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MASTERARBEIT

Titel der Masterarbeit

Fully Packed Loop Configurations: The Razumov-Stroganov-Cantini-Sportiello theorem

Verfasser

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Contents

Introduction

In this paper, we shall prove the generalised Razumov-Stroganov-Cantini-Sportiello theorem.

One may think of a *link pattern of size* n as a circle with the vertices $1, \dots, 2n$ at $\exp(\frac{2\pi ik}{2n}), k = 1, \dots, 2n$, and a perfect matching of these vertices by n edges such that all edges are inside the circle and two edges never cross each other (see e. g. Figure 1).

On $\mathcal{LP}(2n)$, the set of link patterns of size n, one defines $2n$ different operators $e_i \colon \mathcal{LP}_n \to \mathcal{LP}_n, i = 1, \dots, 2n$, where e_i unconnects $i, i+1$ and the respective "partners" j, k of a link pattern and connects i to $i+1$ and j to k instead.
Of course, $\mathcal{LP}(2n)$ is a finite set with elements which can be listed as $\pi_1, \dots, \pi_m$.

Now, for several reasons (e. g. reasons arising in physics), one is interested in the (as it turns out, unique) eigenvector $\overrightarrow{\mu_{2n}}$ in $\mathbb{C}^{|\mathcal{LP}(2n)|}$ of an operator called *Hamiltonian* H_0, where H_0 is defined by

$$H_0 = \sum_{i=1}^{2n} (e_i - 1).$$

One reason of being interested in $\overrightarrow{\mu_{2n}}$ is the study of the *dense O(1) loop model* and configurations of ice molecules in statistical mechanics; in this paper, however, we will motivate this topic by means of percolation theory.

The Razumov-Stroganov-Cantini-Sportiello theorem now links the vector $\overrightarrow{\mu_{2n}}$ to some other well-studied objects in combinatorics, the so-called *fully packed loop configurations (FPLs)*. An FPL of size n is a bicoloration of the edges of the graph with vertex set $\{(x, y) \colon 0 < x, y \leq n\}$, where (x, y) is paired to $(x, y+1)$ and to $(x+1, y)$. When defining them properly, it is immediate to define a function Π assigning a link pattern to an FPL in a very obvious way.

The ordinary Razumov-Stroganov-Cantini-Sportiello theorem states, that, writing $\mathcal{FPL}_\mathrm{n}$ for the set of FPLs of size n, $\overrightarrow{\mu_{2n}}$ is given by

$$\overrightarrow{\mu_{2n}} = \begin{pmatrix} |\{\phi \in \mathcal{FPL}_\mathrm{n} \colon \Pi(\phi) = \pi_1\}| \\ \vdots \\ |\{\phi \in \mathcal{FPL}_\mathrm{n} \colon \Pi(\phi) = \pi_m\}| \end{pmatrix}.$$

This somewhat surprising theorem was conjectured by A. V. Razumov and Yu. G. Stroganov in [9] (2001) and proven and generalized by A. Cantini and L. Sportiello in 2010 ([2], [1]) to some generalized sets of link patterns and FPLs.
We shall give the proof of this generalized theorem, following [1], in great detail.

This thesis is organized as follows:

In Chapter 1, we will motivate the problem by introducing *percolation models* and proving the equivalence of finding $\overrightarrow{\mu_{2n}}$ to some natural question from percolation theory.

The short Chapter 2 is intended to give the reader some concrete examples of link

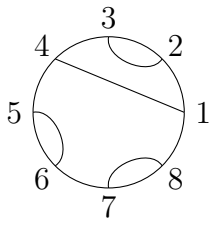


Figure 1: A link pattern in $\mathcal{LP}(8)$.

patterns and FPLs and a proof of the (easy to handle) case $n = 3$ of the ordinary Razumov-Stroganov-Cantini-Sportiello theorem.

In Chapter 3, we shall gather all the ingredients for proving the refined Razumov-Stroganov-Cantini-Sportiello theorem such as generalized link patterns, prove the uniqueness of $\overrightarrow{\mu_N}$ (which will be a generalization of $\overrightarrow{\mu_{2n}}$) and deduce a condition equivalent to satisfying $H_0\overrightarrow{\mu_N} = 0$.

Chapter 4 introduces the generalizations of FPLs we will need and also a map called *Wieland half-gyration*, which turns out to be crucial for all following considerations.

Finally, in Chapter 5, we will give a (combinatorial) proof of the generalized Razumov-Stroganov-Cantini-Sportiello theorem; we will first prove that a certain vector $\vec{\psi}$ satisfies $H_0\vec{\psi} = 0$ and then $\vec{\psi} = \overrightarrow{\mu_N}$.

The reader is assumed to have some basic knowledge about linear algebra, analysis, probability theory, complex analysis and graph theory.

I want to express my gratitude to my supervisor Professor Ilse Fischer for introducing me to the fascinating world of link patterns and FPLs and her support while writing this thesis – I'm especially grateful for her readiness to openly discuss problems whenever they arised. Furthermore, I want to thank my family and friends for their constant help during my studies – not only in financial ways but also for being there when I needed them.

Last but not least, thanks to my colleagues, especially Noëma Nicolussi, for extensive discussions and useful suggestions on the Razumov-Stroganov-Cantini-Sportiello theorem.

1 Percolation

1.1 Introduction

Percolation is the mathematical model of a fluid flowing through some porous material. Although the model is very easy to state, it is the field of extensive research by mathematicans and physicans since its introduction by Broadbent and Hemmersley in 1954.

One usually considers the model of *bond percolation*: One takes the square lattice $\mathbb{Z}^2$ and each edge is independently chosen to be "open" with some probability $p \in (0,1)$ or "closed" (with probability $1-p$). This gives a random graph, see Figure 2.

This model has its origins in physics in which percolation is regarded as a model for some porous material: Some fluid is poured on the material. An open edge allows the fluid to flow through whereas it can not cross a closed edge. So the fluid can flow all over the whole material if and only if there is no infinitely connected component in the bond percolation graph. Therefore, a natural (and probably the most well-known) question about bond percolation is the following one:

For a given probability p, what is the probability $\theta(p)$ that a resulting random graph has an infinite component?

Bond percolation is easily extended to $\mathbb{Z}^n$ where one can ask the same question. Kolmogorov's zero-one law implies that $\theta(p)$ is always 0 or 1, depending on p; it also can be proven easily that there is a threshold value p_c (depending on the dimension n) such that for $p < p_c, \theta(p) = 0$ and for $p > p_c, \theta(p) = 1$.

For $n = 2, p_c$ is known to be $\frac{1}{2}$ (although the proof is highly non-trivial) and it is also known that $\theta(\frac{1}{2}) = 0$. If $n \geq 19$, the value p_c is also known; however, for $2 < n < 18$, this question is open. There are also many other conjectures about percolation. We will now state a similar percolation model, called *loop percolation*.

Many of our considerations take place on $\mathbb{Z}$ (or a subset of $\mathbb{Z}^2$); we shall say that (x,y) and (a,b) are neighbor vertices if their distance is 1, i. e. $(a,b) = (x, y+1)$ or $(a,b) = (x+1, y)$. The *square lattice* then is the graph on $\mathbb{Z}$ with edges between any two neighbor vertices; by a quadratic grid of size n (see Chapter 2) we shall mean the graph

$$\{(x,y) \in \mathbb{Z}^2 \colon 0 < x, y \leq n\} \text{ for some } n \in \mathbb{N}$$

with edges between any two neighbor vertices; *rectangular $L_x \times L_y$-grids* which we will use in Chapter 4 are defined analogously as

$$\{(x,y) \in \mathbb{Z}^2 \colon 0 < x \leq L_x, 0 < y \leq L_y\} \text{ for some } L_x, L_y \in \mathbb{N}$$

(again with edges between any two neighbor vertices).

Loop percolation is defined as follows:

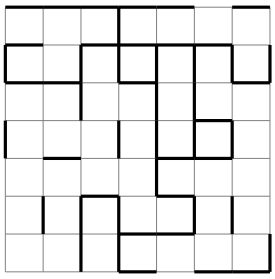


Figure 2: A portion of a percolation graph on $\mathbb{Z}^2$ with $p = \frac{1}{2}$ (note that this graph is infinite in all four directions).

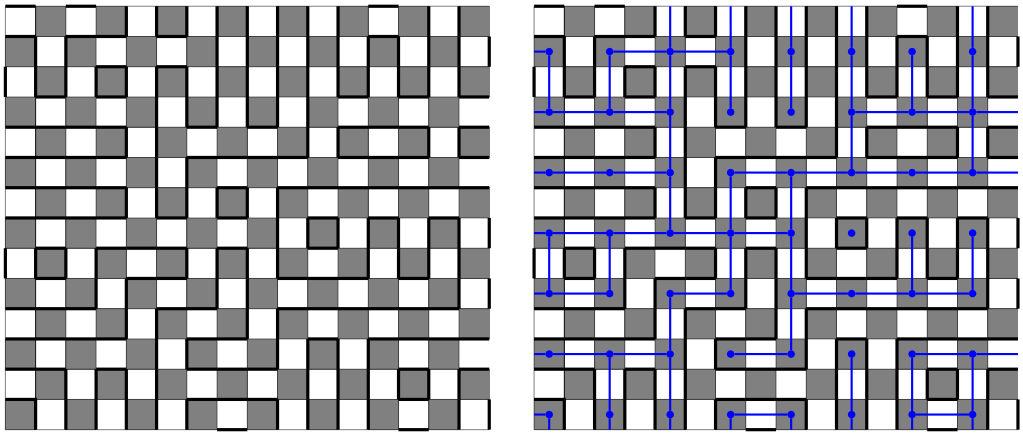


Figure 3: Left: A portion of a loop percolation graph on $\mathbb{Z}^2$ with $p = \frac{1}{2}$ (note that this graph is infinite in all four directions). Right: The corresponding portion of the associated percolation graph (blue).

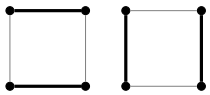


Figure 4: Left: Edges included with probability p, right: edges included with probability $1-p$.

Definition 1.1 (Loop Percolation on $\mathbb{Z}^2$)**.** Consider another random graph LP on the square lattice:

Color the squares of $\mathbb{Z}^2$ in a checkerboard manner with black and white such that the face bounded by $(0,0),(0,1),(1,0),(1,1)$ is white.

Now, for each white square, one includes independently with probability p its bottom and top edge into the set E and with probability $1-p$ its left and right edge (see Figure 4). This gives a random graph $LP = (\mathbb{Z}^2, E)$, called *loop percolation graph*.

Figures 2 and 3 (left) are examples for a percolation and a loop percolation graph. The two considered models turn out to be in fact equivalent:

When considering a loop percolation graph, one may put a vertex into each black square in every other row. Now two vertices of the form (x,y) and $(x,y+2)$ or (x,y) and $(x+2,y)$ shall be connected if and only if this connecting edge would not cross an edge of the loop percolation graph. This gives a normal percolation graph – one can easily reconstruct the associated percolation graph (see Figure 3).

1.2 Cylindrical loop percolation and link patterns

We now consider a special variant of loop percolation:

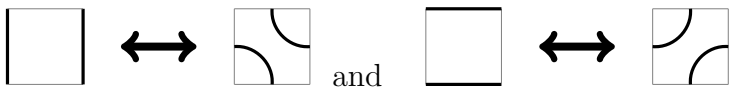
Definition 1.2 (Cylindrical loop percolation on $\mathbb{Z}^2$)**.** Let $n \in \mathbb{N}, n \geq 1$. We define the cylindrical loop percolation graph LP_n with vertex set

$$V(LP_n) = \{(x,y) \in \mathbb{Z}^2 \colon x+y \geq 0, -2(n-1) \leq x-y \leq 2n\},$$

where we identify for every $k \geq 0$ the two vertices $(-n+1+k, n-1+k)$ and $(n+k, -n+k)$. Again the squares are alternatively colored white and black; as in the usual loop percolation graph the edges are chosen for each white square.

This definition gives a diagonal strip, infinite in (only) one direction, see Figure 5 (the link pattern corresponding to the graph in Figure 5 is given by $\pi(1) = 2, \pi(3) = 4, \pi(5) = 6$).

We perform the following bijection: We rotate the graph counter-clockwise by 45 degrees – then, we remove each black square and replace each white square by another type of square, called *plaquette*, according to the following figure:



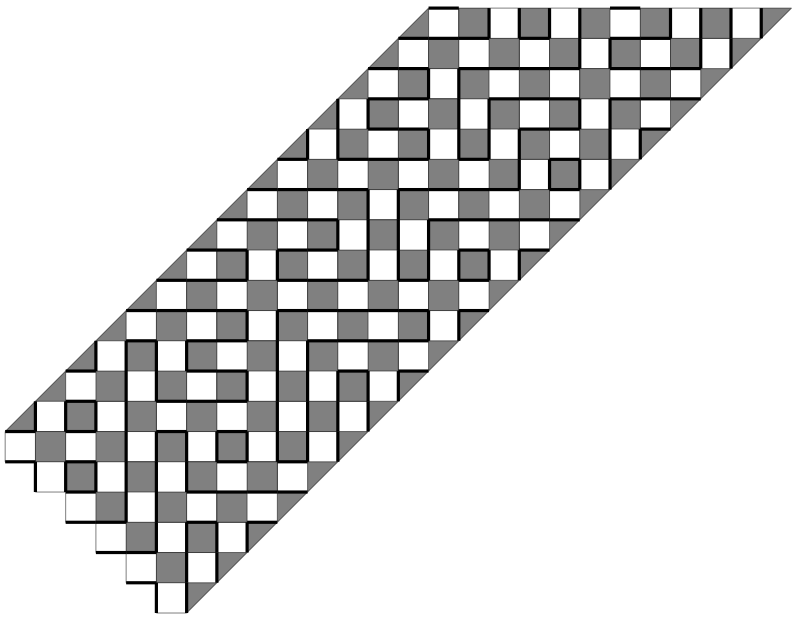


Figure 5: A portion of a cylindric loop percolation graph LP_n with $n = 3$ (note that this graph is infinite (only) in top-right direction).

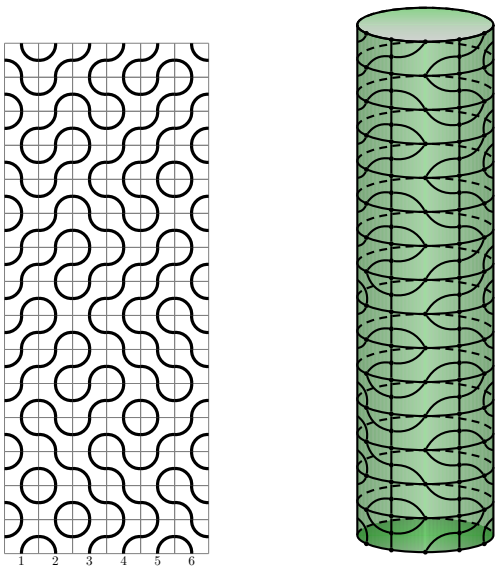


Figure 6: The cylindrical loop percolation graph shown in Figure 5 "after rotating" (note that this graph is upwards infinite) in "normal shape" and wrapped around a cylinder.

The resulting graph - containing a lot of paths - can be wrapped around a cylinder as in Figure 6. This explains the name "cylindrical loop percolation".

The model we will be concerned with is the following: Consider the cylindrical loop percolation graph as in Figure 6 (left). We label the midpoints of the squares in the bottom row from 1 to $2n$ – we can now follow a path starting in $1 \leq i \leq 2n$ until it reemerges in $1 \leq j \leq 2n$. This gives us (almost surely, cf. Lemma 1.5) a link pattern as in the following definition:

Definition 1.3. A *link pattern of size* n is a function $\pi\colon \{1, \dots, 2n\} \to \{1, \dots, 2n\}$ such that

(1) $\pi \circ \pi = \mathrm{id}$

(2) $\pi(k) \neq k$ for all $k \in \{1, \dots, 2n\}$ and

(3) there are no numbers $a < b < c < d$ (understood modulo $2n$) such that $\pi(a) = c$ and $\pi(b) = d$.

The set of all link patterns of size n will be denoted by $\mathcal{LP}(2n)$.
When considering link patterns, notations are (except in the following proof) to be understood modulo $2n$, for example holds $2n + 1 = 1$ or $2n - 1 < 2n < 1$.

Proposition 1.4. *The set $\mathcal{LP}(2n)$ is finite – more precisely, its cardinality is C_n, where $C_n := \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.*

Proof. It is well-known that the set of Dyck paths of length n, which we will denote by $\mathrm{D}(n)$ (i. e., lattice paths starting in $(0, 0)$, ending in $(2n, 0)$, with steps of the form $(1, 1)$ (an "up step") and $(1, -1)$ (a "down step") s. t. the path never crosses the x-axis), has cardinality C_n.

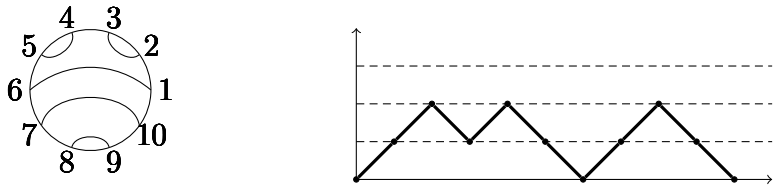


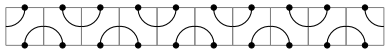
Figure 7: A link pattern and the associated Dyck path.

A bijection between $\mathcal{LP}(2n)$ and $\mathrm{D}(n)$ is given as follows: For a link pattern π, create a Dyck path D: If i is connected in π to $j > i$, the i-th step of D is an "up step", otherwise a "down step" (see Figure 7). □

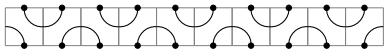
It is not immediately clear that every cylindrical loop percolation graph "has" a corresponding link pattern; however, the definition of a link pattern is meaningful because of the following lemma:

Lemma 1.5. *In the graph LP_n, almost surely all paths are finite.*

Proof. For the moment, we will call a row in which the plaquettes alternate between the two types an *alternating row*, i. e. an alternating row has the form



or


.

If LP_n has an alternating row, all paths below must be finite (since they can not cross that row).
Since an alternating row may have either form, the probability of a row being alternating is given by $2p^n(1-p)^n$. Therefore, the probability that none of the first i rows is alternating is given by α^i, where $\alpha := 1 - 2p^n(1-p)^n$. Since $\alpha < 1$,

$$\lim_{i\to\infty} \alpha^i = 0$$

and hence, almost surely, there is an alternating row in LP_n.

□

We will think of link patterns in two ways:

(1) A link pattern can be seen as a circle with the vertices $1, \ldots, 2n$ on the circle line where each vertex is connected to exactly one other vertex (items (1) and (2)) - if these connecting edges are drawn inside the circle then there are no two edges crossing each other (item (3)). (We will mostly use this interpretation). Figure 8 (left) gives an example for this representation of a link pattern of size 5.

(2) The right picture of Figure 8 gives an example of the other way to represent a link pattern: Here the matched points $1, \ldots, 2n$ are not placed on a circle, but on a straight line.

