{C}$ mod μ, then $\alpha = \gamma$ mod μ.*

Proof. If $\mathfrak{C} = \mathfrak{B}$, then $\varepsilon := \{\{x\} | \ x \in X\}$ is a measurable partition of X (see Example 1.1) with $\widetilde{\sigma}(\varepsilon) = \left\{\bigcup_{l\in L}\{x_l\} | \bigcup_{l\in L}\{x_l\} \in \mathfrak{B}, x_l \in X, L \text{ an arbitrary index set}\right\} = \mathfrak{B}$.
Now let $\mathfrak{C}$ be an arbitrary σ-finite sub-σ-algebra of $\mathfrak{B}$. Then by Theorem 1.4, there is a σ-finite standard measure space $(Y, \mathfrak{D}, \nu)$ and a measurable measure-preserving map $\phi : X_0 \longrightarrow Y_0$ with $\phi^{-1}(\mathfrak{D}) = \mathfrak{C} \bmod \nu$, $X_0 = X \bmod \nu$ and $Y_0 = Y \bmod \nu$. Thus by the first part of the proof there is a measurable ν-partition ζ of Y with $\widetilde{\sigma}(\zeta) = \mathfrak{D} \bmod \nu$. We will show that $\gamma := \{\phi^{-1}(Z) : Z \in \zeta\}$ is a measurable μ-partition for $(X, \mathfrak{C})$ with $\widetilde{\sigma}(\gamma) = \mathfrak{C} \bmod \mu$. Clearly, $\widetilde{\sigma}(\phi^{-1}(\zeta)) = \phi^{-1}(\widetilde{\sigma}(\zeta)) = \phi^{-1}(\mathfrak{D}) = \mathfrak{C} \bmod \mu$, the elements of γ are pairwise disjoint and $\bigcup_{Z\in\zeta}\phi^{-1}(Z) = \phi^{-1}\left(\bigcup_{Z\in\zeta} Z\right) = \phi^{-1}(Y) = X \bmod \mu$. Further, for every $C \in \gamma$ there is a $Z \in \zeta$ such that $C = \phi^{-1}(Z)$. Let $\{F_n \ : n \in \mathbb{N}\}$ be a countable family of $\mathfrak{D} \cap Y_0$-measurable sets which generates ζ, then there is an index set $I_Z \subseteq \mathbb{N}$ such that $Z = \bigcap_{n\in I_Z} F_n \cap \bigcap_{n\notin I_Z} F_n^c$. Thus $C = \phi^{-1}(Z) = \bigcap_{n\in I_Z}\phi^{-1}(F_n) \cap \bigcap_{n\notin I_Z}(\phi^{-1}(F_n))^c$, hence $\{\phi^{-1}(F_n) : n \in \mathbb{N}\}$ is a countable family of $\mathfrak{C}$-measurable sets which generates γ.
To show the uniqueness, let α be another measurable μ-partition of X with $\widetilde{\sigma}(\alpha) = \mathfrak{C} \bmod \mu$. Every element C of γ lies in $\mathfrak{C}$, hence there are $A_l \in \alpha$, l in some index set L such that $C = \bigcup_{l\in L} A_l \bmod \mu$, but $A_l \in \mathfrak{C} = \widetilde{\sigma}(\gamma)$, hence there are some $C_j^l \in \gamma$ for j in some index set J such that $A_l = \bigcup_{j\in J} C_j^l \bmod \mu$, i.e. $C = \bigcup_{l\in L}\bigcup_{j\in J} C_j^l \bmod \mu$, which implies that $C_j^l = C$ and $|L| = 1 = |J|$, thus $C = A_l \bmod \mu$, which gives $\gamma \subseteq \alpha \bmod \mu$. Analogously, one can show that $\gamma \supseteq \alpha \bmod \mu$. Hence, $\gamma = \alpha \bmod \mu$. □

Note, that if $\widetilde{\sigma}(\gamma) = \mathfrak{C} \bmod \mu$, then every element of γ of positive measure is mod μ equal to an atom of $\mathfrak{C}$.

1.4.1 The space of a μ-partition

Corresponding to every measurable μ-partition β of X, one can construct a new measure space $(X_\beta, \mathfrak{B}_\beta, \mu_\beta)$ by setting

$$X_\beta := \beta, \qquad \mathfrak{B}_\beta := \pi_\beta \widetilde{\sigma}(\beta), \qquad \mu_\beta := \mu \circ \pi_\beta^{-1},$$

where $\pi_\beta : X \longrightarrow \beta$ is the map which assigns to each element x of X the unique element β_x of β, which contains x , i.e. $\pi_\beta(x) := \beta_x$ (in case x is not contained in an element of β we just set $\beta_x := \emptyset$ and define $\emptyset \in \beta$), and, as above, $\widetilde{\sigma}(\beta)$ is the σ-algebra consisting of arbitrary (possibly uncountable) measurable unions of elements of β.

The map π_β is $\mathfrak{B}$-$\mathfrak{B}_\beta$-measurable: Every element of $\mathfrak{B}_\beta = \pi_\beta\widetilde{\sigma}(\beta)$ is of the form $\pi_\beta\left(\bigcup_{w\in W} B_w\right)$, for $B_w \in \beta$, W some arbitrary index set such that $\bigcup_{w\in W} B_w \in \mathfrak{B}$, hence $\pi_\beta^{-1}\left(\pi_\beta\left(\bigcup_{w\in W} B_w\right)\right) = \pi_\beta^{-1}(\{B_w : w \in W\}) = \bigcup_{w\in W} B_w \in \widetilde{\sigma}(\beta) \subseteq \mathfrak{B}$. In particular, π_β is $\widetilde{\sigma}(\beta)$-$\mathfrak{B}_\beta$-measurable.

Note, that $\mu_\beta(\beta) = \mu(\pi_\beta^{-1}(\beta)) = \mu\left(\bigcup_{A\in\beta} A\right) = \mu(X)$. Moreover, one can show, that if $(X, \mathfrak{B})$ is a standard measurable space, then so is $(X_\beta, \mathfrak{B}_\beta)$, according to [16], p. 31. Hence, if $\widetilde{\sigma}(\beta)$ is σ-finite and $(X, \mathfrak{B}, \mu)$ is a standard space, then $(X_\beta, \mathfrak{B}_\beta, \mu_\beta)$ is a σ-finite standard measure space. When speaking about such a measure space we will always assume that the μ-partition is such that the corresponding σ-algebra is σ-finite.

If β is such that $T^{-1}\beta \preceq \beta$ and $\widetilde{\sigma}(\beta)$ is σ-finite, then we can define a transformation T_β on $(X_\beta, \mathfrak{B}_\beta, \mu_\beta)$, by

$$T_\beta(B) = D \;:\Leftrightarrow\; D \in \beta \text{ such that } TB \subseteq D,$$

for $B \in \beta$.
Indeed, this is a well-defined map, since by $T^{-1}\beta \preceq \beta$, for every $B \in \beta$ there exists a $D \in \beta$ with $B \subseteq T^{-1}D \;\Rightarrow\; TB \subseteq TT^{-1}D \subseteq D$, and if there were $D, D' \in \beta$, $D \neq D'$, such that $TB \subseteq D$ and $TB \subseteq D'$, then $TB \subseteq D \cap D' = \emptyset$, a contradiction if $B \neq \emptyset$ (for $B = \emptyset$ we set $T_\beta(B) := \emptyset$).

Proposition 1.2. *The properties measurablility, non-singularity, measure-preservingness, conservativity and ergodicity carry over from T to T_β (i.e. if T is conservative, then so is T_β, and analogous for the other properties).*

Proof. Let $\{D_l : D_l \in \beta, l \in L\} \in \mathfrak{B}_\beta$ for some arbitrary index set L, then

$$T_\beta^{-1}\{D_l | D_l \in \beta, l \in L\} = \{B \in \beta | \exists l \in L : TB \subseteq D_l\} = \pi_\beta\Big(\bigcup_{B\in\beta \,:\, \exists l\in L \,:\, TB\subseteq D_l} B\Big) =$$

$$= \pi_\beta\Big(\bigcup_{B\in\beta \,:\, \exists l\in L \,:\, B\cap T^{-1}D_l = B} B\Big) = \pi_\beta\Big(\bigcup_{B\in\beta}\Big(B \cap \bigcup_{l\in L} T^{-1}D_l\Big)\Big) =$$

$$= \pi_\beta\Big(\Big(\bigcup_{B\in\beta} B\Big) \cap \Big(\bigcup_{l\in L} T^{-1}D_l\Big)\Big) = \pi_\beta\Big(T^{-1}\Big(\bigcup_{l\in L} D_l\Big)\Big) = \pi_\beta(T^{-1}(\pi_\beta^{-1}\{D_l : l \in L\})),$$

since $B \cap T^{-1}D_l = B$ or $= \emptyset$ by the assumption $\beta \succeq T^{-1}\beta$, and $TB \subseteq D_l \;\Leftrightarrow\; B \subseteq T^{-1}D_l \Leftrightarrow\; B \cap T^{-1}D_l = B$ for $B \neq \emptyset$, because also $D \supseteq TB \;\Rightarrow\; T^{-1}D \supseteq T^{-1}TB \supseteq B$.
With the above identity

$$T_\beta^{-1} = \pi_\beta \circ T^{-1} \circ \pi_\beta^{-1}, \tag{1}$$

we can easily compute that many properties of T carry over to T_β.

The map T_β is measurable, since $\{D_l | D_l \in \beta, l \in L\} \in \mathfrak{B}_\beta$ if and only if $\bigcup_{l\in L} D_l \in \mathfrak{B}$, hence by measurability of T, we have $T^{-1}\big(\bigcup_{l\in L} D_l\big) \in \mathfrak{B}$, thus since $T^{-1}D_l \in T^{-1}\beta$, we get $T^{-1}\big(\bigcup_{l\in L} D_l\big) = \bigcup_{l\in L} T^{-1}D_l \in \widetilde{\sigma}(T^{-1}\beta)$ by definition. Therefore, by (1),

$$T_\beta^{-1}\{D_l | D_l \in \beta, l \in L\} = \pi_\beta\underbrace{\Big(T^{-1}\Big(\bigcup_{l\in L} D_l\Big)\Big)}_{\in\widetilde{\sigma}(T^{-1}\beta)} \in \mathfrak{B}_{T^{-1}\beta} \subseteq \mathfrak{B}_\beta,$$

since $T^{-1}\beta \preceq \beta$.

For the verification of the measure-depending properties, first note, that

$$\pi_\beta^{-1}\Big(\pi_\beta\Big(\bigcup_{B_i\in\beta,\, i\in I} B_i\Big)\Big) = \bigcup_{B_i\in\beta,\, i\in I} B_i, \quad \text{for } B_i \in \beta \text{ with } \bigcup_{i\in I} B_i \in \widetilde{\sigma}(\beta),$$

viewing the image and the preimage of π_β as functions $\pi_\beta : \widetilde{\sigma}(\beta) \longrightarrow \mathfrak{B}_\beta$ and $\pi_\beta^{-1} : \mathfrak{B}_\beta \longrightarrow \widetilde{\sigma}(\beta)$, one may write the above as

$$\pi_\beta^{-1} \circ \pi_\beta = id_{\widetilde{\sigma}(\beta)} \quad \text{on } \widetilde{\sigma}(\beta). \tag{2}$$

Moreover, $\pi_\beta \circ \pi_\beta^{-1} = id_{\mathfrak{B}_\beta}$ on $\mathfrak{B}_\beta$, by surjectivity of π_β. Note, that (2) also holds on $\widetilde{\sigma}(T^{-1}\beta)$, since $\widetilde{\sigma}(T^{-1}\beta) \subseteq \widetilde{\sigma}(\beta)$, by $T^{-1}\beta \preceq \beta$.
Using (1) and (2), we deduce

$$\mu_\beta \circ T_\beta^{-1} = \mu \circ \pi_\beta^{-1} \circ \pi_\beta \circ T^{-1} \circ \pi_\beta^{-1} = \mu \circ T^{-1} \circ \pi_\beta^{-1} \text{ on } \mathfrak{B}_\beta = \pi_\beta(\widetilde{\sigma}(\beta)),$$

i.e.

$$(\mu_\beta \circ T_\beta^{-1})(\{D_l : D_l \in \beta, l \in L\}) = (\mu \circ T^{-1})\Big(\bigcup_{D_l \in \beta,\ l \in L} D_l\Big),$$

for $\{D_l : D_l \in \beta, l \in L\} \in \mathfrak{B}_\beta$. Hence, T is measure-preserving if and only if T_β is measure-preserving.

Further, by (1), $T_\beta^{-2} = (\pi_\beta \circ T^{-n} \circ \pi_\beta^{-1})^{-1} = (\pi_\beta^{-1})^{-1} \circ T^{-n} \circ \pi_\beta^{-1} = \pi_\beta \circ T^{-n} \circ \pi_\beta^{-1}$, since $(\pi_\beta^{-1})^{-1} = \pi_\beta$, by uniqueness of the inverse function. Hence, for $n \in \mathbb{N}$, we get

$$T_\beta^{-n} = \pi_\beta \circ T^{-n} \circ \pi_\beta^{-1}. \tag{3}$$

Note, that $\widetilde{\sigma}(T^{-n}\beta) \subseteq \widetilde{\sigma}(\beta)$, hence again $\pi_\beta^{-1} \circ \pi_\beta = id_{\widetilde{\sigma}(\beta)}$ on $\widetilde{\sigma}(T^{-n}\beta)$.
Now, let T be conservative, i.e. every $C \in \mathfrak{B}$ is recurrent, thus $\mu\big(C \cap \bigcap_{n\geq 1} T^{-n}C^c\big) = 0$.
Then,

$$\mu_\beta\Big(\{D_l : l \in L\} \cap \bigcap_{n\geq 1} T_\beta^{-n}(\{D_l : l \in L\}^c)\Big) =$$

$$= \mu\Big(\pi_\beta^{-1}(\{D_l : l \in L\}) \cap \pi_\beta^{-1}\Big(\bigcap_{n\geq 1} \pi_\beta(T^{-n}(\pi_\beta^{-1}(\{D_l : l \in L\}^c)))\Big)\Big) =$$

$$= \mu\Big(\Big(\bigcup_{l\in L} D_l\Big) \cap \bigcap_{n\geq 1} T^{-n}\Big(\Big(\bigcup_{l\in L} D_l\Big)^c\Big)\Big) = 0,$$

by recurrence of $\bigcup_{l\in L} D_l \in \mathfrak{B}$ w.r.t. T. Hence, T_β is conservative.
Similar, if T is ergodic, then so is T_β: Let T be ergodic, i.e. $\mu(A \,\Delta\, T^{-1}A) = 0 \Rightarrow \mu(A) = 0$ or $\mu(A^c) = 0$. Let $\{D_l : D_l \in \beta, l \in L\} \in \mathfrak{B}_\beta$ be such that $0 = \mu_\beta(\{D_l : l \in L\} \,\Delta\, T_\beta^{-1}\{D_l : l \in L\}) = \mu\big(\bigcup_{l\in L} D_l \,\Delta\, T^{-1}\big(\bigcup_{l\in L} D_l\big)\big)$, hence by ergodicity of T, we get that either $0 = \mu\big(\bigcup_{l\in L} D_l\big) = \mu_\beta(\{D_l : l \in L\})$ or $0 = \mu\big(\big(\bigcup_{l\in L} D_l\big)^c\big) = \mu_\beta((\{D_l : l \in L\})^c)$. Thus, T_β is ergodic, too. $\square$

Let $(X, \mathfrak{B}, \mu, T)$ be conservative and $T^{-1}\beta \preceq \beta$. Since in that case $(X_\beta, \mathfrak{B}_\beta, \mu_\beta, T_\beta)$ is conservative, too, we can consider the first return map $(T_\beta)_{\pi_\beta(A)}$ of T_β on $\pi_\beta(A) \in \mathfrak{B}_\beta = \pi_\beta(\widetilde{\sigma}(\beta))$ for (fixed) $A \in \widetilde{\sigma}(\beta)$, $A = \bigcup_{l\in L} B_l$, for $B_l \in \beta$. On the other hand, the first return map T_A of T on $A \in \widetilde{\sigma}(\beta) \subseteq \mathfrak{B}$ fulfills $T_A^{-1}(\beta \cap A) \preceq \beta \cap A$. Indeed, given $B \in \beta \cap A$ then $T_A^{-1}B = \bigcup_{m\geq 1} T^{-m}B \cap A \cap T^{-1}A^c \cap \ldots \cap T^{-m+1}A^c \cap T^{-m}A = \bigcup_{m\geq 1} T^{-m}B \cap \bigcup_{l\in L} B_l \cap T^{-1}\big(\bigcup_{B_i\in\beta, i\notin L} B_i\big) \cap \ldots \cap T^{-m+1}\big(\bigcup_{B_i\in\beta, i\notin L} B_i\big) \cap T^{-m}\big(\bigcup_{l\in L} B_l\big)$ $= \bigcup_{B_j\in\beta: \exists m\geq 1, T^{-m}B\cap B_j \cap \bigcap_{k=1}^{m-1} T^{-k}\left(\bigcup_{B_i\in\beta, i\notin L} B_i\right)\neq\emptyset} B_j$, since either $B_j \cap T^{-k}B_i = \emptyset$ or $B_j \cap T^{-k}B_i = B_j$ due to the assumption $T^{-1}\beta \preceq \beta$, which obviously implies $T^{-k}\beta \preceq \beta$.
Having $T_A^{-1}(\beta \cap A) \preceq \beta \cap A$, we can consider the map $(T_A)_{\beta\cap A}$ defined like before.
Moreover,

$$(T_A)_{\beta\cap A} = (T_\beta)_{\pi_\beta(A)}.$$

To verify this identity, let $B \in \beta$, $B \subseteq A$ be fixed. For $x \in B$, $T^{\varphi_{\bigcup_{l\in L} B_l}(x)} x \in \bigcup_{l\in L} B_l$, i.e. $T^{\varphi_{\bigcup_{l\in L} B_l}(x)} x \in B_u$, for some $u \in L$, but then $T^{-1}\beta \preceq \beta$ implies that $T^{\varphi_{\bigcup_{l\in L} B_l}(x)} B \subseteq B_u$, which can be written as $T_\beta^{\varphi_{\bigcup_{l\in L} B_l}(x)} B = B_u$. (Obviously, $(T^{\varphi_{\bigcup_{l\in L} B_l}(x)})_\beta B = (T_\beta)^{\varphi_{\bigcup_{l\in L} B_l}(x)} B$, since $(T_\beta)^k B = D \in \beta \Leftrightarrow$[3] $T^k B \subseteq D$ for every $k \in \mathbb{N}$.) Thus, $\varphi_{\bigcup_{l\in L} B_l}(x) = \varphi_{\{B_l : l\in L\}}(B)$ for every $x \in B$, since $\varphi_{\bigcup_{l\in L} B_l}(x)$ is minimal such that $T^{\varphi_{\bigcup_{l\in L} B_l}(x)} x \in \bigcup_{l\in L} B_l$. Hence, $(T_\beta)_{\pi_\beta(A)} = T_\beta^{\varphi_{\{B_l : l\in L\}}(B)} B = B_u$ and on the other hand, $(T_A)_{\beta\cap A}(B) = (T_A)_\beta(B) = B_u$ if and only if $B_u \supseteq T_A B = \{T^{\varphi_{\bigcup_{l\in L} B_l}(x)} x : x \in B\} = \{T^{\varphi_{\{B_l : l\in L\}}(B)} x : x \in B\} = T^{\varphi_{\{B_l : l\in L\}}(B)} B$, which is exactly $B_u = T_\beta^{\varphi_{\{B_l : l\in L\}}(B)} B$, which completes the argument.

1.4.2 The canonical system of measures

Consider a σ-finite standard space $(X, \mathfrak{B}, \mu)$ and a measurable μ-partition γ on X such that $\widetilde{\sigma}(\gamma)$ is σ-finite. Due the the previous section, we can speak of almost every $C \in \gamma$, by considering μ_γ-null-sets in $(\gamma, \mathfrak{B}_\gamma)$.

The theory of canonical systems of measures, studied in-depth in [16], is crucial for Parry's definition of entropy, respectively, for the definition of conditional entropy.

Theorem 1.10. *Let $(X, \mathfrak{B}, \mu)$ be a standard probability space and γ be a measurable μ-partition of X. Then for μ_γ-almost all $C \in \gamma$ there is a σ-algebra $\mathfrak{B}_C$ on C and a measure μ_C on $\mathfrak{B}_C$, such that $(C, \mathfrak{B}_C, \mu_C)$ is a standard probability space, and for all $B \in \mathfrak{B}$ we have*

- *$B \cap C \in \mathfrak{B}_C$, for μ_γ-almost all $C \in \gamma$,*
- *the map $C \mapsto \mu_C(C \cap B)$ is measurable w.r.t. $\mathfrak{B}_\gamma$, and*
- *$\mu(B) = \int_{X_\gamma} \mu_C(C \cap B)\, d\mu_\gamma(C)$.*

The system of measures $\{\mu_C\}_{C\in\gamma}$ obtained in this way is called canonical system w.r.t. γ.
Moreover, this system is uniquely determined modulo μ_γ, i.e. if there is another system $\{\mu'_C\}_{C\in\gamma}$ with the above properties, then for μ_γ-a.e. $C \in \gamma$ we have $\mu_C = \mu'_C$.

To prove the statement, we just need to apply the Theorem of Disintegration (Theorem 1.2).

Proof. The map $\pi_\gamma : X \longrightarrow \gamma$, $x \mapsto \gamma_x$, considered before, is a measurable, measure-preserving map between two standard spaces, thus we can apply Theorem 1.2 and achieve for μ_γ-almost every $C \in \gamma$ a probability measure μ_C on $(X, \mathfrak{B})$ with $1 = \mu_C(\pi_\gamma^{-1}(\{C\})) = \mu_C(C)$, which means that we can regard μ_C as a measure on $(C, \mathfrak{B} \cap C) =: (C, \mathfrak{B}_C)$ (by Remark 1.1 (1) a standard space).
Furthermore, the map $C \mapsto \mu_C$ is measurable w.r.t. $(\gamma, \mathfrak{B}_\gamma)$ and $(\mathsf{P}(X, \mathfrak{B}), \mathfrak{G})$, i.e. for every (fixed) $B \in \mathfrak{B}$ we have $\{C : \mu_C \in N_B^{-1}(\{[a,b]\})\} = \{C : \mu_C(B) \in [a,b]\} \in \mathfrak{B}_\gamma$ for

[3] By induction: For $(T_\beta)^2(B) = T_\beta(T_\beta(B)) = D_2$ and $T_\beta(B) = D_1$ with $D_i \in \beta$, $i \in \mathbb{N}$, we get by definition, that $T(T_\beta(B)) \subseteq D_2$ and $T(B) \subseteq D_1$, hence $T(T(B)) \subseteq T(D_1) = T(T_\beta(B)) \subseteq D_2$. Now, assume that we the claim is true for $k-1$-iterations, $k \in \mathbb{N}$, then $(T_\beta)^k B = D_k \Leftrightarrow$: $T((T_\beta)^{k-1}(B)) \subseteq D_k$, now for $(T_\beta)^{k-1}(B) = D_{k-1}$ we know that $TB \subseteq D_{k-1}$, hence $T(T^{k-1}(B)) \subseteq T(D_{k-1}) = T((T_\beta)^{k-1}(B)) \subseteq D$.

$a, b \in [0,1]$, which is measurablility of $C \mapsto \mu_C(B) = \mu_C(C \cap B)$.
Moreover, for all $B \in \mathfrak{B}$ fixed, we see that

$$\mu\Big(B \cap \pi_\gamma^{-1}\Big(\pi_\gamma\Big(\bigcup_{w \in W} C_w\Big)\Big)\Big) = \mu\Big(B \cap \bigcup_{w \in W} C_w\Big) = \int_{\{C_w : w \in W\}} \mu_C(B)\, d\mu_\gamma(C),$$

for every $\bigcup_{w \in W} C_w \in \widetilde{\sigma}(\gamma)$, $C_w \in \gamma$, W some index set, and especially,

$$\mu(B) = \mu\Big(B \cap \bigcup_{C \in \gamma} C\Big) = \int_\gamma \mu_C(B \cap C)\, d\mu_\gamma(C).$$

To show the uniqueness, let $\{\mu'_C\}_{C \in \gamma}$ be another canonical system of γ. Then for every $M \subseteq \gamma$, $M \in \mathfrak{B}_\gamma$ and every $A \in \mathfrak{B}$ fixed, we obtain

$$\int_M \mu_C(A \cap C)\, d\mu_\gamma(C) = \int_\gamma \mu_C(A \cap C) 1_M(C)\, d\mu_\gamma(C) = \int_\gamma \mu_C(A \cap C \cap \pi_\gamma^{-1}(M))\, d\mu_\gamma(C) =$$

$$= \mu(A \cap \pi_\gamma^{-1}(M)) = \int_\gamma \mu'_C(A \cap C \cap \pi_\gamma^{-1}(M))\, d\mu_\gamma(C) = \int_M \mu'_C(A \cap C)\, d\mu_\gamma(C),$$

since $1_M(C) = 1 \Leftrightarrow C \cap \pi_\gamma^{-1}(M) = C$ and $1_M(C) = 0 \Leftrightarrow C \cap \pi_\gamma^{-1}(M) = \emptyset$, for all $C \in \gamma$. We can take $M := \{C \in \gamma | \mu_C(A \cap C) \leq \mu'_C(A \cap C)\}$, because by assumption $C \mapsto \mu_C(A \cap C)$ and $C \mapsto \mu'_C(A \cap C)$ are $\mathfrak{B}_\gamma$-measurable, and therefore M is $\mathfrak{B}_\gamma$-measurable. This gives

$$\int_M \mu'_C(A \cap C) - \mu_C(A \cap C)\, d\mu_\gamma(C) = 0 \text{ and } \mu'_C(A \cap C) - \mu_C(A \cap C) \geq 0 \text{ for } C \in M$$

and analogously for M^c, thus

$$\mu_C(A \cap C) = \mu'_C(A \cap C) \text{ for } \mathfrak{B}_\gamma\text{-almost every } C \in \gamma.$$

Now, $A \in \mathfrak{B}$ was arbitrary chosen, and for fixed $C \in \gamma$ we have $\mathfrak{B}_C = \{A \cap C | A \in \mathfrak{B}\}$, hence $\mu_C = \mu'_C$, for $\mathfrak{B}_\gamma$-almost every $C \in \gamma$. □

Let us now state some useful properties of the canonical system of measures, obtained above.

Proposition 1.3. *Let $(X, \mathfrak{B}, \mu)$ be a σ-finite standard measure space and γ a measurable μ-partition of X.*

1. *The theorem above is also true for σ-finite spaces, if $\widetilde{\sigma}(\gamma)$ is σ-finite, i.e. then there is a (unique) canonical system of probability measures $\{\mu_C\}_{C \in \gamma}$ which fulfills the properties given in Theorem 1.10.*

2. *If $\mu(C) \neq 0$, for $C \in \gamma$, then the canonical measure μ_C is just the conditional measure w.r.t. C:*
$$\mu_C(B \cap C) = \mu(B|C),$$
for all $B \in \mathfrak{B}$.

3. *In fact, for $A \in \mathfrak{B}$ fixed we obtain*
$$\mu_{\pi_\gamma(x)}(A \cap \pi_\gamma(x)) = \mu(A|\widetilde{\sigma}(\gamma))(x)$$
for almost all $x \in X$, when viewing the canonical measure as map from X to $[0,1]$, given by $x \mapsto \mu_{\pi_\gamma(x)}(A \cap \pi_\gamma(x))$.

4. *For all $A \in \mathfrak{B}$,*

$$\mu(A) = 0 \iff \mu_C(A \cap C) = 0 \text{ for } \mu_\gamma\text{-a.e. } C \in \gamma.$$

Further, for $B, B' \in \mathfrak{B}$ with $B = B'$ mod μ,

$$\mu_C(C \cap B) = \mu_C(C \cap B') \text{ for } \mu_\gamma\text{-a.e. } C \in \gamma.$$

In particular,

$$\text{mod } \mu \Leftrightarrow \text{mod } \mu_C \text{ for } \mu_\gamma\text{-a.e. } C \in \gamma.$$

5. *Let γ' be another measurable μ-partition of X which is modulo μ equal to γ. Then the corresponding canonical systems are equal: For every fixed $A \in \mathfrak{B}$, we have*

$$\mu_{\pi_\gamma(x)}(A \cap \pi_\gamma(x)) = \mu_{\pi_{\gamma'}(x)}(A \cap \pi_{\gamma'}(x)) \text{ for } \mu\text{-a.e. } x \in X\ .$$

Since the domain of a measure is defined by the measure itself, we sometimes drop the restricting set and just write

$$\mu_{\pi_\gamma(x)}(A) = \mu_{\pi_{\gamma'}(x)}(A) \text{ for } \mu\text{-a.e. } x \in X\ .$$

6. *Consider another σ-finite standard measure space $(Y, \mathfrak{D}, \nu)$ and a measurable, measure-preserving map $\phi : X \longrightarrow Y$ and let ζ be a measurable ν-partition of Y such that $\widetilde{\sigma}(\zeta)$ is σ-finite. Then, ϕ preserves the measures of the corresponding canonical systems:*

$$\nu_{\pi_\zeta(\phi(x))} = \mu_{\pi_{\phi^{-1}(\zeta)}(x)} \circ \phi^{-1}|_{\pi_\zeta(\phi(x))} \text{ for } \mu\text{-a.e. } x \in X.$$

Moreover,

$$\phi^{-1} \circ \pi_\zeta \circ \phi = \pi_{\phi^{-1}(\zeta)} \ \mu\text{-almost everywhere.}$$

In fact, by (5), instead of ϕ we rather can take a map $\phi' : X_0 \longrightarrow Y_0$, for $X_0 = X$ mod μ and $Y_0 = Y$ mod ν with the above properties.
In particular, if $(Y, \mathfrak{D}, \nu)$ and ϕ are as in Theorem 1.4 w.r.t. the σ-finite sub-σ-algebra $\widetilde{\sigma}(\gamma)$, then $\phi^{-1}(\zeta) = \gamma$ mod μ and

$$\nu_{\pi_\zeta(\phi(x))} = \mu_{\pi_\gamma(x)} \circ \phi^{-1}|_{\pi_\zeta(\phi(x))} \text{ for } \mu\text{-a.e. } x \in X.$$

Consider the case of a measure-preserving automorphism $T : X \longrightarrow X$. Then we obtain

$$\mu_{TC} = \mu_C \circ T^{-1} \text{ for almost all } C \in \gamma.$$

Proof. 1. Let $\widetilde{\sigma}(\gamma)$ be σ-finite. Then the elements of γ can be collected to countable many sets of positive measure: Denote $\gamma = \{C_w | w \in W\}$ for a (possibly uncountable) index set W, then there exists $U_i \subseteq W$, i in a countable index set I, such that

$$D_i := \bigcup_{u \in U_i} C_u \in \mathfrak{B},\ \mu(D_i) < \infty,\ \forall i \in I \text{ and } \bigcup_{i \in I} D_i = X \text{ mod } \mu.$$

We can choose the sets D_i, $i \in I$, to be disjoint and of positive measure. Then $(D_i, \mathfrak{B} \cap D_i, \mu^{D_i})$, with $\mu^{D_i} := \mu(\cdot|D_i)$, is a standard probability space (ref. Remark 1.1) and $\gamma \cap D_i = \{C \in \gamma | C \subseteq D_i\}$ a measurable μ-partition on this space (see Example 1.1 (4)). Thus we can apply Theorem 1.10 to obtain a system of canonical measures

$$\{\mu_C^{D_i}\}_{C\in\gamma\cap D_i} \quad \text{on } (D_i, \mathfrak{B}\cap D_i, \mu^{D_i}).$$

The measures $\mu_C^{D_i}$ actually do not depend on the choice of the collection $\{D_i\}_{i\in I}$ with the above properties: Given another collection $D'_j = \bigcup_{v\in V_j} C_v$, for $j \in J$ countable, $\mu(D'_j) < \infty$ with $\bigcup_{j\in J} D'_j = X$ (mod μ). For D_i ($i \in I$) fixed, we know by Theorem 1.10, that $(C, \mathfrak{B}\cap C, \mu_C^{D_i})$ is unique for $\mu^{D_i}_{\gamma\cap D_i}$-almost every $C \in \gamma\cap D_i$. Clearly, there is a $j \in J$ such that $D_i \cap D'_j \neq \emptyset$. We are going to show that

$$\mu(D_i)\cdot\mu^{D_n}_{\gamma\cap D_n} = \mu(D'_j)\cdot\mu^{D'_j}_{\gamma\cap D'_j} \quad \text{on } \gamma\cap D_i\cap D'_j, \tag{4}$$

then we get $\mu_C^{D_i} = \mu_C^{D'_j}$ for $\mu^{D_i}_{\gamma\cap D_i}$-a. e.$\Leftrightarrow\mu^{D'_j}_{\gamma\cap D'_j}$-a.e. $C \in \gamma\cap D_i\cap D'_j$. And since $\{D_l \cap D'_k | l \in I, k \in J\}$ is a countable μ-partition of X, we get a unique canonical measure for every $C \in \gamma$, which we can now denote by $\mu_C := \mu_C^{D_l}$, for $l \in I$ such that $C \subseteq D_l$. Still, $E_A : C \mapsto \mu_C(A\cap C)$ is measurable, for every fixed $A \in \mathfrak{B}$, since

$$E_A^{-1}(O) = \bigcup_{i\in I} \underbrace{\gamma\cap D_i}_{\in\mathfrak{B}_\gamma}\cap \underbrace{E_A|_{D_i}^{-1}(O)}_{\in(\mathfrak{B}\cap D_i)_{\gamma\cap D_i}\subseteq\mathfrak{B}_\gamma} \in \mathfrak{B}_\gamma,$$

for every $O \in \mathfrak{B}(\overline{\mathbb{R}})$.

With the notation of section 1.4, equation (4) follows easily: $\mu^{D_i}_{\gamma\cap D_i} = \frac{1}{\mu(D_i)}\mu\circ\pi^{-1}_{\gamma\cap D_i}$, hence $\mu(D_i)\cdot\mu^{D_i}_{\gamma\cap D_i}|_{\gamma\cap D_i\cap D'_j} = \mu\circ\pi^{-1}_{\gamma\cap D_i\cap D'_j} = \mu(D'_j)\cdot\mu^{D'_j}_{\gamma\cap D'_j}|_{\gamma\cap D_i\cap D'_j}$.

2. Let $B \in \mathfrak{B}$ be fixed. The map π_γ is measurable, so we can use the transformation formula, and by the theorem above we obtain

$$\mu(B) = \int_{X_\gamma} \mu_C(C\cap B)\, d(\mu\circ\pi_\gamma^{-1})(C) = \int_X \mu_{\pi_\gamma(x)}(\pi_\gamma(x)\cap B)\, d\mu(x).$$

Thus for a fixed $C \in \gamma$ we get

$$\mu(C\cap B) = \int_X \mu_{\pi_\gamma(x)}(\pi_\gamma(x)\cap B\cap C)\, d\mu(x) = \int_X \mu_{\pi_\gamma(x)}(\pi_\gamma(x)\cap B)1_C(x)\, d\mu(x) =$$

$$= \mu_C(C\cap B)\mu(C),$$

i.e., if $\mu(C) \neq 0$, then $\mu_C(C\cap B) = \mu(B|C)$.

3. By definition of the conditional expectation, we obtain for every fixed $A \in \mathfrak{B}$ that $\int_X \mu(A|\widetilde{\sigma}(\gamma))(x)\, d\mu(x) = \int_X 1_A(x)\, d\mu(x) = \mu(A)$, since $X \in \widetilde{\sigma}(\gamma)$. Also by definition, $x \mapsto \mu(A|\widetilde{\sigma}(\gamma))(x)$ is $\widetilde{\sigma}(\gamma)$-measurable. Now, $\pi_\gamma(x) \mapsto \mu_{\pi_\gamma(x)}(A\cap\pi_\gamma(x))$ is $\pi_\gamma\widetilde{\sigma}(\gamma)$-measurable if and only if $\mu_\cdot(A\cap\cdot)\circ\pi_\gamma : x \mapsto \mu_{\pi_\gamma(x)}(A\cap\pi_\gamma(x))$ is $\widetilde{\sigma}(\gamma)$-measurable. Indeed, let $\pi_\gamma(x) \mapsto \mu_{\pi_\gamma(x)}(A\cap\pi_\gamma(x))$ be $\pi_\gamma\widetilde{\sigma}(\gamma)$-measurable, then, since π_γ is $\widetilde{\sigma}(\gamma)$-$\pi_\gamma\widetilde{\sigma}(\gamma)$-measurable, we get that $\mu_\cdot(A\cap\cdot)\circ\pi_\gamma : x \mapsto \pi_\gamma(x) \mapsto \mu_{\pi_\gamma(x)}(A\cap\pi_\gamma(x))$ is $\widetilde{\sigma}(\gamma)$-measurable. On the other hand, if $x \mapsto \mu_{\pi_\gamma(x)}(A\cap\pi_\gamma(x))$ is $\widetilde{\sigma}(\gamma)$-measurable, i.e. $\pi_\gamma^{-1}(\mu_\cdot(A\cap\cdot)^{-1}([a,b])) \in \widetilde{\sigma}(\gamma)$ for $a \leq b \in [0,1]$, then

$\pi_\gamma\widetilde{\sigma}(\gamma) \ni \pi_\gamma(\pi_\gamma^{-1}(\mu_\cdot(A\cap\cdot)^{-1}([a,b]))) = \mu_\cdot(A\cap\cdot)^{-1}([a,b])$, by surjectivity of π_γ. Therefore in the definition of a canonical system of measures we can replace the condition that $\pi_\gamma(x) \mapsto \mu_{\pi_\gamma(x)}(A\cap\pi_\gamma(x))$ is $\pi_\gamma\widetilde{\sigma}(\gamma)$-measurable by the condition that $\mu_\cdot(A\cap\cdot)\circ\pi_\gamma : x \mapsto \mu_{\pi_\gamma(x)}(A\cap\pi_\gamma(x))$ is $\widetilde{\sigma}(\gamma)$-measurable. The latter condition is fulfilled by $\mu(A|\widetilde{\sigma}(\gamma))$, thus by uniqueness of the canonical system, we get

$$\mu(\cdot|\widetilde{\sigma}(\gamma))(x) = \mu_{\pi_\gamma(x)}$$

for μ_γ-almost all $\pi_\gamma(x) \in \gamma$, hence the equation is in particular true for μ-almost all $x \in X$, since $\mu_\gamma(\{C_l : C_l \in \gamma, l \in L\}) = \mu\left(\bigcup_{C_l\in\gamma, l\in L} C_l\right)$.

4. Let $A \in \mathfrak{B}$, then

$$\int_\gamma \mu_C(C\cap A)\, d\mu_\gamma(C) = \mu(A) = 0 \;\Leftrightarrow\; \mu_C(C\cap A) = 0 \text{ for } \mu_\gamma\text{-a.e. } C\in\gamma.$$

For $B = B'$ mod μ set $N_1 := B\setminus B'$ and $N_2 := B'\setminus B$, then by the above $\mu_C(C\cap N_j) = 0$ for $j = 1,2$, hence

$$\mu_C(C\cap B) = \mu_C(C\cap(B\setminus N_1\cup N_1)) = \mu_C(C\cap B\setminus N_1) + \mu_C(C\cap N_1) =$$

$$= \mu_C(C\cap B'\setminus N_2) = \mu_C(C\cap B').$$

5. Let $A \in \mathfrak{B}$ fixed and denote

$$M := \{x\in X | \mu_{\pi_\gamma(x)}(A\cap\pi_\gamma(x)) \le \mu_{\pi_{\gamma'}(x)}(A\cap\pi_{\gamma'}(x))\}.$$

(This set is clearly measurable, since $x \mapsto \pi_\gamma(x)$ and $\pi_\gamma(x) \mapsto \mu_{\pi_\gamma(x)}$ are measurable, and analogous for γ'.) Let $x\in M$ and $C := \pi_\gamma(x)$ and let $C' \in \gamma'$ such that $C = C'$ mod μ, then either $x \in C\triangle C'$, which is a null-set, or $x\in C'$. Now, for μ-a.e. $y\in C$ we know that $y \in C'$, hence in the case $x\in C'$ we see that $\pi_{\gamma'}(x) = \pi_{\gamma'}(y)$ and therefore μ-a.e. $y\in C$ lies in M, thus

$$\pi_\gamma(x) \subseteq M \text{ mod } \mu, \text{ for a.e. } x\in M,$$

more precisely,

$$\pi_\gamma(x)\setminus(\pi_\gamma(x)\triangle\pi_{\gamma'}(x)) \subseteq M \text{ for a.e. } x\in M.$$

Analogously, $\pi_{\gamma'}(x)\setminus(\pi_\gamma(x)\triangle\pi_{\gamma'}(x)) \subseteq M$ for a.e. $x\in M$. Thus by (4), for a.e. $x\in M$ we get

$$\mu_{\pi_\gamma(x)}(\pi_\gamma(x)\cap M\cap A) = \mu_{\pi_\gamma(x)}\Big(\pi_\gamma(x)\setminus\Big(\bigcup_{C\in\gamma} C \triangle \bigcup_{C'\in\gamma'} C'\Big)\cap A\Big) =$$

$$= \mu_{\pi_\gamma(x)}(\pi_\gamma(x)\cap A).$$

Now,

$$\mu(A\cap M) = \int_\gamma \mu_C(A\cap C\cap M))\, d(\mu_\gamma\circ\pi_\gamma^{-1})(C) =$$

$$= \int_X \mu_{\pi_\gamma(x)}(A\cap\pi_\gamma(x)\cap M)\, d\mu(x) = \int_X \mu_{\pi_\gamma(x)}(A\cap\pi_\gamma(x))\cdot 1_M(x)\, d\mu(x).$$

In the same way we deduce that

$$\mu(A\cap M) = \int_X \mu_{\pi_{\gamma'}(x)}(A\cap\pi_{\gamma'}(x))\cdot 1_M(x)\, d\mu(x).$$

Hence,

$$\int_M \mu_{\pi_\gamma(x)}(A\cap\pi_\gamma(x))\, d\mu(x) = \int_M \mu_{\pi_{\gamma'}(x)}(A\cap\pi_{\gamma'}(x))\, d\mu(x),$$

which implies by the choice of M, that

$$\mu_{\pi_\gamma(x)}(A\cap\pi_\gamma(x)) = \mu_{\pi_{\gamma'}(x)}(A\cap\pi_{\gamma'}(x)) \text{ for } \mu\text{-a.e. } x\in M\ .$$

And we can deduce the same result on M^c. So we have shown, that

$$\mu_{\pi_\gamma(x)}(A\cap\pi_\gamma(x)) = \mu_{\pi_{\gamma'}(x)}(A\cap\pi_{\gamma'}(x)) \text{ for } \mu\text{-a.e. } x\in X\ .$$

6. First note that $\phi^{-1}(\zeta)$ is a measurable μ-partition of X. The last statement will follow by the first and (5), if we verify that

$$\phi^{-1}(\zeta) = \gamma \bmod \mu,$$

but this follows by uniqueness of the generator (Lemma 1.2), since $\widetilde{\sigma}(\phi^{-1}(\zeta)) = \phi^{-1}(\widetilde{\sigma}(\zeta)) = \phi^{-1}(\mathfrak{D}) = \mathfrak{C} \bmod \mu$ (because by Theorem 1.4 $\widetilde{\sigma}(\zeta) = \mathfrak{D}$ mod $\nu = \mu\circ\phi^{-1}$).
Our first claim,

$$\nu_{\pi_\zeta(\phi(x))} = \mu_{\pi_{\phi^{-1}(\zeta)}(x)}\circ\phi^{-1}|_{\pi_\zeta(\phi(x))} \ \text{ for } \mu\text{-a.e. } x\in X,$$

can be rewritten as

$$\nu_{\pi_\zeta(\phi(x))} = \mu_{\phi^{-1}(\pi_\zeta(\phi(x))}\circ\phi^{-1}|_{\pi_\zeta(\phi(x))} \ \text{ for } \mu\text{-a.e. } x\in X$$

since

$$\pi_{\phi^{-1}(\zeta)}(x) = \phi^{-1}(\pi_\zeta(\phi(x)) \ \text{ for } \mu\text{-a.e. } x\in X, \tag{5}$$

because $x\in\phi^{-1}(\zeta)\Leftrightarrow\phi(x)\in\zeta$, hence $\pi_{\phi^{-1}(\zeta)}(x) = (\phi^{-1}(\zeta))_x = \phi^{-1}(\zeta_{\phi(x)}) = \phi^{-1}(\pi_\zeta(\phi(x)))$. By uniqueness, it is sufficient to show that for fixed $D\in\mathfrak{D}$,

$$\zeta\ni Z\mapsto\mu_{\phi^{-1}(Z)}(\phi^{-1}(Z\cap D)) \text{ is measurable}$$

and

$$\nu(D) = \int_\zeta \mu_{\phi^{-1}(Z)}(\phi^{-1}(Z\cap D))\, d\nu_\zeta(Z).$$

By definition, $E_D : \phi^{-1}(Z)\mapsto\mu_{\phi^{-1}(Z))}(\phi^{-1}(Z\cap D))$ is measurable w.r.t. $\mathfrak{B}_{\phi^{-1}(\zeta)}$, i.e. for every $a\le b$, $a,b\in[0,1]$ fixed, we have

$$E_D^{-1}([a,b]) =: \{\phi^{-1}(Z_k) : k\in K\} \ \in \ \mathfrak{B}_{\phi^{-1}(\zeta)} = \pi_{\phi^{-1}(\zeta)}(\widetilde{\sigma}(\phi^{-1}(\zeta))),$$

where K shall be the index set of those $Z_k\in\zeta$ for which $\mu_{\phi^{-1}(Z_k))}(\phi^{-1}(Z_k\cap D))\in[a,b]$. This means that

$$\bigcup_{k\in K}\phi^{-1}(Z_k) = \phi^{-1}\Big(\bigcup_{k\in K} Z_k\Big) \ \in \ \widetilde{\sigma}(\phi^{-1}(\zeta)) = \phi^{-1}(\widetilde{\sigma}(\zeta)),$$

thus

$$\bigcup_{k\in K} Z_k \;\in\; \widetilde{\sigma}(\zeta).$$

(More precisely, we first obtain that $\phi^{-1}\big(\bigcup_{k\in K} Z_k\big) = \phi^{-1}\big(\bigcup_{l\in L} Z_l\big)$, for some $\bigcup_{l\in L, Z_l\in\zeta} Z_l \in \widetilde{\sigma}(\zeta)$, but this implies that $\bigcup_{k\in K}\phi^{-1}(Z_k) = \phi^{-1}\big(\bigcup_{k\in K} Z_k \cap \bigcup_{l\in L} Z_l\big)$ $= \bigcup_{j\in K\cap L}\phi^{-1}(Z_j)$, since the elements of ζ are disjoint, hence $L = K$).
Therefore,

$$\{Z_k : k\in K\} \;\in\; \pi_\zeta(\widetilde{\sigma}(\zeta) = \mathfrak{B}_\zeta,$$

i.e. $Z \mapsto \mu_{\phi^{-1}(Z))}(\phi^{-1}(Z\cap D))$ is measurable w.r.t. $\mathfrak{B}_\zeta$.
Now, using the transformation formula, we get

$$\int_\zeta \mu_{\phi^{-1}(Z)}(\phi^{-1}(Z\cap D))\; d(\nu\circ\pi_\zeta^{-1})(Z) =$$

$$= \int_Y \mu_{\phi^{-1}(\pi_\zeta(y))}(\phi^{-1}(\pi_\zeta(y))\cap\phi^{-1}(D))\; d(\mu\circ\phi^{-1})(y) =$$

$$= \int_X \mu_{\phi^{-1}(\pi_\zeta(\phi(x)))}(\phi^{-1}(\pi_\zeta(\phi(x)))\cap\phi^{-1}(D))\; d(\mu)(x) =$$

$$= \int_X \mu_{\pi_{\phi^{-1}(\zeta)}(x)}(\pi_{\phi^{-1}(\zeta)}(x)\cap\phi^{-1}(D))\; d(\mu)(x) = \mu(\phi^{-1}(D)) = \nu(D),$$

again due to identity (5).