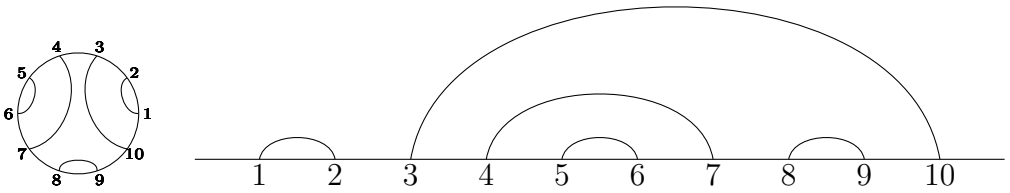


Figure 8: Two ways on how to think of link patterns.

Now we have described how to obtain a link pattern from a cylindrical loop percolation graph.
It is now natural to ask:

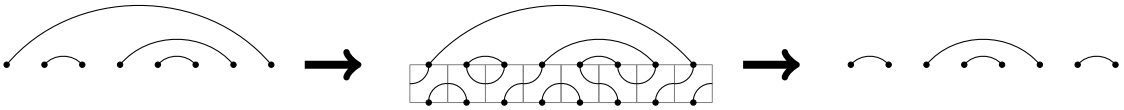


Figure 9: The process of composing a link pattern with a row $a \in \{0,1\}^{2n}$.

What is the probability that a random cylindrical loop percolation graph corresponds to a given link pattern π? We now list the link patterns of size n as $\pi_1, \pi_2, \ldots, \pi_{|\mathcal{LP}(2n)|}$ and consider a vector $\overrightarrow{\mu_{2n}} \in \mathbb{C}^{|\mathcal{LP}(2n)|}$, where the i-th component of $\overrightarrow{\mu_{2n}}$ is the probability that a cylindrical loop percolation graph corresponds to π_i.

It is convenient to refer to the two plaquettes

as "type 0-plaquette" and "type 1-plaquette", respectively. One may now define a composition of a link pattern with a row a (which we can encode as a vector $a \in \{0,1\}^{2n}$); one thinks of π as in the "line representation" and draws the row a below; this gives a new link pattern $\pi \circ a$ as in Figure 9.

We can now define a Markov chain with states $\pi_1, \ldots, \pi_m \in \mathcal{LP}(2n)$. A transition $\pi \to \pi'$ is given by $\pi \circ a$, where a is a random chosen row; more precisely, the components of a are chosen independently, where a component is 0 with probability p and 1 with probability $1-p$.

Therefore, the transition property $P(\pi \to \pi')$ of this Markov chain is given by

$$P(\pi \to \pi') = \frac{\sum_{a \in \{0,1\}^{2n}} P(\pi \circ a = \pi')}{2^{2n}}$$

(for example, if $p = \frac{1}{2}$, this equals $\frac{\#a \in \{0,1\}^{2n} \colon \pi \circ a = \pi'}{2^{2n}}$). We write $T_n^{(p)}$ for the transition matrix.

Lemma 1.6. $T_n^{(p)} \overrightarrow{\mu_{2n}} = \overrightarrow{\mu_{2n}}$.

Proof. Let $\mathrm{Cyl}(2n, \pi)$ be the probability that a cylindrical loop percolation graph of size $2n$ "has" the link pattern π. We shall introduce a notion we will later use very frequently: If, thinking of a cylindrical loop percolation graph as a sequence of 0-1-vectors $(a_1, a_2, \ldots)$, this graph has the link pattern π, we write $\Pi(a_1, a_2, \ldots) = \pi$. We get

$$\begin{aligned}
\mathrm{Cyl}(2n, \pi) &= P((a_1, a_2, \ldots) \in (\{0,1\}^{2n})^{\mathbb{N}} \colon \Pi(a_1, a_2, \ldots) = \pi) \\
&= \sum_{\sigma \in \mathcal{LP}(2n)} P(\sigma \to \pi) P((a_2, a_3, \ldots) \in (\{0,1\}^{2n})^{\mathbb{N}} \colon \Pi(a_2, a_3, \ldots) = \sigma) \\
&= \sum_{\sigma \in \mathcal{LP}(2n)} P(\sigma \to \pi)\, \mathrm{Cyl}(2n, \pi).
\end{aligned}$$

Observing now that

$$T_n^{(p)} = (P(\pi_i \to \pi_j))_{i,j\in\{1,\ldots,m\}}$$

(with $m := |\mathcal{LP}(2n)|$) and

$$\overrightarrow{\mu_{2n}} = \begin{pmatrix} \mathrm{Cyl}(2n, \pi_1) \\ \vdots \\ \mathrm{Cyl}(2n, \pi_m) \end{pmatrix}$$

completes the proof.

□

Before we can prove a crucial theorem, we need to define $2n$ certain operators on $\mathcal{LP}(2n)$ (see Figure 10):

Definition 1.7. The operator $e_i\colon\ \mathcal{LP}(2n) \to \mathcal{LP}(2n), 1 \le i \le 2n$, is given by

- $e_i(\pi) = \pi$, if $\pi(i) = i + 1$ (that is, i and $i + 1$ are connected in π)
- $e_i(\pi) = \pi'$, if $\pi(i) \neq i + 1$; here π' shall denote the link pattern with $\pi'(i) = i + 1, \pi'(\pi(i)) = \pi(i + 1)$ (and $\pi'(j) = \pi(j)$ for all $j \notin \{i, i + 1, \pi(i), \pi(i + 1)\}$). That is, if i and $i + 1$ are not connected in π (let i be connected to k and $i + 1$ to ℓ, say), in $e_i(\pi)$ all connections stay the same but those where i and $i + 1$ are involved; i gets connected to $i + 1$ and k to ℓ. That is, if i and $i + 1$ are not connected in π (let i be connected to k and $i+1$ to ℓ, say), in $e_i(\pi)$ all connections stay the same but those where i and $i + 1$ are involved; i gets connected to $i + 1$ and k to ℓ.

Figure 10: Left: A link pattern $\pi \in \mathcal{LP}(2n)$, right: $e_1(\pi)$. To obtain $e_1(\pi)$, one connects 1 and 2 and their previous "partners", i. e., 3 and 4.

Remark. We will, by abuse of notation - in this and in all following chapters - identify linear operators and the corresponding matrices.

Some other examples of the operators e_i on $\mathcal{LP}(6)$ can be found in Chapter 2 in the example after Theorem 2.3. It will be customary to work in the vector space $\mathbb{C}^{|\mathcal{LP}(2n)|}$

– by abuse of notation, we will refer to this space as $\mathbb{C}^{\mathcal{LP}(2n)}$. Let $\pi_1, \pi_2, \ldots, \pi_m$ be a list of $\mathcal{LP}(2n)$, then we will use the notations

$$\overrightarrow{\pi_1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \overrightarrow{\pi_2} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \overrightarrow{\pi_m} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

for the standard basis of $\mathbb{C}^{\mathcal{LP}(2n)}$. We thus identify the i-th item in the list with the i-th coordinate.

Remark. Operators acting on $\mathcal{LP}(2n)$ (like the e_i) can be extended to linear operators on $\mathbb{C}^{\mathcal{LP}(2n)}$ – if S is a map $\mathcal{LP}(2n) \to \mathcal{LP}(2n)$, we define its linear extension (which we will also denote by S) on $\mathbb{C}^{\mathcal{LP}(2n)}$ by

$$S\left(\sum_{\pi \in \mathcal{LP}(2n)} c_\pi \overrightarrow{\pi}\right) = \sum_{\pi \in \mathcal{LP}(2n)} c_\pi \overrightarrow{S(\pi)}.$$

Definition 1.8. We define a linear operator H_0 on $\mathbb{C}^{\mathcal{LP}(2n)}$, called *Hamiltonian*, by

$$H_0 := \sum_{i=1}^{2n} (e_i - 1).$$

Theorem 1.9. *The vector $\overrightarrow{\mu_{2n}}$ satisfies*

$$H_0 \overrightarrow{\mu_{2n}} = 0$$

(i. e., $\overrightarrow{\mu_{2n}}$ is an eigenvector of H_0 with eigenvalue 0).

Proof. We define for every $a \in \{0,1\}^{2n}$ an operator f_a on $\mathcal{LP}(2n)$ by setting $f_a(\pi) = \pi'$, where π' is the link pattern obtained by composing the diagram of π with the row of plaquettes encoded by a.
We therefore can write

$$(T_n^{(p)})_{\pi,\pi'} = \sum_{\substack{a \in \{0,1\}^{2n}, \\ f_a(\pi) = \pi'}} p^{\sum_{j=1}^{2n} a_j} (1-p)^{\sum_{j=1}^{2n} (1-a_j)}. \tag{1}$$

Define $2n+1$ "special" vectors in $\{0,1\}^{2n}$:

$$\begin{aligned} 0 &:= (0, 0, \ldots, 0), \\ a_i &:= (\delta_{1,i}, \delta_{2,i}, \ldots, \delta_{2n,i}) \text{ for } 1 \leq i \leq 2n, \end{aligned}$$

where δ denotes the Kronecker delta.

Figure 11: The link pattern π is, by adding the row $0 = (0, \ldots, 0)$ to its cylinder, transferred into $R(\pi)$.

Figure 12: The link pattern π is, by adding the row a_6 to its cylinder, transferred into $Re_6(\pi)$.

Now we split the sum in (1) into three parts:

$$\begin{aligned}(T_n^{(p)})_{\pi,\pi'} &= \delta_{f_0(\pi),\pi'} p^0 (1-p)^{2n} \\ &+ \sum_{i=1}^{2n} \delta_{f_{a_i}(\pi),\pi'} p^1 (1-p)^{2n-1} \\ &+ \sum_{\substack{a\in\{0,1\}^{2n},\\ f_a(\pi)=\pi',\\ a\notin\{0,a_1,\ldots,a_{2n}\}}} p^{\sum_{j=1}^{2n} a_j} (1-p)^{\sum_{j=1}^{2n}(1-a_j)}.\end{aligned}$$

We have done so because we would like to differentiate $T_n^{(p)}$ with respect to p at $p = 0$; the third sum will thus not give any contribution at all. We thus obtain

$$\frac{\mathrm{d}}{\mathrm{d}\,p}\left((T_n^{(p)})_{\pi,\pi'}\right)|_{p=0} = -2n\delta_{f_0(\pi),\pi'} + \sum_{i=1}^{2n} \delta_{f_{a_i}(\pi),\pi'}.$$

We investigate the action of the functions f_0 and f_{a_i}.
If in π the vertices i and j are connected, then so are in $f_0(\pi)$ the vertices $i-1$ and $j-1$ – that is, $f_0(\pi) = R(\pi)$.
The example in Figure 11 illustrates this thought. Similarly, we have $f_{a_i}(\pi) = Re_i(\pi) = e_{i-1}R(\pi)$ (see Figure 12).
So, for $Q_n := \frac{\mathrm{d}}{\mathrm{d}\,p}(T_n^{(p)})|_{p=0}$, we have

$$Q_n = -2nR + \sum_{i=1}^{2n} e_{i-1}R$$

and consequently get

$$Q_n R^{-1} = -2n + \sum_{i=1}^{2n} e_{i-1} = \sum_{i=1}^{2n} (e_i - 1) = H_0.$$

We have

$$T_n^{(p)} \overrightarrow{\mu_{2n}} = \overrightarrow{\mu_{2n}}$$

and thus

$$Q_n \overrightarrow{\mu_{2n}} = 0.$$

Since $R^{-1}\overrightarrow{\mu_{2n}} = \overrightarrow{\mu_{2n}}$,

$$H_0 \overrightarrow{\mu_{2n}} = Q_n R^{-1} \overrightarrow{\mu_{2n}} = Q_n \overrightarrow{\mu_{2n}} = 0.$$

□

Remark. We will see later that the eigenspace generated by $\overrightarrow{\mu_{2n}}$ is one-dimensional. Therefore, the vector $\overrightarrow{\mu_{2n}}$ is independent of the chosen probability p, i. e. the notion $\overrightarrow{\mu_{2n}}$ (without any mention of p) is justified.

We thus want to compute the (we will later see, unique up to normalization) eigenvector w. r. t. the eigenvalue 0 of the Hamiltonian.
For stating the Razumov-Stroganov-Cantini-Sportiello theorem, the main result of this thesis, we shall define (in Chapter 2) another class of objects, the so-called *fully packed loop configurations*.

We will see that the components of $\overrightarrow{\mu_{2n}}$ can be computed using these fully packed loop configurations; in contrary to the cylindrical loop percolation graphs, there are only finitely many fully packed loop configurations of a given size.

2 The ordinary Razumov-Stroganov-Cantini-Sportiello theorem

In this chapter, we shall state the "ordinary" Razumov-Stroganov-Cantini-Sportiello theorem. (We will later state and prove a refined version.)
Moreover, we prove this theorem for $n = 3$ to give the reader an idea on what is going on.

2.1 Fully packed loop configurations and the Razumov-Stroganov-Cantini-Sportiello theorem

Let G_n be the square grid of size n (the *internal vertices*) together with the set of *external vertices*

$$\{(1,0),(2,0),\dots,(n,0),(n+1,1),(n+1,2),\dots,(n+1,n),$$
$$(n,n+1),(n-1,n+1),\dots,(1,n+1),(0,n),(0,n-1),\dots,(0,1)\}$$

such that each external vertex is incident to its internal neighbor vertex (i. e. as in Figure 13 (left)).
This is the playground which we will color in a certain way to obtain fully packed loop configurations (see Figure 13 (right) for an example):

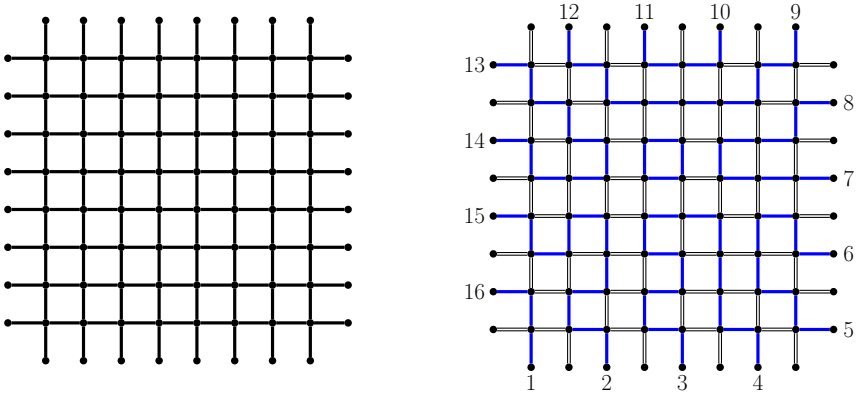


Figure 13: The "playground" of size 8 (left) and an FPL of size 8 (right).

Definition 2.1. A *fully packed loop configuration* (FPL) ϕ of size n is the graph G_n defined above equipped with an edge-coloring function $c\colon E \to \{b, w\}$ satisfying the following properties:

(1) Each internal vertex is incident to two black and two white edges (we call an edge e black, if $c(e) = b$ and white, if $c(e) = w$).

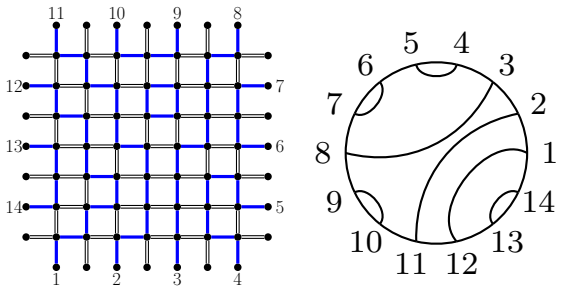


Figure 14: A fully packed loop configuration ϕ and its corresponding link pattern $\Pi(\phi)$.

(2) Every other edge incident with an external vertex is colored black; starting with a black edge at vertex $(1, 0)$. Despite of not being colored, we refer to an external vertex adjacent to a black edge as a *black vertex* and an external vertex adjacent to a white edge as a *white vertex*.

Here and in the following, black vertices in FPLs will always be blue in illustrations.

Definition 2.2. The set of all FPLs of size n will (in this chapter) be called $\mathcal{FPL}_n$.

Example. Figure 14 (left) gives an example of an FPL of size 7.

Remark.

(1) Calling A_n the number of FPLs of size n, the sequence (A_n) is given by

$$(A_n)_{n\geq 1} = (1, 2, 7, 42, 429, 7436, 218348, \ldots).$$

(2) There exists a formula for A_n, proven in [13] by Doron Zeilberger:

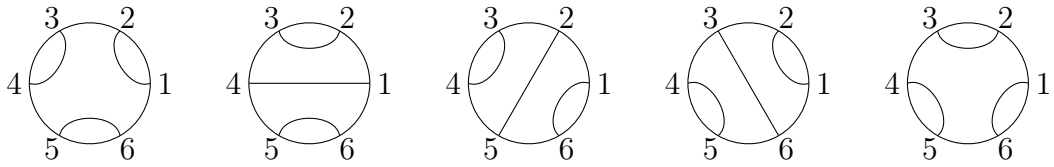
$$A_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

(3) There exist several other interesting objects standing in bijection to FPLs of order n, such as *alternating sign matrices* (quadratic matrices consisting only of 0's, 1's and -1's such that the nonzero entries alternate in each row and column and each row and column sums up to 1). The interested reader is referred to [10].

Consider the subgraph ϕ_b induced by the black edges of an FPL ϕ. Since each internal vertex in ϕ_b has degree 2, ϕ_b decomposes into circles and paths, where each path connects two external vertices.

We label the black external vertices in ϕ from 1 to $2n$ starting with 1 at vertex (1,0); this gives rise to a link pattern. We call $\Pi\colon \mathcal{FPL}_n \to \mathcal{LP}(2n)$ the map assigning a link pattern to a FPL in that way. See Figure 14 for an example.

We remind the reader of the definitions of the Hamiltonian H_0 and the operators $e_i, 1 \leq i \leq 2n$ on $\mathcal{LP}(2n)$ and state the former Razumov-Stroganov conjecture, proven 2010 by Cantini and Sportiello, the main result of this thesis.



π_1 π_2 π_3 π_4 π_5

Figure 15: The five link patterns in $\mathcal{LP}(6)$.

Theorem 2.3 (Razumov-Stroganov-Cantini-Sportiello theorem)**.** *The vector*

$$\overrightarrow{\mu_{2n}} = \begin{pmatrix} |\{\phi \in \mathcal{FPL}_n \colon \Pi(\phi) = \pi_1\}| \\ \vdots \\ \big|\{\phi \in \mathcal{FPL}_n \colon \Pi(\phi) = \pi_{|\mathcal{LP}(2n)|}\}\big| \end{pmatrix}$$

spans the eigenspace of H_0 w. r. t. the eigenvalue 0*.*

Remark. Since (up to multiplication with a scalar) only $\overrightarrow{\mu_{2n}}$ satisfies $H_0\overrightarrow{\mu_{2n}} = 0$ (as we will see later), this vector (after normalising) is also the vector considered in Chapter 1 satisfying $T_n^{(p)}\overrightarrow{\mu_{2n}} = \overrightarrow{\mu_{2n}}$.

In the remaining part of this chapter, we will illustrate this theorem for the case $n = 3$.