□

Parallel to the above arguments, one can show

Theorem 1.11. *Let $(X,\mathfrak{B},\mu)$ be a σ-finite complete standard measure space without atoms such that $\mu(X) = \infty$ and $\mathfrak{C}$ a σ-finite complete sub-σ-algebra of $\mathfrak{B}$ - thus by Theorem 1.1 there is an essential isomorphism $\phi : (X,\mathfrak{C},\mu) \longrightarrow (\mathbb{R},\mathfrak{B}(\mathbb{R}),\lambda)$ - then there exists a standard measurable space $(Y,\mathfrak{Y})$ and a family of probability measures $\{m_r\}_{r\in\mathbb{R}}$ on this space, such that there is an essential isomorphism*

$$\theta : (X,\mathfrak{B},\mu) \longrightarrow (\mathbb{R}\times Y, \mathfrak{B}(\mathbb{R})\otimes\mathfrak{Y}, \overline{\mu})$$

with

$$\overline{\mu}(E\times A) := \int_E m_r(A)\; d\lambda(r) = \int_{\mathbb{R}} \delta_r\otimes m_r(E\times A)\; d\lambda(r),\; \forall E\in\mathfrak{B}(\mathbb{R}),\; A\in\mathfrak{Y},$$

and

$$\theta^{-1}(\mathfrak{B}(\mathbb{R})\times Y) = \phi^{-1}(\mathfrak{B}(\mathbb{R})) = \mathfrak{C}.$$

Note, that the map $r\mapsto\delta_r\otimes m_r(Q)$ is measurable for fixed $Q\in\mathfrak{B}(\mathbb{R})\otimes\mathfrak{Y}$, hence the above integration is defined.

There are some basic properties of measures of the form like $\overline{\mu}$ above, which will be useful later on:

Remark 1.5. Let $(X,\mathfrak{X},\eta)$ be a σ-finite measure space such that $\varepsilon := \{\{x\} | x\in X, \{x\}\in\mathfrak{X}\}$ is a measurable μ-partition of X (e.g. let $\mathfrak{X}$ be complete w.r.t. η), hence $\mathfrak{X} = \widetilde{\sigma}(\varepsilon)$, and let $(Z,\mathfrak{A})$ be another measurable space on which a system of probability measures P_x are defined for η-a.e. $x\in X$. Then

$$\nu(F) := \int_X \delta_x\otimes P_x(F)\; d\eta(x),\; \forall F\in\mathfrak{X}\otimes\mathfrak{A}$$

defines a measure on the product space $(X\times Z, \mathfrak{X}\otimes\mathfrak{A})$, which[4] satisfies the following properties.

1. The measure ν is σ-finite: This follows directly by
$$\nu(E\times D)\leq \eta(E),$$
as $P_x\leq 1$ for $E\in\mathfrak{X}$, $D\in\mathfrak{A}$, and η since is σ-finite.
Further, if η is atomless, then so is ν: Atoms have to be of the form $\{x\}\times D$, for some $D\in\mathfrak{A}$, $x\in X$, but
$$\nu(\{x\}\times D)=\int_{\{x\}}P_x(D)\;d\eta(x)=P_x(D)\cdot\eta(\{x\})=0.$$

2. The special form of ν gives
$$\int_{X\times Z}g(x,z)\;d\nu(x,z)=\int_X\int_Z g(x,z)\;dP_x(z)\;d\eta(x),$$
for every measurable map $g: X\times Z\longrightarrow[0,\infty]$.

Proof. If $g=1_F$, for $F\in\mathfrak{X}\otimes\mathfrak{A}$, then
$$\int_{X\times Z}1_F(x,y)\;d\nu(x,y)=\nu(F)=\int_X(\delta_x\otimes P_x)(F)\;d\eta(x)=$$
$$=\int_X\int_Z\int_X 1_F(z,y)\;d\delta_x(z)\;dP_x(y)\;d\eta(x)=\int_X\int_Z 1_F(x,y)\;dP_x(y)\;d\eta(x),$$
by using Fubini's Theorem (see e.g. [12], Satz 14.16). By linearity we can deduce the claim for elementary functions (i.e. function of the from $\sum_{n=1}^m c_n 1_{C_n}$ for C_n measurable, $c_n\in\mathbb{R}$ $n,m\in\mathbb{N}$) and by approximation (this is standard method of integration theory, which can be found in [12] for instance) we get the general statement. ☐

3. The canonical measures of ν w.r.t. $\varepsilon\times Z$ are nothing but the measures $\{\delta_x\otimes P_x\}_{x\in X}$, i.e.
$$\nu_{\{x\}\times Z}=\delta_x\otimes P_x|_{\{x\}\times Z}\;\text{ for }\eta\text{-a.e. }x\in X.$$

Proof. First note that $\pi_\varepsilon: x\mapsto\{x\}$, $x\in X$ is an isomorphism (π_ε^{-1} is measurable since $\mathfrak{X}=\widetilde{\sigma}(\varepsilon)=\sigma(S_n:n\in\mathbb{N})$ for $\{S_n:n\in\mathbb{N}\}$ a generating set of ε and clearly $\pi_\varepsilon(S_n)\in\pi_\varepsilon(\widetilde{\sigma}(\varepsilon))=\mathfrak{X}_\varepsilon$).
By uniqueness of a canonical system we just have to check the defining properties: For fixed $Q\in\mathfrak{X}\otimes\mathfrak{A}$
$$\{y\}\times Z\mapsto\{y\}\mapsto y\mapsto\delta_y\otimes P_y(Q)\text{ is measurable,}$$
since the projection p_1 to the first coordinate and π_ε^{-1} are measurable. It is left to show that
$$\nu(Q)=\int_{\varepsilon\times Z}\delta_y\otimes P_y((\{y\}\times Z)\cap Q)\;d(\nu\circ\pi_{\varepsilon\times Y}^{-1})(\{y\}\times Z).$$

[4] In particular, all this is true for every product measure $\nu=\mu_1\otimes\mu_2$, with μ_1 a σ-finite measure on $(X,\mathfrak{X})$ and μ_2 a probability measure on $(Z,\mathfrak{A})$, since $\mu_1\otimes\mu_2(F)=\int_X\mu_2(\{z\in Z:(x,z)\in F\})\;d\mu_1(x)=\int_X\delta_x\otimes\mu_2(F)\;d\mu_1(x)$ for every $F\in\mathfrak{X}\otimes\mathfrak{A}$ (see e.g. [7]), hence, with the notation above, $P_x:=\mu_2$, $\forall x\in X$.

By the transformation formula and the fact that $y = \pi_\varepsilon^{-1}(p_1(\{y\} \times Z))$, the right hand side is

$$\int_{X\times Z} \delta_{\pi_\varepsilon^{-1}(p_1(\pi_{\varepsilon\times Z}(x,z)))} \otimes P_{\pi_\varepsilon^{-1}(p_1(\pi_{\varepsilon\times Z}(x,z)))}(\pi_{\varepsilon\times Z}(x,z) \cap Q)\; d\nu(x,z) =$$

$$= \int_{X\times Z} \delta_x \otimes P_x((\{x\} \times Z) \cap Q)\; d\nu(x,z) = \int_{X\times Z} \delta_x \otimes P_x(Q)\; d\nu(x,z),$$

which by (2) is equal to

$$\int_X \int_Z \delta_y \otimes P_y(Q)\; dP_y(y)\; d\eta(y) = \int_X \delta_y \otimes P_y(Q)\; d\eta(y) = \nu(Q),$$

since P_y is a probability measure. □

1.5 Some examples of infinite measure systems

There are well-studied ([6], [14], [21], [26] and many more) examples of dynamical systems with infinite measure. In this section we will roughly introduce a few of them, and turn back to them later on, when considering the entropy.

There is a wide class of dynamical systems, which turned out to behave quite nicely. This class has been studied by Maximilian Thaler, [21], therefore, we will call them just Thaler-maps.

Definition ([21])**.** A transformation T on $[0,1]$ is called *Thaler-map*, if there exists a countable λ-partition $\beta = \{B_i : i \in I\}$ of $[0,1]$, with $|I| \geq 2$, such that:

1. For all $i \in I$, we have that $T|_{B_i}$ is twice differentiable, the closure of TB_i (w.r.t. the usual metric in $\mathbb{R}$) is equal to $[0,1]$ and B_i contains one and only one fixed point, denoted by y_i. Moreover, $|\{y \in X : T'(y) = 1\}|$ is finite.
2. For every $\varepsilon > 0$ there exists a $\rho(\varepsilon) > 1$ in $\mathbb{R}$ such that $T'(x) \geq \rho(\varepsilon) > 1$ for all $x \in \bigcup_{i\in I} B_i \setminus \bigcup_{i\in I}(y_i - \varepsilon, y_i + \varepsilon)$. Further, there shall exist an $\eta > 0$ such that T' is increasing on $B_i \cap (y_i, y_i + \eta)$ and decreasing on $B_i \cap (y_i - \eta, y_i)$ for $i \in I$.
3. There is a constant $A < \infty$ in $\mathbb{R}$ such that $|\frac{T''(x)}{(T'(x))^2}| \leq A$ for all $x \in \bigcup_{i\in I} B_i$.

There is a strong result about this class of systems, namely, that every Thaler-map has an invariant measure, which is equivalent to λ, and the system is ergodic and conservative w.r.t. that measure. (See [21] for a proof.)

Example 1.2 (Boole's Transformation)**.** The map $Tx := x - \frac{1}{x}$ on $\mathbb{R}$ is called Boole's transformation. The system $(\mathbb{R}, \mathfrak{B}(\mathbb{R}), \lambda, T)$ is measure-preserving, conservative, ergodic and $\lambda(\mathbb{R}) = \infty$. (Ergodicity can be obtained by looking at the induced transformation T_Y with $Y = [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$, confer [27].) Further, one can show that

$$(\mathbb{R}, \mathfrak{B}(\mathbb{R}), \lambda, T) \approx ((0,1), \mathfrak{B}((0,1)), \eta, S)$$

where

$$S(x) := \begin{cases} \dfrac{x(1-x)}{1-x-x^2} & \text{for } x \in (0, \frac{1}{2}), \\ 1 - S(1-y) = \dfrac{2x-1}{3x-x^2-1} & \text{for } x \in (\frac{1}{2}, 1), \end{cases}$$

for $x \in (0,1)$ and

$$\eta([a,b]) := \int_a^b \Big(\frac{1}{(1-x)^2} + \frac{1}{x^2}\Big) dx$$

for $a < b$ in $(0,1)$. Note, that η is S-invariant and, still, $\eta((0,1)) = \infty$, caused by the indifferent fixed points 0 and 1 with $S'(0) = 1$, $S'(1) = 1$. Moreover, $((0,1), \mathfrak{B}((0,1)), \eta, S)$ is a Thaler-map. (See [26] for more details.)

Example 1.3 (Markov shifts)**.**

1. *Markov shifts:* Let $(Y_n)_{n\in\mathbb{N}_0}$ be a Markov chain with discrete time steps with values in a countable state space $\Sigma := \{a_1, a_2, \dots\}$, defined on some probability space $(\Omega, \mathfrak{C}, P)$ i.e. $Y_n : \Omega \longrightarrow \Sigma$ is $\mathfrak{C}$-$\mathscr{P}(\Sigma)$- measurable and $P(Y_{n+1} = b | Y_n = a, Y_{n-1} = a_{i_{n-1}}, \dots, Y_0 = a_{i_0}) = P(Y_{n+1} = b | Y_n = a) = p_{ab}$ for $a, b, a_{i_{n-1}}, \dots, a_{i_0} \in \Sigma$, $\forall n \in \mathbb{N}_0$. The probabilities p_{ab} of going from state a to state b (called transition probabilities) form a stochastic matrix $(p_{a_i a_j})_{i,j\in\mathbb{N}}$. Further, let u_{a_i} be the probability of starting at state a_i such that $u_{a_i} = \sum_{a\in\Sigma} u_a p_{aa_i}$ for every $i \in \mathbb{N}$ (stationary distribution).
   Formally, let us consider $(X, \mathfrak{C}, \mu, S)$ with $X := \Sigma^{\mathbb{N}}$, $\mathfrak{A} := \sigma([a_{i_1} \dots a_{i_n}] : a_{i_1} \dots, a_{i_n} \in \Sigma, n \in \mathbb{N})$ for $[a_{i_1} \dots a_{i_n}] := \{x = (x_1, x_2, \dots) \in X : x_i = a_{i_k}, \forall k \in \{1, \dots, n\}\}$ and $\mu([a_{i_1} \dots a_{i_n}]) := u_{a_{i_1}} p_{a_{i_1} a_{i_2}} \cdots p_{a_{i_{n-1}} a_{i_n}}$ for all $a_{i_1} \dots, a_{i_n} \in \Sigma$, $n \in \mathbb{N}$ (note that this indeed defines a unique measure on $(X, \mathfrak{B})$ since $\{[a_{i_1} \dots a_{i_n}] : a_{i_1}, \dots, a_{i_n} \in \Sigma, n \in \mathbb{N}\}$ is an intersection-stable, σ-finite generator of $\mathfrak{B}$), and S the shift on X, i.e. $S((a_{i_1}, a_{i_2}, a_{i_3}, \dots)) := (a_{i_2}, a_{i_3}, \dots)$.
   This defines a Markov process: $\mu(X_n = b \mid X_1 = a_1, \dots, X_{n-2} = a_{i_{n-2}}, X_{n-1} = a]) = \frac{u_{a_{i_1}} p_{a_{i_1} a_{i_2}} \cdots p_{a_{i_{n-2}} a} p_{ab}}{u_{a_{i_1}} p_{a_{i_1} a_{i_2}} \cdots p_{a_{i_{n-2}} a}} = p_{ab} = \mu(X_n = b | X_{n-1} = a)$, for X_n the random variable assigning to each chain $x = (x_0, x_1, \dots) \in X$ its nth entry x_n.
   Clearly, S is measure-preserving, since for $i_1, \dots, i_n \in \mathbb{N}$, $n \in \mathbb{N}$,

   $$\mu(S^{-1}[a_{i_1} \dots a_{i_n}]) = \mu\Big(\bigcup_{a\in\Sigma} [aa_{i_1} \dots a_{i_n}]\Big) = \sum_{a\in\Sigma} \mu([aa_{i_1} \dots a_{i_n}])) =$$

   $$= \sum_{a\in\Sigma} u_a p_{aa_{i_1}} p_{a_{i_1} a_{i_2}} \cdots p_{a_{i_{n-1}} a_{i_n}} = u_{a_{i_1}} p_{a_{i_1} a_{i_2}} \cdots p_{a_{i_{n-1}} a_{i_n}} = \mu([a_{i_1} \dots a_{i_n}]),$$

   because $u_{a_i} = \sum_{a\in\Sigma} u_a p_{aa_i}$.
   Moreover, if the Markov chain is null-recurrent, then S is conservative (see e.g. [14] and references given there). Further, if we assume that from every state one can go to any other state, that is, the process is irreducible, then S is ergodic.

2. *Renewal chains:* Let us consider a renewal chain on $\mathbb{N}$, in that, starting at state 1 the probability to go to state $m \in \mathbb{N}$ shall be positive, say f_m, and $\sum_{m\in\mathbb{N}} f_m = 1$. Being at state m the probability to go to state $m-1$ shall be 1 for every $n \in \mathbb{N}$. So we obtain the stochastic matrix

   $$(p_{ij})_{i,j\in\mathbb{N}} = \begin{pmatrix} f_1 & f_2 & f_3 & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ \vdots & & & \end{pmatrix},$$

   for p_{ij} denoting the probability of going from state i to state j. Clearly, this defines an irreducible and recurrent process.

The starting probabilities $(u_m)_{m\in\mathbb{N}}$ shall fulfill $u_m = \sum_{i\in\mathbb{N}} u_i p_{im}$ for every $m \in \mathbb{N}$, hence in our setting,

$$u_m = \sum_{i\in\mathbb{N}} u_i p_{im} = u_1 f_m + u_{m+1}, \quad \forall m \in \mathbb{N}.$$

We want to start at state 1 with probability 1, therefore the above recursion is $u_{m+1} = u_m - f_m$. So, for $m \geq 2$, we have $u_{m+1} = u_{m-1} - f_{m-1} - f_m = \ldots = u_1 - \sum_{k=1}^{m} f_k = 1 - \sum_{k=1}^{m} f_k = \sum_{l=1}^{\infty} f_l - \sum_{k=1}^{m} f_k = \sum_{l=m+1}^{\infty} f_l$, i.e.

$$u_m = \sum_{l\geq m} f_l.$$

Let $(X, \mathfrak{B}, \mu, S)$ be the corresponding shift space as in Example (1) above. We have

$$\mu(X) = \mu\Big(\bigcup_{n\in\mathbb{N}} [n]\Big) = \sum_{n\in\mathbb{N}} \mu([n]) = \sum_{n\in\mathbb{N}} u_n =$$

$$= \sum_{n=1}^{\infty} \sum_{k\geq n} f_k = \sum_{k\geq 1} f_k + \sum_{k\geq 2} f_k + \sum_{k\geq 3} f_k + \ldots = f_1 + 2f_2 + 3f_3 + \ldots = \sum_{k=1}^{\infty} k f_k.$$

We are interested in the case when $\mu(X) = \infty$, hence we want $\sum_{k=1}^{\infty} k f_k$ to be infinite. So, set, for example,

$$f_k := ck^{-1-\alpha} \text{ for some } \alpha \in (0,1) \text{ and } c \in (0,\infty) \text{ constant.}$$

(Of course, it would be enough to assume that f_k is asymptotically equivalent to $ck^{-1-\alpha}$.) Then, clearly, $\sum_{k=1}^{\infty} k f_k = c \sum_{k=1}^{\infty} \frac{1}{k^{\alpha}} = \infty$. Moreover, this process then is null-recurrent, since f_k is also the probability to return to 1 in exactly k steps, thus if 1 is recurrent and $\sum_{k=1}^{\infty} k f_k = \infty$, then 1 is null-recurrent by [8], hence the system is null-recurrent, since 1 is recurrent and 1 is the only state to start with (with positive probability). Thus, the shift S is conservative and ergodic.
Let us consider $Y := [1] = \{(1, a_2, a_3 \ldots) \in X : a_i \in \mathbb{N}, i = 2, 3 \ldots\}$, the set of all chains starting at the state 1. This set has positive finite measure: $\mu(Y) = \mu([1]) = \sum_{a\in\mathbb{N}} u_1 p_{1a} = u_1 = 1$. Moreover, since the system is conservative and ergodic, we get that Y is a sweep-out set (confer Remark 1.3). So, $(Y, \mathfrak{B} \cap Y, \mu|_Y, S_Y)$ is a conservative and ergodic dynamical system with a probability measure, by Proposition 1.1.

3. *Random walk on $\mathbb{Z}$:* Let us now consider a Markov chain which behaves very randomly, namely the random walk on $\mathbb{Z}$ with $p_{ii+1} = \frac{1}{2} = p_{ii-1}$ and $u_i = 1$ for every state i in $\mathbb{Z}$. (Then, obviously, the condition $u_i = \sum_{j\in\mathbb{Z}} u_j p_{ji} = \frac{1}{2} u_{i-1} + \frac{1}{2} u_{i+1}$ is fulfilled.) This process is obviously irreducible. Moreover, it is well known, that this process is null-recurrent. Hence the corresponding shift space is conservative and ergodic with infinite measure. (Clearly, $\mu(X) = \sum_{i\in\mathbb{Z}} u_i = \infty$.)

1.6 Prerequisites from probability theory

Now let us consider a probability space $(\Omega, \mathfrak{A}, P)$. A family of real-valued random variables $(X_n)_{n\in\mathbb{N}_0}$ (i.e. $X_n : \Omega \longrightarrow \mathbb{R}$ measurable) is called a *(sub-)martingale* w.r.t. to a filtration $(\mathfrak{F}_n)_{n\in\mathbb{N}_0}$ (i.e. $\mathfrak{F}_n$ are sub-σ-algebras of $\mathfrak{A}$ and $\mathfrak{F}_n \subseteq \mathfrak{F}_{n+1}$, $\forall n \in \mathbb{N}_0$), if X_n is $\mathfrak{F}_n$-measurable for every $n \in \mathbb{N}_0$, $E[|X_n|] < \infty$, $\forall n \in \mathbb{N}_0$, and $E[X_n|\mathfrak{F}_m] = X_m$ (resp. $E[X_n|\mathfrak{F}_m] \geq X_m$), for all $m < n$, $m \in \mathbb{N}_0$. For such concepts we have the following convergence result, proven e.g. in [12], Satz 11.4.

Theorem 1.12 (Martingale convergence)**.** *For every sub-martingale $(X_n)_{\in\mathbb{N}_0}$ w.r.t. a filtration $(\mathfrak{F}_n)_{n\in\mathbb{N}_0}$, with $\sup_{n\in\mathbb{N}_0} E[\max(X_n,0)] < \infty$, there is a $\sigma\big(\bigcup_{n\in\mathbb{N}_0}\mathfrak{F}_n\big)$-measurable random variable X_∞ with $E[|X_\infty|] < \infty$ and $X_n \longrightarrow X_\infty$ a.e. for $n \to \infty$.*

1.6.1 Random measures

We will shortly introduce the concept of random measures which will be useful later on. For a detailed elaboration we refer to [29] or [12].

Definition. Let $(\Omega,\mathfrak{A},P)$ be a probability space and $(X,\mathfrak{B})$ a measure space. A map

$$Z : \Omega \times \mathfrak{B} \longrightarrow \overline{\mathbb{R}}$$

is called a *random measure* on $(X,\mathfrak{B})$, if

- for every fixed $B \in \mathfrak{B}$ the map $\omega \mapsto Z(\omega,B)$ for $\omega \in \Omega$ is measurable, i.e. $Z(\cdot,A) =: Z(A)$ is a random variable, and
- for every fixed $\omega \in \Omega$ the map $B \mapsto Z(\omega,B)$ for $B \in \mathfrak{B}$ defines a measure $Z(\omega,\cdot) =: Z_\omega$ on $(X,\mathfrak{B})$.

Random measures on the real line can be identified with choosing points randomly as the next Theorem (given e.g. in [29]) points out

Theorem 1.13. *Let Z be a random measure on $([0,\infty),\mathfrak{B}([0,\infty)))$ which takes values in $\overline{\mathbb{N}}_0$ such that $Z([0,t]) < \infty$ almost everywhere for $t \in [0,\infty)$, then there are countably many random variables $X_1, X_2, \ldots$ taking values in $([0,\infty],\mathfrak{B}([0,\infty]))$ with*

$$Z([0,t]) = \sum_{n=1}^{\infty} 1_{[0,t]} \circ X_n = \sum_{n=1}^{\infty} \delta_{X_n(\cdot)}([0,t]) \ \ \textit{almost everywhere,}$$

for $t \in [0,\infty)$.
(For a measure space $(X,\mathfrak{B})$, let δ_x denote the Dirac measure for $x \in X$, as usual defined as $\delta_x(B) := 1$ if $x \in B$ and equal to zero if $x \notin B$, for $B \in \mathfrak{B}$.)

The proof follows [29]:

Proof. Let $(\Omega,\mathfrak{A},P)$ denote the underlying probability space of Z. For $\omega \in \Omega$ define

$$X_n(\omega) := \inf\{s \in [0,\infty] : Z(\omega,[0,s]) \geq n\},$$

for each $n \in \mathbb{N}$. We set $\inf\emptyset := \infty$. The map X_n is a random variable on $(\Omega,\mathfrak{A},P)$ since for $t \in [0,\infty]$ we have

$$\{X_n \in [0,t]\} = \{\inf\{s \in [0,\infty] : Z([0,s]) \geq n\} \leq t\} = \{\exists s \in [0,t] : Z([0,s]) \geq n\} =$$
$$= \{Z([0,t]) \geq n\} \ \in \mathfrak{A}$$

(since $Z([0,t])$ is a random variable), because $Z([0,s]) \leq Z([0,t])$ for $s \leq t$, as Z_ω is a measure.
Further, $X_n \leq X_{n+1}$, for every $n \in \mathbb{N}$: This is obvious, since $Z([0,t]) \leq n+1 \Rightarrow Z([0,t]) \leq n$.
Consider $\Omega' := \Omega \setminus \{\omega \in \Omega | \exists t \in [0,\infty) : Z(\omega,[0,t]) = \infty\}$. This set is measurable because, since $\{\omega \in \Omega | Z(\omega,[0,t]) < \infty\} \supseteq \{\omega \in \Omega | Z(\omega,[0,m]) < \infty\}$ for $m \in \mathbb{N}$ with $m \geq t$, we can write $\Omega' = \bigcap_{m\in\mathbb{N}}\{\omega \in \Omega | Z(\omega,[0,m]) < \infty\} \in \mathfrak{A}$. And by assumption we

get $\Omega = \Omega'$ mod P. Since by assumption Z takes only values in $\overline{\mathbb{N}}_0$, we have for $\omega \in \Omega'$ that $Z(\omega,[0,t]) \in \mathbb{N}$, for fixed $t \in [0,\infty)$, thus for $\omega \in \Omega'$ we can write

$$\sum_{n=1}^{\infty} 1_{\{X_n \le t\}}(\omega) = \sum_{n=1}^{\infty} 1_{\{Z([0,t]) \ge n\}}(\omega) =$$

$$= 1_{\{\omega \in \Omega : Z(\omega,[0,t]) \ge 1\}}(\omega) + 1_{\{\omega \in \Omega : Z(\omega,([0,t]) \ge 2\}}(\omega) + \ldots = Z(\omega,[0,t]),$$

i.e. $Z_\omega = \sum_{n=1}^{\infty} \delta_{X_n(\omega)}$ on $\{[0,t] : t \in [0,\infty)\}$, which generates $\mathfrak{B}([0,\infty))$ and is stable under intersections and Z_ω is σ-finite on it. Hence by uniqueness of measures ([28], Satz 2.4) we get $Z_\omega = \sum_{n=1}^{\infty} \delta_{X_n(\omega)}$ on $\mathfrak{B}([0,\infty))$ for $\omega \in \Omega'$, thus for almost every $\omega \in \Omega$ we get $Z_\omega = \sum_{n=1}^{\infty} \delta_{X_n(\omega)}$. □

For random measures on $\mathbb{R}$, we just need to extend the defining sequence of random variables canonically: Let Z be an $\overline{\mathbb{N}}_0$-valued random measure on $(\mathbb{R},\mathfrak{B}(\mathbb{R}))$ with $Z([a,b]) < \infty$ almost surely for every $a \le b \in \mathbb{R}$, then

$$Z = \sum_{k \in \mathbb{Z}\setminus\{0\}} \delta_{X_k} \text{ almost everywhere,}$$

for countably many random variables X_k taking values in $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$. Indeed, if we additionally define

$$X_{-n}(\omega) := \sup\{r \in [-\infty,0) : Z(\omega,[r,0)) \ge n\}$$

for $n \in \mathbb{N}$, $\omega \in \Omega$, then, as above, we get for $a \le b$ in $\mathbb{R}$ and $\omega \in \Omega' := \{\omega \in \Omega | Z(\omega,[a,b]) < \infty,\ \forall a \le b \in \mathbb{R}\} = \bigcap_{m\in\mathbb{N}}\{\omega \in \Omega | Z(\omega,[0,m]) < \infty\} \cap \bigcap_{m\in\mathbb{N}}\{\omega \in \Omega | Z(\omega,[-m,0]) < \infty\}$

$$\sum_{k \in \mathbb{Z}\setminus\{0\}} \delta_{X_k(\omega)}([a,b]) = \sum_{n\in\mathbb{N}} \delta_{X_n(\omega)}([a,b] \cap [0,\infty)) + \sum_{n\in\mathbb{N}} \delta_{X_{-n}(\omega)}([a,b] \cap (-\infty,0)) =$$

$$= Z(\omega,[a,b] \cap [0,\infty)) + Z(\omega,[a,b] \cap (-\infty,0)) = Z(\omega,[a,b]).$$

In fact, we get

Corollary 1.2. *The above Theorem is still true for random measures on standard spaces, i.e. let Z be an $\overline{\mathbb{N}}_0$-valued random measure on some standard measurable space $(X,\mathfrak{B})$ such that $\int_\Omega Z(\omega)\, dP(\omega)$ - called the* intensity *of Z - is σ-finite, then there are countably many random variables X_i for $i \in \mathbb{Z}$, with values in $(X \cup \{\chi\}, \sigma(\mathfrak{B} \cup \{\chi\}))$, for some $\chi \notin X$, such that $X \cup \{\chi\}$ is contained in some separable complete space w. r. t. the same metric as X, which determine Z, i.e.*

$$Z = \sum_{i\in\mathbb{Z}} \delta_{X_i}.$$

Proposition 1.4. *The distribution $P \circ Z^{-1}$ of a random measure Z on $(X,\mathfrak{B})$ is uniquely determined, if $P \circ Z^{-1}$ is given on $\left\{\bigcap_{i=1}^{l} N_{B_i}^{-1}(E) : E \in \mathfrak{E},\ B_1,\ldots B_l \in \mathfrak{B} \text{ disjoint }, l \in \mathbb{N}\right\}$, where $\mathfrak{E}$ is an intersection-stable generator of $\mathfrak{B}(\mathbb{R})$ and N_B the evaluation map, $N_B(\nu) = \nu(B)$ for every measure ν on $(X,\mathfrak{B})$, $N_B : \{\nu \text{ measure on } (X,\mathfrak{B})\} \longrightarrow [0,\infty]$.*

This result is proven e.g. in [29].

Since Z_ω is a measure for $\omega \in \Omega$ we can consider integrals w.r.t. Z_ω. Especially, $Z_\omega(B) = \int 1_B\, dZ_\omega$, for $B \in \mathfrak{B}$. This leads to the notion of a random integral:

Definition. Let Z be a random measure on $(X,\mathfrak{B})$ and $f : X \longrightarrow [0,\infty]$ a measurable function. The map

$$Z(f) := \int f\, dZ : \omega \mapsto \int f\, dZ_\omega,$$

for $\omega \in \Omega$, is called the *random integral* of f w.r.t. Z.

One can show that the random integral is measurable.

With the above definition we can adapt the Laplace transform of measures to random measures:

Definition. The *Laplace functional* of Z is defined by

$$\Psi_Z(f) := E\left[e^{-\int f\, dM}\right] = \int e^{-\int f\, dM}\, dP,$$

for $f : (X,\mathfrak{B}) \longrightarrow [0,\infty]$ measurable.

This definition is as useful as the usual one for (non-random) measures (for a proof see e.g. [12], Satz 24.7):

Proposition 1.5. *The Laplace functional of Z uniquely determines the distribution $P \circ Z^{-1}$ of Z.*

2 The Poisson suspension

In this chapter let $(X,\mathfrak{B},\mu)$ be a σ-finite standard measure space.
We will construct a standard probability space out of $(X,\mathfrak{B},\mu)$ together with a transformation which will inherit important properties from T. The possibility to switch from an infinite space to a special probability space bears some advantages (especially for handling the notion of entropy).

2.1 Construction and basic properties

Let us consider the set

$$X^\star := \{\nu : \mathfrak{B} \longrightarrow \overline{\mathbb{N}}_0 | \nu \text{ measure on } X\}.$$

For any fixed $B \in \mathfrak{B}$, we want the function $N_B : X^\star \longrightarrow \overline{\mathbb{N}}_0$, $N_B(\nu) := \nu(B)$, which assigns to each measure $\nu \in X^\star$ its evaluation at B, to be measurable. So we equip $X^\star$ with the generated σ-algebra

$$\mathfrak{B}^\star := \sigma(N_B : B \in \mathfrak{B}).$$

The Poisson distribution with parameter λ will be denoted by Poi_λ, i.e. $\mathrm{Poi}_\lambda(n) = e^{-\lambda}\dfrac{\lambda^n}{n!}$ for $n \in \mathbb{N}_0$. We set $\mathrm{Poi}_\lambda(\infty) := 0$ and $\mathrm{Poi}_\infty := \delta_\infty$.

The following proposition provides a special probability measure on $(X^\star,\mathfrak{B}^\star)$.

Proposition 2.1. *Let $(X,\mathfrak{B},\mu)$ be a σ-finite standard measure space and $(X^\star,\mathfrak{B}^\star)$ constructed as above. There exists a unique probability measure $\mu^\star$ on $(X^\star,\mathfrak{B}^\star)$, such that*

- *for any fixed $l \in \mathbb{N}$, if with $B_1, \dots, B_l \in \mathfrak{B}$ are pairwise disjoint, then the random variables $N_{B_1}, \dots, N_{B_l}$ are independent, and*
- *N_B is Poisson distributed with parameter $\mu(B)$ for every $B \in \mathfrak{B}$, i.e.*

$$\mu^\star[N_B = n] = Poi_{\mu(B)}(n) = e^{-\mu(B)} \frac{\mu(B)^n}{n!}, \quad \forall n \in \overline{\mathbb{N}}_0.$$

Proof. To obtain a probability measure $\mu^\star$ on $X^\star$ with the above properties, we will use a construction by [12] (Satz 24.12) of a probability space $(\Omega, \mathfrak{A}, P)$ and a random measure $Z : \Omega \times \mathfrak{B} \longrightarrow \overline{\mathbb{N}}_0$, such that

$$P \circ Z^{-1} = \mu^\star.$$

We first consider the case $\mu(X) < \infty$. (As usual we will disregard the case $\mu(X) = 0$.) In this setting, we can define a probability measure η on $(X, \mathfrak{B})$ by $\eta(B) := \frac{\mu(B)}{\mu(X)}$, for $B \in \mathfrak{B}$. Roughly speaking, we want to consider randomly many randomly picked points in $(X, \mathfrak{B}, \mu)$. In order to do so we consider the infinite product space

$$(\Omega, \mathfrak{A}, P) := (\overline{\mathbb{N}}_0, \mathscr{P}(\overline{\mathbb{N}}_0), \mathrm{Poi}_{\mu(X)}) \otimes (X, \mathfrak{B}, \eta) \otimes (X, \mathfrak{B}, \eta) \otimes \dots$$

for $\eta := \frac{\mu(\cdot)}{\mu(X)}$ the normalized measure. (The set $\mathscr{P}(M)$ denotes the power set of a set M.) By construction, the projections Y_j of $(\Omega, \mathfrak{A}, P)$ to $(\overline{\mathbb{N}}_0, \mathscr{P}(\overline{\mathbb{N}}_0), \mathrm{Poi}_{\mu(X)})$ for $j = 0$, respectively to $(X, \mathfrak{B}, \eta)$ for $j = 1, 2, \dots$, are independent random variables, which satisfy $P_{Y_0} = \mathrm{Poi}_{\mu(X)}$ and $P_{Y_j} = \eta$, for $j \geq 1$.
Now, we define

$$Z(B) := \sum_{j=1}^{Y_0} 1_B \circ Y_j$$

for $B \in \mathfrak{B}$, where we define $\sum_{j=1}^{0} := 0$ (and we set $\prod_{j=1}^{0} := 1$ for later purposes). This is an $\overline{\mathbb{N}}_0$-valued random variable on $(\Omega, \mathfrak{A}, P)$ and $Z(\omega) = \sum_{j=1}^{Y_0(\omega)} \delta_{Y_j(\omega)}$ is a measure on $(X, \mathfrak{B})$ for every $\omega \in \Omega$.
We show that $Z(B)$ is Poisson distributed for fixed $B \in \mathfrak{B}$. To this end, we can use characteristic functions (since $\overline{\mathbb{N}}_0$ can be embedded in $\mathbb{R}$ by mapping ∞ to a fixed real but not natural number, hence we are allowed to treat $Z(B)$ and Y_0 like real valued random variables). The random variables $1_B \circ Y_1, 1_B \circ Y_2, \dots$ are independent and all Bernoulli distributed with parameter $\eta(B)$, because $Y_1, Y_2, \dots$ are independent and $P[1_B \circ Y_j = 1] = P[Y_j \in B] = P_{Y_j}(B) = \eta(B)$ and $P[1_B \circ Y_j = 0] = P[Y_j \notin B] = \eta(B^c) = 1 - \eta(B)$ for $j = 1, 2, \dots$. So by $P[Y_0 = \infty] = Poi_{\mu(X)}(\infty) = 0$ and the convention $0 \cdot \infty := 0$, we obtain

$$\varphi_{Z(B)}(t) := E[e^{i<t, Z(B)>}] = \sum_{m=0}^{\infty} E[1_{[Y_0 = m]} e^{it(1_B \circ Y_1 + \dots + 1_B \circ Y_m)}] + P[Y_0 = \infty] E[e^{\infty}] =$$

$$= \sum_{m=0}^{\infty} P[Y_0 = m] \prod_{j=1}^{m} E[e^{it 1_B \circ Y_j}] = \sum_{m=0}^{\infty} \mathrm{Poi}_{\mu(X)}(m) \prod_{j=1}^{m} \varphi_{1_B \circ Y_j}(t) =$$

$$= \sum_{m=0}^{\infty} e^{-\mu(X)} \frac{\mu(X)^m}{m!} \big(\varphi_{1_B \circ Y_1}(t)\big)^m = \exp\big(\mu(X) \varphi_{1_B \circ Y_1}(t) - \mu(X)\big) =$$

$$= \exp\big(\mu(X)(1 - \eta(B) + \eta(B) e^{it} - 1)\big) = \exp\big(\mu(B)(e^{it} - 1)\big) \qquad \forall t \in \mathbb{R},$$

which is the characteristic function of the Poisson distribution with parameter $\mu(B)$, hence we get

$$P_{Z(B)} = \mathrm{Poi}_{\mu(B)},$$

since characteristic functions uniquely determine a distribution.
We claim that $Z(B_1), \dots, Z(B_l)$ are independent, for pairwise disjoint sets $B_1, \dots, B_l \in \mathfrak{B}$, for fixed $l \in \mathbb{N}$. Indeed, the tuples $(1_{B_1} \circ Y_1, \dots, 1_{B_l} \circ Y_1), (1_{B_1} \circ Y_2, \dots, 1_{B_l} \circ Y_2), \dots$ defined on Ω are independent, identically distributed and independent of Y_0, so a similar computation as above leads to

$$\varphi_{(Z(B_1),\dots,Z(B_l))}(t_1, \dots, t_l) = \exp\big(\mu(X)(\varphi_{(1_{B_1}\circ Y_1,\dots,1_{B_l}\circ Y_1)}(t_1, \dots, t_l) - 1)\big) =$$

$$= \exp\Big(\mu(X)\Big(\int_\Omega \exp\Big(i \sum_{k=1}^{l} t_k 1_{B_k} \circ Y_1\Big)\ dP - 1\Big)\Big) =$$

$$= \exp\Big(\mu(X)\Big(\sum_{k=1}^{l} e^{it_k} P(Y_1^{-1}(B_k)) + P\Big(Y_1^{-1}\Big(\Big(\bigcup_{k=1}^{l} B_k\Big)^c\Big)\Big) - 1\Big)\Big) =$$

$$= \exp\Big(\mu(X)\Big(\sum_{k=1}^{l} \eta(B_k)(e^{it_k} - 1)\Big)\Big) = \exp\Big(\sum_{k=1}^{l} \mu(B_k)(e^{it_k} - 1)\Big) =$$

$$= \prod_{k=1}^{l} \varphi_{Z(B_k)}(t_k),$$

since $Z(B_k)$ is Poisson distributed with parameter $\mu(B_k)$, for all $k = 1, \dots, l$. (In the third equality disjointedness of $B_1, \dots, B_l$ is used.) As a standard result of probability theory (see for instance [2]), independence follows.

The map $Z : \Omega \longrightarrow X^\star$, which assigns to each element ω of Ω the $\overline{\mathbb{N}}_0$-valued measure $Z(\omega)$, is measurable, since $Z^{-1}(N_B^{-1}(\{n\})) = \{\omega \in \Omega | N_B(Z(\omega)) = n\} = \{\omega \in \Omega | Z(\omega)(B) = n\} = Z(B)^{-1}(\{n\}) \in \mathfrak{A}$, for all $n \in \overline{\mathbb{N}}_0$ and all $B \in \mathfrak{B}$ (and preimages respect set-operations). In particular, we have

$$N_B \circ Z = Z(B). \tag{6}$$

Now, we are finally ready to define a probability measure on $(X^\star, \mathfrak{B}^\star)$ by

$$\mu^\star := P_Z,$$

i.e. $\mu^\star : X^\star \xrightarrow{Z^{-1}} \Omega \xrightarrow{P} [0,1]$.
From what we have shown before we can easily deduce the claimed properties: For all $n \in \overline{\mathbb{N}}_0$ and $B \in \mathfrak{B}$ we see that

$$\mu^\star[N_B = n] = P(Z^{-1}(N_B^{-1}(\{n\}))) = P(Z(B)^{-1}(\{n\})) = \mathrm{Poi}_{\mu(B)}(n),$$

due to (1) and the construction of Z on $(\Omega, \mathfrak{A}, P)$, which also gives

$$\mu^\star[N_{B_1} = n_1, \dots, N_{B_l} = n_l] = P\Big(Z^{-1}\Big(\bigcap_{k=1}^{l} N_{B_k}^{-1}(\{n_k\})\Big)\Big) = P\Big(\bigcap_{k=1}^{l} Z(B_k)^{-1}(\{n_k\})\Big) =$$

$$= \prod_{k=1}^{l} P(Z(B_k)^{-1}(\{n_k\})) = \prod_{k=1}^{l} P(Z^{-1}(N_{B_k}^{-1}(\{n_k\}))) = \prod_{k=1}^{l} \mu^\star[N_{B_k} = n_k],$$

for $B_1,\dots,B_l$ pairwise disjoint sets in $\mathfrak{B}$ and $n_1,\dots,n_l \in \overline{\mathbb{N}}_0$ with $l \in \mathbb{N}$ fixed.

We now consider the general case $\mu(X) \in (0,\infty]$.

Since $(X,\mathfrak{B},\mu)$ is σ-finite, there exists an increasing sequence $(C_m)_{m\in\mathbb{N}}$ in $\mathfrak{B}$, such that $\mu(C_m) < \infty$ and $\cup_{m=1}^{\infty} C_m = X$. Define

$$\mu_1(B) := \mu(C_1 \cap B) \quad \text{and} \quad \mu_m(B) := \mu((C_m \setminus C_{m-1}) \cap B),$$

for $m \geq 2$, $B \in \mathfrak{B}$. Applying the above construction to this finite measures on X, we obtain for every $m \in \mathbb{N}$ a probability measure $\mu_m^\star$ on $(X^\star,\mathfrak{B}^\star)$ such that $\mu_m^\star \circ N_B = \mathrm{Poi}_{\mu_m(B)}$ for all $B \in \mathfrak{B}$ and $\mu_m^\star[N_{B_1} = n_1,\dots,N_{B_l} = n_l] = \prod_{k=1}^{l} \mu_m^\star[N_{B_k} = n_k]$ for all disjoint sets $B_1,\dots,B_l \in \mathfrak{B}$, $l \in \mathbb{N}$ and for all $n_1,\dots,n_k \in \overline{\mathbb{N}}_0$. Denote by W_m the projection of $(\Xi,\mathfrak{C},Q) := \bigotimes_{n\in\mathbb{N}}(X^\star,\mathfrak{B}^\star,\mu_n^\star)$ to $(X^\star,\mathfrak{B}^\star,\mu_m^\star)$, for all $m = 1,2,\dots$. Analogous to (1) we have $N_B \circ W_m = W_m(B)$, when setting $W_m(B)(x) := W_m(x)(B)$ for $x \in \Xi$ and $B \in \mathfrak{B}$. Define $W := \sum_{m=1}^{\infty} W_m$ (again a random measure), then by

$$\mu^\star := Q_W,$$

we get a probability measure on $(X^\star,\mathfrak{B}^\star)$, which satisfies the postulated properties. Indeed, independence of $W_1,W_2,\dots$ yields independence of $W_1(B),W_2(B),\dots$, since $W_m(B) = N_B \circ W_m$, $m \in \mathbb{N}$ and N_B is measurable, for fixed $B \in \mathfrak{B}$. Therefore

$$\mu^\star \circ N_B^{-1} = Q \circ W^{-1} \circ N_B^{-1} = Q_{W(B)} = Q_{\sum_{m=1}^{\infty} W_m(B)} = Q_{W_1(B)} * Q_{W_2(B)} * \dots =$$

$$= \mu_1^\star \circ N_B^{-1} * \mu_2^\star \circ N_B^{-1} * \dots = \mathrm{Poi}_{\mu_1(B)} * \mathrm{Poi}_{\mu_2(B)} * \dots = \mathrm{Poi}_{\sum_{m=1}^{\infty} \mu_m(B)} = \mathrm{Poi}_{\mu(B)},$$

for all $B \in \mathfrak{B}$.

Furthermore, $W_m(B_1),\dots,W_m(B_l)$ are independent for disjoint sets $B_1,\dots,B_l \in \mathfrak{B}$ and fixed $m \in \mathbb{N}$, because

$$Q\Big(\bigcap_{k=1}^{l} W(B_k)^{-1}(\{n_k\})\Big) = Q\Big(W_m^{-1}\Big(\bigcap_{k=1}^{l} N_{B_k}^{-1}(\{n_k\})\Big)\Big) = \mu_m^\star\Big(\bigcap_{k=1}^{l} N_{B_k}^{-1}(\{n_k\})\Big) =$$

$$= \prod_{k=1}^{l} \mu_m^\star(N_{B_k}^{-1}(\{n_k\})) = \prod_{k=1}^{l} Q(W_m(B_k)^{-1}(\{n_k\})).$$

Thus $\sum_{m=1}^{\infty} W_m(B_1),\dots,\sum_{m=1}^{\infty} W_m(B_l)$ are independent for disjoint sets $B_1,\dots,B_l \in \mathfrak{B}$, since by definition $W_1(B_{i_1}),W_2(B_{i_2}),\dots$ are independent for all $i_1,i_2\dots \in \{1,\dots,l\}$. So we compute

$$\mu^\star\Big(\bigcap_{k=1}^{l} N_{B_k}^{-1}(\{n_k\})\Big) = Q\Big(\bigcap_{k=1}^{l} W(B_k)^{-1}(\{n_k\})\Big) = \prod_{k=1}^{l} Q(W(B_k)^{-1}(\{n_k\})) =$$

$$= \prod_{k=1}^{l} \mu^\star(N_{B_k}^{-1}(\{n_k\})) \qquad \forall n_1,\dots,n_l \in \overline{\mathbb{N}}_0.$$

Hence, $N_{B_1},\dots,N_{B_l}$ are independent with respect to $\mu^\star$.