2.2 Proof of the theorem for $n = 3$

Example (Razumov-Stroganov-Cantini-Sportiello theorem for $n = 3$). Enumerating $\mathcal{LP}(6)$ as in Figure 15, the actions of $R, e_1, \ldots, e_6$ are given by

	$\pi = \pi_1$	$\pi = \pi_2$	$\pi = \pi_3$	$\pi = \pi_4$	$\pi = \pi_5$
$R(\pi)$	π_5	π_4	π_2	π_3	π_1
$e_1(\pi)$	π_1	π_1	π_1	π_4	π_4
$e_2(\pi)$	π_2	π_2	π_5	π_5	π_5
$e_3(\pi)$	π_1	π_1	π_3	π_1	π_3
$e_4(\pi)$	π_4	π_5	π_5	π_4	π_5
$e_5(\pi)$	π_1	π_2	π_1	π_1	π_2
$e_6(\pi)$	π_3	π_5	π_3	π_5	π_5

Therefore, the matrices $R, e_1, \ldots, e_6$ are given by

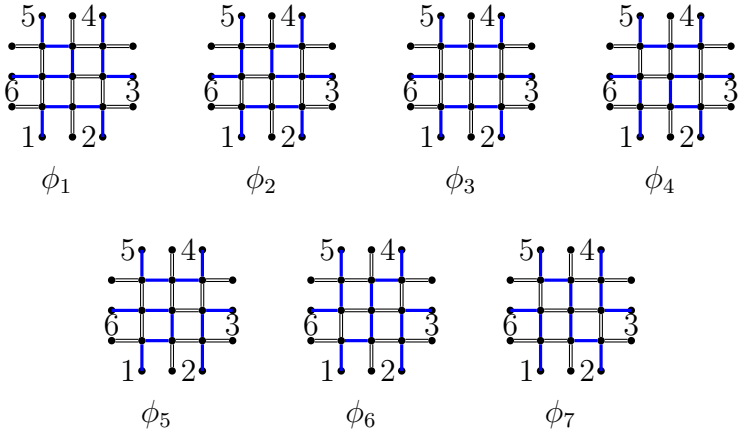


Figure 16: The seven FPLs in $\mathcal{FPL}_3$.

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, e_1 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix},$$

$$e_3 = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix},$$

$$e_5 = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, e_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

and the Hamiltonian by

$$H_0 = \sum_{i=1}^{2\cdot 3} (e_i - 1) = \begin{pmatrix} 3 & 2 & 2 & 2 & 0 \\ 1 & 2 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 2 & 1 \\ 0 & 2 & 2 & 2 & 3 \end{pmatrix} - 6\,\mathrm{Id} = \begin{pmatrix} -3 & 2 & 2 & 2 & 0 \\ 1 & -4 & 0 & 0 & 1 \\ 1 & 0 & -4 & 0 & 1 \\ 1 & 0 & 0 & -4 & 1 \\ 0 & 2 & 2 & 2 & -3 \end{pmatrix}.$$

Figure 16 lists all FPLs of size 7; one readily checks that

$$\Pi(\phi_1) = \pi_1, \Pi(\phi_2) = \pi_1, \Pi(\phi_3) = \pi_4, \Pi(\phi_4) = \pi_5, \Pi(\phi_5) = \pi_5, \Pi(\phi_6) = \pi_2, \Pi(\phi_7) = \pi_3.$$

Therefore,

$$\vec{\mu_6} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}.$$

One checks that

$$H_0\vec{\mu_6} = \begin{pmatrix} -3 & 2 & 2 & 2 & 0 \\ 1 & -4 & 0 & 0 & 1 \\ 1 & 0 & -4 & 0 & 1 \\ 1 & 0 & 0 & -4 & 1 \\ 0 & 2 & 2 & 2 & -3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

that is, the Razumov-Stroganov-Cantini-Sportiello theorem holds for $n = 3$.

3 An equivalent condition for satisfying the Razumov-Stroganov-Cantini-Sportiello theorem

This chapter has three goals:

- We state refined versions of link patterns needed for the refined Razumov-Stroganov-Cantini-Sportiello theorem.
- We prove existence and uniqueness of the Hamiltonian's eigenvector.
- We deduce a condition equivalent to satisfying the Razumov-Stroganov-Cantini-Sportiello theorem.

We will, here and in the remaining part, mostly follow [1].

3.1 Definition of the refined link patterns

The reader may recall the definition of $\mathcal{LP}(2n)$ in Chapter 1 (Definition 1.3). For the refined Razumov-Stroganov-Cantini-Sportiello theorem, we need, in addition, two refined versions of $\mathcal{LP}(2n)$.

Definition 3.1.

(1) An element of $\mathcal{LP}^*(2n-1)$ is a function

$$\pi\colon \{1,\dots,2n-1\}\cup\{x\} \to \{1,\dots,2n-1\}\cup\{x\}$$

such that (cf. Definition 1.3)

(1) $\pi\circ\pi = \mathrm{id}$

(2) $\pi(k)\neq k$ for all $k\in\{1,\dots,2n-1\}\cup\{x\}$ and

(3) there are no numbers $a,b,c,d\in\{1,\dots,2n-1\}, a<b<c<d$ (understood modulo $2n-1$) such that $\pi(a)=c$ and $\pi(b)=d$.

(2) An element of $\mathcal{LP}(2n)$, viewed in circle-shape, consists of several areas (bounded by the circle line and the connecting edges). An element $\pi\in\mathcal{LP}^*(2n)$ shall be an element of $\mathcal{LP}(2n)$ where we choose one of these areas and mark it (in the circle representation with a puncture).

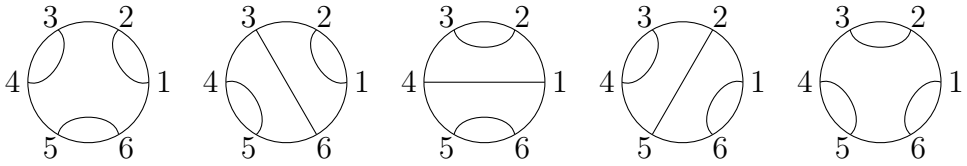
(3) We often prove statements for $\mathcal{LP}(2n), \mathcal{LP}^*(2n), \mathcal{LP}^*(2n-1)$ simultaneously and shall write N instead of $2n$ or $2n-1$. The notation $\mathcal{LP}(N)$ can mean any of these three sets. If π belongs to any of the three sets, we refer to it as a *link pattern*.

Again, if we consider link patterns, calculations are to be understood modulo N.

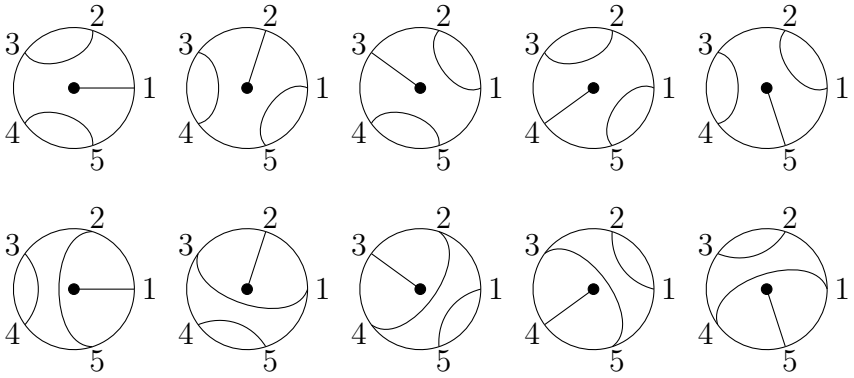
Remark. An element of $\mathcal{LP}^*(2n-1)$ can be viewed as a circle with a puncture which is connected to one of the boundary vertices and connections between the other vertices as in $\mathcal{LP}(2n)$ (see the following example).

Example. We regard these three sets for $n = 3$.

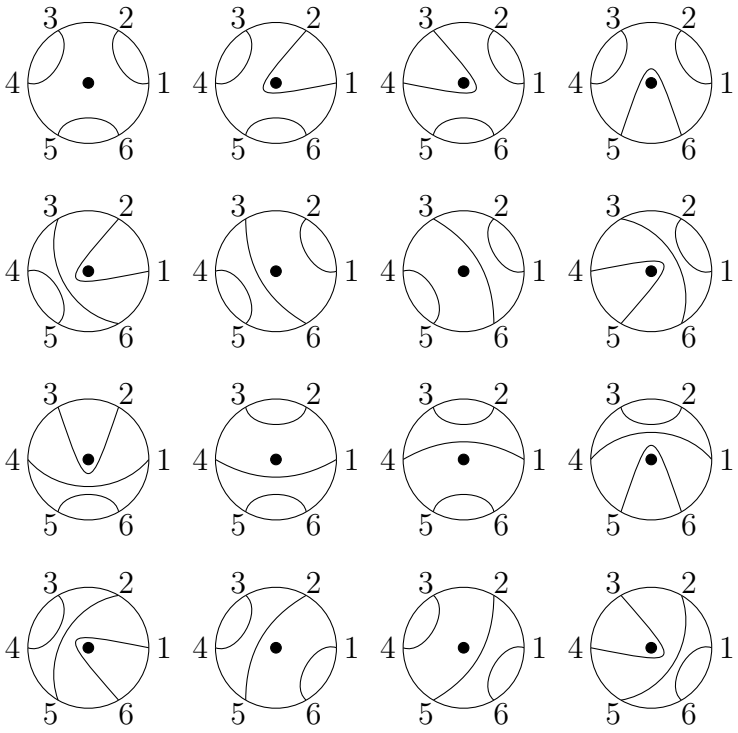
(1) The set $\mathcal{LP}(2n)$ consists of the following five elements:

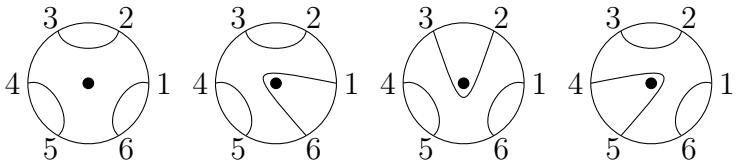


(2) The set $\mathcal{LP}^*(2n-1)$ consists of the following ten elements:



(3) The set $\mathcal{LP}^*(2n)$ consists of the following twenty elements:





Remark. The sets $\mathcal{LP}(2n), \mathcal{LP}^*(2n-1)$ and $\mathcal{LP}^*(2n)$ have cardinalities $C_n, (2n-1)C_{n-1}$ and $(n+1)C_n$, respectively, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ denotes the n-th Catalan number (cf. Proposition 1.4).

Remark. Each connection line between two vertices in a link pattern splits the pattern up into two link patterns (out of which one may be empty) of smaller size. This allows us to do induction proofs on the size of a link pattern.

We now introduce some more notation:

Definition 3.2.

(1) Let $\pi \in \mathcal{LP}(N), i, j \in \{1, \dots, N\}$. We write

$$i \sim j$$

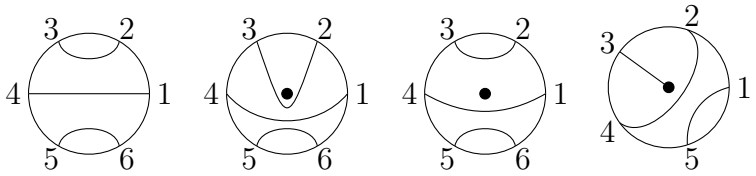
if $\pi(i) = j$ (that is, i and j are connected in π). Otherwise, we write $i \not\sim j$.

(2) We need an extra definition for the case $\mathcal{LP}^*(2n)$:
If in $\pi \in \mathcal{LP}^*(2n)$ the vertices i and j are connected, the connecting edge decomposes the link pattern in two topological components; the first one consists of the vertices m with $i < m < j$, the second one consists of the vertices m with $j < m < i$. Exactly one of these components contains the puncture; we write $i \dot{\sim} j$, if the puncture is contained in the first one of these two components (see the next example) and $j \dot{\sim} i$ otherwise.

(3) In each of the three sets, $i \frown i+1$ shall mean that

- for $\mathcal{LP}(2n), \mathcal{LP}^*(2n-1)$: $i \sim i+1$,
- for $\mathcal{LP}^*(2n)$: $i \sim i+1$ and $i+1 \dot{\sim} i$ (that is, the short arc between i and $i+1$ does not surround the puncture).

Example. To get used to the definitions, we consider the following link patterns $\pi_1, \pi_2, \pi_3, \pi_4$ and note some of their properties:



In π_1, π_2 and π_3, we have $1 \sim 4, 4 \sim 1, 2 \sim 3, 3 \sim 2, 5 \sim 6, 6 \sim 5$ and $i \not\sim j$ for every other pair (i, j). In π_1 and π_3, we have, in addition, for $i \in \{2, 5\}$ the relation $i \frown i+1$;

in π_2 this does only hold for $i = 5$.
In π_4 we have $1 \sim 5, 5 \sim 1, 2 \sim 4, 4 \sim 2$ and $i \not\sim j$ for every other pair (i, j) as well as $5 \frown 1$.
The relation $i \dot{\sim} j$ does hold exactly for the following pairs (i, j):

- in π_2: $(1, 4), (2, 3), (6, 5)$,
- in π_4: $(1, 4), (3, 2), (6, 5)$.

3.2 Generalisations of R, e_i and the Hamiltonian

We first extend the operators R and e_i to the "new" sets:

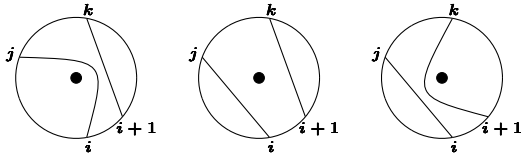
Definition 3.3. We define operators $R, e_i, i \in \{1, \ldots, N\}$, acting on the sets $\mathcal{LP}(2n), \mathcal{LP}^*(2n)$, $\mathcal{LP}^*(2n-1)$ as follows:

(1) Rotation operator R: This operator rotates the outer vertices of π one step counterclockwise. If in $\pi \in \mathcal{LP}(N)$ the vertices i and j are connected, then so are $i-1$ and $j-1$ in $R(\pi)$.
The "structure" of the link pattern does not change: If in $\pi \in \mathcal{LP}^*(2n-1)$, i is connected to the puncture, then so is $i-1$ in $R(\pi)$.
Also, if in $\pi \in \mathcal{LP}^*(2n), i \dot{\sim} j$, then $i-1 \dot{\sim} j-1$ in $R(\pi)$.

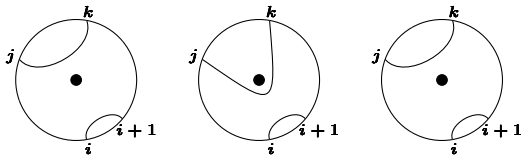
(2) Operators e_i:
- If it holds in π that $i \frown i+1$, then $e_i \pi = \pi$.
- Otherwise, if in π the vertex i is connected to j and $i+1$ to k, then in $e_i\pi$ the vertices i and $i+1$ are connected (such that $i \frown i+1$) and so are j and k.
This definition may be ambiguous if $\pi \in \mathcal{LP}^*(2n)$ – then, in some cases, the connection between j and k can be drawn at both sides of the puncture. In this case:

$$\text{in } e_i\pi \text{ holds } j \dot{\sim} k :\Leftrightarrow \text{ in } \pi \text{ holds } i \dot{\sim} j, \text{ but not } i+1 \dot{\sim} k$$

Remark. The last item of the previous definition may seem strange at first glance; it means the following: If in $\pi \in \mathcal{LP}^*(2n)$ does not hold $i \sim i+1$, then the connections $\{i, j\}$ and $\{i+1, k\}$ split π into three components, the puncture may be in any of the three components:



In $e_i\pi$ the vertices j and k get connected - the puncture shall be in the thereby resulting sector if and only if the puncture in π was in the middle sector.
Therefore, these link patterns get mapped to the following ones:



It may be customary to give an alternative definition of the operators R and $e_i, 1 \leq i \leq N$:

The actions of R and e_i are given by the left and right diagram in Figure 17; $\pi \in \mathcal{LP}(N)$ is drawn inside the disk and $R(\pi)$ or $e_i(\pi)$ are given by the juxtaposition of π with the diagram shown in Figure 17. If loops are produced by this procedure, we just ignore them.

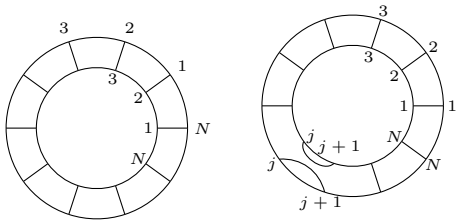
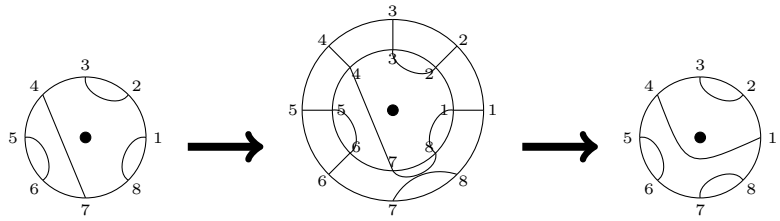
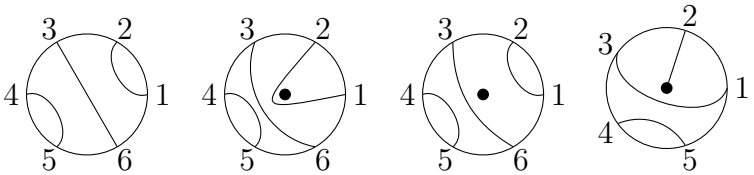


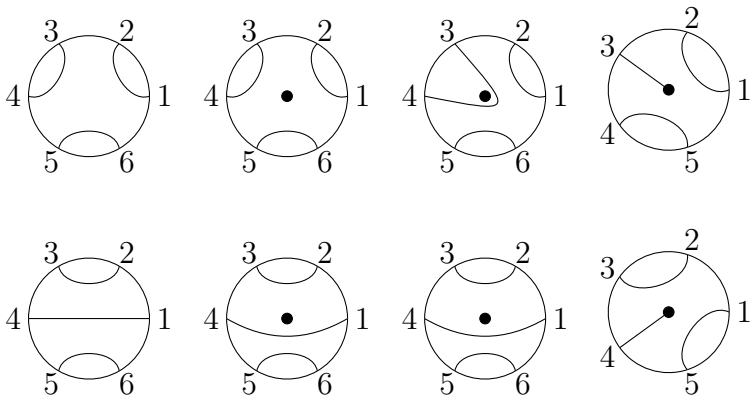
Figure 17: Left: the action of the operator R; right: the action of the operator e_j.

Example. We illustrate the function of this diagram with an example and apply the operator e_7 to a link pattern:



Example. We again consider $\pi_1, \pi_2, \pi_3, \pi_4$ from the previous example and apply the operators R (first row), e_1 (second row) and e_2 (third row) to these link patterns:





As in Chapter 1, we introduce the vector space $\mathbb{C}^{|\mathcal{LP}(N)|}$ as $\mathbb{C}^{\mathcal{LP}(N)}$: Let $\pi_1, \pi_2, \ldots, \pi_m$ be the elements of $\mathcal{LP}(\mathrm{N})$, then we will use the notations

$$\overrightarrow{\pi_1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \overrightarrow{\pi_2} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \overrightarrow{\pi_m} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

for the standard basis of $\mathbb{C}^{\mathcal{LP}(N)}$.

We want to establish some relations between the e_i and R which we will use frequently. To this end, we introduce the *Temperley-Lieb algebra*:

Definition 3.4. We call the algebra generated by R and $e_i, 1 \leq i \leq N$, the *(Cyclic) Temperley-Lieb Algebra* and denote this algebra by $\mathrm{CTL_N}$.