The uniqueness of $\mu^\star$ follows by Proposition 1.4, since $\mu^\star\Big(\bigcap_{k=1}^{l} N_{B_k}^{-1}(\{n_k\})\Big) = \prod_{k=1}^{l} \mu^\star(N_{B_k}^{-1}(\{n_k\})) = \prod_{k=1}^{l} \mathrm{Poi}_{\mu(B_k)}(n_k)$ for all disjoint $B_1,\dots,B_l \in \mathfrak{B}$, $n_1,\dots,n_l \in \overline{\mathbb{N}}_0$ with $l \in \mathbb{N}$. $\square$

The above constructed measure space $(X^\star,\mathfrak{B}^\star,\mu^\star)$ will be called *Poisson suspension* of $(X,\mathfrak{B},\mu)$ in the following.

Remark 2.1.

1. Note, that the intensity of the random measure Z, respectively W in the proof above is nothing else than the measure μ, since

$$\int_\Omega Z(B)(\omega)\, dP(\omega) = \int_{\overline{\mathbb{N}}_0} n\, d(P\circ Z(B)^{-1})(n) = \sum_{n\in\overline{\mathbb{N}}_0} n\cdot Poi_{\mu(B)}(n) =$$

$$= e^{-\mu(B)} \sum_{n\in\mathbb{N}} \frac{\mu(B)^n}{(n-1)!} = e^{-\mu(B)} \sum_{n\in\mathbb{N}_0} \frac{\mu(B)^n}{n!}\mu(B) =$$

$$= e^{-\mu(B)} e^{\mu(B)} \mu(B) = \mu(B),$$

if $\mu(B)\neq\infty$, and $\int_\Omega Z(B)(\omega)\ dP(\omega) = \infty\cdot Poi_{\mu(B)}(\infty) = \infty$, if $\mu(B)=\infty$, for $B\in\mathfrak{B}$. The same is true for W (w.r.t. the space $(\Theta,\mathfrak{C},Q)$ of the proof above). From now on we will only write Z and $(\Omega,\mathfrak{A},P)$, meaning both Z on $(\Omega,\mathfrak{A},P)$ and W on $(\Xi,\mathfrak{C},Q)$. Using the transformation formula and $\mu^\star := P\circ Z^{-1}$, we can rewrite the above integration as

$$\int_{X^\star} \nu\, d\mu^\star(\nu) = \int_\Omega Z(\omega)\, dP(\omega) = \mu.$$

Since μ is σ-finite, by Corollary 1.2, the random measure Z is given by

$$Z(\omega) = \sum_{i\in\mathbb{Z}} \delta_{X_i(\omega)}$$

for random variables X_i with values in $(X,\mathfrak{B})$.
Note, that $Z(A_1),\dots,Z(A_n)$ are independent random variables, for $A_1,\dots,A_l\in\mathfrak{B}$ disjoint and $Z(A)$ is Poisson distributed with parameter $\mu(A)$ for $A\in\mathfrak{B}$.

2. Further, $\mu^\star$-a.e. element ν of $X^\star$ is given by $Z(\omega)$ (respectively $W(\omega)$) for some $\omega\in\Omega$, i.e. by (1) we get

$$\nu = Z(\omega) = \sum_{j\in\mathbb{Z}} \delta_{X_j(\omega)}.$$

Indeed, $\{Z(\omega):\omega\in\Omega\}\subseteq X^\star$ and $\mu^\star(\{Z(\omega):\omega\in\Omega\}) = P(Z^{-1}(\{Z(\omega):\omega\in\Omega\})) = P(\Omega) = 1$, hence $X^\star = \{Z(\omega):\omega\in\Omega\}$ mod $\mu^\star$.

3. Observe, that for $B\in\mathfrak{B}$

$$\mu(B) = 0 \ \Leftrightarrow\ \mu^\star(\{\nu\in X^\star | \nu(B)=0\}) = 1,$$

i.e. $\mu(B)=0$ if and only if $\mu^\star$-a.e. $\nu\in X^\star$ fulfills $\nu(B)=0$. This follows directly by the definition of $\mu^\star$:

$$\mu^\star(N_B^{-1}(\{0\})^c) = \mu^\star\Big(\bigcup_{n\in\overline{\mathbb{N}}} N_B^{-1}(\{n\})\Big) = \sum_{n\in\mathbb{N}} e^{-\mu(B)}\frac{\mu(B)^n}{n!} = 0 \ \Leftrightarrow\ \mu(B)=0$$

for $B\in\mathfrak{B}$ with $\mu(B)\neq\infty$. If $\mu(B)=\infty$, then we obtain $\mu^\star(N_B^{-1}(\{0\})^c) = \sum_{n\in\overline{\mathbb{N}}} Poi_\infty(n) = 1 \neq 0$.
This implies that for $A,B\in\mathfrak{B}$ we have $A=B$ mod μ if and only if $A=B$ mod ν for $\mu^\star$-a.e. $\nu\in X^\star$.

The following Lemma (given e.g. in [10], Lemma 2.1) guarantees that almost every measure of $X^\star$ assigns at most mass 1 to each point of X, provided the space bears no atoms.

Lemma 2.1. *Let $(X,\mathfrak{B},\mu)$ be a σ-finite standard space without atoms and $(X^\star,\mathfrak{B}^\star,\mu^\star)$ its Poisson suspension, then*

$$\mu^\star(\{\nu \in X^\star | \exists x \in X \ : \ \nu(\{x\}) \geq 2\}) = 0.$$

Remark 2.2. The above Lemma in particular implies, that in that case for $\mu^\star$-a.e. $\nu \in X^\star$ the above notation $\nu = \sum_{i\in\mathbb{Z}} \delta_{X_i(\omega)}$ is unique, since $\nu(\{x\}) \leq 1$ for all $x \in X$ implies that $\nu(\{X_i(\omega)\}) = 1$, and therefore it cannot happen that $X_i(\omega) = X_j(\omega)$ for $i \neq j$ in $\mathbb{Z}$, i.e. the random variables X_i are up to permutation of the indices uniquely determined. The sets $\{X_i(\omega)\}$ are precisely the atoms of ν.
Since $\{X_i(\omega)\}_{i\in\mathbb{Z}}$ only depends on ν, we can also write $x_i(\nu) \in X$ instead of $X_i(\omega)$, i.e.

$$\nu = \sum_{i\in\mathbb{Z}} \delta_{x_i(\nu)}.$$

Remark 2.3. Note that the Possion extension $(X^\star,\mathfrak{B}^\star,\mu^\star)$ of a standard measure space $(X,\mathfrak{B},\mu)$ is again a standard measure space, according to [18].

Lemma 2.2. *Let Z be a random measure corresponding to $\mu^\star$ (i.e. with the notation of the proof of Proposition 2.1 we mean either Z or W), then the Laplace functional of Z is given by*

$$\Psi_Z(f) = \exp\Big(-\int (1-e^{-f})\ d\mu\Big)$$

for every $f : X \longrightarrow [0,\infty]$.

For a proof we refer to [12], Satz 24.10.

Remark 2.4. Note that the Laplace functional

$$\Psi_Z(f) = \int_\Omega \Big[\exp\Big(-\int_X f(x)\ dZ(\omega)(x)\Big)\Big]\ dP(\omega)$$

of a random measure Z, corresponding to $\mu^\star = P \circ Z^{-1}$ as above, can also be written as

$$\Psi_{\mu^\star}(f) := \Psi_Z(f) = \int_{X^\star} \Big[\exp\Big(-\int_X f(x)\ d\nu(x)\Big)\Big]\ d\mu^\star(\nu),$$

by using the transformation formula[5] w.r.t. Z, and since by Remark 2.1 (2), $Z(\Omega) = X^\star$ mod $\mu^\star$. In this case we will call the Laplace functional of Z also the *Laplace functional of $\mu^\star$*.
Regarding this, the above Lemma can be stated as

$$\int_{X^\star} \Big[\exp\Big(-\int_X f(x)\ d\nu(x)\Big)\Big]\ d\mu^\star(\nu) = \exp\Big(\int_X \Big[\exp(-f(x))-1\Big]\ d\mu(x)\Big).$$

This identity will be very useful later on.

[5]Note that the map $g : \nu \mapsto \int f\ d\nu$ is measurable: For $f = 1_B$ with $B \in \mathfrak{B}$ we have $g^{-1}([a,b]) = \{\nu \in X^\star : \int 1_B\ d\nu \in [a,b]\} = \bigcup_{n\in\overline{\mathbb{N}}_0, n\in[a,b]} N_B^{-1}(\{n\})\ \in \mathfrak{B}^\star$ and and if $f = \sum_{i\in I} c_i \cdot 1_{B_i}$, for $B_i \in \mathfrak{B}$, $i \in I$ countable, then $g^{-1}([a,b]) = \bigcup_{n_i\in\overline{\mathbb{N}}_0, \sum_{i\in I} c_i \cdot n_i \in[a,b]} \bigcap_{i\in I} N_{B_i}^{-1}(\{n_i\})\ \in \mathfrak{B}^\star$, and the general case follows by the usual approximation argument for measurable functions.

The Poisson suspension respects isomorphisms:

Remark 2.5. Let $(X,\mathfrak{B},\mu)$, $(Z,\mathfrak{A},\eta)$ be two σ-finite standard spaces, then

$$(X,\mathfrak{B},\mu)\approx(Z,\mathfrak{A},\eta) \;\Rightarrow\; (X^\star,\mathfrak{B}^\star,\mu^\star)\approx(Z^\star,\mathfrak{A}^\star,\eta^\star),$$

where $\approx$ stands for (essential) isomorphic to. Indeed, let $\psi:(X,\mathfrak{B},\mu)\longrightarrow(Z,\mathfrak{A},\eta)$ be an (essential) isomorphism, then

$$\psi^\star: X^\star\longrightarrow Y^\star,\;\; \psi^\star(\nu):=\nu\circ\psi^{-1} \text{ for } \nu\in X^\star$$

provides an (essential) isomorphism: First note that, though ψ might be only up to a null-set defined on Y and X, we can extend $\nu\circ\psi^{-1}$, for $\mu^\star$-a.e. $\nu\in X^\star$ and $\eta\circ\psi$, for $\eta^\star$-a.e. $\eta\in Y^\star$, to measures on whole Y and X, respectively, since by Remark 2.1 (3)

$$\mu(N)=0 \;\Leftrightarrow\; \nu(N)=0 \text{ for } \mu^\star\text{-a.e. } \nu\in X^\star,$$

for $N\in\mathfrak{B}$ and analogously for λ. Clearly, $\psi^\star$ is bijective, because ψ is so, thus for $\eta\in Y^\star$ we have $\psi^{\star-1}(\eta)=\psi^{\star-1}(\eta\circ\psi\circ\psi^{-1})=\eta\circ\psi$. Moreover, $\psi^\star$ and $\psi^{\star-1}$ are measurable, as ψ and ψ^{-1} are: for every $R\in\mathfrak{A}$ and $n\in\overline{\mathbb{N}}_0$, we compute $\psi^{\star-1}(N_R^{-1}(\{n\}))=\{\nu\in X^\star:\psi^\star(\nu)(R)=n\}=N^{-1}_{\psi^{-1}(R)}(\{n\})\in\mathfrak{B}^\star$, since $\psi^{-1}(R)\in\mathfrak{B}$, and for every $B\in\mathfrak{B}$ we have $(\psi^{\star-1})^{-1}(N_B^{-1}(\{n\}))=\{\eta\in Y^\star:\psi^{\star-1}(\eta)(B)=n\}=N^{-1}_{\psi(B)}(\{n\})$, since $\psi(B)\in\mathfrak{A}$. Further, $\psi^\star$ is measure-preserving: Let $n_i\in\overline{\mathbb{N}}_0$ and $R_i\in\mathfrak{A}$ be disjoint for i in a countable index set I, then $\mu^\star\big(\psi^{\star-1}\big(\bigcap_{i\in I}N_{R_i}^{-1}(\{n_i\})\big)\big)=\prod_{i\in I}e^{-\mu(\psi^{-1}(R_i))}\frac{\mu(\psi^{-1}(R_i))^{n_i}}{n_i!}=\prod_{i\in I}e^{-\lambda(R_i)}\frac{\lambda(R_i)^{n_i}}{n_i!}=\lambda^\star\big(\bigcap_{i\in I}N_{R_i}^{-1}(\{n_i\})\big)$, hence, by Proposition 1.4, $\mu^\star\circ\psi^{\star-1}=\lambda^\star$.

2.1.1 The corresponding transformation on $X^\star$

Let $(X,\mathfrak{B},\mu,T)$ be a non-singular dynamical system.
A reasonable way of defining a transformation $T^\star$ on $X^\star$ is by

$$T^\star(\nu):=\nu\circ T^{-1},\quad \forall\nu\in X^\star.$$

So, $T^\star$ is measurable, since by

$$N_B(T^\star(\nu))=N_B(\nu\circ T^{-1})=\nu(T^{-1}(B))=N_{T^{-1}B}(\nu),$$

for all $B\in\mathfrak{B}$, $\nu\in X^\star$, we get $T^{\star-1}(N_B^{-1}(\{n\}))=N^{-1}_{T^{-1}B}(\{n\})\in\mathfrak{B}^\star$, for every $n\in\mathbb{N}_0$.

Moreover, we have the following crucial property: *If T is measure-preserving, then so is $T^\star$.* Indeed, for all $B_1,\dots,B_l\in\mathfrak{B}$ pairwise disjoint, $n_1,\dots,n_l\in\overline{\mathbb{N}}_0$ we have

$$\mu^\star\Big(T^{\star-1}\Big(\bigcap_{k=1}^{l}N_{B_k}^{-1}(\{n_k\})\Big)\Big)=\mu^\star\Big(\bigcap_{k=1}^{l}T^{\star-1}(N_{B_k}^{-1}(\{n_k\}))\Big)=$$

$$=\mu^\star\Big(\bigcap_{k=1}^{l}N^{-1}_{T^{-1}B_k}(\{n_k\}))\Big)=\prod_{k=1}^{l}\mu^\star(N^{-1}_{T^{-1}B_k}(\{n_k\})))=\prod_{k=1}^{l}\mathrm{Poi}_{\mu(T^{-1}B_k)}(n_k)=$$

$$=\prod_{k=1}^{l}\mathrm{Poi}_{\mu(B_k)}(n_k)=\prod_{k=1}^{l}\mu^\star(N_{B_k}^{-1}(\{n_k\}))=\mu^\star\Big(\bigcap_{k=1}^{l}N_{B_k}^{-1}(\{n_k\})\Big),$$

since $T^{-1}B_1, \ldots, T^{-1}B_l$ are pairwise disjoint, too. Thus by Proposition 1.4 we get

$$\mu^\star = \mu^\star \circ T^{\star -1}.$$

(Trivially, this also shows non-singularity of $T^\star$ for measure-preserving T, and we will only be concerned with such T.)

The above obtained dynamical system $(X^\star, \mathfrak{B}^\star, \mu^\star, T^\star)$ is called the *Poisson suspension* of $(X, \mathfrak{B}, \mu, T)$.

2.2 Ergodicity of the Poisson suspension

From now on we assume $(X, \mathfrak{B}, \mu)$ to be a standard space. Our next goal is to prove that ergodicity of $(X, \mathfrak{B}, \mu, T)$ carries over to $(X^\star, \mathfrak{B}^\star, \mu^\star, T^\star)$. First we will establish this in case of T being an automorphism, following [6] and then release this restriction, which is done in [25].

Theorem 2.1 ([6])**.** *Let $(X, \mathfrak{B}, \mu)$ be a σ-finite standard measure space with $\mu(X) = \infty$ and T a measure-preserving automorphism on X, which is ergodic and conservative. Then the corresponding Poisson suspension $(X^\star, \mathfrak{B}^\star, \mu^\star, T^\star)$ is weakly mixing (and therefore in particular ergodic as well).*

In order to prove this theorem, let us first introduce the following notation and a Lemma of [6]: For every $C \in \mathfrak{B}$ with $\mu(C) < \infty$ we consider the sub-σ-algebra

$$(\mathfrak{B} \cap C)^\star := \sigma(N_A : A \in \mathfrak{B} \cap C)$$

of $\mathfrak{B}^\star$, for $C \in \mathfrak{B}$ and following [6] we define

$$\rho((\mathfrak{B} \cap C)^\star, (\mathfrak{B} \cap D)^\star) :=$$

$$= \sup\{|\mu^\star(M \cap M') - \mu^\star(M)\mu^\star(M')| : M \in (\mathfrak{B} \cap C)^\star, M' \in (\mathfrak{B} \cap D)^\star\},$$

for $C, D \in \mathfrak{B}$, $\mu(C) < \infty$, $\mu(D) < \infty$. For n going to infinity, $|\mu^\star(M \cap T^{-n}M') - \mu^\star(M)\mu^\star(M')|$ characterizes "how far" the sets M and M' are "away from mixing", roughly spoken.

Lemma 2.3. *Under the assumption of the theorem above, for $C, E \in \mathfrak{B}$ such that $\mu(C) < \infty$ and $\mu(E) < \infty$, we obtain, that $\lim_{n\to\infty} \mu(C \cap T^{-n}E) = 0$ implies $\lim_{n\to\infty} \rho((\mathfrak{B} \cap C)^\star, (\mathfrak{B} \cap T^{-n}E)^\star) = 0$.*

Proof. The first step of the proof is to verify, that

$$\rho((\mathfrak{B} \cap D_1)^\star, (\mathfrak{B} \cap (C \cup D_2))^\star) = \rho((\mathfrak{B} \cap D_1)^\star, (\mathfrak{B} \cap C)^\star) = \tag{7}$$

$$= \rho((\mathfrak{B} \cap D_1)^\star, (\mathfrak{B} \cap (C \setminus D_2))^\star)$$

for $D_1, D_2 \in \mathfrak{B}$ with $\mu(D_1 \cap D_2) = \emptyset$ and $\mu(D_1) < \infty$, $\mu(D_2) < \infty$.
Clearly, $\rho((\mathfrak{B} \cap D_1)^\star, (\mathfrak{B} \cap (C \cup D_2))^\star) \geq \rho((\mathfrak{B} \cap D_1)^\star, (\mathfrak{B} \cap C)^\star) \geq \rho((\mathfrak{B} \cap D_1)^\star, (\mathfrak{B} \cap (C \setminus D_2))^\star)$, hence it is sufficient to show that

$$\rho((\mathfrak{B} \cap D_1)^\star, (\mathfrak{B} \cap (C \setminus D_2))^\star) \geq \rho((\mathfrak{B} \cap D_1)^\star, (\mathfrak{B} \cap (C \cup D_2))^\star). \tag{8}$$

Define

$$\mathscr{M} := \Big\{ \bigcup_{j\in J} M_j \cap W_j \Big| M_j \in (\mathfrak{B}\cap D_2)^\star \text{ pairwise disjoint}, W_j \in (\mathfrak{B}\cap(C\setminus D_2))^\star, J \text{ finite} \Big\}.$$

This set is an algebra[6]. To see this, let $M_j^i \in (\mathfrak{B}\cap D_2)^\star$, $W_j^i \in (\mathfrak{B}\cap(C\setminus D_2))^\star$ for $j = 1,\dots,k(i)$ for $k(i) \in \mathbb{N}$, $i = 1,\dots,m$ for $m \in \mathbb{N}$ and $M_1^i, M_2^i, \dots, M_{k(i)}^i$ be pairwise disjoint for fixed $i \in \{1,\dots,m\}$. Then

$$\bigcap_{i=1}^{m} \Big(\bigcup_{n=1}^{k(i)} M_n^i \cap W_n^i \Big) =$$

$$= ((M_1^1 \cap W_1^1) \cup \ldots \cup (M_{k(1)}^1 \cap W_{k(1)}^1)) \cap \ldots \cap ((M_1^m \cap W_1^m) \cup \ldots \cup (M_{k(m)}^m \cap W_{k(m)}^m)) =$$

$$= \bigcup_{n_1\in\{1,\dots,k(1)\}} \cdots \bigcup_{n_m\in\{1,\dots,k(m)\}} M_{n_1}^1 \cap \ldots \cap M_{n_m}^m \cap W_{n_1}^1 \cap \ldots \cap W_{n_m}^m$$

which lies in $\mathscr{M}$, since $\bigcap_{i=1}^m M_{n_i}^i \in (\mathfrak{B}\cap D_2)^\star$, $\bigcap_{i=1}^m W_{n_i}^i \in (\mathfrak{B}\cap(C\setminus D_2))^\star$ and for fixed $n_i, h_i \in \{1,\dots,k(i)\}$, $i = 1,\dots,m$, such that there is at least one $l \in \{1,\dots,m\}$ such that $h_l \neq n_l$, we have $\big(\bigcap_{i=1}^m M_{n_i}^i\big) \cap \big(\bigcap_{i=1}^m M_{h_i}^i\big) = \emptyset$, because $M_{n_l}^l \cap M_{h_l}^l = \emptyset$ by assumption. Using this finite intersection stability, we also obtain that the complement of every element $\bigcup_{j\in J} M_j \cap W_j$ of $\mathscr{M}$ is contained in $\mathscr{M}$. This follows by writing

$$\Big(\bigcup_{j\in J} M_j \cap W_j\Big)^c = \bigcap_{j\in J} M_j^c \cup W_j^c = \bigcap_{j\in J} M_j^c \cup (M_j^c \cap W_j^c) \cup (M_j \cap W_j^c) =$$

$$= \bigcap_{j\in J} (M_j^c \cap X^\star) \cup (M_j \cap W_j^c).$$

Clearly $X^\star \in \mathscr{M}$.
The algebra $\mathscr{M}$ satisfies $\sigma(\mathscr{M}) = (\mathfrak{B}\cap(C\cup D_2))^\star$. Indeed, $\sigma(\mathscr{M}) = \sigma(N_B | B \subseteq D_2 \vee B \subseteq C\setminus D_2, B \in \mathfrak{B}) = \sigma(N_B | B \subseteq C\cup D_2, B \in \mathfrak{B})$, since for $B\cap(C\setminus D_2) \neq \emptyset \wedge B\cap D_2 \neq \emptyset$, we can write $N_B^{-1}(\{n\}) = \bigcup_{k,m\in\overline{\mathbb{N}}_0, k+m=n} N_{B\cap(C\setminus D_2)}^{-1}(\{k\}) \cap N_{B\cap D_2}^{-1}(\{m\})$ for $B \in \mathfrak{B}$, $n \in \mathbb{N}$.
Assume $V \in (\mathfrak{B}\cap D_1)^\star$ and $V' \in \mathscr{M}$ to be given, i.e. $V' = \bigcup_{j\in J} M_j \cap W_j$, for certain pairwise disjoint sets $M_j \in (\mathfrak{B}\cap D_2)^\star$ and $W_j \in (\mathfrak{B}\cap(C\setminus D_2))^\star$, where J is a finite index set. By construction of the Poisson suspension M_j and W_j are independent (w.r.t. $\mu^\star$), because $D_2 \cap (C\setminus D_2) = \emptyset$. The same is true for M_j and $W_j \cap V$, since $D_2 \cap ((C\setminus D_2)\cup D_1) = \emptyset$. So we see that

$$|\mu^\star(V\cap V') - \mu^\star(V)\mu^\star(V')| \leq \sum_{j\in J} |\mu^\star(V\cap M_j \cap W_j) - \mu^\star(V)\mu^\star(M_j\cap W_j)| =$$

$$= \sum_{j\in J} \mu^\star(M_j)|\mu^\star(V\cap W_j) - \mu^\star(V)\mu^\star(W_j)| \leq$$

$$\leq \sum_{j\in J} \mu^\star(M_j)\rho((\mathfrak{B}\cap D_1)^\star, (\mathfrak{B}\cap(C\setminus D_2))^\star) =$$

$$= \mu^\star\Big(\bigcup_{j\in J} M_j\Big)\rho((\mathfrak{B}\cap D_1)^\star, (\mathfrak{B}\cap(C\setminus D_2))^\star) \leq \rho((\mathfrak{B}\cap D_1)^\star, (\mathfrak{B}\cap(C\setminus D_2))^\star).$$

[6]An algebra $\mathscr{F}$ is a family of subsets of X such that $X \in \mathscr{F}$, $\forall D,E \in \mathscr{F}$: $D\cap E \in \mathscr{F}$ and $D^c \in \mathscr{F}$.

Since $\mathscr{M}$ is an algebra, every element of $U \in \sigma(\mathscr{M}) = (\mathfrak{B} \cap (C \cup D_2))^\star$ can be approximated by a (w.l.o.g. monotonically increasing) sequence $(V'_n)_{n\in\mathbb{N}}$ in $\mathscr{M}$, such that $\lim\limits_{n\to\infty} \mu^\star(V'_n) = \mu^\star(U)$ (see e.g. [19], 5.4.2 Folgerung). Thus

$$|\mu^\star(V \cap U) - \mu^\star(V)\mu^\star(U)| = \lim_{n\to\infty} |\mu^\star(V \cap V'_n) - \mu^\star(V)\mu^\star(V'_n)| \leq$$

$$\leq \rho((\mathfrak{B} \cap D_1)^\star, (\mathfrak{B} \cap (C \setminus D_2))^\star).$$

Because this inequality holds for all $V \in (\mathfrak{B} \cap D_1)^\star$ and $U \in (\mathfrak{B} \cap (C \cup D_2))^\star$, we can take the supremum and get (8):

$$\rho((\mathfrak{B} \cap D_1)^\star, (\mathfrak{B} \cap (C \cup D_2))^\star) =$$

$$= \sup\{|\mu^\star(V \cap U) - \mu^\star(V)\mu^\star(U)| : V \in (\mathfrak{B} \cap D_1)^\star, U \in (\mathfrak{B} \cap (C \cup D_2))^\star\} \leq$$

$$\leq \rho((\mathfrak{B} \cap D_1)^\star, (\mathfrak{B} \cap (C \setminus D_2))^\star),$$

hence (7) is proven.

Now applying (7) twice, we see, that

$$\rho((\mathfrak{B} \cap C)^\star, (\mathfrak{B} \cap T^{-n}E)^\star) = \rho((\mathfrak{B} \cap C)^\star, (\mathfrak{B} \cap T^{-n}E \setminus C^c)^\star) =$$

$$= \rho((\mathfrak{B} \cap C)^\star, (\mathfrak{B} \cap T^{-n}E \cap C)^\star) = \rho((\mathfrak{B} \cap T^{-n}E \cap C)^\star, (\mathfrak{B} \cap C)^\star) =$$

$$= \rho((\mathfrak{B} \cap T^{-n}E \cap C)^\star, (\mathfrak{B} \cap C \setminus (T^{-n}E)^c)^\star) = \rho((\mathfrak{B} \cap T^{-n}E \cap C)^\star, (\mathfrak{B} \cap T^{-n}E \cap C)^\star).$$

Thus it is left to show, that $\lim\limits_{n\to\infty} \rho((\mathfrak{B} \cap T^{-n}E \cap C)^\star, (\mathfrak{B} \cap T^{-n}E \cap C)^\star) = 0$, if $\lim\limits_{n\to\infty} \mu(C \cap T^{-n}E) = 0$.

For $B \in \mathfrak{B}$ with $B \subseteq T^{-n}E \cap C$, we have $N^{-1}_{T^{-n}E\cap C}(\{0\}) \subseteq N_B^{-1}(\{0\})$ and $N_B^{-1}(\{n\}) \cap N^{-1}_{T^{-n}E\cap C}(\{0\}) = \emptyset$, if $n \neq 0$, $n \in \overline{\mathbb{N}}$, since $N_B(\nu) = \nu(B) \leq \nu(T^{-n}E \cap C) = N_{T^{-n}E\cap C}(\nu)$ for all $\nu \in X^\star$. Every $M \in (\mathfrak{B} \cap T^{-n}E \cap C)^\star$ is an (at most countable) combination of countable unions and intersections of elements $N_B^{-1}(\{n\})$ with $B \subseteq T^{-n}E \cap C$, $B \in \mathfrak{B}$, $n \in \overline{\mathbb{N}}_0$, because $(N_B^{-1}(\{n\}))^c = \bigcup_{m\in\overline{\mathbb{N}}_0\setminus\{n\}} N_B^{-1}(\{m\})$. Therefore there are only two possible cases for $M \in (\mathfrak{B} \cap T^{-n}E \cap C)^\star$, namely either $N^{-1}_{T^{-n}E\cap C}(\{0\}) \subseteq M$ or $M \cap N^{-1}_{T^{-n}E\cap C}(\{0\}) = \emptyset$, i.e. $M \subseteq (N^{-1}_{T^{-n}E\cap C}(\{0\}))^c$. Let $M_1, M_2 \in (\mathfrak{B} \cap T^{-n}E \cap C)^\star$ be sets which attain the supremum of $\rho((\mathfrak{B} \cap T^{-n}E \cap C)^\star, (\mathfrak{B} \cap T^{-n}E \cap C)^\star)$. In every of the four cases $\mu^\star(M_i) \geq \mu^\star(N^{-1}_{T^{-n}E\cap C}(\{0\}))$ or $\mu^\star(M_i) \leq 1 - \mu^\star(N^{-1}_{T^{-n}E\cap C}(\{0\}))$ for $i = 1, 2$, we obtain

$$\rho((\mathfrak{B} \cap T^{-n}E \cap C)^\star, (\mathfrak{B} \cap T^{-n}E \cap C)^\star) \leq 1 - \mu^\star(N^{-1}_{T^{-n}E\cap C}(\{0\})) =$$

$$= 1 - \exp(-\mu(T^{-n}E \cap C)),$$

since $\mu^\star(M_1 \cap M_2) - \mu^\star(M_1)\mu^\star(M_2) \leq \mu^\star(M_k)(1 - \mu^\star(M_h)) \leq \begin{cases} \mu^\star(M_k) \\ 1 - \mu^\star(M_h), \end{cases}$ for $k, h \in \{1, 2\}$ such that $k \neq h$. (Respectively, if $\mu^\star(M_1 \cap M_2) - \mu^\star(M_1)\mu^\star(M_2) < 0$, then the inequality follows by $\mu^\star(M_1)\mu^\star(M_2) - \mu^\star(M_1 \cap M_2) \leq \mu^\star(M_i)$, for $i = 1, 2$, except when both $M_1 \supseteq N^{-1}_{T^{-n}E\cap C}(\{0\})$ and $M_2 \supseteq N^{-1}_{T^{-n}E\cap C}(\{0\})$, but then $\mu^\star(M_1)\mu^\star(M_2) - \mu^\star(M_1 \cap M_2) \leq 1 - \mu^\star(M_1 \cap M_2) \leq 1 - \mu^\star(N^{-1}_{T^{-n}E\cap C}(\{0\}))$.)

Now since $\lim\limits_{n\to\infty} \mu(T^{-n}E \cap C) = 0$ implies $\lim\limits_{n\to\infty} \exp(-\mu(T^{-n}E \cap C)) = 1$, we get

$$\rho((\mathfrak{B} \cap T^{-n}E \cap C)^\star, (\mathfrak{B} \cap T^{-n}E \cap C)^\star) \to 0 \qquad \text{for } n \to \infty.$$

$\square$

Proof of Theorem 2.1. Let $C \in \mathfrak{B}$ be such that $\mu(C) < \infty$. By Corollary 1.1 we know that

$$\frac{1}{n}\sum_{k=0}^{n-1}\mu(C\cap T^{-k}C)\to 0 \qquad \text{for } n\to\infty.$$

Moreover, the sequence $\{\mu(C\cap T^{-k}C)\}_{k\in\mathbb{N}}$ is bounded by $\mu(C)<\infty$. Therefore, we can apply a well-known result (see e.g. [22], Theorem 1.20), which states that the above convergence is equivalent to the existence of a subsequence $(k_l)_{l\in\mathbb{N}}$ in $\mathbb{N}^{\mathbb{N}}$ such that

$$\mu(C\cap T^{-k_l}C)\to 0 \ \text{ for } k_l\to\infty$$

with $\frac{|\{0,1,\dots,n-1\}\setminus\{k_l:l\in\mathbb{N}\}|}{n}\to 0$ for $n\to\infty$. Thus by Lemma 2.3,

$$\rho((\mathfrak{B}\cap C)^\star,(\mathfrak{B}\cap T^{-k_l}C)^\star)\to 0 \ \text{ for } k_l\to\infty$$

for $(k_l)_{l\in\mathbb{N}}$ with $\lim\limits_{n\to\infty}\frac{|\{0,1,\dots,n-1\}\setminus\{k_l:l\in\mathbb{N}\}|}{n}=0$. So, again by Theorem 1.20 of [22], we obtain

$$\frac{1}{n}\sum_{k=0}^{n-1}\rho((\mathfrak{B}\cap C)^\star,(\mathfrak{B}\cap T^{-k}C)^\star)\to 0 \ \text{ for } n\to\infty. \tag{9}$$

Now, let $M,W\in(\mathfrak{B}\cap C)^\star$ be arbitrary fixed, then $T^{\star -k}W\in(\mathfrak{B}\cap T^{-k}C)^\star$, because $T^{\star -k}N_B^{-1}(\{n\})=\{\nu\in X^\star:\nu(T^{-k}(B))=n\}=N_{T^{-k}B}^{-1}(\{n\})$ for every $B\in\mathfrak{B}$, $n\in\overline{\mathbb{N}}_0$, hence

$$T^{\star -k}\{N_D:D\in\mathfrak{B}\cap C\}=\{N_{T^{-k}B\cap T^{-k}C}:B\in\mathfrak{B}\}\subseteq\sigma(N_E:E\in\mathfrak{B}\cap T^{-k}C)$$

since $T^{-k}\mathfrak{B}\subseteq\mathfrak{B}$, thus

$$T^{\star -k}\sigma(\{N_D:D\in\mathfrak{B}\cap C\})=\sigma(T^{\star -k}\{N_D:D\in\mathfrak{B}\cap C\})\subseteq\sigma(N_E:E\in\mathfrak{B}\cap T^{-k}C).$$

Therefore by $T^\star$-invariance

$$\frac{1}{n}\sum_{k=0}^{n-1}|\mu^\star(M\cap T^{\star -k}W)-\mu^\star(M)\mu^\star(W)|=$$

$$=\frac{1}{n}\sum_{k=0}^{n-1}|\mu^\star(M\cap T^{\star -k}W)-\mu^\star(M)\mu^\star(T^{\star -k}W)|\le$$

$$\le\frac{1}{n}\sum_{k=0}^{n-1}\rho((\mathfrak{B}\cap C)^\star,(\mathfrak{B}\cap T^{-n}C)^\star)\to 0 \text{ for } n\to\infty,$$

by (9). Let $C_n\in\mathfrak{B}$ be such that $\mu(C_n)<\infty$, $C_n\subseteq C_{n+1}$, $\forall n\in\mathbb{N}$ and $\bigcup_{n\in\mathbb{N}}C_n=X$. If we have shown that $\{(\mathfrak{B}\cap C_n)^\star\}_{n\in\mathbb{N}}$ is an increasing sequence of σ-algebras such that the algebra $\bigcup_{n\in\mathbb{N}}(\mathfrak{B}\cap C_n)^\star$ generates $\mathfrak{B}^\star$, i.e. $\sigma(\bigcup_{n\in\mathbb{N}}(\mathfrak{B}\cap C_n)^\star)=\mathfrak{B}^\star$, then the proof is complete, since it is sufficient to verify the (weakly) mixing property only on a generating (semi-)algebra (see [22], Theorem 1.17), which is done by replacing C by C_n in the computation above. Clearly, $(\mathfrak{B}\cap C_n)^\star\subseteq(\mathfrak{B}\cap C_{n+1})^\star$ for every $n\in\mathbb{N}$. Since $\sigma(\bigcup_{n\in\mathbb{N}}(\mathfrak{B}\cap C_n)^\star)\subseteq\mathfrak{B}^\star$ it is enough to show that every N_B is measurable w.r.t. $\sigma(\bigcup_{n\in\mathbb{N}}(\mathfrak{B}\cap C_n)^\star)$, for $B\in\mathfrak{B}$ arbitrary fixed. By continuity of the measure we have $\lim\limits_{n\to\infty}N_{B\cap C_n}(\nu)=\lim\limits_{n\to\infty}\nu(B\cap C_n)=\nu(B)=N_B(\nu)$ for every $\nu\in X^\star$, hence $N_B=$

$\lim\limits_{n\to\infty} N_{B\cap C_n}$ is $\sigma((\mathfrak{B}\cap C_n)^\star : n\in\mathbb{N})$-measurable, since $N_{B\cap C_n}$ is for every $n\in\mathbb{N}$. So, it is left to show that $\bigcup_{n\in\mathbb{N}}(\mathfrak{B}\cap C_n)^\star$ is an algebra. Obviously, $X\in\bigcup_{n\in\mathbb{N}}(\mathfrak{B}\cap C_n)^\star$. Given $D,E\in\bigcup_{n\in\mathbb{N}}(\mathfrak{B}\cap C_n)^\star$, then there exists $n,m\in\mathbb{N}$ such that $D\in(\mathfrak{B}\cap C_n)^\star$ and $E\in(\mathfrak{B}\cap C_m)^\star$, hence $D\cap E\in(\mathfrak{B}\cap C_{\max\{n,m\}})^\star\subseteq\bigcup_{n\in\mathbb{N}}(\mathfrak{B}\cap C_n)^\star$ and $D^c\in(\mathfrak{B}\cap C_n)^\star\subseteq\bigcup_{n\in\mathbb{N}}(\mathfrak{B}\cap C_n)^\star$. □

Theorem 2.2. *Let $(X,\mathfrak{B},\mu)$ be as in Theorem 2.1 and T a measure-preserving transformation on X which is conservative and ergodic. Then the Poisson suspension $(X^\star,\mathfrak{B}^\star,\mu^\star,T^\star)$ is ergodic.*

Proof. Let $(\widetilde{X},\widetilde{\mathfrak{B}},\widetilde{\mu},\widetilde{T})$ be the natural extension of $(X,\mathfrak{B},\mu,T)$. Since $(X,\mathfrak{B},\mu)$ is conservative and ergodic, the same is true for $(\widetilde{X},\widetilde{\mathfrak{B}},\widetilde{\mu},\widetilde{T})$ (recall Theorem 1.9). By Theorem 2.1 the Poisson suspension $(\widetilde{X}^\star,\widetilde{\mathfrak{B}}^\star,\widetilde{\mu}^\star,\widetilde{T}^\star)$ of $(\widetilde{X},\widetilde{\mathfrak{B}},\widetilde{\mu},\widetilde{T})$ is ergodic.
Following [25], we will show that the Poisson suspension of the natural extension is the natural extension of the Poisson suspension, i.e.

$$(\widetilde{X}^\star,\widetilde{\mathfrak{B}}^\star,\widetilde{\mu}^\star,\widetilde{T}^\star)\cong(\widetilde{X^\star},\widetilde{\mathfrak{B}^\star},\widetilde{\mu^\star},\widetilde{T^\star}). \tag{10}$$

Note, that the Poisson suspension is a standard space, by Remark 2.3, so we can build its natural extension. If we have verified (10), then the proof is complete because ergodicity of an extension immediately implies ergodicity of the underlying system: Suppose that $T^\star$ is not ergodic, but an extension $\widetilde{T^\star}$ of $T^\star$ is. Then there exists an $M\in\mathfrak{B}^\star$ with $\mu^\star(M)\in(0,1)$ such that $T^{\star-1}M=M \bmod \mu^\star$. By the properties of the factor map $\tau:\widetilde{X^\star}\longrightarrow X^\star$ of the extension, we have $\widetilde{M}:=\tau^{-1}M\in\widetilde{\mathfrak{B}^\star}$ and $\widetilde{\mu^\star}(\widetilde{M})=\widetilde{\mu^\star}(\tau^{-1}M)=\mu^\star(M)\in(0,1)$ and $\widetilde{T^\star}^{-1}\widetilde{M}=\widetilde{T^\star}^{-1}\tau^{-1}M=\tau^{-1}T^{\star-1}M=\tau^{-1}M=\widetilde{M}$ modulo $\widetilde{\mu^\star}$, a contradiction.

Let us consider the map

$$\tau:\widetilde{X}^\star\longrightarrow X^\star,\ \tau(\widetilde{\nu}):=\widetilde{\nu}\circ\pi^{-1},\ \ \forall\widetilde{\nu}\in\widetilde{X}^\star,$$

where $\pi:\widetilde{X}\longrightarrow X$ is the factor map of the natural extension of $(X,\mathfrak{B},\mu,T)$, i.e. $\pi^{-1}\mathfrak{B}\subseteq\widetilde{\mathfrak{B}}$, $\widetilde{\mu}\circ\pi^{-1}=\mu$, $\pi\circ\widetilde{T}=T\circ\pi$ and $\sigma(\widetilde{T}^m\pi^{-1}B:B\in\mathfrak{B},m\in\mathbb{N})=\widetilde{\mathfrak{B}}$ (mod $\widetilde{\mu}$).
We will show, that τ is a natural extension, then by uniqueness (up to isomorphic systems), we get (10). Hence we have to check that

1. $\tau^{-1}\mathfrak{B}^\star\subseteq\widetilde{\mathfrak{B}}^\star$
2. $\widetilde{\mu}^\star\circ\tau^{-1}=\mu^\star$
3. $\tau\circ\widetilde{T}^\star=T^\star\circ\tau$
4. $\sigma((\widetilde{T}^\star)^m\tau^{-1}G:G\in\mathfrak{B}^\star,m\in\mathbb{N})=\widetilde{\mathfrak{B}}^\star$ (mod $\widetilde{\mu}^\star$).

The sets $\{N_B^{-1}(\{n\})|B\in\mathfrak{B},n\in\overline{\mathbb{N}}_0\}$ and $\{\widetilde{N}_{\widetilde{B}}^{-1}(\{n\})|\widetilde{B}\in\widetilde{\mathfrak{B}},n\in\overline{\mathbb{N}}_0\}$ generate $\mathfrak{B}^\star$ and $\widetilde{\mathfrak{B}}^\star$, respectively, where $\widetilde{N}_{\widetilde{B}}(\widetilde{\nu}):=\widetilde{\nu}(\widetilde{B}),\forall\widetilde{\nu}\in\widetilde{X}^\star$. Hence, (1) follows from $\tau^{-1}N_B^{-1}(\{n\})=\{\widetilde{\nu}\in\widetilde{X}^\star|\tau(\widetilde{\nu})(B)=n\}=\{\widetilde{\nu}\in\widetilde{X}^\star|\widetilde{\nu}(\pi^{-1}B)=n\}=\widetilde{N}_{\pi^{-1}B}^{-1}(\{n\})\in\widetilde{\mathfrak{B}}^\star$, because $\pi^{-1}B\in\widetilde{\mathfrak{B}}$. For every $B\in\mathfrak{B}$, $\widetilde{\nu}\in\widetilde{X}^\star$ we obtain $\tau(\widetilde{T}^\star(\widetilde{\nu}))(B)=\widetilde{T}^\star(\widetilde{\nu})(\pi^{-1}B)=\widetilde{\nu}(\widetilde{T}^{-1}(\pi^{-1}B))=\widetilde{\nu}(\pi^{-1}(T^{-1}B))=T^\star(\tau(\widetilde{\nu})(B))$, which establish (3). Again using the properties of the extension π and of the Poisson suspension itself, we can deduce that $\widetilde{\mu}^\star(\tau^{-1}(\bigcap_{k=1}^l N_{B_k}^{-1}(\{n_k\})))=\widetilde{\mu}^\star(\bigcap_{k=1}^l\widetilde{N}_{\pi^{-1}B_k}^{-1}(\{n_k\}))=\prod_{k=1}^l\mathrm{Poi}_{\widetilde{\mu}(\pi^{-1}B_k)}(n_k)=$

$\prod_{k=1}^{l} \mathrm{Poi}_{\mu(B_k)}(n_k) \prod_{k=1}^{l} \mu^\star(N_{B_k}^{-1}(n_k)) = \mu^\star(\bigcap_{k=1}^{l} N_{B_k}^{-1}(n_k))$, for $B_1,\dots,B_l \in \mathfrak{B}$ pairwise disjoint and $n_1,\dots,n_l \in \overline{\mathbb{N}}_0$. Thus, $\widetilde{\mu}^\star \circ \tau^{-1}$ equals $\mu^\star$ on an intersection-stable generator of $\mathfrak{B}^\star$, which implies (2).

Using the definition of the Poisson suspension, we compute $(\widetilde{T}^\star)^m \tau^{-1} N_B^{-1}(\{n\}) = (\widetilde{T}^\star)^m \widetilde{N}_{\pi^{-1}B}^{-1}(\{n\}) = \{\widetilde{\nu} \circ \widetilde{T}^{-m} | \widetilde{\nu} \in \widetilde{X}^\star, \ \ \widetilde{\nu}(\pi^{-1}B) = n\} = \{\widetilde{\rho} \in \widetilde{X}^\star | \widetilde{\rho}(\widetilde{T}^m(\pi^{-1}B)) = n\} = \widetilde{N}_{\widetilde{T}^m(\pi^{-1}B)}^{-1}(\{n\})$, for some $m \in \mathbb{N}$, $n \in \overline{\mathbb{N}}_0$ and $B \in \mathfrak{B}$, since $\widetilde{T}$ is an automorphism. Now because $\sigma(\widetilde{T}^m \pi^{-1} B : B \in \mathfrak{B}, m \in \mathbb{N}) = \widetilde{\mathfrak{B}}$ (mod $\widetilde{\mu}$), we can deduce (4). $\square$

Note, that the Poisson suspension is conservative by the Theorem of Poincaré, since $\mu^\star$ is finite and $T^\star$-invariant.

3 Entropy: Definition and basic properties

To quantify how chaotic and unpredictable a dynamical system behaves, the notion of entropy is used.

First let us have a look at the entropy of a μ-partition.

3.1 Static entropy

We will try to provide the following definitions in a rather general setting. For the definition of the information function we follow [10]. The concept of conditional entropy for not necessarily countable partitions introduced in the following has already been considered by Kornfeld and Sinai, [13], and by Rokhlin, [17], for probability spaces, and by Parry, [15], for σ-finite standard measure spaces.

By log we will denote the natural logarithm with base e.

Definition. Let $(X, \mathfrak{B}, \mu)$ be a σ-finite measure space and α a measurable μ-partition of X. The so-called *information function* of α is defined by

$$I_\mu(\alpha)(x) := \begin{cases} \log\left(\dfrac{1}{\mu(\alpha_x)}\right) = -\log(\mu(\alpha_x)) & \text{for } \mu(\alpha_x) \in (0,\infty), \\ \infty & \text{for } \mu(\alpha_x) = 0, \\ 0 & \text{for } \mu(\alpha_x) = \infty \end{cases}$$

for $x \in X$ (where we define $\alpha_x := \emptyset$, if $x \notin \bigcup_{A\in\alpha} A$). The *(static) entropy of* α is defined as the expectation of the information function, if α is equal to a countable μ-partition modulo μ:

$$H_\mu(\alpha) = \begin{cases} \int\limits_X I_\mu(\alpha)(x)\, d\mu(x), & \text{if } \alpha = \text{a countable } \mu\text{-partition of } X \text{ modulo } \mu \\ \infty, & \text{else.} \end{cases}$$

Note, that in the first case we actually obtain

$$H_\mu(\alpha) = \int\limits_X I_\mu(\alpha)(x)\, d\mu(x) = \sum_{\substack{A\in\alpha \\ \mu(A)<\infty}} \mu(A)\log(\mu(A)).$$

To exclude the possibility of having negative entropy, we will always require that α is fine enough such that for every $A \in \alpha$ we either have $\mu(A) \leq 1$ or $\mu(A) = \infty$ when using the above definition.