Proposition 3.5 (Properties of the Temperley-Lieb generators)**.** *The maps $R, e_i, 1 \leq i \leq N$, have the following properties:*

(1) $e_i = Re_{i+1}R^{-1}$

(2) $e_i^2 = e_i$

(3) $e_i e_j = e_j e_i$, *if* $|i - j| \neq 1$

(4) $e_i e_{i\pm 1} e_i = e_i$.

Proof. We start with the cases $\mathcal{LP}(2n)$ and $\mathcal{LP}^*(2n-1)$. In the following, j, k, ℓ, m, h may also denote the puncture (in the case $\mathcal{LP}^*(2n-1)$). Figures 18 and 19 illustrates the proofs of items (1) and (3).

(1) Say $i \sim j, i+1 \sim k$ are connected in π.
Then $i+1 \sim j+1, i+2 \sim k+1$ in $R^{-1}\pi$, therefore $i+1 \sim i+2, j+1 \sim k+1$ in

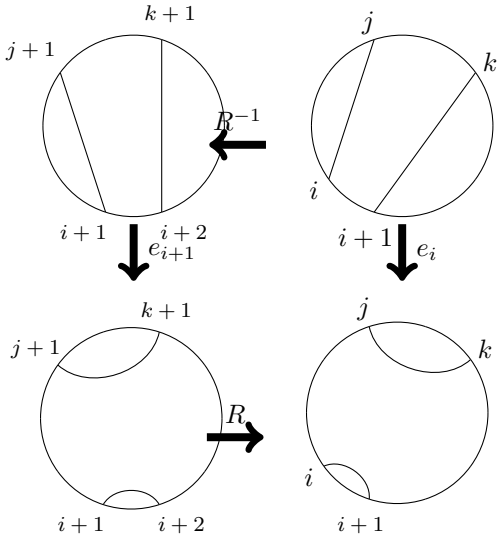


Figure 18: Illustration of item (1) of Proposition 3.5.

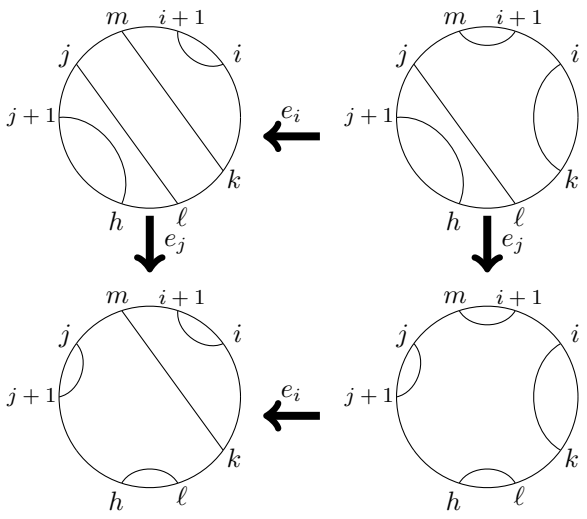


Figure 19: Illustration of item (3) of Proposition 3.5.

$e_{i+1}R^{-1}\pi$ and thus $i \sim i+1, j \sim k$ in $Re_{i+1}R^{-1}\pi$.
On the other hand, the relations

$$i \sim i+1, j \sim k, \ell \sim m$$

also hold in $e_i\pi$. Since in neither in $e_i\pi$ nor in $Re_{i+1}R^{-1}$ any other edges are "affected", this proves (1).

(2) In $e_i\pi$, we have $i \frown i+1$, so another application of e_i does (by its very definition) not change anything.

(3) Say we have the relations $i \sim k, j \sim \ell, i+1 \sim m, j+1 \sim h$ in π.
Then $i \sim i+1, j \sim \ell, k \sim m, j+1 \sim h$ in $e_i\pi$ and $i \sim i+1, j \sim j+1, k \sim m, \ell \sim h$ in $e_je_i\pi$.
On the other hand, $j \sim j+1, i \sim k, \ell \sim h, i+1 \sim m$ in $e_j\pi$ and thus also $i \sim i+1, j \sim j+1, k \sim m, \ell \sim h$ in $e_ie_j\pi$. Again, any other edges stay the same during both procedures.

(4) Let $i \sim j, i+1 \sim k, i+2 \sim \ell$ in π be connected, say. We again only pay attention to the edges changing during the applications of the e_k. Then $i \sim i+1, j \sim k, i+2 \sim \ell$ in $e_i\pi$ and thus $i \sim \ell, i+1 \sim i+2, j \sim k$ in $e_{i+1}e_i\pi$, so $i \sim i+1, \ell \sim i+2, j \sim k$ in $e_ie_{i+1}e_i\pi$, which therefore equals $e_i\pi$.
Similarly, if $i-1 \sim j, i \sim k, i+1 \sim \ell$ in π, then $i-1 \sim j, i \sim i+1, k \sim \ell$ in $e_i\pi$, therefore $i-1 \sim i, j \sim j+1, k \sim \ell$ in $e_{i-1}e_i\pi$ and thus $i-1 \sim j, i \sim i+1, k \sim \ell$ in $e_ie_{i-1}e_i\pi$, which again equals $e_i\pi$.

The proofs for the case $\mathcal{LP}^*(2n)$ are essentially the same; although one has to make more distinctions about the location of the puncture. □

Now that we have extended the operators R and e_i to $\mathcal{LP}^*(2n-1)$ and $\mathcal{LP}^*(2n)$, we do the same for H_0 in a canonical way:

Definition 3.6. The *Hamiltonian* $H_0 \in \mathrm{CTL_N}$ is defined as

$$H_0 := \sum_{i=1}^{N}(e_i - 1).$$

We will prove later (see Corollary 3.19) that H_0 has a unique (up to normalization) right-nullvector. We call this vector $\overrightarrow{\mu_N}$ its *ground state*, that is, $\overrightarrow{\mu_N}$ is the unique vector in $\mathbb{C}^{\mathcal{LP}(N)}$ satisfying

$$H_0\overrightarrow{\mu_N} = 0. \qquad (\dagger)$$

In Chapter 4, we will define refined versions of FPLs which will be necessary for stating the Razumov-Stroganov-Cantini-Sportiello theorem for $\mathcal{LP}^*(2n-1)$ and $\mathcal{LP}^*(2n)$.

3.3 Excursion: Tools for proving our results

Definition 3.7. A real $n \times n$ matrix A is called

(1) *right stochastic* if each row of A sums to 1.

(2) *left stochastic* if each column of A sums to 1.

Remark. The matrix A is right stochastic if and only if A^T is left stochastic.

Definition 3.8. Let A be a real $n \times n$ matrix with $a_{ij} \geq 0$ for all $i, j \in \{1, \ldots, n\}$. We call A *irreducible* if for all $i, j \in \{1, \ldots, n\}$ there is a sequence $s_0, \ldots, s_m$ with $s_0 = i, s_m = j$ and $a_{s_k, s_{k+1}} > 0$ for all $k \in \{0, \ldots, m-1\}$.

Lemma 3.9. *Let A be an irreducible $n \times n$ matrix and let $i, j \in \{1, \ldots, n\}$.*
Then there is a $k > 0$ such that

$$(A^k)_{ij} > 0.$$

Proof. Let $s_0, \ldots, s_m, s_0 = i, s_m = j$ be a sequence witnessing the irreducibility of i and j. Then

$$\begin{aligned}(A^m)_{ij} &= \sum_{(t_1, \ldots, t_{m-1}) \in [n]^{m-1}} a_{i,t_1} \cdot a_{t_1,t_2} \cdot \ldots \cdot a_{t_{m-1},j} \\ &\geq a_{s_0,s_1} \cdot a_{s_1,s_2} \cdot \ldots \cdot a_{s_{m-1},s_m} > 0.\end{aligned}$$

□

Lemma 3.10. *The largest eigenvalue of a (left or right) stochastic matrix A his* 1.

Proof. It suffices to prove this claim for a right stochastic matrix A. For if $\chi_A(x), \chi_{A^T}(x)$ are the characteristic polynomials of A, A^T, respectively, then, because the determinant function is stable under transposition,

$$\begin{aligned}\chi_{A^T}(x) &= \det(A^T - \lambda \operatorname{Id}) = \det((A^T - \lambda \operatorname{Id})^T) \\ &= \det(A - \lambda \operatorname{Id}^T) = \det(A - \lambda \operatorname{Id}) = \chi_A(x);\end{aligned}$$

that is, the eigenvalues of A^T and A equal each other (as roots of the characteristic polynomials).
We show that A has eigenvalue 1:

(1) The matrix A is right stochastic, i. e. its entries are non-negative and each row sums up to 1. So we get

$$A \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix},$$

i. e. 1 is eigenvalue of A.

(2) We show: 1 is the largest eigenvalue of A:
Suppose $\lambda > 1$ was an eigenvalue of A. Then for an $x \in \mathbb{R}^n$, we had $Ax = \lambda x$. Let $\bar{x} = \max\{x_1, x_2, \ldots, x_n\}$. Since the rows of A sum up to 1 and all components of A are non-negative, all components y_i of $y = Ax$ satisfy $y_i \leq \bar{x}$.
On the other hand, since $\lambda > 1, \lambda\bar{x} > \bar{x}$. But this implies $Ax = y \neq \lambda x$, a contradiction.

□

Theorem 3.11 (Perron-Frobenius theorem, special case)**.** *Let A be a left real stochastic irreducible matrix of dimension $n \times n$. Then 1 is a simple eigenvalue of A and A^T. Moreover, the eigenvector corresponding to 1 of A and A^T can be chosen such that all components are positive.*

Remark. One can prove a more general theorem dealing with quadratic matrices not being necessarily stochastic. The reader is referred to [11].

Proof.

(1) We show: If B is an $n \times n$ matrix, $B \neq A$, with $0 \leq b_{ij} \leq a_{ij}, 0 \leq i, j \leq n$ and σ an eigenvalue of B, then $\sigma < 1$:
Let σ be the eigenvalue of S associated with x, i. e. we have $Sx = \sigma x$.
Suppose $\sigma \geq 1$.
As before, let $\bar{x} = \max\{x_1, \ldots, x_n\}$. Since $B \neq A$, there is a row i of B, whose entries sum up to a smaller value than 1.
So it follows for the component y_i of $y = Sx$, that $y_i < \bar{x}$.
On the other hand, since $\sigma \geq 1, \sigma\bar{x} \geq \bar{x}$.
But this implies $Sx = y \neq \sigma x$, a contradiction.

(2) Let $A_{(i)}$ be the matrix A with the i-th row and i-th column replaced by zeroes. Since A was assumed irreducible, no row of A can consist completely of zeros. So $A_{(i)} \neq A$ (and therefore, by (1), each eigenvalue of $A_{(i)}$ is stricly smaller 1).

(3) Let

$$\Lambda = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}.$$

Then $\det(\Lambda_i - A_i) = \frac{\partial}{\partial \lambda_i} \det(\Lambda - A)$, where Λ_i, A_i are the $(n-1) \times (n-1)$ matrices resulting from deleting the i-th row and j-th column of Λ, A, respectively.
To prove this, we expand $\det(\Lambda - A)$ along the i-th row and obtain (here B_{ij}

denotes the matrix obtained by deleting the i-th row and j-th column of $\Lambda - A$):

$$\begin{aligned}\frac{\partial}{\partial \lambda_i}\det(\Lambda - A) &= \frac{\partial}{\partial \lambda_i}\left(\sum_{j=1}^{n}(-1)^{i+j}(\Lambda - A)_{ij}\det(B_i j)\right)\\ &= \frac{\partial}{\partial \lambda_i}((-1)^{2i}(\lambda_i - a_{ii}\det(B_{ii}))) + \frac{\partial}{\partial \lambda_i}\left(\sum_{j\neq i}(-1)^{i+j}(-a_{ij})\det(B_{ij})\right)\\ &= \frac{\partial}{\partial \lambda_i}((\lambda_i - a_{ii})\det(\Lambda_i - A_i))\\ &= \frac{\partial}{\partial \lambda_i}(\lambda_i \det(\Lambda_i - A_i)) = \det(\Lambda_i - A_i).\end{aligned}$$

This is what we wanted to show.

(4) We want to show:

$$\frac{\mathrm{d}}{\mathrm{d}\,\lambda}\det(\lambda\,\mathrm{Id_n}\text{-A}) = \sum_{i=1}^{n}\det(\lambda\,\mathrm{Id}_{n-1} - A_i).$$

Define $f\colon \mathbb{R}^n \to \mathbb{R}, f(\lambda_1,\ldots,\lambda_n) = \det(\Lambda - A); g\colon \mathbb{R} \to \mathbb{R}^n, g(\lambda) = (\lambda,\ldots,\lambda)$; $h_i\colon \mathbb{R}^n \to \mathbb{R}, h_i(\lambda_1,\ldots,\lambda_n) = \det(\Lambda_i - A_i)$ (where Λ, Λ_i are defined as in (3)). Then, by (3)

$$h_i(g(\lambda)) = \frac{\partial}{\partial \lambda_i} f(g(\lambda)).$$

It follows

$$\begin{aligned}\sum_{i=1}^{n}\det(\lambda\,\mathrm{Id} - A_i) &= \sum_{i=1}^{n} h_i(g(\lambda)) = \sum_{i=1}^{n}\frac{\partial}{\partial \lambda_i} f(g(\lambda))\\ &= \left(\tfrac{\partial}{\partial \lambda_1} f(g(\lambda)),\ldots,\tfrac{\partial}{\partial \lambda_n} f(g(\lambda))\right)\cdot\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}\\ &= \mathrm{J}_f(g(\lambda))\cdot \mathrm{J}_g(\lambda) = J_{f\circ g}(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\,\lambda} f(g(\lambda))\\ &= \frac{\mathrm{d}}{\mathrm{d}\,\lambda}\det(\lambda\,\mathrm{Id} - A)\end{aligned}$$

by an application of the higher-dimensional chain rule (where $\mathrm{J}_h(x)$ is the Jacobian of the function h in x).

(5) We show: The characteristic polynomial $\chi_{A_i}(x)$ of each A_i is positive for $x = 1$: The leading coefficent of each $\chi_{A_i}(x)$ is 1, so it follows that $\lim\limits_{x\to\infty}\chi_{A_i}(x) = +\infty$.

By (2), each $A_{(i)}$ has eigenvalues strictly less than 1.
Since each eigenvalue of A_i is also eigenvalue of $A_{(i)}$ (for if σ is eigenvalue of A_i, then there is an $x \in \mathbb{R}^{n-1}$ such that $A_i x = \sigma x$ – now σ is eigenvalue of $A_{(i)}$ with eigenvector x^+, where $x^+_j = \begin{cases} x_j, & \text{if } j > i \\ 0 & \text{if } j = i \\ x_{j-1}, & \text{if } j > i \end{cases}$; so every eigenvalue of A_i is strictly smaller than 1).
Therefore, and because of $\lim_{x\to\infty} \chi_{A_i}(x) = +\infty$, it follows that $\chi_{A_i}(1) > 0$.

(6) We have

$$\frac{\partial}{\partial \lambda} \det(\lambda \operatorname{Id}_n - A) \overset{(4)}{=} \sum_{i=1}^{n} \det(\lambda \operatorname{Id}_{n-1} - A_i),$$

and at $\lambda = 1$ we obtain

$$\left.\frac{\partial}{\partial \lambda}\right|_{\lambda=1} \det(\lambda \operatorname{Id}_n - A) = \sum_{i=1}^{n} \det(\operatorname{Id}_{n-1} - A_i) \overset{(5)}{>} 0.$$

So the zero of the characteristic polynomial of A at $\lambda = 1$ is simple; therefore 1 has algebraic (and hence also geometric) multiplicity 1. This is what we wanted to show.

(7) It remains to show the part about the positivity of the eigenvectors:
The corresponding eigenvector in A is $\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$, therefore we investigate $T := A^T$.
By the lemma, for each pair (i, j) of an irreducible matrix B there is an $\ell \in \mathbb{N}$ with $(B^\ell)_{ij} > 0$. The transposition of A clearly preserves the irreducibiliy of A. Therefore, and because of the non-negativity of the components of A^T it follows that for sufficiently large k the matrix

$$(\operatorname{Id} + T)^k = \operatorname{Id} + kT + \frac{k(k-1)}{2} T^2 + \cdots$$

(binomial expansion) has strictly positive components..
Let y be the eigenvector corresponding to 1 of $T = A^T$, i. e. $Ty = y$.
We can assume (after multiplication with a scalar) that all components of y are non-negative.

We shall write $0 < z$, if all components of z are greater 0. We have

$$\begin{aligned} 0 < (\mathrm{Id} + T)^k y &= (\mathrm{Id} + kT + \frac{k(k-1)}{2} T^2 + \ldots) y \\ &= (y + ky + \frac{k(k-1)}{2} y + \ldots) = (1+1)^k y \end{aligned}$$

and this proves the remaining part.

□

We will later also need the following "continuity result":

Theorem 3.12. *Let*

$$f(z) = a_0 + a_1 z + \ldots + a_n z^n = \prod_{j=1}^{m} (z - z_j)^{k_j},$$

$a_i \in \mathbb{C}, a_n \neq 0$, be a polynomial with the zero z_j having multiplicity k_j.
Let K_i be the disk with center z_i and radius R_i with R_i chosen such that

$$0 < R_i < \min_{j \in \{1,\ldots,m\} \setminus \{i\}} |z_i - z_j| \,. \tag{2}$$

Then there is an $\varepsilon > 0$ such that for each $\delta_s \in \mathbb{C}, |\delta_s| \leq \varepsilon$ ($s = 0, 1, \ldots, n-1$) the polynomial

$$\tilde{f}(z) = (a_0 + \delta_0) + (a_1 + \delta_1) z + \ldots + (a_{n-1} + \delta_{n-1}) z^{n-1} + a_n z^n$$

has exactly k_i zeros in the disk K_i.