Note, that by the convention $\infty \cdot 0 = 0$ we get $\mu(\alpha_x) I_\mu(\alpha)(x) = 0$ if $\mu(\alpha_x) = 0$ or $\mu(\alpha_x) = \infty$.
The condition, that α is modulo μ equal to a countable μ-partition α' of X actually means that α consists of countably many sets, which already build a μ-partition of X, and arbitrary many (possible uncountably many) null-sets whose union is a null-set[7]. In this case we clearly have $H_\mu(\alpha) = H_\mu(\alpha')$.
The integration in the first case of the definition makes sense, because if α is a countable μ-partition together with some (possibly uncountably many) null sets, then $I_\mu(\alpha) : X \longrightarrow \mathbb{R} \cup \{\infty\}$ is measurable: Given $a \leq b \in \mathbb{R}$, we have $I_\mu(\alpha)^{-1}((a,b))$ is an at most countable union of elements of α and therefore contained in $\mathfrak{B}$, and $I_\mu(\alpha)^{-1}(\{\infty\})$ is the union of all null-set in α and the null set $X \setminus (\bigcup_{A \in \alpha} A)$, hence measurable.

Remark 3.1. The only way for a μ-partition α to be not countable modulo μ, is that it bears uncountably many null-sets, whose union is no null set any more. This is a consequence of the fact that in a σ-finite setting it cannot happen that there are uncountably many disjoint measurable sets of positive measure: Consider $D_j \in \mathfrak{B}$ disjoint with $\mu(D_j) =: n_j < \infty$ for all j in a countable index set J and $\bigcup_{j \in J} D_j = X$ mod μ. Let $j \in J$ be fixed. If there are uncountably many disjoint sets $E_l \in \mathfrak{B} \cap D_j$, $l \in L$, in $(D_j, \mathfrak{B} \cap D_j, \mu|_{D_j})$ with $\mu(E_l) > 0$ for every $l \in L$ (L an uncountable index set), then[8] $\infty = \sum_{l \in L} \mu(E_l) := \sup\limits_{F \subseteq L,\ F \text{ finite}} \sum_{f \in F} \mu(E_f) = \sup\limits_{F \subseteq L,\ F \text{ finite}} \underbrace{\mu(\bigcup\limits_{f \in F} E_f)}_{\leq n_j} \leq n_j$, a contradiction. Thus, $(D_j, \mathfrak{B} \cap D_j, \mu|_{D_j})$ cannot contain uncountably many disjoint measurable sets of positive measure, for every $j \in J$, and since J is countable the same is true for $(X, \mathfrak{B}, \mu)$.
That there are only at most countably many elements in α of positive measure in particular implies that the union of all null-sets in α is measurable.

The Poisson distribution plays a crucial role in our consideration, so let us compute its entropy:

Example 3.1. The entropy of the Poisson distribution is given by the entropy of an arbitrary random variable which is Poisson distributed. Therefore, let $(\Omega, \mathfrak{A}, P)$ be a probability space and Z an $\overline{\mathbb{N}}_0$-valued random variable on it which is Poisson distributed with parameter $\lambda \in [0, \infty)$. The entropy of a random variable taking values in an at most countable set is defined as the entropy of the P-partition given by the random variable's preimages. So we consider the partition $\alpha := \{Z^{-1}(\{n\}) | n \in \overline{\mathbb{N}}_0\}$. Then

$$H(Poi_\lambda) := H_P(Z) := H_P(\alpha) = -\sum_{n \in \overline{\mathbb{N}}_0} P(Z^{-1}(\{n\})) \log(P(Z^{-1}(\{n\}))) =$$

$$= -\sum_{n \in \overline{\mathbb{N}}_0} Poi_\lambda(n) \log(Poi_\lambda(n)) = -\sum_{n=0}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} \log\Big(\frac{\lambda^n e^{-\lambda}}{n!}\Big) =$$

[7]Note that the union of all null-sets in such a μ-partition is always measurable, since it is the complement of the at most countable union of sets of positive measure in the μ-partition (and maybe a null-set of elements, which are not contained in elements of the μ-partition).

[8]Intuitively it is clear that a sum of uncountably many positive elements is infinite. Formally, this follows from then revers direction oft the fact, that if $\sum_{F \in \alpha} g(F) < \infty$, for some non-negative function g, then $\{F \in \alpha | g(F) \neq 0\}$ is countable. Proof: Let $m \in \mathbb{N}$ be arbitrary fixed and set $L_m := \{F \in \alpha | g(F) > \frac{1}{m}\}$. Then $\sum_{F \in \alpha} g(F) > \frac{1}{m}|L_m|$, thus $|L_m| < m \sum_{F \in \alpha} g(F) < \infty$, hence $\{F \in \alpha | g(F) > 0\} = \bigcup_{m \in \mathbb{N}} L_m$ is a countable union of finite sets, which is our claim.

$$= -\sum_{n=0}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} (n \log(\lambda) - \lambda - \log(n!)) =$$

$$= -\sum_{n=1}^{\infty} \frac{\lambda^n}{(n-1)!} e^{-\lambda} \log(\lambda) + \sum_{n=0}^{\infty} \frac{\lambda^{n+1}}{n!} e^{-\lambda} + \sum_{n=0}^{\infty} \frac{\lambda^n e^{-\lambda} \log(n!)}{n!} =$$

$$= -\lambda \log(\lambda) + \lambda + \sum_{n=0}^{\infty} \frac{\lambda^n e^{-\lambda} \log(n!)}{n!}.$$

Definition. Let α and γ be measurable μ-partitions of X and let γ be countable, then the *conditional information* of α given γ is set to be

$$I_\mu(\alpha|\gamma)(x) := \begin{cases} I_{\mu(\cdot|\gamma_x)}(\alpha)(x) & \text{for } \mu(\gamma_x) \neq \infty, \\ I_\mu(\alpha \vee \{\gamma_x, X \setminus \gamma_x\})(x) & \text{else.} \end{cases}$$

For formal reasons, if $\mu(\gamma_x) = 0$, we define $\mu(\cdot|\gamma_x)$ to be just some probability measure with $\mu(\gamma_x|\gamma_x) = 1$. If X is a standard space and $\widetilde{\sigma}(\gamma)$ is σ-finite, then we use canonical measures and set $\mu(\cdot|\gamma_x) := \mu_{\gamma_x}$. Actually, we do not have to care much about this cases, since they vanish in the entropy as long as γ is countable: The *conditional (static) entropy* of α given γ, for γ countable, is defined as

$$H_\mu(\alpha|\gamma) := \int_X I_\mu(\alpha|\gamma)(x)\, d\mu(x) =$$

$$= \sum_{\substack{A\in\alpha,\ C\in\gamma \\ \mu(C)\neq\infty,\ \mu(A|C)\neq\infty}} -\mu(A\cap C)\log(\mu(A|C)) + \sum_{\substack{A\in\alpha,\ C\in\gamma \\ \mu(C)=\infty,\ \mu(A\cap C)\neq\infty}} -\mu(A\cap C)\log(\mu(A\cap C)),$$

$$\text{if } \alpha \text{ is modulo } \mu \text{ equal to a countable } \mu\text{-partition,}$$

and as

$$H_\mu(\alpha|\gamma) := \infty, \text{ if } \alpha \text{ is not equal to a countable } \mu\text{-partition modulo } \mu.$$

Analogous to the previous definition we require that α and β are such that for all $A \in \alpha$, $C \in \gamma$ we have that $\mu(A \cap C) \leq 1$ or $\mu(A \cap C) = \infty$.

For the case, when γ is not necessarily countable, we have to pay more attention to null-sets, since their union need not be a null set anymore. We will use the previously developed theory of canonical systems of measures. In order to do so, we now have to restrict our considerations to the case when $(X, \mathfrak{B}, \mu)$ is a standard measure space and $\widetilde{\sigma}(\gamma)$ a σ-finite sub-σ-algebra (confer Proposition 1.3 (1)). Then, according to Theorem 1.10, there exists a system of probability measures $\{\mu_C\}_{\mu_\gamma\text{-a.e. } C\in\gamma}$ and we can define conditional entropy as follows:

Definition. Let $(X, \mathfrak{B}, \mu)$ be a σ-finite standard measure space and α, γ measurable μ-partitions of X such that $\widetilde{\sigma}(\gamma)$ is σ-finite, then the *conditional (static) entropy* of α given γ is given by

$$H_\mu(\alpha|\gamma) := \int_{X_\gamma} H_{\mu_C}(\alpha \cap C)\ d\mu_\gamma(C) = \int_X H_{\mu_{\pi_\gamma(x)}}(\alpha \cap \pi_\gamma(x))\ d\mu(x).$$

This definition makes sense (and we were allowed to use the transformation formula for the second equality), since for μ_γ-a.e. $C \in X_\gamma$ we have that $\alpha \cap C$ is a measurable μ-partition of $(C, \mathfrak{B}_C)$ (see Example 1.1 (4)) and $C \mapsto H_{\mu_C}(\alpha \cap C)$ is measurable. Indeed, let us denote

$$G:\ C \mapsto H_{\mu_C}(\alpha \cap C) =$$

$$= \begin{cases} \sum\limits_{A \in \alpha,\ \mu_C(A \cap C) \neq 0} \mu_C(A \cap C) \log(\mu_C(A \cap C)), & \text{if } \alpha \cap C = \text{a countable } \mu\text{-partition of} \\ & C \text{ modulo } \mu_C \\ \infty, & \text{else,} \end{cases}$$

defined for μ_γ-a.e. $C \in X_\gamma$, then by Theorem 1.10 we know that the map

$$E_A:\ C \mapsto \mu_C(A \cap C)$$

is $\mathfrak{B}_\gamma$-measurable, for every $A \in \alpha$ (and μ_γ-almost all $C \in \gamma$), thus for every set $U \in \mathfrak{B}(\mathbb{R})$, which does not contain ∞, we clearly have $G^{-1}(U) \in \mathfrak{B}_\gamma$ (since the sum in the first case is countable and the logarithm is continuous). While,

$$G^{-1}(\{\infty\}) =$$

$$= \{C \in \gamma' | \alpha \cap C = \text{a countable } \mu\text{-partition of } C \text{ modulo } \mu_C \text{ and}$$

$$\sum_{A \in \alpha} \mu_C(A \cap C) \log(\mu_C(A \cap C)) = \infty\} \cup \{C \in \gamma' |\ \alpha \cap C \text{ is not equal to a countable } \mu\text{-par-}$$

$$\text{tition of } C \text{ modulo } \mu_C\ \},$$

where γ' denotes the set of elements of γ for which μ_C exists. With the same argument as above, we get that the first set of the above union is measurable w.r.t. $\mathfrak{B}_\gamma$, so let us treat the second one. Let $C \in \gamma'$ be such that $\alpha \cap C$ is not equal to a countable μ-partition of C modulo μ_C, i.e. there are uncountably many $A_l \in \alpha$, $l \in L$, such that $\mu_C(\bigcup_{l \in L \setminus J} A_l \cap C) > 0$ for every countable $J \subset L$ with $\bigcup_{l \in L \setminus J} A_l \cap C \in \mathfrak{B} \cap C$. Since by Remark 3.1 there are at most countably many elements of positive measure in α, we can neglect them in the above collection of $A_l \in \alpha$, $l \in L$ and further add all (remaining) null-sets of α. Hence,

$$\{C \in \gamma' |\ \alpha \cap C \text{ is not equal to a countable } \mu\text{-partition of } C \text{ modulo } \mu_C\ \} =$$

$$= \Big\{C \in \gamma' \Big| \mu_C\Big(\bigcup_{A \in \alpha,\ \mu(A)=0} A \cap C\Big) > 0\Big\} = E^{-1}_{\bigcup_{A \in \alpha,\ \mu(A)=0} A}((0,1]) \in \mathfrak{B}_\gamma.$$

The definition of conditional entropy for measurable μ-partitions coincides with the one given before for countable μ-partitions: Assume α and γ to be countable, then

$$\int_\gamma H_{\mu_C}(\alpha \cap C)\ \ d(\mu \circ \pi_\gamma^{-1})(C) = \sum_{C \in \gamma} \mu(\pi_\gamma^{-1}(\{C\}))\ H_{\mu_C}(\alpha \cap C) =$$

$$= \sum_{C \in \gamma, \mu(C) \neq 0} \mu(C)\ H_{\mu(\cdot|C)}(\alpha),$$

due to Proposition 1.3 (2), which by (4) of the next Proposition 3.1 is exactly the conditional entropy for countable μ-partitions.

The following example illustrates the idea of defining the conditional entropy that way and shows that even for uncountable μ-partitions whose elements are all null sets, the conditional entropy turns out to be finite.

Example 3.2. Let $X = [0,1] \times [0,1]$, $\mathfrak{B} = \mathfrak{B}([0,1]) \otimes \mathfrak{B}([0,1])$, $\mu = \lambda \otimes \lambda$, where λ denotes the one-dimensional Lebesgue measure and set $I := [0,1]$. We are considering the partition $\gamma := \{\{x\} \times I \ : \ x \in I,\}$ of X. This partition is uncountable and every element has measure zero. But γ is measurable, since e.g. $\Delta := \{J \times I \ :$ J subinterval of I with rational endpoints$\}$ generates γ countably. Further, we take the measurable μ-partition $\alpha := \{\{x\} \times (0,x] \ : x \in I\} \cup \{\{x\} \times (x,1] \ : x \in I\}$ of X. This μ-partition is finer than γ, and for every fixed $x \in I$ the set $\{\{x\} \times (0,x] \ , \ \{x\} \times (x,1]\} =:$ A_x is a μ-partition of γ_x.
We are going to show that $H_\mu(\alpha|\gamma)$ is finite, though $H_\mu(\alpha) = H_\mu(\gamma) = \infty$.
Using the Theorem of Fubini, the fact that $\pi_\gamma(x,y) = \{x\} \times I =: \gamma_x$ does not depend on y and that $\lambda(I) = 1$, we compute

$$H_\mu(\alpha|\gamma) = \int_{I \times I} H_{\mu_{\gamma_x}}(\alpha \cap \gamma_x) \ d(\lambda \otimes \lambda)(x,y) = \int_I H_{\mu_{\gamma_x}}(A_x) \ d\lambda(x) =$$

$$= \int_I (-\mu_{\gamma_x}(\{x\} \times (0,x]) \log(\mu_{\gamma_x}(\{x\} \times (0,x]))$$
$$-\mu_{\gamma_x}(\{x\} \times (x,1]) \log(\mu_{\gamma_x}(\{x\} \times (x,1]))) \ d(\lambda)(\gamma_x).$$

Now, by Remark 1.5 (3), we know that $\mu_{\gamma_x} = (\lambda \otimes \lambda)_{\{x\} \times I} = \delta_x \otimes \lambda$, thus the above is equal to

$$\int_I (-\lambda((0,x]) \log(\lambda((0,x])) - \lambda((x,1]) \log(\lambda((x,1]))) \ d\lambda(x) =$$

$$= \int_I (-x \log(x) - (1-x) \log(1-x)) \ d\lambda(x) = \int_I H(\mathrm{Ber}_{(x,1-x)}) \ d\lambda(x),$$

where $H(\mathrm{Ber}_{(x,1-x)})$ shall denote the entropy of the Bernoulli distribution with parameter x, i.e. $H(\mathrm{Ber}_{(x,1-x)}) := H_P(\{Z^{-1}(\{0\}), Z^{-1}(\{0\})\})$ for Z a Bernoulli distributed random variable (with parameter x).
In particular $\int_I H(\mathrm{Ber}_{(x,1-x)}) \ d\lambda(x) \leq \lambda(I) \cdot \max_{x \in I}(H(\mathrm{Ber}_{(x,1-x)})) = H(\mathrm{Ber}_{(\frac{1}{2},\frac{1}{2})}) =$ $-\log(\frac{1}{2}) = \log(2) < \infty$ (see e.g. [4] for the maximum result).

Remark 3.2. There is one more way of defining the conditional information, namely, if α is a countable μ-partition of a standard space $(X, \mathfrak{B})$ and $\widetilde{\sigma}(\gamma)$ is σ-finite, then the *conditional entropy of* α *given* $\widetilde{\sigma}(\gamma)$ is defined by

$$I_\mu(\alpha|\widetilde{\sigma}(\gamma))(x) := \sum_{A \in \alpha} 1_A(x)(-\log(\mu(A|\widetilde{\sigma}(\gamma))(x)))$$

for $x \in X$. This definition has been considered by Krengel, [14].
This definition clearly coincides with the previous one modulo μ, since for almost every $x \in A$ we have $I_\mu(\alpha|\widetilde{\sigma}(\gamma))(x) = -\log(\mu(A|\widetilde{\sigma}(\gamma))(x)) = -\log(\mu_{\gamma_x}(A \cap \gamma_x)) =$ $-\log(\mu_{\gamma_x}(\alpha_x \cap \gamma_x)) = I_\mu(\alpha|\gamma)(x)$, using Proposition 1.3 (3) and the fact that $\mu(\gamma_x) < \infty$ due to σ-finiteness of $\widetilde{\sigma}(\gamma)$.
Moreover,

$$\int_X I_\mu(\alpha|\widetilde{\sigma}(\gamma))(x) \ d\mu(x) = H_\mu(\alpha|\gamma)$$

for α countable. Indeed, by definition of the conditional expectation,

$$\int_X 1_A \log(\mu(A|\widetilde{\sigma}(\gamma))(x)) \ d\mu(x) = \int_X E_\mu[1_A \log(\mu(A|\widetilde{\sigma}(\gamma))(x))|\widetilde{\sigma}(\gamma)] \ d\mu(x) =$$

$$\int_X E_\mu[1_A|\widetilde{\sigma}(\gamma)](x)\log(\mu(A|\widetilde{\sigma}(\gamma))(x))\,d\mu(x) = \int_X \mu_{\gamma_x}(A\cap\gamma_x)\log(\mu_{\gamma_x}(A\cap\gamma_x))\,d\mu(x) =$$
$$= H_\mu(\alpha|\gamma),$$
since $\log\circ\mu(A|\widetilde{\sigma}(\gamma))$ is $\widetilde{\sigma}(\gamma)$-measurable.

3.1.1 Properties of static entropy

For the rest of the thesis, when considering the conditional entropy given a μ-partition γ of X and γ is not equal to a countable μ-partition of X mod μ, then we will always assume that $(X,\mathfrak{B},\mu)$ is a standard measure space and $\widetilde{\sigma}(\gamma)$ is σ-finite.

Remark 3.3. The above definitions are *robust w.r.t. the measure* in that, if α, α', γ, γ' are measurable μ-partitions, such that $\alpha=\alpha'$ mod μ and $\gamma=\gamma'$ mod μ, then

$$I_\mu(\alpha)(x) = I_\mu(\alpha')(x) \text{ and } I_\mu(\alpha|\gamma)(x) = I_\mu(\alpha'|\gamma')(x) \text{ for almost every } x\in X,$$

and
$$H_\mu(\alpha) = H_\mu(\alpha'),\ H_\mu(\alpha|\gamma) = H_\mu(\alpha'|\gamma').$$
If γ is countable, then the claim follows by robustness of the canonical system of measures (Proposition 1.3 (1) ,(4), (5)) :

$$H_\mu(\alpha|\gamma) = \int_\gamma H_{\mu_C}(\alpha\cap C)\,d\mu_\gamma(C) = \int_X \sum_{A\in\alpha}\mu_{\pi_\gamma(x)}(A\cap\pi_\gamma(x))\log(\mu_{\pi_\gamma(x)}(A\cap\pi_\gamma(x)))\,d\mu(x) =$$

$$= \int_X \sum_{A'\in\alpha'}\mu_{\pi_{\gamma'}(x)}(A'\cap\pi_{\gamma'}(x))\log(\mu_{\pi_{\gamma'}(x)}(A'\cap\pi_{\gamma'}(x)))\,d\mu(x) = H_\mu(\alpha'|\gamma'),$$

if $\alpha\cap C$ is mod μ_C equal to a countable μ-partition for μ_γ-a.e. $C\in\gamma$. If this is not the case, then
$$H_\mu(\alpha|\gamma) = \infty = H_\mu(\alpha'|\gamma'),$$
since if there is a $C\in\gamma$ not contained in a μ_γ-null-set such that $\alpha\cap C$ is not equal to a countable μ-partition mod μ_C, then for $C'\in\gamma'$ with $C'=C$ mod μ, we get that C' is not contained in a $\mu_{\gamma'}$-null-set (since $0\neq\mu_\gamma(C)=\mu(\pi_\gamma^{-1}(C))=\mu(C)=\mu(C')=\mu_{\gamma'}(C')$) and $\alpha\cap C'$ is not equal to a countable μ-partition modulo $\mu_{C'}$, because μ_C, μ_C' are just the conditional measure w.r.t C, rep. C' (Proposition 1.3 (2)), since $\mu(C)=\mu(C')\neq 0$, which gives $\mu_C=\mu_{C'}$.

Due to this Remark, we do not have to distinguish μ-partitions, which differ only on a null-set, when considering the entropy. Thus, we will e.g. write $H_\mu(\alpha)$ instead of being precise and writing $H_\mu(\alpha\cap X_0)$ for $X=X_0$ mod μ in some cases.

Notation. For σ-finite sub-σ-algebras $\mathfrak{A}$, $\mathfrak{C}$ of $\mathfrak{B}$ one can always find generating measurable μ-partitions α, γ of X which are unique mod μ (ref. Lemma 1.2) and thus by the remark above the following notation is (almost everywhere) well-defined: We will write $I_\mu(\mathfrak{A}) := I_\mu(\alpha)$, $I_\mu(\mathfrak{A}|\mathfrak{C}) := I_\mu(\alpha|\gamma)$, $H_\mu(\mathfrak{A}) := H_\mu(\alpha)$ and $H_\mu(\mathfrak{A}|\mathfrak{C}) := H_\mu(\alpha|\gamma)$.

Remark 3.4. Let $(X,\mathfrak{B},\mu)$, $(Y,\mathfrak{D},\nu)$ be two σ-finite standard measure spaces and $\phi: X_0\longrightarrow Y_0$ a measure-preserving map with $X=X_0$ mod μ and $Y=Y_0$ mod ν. Then for σ-finite sub-σ-algebras $\mathfrak{G}$, $\mathfrak{H}$ of $\mathfrak{D}$ such[9] that $\mathfrak{G} \supseteq \mathfrak{H}$, we get that "$\phi$ preserves the entropy":

$$H_\mu(\phi^{-1}(\mathfrak{G})|\phi^{-1}(\mathfrak{H})) = H_\nu(\mathfrak{G}|\mathfrak{H}).$$

In particular, if we are in the situation of Theorem 1.4, i.e. we have σ-finite sub-σ-algebras $\mathfrak{A}$, $\mathfrak{C}$ of $\mathfrak{B}$, $\mathfrak{A} \supseteq \mathfrak{C}$, then we already know that there is a σ-finite standard measure space $(Y, \mathfrak{A}', \nu)$ and a measure-preserving map $\phi : X_0 \longrightarrow Y_0$, with $Y_0 = Y$ mod ν, $X_0 = X$ mod μ such that $\phi^{-1}(\mathfrak{A}') = \mathfrak{A}$ mod μ. Thus by the claim above, we get

$$H_\mu(\mathfrak{A}|\mathfrak{C}) = H_\nu(\mathfrak{A}'|\mathfrak{C}'),$$

where $\mathfrak{C}' := \{C' \in \mathfrak{A}' | \phi^{-1}(C') \in \mathfrak{C} \text{ mod } \mu\}$, which is a σ-finite sub-σ-algebra[10] of $\mathfrak{A}'$ with $\phi^{-1}(\mathfrak{C}') = \mathfrak{C}$ mod μ (since every element of $\mathfrak{C}$ can be written mod μ as preimage of an element of $\mathfrak{A}'$).

Indeed, let α be the measurable ν-partition which generates $\mathfrak{G}$ and γ those of $\mathfrak{H}$. By Remark 3.3 we can w.l.o.g. assume that $X_0 = X$ and $Y_0 = Y$. Clearly, $\phi^{-1}(\alpha)$ and $\phi^{-1}(\gamma)$ are measurable μ-partitions which generate $\phi^{-1}(\mathfrak{G})$ and $\phi^{-1}(\mathfrak{H})$, respectively. Thus, using the transformation formula, we see that

$$H_\nu(\alpha|\gamma) = \int_Y H_{\nu_{\pi_\gamma(y)}}(\alpha \cap \pi_\gamma(y))\, d(\mu \circ \phi^{-1})(y) = \int_X H_{\nu_{\pi_\gamma(\phi(x))}}(\alpha \cap \pi_\gamma(\phi(x)))\, d\mu(x).$$

Now, because for μ-almost every $x \in X$ we know that $\nu_{\pi_\gamma(\phi(x))} = \mu_{\pi_{\phi^{-1}(\gamma)}(x)} \circ \phi^{-1}$ and $\phi^{-1}(\pi_\gamma(\phi(x))) = \pi_{\phi^{-1}(\gamma)}(x)$ due to Proposition 1.3 (6), the above is equal to

$$\int_X H_{\mu_{\pi_{\phi^{-1}(\gamma)}(x)}}(\phi^{-1}(\alpha) \cap \phi^{-1}(\pi_\gamma(\phi(x)))\, d\mu(x) =$$

$$= \int_X H_{\mu_{\pi_{\phi^{-1}(\gamma)}(x)}}(\phi^{-1}(\alpha) \cap \pi_{\phi^{-1}(\gamma)}(x))\, d\mu(x) = H_\mu(\phi^{-1}(\alpha)|\phi^{-1}(\gamma)).$$

I.e. $H_\nu(\mathfrak{G}|\mathfrak{H}) = H_\mu(\phi^{-1}(\mathfrak{G})|\phi^{-1}(\mathfrak{H}))$.

Let us list some basic properties of static entropy:

Proposition 3.1. *Let $(X, \mathfrak{B}, \mu, T)$ be a σ-finite dynamical system and α, β, γ measurable μ-partitions of X.*

1. *If α is finite then so is $H_\mu(\alpha)$.*
2. *$I_\mu(\alpha|\{X, \emptyset\}) = I_\mu(\alpha)$ and $H_\mu(\alpha|\{X, \emptyset\}) = H_\mu(\alpha)$, if μ is a probability measure or $\mu(X) = \infty$.*
3. *$H_\mu(\alpha|\beta) = H_\mu(\alpha \vee \beta|\beta)$.*
4. *$H_\mu(\alpha|\beta) = \sum_{B \in \beta} \mu(B)\, H_{\mu(\cdot|B)}(\alpha)$, if β is equal to a countable μ-partition modulo μ and $\mu(B) < \infty$, $\forall B \in \beta$.*

[9] Actually, the condition $\mathfrak{G} \supseteq \mathfrak{H}$ is no restriction, since by Proposition 3.1 $H_\mu(\mathfrak{G}|\mathfrak{H}) = H_\mu(\mathfrak{G} \vee \mathfrak{H}|\mathfrak{H})$, where we denote $\mathfrak{G} \vee \mathfrak{H} := \sigma(\{G \cap H : G \in \mathfrak{G}, H \in \mathfrak{H}\})$, thus we could just replace $\mathfrak{G}$ by $\mathfrak{G} \vee \mathfrak{H}$.

[10] Clearly, $\mathfrak{C}'$ is a σ-algebra, since $\mathfrak{C}$ is one and set-operations are respected by preimages. Moreover, $\mathfrak{C}'$ is σ-finite: By σ-finiteness of $\mathfrak{C}$ there are $C_n \in \mathfrak{C}$ with $\mu(C_n) < \infty$ ($n \in \mathbb{N}$) and $\bigcup_{n \in \mathbb{N}} C_n = X$ mod μ, then because $\phi^{-1}(\mathfrak{A}') = \mathfrak{A} \supseteq \mathfrak{C}$ mod μ, for every $n \in \mathbb{N}$, there is a $C'_n \in \mathfrak{A}'$ with $\phi^{-1}(C'_n) = C_n$ mod μ. Clearly, $C'_n \in \mathfrak{C}'$, further $\nu(C'_n) = \mu(\phi^{-1}(C'_n)) = \mu(C_n) < \infty$ for all $n \in \mathbb{N}$, and $\nu(Y \setminus (\bigcup_{n \in \mathbb{N}} C'_n)) = \mu(\phi^{-1}(Y) \cap (\bigcup_{n \in \mathbb{N}} \phi^{-1}(C'_n))^c) = \mu(X \setminus (\bigcup_{n \in \mathbb{N}} C_n)) = 0$.

5. $I_\mu(\alpha \vee \beta) = I_\mu(\alpha|\beta) + I_\mu(\beta)$, $H_\mu(\alpha \vee \beta) = H_\mu(\alpha|\beta) + H_\mu(\beta)$ *and* $I_\mu(\alpha \vee \beta|\gamma)(x) = I_\mu(\beta|\gamma)(x) + I_\mu(\alpha|\beta \vee \gamma)(x)$, *for* $x \in X$ *such that* $\mu(\gamma_x) \neq 0$, *and* $H_\mu(\alpha \vee \beta|\gamma) = H_\mu(\beta|\gamma) + H_\mu(\alpha|\beta \vee \gamma)$, *where we assume that* μ *is a probability measure in case* γ *is not equal to a countable* μ*-partition modulo* μ.

6. $\alpha \preceq \gamma \Leftrightarrow H_\mu(\alpha|\gamma) = 0$, *i.e. roughly speaking, if we have already the whole (or even more) information of what we want to know, there is no uncertainty (or chaos) left, hence the entropy is zero, and vice-versa.*
Moreover, if μ *is a probability measure, then* $H_\mu(\alpha) = 0 \Leftrightarrow \alpha = \{X, \emptyset\}$ *mod* μ.

7. *If* μ *is a probability measure, then, clearly,* $H_\mu(\alpha) \geq 0$ *and* $H_\mu(\alpha|\gamma) \geq 0$. *For* σ*-finite* μ, *we still have* $H_\mu(\alpha|\gamma) \geq 0$, *as long as* $\widetilde{\sigma}(\gamma)$ *is* σ*-finite.*
Moreover, $\alpha \preceq \beta \Rightarrow H_\mu(\alpha) \leq H_\mu(\beta)$ *for* μ σ*-finite.*
If γ *is mod* μ *equal to a countable* μ*-partition or* μ *is a probability measure, then* $\alpha \preceq \beta \Rightarrow H_\mu(\alpha|\gamma) \leq H_\mu(\beta|\gamma)$.
If $\mu(X) = 1$, *then we can deduce that* $\beta \preceq \gamma \Rightarrow H_\mu(\alpha|\beta) \geq H_\mu(\alpha|\gamma)$ *and, especially,* $H_\mu(\alpha) = H_\mu(\alpha|\{X, \emptyset\}) \geq H_\mu(\alpha|\gamma)$.

8. *If* T *is measure-preserving, then*

$$I_\mu(\bigvee_{k=0}^{n-1} T^{-k}\alpha)(x) = I_\mu(\alpha)(T^{n-1}x) + \sum_{m=1}^{n-1} I_\mu(\alpha|\bigvee_{k=1}^{m} T^{-k}\alpha)(T^{n-1-m}x)$$

for $x \in X$ *such that* $\mu((\bigvee_{k=1}^{n-1} T^{-k}\alpha)_x) \neq 0$. *If* μ *is a probability or* $\mu(X) = \infty$, *we set* $\alpha_1^0 := \{X, \emptyset\}$ *and write* $\sum_{m=0}^{n-1} I_\mu(\alpha|\bigvee_{k=1}^{m} T^{-k}\alpha) \circ T^{n-1-m}$ *for the right hand side above.*
If T *is a measure-preserving automorphism, then almost everywhere we have*

$$I_\mu(\bigvee_{k=0}^{n-1} T^{k}\alpha) = I_\mu(\alpha) \circ T^{-(n-1)} + \sum_{m=1}^{n-1} I_\mu(\alpha|\bigvee_{k=1}^{m} T^{k}\alpha) \circ T^{m-(n-1)},$$

respectively

$$I_\mu(\bigvee_{k=0}^{n-1} T^{k}\alpha) = \sum_{m=0}^{n-1} I_\mu(\alpha|\bigvee_{k=1}^{m} T^{k}\alpha) \circ T^{m-(n-1)}$$

for $\bigvee_{k=1}^{0} T^k\alpha := \{X, \emptyset\}$ *in case* $\mu(X) = 1$ *or* $\mu(X) = \infty$.
If α *is countable, then*

$$\int_X I_\mu(\bigvee_{k=0}^{n-1} T^{-k}\alpha)\, d\mu = \int_X I_\mu(\alpha)\, d\mu + \sum_{m=1}^{n-1} \int_X I_\mu(\alpha|\bigvee_{k=1}^{m} T^{-k}\alpha)\, d\mu.$$

Before proving this Proposition, let us remark a simple property about refinements of μ-partitions:

Remark 3.5. Note, that $\alpha \vee \beta$ is equal to a countable μ-partition modulo μ if and only if α and β are equal to countable μ-partitions mod μ.

Proof of Proposition 3.1. 1. This is obvious by just noting that if $\mu(\alpha_x) = \infty$ then $I_\mu(\alpha)(x) = 0$ by definition, and if $I_\mu(\alpha)(x) = \infty$ then $\mu(\alpha_x) = 0$, and by definition $\infty \cdot 0 = 0$.

2. By definition, if $\mu(X)=\infty$, then we get $I_\mu(\alpha|\{X,\emptyset\}) = I_\mu(\alpha\cap X) = I_\mu(\alpha)$. If $\mu(X)=1$, then $\mu(\cdot|X)=\mu$ and therefore $I_\mu(\alpha|\{X,\emptyset\}) = I_{\mu(\cdot|X)}(\alpha) = I_\mu(\alpha)$.
For α equal to a countable μ-partition mod μ and $\mu(X)\neq\infty$, we compute

$$H_\mu(\alpha|\{X,\emptyset\}) = -\sum_{A\in\alpha}\mu(A\cap X)\log\Big(\frac{\mu(A\cap X)}{\mu(X)}\Big) =$$

$$= -\sum_{A\in\alpha}\big(\mu(A)\log(\mu(A)) - \mu(A)\log(\mu(X))\big) = H_\mu(\alpha)+\mu(X)\log(\mu(X)),$$

and the second term of this sum is equal to zero for $\mu(X)=1$. (Note, that we always assume that $\mu(X)\neq 0$.) If α is not equal to a countable μ-partition mod μ, then we get $H_\mu(\alpha|\{X,\emptyset\}) = \infty = H_\mu(\alpha)$.
Now, if $\mu(X)=\infty$, then by definition,

$$H_\mu(\alpha|\{X,\emptyset\}) = -\sum_{A\in\alpha,\ \mu(A)\neq\infty}\mu(A\cap X)\log(\mu(A\cap X)) = H_\mu(\alpha),$$

if α is mod μ equal to a countable μ-partition, and $H_\mu(\alpha|\{X,\emptyset\}) = \infty = H_\mu(\alpha)$, if not.

3. If $\widetilde{\sigma}(\beta)$ is σ-finite, then

$$H_\mu(\alpha\vee\beta|\beta) = \int_\beta H_{\mu_B}(\alpha\vee\beta)\ d\mu_\beta(B) =$$

$$= -\int_\beta\sum_{A\in\alpha,\ D\in\beta}\mu_B(A\cap D\cap B)\log(\mu_B(A\cap D\cap B)\ d\mu_\beta(B) =$$

$$= -\int_\beta\sum_{A\in\alpha}\mu_B(A\cap B)\log(\mu_B(A\cap B)\ d\mu_\beta(B) = H_\mu(\alpha|\beta),$$

if $\alpha\cap B$ is equal to a countable μ-partition modulo μ_B for μ_β-a.e. $B\in\beta$, and if this is not the case, then $H_\mu(\alpha\vee\beta|\beta) = \infty = H_\mu(\alpha|\beta)$.
If β is equal to a countable μ-partition modulo μ and $\widetilde{\sigma}(\beta)$ not necessary σ-finite, then

$$H_\mu(\alpha\vee\beta|\beta) = -\sum_{B\in\beta,\mu(B)\neq\infty}\sum_{A\in\alpha,D\in\beta}\mu(A\cap D|B)\log(\mu(A\cap D|B))$$

$$-\sum_{\substack{B\in\beta,\mu(B)=\infty\\ \mu(A\cap B)\neq\infty}}\sum_{A\in\alpha,D\in\beta}\mu(A\cap D\cap B)\log(\mu(A\cap D\cap B)) =$$

$$= -\sum_{B\in\beta,\mu(B)\neq\infty}\sum_{A\in\alpha}\mu(A|B)\log(\mu(A|B))$$

$$-\sum_{\substack{B\in\beta,\mu(B)=\infty\\ \mu(A\cap B)\neq\infty}}\sum_{A\in\alpha}\mu(A\cap B)\log(\mu(A\cap B)) = H_\mu(\alpha|\beta),$$

if α is modulo μ equal to a countable μ-partition, and else again $H_\mu(\alpha\vee\beta|\beta) = \infty = H_\mu(\alpha|\beta)$.

4. If α is equal to a countable μ-partition modulo μ, then

$$H_\mu(\alpha|\beta) = -\sum_{B\in\beta,}\sum_{A\in\alpha}\mu(A\cap B)\log(\mu(A|B)) =$$

$$= -\sum_{B\in\beta}\sum_{A\in\alpha}\mu(B)\mu(A|B)\log(\mu(A|B)) = \sum_{B\in\beta}\mu(B)H_{\mu(\cdot|B)}(\alpha).$$

If α is not equal to a countable μ-partition modulo μ, then we have by definition $H_\mu(\alpha|\beta) = \infty$ and also due to countability of β there has to be an $B \in \beta$ with $\mu(B) > 0$ such that $B \cap \alpha$ is not countable modulo μ (else α would be a countable union of mod μ countable μ-partitions, a contradiction). Hence, $H_{\mu(\cdot|B)}(\alpha) = \infty$ by definition.

5. First, let $x \in X$ such that $\mu(\alpha_x \cap \beta_x) \neq 0$ and $\mu(\alpha_x \cap \beta_x) \neq \infty$. (Note that this in particular implies that $\mu(\beta_x) \neq 0$.) If $\mu(\beta_x) \neq \infty$, then

$$I_\mu(\alpha \vee \beta)(x) = -\log(\mu(\alpha_x \cap \beta_x)) = -\log\Big(\frac{\mu(\alpha_x \cap \beta_x)}{\mu(\beta_x)}\mu(\beta_x)\Big) =$$

$$= -\log\Big(\frac{\mu(\alpha_x \cap \beta_x)}{\mu(\beta_x)}\Big) - \log(\mu(\beta_x)) = I_\mu(\alpha|\beta)(x) + I_\mu(\beta)(x).$$

If $\mu(\beta_x) = \infty$, we get by definition $I_\mu(\alpha|\beta)(x) = I_\mu(\alpha \vee \beta)(x)$ and since $I_\mu(\beta)(x) = 0$, the claimed equality follows. (Recall that the information function cannot take the value $-\infty$ by definition.)
Now, if $x \in X$ is such that $\mu(\alpha_x \cap \beta_x) = 0$, then $I_\mu(\alpha \vee \beta)(x) = \infty$, and $I_\mu(\alpha|\beta)(x) = \infty$, if $\mu(\beta_x) \neq 0$, or, if this is not the case, then $I_\mu(\beta)(x) = \infty$, hence $I_\mu(\alpha \vee \beta)(x) = \infty = I_\mu(\alpha|\beta)(x) + I_\mu(\beta)(x)$.
If $\mu(\alpha_x \cap \beta_x) = \infty$ for $x \in X$, then clearly $\mu(\alpha_x) = \infty$ and $\mu(\beta_x) = \infty$. Thus, by definition $I_\mu(\alpha \vee \beta)(x) = 0$, $I_\mu(\beta)(x) = 0$ and $I_\mu(\alpha|\beta)(x) = I_\mu(\alpha \vee \beta)(x) = 0$. Hence, we have verified that

$$I_\mu(\alpha \vee \beta)(x) = I_\mu(\alpha|\beta)(x) + I_\mu(\beta)(x) \text{ for all } x \in X.$$

If $\alpha \vee \beta$ is modulo μ equal to a countable μ-partition, then so are α and β, by Remark 3.5, and we obtain

$$H_\mu(\alpha \vee \beta) = \int_X I_\mu(\alpha \vee \beta)(x)\, d\mu(x) =$$

$$= \int_X I_\mu(\alpha|\beta)(x)\, d\mu(x) + \int_X I_\mu(\beta)(x)\, d\mu(x) = H_\mu(\alpha|\beta) + H_\mu(\beta).$$

Let us consider the case when $\alpha \vee \beta$ is not equal to a countable μ-partition modulo μ, then α or β have to be unequal to a countable μ-partition modulo μ. If β is not equal to a countable μ-partition modulo μ, then we immediately get $H_\mu(\alpha \vee \beta) = \infty = H_\mu(\alpha|\beta) + H_\mu(\beta)$. So, let α be uncountable modulo μ and β be equal to a countable μ-partition modulo μ, but then by definition $H_\mu(\alpha|\beta) = \infty$ and we get the same equality as above.
The equation $I_\mu(\alpha \vee \beta|\gamma) = I_\mu(\beta|\gamma) + I_\mu(\alpha|\beta \vee \gamma)$ follows by the first part of the proof:

$$I_\mu(\alpha \vee \beta \vee \gamma) = I_\mu(\alpha|\beta \vee \gamma) + I_\mu(\beta \vee \gamma) = I_\mu(\alpha|\beta \vee \gamma) + I_\mu(\beta|\gamma) + I_\mu(\gamma),$$

and if $x \in X$ is such that $\mu(\gamma_x) \neq 0$ which is equivalent to $I_\mu(\gamma)(x) < \infty$, then this gives

$$I_\mu(\alpha \vee \beta \vee \gamma)(x) - I_\mu(\gamma)(x) = I_\mu(\alpha|\beta \vee \gamma)(x) + I_\mu(\beta|\gamma)(x),$$

which is exactly

$$I_\mu(\alpha \vee \beta|\gamma)(x) = I_\mu(\alpha|\beta \vee \gamma)(x) + I_\mu(\beta|\gamma)(x),$$

again by the first part of the proof.
Hence, if γ is equal to a countable μ-partition modulo μ, we can neglect the null-sets in γ and obtain

$$\int_X I_\mu(\alpha \vee \beta|\gamma)(x)\, d\mu(x) = \int_X I_\mu(\alpha|\beta \vee \gamma)(x)\, d\mu(x) + \int_X I_\mu(\beta|\gamma)(x)\, d\mu(x).$$

If additionally β is equal to a countable μ-partition modulo μ, too, then we have shown that $H_\mu(\alpha \vee \beta|\gamma) = H_\mu(\alpha|\beta \vee \gamma) + H_\mu(\beta|\gamma)$. If β is not equal to a countable μ-partition modulo μ, but γ is, then $H_\mu(\alpha \vee \beta|\gamma) = \infty$ and $H_\mu(\beta|\gamma) = \infty$.
Now, let us consider the case when γ is not countable modulo μ. We assume that $\widetilde{\sigma}(\gamma)$ and $\sigma(\gamma \vee \beta)$ are σ-finite. Then, by the first part of the proof,

$$H_\mu(\alpha \vee \beta|\gamma) = \int_\gamma H_{\mu_C}(\alpha \vee \beta)\, d\mu_\gamma(C) = \int_\gamma (H_{\mu_C}(\alpha|\beta) + H_{\mu_C}(\beta))\, d\mu_\gamma(C) =$$

$$= \int_\gamma H_{\mu_C}(\alpha|\beta)\, d\mu_\gamma(C) + H_\mu(\beta|\gamma).$$

If $\beta \cap C_l$ is not countable modulo μ_{C_l} for some (non-negligible) sets $C_l \in \gamma$, $l \in L$ with $\mu(\bigcup_{l \in L} C_l) \neq 0$, then $H_{\mu_{C_l}}(\alpha \vee \beta) = \infty$ and $H_{\mu_{C_l}}(\beta) = \infty$ and

$$H_\mu(\alpha \vee \beta|\gamma) = \int_\gamma H_{\mu_C}(\alpha \vee \beta)\, d\mu_\gamma(C) = \infty = H_\mu(\alpha|\beta \vee \gamma) + \int_\gamma H_{\mu_C}(\beta)\, d\mu_\gamma(C) =$$

$$= H_\mu(\alpha|\beta \vee \gamma) + H_\mu(\beta|\gamma).$$

If $\alpha \cap C$ is not equal to a countable μ-partition modulo μ_{C_l} for some $C_l \in \gamma$ with $\mu(\bigcup_{l \in L} C_l) > 0$, then, since β is countable, there has to be a $B \in \beta$ with $\mu(B) > 0$ such that $\alpha \cap C_l \cap B$ is equal to a not countable μ-partition mod $\mu_{C_l \cap B}$ for some $C_l \in \gamma$ with $\mu(\bigcup_{l \in L} C_l) > 0$. (Recall, that by Proposition 1.3 (4) mod μ_C for μ_γ-a.e. $C \in \gamma \Leftrightarrow$ mod $\mu \Leftrightarrow \mu_{C \cap B}$ for $\mu_{\gamma \vee \beta}$-a.e. $C \cap B \in \gamma \vee \beta$.) Thus $H_\mu(\alpha|\beta \vee \gamma) = \infty = \int_\gamma H_{\mu_C}(\alpha|\beta)\, d\mu_\gamma(C)$.
For proof of the case when $\alpha \cap C$ and $\beta \cap C$ are equal to countable μ-partitions modulo μ_C for almost all $C \in \gamma$, but γ is uncountable, we refer the reader to [15], Theorem 5.6. where this case is treated in probability spaces.

6. For μ a probability measure, we clearly have $H_\mu(\{X, \emptyset\}) = -\mu(X)\log(\mu(X)) - 0 \cdot \infty = -\log(1) = 0$. Moreover, if $H_\mu(\alpha) = 0$, then, by definition, α is equal to a countable μ-partition mod μ, hence $H_\mu(\alpha) = -\sum_{A \in \alpha, \mu(A) \neq \infty, 0} \mu(A)\log(\mu(A)) = 0$, thus $\mu(A) = 1$ for all $A \in \alpha$, but this implies that $A = X$ modulo μ, $\forall A \in \alpha$.
Let μ be arbitrary now. By definition, $H_\mu(\alpha|\gamma) = \int_\gamma H_{\mu_C}(\alpha \cap C)\, d\mu_\gamma(C)$ and μ_C is a probability measure on C. Hence by the above $H_{\mu_C}(\alpha \cap C) = 0 \Leftrightarrow \alpha \cap C = \{C, \emptyset\}$ mod $\mu \Leftrightarrow \alpha \preceq \gamma$.

7. Note that if μ is a probability measure, then $H_\mu(\alpha) = \sum_{A\in\alpha} -\mu(A)\underbrace{\log(\underbrace{\mu(A)}_{\leq 1})}_{\leq 0} \geq 0$ or $H_\mu(\alpha) = \infty \geq 0$. Thus, if $\widetilde{\sigma}(\gamma)$ is σ-finite, then

$$H_\mu(\alpha|\gamma) = \int \underbrace{H_{\mu_C}(\alpha)}_{\geq 0}\, d\mu_\gamma(C) \geq 0,$$

since μ_C is a probability measure for (a.e.) $C \in \gamma$.
Let $\alpha \preceq \beta$. If α is not equal to a countable μ-partition modulo μ, then β is not equal to a countable μ-partition mod μ, and we get $H_\mu(\beta) = \infty = H_\mu(\alpha)$. If α is equal to a countable μ-partition modulo μ, then, due to (5), $H_\mu(\beta) = H_\mu(\beta \vee \alpha) = H_\mu(\alpha) + H_\mu(\beta|\alpha)$, and by the above $H_\mu(\beta|\alpha) \geq 0$ since being equal to a countable μ-partition modulo μ, $\widetilde{\sigma}(\alpha)$ is clearly σ-finite, hence $H_\mu(\beta) \geq H_\mu(\alpha)$.
If γ is countable or μ is a probability measure, then again by (5), $H_\mu(\alpha \vee \beta|\gamma) = H_\mu(\alpha|\gamma) + H_\mu(\beta|\alpha \vee \gamma)$. If $\alpha \preceq \beta$, then $\alpha \vee \beta = \beta$, thus

$$H_\mu(\beta|\gamma) = H_\mu(\alpha \vee \beta|\gamma) = H_\mu(\alpha|\gamma) + H_\mu(\beta|\alpha \vee \gamma),$$

hence $H_\mu(\beta|\gamma) \leq H_\mu(\alpha|\gamma)$, since $H_\mu(\beta|\alpha \vee \gamma) > 0$.
For the proof of $\beta \preceq \gamma \Rightarrow H_\mu(\alpha|\beta) \geq H_\mu(\alpha|\gamma)$ we refer to [15], Theorem 5.1.

8. Let $n \in \mathbb{N}$ be fixed and $x \in X$ such that $\mu((\bigvee_{k=1}^{n-1} T^{-k}\alpha)_x) \neq 0$. By repeated application of (5) we get

$$I_\mu(\bigvee_{k=0}^{n-1} T^{-k}\alpha)(x) = I_\mu\Big(T^{-n+1}\alpha \vee \bigvee_{k=0}^{n-2} T^{-k}\alpha\Big)(x) =$$

$$= I_\mu(T^{-n+1}\alpha)(x) + I_\mu\Big(\bigvee_{k=0}^{n-2} T^{-k}\alpha|T^{-n+1}\alpha\Big)(x) =$$

$$= I_\mu(\alpha)(T^{n-1}x) + I_\mu\Big(T^{-n+2}\alpha \vee \bigvee_{k=0}^{n-3} T^{-k}\alpha|T^{-n+1}\alpha\Big)(x) =$$

$$= I_\mu(\alpha)(T^{n-1}x) + I_\mu(T^{-n+2}\alpha|T^{-n+1}\alpha)(x) +$$

$$I_\mu(\bigvee_{k=0}^{n-3} T^{-k}\alpha|T^{-n+1}\alpha \vee T^{-n+2}\alpha)(x) =$$

$$= I_\mu(\alpha)(T^{n-1}x) + I_\mu(\alpha|T^{-1}\alpha)(T^{n-2}x) +$$

$$I_\mu\Big(T^{-n+3}\alpha \vee \bigvee_{k=0}^{n-4} T^{-k}\alpha|T^{-n+1}\alpha \vee T^{-n+2}\alpha\Big)(x) =$$

$$= \ldots =$$

$$= I_\mu(\alpha)(T^{n-1}x) + I_\mu(\alpha|T^{-1}\alpha)(T^{n-2}x) + I_\mu(\alpha|T^{-1}\alpha \vee T^{-2}\alpha)(T^{n-3}x) +$$

$$\ldots + I_\mu\Big(\alpha|\bigvee_{k=1}^{n-1} T^{-k}\alpha\Big)(x) =$$

$$= I_\mu(\alpha)(T^{n-1}x) + \sum_{m=1}^{n-1} I_\mu\Big(\alpha | \bigvee_{k=1}^{m} T^{-m}\alpha\Big)(T^{n-1-m}x),$$

where we have used that, due to T-invariance, $I_\mu(T^{-n+1}\alpha)(x) = I_\mu(\alpha)(T^{n-1}x)$ and similar for the conditional information. To verify this, set $\alpha_{T^{n-1}x} =: A$. Clearly, $T^{n-1}x \in A \Leftrightarrow x \in T^{-n+1}A$, hence $T^{-n+1}A = (T^{-n+1}\alpha)_x$. Now, the information function only depends on $\mu(A)$, but $\mu(A) = \mu(T^{-n+1}A)$, therefore

$$I_\mu(T^{-n+1}\alpha)(x) = I_\mu(\alpha)(T^{n-1}x).$$

This also implies that

$$I_\mu(T^{-n+2}\alpha|T^{-n+1}\alpha)(x) = I_{\mu(\cdot|(T^{-n+1}\alpha)_x)}(T^{-n+2}\alpha)(x) =$$

$$= I_{\mu(\cdot|(T^{-1}\alpha)_{T^{n-2}x})}(\alpha)(T^{n-2}x) = I_\mu(\alpha|T^{-1}\alpha)(T^{n-2}x)$$

for $\mu((T^{-n+1}\alpha)_x) \neq \infty$, because

$$\mu((T^{-n+2}\alpha)_x|(T^{-n+1}\alpha)_x) = \frac{\mu(T^{-n+2}(\alpha_{T^{n-2}x}) \cap T^{-n+2}((T^{-1}\alpha)_{T^{n-2}x}))}{\mu(T^{-n+2}((T^{-1}\alpha)_{T^{n-2}x}))} =$$

$$= \frac{\mu(\alpha_{T^{n-2}x} \cap (T^{-1}\alpha)_{T^{n-2}x}))}{\mu((T^{-1}\alpha)_{T^{n-2}x})} = \mu(\alpha_{T^{n-2}x}|(T^{-1}\alpha)_{T^{n-2}x})),$$

and for $\mu((T^{-n+1}\alpha)_x) = \infty$, we have

$$I_\mu(T^{-n+2}\alpha|T^{-n+1}\alpha)(x) = I_\mu((T^{-n+1}\alpha)_x \cap T^{-n+2}\alpha)(x) =$$

$$= I_\mu((T^{-1}\alpha)_{T^{n-2}x} \cap \alpha)(T^{n-2}x) = I_\mu(\alpha|T^{-1}\alpha)(T^{n-2}x).$$

If α is countable, then so is $\bigvee_{k=1}^{n-1} T^{-k}\alpha$, and we can neglect all null-sets of it. Thus, by integrating and using the transformation formula and T-invariance, the last claim follows.

□

The following Lemma provides us with convergence in information.

Lemma 3.1. *Let $(\Omega, \mathfrak{A}, P)$ be a probability space, α a P-partition of Ω with $H_P(\alpha) < \infty$ and $\mathfrak{A}_n$ sub-σ-algebras of $\mathfrak{A}$, such that $\mathfrak{A}_n \subseteq \mathfrak{A}_{n+1}$ for all $n \in \mathbb{N}_0$ and $\mathfrak{A} = \sigma(\bigcup_{n\in\mathbb{N}_0} \mathfrak{A}_n)$. Then*

$$I_P(\alpha|\mathfrak{A}_\mathrm{n}) \to I_P(\alpha|\mathfrak{A}) \qquad \text{in } L_1(P) \text{ and } P\text{-a.e.}$$

as $n \to \infty$.
In particular,

$$\lim_{n\to\infty} \int_\Omega I_P(\alpha|\mathfrak{A}_\mathrm{n})(\omega)\, dP(\omega) = \int_\Omega I_P(\alpha|\mathfrak{A})(\omega)\, dP(\omega).$$

Proof. By the assumption $H_P(\alpha) < \infty$ we know that α is modulo P equal to a countable P-partition, so by Remark 3.3 we can w.l.o.g. assume that α is countable.
For fixed $A \in \alpha$ the process $X_n := E[1_A|\mathfrak{A}_n] = P(A|\mathfrak{A})$ is a martingal w.r.t. $\mathfrak{A}_n$, for $n \in \mathbb{N}_0$, because conditional expectation is adapted by definition and for every $k \in \mathbb{N}$ we have $E[X_{n+k}|\mathfrak{A}_n] = E[E[1_A|\mathfrak{A}_{n+k}]|\mathfrak{A}_n] = E[1_A|\mathfrak{A}_n] = X_n$. Since $\sup\{E[X_n] : n \in \mathbb{N}_0\} <$

∞, we can use the martingale convergence theorem 1.12 and obtain a $\sigma(\bigcup_{n\in\mathbb{N}_0}\mathfrak{A}_n)$-measurable random variable X_∞, such that $X_n \to X_\infty$ a.e. for $n \to \infty$. Clearly, $X_\infty = E[1_A|\mathfrak{A}]$, because for every $B \in \sigma(\bigcup_{n\in\mathbb{N}_0}\mathfrak{A}_n) = \mathfrak{A}$, we have $E[X_\infty 1_B] = \lim\limits_{n\to\infty} E[X_n 1_B] = \lim\limits_{n\to\infty} E[E[1_A|\mathfrak{A}_n]1_B] = \lim\limits_{n\to\infty} E[1_A 1_B] = E[1_A 1_B]$. Thus

$$P(A|\mathfrak{A}_n) \to P(A|\mathfrak{A}) \qquad P\text{-a.e.}$$

For every $x \in A$, $P(\alpha_x|\mathfrak{A}_{nx}) = P(A|\mathfrak{A}_n)(x)$, where $\mathfrak{A}_{nx}$ denotes the element of the corresponding P-partition which contains x, hence $I_P(\alpha|\mathfrak{A}_n) = -\log\circ P(A|\mathfrak{A}_n)$ and therefore

$$I_P(\alpha|\mathfrak{A}_n) \to I_P(\alpha|\mathfrak{A}) \qquad P\text{-a.e.}$$

Convergence in $L_1(P)$ then follows by dominated convergence, if we show

$$\int \sup_{n\in\mathbb{N}_0} I_P(\alpha|\mathfrak{A}_n)\, dP \le H_P(\alpha) + 1, \tag{11}$$

since by assumption $H_P(\alpha) < \infty$, hence the function $\sup_{n\in\mathbb{N}_0} I_P(\alpha|\mathfrak{A}_n)$ dominates the map $I_P(\alpha|\mathfrak{A}_n)$ for every $n \in \mathbb{N}_0$ and satisfies $\int \sup_{n\in\mathbb{N}_0} I_P(\alpha|\mathfrak{A}_n)\, dP < \infty$.
To prove (11), we define $f(x) := \sup_{n\in\mathbb{N}_0} I_P(\alpha|\mathfrak{A}_n)(x)$ for $x \in \Omega$ and $F(a) := P(\{x \in \Omega : f(x) > a\}) = P(f^{-1}((a,\infty]))$ for $a \in [0,\infty)$. Then

$$\int_\Omega f\, dP = \int_0^\infty F(a)\, da.$$

Indeed, $\int_0^\infty F(a)\, da = \int_0^\infty \int_\Omega 1_{f^{-1}(a,\infty]}\, dP\, da = \int_\Omega \int_0^\infty 1_{(a,\infty]} \circ f\, da\, dP$ and $1_{(a,\infty]}(f(x)) = \begin{cases} 1, & f(x) > a \\ 0, & f(x) \le a \end{cases} = 1_{[0,f(x))}(a)$ for $x \in \Omega$ because $a \ge 0$, thus $\int_0^\infty 1_{(a,\infty]}(f(x))\, da = \int_0^\infty 1_{[0,f(x))}(a)\, da = f(x)$.
Furthermore,

$$F(a) = P(\{x \in \Omega | \sup_{n\in\mathbb{N}_0} [-\log(P(\alpha_x|\mathfrak{A}_{nx}))] > a\}) =$$

$$= \sum_{A\in\alpha} P(\{x \in A | \inf_{n\in\mathbb{N}_0} P(\alpha_x|\mathfrak{A}_{nx}) < e^{-a}\}) =$$

$$= \sum_{A\in\alpha}\sum_{n=0}^\infty P(A \cap \{x \in \Omega | P(\alpha_x|\mathfrak{A}_{nx}) < e^{-a},\ P(\alpha_x|\mathfrak{A}_{kx}) \ge e^{-a}\ \forall k < n\}) =$$

$$= \sum_{A\in\alpha}\sum_{n=0}^\infty \int_\Omega 1_A 1_{C_n}\, dP = \sum_{A\in\alpha}\sum_{n=0}^\infty \int_{C_n} P(A|\mathfrak{A}_n)\, dP$$

by definition of the conditional expectation, where we set $C_n := \{x \in \Omega | P(\alpha_x|\mathfrak{A}_{nx}) < e^{-a},\ P(\alpha_x|\mathfrak{A}_{kx}) \ge e^{-a}\ \forall k < n\}$, which is $\mathfrak{A}_n$-measurable. Now, by definition of C_n, $\sum_{n=0}^\infty \int_{C_n} P(A|\mathfrak{A}_n)\, dP \le \sum_{n=0}^\infty e^{-a} P(C_n) = e^{-a} P(\bigcup_{n=0}^\infty C_n) \le e^{-a}$ and therefore

$$F(a) \le \sum_{A\in\alpha} \min(P(A), e^{-a}).$$

Combining the above (in)equations, we finally get

$$\int_\Omega f\, dP = \int_0^\infty F(a)\, da \le \int_0^\infty \sum_{A\in\alpha} \min(P(A), e^{-a})\, da = \sum_{A\in\alpha} \int_0^\infty \min(P(A), e^{-a})\, da =$$

$$= \sum_{A\in\alpha} \int_0^{-\log(P(A))} P(A)da + \int_{-\log(P(A))}^{\infty} e^{-a}da = \sum_{A\in\alpha} (-P(A)\log(P(A)) + P(A)) =$$

$$= H_P(\alpha) + 1,$$

since $H_P(\alpha) = \int_\Omega I_P(\alpha)\, dP = \sum_{A\in\alpha} \int_A -\log(P(A))\, dP = \sum_{A\in\alpha} -P(A)\log(P(A))$. Thus (4) is shown and therefore the proof is complete. $\square$

3.2 Entropy of dynamical systems with a probability measure

Now, let us define the entropy of a dynamical system in the sense of Kolmogorv and Sinaj. For this end we have to restrict our considerations to measure spaces with a finite measure. For simplicity, let us take a probability measure. But before doing so, let us introduce the following notation:

Notation. Let T be a non-singular transformation on $(X, \mathfrak{B}, \mu)$ and α a μ-partition of X. The common refinement of the preimages of α under T will be denoted as follows:

$$\alpha_m^n := \bigvee_{k=m}^{n} T^{-k}\alpha$$

for $m,n \in \overline{\mathbb{N}}_0$, $m \leq n$.

Definition. Let T be a measure-preserving transformation on a measure space $(X, \mathfrak{B}, \mu)$ with $\mu(X) = 1$. The *(dynamical) entropy* of the system $(X, \mathfrak{B}, \mu, T)$ with respect to α is given by

$$h(X, \mathfrak{B}, \mu, T, \alpha) := \lim_{n\to\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1})$$

If α is countable then

$$h(X, \mathfrak{B}, \mu, T, \alpha) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}) = \inf_{n\in\mathbb{N}} \frac{1}{n} H_\mu(\alpha_0^{n-1}),$$

by the next Corollary 3.1.
Finally, the (Kolmogorov-Sinai) *entropy* of the dynamical system is defined by

$$h(X, \mathfrak{B}, \mu, T) := \sup\{h(X, \mathfrak{B}, \mu, T, \alpha) :\ \alpha \text{ a } \mu\text{-partition of } X \text{ with } H_\mu(\alpha) < \infty\}.$$

Note, that in fact we always can take a countable μ-partition α in the definition above, since the condition $H_\mu(\alpha) < \infty$ implies that α is modulo μ equal to a countable μ-partition α' of X, and $h(X, \mathfrak{B}, \mu, T, \alpha) = h(X, \mathfrak{B}, \mu, T, \alpha)$ by Remark 3.3.

Lemma 3.2. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. If $(a_n)_{n\in\mathbb{N}}$ is* subadditive*, that is $a_{n+m} \leq a_n + a_m$ for all $n,m \in \mathbb{N}$, then $\lim\limits_{n\to\infty} \frac{1}{n}a_n = \inf\limits_{n\in\mathbb{N}} \frac{1}{n}a_n$.*
If $a_n \in [0,\infty)$, $\forall n \in \mathbb{N}$, such that $a_{n+1} - a_n \geq a_{n+2} - a_{n-1}$, $\forall n \in \mathbb{N}$ and $a_1 \geq a_2 - a_1$, i.e. the sequence has decreasing increments*, then the sequence is subadditive and fulfills $\lim\limits_{n\to\infty} \frac{1}{n}a_n = \inf\limits_{n\in\mathbb{N}} \frac{1}{n}a_n = \lim\limits_{n\to\infty}(a_{n+1} - a_n)$.*

For proofs of these well-known statements we refer to [4], Fact 2.1.1, respectively to [11], Lemma 9.13, for the first claim.

Corollary 3.1. *Let $(X, \mathfrak{B}, \mu)$ be a probability space, T a measure-preserving transformation on it and α a countable μ-partition of X, then $H(\alpha_0^{m+n-1}) \leq H(\alpha_0^{m-1}) + H(\alpha_0^{n-1})$ and $\lim\limits_{n\to\infty} \frac{1}{n} H(\alpha_0^{n-1}) = \inf_{n\in\mathbb{N}} \frac{1}{n} H(\alpha_0^{n-1})$.*

Proof. Using Proposition 3.1 (5) and (7), we compute

$$H_\mu(\alpha_0^{m+n-1}) = H_\mu(\alpha_0^{m-1} \vee \alpha_m^{m+n-1}) = H_\mu(\alpha_0^{m-1}) + H_\mu(\alpha_m^{m+n-1}|\alpha_0^{m-1}) \leq$$

$$\leq H_\mu(\alpha_0^{m-1}) + H_\mu(\alpha_m^{m+n-1}) = H_\mu(\alpha_0^{m-1}) + H_\mu(T^{-m}\alpha_0^{n-1}) =$$

$$= H_\mu(\alpha_0^{m-1}) + H_\mu(T^{-m}\alpha_0^{n-1}),$$

due to T-invariance.
If $H_\mu(\alpha) = \infty$, then $H_\mu(\alpha_0^{n-1}) \geq H_\mu(\alpha) = \infty$ for all $n \in \mathbb{N}$, hence

$$\lim_{n\to\infty} \frac{1}{n} H(\alpha_0^{n-1}) = \infty = \inf_{n\in\mathbb{N}} \frac{1}{n} H(\alpha_0^{n-1}).$$

If $H_\mu(\alpha) < \infty$, then we also have $H_\mu(\alpha_0^{n-1}) < \infty$ for every $n \in \mathbb{N}$, since otherwise we could choose a minimal $n \in \mathbb{N}$ such that $H_\mu(\alpha_0^{n-1}) = \infty$, but by subadditivity we know that $H_\mu(\alpha_0^{n-1}) = H_\mu(\alpha_0^{n-2}) + H_\mu(\alpha) < \infty$, a contradiction. Therefore, if $H_\mu(\alpha) < \infty$, then $a_n := H_\mu(\alpha_0^{n-1})$, $n \in \mathbb{N}$, defines a subadditive sequence in $[0,\infty)^{\mathbb{N}}$ and by applying Lemma 3.2 we obtain $\lim\limits_{n\to\infty} \frac{1}{n} H(\alpha_0^{n-1}) = \inf_{n\in\mathbb{N}} \frac{1}{n} H(\alpha_0^{n-1})$. $\square$

Remark 3.6. If μ is a T-invariant probability measure, and α is a measurable μ-partition of X such that $H_\mu(\alpha) < \infty$, then we can rewrite the entropy as

$$h(X,\mathfrak{B},\mu,T,\alpha) = H_\mu\Big(\alpha|\bigvee_{k=1}^{\infty} T^{-k}\alpha\Big) = H_\mu\Big(\bigvee_{k=0}^{\infty} T^{-k}\alpha|\bigvee_{k=1}^{\infty} T^{-k}\alpha\Big).$$

Indeed, applying Proposition 3.1 (5) and using the fact that $H_\mu \circ T^{-1} = H_\mu$, due to T-invariance of μ, we compute

$$h(X,\mathfrak{B},\mu,T,\alpha) = \lim_{n\to\infty} \frac{1}{n} H_\mu\Big(\alpha \vee \bigvee_{k=1}^{n-1} T^{-k}\alpha\Big) =$$

$$= \lim_{n\to\infty} \frac{1}{n}\Big(H_\mu\Big(\alpha|\bigvee_{k=1}^{n-1} T^{-k}\alpha\Big) + H_\mu\Big(T^{-1}\Big(\bigvee_{k=0}^{n-2} T^{-k}\alpha\Big)\Big)\Big) =$$

$$= \lim_{n\to\infty} \frac{1}{n}\Big(H_\mu\Big(\alpha|\bigvee_{k=1}^{n-1} T^{-k}\alpha\Big) + H_\mu\Big(\bigvee_{k=0}^{n-2} T^{-k}\alpha\Big)\Big) =$$

$$= \lim_{n\to\infty} \frac{1}{n}\Big(H_\mu\Big(\alpha|\bigvee_{k=1}^{n-1} T^{-k}\alpha\Big) + H_\mu\Big(\alpha|\bigvee_{k=1}^{n-2} T^{-k}\alpha\Big) + H_\mu\Big(T^{-1}\Big(\bigvee_{k=0}^{n-3} T^{-k}\alpha\Big)\Big)\Big) =$$

$$= \ldots = \lim_{n\to\infty} \frac{1}{n}\Big(\sum_{j=0}^{n-1} H_\mu\Big(\alpha|\bigvee_{k=1}^{j} T^{-k}\alpha\Big)\Big),$$

where we define $\bigvee_{k=1}^{0} T^{-k}\alpha := \{X,\emptyset\}$. If $H_\mu(\alpha) < \infty$ (which in particular implies that $H_\mu(\alpha|\bigvee_{k=1}^{j} T^{-k}\alpha) \leq H_\mu(\alpha) < \infty$), then the sequence $A_n := \sum_{j=0}^{n-1} H_\mu(\alpha|\bigvee_{k=1}^{j} T^{-k}\alpha)$ has decreasing increments: Clearly, by Proposition 3.1 (7), we get that $A_{n+1} - A_n = H_\mu(\alpha|\bigvee_{k=1}^{n} T^{-k}\alpha) \geq H_\mu(\alpha|\bigvee_{k=1}^{n+1} T^{-k}\alpha) = A_{n+2} - A_{n+1}$, and we see that $A_1 = H_\mu(\alpha) \geq$

$H_\mu(\alpha|T^{-1}\alpha) = H_\mu(\alpha) + H_\mu(\alpha|T^{-1}\alpha) - H_\mu(\alpha) = A_2 - A_1$. So, by Lemma 3.2, we obtain $\lim\limits_{n\to\infty} \frac{1}{n} A_n = \lim\limits_{n\to\infty} (A_n - A_{n-1})$, hence

$$\lim_{n\to\infty} \frac{1}{n} \Big(\sum_{j=0}^{n-1} H_\mu(\alpha| \bigvee_{k=1}^{j} T^{-k}\alpha) \Big) = \lim_{n\to\infty} H_\mu(\alpha| \bigvee_{k=1}^{n-1} T^{-k}\alpha) = H_\mu(\alpha| \bigvee_{k=1}^{\infty} T^{-k}\alpha),$$

where the convergence is a consequence of the fact, that $(H_\mu(\alpha| \bigvee_{k=1}^{n-1} T^{-k}\alpha))_{n\in\mathbb{N}}$ is a decreasing sequence in the (extended) real line, which is bounded from below by $H_\mu(\alpha| \bigvee_{k=1}^{\infty} T^{-k}\alpha)$.
By Proposition 3.1 (3) we can write $H_\mu(\alpha| \bigvee_{k=1}^{\infty} T^{-k}\alpha)$ as $H_\mu(\bigvee_{k=0}^{\infty} T^{-k}\alpha| \bigvee_{k=1}^{\infty} T^{-k}\alpha)$.

Proposition 3.2. *Let $(X, \mathfrak{B}, \mu, T)$ be a measure-preserving dynamical system with $\mu(X) = 1$ and α, β be measurable μ-partitions of X.*

1. *$\alpha \preceq \beta$ implies $h(X, \mathfrak{B}, \mu, T, \alpha) \leq h(X, \mathfrak{B}, \mu, T, \beta)$.*
2. *We always have $h(X, \widetilde{\sigma}(\alpha), \mu, T) \leq h(X, \widetilde{\sigma}(\alpha), \mu, T, \alpha) = h(X, \mathfrak{B}, \mu, T, \alpha)$. If $H_\mu(\alpha) < \infty$ then we obtain equality: $h(X, \widetilde{\sigma}(\alpha), \mu, T) = h(X, \mathfrak{B}, \mu, T, \alpha)$.*
3. *If $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$ are two σ-finite sub-σ-algebras of $\mathfrak{B}$, then we get $h(X, \mathfrak{B}_1, \mu, T) \leq h(X, \mathfrak{B}_1, \mu, T) \leq h(X, \mathfrak{B}, \mu, T, \alpha)$. Especially, if there is an increasing sequence of σ-finite sub-σ-algebras $\mathfrak{B}_n$ of $\mathfrak{B}$ with $n \in \mathbb{N}$ such that $\sigma(\bigcup_{n\in\mathbb{N}} \mathfrak{B}_n) = \mathfrak{B}$ mod μ, then $\lim\limits_{n\to\infty} h(X, \mathfrak{B}_n, \mu, T) = h(X, \mathfrak{B}, \mu, T)$.*

Proof. 1. Clearly, $\alpha \preceq \beta$ implies that $\alpha_0^{n-1} \preceq \beta_0^{n-1}$ for every $n \in \mathbb{N}$, hence the claim follows by Proposition 3.1 (7).

2. Since
$$h(X, \mathfrak{B}, \mu, T, \alpha) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\alpha) =$$
$$= \begin{cases} \lim_{n\to\infty} \frac{1}{n} \Big(\sum\limits_{A\in\alpha} \mu(A) \log(\mu(A)) \Big), & \text{if } \alpha = \text{a countable } \mu\text{-partition mod } \mu \\ \infty, & \text{else} \end{cases}$$
depends only[11] on those elements of $\mathfrak{B}$, which also lie in α , it is no restriction to consider $\widetilde{\sigma}(\alpha)$ instead of $\mathfrak{B}$, i.e.
$$h(X, \widetilde{\sigma}(\alpha), \mu, T, \alpha) = h(X, \mathfrak{B}, \mu, T, \alpha).$$
By definition, $h(X, \widetilde{\sigma}(\alpha), \mu, T) = \sup\limits_{\substack{\beta\ \mu\text{-partition of } (X, \widetilde{\sigma}(\alpha), \mu) \\ H_\mu(\beta)<\infty}} h(X, \widetilde{\sigma}(\alpha), \mu, T, \beta)$, but for every countable μ-partition $\beta \subseteq \widetilde{\sigma}(\alpha)$ we obviously have $\beta \preceq \alpha$, hence by (1), we obtain $h(X, \widetilde{\sigma}(\alpha), \mu, T, \beta) \leq h(X, \widetilde{\sigma}(\alpha), \mu, T, \alpha)$, for every countable μ-partition β of $(X, \widetilde{\sigma}(\alpha), \mu)$, and therefore
$$h(X, \widetilde{\sigma}(\alpha), \mu, T) = \sup_{\substack{\beta\ \mu\text{-partition of } (X, \widetilde{\sigma}(\alpha), \mu) \\ H_\mu(\beta)<\infty}} h(X, \widetilde{\sigma}(\alpha), \mu, T, \beta) \leq$$
$$\leq h(X, \widetilde{\sigma}(\alpha), \mu, T, \alpha).$$

[11] Recall, that the condition that α is equal to a countable μ-partition, whose elements are almost those of α, can in fact be reduces to α having countably many elements, which already build a μ-partition of X.

Thus, if $H_\mu(\alpha) < \infty$, we immediately get

$$h(X, \widetilde{\sigma}(\alpha), \mu, T) = h(X, \widetilde{\sigma}(\alpha), \mu, T, \alpha).$$

3. Using (2), we see that

$$h(X, \mathfrak{B}_1, \mu, T) = \sup_{\beta \subseteq \mathfrak{B}_1,\, H_\mu(\beta) < \infty} h(X, \mathfrak{B}_1, \mu, T, \beta) =$$

$$= \sup_{\beta \subseteq \mathfrak{B}_1,\, H_\mu(\beta) < \infty} h(X, \mathfrak{B}, \mu, T, \beta) \leq \sup_{\beta \subseteq \mathfrak{B}_2,\, H_\mu(\beta) < \infty} h(X, \mathfrak{B}, \mu, T, \beta) =$$

$$= h(X, \mathfrak{B}_1, \mu, T),$$

since, trivially, every $\beta_1 \subseteq \mathfrak{B}_1$ is also a subset of $\mathfrak{B}_2$.
Now, since $(h(X, \mathfrak{B}_n, \mu, T))_{n \in \mathbb{N}}$ is a monotonously increasing sequence of positive extended real numbers, which is bounded from above by $h(X, \mathfrak{B}, \mu, T)$, the sequence converges (or is equal to infinity but in this case also $h(X, \mathfrak{B}, \mu, T) = \infty$).

□

One of the main reasons, why entropy is of such big interest is, that it is an invariant for isomorphic systems. This will be stated in the next proposition, together with other useful properties.

Proposition 3.3. *Let $(X, \mathfrak{B}, \mu, T)$ and $(Y, \mathfrak{D}, \eta, S)$ be two σ-finite measure-preserving dynamical systems.*

1. *If $(X, \mathfrak{B})$ is a standard measurable space and $(X, \mathfrak{B}, \mu, T)$ and $(Y, \mathfrak{D}, \eta, S)$ are isomorphic, then every measurable μ-partition γ of X, with $\widetilde{\sigma}(\gamma)$ being σ-finite, satisfies*
$$H_\mu(\gamma | T^{-1}\gamma) = H_\eta(f(\gamma) | S^{-1} f(\gamma)),$$
where $f : X \longrightarrow Y$ denotes an isomorphism between $(X, \mathfrak{B}, \mu, T)$ and $(Y, \mathfrak{D}, \eta, S)$.

2. *Let μ and η be probability measures. If $(X, \mathfrak{B}, \mu, T)$ is a factor of $(Y, \mathfrak{D}, \eta, S)$, then*
$$h(X, \mathfrak{B}, \mu, T) \leq h(Y, \mathfrak{D}, \eta, S).$$
If the two systems are isomorphic, then
$$h(X, \mathfrak{B}, \mu, T) = h(Y, \mathfrak{D}, \eta, S).$$

3. *If $\mu(X) = 1$ and β is a measurable μ-partition of X such that $\beta, T^{-1}\beta, \ldots T^{-n}\beta$ are independent, then $\frac{1}{n} H_\mu(\bigvee_{k=0}^{n-1} T^{-k}\beta) = H_\mu(\beta)$, in particular for $H_\mu(\beta) < \infty$, we get*
$$h(X, \mathfrak{B}, \mu, T, \beta) = H_\mu(\beta).$$

4. *If β is a measurable μ-partition such that $\widetilde{\sigma}(\bigvee_{k=0}^{\infty} T^{-k}\beta) = \mathfrak{B}$ modulo μ with $H_\mu(\beta) < \infty$, and μ is a probability measure, then*
$$h(X, \mathfrak{B}, \mu, T) = h(X, \mathfrak{B}, \mu, T, \beta).$$

We will only prove the first statement, the other facts are well-know and simple to prove (see e.g. [4]).

Proof. With the notation as above, due to Proposition 1.3 (6) we get, by using the transformation formula,

$$H_\mu(\gamma|T^{-1}\gamma) = \int_X H_{\pi_{T^{-1}\gamma}(x)}(\gamma \cap \pi_{T^{-1}\gamma}(x))\, d\mu(x) =$$

$$= \int_Y H_{\mu_{\pi_{T^{-1}\gamma}(f^{-1}(y))}}(\gamma \cap \pi_{T^{-1}\gamma}(f^{-1}(y)))\, d\eta(y) =$$

$$= \int_Y H_{\eta_{\pi_{f(T^{-1}\gamma)}(y)}}(\gamma \cap \pi_{f(T^{-1}\gamma)}(y))\, d\eta(y) =$$

$$= \int_Y H_{\eta_{\pi_{S^{-1}f(\gamma)}(y)}}(\gamma \cap \pi_{S^{-1}f(\gamma)}(y))\, d\eta(y) = H_\eta(f(\gamma)|S^{-1}f(\gamma)),$$

if $\gamma \cap \pi_{T^{-1}\gamma}(x)$ is equal to a countable partition of $\pi_{T^{-1}\gamma}(x)$ modulo μ (according to Proposition 1.3 (4). If this is not the case, then $f(\gamma \cap \pi_{T^{-1}\gamma}(x)$ is not equal to a countable partition mod μ, neither, so we obtain $H_\mu(\gamma|T^{-1}\gamma) = \infty = H_\eta(f(\gamma)|S^{-1}f(\gamma))$. □

3.3 Tools for infinite measure systems

Let $(X, \mathfrak{B}, \mu, T)$ be a σ-finite measure-preserving, conservative dynamical system.
We define a special kind of μ-partitions of X (also considered in [10] and [15]), which allows us to reduce the static entropy of an infinite space to a finite one:

Definition. A μ-partition α of X is called *local* with *core* $A \in \mathfrak{B}$, $0 < \mu(A) < \infty$, if $A^c \in \alpha$ and $H_\mu(\alpha) < \infty$.

Note that $H_\mu(\alpha) < \infty$ implies that α is mod μ equal to a countable μ-partition, hence by Remark 3.3 we may w.l.o.g. assume that α is countable, when studying properties of the entropy of α. The advantage of this definition is that, if $\mu(A^c) = \infty$, then $H_\mu(\alpha) = \sum_{B\in\alpha,\mu(B)<\infty} -\mu(B)\log(\mu(B)) = H_{\mu|_A}(\alpha \setminus \{A^c\})$, i.e. we are actually left with treating the entropy of the finite-measure space $(A, \mathfrak{B} \cap A, \mu|_A)$.
Be aware, that if $\mu(X) = \infty$ and α is a local μ-partition with core A, then $\widetilde{\sigma}(\alpha)$ is not σ-finite unless we restrict it to A.
Local μ-partitions always exist in σ-finite measure spaces (not empty modulo measure), because, as long as there is an $A \in \mathfrak{B}$ with positive finite measure, we can construct a local μ-partition with core A: As a trivial example, just take $\alpha := \{A, A^c\}$, then $H_\mu(\alpha) = \mu(A)\log(\mu(A)) < \infty$, if $\mu(X) = \infty$ and, clearly, $H_\mu(\alpha) = \mu(A)\log(\mu(A)) + \mu(A^c)\log(\mu(A^c)) < \infty$, if $\mu(X) < \infty$.

Of course, the (abstract) concept of a local μ-partition is primarily of interest in infinite measure systems. Therefore, when dealing with local μ-partitions, we will mostly restrict our considerations to infinite measure systems.

By conservativity every measurable set A of positive measure is recurrent, and so we can partition A by the time steps which are needed to take the elements of A again back to A the original set (considered also in [10] and [15]):

Definition. For $A \in \mathfrak{B}$ with $0 < \mu(A) < \infty$ the *first return time μ-partition of* A is defined as

$$\rho_{T,A} := \left\{ A \cap T^{-n}A \cap \bigcap_{j=1}^{n-1} T^{-j}A^c : n \in \mathbb{N} \right\} = \{\{\varphi_{T,A} = n\} : n \in \mathbb{N}\},$$

where we set $\bigcap_{j=1}^{0} := X$. (By $\varphi_{T,A}$ we denote the first return map w.r.t. T, confer section 1.3.2.)

If T is an automorphism, we obtain also the *last visit time μ-partition of A*

$$\rho_{T^{-1},A} := \Big\{ A \cap T^n A \cap \bigcap_{j=1}^{n-1} T^j A^c : n \in \mathbb{N} \Big\}.$$

Note that these μ-partitions are countable.

The following definition of [10] (similar in [15] and [14]) will be a crucial condition for the equality of the different notions of entropy.

Definition. Let $(X, \mathfrak{B}, \mu, T)$ be as in the definition above. A set $A \in \mathfrak{B}$ with $\mu(A) \in (0, \infty)$ is called *quasi finite* if

$$H_{\mu|_A}(\rho_{T,A}) < \infty.$$

And the whole system $(X, \mathfrak{B}, \mu, T)$ is called *quasi finite* if it contains a quasi finite sweep-out set of positive finite measure. Further, we say that a local μ-partition α of X with core $A \in \mathfrak{B}$ is *quasi finite*, if A is a quasi finite set and $\rho_{T,A} \preceq \alpha \setminus \{A^c\}$.
If T is an automorphism, we call the local μ-partition α with core A *inverse quasi finite*, if A is a quasi finite set and $\rho_{T^{-1},A} \preceq \alpha \setminus \{A^c\}$.

Remark 3.7. Note, that if T is an automorphism, then

$$H_{\mu|_A}(\rho_{T,A}) < \infty \;\Leftrightarrow\; H_{\mu|_A}(\rho_{T^{-1},A}) < \infty,$$

since $T^n(\{\varphi_{T,A} = n\}) = T^n(A \cap T^{-n}A \cap T^{-1}A^c \cap \ldots \cap T^{-n+1}A^c = T^n A \cap A \cap T^{n-1}A^c \cap T^1 A^c = \{\varphi_{T^{-1},A} = n\}$, hence by using T-invariance we see that $\mu(\{\varphi_{T,A} = n\}) = \mu(T^{-n}(T^n\{\varphi_{T,A} = n\})) = \mu(T^n\{\varphi_{T,A} = n\}) = \mu(\{\varphi_{T^{-1},A} = n\})$.

Similar to the case of probability measures, we consider the quantity

$$\hat{h}(X, \mathfrak{B}, \mu, T, \alpha) := \liminf_{n\to\infty} \frac{1}{n} H_\mu\Big(\bigvee_{k=0}^{n-1} T^k \alpha\Big),$$

following [10].
Note that if α is countable and T is a measure-preserving automorphism, then

$$H_\mu\Big(\bigvee_{k=0}^{n-1} T^{-k}\alpha\Big) = H_\mu\Big(\bigvee_{k=0}^{n-1} T^k \alpha\Big)$$

for every $n \in \mathbb{N}$, because

$$H_\mu\Big(\bigvee_{k=0}^{n-1} T^{-k}\alpha\Big) =$$

$$= - \sum_{\substack{A_k \in \alpha, k=0,\ldots,n-1 \\ \mu(A_0 \cap \ldots \cap T^{-n+1}A_{n-1}) \neq \infty}} \mu(A_0 \cap \ldots \cap T^{-n+1}A_{n-1}) \log(\mu(A_0 \cap \ldots \cap T^{-n+1}A_{n-1})) =$$

$$= - \sum_{\substack{A_k \in \alpha, k=0,\ldots,n-1 \\ (\mu \circ T^{n-1})(A_0 \cap \ldots \cap T^{-n+1}A_{n-1}) \neq \infty}} (\mu \circ T^{n-1})(A_0 \cap \ldots \cap T^{-n+1}A_{n-1}) \cdot$$

$$\log((\mu \circ T^{n-1})(A_0 \cap \ldots \cap T^{-n+1}A_{n-1})) =$$

$$= - \sum_{\substack{A_k \in \alpha, k=0,\dots,n-1 \\ \mu(T^{n-1}A_0 \cap \ldots \cap A_{n-1}) \neq \infty}} \mu(T^{n-1}A_0 \cap \ldots \cap A_{n-1}) \log(\mu(T^{n-1}A_0 \cap \ldots \cap A_{n-1})) =$$

$$= H_\mu\Big(\bigvee_{k=0}^{n-1} T^k \alpha\Big).$$

If we additionally require that μ is a probability measure, then $\hat{h}(X, \mathfrak{B}, \mu, T, \alpha)$ is nothing else than the dynamical entropy $h(X, \mathfrak{B}, \mu, T, \alpha)$, since in that case

$$\hat{h}(X, \mathfrak{B}, \mu, T, \alpha) = \liminf_{n\to\infty} \frac{1}{n} H_\mu\Big(\bigvee_{k=0}^{n-1} T^k \alpha\Big) = \liminf_{n\to\infty} \frac{1}{n} H_\mu\Big(\bigvee_{k=0}^{n-1} T^{-k} \alpha\Big) =$$

$$= \lim_{n\to\infty} \frac{1}{n} H_\mu\Big(\bigvee_{k=0}^{n-1} T^{-k} \alpha\Big) = h(X, \mathfrak{B}, \mu, T, \alpha),$$

because by Lemma 3.2 we know that the series converges for $\mu \leq 1$.

We can deduce the same identity for $\hat{h}(X, \mathfrak{B}, \mu, T, \alpha)$ as for $h(X, \mathfrak{B}, \mu, T, \alpha)$ in Remark 3.6 in the following case:

Lemma 3.3. *Let $(X, \mathfrak{B}, \mu, T)$ be a σ-finite dynamical system with $\mu(X) = \infty$ such that T is a measure-preserving automorphism and let α be a quasi finite local μ-partition of X, whose core $A \in \mathfrak{B}$ is a sweep-out set w.r.t. the inverse of T, i.e. $X = \bigcup_{n\in\mathbb{N}_0} T^n A$ mod μ, then*

$$\hat{h}(X, \mathfrak{B}, \mu, T, \alpha) = H_\mu\Big(\alpha | \bigvee_{k=1}^{\infty} T^k \alpha\Big) = H_\mu\Big(\bigvee_{k=0}^{\infty} T^k \alpha | \bigvee_{k=1}^{\infty} T^k \alpha\Big).$$

Moreover,

$$H_\mu\Big(\alpha | \bigvee_{k=1}^{\infty} T^k \alpha\Big) = H_{\mu|_A}\Big(\alpha \cap A | \Big(\bigvee_{k=1}^{\infty} T^k \alpha\Big) \cap A\Big).$$

If we assume that α is an inverse quasi finite local μ-partition whose core is a sweep-out set (w.r.t. T), then we get

$$\hat{h}(X, \mathfrak{B}, \mu, T, \alpha) = \hat{h}(X, \mathfrak{B}, \mu, T^{-1}, \alpha) = H_\mu\Big(\bigvee_{k=0}^{\infty} T^{-k} \alpha | \bigvee_{k=1}^{\infty} T^{-k} \alpha\Big).$$

The proof follows [10].