Proof. Let $H(z) := \delta_0 + \delta_1 z + \ldots + \delta_{n-1} z^{n-1}$. Fix an $i \in \{1, \ldots, m\}$.
In the disk K_i we have

$$\begin{aligned} |H(z)| &= \left|\delta_0 + \delta_1 z + \ldots + \delta_{n-1} z^{n-1}\right| \\ &\leq |\delta_0| + |\delta_1|\,|z| + \ldots + |\delta_{n-1}|\,|z|^{n-1} \\ &\leq \varepsilon(1 + |z| + \ldots + |z|^{n-1}) \\ &= \varepsilon(1 + |(z - z_i) + z_i| + \ldots + |(z - z_i) + z_i|^{n-1}) \\ &\leq \varepsilon(1 + (R_i + |z_i|) + \ldots + (R_i + |z_i|)^{n-1}) = \underbrace{\varepsilon \left(\sum_{j=0}^{n-1} (R_i + |z_i|)^j \right)}_{:=\alpha_i}. \end{aligned}$$

Furthermore, the triangle inequality for $j \neq i$ yields

$$|z_i - z_j| \leq |z_i - z| + |z - z_j| \leq R_i + |z - z_j| \,,$$

thus

$$|z - z_i| \geq |z_i - z_j| - R_i. \tag{3}$$

We now estimate $f(z)$ on K_i:

$$\begin{aligned}
|f(z)| &= |a_0 + a_1 z + \ldots + a_n z^n| \\
&= |a_n| \left| \prod_{j=0}^{m} (z - z_j)^{k_j} \right| = |a_n| \prod_{\substack{j=1, \\ j \neq i}}^{m} |z - z_j|^{k_j} \\
&= |a_n| \, |z - z_i|^{k_i} \prod_{\substack{j=1, \\ j \neq i}}^{m} |z - z_j|^{k_j} = |a_n| \, R_i^{k_i} \prod_{j=1}^{m} |z - z_j|^{k_j} \\
&\overset{(3)}{\geq} \underbrace{|a_n| \, R_i^{k_i} \prod_{\substack{j=1, \\ j \neq i}}^{m} \underbrace{(|z - z_j| - R_i)^{k_j}}_{>0 \text{ by } (2)}}_{>0} =: \omega_i.
\end{aligned}$$

Now choosing $\varepsilon < \frac{\omega_i}{\alpha_i}$ gives

$$|H(z)| \leq \varepsilon \alpha_i < \frac{\omega_i}{\alpha_i} \cdot \alpha_i = \omega_i \leq |f(z)| \, .$$

By Rouché's theorem, $\tilde{f}(z) = f(z) + H(z)$ and $f(z)$ have the same amount of zeros inside K_i.
But, by (2), z_i is the only zero (with multiplicity k_i) in K_i.
Therefore, $\tilde{f}(z)$ has exactly k_i roots in K_i. □

3.4 Scattering Equations

We continue by rewriting the Hamiltonian:

Lemma 3.13. *Let $i \in \{1, \ldots, N\}$. Then*

$$H_0 = \sum_{k=0}^{N-1} R^k (e_i - 1) R^{-k}.$$

Proof. Fix $i \in \{1, \ldots, N\}$.
By Proposition 3.5, we have $R e_i = e_{i+1} R$, which inductively implies $R^k e_i = e_{i+k} R^k$.
This yields

$$\sum_{k=0}^{N-1} R^k (e_i - 1) R^{-k} = \sum_{k=0}^{N-1} (e_{i+k} - 1) = \sum_{i=1}^{N} (e_i - 1) = H_0.$$

□

Definition 3.14. We define operators $X_i(t) \in \mathrm{CTL_N}, 1 \leq i \leq N$ for $0 \leq t \leq 1$, by

$$X_i(t) = t \cdot \mathrm{Id} + (1-t)e_i.$$

We define for each i a so-called *scattering equations* by

$$X_i(t)\overrightarrow{\psi^{(i)}(t)} = R\overrightarrow{\psi^{(i)}(t)}. \qquad (\star)$$

(where $\overrightarrow{\psi^{(i)}(t)}$ is unknown). Our next goal is to show that a vector $\overrightarrow{\psi^{(i)}(t)}$ satisfying $(\star)$ is a solution of $(\dagger)$.

Proposition 3.15 (Properties of X_i)**.** *The operator $X_i(t)$ has the following properties:*

(1) $X_i(1) = \mathrm{Id}$

(2) $R^k X_i(t) R^{-k} = X_{i+k}(t)$

(3) $\frac{dX_i(t)}{dt} = 1 - e_i$.

Proof.

(1) $X_i(1) = \mathrm{Id} + (1-1)e_i = \mathrm{Id}$.

(2) We apply property (1) of Proposition 3.5:

$$\begin{aligned} R^k X_i(t) R^{-k} &= R^k t\,\mathrm{Id}\, R^{-k} + R^k(1-t)e_i R^{-k} \\ &= t\,\mathrm{Id}\, R^k R^{-k} + (1-t)R^k e_i R^{-k} \\ &= t\,\mathrm{Id} + (1-t)e_{i+k} R^k R^{-k} \\ &= t\,\mathrm{Id} + (1-t)e_{i+k} = X_{i+k}(t). \end{aligned}$$

(3) $\frac{dX_i(t)}{dt} = \mathrm{Id} - e_i = 1 - e_i$.

□

Theorem 3.16. *If $\overrightarrow{\psi^{(i)}(t)}$ satisfies the scattering equation $(\star)$, then*

$$H_0\overrightarrow{\psi^{(i)}(1)} = 0,$$

i. e. $\overrightarrow{\psi^{(i)}(1)}$ satisfies $(\dagger)$.
In particular, the $\overrightarrow{\psi^{(i)}(1)}, 1 \leq i \leq 2n$, all equal each other.

Proof. By the previous proposition, part (2),

$$X_{i+k}(t) = R^k X_i(t) R^{-k}.$$

Thus it follows

$$\begin{aligned} &X_{i+\ell}(t)X_{i+\ell-1}(t)\dots X_{i+1}(t)X_i(t) \\ &\qquad = R^\ell X_i(t) R^{-\ell} R^{\ell-1} X_i(t) R^{-(\ell-1)} \dots R X_i(t) R^{-1} X_i(t) \\ &\qquad = R^{\ell+1} R^{-1} X_i(t) R^{-1} X_i(t) R^{-1} \dots R^{-1} X_i(t) R^{-1} X_i(t) \\ &\qquad = R^{\ell+1} (R^{-1} X_i(t))^{\ell+1}. \end{aligned}$$

We set $\ell = N-1$ and obtain $X_{i+N-1}(t)\dots X_i(t) = R^N (R^{-1}X_i(t))^N$.
If $\overrightarrow{\psi^{(i)}(t)}$ satisfies the equation $(\star)$, i. e.

$$X_i \overrightarrow{\psi^{(i)}(t)} = R \overrightarrow{\psi^{(i)}(t)},$$

we get

$$R^{-1} X_i \overrightarrow{\psi^{(i)}(t)} = \overrightarrow{\psi^{(i)}(t)}.$$

We define the *scattering matrix* $S_i(t), 1 \le i \le N$, by

$$S_i(t) = X_{i+N-1}(t)\dots X_{i+1}(t) X_i(t);$$

this implies, since $R^N = 1$,

$$\begin{aligned} S_i(t) &= X_{i+N-1}(t) X_{i+N-2}(t) \dots X_{i+1}(t) X_i(t) \overrightarrow{\psi^{(i)}(t)} \\ &= R^N (R^{-1} X_i)^N \overrightarrow{\psi^{(i)}(t)} = R^N \overrightarrow{\psi^{(i)}(t)} = \overrightarrow{\psi^{(i)}(t)}, \end{aligned}$$

which yields

$$(S_i(t) - 1) \overrightarrow{\psi^{(i)}(t)} = 0.$$

We have $S_i(1) = X_{i+N-1}(1)\dots X_{i+1}(1)X_i(1) = 1 \cdot \dots \cdot 1 = 1$ and $\frac{dX_j(t)}{dt} = 1 - e_j$, and therefore

$$\begin{aligned} \left.\frac{dS_i(t)}{dt}\right|_{t=1} &= \left(\sum_{j=i}^{i+N-1} X_{i+N-1}(t)\dots X_{j+1}(t) \frac{dX_j(t)}{dt} X_{j-1}(t)\dots X_i(t) \right) \Bigg|_{t=1} \\ &= \sum_{j=i}^{i+N-1} (1 - e_j) = -H_0, \end{aligned}$$

what proves the statement. □

In the following, we consider only $X_1(t)$ and the corresponding equation

$$(t\,\mathrm{Id} + (1-t)e_1)\overrightarrow{\psi^{(i)}}(t) = R\overrightarrow{\psi^{(i)}}(t).$$

Proposition 3.17 (Properties of $\frac{1}{N}H_0 + \mathrm{Id}$)**.** *Let $A := \frac{1}{N}H_0 + \mathrm{Id}$. Then the following properties hold:*

(1) A is left stochastic.

(2) A has eigenvalue 1 corresponding to the eigenvector v with multiplicity 1 if and only if H_0 has eigenvalue 0 corresponding to the eigenvector v with multiplicity 1.

Proof.

(1) It suffices to show that all entries in A are non-negative and all columns of H_0 sum up to 0. We start with the latter.
The j-th column of H_0 is given by

$$\begin{pmatrix} \#k \in \{0,\dots,N-1\} \colon e_k(\pi_j) = \pi_1 \\ \#k \in \{0,\dots,N-1\} \colon e_k(\pi_j) = \pi_2 \\ \vdots \\ \#\big(k \in \{0,\dots,N-1\} \colon e_k(\pi_j) = \pi_j\big) - N \\ \vdots \\ \#k \in \{0,\dots,N-1\} \colon e_k(\pi_j) = \pi_N \end{pmatrix} \leftarrow \textbf{j-th row}$$

Therefore, the sum of the entries of the j-th column equals

$$\#\big(k \in \{0,\dots,N-1\} \colon e_k(\pi_j) \in \{\pi_1,\dots,\pi_N\}\big) - N = N - N = 0.$$

Concerning the former:
The only (possibly) negative entries of H_0 are on its main diagonal. But each diagonal element h of H_0 satisfies $-N \leq h$, therefore every diagonal element h' of $\frac{1}{N}H_0$ satisfies the inequality $-1 \leq h'$ and hence each diagonal element a of $A = \frac{1}{N}H_0 + \mathrm{Id}$ satisfies $0 \leq a$.

(2) We have

$$\left(\frac{1}{N}H_0 + 1\right)v = v \Leftrightarrow \frac{1}{N}H_0 v = 0 \Leftrightarrow H_0 v = 0.$$

Since the matrix A is only a scaling of H_0, these two eigenvalues have in both matrices the same algebraic and geometric multiplicity.

□

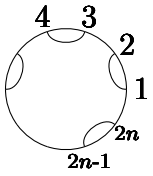
Lemma 3.18. *The matrix $A = \frac{1}{N}H_0 + \mathrm{Id}$ is irreducible.*

Proof. By definition of H_0, we have:

$$a_{ij} \neq 0 \text{ if and only if there is an } e_k \colon e_k(\pi_j) = \pi_i.$$

So we want to show: For each link pattern π_j there is a finite sequence $e_{k_\ell}, e_{k_{\ell-1}}, \ldots, e_{k_1}$ such that $e_{k_\ell} e_{k_{\ell-1}} \ldots e_{k_1} \pi_j = \pi_i$. We first show this fact for the case $\mathcal{LP}(2n)$ and describe afterwards the changes for the cases $\mathcal{LP}^*(2n-1)$ and $\mathcal{LP}^*(2n)$.

(1) We can assume that π_j is the following "very simple" link pattern π:



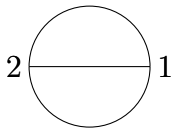
For if $\pi_j \neq \pi$, then

$$e_{2n-1} \ldots e_3 e_1 \pi_j = \pi$$

(since e_1 connects the vertices 1 and 2, e_3 connects 3 and 4 and preserves the already connected vertices 1 and 2 etc.).
So, if we have found a sequence $e_{k_\ell}, \ldots, e_{k_1}$ with $e_{k_\ell} \ldots e_{k_1} \pi = \pi_i$, we get by concaternating

$$e_{k_\ell} \ldots e_{k_1} e_{2n-1} \ldots e_1 \pi_j = \pi_i.$$

(2) We thus want to show that there exists a sequence $e_1, \ldots, e_\ell$ with $e_\ell \ldots e_1 \pi = \pi_i$ and show this by induction on n.
The induction basis ($n = 1$) is clear, as there is only one link pattern in $\mathcal{LP}(2)$, namely



.

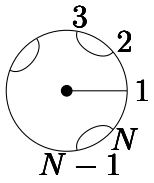
$n \to n+1$: Suppose $\pi_i(1) = m$.
Then

$$\begin{aligned} \text{in } \pi &: 1 \sim 2, 3 \sim 4, 5 \sim 6, \ldots, 2n-1 \sim 2n, \\ \text{in } e_2\pi &: 1 \sim 4, 2 \sim 3, 5 \sim 6, \ldots, 2n-1 \sim 2n, \\ \text{in } e_4 e_2 \pi &: 1 \sim 6, 2 \sim 3, 4 \sim 5, \ldots, 2n-1 \sim 2n, \end{aligned}$$

in $e_6 e_4 e_2$ are 1 and 8 connected etc.
Since m has to be even, in $e_{m-2} e_{m-4} \ldots e_6 e_4 e_2 \pi$ are the vertices 1 and m connected.

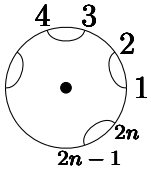
By connecting 1 and m, the link pattern breaks down into two components. Thus the statement follows from the induction hypothesis.

(3) The proof in the case $\mathcal{LP}^*(2n-1)$ is essentially the same, but we use as "very simple" link pattern π the following one obtained via $e_{N-1}\dots e_2\pi_j$:



(i. e. the vertex 1 is connected to the center and each even vertex M with $M+1$). If in π_i the vertex 1 is connected to the center, we apply the induction hypothesis (for $\mathcal{LP}(2n)$) for the link pattern with the vertices $2,\dots,N$; otherwise in π_i the vertex 1 is connected to a vertex m, in this case we connect by the same procedure as above the vertices 1 und m. This connection breaks down the link pattern in two link patterns as before, on which we apply the induction hypothesis (for the cases $\mathcal{LP}^*(2n-1)$ and $\mathcal{LP}(2n)$).

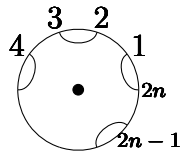
(4) The case $\mathcal{LP}^*(2n)$ is somewhat more complicated, since in this case two vertices i and j are not just connected; here we rather have $i\dot{\sim}j$ or $j\dot{\sim}i$. Again we only have to show that the vertex 1 may be "suitable connected" and can then apply the induction hypothesis. Let m be the vertex with $1\sim m$.
First case: $m\dot{\sim}1$. Our "very simple link pattern" π then should be $e_{2n-1}\dots e_1\pi_j$:



We have the following "connection-relations":

$$\begin{aligned}
\text{in } \pi &: 2\dot{\sim}1, 4\dot{\sim}3, 6\dot{\sim}5, \dots, 2n\dot{\sim}2n-1,\\
\text{in } e_2\pi &: 3\dot{\sim}2, 4\dot{\sim}1, 6\dot{\sim}5, \dots, 2n\dot{\sim}2n-1,\\
\text{in } e_4e_2\pi &: 5\dot{\sim}4, 6\dot{\sim}1, 3\dot{\sim}2, \dots, 2n\dot{\sim}2n-1
\end{aligned}$$

etc., in $e_{m-2}e_m\dots e_2\pi$ we have $m\dot{\sim}1$.
Second case: $1\dot{\sim}m$. Then we consider the following link pattern $\pi' = R\pi$ instead of π, obtained by $e_{2n}\dots e_2\pi_j$:



As before, we have the following relations:

$$\begin{aligned}
\text{in } \pi' &: 3\dot{\sim}2, 5\dot{\sim}4, m+5\dot{\sim}m+4, m+3\dot{\sim}m+2, m+1\dot{\sim}m, \ldots, 1\dot{\sim}2n,\\
\text{in } e_{m+1}\pi' &: 3\dot{\sim}2, 5\dot{\sim}4, m+5\dot{\sim}m+4, m+2\dot{\sim}m+1, m+3\dot{\sim}m, \ldots, 1\dot{\sim}2n,\\
\text{in } e_{m+3}e_{m+1}\pi' &: 3\dot{\sim}2, 5\dot{\sim}4, m+2\dot{\sim}m+1, m+4\dot{\sim}m+3, m+5\dot{\sim}m, \ldots, 1\dot{\sim}2n.
\end{aligned}$$

If we continue in this vein, we have in $e_{m+5}e_{m+3}e_{m+1}\pi'$ the connection $m+7\dot{\sim}m$ etc. and thus the link pattern $e_{2n-1}e_{2n-3}\ldots e_{m+3}e_{m+1}\pi'$ satisfies $1\dot{\sim}m$.

□

Corollary 3.19. *The vector $\overrightarrow{\mu_N}$ as in Definition 3.6 exists and is unique (if normalized).*

Proof. Apply the Perron-Frobenius theorem to $A = \frac{1}{N}H_0 + \mathrm{Id}$ and use the previous proposition and the previous lemma. □

Lemma 3.20. *Fix $i \in \{0, \ldots, N-1\}, 0 \leq t \leq 1$.*
Then there is a unique (up to multiplication with a scalar) solution of

$$X_i(t)\overrightarrow{\psi^{(i)}(t)} = R\overrightarrow{\psi^{(i)}(t)}. \tag{4}$$

Proof. The fact that there is a unique solution to (4) is equivalent to the existence of a unique solution to the equation

$$R^{-1}X_i(t)\overrightarrow{\psi^{(i)}(t)} = \overrightarrow{\psi^{(i)}(t)}. \tag{5}$$

But if we can show that the matrix $R^{-1}X_i(t)$ is left stochastic and irreducible, this follows from the Perron-Frobenius theorem. We can do so for $t \in (0,1)$, for $t \in \{0,1\}$ we will use a continuity argument.

(1) We show that $R^{-1}X_i(t)$ is left stochastic:

As in the proof of Proposition 3.17, the j-th column of $X_i(t)$ has the form

$$j\textbf{-th row} \to \begin{pmatrix} 0 \\ \vdots \\ 0 \\ t \\ 0 \\ \vdots \\ 0 \end{pmatrix} + (1-t) \begin{pmatrix} M_1 \\ \vdots \\ M_{j-1} \\ M_j \\ M_{j+1} \\ \vdots \\ M_N \end{pmatrix},$$

where

$$M_\ell := \#k \in \{0, \dots, N-1\} \colon e_k(\pi_j) = \pi_\ell,$$

the column thus sums up to $t + (1-t) \cdot 1 = 1$.
Since the left-multiplication with R^{-1} only permutes the rows of $X_i(t)$, this also holds for $R^{-1}X_i(t)$.

(2) We prove that $R^{-1}X_i(t), 0 < 1 < t$ is irreducible.
We have:

$$(R^{-1}X_i(t))_{j\ell} = \#k \in \{0, \dots, N-1\} \colon e_k(\pi_\ell) = R^{-1}(\pi_j)$$

and this is (since $1 - t \neq 0$) always greater 0 (compare the proof of Lemma 3.18). But we have proven the latter fact already in Lemma 3.18.

(3) The proof given in (2) does not extend to $t = 1$ (since X_i is not irreducible then). However, let $\chi_t(z)$ be the characteristic polynomial of $R^{-1}X_i(t)$.
By the Perron-Frobenius theorem, for $0 < t < 1$, the zero of $\chi_t(z)$ at $z = 1$ is simple.
Let $t \to 1$, by Theorem 3.12 can $\chi_1(z)$ also have a at most simple zero at $z = 1$; being left stochastic guarantees the existence of such a zero.