Proof. By Proposition 3.1 (8) we know that

$$I_\mu\Big(\bigvee_{k=0}^{n-1} T^k \alpha\Big) = \sum_{m=0}^{n-1} I_\mu\Big(\alpha | \bigvee_{k=1}^{m} T^k \alpha\Big) \circ T^{m-(n-1)},$$

where $\bigvee_{k=1}^{0} T^k \alpha := \{X, \emptyset\}$. Hence,

$$\hat{h}(X, \mathfrak{B}, \mu, T, \alpha) = \liminf_{n\to\infty} \frac{1}{n} \int_X I_\mu\Big(\bigvee_{k=0}^{n-1} T^k \alpha\Big)(x)\, d\mu(x) =$$

$$= \liminf_{n\to\infty} \frac{1}{n} \Big(\sum_{m=0}^{n-1} \int_X I_\mu \Big(\alpha | \bigvee_{k=1}^{m} T^k \alpha \Big) (T^{m-(n-1)} x) \, d\mu(x) \Big) =$$

$$= \liminf_{n\to\infty} \frac{1}{n} \Big(\sum_{m=0}^{n-1} \int_X I_\mu \Big(\alpha | \bigvee_{k=1}^{m} T^k \alpha \Big) (x) \, d\mu(x) \Big),$$

by the transformation formula and T-invariance.
We will show that

$$\int_X I_\mu \Big(\alpha | \bigvee_{k=1}^{n-1} T^k \alpha \Big)(x) \, d\mu(x) \to \int_X I_\mu \Big(\alpha | \bigvee_{k=1}^{\infty} T^k \alpha \Big)(x) \, d\mu(x) \text{ for } n \to \infty,$$

which implies that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} \int_X I_\mu \Big(\alpha | \bigvee_{k=1}^{m} T^k \alpha \Big)(x) \, d\mu(x) = \int_X I_\mu \Big(\alpha | \bigvee_{k=1}^{\infty} T^k \alpha \Big)(x) \, d\mu(x),$$

and this is precisely the desired equality.
We will compute that

$$I_\mu \Big(\alpha | \bigvee_{k=1}^{n} T^k \alpha \Big)(x) = 0 \;\text{ for every } x \in A^c \text{ with } \mu \Big(\alpha_x \cap \Big(\bigvee_{k=1}^{n} T^k \alpha \Big)_x \Big) \neq 0.$$

Clearly, elements for which $\mu\big(\alpha_x \cap \big(\bigvee_{k=1}^n T^k\alpha\big)_x\big) = 0$ vanish in the entropy (since α and $\bigvee_{k=1}^n T^k\alpha$ are countable, so the definition of conditional entropy uses just set-theoretical conditional measures).
Let $x \in A^c$ be fixed such that $\big(\bigvee_{k=1}^n T^k\alpha\big)_x \neq \emptyset \bmod \mu$, and let us write $\big(\bigvee_{k=1}^n T^k\alpha\big)_x = T^1 A_1 \cap \ldots \cap T^n A_n$ for $A_1, \ldots, A_n \in \alpha$. Since this set is not a null-set, it is a subset (mod μ) of $\bigcup_{j\in\mathbb{N}_0} T^j A$, because A is a sweep out set w.r.t. T^{-1}. Hence, there has to be an $i \in \mathbb{N}$ with $(T^1 A_1 \cap \ldots \cap T^n A_n) \cap T^i A \neq \emptyset \bmod \mu$. (Note, that we can assume that $i \geq 1$, since $T^1 A_1 \cap \ldots \cap T^n A_n$ cannot be covered by A, since by assumption, $A^c \cap T^1 A_1 \cap \ldots \cap T^n A_n \neq \emptyset \bmod \mu$.) If n is large enough, then $i \leq n$, thus $A_i \cap A \neq \emptyset \bmod \mu$, i.e. $A_i \subseteq A$, because α is a local μ-partition with core A. In particular, for n large enough,

$$\mu(T^1 A_1 \cap \ldots \cap T^n A_n) \leq \mu(T^i A_i) = \mu(A_i) \leq \mu(A) < \infty,$$

thus by definition

$$I_\mu \Big(\alpha | \bigvee_{k=1}^{n} T^k \alpha \Big)(x) = I_{\mu(\cdot | (T^1 A_1 \cap \ldots T^n A_n))}(\alpha)(x).$$

Moreover, by assumption $\rho_{T,A} \preceq \alpha|_A$, hence if $A_i \cap T^{-i} A^c \neq \emptyset \bmod \mu$, then all elements of A_i have to lie in A^c after i (time) steps of T, i.e. $A_i \subseteq T^{-i} A^c \bmod \mu$. In particular, if $\mu(A^c \cap (T^1 A_1 \cap \ldots T^n A_n)) \neq 0$, then $A_i \subseteq T^{-i} A^c \bmod \mu$, which is equivalent to $T^i A_i \subseteq A^c$. Hence, in that case we obtain

$$A^c \cap (T^1 A_1 \cap \ldots \cap T^i A_i \cap \ldots \cap T^n A_n) = T^1 A_1 \cap \ldots \cap T^i A_i \cap \ldots \cap T^n A_n$$

modulo μ, which implies

$$\mu(A^c | T^1 A_1 \cap \ldots \cap T^n A_n) = \frac{\mu(A^c \cap (T^1 A_1 \cap \ldots \cap T^n A_n))}{\mu(T^1 A_1 \cap \ldots \cap T^n A_n)} = 1,$$

and therefore

$$I_\mu\Big(\alpha|\bigvee_{k=1}^{n}T^k\alpha\Big)(x)=I_{\mu(\cdot|(T^1A_1\cap\ldots T^nA_n))}(\alpha)(x)=$$

$$=-\log(\mu(A^c\cap(T^1A_1\cap\ldots\cap T^nA_n)|T^1A_1\cap\ldots\cap T^nA_n))=0.$$

Analogously, we deduce that

$$I_\mu\Big(\alpha|\bigvee_{k=1}^{\infty}T^k\alpha\Big)(x)=0$$

for $x\in A^c$ such that $\big(\alpha\vee\bigvee_{k=1}^{\infty}T^k\alpha\big)_x\neq\emptyset \mod \mu$. (Note, that we use the definition of conditional entropy given in Remark 3.2 here, since $\bigvee_{k=1}^{\infty}T^k\alpha$ could be uncountable modulo μ.)
Now, the case $\mu(A^c\cap(T^1A_1\cap\ldots\cap T^nA_n))=0$ (and in particular the case $\mu(T^1A_1\cap\ldots\cap T^nA_n)=0$) vanishes in the definition of the static conditional entropy. Thus, all in all we have shown that

$$\int_{A^c}I_\mu\Big(\alpha|\bigvee_{k=1}^{n}T^k\alpha\Big)(x)\,d\mu(x)=0$$

for n sufficiently large. Analogously, $\int_{A^c}I_\mu\Big(\alpha|\bigvee_{k=1}^{\infty}T^k\alpha\Big)(x)\,d\mu(x)=0$. Therefore,

$$H_\mu\Big(\alpha|\bigvee_{k=1}^{n}T^k\alpha\Big)=\int_X I_\mu\Big(\alpha|\bigvee_{k=1}^{n}T^k\alpha\Big)(x)\,d\mu(x)=\int_A I_\mu\Big(\alpha|\bigvee_{k=1}^{n}T^k\alpha\Big)(x)\,d\mu(x)=$$

$$=H_{\mu|_A}\Big(\alpha\cap A|\Big(\bigvee_{k=1}^{n}T^k\alpha\Big)\cap A\Big),$$

and

$$H_\mu\Big(\alpha|\bigvee_{k=1}^{\infty}T^k\alpha\Big)=\int_X I_\mu\Big(\alpha|\bigvee_{k=1}^{\infty}T^k\alpha\Big)(x)\,d\mu(x)=\int_A I_\mu\Big(\alpha|\bigvee_{k=1}^{\infty}T^k\alpha\Big)(x)\,d\mu(x)=$$

$$=H_{\mu|_A}\Big(\alpha\cap A|\Big(\bigvee_{k=1}^{\infty}T^k\alpha\Big)\cap A\Big).$$

Now, by $\mu(A)<\infty$ we can use Lemma 3.1 on $(A,\mathfrak{B}\cap A,\frac{\mu}{\mu(A)})$, because

$$H_{\frac{\mu}{\mu(A)}}(\alpha|_A)=-\sum_{D\in\alpha|_A}\frac{\mu(D)}{\mu(A)}\log\Big(\frac{\mu(D)}{\mu(A)}\Big)=$$

$$=\frac{1}{\mu(A)}H_\mu(\alpha|_A)+\sum_{D\in\alpha|_A}\frac{\mu(D)}{\mu(A)}\log(\mu(A))=\frac{1}{\mu(A)}H_\mu(\alpha)+\log(\mu(A))<\infty,$$

since α is a local μ-partition. So, Lemma 3.1 gives

$$\lim_{n\to\infty}H_{\frac{\mu}{\mu(A)}}\Big(A\cap\alpha|\Big(A\cap\bigvee_{k=1}^{n}T^k\alpha\Big)\Big)=H_{\frac{\mu}{\mu(A)}}\Big(A\cap\alpha|\Big(A\cap\bigvee_{k=1}^{\infty}T^k\alpha\Big)\Big).$$

Further,

$$\int_A I_{\frac{\mu}{\mu(A)}}\Big(\alpha|\Big(A\cap\bigvee_{k=1}^{n}T^k\alpha\Big)\Big)(x)\,d\frac{\mu}{\mu(A)}(x)=$$

$$=\sum_{\substack{D\in\alpha|_A\\ A_k\in\alpha,\ k=1,\dots,n}}-\frac{\mu(D\cap T^1A_1\cap\ldots\cap T^nA_n)}{\mu(A)}\log\Big(\frac{\frac{\mu(D\cap T^1A_1\cap\ldots\cap T^nA_n)}{\mu(A)}}{\frac{\mu(T^1A_1\cap\ldots\cap T^nA_n)}{\mu(A)}}\Big)=$$

$$=\frac{1}{\mu(A)}\int_A I_\mu\Big(\alpha|\bigvee_{k=1}^{n}T^k\alpha\Big)(x)\,d\mu(x)$$

Analogous for the infinite refinement:

$$\int_A I_{\frac{\mu}{\mu(A)}}\Big(\alpha|\Big(A\cap\bigvee_{k=1}^{\infty}T^k\alpha\Big)\Big)\,d\frac{\mu}{\mu(A)}=\frac{1}{\mu(A)}\int_A I_\mu\Big(\alpha|\bigvee_{k=1}^{\infty}T^k\alpha\Big)\,d\mu.$$

Putting those things together, we obtain

$$\lim_{n\to\infty}\int_X I_\mu\Big(\alpha|\bigvee_{k=1}^{n}T^k\alpha\Big)(x)\,d\mu(x)=\lim_{n\to\infty}\int_A I_\mu\Big(\alpha|\bigvee_{k=1}^{n}T^k\alpha\Big)(x)\,d\mu(x)=$$

$$=\lim_{n\to\infty}\mu(A)\int_A I_{\frac{\mu}{\mu(A)}}\Big(\alpha|\Big(A\cap\bigvee_{k=1}^{n}T^k\alpha\Big)\Big)(x)\,d\frac{\mu}{\mu(A)}(x)=$$

$$=\mu(A)\int_A I_{\frac{\mu}{\mu(A)}}\Big(\alpha|\Big(A\cap\bigvee_{k=1}^{\infty}T^k\alpha\Big)\Big)(x)\,d\frac{\mu}{\mu(A)}(x)=\int_A I_\mu\Big(\alpha|\bigvee_{k=1}^{\infty}T^k\alpha\Big)(x)\,d\mu(x)=$$

$$=\int_X I_\mu\Big(\alpha|\bigvee_{k=1}^{\infty}T^k\alpha\Big)(x)\,d\mu(x).$$

Replacing T by T^{-1} we obtain the claim for inverse quasi finite local μ-partitions.

□

4 Concepts of entropy for infinite measure systems

From now on let us assume that $(X,\mathfrak{B},\mu)$ is a σ-finite standard measure space with a measure-preserving, conservative and ergodic transformation T.

There are different concepts of entropy of a dynamical system with infinite measure. We will now introduce Krengel's, Parry's and Poisson entropy. Let us start with the most recent one, introduced 2005 by Emmanuel Roy:

4.1 Poisson entropy

In section 2 we have construed a probability space from an infinite measure space. For this probability system we can use the Kolmogorov Sinai entropy and obtain the following

Definition. Let $(X^\star, \mathfrak{B}^\star, \mu^\star, T^\star)$ be the Poisson suspension of $(X, \mathfrak{B}, \mu, T)$ with $\mu(X) = \infty$. Then the entropy of $(X^\star, \mathfrak{B}^\star, \mu^\star, T^\star)$ is called the *Poisson entropy of* $(X, \mathfrak{B}, \mu, T)$, i.e.

$$h_{Poi}(X, \mathfrak{B}, \mu, T) := h(X^\star, \mathfrak{B}^\star, \mu^\star, T^\star).$$

Notation. Corresponding to a given countable μ-partition α of X one can construct a $\mu^\star$-partition $\alpha^\star$ of $X^\star$, by using the evaluation map:

$$\alpha^\star := \Big\{ \bigcap_{A \in \alpha} N_A^{-1}(\{n_A\}) : (n_A)_{A\in\alpha} \in \overline{\mathbb{N}}_0^{|\alpha|} \Big\}.$$

The static entropy of the $\mu^\star$-partition $\alpha^\star$ as above is strongly related to the entropy of the underlying system in the following sense:

Lemma 4.1. *Let $(X, \mathfrak{B}, \mu)$ be a σ-finite measure space with $\mu(X) = \infty$ and α a local μ-partition of X, then $\alpha^\star$ is equal to a countable $\mu^\star$-partition modulo $\mu^\star$, and*

$$H_{\mu^\star}(\alpha^\star) = \sum_{A\in\alpha} H(Poi_{\mu(A)}).$$

More precisely,

$$H_{\mu^\star}(\alpha^\star) = H_\mu(\alpha) + \sum_{i=1}^{\infty} \mu(A_i) + \sum_{i=1}^{\infty}\sum_{n=0}^{\infty} \frac{\mu(A_i)^n e^{-\mu(A_i)} \log(n!)}{n!},$$

where $A_1, A_2, \dots$ shall denote all elements of α beside the complement of the core of α. In particular, $H_{\mu^\star}(\alpha^\star) < \infty$.

Proof. By the assumption of being local, we know that α is mod μ equal to a countable μ-partition. So w.l.o.g. let α be countable. Let A denote the core of α, i.e. A is a positive finite measure set with $A^c \in \alpha$, $\mu(A^c) = \infty$, and let A_i, $i = 1, 2, \dots$, denote those elements of α which are subsets of A. Now, $\mu^\star$-a.e. element of $\alpha^\star$ is of the form

$$\bigcap_{i=1}^{\infty} N_{A_i}^{-1}(\{n_i\}) \cap N_{A^c}^{-1}(\{\infty\}), \text{ for } n_1, n_2, \dots \text{ in } \mathbb{N}_0 \text{ such that } \sum_{i=1}^{\infty} n_i < \infty,$$

since all elements in $\alpha^\star$ different from the above form are contained in the $\mu^\star$-null set $N_A^{-1}(\{\infty\}) \cup \bigcup_{n\in\mathbb{N}_0} N_{A^c}^{-1}(\{n\})$. Now, the condition, that $\sum_{i=1}^{\infty} n_i < \infty$ actually means that only finitely many elements among $n_1, n_2, \dots$ can be non-zero. And the set of all countably infinite sequences of natural numbers, with at most finitely many non-zero entries, is countable:

$$|\{(n_i)_{i=1}^{\infty} \in \mathbb{N}_0^{\mathbb{N}} : \sum_{i=1}^{\infty} n_i < \infty\}| = |\bigcup_{k=0}^{\infty} \bigcup_{J_k \subset \mathbb{N}, |J_k| = k} \{(n_i)_{i=1}^{\infty} \in \mathbb{N}_0^{\mathbb{N}} : n_l = 0, \forall l \notin J_k\}| =$$

$$= |\bigcup_{k=0}^{\infty} \{J_k \subset \mathbb{N}, |J_k| = k\}| = |\bigcup_{k=0}^{\infty} \mathbb{N}^k|.$$

Hence $\alpha^\star$ is modulo $\mu^\star$ equal to the countable $\mu^\star$-partition $\{\bigcap_{i=1}^{\infty} N_{A_i}^{-1}(\{n_i\}) \cap N_{A^c}^{-1}(\{\infty\}) : (n_i)_{i=1}^{\infty} \in \mathbb{N}_0^{\mathbb{N}}$ such that $\sum_{i=1}^{\infty} n_i < \infty\}$.

Now, we can compute the entropy of $\alpha^\star$ as follows: Let us set $A_0 := A^c$. Using independence of $N_{A_0}, N_{A_1}, \ldots$ w.r.t. $\mu^\star$, we get

$$H_{\mu^\star}(\alpha^\star) = \sum_{(n_i)_{i=0}^\infty \in \overline{\mathbb{N}}_0^{\mathbb{N}_0}} \mu^\star\Big(\bigcap_{k=0}^\infty N_{A_k}^{-1}(\{n_k\})\Big)\Big(-\log\Big(\mu^\star\Big(\bigcap_{j=0}^\infty N_{A_j}^{-1}(\{n_j\})\Big)\Big)\Big) =$$

$$= \sum_{(n_i)_{i=0}^\infty \in \overline{\mathbb{N}}_0^{\mathbb{N}_0}} \prod_{k=0}^\infty \mu^\star(N_{A_k}^{-1}(\{n_k\}))\Big(\sum_{j=0}^\infty(-\log(\mu^\star(N_{A_j}^{-1}(\{n_j\}))))\Big) =$$

$$= \sum_{j=0}^\infty \sum_{n_j\in\overline{\mathbb{N}}_0}(-\log(\mu^\star(N_{A_j}^{-1}(\{n_j\}))))\mu^\star(N_{A_j}^{-1}(\{n_j\}))\cdot$$

$$\sum_{(n_i)_{i\in\mathbb{N}_0\setminus\{j\}}\in\overline{\mathbb{N}}_0^{\mathbb{N}}} \prod_{k\in\mathbb{N}_0\setminus\{j\}} \mu^\star(N_{A_k}^{-1}(\{n_k\})),$$

where we were allowed to reorder the summations, since we clearly have that $\mu^\star(N_{A_k}^{-1}(\{n_k\}))(-\log(\mu^\star(N_{A_j}^{-1}(\{n_j\})))) \geq 0$, hence if the sum converges, then it converges absolutely. Now, $\sum_{(n_i)_{i\in\mathbb{N}_0\setminus\{j\}}\in\overline{\mathbb{N}}_0^{\mathbb{N}}} \prod_{k\in\mathbb{N}_0\setminus\{j\}} \mu^\star(N_{A_k}^{-1}(\{n_k\})) = 1$, because $\sum_{n_i\in\overline{\mathbb{N}}_0}\mu^\star(N_{A_i}^{-1}(\{n_i\})) = 1$ for all $i\in\mathbb{N}_0\setminus\{j\}$, thus

$$H_{\mu^\star}(\alpha^\star) = \sum_{j=0}^\infty \sum_{n_j\in\overline{\mathbb{N}}_0}(-\log(\mu^\star(N_{A_j}^{-1}(\{n_j\}))))\mu^\star(N_{A_j}^{-1}(\{n_j\})) =$$

$$= \sum_{n_0\in\overline{\mathbb{N}}_0}\mu^\star(N_{A_0}^{-1}(\{n_0\}))(-\log(\mu^\star(N_{A_0}^{-1}(\{n_0\}))))+$$

$$\sum_{n_1\in\overline{\mathbb{N}}_0}\mu^\star(N_{A_1}^{-1}(\{n_1\}))(-\log(\mu^\star(N_{A_1}^{-1}(\{n_1\}))))+\ldots=$$

$$= \sum_{B\in\alpha}\sum_{n\in\overline{\mathbb{N}}_0} Poi_{\mu(B)}(n)(-\log(Poi_{\mu(B)}(n))) = \sum_{B\in\alpha} H(Poi_{\mu(B)}).$$

More precisely, since $\sum_{n\in\overline{\mathbb{N}}_0} -Poi_\infty(n)\log(Poi_\infty(n)) = 0$, we have

$$H_{\mu^\star}(\alpha^\star) = \sum_{B\in\alpha,\mu(B)\neq\infty} H(Poi_{\mu(B)}) = \sum_{B\in\alpha\setminus\{A^c\}} H(Poi_{\mu(B)}).$$

Further, by Example 3.2, we get

$$H_{\mu^\star}(\alpha^\star) = \sum_{A\in\alpha,\mu(A)\neq\infty}\Big(-\mu(A)\log(\mu(A))+\mu(A)+\sum_{n=0}^\infty\frac{\mu(A)^n e^{-\mu(A)}\log(n!)}{n!}\Big) =$$

$$= H_\mu(\alpha)+\sum_{i=1}^\infty \mu(A_i)+\sum_{i=1}^\infty\sum_{n=0}^\infty\frac{\mu(A_i)^n e^{-\mu(A_i)}\log(n!)}{n!}.$$

Clearly, $H_\mu(\alpha)<\infty$ by definition of a local μ-partition, and $\sum_{i=1}^\infty \mu(A_i) = \mu(A) < \infty$. Further, for $n\in\mathbb{N}$, $n\geq 2$, we have $\frac{\log(n!)}{n!} = \frac{\sum_{j=1}^n\log(j)}{n!} \leq \frac{(n-1)\log(n)}{n!} \leq \frac{(n-1)n}{n!} = \frac{1}{(n-2)!}$, and therefore

$$\sum_{i=1}^\infty\sum_{n=0}^\infty\frac{\mu(A_i)^n e^{-\mu(A_i)}\log(n!)}{n!} \leq \sum_{i=1}^\infty\sum_{n=2}^\infty\frac{\mu(A_i)^n e^{-\mu(A_i)}}{(n-2)!} =$$

$$= \sum_{i=1}^{\infty} \mu(A_i)^2 \sum_{n=0}^{\infty} \frac{\mu(A_i)^n}{n!} e^{-\mu(A_i)} = \sum_{i=1}^{\infty} \mu(A_i)^2 \leq \Big(\sum_{i=1}^{\infty} \mu(A_i)\Big)^2 = (\mu(A))^2 < \infty.$$

Hence, all in all, we get $H_{\mu^\star}(\alpha^\star) < \infty$. □

Remark 4.1. For countable μ-partitions α we have

$$\widetilde{\sigma}(\alpha^\star) \supseteq (\widetilde{\sigma}(\alpha))^\star,$$

where, as before, $(\widetilde{\sigma}(\alpha))^\star = \sigma(N_E : E \in \widetilde{\sigma}(\alpha))$ and $\widetilde{\sigma}(\alpha^\star) = \{\bigcup_{M_l \in \alpha^\star, l \in L} M_l \in \mathfrak{B}^\star :$ L arbitrary index set$\}$. If α is a local μ-partition, then

$$\widetilde{\sigma}(\alpha^\star) = (\widetilde{\sigma}(\alpha))^\star \mod \mu^\star.$$

(Note that actually $\widetilde{\sigma}(\alpha) = \sigma(\alpha)$ since α is countable.)

Proof. To see that $(\sigma(\alpha))^\star \subseteq \widetilde{\sigma}(\alpha^\star)$, let $E = \bigcup_{l \in L} A_l \in \sigma(\alpha)$ for $A_l \in \alpha$ and $n \in \overline{\mathbb{N}}_0$ be arbitrary. Then

$$N_E^{-1}(\{n\}) = \bigcup_{\substack{m_l \in \overline{\mathbb{N}}_0, l \in L \\ \sum_{l \in L} m_l = n}} \bigcap_{l \in L} N_{A_l}^{-1}(\{m_l\}) =$$

$$= \bigcup_{\substack{m_l \in \overline{\mathbb{N}}_0, l \in L \cup L^c \\ \sum_{l \in L} m_l = n}} \underbrace{\bigcap_{l \in L} N_{A_l}^{-1}(\{m_l\}) \cap \bigcap_{l \in L^c} N_{A_l}^{-1}(\{m_l\})}_{\in \alpha^\star},$$

where L^c shall denote the index set of the remaining elements of α when taking away those with index in L. So, $N_E^{-1}(\{n\})$ is a union of elements of $\alpha^\star$ and $N_E^{-1}(\{n\}) \in \mathfrak{B}^\star$, thus $N_E^{-1}(\{n\}) \in \widetilde{\sigma}(\alpha^\star)$. Thus $\sigma(N_E : E \in \sigma(\alpha)) \subseteq \widetilde{\sigma}(\alpha^\star)$.

If α is a local μ-partition, then $\alpha^\star$ is equal to a countable $\mu^\star$-partition $\alpha^{\star\prime}$ of $X^\star$ mod $\mu^\star$, by Lemma 4.1. W.l.o.g. let $\alpha^{\star\prime} \subseteq \alpha^\star$. Thus, $\widetilde{\sigma}(\alpha^\star) = \widetilde{\sigma}(\alpha^{\star\prime}) = \sigma(\alpha^{\star\prime})$ modulo $\mu^\star$. Now, since α is countable we clearly have $\alpha^\star \subseteq \sigma(N_E : E \in \sigma(\alpha))$. Hence, $\sigma(\alpha^{\star\prime}) \subseteq \sigma(N_E : E \in \sigma(\alpha))$, for $\alpha^{\star\prime} \subseteq \alpha^\star$. Therefore,

$$\widetilde{\sigma}(\alpha^\star) = \sigma(\alpha^{\star\prime}) \subseteq \sigma(N_E : E \in \sigma(\alpha)) \bmod \mu^\star.$$

Thus, with the above, $\widetilde{\sigma}(\alpha^\star) = \sigma(N_E : E \in \sigma(\alpha)) \bmod \mu^\star$. □

The following Lemma of [10] (Lemma 4.2) provides equality of the conditional static entropy for the Poisson suspension and its underlying system.

Lemma 4.2. *Let $(X, \mathfrak{B}, \mu)$ be a complete standard measure space and μ σ-finite without atoms such that $\mu(X) = \infty$. Then, for σ-finite sub-σ-algebras $\mathfrak{A}$ and $\mathfrak{C}$ of $\mathfrak{B}$, $\mathfrak{C} \subseteq \mathfrak{A}$, both without atoms, we obtain*

$$H_{\mu^\star}(\mathfrak{A}^\star | \mathfrak{C}^\star) = H_\mu(\mathfrak{A} | \mathfrak{C}),$$

where $\mathfrak{A}^\star := \sigma(N_A : A \in \mathfrak{A})$, $\mathfrak{C}^\star := \sigma(N_C : C \in \mathfrak{C})$ and $\mu^\star$ is the probability measure of the Poisson suspension $(X^\star, \mathfrak{B}^\star)$ of $(X, \mathfrak{B}, \mu)$.

(Note, that $\mathfrak{A}$ having no atoms implies that all elements of the generating μ-partition are null sets and therefore the μ-partition cannot be equal to a countable μ-partition modulo μ, hence $H_\mu(\mathfrak{A}) = \infty$. And analogously, $H_\mu(\mathfrak{C}) = \infty$.)

We are following the proof of [10]:

Proof. W.l.o.g. we can assume, that $\mathfrak{A} = \mathfrak{B}$, since by Theorem 1.4 there is a σ-finite standard measure space $(Y, \mathfrak{A}', \nu)$ and a σ-finite sub-σ-algebra $\mathfrak{C}'$ of $\mathfrak{A}'$ such that by Remark 3.4, $H_\mu(\mathfrak{A}|\mathfrak{C}) = H_\nu(\mathfrak{A}'|\mathfrak{C}')$, i.e. it is sufficient to show the claim for the Borel-σ-algebra $\mathfrak{A}'$ on this standard space, but for simplicity we just use the notation of the original standard space $(X, \mathfrak{B}, \mu)$. Note that we do not lose generality by the assumption of completeness, since adding null-sets which are contained in null-sets already taken into account does not change the entropy[12]. So let us assume that $\mathfrak{C}$ is complete. Then by Theorem 1.11 there is a standard measurable space $(Y, \mathfrak{D})$ and a family of (measurable) measures $\{m_t\}_{t\in\mathbb{R}}$ on this space, such that there is an essential isomorphism

$$\theta = (\psi, f) : (X, \mathfrak{B}, \mu) \longrightarrow (\mathbb{R} \times Y, \mathfrak{B}(\mathbb{R}) \otimes \mathfrak{D}, \overline{\mu}) \text{ with } \overline{\mu}(E \times D) := \int_E m_t(D)\, d\lambda(t)$$

for $E \in \mathfrak{B}(\mathbb{R})$, $D \in \mathfrak{D}$ and

$$\theta^{-1}(\mathfrak{B}(\mathbb{R}) \times Y) = \psi^{-1}(\mathfrak{B}(\mathbb{R})) = \mathfrak{C}.$$

Clearly, $(\mathbb{R} \times Y, \mathfrak{B}(\mathbb{R}) \otimes \mathfrak{D}, \overline{\mu})$ is a standard measure space, so we can consider its Poisson suspension $((\mathbb{R} \times Y)^\star, (\mathfrak{B}(\mathbb{R}) \otimes \mathfrak{D})^\star, \overline{\mu}^\star)$. Due to Remark 2.2 $\overline{\mu}^\star$-a.e. element ν of $(\mathbb{R} \times Y)^\star$ can be written as

$$\nu = \sum_{i\in\mathbb{Z}} \delta_{(t_i(\nu), y_i(\nu))},$$

for $t_i(\nu) \in \overline{\mathbb{R}}$ and $y_i(\nu) \in Y$, moreover let $(t_i)_{i\in\mathbb{Z}}$ be ordered such that $t_i < t_{i+1}$ for all $i \in \mathbb{Z}$ (unless both are equal to $+\infty$ or $-\infty$) and $t_i \le 0$ for $i \le 0$ and $t_i > 0$ for $i \ge 1$. Note, that choosing an ordering guarantees uniqueness of the sequence of tuples $((t_i, y_i))_{i\in\mathbb{Z}}$ for fixed ν, confer Remark 2.2.
We will show that

$$((\mathbb{R} \times Y)^\star, (\mathfrak{B}(\mathbb{R}) \otimes \mathfrak{D})^\star, \overline{\mu}^\star) \approx (\mathbb{R}^\star \times Y^{\mathbb{Z}}, \mathfrak{B}(\mathbb{R})^\star \otimes \mathfrak{D}^{\otimes\mathbb{Z}}, P)$$

with

$$P(M \times N) := \int_M \bigotimes_{i\in\mathbb{Z}} m_{t_i(\eta)}(N)\, d\lambda^\star(\eta), \ \ \forall M \in \mathfrak{B}(\mathbb{R})^\star,\ N \in \mathfrak{D}^{\otimes\mathbb{Z}}.$$

Note that $m_{t_i(\eta)}$ is defined for $\lambda^\star$-a.e. $\eta \in \mathbb{R}^\star$, because $t_i(\eta) \neq \pm\infty$ for $\lambda^\star$-a.e. $\eta \in \mathbb{R}^\star$: Those $\eta \in \mathbb{R}^\star$, which are represented by a finite sum, form a null set: $\lambda^\star(\{\eta \in \mathbb{R}^\star : \eta = \sum_{i\in\mathbb{Z}} \delta_{t_i(\eta)} < \infty\}) = \sum_{n\in\mathbb{N}_0} \lambda^\star(N_{\mathbb{R}}^{-1}(\{n\})) = \sum_{n\in\mathbb{N}_0} Poi_\infty(n) = 0$ as $\lambda(\mathbb{R}) = \infty$. (Obviously, P is a probability, as m_{t_i} and $\lambda^\star$ are probabilities.)
To this end, let us consider the map

$$\Phi : \nu = \sum_{i\in\mathbb{Z}} \delta_{(t_i, y_i)} \mapsto (\sum_{i\in\mathbb{Z}} \delta_{t_i}, (y_i)_{i\in\mathbb{Z}})$$

[12] Let ζ be the measurable μ-partition which generates a sub-σ-algebra $\mathfrak{E}$ of a complete σ-finite standard measure space $(X, \mathfrak{B}, \mu)$ and let $\overline{\zeta}$ be the measurable μ-partition which generates the completion $\overline{\mathfrak{E}}$ of $\mathfrak{E}$. Hence $\overline{\zeta}$ is obtained by replacing all null sets in ζ by all one-element subsets of it, thus $\overline{\zeta} = \zeta \bmod \mu$, since for every $Z \in \zeta$ either $\mu(Z) \neq 0$ which implies $Z \in \overline{\zeta}$ or $\mu(Z) = 0$, but then $\{x\} \in \overline{\zeta}$ for $x \in Z$ and $\{x\} = Z \bmod \mu$, i.e. $\overline{\zeta} \supseteq \zeta \bmod \mu$, and analogously $\overline{\zeta} \subseteq \zeta \bmod \mu$. Therefore, we get $H_\mu(\zeta) = H_\mu(\overline{\zeta})$. Further for another measurable μ-partition ξ and $\overline{\xi}$ the μ-partition which generates the completion of $\widetilde{\sigma}(\xi)$ (again $\xi = \overline{\xi} \bmod \mu$) by Proposition 1.3 (5) we see $H_\mu(\overline{\xi}|\overline{\zeta}) = \int H_{\mu_{\overline{\zeta}_x}}(\overline{\xi} \cap \overline{\zeta}_x)\, d\mu(x) = \int H_{\mu_{\zeta_x}}(\overline{\xi} \cap \zeta_x)\, d\mu(x) = \int H_{\mu_{\zeta_x}}(\xi \cap \zeta_x)\, d\mu(x) = H_\mu(\xi|\zeta)$, since for every $E \in \xi$, either $\mu(E) = 0$, but then $\mu_{\zeta_x}(E \cap \zeta_x) = 0$ by Proposition 1.3 (4), or $\mu(E) \neq 0$ which implies $E \in \overline{\xi}$.

for $t_i \in \overline{\mathbb{R}}$ and $y_i \in Y$ depending on $\nu \in X^\star$. By the above,

$$\Phi : (\mathbb{R} \times Y)^\star \setminus G \longrightarrow \mathbb{R}^\star \times Y^{\mathbb{Z}}$$

for some $\overline{\mu}^\star$-null set G. For every $M \in \mathfrak{B}(\mathbb{R})^\star$ we have

$$P(M \times Y^{\mathbb{Z}}) = \int_M \bigotimes_{i\in\mathbb{Z}} m_{t_i(\eta)}(Y^{\mathbb{Z}})\, d\lambda^\star(\eta) = \lambda^\star(M).$$

In particular, this implies that if M is a $\lambda^\star$-null set, then $M \times N$ is a P-null set for every $N \in \mathfrak{D}^{\otimes\mathbb{Z}}$, since

$$0 = \lambda^\star(M) = P(M \times Y^{\mathbb{Z}}) \geq P(M \times N) \text{ for every } N \in \mathfrak{D}^{\otimes\mathbb{Z}}.$$

Now, $(\mathbb{R}, \mathfrak{B}(\mathbb{R}), \lambda)$ is a σ-finite standard space without atoms, thus by Remark 2.2, $\lambda^\star$-almost every element $\eta \in \mathbb{R}^\star$ is uniquely determined by a sequence $(t_i)_{i\in\mathbb{Z}}$ in $\overline{\mathbb{R}}^{\mathbb{Z}}$, ordered as above, with $\eta = \sum_{i\in\mathbb{Z}} \delta_{t_i}$. (From now on we will only consider sequences ordered as above, without stating it again.) So, by the above, P-a.e. $(\eta, (y_i)_{i\in\mathbb{Z}}) \in \mathbb{R}^\star \times Y^{\mathbb{Z}}$ is of the form $(\sum_{i\in\mathbb{Z}} \delta_{t_i}, (y_i)_{i\in\mathbb{Z}})$, for a uniquely determined sequence $(t_i)_{i\in\mathbb{Z}}$ with $t_i \in \overline{\mathbb{R}}$.
This immediately implies (a.e.-) injectivity of the map Φ: If

$$\Big(\sum_{i\in\mathbb{Z}} \delta_{t_i}, (y_i)_{i\in\mathbb{Z}}\Big) = \Big(\sum_{i\in\mathbb{Z}} \delta_{s_i}, (z_i)_{i\in\mathbb{Z}}\Big)$$

for (ordered) sequences $(t_i)_{i\in\mathbb{Z}}, (s_i)_{i\in\mathbb{Z}} \in \overline{\mathbb{R}}^{\mathbb{Z}}$, $y_i, z_i \in Y$, then $y_i = z_i$ $\forall i \in \mathbb{Z}$ and, by uniqueness of the sequence, $t_i = s_i$ for all $i \in \mathbb{Z}$, hence Φ is injective everywhere beside on a P-null set of $(\mathbb{R}^\star \times Y^{\mathbb{Z}}, \mathfrak{B}(\mathbb{R})^\star \otimes \mathfrak{D}^{\otimes\mathbb{Z}})$.
Clearly, Φ is a.e. surjective: Given two sequences $(t_i)_{i\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ and $(y_i)_{i\in\mathbb{Z}} \in Y^{\mathbb{Z}}$, then $\nu := \sum_{i\in\mathbb{Z}} \delta_{(t_i,y_i)}$ defines an $\overline{\mathbb{N}}_0$-valued measure on $\mathbb{R} \times Y$. Hence, since P-a.e. element of $\mathbb{R}^\star \times Y^{\mathbb{Z}}$ is given by such sequences, Φ reaches P-a.e. element.
Moreover, given $B \in \mathfrak{B}$, $n_l \in \overline{\mathbb{N}}_0$, l in some countable index set L and $D_i \in \mathfrak{D}$, $i \in \mathbb{Z}$, then

$$\Phi^{-1}\Big(\bigcup_{l\in L} {N_B}^{-1}(\{n_l\}) \times \prod_{i\in\mathbb{Z}} D_i\Big) =$$

$$= \Big\{ \sum_{i\in\mathbb{Z}} \delta_{(t_i,y_i)} \in (\mathbb{R}\times Y)^\star \setminus G \Big| \exists l \in L : \sum_{i\in\mathbb{Z}} \delta_{t_i}(B) = n_l, y_i \in D_i, \forall i \in \mathbb{Z} \Big\} =$$

$$= \Big\{ \sum_{i\in\mathbb{Z}} \delta_{(t_i,y_i)} \in (\mathbb{R}\times Y)^\star \setminus G \Big| \exists l \in L : \sum_{i\in\mathbb{Z}} \delta_{(t_i,y_i)}(B \times Y) = n_l,$$

$$\sum_{i\in\mathbb{Z}} \delta_{(t_i,y_i)}(\{t_j\} \times D_j) = 1, \forall j \in \mathbb{Z} \Big\} =$$

$$= \bigcup_{l\in L} \Big(N^{-1}_{B\times Y}(\{n_l\}) \cap \bigcap_{j\in\mathbb{Z}} N^{-1}_{\{t_j\}\times D_j}(\{1\}) \Big) \quad \in \sigma(N_C : C \in \mathfrak{B}(\mathbb{R}) \otimes \mathfrak{D}) = (\mathfrak{B}(\mathbb{R}) \otimes \mathfrak{D})^\star$$

This shows measurablility of Φ, because preimages respect set operations and the set $\{\bigcup_{n\in L} N_B^{-1}(\{n\}) \times \prod_{i\in\mathbb{Z}} D_i \ : \ D_i \in \mathfrak{D}, B \in \mathfrak{B}, L \subseteq \overline{\mathbb{N}}_0\}$ generates $\mathfrak{B}(\mathbb{R})^\star \otimes \mathfrak{D}^{\otimes\mathbb{Z}}$, since there is a sequence $(A_n)_{n\in\mathbb{N}_0}$ of element of this set such that $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}_0$ and $\bigcup_{n=0}^{\infty} A_n = \mathfrak{B}(\mathbb{R})^\star$, e.g. $A_0 := N_B^{-1}(\{\infty\})$, $A_n := \bigcup_{k=0}^{n-1} N_B^{-1}(\{k\}) \cup N_B^{-1}(\{\infty\})$ (see e.g. Bemerkung 6.1 of [28]).
Similarly, we see that Φ^{-1} is measurable.

To prove that Φ is measure-preserving, we will use the Laplace functional of $\overline{\mu}^\star$, which, by Lemma 1.5, uniquely determines $\overline{\mu}^\star$, i.e. if we can show, that

$$\Psi_{P\circ\Phi}(f) = \Psi_{\overline{\mu}^\star}(f),$$

for every non-negative measurable function f of $(\mathbb{R}\times Y, \mathfrak{B}(\mathbb{R})\otimes\mathfrak{D})$, then

$$P\circ\Phi = \overline{\mu}^\star.$$

Recall, that by Remark 2.4 the Laplace functional of the Poisson suspension of a standard measure space $(Z,\mathfrak{A},Q)$ fulfills the following identity:
(12)

$$\Psi_{Q^\star}(h) = \int_{Z^\star}\Big[\exp\Big(-\int_Z h(\omega)\,d\nu(\omega)\Big)\Big]\,dQ^\star(\nu) = \exp\Big(\int_Z\Big[\exp(-h(\omega))-1\Big]\,dQ(\omega)\Big),$$

for every measurable function $h : Z \longrightarrow [0,\infty]$.
Let f be a non-negative measurable function of $(\mathbb{R}\times Y, \mathfrak{B}(\mathbb{R})\otimes\mathfrak{D}, \overline{\mu})$, then, again by Remark 2.4, and using the transformation formula w.r.t. Φ^{-1}, we get

$$\Psi_{P\circ\Phi}(f) = \int_{(\mathbb{R}\times Y)^\star}\exp\Big[-\int_{\mathbb{R}\times Y} f(x,y)\,d\nu(x,y)\Big]\,d(P\circ\Phi)(\nu) =$$

$$= \int_{\mathbb{R}^\star\times Y^{\mathbb{Z}}}\exp\Big[-\int_{\mathbb{R}\times Y} f(x,y)\,d(\Phi^{-1}(\eta,(y_i)_{i\in\mathbb{Z}}))(x,y)\Big]\,dP(\eta,(y_i)_{i\in\mathbb{Z}}) =$$

$$= \int_{\mathbb{R}^\star\times Y^{\mathbb{Z}}}\exp\Big[-\int_{\mathbb{R}\times Y} f(x,y)\,d(\sum_{i\in\mathbb{Z}}\delta_{(t_i(\eta),y_i)})(x,y)\Big]\,dP(\eta,(y_i)_{i\in\mathbb{Z}}) =$$

$$= \int_{\mathbb{R}^\star\times Y^{\mathbb{Z}}}\exp\Big[-\sum_{i\in\mathbb{Z}} f(t_i(\eta),y_i)\Big]\,dP(\eta,(y_i)_{i\in\mathbb{Z}}).$$

By definition $P = \int_{\mathbb{R}^\star}\delta_\eta\otimes\bigotimes m_{t_i(\eta)}\,d\lambda^\star(\eta)$. Due to this form, we can split the integration (confer Remark 1.5 (2)), and pursue the computation (using the Theorem of Fubini):

$$\Psi_{P\circ\Phi}(f) = \int_{\mathbb{R}^\star}\int_{Y^{\mathbb{Z}}}\exp\Big[-\sum_{i\in\mathbb{Z}} f(t_i(\eta),y_i)\Big]\,d\bigotimes_{i\in\mathbb{Z}} m_{t_i(\eta)}((y_i)_{i\in\mathbb{Z}})\,d\lambda^\star(\eta) =$$

$$= \int_{\mathbb{R}^\star}\cdots\int_Y\int_Y\cdots\prod_{i\in\mathbb{Z}}\exp[-f(t_i(\eta),y_i)]\ldots\,dm_{t_0}(y_0)\,dm_{t_1}(y_1)\ldots\,d\lambda^\star(\eta) =$$

$$= \int_{\mathbb{R}^\star}\prod_{i\in\mathbb{Z}}\Big(\int_Y\exp\Big[-f(t_i(\eta),y)\Big]\,dm_{t_i(\eta)}(y)\Big)\,d\lambda^\star(\eta) =$$

$$= \int_{\mathbb{R}^\star}\exp\Big(\sum_{i\in\mathbb{Z}}\Big(\log\Big(\int_Y\exp[-f(t_i(\eta),y)]\,dm_{t_i(\eta)}(y)\Big)\Big)\Big)\,d\lambda^\star(\eta) =$$

$$= \int_{\mathbb{R}^\star}\exp\Big(\int_{\mathbb{R}}\Big(\log\Big(\int_Y\exp[-f(t,y)]\,dm_t(y)\Big)\Big)\,d(\sum_{i\in\mathbb{Z}}\delta_{t_i(\eta)})(t)\Big)\,d\lambda^\star(\eta)$$

$$= \int_{\mathbb{R}^\star}\exp\Big(\int_{\mathbb{R}}\Big(\log\Big(\int_Y\exp[-f(t,y)]\,dm_t(y)\Big)\Big)\,d\eta)(t)\Big)\,d\lambda^\star(\eta),$$

where we used the Theorem of Fubini, and by (12) this is equal to

$$\exp\Big(\int_{\mathbb{R}}\Big(\exp\Big[\log\Big[\int_Y\exp[-f(t,y)]\,dm_t(y)\Big]\Big]-1\Big)\,d\lambda(t) =$$

$$= \exp\Big(\int_{\mathbb{R}}\Big(\int_Y \exp[-f(t,y)]\; dm_t(y)\Big) - 1\Big)\; d\lambda(t) =$$

$$= \exp\Big(\int_{\mathbb{R}}\Big(\int_Y (\exp[-f(t,y)] - 1)\; dm_t(y)\Big)\; d\lambda(t) =$$

$$= \exp\Big(\int_{\mathbb{R}\times Y} (\exp[-f(t,y)] - 1)\; d\overline{\mu}(t,y)\Big) = \Psi_{\overline{\mu}^\star}(f),$$

again by (12). (Recall that $\overline{\mu} = \int_{\mathbb{R}} \delta_t \otimes m_t \; d\lambda(t)$, so we could glue the integration together like in Remark 1.5 (2).)
Thus we finally have verified that Φ is an essential isomorphism, hence

$$((\mathbb{R}\times Y)^\star, (\mathfrak{B}(\mathbb{R})\otimes\mathfrak{D})^\star, \overline{\mu}^\star) \approx (\mathbb{R}^\star \times Y^{\mathbb{Z}}, \mathfrak{B}(\mathbb{R})^\star \otimes \mathfrak{D}^{\otimes\mathbb{Z}}, P).$$

Now, Remark 2.5 states that

$$(X,\mathfrak{B},\mu) \approx (\mathbb{R}\times Y, \mathfrak{B}(\mathbb{R})\otimes\mathfrak{D}, \overline{\mu}) \;\Rightarrow\; (X^\star,\mathfrak{B}^\star,\mu^\star) \approx ((\mathbb{R}\times Y)^\star, (\mathfrak{B}(\mathbb{R})\otimes\mathfrak{D})^\star, \overline{\mu}^\star),$$

thus combining this with the above, we gain

$$(X^\star,\mathfrak{B}^\star,\mu^\star) \approx (\mathbb{R}^\star \times Y^{\mathbb{Z}}, \mathfrak{B}(\mathbb{R})^\star \otimes \mathfrak{D}^{\otimes\mathbb{Z}}, P),$$

with corresponding isomorphism

$$\Theta := \Phi \circ \theta^\star, \text{ where } \theta^\star(\gamma) := \gamma \circ \theta^{-1}, \; \forall \gamma \in X^\star.$$

Recall that $\theta^{-1}(\mathfrak{B}(\mathbb{R})\times Y) = \mathfrak{C}$, thus we can deduce that

$$\Theta^{-1}(\mathfrak{B}(\mathbb{R})^\star \times Y^{\mathbb{Z}}) = \mathfrak{C}^\star.$$

Indeed,

$$\Theta^{-1}(\mathfrak{B}(\mathbb{R})^\star \times Y^{\mathbb{Z}}) = \theta^{\star -1}(\Phi^{-1}(\{E\times Y^{\mathbb{Z}} : E \in \sigma(N_B : B\in\mathfrak{B}(\mathbb{R}))\})) =^{13}$$

$$= \theta^{\star -1}(\Phi^{-1}(\sigma(N_B \times Y^{\mathbb{Z}} : B\in\mathfrak{B}(\mathbb{R})))) = \theta^{\star -1}(\sigma(\Phi^{-1}(N_B \times Y^{\mathbb{Z}} : B\in\mathfrak{B}(\mathbb{R})))) =^{14}$$

$$= \sigma(\theta^{\star -1}(N_{B\times Y} : B\in\mathfrak{B}(\mathbb{R}))) = \sigma(N_{B\times Y}\circ\theta : B\in\mathfrak{B}(\mathbb{R})) = \sigma(N_{\theta^{-1}(B\times Y)} : B\in\mathfrak{B}(\mathbb{R})) =$$

$$= \sigma(N_C : C\in\mathfrak{C}) = \mathfrak{C}^\star.$$

So, using Remark 3.4, we obtain

$$H_{\mu^\star}(\mathfrak{B}^\star|\mathfrak{C}^\star) = H_{\mu^\star}(\Theta^{-1}(\mathfrak{B}(\mathbb{R})^\star\otimes\mathfrak{D}^{\otimes\mathbb{Z}})|\Theta^{-1}(\mathfrak{B}(\mathbb{R})^\star\times Y^{\mathbb{Z}})) =$$

$$= H_P(\mathfrak{B}(\mathbb{R})^\star\otimes\mathfrak{D}^{\otimes\mathbb{Z}}|\mathfrak{B}(\mathbb{R})^\star\times Y^{\mathbb{Z}}).$$

Because $\varepsilon_{\mathbb{R}} = \{\{t\} : t\in\mathbb{R}\}$ generates $\mathfrak{B}(\mathbb{R})$, we get that

$$\widetilde{\sigma}(\varepsilon_{\mathbb{R}^\star}) = \mathfrak{B}(\mathbb{R})^\star,$$

[13](This quality holds since, clearly, $(N_{B_i}\times Y^{\mathbb{Z}})^c = (\mathbb{R}^\star\times\emptyset)\cup(N_B^c\times Y^{\mathbb{Z}}) = N_B^c\times Y^{\mathbb{Z}}$, $\bigcap_{j\in J}(N_{B_j}\times Y^{\mathbb{Z}}) = (\bigcap_{j\in J} N_{B_j})\times Y^{\mathbb{Z}}$ and $\bigcup_{j\in J}(N_{B_j}\times Y^{\mathbb{Z}}) = (\bigcup_{j\in J} N_{B_j})\times Y^{\mathbb{Z}}$ for $B_j\in\mathfrak{B}(\mathbb{R})$, j in a countable index set J.)

[14]Note, that $\Phi^{-1}(N_B^{-1}(\{n\})\times Y^{\mathbb{Z}}) = \{\sum_{i\in\mathbb{Z}}\delta_{(t_i,y_i)} \in (\mathbb{R}\times Y)^\star : \sum_{i\in\mathbb{Z}}\delta_{t_i}(B) = n\} = \{\sum_{i\in\mathbb{Z}}\delta_{(t_i,y_i)} \in (\mathbb{R}\times Y)^\star : \sum_{i\in\mathbb{Z}}\delta_{(t_i,y_i)}(B\times Y) = n\} = N_{B\times Y}^{-1}(\{n\})$, for every $B\in\mathfrak{B}(\mathbb{R})$, $n\in\overline{\mathbb{N}}_0$.

where $\varepsilon_{\mathbb{R}^\star} := \{\{\eta\} : \eta \in \mathbb{R}^\star\}$: A.e. element η of $\mathbb{R}^\star$ is of the form $\eta = \sum_{i\in\mathbb{Z}} \delta_{t_i}$ for some $t_i \in \overline{\mathbb{R}}$, thus

$$\{\eta\} = \bigcap_{i\in\mathbb{Z}, t_i\neq\infty} N^{-1}_{\{t_i\}}(\{1\}) \cap N^{-1}_{\mathbb{R}\setminus\bigcup_{i\in\mathbb{Z}} t_i}(\{0\}) \in \mathfrak{B}(\mathbb{R})^\star,$$

hence $\varepsilon_{\mathbb{R}^\star}$ builds a $\mu^\star$-partition of $\mathbb{R}^\star$ (with $\widetilde{\sigma}(\varepsilon^\star) \subseteq \mathfrak{B}(\mathbb{R})^\star$) and since it is the finest $\mu^\star$-partition, we see $\widetilde{\sigma}(\varepsilon_{\mathbb{R}^\star}) = \mathfrak{B}(\mathbb{R})^\star$. Thus, using Remark 1.5, we compute

$$H_P(\mathfrak{B}(\mathbb{R})^\star \otimes \mathfrak{D}^{\otimes\mathbb{Z}} | \mathfrak{B}(\mathbb{R})^\star \times Y^{\mathbb{Z}}) =$$

$$= \int_{\mathbb{R}^\star\times Y^{\mathbb{Z}}} H_{P_{\pi_{\varepsilon_{\mathbb{R}^\star}\times Y^{\mathbb{Z}}}(\eta,(y_i)_{i\in\mathbb{Z}})}}((\mathfrak{B}(\mathbb{R})^\star \otimes \mathfrak{D}^{\otimes\mathbb{Z}}) \cap \pi_{\varepsilon_{\mathbb{R}^\star}\times Y^{\mathbb{Z}}}(\eta,(y_i)_{i\in\mathbb{Z}}))\, dP(\eta,(y_i)_{i\in\mathbb{Z}}) =$$

$$= \int_{\mathbb{R}^\star}\int_{Y^{\mathbb{Z}}} H_{\delta_\eta \otimes \bigotimes_{i\in\mathbb{Z}} m_{t_i(\eta)}}((\mathfrak{B}(\mathbb{R})^\star \otimes \mathfrak{D}^{\otimes\mathbb{Z}}) \cap (\{\eta\}\times Y^{\mathbb{Z}}))\, d\bigotimes_{i\in\mathbb{Z}} m_{t_i(\eta)}((y_i)_{i\in\mathbb{Z}})\, d\lambda^\star(\eta) =^{15}$$

$$= \int_{\mathbb{R}^\star}\int_{Y^{\mathbb{Z}}} H_{\bigotimes_{i\in\mathbb{Z}} m_{t_i(\eta)}}(\mathfrak{D}^{\otimes\mathbb{Z}})\, d\bigotimes_{i\in\mathbb{Z}} m_{t_i(\eta)}((y_i)_{i\in\mathbb{Z}})\, d\lambda^\star(\eta) = \int_{\mathbb{R}^\star} H_{\bigotimes_{i\in\mathbb{Z}} m_{t_i(\eta)}}(\mathfrak{D}^{\otimes\mathbb{Z}})\, d\lambda^\star(\eta),$$

since the map under the integral does no longer depend on $(y_i)_{i\in\mathbb{Z}}$ and $\bigotimes_{i\in\mathbb{Z}} m_{t_i(\eta)}(Y^{\mathbb{Z}}) = 1$.
Let $\eta = \sum_{i\in\mathbb{Z}} \delta_{t_i} \in \mathbb{R}^\star$ be fixed for the moment, and let for each $i \in \mathbb{Z}$, ζ_i denote the measurable m_{t_i}-partition of $(Y, \mathfrak{D}, m_{t_i})$ which generates $\mathfrak{D}$, then

$$\Lambda := \{\ldots \times A_{j_0} \times A_{j_1} \times \ldots : A_{j_i} \in \zeta_i, j \in \mathbb{Z}\}$$

is a measurable $\bigotimes_{i\in\mathbb{Z}} m_{t_i}$-partition of $Y^{\mathbb{Z}}$ which generates $\mathfrak{D}^{\mathbb{Z}}$.
We first consider the case that every m_{t_i}-partition ζ_i is equal to a countable m_{t_i}-partition modulo m_{t_i}. Then, w.l.o.g., we can assume that ζ_i contains no m_{t_i}-null sets for every $i \in \mathbb{Z}$, which implies that Λ is equal to a countable $\bigotimes_{i\in\mathbb{Z}} m_{t_i}$-partition modulo $\bigotimes_{i\in\mathbb{Z}} m_{t_i}$. Indeed, if Λ contains uncountably many sets, then by Remark 3.1 they - up to countably many - have to be $\bigotimes_{i\in\mathbb{Z}} m_{t_i}$-null sets, but there are none, since $\bigotimes_{i\in\mathbb{Z}} m_{t_i}(\prod_{i\in\mathbb{Z}} A_{j_i}) = 0$ if and only if there is an $i \in \mathbb{Z}$ such that $m_{t_{j_i}}(A_{j_i}) = 0$, which is forbidden. Thus, similar to the proof of Lemma 4.1,

$$H_{\bigotimes_{i\in\mathbb{Z}} m_{t_i}}(\mathfrak{D}^{\otimes\mathbb{Z}}) = -\sum_{F\in\Lambda} \bigotimes_{k\in\mathbb{Z}} m_{t_k}(F) \log(\bigotimes_{i\in\mathbb{Z}} m_{t_i}(F)) =$$

$$= \ldots \sum_{A_{j_0}\in\zeta_0} \sum_{A_{j_1}\in\zeta_1} \cdots \Big(-\prod_{k\in\mathbb{Z}} m_{t_k}(A_{j_k}) \Big(\sum_{i\in\mathbb{Z}} \log(m_{t_i}(A_{j_i})) \Big) \Big) =$$

$$\ldots \sum_{A_{j_0}\in\zeta_0} \sum_{A_{j_1}\in\zeta_1} \cdots \Big(\ldots - \prod_{k\in\mathbb{Z}} m_{t_k}(A_{j_k}) \log(m_{t_0}(A_{j_0})) - \prod_{k\in\mathbb{Z}} m_{t_k}(A_{j_k}) \log(m_{t_1}(A_{j_1})) - \ldots \Big) =$$

$$= \ldots \sum_{A_{j_0}\in\zeta_0} \sum_{A_{j_2}\in\zeta_2} \cdots \Big(\ldots - \underbrace{\sum_{A_{j_1}\in\zeta_1} m_{t_1}(A_{j_1})}_{=1} \prod_{k\in\mathbb{Z}\setminus\{1\}} m_{t_k}(A_{j_k}) \log(m_{t_0}(A_{j_0}))$$

[15] Clearly, $H_{\delta_{\{\eta\}}\otimes Q}(\{\eta\}\times\zeta) = \sum_{F\in\zeta} \delta_{\{\eta\}} \otimes Q(\{\eta\}\times F) \cdot \log(\delta_{\{\eta\}} \otimes Q(\{\eta\}\times F)) = \sum_{F\in\zeta} Q(F) \cdot \log(Q(F)) = H_Q(\zeta)$ or $H_{\delta_{\{\eta\}}\otimes Q}(\{\eta\}\times\zeta) = \infty = H_Q(\zeta)$, for Q some σ-finite measure on $\widetilde{\sigma}(\zeta)$ with ζ a measurable Q-partition.

$$- \sum_{A_{j_1} \in \zeta_1} m_{t_1}(A_{j_1}) \prod_{i \in \mathbb{Z}} m_{t_i}(A_{j_i}) \log(m_{t_1}(A_{j_1})) - \dots \Big) =$$

$$= \dots = \sum_{i \in \mathbb{Z}} \sum_{A_{j_i} \in \zeta_i} -m_{t_i}(A_{j_i}) \log(m_{t_i}(A_{j_i})) = \sum_{i \in \mathbb{Z}} H_{m_{t_i}}(\mathfrak{D}).$$

In case, there is an $l \in \mathbb{Z}$ such that ζ_l is not equal to a countable m_{t_l}-partition modulo m_{t_l}, then[16] Λ is not equal to a countable $\bigotimes_{i \in \mathbb{Z}} m_{t_i}$-partition modulo $\bigotimes_{i \in \mathbb{Z}} m_{t_i}$, hence $H_{\bigotimes_{i \in \mathbb{Z}} m_{t_i}}(\mathfrak{D}^{\otimes \mathbb{Z}}) = \infty = \sum_{i \in \mathbb{Z}} H_{m_{t_i}}(\mathfrak{D})$.
With the above, we get

$$H_P(\mathfrak{B}(\mathbb{R})^\star \otimes \mathfrak{D}^{\otimes \mathbb{Z}} | \mathfrak{B}(\mathbb{R})^\star \times Y^{\mathbb{Z}}) = \int_{\mathbb{R}^\star} H_{\bigotimes_{i \in \mathbb{Z}} m_{t_i(\eta)}}(\mathfrak{D}^{\otimes \mathbb{Z}}) \, d\lambda^\star(\eta) =$$

$$= \int_{\mathbb{R}^\star} \sum_{i \in \mathbb{Z}} H_{m_{t_i(\eta)}}(\mathfrak{D}) \, d\lambda^\star(\eta) = \int_{\mathbb{R}^\star} \int_{\mathbb{R}} H_{m_t}(\mathfrak{D}) \, d\Big(\sum_{i \in \mathbb{Z}} \delta_{t_i(\eta)} \Big)(t) \, d\lambda^\star(\eta) =$$

$$= \int_{\mathbb{R}^\star} \int_{\mathbb{R}} H_{m_t}(\mathfrak{D}) \, d\eta(t) \, d\lambda^\star(\eta) = \int_{\mathbb{R}} H_{m_t}(\mathfrak{D}) \, d\lambda(t),$$

since by Remark 2.1 (1)

$$\lambda = \int_{\mathbb{R}^\star} \eta \, d\lambda^\star(\eta),$$

and similar to Remark 1.5 (2), we get $\int_{\mathbb{R}} f(t) \, d\lambda(t) = \int_{\mathbb{R}^\star} \int_{\mathbb{R}} f(t) \, d\eta(t) \, d\lambda^\star(\eta)$, for any measurable non-negative map f on $\mathbb{R}$.
While, analogous to a previous computation, we have

$$H_\mu(\mathfrak{B}|\mathfrak{C}) = H_\mu(\theta^{-1}(\mathfrak{B}(\mathbb{R}) \otimes \mathfrak{D}) | \theta^{-1}(\mathfrak{B}(\mathbb{R}) \times Y)) = H_{\overline{\mu}}(\mathfrak{B}(\mathbb{R}) \otimes \mathfrak{D} | \mathfrak{B}(\mathbb{R}) \times Y) =$$

$$= \int_{\mathbb{R} \times Y} H_{\overline{\mu}_{\{t\} \times Y}}(\{t\} \times \mathfrak{D}) \, d\overline{\mu}(t,y) = \int_{\mathbb{R}} H_{m_t}(\mathfrak{D}) \, d\lambda(t),$$

which is exactly the expression above. Hence, we finally obtain

$$H_{\mu^\star}(\mathfrak{B}^\star | \mathfrak{C}^\star) = H_P(\mathfrak{B}(\mathbb{R})^\star \otimes \mathfrak{D}^{\otimes \mathbb{Z}} | \mathfrak{B}(\mathbb{R})^\star \times Y^{\mathbb{Z}}) = \int_{\mathbb{R}} H_{m_t}(\mathfrak{D}) \, d\lambda(t) = H_\mu(\mathfrak{B}|\mathfrak{C}).$$

□

Lemma 4.3 ([10]). *Given an ergodic, conservative, measure-preserving, σ-finite dynamical system $(X, \mathfrak{B}, \mu, T)$ such that $\mu(X) = \infty$ and T is an automorphism, then*

$$\lim_{n \to \infty} \sup \Big\{ \mu(B) \Big| B \in \bigvee_{k=0}^{n-1} T^{-k}\alpha, \; \mu(B) \in (0, \infty) \Big\} = 0,$$

for every local μ-partition α of X.

Proof. Let α be a local μ-partition of X with core $A \in \mathfrak{B}$, $\mu(A) < \infty$. We already know (see Remark 1.3) that under the stated assumptions every set of positive measure is a sweep-out set, i.e. $X = \bigcup_{n \geq 0} T^{-n}A \bmod \mu$. Since T is an automorphism, we can adapt this result to the system $(X, \mathfrak{B}, \mu, T^{-1})$ (note that $T^{-1}\mathfrak{B} = \mathfrak{B} \bmod \mu$) and

[16]To see this, let $\{A_{j_u}\}_{u \in U}$, $A_{j_u} \in \zeta_l$, be an uncountable family of m_{t_l}-null-sets with $m_{t_l}(\bigcup_{u \in U} A_{j_u}) \neq 0$, and let $A_{j_i} \in \zeta_i$ be such that $m_i(A_{j_i}) \neq 0$ (such sets clearly exist, since ζ_i is a m_{t_i}-partition and $m_i(Y) \neq 0$ by definition) for $i \in \mathbb{Z} \setminus \{l\}$, then $L_u := \prod_{i<l} A_{j_i} \times A_{j_u} \times \prod_{i>l} A_{j_i} \in \Lambda$ with $\bigotimes_{i \in \mathbb{Z}} m_{t_i}(L_u) = 0$ for all $u \in U$, but $\bigotimes_{i \in \mathbb{Z}} m_{t_i}(\bigcup_{u \in U} L_u) = \prod_{i \in \mathbb{Z} \setminus \{l\}} m_{t_i}(A_{j_i}) \cdot m_{t_l}(\bigcup_{u \in U} A_{j_u}) \neq 0$.

get $X = \bigcup_{n\geq 0} T^n A \bmod \mu$. Therefore, the first return time map $\varphi_{T,A}(x) := \min\{n \geq 1 | T^n(x) \in A\}$ and $\psi_A(x) := \{n \geq 1 | T^{-n}x \in A\}$ are both finite for almost every $x \in A$. By $\mu(A) < \infty$, we get $\lim_{n\to\infty} \mu(A \cap \{\varphi_{T,A} > n\}) = \mu(A \cap \{\varphi_{T,A} = \infty\}) = 0$ and analogously for ψ_A. Thus, given $\varepsilon > 0$, we can choose $N \in \mathbb{N}$ such that $\mu(A \cap \{\varphi_{T,A} > N\}) < \varepsilon$ and $\mu(A \cap \{\psi_A > N\}) < \varepsilon$.

Now, let $B \in \bigvee_{k=0}^{Nn} T^{-k}\alpha$ for arbitrary fixed $n \in \mathbb{N}$ with $0 < \mu(B) < \infty$, i.e. $B = \bigcap_{k=0}^{Nn} T^{-k}A_k$, for some $A_k \in \alpha$. (Such an element exists, as long as $\bigvee_{k=0}^{Nn} T^{-k}A_k$ contains elements of positive measure, since $\mu(B) \leq \mu(T^{-k}A_k) = \mu(A_k)$, and α contains elements of finite measure by assumption. If there are only null sets in $\bigvee_{k=0}^{Nn} T^{-k}A_k$, then we immediately get $\sup\{\mu(B) | B \in \bigvee_{k=0}^{Nn} T^{-k}A_k, \mu(B) \in (0,\infty)\} = \sup_{x\in[0,\infty]}(\emptyset) = 0$ and the proof is done.) Denote by $k_1 \leq \ldots \leq k_l$ (with $l \in \mathbb{N}$) those indices of the intersection, for which $A_{k_1}, \ldots A_{k_l}$ are subsets of A. (Since A is a sweep-out set, there exist such indices, if only n is large enough.) By definition of a local μ-partition, all other elements A_k with $k \notin \{k_1, \ldots, k_l\}$ are equal to A^c:

$$B = A^c \cap T^{-1}A^c \cap \ldots \cap T^{-k_1+1}A^c \cap T^{k_1}A_{k_1} \cap \ldots\ldots \cap T^{-Nn}A^c,$$

for example if $1 < k_1$ and $k_l \neq Nn$.

We will distinguish three cases. If $k_1 > N$ or if $k_{i+1} - k_i > N$ for some $i \in \{1, \ldots, l-1\}$, then $B \subseteq T^{-k_1}[A \cap \{\psi_A > N\}]$, and $B \subseteq T^{-k_{i+1}}[A \cap \{\psi_A > N\}]$, respectively, since for any element in B after k_1 (resp. k_{i+1}) time steps forward, it takes more than k_1 (resp. k_{i+1}), and especially more than N, time steps backwards to reach A. Hence by T-invariance, we get $\mu(B) \leq (A \cap \{\psi_A > N\}) < \varepsilon$, in this case.

If $Nn - k_l > N$, then we have $B \subseteq T^{-k_l}[A \cap \{\varphi_{T,A} > N\}]$ and therefore $\mu(B) \leq (A \cap \{\varphi_{T,A} > N\}) < \varepsilon$, too.

The remaining case is that every of the above distances is less than or equal to N, which implies $B \subseteq T^{-k_1}[A \cap \{\varphi_{T,A} \leq N\} \cap \{\varphi_{T,A} \circ T_A \leq N\} \cap \ldots \cap \{\varphi_{T,A} \circ T_A^n \leq N\}]$, since in this case we have $l \geq n$. We will show that

$$\lim_{n\to\infty} \mu(C_n) = 0 \ \text{ for } C_n := A \cap \{\varphi_{T,A} \leq N\} \cap \{\varphi_{T,A} \circ T_A \leq N\} \cap \ldots \cap \{\varphi_{T,A} \circ T_A^n \leq N\}],$$

which gives $\mu(B) < \varepsilon$ for n large enough and completes the proof. Note that $\lim_{n\to\infty}(C_n) = \mu(C_\infty)$ for $C_\infty := \bigcap_{n=1}^{\infty} C_n$, since $(C_n)_{n\in\mathbb{N}}$ is a decreasing sequence and $\mu(C_1) \leq \mu(A) < \infty$. To obtain a contradiction, let us suppose that $\mu(C_\infty) > 0$. Since A is a finite measure sweep-out set, T_A is conservative, ergodic and measure-preserving ([27]). So, by Remark 1.3, C_∞ is a sweep-out set w.r.t. T_A, i.e. $\bigcup_{m=0}^{\infty} T_A^{-m} C_\infty = A \bmod \mu$, hence

$$\bigcup_{m=0}^{\infty} A \cap \{\varphi_{T,A} \circ T_A^m \leq N\} \cap \{\varphi_{T,A} \circ T_A^{m+1} \leq N\} \cap \ldots = A \mod \mu.$$

But $\mu(A \cap \{\varphi_{T,A} > N\}) > 0$, because by Lemma 1.1 $\mu(A \cap \{\varphi_{T,A} > l\}) = \mu(A^c \cap \{\varphi_{T,A} = l\})$ for all $l \in \mathbb{N}$, and $\sum_{l\in\mathbb{N}} \mu(A^c \cap \{\varphi_{T,A} = l\}) = \mu(A^c) = \infty$, thus infinitely many elements $A^c \cap \{\varphi_{T,A} = l\}$ have to have positive measure, so in particular there has to be an $l \geq N$ such that $\mu(A^c \cap \{\varphi_{T,A} = l\}) > 0$, hence $0 < \mu(A^c \cap \{\varphi_{T,A} = l\}) = \mu(A \cap \{\varphi_{T,A} > l\}) \leq \mu(A \cap \{\varphi_{T,A} > N\})$. So, since

$$A \cap \{\varphi_{T,A} > N\} \subseteq \bigcup_{m=0}^{\infty} A \cap \{\varphi_{T,A} \circ T_A^m \leq N\} \cap \{\varphi_{T,A} \circ T_A^{m+1} \leq N\} \cap \ldots \mod \mu,$$

there has to be a $k \in \mathbb{N}$ such that

$$(A \cap \{\varphi_{T,A} > N\}) \cap (A \cap \{\varphi_{T,A} \circ T_A^k \leq N\} \cap \{\varphi_{T,A} \circ T_A^{k+1} \leq N\} \cap \ldots) \neq \emptyset \mod \mu.$$

Let k be minimal with that property, i.e. let $A \cap \{\varphi_{T,A} > N\} \cap \{\varphi_{T,A} \circ T_A^m \leq N\} = \emptyset$ mod μ for all $m < k$. We claim that

$$W := A \cap \{\varphi_{T,A} > N\} \cap \{\varphi_{T,A} \circ T_A^k \leq N\} \cap \{\varphi_{T,A} \circ T_A^{k+1} \leq N\} \cap \dots$$

is a wandering set w.r.t. T_A. This would yield a contradiction, since $\mu(W) > 0$ and T_A is conservative. By minimality of k, $A \cap \{\varphi_{T,A} > N\} \cap \{\varphi_{T,A} \circ T_A^m > N\} = A \cap \{\varphi_{T,A} > N\}$ mod μ for all $m < k$. Hence, we can rewrite W in particular as

$$W = A \cap \{\varphi_{T,A} > N\} \cap \{\varphi_{T,A} \circ T_A^{k-1} > N\} \cap \{\varphi_{T,A} \circ T_A^k \leq N\} \cap \dots \text{ mod } \mu.$$

Thus,

$$W \cap T_A^{-1} W = W \cap (A \cap \{\varphi_{T,A} \circ T_A^{-1} > N\} \cap \{\varphi_{T,A} \circ T_A^k > N\} \cap \{\varphi_{T,A} \circ T_A^{k+1} \leq N\} \cap \dots) =$$

$$= \emptyset$$

modulo μ. Similar, for $j < i$ in $\mathbb{N}$, we get

$$T_A^{-i} W \cap T_A^{-j} W =$$

$$= (A \cap \{\varphi_{T,A} \circ T_A^i > N\} \cap \{\varphi_{T,A} \circ T_A^{i+k} \leq N\} \cap \dots \cap \{\varphi_{T,A} \circ T_A^{j+k-1} \leq N\} \cap \dots)$$

$$\cap (A \cap \{\varphi_{T,A} \circ T_A^{j+k-1} > N\} \cap \{\varphi_{T,A} \circ T_A^{j+k} \leq N\} \cap \dots) =$$

$$= \emptyset,$$

since by $i < j$ we have $i + k < j + k$, hence $i + k \leq j + k - 1$. So, we have shown that W is a wandering set of positive measure, a contradiction. $\square$

Proposition 4.1 ([10])**.** *Given a measure-preserving, conservative and ergodic automorphism T of $(X, \mathfrak{B}, \mu)$, where $\mu(X) = \infty$, and let α be a local μ-partition of X, then*

$$h(X^\star, (\widetilde{\sigma}(\alpha_0{}^\infty))^\star, \mu^\star, T^\star) \leq \hat{h}(X, \mathfrak{B}, \mu, T^{-1}, \alpha).$$

Proof. Let $p, n \in \mathbb{N}$ be arbitrary fixed. First note that $((\alpha_0^p)^\star)_0^{n-1} \preceq (\alpha_0^{n-1+p})^\star$, since for every given sequence $(j_{A_0 \cap \dots \cap T^{-n-p+1} A_{n+p-1}})_{A_0, \dots, A_{n+p-1} \in \alpha} \in \overline{\mathbb{N}}_0^{|\alpha_0^{n+p-1}|}$, we have

$$\bigcap_{\substack{A_i \in \alpha \\ i=0,\dots,p}} N^{-1}_{A_0 \cap \dots \cap T^{-n+1-p} A_{n-1+p}}(\{j_{A_0 \cap \dots \cap T^{-n+1-p} A_{n-1+p}}\}) \subseteq$$

$$\bigcap_{\substack{A_i \in \alpha \\ i=0,\dots,p}} N^{-1}_{A_0 \cap \dots \cap T^{-p} A_p}(\{k_0\}) \cap \bigcap_{\substack{A_i \in \alpha \\ i=1,\dots,p+1}} N^{-1}_{T^{-1} A_1 \cap \dots \cap T^{-p-1} A_{p+1}}(\{k_1\}) \cap \dots$$

$$\dots \cap \bigcap_{\substack{A_i \in \alpha \\ i=n-1,\dots,n-1+p}} N^{-1}_{T^{-n+1} A_{n-1} \cap \dots \cap T^{-n+1-p} A_{n-1+p}}(\{k_{n-1}\}),$$

for $k_0 := \sum_{\substack{B_i \in \alpha \\ i=1,\dots,n-1}} j_{A_0 \cap \dots \cap T^{-p} A_p \cap T^{-p-1} B_1 \cap \dots \cap T^{-p-n+1} B_{n-1}}$ and
$k_l := \sum_{\substack{B_i \in \alpha \\ i=0,\dots,n-2}} j_{B_0 \cap \dots \cap T^{-l+1} B_{l-1} \cap T^{-l} A_l \cap \dots \cap T^{-l-p} A_{l+p} \cap T^{-l-p-1} B_l \cap \dots \cap T^{-n+1-p} B_{n-1}}$ for every $l \in$

$\{1,\ldots,n-1\}$.
Thus by Proposition 3.1 (7) we get

$$H_{\mu^\star}((((\alpha_0^p)^\star)_0^{n-1}) \le H_{\mu^\star}((\alpha_0^{n-1+p})^\star). \tag{13}$$

And by Lemma 4.1 we know that

$$H_{\mu^\star}((\alpha_0^{n-1+p})^\star) = \sum_{D\in\alpha_0^{n-1+p}} H(Poi_{\mu(D)}) = \sum_{\substack{D\in\alpha_0^{n-1+p}\\ \mu(D)\neq\infty}} H(Poi_{\mu(D)}),$$

since $H(Poi_\infty) = \sum_{k\in\overline{\mathbb{N}}_0} Poi_\infty(k)\log(Poi_\infty(k)) = 0$. We have already computed, that

$$H(Poi_\lambda) = -\lambda\log(\lambda) + \lambda + \sum_{k=0}^{\infty} \frac{e^{-\lambda}\log(k!)}{k!}\lambda^k,$$

for $\lambda \in \mathbb{R}_0^+$. Therefore, $H(Poi_\lambda)$ is asymptotically equivalent to $-\lambda\log(\lambda)$ for $\lambda \to 0^+$, denoted by

$$H(Poi_\lambda) \sim -\lambda\log(\lambda) \text{ for } \lambda \to 0^+,$$

where $\lambda \to 0^+$ denotes the approach to 0 from the right (positive) hand side. Indeed,

$$\frac{H(Poi_\lambda)}{-\lambda\log(\lambda)} = 1 + \frac{1}{-\log(\lambda)} + \sum_{k=0}^{\infty} \frac{\log(k!)\lambda^{k-1}}{k!} e^{-\lambda} \frac{1}{-\log(\lambda)} \to 1 \text{ for } \lambda \to 0^+,$$

since[17] $\sum_{k=0}^{\infty} \frac{\log(k!)\lambda^{k-1}}{k!} e^{-\lambda} \frac{1}{-\log(\lambda)} \le \sum_{k=2}^{\infty} \frac{\lambda^{k-1}}{(k-2)!} e^{-\lambda} \frac{1}{-\log(\lambda)} = \lambda \frac{1}{-\log(\lambda)} \to 0$ for $\lambda \to 0^+$ (confer proof of Lemma 4.1), hence $\sum_{k=0}^{\infty} \frac{\log(k!)\lambda^{k-1}}{k!} e^{-\lambda} \frac{1}{-\log(\lambda)} \to 0$ for $\lambda \to 0^+$, since $\frac{\log(k!)\lambda^{k-1}}{k!} e^{-\lambda} \frac{1}{-\log(\lambda)} \ge 0$. Further, $\frac{H(Poi_\lambda)}{-\lambda\log(\lambda)} \to 1$ for $\lambda \to 0^+$, can (by definition) be formulated as $\forall\varepsilon > 0$ there is an $\lambda_0(\varepsilon) \in \mathbb{R}^+$ such that $\forall\lambda \le \lambda_0(\varepsilon)$ with $\lambda > 0$: $|\frac{H(Poi_\lambda)}{-\lambda\log(\lambda)} - 1| < \varepsilon$, i.e.[18]

$$(1-\varepsilon)(-\lambda\log(\lambda)) < H(Poi_\lambda) < (1+\varepsilon)(-\lambda\log(\lambda)).$$

Now, by Lemma 4.3, we can choose n so large, that $\mu(D) \le \lambda_0(\varepsilon)$ for all $D \in \alpha_0^{n-1+p}$ with $\mu(D) \neq \infty, 0$. (Clearly, $\mu(D)$ approaches 0 from the right hand side.) Hence,

$$(1-\varepsilon) \sum_{\substack{D\in\alpha_0^{n-1+p}\\ \mu(D)\neq\infty,0}} -\mu(D)\log(\mu(D)) \le \sum_{\substack{D\in\alpha_0^{n-1+p}\\ \mu(D)\neq\infty,0}} H(Poi_{\mu(D)}) \le$$

$$\le (1+\varepsilon) \sum_{\substack{D\in\alpha_0^{n-1+p}\\ \mu(D)\neq\infty,0}} -\mu(D)\log(\mu(D)))$$

---

[17]Note, that we are allowed to interchange the limit in λ with the infinite sum: The power series $\sum_{k=0}^{\infty} \frac{\log(k!)}{k!}\lambda^k$ converges for $\lambda < 1$, since $\frac{\log(k!)}{k!} \le 1$, and thus converges uniformly on compact subsets of $[0,1)$, to which we can restrict our computation, since we are only interested in λ close to zero, anyway.

[18]If $\frac{H(Poi_\lambda)}{-\lambda\log(\lambda)} - 1 > 0$, then $|\frac{H(Poi_\lambda)}{-\lambda\log(\lambda)} - 1| < \varepsilon$ gives $H(Poi_\lambda) < (1+\varepsilon)(-\lambda\log(\lambda))$, since $-\lambda\log(\lambda) \neq 0$. For $\frac{H(Poi_\lambda)}{-\lambda\log(\lambda)} - 1 \le 0$ we immediately have $H(Poi_\lambda) \le -\lambda\log(\lambda) < (1+\varepsilon)(-\lambda\log(\lambda))$. The other inequality follows analogously.

for n large enough. Thus, for $n \to \infty$,

$$H_{\mu^\star}((\alpha_0^{n-1+p})^\star) = \sum_{\substack{D\in\alpha_0^{n-1+p} \\ \mu(D)\neq\infty,0}} H(Poi_{\mu(D)}) \sim \sum_{\substack{D\in\alpha_0^{n-1+p} \\ \mu(D)\neq\infty,0}} -\mu(D)\log(\mu(D)) = H_\mu(\alpha_0^{n-1+p}).$$

So,

$$\lim_{n\to\infty} \frac{1}{n} H_{\mu^\star}((\alpha_0^{n-1+p})^\star) = \liminf_{n\to\infty} \frac{1}{n} H_{\mu^\star}((\alpha_0^{n-1+p})^\star) = \liminf_{n\to\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1+p}) =$$

$$= \liminf_{n\to\infty} \frac{n+p}{n} \frac{1}{n+p} H_\mu(\alpha_0^{n+p-1}) = \hat{h}(X,\mathfrak{B},\mu,T,\alpha).$$

Putting this together with inequality (13), we get

$$\lim_{n\to\infty} \frac{1}{n} H_{\mu^\star}(((\alpha_0^p)^\star)_0^{n-1}) \leq \hat{h}(X,\mathfrak{B},\mu,T,\alpha).$$

Now we want to take p to infinity: We just have shown, that there is an $N \in \mathbb{N}$, such that for all $n \in \mathbb{N}$ with $n \geq N$, we obtain

$$\hat{h}(X,\mathfrak{B},\mu,T,\alpha) \geq \liminf_{p\to\infty} \frac{1}{n} \int_{X^\star} I_{\mu^\star}(((\alpha_0^p)^\star)_0^{n-1})(z)\, d\mu^\star(z) \geq$$

$$\geq \frac{1}{n} \int_{X^\star} \liminf_{p\to\infty} I_{\mu^\star}(((\alpha_0^p)^\star)_0^{n-1})(z)\, d\mu^\star(z) = \frac{1}{n} \int_{X^\star} I_{\mu^\star}(((\alpha_0^\infty)^\star)_0^{n-1})(z)\, d\mu^\star(z),$$

where we used countability of $(\alpha_0^p)^\star$ (and thus of $((\alpha_0^p)^\star)_0^{n-1}$), to write the entropy as integral, and the Lemma of Fatou (see e.g. [7]), since $(I_{\mu^\star}(((\alpha_0^p)^\star)_0^{n-1}))_{p\in\mathbb{N}}$ are non-negative measurable functions, because $\mu^\star$ is a probability measure, and the last equality follows by continuity of the measure from above and continuity of the logarithm.
Hence

$$\hat{h}(X,\mathfrak{B},\mu,T,\alpha) \geq \lim_{n\to\infty} \frac{1}{n} H_{\mu^\star}(((\alpha_0^\infty)^\star)_0^{n-1}) = h(X^\star,\mathfrak{B}^\star,\mu^\star,T^\star,(\alpha_0^\infty)^\star).$$

If $(\alpha_0^\infty)^\star$ is equal to a countable $\mu^\star$-partition (w.l.o.g. is countable itself), then $\widetilde{\sigma}((\alpha_0^\infty)^\star) \supseteq (\widetilde{\sigma}(\alpha_0^\infty))^\star$ by Remark 4.1, thus using Proposition 3.2 (2), (3) we obtain

$$\hat{h}(X,\mathfrak{B},\mu,T,\alpha) \geq h(X^\star,\mathfrak{B}^\star,\mu^\star,T^\star,(\alpha_0^\infty)^\star) \geq h(X^\star,\widetilde{\sigma}((\alpha_0^\infty)^\star),\mu^\star,T^\star) \geq$$

$$\geq h(X^\star,(\widetilde{\sigma}(\alpha_0^\infty))^\star,\mu^\star,T^\star).$$

If $(\alpha_0^\infty)^\star$ cannot be written as a countable $\mu^\star$-partition of $X^\star$ by neglecting a null-set, then by definition $h(X^\star,\mathfrak{B}^\star,\mu^\star,T^\star,(\alpha_0^\infty)^\star) = \infty$, which trivially implies that

$$h(X^\star,\mathfrak{B}^\star,\mu^\star,T^\star,(\alpha_0^\infty)^\star) \geq h(X^\star,(\widetilde{\sigma}(\alpha_0^\infty))^\star,\mu^\star,T^\star),$$

hence we get the same inequality as above. $\square$

4.2 Parry's entropy

In [15] Parry used the concept of conditional entropies and a well-known identity for the entropy of a probability space to extend the notion of entropy to infinite measure-preserving dynamical systems.

Definition. The *Parry entropy* is defined as

$$h_{Pa}(X,\mathfrak{B},\mu,T) := \sup_{\substack{T^{-1}\mathfrak{C}\subseteq\mathfrak{C},\\ \mathfrak{C}\ \sigma\text{-finite sub-}\sigma\text{-algebra of }\mathfrak{B}}} H_\mu(\mathfrak{C}|T^{-1}\mathfrak{C}) =$$

$$= \sup_{\substack{T^{-1}\gamma\preceq\gamma,\\ \gamma\text{ measurable }\mu\text{-partition of }X,\ \widetilde{\sigma}(\gamma)\ \sigma\text{-finite}}} H_\mu(\gamma|T^{-1}\gamma).$$

Descriptively spoken, one may view this value as the entropy of the present, conditioned by knowing the past. Hence, actually, we consider only one time step in our dynamical system. In this context, one could interpret the condition $T^{-1}\mathfrak{C}\subseteq\mathfrak{C}$ in the following way: Whenever we go one time step further we gain more information about the system, which seems rather natural.

Remark 4.2. This definition coincides with the previous definition of entropy, if μ is a T-invariant probability measure. Indeed, given a measurable μ-partition γ of X with $H_\mu(\gamma)<\infty$, then by Remark 3.6 we know that

$$h(X,\mathfrak{B},\mu,T,\gamma) = H_\mu(\bigvee_{k=0}^{\infty} T^{-k}\gamma | \bigvee_{k=1}^{\infty} T^{-k}\gamma) = H_\mu(\bigvee_{k=0}^{\infty} T^{-k}\gamma | T^{-1}(\bigvee_{k=0}^{\infty} T^{-k}\gamma)),$$

hence

$$h(X,\mathfrak{B},\mu,T) = \sup_{\gamma\,:\,H_\mu(\gamma)<\infty} h(X,\mathfrak{B},\mu,T,\gamma) = \sup_{\gamma\,:\,H_\mu(\gamma)<\infty} H_\mu(\bigvee_{k=0}^{\infty} T^{-k}\gamma | T^{-1}(\bigvee_{k=0}^{\infty} T^{-k}\gamma))$$

Now, setting $\mathfrak{C} := \widetilde{\sigma}(\bigvee_{k=0}^{\infty} T^{-k}\gamma)$, we obtain $T^{-1}\mathfrak{C} = \widetilde{\sigma}(\bigvee_{k=1}^{\infty} T^{-k}\gamma)\subseteq\mathfrak{C}$, thus

$$h(X,\mathfrak{B},\mu,T,\gamma) = H_\mu(\mathfrak{C}|T^{-1}\mathfrak{C}) \le \sup_{\mathfrak{D}\supseteq T^{-1}\mathfrak{D}} H_\mu(\mathfrak{D}|T^{-1}\mathfrak{D}) = h_{Pa}(X,\mathfrak{B},\mu,T),$$

hence

$$h(X,\mathfrak{B},\mu,T) = \sup\{h(X,\mathfrak{B},\mu,T,\gamma) : \gamma \text{ such that } H_\mu(\gamma)<\infty\} \le h_{Pa}(X,\mathfrak{B},\mu,T).$$

On the other hand, let γ be such that $T^{-1}\gamma\preceq\gamma$ and for the moment assume that $H_\mu(\gamma)<\infty$. Since $T^{-1}\gamma\preceq\gamma \Leftrightarrow T^{-1}\gamma\vee\gamma=\gamma \Leftrightarrow \bigvee_{k=0}^{\infty} T^{-k}\gamma=\gamma$, we have

$$H_\mu(\gamma|T^{-1}\gamma) = H_\mu(\bigvee_{k=0}^{\infty} T^{-k}\gamma | \bigvee_{k=1}^{\infty} T^{-k}\gamma) = h(X,\mathfrak{B},\mu,T,\gamma) \le \sup_{\gamma\,:\,H_\mu(\gamma)<\infty} h(X,\mathfrak{B},\mu,T,\gamma) =$$

$$= h(X,\mathfrak{B},\mu,T).$$

Now, we follow [15], Theorem 5.14, and consider[19] a refining sequence of countable μ-partitions γ_n, $n\in\mathbb{N}$, with $H_\mu(\gamma)<\infty$ such that $\bigvee_{n=1}^{\infty}\gamma_n=\gamma$. Then, using Proposition

[19]Such a sequence always exists: Let $\Delta := \{E_1,E_2,\dots\}$ be a generating set of γ, like in the definition of a measurable μ-partition. Then, we can e.g. define $\gamma_1 := \{E_1,E_1^c\}$, $\gamma_2 := \{E_1\cap E_2, E_1^c\cap E_2, E_1\cap E_2^c, E_1^c\cap E_2^c\}$, and so on. Clearly, all this μ-partitions have finite static entropy, since they are finite itself. And $\bigvee_{n=0}^{\infty}\gamma_n = \{\bigcap_{n=1}^{\infty} R_n\ :\ R_n = E_n \text{ or } R_n = E_n^c\} = \gamma$.

3.1 (7),

$$H_\mu(\gamma_n|T^{-1}\gamma) = H_\mu(\gamma_n|\bigvee_{k=1}^{\infty} T^{-k}\gamma) \le H_\mu(\gamma_n|\bigvee_{k=1}^{\infty} T^{-k}\gamma_n) = h(X,\mathfrak{B},\mu,T,\gamma_n) \le h(X,\mathfrak{B},\mu,T)$$

for every $n \in \mathbb{N}$. Now, also by Proposition 3.1 (7), we see $H_\mu(\gamma_n|T^{-1}\gamma) \le H_\mu(\gamma_{n+1}|T^{-1}\gamma)$ $\le H_\mu(\gamma|T^{-1}\gamma)$, thus

$$\lim_{n\to\infty} H_\mu(\gamma_n|T^{-1}\gamma) = H_\mu(\gamma|T^{-1}\gamma).$$

Combing this with the above, we get

$$H_\mu(\gamma|T^{-1}\gamma) = \lim_{n\to\infty} H_\mu(\gamma_n|T^{-1}\gamma) \le h(X,\mathfrak{B},\mu,T).$$

Since γ was an arbitrary measurable μ-partition with $T^{-1}\gamma \preceq \gamma$, we finally get

$$h_{Pa}(X,\mathfrak{B},\mu,T) = \sup_{\gamma\,:\,T^{-1}\gamma\preceq\gamma} H_\mu(\gamma|T^{-1}\gamma) \le h(X,\mathfrak{B},\mu,T).$$

Remark 4.3. Analogous to Remark 3.2 (3) we have

$$h_{Pa}(X,\mathfrak{B}_1,\mu,T) \le h_{Pa}(X,\mathfrak{B}_2,\mu,T) \le h_{Pa}(X,\mathfrak{B},\mu,T)$$

for σ-finite sub-σ-algebras $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$ of $\mathfrak{B}$ and

$$\lim_{n\to\infty} h_{Pa}(X,\mathfrak{B}_n,\mu,T) = h_{Pa}(X,\mathfrak{B},\mu,T)$$

for a monotonously increasing sequence of σ-finite sub-σ-algebras $(\mathfrak{B}_n)_{n\in\mathbb{N}}$ with $\sigma(\bigcup_{n\in\mathbb{N}}\mathfrak{B}_n) = \mathfrak{B} \bmod \mu$. This is obvious, since

$$h_{Pa}(X,\mathfrak{B}_1,\mu,T) = \sup_{\substack{\mathfrak{C}\sigma\text{-finite sub-}\sigma\text{-algabra of }\mathfrak{B}_1\\ T^{-1}\mathfrak{C}\subseteq\mathfrak{C}}} H_\mu(\mathfrak{C}|T^{-1}\mathfrak{C}) \le$$

$$\le \sup_{\substack{\mathfrak{C}\ \sigma\text{-finite sub-}\sigma\text{-algabra of }\mathfrak{B}_2\\ T^{-1}\mathfrak{C}\subseteq\mathfrak{C}}} H_\mu(\mathfrak{C}|T^{-1}\mathfrak{C}) = h_{Pa}(X,\mathfrak{B}_2,\mu,T).$$

Theorem 4.1. *If T is a conservative, ergodic, measure-preserving transformation on a σ-finite standard measure space $(X,\mathfrak{B},\mu)$ with $\mu(X) = \infty$, then*

$$h_{Pa}(X,\mathfrak{B},\mu,T) \le h(X^\star,\mathfrak{B}^\star,\mu^\star,T^\star).$$

Proof. Given a σ-finite sub-invariant σ-algebra $\mathfrak{C}$ of $\mathfrak{B}$, i.e. $T^{-1}\mathfrak{C} \subseteq \mathfrak{C}$, we know by Remark 1.3, that it has no atoms. Since null-sets (which are contained in a null-set) do not have any impact on the entropy[20], we may w.l.o.g. assume that $\mathfrak{B}$ is complete. Thus Lemma 4.2 gives

$$H_\mu(\mathfrak{C}|T^{-1}\mathfrak{C}) = H_{\mu^\star}(\mathfrak{C}^\star|T^{\star-1}\mathfrak{C}^\star),$$

since $(T^{-1}\mathfrak{C})^\star = \sigma(N_A : A \in T^{-1}\mathfrak{C}) = \sigma(N_{T^{-1}C} : C \in \mathfrak{C}) = \sigma(T^{\star-1}N_C C \in \mathfrak{C}) = T^{\star-1}\mathfrak{C}^\star$, because $N_{T^{-1}C}^{-1}(\{n\}) = \{\nu \in X^\star : \nu(T^{-1}C) = n\} = \{\nu \in X^\star : (T^\star(\nu))(C) = n\} =$

[20]Confer footnote in proof of Lemma 4.2

$T^{\star -1}(N_C^{-1}(\{n\}))$ for $C \in \mathfrak{C}$, $n \in \overline{\mathbb{N}}_0$.
Thus,

$$H_\mu(\mathfrak{C}|T^{-1}\mathfrak{C}) \leq \sup_{T^{\star -1}\mathfrak{D}^\star \subseteq \mathfrak{D}^\star} H_{\mu^\star}(\mathfrak{D}^\star|T^{\star -1}\mathfrak{D}^\star),$$

where $\mathfrak{D}^\star$ shall be a sub-σ-algebra of $\mathfrak{B}^\star$. Therefore,

$$h_{Pa}(X,\mathfrak{B},\mu,T) = \sup_{T^{-1}\mathfrak{C}\subseteq\mathfrak{C}} H_\mu(\mathfrak{C}|T^{-1}\mathfrak{C}) \leq \sup_{T^{\star -1}\mathfrak{D}^\star \subseteq \mathfrak{D}^\star} H_{\mu^\star}(\mathfrak{D}^\star|T^{\star -1}\mathfrak{D}^\star).$$

And since $\mu^\star$ is a $T^\star$-invariant probability measure, we obtain by Remark 4.2, that $\sup_{T^{\star -1}\mathfrak{D}^\star \subseteq \mathfrak{D}^\star} H_{\mu^\star}(\mathfrak{D}^\star|T^{\star -1}\mathfrak{D}^\star) = h(X^\star,\mathfrak{B}^\star,\mu^\star,T^\star)$. □

4.3 Krengel's entropy

Krengel's idea ([14]) was to fix measurable set and to look at the dynamics of the set given by the first return map of that set.

Definition. *Krengel's entropy* is defined by

$$h_{Kr}(X,\mathfrak{B},\mu,T) := \sup_{A\in\mathfrak{B},\ 0<\mu(A)<\infty} \mu(A)\cdot h(A,\mathfrak{B}\cap A,\frac{\mu|_A}{\mu(A)},T_A),$$

where T_A is the first return map of T on A (see section 1.3.2).

A proof of the following well-known Proposition can be found in [14] and references given there.

Proposition 4.2. *In fact, we have*

$$h_{Kr}(X,\mathfrak{B},\mu,T) = \mu(A)\cdot h(A,\mathfrak{B}\cap A,\frac{\mu|_A}{\mu(A)},T_A)$$

for every arbitrary $A \in \mathfrak{B}$ of positive finite measure.

In particular, this Proposition shows that Krengel's entropy indeed extends the Kolmogorov-Sinai entropy: For $\mu(X) = 1$ the above identity holds for the positive finite measure set X, i.e. $h_{Kr}(X,\mathfrak{B},\mu,T) = \mu(X)\cdot h(X,\mathfrak{B}\cap X,\frac{\mu|_X}{\mu(X)},T_X) = h(X,\mathfrak{B},\mu,T)$.

Note, that the Krengel entropy of two isomorphic dynamical systems is equal (see [14]).

4.3.1 Computing Krengel's entropy: Some examples

Due to the following theorem of [21], the well-known Rokhlin-formular for the entropy of probability systems carries over to the infinite case, at least for a large class of systems:

Theorem 4.2. *Let T be a Thaler-map on $[0,1]$ and μ be a T-invariant measure on $([0,1],\mathfrak{B}([0,1]))$, then*

$$h_{Kr}([0,1],\mathfrak{B}([0,1]),\mu,T) = \int_{[0,1]} \log(T'(x))\,d\mu(x).$$

For a proof we refer to [21].