□

Proposition 3.21. *Let $\overrightarrow{\psi(t)} \in \mathbb{C}^{\mathcal{LP}(N)}$. Then $\overrightarrow{\psi(t)}$ satisfies the scattering equation ($\star$) if and only if $\overrightarrow{\psi(t)}$ satisfies the two equations*

$$e_1(1-R)\overrightarrow{\psi(t)} = 0 \text{ and} \tag{6}$$

$$(1-e_1)(t1-R)\overrightarrow{\psi(t)} = 0. \tag{7}$$

Proof. If $\overrightarrow{\psi(t)}$ satisfies ($\star$), we have

$$(t\,\mathrm{Id} + (t-1)e_1 - R)\overrightarrow{\psi(t)} = 0. \tag{8}$$

By applying e_1 and $(1-e_1)$, respectively, we obtain

$$e_1(t\,\mathrm{Id} - R - (t-1)e_1)\overrightarrow{\psi(t)} = 0 \tag{9}$$

$$(1-e_1)(t\,\mathrm{Id} - R - (t-1)e_1)\overrightarrow{\psi(t)} = 0. \tag{10}$$

Conversely, adding these two equations yields the scattering equation (8).
Since by the elementary Temperley-Lieb properties, $e_1^2 = e_1$, the equation (9) is equivalent to

$$0 = (e_1 t - e_1 R - (t-1)e_1)\overrightarrow{\psi(t)} = e_1(1-R)\overrightarrow{\psi(t)}$$

and (10) to

$$\begin{aligned} 0 &= (t\,\mathrm{Id} - R - (t-1)e_1 - te_1 + e_1 R + (t-1)e_1)\overrightarrow{\psi(t)} \\ &= (t\,\mathrm{Id} - R - te_1 + e_1 R)\overrightarrow{\psi(t)} = (1-e_1)(t1-R)\overrightarrow{\psi(t)}, \end{aligned}$$

completing the proof of the proposition. □

Theorem 3.22. *The vector $\overrightarrow{\psi(t)}$ satisfies the scattering equation ($\star$) at $i=1$ if and only if the following two conditions are satisfied:*

(1) $e_1(1-R)\overrightarrow{\psi(t)} = 0$

(2) For every $\pi_j \in \mathcal{LP}(N)$ with $1 \not\prec 2$ we have

$$t\psi_i(t) = \psi_{R^{-1}(i)}(t).$$

Here, $\psi_i(t)$ denotes the i-th component of $\overrightarrow{\psi(t)}$ and $\psi_{R^{-1}(i)}(t)$ the j-th component of $\overrightarrow{\psi(t)}$, where j is defined by

$$\psi_j = R^{-1}(\psi_i).$$

Proof. We rewrite the second equation (7) of the previous proposition:

$$\begin{aligned}
0 = (1-e_1)(t1-R)\overrightarrow{\psi(t)} =& (1-e_1)\left(\sum_{i=1}^{|\mathcal{LP}(N)|} t\psi_i(t)\vec{\pi_i} - \sum_{i=1}^{|\mathcal{LP}(N)|} \psi_i(t)R\vec{\pi_i}\right) \\
=& (1-e_1)\left(\sum_{i=1}^{|\mathcal{LP}(N)|} t\psi_i(t)\vec{\pi_i} - \sum_{i=1}^{|\mathcal{LP}(N)|} \psi_{R^{-1}(i)}(t)\vec{\pi_i}\right) \\
=& \sum_{i=1}^{|\mathcal{LP}(N)|} (t\psi_i(t) - \psi_{R^{-1}(i)}(t))(1-e_1)\vec{\pi_i} \\
=& \sum_{\substack{i\in\{1,\ldots,|\mathcal{LP}(N)|\},\\ 1\frown 2}} (t\psi_i(t) - \psi_{R^{-1}(i)}(t))(1-e_1)\vec{\pi_i} \\
&+ \sum_{\substack{i\in\{1,\ldots,|\mathcal{LP}(N)|\},\\ 1\not\frown 2}} (t\psi_i(t) - \psi_{R^{-1}(i)}(t))(1-e_1)\vec{\pi_i}
\end{aligned}$$

If $1 \frown 2$ in π_i, then $\pi_i = e_1\pi_i \Rightarrow (1-e_1)\vec{\pi_i} = 0$.
On the other hand, if $1 \not\frown 2$ in π_i, then $\pi_i \neq e_1\pi_i$; so satisfying (7) is equivalent to

$$t\psi_i(t) - \psi_{R^{-1}(i)}(t) = 0 \text{ for all } i \in \{1, \ldots, |\mathcal{LP}(N)|\} \text{ with } 1 \not\frown 2.$$

□

4 Generalized FPLs and the refined theorem

In this chapter, we will define generalizations of fully packed loop configurations and the corresponding version of the Razumov-Stroganov-Cantini-Sportiello theorem.

4.1 Generalized FPLs and corresponding link patterns

We first need to define the domains we will work with. For the ordinary FPLs in Chapter 2, our domains were quadratic squares of size n; here we weaken this conditions to rectangular domains where a square of vertices is removed on some corners. Figure 20 shows the procedure of creating a suitable domain.

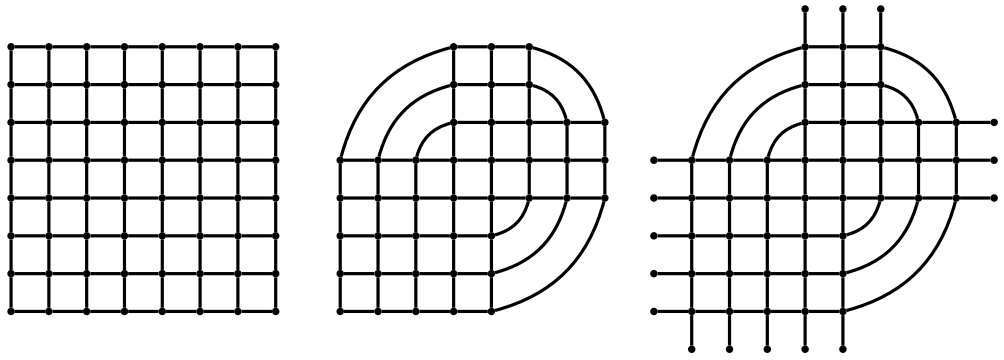
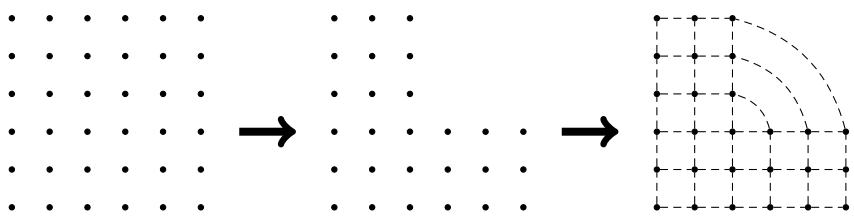


Figure 20: Producing a suitable domain in three steps: (1) Taking a rectangular grid, (2) removing squares on some corners (and connecting the remaining vertices), (3) adding external vertices.

More formally, a domain is defined as follows:

Definition 4.1. In the following, a *domain* Λ is a graph (V, E) consisting of vertices created by the following procedure:
Take a rectangular $L_x \times L_y$ grid. Now, calling the corners A_1, A_2, A_3, A_4, remove an $a_i \times a_i$ square of vertices from the corner A_i. Here, the parameters a_1, a_2, a_3, a_4 must be chosen such that the squares removed do not overlap (but they might share a perimeter).

Therefore, $2a_i$ external vertices are cut out; the remaining vertices are connected pairwise "from the outside inwards" as in the following picture:



A such domain $\Lambda = \Lambda(L_x, L_y, a_1, a_2, a_3, a_4)$ therefore is given by six integer parameters $L_x, L_y, a_1, a_2, a_3, a_4$.

The domain Λ has $L_x - a_1 - a_2$ vertices of the form $(1, y)$, $L_y - a_2 - a_3$ vertices of the form $(x, 1)$, $L_x - a_3 - a_4$ vertices of the form (L_x, y) and $L_y - a_4 - a_1$ vertices of the form (x, L_y).
For every such vertex v, we add another vertex which is only adjacent to v. For $v \in \{A_1, A_2, A_3, A_4\}$, if present, we add two vertices, each only adjacent to the respective A_i.
We will refer to these new vertices as the *external vertices of* Λ. So in total, we have $2n$ external vertices, where n is defined by

$$n = L_x + L_y - a_1 - a_2 - a_3 - a_4.$$

We enumerate the external vertices form 1 up to $2n$, counter-clockwise, starting from the bottom-left. (Note that we have labeled *all* external vertices unlike in Chapter 2, where we did so only for the black vertices.)

This procedure gives us a suitable domain where we can define the generalised fully packed loop configurations.

Definition 4.2. Let $\Lambda = \Lambda(L_x, L_y, a_1, a_2, a_3, a_4)$ be a domain as in Definition 4.1.
A *fully packed loop configuration (FPL)* ϕ *on* Λ is a graph with the same vertex and edge set as Λ equipped with an edge coloring function $c\colon E(\phi) \to \{b, w\}$ such that

- each vertex of degree 4 has two black and two white adjacent edges (that is, edges e with $c(e) = b$ and $c(e) = w$, respectively)
- each vertex of degree 3 has one black and two white adjacent edges or vice versa
- each vertex of degree 2 has one black and one white adjacent edge (these degree conditions guarantee the black/white subgraph of ϕ to decompose into paths and circles)
- the *boundary conditions are alternating*: if $c(i) = b$, then $c(i \pm 1) = w$ for all external vertices i. (Note that in contrary to Definition 2.1, we do not require 1 to be black.) Again, any calculations are to be understood modulo $2n$.

The set of all FPLs on the domain Λ will be called $\mathcal{FPL}(\Lambda)$. We refer to the vertices A_1, A_2, A_3, A_4, if present in ϕ, as the *corners* of ϕ and define c_Λ to be the number of corners in ϕ (which, of course, only depends on Λ).

We will, in the following, assume $c_\Lambda \neq 0$ and furthermore assume without loss of generality $a_1 = 0$, i. e., the corner A_1 is present in Λ. We refer to A_1 as the *reference corner* of Λ and call $L = L(\Lambda)$ the *length of the reference side*, that is, the number of vertices form A_1 up to the next corner (which may be again A_1, if $c_\Lambda = 1$). The vertices $1, \ldots, L$ form the *reference side of* Λ.

We have to explain how these new FPLs relate to the sets $\mathcal{LP}(2n)$, $\mathcal{LP}^*(2n-1)$ and $\mathcal{LP}^*(2n)$. For this purpose, we define two important subsets of $\mathcal{FPL}(\Lambda)$:

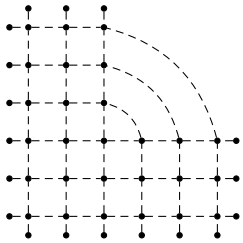
If $L_x - 1 = L_y = a_2 + a_4$ or $L_y = L_x - 1 = a_2 + a_4$, the domain Λ has an internal edge for which both adjacent faces have at most 3 sides.
We can split this edge into two and produce thereby a vertex of degree 2 which will be the puncture in a link pattern in $\mathcal{LP}^*(2n-1)$.

If $L_x = L_y = a_2 + a_4$ there is in Λ at most one face with 1 or 2 sides.
One may put a puncture in this special face; the so-punctured domain will allow for the $\mathcal{LP}^*(2n)$ correspondence.

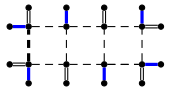
Figures 21 and 22 give examples of generalized FPLs.

Therefore, some domains allow for two correspondences (namely $\mathcal{LP}(2n)$ and $\mathcal{LP}^*(2n-1)$ or $\mathcal{LP}(2n)$ and $\mathcal{LP}^*(2n)$). We thus make an implicit choice by choosing Λ - if the domain shall be punctured or not. It is then immediate how to assign an link pattern π to an FPL - one only has to decide whether the black or white paths shall be present in π and which path shall be numbered 1 in π.
We shall write $\mathcal{LP}(\Lambda)$ for the set of link pattern corresponding to the FPLs on Λ.

Remark. FPLs do not exist on every domain constructed in this way!
On some domains FPLs do not exist for a more obvious reason, namely because $L_x + L_y - a_1 - a_2 - a_3 - a_4$ is odd and there is no puncture in $\Lambda = \Lambda(L_x, L_y, a_1, a_2, a_3, a_4)$. Then there exists no possible matching of the black vertices - simply because there is an odd number of black external vertices.
An example would be the domain $\Lambda = \Lambda(6, 6, 0, 0, 3, 0)$:



The condition that $L_x + L_y - a_1 - a_2 - a_3 - a_4$ is even is however not sufficient, consider e. g. $\Lambda = \Lambda(4, 2, 0, 0, 0, 0)$ and suppose there was an FPL in $\mathcal{FPL}_b(\Lambda)$. Then one would have to color


,

this would however not work properly. (One could start e. g. coloring the marked edge on the left either black or white and then recall the degree constraints – each internal vertex has two black and two white adjacent edges.)

In the following, whenever we write Λ, we mean a domain as defined in Definition 4.1.

Definition 4.3. We define an operator $\Pi(\phi, v)$, where ϕ is an FPL and v a black vertex of ϕ: $\Pi(\phi, v)$ is the link pattern π associated to the black subgraph of the FPL ϕ, where the vertex v corresponds to 1 in π, $v + 2$ to 2, $v + 4$ to 3, and so on.

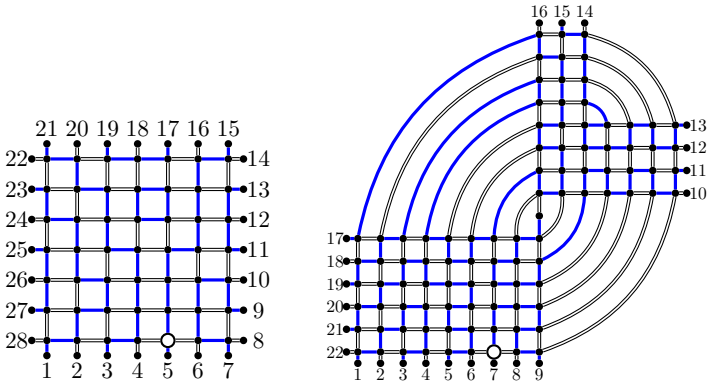


Figure 21: Left: An FPL ϕ on $\Lambda = \Lambda(7,7,0,0,0,0)$ with four corners (i. e. $c_\Lambda = 0$). We further have $h(\phi) = 5, L = 7$ and $n = 14$. Right: An FPL ϕ on $\Lambda = \Lambda(15,14,0,6,4,8)$ with only one corner A_1 (i. e. $c_\Lambda = 1$). The further characteristics are $h(\phi) = 7, L = 22, 2n-1 = 11$. Because of $a_2 + a_4 = L_y = L_x - 1$, Λ allows for the $\mathcal{LP}^*(2n-1)$ correspondence; therefore a vertex is added in the middle (and thereby the edge in the middle is split up into two).

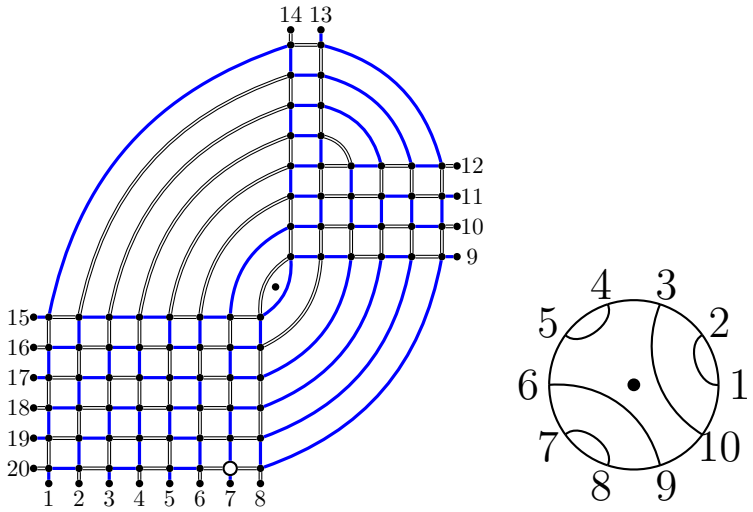


Figure 22: Left: An FPL ϕ on $\Lambda = \Lambda(14,14,0,6,4,8)$ with one corner (i. e. $c_\Lambda = 1$) allowing for the $\mathcal{LP}^*(2n)$ correspondence (since $a_2 + a_4 = L_x = L_y$). We further have $h(\phi) = 7, L = 20$ and $n = 10$. Right: The link pattern $\pi = \Pi(\phi, 1)$. (The link pattern $R^i\pi$ is, as always, given as $\Pi(\phi, v - 2i)$.)

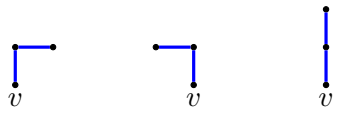


Figure 23: A black vertex v of type a, b, c, respectively.

See Figure 22 for an example. $\mathcal{FPL}(\Lambda)$ clearly splits into two domains:

Definition 4.4. $\mathcal{FPL}_+(\Lambda)$ denotes the subset of $\mathcal{FPL}(\Lambda)$ where 1 is colored black; analogously $\mathcal{FPL}_-(\Lambda)$ denotes the subset of $\mathcal{FPL}(\Lambda)$ where 1 is colored white.

Remark. Trivially, $|\mathcal{FPL}_+(\Lambda)| = |\mathcal{FPL}_-(\Lambda)| = \frac{|\mathcal{FPL}(\Lambda)|}{2}$.

A special case of Π is the following one:

Definition 4.5. $\Pi_+(\phi) := \Pi(\phi, 1)$ for $\phi \in \mathcal{FPL}_+(\Lambda)$.
That is, Π_+ assigns to $\phi \in \mathcal{FPL}_+(\Lambda)$ the corresponding link pattern, where the vertex 1 in the link pattern corresponds to the black vertex 1 in ϕ.

We can finally state the refined Razumov-Stroganov-Cantini-Sportiello theorem:

Theorem 4.6 (Refined Razumov-Stroganov-Cantini-Sportiello theorem)**.** *The vector*

$$\overrightarrow{\mu_N} := \sum_{\phi \in \mathcal{FPL}_+(\Lambda)} \overrightarrow{\Pi_+(\phi)}$$

satisfies (†), i. e. $H_0 \overrightarrow{\mu_N} = 0$.

Remark. If $\Lambda = \Lambda(n, n, 0, 0, 0, 0)$ for some n, $\overrightarrow{\mu_N}$ equals $\overrightarrow{\mu_{2n}}$ defined in Chapter 2.

Another important definition will be the one of a refinement position.

Consider a black external vertex v of an FPL ϕ and the corresponding black path. The edge adjacent to v will go towards the center of ϕ; but then there are three possibilities of what the next edge of the corresponding black path is (cf. Figure 23):

- the path goes to the right; we then call v a "type a" vertex
- the path goes to the left; we then call v a "type b" vertex
- the path goes up; we then call v a "type c" vertex.

Lemma 4.7. *Consider a side of an FPL starting with a black vertex. Then*

- *the first vertex (on this side) is of type a or c*
- *vertices of type c are followed by vertices of type b*

- *the last vertex is of type b or c.*

Proof. The lemma is proven if we can rule out the following five cases:

(1) The first vertex can not be of type b.

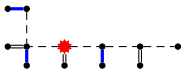
(2) The last vertex can not be of type a.

(3) A vertex of type b cannot be followed by a vertex of type c.

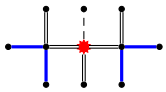
(4) A vertex of type b cannot be followed by a vertex of type a.

(5) A vertex of type c cannot be followed by a vertex of type a.

The first vertex can not be of type b because of the alternating boundary conditions:

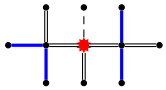


Analogously, the last vertex can not be of type a. This rules out (1) and (2).
Now consider a vertex of type b. If it was followed by a type a vertex, the situation would look like this:

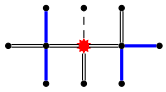


But then the marked vertex would have at most one adjacent black edge, what is impossible (since a vertex of degree 4 must have 2 adjacent black edges). This rules out (3).