Example 4.1 (Boole's transformation). Let us consider Boole's transformation $Tx = x - \frac{1}{x}$ on $\mathbb{R}$ and its isomorphic system $((0,1), \mathfrak{B}((0,1)), \eta, S)$ with

$$\eta([a,b]) = \int_a^b \Big(\frac{1}{(1-x)^2} + \frac{1}{x^2}\Big)dx, \;\; S(x) = \begin{cases} \dfrac{x(1-x)}{1-x-x^2} & \text{for } x \in (0, \frac{1}{2}), \\ \dfrac{2x-1}{3x-x^2-1} & \text{for } x \in (\frac{1}{2}, 1) \end{cases}.$$

Since η is S-invariant and S is a Thaler-map we can use the theorem above and obtain

$$h_{Kr}((\mathbb{R}, \mathfrak{B}(\mathbb{R}), \lambda, T) = h_{Kr}((0,1), \mathfrak{B}((0,1)), \eta, S) = \int_{(0,1)} \log(S'(x))\, d\eta(x) =$$

$$= \int_{(0,\frac{1}{2})} \log\Big(\frac{1-2x+2x^2}{(1-x-x^2)^2}\Big)\Big(\frac{1}{(1-x)^2} + \frac{1}{x^2}\Big)\, dx +$$

$$\int_{(\frac{1}{2},1)} \log\Big(\frac{1-2x+2x^2}{(3x-x^2-1)^2}\Big)\Big(\frac{1}{(1-x)^2} + \frac{1}{x^2}\Big)\, dx =$$

$$= 2\pi.$$

Example 4.2 (Renewal chains). Let us consider an irreducible null-recurrent renewal chain on $\mathbb{N}$ with $p_{1m} = f_m$,$p_{mm-1} = 1$ for $m \geq 2$ and $u_m = \sum_{l \geq m} f_l$ for every $m \in \mathbb{N}$. Further, let S be the shift of the corresponding space $(X, \mathfrak{B}, \mu)$ (confer Example 1.3). The cylinder set $Y := [1]$ is a sweep-out set with $\mu(Y) = 1$, thus

$$h_{Kr}(X, \mathfrak{B}, \mu, S) = \mu(Y) \cdot h(Y, \mathfrak{B} \cap Y, \frac{\mu}{\mu(Y)}, S_Y) = h(Y, \mathfrak{B} \cap Y, \mu, S_Y).$$

Let us consider the countable partition $\beta := \{[1m] : m \in \mathbb{N}\}$ of Y. In fact, β is (modulo μ) the first return time partition $\rho_{S,[1]}$ of the set $[1]$. This will be useful later on. Now,

$$\sigma(\bigvee_{k=0}^{\infty} S_Y^{-k}\beta) = \mathfrak{B} \cap Y = \sigma([1a_2 \ldots a_n] : a_2, \ldots a_n \in \mathbb{N}, n \in \mathbb{N}) \bmod \mu,$$

because for $a_2, \ldots a_n \in \mathbb{N}, n \in \mathbb{N}$ with $\mu([1a_2 \ldots a_n]) \neq 0$ there exist $m_1, m_2, \ldots, m_l \in \mathbb{N}, l \in \mathbb{N}$ such that modulo μ we get the following equality

$$[1a_2 \ldots a_n] = [1m_1 m_1 - 1 \ldots 1m_2 m_2 - 1 \ldots 1m_3 \ldots m_l] =$$

$$= [1m_1] \cap S_Y^{-1}[1m_2] \cap S_Y^{-2}[1m_3] \cap \ldots \cap S_Y^{-l+1}[1m_l].$$

(Note that we could assume that the cylinder ends at m_l because $[\ldots m_l] = [\ldots m_l m_l - 1 \ldots m_l - k] = [\ldots m_l \ldots 1]$ modulo μ for every $k < m_l$.)
Further,

$$\beta, S_Y^{-1}\beta, \ldots, S_Y^{-n}\beta \text{ are independent.}$$

Indeed, let $k < l \in \mathbb{N}$ and $m_{k+1}, m_{l+1} \in \mathbb{N}$ be fixed, then

$$S_Y^{-k}[1m_{k+1}] \cap S_Y^{-l}[1m_{l+1}] =$$

$$= \bigcup_{i=1}^{\infty} \bigcup_{\sum_{j=1}^{k} \tilde{n_j} = i} [1\tilde{n_1} \ldots 1\tilde{n_2} \ldots 1\tilde{n_k} \ldots 1m_{k+1}] \cap \bigcup_{i=1}^{\infty} \bigcup_{\sum_{j=1}^{l} n_j = i} [1n_1 \ldots 1n_2 \ldots 1n_l \ldots 1m_{l+1}] =$$

$$= \bigcup_{i=1}^{\infty} \bigcup_{\sum_{j=1}^{l} n_j = i} [1n_1 \dots 1n_2 \dots 1m_{k+1} \dots 1n_l \dots 1m_{l+1}] =$$

$$= \bigcup_{n_1,\dots,n_l \in \mathbb{N}} [1n_1 \dots 1n_2 \dots 1m_{k+1} \dots 1n_l \dots 1m_{l+1}]$$

(since the intersection is only not empty if $n_i = \tilde{n}_i$, $\forall i \leq k$), hence

$$\mu(S_Y^{-k}[1m_{k+1}] \cap S_Y^{-l}[1m_{l+1}]) = \sum_{n_1,\dots,n_l \in \mathbb{N}} \mu([1n_1 \dots 1n_2 \dots 1m_{k+1} \dots 1n_l \dots 1m_{l+1}]) =$$

$$= \sum_{n_1,\dots,n_l \in \mathbb{N}} u_1 p_{1n_1} 1 \dots 1 p_{1n_2} \dots p_{1m_{k+1}} \dots p_{1n_l} \dots p_{1m_{l+1}} = p_{1m_{k+1}} p_{1m_{l+1}} =$$

$$= \mu(S_Y^{-k}[1m_{k+1}]) \mu(S_Y^{-l}[1m_{l+1}]),$$

because $\sum_{n \in \mathbb{N}} p_{1n} = 1$.
Now, since β is a generator of $\mathfrak{B} \cap Y$ and $\beta, S_Y^{-1}\beta, \dots S_Y^{-n}\beta$ are independent for every $n \in \mathbb{N}$, we can use Theorem 3.3 and obtain

$$h(Y, \mathfrak{B} \cap Y, \mu, S_Y) = h(Y, \mathfrak{B} \cap Y, \mu, S_Y, \beta) = H_\mu(\beta).$$

And

$$H_\mu(\beta) = -\sum_{m \in \mathbb{N}} p_{1m} \log(p_{1m}) = -\sum_{m \in \mathbb{N}} f_m \log(f_m),$$

so if we choose f_m such that $f_m = c\frac{1}{m^{1+\alpha}}$ for $\alpha \in (0,1)$, $c \in (0,\infty)$ fixed, we get

$$H_\mu(\beta) = -\sum_{m \in \mathbb{N}} c \frac{1}{m^{1+\alpha}} \log\Big(c \frac{1}{m^{1+\alpha}}\Big) =$$

$$= c\Big(-\sum_{m \in \mathbb{N}} \frac{1}{m^{1+\alpha}} \log(c) + \sum_{m \in \mathbb{N}} \frac{1}{m^{1+\alpha}} (1+\alpha) \log(m)\Big).$$

To ensure that $\sum_{n \in \mathbb{N}} f_n = 1$, one can just take $c = \big(\sum_{n \in \mathbb{N}} \frac{1}{n^{1+\alpha}}\big)^{-1}$, then

$$H_\mu(\beta) = \Big(\sum_{n \in \mathbb{N}} \frac{1}{n^{1+\alpha}}\Big)^{-1} \Big(\sum_{m \in \mathbb{N}} \frac{1}{m^{1+\alpha}} \log\Big(\sum_{n \in \mathbb{N}} \frac{1}{n^{1+\alpha}}\Big) + \sum_{m \in \mathbb{N}} \frac{1}{m^{1+\alpha}} (1+\alpha) \log(m)\Big) =$$

$$= \log\Big(\sum_{n \in \mathbb{N}} \frac{1}{n^{1+\alpha}}\Big) + \Big(\sum_{n \in \mathbb{N}} \frac{1}{n^{1+\alpha}}\Big)^{-1} \sum_{m \in \mathbb{N}} \frac{1}{m^{1+\alpha}} (1+\alpha) \log(m).$$

Thus,

$$h_{Kr}(X, \mathfrak{B}, \mu, S) = H_\mu(\beta) = \log\Big(\sum_{n \in \mathbb{N}} \frac{1}{n^{1+\alpha}}\Big) + (1+\alpha)\Big(\sum_{n \in \mathbb{N}} \frac{1}{n^{1+\alpha}}\Big)^{-1} \sum_{m \in \mathbb{N}} \frac{\log(m)}{m^{1+\alpha}}.$$

In general, Krengel's entropy of any null-recurrent Markov-chain is given by an explicit formula, proven in [14]:

Theorem 4.3. *Let S be the Markov shift, as in Example 1.3, corresponding to a null-recurrent Markov chain on the countable state space $\Sigma = \{a_1, a_2, \dots\}$, defined by the transition matrix $(p_{a_i a_j})_{i,j \in \mathbb{N}}$ and the stationary distribution $(u_{a_i})_{i \in \mathbb{N}}$ with $u_{a_i} = \sum_{j \in \mathbb{N}} u_{a_j} p_{a_j a_i}$, $\forall i \in \mathbb{N}$. Then,*

$$h_{Kr}(X, \mathfrak{B}, \mu, S) = -\sum_{a \in \Sigma} u_a \sum_{b \in \Sigma} p_{ab} \log(p_{ab}),$$

where $X := \Sigma^{\mathbb{N}}$, $\mathfrak{B}$ *is the* σ*-algebra generated by the cylinders* $[a_{i_1} \dots a_{i_n}] := \{(x_1, x_2, \dots) \in \Sigma^{\mathbb{N}} : x_k = a_{i_k}, \forall k \in \{1, \dots, n\}\}$ *for* $i_1, \dots i_n \in \mathbb{N}$, $n \in \mathbb{N}$, *and* μ *the measure given by* $\mu([a_{i_1} \dots a_{i_n}]) := u_{a_{i_1}} p_{a_{i_1} a_{i_2}} \cdots p_{a_{i_{n-1}} a_{i_n}}$, *and* S *is the shift on* X.

Example 4.3 (Renewal chain)**.** With the notation of Example 4.2, the above Theorem gives

$$h_{Kr}(X, \mathfrak{B}, \mu, S) = -\sum_{a \in \mathbb{N}} u_a \sum_{b \in \mathbb{N}} p_{ab} \log(p_{ab}) =$$

$$= -\sum_{b \in \mathbb{N}} p_{1b} \log(p_{1b}) - \sum_{a \in \mathbb{N}, a > 1} \sum_{l \geq a} f_l p_{aa-1} \underbrace{\log(p_{aa-1})}_{=0} = -\sum_{b \in \mathbb{N}} f_b \log(f_b),$$

which was exactly the entropy of the induced system in Example 4.2.

Example 4.4 (Random walk on $\mathbb{Z}$)**.** Let $p_{aa+1} = \frac{1}{2} = p_{aa-1}$ and $u_a = 1$ for all $a \in \mathbb{Z}$. By Theorem 4.3,

$$h_{Kr}(X, \mathfrak{B}, \mu, S) = -\sum_{a \in \mathbb{Z}} u_a \sum_{b \in \mathbb{Z}} p_{ab} \log(p_{ab}) =$$

$$= -\sum_{a \in \mathbb{Z}} (p_{aa-1} \log(p_{aa-1}) + p_{aa+1} \log(p_{aa+1})) = \infty \cdot \log(2) = \infty.$$

Remark 4.4. The entropy of a Thaler-map and the entropy of an irreducible null-recurrent Markov chain are always positive. Indeed, since the image of a Thaler-map T is equal to $[0,1]$ when taking the closure, its slope must be at least as big as the slope of the map $x \mapsto x$ on almost every point, hence $T' \geq 1$, thus $\log(T') \geq 0$. Moreover, since a Thaler-map has at most finitely many points with slope 1, we get $\int_{[0,1]} \log(T'(x))\, d\mu(x) > 0$.
For an irreducible null-recurrent Markov chain, we have $\sum_{a \in \Sigma} u_a \sum_{b \in \Sigma} p_{ab}(-\log(p_{ab})) > 0$ if there is an $a \in \Sigma$ and an $b \in \Sigma$ such that $p_{ab} \in (0,1)$. If for every $a, b \in \Sigma$ we would have $p_{ab} = 0$ or $p_{ab} = 1$, then the process cannot be[21] null-recurrent and irreducible.

4.4 Equality of Poisson entropy, Krengel's entropy and Parry's entropy

Now we are finally able to formulate and prove the main result about the relations of the different notions of entropy, which is due to [10] and [15].

In [15] Parry has proven that his notion of entropy and Krengel's coincide in the following case (see [15], Theorem 10.10, for a proof)

Theorem 4.4. Let T be an ergodic, conservative, measure-preserving automorphism on a σ-finite standard measure space $(X, \mathfrak{B}, \mu)$. If there exists a set $A \in \mathfrak{B}$ such that $0 < \mu(A) < \infty$ which is quasi finite, then

$$h_{Pa}(X, \mathfrak{B}, \mu, T) = h_{Kr}(X, \mathfrak{B}, \mu, T).$$

Due to [10] (Theorem 9.1) we obtain equality of all three definitions of entropy under these assumptions:

[21] Assume that for every $a, b \in \Sigma$, p_{ab} is equal to 1 or to 0. By $\sum_{a \in \Sigma} p_{ba} = 1$ for every $b \in \Sigma$, if $p_{ab} = 1$, then $p_{ac} = 0$ for all $c \in \Sigma$, $c \neq b$. By irreducibility, every element has to be visited somehow. So, $p_{a_{j_1} a_{j_2}} = 1$, $p_{a_{j_2} a_{j_3}} = 1$, … for $a_{j_1}, a_{j_2}, \dots \in \Sigma$ all distinct and such that $\{a_{j_k} : k \in \mathbb{N}\} = \Sigma$. But, since Σ is infinite by null-recurrence, there is no positive probability of going back to a state again, a contradiction to recurrence.

Theorem 4.5. *Let $(X,\mathfrak{B},\mu)$ be a σ-finite standard measure space with $\mu(X)=\infty$ and T a measure-preserving, conservative, ergodic automorphism on X, such that there is a quasi finite set of finite positive measure. Then the different notions of entropy are equal:*

$$h(X^\star,\mathfrak{B}^\star,\mu^\star,T^\star) = h_{Pa}(X,\mathfrak{B},\mu,T) = h_{Kr}(X,\mathfrak{B},\mu,T).$$

We are following [10].

Proof. By Theorem 4.4 we already know that Parry's and Krengel's entropy coincide under these assumptions. Thus, we only need to show that Parry's entropy and the entropy of the corresponding Poisson suspension are equal.
Let $A\in\mathfrak{B}$, $\mu(A)\in(0,\infty)$, be a quasi finite set and let α be a local μ-partition of X, whose core is A. (Recall, that to every measurable set of positive finite measure we can find a local μ-partition such that this set is its core.) W.l.o.g. let $\alpha\setminus\{A^c\}\succeq\rho_{T^{-1},A}$. (Indeed, if $\alpha\setminus\{A^c\}$ is not finer than $\rho_{T^{-1}A}$, then we just replace it by $(\alpha\setminus\{A^c\})\vee\rho_{T^{-1},A}$.) Moreover, let α be such that its elements have measure smaller or equal to 1 or equal to infinity. By Proposition 4.1 we know, that

$$h\Big(X^\star,\Big(\widetilde{\sigma}\Big(\bigvee_{k=0}^{\infty}T^{-k}\alpha\Big)\Big)^\star,\mu^\star,T^\star\Big)\leq\hat{h}(X,\mathfrak{B},\mu,T^{-1},\alpha)$$

and by Lemma 3.3 we have

$$\hat{h}(X,\mathfrak{B},\mu,T^{-1},\alpha)=H_\mu\Big(\bigvee_{k=0}^{\infty}T^{-k}\alpha|\bigvee_{k=1}^{\infty}T^{-k}\alpha\Big),$$

since, due to ergodicity and conservativity, A is a sweep-out set by Remark 1.3.

Note, that $\widetilde{\sigma}\big(\bigvee_{k=0}^{\infty}T^{-k}\alpha\big)$ is σ-finite, since $X=\bigcup_{n\geq0}T^{-n}A$ mod μ, $\mu(T^{-n}A)=\mu(A)<\infty$ and $T^{-n}A\in\widetilde{\sigma}\big(\bigvee_{k=0}^{\infty}T^{-k}\alpha\big)$. (Indeed, $T^{-n}A=\bigcup_{A_k\in\alpha:A_n\neq A^c,k\in\mathbb{N}_0}\bigcap_{i=0}^{\infty}T^{-i}A_i$, which is a uncountable union of elements of α_0^∞ and, moreover, $T^{-n}A\in\mathfrak{B}$, thus $T^{-n}A\in\widetilde{\sigma}(\alpha_0^\infty)$.)

Further,

$$H_\mu\Big(\bigvee_{k=0}^{\infty}T^{-k}\alpha|\bigvee_{k=1}^{\infty}T^{-k}\alpha\Big)=H_\mu\Big(\bigvee_{k=0}^{\infty}T^{-k}\alpha|T^{-1}\Big(\bigvee_{k=0}^{\infty}T^{-k}\alpha\Big)\Big)\leq$$

$$\leq\sup_{\substack{\mathfrak{C}\ \sigma\text{-finite sub-}\sigma\text{-algebra of }\widetilde{\sigma}(\bigvee_{k=0}^{\infty}T^{-k}\alpha)\\ T^{-1}\mathfrak{C}\subseteq\mathfrak{C}}}H_\mu(\mathfrak{C}|T^{-1}\mathfrak{C})=h_{Pa}\Big(X,\widetilde{\sigma}\Big(\bigvee_{k=0}^{\infty}T^{-k}\alpha\Big),\mu,T\Big)$$

since, clearly, $T^{-1}\big(\widetilde{\sigma}\big(\bigvee_{k=0}^{\infty}T^{-k}\alpha\big)\big)=\widetilde{\sigma}\big(\bigvee_{k=1}^{\infty}T^{-k}\alpha\big)\subseteq\widetilde{\sigma}\big(\bigvee_{k=0}^{\infty}T^{-k}\alpha\big)$.
Moreover, by Theorem 4.1 we have

$$h_{Pa}\Big(X,\widetilde{\sigma}\Big(\bigvee_{k=0}^{\infty}T^{-k}\alpha\Big),\mu,T\Big)\leq h\Big(X^\star,\Big(\widetilde{\sigma}\Big(\bigvee_{k=0}^{\infty}T^{-k}\alpha\Big)\Big)^\star,\mu^\star,T^\star\Big).$$

Hence, we have shown that

$$h\Big(X^\star,\Big(\widetilde{\sigma}\Big(\bigvee_{k=0}^{\infty}T^{-k}\alpha\Big)\Big)^\star,\mu^\star,T^\star\Big)\leq\hat{h}(X,\mathfrak{B},\mu,T^{-1},\alpha)=$$

$$= H_\mu\Big(\bigvee_{k=0}^{\infty} T^{-k}\alpha | \bigvee_{k=1}^{\infty} T^{-k}\alpha\Big) \leq h_{Pa}\Big(X, \widetilde{\sigma}\Big(\bigvee_{k=0}^{\infty} T^{-k}\alpha\Big), \mu, T\Big) \leq$$

$$\leq h\Big(X^\star, \Big(\widetilde{\sigma}\Big(\bigvee_{k=0}^{\infty} T^{-k}\alpha\Big)\Big)^\star, \mu^\star, T^\star\Big),$$

thus

$$h\Big(X^\star, \Big(\widetilde{\sigma}\Big(\bigvee_{k=0}^{\infty} T^{-k}\alpha\Big)\Big)^\star, \mu^\star, T^\star\Big) = H_\mu\Big(\bigvee_{k=0}^{\infty} T^{-k}\alpha | \bigvee_{k=1}^{\infty} T^{-k}\alpha\Big) = \tag{14}$$

$$= h_{Pa}\Big(X, \widetilde{\sigma}\Big(\bigvee_{k=0}^{\infty} T^{-k}\alpha\Big), \mu, T\Big).$$

Let us now construct a sequence of refining μ-partitions $(\alpha_m)_{m\in\mathbb{N}_0}$ which are all inverse quasi finite with core A, such that $\sigma\big(\bigcup_{m=0}^{\infty} \widetilde{\sigma}\big(\bigvee_{k=0}^{\infty} T^{-k}(\alpha_m)\big)\big) = \mathfrak{B} = \widetilde{\sigma}(\varepsilon) \bmod \mu$. By measureability of ε (confer Example 1.1), there is a countable set $\Delta = \{D_1, D_2, \ldots\}$ such that $\{x\} = \bigcap_{n\in\mathbb{N}} S_n$ for all $x \in X$ with either $S_n = D_n$ or $S_n = D_n^c$ for all $n \in \mathbb{N}$. Let us take an explicit set for Δ, namely let us consider an isomorphism $\psi : X \longrightarrow \mathbb{R}$ (which exists by Theorem 1.1) and take $\Delta = \{D_1, D_2, \ldots\}$ such that its images under ψ are the Dyadic intervals in $\mathbb{R}$, i.e. $D_m := \psi^{-1}\big(\bigcup_{n\in\mathbb{Z}} [\frac{n}{2^m}, \frac{n-1}{2^m})\big)$ for every $m = 1, 2, \ldots$. We define:

$$\alpha_0 := \alpha$$

$$\alpha_1 := (\alpha|_A \cap D_1) \cup (\alpha|_A \cap D_1^c) \cup \{A^c\}$$

$$\alpha_2 := (\alpha|_A \cap D_1 \cap D_2) \cup (\alpha|_A \cap D_1^c \cap D_2) \cup (\alpha|_A \cap D_1 \cap D_2^c) \cup (\alpha|_A \cap D_1^c \cap D_2^c) \cup \{A^c\}$$

$$\vdots$$

$$\alpha_m := \bigcup_{S_n = D_n \text{ or } S_n = D_n^c} \Big(\alpha|_A \cap \bigcap_{n=1}^{m} S_n\Big) \cup \{A^c\}.$$

Then, clearly, these are (countable) local μ-partitions with core A such that $\rho_{T^{-1},A} \preceq \alpha = \alpha_0 \preceq \alpha_1 \preceq \alpha_2 \preceq \ldots$. So, α_m corresponds to the dyadic intervals of $\psi(A)$ of level m, i.e. $\psi(\alpha_m) = \big(\psi(A) \cap \bigcap_{n=1}^{m} \bigcup_{n\in\mathbb{Z}} [\frac{n}{2^m}, \frac{n-1}{2^m})\big) \cup \{\psi(A^c)\}$. The dyadic intervals generate the Borel-σ-algebra of $\mathbb{R}$, therefore,

$$\sigma(\alpha_m|_A : m \in \mathbb{N}_0) = \mathfrak{B} \cap A \bmod \mu.$$

For $k \in \mathbb{N}_0$ fixed we have

$$\sigma(T^{-k}(\alpha_m|_A) : m \in \mathbb{N}_0) = T^{-k}(\mathfrak{B} \cap A) = \mathfrak{B} \cap T^{-k}A \bmod \mu,$$

since T is an automorphism. Hence

$$\sigma\Big(\bigcup_{k\in\mathbb{N}_0} \sigma(T^{-k}(\alpha_m|_A) : m \in \mathbb{N}_0)\Big) = \sigma\Big(\bigcup_{k\in\mathbb{N}_0} \mathfrak{B} \cap T^{-k}A\Big) = \mathfrak{B} \bmod \mu,$$

because A is a sweep-out set. In particular, we obtain that

$$\sigma\Big(\bigcup_{m\in\mathbb{N}_0} \widetilde{\sigma}\Big(\bigvee_{k=0}^{\infty} T^{-k}\alpha_m\Big)\Big) =$$

$$= \sigma\Big(\Big\{\bigcup_{l\in L}\bigcap_{k=0}^{\infty} T^{-k}A_{l_k}^{(m)}\Big|A_{l_k}^{(m)} \in \alpha_m, \bigcup_{l\in L}\bigcap_{k=0}^{\infty} T^{-k}A_{l_k}^{(m)} \in \mathfrak{B}, L \text{ some index set}, m \in \mathbb{N}_0\Big\}\Big) \supseteq$$

$$\supseteq \sigma\Big(\bigcup_{k\in\mathbb{N}_0} \sigma(T^{-k}(\alpha_m|_A) : m \in \mathbb{N}_0)\Big) = \mathfrak{B} \bmod \mu,$$

thus

$$\sigma\Big(\bigcup_{m\in\mathbb{N}_0} \widetilde{\sigma}\Big(\bigvee_{k=0}^{\infty} T^{-k}\alpha_m\Big)\Big) = \mathfrak{B} \bmod \mu.$$

Similarly, one can deduce that

$$\sigma\Big(\bigcup_{m\in\mathbb{N}_0} \Big(\widetilde{\sigma}\Big(\bigvee_{k=0}^{\infty} T^{-k}\alpha_m\Big)^{\star}\Big)\Big) = \mathfrak{B}^{\star} \bmod \mu^{\star}.$$

Therefore, Proposition 3.2 (3) gives

$$\lim_{m\to\infty} h\Big(X^{\star}, \Big(\widetilde{\sigma}\Big(\bigvee_{k=0}^{\infty} T^{-k}\alpha_m\Big)\Big)^{\star}, \mu^{\star}, T^{\star}\Big) = h(X^{\star}, \mathfrak{B}^{\star}, \mu^{\star}, T^{\star}).$$

Now, using equation (14) with α_m instead of α, we can deduce the desired equality:

$$h(X^{\star}, \mathfrak{B}^{\star}, \mu^{\star}, T^{\star}) = \lim_{m\to\infty} h\Big(X^{\star}, \Big(\widetilde{\sigma}\Big(\bigvee_{k=0}^{\infty} T^{-k}\alpha_m\Big)\Big)^{\star}, \mu^{\star}, T^{\star}\Big) =$$

$$= \lim_{m\to\infty} h_{Pa}\Big(X, \widetilde{\sigma}\Big(\bigvee_{k=0}^{\infty} T^{-k}\alpha_m\Big), \mu, T\Big) = h_{Pa}(X, \mathfrak{B}, \mu, T),$$

by Remark 4.3. □

If all assumptions of the theorem above are fulfilled beside the one of T being an automorphism, we can pass to the natural extension to apply the theorem. In order to receive a result for the underlying space, we need the following remark:

Remark 4.5. *Let T be a measure-preserving, conservative, ergodic transformation on a σ-finite standard measure space $(X, \mathfrak{B}, \mu)$ with $\mu(X) = \infty$ and let $(\widetilde{X}, \widetilde{\mathfrak{B}}, \widetilde{\mu}, \widetilde{T})$ be its natural extension, w.l.o.g. constructed like in Theorem 1.8. We consider $B' := \tau^{-1}(B) = \{(x_0, x_1, x_2, \dots) \in \widetilde{X} | x_0 \in B\}$ corresponding to a given set $B \in \mathfrak{B}$, where $\tau : \widetilde{X} \longrightarrow X$ denotes the factor map of the natural extension.*

1. *A set $A \in \mathfrak{B}$ is a quasi finite set of positive finite measure if and only if A' is a quasi finite set in $(\widetilde{X}, \widetilde{\mathfrak{B}}, \widetilde{\mu}, \widetilde{T})$ of positive finite measure.*

2. *Since the natural extension of the Poisson suspension is the Poisson suspension of the natural extension due to [25] (see proof of Theorem 2.2), we obtain*

$$h_{Poi}(\widetilde{X}, \widetilde{\mathfrak{B}}, \widetilde{\mu}, \widetilde{T}) = h(\widetilde{X}^{\star}, \widetilde{\mathfrak{B}}^{\star}, \widetilde{\mu}^{\star}, \widetilde{T}^{\star}) = h(\widetilde{X^{\star}}, \widetilde{\mathfrak{B}^{\star}}, \widetilde{\mu^{\star}}, \widetilde{T^{\star}}) = h_{Poi}(X, \mathfrak{B}, \mu, T),$$

since the entropy is invariant under isomorphisms (Proposition 3.3) and the dynamical entropy of a probability system equals the dynamical entropy of the natural extension of that system (see e.g. [4], Fact 4.3.2).

3. *Let us have a closer look at the Krengel entropy of the natural extension: For $B \in \mathfrak{B}$ fixed such that $\mu(B) < \infty$ we see that*

$$h_{Kr}(\widetilde{X}, \widetilde{\mathfrak{B}}, \widetilde{\mu}, \widetilde{T}) = \widetilde{\mu}(B') \cdot h(B', \widetilde{\mathfrak{B}} \cap B', \frac{\widetilde{\mu}}{\widetilde{\mu}(B')}, \widetilde{T}_{B'}) =$$

$$= \mu(B) \cdot h(\widetilde{B}, \widetilde{\mathfrak{B}} \cap \widetilde{B}, \frac{\widetilde{\mu}}{\mu(B)}, \widetilde{T}_B) = \mu(B) \cdot h(B, \mathfrak{B} \cap B, \frac{\mu}{\mu(B)}, T_B) = h_{Kr}(X, \mathfrak{B}, \mu, T),$$

since the induced system of a natural extension is isomorphic to the natural extension of the induces system, i.e.

$$(B', \widetilde{\mathfrak{B}} \cap B', \frac{\widetilde{\mu}}{\mu(B)}, (\widetilde{T})_{B'}) \approx (\widetilde{B}, \widetilde{\mathfrak{B} \cap B}, \frac{\widetilde{\mu}}{\mu(B)}, \widetilde{(T_B)}).$$

4. *Moreover,*

$$h_{Pa}(\widetilde{X}, \widetilde{\mathfrak{B}}, \widetilde{\mu}, \widetilde{T}) \geq h_{Pa}(X, \mathfrak{B}, \mu, T).$$

Proof of (1). Let us state some simple properties of B', C' corresponding to $B, C \in \mathfrak{B}$ given: Clearly,

$$\widetilde{T}^{-k}(B') = \{(x_0, x_1, x_2, \ldots) \in \widetilde{X} | (T^k x_0, T^{k-1} x_0, \ldots, x_0, x_1, \ldots) \in B'\} = (T^{-k}B)',$$

$$(B')^c = \{(x_0, x_1, \ldots) \in \widetilde{X} | x_0 \notin B\} = (B^c)',$$

and

$$B' \cap C' = (B \cap C)'.$$

Therefore, for a given set $A \in \mathfrak{B}$, we deduce that

$$\rho_{\widetilde{T},A'} = \left\{A' \cap \widetilde{T}^{-n}A' \cap \bigcap_{k=1}^{n-1} \widetilde{T}^{-k}(A')^c \middle| n \in \mathbb{N}\right\} = \left\{\left(A \cap T^{-n}A \cap \bigcap_{k=1}^{n-1} T^{-k}A^c\right)' \middle| n \in \mathbb{N}\right\}.$$

Thus, since $\widetilde{\mu}(B') = \mu(B)$ for every $B \in \mathfrak{B}$, we get

$$H_{\widetilde{\mu}}(\rho_{\widetilde{T},A'}) = H_\mu(\rho_{T,A}).$$

Hence, A' is quasi finite if and only if A is quasi finite. Moreover, $\widetilde{\mu}(A') = \mu(A)$. $\square$

Proof of (4). Using the transformation formula and T-invariance, we compute

$$h_{Pa}(X, \mathfrak{B}, \mu, T) = \sup_{\substack{\gamma \text{ measurable partition of } X, \\ \widetilde{\sigma}(\gamma)\ \sigma\text{-finite},\ T^{-1}\gamma \preceq \gamma}} H_\mu(\gamma | T^{-1}\gamma) =$$

$$= \sup_{\substack{\gamma \text{ measurable partition of } X \\ \widetilde{\sigma}(\gamma)\ \sigma\text{-finite},\ T^{-1}\gamma \preceq \gamma}} \int_X H_{\mu_{\pi_{T^{-1}\gamma}(x)}}(\gamma \cap \pi_{T^{-1}\gamma}(x))\, d \underbrace{\mu}_{=\widetilde{\mu} \circ \tau^{-1}} (x) =$$

$$= \sup_{\substack{\gamma \text{ measurable partition of } X \\ \widetilde{\sigma}(\gamma)\ \sigma\text{-finite},\ T^{-1}\gamma \preceq \gamma}} \int_{\widetilde{X}} H_{\mu_{\pi_{T^{-1}\gamma} \circ \tau(y)}}(\gamma \cap (\pi_{T^{-1}\gamma} \circ \tau(y)))\, d\widetilde{\mu}(y).$$

Now by Proposition 1.3 (6),

$$\mu_{\pi_{T^{-1}\gamma} \circ \tau(y)} = \mu_{\pi_{\tau^{-1}T^{-1}\gamma}(y)} \circ \tau^{-1} \text{ and } \tau^{-1} \circ \pi_{T^{-1}\gamma} \circ \tau = \pi_{\tau^{-1}T^{-1}\gamma}.$$

Thus the supremum above is equal to

$$\sup_{\substack{\gamma \text{ measurable partition of } X\\ \widetilde{\sigma}(\gamma)\ \sigma\text{-finite},\ T^{-1}\gamma\preceq\gamma}} \int_{\widetilde{X}} H_{\mu_{\pi_{\tau^{-1}T^{-1}\gamma}(y)}\circ\tau^{-1}}(\gamma\cap(\pi_{T^{-1}\gamma}\circ\tau(y)))\, d\widetilde{\mu}(y) =$$

$$= \sup_{\substack{\gamma \text{ measurable partition of } X\\ \widetilde{\sigma}(\gamma)\ \sigma\text{-finite},\ T^{-1}\gamma\preceq\gamma}} \int_{\widetilde{X}} H_{\mu_{\pi_{\tau^{-1}T^{-1}\gamma}(y)}}((\tau^{-1}\gamma)\cap(\tau^{-1}\circ\pi_{T^{-1}\gamma}\circ\tau(y)))\, d\widetilde{\mu}(y) =$$

$$= \sup_{\substack{\gamma \text{ measurable partition of } X\\ \widetilde{\sigma}(\gamma)\ \sigma\text{-finite},\ T^{-1}\gamma\preceq\gamma}} \int_{\widetilde{X}} H_{\mu_{\pi_{\tau^{-1}T^{-1}\gamma}(y)}}((\tau^{-1}\gamma)\cap\pi_{\tau^{-1}T^{-1}\gamma})\, d\widetilde{\mu}(y) =$$

$$= \sup_{\substack{\gamma \text{ measurable partition of } X\\ \widetilde{\sigma}(\gamma)\ \sigma\text{-finite},\ T^{-1}\gamma\preceq\gamma}} H_{\widetilde{\mu}}(\tau^{-1}\gamma|\tau^{-1}T^{-1}\gamma) =$$

$$= \sup_{\substack{\tau^{-1}\gamma \text{ measurable partition of } X\\ \widetilde{\sigma}(\tau^{-1}\gamma)\ \sigma\text{-finite},\ \widetilde{T}^{-1}\tau^{-1}\gamma\preceq\tau^{-1}\gamma}} H_{\widetilde{\mu}}(\tau^{-1}\gamma|\widetilde{T}^{-1}\tau^{-1}\gamma),$$

since $\widetilde{\sigma}(\gamma)$ is σ-finite if and only if $\widetilde{\sigma}(\tau^{-1}\gamma)=\tau^{-1}\widetilde{\sigma}(\gamma)$ is σ-finite (w.r.t. $\widetilde{\mu}$) and $T^{-1}\gamma\preceq\gamma$ if and only if $\widetilde{T}^{-1}\tau^{-1}\gamma\preceq\tau^{-1}\gamma$. Thus we have shown that

$$h_{Pa}(X,\mathfrak{B},\mu,T) = \sup_{\substack{\tau^{-1}\gamma \text{ measurable partition of } \widetilde{X}\\ \widetilde{\sigma}(\tau^{-1}\gamma)\ \sigma\text{-finite},\ \widetilde{T}^{-1}\tau^{-1}\gamma\preceq\tau^{-1}\gamma}} H_{\widetilde{\mu}}(\tau^{-1}\gamma|\widetilde{T}^{-1}\tau^{-1}\gamma).$$

This obviously gives

$$h_{Pa}(X,\mathfrak{B},\mu,T)\le h_{Pa}(\widetilde{X},\widetilde{\mathfrak{B}},\widetilde{\mu},\widetilde{T}).$$

□

Corollary 4.1. *Let $(X,\mathfrak{B},\mu,T)$ be a measure-preserving, conservative, ergodic dynamical system with $\mu(X)=\infty$ such that $(X,\mathfrak{B},\mu)$ is a σ-finite standard measure space. If there exists a quasi finite measurable set of finite positive measure in $(X,\mathfrak{B},\mu,T)$, then*

$$h(X^\star,\mathfrak{B}^\star,\mu^\star,T^\star)=h_{Kr}(X,\mathfrak{B},\mu,T).$$

Proof. We need just to apply Theorem 4.5 to the[22] natural extension $(\widetilde{X},\widetilde{\mathfrak{B}},\widetilde{\mu},\widetilde{T})$ of $(X,\mathfrak{B},\mu,T)$. Recall that the natural extension is a σ-finite standard measure space together with a conservative, ergodic, measure-preserving automorphism and $\widetilde{\mu}(\widetilde{X})=\mu(X)=\infty$ (confer section 1.3.3). Moreover, by Remark 4.5, the set $A'=\{(x_0,x_1,\dots)\in\widetilde{X}|x_0\in A\}$ is quasi finite for $(\widetilde{X},\widetilde{\mathfrak{B}},\widetilde{\mu},\widetilde{T})$ and of positive finite measure. Thus, the assumptions of Theorem 4.5 are satisfied and we obtain

$$h_{Poi}(\widetilde{X},\widetilde{\mathfrak{B}},\widetilde{\mu},\widetilde{T})=h_{Pa}(\widetilde{X},\widetilde{\mathfrak{B}},\widetilde{\mu},\widetilde{T})=h_{Kr}(\widetilde{X},\widetilde{\mathfrak{B}},\widetilde{\mu},\widetilde{T}).$$

Now, again by Remark 4.5,

$$h_{Poi}(\widetilde{X},\widetilde{\mathfrak{B}},\widetilde{\mu},\widetilde{T})=h_{Poi}(X,\mathfrak{B},\mu,T)$$

and

$$h_{Kr}(\widetilde{X},\widetilde{\mathfrak{B}},\widetilde{\mu},\widetilde{T})=h_{Kr}(X,\mathfrak{B},\mu,T)$$

Therefore,

$$h_{Poi}(X,\mathfrak{B},\mu,T)=h_{Poi}(\widetilde{X},\widetilde{\mathfrak{B}},\widetilde{\mu},\widetilde{T})=h_{Kr}(\widetilde{X},\widetilde{\mathfrak{B}},\widetilde{\mu},\widetilde{T})=h_{Kr}(X,\mathfrak{B},\mu,T).$$

□

[22] W.l.o.g. we may take the one formulated in Theorem 1.8.

4.4.1 Application to some examples

Example 4.5 (Boole's transformation). The set $Y = [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$ is quasi finite for Boole's transformation T: Clearly, $\lambda(Y) = \sqrt{2} < \infty$. We claim that the elements of the first return time partition $\rho_{T,Y} = \{Y \cap \{\varphi_{T,Y} = n\} : n \in \mathbb{N}\}$ fulfill

$$\lambda(Y \cap \{\varphi_{T,Y} = n\}) \sim \frac{1}{\sqrt{2n^3}} \text{ for } n \to \infty. \tag{15}$$

Then

$$H_\lambda(\rho_{T,Y}) = -\sum_{n\in\mathbb{N}} \lambda(Y \cap \{\varphi_{T,Y} = n\}) \log \lambda(Y \cap \{\varphi_{T,Y} = n\})) \sim \sum_{n\in\mathbb{N}} \frac{1}{\sqrt{2n^3}} \log(\sqrt{2n^3})$$

for $n \to \infty$, and $\sum_{n\in\mathbb{N}} \frac{1}{\sqrt{2n^3}} \log(\sqrt{2n^3}) < \infty$. Hence

$$H_\lambda(\rho_{T,Y}) < \infty.$$

Thus, due to Corollary 4.1

$$h(\mathbb{R}^\star, \mathfrak{B}(\mathbb{R})^\star, \mu^\star, T^\star) = h_{Kr}(\mathbb{R}, \mathfrak{B}(\mathbb{R}), \mu, T) = 2\pi,$$

by Example 4.1.

To verify (15) we follow [26]: We define a sequence of elements x_n in $\mathbb{R}^+$, starting at $x_0 := \frac{1}{\sqrt{2}}$, by $Tx_n = x_{n-1}$ for all $n \in \mathbb{N}$, i.e. $[x_{n-1}, x_n] = \mathbb{R}^+ \cap \{\varphi_{T,Y} = n\}$. Thus, by symmetry of T and Lemma 1.1, we get $\lambda(Y \cap \{\varphi_{T,Y} > n\}) = \lambda(Y^c \cap \{\varphi_{T,Y} = n\}) = 2(x_n - x_{n-1})$ for all $n \in \mathbb{N}$. By definition of T and $(x_n)_{n\in\mathbb{N}}$, we have $x_{n-1} = Tx_n = x_n - \frac{1}{x_n}$, which yields $x_n^2 - x_{n-1}^2 = x_n^2 - (x_n^2 - 2 + \frac{1}{x_n^2}) = 2 - \frac{1}{x_n^2}$ for every $n \in \mathbb{N}$. Therefore

$$\frac{x_n^2}{n} = \frac{x_0^2}{n} + \frac{1}{n}\sum_{k=1}^{n}(x_k^2 - x_{k-1}^2) = \frac{x_0^2}{n} + 2 + \frac{1}{n}\sum_{k=1}^{n}\frac{1}{x_k^2} \to 2 \text{ for } n \to \infty,$$

since $x_n \to \infty$ for $n \to \infty$. Thus

$$x_n \sim \sqrt{2n} \text{ for } n \to \infty.$$

(This follows by a simple case distinction from $2(1-\varepsilon) \le \frac{x_n^2}{n} \le 2(1-\varepsilon)$ for $\varepsilon > 0$ arbitrary small (w.l.o.g. $\varepsilon < 1$) and $x_n \ge 0$.) Then,

$$x_n - x_{n-1} = x_n - (x_n - \frac{1}{x_n}) = \frac{1}{x_n} \sim \frac{1}{\sqrt{2n}},$$

hence

$$\lambda(Y \cap \{\varphi_{T,Y} > n\}) = 2(x_n - x_{n-1}) \sim \sqrt{\frac{2}{n}}.$$

Now, $\lambda(Y \cap \{\varphi_{T,Y} > n\}) = \sum_{k>n} \lambda(Y \cap \{\varphi_{T,Y} = k\}) := g(n)$ and $g(n-1) - g(n) = \lambda(Y \cap \{\varphi_{T,Y} = n\})$. So,

$$\lambda(Y \cap \{\varphi_{T,Y} = n\}) \sim \sqrt{\frac{2}{n-1}} - \sqrt{\frac{2}{n}},$$

and $\sqrt{\frac{2}{n-1}} - \sqrt{\frac{2}{n}} = (-\frac{1}{2}) \cdot \sqrt{\frac{2}{n^3}} \cdot ((n-1) - n) = \frac{1}{\sqrt{2n^3}}$ by the Mean Value Theorem, hence

$$\lambda(Y \cap \{\varphi_{T,Y} = n\}) \sim \frac{1}{\sqrt{2n^3}}.$$

Example 4.6 (Renewal chain)**.** Given a shift space $(X,\mathfrak{B},\mu,S)$ of a renewal chain as in Example 1.3 (2) and Example 4.2 with $f_m = \left(\sum_{n\in\mathbb{N}} \frac{1}{n^{1+\alpha}}\right)^{-1} \frac{1}{m^{1+\alpha}}$ for $\alpha \in (0,1)$ arbitrary fixed. Then, the cylinder set $[1]$ is quasi finite: In Example 4.2, we have already computed the entropy of the first return time μ-partition $\rho_{S,[1]} = \{[1n] : n \in \mathbb{N}\}$ (denoted with β in Example 4.2):

$$H_\mu(\rho_{S,[1]}) = -\sum_{n\in\mathbb{N}} \mu([1n])\log(\mu([1n])) =$$

$$= \log\Big(\sum_{m\in\mathbb{N}} \frac{1}{m^{1+\alpha}}\Big) + (1+\alpha)\Big(\sum_{m\in\mathbb{N}} \frac{1}{m^{1+\alpha}}\Big)^{-1} \sum_{n\in\mathbb{N}} \frac{\log(n)}{n^{1+\alpha}} < \infty,$$

because $\sum_{n\in\mathbb{N}} \frac{1}{n^{1+\alpha}} < \infty$ and $\sum_{m\in\mathbb{N}} \frac{\log(m)}{m^{1+\alpha}} < \infty$.
Therefore, by Theorem 4.5 and Remark 4.5, we see that

$$h_{Pa}(\widetilde{X},\widetilde{\mathfrak{B}},\widetilde{\mu},\widetilde{S}) = h(X^\star,\mathfrak{B}^\star,\mu^\star,S^\star) = h_{Kr}(X,\mathfrak{B},\mu,S) = -\sum_{n\in\mathbb{N}} f_n \log(f_n),$$

referring to Example 4.3.

Using Theorem 4.5 and Theorem 4.3 one can show that the Poisson entropy of an invertible Markov shift is given by the same formula as the Krengel entropy without assuming the existence of a quasi finite set (see [10], Corollary 9.2, for a proof):

Theorem 4.6. *Let S be an invertible*[23] *Markov shift on $(X,\mathfrak{B},\mu)$ with $X = \Sigma^{\mathbb{Z}}$ for Σ countable, which corresponds to a null-recurrent, irreducible Markov chain, and let the transition probabilities denote by p_{ab} for $a,b \in \Sigma$ and the stationary distribution by $(u_a)_{a\in\Sigma}$, then*

$$h_{Pa}(X,\mathfrak{B},\mu,S) = h_{Kr}(X,\mathfrak{B},\mu,S) = h(X^\star,\mathfrak{B}^\star,\mu^\star,S^\star) = -\sum_{a\in\Sigma} u_a \sum_{b\in\Sigma} p_{ab}\log(p_{ab}).$$

In particular, we obtain for the two-sided random walk on $\mathbb{Z}$

Example 4.7 (Two sided random walk on $\mathbb{Z}$)**.** It is not obvious if there exists a quasi finite set for the two-sided shift space $(X,\mathfrak{B},\mu,S)$ corresponding to a random walk, but applying the Theorem above, we obtain (for $u_a = 1$, $p_{ab} = \frac{1}{2}$ for $b = a-1$ or $b = a+1$, $\forall a \in \mathbb{Z}$)

$$h_{Pa}(X,\mathfrak{B},\mu,S) = h(X^\star,\mathfrak{B}^\star,\mu^\star,S^\star) = h_{Kr}(X,\mathfrak{B},\mu,S) = \infty.$$

[23] The construction of this process is analogous to that of Example 1.3 (1), see e.g. [10] for a detailed definition.

References

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Appendix

Abstract (english)

Main topic of this master thesis is the notion of entropy for measure-preserving dynamical systems with an infinite measure. We will introduce Parry's Krengel's and Roy's idea of defining this quantity and state some results about their relation. Finally, we will prove that these different definitions coincide for ergodic, conservative, measure-preserving automorphisms under a weak additional assumption, by following [10]. Additionally, we note that the equality of Krengel's and Roy's entropy can be extended to not necessarily invertible transformations.
We will investigate Poisson suspensions, canonical systems of measures (studied by [16]) and some examples.

Abstract (deutsch)

In der vorliegenden Masterarbeit werden Entropie-Begriffe für maßtreue dynamische Systeme mit unendlichem Maß, anhand von Krengel-, Parry- und Poisson-Entropie, behandelt, wobei besonderes Augenmerk auf deren Beziehung zueinander gelegt wird. Wir werden uns an [10] orientieren und einen Satz beweisen, der die Gleichheit dieser Entropien für eine große Klasse von ergodischen, konservativen und maßtreuen Automorphismen auf Standard Maßräumen liefert. Bezüglich Krengel- und Poisson Entropie lässt sich dieser Satz auf allgemeine (nicht notwendigerweise invertierbare) Transformation übertragen.
Weitere Theorien, die hier ausgeführt werden, sind Poisson Suspensionen und kanonische Systeme von Maßen gemäß Rokhlin, [16]. Außerdem werden wir die Krengel-Entropie von einigen Beispielen mit unendlichem Maß berechnen und diese auf die Anwendbarkeit des oben erwähnten Satzes hin untersuchen.

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Academic Studies

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