Cases (4) and (5) are analysed similarly: If a vertex of type b was followed by a type c vertex, there would be the following situation:

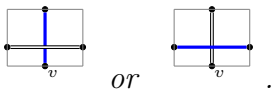


Finally consider a vertex of type c. If it was followed by a type a vertex, the situation would look like this:



Both are contradictions for the marked vertices by the degree constraints. □

Corollary 4.8. *On each side of an FPL (and in particular on the reference side) there is exactly one boundary vertex v (black or white) which has the form*



.

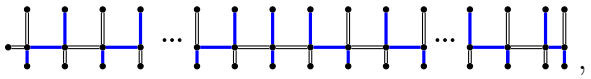
Proof. By the last lemma, the sequence of black vertices is either

$$(a, \ldots, a, c, b \ldots, b)$$

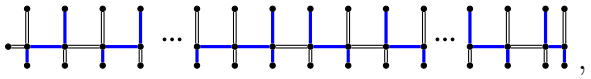
or

$$(a, \ldots, a, b \ldots, b)$$

(with a possibly empty sequence of a's and b's). The first case corresponds to

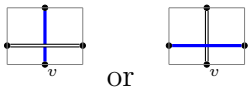


(that is, the row contains exactly one  and not ); the second case to



(that is, the row contains exactly one  and not ). □

Definition 4.9. Let $\phi \in \mathcal{FPL}(\Lambda)$. We have seen that there is a unique tile of the form



on the reference side of ϕ; we refer to the vertex v as the *refinement position of* ϕ and say, ϕ has a black or white refinement position, respectively. We shall write $h(\phi)$ for this vertex.

Definition 4.10. $\mathcal{FPL}_b(\Lambda)$ is the subset of $\mathcal{FPL}(\Lambda)$ consisting of all FPLs with black refinement position; analogously is $\mathcal{FPL}_w(\Lambda)$ the subset of $\mathcal{FPL}(\Lambda)$ consisting of all FPLs with white refinement position.

Remark. There is no a priori relation between $\mathcal{FPL}_b(\Lambda)$ and $\mathcal{FPL}_+(\Lambda)$ (except for $|\mathcal{FPL}_b(\Lambda)| = |\mathcal{FPL}_+(\Lambda)|$).

When we assign a link pattern to an FPL, it will be often convenient to start counting from the refinement position:

Definition 4.11. $\Pi_b(\phi) := \Pi(\phi, h(\phi))$.

4.2 Wieland half-gyration

We need to define an important map on $\mathcal{FPL}(\Lambda)$, the so-called *Wieland half-gyration*. Introduced by Wieland 2000 in [12], it is the core of many works concerning FPLs.

Let ϕ be an FPL, than color the faces of the underlying domain in a checkerboard manner. This can be done in two ways; therefore we make the following agreement:

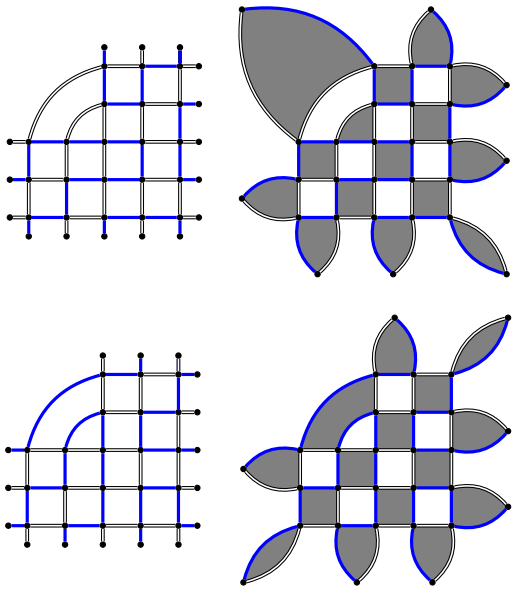


Figure 24: First row: An FPL in $\mathcal{FPL}_+(\Lambda)$ and the associated pairing. Second row: An FPL in $\mathcal{FPL}_-(\Lambda)$ and the associated pairing. The domain Λ is given by $\Lambda(5,5,0,0,0,2)$.

If $\phi \in \mathcal{FPL}_+(\Lambda)$, the bottom-left face is colored white; if $\phi \in \mathcal{FPL}_-(\Lambda)$, the bottom-left face is colored black.

Now extend this coloring to the external vertices in the following way:

If $\phi \in \mathcal{FPL}_+(\Lambda)$, then we "pair" the vertices $2i$ and $2i+1$ for every $i \in \{1,\dots,n\}$ (that is, we identify the vertices $2i$ and $2i+1$); if $\phi \in \mathcal{FPL}_-(\Lambda)$, we "pair" the vertices $2i-1$ and $2i$ for every $i \in \{1,\dots,n\}$ and identify the resulting FPL with ϕ.

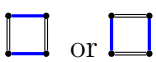
By identifying two vertices, another face is produced; we color all new produced faces black.

We refer to the set of black faces as Γ_+, if $\phi \in \mathcal{FPL}_-(\Lambda)$ and Γ_+, if $\phi \in \mathcal{FPL}_-(\Lambda)$.

Definition 4.12. To define Wieland half-gyration, we first need to define two maps

$$H_+\colon \mathcal{FPL}_+(\Lambda) \to \mathcal{FPL}_-(\Lambda), H_-\colon \mathcal{FPL}_-(\Lambda) \to \mathcal{FPL}_+(\Lambda).$$

The maps H_+ and H_- are defined similarly, but H_+ acts on each black face $\gamma \in \Gamma_+$, whereas H_- acts on each black face $\gamma \in \Gamma_-$ in the following way:
If the face γ is of size 4 and its edges colored alternatively, that is, has the form



or has size 2 and is punctured and colored alternatively, it stays unchanged, in every other case its colors are inverted.
Figure 26 lists all possible configurations of a circle γ up to rotation and switching black

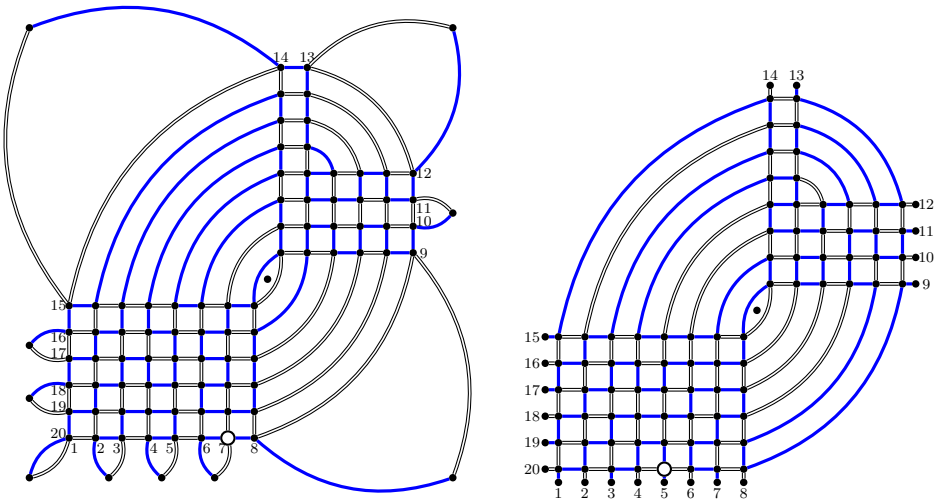


Figure 25: An FPL ϕ (left) and $H(\phi)$ (right).

and white and $H_{\pm}(\gamma)$ directly under γ.
Since these circles are disjoint, we can let $H_{\pm}$ act on all circles simultaneously; $H(\phi)$ shall be the thereby resulting FPL.
We now define the Wieland half-gyration H by

$$H(\phi) := \begin{cases} H_+(\phi) & \text{if } \phi \in \mathcal{FPL}_+(\Lambda) \\ H_-(\phi) & \text{if } \phi \in \mathcal{FPL}_-(\Lambda) \end{cases}$$

Remark.

(1) Each vertex has at most two black adjacent faces (i. e. faces affected by H). One may check that for each combination of these two faces, the degree constraints after the application of H are still met, i. e. that $H(\phi)$ is indeed an FPL.

(2) By the very definition of $H_{\pm}$, vertices of unpunctured faces of size 2 always change their colors.
Therefore, all external vertices of an FPL change their colors by applying H (this can be seen as a justification that H_+, H_- indeed map into $\mathcal{FPL}_-(\Lambda), \mathcal{FPL}_+(\Lambda)$, respectively).

(3) We have

$$H^k(\phi) = H_{(-1)^k} H_{(-1)^{k-1}} \dots H_- H_+ \phi$$

if $\phi \in \mathcal{FPL}_+(\Lambda)$ and

$$H^k(\phi) = H_{(-1)^{k+1}} H_{(-1)^k} \dots H_+ H_- \phi$$

if $\phi \in \mathcal{FPL}_-(\Lambda)$.

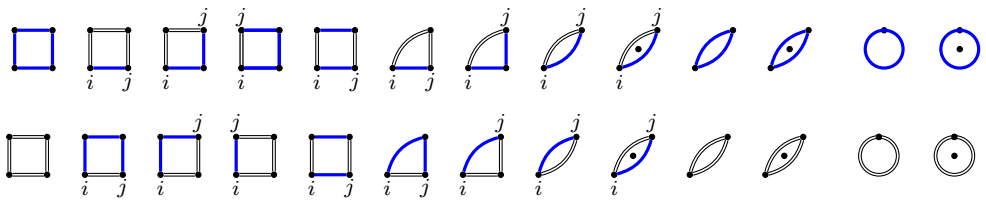


Figure 26: Top row: Circles before the application of H, bottom row: the corresponding circles after the application of H.

(4) Often, one is interested in H^2 and calls this map the *Wieland gyration*, therefore the name *half-gyration* for H.

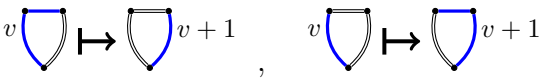
(5) After applying H, there is no need for the external vertices to stay paired (and if we want to apply H again, we need to do the "other pairing"). In illustrations, we let sometimes the vertices stay paired and sometimes not - there is no difference whatsoever.

Lemma 4.13 (Wieland half-gyration lemma)**.** *Let $\phi \in \mathcal{FPL}(\Lambda)$ and v be an external black vertex of ϕ. Then*

$$\Pi(H(\phi), v+1) = \Pi(\phi, v).$$

Proof. We show that two external black vertices v and w are connected by a black path if and only if $v+1$ and $w+1$ are connected by a black path in $H(\phi)$.
We thus have to show that for each circle affected by H the following holds:
If a black path enters the circle at the vertex i and leaves it at the vertex j, in $H(\phi)$ also a black path enters the circle at i and leaves it at j.
But this is clear by inspecting Figure 26.

So we can identify the black paths in ϕ and $H(\phi)$. It remains to show that a black path connecting v and w in ϕ connects $v+1$ and $w+1$ in $H(\phi)$:
Since v is paired to $v+1$ and



under the application of H (that is, a path ending in ϕ in the vertex v ends in $H(\phi)$ in the vertex $v+1$), the lemma follows. (By inspecting the "punctured" cases in Figure 26, one gets the lemma for $\mathcal{LP}^*(2n)$.) □

Corollary 4.14. *Let $\phi \in \mathcal{FPL}(\Lambda)$ and v be an external black vertex of ϕ. Then*

$$\Pi(H^2(\phi), v) = R\Pi(\phi, v).$$

Proof. The set of black external edges of ϕ is given by $\{\ldots, v-2, v, v+2, \ldots\}$ and thus

$$\Pi(\phi, v+2) = R^{-1}\Pi(\phi, v).$$

By the previous lemma,

$$\begin{aligned}\Pi(H^2(\phi), v) &= \Pi(H(\phi), v-1) \\ &= \Pi(\phi, v-2) = R\Pi(\phi, v).\end{aligned}$$

□

5 The proof of the Razumov-Stroganov-Cantini-Sportiello theorem

This chapter is devoted to two things:
We first prove that a certain vector $\overrightarrow{\psi_\Lambda(t)}$ satisfies the two conditions stated in Theorem 3.22, i. e. $\overrightarrow{\psi_\Lambda(t)}$ is the unique (up to normalization) eigenvector for the Hamiltonian H_0. Then, we show that $\overrightarrow{\psi_\Lambda(t)}$ equals the vector $\overrightarrow{\mu_N}$ defined in Theorem 4.6.

Fix a domain Λ. In this chapter, an FPL's refinement positions plays a crucial role; therefore we will mark it in illustrations with a vertex bigger than the other vertices. Sometimes it is convenient to work in the vector space $\mathbb{C}^{\mathcal{FPL}_b(\Lambda)}$, defined by

$$\mathbb{C}^{\mathcal{FPL}_b(\Lambda)} := \mathbb{C}^{|\mathcal{FPL}_b(\Lambda)|}$$

with standard basis

$$\overrightarrow{\phi_1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \overrightarrow{\phi_m} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

where $\phi_1, \ldots, \phi_m$ is an enumeration of $\mathcal{FPL}_b(\Lambda)$.
Any map F defined on $\mathcal{FPL}_b(\Lambda)$ can be extended (as in Chapter 1) to a linear map on $\mathbb{C}^{\mathcal{FPL}_b(\Lambda)}$ by

$$F\left(\sum_{\phi \in \mathcal{FPL}_b(\Lambda)} c_\phi \overrightarrow{\phi}\right) = \sum_{\phi \in \mathcal{FPL}_b(\Lambda)} c_\phi \overrightarrow{F(\phi)}.$$

5.1 A certain vector satisfies the Razumov-Stroganov-Cantini-Sportiello theorem

The vector $\overrightarrow{\psi_\Lambda(t)}$ shall assign to each FPL the corresponding link pattern, weighted with a factor corresponding to its refinement position, more precisely:

Definition 5.1. We define the vector $\overrightarrow{\psi_\Lambda(t)}$ by

$$\overrightarrow{\psi_\Lambda(t)} := \sum_{\phi \in \mathcal{FPL}_b(\Lambda)} t^{h(\phi)-1} \overrightarrow{\Pi_b(\phi)},$$

where $h(\phi)$ denotes the refinement position of $\phi \in \mathcal{FPL}_b(\Lambda)$.

Our goal is to show Theorem 5.2, which, by Theorem 3.22, follows from Theorem 5.3 and Theorem 5.4.

Theorem 5.2. *The vector* $\overrightarrow{\psi_\Lambda(t)}$ *satisfies the scattering equation* $(\star)$ *at* $i = 1$, *i. e.*

$$X_1(1)\overrightarrow{\psi_\Lambda(1)} = R\overrightarrow{\psi_\Lambda(1)}.$$

Theorem 5.3. $e_1(1 - R)\overrightarrow{\psi_\Lambda(t)} = 0$.

Theorem 5.4. *Let* $\pi_i \in \mathcal{LP}(\Lambda)$ *with* $1 \not\prec 2$. *Then the* i*-th coordinate* $\psi_i(t)$ *of* $\overrightarrow{\psi_\Lambda(t)}$ *satisfies*

$$t\psi_i(t) = \psi_{R^{-1}(i)}(t)$$

(where $R^{-1}(i)$ *is defined as in Theorem 3.22).*

It will be convenient to use the following notions:

Definition 5.5. Let $\mathcal{FPL}_b^{[i]}(\Lambda)$ be the subset of $\mathcal{FPL}_b(\Lambda)$ consisting of all FPLs with refinement position i (and we define $\mathcal{FPL}_w^{[i]}(\Lambda)$ alike).

(1) We define a vector $\overrightarrow{\psi_\Lambda^{[i]}} \in \mathbb{C}^{\mathcal{LP}(\Lambda)}$ (which is a non-weighted restriction of $\overrightarrow{\psi_\Lambda}$ to $\mathcal{FPL}_b(\Lambda)$) by

$$\overrightarrow{\psi_\Lambda^{[i]}} := \sum_{\phi \in \mathcal{FPL}_b^{[i]}(\Lambda)} \overrightarrow{\Pi_b(\phi)}.$$

(2) We define another vector $\overrightarrow{s_\Lambda^{[i]}} \in \mathbb{C}^{\mathcal{FPL}_b(\Lambda)}$ by

$$\overrightarrow{s_\Lambda^{[i]}} := \sum_{\phi \in \mathcal{FPL}_b^{[i]}(\Lambda)} \vec{\phi}.$$

We make the following easy observations:

Proposition 5.6.

(1) $\Pi_b \overrightarrow{s_\Lambda^{[i]}} = \overrightarrow{\psi_\Lambda^{[i]}}$.

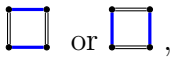
(2) $\overrightarrow{\psi_\Lambda(t)} = \sum_{i=1}^{L} t^{i-1} \overrightarrow{\psi_\Lambda^{[i]}}$.

Now we can prove Theorems 5.3 and 5.4:

Proof of Theorem 5.3. Fix Λ and a refinement position $i \in \{1, \ldots, L\}$.
We define two operators $\tilde{e_1}, \tilde{e_N}$ on $\mathcal{FPL}_+(\Lambda)$ and $\mathcal{FPL}_-(\Lambda)$.
Let $\phi \in \mathcal{FPL}_+(\Lambda)$, then $\tilde{e_1}(\phi)$ (or $\tilde{e_N}(\phi)$) is the FPL resulting from the following procedure:

- All faces but the face on the top-right (or top-left) of the refinement position stay unchanged.

- All colors of the face on the top-right (or top-left) of the refinement position are inverted, if the face is of the form



otherwise it also stays unchanged.

On $\mathcal{FPL}_-(\Lambda)$ are the operators $\tilde{e_1}$ and $\tilde{e_N}$ defined alike, but the terms "top-left" and "top-right" are interchanged.

The choice of the names $\tilde{e_1}$ and $\tilde{e_N}$ is not coincidentally; indeed we have for $\tilde{e_1}$:

and

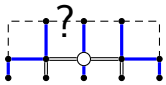
and analogously for $\tilde{e_N}$:

and

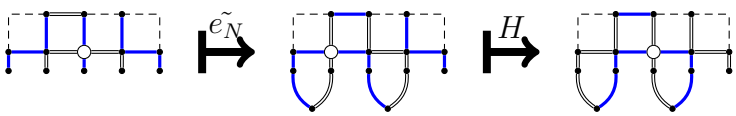
that is, we have

$$\Pi_b(\tilde{e_1}(\phi)) = e_1\Pi_b(\phi), \Pi_b(\tilde{e_N}(\phi)) = e_N\Pi_b(\phi).$$

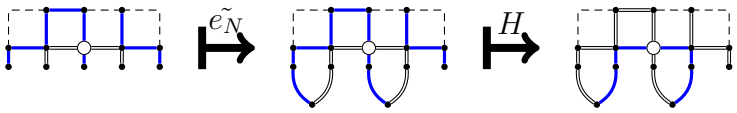
Claim: The functions $H\tilde{e_N}$ and $H^{-1}\tilde{e_1}$ are bijections $\mathcal{FPL}_{\mathrm{b}}^{[\mathrm{i}]}(\Lambda) \to \mathcal{FPL}_{\mathrm{w}}^{[\mathrm{i}]}(\Lambda)$.
Since the functions $H, \tilde{e_1}, \tilde{e_N}$ are invertible, they are indeed bijections; it thus only remains to show that their image is $\mathcal{FPL}_{\mathrm{w}}^{[\mathrm{i}]}(\Lambda)$.
Let $\phi \in \mathcal{FPL}_{\mathrm{b}}^{[\mathrm{i}]}(\Lambda)$, then ϕ looks near its refinement position like this:



We distinguish two cases depending on the color of the edge marked with "?".
Case 1: If this edge is white, applying $H\tilde{e_N}$ results in the following FPL (note that the refinement position changes from i to $i-1$ and back to i):



Case 2: If the edge is black, then applying $H\tilde{e_N}$ results in the following FPL:



In both cases, $H\tilde{e_N}(\phi) \in \mathcal{FPL}_w^{[i]}(\Lambda)$.
From that and the invertibility of H and $\tilde{e_N}$ we obtain the claim concerning $H\tilde{e_N}$.
An analogous proof holds for $H^{-1}\tilde{e_1}$; therefore the two maps are indeed bijections $\mathcal{FPL}_{\mathrm{b}}^{[\mathrm{i}]}(\Lambda) \to \mathcal{FPL}_{\mathrm{w}}^{[\mathrm{i}]}(\Lambda)$.

So it follows

$$H\tilde{e_N}\overrightarrow{s_\Lambda^{[i]}} = \sum_{\phi \in \mathcal{FPL}_w^{[i]}(\Lambda)} \vec{\phi} \text{ and } H^{-1}\tilde{e_1}\overrightarrow{s_\Lambda^{[i]}} = \sum_{\phi \in \mathcal{FPL}_w^{[i]}(\Lambda)} \vec{\phi}$$

and therefore

$$H^2\tilde{e_N}\overrightarrow{s_\Lambda^{[i]}} = \tilde{e_1}\overrightarrow{s_\Lambda^{[i]}}.$$

By recalling Wieland's half-gyration lemma 4.13, we obtain

$$\begin{aligned} e_1\overrightarrow{\psi_\Lambda^{[i]}} &= e_1\Pi_b(\overrightarrow{s_\Lambda^{[i]}}) = \Pi_b(\tilde{e_1}\overrightarrow{s_\Lambda^{[i]}}) \\ &= \Pi_b(H^2\tilde{e_N}\overrightarrow{s_\Lambda^{[i]}}) = R\Pi_b(\tilde{e_N}\overrightarrow{s_\Lambda^{[i]}}) \\ &= Re_N\Pi_b(\overrightarrow{s_\Lambda^{[i]}}) = Re_N\overrightarrow{\psi_\Lambda^{[i]}}. \end{aligned}$$

This implies

$$e_1(1-R)\overrightarrow{\psi_\Lambda^{[i]}} = (e_1 - e_1R)\overrightarrow{\psi_\Lambda^{[i]}} = (e_1 - Re_N)\overrightarrow{\psi_\Lambda^{[i]}} = 0$$

and thus, by Proposition 5.6,

$$e_1(1-R)\overrightarrow{\psi_\Lambda(t)} = 0,$$

as required. □

Proof of Theorem 5.4. Choose $i \in \{1, \ldots, \mathcal{LP}(N)\}$ such that $1 \not\prec 2$ in π_i; let $\psi_i(t)$ be the i-th coordinate of $\overrightarrow{\psi_\Lambda(t)}$.
We have by definition of $\overrightarrow{\psi_\Lambda(t)}$

$$t\psi_i(t) = t(a_0t^0 + a_1t^1 + \ldots + a_{L-1}t^{L-1}) = a_0t^1 + a_1t^2 + \ldots + a_{L-1}t^{L-1},$$

where

$$a_j = \#(\phi \in \mathcal{FPL}_{\mathrm{b}}(\Lambda) \colon \Pi_b(\phi) = \pi_i, h(\phi) = j+1).$$

Since we want to show

$$t\psi_i(t) = \psi_{R^{-1}(i)}(t),$$

we have to show that this expression equals

$$\psi_{R^{-1}(i)}(t) = b_0 t^0 + \ldots + b_{L-1} t^{L-1},$$

where

$$b_j = \#(\phi \in \mathcal{FPL}_{\mathrm{b}}(\Lambda) \colon \Pi_b(\phi) = R^{-1}(\pi_i), h(\phi) = j+1).$$

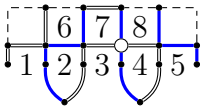
So we have to show

$$\forall j \in \{0, \ldots, L-1\} \colon a_j = b_{j+1},$$

i. e.

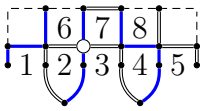
$$\begin{aligned} &\#(\phi \in \mathcal{FPL}_{\mathrm{b}}(\Lambda) \colon \Pi_b(\phi) = \pi_i, h(\phi) = j+1) \\ =&\#(\phi \in \mathcal{FPL}_{\mathrm{b}}(\Lambda) \colon \Pi_b(\phi) = R^{-1}(\pi_i), h(\phi) = j+2). \end{aligned}$$

Let $\phi \in \mathcal{FPL}_{\mathrm{b}}(\Lambda)$ such that $\Pi_b(\phi) = R^{-1}(\pi)$ and $h(\phi) = j+2$. Then $N \not\prec 1$ in $\phi' := H(\phi)$ (because of $\Pi_b(\phi) = R^{-1}(\pi_i)$ and $1 \not\prec 2$ in π_i).
We claim $\Pi_b(\phi') = R(\pi_i)$ and $h(\pi') = j+1$ and start to prove the latter.
The area near ϕ's refinement position looks schematically like (here, we label the faces in ϕ to illustrate the situation better)



.

Note that $N \not\prec 1$ in π_i; so the edge on top of face 7 must be white.
Now we apply H.
Regardless of whether $\phi \in \mathcal{FPL}_+(\Lambda)$ or $\phi \in \mathcal{FPL}_-(\Lambda)$, of the faces in the picture only faces $2, 4$ and 7 are affected. So this part of ϕ becomes



,

that is, $h(H(\phi)) = j+1$.
The former is a calculation involving Lemma 4.14 and Corollary 4.14:

$$\begin{aligned} \Pi_b(\phi') &= \Pi(H(\phi), j+1) \\ &= \Pi(H^2(\phi), j+2) \\ &= R(\phi, j+2) = R\Pi_b(\phi) = RR^{-1}\pi_i = \pi_i. \end{aligned}$$

This proves our claim.

Now let $\phi \in \mathcal{FPL}_{\mathrm{b}}(\Lambda)$ with $\Pi_b(\phi) = \pi_i$ and $h(\phi) = j + 1$.
Then setting $\phi' := H(\phi)$, $\Pi_b(\phi') = R^{-1}(\pi_i)$ and $h(\phi') = j + 2$; the proof of these facts is analogous to the proof above.
Thus H is a bijection between

$$\{\phi \in \mathcal{FPL}_{\mathrm{b}}(\Lambda), \Pi_b(\phi) = \pi_i, h(\phi) = j + 1\}$$

and

$$\{\phi \in \mathcal{FPL}_{\mathrm{b}}(\Lambda), \Pi_b(\phi) = R^{-1}(\pi_i), h(\phi) = j + 2\}$$

so the theorem follows. □

We have proven that $H_0\overrightarrow{\psi_\Lambda(1)} = 0$; the Razumov-Stroganov-Cantini-Sportiello theorem however states $H_0\overrightarrow{\mu_N} = 0$ (with $\overrightarrow{\mu_N}$ defined as in Chapter 4). Hence it remains to prove $\overrightarrow{\psi_\Lambda(1)} = \overrightarrow{\mu_N}$.

5.2 The remaining part

We start by making observations about the refinement position of an FPL.

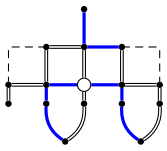
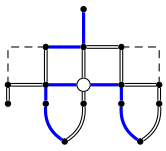
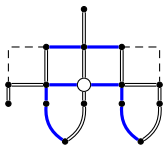
Lemma 5.7. *Let $\phi \in \mathcal{FPL}_{+}(\Lambda)$ and $\phi' := H(\phi)$.*

(1) If $h(\phi)$ is even (=white), then $h(\phi') \in \{h(\phi), h(\phi) + 1\}$.

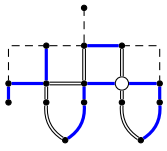
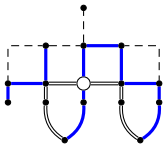
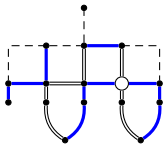
(2) If $h(\phi)$ is odd (=black), then $h(\phi') \in \{h(\phi), h(\phi) - 1\}$.

Proof.

(1) There are three possible cases on how the vertex directly above the refinement position looks like:

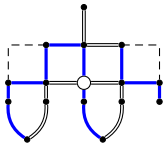
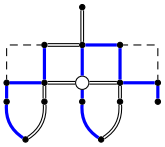
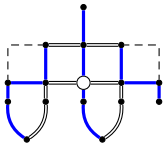
(a)  (b)  (c) 

After applying H, the cases correspond to (recall that since $\phi \in \mathcal{FPL}_{+}(\Lambda)$ and ϕ's refinement position is even, the face on the top-right of the refinement position is affected by H)

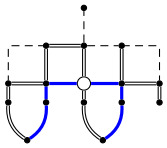
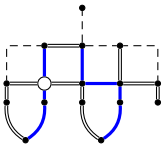
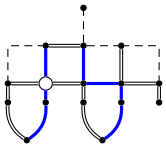
(a)  (b)  (c) 

Note that the edge $h(\phi)+1, h(\phi), H(\phi)+1$ is the new (white, black or white) refinement position, respectively.

(2) In the same vein, for ϕ with $h(\phi)$ odd, we have the three possible cases

(a)  (b)  (c)  .

By recalling that the face on the top-left of the refinement position is affected by H (since $\phi \in \mathcal{FPL}_+(\Lambda)$ and the refinement position is odd), we get by applying H

(a)  (b)  (c)  .

Note that the edge $h(\phi), h(\phi)-1$ or $H(\phi)-1$ is the new (white, black or black) refinement position, respectively.

□

Dual to the previous lemma is the next one:

Lemma 5.8. *Let $\phi \in \mathcal{FPL}_-(\Lambda)$ and $\phi' := H(\phi)$.*

(1) If $h(\phi)$ is odd (=white), then $h(\phi') \in \{h(\phi), h(\phi)+1\}$.

(2) If $h(\phi)$ is even (=black), then $h(\phi') \in \{h(\phi), h(\phi)-1\}$.

Lemma 5.9. *Let $h_t(\phi) := h(H^t(\phi))$ and define $g(t;\phi) := h_t(\phi) - t = h(H^t(\phi)) - t$. Fix $\phi \in \mathcal{FPL}_+(\phi)$; then $g(t;\phi)$ is non-decreasing in t and each odd value has exactly one preimage.*

Proof. By Lemma 5.7, if $g(t)$ is even, then $h_{t+1}(\phi) \in \{h_t(\phi), h_t(\phi)+1\}$, so $g(t+1) \in \{h_t(\phi)-t-1, h_t(\phi)-t\} = \{g(t)-1, g(t)\}$; if $g(t)$ is odd, then $h_{t+1}(\phi) \in \{h_t(\phi), h_t(\phi)-1\}$, so $g(t+1) \in \{h_t(\phi)-t-1, h_t(\phi)-t-2\} = \{g(t)-1, g(t)-2\}$.
Thus, to prove the second fact, it remains to show that g can not be ultimately constant; but this follows from t increasing to infinity and $h_t(\phi)$ being bounded by $L(\phi)$. □

Hence, if ϕ is an an FPL, we can define $t^*(\phi)$ to be the preimage of 1 under g.

Proposition 5.10. *Define a function Θ on $\mathcal{FPL}_+(\Lambda)$ and a function Θ^{-1} on $\mathcal{FPL}_b(\Lambda)$ by*

$$\Theta(\phi) := H^{t^*(\phi)}\phi; \Theta^{-1}(\phi) := H^{-h(\phi)+1}\phi.$$

Then

(a) $\mathrm{im}(\Theta) = \mathcal{FPL}_b(\Lambda)$ *and* $\mathrm{im}(\Theta^{-1}) = \mathcal{FPL}_+(\Lambda)$.

(b) $\Theta(\Theta^{-1}(\phi)) = \phi$ *and* $\Theta^{-1}(\Theta(\phi)) = \phi$.

Thus, Θ *is a bijection between* $\mathcal{FPL}_+(\Lambda)$ *and* $\mathcal{FPL}_b(\Lambda)$.

(c) For $\phi \in \mathcal{FPL}_+(\Lambda)$*, we have*

$$\Pi_+(\phi) = \Pi_b(\Theta(\phi)).$$

Proof.

(a) Let $\phi \in \mathcal{FPL}_+(\Lambda)$. Recall that since $\phi \in \mathcal{FPL}_+(\Lambda), H^{2k+1}(\phi) \in \mathcal{FPL}_-(\Lambda)$ and $H^{2k}(\phi) \in \mathcal{FPL}_+(\Lambda)$. We distinguish two cases:

First case: $h(H^{t^*(\phi)}\phi)$ is even. Then, because of $h(H^{t^*(\phi)}\phi) - t^*(\phi) = 1, t^*(\phi)$ is odd. So, $H^{t^*(\phi)}(\phi) \in \mathcal{FPL}_-(\Lambda)$ and has an even refinement position, so $h(H^{t^*(\phi)}\phi)$ is black.

Second case: $h(H^{t^*(\phi)}\phi)$ is odd. Then, $t^*(\phi)$ is even. So, $H^{t^*(\phi)}(\phi) \in \mathcal{FPL}_+(\Lambda)$ and has an odd refinement position, so $h(H^{t^*(\phi)}\phi)$ is black.

Similarly, one checks that $\Theta^{-1}(\phi) \in \mathcal{FPL}_+(\Lambda)$ for $\phi \in \mathcal{FPL}_b(\Lambda)$:

First case: $h(\phi)$ is even. Then $\phi \in \mathcal{FPL}_-(\Lambda)$ and $-h(\phi)+1$ is odd, so $H^{-h(\phi)+1}\phi \in \mathcal{FPL}_+(\Lambda)$.

Second case: $h(\phi)$ is odd. Then $\phi \in \mathcal{FPL}_+(\Lambda)$ and $-h(\phi)+1$ is even, so $H^{-h(\phi)+1}\phi \in \mathcal{FPL}_+(\Lambda)$.

(b) We have to show $(\Theta(\Theta^{-1}(\phi))) = \phi$ and $(\Theta^{-1}(\Theta(\phi))) = \phi$.
Concerning the former, $\Theta(\Theta^{-1}(\phi)) = \phi$ is equivalent to

$$H^{t^*(H^{-h(\phi)+1}\phi)}H^{-h(\phi)+1}\phi = \Theta(H^{-h(\phi)+1}\phi) = \phi;$$

this, in turn, holds if and only if

$$t^*(H^{-h(\phi)+1}\phi) - h(\phi) + 1 = 0,$$

that is, if and only if

$$t^*(H^{-h(\phi)+1}\phi) = h(\phi) - 1.$$

Let $\psi := H^{-h(\phi)+1}\phi$; thus we want to show $g(h(\phi) - 1; \psi) = 1$.

$$\begin{aligned} g(h(\phi) - 1; \psi) &= h(H^{h(\phi)-1}\psi) - h(\phi) + 1 \\ &= h(H^{h(\phi)-1}H^{-h(\phi)+1}\phi) - h(\phi) + 1 \\ &= h(\phi) - h(\phi) + 1 = 1. \end{aligned}$$

Concerning the latter, we want to show $\Theta^{-1}(\Theta(\phi)) = \phi$. Let $m := t^*(\phi)$ (i. e. we have $h(H^m(\phi)) - m = 1$).
So

$$\begin{aligned}\Theta^{-1}(\Theta(\phi)) &= \Theta^{-1}(H^m\phi) = H^{-h(H^m(\phi))+1}H^m\phi \\ &= H^{-(h(H^m\phi))+1+m}\phi = H^{-1+1}\phi = \phi.\end{aligned}$$

(c) By Wieland gyration, we have

$$\begin{aligned}\Pi_b(\Theta(\phi)) &= \Pi_b(H^{t^*(\phi)}\phi) = \Pi(H^{t^*(\phi)}\phi, h(H^{t^*(\phi)}\phi)) \\ &= \Pi(\phi, \underbrace{h(H^{t^*(\phi)}\phi) - t^*(\phi)}_{=1 \text{ by definition of } t^*(\phi)}) = \Pi(\phi, 1) \\ &= \Pi_+(\phi).\end{aligned}$$

□

Corollary 5.11 (Razumov-Stroganov-Cantini-Sportiello theorem)**.** *Recall the definition of $\overrightarrow{\mu_N}$ given in Chapter 4,*

$$\overrightarrow{\mu_N} := \sum_{\phi \in \mathcal{FPL}_+(\Lambda)} \overrightarrow{\Pi_b(\phi)}.$$

Then, $\overrightarrow{\mu_N} = \overrightarrow{\psi_\Lambda(1)}$, that is, $\overrightarrow{\mu_N}$ satisfies (†).

Proof. We have

$$\begin{aligned}\overrightarrow{\mu_N} = \Pi_+ \sum_{\phi \in \mathcal{FPL}_+(\Lambda)} \overrightarrow{\phi} &\overset{\text{(c)}}{=} \Pi_b \sum_{\phi \in \mathcal{FPL}_+(\Lambda)} \Theta\left(\overrightarrow{\phi}\right) \\ &\overset{\text{(b)}}{=} \Pi_b \sum_{\phi \in \mathcal{FPL}_b(\Lambda)} \overrightarrow{\phi} = \overrightarrow{\psi_\Lambda(1)},\end{aligned}$$

using (c) and (b) of the previous proposition, respectively. □

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Abstract

For several reasons, one is interested in finding the unique eigenvector of the Hamiltonian, a linear operator in CTL_n, the Cyclic Temperley-Lieb algebra of size n (an algebra acting on the set of link patterns of size n).

A. V. Razumov and Yu. G. Stroganov conjectured in 2001 a relation between these eigenvector, the so-called ground-state and enumerations of fully packed loop configurations of size n.

L. Cantini and A. Sportiello managed to prove this conjecture in 2010 and also proved a refined version corresponding to more general fully packed loop configurations.

In this thesis, we shall – after a motivation for the topic by means of percolation theory – prove this refined theorem (by a proof which is of combinatorical nature) in great detail and give examples.

Zusammenfassung

Aus meheren Grïden ist es von Interesse, den eindeutigen Eigenvektor des Hamilton-Operators, eines linearen Operators in CTL_n, der Cyclic Temperley-Lieb-Algebra der Ordnung n (eine Algebra, die auf der Menge der Link Patterns der Ordnung n operiert), zu finden.

A. V. Razumov und Yu. G. Stroganov stellten im Jahr 2001 die Vermutung auf, dass eine bestimmte Verbindung zwischen diesem Eigenvektor, dem sogenannten ground-state, und Abzählungen von fully packed loop configurations der Größe n bestehe.

L. Cantini and A. Sportiello gelang es 2010, diese Vermutung zu beweisen – sie bewiesen außerdem eine verfeinerte Version, die sich auf allgemeinere fully packed loop configurations bezieht.

In dieser Arbeit werden wir – nach einer Motivation des Themas durch das Gebiet der Perkolationstheorie – diese verfeinerte Aussage in großer Ausführlichkeit beweisen (der Beweis wird auf kombinatorische Weise geführt) und Beispiele geben.

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